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Álgebras de Banach generadas por operadores de Toeplitz con símbolos parabólicos cuasi-radiales cuasi-homogéneos

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Banach Algebras Generated by Toeplitz Operators With Parabolic Quasi-radial Quasi-homogeneous Symbols

A thesis presented by

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With love to my family.

Contents

Introduction

The study of Toeplitz operators is a vast and active area of research in mathematics that involves many branches of mathematics and even of mathematical physics. These operators owe their name to Otto von Toeplitz, who approached the following problem: when does the infinite matrix

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_{-1} & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

defines a bounded linear operator in $l_2(\mathbb{Z}_+)$? This problem seems to have been completely solved in 1954 by Hartman and Wintner: the above matrix defines a bounded linear operator if and only if there is a bounded function f in $\mathbb{T}$ such that $a_k = \hat{f}(k)$, where $\hat{f}$ is the discrete Fourier transform.

This result turns out to be rather simple when one adopts the "right" viewpoint. Indeed, consider the (one-dimensional) *Hardy space* H^2 of functions in $L_2(\mathbb{T}, d\sigma)$, where σ is the invariant measure of $\mathbb{T}$, such that $\hat{f}(n) = 0$ whenever n is negative. H is a closed space since it is the intersection of the kernels of the continuous functionals defined by $f \mapsto \hat{f}(n)$. Let $P\colon L_2(\mathbb{T}, d\sigma) \to H^2$ be the orthogonal projection (which, by the way, is called the Szegő projection) and let $\varphi \in L_\infty(\mathbb{T}, d\sigma)$. We define the Toeplitz operator with symbol φ, denoted T_φ, as the compression to H^2 of the multiplication by φ, that is,

$$T_\varphi f = P(\varphi f), \quad f \in H.$$

Then, in this context, the Toeplitz matrix above is just the matrix representation of the operator T_φ, where $a_n = \hat{\varphi}(n)$.

Taking this setting as a model, one can define and study Toeplitz operators in many other spaces. One usually considers spaces of "nice" functions in order to obtain "nice" structures. An example of this (which, by the way, arises naturally from quantum mechanics), is the class of analytic functions. In this thesis we work with to well-known spaces of analytic functions: the Bergman space and the Fock space (see sections 1.4 and 1.5). Throughout the years there has been a lot of research about the behavior of Toeplitz operators in these spaces.

A natural and difficult problem that arises in the study of Toeplitz operators on Bergman spaces is to determine under which conditions one obtains commutative operator algebras. As a matter of example and contrary to the case of the one-dimensional Hardy space H^2, there are several non-trivial commutative algebras generated by Toeplitz operators defined on the weighted Bergman space on the unit disc $\mathcal{A}_\lambda(\mathbb{D})$. A great step in this direction is the work of N. Vasilevski, S. Grudsky and A. Karapetyants, who proved at the beginning of the 21st century, that there are classes of symbols, geometrically defined, such that the C^*-algebras generated by Toeplitz operators with these symbols are commutative on every weighted Bergman space.

Later, N. Vasilevski, S. Grudsky and R. Quiroga Barranco, proved the converse in [5]. A equivalent reformulation (for more details see Preliminaries) of the main result in this paper reads as follows: assuming some natural condition on the "richness" of the symbol set, *the C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if there is a maximal commutative subgroup of the Möbius transformation such that the symbols of the Toeplitz operators are invariant under the action of this subgroup.*

R. Quiroga Barranco and N. Vasilevski extended this fact from the unit disk of $\mathbb{C}$ to the unit ball of $\mathbb{C}^n$ in [10] and [11]. They uased a classification of the maximal subgroups of automorphisms of $\mathbb{B}^n$ and proved that, given one of these subgroups, the C^*-algebra generated by Toeplitz operators with symbols invariant under the action of it is commutative on every weighted Bergman space. The maximal commutative subgroups of automorphisms are the *quasi-elliptic group*, the *quasi-parabolic group*, the *quasi-hyperbolic group*, the *nilpotent group* and the *quasi-nilpotent group*.

As N. Vasilevski wrote in [15], it was firmly expected that the above algebras exhaust all possible algebras of Toeplitz operators on the unit ball which are commutative on each weighted Bergman space. However, as it usually happens when one generalizes a problem arisen in a one-dimensional setting, the multidimensional case turned out to be much more interesting. In fact, inspired by [20], N. Vasilevski presented in [15] a new class of symbols whose induced Toeplitz operators generate commutative operator algebras on each weighted Bergman space. This idea was later applied in [18], where he presented another classes of symbols subordinated to one of the model classes of the above maximal commutative subgroups (except for the nilpotent group) such that the corresponding operator algebras are commutative. All these algebras are Banach. They collapse into C^*-algebras when the dimension is the "trivial" one and, otherwise, the corresponding extended C^*-algebra is non-commutative.

Being commutative (non-C^*) Banach algebras, they represent a rich and interesting mathematical object. Natural problems regarding the structure of these algebras arise, such as to determine their respective maximal ideal spaces, Gelfand transforms, radicals, etc. In this regard, the algebra which has been best understood so far, thanks to a series of papers of N. Vasilevski and W. Bauer ([1], [2], [3]), is the one subordinated to the quasi-elliptic group.

This thesis is a further step in this direction. Following the principal ideas that were used for the elliptic case, we approach the task of describing the commutative Banach algebra subordinated to the quasi-parabolic group for the lowest non-trivial dimension $n = 3$. As in [1], these results are expected to reveal some important features which would be useful to understand the higher dimensional case $n > 3$.

Throughout the thesis we will denote this algebra by $\mathcal{T}(\lambda)$, where λ stands for the weight parameter used to define the Bergman space $\mathcal{A}_\lambda(D_3)$ (for more details, see Preliminaries). One of the results states that $\mathcal{T}(\lambda)$ is generated by two subalgebras: $\mathcal{T}_{qr}$ and $\mathcal{T}_\phi$, where $\mathcal{T}_{qr}$ is the algebra generated by Toeplitz operators with parabolic (2)-quasi-radial symbols and $\mathcal{T}_\phi$ is the algebra generated by the single operator T_ϕ, where ϕ is the simplest quasi-homogeneous function given by

$$\phi(z', z_n) = \frac{z_1 \overline{z_2}}{|z_1|^2 + |z_2|^2},$$

where $(z', z_n) \in D_3$ and $z' = (z_1, z_2)$.

The thesis is divided into four chapters, the contents of which can be described as follows. The first chapter presents the necessary tools so that this work is self-contained. We introduce some well-known results about commutative Banach algebras theory and some others regarding Toeplitz operators. In particular, we present the main results of [1], inasmuch as some results of the parabolic case follow (although not directly) from these ones.

In Chapter 2 we represent the Bergman space as a direct integral of weighted Fock spaces. This representation will be crucial to prove the main results of the thesis. As a matter of example, we can mention the generalized Berezin transform, which, inspired by [1], was used to characterize some elements of the maximal ideal space of $\mathcal{T}(\lambda)$. There is not a natural way of extrapolating the methods of [1] to the present work and the direct integral of Fock spaces turned out to be very useful.

The main result in this chapter is the existence of an operator

$$R\colon L_2(D_n, d\mu_\lambda) \to \int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1}) d\xi,$$

(where D_n stands for the Siegel domain) onto a direct integral of Fock spaces (see Preliminaries for the notation) that maps $L_2(D_n, d\mu_\lambda)$ onto $\int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1}) d\xi$, such that the restriction

$$R|_{\mathcal{A}_\lambda^2(D_n)}\colon \mathcal{A}_\lambda^2(D_n) \to \int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1}) d\xi$$

is an isometric isomorphism and such that for every essentially bounded function a on D_n that depends only on z' and $\operatorname{Im} z_n$, we have

$$R T_a^{(\lambda)} R^* = \int_{\mathbb{R}_+}^{\oplus} T_{\widetilde{a}_\xi}^{(\xi)} d\xi,$$

where

$$\widetilde{a}_\xi(z') = \int_{\mathbb{R}_+} \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} e^{-2\xi v} v^\lambda a(z', v - |z'|^2) dv.$$

Note that each function belonging to one of the classes of symbols defined on the Siegel domain and invariant under the action of one of the above maximal abelian subgroup depends only on z' and $\operatorname{Im} z_n$. Therefore we can apply the results of Chapter 2 and thus, as a corollary of the above representation, we diagonalize the Toeplitz operators defined by these symbols. That is, we give an alternative proof of the main results of [10].

Chapter 4 contains the main results of the thesis. We study the structure of the commutative Banach algebra $\mathcal{T}(\lambda)$. As we already mentioned, we reduce the set of generators and study separately the algebras $\mathcal{T}_{qr}$ and $\mathcal{T}_\phi$.

In the first two sections we analyse the algebra $\mathcal{T}_{qr}$. We show that its maximal ideal space is some compactification of $\mathbb{Z}_+ \times \mathbb{R}_+$ and we analyse the behavior of the points "at infinity".

In Sections 3, 4 and 5 we study the algebra $\mathcal{T}_\phi$ and analyse how this algebra and the previous one act together. Among other things, we prove that, as in the elliptic case, the algebra $\mathcal{T}(\lambda)$ is generated by Toeplitz operators with parabolic quasi-radial quasi-homogeneous symbols and the single operator T_ϕ.

In Sections 7, 8 and 9 we present some technical results that are necessary to characterize the maximal ideal space of $\mathcal{T}(\lambda)$. We use the direct integral representation of the Bergman space and Dirac sequences to show how to extend some multiplicative linear functionals.

The main results are presented in Section 10. Theorem 4.10.4 characterizes the maximal ideal space of $\mathcal{T}(\lambda)$. This theorem reads as follows:

The compact set $M(\mathcal{T}(\lambda))$ of maximal ideals of the algebra $\mathcal{T}(\lambda)$ has the form

$$M(\mathcal{T}(\lambda)) = (\mathbb{Z}_+ \times \mathbb{R}_+ \times \{0\}) \cup \left(M_{\infty,\mathbb{R}_+}(\lambda) \times \overline{D}(0, \frac{1}{2}) \right).$$

1. The Gelfand image of the algebra $\mathcal{T}(\lambda)$ is isomorphic to $\mathcal{T}(\lambda)/\operatorname{Rad}\mathcal{T}(\lambda)$ and coincides with the algebra

$$\mathcal{A}_{qr} \cup [C(M_{\infty,\mathbb{R}_+})\hat{\otimes}_e C_\alpha(\overline{D}(0, \frac{1}{2}))],$$

which is identified with the set of all pairs

$$(\gamma, f) \in \mathcal{A}_{qr} \times [C(M_{\infty,\mathbb{R}_+})\hat{\otimes}_e C_\alpha(\overline{D}(0, \frac{1}{2}))]$$

satisfying the following compatibility condition $\gamma(\mu) = f(\mu, 0)$, for all $\mu \in M_{\infty,\mathbb{R}_+}(\lambda)$.

2. The Gelfand transform is generated by the following mapping:

$$\sum_{p=0}^{m} T_{a_p} T_\phi^p \mapsto \begin{cases} \gamma_{a_0}(k, \xi), & \text{if } (k, \xi, 0) \in \mathbb{Z} \times \mathbb{R}_+ \times \{0\}, \\ \sum_{p=0}^{m} \gamma_{a_p}(\mu)\zeta^p, & \text{if } (\mu, \zeta) \in M_{\infty,\mathbb{R}_+} \times \overline{D}(0, \frac{1}{2}). \end{cases}$$

$M_{\infty,\mathbb{R}_+}$ represents those points "at infinity" of the maximal ideal space of $\mathcal{T}_{qr}$ that can be reached by nets of the form $\{(k_\alpha, \xi_\alpha)\}$ such that $k_\alpha \to \infty$.

Finally, we use these results to prove in Section 11 that $\mathcal{T}(\lambda)$ is a inverse-closed algebra and, therefore, that the spectrum of an operator in $\mathcal{T}(\lambda)$ as an element of this Banach algebra coincides with its spectrum as an element of $\mathscr{L}(\mathcal{A}_\lambda(D_3))$.

Chapter 1

Preliminaries

In this Chapter we introduce those objects and results which will be used later. Some of them are well-known and can be consulted in many books and papers

For the theory of commutative Banach algebras we used [6], [4] and [8]. A general reference for Dirac sequences and their basic properties, which are not difficult to proof, could be [7].

For Toeplitz operators on Bergman and Fock spaces there are many references. A general treatment can be consulted in [13], [21] and [22]. The study of C^*-algebras generated by Toeplitz operators commutative on each weighted Bergman space and maximal subgroups of automorphisms of the unit ball is developed mainly in [5], [10] and [11]. The commutative Banach algebras subordinated to these maximal subgroups of automporhisms are introduced and, for the elliptic case, deeply analysed in [14], [15], [16], [17], [1], [2], [3] and [18].

1.1 Commutative Banach Algebras

We mention without proof some very well-known results concerning commutative Banach algebras. All algebras will be assumed to be unital.

Let $\mathcal{A}$ be a commutative Banach algebra. The set of multiplicative linear functionals of $\mathcal{A}$, denoted by $\mathcal{M}(\mathcal{A})$ is called the *maximal ideal space* of $\mathcal{A}$. By Banach-Alaoglu theorem, $\mathcal{M}(\mathcal{A})$ is a compact Hausdorff space. The following result justifies its name.

1.1.1 Proposition. *Let $\mathcal{A}$ be a commutative Banach algebra. Then there is a bijection between the maximal ideal space of $\mathcal{A}$ and the set of maximal ideals of $\mathcal{A}$ given by*

$$\varphi \mapsto \ker \varphi, \quad \varphi \in \mathcal{M}(\mathcal{A}).$$

1.1.2 Definition. Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$.

- The *spectrum* of a (with respect to $\mathcal{A}$) is the set

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} \colon \lambda - a \text{ is not invertible}\}.$$

- The *spectral radius* of an element $a \in \mathcal{A}$ is the (always finite) number

$$r(a) = \sup\{|\lambda| \colon \lambda \in \sigma_{\mathcal{A}}(a)\}.$$

We also recall the Gelfand-Beurling norm for the spectral radius:

1.1.3 Proposition. *Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. Then*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

When $\mathcal{A}$ is a C^*-algebra of bounded operators on some Hilbert space, $\sigma_{\mathcal{A}}(T)$ is just the spectrum of the operator $T \in \mathcal{A}$. In this case we will write $\operatorname{sp}(T) = \sigma_{\mathcal{A}}(T)$.

1.1.4 Definition. Given a commutative Banach algebra $\mathcal{A}$, the function

$$\Gamma \colon \mathcal{A} \to C(\mathcal{M}(\mathcal{A}))$$

given by

$$\Gamma(a)(\phi) = \phi(a), \quad \phi \in \mathcal{M}(\mathcal{A}),$$

is a homomorphism called the *Gelfand map*. For an element $a \in \mathcal{A}$, the function $\Gamma(a)$ is called the *Gelfand transform* of a.

1.1.5 Proposition. *Let $\mathcal{A}$ and Γ as above. Then the following properties hold:*

1. $\sigma_{\mathcal{A}}(a) = \Gamma(a)(\mathcal{M}(\mathcal{A}))$*, for every* $a \in \mathcal{A}$.
2. Γ *is norm-decreasing.*
3. Γ *is isometric if and only if* $\|a\|^2 = \|a^2\|$*, for all* $a \in \mathcal{A}$

We note that the Gelfand map is not necessarily injective or surjective. However, it completely characterizes (unital) commutative C^*-algebras, as the following well-known result shows:

1.1.6 Proposition. *Let $\mathcal{A}$ be a commutative C^*-algebra. Then the Gelfand map*

$$\Gamma\colon \mathcal{A} \to C(\mathcal{M}(\mathcal{A}))$$

is a $*$-isomorphism.

1.1.7 Definition. Let $\mathcal{A}$ be a commutative Banach algebra. The *radical* of $\mathcal{A}$, $\mathrm{Rad}(\mathcal{A})$, is the intersection of all maximal ideals. That is,

$$\mathrm{Rad}(\mathcal{A}) = \bigcap \{\ker \varphi \colon \varphi \in \mathcal{M}(\mathcal{A})\}.$$

The algebra $\mathcal{A}$ is called *semisimple* if $\mathrm{Rad}(\mathcal{A}) = \{0\}$ and *radial* if $\mathrm{Rad}(\mathcal{A}) = \mathcal{A}$.

1.1.8 Corollary. *Let $\mathcal{A}$ be a Banach algebra. Then $a \in \mathrm{Rad}(\mathcal{A})$ if and only if a is* topologically nilpotent*, that is, if and only if*

$$\lim_{n\to\infty} \|a^n\|^{1/n} = 0.$$

1.2 Compactifications

Given a topological space S, we denote by $CB(S)$ the algebra of bounded continuous complex-valued functions on S. The algebra $CB(S)$ turns out to be a commutative C^*-algebra with the norm

$$\|f\|_S = \sup_{s\in S} |f(s)|$$

and the involution

$$f^* = \overline{f}.$$

By Proposition 1.1.6, $CB(S)$ and any of its C^*-subalgebras is of the form $C(S')$, for some compact Hausdorff space S'. We will see that this space is in fact related to S.

1.2.1 Definition. A *compactification* of the topological space S is a compact Hausdorff space X and a continuous one-to-one function τ of S onto a dense subset $\tau(S)$ of X.

We often identify $s \in S$ with $\tau(s) \in X$. Under this identification, every compactification X of S determines, by restricting the functions of $C(X)$ to S, a closed separating self-adjoint subalgebra $\mathcal{A}$ of $CB(S)$ which contains the constants.

The converse also holds. That is, if $\mathcal{A}$ is a C^*-subalgebra of $CB(S)$, then the maximal ideal space of $\mathcal{A}$, $\mathcal{M}(\mathcal{A})$, is a compactification of S.

We have thus the following proposition:

1.2.2 Proposition. *Let S be a topological space. There is a bijective correspondece between compactifications X of S and closed separating self-adjoint subalgebras A of $CB(S)$ which contain the constants. The algebra $\mathcal{A}$ associated with the compactification X consists of the functions in $CB(S)$ which extend continuously to X. The compactification X associatd with $\mathcal{A}$ is the maximal ideal space of $\mathcal{A}$.*

1.3 Dirac Sequences

We introduce a useful tool that we will need later.

1.3.1 Definition. A *Dirac sequence* on $\mathbb{R}^n$ is a sequence of real-valued continuous functions $(\varphi_k)_{k=1}^{\infty}$ satisfying the following properties:

1. $\varphi \geq 0$ for all $k \geq 1$.
2. For all $k \geq 1$ we have
$$\int \varphi_k(x)dx = 1.$$
3. Given $\varepsilon, \delta > 0$ there exists k_0 such that
$$\int_{|x|\geq\delta} \varphi_k(x)dx < \varepsilon,$$
for all $k \geq k_0$.

A Dirac sequence can be used to approximate a function as the following results shows.

1.3.2 Proposition. *Let f be a bounded measurable function on $\mathbb{R}^n$. Let K be a compact set on which f is continuous. Let $(\varphi_k)_{k=1}^{\infty}$ be a Dirac sequence. Then $\varphi_k * f$ converges uniformly to f on A.*

Here, $g * f$ denotes the usual *convolution* given by

$$g * f(x) = \int_{\mathbb{R}} g(t)f(x-t)dt.$$

Among other properties, this operation is bilinear and commutative.

Note that, in particular, if f is continuous on a point $x \in \mathbb{R}^n$, then

$$\lim_{k\to\infty} f * \varphi_k(x) = f(x).$$

1.3.3 Proposition. *Let* $(\varphi_k)_{k=1}^{\infty}$ *a Dirac sequence. Let* $1 \le p < \infty$ *and* $f \in L_p(\mathbb{R})$. *Then*

$$\|f * \varphi_k - f\|_p \to 0, \quad k \to \infty.$$

1.4 Bergman Spaces

Now we introduce the spaces we will be working with. Let $\mathbb{B}^n$ be the unit ball of $\mathbb{C}^n$ and fix $\lambda > -1$. Consider the weighted Lebesgue measure $d\mu'_\lambda$ defined by

$$d\mu'_\lambda(z) = c_\lambda(1 - |z|^2)^\lambda d\nu(z),$$

where

$$c_\lambda = \frac{\Gamma(n + \lambda + 1)}{n!\Gamma(\lambda + 1)}$$

is a normalizing constant so that $d\mu'$ is a probability measure on $\mathbb{B}^n$.

The weighted Bergman space, denoted by $\mathcal{A}^2_\lambda(\mathbb{B}^n)$, is the subspace of $L_2(\mathbb{B}^n, d\mu'_\lambda)$ of holomorphic functions. This space turns out to be a reproducing kernel Hilbert space. Its kernel, called the weighted *Bergman kernel*, is given by

$$K_{\mathbb{B}^n,\lambda}(z, \zeta) = \frac{1}{(1 - z \cdot \overline{\zeta})^{n+\lambda+1}}$$

and the Bergman projection $B_{\mathbb{B}^n,\lambda}$ of $L_2(\mathbb{B}^n, \mu'_\lambda)$ onto $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ has the form

$$(B_{\mathbb{B}^n,\lambda} f(z)) = \int_{\mathbb{B}^n} f(\zeta) \frac{(1 - |\zeta|)^\lambda}{(1 - z \cdot \overline{\zeta})^{n+\lambda+1}} c_\lambda dv(\zeta).$$

Sometimes it is easier to work with the unbounded realisation of the unit ball, that is, with the Siegel domain D_n. This domain is defined as follows:

$$D_n = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \colon \operatorname{Im} z_n - |z'|^2 > 0\}.$$

We note that, for $n = 1$, D_n is just the upper half-plane, which is the biholomorphic image, by means of the (inverse) Cayley transform, of the unit disk of $\mathbb{C}$. The Cayley transform for the Siegel domain D_n is the function ω given by $\zeta = \omega(z)$, where

$$\zeta_k = i\frac{z_k}{1+z_n}, \quad k = 1, \dots, n-1,$$
$$\zeta_n = i\frac{1-z_n}{1+z_n}.$$

It is well known (and easy to check directly) that ω maps biholomorphically $\mathbb{B}^n$ to D_n.

The inverse transform $z = \omega^{-1}(\zeta)$ is given by

$$z_k = -\frac{2i\zeta_k}{1-i\zeta_n}, \quad k = 1, \dots, n-1, \tag{1.1}$$
$$z_n = \frac{1+i\zeta_n}{1-i\zeta_n}. \tag{1.2}$$

Denote by $d\nu(z) = dx_1 dx_2 \cdots dx_n dy_n$, where $z_m = x_m + iy_m$, $m = 1, \dots, n$, the standard Lebesgue measure in $\mathbb{C}^n$, and introduce the following one-parameter family of weighted measures

$$d\mu_\lambda(z) = \frac{c_\lambda}{4}(\operatorname{Im} z_n - |z'|^2)^\lambda d\nu(z)$$

1.5 Fock Spaces

We introduce now the weighted Fock space on $\mathbb{C}^n$. Given a (weight) parameter $\alpha \in \mathbb{R}_+$ consider $L_2(\mathbb{C}^n, dv_\alpha)$, where

$$dv_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} dv(z), \quad z \in \mathbb{C}^n.$$

The Fock space $F^2_\alpha(\mathbb{C}^n)$ is the subspace of $L_2(\mathbb{C}^n, dv_\alpha)$ which consists of analytic functions. As the Bergman space, the Fock space turns out to be a reproducing kernel Hilbert space. Its reproducing kernel is given by

$$K_w(z) = e^{\alpha z \cdot \overline{w}}.$$

The orthogonal projection $P_\alpha \colon L_2(\mathbb{C}^n, dv_\alpha) \to F^2_\alpha(\mathbb{C}^n)$ is called the *Bargmann projection* and, using the reprodruncing kernel, it is easy to see that P_α is given by the integral operator

$$P_\alpha f(z) = \int_{\mathbb{C}^n} f(w) e^{\alpha z \cdot \overline{w}} dv_\alpha(w).$$

The monomials $e_p(z) = \frac{z^p}{\sqrt{p!}}$, $p \in \mathbb{Z}^n_+$, form an orthonormal basis for the Fock space.

1.6 Toeplitz Operators

We define Toeplitz operators, as always, as the compression of a multiplication operator. That is, let H be some Hilbert space $L_2(X,\mu)$, (who X and μ are is not really important at the moment), let K be a closed subspace of H and $P\colon H \to K$ the orthogonal projection (in this thesis K will be either a Bergman space or a Fock space and P its corresponding Bergman or Bargmann projection). For a bounded measurable function φ, define the Toeplitz operator T_φ by the rule

$$Tf = P(\varphi f), \quad f \in K.$$

The function φ is called the *symbol* of the Toeplitz opertor T_φ.

One can easily deduce from the definition some elementary properties:

1. The application $\varphi \mapsto T_\varphi$ is linear and preserves involutions, i. e., $T_{\overline{\varphi}} = T_\varphi^*$. (This application is, in general, not multiplicative).
2. $\|T_\varphi\| \leq \|\varphi\|_\infty$.

One general task in the study of Toeplitz operators is to study the operator algebras which they generate in terms of their symbols. This is quite a difficult problem for arbitrary symbols and one usually restricts attention to special classes of symbols.

1.7 Commutative C^*-Algebras Generated by Toeplitz Operators

A natural question that arises in the study of Toeplitz operators on Bergman spaces is when do Toeplitz operators commute or under which conditions do we obtain commutative operator algebras of Toeplitz operators. Indeed, commutative algebras are more manageable, inasmuch as we have the Gelfand theory; otherwise one cannot say too much about an operator algebra. Moreover, as we already mentioned, the commutativity of algebras of Toeplitz operators on Bergman spaces, contrary to the case of the one-dimensional Hardy space, is not trivial.

In the context of Bergman spaces on the unit disc $\mathbb{D}$, it was discovered by S. Grudsky, A. Karapetyants and N. Vasilevski that there are special classes of symbols such that the respective C^*-algebras generated by Toeplitz operators are commutative on every weighted Bergman space. These classes of symbols can be defined in terms of the geometric properties of the unit disc.

S. Grudsky, R. Quiroga Barranco and N. Vasilevski proved in [5] that, discarding the trivial case of the C^*-algebra generated by the identity and a self-adjoint Toeplitz operator, the above classes are the only possible sets of symbols which might generate the commutative C^*-algebras of Toeplitz operators on each weighted Bergman space. The main result states that, assuming only some natural conditions on the "richness" of the symbol set, *the C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if there is a pencil of hyperbolic geodesics such that the symbols of the Toeplitz operators are constant on the cycles of this pencil.*

This result admits the following equivalent reformulation: assuming some natural condition on the "richness" of the symbol set, *the C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if there is a maximal commutative subgroup of the Möbius transformation such that the symbols of the Toeplitz operators are invariant under the action of this subgroup.*

Using this reformulation, R. Quiroga Barranco and N. Vasilevski extended this result from the unit disc of $\mathbb{C}$ to the unit ball of $\mathbb{C}^n$. They proved in [10] that each maximal abelian subgroup of automorphisms of the unit ball induces a commutative C^*-algebra of Toeplitz operators on each weighted Bergman space. Moreover, they explicitly constructed, for each case, a unitary operator that diagonalizes the corresponding Toeplitz operator and gave explicit expressions for the corresponding eigenvalue functions.

The following list classifies five essentially different types of commutative subgroups of biholomorphisms of the unit ball $\mathbb{B}^n$, or its unbounded realisation, the Siegel domain D_n. In the second part of the paper, [11], R. Quiroga Barranco and N. Vasilevski proved that these subgroups are maximal commutative subgroups of biholomorphisms and that each maximal commutative subgroup of biholomorphisms is conjugate to one from the list, while neither two from the list are conjugate.

- **Quasi-elliptic** group of biholomorphisms of the unit ball $\mathbb{B}^n$ if isomorphic to $\mathbb{T}^n$ with the following action:

  $$t\colon z = (z_1, \ldots, z_n) \in \mathbb{B}^n \longmapsto tz = (t_1 z_1, \ldots, t_n z_n) \in \mathbb{B}^n,$$

  for each $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$.

- **Quasi-parabolic** group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$ with the following group action:

  $$(t, h) \colon (z', z_n) \in D_n \longmapsto (tz', z_n + h) \in D_n,$$

  for each $(t, h) \in \mathbb{T}^{n-1} \times \mathbb{R}$.

- **Quasi-hyperbolic** group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}_+$ with the following group action:

  $$(t, r) \colon (z', z_n) \in D_n \longmapsto (r^{1/2}tz', rz_n) \in D_n,$$

  for each $(t, r) \in \mathbb{T}^{n-1} \times \mathbb{R}_+$.

- **Nilpotent** group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}$

  $$(b, h) \colon (z', z_n) \in D_n \longmapsto (z' + b, z_n + h + 2iz' \cdot b + i|b|^2) \in D_n,$$

  for each $(b, h) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

- **Quasi-nilpotent** group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$, $0 < k < n - 1$, with the following group action:

  $$(t, b, h) \colon (z', z'', z_n) \in D_n \longmapsto (tz', z'' + b, z_n + h + 2iz'' \cdot b + i|b|^2) \in D_n,$$

  for each $(t, b, h) \in \mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$.

In Chapter 3 we give another proof of this diagonalization for the quasi-parabolic group, the hyperbolic group, the nilpotent group and the quasi-nilpotent group. Since we do not really use these eigenvalue functions until that chapter, we do not present them here.

1.8 Commutative Banach Algebras Generated by Toeplitz Operators

Surprisingly, it turned out that, for $n > 1$ there exist many other, not geometrically defined, classes of symbols which generate commutative Toeplitz operator algebras on each weighted Bergman space. These classes of symbols were always subordinated to one of the model classes of the maximal commutative subgroups we cited in the previous section (with the exception of the nilpotent subgroup). The corresponding commutative operator algebras were Banach, and being extended to C^* they became non-commutative.

1.8.1 Elliptic case

W. Bauer and N. Vasilevski studied in a series of papers the case of the commutative Banach algebra generated by Toeplitz operators with quasi-radial quasi-homogeneous symbols (i.e. an algebra subordinated to the quasi-elliptic group). They started from the lowest dimensional case $n = 2$ in [1] and studied the general dimension case in [2] and [3]. The general aim was to develop the Gelfand theory of these algebras.

For a general dimension n, the definition of this algebra is similar to the corresponding definition in the parabolic case, which will be introduced in the next section. Therefore, we present here only the particular case $n = 2$

For $n = 2$, W. Bauer and N. Vasilevski were able to give explicit descriptions of the maximal ideal space, the Gelfand transform and the radical. The corresponding (unique) commutative Toeplitz operator algebra is Banach (not C^*!). Since this algebra only appears explicitly in this section we keep the notation of [1] and denote this algebra by $\mathcal{T}(\lambda)$. For the rest of the thesis we will use this notation just for the parabolic case.

The algebra $\mathcal{T}(\lambda)$ can be described as follows: Let $H := \mathcal{A}^2_\lambda(\mathbb{B}^2)$ be the weighted Bergman space over $\mathbb{B}^2$ with parameter $\lambda > -1$, and write $\mathcal{T}_{rad}(\lambda)$ for the commutative C^*-subalgebra of $\mathscr{L}(H)$ generated by all Toeplitz operators T_a with radial bounded measurable symbols a on $\mathbb{B}^2$ (i.e. $a(z) = a(|z|)$).

For a bounded measurable function $a(r)$ we have

$$T_a z^\alpha = \gamma_{a,\lambda}(|\alpha|) z^\alpha, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_+,$$

where, using the usual multi-index notation, $|\alpha| = \alpha_1 + \alpha_2$ and

$$\gamma_{a,\lambda}(|\alpha|) = \frac{\Gamma(|\alpha| + \lambda + 3)}{\Gamma(\lambda + 1)\Gamma(|\alpha| + 2)} \int_0^1 a(\sqrt{r})(1 - r)^\lambda r^{|\alpha|+1} dr.$$

We denote by D_γ the multiplication operator with symbol γ. The C^*-algebra $\mathcal{T}_{rad}(\lambda)$ can be identified with a certain subalgebra of slowly oscillating sequences. We denote it by $SO(\lambda)$.

Let $M(\mathcal{T}_{rad}(\lambda))$ be the compact set of maximal ideals of $\mathcal{T}_{rad}(\lambda)$. By Proposition 1.2.2, $M(\mathcal{T}_{rad}(\lambda))$ is some compactification of $\mathbb{Z}_+$. Therefore we can decompose

$$M(\mathcal{T}_{rad}(\lambda)) = \mathbb{Z}_+ \cup M_\infty(\lambda).$$

Further, we denote by $\mathcal{T}_\phi$ the unital Banach algebra generated by a single generator T_ϕ, where ϕ is the "simplest" quasi-homogeneous symbol on $\mathbb{B}^2$ given by:

$$\phi(z) = \frac{z_1\overline{z_2}}{|z_1|^2+|z_2|^2} = \zeta^{(1,0)}\overline{\zeta}^{(0,1)},$$

where $\zeta = z/|z| \in \mathbb{S}^2$.

The operator T_ϕ acts on the basic elements (normalized monomials) of $\mathcal{A}^2_\lambda(\mathbb{B}^2)$,

$$e_\alpha(z) = \sqrt{\frac{\Gamma(|\alpha|+\lambda+4)}{\alpha!\Gamma(\lambda+3)}}z^\alpha, \quad \alpha \in \mathbb{Z}^2_+,$$

by the following rule:

$$T_\phi e_\alpha = \begin{cases} \dfrac{\sqrt{(\alpha_1+1)\alpha_2}}{2+|\alpha|}e_{(\alpha_1+1,\alpha_2-1)}, & \alpha_2 \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

More generally, let $\phi_p = \phi^p$. Then

$$T_{\phi_p} e_\alpha = \begin{cases} \dfrac{\alpha_2(\alpha_2-1)\cdots(\alpha_2-p+1)}{(p+1+|\alpha|)\cdots(2+|\alpha|)}e_{(\alpha_1+p,\alpha_2-p)}, & \alpha_2 \geq p, \\ 0, & \text{otherwise.} \end{cases}$$

By Corollary 4.3 from [15], for any bounded measurable function $a(r)$ and $p \in \mathbb{Z}_+$ we have

$$T_{a\phi_p} = T_{\phi_p}T_a = T_aT_{\phi_p}.$$

As a consequence, the algebra $\mathcal{T}(\lambda)$, generated by Toeplitz operators with radial symbols and the Toeplitz operators T_{ϕ_p}, is a commutative Banach algebra and is generated by the operators T_a, with $a \in L_\infty[0,1)$ and T_{ϕ_p}, $p \in \mathbb{Z}_+$. Moreover, Corollary 3.5 from [1] states that $\mathcal{T}(\lambda)$ is generated just by Toeplitz operators T_a with bounded measurable radial symbols $a(r)$ and the single Toeplitz operator T_ϕ.

We mention two of the main results of [1]:

1.8.1 Theorem ([1, Theorem 3.6]). *The Banach algebra $\mathcal{T}_\phi$ is isomorphic via the Gelfand transform to the algebra $C_\alpha(\overline{D}(0,\frac{1}{2}))$, which consists of all functions analytic in $D(0,\frac{1}{2})$ and continuous on $\overline{D}(0,\frac{1}{2})$.*

1.8.2 Theorem ([1, Theorem 5.4]). *The compact set $M(\mathcal{T}(\lambda))$ of maximal ideals of the algebra $\mathcal{T}(\lambda)$ has the form*

$$M(\mathcal{T}(\lambda)) = \mathbb{Z}_+ \times \{0\} \cup M_\infty(\lambda) \times \overline{D}(0,\frac{1}{2}).$$

(i) The Gelfand image of the algebra $\mathcal{T}(\lambda)$ is isomorphic to $\mathcal{T}(\lambda)/\operatorname{Rad}\mathbb{T}(\lambda)$ and coinides with the algebra

$$SO(\lambda) \cup \left[C(M_\infty(\lambda))\hat{\otimes}_\epsilon C_\alpha(\overline{D}(0,\frac{1}{2}))\right]$$

satisfying the following compatibility condition $\gamma(\mu) = f(\mu,0)$, for all $\mu \in M_\infty(\lambda)$ Here $\hat{\otimes}_\epsilon$ denotes the injective tensor product, and we identify $\gamma(\mu)$ with the value of the functional $\mu \in M_\infty(\lambda)$ on the element $\gamma \in SO(\lambda)$.

(ii) The Gelfand transform is generated by the following mapping of elements of $\mathcal{D}(\lambda)$, the dense set of all finite sums of finite products of elements of $\mathcal{T}(\lambda =$

$$\sum_{j=0}^{n} D_{\gamma_j} T_\phi^j \longmapsto \begin{cases} \gamma_0(k), & \text{if } (k,0) \in \mathbb{Z}_+ \times \{0\}, \\ \sum_{j=0}^{n} \gamma_j(\mu)\zeta^j, & \text{if } (\mu,\zeta) \in M_\infty(\lambda) \times \overline{D}(0,\frac{1}{2}). \end{cases}$$

1.8.2 Parabolic case

We first recall some facts from [14] which are also consequence from the results of [10]. Fix a weight parameter $\lambda > -1$.

There is a surjective operator

$$R\colon L_2(D_n,\widetilde{\mu}) \to l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$$

such that the restriction onto the Bergman space $\mathcal{A}_\lambda^2(D_n)$

$$R|_{\mathcal{A}_\lambda^2} : \mathcal{A}_\lambda^2(D_n) \to l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$$

is an isometric isomorphism. The adjoint

$$R^*\colon l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \to L_2(D_n, \widetilde{\mu})$$

is an isometric isomorphism.

Furthermore,

$$RR^* = I\colon l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \to l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$$

$$R^*R = B_{D_n,\lambda}\colon L_2(D_n, \widetilde{\mu}) \to \mathcal{A}_\lambda^2(D_n),$$

where $B_{D_n,\lambda}$ is the Bergman projection on the Siegel domain D_n.

The general aim of this thesis is to study the parabolic case for the simplest case $n = 3$. We describe how these algebras are constructed for a general $n \geq 2$.

In what follow we will use the standard multi-index notation. That is, for a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1}$:

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}, \\ \alpha! &= \alpha_1!\alpha_2! \cdots \alpha_{n-1}!, \\ z^\alpha &= z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_{n-1}^{\alpha_{n-1}}. \end{aligned}$$

Two multi-indices α and β are called orthogonal,$\alpha \perp \beta$, if

$$\alpha \cdot \beta = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_{n-1}\beta_{n-1} = 0.$$

Let $k = (k_1, \dots, k_m)$ be a tuple of positive integers with $k_1 + \cdots + k_m = n - 1$. We rearrange the $n-1$ coordinates of $z \in \mathbb{C}^{n-1}$ in m groups, each one of which has k_j, $j = 1, \dots, m$, entries and introduce the notation

$$z_{(1)} = (z_{1,1}, ..., z_{1,k_1}), \quad z_{(2)} = (z_{2,1}, ..., z_{2,k_2}), \quad \dots, \quad z_{(m)} = (z_{m,1}, ..., z_{m,k_m}).$$

We represent then each $z_{(j)} = (z_{j,1}, \dots, z_{j,k_j}) \in \mathbb{C}^{k_j}$ in the form

$$z_{(j)} = r_j \zeta_{(j)}$$

, where $r_j = \sqrt{|z_{j,1}|^2 + \cdots + |z_{j,k_j}|^2}$ and $\zeta_{(j)} \in \mathbb{S}^{k_j} := \partial\mathbb{B}^k$.

A bounded measurable function $a = a(w)$, $w \in D_n$, will be called *parabolic k-quasi-radial* if it depends only on $r_1, \ldots, r_m$ and $y_n = \operatorname{Im} w_n$. We denote by $\mathcal{R}_k$ the set of k-quasi-radial functions.

We will always assume first, that $k_1 \leq k_2 \leq \cdots \leq k_m$, and second, that

$$z_{1,1} = z_1, z_{1,2} = z_2, \ldots, z_{1,k_1} = z_{k_1}, z_{2,1} = z_{k_1+1}, ...,$$
$$z_{2,k_2} = z_{-}k_1 + k_2, \ldots, z_{m,k_m} = z_{n-1}.$$

We use the representations $z_{(j)} = r_j \zeta_{(j)}$, $j = 1, \ldots, m$, to define the vector

$$\zeta = (\zeta_{(1)}, \zeta_{(2)}, ..., \zeta_{(m)}) \in \mathbb{S}^{k_1} \times \mathbb{S}^{k_2} \times \cdots \times \mathbb{S}^{k_m}.$$

Let $p, q \in \mathbb{Z}_+^{n-1}$ be a pair of orthogonal ($p \perp q$) multi-indices. A function $\varphi \in L_\infty(D_n)$ is called *parabolic quasi-homogeneous* (or *parabolic k-quasi-homogeneous*) function if it has the form

$$\varphi(z) = \varphi(z_{(1)}, z_{(2)}, \ldots, z_{(m)}) = a(r_1, r_2, \ldots, r_m, y_n)\zeta^p \overline{\zeta}^q, \tag{1.3}$$

where $a(r_1, r_2, \ldots, r_m, y_n) \in \mathcal{R}_k$. We will call the pair (p, q) the quasi-homogeneous degree of the parabolic k-quasi-homogeneous function
$a(r_1, r_2, \ldots, r_m, y_n)\zeta^p \overline{\zeta}^q$.

As it was shown in [14], we can construct commutative Banach algebras generated by Toeplitz operators with parabolic quasi-radial quasi-homogeneous symbols satisfying certain conditions as follows.

To avoid the repetition of the unitary equivalent algebras and to simplify the classification of the (non-unitary equivalent) algebras we rearrange the variables z_l and correspondingly the components of multi-indices in p and q so that

(i) for each j with $k_j > 1$, we have

$$p_{(j)} = (p_{j,1}, \ldots, p_{j,h_j}, 0, \ldots, 0), \quad q_{(j)} = (0, \ldots, 0, q_{j,h_{j+1}}, \ldots, q_{j,k_j}); \tag{1.4}$$

(ii) if $k_{j'} = k_{j''}$ with $j' < j''$, then $h_{j'} \leq h_{j''}$.

Now, given $k = (k_1, \ldots, k_m)$, we start with a m-tuple $h = (h_1, \ldots, h_m)$, where $h_j = 0$ if $k_j = 1$ and $1 \leq h_j \leq k_j - 1$ if $k_j \geq 1$; in the last case, if $k_{j'} = k_{j''}$ with $j' < j''$, then $h_{j'} \leq h_{j''}$.

We denote by $\mathcal{R}_k(h)$ the linear space generated by all parabolic k-quasi-radial quasi-homogeneous functions $a(r_1, r_2, \ldots, r_m, y_n)\zeta^p\overline{\zeta}^q$, where $a(r_1, r_2, \ldots, r_m, y_n) \in \mathcal{R}_k$, and the components $p(j)$ and $q(j)$, $j = 1, 2, \ldots, m$, of the multi-indices p and q are of the form (1.4) with

$$p_{j,1} + \cdots + p_{j,h_j} = q_{j,h_{j+1}} + \cdots + q_{j,k_j},$$

$p_{j,1}, \ldots, p_{j,h_j}, q_{j,h_{j+1}}, \ldots, q_{j,k_j} \in \mathbb{Z}_+$.

1.8.3 Lemma ([14, Lemma 3.1]). *Given a parabolic quasi-radial function $a = a(r_1, \ldots, r_m, y_n)$, we have*

$$RT_aR^* \colon \{c_\alpha(\xi)\}_{\alpha \in \mathbb{Z}_+^{n-1}} \longmapsto \{\gamma_{a,k}(\alpha, \xi)c_\alpha(\xi)\}_{\alpha \in \mathbb{Z}_+^{n-1}},$$

where

$$\begin{aligned}\gamma_{a,k}(\alpha, \xi) = &\frac{1}{\Gamma(\lambda+1)\Pi_{j=1}^m(k_j - 1 + |\alpha_{(j)}|)!} \\ &\times \int_{\mathbb{R}_+^{m+1}} a\left(\sqrt{\frac{r_1}{2\xi}}, \cdots, \sqrt{\frac{r_m}{2\xi}}, \frac{v + r_1 + \cdots + r_m}{2\xi}\right) \\ &\times v^\lambda e^{-(v+r_1+\cdots+r_m)} dv \prod_{j=1}^m r_j^{|\alpha_{(j)}|+k_j-1} dr_j, \quad \xi \in \mathbb{R}_+.\end{aligned}$$

For each $\alpha \in \mathbb{Z}_+^{n-1}$, we denote by $\hat{e}_\alpha = \{\delta_{\alpha,\beta}\}_{\beta \in \mathbb{Z}_+^{n-1}}$ the α's element of the standard orthonormal basis in $l_2(\mathbb{Z}_+^{n-1})$. Given $c(\xi) \in L_2(\mathbb{R}_+)$ let

$$\hat{e}_\alpha(c(\xi)) = \hat{e}_\alpha \otimes c(\xi) = \{\delta_{\alpha,\beta} c(\xi)\}_{\beta \in \mathbb{Z}_+^{n-1}}$$

be the corresponding one-component element of $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$.

1.8.4 Lemma ([14, Lemma 3.2]). *Given a parabolic k-quasi-radial quasi-homogeneous symbol of the form* (1.3), *we have*

$$RT_\varphi R^* \colon \hat{e}_\alpha(c(\xi)) \longmapsto \begin{cases} 0, & \text{if there exists } l \text{such that} \alpha_l + p_l - q_l < 0 \\ \widetilde{\gamma}_{a,k,p,q}(\alpha, \xi)\hat{e}_{\alpha+p-q}(c(\xi)), & \text{if } \forall l \quad \alpha_1 + p_l - q_l \geq 0, \end{cases}$$

where

$$\widetilde{\gamma}_{a,k,p,q}(\alpha,\xi) = \frac{2^m(\alpha+p)!}{\sqrt{\alpha!(\alpha+p-q)!}\Gamma(\lambda+1)} \prod_{j=1}^{m} \frac{1}{(k_j-1+|\alpha_{(j)}+p_{(j)})!}$$
$$\times \int_{\mathbb{R}_+^{m+1}} a\left(\frac{r_1}{\sqrt{2\xi}},\cdots,\frac{r_m}{\sqrt{2\xi}},\frac{v+r^2}{2\xi}\right)$$
$$\times v^{\lambda}e^{-(v+r^2)}dv \prod_{j=1}^{m} r_j^{2|\alpha_{(j)}|+|p_{(j)}|-|q_{(j)}|+2k_j-1} dr_j, \quad \xi \in \mathbb{R}_+.$$

According to Corolary 4.6 and its remarks in [14], the Banach algebra generated by Toeplitz operators with symbols in $\mathcal{R}_k(h)$ is commutative on every weighted Bergman space $\mathcal{A}^2_\lambda(D_n)$, $\lambda > -1$. For $n = 2$ these algebras collapse to the single C^*-algebra generated by Toeplitz operators with quasi-parabolic symbols and for $n > 2$ these algebras are just Banach and, extending them to C^*-algebras, they become non commutative.

In this thesis we will study the simplest case $n = 3$ and $k = (2)$. This implies that $h = (1) = 1$ and thus the symbols we will work with are of the form $a(r,y)\zeta^{(p,0)}\overline{\zeta}^{(0,p)}$. We denote by $\mathcal{T}(\lambda)$ the Banach algebra generated by the Toeplitz operators $T_{a(r,y)\zeta^{(p,0)}\overline{\zeta}^{(0,p)}}$, where $a = a(r, \operatorname{Im} z_3) \in L_\infty(D_3)$, $r = \sqrt{|z_1|^2+|z_1|^2}$, and $p \in \mathbb{Z}_+$.

Chapter 2

Bergman Space Representation

2.1 Bergman Space Representation

We denote by D_n the Siegel in $\mathbb{C}^n$ defined by

$$D_n = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \colon \operatorname{Im} z_n - |z'|^2 > 0\}. \tag{2.1}$$

Let $\mathcal{D} = \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$. The mapping

$$\kappa \colon (z', u, v) \in \mathcal{D} \longmapsto (z', u + iv + i|z'|^2) \in D_n \tag{2.2}$$

is a diffeomorphism between $\mathcal{D}$ and D_n with inverse

$$\kappa \colon (z', z_n) \longmapsto (z', \operatorname{Re} z_n, \operatorname{Im} z_n - |z'|^2).$$

Denote by $d\nu(z) = dx_1 dx_2 \cdots dx_n dy_n$, where $z_m = x_m + iy_m$, $m = 1, \ldots, n$, the standard Lebesgue measure in $\mathbb{C}^n$, and introduce the following one-parameter family of weighted measures

$$d\mu_\lambda(z) = \frac{c_\lambda}{4} (\operatorname{Im} z_n - |z'|^2)^\lambda d\nu(z)$$

where the normalizing constant is given by

$$c_\lambda = \frac{\Gamma(n + \lambda + 1)}{\pi^n \Gamma(\lambda + 1)}. \tag{2.3}$$

Denote by $\mathcal{A}^2_\lambda(D_n)$ the weighted Bergman space being the (closed) subspace of $L_2(D_n, d\mu_\lambda)$ which consists of analytic functions. It is well known that the weighted

Bergman projection $B_{D_{n,\lambda}}$ of $L_2(D_n, d\mu_\lambda)$ onto the Bergman space $\mathcal{A}^2_\lambda(D_n)$ is given by

$$(B_{D_{n,\lambda}}f)(z) = \int_{D_n} \frac{f(\zeta)}{\left(\frac{z_n - \bar{\zeta}_n}{2i} - z' \cdot \overline{\zeta'}\right)^{n+\lambda+1}} d\mu_\lambda(\zeta).$$

Return now to the domain $\mathcal{D} = \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$ and introduce the space $L_2(\mathcal{D}, d\eta_\lambda)$ where the measure $d\eta_\lambda$ is given by the formula

$$d\eta_\lambda(w) = \eta_\lambda(w) = \frac{c_\lambda}{4} v^\lambda d\nu(z), \quad \lambda > -1,$$

and the constant c_λ is given by (2.3).

The operator $U_0\colon L_2(D_n, d\mu_\lambda) \longmapsto L_2(\mathcal{D}, d\eta_\lambda)$, defined by

$$(U_0 f)(w) = f(\kappa(w)),$$

where the mapping κ is given by (2.2), is obviously unitary.

The image $\mathcal{A}_0 = U_0(\mathcal{A}^2_\lambda(D_n)))$ coincides with the set of all $L_2(\mathcal{D}, d\eta_\lambda)$-functions that satisfy the equations

$$\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\varphi = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial \overline{z}_k} - i\frac{\partial}{\partial u} z_k\right)\varphi = 0, \quad k = 1, \dots, n-1.$$

We introduce the unitary operator $U_1 = I \otimes F \otimes I$ acting on $L_2(\mathcal{D}, d\eta_\lambda) = L_2(\mathbb{C}^{n-1}, dv_\lambda) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, d\eta_\lambda)$, where

$$(Ff)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-i\xi u} du$$

is the Fourier transform on $L_2(\mathbb{R})$.

Then the image $\mathcal{A}_1(\mathcal{D}) = U_1(\mathcal{A}_0(\mathcal{D}))$ consists of all $L_2(\mathcal{D}, d\eta_\lambda)$-functions of the form

$$\varphi(z', \xi, v) = \chi_{\mathbb{R}_+}(\xi)\psi(z', \xi)e^{-|\xi|v}, \tag{2.4}$$

where a functions ψ has to satisfy the equations

$$\left(\frac{\partial}{\partial \overline{z}_k} + \xi z_k\right)\psi(z', \xi) = 0, \quad k = 1, \dots, n-1. \tag{2.5}$$

We introduce now the weighted Fock space on $\mathbb{C}^{n-1}$. Given a (weight) parameter $\alpha \in \mathbb{R}_+$ consider $L_2(\mathbb{C}^{n-1}, dv_\alpha)$, where

$$dv_\alpha(z') = \left(\frac{\alpha}{\pi}\right)^{n-1} e^{-\alpha|z'|^2} dv(z'), \quad z' \in \mathbb{C}^{n-1}.$$

Then the Fock space $F^2_\alpha(\mathbb{C}^{n-1})$ is the closed subspace of $L_2(\mathbb{C}^{n-1}, dv_\alpha)$ which consists of analytic functions. We denote by P_α the orthogonal Bargmann projection of $L_2(\mathbb{C}^{n-1}, dv_\alpha)$ onto $F^2_\alpha(\mathbb{C}^{n-1})$.

For each $\xi \in \mathbb{R}$, we introduce the operator

$$(V_\xi f)(z') = \left(\frac{2|\xi|}{\pi}\right)^{-\frac{n-1}{2}} e^{|\xi||z'|^2} f(z') \tag{2.6}$$

which maps unitarily $L_2(\mathbb{C}^{n-1})$ onto $L_2(\mathbb{C}^{n-1}, dv_{2|\xi|})$.

Note that if $f \in L_2(\mathbb{C}^{n-1}, dv_{2|\xi|})$, then

$$(V_\xi^{-1} f)(z') = \left(\frac{2|\xi|}{\pi}\right)^{\frac{n-1}{2}} e^{-|\xi||z'|^2} f(z').$$

So, for each $\xi \in \mathbb{R}_+$ and $f \in L_2(\mathbb{C}^{n-1}, dv_{2\xi})$, we have

$$\begin{aligned}\left(\frac{\partial}{\partial \overline{z}_k} + \xi z_k\right)(V_\xi^{-1} f)(z') &= \left(\frac{2\xi}{\pi}\right)^{\frac{n-1}{2}} \left(-\xi z_k e^{-\xi|z'|^2} f(z') + e^{-\xi|z'|^2} \frac{\partial f}{\partial \overline{z}_k}(z') + \xi z_k e^{-\xi|z'|^2} f(z')\right) \\ &= \left(\frac{2\xi}{\pi}\right)^{\frac{n-1}{2}} e^{-\xi|z'|^2} \frac{\partial f}{\partial \overline{z}_k}(z').\end{aligned}$$

Thus,

$$V_\xi \left(\frac{\partial}{\partial \overline{z}_k} + \xi z_k\right) V_\xi^{-1} = \frac{\partial}{\partial \overline{z}_k}, \quad k = 1, \ldots, n-1. \tag{2.7}$$

It is convenient to represent $L_2(\mathcal{D}, d\eta_\lambda)$ in the form

$$L_2(\mathcal{D}, d\eta_\lambda) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda) \otimes L_2(\mathbb{C}^{n-1}) = L_2(\mathbb{R}_+, \eta_\lambda) \otimes \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{C}^{n-1}, dv_{2\xi}) d\xi.$$

Using this representation we define the operator

$$V = I \oplus \int_{\mathbb{R}}^{\oplus} V_\xi d\xi,$$

which maps unitarily

$$L_2(\mathbb{R}_+,\eta_\lambda)\otimes\int_{\mathbb{R}}^{\oplus}L_2(\mathbb{C}^{n-1})d\xi \quad \text{onto} \quad L_2(\mathbb{R}_+,\eta_\lambda)\otimes\int_{\mathbb{R}}^{\oplus}L_2(\mathbb{C}^{n-1},dv_{2|\xi|})d\xi.$$

Using (2.4), if we take a function $\varphi(z',\xi,v)=\chi_{\mathbb{R}_+}(\xi)\psi(z',\xi)e^{-\xi v}$ in $\mathcal{A}_1(\mathcal{D})$, we have

$$V\varphi(z',\xi,v)=\chi_{\mathbb{R}_+}(\xi)e^{-\xi v}\psi'(z',\xi),$$

where $\psi'(z',\xi)=\left(\frac{2|\xi|}{\pi}\right)^{-\frac{n-1}{2}}e^{|\xi||z'|^2}\psi(z',\xi)$ and, by Fubini's Theorem,

$$\begin{aligned}\|V\varphi\|^2&=\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}e^{-2\xi v}\|\psi'(\cdot,\xi)\|^2d\xi\right)\frac{c_\lambda}{4}v^\lambda dv\\&=\int_{\mathbb{R}}h(\xi)^2\|\psi'(\cdot,\xi)\|^2d\xi\\&=\int_{\mathbb{R}}\|h(\xi)\psi'(\cdot,\xi)\|^2d\xi.\end{aligned}\tag{2.8}$$

where $h(\xi)^2=\int_{\mathbb{R}_+}e^{-2\xi v}\frac{c_\lambda}{4}v^\lambda dv$.

Since $\left(\dfrac{\partial}{\partial\overline{z}_k}+\xi z_k\right)\psi=0$, by (2.7), the function ψ' is analytic. Moreover, by (2.8), we see that the function $(z',\xi)\mapsto h(\xi)\psi'(z',\xi)$ belongs to $\int_{\mathbb{R}}^{\oplus}L_2(\mathbb{C}^{n-1},dv_{2\xi})d\xi$.

Thus, if we let $c(\xi)=h(\xi)^{-1}$, we can write

$$\varphi(z',\xi,v)=\chi_{\mathbb{R}_+}c(\xi)e^{-\xi v}(h(\xi)\psi'(z',\xi,v)).$$

This proves that $\mathcal{A}_V=V(\mathcal{A}_1(\mathcal{D}))$ consists of all functions of the form

$$\varphi(z',\xi,v)=\chi_{\mathbb{R}_+}(\xi)c(\xi)e^{-\xi v}\psi(\xi,z'),\tag{2.9}$$

where $\|\varphi\|=\|\psi\|$.

Since

$$\int_{\mathbb{R}_+}e^{-2\xi v}v^\lambda dv=\frac{\Gamma(\lambda+1)}{(2\xi)^{\lambda+1}},$$

we can summarize the previous lines in the following result.

2.1.1 Lemma. *The unitary operator $U=VU_1U_0$ maps the Bergman space $\mathcal{A}^2_\lambda(D_n)$ onto the space $\mathcal{A}_V$ which is the closed subspace of*

$$L_2(\mathbb{R}_+,d\eta_\lambda)\otimes\int_{\mathbb{R}}^{\oplus}L_2(\mathbb{C}^{n-1},dv_{2\xi})d\xi$$

and consists of all functions of the form

$$\varphi(z', \xi, v) = \chi_{\mathbb{R}_+}(\xi) \left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda \Gamma(\lambda+1)} \right)^{\frac{1}{2}} e^{-\xi v} \psi(\xi, z'),$$

where

$$\psi(\xi, z') \in \int_{\mathbb{R}_+}^{\oplus} F^2_{2\xi}(\mathbb{C}^{n-1}) d\xi.$$

Introduce now the isometric embedding

$$R_0 \colon \int_{\mathbb{R}_+}^{\oplus} F^2_{2\xi}(\mathbb{C}^{n-1}) d\xi \to L_2(\mathbb{R}_+, d\eta_\lambda) \otimes \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{C}^{n-1}, dv_{2\xi}) d\xi$$

by the rule

$$R_0 \colon \psi(\xi, z') \longmapsto \chi_{\mathbb{R}_+}(\xi) \left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda \Gamma(\lambda+1)} \right)^{\frac{1}{2}} e^{-\xi v} \psi(\xi, z'),$$

where the function $\psi(\xi, z')$ is extended by zero for $\xi \in \mathbb{R} \backslash \mathbb{R}_+$ for each $z' \in \mathbb{C}^{n-1}$.

Since, by Fubini's Theorem,

$$\begin{aligned}
\langle R_0 \psi, \varphi \rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_{\mathbb{C}^{n-1}} \chi_{\mathbb{R}_+}(\xi) c(\xi) e^{-\xi v} \psi(\xi, z') \overline{\varphi(z', \xi, v)} \frac{c_\lambda}{4} v^\lambda dv_{2|\xi|} dv d\xi \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{C}^{n-1}} \psi(\xi, z') \overline{\left(\int_{\mathbb{R}_+} c(\xi) e^{-\xi v} \varphi(z', \xi, v) \frac{c_\lambda}{4} v^\lambda dv \right)} dv_{2|\xi|} d\xi \\
&= \int_{\mathbb{R}_+} \langle \psi(\xi, \cdot), \varphi'(\xi, \cdot) \rangle_{L_2(\mathbb{C}^{n-1}, dv_{2\xi})} d\xi \\
&= \int_{\mathbb{R}_+} \langle \psi(\xi, \cdot), P_{2\xi} \varphi'(\xi, \cdot) \rangle_{L_2(\mathbb{C}^{n-1}, dv_{2\xi})} d\xi \\
&= \langle \psi, (P_{2\xi} \otimes I) \varphi' \rangle,
\end{aligned}$$

where

$$\varphi'(\xi, z') = \int_{\mathbb{R}_+} c(\xi) e^{-\xi v} \varphi(z', \xi, v) \frac{c_\lambda}{4} v^\lambda dv,$$

we conclude that

$$R_0^* \colon f \longmapsto \int_{\mathbb{R}_+} \left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda \Gamma(\lambda+1)} \right)^{\frac{1}{2}} e^{-\xi v} (I \otimes P_{2\xi}) f(v, \xi, z') \frac{c_\lambda}{4} v^\lambda dv \tag{2.10}$$

Since R_0 is isometric, we have

$$R_0^* R_0 = I \colon \int_{\mathbb{R}_+}^{\oplus} F^2_{2\xi}(\mathbb{C}^{n-1}) d\xi \to \int_{\mathbb{R}_+}^{\oplus} F^2_{2\xi}(\mathbb{C}^{n-1}) d\xi.$$

Moreover, R_0^* is surjective, since R_0 is injective, and by (2.9), R_0 maps $\int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1})d\xi$ onto $\mathcal{A}_V$. This implies that the image of $R_0R_0^*$ is A_V. Also, $(R_0R_0^*)^2 = R_0R_0^*$ and $(R_0R_0^*)^* = R_0R_0^*$, therefore we conclude that

$$R_0R_0^* = P_V\colon L_2(\mathbb{R}_+, d\eta_\lambda) \otimes \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{C}^{n-1}, dv_{2|\xi|})d\xi \to \mathcal{A}_V,$$

where P_V is the orthogonal projection of

$$L_2(\mathbb{R}_+, d\eta_\lambda) \otimes \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{C}^{n-1}, dv_{2|\xi|})d\xi$$

onto $\mathcal{A}_V$.

Thus finally we have

2.1.2 Theorem. *The operator $R = R_0^*U$ maps $L_2(D_n, d\mu_\lambda)$ onto $\int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1})d\xi$, and the restriction*

$$R|_{\mathcal{A}_\lambda^2(D_n)}\colon \mathcal{A}_\lambda^2(D_n) \to \int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1})d\xi$$

is an isometric isomorphism.

The adjoint operator

$$R^* = U^*R_0\colon \int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1})d\xi \to \mathcal{A}_\lambda^2(D_n) \subset L_2(D_n, d\mu_\lambda)$$

is the isometric isomorphism of $\int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1})d\xi$ onto the subspace $\mathcal{A}_\lambda^2(D_n)$ of $L_2(D_n, d\mu_\lambda)$,

Furthermore

$$RR^* = I\colon \int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1})d\xi \to \int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1})d\xi,$$

$$R^*R = B_{D_{n,\lambda}}\colon L_2(D_n, d\mu_\lambda) \to \mathcal{A}_\lambda^2(D_n),$$

where $B_{D_{n,\lambda}}$ is the Bergman projection of $L_2(D_n, d\mu_\lambda)$ onto $\mathcal{A}_\lambda^2(D_n)$.

2.2 Toeplitz operators

Consider a function $a \in L^\infty(D_n, d\mu_\lambda)$. Let $M_a\colon L_2(D_n, d\mu_\lambda) \longmapsto L_2(D_n, d\mu_\lambda)$ be the multiplication operator with symbol a and $T_a^{(\lambda)} = B_{D_{n,\lambda}} M_a|_{\mathcal{A}_\lambda^2(D_n)}$ the Toeplitz operator with symbol a.

Then, since $B_{D_{n,\lambda}} = R^*R$ and $I = RR^*$, we have

$$RT_a^{(\lambda)}R^* = R((R^*R)M_a)R^* = RM_aR^* = R_0^*VU_1U_0M_aU_1^*V^*R_0.$$

Note that for every $f \in L_2(\mathcal{D}, d\eta_\lambda)$,

$$U_0 M_a U_0^* f = U_0(a \cdot (f \circ \kappa^{-1})) = (a \circ \kappa) f = M_{a_0} f,$$

where M_{a_0} is the multiplication operator in $L_2(\mathcal{D}, d\eta_\lambda)$ with symbol $a_0 = a \circ \kappa$.

Suppose now that a does not depend on $\operatorname{Re} z_n$, that is,

$$a(z', z_n) = a(z', z_n + t), \quad t \in \mathbb{R}. \tag{2.11}$$

In this case, a_0 doesn't depend on u and we can write $a_0 = a_0(z', v) \in L_\infty(\mathbb{C}^{n-1} \times \mathbb{R}_+)$.

This implies that, for every $f \in L_2(\mathcal{D}, d\eta_\lambda) \cap L_1(\mathcal{D}, d\eta_\lambda)$,

$$\begin{aligned}
(M_{a_0} U_1^* f)(z', u, v) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} a_0(z', v) f(t) e^{iut} dt \\
&= a_0(z', v) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{iut} dt \\
&= a_0(z', v)(F^* f)(z', u, v)
\end{aligned}$$

and then

$$\begin{aligned}
(U_1 M_{a_0} U_1^* f)(z', u, v) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} a_0(z', v)(F^* f)(z', t, v) e^{-iut} dt \\
&= a_0(z', v) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (F^* f)(z', t, v) e^{-iut} dt \\
&= a_0(z', v)(F F^* f)(z', t, v) \\
&= a_0(z', v) f(z', t, v).
\end{aligned}$$

This proves that $U_1 M_{a_0} U_1^* = M_{a_0}$.

Furthermore, for every $f \in L_2(\mathcal{D}, d\eta_\lambda) = L_2(\mathbb{R}_+, d\eta_\lambda) \otimes \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{C}^{n-1}, dv_{2\xi}) d\xi$ we have

$$\begin{aligned}
V M_{a_0} V^* \colon f &\longmapsto \left(\frac{2|\xi|}{\pi}\right)^{-\frac{n-1}{2}} e^{|\xi||z'|^2} f(z', \xi, v) \\
&\longmapsto a_0(z', v) \left(\frac{2|\xi|}{\pi}\right)^{-\frac{n-1}{2}} e^{|\xi||z'|^2} f(z', \xi, v) \\
&\longmapsto a_0(z', v) f(z', \xi, v),
\end{aligned}$$

which implies $V M_{a_0} V^* = M_{a_0}$.

Thus, we conclude that

$$\begin{aligned} R^*T_aR &= R_0^*VU_1(U_0M_aU_1^*)V^*R_0 \\ &= R_0^*V(U_1M_{a_0}U_1^*)V^*R_0 \\ &= R_0^*(VM_{a_0}V^*)R_0 \\ &= R_0^*M_{a_0}R_0. \end{aligned}$$

We take now $\psi \in \int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^{n-1})d\xi$ and see how $R_0^*M_{a_0}R_0$ acts on it. It's clear that

$$(M_{a_0}R_0\psi)(v,\xi,z') = a_0(z',v)\chi_{\mathbb{R}_+}(\xi)\left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda\Gamma(\lambda+1)}\right)^{\frac{1}{2}} e^{-\xi v}\psi(\xi,z'), \tag{2.12}$$

where ψ is extended as before.

Recall that the Bargmann projection $P_{2|\xi|}$ of $L_2(\mathbb{C}^{n-1}, dv_{2|\xi|})$ onto $F_{2|\xi|}^2(\mathbb{C}^{n-1})$ is given by the integral operator

$$P_{2|\xi|}f(z) = \int_{\mathbb{C}^{n-1}} f(w')e^{2|\xi|z'\overline{w'}}dv_{2|\xi|}(w'), \quad f \in L_2(\mathbb{C}^{n-1}, dv_{2|\xi|}),$$

where $K_{w'}(z') = K(z',w') = e^{2|\xi|z'\overline{w'}}$ is the reproducing kernel of $F_{2|\xi|}^2(\mathbb{C}^{n-1})$.

It follows that if $f \in L_2(\mathbb{R}_+, d\eta_\lambda) \otimes \int_{\mathbb{R}}^{\oplus} L_2(\mathbb{C}^{n-1}, dv_{2|\xi|})d\xi$, fixing ξ,

$$(I \otimes P_{2|\xi|})f(v,\xi,z') = \int_{\mathbb{C}^{n-1}} f(v,\xi,w')K(z',w')dv_{2|\xi|}(w').$$

From (2.12) we obtain

$$\begin{aligned} &(I\otimes P_{2|\xi|})(M_{a_0}R_0)\psi(v,\xi,z') \\ &\quad = \int_{\mathbb{C}^{n-1}} a_0(w',v)\chi_{\mathbb{R}_+}(\xi)c(\xi)e^{-\xi v}\psi(\xi,w')K(z',w')dv_{2|\xi|}(w') \\ &\quad = \chi_{\mathbb{R}_+}(\xi)c(\xi)e^{-\xi v}\int_{\mathbb{C}^{n-1}} a_0(w',v)\psi(\xi,w')K(z',w')dv_{2|\xi|}(w') \end{aligned}$$

Finally, using (2.10), (2.12) and Fubini's Theorem, we have

$$\begin{aligned}
&(R_0^* M_{a_0} R_0 \psi)(\xi, z') \\
&\quad = \int_{\mathbb{R}_+} \chi_{\mathbb{R}_+}(\xi) c(\xi)^2 e^{-2\xi v} \left(\int_{\mathbb{C}^{n-1}} a_0(w', v) \psi(\xi, w') K(z', w') dv_{2|\xi|}(w') \right) \frac{c_\lambda}{4} v^\lambda dv \\
&\quad = \int_{\mathbb{C}^{n-1}} \left(\int_{\mathbb{R}_+} \chi_{\mathbb{R}_+}(\xi) c(\xi)^2 e^{-2\xi v} a_0(w', v) \frac{c_\lambda}{4} v^\lambda dv \right) \psi(\xi, w') K(z', w') dv_{2|\xi|}(w') \\
&\quad = \int_{\mathbb{C}^{n-1}} \widetilde{a}_\xi(w') \psi(\xi, w') K(z', w') dv_{2|\xi|}(w') \\
&\quad = T^{(\xi)}_{\widetilde{a}_\xi}(z'),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{a}_\xi(z') &= \int_{\mathbb{R}_+} \left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda \Gamma(\lambda+1)} \right) e^{-2\xi v} a_0(z', v) \frac{c_\lambda}{4} v^\lambda dv \\
&= \int_{\mathbb{R}_+} \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} e^{-2\xi v} v^\lambda a_0(z', v) dv \\
&= \int_{\mathbb{R}_+} \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} e^{-2\xi v} v^\lambda a \circ \kappa(z', u, v) dv \qquad (2.13)
\end{aligned}$$

and $T^{(\xi)}_{\widetilde{a}_\xi}$ is the Toeplitz operator with symbol $\widetilde{a}_\xi$ acting on $F^2_{2\xi}(\mathbb{C}^{n-1})$.

We conclude that

$$RT_a^{(\lambda)} R^* = \int_{\mathbb{R}_+}^{\oplus} T^{(\xi)}_{\widetilde{a}_\xi} d\xi. \qquad (2.14)$$

2.3 The symbol $\widetilde{a}_\xi$

Recall that

$$\widetilde{a}_\xi(z') = \int_{\mathbb{R}_+} \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} e^{-2\xi v} v^\lambda a_0(z', v) dv.$$

2.3.1 $\|\widetilde{a}_\xi\|_\infty$ and $\|a\|_\infty$

If $a_0(z', v) = b_1(z') b_2(v)$, for some functions $b_1 \in L_\infty(\mathbb{C}^{n-1})$, $b_2 \in L_\infty(\mathbb{R}_+)$, then

$$\widetilde{a}_\xi(z') = b_1(z') \int_{\mathbb{R}_+} \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} e^{-2\xi v} v^\lambda b_2(v) dv.$$

Since

$$\int_{\mathbb{R}_+} v^s e^{-tv} dv = \frac{\Gamma(s+1)}{t^{s+1}}, \qquad (2.15)$$

we have $\widetilde{a}_\xi(z') = b_1(z')$ when $b_2(v) = 1$. In this case $\|\widetilde{a}_\xi\|_\infty = \|a_0\|_\infty$.

Let $\alpha \geq 0$, $\beta > 0$ and suppose $b_2(v) = v^\alpha e^{-\beta v}$. Then we have $b_2'(v) = (\alpha v^{-1} - \beta)v^\alpha e^{-\beta v}$, so that b_2 has a maximum at $v = \alpha/\beta$. Therefore

$$\|b_2\|_\infty = b_2(\alpha/\beta) = (\frac{\alpha}{\beta e})^\alpha$$

and

$$\|a_0\|_\infty = \|b_1\|_\infty \|b_2\|_\infty = \|b_1\|_\infty (\frac{\alpha}{\beta e})^\alpha.$$

Also, by (2.15),

$$\begin{aligned}\widetilde{a}_\xi(z') &= b_1(z') \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} \int_{\mathbb{R}^+} v^{\lambda+\alpha} e^{-(2\xi+\beta)v} dv \\ &= b_1(z') \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} \frac{\Gamma(\lambda+\alpha+1)}{(2\xi+\beta)^{\lambda+\alpha+1}}.\end{aligned}$$

Then in general $\|a_\xi\|_\infty \neq \|a_0\|_\infty$.

However, we always have

$$|\widetilde{a}_\xi(z')| \leq \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} \int_{\mathbb{R}^+} v^\lambda e^{-2\xi v} \|a_0\|_\infty dv = \|a_0\|_\infty,$$

which implies $\|\widetilde{a}_\xi\| \leq \|a\|_\infty$.

2.3.2 Limits at 0 and at ∞

Suppose $a_0(z', v) = b_1(z') b_2(v)$ such that

$$b_2(t) \to A, \quad t \to \infty,$$

$$b_2(t) \to B, \quad t \to 0^+.$$

Note that, applying the change of variable $y = 2\xi v$,

$$\begin{aligned}\widetilde{a}_\xi(z') &= b_1(z') \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+} e^{-2\xi v} v^\lambda b_2(v) dv \\ &= \frac{b_1(z')}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+} e^{-y} y^\lambda b_2(\frac{y}{2\xi}) dy.\end{aligned}$$

Since

$$|e^{-v} v^\lambda b_2(\frac{v}{2\xi})| \leq \|b_2\|_\infty |e^{-v} v^\lambda|,$$

where the last function is integrable and independent of ξ, by Lebesgue's dominated convergence theorem we have

$$\lim_{\xi\to 0^+} \widetilde{a}_\xi(z') = A\frac{b_1(z')}{\Gamma(\lambda+1)}\int_{\mathbb{R}_+} e^{-y}y^\lambda dy = Ab_1(z'), \quad z' \in \mathbb{C}^{n-1}$$

and

$$\lim_{\xi\to\infty} \widetilde{a}_\xi(z') = B\frac{b_1(z')}{\Gamma(\lambda+1)}\int_{\mathbb{R}_+} e^{-y}y^\lambda dy = Bb_1(z'), \quad z' \in \mathbb{C}^{n-1}.$$

2.3.3 Continuity

Consider the logarithmic metric ρ on $\mathbb{R}_+$ defined by

$$\rho(x,y) = |\ln(x) - \ln(y)|, \quad x, y \in \mathbb{R}_+.$$

Suppose that $\lambda \geq 0$ and $a(z',v) = b_1(z')b_2(v)$. Then $\xi \mapsto \widetilde{a}_\xi(z')$ is uniformly continuous with respect to the metric ρ for every $z' \in \mathbb{C}^{n-1}$.

Indeed, reasoning as in [19, Theorem 4.4], we can assume, without loss of generality, that $\xi_2 > \xi_1$. Then

$$\begin{aligned}(2\xi_1)^{\lambda+1}v^\lambda e^{-2\xi_1 v} \geq (2\xi_2)^{\lambda+1}v^\lambda e^{-2\xi_2 v} \quad &\text{iff} \quad \xi_1^{\lambda+1}e^{-2\xi_1 v} \geq \xi_2^{\lambda+1}e^{-2\xi_2 v}\\ &\text{iff} \quad (\lambda+1)\ln\xi_1 - 2\xi_1 v \geq (\lambda+1)\ln\xi_2 - 2\xi_2 v\\ &\text{iff} \quad v \geq \frac{(\lambda+1)}{2}\frac{1}{\xi_2-\xi_1}\ln\frac{\xi_2}{\xi_1} := v_0\end{aligned}$$

Thus we have

$$\begin{aligned}&\left|\int_{\mathbb{R}_+}(2\xi_1)^{\lambda+1}v^\lambda e^{-2\xi_1 v}b_2(v)dv - \int_{\mathbb{R}_+}(2\xi_2)^{\lambda+1}v^\lambda e^{-2\xi_2 v}b_2(v)dv\right|\\ &\qquad \leq \|b_2\|_\infty \int_{\mathbb{R}_+}|(2\xi_1)^{\lambda+1}v^\lambda e^{-2\xi_1 v} - (2\xi_2)^{\lambda+1}v^\lambda e^{-2\xi_2 v}|dv\\ &\qquad = \|b_2\|_\infty\left(\int_{v_0}^\infty(2\xi_1)^{\lambda+1}v^\lambda e^{-2\xi_1 v}dv - \int_{v_0}^\infty(2\xi_2)^{\lambda+1}v^\lambda e^{-2\xi_2 v}dv\right.\\ &\qquad \left. + \int_0^{v_0}(2\xi_2)^{\lambda+1}v^\lambda e^{-2\xi_2 v}dv - \int_0^{v_0}(2\xi_1)^{\lambda+1}v^\lambda e^{-2\xi_1 v}dv\right).\end{aligned}$$

Applying in each integral the change of variables $y = 2\xi_1 v$ or $y = 2\xi_2 v$, respectively, the sum inside parenthesis equals

$$\begin{aligned}\int_{2\xi_1 v_0}^{\infty} y^\lambda e^{-y} dy - \int_{2\xi_2 v_0}^{\infty} y^\lambda e^{-y} dy + \int_0^{2\xi_2 v_0} y^\lambda e^{-y} dy - \int_0^{2\xi_1 v_0} y^\lambda e^{-y} dy \\ = 2\int_{2\xi_1 v_0}^{2\xi_2 v_0} y^\lambda e^{-y} dy \leq 4M_\lambda(\xi_2 - \xi_1)v_0 = 2M_\lambda(\lambda+1)\ln\frac{\xi_2}{\xi_1} \\ = 2M_\lambda(\lambda+1)\rho(\xi_2, \xi_1),\end{aligned}$$

where

$$M_\lambda = \sup_{y\in\mathbb{R}_+} y^\lambda e^{-y} = \lambda^\lambda e^{-\lambda}, \quad \text{if } \lambda > 0,$$

and

$$M_0 = 1.$$

Therefore, we conclude that

$$\begin{aligned}|\widetilde{a}_{\xi_1}(z') - \widetilde{a}_{\xi_2}(z')| &\leq 2M_\lambda \frac{\lambda+1}{\Gamma(\lambda+1)} |b_1(z')| \|b_2\|_\infty \rho(\xi_2, \xi_1) \\ &\leq 2M_\lambda \frac{\lambda+1}{\Gamma(\lambda+1)} \|b_1\|_\infty \|b_2\|_\infty \rho(\xi_2, \xi_1).\end{aligned}$$

Doing a similar analysis for $-1 < \lambda < 0$ we conclude the following result.

2.3.1 Lemma. *Let $t > -1$ and $r_1, r_2 > 0$. Then*

$$\int_{\mathbb{R}_+0} |(r_1)^{t+1} v^t e^{-r_1 v} - (r_2)^{t+1} v^t e^{-r_2 v}| dv \leq 2t^t e^{-t}(t+1)|\log(r_1/r_2)|.$$

Chapter 3

Commutative C^*-Algebras

For every $\alpha > 0$ introduce now the linear operator

$$R_\alpha \colon F_1^2(\mathbb{C}^{n-1}) \to F_\alpha^2(\mathbb{C}^{n-1}),$$

given by

$$R_\alpha f(z') = f(\alpha^{1/2} z'), \quad z' \in \mathbb{C}^{n-1}.$$

It follows from the Change of Variables Theorem that R_α is a unitary operator with $R_\alpha^* = R_\alpha^{-1} = R_{\alpha^{-1}}$.

3.0.1 Lemma. *If $\varphi \in L_\infty(\mathbb{C}^{n-1})$ then*

$$R_\alpha^* T_\varphi^\alpha R_\alpha = T_{\varphi \circ \tau_\alpha}^1,$$

where $\tau_\alpha(z') = \alpha^{-1/2} z'$ and T_φ^α and $T_{\varphi\circ\tau_\alpha}^1$ denote the Toeplitz operators with symbols φ and $\varphi \circ \tau_\alpha$ acting on $F_\alpha^2(\mathbb{C}^{n-1})$ and $F_1^2(\mathbb{C}^{n-1})$, respectively.

Proof. We have, by a change of variables,

$$\begin{aligned}
R_\alpha^* T_\varphi^\alpha R_\alpha = R_\alpha^* P_\alpha M_\varphi R_\alpha \colon f &\longmapsto f(\alpha^{1/2} z') \\
&\longmapsto \varphi(z') f(\alpha^{1/2} z') \\
&\longmapsto \int_{\mathbb{C}^{n-1}} \varphi(w') f(\alpha^{1/2} w') e^{\alpha z' \overline{w'}} \left(\frac{\alpha}{\pi}\right)^{n-1} e^{-\alpha |w'|^2} d\nu(w') \\
&= \int_{\mathbb{C}^{n-1}} \varphi(\alpha^{-1/2} w') f(w') e^{\alpha^{1/2} z' \overline{w'}} \frac{e^{-|w'|^2}}{\pi^{n-1}} d\nu(w') \\
&\longmapsto \int_{\mathbb{C}^{n-1}} \varphi(\alpha^{-1/2} w') f(w') e^{z' \overline{w'}} \frac{e^{-|w'|^2}}{\pi^{n-1}} d\nu(w') \\
&= T^1_{\varphi \circ \tau_\alpha}.
\end{aligned}$$

□

3.1 Quasi-parabolic case

We will call a function $a(z)$, $z \in D_n$, *quasi-parabolic* if $a(z) = a(r, y_n) = a(r_1, \dots, r_{n-1}, \operatorname{Im} z_n)$. Note that such a satisfies condition (2.11) and $a_0(z', v) = a(\kappa(z', v)) = a(z', v + |z'|^2)$. Thus we can write $a_0 = a_0(r, v) = a(r, v + |r|^2)$.

Then (2.14) holds and by (2.13) we have

$$\widetilde{a}_\xi(z') = \widetilde{a}_\xi(r) = \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+} e^{-2\xi v} v^\lambda a(r, v + |r|^2) dv. \tag{3.1}$$

We will show that the operator $T_a^{(\lambda)}$ is unitarily equivalent to a specific multiplication operator.

Let us first recall some facts about radial operators. A function $\varphi \in L_\infty(\mathbb{C})$ is called *radial* if there is some function $\psi \in L_\infty(\mathbb{R}_+)$ such that $\varphi(z) = \psi(|z|)$.

It is well known (see, for example, [9]) that if $\varphi \in L_\infty(\mathbb{C})$ is a radial function, then the Toeplitz operator T_φ with symbol φ acting on the one-dimensional Fock space $F^2(\mathbb{C})$ is diagonal with respect to the orthonormal basis consisting of the normalized monomials $e_k(z) = z^k/\sqrt{k!}$, $n \in \mathbb{Z}_+$, with eigenvalues

$$\gamma_\varphi(x) = \frac{1}{n!} \int_{\mathbb{R}_+} a(\sqrt{r}) e^{-r} r^n dr, \quad n \in \mathbb{Z}_+. \tag{3.2}$$

We extend this result to the Fock space $F_\alpha^2(\mathbb{C}^m)$.

3.1.1 Lemma. *Let $\varphi \in L_\infty(\mathbb{C}^m)$ a function such that $\varphi(z) = \varphi(r_1, \ldots, r_m)$ and T_φ^α the Toeplitz operator with symbol φ acting on $F_\alpha^2(\mathbb{C}^m)$. Then T_φ^α is diagonal with respect to the orthonormal basis consisting of the monomials $e_p(z) = \sqrt{\frac{\alpha^{|p|}}{p!}} z^p$, $p \in \mathbb{Z}_+^m$ and the eigenvalues are given by*

$$\gamma_\varphi^{(\alpha)}(p) = \frac{\alpha^{|p|+m}}{p!} \int_{\mathbb{R}_+^m} \varphi(\sqrt{r}) e^{-\alpha(r_1+\cdots+r_m)} r^p dr, \quad p \in \mathbb{Z}_+^m, \tag{3.3}$$

where $\sqrt{r} = (\sqrt{r_1}, \ldots, \sqrt{r_m})$.

Proof. It is well known that the monomials $e_p(z) = z^p/\sqrt{p!}$ form an orthonormal basis for the space $F_1^2(\mathbb{C}^m)$. Then the functions $R_\alpha e_p(z) = \sqrt{\frac{\alpha^{|p|}}{p!}} z^p$, $p \in \mathbb{Z}_+^m$, form an orthonormal basis por $F_\alpha^2(\mathbb{C}^m)$ and, by Lemma 3.0.1, the operator T_φ^α is diagonal with respect to this basis if and only if $T_{\varphi\circ\tau_\alpha}^1$ is diagonal with respect to the monomials $\{e_p\}_{p\in\mathbb{Z}_+^m}$.

Thus, we only need to prove the result for the case $\alpha = 1$. Let P_1 and P be the Bargmann projection in $L_2(\mathbb{C}^m, dv_1)$ and $L^2(\mathbb{C}, dg)$, respectively, where $dg(z) = \frac{1}{\pi} e^{-|z|^2} dA(z)$, and note that

$$P_1 f(z) = \int_{\mathbb{C}^m} f(w) e^{z\overline{w}} dv_1(w) = (P \otimes \cdots \otimes P) f(z).$$

This also implies that $F_1^2(\mathbb{C}^m) = F^2(\mathbb{C}) \otimes \cdots \otimes F^2(\mathbb{C})$.

Now let $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_m \in L_\infty(\mathbb{C}) \otimes \cdots \otimes L_\infty(\mathbb{C})$, where each φ_k is a radial function. Then

$$T_\varphi^1 = P_1 M_\varphi = (P \otimes \cdots \otimes P)(M_{\varphi_1} \otimes \cdots \otimes M_{\varphi_m}) = T_{\varphi_1} \otimes \cdots \otimes T_{\varphi_m},$$

where T_{φ_k} denotes the Toeplitz operator with (radial) symbol φ_k acting on $F^2(\mathbb{C})$. Using the facts stated before the lemma we see that for every $k = 1, \ldots, m$, T_{φ_k} is unitarily equivalent to the operator $\gamma_{\varphi_k} I$, acting on $L_2(\mathbb{Z}_+)$.

Therefore T_φ^1 is unitarily equivalent to the multiplication operator

$$\gamma_{\varphi_1} I \otimes \cdots \otimes \gamma_{\varphi_m} I = \gamma_\varphi I,$$

acting on $L_2(\mathbb{Z}_+^m)$, where $\gamma_\varphi = \gamma_{\varphi_1} \otimes \cdots \otimes \gamma_{\varphi_m}$. That is,

$$\begin{aligned}\gamma_\varphi(p) &= \frac{1}{p!} \int_{\mathbb{R}_+^m} \varphi(\sqrt{r_1}) \cdots \varphi_{n-1}(\sqrt{r_m}) e^{-(r_1+\cdots+r_m)} r^p dr_1 \cdots dr_m \\ &= \frac{1}{p!} \int_{\mathbb{R}_+^m} \varphi(\sqrt{r}) e^{-(r_1+\cdots+r_m)} r^p dr,\end{aligned}$$

where $\sqrt{r} = (\sqrt{r_1}, \ldots, \sqrt{r_m})$. This proves the formula for $\alpha = 1$.

For a general α we simply replace φ with $\varphi \circ \tau_\alpha$ and apply the Change of Variables Theorem to obtain the desired result. □

Returning to the original problem, for every $\xi > 0$ introduce the linear operator $Q_\xi \colon F^2_{2\xi}(\mathbb{C}^{n-1}) \to l_2(\mathbb{Z}_+^{n-1})$ defined as the unitary operator such that $Q_\xi(R_{2\xi} e_p) = (\delta_{q,p})_{q \in \mathbb{Z}_+^{n-1}}$ and define $Q = \int_{\mathbb{R}_+}^{\oplus} Q_\xi d\xi$. Note that by the preceding lemma, if $\varphi = \varphi(r_1, \ldots, r_{n-1}) \in L_\infty(\mathbb{C}^{n-1})$, then $Q_\xi T_\varphi^{2\xi} Q_\xi^*$ is the multiplication operator $\gamma_\varphi^{(2\xi)} I$ acting on $l_2(\mathbb{Z}_+^{n-1})$, where $\gamma_\varphi^{(2\xi)}$ is given by this Lemma.

Using this, we give an alternative proof to Theorem 10.2 in [10].

3.1.2 Theorem. *Let $a = a(r, y_n)$ be a bounded measurable quasi-parabolic function. Then the Toeplitz operator $T_a^{(\lambda)}$ acting on $\mathcal{A}^2_\lambda(D_n)$ is unitary equivalent to the multiplication operator $\gamma_a I = QRT_a^{(\lambda)} R^* Q^*$ acting on $l_2(\mathbb{Z}_+^{n-1}) \otimes L_2(\mathbb{R}_+)$. The sequence $\gamma_a = \{\gamma_a(p, \xi)\}_{p \in \mathbb{Z}_+^{n-1}}$, $\xi \in \mathbb{R}_+$, is given by*

$$\gamma_a(p, \xi) = \frac{(2\xi)^{|p|+\lambda+n}}{p!\Gamma(\lambda+1)} \int_{\mathbb{R}_+^n} a(\sqrt{r}, v + r_1 + \cdots + r_{n-1}) r^p e^{-2\xi(v+r_1+\cdots+r_{n-1})} v^\lambda dr dv.$$

Proof. By the remarks at the beginning of the section we have $R^* T_a^{(\lambda)} R = \int_{\mathbb{R}_+}^{\oplus} T_{\widetilde{a}_\xi}^{(\xi)} d\xi$, where $\widetilde{a}_\xi$ es given by (3.1).

Since $\widetilde{a}_\xi = \widetilde{a}_\xi(r_1, \ldots, r_{n-1})$, we have

$$QRT_a^{(\lambda)} R^* Q^* = \int_{\mathbb{R}_+}^{\oplus} Q_\xi T_{\widetilde{a}_\xi}^{(\xi)} Q_\xi^* d\xi = \int_{\mathbb{R}_+}^{\oplus} \gamma_{\widetilde{a}_\xi}^{(2\xi)} I d\xi,$$

which is the multiplication operator $\gamma_a I$ acting on $\int_{\mathbb{R}_+}^{\oplus} l_2(\mathbb{Z}_+^{n-1})d\xi = l^2(\mathbb{Z}_+^{n-1}) \otimes L^2(\mathbb{R}_+)$, where, by (3.3) and (3.1),

$$\begin{aligned}\gamma_a(p,\xi) = \gamma_{\widetilde{a}_\xi}^{(2\xi)}(p) &= \frac{(2\xi)^{|p|+n-1}}{p!}\int_{\mathbb{R}_+^{n-1}} \widetilde{a}_\xi(\sqrt{r})e^{-2\xi(r_1+\dots+r_{n-1})}r^p dr \\ &= \frac{(2\xi)^{|p|+n-1}}{p!}\int_{\mathbb{R}_+^{n-1}} \left(\frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)}\int_{\mathbb{R}_+} e^{-2\xi v}v^\lambda a(\sqrt{r}, v+r_1+\dots+r_{n-1})dv.\right) \\ &\cdot e^{-2\xi(r_1+\dots+r_{n-1})}r^p dr \\ &= \frac{(2\xi)^{|p|+\lambda+n}}{p!\Gamma(\lambda+1)}\int_{\mathbb{R}_+^n} a(\sqrt{r}, v+r_1+\dots+r_{n-1})r^p e^{-2\xi(v+r_1+\dots+r_{n-1})}v^\lambda dr dv.\end{aligned}$$

□

3.2 Nilpotent case

We will call a function $a(z)$, $z \in D_n$, *nilpotent* if $a(z) = a(y', \operatorname{Im} z_n - |z'|^2)$, with $y' = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_{n-1})$. Note that such an a satisfies condition (2.11) and $a_0(z', v) = a(\kappa(z', v)) = a(z', v + |z'|^2)$. Thus we can write $a_0 = a_0(y', v)$ and (2.14) holds.

We will show that the operator $T_a^{(\lambda)}$ is unitarily equivalent to a specific multiplication operator. We make use of the following facts (see [18]):

A function $\varphi \in L_\infty(\mathbb{C}^{n-1})$ is said to be *horizontal* if for every $h \in \mathbb{R}^{n-1}$

$$\varphi(z - ih) = \varphi(z), \quad \text{for almost all } z \in \mathbb{C}^{n-1}.$$

A function $\varphi \in L_\infty(\mathbb{C}^{n-1})$ is horizontal if and only if there exists $\psi \in L_\infty(\mathbb{R}^{n-1})$ such that

$$\varphi(z) = \psi(\operatorname{Re} z), \quad \text{a. e. } z' \in \mathbb{C}^{n-1}.$$

A Toeplitz operator T_φ^1 with horizontal symbol φ acting on the Fock space $F_1^2(\mathbb{C}^{n-1})$ is unitarily equivalent to the multiplication operator $\gamma_\varphi I$ acting on $L_2(\mathbb{R}^{n-1})$.

It turns out that a similar result holds for any function $\varphi \in L_\infty(\mathbb{C}^{n-1})$ such that, for every h in a Lagrangian plane $\mathcal{L} \subset \mathbb{R}^{2(n-1)}$,

$$\varphi(z - h) = \varphi(z), \quad \text{for almost all} \quad z \in \mathbb{C}^{n-1}.$$

In the case $\mathcal{L} = \mathbb{R}^{n-1} \times \{0\}$, we have $\varphi(z - h) = \varphi(z)$, $h \in \mathbb{R}^{n-1}$, and we say that φ is *vertical*. There exists a unitary operator $Q_1 \colon F_1^2(\mathbb{C}^{n-1}) \to L_2(\mathbb{R}^{n-1})$ such that for every vertical function $\varphi \in L_\infty(\mathbb{C}^{n-1})$ we have

$$Q_1 T^1_\varphi Q_1^* = \gamma_\varphi I,$$

$\varphi(z) = \varphi(\operatorname{Im} z_1, \ldots, \operatorname{Im} z_{n-1})$ and thus the Toeplitz operator T^1_φ is unitarily equivalent to the multiplication operator $\gamma_\varphi I$ acting on $L_2(\mathbb{R}^{n-1})$ with

$$\gamma_\varphi(x) = \pi^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} \varphi\left(-\frac{y}{\sqrt{2}}\right) e^{-|x-y|^2} dy, \quad x \in \mathbb{R}^{n-1}.$$

Moreover, let $\alpha > 0$. If $\varphi \in L_\infty(\mathbb{C}^{n-1})$ is vertical then $\varphi \circ \tau_\alpha$ is also vertical and, by Lemma 3.0.1, the Toeplitz operator T^α_φ acting on $F^2_\alpha(\mathbb{C}^{n-1})$ is unitarily equivalent to the multiplication operator

$$Q_1 R^*_\alpha T^\alpha_\varphi R_\alpha Q_1^* = Q_1 T^1_{\varphi\circ\tau_\alpha} Q_1^* = \gamma_{\varphi\circ\tau_\alpha} I = \gamma^{(\alpha)}_\varphi I,$$

acting on Ł$_2(\mathbb{R}^{n-1})$, where

$$\gamma^{(\alpha)}_\varphi(x) = \pi^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} \varphi\left(-\frac{y}{\sqrt{2\alpha}}\right) e^{-|x-y|^2} dy, \quad x \in \mathbb{R}^{n-1}. \tag{3.4}$$

Returning to the original problem, for every $\xi > 0$ introduce the unitary operator $Q_\xi = Q_1 R^*_{2\xi} \colon F^2_{2\xi}(\mathbb{C}^{n-1}) \to L_2(\mathbb{R}^{n-1})$ and define $Q = \int^\oplus_{\mathbb{R}_+} Q_\xi d\xi$. Note that by the preceding lemma, if $\varphi = \varphi(y') \in L_\infty(\mathbb{C}^{n-1})$, then $Q_\xi T^{2\xi}_\varphi Q^*_\xi$ is the multiplication operator $\gamma^{(2\xi)}_\varphi I$ acting on $L_2(\mathbb{R}^{n-1})$, where $\gamma^{(2\xi)}_\varphi$ is given by the preceding Lemma.

Using this, we give an alternative proof to Theorem 10.3 in [10].

3.2.1 Theorem. *Let $a = a(y', \operatorname{Im} z_n - |z'|^2)$ be a bounded measurable nilpotent function. Then the Toeplitz operator $T^{(\lambda)}_a$ acting on $\mathcal{A}^2_\lambda(D_n)$ is unitary equivalent to the multiplication operator $\gamma_a I = QRT^{(\lambda)}_a R^* Q^*$ acting on $L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}_+)$. The function $\gamma_a = \gamma_a(x', \xi)$, where $x' \in \mathbb{R}^{n-1}$ and $\xi \in \mathbb{R}_+$, is given by*

$$\gamma_a(x', \xi) = \frac{(2\xi)^{\lambda+1}}{\pi^{\frac{n-1}{2}}\Gamma(\lambda+1)} \int_{\mathbb{R}^{n-1}\times\mathbb{R}_+} a(\frac{1}{2\sqrt{\xi}}(-x'+y'), v) e^{-2\xi v - |y'|^2} v^\lambda dy' dv$$

Proof. Reasoning as in the quasi-parabolic case, we have

$$QRT^{(\lambda)}_a R^* Q^* = \int^\oplus_{\mathbb{R}_+} Q_\xi T^{(\xi)}_{\widetilde{a}_\xi} Q^*_\xi d\xi = \int^\oplus_{\mathbb{R}_+} \gamma^{(2\xi)}_{\widetilde{a}_\xi} I d\xi,$$

which is the multiplication operator $\gamma_a I$ acting on $\int_{\mathbb{R}_+}^{\oplus} L_2(\mathbb{R}^{n-1})d\xi = L_2(\mathbb{R}^{n-1})\otimes L^2(\mathbb{R}_+)$, where, by (2.13), (3.4) and a change of variables,

$$\begin{aligned}\gamma_a(x',\xi) = \gamma_{\widetilde{a}_\xi}^{(2\xi)}(x') &= \pi^{-(n-1)/2}\int_{\mathbb{R}^{n-1}} \widetilde{a}_\xi\left(\frac{y'}{2\sqrt{\xi}}\right) e^{-|x'-y'|^2}dy' \\ &= \frac{(2\xi)^{\lambda+1}}{\pi^{\frac{n-1}{2}}\Gamma(\lambda+1)}\int_{\mathbb{R}^{n-1}\times\mathbb{R}_+} a(-\frac{y'}{2\sqrt{\xi}},v)e^{-2\xi v-|x'-y'|^2}v^\lambda dy'dv \\ &= \frac{(2\xi)^{\lambda+1}}{\pi^{\frac{n-1}{2}}\Gamma(\lambda+1)}\int_{\mathbb{R}^{n-1}\times\mathbb{R}_+} a(\frac{1}{2\sqrt{\xi}}(-x'+y'),v)e^{-2\xi v-|y'|^2}v^\lambda dy'dv.\end{aligned}$$

□

3.3 Quasi-nilpotent case

For an integer $1 \le k \le n-2$, we use the notation $z = (z', w', z_n)$ for points of D_n, where $z' \in \mathbb{C}^k$ and $w \in \mathbb{C}^{n-k-1}$. We will call a function $a(z)$, $z \in D_n$, *quasi-nilpotent* if $a(z) = a(r, y', \operatorname{Im} z_n - |w'|^2)$, where $r = (r_1, \dots, r_k)$, $r_l = |z_l|$ and $y' = \operatorname{Im} w'$. Using this notation we also have

$$\kappa(z', w', u, v) = (z', w', u + iv + i|z'|^2 + i|w'|^2),$$

$(z', w', u, v) \in \mathcal{D}$.

Note that such an a satisfies condition (2.11), $a_0(z', v) = a(\kappa(z', v)) = a(r, y', v + |z'|^2)$ and (2.14) holds.

Using the facts about Toeplitz operators with radial and vertical symbols stated in the preceding sections we can give an alternative proof to Theorem 10.4 in [10].

3.3.1 Theorem. *Let $a = a(r, y', \operatorname{Im} z_n - |w'|^2)$ be a measurable quasi-nilpotent function. Then the Toeplitz operator $T_a^{(\lambda)}$ acting on $\mathcal{A}_\lambda^2(D_n)$ is unitary equivalent to the multiplication operator $\gamma_a I$ acting on $l_2(\mathbb{Z}_+^k) \otimes L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)$.*

The sequence $\gamma_a = \{\gamma_a(p, x', \xi)\}_{p\in\mathbb{Z}_+^k}$, $(x', \xi) \in \mathbb{R}^{n-k-1} \times \mathbb{R}_+$, is given by

$$\begin{aligned}\gamma_a(p,x',\xi) =&\pi^{-\frac{n-k-1}{2}}\frac{(2\xi)^{|p|+\lambda+k+1}}{p!\Gamma(\lambda+1)} \\ &\cdot\int_{\mathbb{R}_+^k\times\mathbb{R}^{n-k-1}\times\mathbb{R}_+} a(\sqrt{r},\frac{1}{2\sqrt{\xi}}(-x'+y'),v+r_1+\dots+r_k) \\ &\cdot r^p e^{-2\xi(v+r_1+\cdot+r_k)-|y'|^2}v^\lambda dr dy' dv,\end{aligned}$$

where $\sqrt{r} = (\sqrt{r_1}, \ldots, \sqrt{r_k})$.

Proof. It suffices to show the result for the case $a(r, y', v+|z'|^2) = b_1(r, v+|z'|^2) \otimes b_2(y')$.

By (2.13), we have

$$\begin{aligned}\widetilde{a}_\xi(z', w') &= \left(\int_{\mathbb{R}_+} \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} e^{-2\xi v} v^\lambda b_1(r, v + |z'|^2) dv \right) \cdot b_2(y') \\ &= b_1' \otimes b_2(r, y'),\end{aligned}$$

where

$$b_1'(r) = \int_{\mathbb{R}_+} \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} e^{-2\xi v} v^\lambda b_1(r, v + |z'|^2) dv.$$

Reasoning as in the proof of Lemma 3.1.1, we have

$$F^2_{2\xi}(\mathbb{C}^{n-1}) = F^2_{2\xi}(\mathbb{C}^k) \otimes F^2_{2\xi}(\mathbb{C}^{n-k-1})$$

and

$$T^{(\xi)}_{\widetilde{a}_\xi} = T_{b_1'} \otimes T_{b_2},$$

where $T_{b_1'}$ and T_{b_2} are the Toeplitz operators with symbols b_1' and b_2 acting on the Fock spaces $F^2_{2\xi}(\mathbb{C}^k)$ and $F^2_{2\xi}(\mathbb{C}^{n-k-1})$, respectively.

Since $b_1' = b_1'(r_1, \ldots, r_k)$ we can apply the remarks before Theorem 3.1.2 with k instead of $n-1$. Thus, there is a unitary operator

$$Q^1_\xi \colon F^2_{2\xi}(\mathbb{C}^k) \to l_2(\mathbb{Z}^k_+)$$

such that $Q^1 T_{b_1'} (Q^1)^* = \gamma_1 I$, where, by (3.3),

$$\begin{aligned}\gamma_1(p) &= \frac{(2\xi)^{|p|+k}}{p!} \int_{\mathbb{R}^k_+} b_1'(\sqrt{r}) e^{-2\xi(r_1+\cdots+r_k)} r^p dr \\ &= \frac{(2\xi)^{|p|+\lambda+k+1}}{p!\Gamma(\lambda+1)} \int_{\mathbb{R}^k_+ \times \mathbb{R}_+} b_1(\sqrt{r}, v + r_1 + \cdots + r_k) e^{-2\xi(v+r_1+\cdots+r_k)} r^p v^\lambda dr dv,\end{aligned}$$

$p \in \mathbb{Z}^k_+$.

On the other hand, we have $b_2 = b_2(y')$, so that b_2 is a vertical symbol. Thus, applying the remarks before Theorem 3.2.1 with $n-k-1$ instead of $n-1$, there is a unitary operator

$$Q^2_\xi \colon F^2_{2\xi}(\mathbb{C}^{n-k-1}) \to L_2(\mathbb{R}^{n-k-1})$$

such that $Q^2 T_{b_2}(Q^2)^* = \gamma_2 I$, where, by (3.4),

$$\begin{aligned}\gamma_2(x') &= \pi^{-\frac{n-k-1}{2}} \int_{\mathbb{R}^{n-k-1}} b_2(-\frac{y'}{2\sqrt{\xi}}) e^{-|x'-y'|^2} dy' \\ &= \pi^{-\frac{n-k-1}{2}} \int_{\mathbb{R}^{n-k-1}} b_2(\frac{-x'+y'}{2\sqrt{\xi}}) e^{-|y'|^2} dy', \quad x' \in \mathbb{R}^{n-k-1}.\end{aligned}$$

Therefore, if we put $Q_\xi = Q_\xi^1 \otimes Q_\xi^2$, we have

$$Q_\xi T^{(\lambda)}_{\widetilde{a}_\xi} Q_\xi^* = \gamma_1 I \otimes \gamma_2 I = (\gamma_1 \otimes \gamma_2) I,$$

where

$$\begin{aligned}\gamma_1 \otimes \gamma_2(p, x') =& \pi^{-\frac{n-k-1}{2}} \frac{(2\xi)^{|p|+\lambda+k+1}}{p!\Gamma(\lambda+1)} \\ &\cdot \int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} b_1(\sqrt{r}, v + r_1 + \cdots + r_k) b_2(\frac{1}{2\sqrt{\xi}}(-x'+v')) \\ &\cdot r^p e^{-2\xi(v+r_1+\cdot+r_k)-|y'|^2} v^\lambda dr dy' dv, \\ =& \pi^{-\frac{n-k-1}{2}} \frac{(2\xi)^{|p|+\lambda+k+1}}{p!\Gamma(\lambda+1)} \\ &\cdot \int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} a(\sqrt{r}, \frac{1}{2\sqrt{\xi}}(-x'+y'), v + r_1 + \cdots + r_k) \\ &\cdot r^p e^{-2\xi(v+r_1+\cdot+r_k)-|y'|^2} v^\lambda dr dy' dv.\end{aligned}$$

As in the preceding cases, to conclude the proof, we only need to consider the unitary operator $Q = \int_{\mathbb{R}_+}^{\oplus} Q_\xi d\xi$ which maps $\int_{\mathbb{R}_+}^{\oplus} F^2_{2\xi}(\mathbb{C}^{n-1}) d\xi$ onto

$$\int_{\mathbb{R}_+}^{\oplus} l_2(\mathbb{Z}^k_+) \otimes L_2(\mathbb{R}^{n-k-1}) d\xi = l_2(\mathbb{Z}^k_+) \otimes L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+).$$

□

Chapter 4

The Banach Algebra $\mathcal{T}(\lambda)$

In this chapter we study the structure of the commutative Banach algebra generated by Toeplitz operators with parabolic quasi-radial quasi-homogeneous symbols $\mathcal{T}(\lambda)$ for the case $n = 3$. This algebra was introduced in Preliminaries (1.8.2).

4.1 Parabolic (2)-quasi-radial symbols

First we study operators with parabolic (2)-quasi-radial symbols. According to Lemma 1.8.3, for a parabolic (2)-quasi-radial function $a(r, y)$ we have $RT_aR^* = \gamma_a I$, where

$$\begin{aligned}\gamma_a(|\alpha|, \xi) &= \frac{1}{\Gamma(\lambda+1)(|\alpha|+1)!} \int_{\mathbb{R}_+^2} a\left(\sqrt{\frac{r}{2\xi}}, \frac{v+r}{2\xi}\right) v^\lambda e^{-(v+r)} r^{|\alpha|+1} dv dr \\ &= \frac{1}{\Gamma(\lambda+1)(|\alpha|+1)!} \int_{\mathbb{R}_+^2} a\left(\sqrt{r}, v+r\right) (2\xi)^{\lambda+|\alpha|+3} v^\lambda e^{-2\xi(v+r)} r^{|\alpha|+1} dv dr,\end{aligned} \tag{4.1}$$

and R is the operator from $L_2(D_3, d\mu_\lambda)$ onto $l_2(\mathbb{Z}_+^2, L_2(\mathbb{R}_+))$ defined in Preliminaries (1.8.2) whose restriction $R|_{\mathcal{A}_\lambda^2(D_n)}$ is a unitary operator (and coincides with the operator $RQ|_{\mathcal{A}_\lambda^2(D_n)}$ defined in Section 3.1). Let $\mathcal{T}_{qr}$ be the Banach algebra generated by Toeplitz operator with parabolic $(2)-$quasi-radial symbols.

In order to simplify the notation, we will write $\widetilde{T}_\varphi$ for the operator $RT_\varphi R^*$, $\widetilde{\mathcal{T}(\lambda)}$ for the Banach algebra $R\mathcal{T}(\lambda)R^*$ and, in general, $\widetilde{\mathcal{A}} = R\mathcal{A}R^*$ for a given algebra $\mathcal{A} \subset \mathcal{T}(\lambda)$. Thus, $\widetilde{\mathcal{T}_{qr}}$ is the algebra generated by the multiplication operators $\gamma_a I$.

Fix $a \in L_\infty(\mathbb{R}_+)$ and let $|\alpha| \in \mathbb{Z}_+$, $\xi_1, \xi_2 \in \mathbb{R}_+$. Note that

$$\begin{aligned}
&|\gamma_a(|\alpha|, \xi_1) - \gamma_a(|\alpha|, \xi_2)| \\
&\qquad \leq C \int_{\mathbb{R}_+^2} |(2\xi_1)^{\lambda+|\alpha|+3} v^\lambda e^{-2\xi_1(v+r)} r^{|\alpha|+1} - (2\xi_2)^{\lambda+|\alpha|+3} v^\lambda e^{-2\xi_2(v+r)} r^{|\alpha|+1}| dv dr \\
&\qquad \leq C \left(\int_{\mathbb{R}_+} (2\xi_1)^{\lambda+1} v^\lambda e^{-2\xi_1 v} dv \int_{\mathbb{R}_+} |(2\xi_1)^{|\alpha|+2} r^{|\alpha|+1} e^{-2\xi_1 r} - (2\xi_2)^{|\alpha|+2} r^{|\alpha|+1} e^{-2\xi_2 r}| dr \right. \\
&\qquad \left. + \int_{\mathbb{R}_+} (2\xi_2)^{|\alpha|+2} r^{|\alpha|+1} e^{-2\xi_2 v} dr \int_{\mathbb{R}_+} |(2\xi_1)^{\lambda+1} v^\lambda e^{-2\xi_1 v} - (2\xi_2)^{\lambda+1} v^\lambda e^{-2\xi_2 v}| dv \right) \\
&\qquad = \|a\|_\infty \left(\frac{1}{(|\alpha|+1)!} \int_{\mathbb{R}_+} |(2\xi_1)^{|\alpha|+2} r^{|\alpha|+1} e^{-2\xi_1 r} - (2\xi_2)^{|\alpha|+2} r^{|\alpha|+1} e^{-2\xi_2 r}| dr \right. \\
&\qquad \left. + \frac{1}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+} |(2\xi_1)^{\lambda+1} v^\lambda e^{-2\xi_1 v} - (2\xi_2)^{\lambda+1} v^\lambda e^{-2\xi_2 v}| dv \right),
\end{aligned}$$

where $C = \dfrac{\|a\|_\infty}{\Gamma(\lambda+1)(|\alpha|+1)!}$.

By Lemma 2.3.1 we have

$$|\gamma_a(|\alpha|, \xi_1) - \gamma_a(|\alpha|, \xi_2)| \leq \|a\|_\infty \left(\frac{|\alpha|^{|\alpha|} e^{-|\alpha|}}{(|\alpha|+1)!} + \frac{\lambda^\lambda e^{-\lambda}}{(\lambda+1)!} \right) \rho(\xi_2, \xi_1).$$

Therefore, for all sufficiently large integers $|\alpha|$ (independently of a), by Stirling's formula, there is some constant C such that

$$|\gamma_a(|\alpha|, \xi_1) - \gamma_a(|\alpha|, \xi_2)| \leq C\sqrt{|\alpha|} \rho(\xi_2, \xi_1).$$

That is, fixing $|\alpha| \in \mathbb{Z}_+$, $\xi \mapsto \gamma_a(|\alpha|, \xi)$ turns out to be a very slowly oscillating function.

On the other hand, let $\xi \in \mathbb{R}_+$ and $k_1, k_2 \in \mathbb{Z}^+$ with $k_2 \geq k_1$. We have, by (4.1),

$$\begin{aligned}
|\gamma_a(k_1, \xi) - \gamma_a(k_2, \xi)| &\leq \frac{\|a\|_\infty}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+^2} \left| \frac{r^{k_1+1}}{(k_1+1)!} - \frac{r^{k_2+1}}{(k_2+1)!} \right| v^\lambda e^{-(v+r)} dv dr \\
&= \|a\|_\infty \int_{\mathbb{R}_+} \left| \frac{r^{k_1+1}}{(k_1+1)!} - \frac{r^{k_2+1}}{(k_2+1)!} \right| e^{-r} dr \\
&= 2\|a\|_\infty \int_0^{r_0} \left(\frac{r^{k_1+1}}{(k_1+1)!} - \frac{r^{k_2+1}}{(k_2+1)!} \right) e^{-r} dr,
\end{aligned}$$

where

$$r_0 = \left(\frac{(k_2+1)!}{(k_1+1)!}\right)^{\frac{1}{k_2-k_1}}.$$

Integrating by parts $k_2 - k_1$ we get

$$\int_0^{r_0} \frac{r^{k_2+1}}{(k_2+1)!}e^{-r} = -\frac{r_0^{k_2+1}}{(k_2+1)!}e^{-r_0} - \frac{r_0^{k_2}}{k_2!}e^{-r_0} - \cdots - \frac{r_0^{k_1+2}}{(k_1+2)!}e^{-r_0} + \int_0^{r_0} \frac{r^{k_1+1}}{(k_1+1)!}e^{-r}dr$$

and, substituting,

$$|\gamma_a(k_1,\xi) - \gamma_a(k_2,\xi)| \le 2\|a\|_\infty \left(\frac{r_0^{k_2+1}}{(k_2+1)!}e^{-r_0} + \frac{r_0^{k_2}}{k_2!}e^{-r_0} + \cdots + \frac{r_0^{k_1+2}}{(k_1+2)!}e^{-r_0}\right).$$

Since

$$\sup_{t\in\mathbb{R}_+} r^j e^{-r} = j^j e^{-j},$$

using Stirling's approximation we obtain a constant $C > 0$ such that for all sufficiently large k_1 (independently of a),

$$\begin{aligned}
\frac{r_0^{k_2+1}}{(k_2+1)!}e^{-r_0} + \frac{r_0^{k_2}}{k_2!}e^{-r_0} + \cdots + \frac{r_0^{k_1+2}}{(k_1+2)!}e^{-r_0} &\le C\frac{1}{\sqrt{k_2+1}} + \cdots + \frac{1}{\sqrt{k_1+2}} \\
&\le \frac{C}{2}\int_{k_1+1}^{k_2+1}\frac{1}{\sqrt{x}} \le \frac{C}{2}\int_{k_1}^{k_2}\frac{dx}{\sqrt{x}} \\
&= C(\sqrt{k_2} - \sqrt{k_1}).
\end{aligned}$$

We conclude that

$$|\gamma_a(k_1,\xi) - \gamma_a(k_2,\xi)| \le C(\sqrt{k_2} - \sqrt{k_1}).$$

Thus, fixing $\xi \in \mathbb{R}_+$, $k \mapsto \gamma_a(k,\xi)$ is a square-root-slowly oscillating function.

Moreover, it follows from the inequalities above that for arbitrary $(k_1,\xi_1),(k_2,\xi_2) \in \mathbb{Z}_+ \times \mathbb{R}_+$ we have

$$|\gamma_a(k_1,\xi_1) - \gamma_a(k_2,\xi_2)| \le C\left(\sqrt{\min(k_1,k_2)}\rho(\xi_1,\xi_2) + |\sqrt{k_2} - \sqrt{k_1}|\right). \tag{4.2}$$

In particular, γ_a is continuous at every point of $\mathbb{Z}_+ \times \mathbb{R}_+$ and, since $\gamma_a \le \|a\|_\infty$, we can identify $\widetilde{\mathcal{T}_{qr}}$ with a subalgebra $\mathcal{A}_{qr}$ of $C_B(\mathbb{Z}_+ \times \mathbb{R}_+)$, the Banach algebra of bounded continuous functions on $\mathbb{Z}_+ \times \mathbb{R}_+$.

Let us analyse this algebra by means of some special symbols. First consider the bounded function $\psi(r,y) = e^{-\frac{y}{y-r^2}+1}$.

We calculate $\gamma_\psi(|\alpha|,\xi)$. Since $\psi(\sqrt{\frac{r}{2\xi}}, \frac{v+r}{2\xi}) = e^{-(\frac{r}{v})}$, we have

$$\begin{aligned}\gamma_\psi(|\alpha|,\xi) = \hat{\gamma}_\psi(|\alpha|) &= \frac{1}{\Gamma(\lambda+1)(|\alpha|+1)!}\int_{\mathbb{R}^2_+} e^{-(\frac{r}{v})} v^\lambda e^{-(v+r)} r^{|\alpha|+1} dvdr \\ &= \frac{1}{\Gamma(\lambda+1)}\int_{\mathbb{R}_+}\left(1+\frac{1}{v}\right)^{-(|\alpha|+2)} v^\lambda e^{-v} dv.\end{aligned}$$

The first factor inside the last integral is always less than 1, thus it is dominated by the integrable function $v \mapsto v^\lambda e^{-v}$. Therefore, by Lebesgue's dominated convergence theorem we have

$$\lim_{|\alpha|\to\infty} \hat{\gamma}_\psi(|\alpha|) = 0.$$

Moreover, $\hat{\gamma}_\psi(|\alpha|)$ is strictly decreasing since

$$\left(1+\frac{1}{v}\right)^{-(|\alpha|+2)} < \left(1+\frac{1}{v}\right)^{-(|\beta|+2)}$$

whenever $|\alpha| > |\beta|$. Therefore, this function separates the points of $\mathbb{Z}_+$ and, by Stone-Weierstrass theorem, it generates (together with the identity) the whole C^* algebra of convergent sequences c.

Since $\gamma_\psi = \hat{\gamma}_\psi \otimes 1 \in C_B(\mathbb{Z}_+ \times \mathbb{R}_+)$, by the preceding remarks we conclude that the single operator $\widetilde{T}_\psi$ generates the C^*-algebra of multiplication operators of the form $\{\gamma(|\alpha|)\otimes 1\}_{\alpha\in\mathbb{Z}^2_+}$.

For each $\alpha \in \mathbb{Z}^2_+$, we denote by $\hat{e}_\alpha = \{\delta_{\alpha,\beta}\} \in \mathbb{Z}^2_+$ the α's element of the standard orthonormal basis in $l_2(\mathbb{Z}^2_+)$. Given $c(\xi) \in L_2(\mathbb{R}_+)$, let

$$\hat{e}_\alpha(c(\xi)) = \hat{e}_\alpha \otimes c(\xi) = \{\delta_{\alpha,\beta} c(\xi)\}_{\beta\in\mathbb{Z}^2_+} \tag{4.3}$$

be the corresponding one-component element of $l_2(\mathbb{Z}^2_+, L_2(\mathbb{R}_+))$.

For each $k \in \mathbb{Z}_+$, we denote by H_k the following subspace of $l_2(\mathbb{Z}^{n-1}_+, L_2(\mathbb{R}_+))$:

$$H_k = \operatorname{span}\{\hat{e}_\alpha(c(\xi))\colon |\alpha| = k, c \in L_2(\mathbb{R}_+)\}. \tag{4.4}$$

Let P_k be the orthogonal projection from $l_2(\mathbb{Z}^{n-1}_+, L_2(\mathbb{R}_+))$ onto H_k. Note that $P_k = \{\delta_{|\alpha|,k} \otimes 1\}_{\alpha\in\mathbb{Z}^2_+} I$.

We summarize the preceding remarks:

4.1.1 Corollary. *Let $\gamma \in c$. Then $\{\gamma(|\alpha|) \otimes 1\}_{\alpha \in \mathbb{Z}_+^2} I$ is an element of $\widetilde{\mathcal{T}(\lambda)}$. In particular, $P_k \in \widetilde{\mathcal{T}(\lambda)}$, for every $k \in \mathbb{Z}_+$.*

We can also obtain continuous functions depending only on ξ as follows. Let $\psi_2(r, y) = \chi_{\mathbb{R}_+}(y - r^2)e^{-y-r^2}$. Then

$$\begin{aligned}\gamma_{\psi_2}(|\alpha|, \xi) &= \frac{1}{\Gamma(\lambda+1)(|\alpha|+1)!} \int_{\mathbb{R}_+^2} e^{-\frac{v}{2\xi}} r^{|\alpha|+1} v^{\lambda} e^{-(v+r)} dv dr \\ &= \frac{1}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+} v^{\lambda} e^{-(1+\frac{1}{2\xi})v} dv \\ &= \left(1 + \frac{1}{2\xi}\right)^{-(\lambda+1)}.\end{aligned}$$

Note that $\xi \mapsto \left(1 + \frac{1}{2\xi}\right)^{-(\lambda+1)}$ is a real-valued strictly decreasing function with finite limits at 0 and ∞. Again by Stone-Weierstrass theorem, this function, together with the identity, generates the whole C^*-algebra of continuous functions on $[0, \infty]$.

These function we have just analyzed separate the points of $\mathbb{Z}_+ \times \mathbb{R}$. Thus, the algebra $\mathcal{A}_{qr}$ also separates them and, by Theorem 8.1 in [4] we conclude that $\mathcal{A}_{qr} \cong C(X)$, where X (i.e., the maximal ideal space $M(\mathcal{T}_{qr})$ of $\mathcal{T}_{qr}$) is some compactification of $\mathbb{Z}_+ \times \mathbb{R}_+$. We identify $\mathbb{Z}_+ \times \mathbb{R}_+$ with the evaluation functionals in the usual way.

Let $M_\infty(\lambda) = X \setminus (\mathbb{Z}_+ \times \mathbb{R}_+)$. Since $\mathbb{Z}_+ \times \mathbb{R}_+$ is dense in X, for every $\mu \in M_\infty(\lambda)$ there is a net $(k_\alpha, \xi_\alpha)_{\alpha \in \Omega}$ that converges to μ. If we denote by ψ_μ the multiplicative functional associated with μ, we have

$$\gamma_a(\mu) = \varphi_\mu(\gamma_a) = \lim_{\alpha \in \Omega} \gamma_a(k_\alpha, \xi_\alpha),$$

for every $\gamma_a \in \mathcal{A}_{qr}$.

Note that $|k_\alpha| \to \infty$ or $|\xi_\alpha| \to \infty$, otherwise there would be a subnet converging to a point in $\mathbb{Z}_+ \times \mathbb{R}_+$, which must be μ.

Replacing (k_α, ξ_α) by a subnet if necessary and by a similar argument as the given above, we can assume that one of the following (mutually exclusive) cases holds:

1. $k_\alpha = k_0$ and $\xi \to \infty$, for some $k_0 \in \mathbb{Z}_+$,

2. $k \to \infty$ and $\xi \to \xi_0$, for some $\xi_0 \in \mathbb{R}_+$,

3. $k \to \infty$ and $\xi \to \infty$.

This implies that $\psi_\mu(\gamma_a) = 0$, for every γ_a such that $\lim_\alpha \gamma_a(k_\alpha, \xi_\alpha) = 0$.

Conversely, if $(k_0, \xi_0) \in \mathbb{Z}_+ \times \mathbb{R}_+$, for every net (k_α, ξ_α) that tends to infinity in some of the ways described above, there is a function $\gamma_a \in \mathcal{A}_{qr}$ such that $\gamma_a(k_\alpha, \xi_\alpha) \to 0$ and $\psi_{(k_0,\xi_0)}(\gamma_a) = \gamma_a(k_0, \xi_0) \neq 0$.

Accordingly, we define the following sets:

1. $M_{k,\infty}$: the set of all multiplicative functionals ψ such that $\psi(\gamma_a) = 0$ for every $\gamma_a \in \mathcal{A}$ with $\gamma_a(k, \xi) \to \infty$ as $\xi \to \infty$,

2. $M_{\infty,\xi}$: the set of all multiplicative functionals ψ such that $\psi(\gamma_a) = 0$ for every $\gamma_a \in \mathcal{A}$ such that $\gamma_a(k, \xi') \to 0$ as $k \to \infty$ and $\xi' \to \xi$,

3. $M_{\infty,\infty}$ the set of all multiplicative functionals ψ such that $\psi(\gamma_a) = 0$ for every $\gamma_a \in \mathcal{A}$ such that $\gamma_a(k, \xi) \to 0$ as $k \to \infty$ and $\xi \to \infty$.

By the preceding remarks we can decompose $M_\infty(\lambda)$ as the disjoint union

$$M_\infty(\lambda) = \left(\bigsqcup_{k \in \mathbb{Z}_+} M_{k,\infty} \right) \bigsqcup \left(\bigsqcup_{\xi \in \mathbb{R}_+} M_{\infty,\xi} \right) \bigsqcup M_{\infty,\infty}. \tag{4.5}$$

We will denote

$$M_{\infty,\mathbb{R}_+}(\lambda) = \left(\bigsqcup_{\xi \in \mathbb{R}_+} M_{\infty,\xi} \right) \bigsqcup M_{\infty,\infty}.$$

4.2 The set $M_{\infty,\infty}$

We study more deeply the set $M_{\infty,\infty}$. Let $\mu \in M_{\infty,\infty}$. By construction there is a net $\{(k_\alpha, \xi_\alpha)\}_{\alpha \in \Lambda}$ converging to μ and such that $k_\alpha \to \infty$ and $\xi_\alpha \to \infty$.

Consider the symbol $a_1(r, y) = e^{-r^2}$ we used in the previous section. We have

$$\gamma_1(k, \xi) = \left(1 + \frac{1}{2\xi}\right)^{-k-2}.$$

Since

$$\gamma_1(\mu) = \lim_{\alpha \in \Lambda} \gamma_1(k_\alpha, \xi_\alpha)$$

is some positive real number,

$$\log(\gamma_1(\mu)) = -\lim_{\alpha\in\Lambda}(k_\alpha + 2)\log\left(1 + \frac{1}{2\xi_\alpha}\right)$$

is either a positive real number or ∞.

The series
$$\log\left(1 + \frac{1}{2\xi}\right) = \sum_{n=1}(-1)^{n+1}\left(\frac{1}{2\xi}\right)^n \frac{1}{n}$$
converges uniformly with respect to ξ for sufficiently large values. Thus we can write

$$(k_\alpha + 2)\log\left(1 + \frac{1}{2\xi_\alpha}\right) = \frac{k_\alpha + 2}{2\xi_\alpha}\left(1 - \left(\frac{1}{2\xi_\alpha}\right)\frac{1}{2} + \left(\frac{1}{2\xi_\alpha}\right)^2\frac{1}{3} + \cdots\right).$$

Since
$$\sum_{n=2}\left(\frac{1}{2\xi}\right)^{n-1}\frac{1}{n} < \sum_{n=2}\left(\frac{1}{2\xi}\right)^{n-1},$$
where the last series converges to zero uniformly as $\xi \to \infty$, it follows that the limit

$$\lim_{\alpha\in\Lambda}\frac{k_\alpha + 2}{2\xi_\alpha}$$

exists and, therefore,
$$\lim_{\alpha\in\Lambda}\frac{k_\alpha + 2}{\xi_\alpha} = \lim_{\alpha\in\Lambda}\frac{k_\alpha}{\xi_\alpha}$$
also exists and is equal to an element of $[0, \infty]$.

Hence there is a unique θ_μ such that

$$\tan(\theta_\mu) = \lim_{\alpha\in\Lambda}\frac{k_\alpha}{\xi_\alpha}$$

and we can decompose
$$M_{\infty,\infty} = \bigsqcup_{\theta\in[0,\frac{\pi}{2}]} M_\theta,$$
where
$$M_\theta = \{\mu \in M_{\infty,\infty} \colon \theta_\mu = \theta\}.$$

Note that, writing (k_α, ξ_α) in polar coordinates, that is,

$$\xi_\alpha = r_\alpha \cos\theta_\alpha,$$

$$k_\alpha = r_\alpha \sin\theta_\alpha,$$

where $r_\alpha = \sqrt{k_\alpha^2 + \xi_\alpha^2}$ and $\theta_\alpha = \arctan\frac{k_\alpha}{\xi_\alpha}$, we have

$$\theta_\mu = \lim_{\alpha\in\Lambda} \theta_\alpha.$$

Moreover, writing two elements of $\mathbb{Z}_+ \times \mathbb{R}_+$ in polar coordinates (r_1, θ_1), (r_2, θ), the inequalities obtained in (4.2) become

$$|\gamma_a(r_1,\theta_1) - \gamma_a(r_2,\theta_2)| \le C\left(\sqrt{\min(r_1\cos\theta_1, r_2\cos\theta_2)}\rho(\sin\theta_1, \sin\theta_2) \right.$$
$$\left. + |\sqrt{r_1\cos\theta_1} - \sqrt{r_2\cos\theta_2}|\right),$$

for all sufficiently large r_1, r_2.

In particular, fixing r, the function $\theta \longmapsto \gamma_a(r,\theta)$ is uniformly continuous with respect to the metric $(\theta_1,\theta_2) \mapsto \rho(\sin\theta_1, \sin\theta_2) + |\sqrt{\cos\theta_1} - \sqrt{\cos\theta_2}|$.

On the other hand, fixing θ the function $r \longmapsto \gamma_a(r,\theta)$ is square-root-slowly oscillating. Discarding the limit cases $\theta = 0$ and $\theta = \pi/2$, one expects that a point in M_θ can be reached by some net with constant angle θ. Thus the sets M_θ seem to be homeomorphic, since they induce the same function algebras.

This can be proved as follows. We will show that all sets M_θ are homeomorphic among them for any $\theta \in (0, \frac{\pi}{2})$.

Let $t \ge 0$ and define the function

$$\Phi_t \colon \mathbb{Z}_+ \times \mathbb{R}_+ \to \mathbb{Z}_+ \times \mathbb{R}_+$$

by

$$\Phi_t(k,\xi) = (k, t\xi).$$

Φ_t is clearly a bijection with inverse $\Phi_{t^{-1}}$. We can extend Φ_t to the compactification X of $\mathbb{Z}_+ \times \mathbb{R}_+$ as follows.

Suppose $\{(k_\alpha, \xi_\alpha)\}_{\alpha\in\Omega}$ is a net in $\mathbb{Z}_+ \times \mathbb{R}_+$ that converges to some X. Then $\gamma_a(k_\alpha, r_\alpha)$ converges to $\gamma_a(\mu)$ for every symbol $a \in L_\infty(\mathbb{R}_+ \times \mathbb{R}_+)$.

In particular, $\gamma_t(k_\alpha, \xi_\alpha)$ must be convergent, for every symbol a, where we define

$$a_t(x,y) = a\left(\frac{x}{\sqrt{t}}, \frac{y}{t}\right).$$

By (4.1), γ_t is given by

$$\begin{aligned}\gamma_t(k,\xi) &= \frac{1}{\Gamma(\lambda+1)(k+1)!}\int_{\mathbb{R}^2_+} a\left(\sqrt{\frac{r}{2t\xi}}, \frac{v+r}{2t\xi}\right) v^\lambda r^{k+1} e^{-(v+r)} dv dr \\ &= \gamma_a(k, t\xi) = \gamma_a(\Phi_t(k,\xi)).\end{aligned}$$

Therefore, $\gamma_a(\Phi_t(k_\alpha,\xi_\alpha))$ is convergent for every $a \in L_\infty(\mathbb{R}_+ \times \mathbb{R}_+)$. Since X is compact, there is some $\nu \in X$ such that $(k_\alpha, \xi_\alpha) \to \nu$. We can thus define $\Phi_t(\mu) = \nu$.

The extension $\Phi_t \colon X \to X$ is continuous since, by similar arguments,

$$\gamma_a(\Phi_t(k_\alpha,\xi_\alpha)) \to \gamma_a(\Phi_t(\mu))$$

for every $\gamma_a \in \mathcal{A}$ and every net (k_α, ξ_α) in X converging to $\mu \in X$. Since its inverse $\Phi_{t^{-1}}$ can be extended the same way, Φ_t is indeed a homeomorphism.

Fix $\theta \in (0, \frac{\pi}{2})$ and let $\mu \in M_\theta$ and (k_α, ξ_α) a net converging to μ. Then we have

$$\theta_{\Phi_t(\mu)} = \arctan \lim_\alpha \frac{k_\alpha}{t\xi_\phi} = \arctan(\frac{\tan\theta}{t}).$$

That is, $\Phi_t(M_\theta) \subset M_{\arctan(\frac{\tan\theta}{t})}$ and, indeed, $\Phi_t(M_\theta) = M_{\arctan(\frac{\tan\theta}{t})}$, as can be seen by applying $\Phi_{t^{-1}}$ to the elements of $M_{\arctan(\frac{\tan\theta}{t})}$.

Therefore, the restriction $\Phi_t|_{M_\theta} \colon M_\theta \to M_{\arctan(\frac{\tan\theta}{t})}$ is a homeomorphism and, varying $t \in (0,\infty)$, we obtain what we claimed.

4.3 Quasi-Homogeneous Symbols

For every $p \in \mathbb{N}$ we denote by ϕ_p the function $\phi_p(\zeta) = \zeta^{(p,0)}\overline{\zeta}^{(0,p)}$; we simply write ϕ for ϕ_1.

Using Lemma 1.8.4 we have

$$\widetilde{T}_{\phi_p} = RT_{\phi_p}R^* \colon \hat{e}_{(\alpha_1,\alpha_2)}(c(\xi)) \longmapsto \begin{cases} 0, & \text{if } a_2 - p < 0, \\ \widetilde{\gamma}_{\phi_p}(\alpha,\xi)\hat{e}_{(\alpha_1+p,\alpha_2-+)}(c(\xi)), & \text{if } a_2 - p \geq 0, \end{cases} \tag{4.6}$$

where

$$\begin{aligned}
\widetilde{\gamma}_{\phi_p}(\alpha,\xi) &= \frac{2(\alpha_1+p)!\alpha_2!}{(|\alpha|+p+1)!\sqrt{\alpha_1!\alpha_2!(\alpha_1+p)!(\alpha_2-p)!}\Gamma(\lambda+1)} \\
&\times \int_{\mathbb{R}^2_+} v^\lambda e^{-(v+r^2)} r^{2(|\alpha|+1)+1} dv dr \\
&= \frac{(\alpha_1+p)!\alpha_2!}{(|\alpha|+p+1)!\sqrt{\alpha_1!\alpha_2!(\alpha_1+p)!(\alpha_2-p)!}} \int_{\mathbb{R}_+} 2e^{-r^2} r^{2(|\alpha|+1)+1} dr \\
&= \frac{(\alpha_1+p)!\alpha_2!(|\alpha|+1)!}{(|\alpha|+p+1)!\sqrt{\alpha_1!\alpha_2!(\alpha_1+p)!(\alpha_2-p)!}} \\
&= \frac{(|\alpha|+1)!}{(|\alpha|+p+1)!}\sqrt{(a_1+1)\cdots(a_1+p)(a_2-p+1)\cdots(a_2-1)a_2}.
\end{aligned}$$

In particular, we have

$$\widetilde{\gamma}_\phi(\alpha,\xi) = \frac{\sqrt{(\alpha_1+1)\alpha_2}}{|\alpha|+2}.$$

Applying this formula several times we get

$$\widetilde{T}^p_\phi = RT^p_\phi R^* \colon \hat{e}_{(\alpha_1,\alpha_2)}(c(\xi)) \longmapsto \begin{cases} 0, & \text{if } a_2-p<0, \\ \widetilde{\gamma}^{(p)}_\phi(\alpha,\xi)\hat{e}_{(\alpha_1+p,\alpha_2-p)}(c(\xi)), & \text{if } a_2-p\geq 0, \end{cases} \tag{4.7}$$

where

$$\widetilde{\gamma}^{(p)}_\phi(\alpha,\xi) = \frac{\sqrt{(a_1+1)\cdots(a_1+p)(a_2-p+1)\cdots(a_2-1)a_2}}{(|\alpha|+2)^p}. \tag{4.8}$$

By comparing the action of $RT^p_\phi R^*$ and $RT_{\phi_p}R^*$ we find that

$$\widetilde{\gamma}^{(p)}_\phi = \frac{(|\alpha|+p+1)!}{(|\alpha|+1)!(|\alpha|+2)^p}\widetilde{\gamma}_{\phi_p}.$$

That is,

$$\widetilde{T}_{\phi_p} = D_{\widetilde{d}_p}\widetilde{T}^p_\phi, \tag{4.9}$$

where $D_{\widetilde{d}_p} = \{\widetilde{d}_p(|\alpha|)\otimes 1\}_{\alpha\in\mathbb{Z}^2_+} I$,

$$\widetilde{d}_p = \frac{(|\alpha|+2)^p}{(|\alpha|+p+1)_{(p)}},$$

and $(x)_{(p)} = x(x-1)\dots(x-p+1)$ is a kind of Pochhammer symbol (compare with [1]).

We note that $\widetilde{d}_p(|\alpha|) \to 1$ as $|\alpha| \to \infty$, so by Corollary 4.1.1, we have analogous results to those in [1]:

4.3.1 Theorem. *For each $p \in \mathbb{N}$, the Toeplitz operator T_{ϕ_p} belongs to the unital algebra generated by the operators T_ψ and T_{ψ_2}.*

4.3.2 Corollary. *The Banach algebra $\mathcal{T}(\lambda)$ is generated, in fact, just by Toeplitz operators T_a with bounded measurable symbols $a(r, y)$ and the single Toeplitz operator T_ϕ.*

4.4 Invariant Subspaces

We have

$$l_2(\mathbb{Z}_+^2, L_2(\mathbb{R}_+)) = \bigoplus_{k=0}^{\infty} H_k,$$

where H_k is the subspace defined in (4.4).

Each subspace H_k is invariant for all operators from $\widetilde{\mathcal{T}(\lambda)}$. From (4.7) we observe that the operator $\widetilde{T}_\phi$ restricted to H_k is nilpotent:

$$(\widetilde{T}_\phi|_{H_k})^{k+1} = 0.$$

This implies that, for all $p \in \mathbb{N}$,

$$\bigoplus_{k=0}^{p-1} H_k \subset \ker \widetilde{T}_\phi^p. \tag{4.10}$$

Reasoning as in [1] we conclude that the algebra $\mathcal{T}(\lambda)$ is not semi-simple:

4.4.1 Lemma. *The algebra $\mathcal{T}(\lambda)$ is not semi-simple. The radical* $\operatorname{Rad}\widetilde{\mathcal{T}(\lambda)}$ *contains, in particular, all operators of the form $A = D_\gamma \widetilde{T}_{\phi_p}$, where $D_\gamma = \{\gamma(|\alpha|) \otimes 1\}_{\alpha \in \mathbb{Z}^2} I$, $\gamma \in c_0$ and $p \in \mathbb{N}$.*

Proof. It suffices to show that this kind of operators is topologically nilpotent and, by (4.9), is sufficient to prove this for the case $p = 1$.

Recall that the orthogonal projections P_k and the operator D_γ belong to the commutative algebra $\widetilde{\mathcal{T}(\lambda)}$. Since $I - (P_0 + \cdots + P_k) \in \widetilde{\mathcal{T}(\lambda)}$ is a projection and the image of $P_0 + \cdots + P_k$ is a subset of $\ker \widetilde{T}_\phi^p$ we have

$$A^k = D_\gamma^k \widetilde{T}_\phi^k = D_\gamma^k \widetilde{T}_\phi^k (I - (P_0 + \cdots + P_{k-1})) = [D_\gamma (I - (P_0 + \cdots + P_{k-1})]^k \widetilde{T}_\phi^k.$$

Note that

$$(\{\gamma(|\alpha|) \otimes 1\}_{\alpha \in \mathbb{Z}^2} I)(I - (P_0 + \cdots + P_{k-1})) = \{\gamma_k(|\alpha|) \otimes 1\}_{\alpha \in \mathbb{Z}^2_+} I,$$

where

$$\gamma_k(|\alpha|) = \begin{cases} 0, & |\alpha| \le k, \\ \gamma(|\alpha|), & |\alpha| \ge k. \end{cases}$$

Thus,

$$\|(\{\gamma(|\alpha|) \otimes 1\}_{\alpha \in \mathbb{Z}^2} I)(I - (P_0 + \cdots + P_{k-1}))\| = \sup_{l \ge k} |\gamma(l)|$$

and

$$\|A^k\|^{\frac{1}{k}} \ge \|D_\gamma(I - (P_0 + \cdots + P_k))\| \|\widetilde{T}_\phi\| = \sup_{l \ge k} |\gamma(l)| \|\widetilde{T}_\phi\| \to 0.$$

Since $\gamma \in c_0$, we have $\|A^k\|^{\frac{1}{k}} \to 0$ as $k \to \infty$. □

4.5 The algebra $\mathcal{T}_\phi$

In this section we study the unital Banach algebra generated by the single operator T_ϕ. Let $\mathcal{T}_\phi$ be this algebra and, as before, $\widetilde{\mathcal{T}}_\phi = R\mathcal{T}_\phi R^*$.

Note that the closed linear span of the elements $\hat{e}_\alpha(c(\xi))$, which were defined in (4.3), is $l_2(\mathbb{Z}^2_+, L_2(\mathbb{R}_+))$. By (4.7) and (4.8) we have

$$\|\widetilde{T}^p_\phi(\hat{e}_\alpha(c(\xi))\| \le \|\widetilde{\gamma}^{(p)}_\phi\|_\infty \|\hat{e}_\alpha(c(\xi))\|.$$

Hence $\|\widetilde{T}^p_\phi\| \le \|\widetilde{\gamma}^{(p)}_\phi\|_\infty$

The elementary inequality

$$\begin{aligned} |\alpha| + 2 &> \alpha_1 + \alpha_2 + 1 \\ &= (\alpha_1 + k) + (\alpha_2 - k + 1) \\ &\ge 2\sqrt{(\alpha_1 + k)(\alpha_2 - k + 1)} \end{aligned}$$

implies

$$|\widetilde{\gamma}^{(p)}_\phi(\alpha, \xi)| = \left| \frac{\sqrt{(\alpha_1 + 1) \cdots (\alpha_1 + p)(\alpha_2 - p + 1) \cdots (\alpha_2 - 1)\alpha_2}}{(|\alpha| + 2)^p} \right| \le 2^{-p}.$$

That is, $\|\widetilde{T}_{\phi}^{p}\| \leq 2^{-p}$. Moreover, taking $\alpha_1 = \alpha_2$ and making $|\alpha| \to \infty$ we have $|\widetilde{\gamma}_{\phi}^{(p)}(\alpha, \xi)| \to 2^{-p}$. Hence, using the same sequence, we have

$$\|\widetilde{T}^{p}(\hat{e}_{(\alpha_1,\alpha_2)}(c(\xi))\| \to 2^{-p}\|\hat{e}_{(\alpha_1,\alpha_2)}(c(\xi))\|.$$

This proves that $\|T_{\phi}^{p}\| = 2^{-p}$ and, therefore, the spectral radius of $\widetilde{T}_{\phi}$ is equal to $\frac{1}{2}$.

By the results of [10], there is a unitary operator

$$U_{\lambda} \colon L_2(\mathbb{B}^3, \mu_{\lambda}) \to L_2(D_3, \widetilde{\mu}_{\lambda})$$

that maps $\mathcal{A}_{\lambda}^{2}(\mathbb{B}^3)$ onto $\mathcal{A}_{\lambda}^{2}(D_3)$. Moreover, we have $U_{\lambda}T_{\phi}U_{\lambda} = T_{\phi\circ\omega^{-1}}$, for some bijective function ω. Therefore,

$$\overline{D}(0, \frac{1}{2}) = \operatorname{Im}\phi \circ \omega^{-1} \subset \text{ess-sp}\, T_{\phi}$$

and thus

$$\overline{D}(0, \frac{1}{2}) \subset \text{ess-sp}\, \widetilde{T}_{\phi} \subset \operatorname{sp} \widetilde{T}_{\phi} \subset \overline{D}(0, \frac{1}{2}).$$

This proves that $\operatorname{sp} \widetilde{T}_{\phi} = \overline{D}(0, \frac{1}{2})$. Furthermore, the maximal ideal space $M(\widetilde{\mathcal{T}}_{\phi})$ of the commutative Banach algebra $\widetilde{\mathcal{T}}_{\phi}$ coincides with the spectrum of $\widetilde{T}_{\phi}$, i. e., $M(\widetilde{\mathcal{T}}_{\phi}) = \overline{D}(0, \frac{1}{2})$

4.5.1 Theorem. *The Banach algebra $\widetilde{T}_{\phi}$ is isomorphic via the Gelfand transform to the algebra $C_{\alpha}(\overline{D}(0, \frac{1}{2}))$, which consists of all functions analytic in $\overline{D}(0, \frac{1}{2})$ and continuous on $\overline{D}(0, \frac{1}{2})$.*

Proof. The operators $\widetilde{T}_{\phi_p}$ act on $\hat{e}_{\alpha}(c(\xi))$ almost the same way they do in the case of the Banach algebra of Toeplitz operators with quasi-radial quasi-homogeneous symbols (see Section 1.8.1).

More precisely, we have

$$\widetilde{T}_{\phi} = \mathbf{T}_{\phi} \otimes I,$$

where $\mathbf{T}_{\phi}$ acts on $l^2(\mathbb{Z}_{+}^{2})$ exactly the same way the operator T_{ϕ}, presented in section 1.8.1, acts on the basic vectors of $\mathcal{A}_{\lambda}^{2}(\mathbb{B}^2)$.

T_{ϕ} clearly generates the same Banach algebra as $\widetilde{T}_{\phi}$, and this last one generates the same Banach algebra as $\mathbf{T}_{\phi}$ by considering the application $\widetilde{T}_{\phi} = \mathbf{T}_{\phi} \otimes I \longmapsto \mathbf{T}_{\phi}$.

□

4.6 Dense Subalgebra in $\mathcal{T}(\lambda)$

We denote by $\mathcal{D}(\lambda)$ the dense (non-closed) subalgebra of $\widetilde{\mathcal{T}(\lambda)}$ formed by finite sums of finite products of its generators: operators from $\mathcal{T}_{qr}$ and the single operator $\widetilde{T}_\phi$. An operator A from $\widetilde{\mathcal{D}(\lambda)}$ has the form

$$A = \sum_{p=0}^{m} (\gamma_p I) \widetilde{T}_\phi^p.$$

For arbitrary multiplication operators $\gamma_p I$ the above representation is not unique. We describe this ambiguity as follows.

Given a function γ defined on $\mathbb{Z}_+ \times \mathbb{R}_+$ we define the operator $K_\gamma(p)$ as the multiplication operator such that

$$K_\gamma(p)\{c_\alpha(\xi)\}_{\alpha\in\mathbb{Z}_+^2} = \begin{cases} \gamma(|\alpha|,\xi)\{c_\alpha(\xi)\}_{\alpha\in\mathbb{Z}_+^2}, & |\alpha| \leq p-1 \\ 0, & |\alpha| \geq p. \end{cases}$$

We note that $K_\gamma(0) = 0$.

Using this operators we can state the following result, similar to the one given in [1].

4.6.1 Lemma. *We have*

$$\sum_{p=0}^{m} (\gamma_p I) \widetilde{T}_\phi^p = 0 \tag{4.11}$$

if and only if $\gamma_p I = K_{\gamma_p}(p)$, for each $p = 0, 1, \dots, m$.

Proof. Note that

$$\operatorname{Im} K_{\gamma_p}(p) \subset \bigoplus_{k=0}^{p-1} H_k \subset \ker \widetilde{T}_\phi^p, \quad p = 0, \dots, m. \tag{4.12}$$

The "if" part follows from this.

On the other hand, suppose (4.11) holds. By the calculations from Section 4.3 we have

$$\widetilde{T}_\phi^p \hat{e}_\alpha(c(\xi)) = \tau_p(\alpha)\hat{e}_{(\alpha_1,\alpha_2)}(c(\xi)),$$

where $\tau_p(\alpha) \neq 0$ if $\alpha_2 - p \geq 0$, $\alpha \in \mathbb{Z}_+^2$.

Let $n \geq m$. By hypothesis we have

$$0 = \sum_{p=0}^{m} (\gamma_p I) \widetilde{T}_\phi^p \hat{e}_{(0,n)}(c(\xi)) = \sum_{p=0}^{m} \gamma_p(n,\xi) \tau_p(0,n) \hat{e}_{(p,n-p)}(c(\xi)).$$

Since $\tau_p(0,n) \neq 0$, it follows that

$$\gamma_p(n,\xi) = 0, \quad p = 1, \ldots, m$$

and, in particular, $\gamma_m I = K_{\gamma_m}(m)$. Therefore, by (4.12) and (4.11),

$$\sum_{p=0}^{m-1} (\gamma_p I) \widetilde{T}_\phi^p = 0.$$

Repeating the above arguments m times we conclude the proof. □

4.7 Finitely Generated Subalgebras of $\mathcal{A}_{qr}$

We recall some known facts and definitions. Let $\mathcal{A} = \mathcal{A}(x_1, \ldots, x_n)$ be a unital commutative Banach algebra generated by the elements $x_1, \ldots, x_n$, and let $M(\mathcal{A})$ denote is maximal ideal space.

The *joint spectrum* $\sigma(x_1, \ldots, x_n)$ of $x_1, \ldots, x_n$ is the set

$$\sigma(x_1, \ldots, x_n) = \{(m(x_1), \ldots, m(x_n) \colon m \in M(\mathcal{A})\}$$

and it is homeomorphic to $M(\mathcal{A})$ via

$$m \in M(A) \longmapsto (m(x_1), \ldots, m(x_m)) \in \sigma(x_1, \ldots, x_n).$$

We also have

$$\sigma(x_1, \ldots, x_n) = \{(\mu_1, \ldots, \mu_n) \colon J(x_1 - \mu_1 e, \ldots, x_n - \mu_n e) \neq \mathcal{A}\},$$

where $e \in \mathcal{A}$ is the unit element and $J(x_1 - \mu_1 e, \ldots, x_n - \mu_n e)$ denotes the smallest ideal of $\mathcal{A}$ containing the elements $x_j - \mu_j e$, $j = 1, \ldots, n$.

Let $\gamma_1, \ldots, \gamma_m \in \mathcal{A}_{qr}$ and let $\mathcal{A}_D^*(\gamma_1, \ldots, \gamma_m)$ denote the C^* algebra generated by the elements of $D = (\gamma_1, \ldots, \gamma_n)$.

Let $(\mu_1, \ldots, \mu_m) \in \sigma(\gamma_1, \ldots, \gamma_m)$. Then

$$\begin{aligned}\gamma &= (\overline{\gamma_1} - \overline{\mu_1})(\gamma_1 - \mu_1) + \ldots + (\overline{\gamma_n} - \overline{\mu_n})(\gamma_n - \mu_n) \\ &= |\gamma_1 - \mu_1|^2 + \ldots + |\gamma_n - \mu_n|^2\end{aligned}$$

belongs to the ideal $J(x_1 - \mu_1, \ldots, x_n - \mu_n)$. Thus γ is not invertible in $\mathcal{A}_D^*$.

The algebra $\mathcal{A}_{qr}$ can be identified with the subalgebra of $\mathscr{L}(\mathcal{A}_\lambda^2(D_3))$ of multiplication operators with symbols in $\mathcal{A}_{qr}$ and, being this one a C^* algebra, it is inverse closed. Since $\gamma \in \mathcal{A}_D^* \subset \mathcal{A}_{qr}$, γ is not invertible in $\mathscr{L}(\mathcal{A}_\lambda^2(D_3))$.

A bounded multiplication operator is not invertible if and only if its symbol is not bounded away from zero. Hence we have

4.7.1 Corollary. *Either there is $(k, \xi) \in \mathbb{Z}_+ \times \mathbb{R}_+$ such that $\gamma_j(k, \xi) = \mu_j$, $j = 1, \ldots, n$ or there is a sequence (k_l, ξ_l) in $\mathbb{Z}_+^* \times \mathbb{R}_+$ such that*

$$\lim_{l \to \infty} \gamma_j(k_l, \xi_l) = \mu_j, \quad j = 1, \ldots, n.$$

We mention that, if $\psi \in M_{\infty, \mathbb{R}_+}(\lambda)$ then

$$(\psi(\gamma_1), \ldots, \psi(\gamma_n)) = (\mu_1, \ldots, \mu_n) \in \sigma(\gamma_1, \ldots, \gamma_n)$$

and we can always assume that the second option in Corollary 4.7.1 holds and that the respective sequence is such that $k_l \to \infty$.

Indeed, let $(\gamma_0(k))$ be a real-valued decreasing sequence such that $\gamma_0(k) \to 0$. By the conclusions from Section 4.1 we have $\gamma_0 \otimes 1 \in \mathcal{A}_{qr}$ and, by definition, $\psi(\gamma_0 \otimes 1) = 0$. Thus $(0, \mu_1, \ldots, \mu_n) \in \sigma(\gamma_0 \otimes 1, \gamma_1, \ldots, \gamma_n)$ and, applying Corollary 4.7.1 to the elements $\gamma_0 \otimes 1, \gamma_1, \ldots, \gamma_n$, we obtain a sequence with the required properties, since $\gamma_0 \otimes 1$ can only achieve 0 by means of a sequence (k_l, ξ_l) such that $k_l \to \infty$.

4.8 Integral representation and Fock spaces

In order to proceed with the analysis of the algebra $\mathcal{T}(\lambda)$ we need some preliminary considerations on the Berezin transform with respect to certain subspaces of the Fock space $F_{2\xi}^2(\mathbb{C}^2)$.

As it was shown in Sections 2.1 and 2.2, there is a unitary operator

$$S\colon \mathcal{A}_\lambda^2(D_3) \to \int_{\mathbb{R}_+}^{\oplus} F_{2\xi}^2(\mathbb{C}^2) d\xi,$$

such that for any bounded function $a \in L_\infty(D_3)$, $a = a(z_1, z_2, \operatorname{Im} z_3)$, we have

$$ST_aS^* = \int_{\mathbb{R}_+}^{\oplus} T^{(\xi)}_{\widetilde{a}_\xi} d\xi,$$

where $T^{(\xi)}_{\widetilde{a}_\xi}$ is the Toeplitz operator acting on the Fock space $F^2_{2\xi}(\mathbb{C}^2)$ whose symbol is given by

$$\widetilde{a}_\xi(z) = \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+} a(z, iv + i|z|^2) e^{-2\xi v} v^\lambda dv.$$

As was already remarked in the proof of Lemma 3.1.1, for every $\xi > 0$ the family $\{e^{(2\xi)}_\alpha\}_{\alpha\in\mathbb{Z}^2_+}$ of funcions in $F^2_{2\xi}(\mathbb{C}^2)$ given by

$$e^{(2\xi)}_\alpha(z) = \sqrt{\frac{(2\xi)^{|\alpha|}}{\alpha!}} z^\alpha$$

is an orthonormal basis for this space.

We also recall a known equality, which will be used later. Let dS be the (not normalized) surface measure of the unit sphere $\mathbb{S}^2$ and $\alpha, \beta \in \mathbb{Z}^2_+$. Then

$$\int_{\mathbb{S}^2} \zeta^\alpha \overline{\zeta}^\beta dS(\zeta) = \delta_{\alpha,\beta} \frac{2\pi^2 \alpha!}{(|\alpha|+1)!}. \tag{4.13}$$

Let us examine how Toeplitz operators with quasi-parabolic (2)-quasi radial quasi-homogeneous symbols act in $\int^{\oplus}_{\mathbb{R}_+} F^2_{2\xi}(\mathbb{C}^2) d\xi$. In the proof of Theorem 3.1.2 we established an isomorphism between $\mathcal{A}^2_\lambda(D_3)$ and $l_2(\mathbb{Z}^2_+, L_2(\mathbb{R}_+))$ using the direct integral representation and obtaining the same expressions for the eigenvalue functions we used in the previous sections. One could therefore predict the action of the algebra $\widetilde{T}(\lambda)$ on $\int^{\oplus}_{\mathbb{R}_+} F^2_{2\xi}(\mathbb{C}^2) d\xi$. We do this calculations explicitly, though.

Let $a = a(r, y_3) \in L_\infty(D_3)$ be a parabolic (2)-quasi-radial function. Since $\widetilde{a}_\xi$ depends only on $r = \sqrt{r_1^2 + r_2^2}$, by Lemma 3.1.1, we have

$$T^{(\xi)}_{\widetilde{a}_\xi} e^{(2\xi)}_\alpha = \gamma^{(2\xi)}_{\widetilde{a}_\xi}(\alpha) e^{(2\xi)}_\alpha, \quad \alpha \in \mathbb{Z}^2_+.$$

Note that, by integration in polar coordinates,

$$\begin{aligned}
\gamma^{(2\xi)}_{\widetilde{a}_\xi}(\alpha) &= \langle T^{(\xi)}_{\widetilde{a}_\xi} e^{(2\xi)}_\alpha, e^{(2\xi)}_\alpha \rangle = \langle \widetilde{a}_\xi e^{(2\xi)}_\alpha, e^{(2\xi)}_\alpha \rangle \\
&= \frac{(2\xi)^{\lambda+|\alpha|+3}}{\pi^2 \alpha! \Gamma(\lambda+1)} \int_{\mathbb{C}^2 \times \mathbb{R}_+} a\left(\sqrt{r_1^2+r_2^2}, v+r_2^2+r_2^2\right) v^\lambda e^{-2\xi(|z|^2+v)} |z^\alpha|^2 dV(z) dv \\
&= \frac{(2\xi)^{\lambda+|\alpha|+3}}{\pi^2 \alpha! \Gamma(\lambda+1)} \int_{\mathbb{R}^2_+} a\left(r, v+r^2\right) e^{-2\xi(r^2+v)} r^{3+2|\alpha|} v^\lambda dr dv \int_{\zeta \in \mathbb{S}^2} |\zeta^\alpha| dS(\zeta) \\
&= \frac{(2\xi)^{\lambda+|\alpha|+3}}{\Gamma(\lambda+1)(|\alpha|+1)!} \int_{\mathbb{R}^2_+} 2a\left(r, v+r^2\right) e^{-2\xi(r^2+v)} r^{3+2|\alpha|} v^\lambda dr dv \\
&= \frac{(2\xi)^{\lambda+|\alpha|+3}}{\Gamma(\lambda+1)(|\alpha|+1)!} \int_{\mathbb{R}^2_+} a\left(\sqrt{r}, v+r\right) e^{-2\xi(r+v)} r^{|\alpha|+1} v^\lambda dr dv = \gamma_a(|\alpha|, \xi),
\end{aligned}$$

where γ_a is the function obtained in (4.1) we have been working with.

Hence,

$$(ST_a S^*)\{e^{(2\xi)}_\alpha\}_{\xi \in \mathbb{R}_+} = \left(\int^{\oplus}_{\mathbb{R}_+} T^{(\xi)}_{\widetilde{a}_\xi} d\xi \right) \{e^{(2\xi)}_\alpha\}_{\xi \in \mathbb{R}_+} = \{\gamma(|\alpha|, \xi) e^{(2\xi)}_\alpha\}_{\xi \in \mathbb{R}_+},$$

as we would expect.

Consider now $ST_{\phi_p} S^*$, $p \in \mathbb{N}$, where ϕ_p is the quasi-homogeneous symbol used in the previous sections:

$$\phi_p(z) = \zeta_1^p \overline{\zeta_2}^p = \frac{z_1^p \overline{z_2}^p}{(|z_1|^2+|z_2|^2)^p}, \quad z \in D_3.$$

By (2.13),

$$\begin{aligned}
\widetilde{\phi_{p_\xi}}(z') &= \frac{(2\xi)^{\lambda+1)}}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+} \phi_p(z', iv+i|z'|^2) e^{-2\xi v} v^\lambda dv \\
&= \zeta_1^p \overline{\zeta_2}^p \frac{(2\xi)^{\lambda+1)}}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+} e^{-2\xi v} v^\lambda dv \\
&= \zeta_1^p \overline{\zeta_2}^p.
\end{aligned}$$

Due to this, we will write simply ϕ_p for $\widetilde{\phi_{p_\xi}}$.

Integrating in polar coordinates, for $\alpha, \beta \in \mathbb{Z}_+^2$ we have

$$\begin{aligned}
\langle T^{(\xi)}_{\widetilde{\phi_{p_\xi}}} e^{(2\xi)}_\alpha, e^{(2\xi)}_\beta \rangle &= \langle \widetilde{\phi_{p_\xi}} e^{(2\xi)}_\alpha, e^{(2\xi)}_\beta \rangle \\
&= \sqrt{\frac{(2\xi)^{|\alpha|+|\beta|}}{\alpha!\beta!}} \left(\frac{2\xi}{\pi}\right)^2 \int_{\mathbb{C}^2} \frac{z_1^p \overline{z_2}^p}{(|z_1|^2+|z_2|^2)^p} z^\alpha \overline{z}^\beta e^{-2\xi|z|^2} dV(z) \\
&= \sqrt{\frac{(2\xi)^{|\alpha|+|\beta|}}{\alpha!\beta!}} \left(\frac{2\xi}{\pi}\right)^2 \int_{\mathbb{R}_+} r^{3+|\alpha|+|\beta|} e^{-2\xi r^2} dr \int_{\zeta\in\mathbb{S}^2} \zeta^{(\alpha_1+p,\alpha_2)} \overline{\zeta}^{\beta_1,\beta_2+p} dS(\zeta).
\end{aligned}$$

Note that, by (4.13), the last expression equals 0 unless $\beta = (\alpha_1+p, \alpha_2-p)$. In particular, $T^{(\xi)}_{\widetilde{\phi_{p_\xi}}} e^{(2\xi)}_\alpha = 0$ if $\alpha_2 - p < 0$.

Assuming $\beta = (\alpha_1+p, \alpha_2-p)$ we have

$$\begin{aligned}
\langle T^{(\xi)}_{\widetilde{\phi_{p_\xi}}} e^{(2\xi)}_\alpha, e^{(2\xi)}_\beta \rangle &= \frac{(2\xi)^{|\alpha|+2}(\alpha_1+p)!\alpha_2!}{\sqrt{\alpha_1!\alpha_2!(\alpha_1+p)!(\alpha_2-p)!}(|\alpha|+p+1)!} \int_{\mathbb{R}_+} 2r^{3+2|\alpha|} e^{-2\xi r^2} dr \\
&= \frac{(2\xi)^{|\alpha|+2}(\alpha_1+p)!\alpha_2!}{\sqrt{\alpha_1!\alpha_2!(\alpha_1+p)!(\alpha_2-p)!}(|\alpha|+p+1)!} \int_{\mathbb{R}_+} r^{|\alpha|+1} e^{-2\xi r} dr \\
&= \frac{(\alpha_1+p)!\alpha_2!(|\alpha|+1)!}{\sqrt{\alpha_1!\alpha_2!(\alpha_1+p)!(\alpha_2-p)!}(|\alpha|+p+1)!} \\
&= \widetilde{\gamma}_{\phi_p}(\alpha)
\end{aligned} \tag{4.14}$$

Note that this is the same expression we obtained in Section 4.3. In particular, it does not depend on the weight parameter 2ξ.

It follows from these relations that $(T^{(\xi)}_{\widetilde{\phi_{p_\xi}}})^*$ acts by the following rule

$$(T^{(\xi)}_{\widetilde{\phi_{p_\xi}}})^*(e^{(2\xi)}_\alpha) = \begin{cases} 0, & \text{if } a_1 < p \\ \widetilde{\gamma}_{\phi_p}((\alpha_1-p, \alpha_2+p)), & \text{if } a_1 \geq p \end{cases}$$

Furthermore, doing a similar calculation one obtains

$$T^{(\xi)}_{\widetilde{\phi_p\overline{\phi}_{q_\xi}}}(e^{(2\xi)}_\alpha) = \begin{cases} 0, & \text{if } \alpha_1+(p-q) < 0 \text{ or } \alpha_2-(p-q) < 0 \\ \widetilde{\gamma}_{\phi_p\overline{\phi_q}}(\alpha), & \text{if } \alpha_1+(p-q) \geq 0 \text{ and } \alpha_2-(p-q) \geq 0, \end{cases} \tag{4.15}$$

where

$$\widetilde{\gamma}_{\phi_p\overline{\phi_q}}(\alpha) = \frac{(\alpha_1+p)!(\alpha_2+q)!}{\sqrt{\alpha_1!\alpha_2!(\alpha_1+(p-q))!(\alpha_2-(p-q))!}}\frac{(|\alpha|+1)!}{(|\alpha|+p+q+1)!} \tag{4.16}$$

4.9 Berezin Transform and Dirac Sequences

We introduce now the generalized Berezin transform.

Let $\xi > 0$ and $S \subset \mathbb{Z}_+^2$. We denote by H_S the following closed subspace of $F_{2\xi}^2(\mathbb{C}^2)$:

$$H_S = \overline{\operatorname{span}}\{e_\alpha^{(2\xi)} \colon \alpha \in S\}$$

and let $K_S^{(2\xi)}(z,w) = \sum_{\alpha\in S} \overline{e_\alpha^{(2\xi)}(w)} e_\alpha^{(2\xi)}(z)$ be its reproducing kernel.

We have $\|K_S^{(2\xi)}(\cdot,w)\|^2 = K_S^{(2\xi)}(w,w)$.

4.9.1 Definition. Let C be a bounded operator in $F_{2\xi}^2(\mathbb{C}^2)$. We define the *Berezin transform of C with respect to H_S* as the function $B_S^{(2\xi)}[C]$ defined on $\mathbb{C}$ and given by

$$B_S^{(2\xi)}[C](w) = \frac{1}{\|K_S^{(2\xi)}(\cdot,w)\|^2}\langle CK_S^{(2\xi)}(\cdot,w), K_S^{(2\xi)}(\cdot,w)\rangle.$$

In the case of a bounded function ψ we write $B_S^{(2\xi)}[\psi] = B_S^{(2\xi)}[T_\psi]$, where T_ψ is the Toeplitz operator with symbol ψ acting on $F_{2\xi}^2(\mathbb{C}^2)$.

Now let (g_n), $g_n \colon \mathbb{R} \to \mathbb{R}_+$, be a Dirac sequence (see Preliminaries). We recall that

$$f(x) = \lim_{n\to\infty} (g_n * f)(x),$$

for every $f \in L_\infty(\mathbb{R})$ continuous in x.

In particular, for a bounded continuous function $\gamma(\alpha,\xi)$ defined on $\mathbb{Z}_+^2 \times \mathbb{R}_+$ and $\xi_0 > 0$ we have

$$\lim_{n\to\infty}\int_{\mathbb{R}} g_n(\xi_0-t)(\gamma_a(\alpha,t)\chi_{\mathbb{R}_+}(t))dt = \lim_{n\to\infty}\int_{\mathbb{R}_+} g_n(\xi_0-\xi)\gamma_a(\alpha,\xi)d\xi = \gamma_a(\alpha,\xi_0), \tag{4.17}$$

where we define $\gamma_a(\alpha,t) = 0$ for $t < 0$.

Let $(g_n^{(\xi_0)})$ be the sequence of functions defined by $g_n^{(\xi_0)}(t) = \sqrt{g_n(\xi_0-t)}\chi_{\mathbb{R}_+}(t)$. We can consider each $g_n^{(\xi_0)}$ defined only on $\mathbb{R}_+$ and we have $g_n^{(\xi_0)} \in L_2(\mathbb{R}_+)$, with $\|g_n^{(\xi_0)}\| = 1$.

Moreover, (4.17) implies

$$\begin{aligned}\lim_{n\to\infty}\langle\gamma(\alpha,\cdot)g_n^{(\xi_0)},g_n^{(\xi_0)}\rangle &= \lim_{n\to\infty}\int_{\mathbb{R}_+}(g_n^{(\xi_0)}(\xi))^2\gamma(\alpha,\xi)d\xi \\ &= \lim_{n\to\infty}\int_{\mathbb{R}_+} g_n(\xi_0-\xi)\gamma(\alpha,\xi)d\xi = \gamma(\alpha,\xi_0).\end{aligned} \tag{4.18}$$

Let γ be a continuous bounded function on $\mathbb{Z}_+\times\mathbb{R}_+$ and suppose that there is a sequence $((\nu_k,\xi_k))_{k=1}^\infty$ such that

$$\gamma(\nu_k,\xi_k)\to\eta\in\mathbb{C}.$$

For such a sequence we define

$$S_k=\{\alpha\in\mathbb{Z}_+^2\colon |\alpha|=\nu_k\}.$$

Since γ is continuous and bounded, for every $k\in\mathbb{N}$ we have

$$\gamma(\nu_k,\xi_k)=\lim_{n\to\infty}\int_{\mathbb{R}_+}(g_n^{(\xi_k)}(\xi))^2\gamma(\nu_k,\xi)d\xi.$$

Thus, by the diagonal method, we can construct a sequence $(\xi_{n(k)}^{(\xi_k)})_{k=1}^\infty$ such that

$$\mu=\lim_{k\to\infty}\gamma(\nu_k,\xi_k)=\lim_{k\to\infty}\int_{\mathbb{R}_+}(g_{n(k)}^{(\xi_k)}(\xi))^2\gamma(\nu_k,\xi)d\xi. \tag{4.19}$$

Given γ as above and considering its associated sequences we have just constructed, for every $w\in\mathbb{C}^2$ we define the sequence $(f_k^{(w)})_{k=1}^\infty$ given by

$$f_k^{(w)}=\left\{g_{n(k)}^{(\xi_k)}(\xi)\frac{K_{S_k}^{(2\xi)}(\cdot,w)}{\|K_{S_k}^{(2\xi)}(\cdot,w)\|}\right\}_{\xi\in\mathbb{R}_+}\in\int_{\mathbb{R}_+}^{\oplus}F_{2\xi}^2(\mathbb{C}^2)d\xi. \tag{4.20}$$

Note that $f_k^{(w)}$ is a unitary vector:

$$\|f_k^{(w)}\|^2=\int_{\mathbb{R}^+}(g_{n(k)}^{(\xi_k)}(\xi))^2\frac{\langle K_{S_k}^{(2\xi)}(\cdot,w),K_{S_k}^{(2\xi)}(\cdot,w)\rangle_{F_{2\xi}^2(\mathbb{C}^2)}}{\|K_{S_k}^{(2\xi)}(\cdot,w)\|^2}d\xi=1$$

4.9.2 Theorem. *Let $a=a(r,y_3)\in L_\infty(D_3)$ be a (2)-quasi-radial quasi-parabolic function and $\gamma_a\in\mathcal{A}_{qr}$ its associated eigenvalue function. Suppose there is a sequence*

(ν_k, ξ_k) in $\mathbb{Z}_+ \times \mathbb{R}_+$ such that $\gamma(\nu_k, \xi_k) \to \eta \in \mathbb{C}$. Then, for every $w \in \mathbb{C}^2$,

$$\lim_{k\to\infty} \langle ST_a S^* f_k^{(w)}, f_k^{(w)} \rangle = \eta,$$

where $f_k^{(w)}$ is given by (4.20).

Proof. We have

$$\begin{aligned} ST_a S^* f_k^{(w)} &= \left(\int_{\mathbb{R}_+}^{\oplus} T_{\widetilde{a}_\xi}^{(\xi)} d\xi \right) \left\{ g_{n(k)}^{(\xi_k)}(\xi) \frac{K_{S_K}^{(2\xi)}(\cdot, w)}{\|K_{S_K}^{(2\xi)}(\cdot, w)\|} \right\}_{\xi\in\mathbb{R}_+} \\ &= \left\{ \frac{g_{n(k)}^{(\xi_k)}(\xi)}{\|K_{S_K}^{(2\xi)}(\cdot, w)\|} T_{\widetilde{a}_\xi}^{(\xi)} (K_{S_K}^{(2\xi)}(\cdot, w)) \right\}_{\xi\in\mathbb{R}_+}, \end{aligned}$$

Since

$$\begin{aligned} T_{\widetilde{a}_\xi}^{(\xi)} K_{S_K}^{(2\xi)}(z, w) &= \sum_{|\alpha|=\nu_k} \overline{e_\alpha^{(2\xi)}(w)} T_{\widetilde{a}_\xi}^{(\xi)} (e_\alpha^{(2\xi)}(z)) \\ &= \sum_{|\alpha|=\nu_k} \overline{e_\alpha^{(2\xi)}(w)} e_\alpha^{(2\xi)}(z) \gamma_a(|\alpha|, \xi), \end{aligned}$$

we get

$$\begin{aligned} \langle T_{\widetilde{a}_\xi}^{(\xi)} K_{S_K}^{(2\xi)}(\cdot, w), K_{S_K}^{(2\xi)}(\cdot, w) \rangle_{F_{2\xi}^2(\mathbb{C}^2)} &= \sum_{|\beta|,|\alpha|=\nu_k} \overline{e_\alpha^{(2\xi)}(w)} e_\beta^{(2\xi)}(w) \langle T_{\widetilde{a}_\xi}^{(\xi)} e_\alpha^{(2\xi)}, e_\beta^{(2\xi)} \rangle_{F_{2\xi}^2(\mathbb{C}^2)} \\ &= \sum_{|\alpha|=\nu_k} |e_\alpha^{(2\xi)}(w)|^2 \gamma_a(|\alpha|, \xi) \\ &= \gamma_a(\nu_k, \xi) \sum_{|\alpha|=\nu_k} |e_\alpha^{(2\xi)}(w)|^2 \\ &= \gamma_a(\nu_k, \xi) \|K_{S_k}^{(2\xi)}(\cdot, w)\|^2 \end{aligned}$$

Therefore,

$$\begin{aligned} \langle STS^* f_k^{(w)}, f_k^{(w)} \rangle &= \int_{\mathbb{R}_+} \frac{(g_{n(k)}^{(\xi_k)}(\xi))^2}{\|K_{S_k}^{(2\xi)}(\cdot, w)\|^2} \langle T_{\widetilde{a}_\xi}^{(\xi)} K_{S_K}^{(2\xi)}(\cdot, w), K_{S_K}^{(2\xi)}(\cdot, w) \rangle_{F_{2\xi}^2(\mathbb{C}^2)} \\ &= \int_{\mathbb{R}_+} (g_{n(k)}^{(\xi_k)})^2 \gamma(\nu_k, \xi) d\xi \to \eta, \end{aligned}$$

by (4.19). □

4.9.3 Theorem. *Let $w \in \mathbb{C}^2\backslash\{0\}$ and $(\nu_k)_{k=1}^{\infty}$ be a sequence such that $\nu_k \to \infty$ as $k \to \infty$. If φ is a bounded continuous function on $\mathbb{C}^2$ such that $\varphi(\lambda z) = \varphi(z)$, for every $\lambda \in \mathbb{C}\backslash\{0\}$ and $z \in \mathbb{C}^2$, then*

$$B_{S_k}^{(2\xi)}[\varphi](w) = B_{S_k}^{(1)}[\varphi](w) \to \varphi(w),$$

where $B_{S_K}^{(2\xi)}$ is the generalized Berezin transform in $F_{2\xi}^2(\mathbb{C}^2)$ and

$$S_k = \{\alpha \in \mathbb{Z}_+^2 \colon |\alpha| = \nu_k\}.$$

Proof. Note that, by the multinomial theorem

$$K_{S_k}^{(2\xi)}(z,w) = \sum_{|\alpha|=\nu_k} \frac{(2\xi)^{\nu_k}}{\alpha!}\overline{w^\alpha} z^\alpha = \frac{(2\xi)^{\nu_k}}{\nu_k!}\langle z, w\rangle^{\nu_k}.$$

In particular,

$$\frac{|K_{S_k}^{(2\xi)}(z,w)|^2}{\|K(\cdot,w)\|^2} = \frac{|K_{S_k}^{(2\xi)}(z,w)|^2}{K_{S_k}^{(2\xi)}(w,w)} = (2\xi)^{\nu_k}\frac{|\langle z,w\rangle|^{2\nu_k}}{\nu_k!|w|^{2\nu_k}}$$

Thus, by a simple change of variable,

$$\begin{aligned} B_{S_k}^{(2\xi)}[\varphi](w) &= \frac{1}{K_{S_k}^{(2\xi)}(w,w)}\langle \varphi K_{S_k}^{(2\xi)}(\cdot,w), K_{S_k}^{(2\xi)}(\cdot,w)\rangle \\ &= (2\xi)^{\nu_k+2}\int_{\mathbb{C}^2}\varphi(z)\frac{|\langle z,w\rangle|^{2\nu_k}}{\nu_k!|w|^{2\nu_k}}\frac{e^{-2\xi|z|^2}}{\pi^2}dV(z) \\ &= \int_{\mathbb{C}^2}\varphi(\frac{1}{\sqrt{2\xi}}z)\frac{|\langle z,w\rangle|^{2\nu_k}}{\nu_k!|w|^{2\nu_k}}\frac{e^{-|z|^2}}{\pi^2}dV(z) \\ &= \int_{\mathbb{C}^2}\varphi(z)\frac{|\langle z,w/|w|\rangle|^{2\nu_k}}{\nu_k!|w|^{2\nu_k}}\frac{e^{-|z|^2}}{\pi^2}dV(z) \\ &= B_{S_k}^{(1)}[\varphi](w/|w|), \end{aligned}$$

which proves that the Berezin transform doesn't depend on ξ and that it suffices to consider the case $B_{S_k}^{(1)}[\varphi](\zeta) \to \varphi(\zeta)$, for $\zeta \in \mathbb{S}^2$.

Let $\omega = \varphi(\zeta)$ and $\epsilon > 0$. Since φ is continuous there exists a $\delta > 0$ such that

$$|\varphi(\zeta) - \omega| < \varepsilon, \quad |z - \zeta| < \delta.$$

Define the sets

$$O_\zeta = \{\lambda\zeta \colon \lambda \in \mathbb{T}\}$$

and

$$O_\delta = \{z \in \mathbb{C}^2 \backslash \{0\} \colon d(z/|z|, O_\zeta) < \delta\}.$$

Note that if $z \in O_\delta$, then there is some $\lambda \in \mathbb{T}$ such that $|\zeta - \lambda^{-1}(z/|z|)| = |\lambda\zeta - z/|z|| < \delta$ and hence

$$|\omega - \varphi(z)| = |\omega - \varphi(\lambda^{-1}(z/|z|))| < \varepsilon/(2\|\varphi - \omega\|_\infty).$$

Moreover, if $z \notin O_\delta$ and $z \neq 0$ then, for every $\lambda \in \mathbb{T}$, $|z/|z| - \lambda\zeta| \geq \delta$. That is,

$$1 - 2\operatorname{Re}(\overline{\lambda}\langle z/|z|, \zeta\rangle) + 1 \geq \delta^2, \quad \forall \lambda \in \mathbb{T}.$$

Taking $\lambda = \dfrac{\overline{\langle z, \zeta\rangle}}{|\langle z, \zeta\rangle|}$ this inequality becomes

$$|\langle z, \zeta\rangle| \leq |z|(1 - \delta^2).$$

Note that this inequality also holds for the case $z = 0$.

Integrating in polar coordinates we get

$$\begin{aligned}
\frac{1}{\pi^2}\int_{O_\delta^c} |\varphi(z) - \omega| \frac{|\langle z, \zeta\rangle|^{2\nu_k}}{\nu_k!} e^{-|z|^2} dV(z) &\leq \frac{(1-\delta^2)^{\nu_k}}{\nu_k!} \frac{\|\varphi - \omega\|_\infty}{\pi^2} \int_{\mathbb{C}^2} |z|^{2\nu_k} e^{-|z|^2} dV(z) \\
&= \|\varphi - \omega\|_\infty \frac{(1-\delta^2)^{\nu_k}}{\nu_k!} \int_0^\infty 2r^{2\nu_k+3} e^{-r^2} dr \\
&= \|\varphi - \omega\|_\infty (\nu_k + 1)(1-\delta^2)^{\nu_k} \to 0,
\end{aligned}$$

as $\nu_k \to \infty$. Thus there is $N \in \mathbb{N}$ such that the last member of the equality is less than $\varepsilon/2$ for $\nu_k \geq N$ and

$$\begin{aligned}
|B^{(1)}_{S_k}[\varphi](\zeta) - \omega| &= |B^{(1)}_{S_k}[\varphi - \omega](\zeta)| \\
&= \int_{O_\delta} |\varphi(z) - \omega| \frac{|\langle z, \zeta\rangle|^{2\nu_k}}{\nu_k!} \frac{e^{-|z|^2}}{\pi^2} dV(z) \\
&+ \int_{O_\delta^c} |\varphi(z) - \omega| \frac{|\langle z, \zeta\rangle|^{2\nu_k}}{\nu_k!} \frac{e^{-|z|^2}}{\pi^2} dV(z) \\
&< \varepsilon.
\end{aligned}$$

□

4.9.4 Corollary. *Let $a = a(r, y_3)$ be a (2)-quasi-radial quasi-parabolic function and $\gamma_a \in \mathcal{A}_{qr}$ its associated eigenvalue function. Let $w \in \mathbb{C}^2$ and suppose there is a sequence (ν_k, ξ_k) such that $\nu_k \to \infty$ and $\gamma_a(\nu_k, \xi_k) \to \omega \in \mathbb{C}$. If $\varphi = \phi_p\overline{\phi_q}$, for any $p, q \geq 0$ then we have*

$$\langle ST_\varphi T_a S^* f_k^{(w)}, f_k^{(w)} \rangle \to \varphi(w)\omega, \quad k \to \infty.$$

In particular, we also have

$$\langle S(T_\phi)^p T_a S^* f_k^{(w)}, f_k^{(w)} \rangle \to \phi(\zeta)^p \omega.$$

Proof. We have

$$\langle ST_\varphi T_a S^* f_k^{(w)}, f_k^{(w)} \rangle = \int_{\mathbb{R}_+} (g_{n(k)}^{(\xi_k)}(\xi))^2 B_{S_k}^{(2\xi)}[T_{\widetilde{\varphi}_\xi}^{(\xi)} T_{\widetilde{a}_\xi}^{(\xi)}](w) d\xi$$

and

$$\begin{aligned} B_{S_k}^{(2\xi)}[T_{\widetilde{\varphi}_\xi}^{(\xi)} T_{\widetilde{a}_\xi}^{(\xi)}](w) &= \frac{1}{\|K_{S_k}(\cdot, w)\|^2} \sum_{|\beta|,|\alpha|=\nu_k} \overline{e_\alpha^{(2\xi)}}(w) e_\beta^{(2\xi)}(w) \langle T_{\widetilde{\varphi}_\xi}^{(\xi)} T_{\widetilde{a}_\xi}^{(\xi)}(e_\alpha^{(2\xi)}), e_\beta^{(2\xi)} \rangle \\ &= \frac{1}{\|K_{S_k}(\cdot, w)\|^2} \sum_{|\beta|,|\alpha|=\nu_k} \overline{e_\alpha^{(2\xi)}}(w) e_\beta^{(2\xi)}(w) \gamma_a(|\alpha|, \xi) \langle T_{\widetilde{\varphi}_\xi}^{(\xi)}(e_\alpha^{(2\xi)}), e_\beta^{(2\xi)} \rangle \\ &= \gamma_a(\nu_k, \xi) B_{S_k}^{(2\xi)}[\varphi](w). \end{aligned}$$

Therefore, by Theorems 4.9.2 and 4.9.3,

$$\begin{aligned} \langle ST_\varphi T_a S^* f_k^{(w)}, f_k^{(w)} \rangle &= \int_{\mathbb{R}_+} (g_{n(k)}^{(\xi_k)}(\xi))^2 \gamma_a(\nu_k, \xi) B_{S_k}^{(2\xi)}[\varphi](w) d\xi \\ &= B_{S_k}^{(1)}[\varphi](w) \int_{\mathbb{R}_+} (g_{n(k)}^{(\xi_k)}(\xi))^2 \gamma_a(\nu_k, \xi) d\xi \\ &\to \varphi(w)\omega, \end{aligned}$$

as $k \to \infty$.

The second assertion follows immediately from this and the fact that $(\widetilde{T}_{\phi_p}) = (\{\widetilde{d_p}(|\alpha|) \otimes 1\}_{\alpha \in \mathbb{Z}_+^2} I)\widetilde{T}_\phi^p$, where $\widetilde{d_p}(|\alpha|) \to 1$, as $|\alpha| \to \infty$. □

4.10 Gelfand Theory of $\mathcal{T}(\lambda)$

Let $M(\mathcal{T}(\lambda))$ be the maximal ideal space of the commutative Banach algebra $\mathcal{T}(\lambda)$. We note that, by Corollary 4.3.2, $\mathcal{T}(\lambda)$ is generated by the algebras $\mathcal{T}_{qr}$ and $\mathcal{T}_{\phi}$. Hence (see [1]), we have a continuous injection

$$\kappa \colon \psi \in M(\mathcal{T}(\lambda)) \longmapsto (\psi_1, \psi_1) \in M(\mathcal{T}_{qr}) \times M(\mathcal{T}_{\phi}),$$

where $\psi_1 = \psi|_{\mathcal{T}_{qr}}$ and $\psi_2 = \psi|_{\mathcal{T}_{\phi}}$.

As it was shown in Sections 4.1 and 4.5, we have $M(\mathcal{T}_{qr}) = (\mathbb{Z}_+ \times \mathbb{R}_+) \cup M_\infty(\lambda)$ and $M(\mathcal{T}_\phi) = \overline{D}(0, \frac{1}{2})$, where $M_\infty(\lambda)$ denotes those multiplicative functionals defined on $\mathcal{T}_{qr}$ that map to zero those functions which, in some sense, converge to zero at infinity.

Therefore, we identify $M(\mathcal{T}(\lambda))$ with a subset of

$$((\mathbb{Z}_+ \times \mathbb{R}_+) \cup M_\infty(\lambda)) \times \overline{D}(0, \frac{1}{2}).$$

4.10.1 Lemma. *None of the points of the set*

$$\left((\mathbb{Z}_+ \times \mathbb{R}_+) \bigcup \left(\bigsqcup_{k \in \mathbb{Z}_+} M_{k,\infty} \right) \right) \times (\overline{D}(0, \frac{1}{2}) \backslash \{0\})$$

belongs to $M(\mathcal{T}(\lambda))$, where $M_{k,\infty}$ is the set defined at the end of section 4.1.

Proof. If there is a point $\psi = (k, \xi, \zeta) \in \mathbb{Z}_+ \times \mathbb{R}_+ \times \overline{D}(0, \frac{1}{2})$, then for the operator $A = P_k \widetilde{T}_\phi \in \widetilde{\mathcal{T}(\lambda)}$, where P_k is the orthogonal projection defined in section 4.1, we have $\psi(A) = 1 \cdot \zeta \neq 0$. We have $\psi(A) \in \mathrm{sp}(A)$. However, by Lemma 4.4.1, A belongs to the radical of the algebra $\widetilde{\mathcal{T}(\lambda)}$, which is a contradiction.

Similarly, if $\psi = (\mu, \zeta) \in M_{k,\infty} \times \overline{D}(0, \frac{1}{2})$, for some k, then there is a net (k, ξ_α) such that $(k, \xi_\alpha) \to \mu$. Thus, using the same A as above, we have $\psi(A) = \lim_\alpha (k, \xi_\alpha, \zeta)(A) = \zeta \neq 0$, from where it follows the same conclusion. ☐

4.10.2 Lemma. *The set $\mathbb{Z}_+ \times \mathbb{R}_+ \times \{0\}$ belongs to $M(\mathcal{T}(\lambda))$.*

Proof. Let $\psi = (k_0, \xi_0, 0) \in \mathbb{Z}_+ \times \mathbb{R}_+ \times \{0\}$. Denote by $\psi_{(k_0,\xi_0)}$ the multiplicative functional defined on $\widetilde{\mathcal{T}}_{qr}$ by $\psi_{(k_0,\xi_0)}(\gamma_a I) = \gamma_a(k_0, \xi_0)$.

We define ψ on the dense subalgebra $\widetilde{\mathcal{D}(\lambda)}$ by

$$\psi(A) = \gamma_0(k_0, \xi_0),$$

where $A = \sum_{p=0}^{m} (\gamma_p I)\widetilde{T}_\phi^p$. By Lemma 4.6.1, $\psi(A) = 0$ implies $\gamma_0 = 0$, so ψ is well-defined.

We extend this functional as follows. Let (g_n) a sequence in $L^2(\mathbb{R}_+)$ such that $\|g_n\|^2 = 1$ and

$$\lim_{n\to\infty} \langle \gamma_a(k_0, \cdot)g_n, g_n\rangle_{L_2(\mathbb{R}_+)} = \gamma_a(k_0, \xi_0).$$

(We constructed such a g_n in section 4.9).

Define the sequence (f_n) in $l_2(\mathbb{Z}_+^2, L_2(\mathbb{R}_+))$ by

$$f_n = \hat{e}_{(k_0,0)}(g_n(\xi)) = \{\delta_{\alpha,(k_0,0)} g_n(\xi)\}_{\alpha\in\mathbb{Z}_+^2}.$$

Note that

$$\|f_n\|^2 = \sum_{\alpha\in\mathbb{Z}^+} \delta_{\alpha,(k_0,0)} \|g_n\|^2 = \|g_n\|^2 = 1.$$

Since $\widetilde{T}_\phi^p(\hat{e}_{(k_0,0)}) = 0$ for every $p > 0$ (because $\alpha_2 - p = -p < 0$), for an operator $A = \sum_{p=0}^{m} (\gamma_p I)\widetilde{T}_\phi^p \in \widetilde{\mathcal{D}(\lambda)}$ we have

$$\langle Af_n, f_n\rangle = \langle \gamma_0 I f_n, f_n\rangle = \sum_{\alpha\in\mathbb{Z}_+^2} \langle \gamma_0(|\alpha|, \cdot)g_n, g_n\rangle \delta_{\alpha,(k_0,0)} = \langle \gamma_0(k_0, \cdot)g_n, g_n\rangle.$$

Thus the functional $\psi = (k_0, \xi_0, 0)$ is defined on the dense subalgebra of operators of the form $A = \sum_{p=0}^{m} (\gamma_p I)(\widetilde{T}_\phi^p)$ by

$$\lim_{n\to\infty} \langle Af_n, f_n\rangle = \gamma_0(k_0, \xi_0) = \psi(A). \tag{4.21}$$

Finally, we have

$$|\psi(A)| = |\lim_{n\to\infty} \langle Af_n, f_n\rangle| \le \limsup_{n\to\infty} \|A\|\|f_n\|^2 = \|A\|.$$

Therefore, ψ is multiplicative and continuous on a dense subalgebra of $\widetilde{\mathcal{T}(\lambda)}$ so that it extends to a multiplicative functional on $\widetilde{\mathcal{T}(\lambda)}$. □

4.10.3 Lemma. *The set*

$$M_{\infty,\mathbb{R}_+}(\lambda) \times \overline{D}(0, \frac{1}{2})$$

belongs to $M(\mathcal{T}(\lambda))$.

Proof. Let $(\mu,\zeta) \in \left(\left(\bigsqcup_{\xi\in\mathbb{R}_+} M_{\infty,\xi}\right) \bigsqcup M_{\infty,\infty}\right) \times \overline{D}(0,\frac{1}{2})$.

We define $\psi = (\mu,\zeta)$ on the dense subalgebra $c\tilde{D}(\lambda)$ as

$$\psi(A) = \sum_{p=0}^{m} \gamma_p(\mu)\zeta^p,$$

for $A = \sum_{p=0}^{m}(\gamma_p I)\widetilde{T}_\phi^p$. It is well defined since, by Lemma 4.6.1, $\psi(A) = 0$ implies that $\gamma_p(k,\xi) = 0$ for $k \geq p$ and so $\gamma_p(\mu) = 0$, $p = 1,\dots,m$. Note also that ψ is multiplicative.

We show that ψ is continuous. Consider the unital C^* algebra generated by $\gamma_1,\dots,\gamma_p$. The restriction of μ to is a multiplicative functional, thus

$$(\gamma_1(\mu),\dots,\gamma_m(\mu)) \in \sigma(\gamma_1,\dots,\gamma_m)$$

By Corollary 4.7.1 there exists a sequence (ν_k,ξ_k) such that $\nu_k \to \infty$ and

$$\gamma_p(\nu_k,\xi_k) \to \gamma_p(\mu), \quad p = 1,\dots,m.$$

We note that every γ_p together with the sequence (ν_k,ξ_k) satisfies our assumptions in Section 4.9 for the construction of the sequence $(f_k^{(w)})$, with $\zeta = \phi_p(w)$. This sequence is the same of every γ_p, since it only depends on the sequence (ν_k,ξ_k) and the point w.

Thus, by Corollary 4.9.4, we have

$$\lim_{k\to\infty} \langle S\left(\sum_{p=0}^{m} T_{a_p}T_\phi^p\right) S^* f_k^{(w)}, f_k^{(w)}\rangle = \sum_{p=0}^{m}\gamma_p(\mu)\zeta^p = \psi(A).$$

Therefore

$$|\psi(A)| = \lim_{k\to\infty} |\langle S\left(\sum_{p=0}^{m} T_{a_p}T_\phi^p\right) S^* f_k^{(w)}, f_k^{(w)}\rangle| \leq \|A\|$$

and we can extend ψ to a multiplicative functional on $\widetilde{\mathcal{T}(\lambda)}$. □

From Lemmas 4.10.1, 4.10.2, 4.10.3 and the injective tensor product description from [12], Section 3.2, we conclude:

4.10.4 Theorem. *The compact set $M(\mathcal{T}(\lambda))$ of maximal ideals of the algebra $\mathcal{T}(\lambda)$ has the form*

$$M(\mathcal{T}(\lambda)) = (\mathbb{Z}_+ \times \mathbb{R}_+ \times \{0\}) \cup \Big(M_{\infty,\mathbb{R}_+}(\lambda) \times \overline{D}(0, \frac{1}{2})\Big).$$

1. *The Gelfand image of the algebra $\mathcal{T}(\lambda)$ is isomorphic to $\mathcal{T}(\lambda)/\operatorname{Rad}\mathcal{T}(\lambda)$ and coincides with the algebra*

$$\mathcal{A}_{qr} \cup [C(M_{\infty,\mathbb{R}_+})\hat{\otimes}_e C_\alpha(\overline{D}(0, \frac{1}{2}))],$$

   which is identified with the set of all pairs

$$(\gamma, f) \in \mathcal{A}_{qr} \times [C(M_{\infty,\mathbb{R}_+})\hat{\otimes}_e C_\alpha(\overline{D}(0, \frac{1}{2}))]$$

   satisfying the following compatibility condition $\gamma(\mu) = f(\mu, 0)$, for all $\mu \in M_{\infty,\mathbb{R}_+}(\lambda)$.

2. *The Gelfand transform is generated by the following mapping:*

$$\sum_{p=0}^{m} T_{a_p} T_\phi^p \mapsto \begin{cases} \gamma_0(k,\xi), & \text{if } (k,\xi,0) \in \mathbb{Z} \times \mathbb{R}_+ \times \{0\}, \\ \sum_{p=0}^{m} \gamma_p(\mu)\zeta^p, & \text{if } (\mu,\zeta) \in M_{\infty,\mathbb{R}_+} \times \overline{D}(0, \frac{1}{2}). \end{cases}$$

4.11 Inverse closedness

4.11.1 Lemma. *Let $A = \sum_{p=0}(\gamma_p I)\widetilde{T}_\phi^p$ an element of the dense subalgebra $\widetilde{\mathcal{D}(\lambda)}$ and $\psi = (k,\xi,0) \in \mathbb{Z}_+ \times \mathbb{R}_+ \times \{0\} \subset M(T(\lambda))$. If A is invertible in $\mathscr{L}(l^2(\mathbb{Z}_+^2, L_2(\mathbb{R}_+)))$, then $\psi(A) \neq 0$.*

Proof. Let $\mathcal{D}^*$ be the C^*-algebra generated by $\widetilde{\mathcal{D}(\lambda)}$.

We extend ψ to $\mathcal{D}^*$ by assigning $\psi(T_\phi^*) = 0$. By (4.21), ψ is defined in $\mathcal{D}(\lambda)$ by

$$\psi(A) = \lim_{n\to\infty} \langle Af_n, f_n \rangle,$$

where f_n is a unitary vector such that $\widetilde{T}_\phi(f_n) = 0$, $p = 1, 2, \ldots$, and

$$\lim_{n\to\infty} \langle \gamma(k,\cdot)g_n, g_n \rangle_{L_2(\mathbb{R}_+)} = \gamma(k,\xi).$$

Consider the dense subset of $\mathcal{D}^*$ defined by all finite linear combinations of products of operators of the form γI, $\widetilde{T}_\phi$ and $\widetilde{T}_\phi^*$. Since γI commutes with $\widetilde{T}_\phi$ and $\widetilde{T}_\phi^*$, a typical element of this algebra can be written as

$$B = \gamma I + \sum_{i=1}^{m} Q_i(\gamma_i I), \tag{4.22}$$

where Q_i is a finite product of $\widetilde{T}_\phi$ and $\widetilde{T}_\phi^*$.

Note that $\langle (\widetilde{T}_\phi)^*(f_n), f_n\rangle = \langle f_n, \widetilde{T}_\phi f_n\rangle = 0$ and thus we have $\langle Q_i f_n, f_n\rangle = 0$. Which implies that

$$\langle Bf_n, f_n\rangle = \langle \gamma I g_n, g_n\rangle_{L_2(\mathbb{R}_+)}$$

and

$$\lim_{n\to\infty} \langle Bf_n, f_n\rangle = \gamma_b(k,\xi).$$

Since this formula defines a bounded functional, we conclude that ψ can indeed be extended to $\mathcal{D}^*$.

Being a C^*-algebra, $\mathcal{D}^*$ is inverse closed and we have $A^{-1} \in \mathcal{D}^*$. There is some B of the form (4.22) such that $\|A - B\| < 1/(2\|A\|)$.

Then

$$1 = \lim_{n\to\infty} \langle AA^{-1} f_n, f_n\rangle = \lim_{n\to\infty} \langle ABf_n, f_n\rangle + \lim_{n\to\infty} \langle (AA^{-1} - AB) f_n, f_n\rangle.$$

By the Cauchy-Schwarz inequality,

$$|\langle (AA^{-1} - AB) f_n, f_n\rangle| \le \|AA^{-1} - AB\| \le \|A\|\|A^{-1} - B\| < 1/2$$

and, therefore,

$$1/2 < \lim_{n\to\infty} \langle ABf_n, f_n\rangle = \gamma_a(k,\xi)\gamma_b(k,\xi) = \psi(A)\psi(B).$$

This proves that $\psi(A) \neq 0$. □

Recall the unitary operator $R_{2\xi}\colon F_1^2(\mathbb{C}^2) \to F_{2\xi}^2(\mathbb{C}^2)$, given by

$$R_{2\xi} f(z) = f((2\xi)^{1/2} z), \quad z \in \mathbb{C}^2, f \in F_1^2(\mathbb{C}^2).$$

In the last section we remarked that

$$ST_\phi S^* = \int_{\mathbb{R}_+} T^{(\xi)}_{\widetilde{\phi}_\xi} d\xi,$$

where $\widetilde{\phi}_\xi$ doesn't really depend on ξ and acts just like ϕ. Moreover, it can be easily seen that, due to this independence, $T^{(\xi)}_{\widetilde{\phi}_\xi} = R_{2\xi} T(\phi) R^*_{2\xi}$, where $T(\phi)$ is the operator $T^{(1/2)}_{\widetilde{\phi}_{1/2}}$ acting on the unweighted Fock space $F^2_1(\mathbb{C}^2) = F^2(\mathbb{C}^2)$.

By the same arguments we have, in general,

$$ST_{\phi^p\overline{\phi}^q} S^* = \int_{\mathbb{R}_+} R_{2\xi} T(\phi^p\overline{\phi}^q) R^*_{2\xi} d\xi,$$

where $T(\phi^p\overline{\phi}^q)$ is the corresponding Toeplitz operator acting on the Fock space $F^2(\mathbb{C}^2)$.

Let $\mathcal{K}$ be the space of all operators K of the form

$$K = \int_{\mathbb{R}_+} R_{2\xi} K_0 R^*_{2\xi} d\xi, \tag{4.23}$$

where K_0 is a compact operator acting on $F^2(\mathbb{C}^2)$.

4.11.2 Lemma. *Let p, q be non negative integers. Then the semicommutators*

$$S(T_{\phi^p\overline{\phi}^q} - T^p_\phi T^q_{\overline{\phi}})S^* \quad \textit{and} \quad S(T_{\phi^p\overline{\phi}^q} - T^q_{\overline{\phi}} T^p_\phi)S^*$$

belong to $\mathcal{K}$.

Proof. We prove it just for the first operator. The other case is very similar.

Since

$$S(T_{\phi^p\overline{\phi}^q} - T^p_\phi T^q_{\overline{\phi}})S^* = \int_{\mathbb{R}_+} R_{2\xi}(T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\overline{\phi}^q))R^*_{2\xi} d\xi,$$

we need to prove that $T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\overline{\phi}^q)$ is compact in $F^2(\mathbb{C}^2)$

Suppose that $p \geq q$. The case $q \geq p$ follows from taking adjoints.

We show first prove that $T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\overline{\phi^q}) = T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\phi^q)^*$ is compact.

Let $0 < \lambda < 1$ and consider the sets $I_1, I_2, I_3 \in \mathbb{Z}^2_+$ given by

$$I_1 = \{\alpha \in \mathbb{Z}^2_+ \colon \alpha_1 < q\},$$

$$I_2 = \{\alpha \in \mathbb{Z}^2_+ \colon \alpha_1 \geq |\alpha|^\lambda\} \backslash I_1,$$

$$I_3 = \{\alpha \in \mathbb{Z}_+^2 \colon \alpha_1 < |\alpha|^\lambda\} \backslash I_1,$$

and let $H_i = \overline{\operatorname{span}}\{e_\alpha^{(1)} \colon \alpha \in I_i\}$.

By equations (4.14), (4.15) and (4.16) we have the following relations:

1. $(T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\overline{\phi}^q))|_{I_1}(e_\alpha)$, $\alpha \in I_1$, is 0 if $a_2 < p - 1$. Otherwise, it's equal to a basic vector multiplied by

$$\frac{(\alpha_1+1)!(\alpha_2+1)!}{\sqrt{\alpha_1!\alpha_2!(\alpha_1+(p-q))!(\alpha_2-(p-q))!}}\frac{(|\alpha|+1)!}{(|\alpha|+p+q+1)!}.$$

Since $a_1 < q$, the last quantity tends to zero as $|\alpha| \to \infty$. Thus $(T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\overline{\phi}^q))|_{I_1}$ can be aproximated by finite-rank operators.

2. $(T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\overline{\phi}^q))|_{I_2}(e_\alpha)$, $\alpha \in I_2$, is 0 if $a_2 < p - 1$. Otherwise, it's equal to a basic vector multiplied by

$$\frac{(\alpha_1+p)!(\alpha_2+q)!}{\sqrt{\alpha_1!\alpha_2!(\alpha_1+(p-q))!(\alpha_2-(p-q))!}}\frac{(|\alpha|+1)!}{(|\alpha|+p+q+1)!}$$
$$\cdot\left[\frac{(|\alpha|+p+q+1)!(|\alpha|+1)!}{(|\alpha|+p+1)!(|\alpha|+q+1)!}\frac{(\alpha_1-q+1)\cdots\alpha_1}{(\alpha_1+(p-q)+1)\cdots(\alpha_1+p)}-1\right]$$

Since $\alpha_1 \geq |\alpha|^\lambda$, we have $\alpha_1 \to \infty$ when $|\alpha| \to \infty$. Thus the right factor tends to zero as $|\alpha| \to \infty$. The left factor is just $\widetilde{\gamma}_{\phi_p\overline{\phi_q}}$, which is bounded. Therefore, the whole expression tends to zero as $|\alpha|$ tends to infinity. This implies that $(T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\overline{\phi}^q))|_{I_2}$ can be approximated by finite-rank operators.

3. $(T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\overline{\phi}^q))|_{I_3}(e_\alpha)$, $\alpha \in I_3$, is 0 if $a_2 < p - 1$. Otherwise, it's equal to a basic vector multiplied by the same quantity as above. In this case, note that the right factor is bounded (since it tends to zero as $\alpha_1 \to \infty$).

   On the other hand, the left side is equal to

$$\frac{\sqrt{(\alpha_1+1)\cdots(\alpha_1+(p-q))}(\alpha_1+(p-q)+1)\cdots(\alpha_1+p)}{(|\alpha|+2)\cdots(|\alpha|+p+q+1)}$$
$$\cdot\sqrt{(\alpha_2-(p-q)+1)\cdots\alpha_2}(\alpha_2+1)\cdots(\alpha_2+q)$$
$$\leq M\frac{\alpha_1^{\frac{p+q}{2}}\alpha_2^{\frac{p+q}{2}}}{|\alpha|^{-p-q}},$$

   for some constant M.

Since $\alpha_1 < |\alpha|^\lambda$, the last quantity tends to zero as $|\alpha| \to \infty$ and, as before, $(T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\overline{\phi}^q))|_{I_3}$ can be approximated by finite-rank operators.

This proves that $T(\phi^p\overline{\phi}^q) - T(\phi^p)T(\overline{\phi}^q)$ is compact.

Finally, we recall that $T(\phi_p)$ acts on the basis of $F^2_(\mathbb{C}^2)$ as $\widetilde{T}_{\phi_p}$ does on the elements $\hat{e}_\alpha(c(\xi))$. Thus, according to (4.9), $T(\phi_p) - T(\phi)^p$ is a diagonal operator whose eigenvalues (depending only on $|\alpha|$) tend to zero as $|\alpha| \to \infty$. Hence, $T(\phi_p) - T(\phi)^p$ and its adjoint $T(\overline{\phi_p}) - T(\overline{\phi})^p$ are a compact operators. It follows from this that $S(T_{\phi^p\overline{\phi}^q} - T_{\phi^p}T_{\overline{\phi}^q})S^*$ is and operator of the form (4.23).

Therefore, we have

$$S(T_{\phi^p\overline{\phi}^q} - T^p_\phi T^q_{\overline{\phi}})S^* = S(T_{\phi^p\overline{\phi}^q} - T_{\phi^p}T_{\overline{\phi}^q})S^* + K_1 S T^q_{\overline{\phi}} S^* + S T^p_\phi S^* K_2,$$

where K_1 and K_2 belong to $\mathcal{K}$. This proves what we wanted. □

Let $((\nu_k, \xi_k))_{k=1}^\infty$ and $w \in \mathbb{C}^2$ be as considered in section 4.9 and let $(f_k^{(w)})_{k=1}^\infty$ be the sequence given by (4.20). That is,

$$f_k^{(w)} = \left\{ g_{n(k)}^{(\xi_k)}(\xi) \frac{K_{S_k}^{(2\xi)}(\cdot, w)}{\|K_{S_k}^{(2\xi)}(\cdot, w)\|} \right\}_{\xi\in\mathbb{R}_+} \in \int_{\mathbb{R}_+}^{\oplus} F^2_{2\xi}(\mathbb{C}^2) d\xi.$$

4.11.3 Lemma. *For every $\xi > 0$ we have*

$$R^*_{2\xi}\left(\frac{K_{S_k}^{(2\xi)}(\cdot, w)}{\|K_{S_k}^{(2\xi)}(\cdot, w)\|}\right) = \frac{K_{S_k}^{(1)}(\cdot, w)}{\|K_{S_k}^{(1)}(\cdot, w)\|}.$$

Proof. By the Multinomial theorem we have

$$\frac{K_{S_k}^{(2\xi)}(z, w)}{\|K_{S_k}^{(2\xi)}(\cdot, w)\|} = \frac{\sum_{|\alpha|=\nu_k} \frac{(2\xi)^{\nu_k}}{\alpha!} z^\alpha \overline{w}^\alpha}{\left(\sum_{|\alpha|=\nu_k} \frac{(2\xi)^{\nu_k}}{\alpha!} |w^\alpha|^2\right)^{1/2}} = \frac{(2\xi)^{\nu_k/2}}{\sqrt{\nu_k!}} \frac{\langle z, w\rangle^{\nu_k}}{|w|^{\nu_k}}.$$

Since $R^*_{2\xi}$ maps z to $(2\xi)^{-1/2}z$ we obtain

$$R^*_{2\xi}\left(\frac{K_{S_k}^{(2\xi)}(\cdot, w)}{\|K_{S_k}^{(2\xi)}(\cdot, w)\|}\right)(z) = \frac{1}{\sqrt{\nu_k!}} \frac{\langle z, w\rangle^{\nu_k}}{|w|^{\nu_k}} = \frac{K_{S_k}^{(2\xi)}(\cdot, w)}{\|K_{S_k}^{(1)}(\cdot, w)\|}.$$

□

4.11.4 Lemma. *The sequence of normalized reproducing kernels given by*

$$\frac{K_{S_k}^{(1)}(\cdot, w)}{\|K_{S_k}^{(1)}(\cdot, w)\|}, \quad k = 1, 2, \cdots .$$

converges weakly to zero.

Proof. It suffices to prove that

$$\langle \frac{K_{S_k}^{(1)}(\cdot, w)}{\|K_{S_k}^{(1)}(\cdot, w)\|}, g\rangle \to 0, \quad k \to \infty$$

for every g in a total subset A of $F^2(\mathbb{C}^2)$.

Let A be the orthonormal basis of $F^2(\mathbb{C}^2)$. Since $\nu_k \to \infty$, for every $\alpha \in \mathbb{Z}_+^2$ we can choose an integer N such that $\nu_k > |\alpha|$, for every $k \geq N$. In this case

$$\langle \frac{K_{S_k}^{(1)}(\cdot, w)}{\|K_{S_k}^{(1)}(\cdot, w)\|}, e_\alpha\rangle = 0, \quad k \geq N,$$

and we are done. □

4.11.5 Corollary. *Let $(f_k^{(w)})_{k=1}^{\infty}$ be as before and $K \in \mathcal{K}$. Then*

$$\|K(f_k^{(w)})\| \to 0,$$

as $k \to 0$.

Proof. We have, by Lemma 4.11.3,

$$\begin{aligned} K(f_k^{(w)}) &= \left(\int_{\mathbb{R}_+} R_{2\xi} K_0 R_{2\xi}^* d\xi\right)(f_k^{(w)}) \\ &= \left\{ g_{n(k)}^{(\xi_k)}(\xi) R_{2\xi} K_0 \left(R_{2\xi}^* \left(\frac{K_{S_k}^{(2\xi)}(\cdot, w)}{\|K_{S_k}^{(2\xi)}(\cdot, w)\|}\right)\right)\right\}_{\xi \in \mathbb{R}_+} \\ &= \left\{ g_{n(k)}^{(\xi_k)}(\xi) R_{2\xi} K_0 \left(\frac{K_{S_k}^{(1)}(\cdot, w)}{\|K_{S_k}^{(1)}(\cdot, w)\|}\right)\right\}_{\xi \in \mathbb{R}_+} \end{aligned}$$

.

Thus

$$\begin{aligned}\|K(f_k^{(w)})\|^2 &= \int_{\mathbb{R}_+} (g_{n(k)}^{(\xi_k)}(\xi))^2 \left\| R_{2\xi} K_0 \left(\frac{K_{S_k}^{(1)}(\cdot, w)}{\|K_{S_k}^{(1)}(\cdot, w)\|} \right) \right\|^2 d\xi \\ &= \int_{\mathbb{R}_+} (g_{n(k)}^{(\xi_k)}(\xi))^2 \left\| K_0 \left(\frac{K_{S_k}^{(1)}(\cdot, w)}{\|K_{S_k}^{(1)}(\cdot, w)\|} \right) \right\|^2 d\xi \\ &= \left\| K_0 \left(\frac{K_{S_k}^{(1)}(\cdot, w)}{\|K_{S_k}^{(1)}(\cdot, w)\|} \right) \right\|^2 .\end{aligned}$$

Since K_0 is compact and, by Lemma 4.11.4, the sequence of normalized reproducing kernels tends weakly to zero, the last member of the equality tends to zero. □

Given $D = (a_0, \ldots, a_m)$, with a_p a (2)-quasi-radial quasi-parabolic function, $p = 0, 1, \ldots, m$, let $\mathcal{A}_D$ be the Banach algebra generated by $ST_{a_0}S^*, \ldots, ST_{a_m}S^*$, $ST_\phi S^*$ and let $\mathcal{A}_D^*$ the C^*-algebra generated by $ST_{a_0}S^*, \ldots, ST_{a_m}S^*$, $ST_\phi S^*$ and $\mathcal{K}$.

As we did in the proof of Lemma 4.10.3, for every $\psi = (\mu, \zeta) \in M_{\infty, \mathbb{R}_+}(\lambda) \times \overline{D}(0, \frac{1}{2})$, we can construct an associated sequence $(f_k^{(w)})_{k=1}^\infty$ of the form (4.20) and such that $\psi(A)$, $A \in \mathcal{A}_D$, can be calculated in terms of this sequence. In particular, we proved that $\psi(T_\phi) = \phi(w) = \zeta$ and $\psi(T_{a_p}) = \gamma_p(\mu)$.

4.11.6 Lemma. *With the notations introduced above, let $T \in \mathcal{A}_D^*$. Then the limit*

$$\psi'(T) = \lim_{k \to \infty} \langle T f_k^{(w)}, f_k^{(w)} \rangle$$

always exists and defines a multiplicative functional on $\mathcal{A}_D^*$ that extends ψ.

Proof. We first note that the operators $ST_{a_l}S^*$ commute with all the other operators considered. Thus, the algebra $\mathcal{A}_D^*$ is the linear span of all operators of the form

$$QST_{a_l}S^*$$

and

$$ST_{a_n}S^*Q_1KQ_2,$$

where Q_1, Q_1 and Q_2 are finite products of operators of the form $ST_\phi^iS^*$ and $S(T_\phi^j)^*S^*$, $i, j \geq 0$, $l, n \in \{0, 1, \ldots, m\}$, and $K \in \mathcal{K}$.

Since the operators Q_1 and Q_2 can be written as a direct integral we have $Q_1KQ_2 \in \mathcal{K}$. Moreover, by Lemma 4.11.2, $Q = ST_{\phi^p\overline{\phi}^q}S^* + K'$, for some $K' \in \mathcal{K}$.

Thus, $\mathcal{A}_D^*$ is indeed generated by operators of the form

$$ST_{\phi^p\overline{\phi}^q}T_{a_l}S^* + K$$

and

$$ST_{a_n}S^*K_2,$$

where $K_1, K_2 \in \mathcal{K}$ and $l, n \in \{0, 1, \dots, m\}$.

By Cauchy-Schwarz inequality and Corollary 4.11.5,

$$\|\langle ST_{a_n}S^*K_2 f_k^{(w)}, f_k^{(w)}\rangle\| \leq \|ST_{a_n}S^*\|\|K_2 f_k^{(w)}\| \to 0,$$

as $k \to \infty$.

On the other hand, by an similar argument as above and by Corollary 4.9.4,

$$\begin{aligned}\lim_{k\to\infty}\langle(ST_{a_l\phi^p\overline{\phi}^q}S^* + K)f_k^{(w)}, f_k^{(w)}\rangle &= \lim_{k\to\infty}\langle(ST_{a_l\phi^p\overline{\phi}^q}S^*)f_k^{(w)}, f_k^{(w)}\rangle \\ &= \gamma_l(\mu)\phi_p(w)\overline{\phi_q}(w) \\ &= \psi(T_{a_l})\psi(T_\phi)^p\overline{\psi(T_\phi)}^q.\end{aligned}$$

It follows from these relations that the limit exists for every element in this dense subalgebra of $\mathcal{A}_D^*$, that ψ' is multiplicative in this subalgebra and that this functional extends ψ.

Finally, the formula defining ψ' shows that this functional is continuous and can be extended to the whole algebra $\mathcal{A}_D^*$, as we wanted to prove. □

4.11.7 Corollary. *Let $\psi = (\mu, \zeta) \in M_{\infty,\mathbb{R}_+} \times \overline{D}(0, \frac{1}{2})$ and let $A \in \mathcal{A}_D$ be invertible as an element of $\mathscr{L}(\mathcal{A}_\lambda^2(D_3))$. Then $\psi(A) \neq 0$.*

Proof. The C^*-algebra $\mathcal{A}_D^*$ is inversed closed and therefore $A^{-1} \in \mathcal{A}_D$. By Lemma 4.11.1 we can extend ψ to a multiplicative functional on $\mathcal{A}_D$. Therefore, $\psi(A) = \psi'(A) \neq 0$. □

4.11.8 Theorem. *The commutative Banach algebra $\mathcal{T}(\lambda)$ is inverse closed and, in particular, for each $A \in \mathcal{T}(\lambda)$,*

$$\mathrm{sp}_{\mathcal{T}(\lambda)} A = \mathrm{sp}_{\mathscr{L}(\mathcal{A}_\lambda^2(D_3))} A.$$

Proof. Let $A \in \mathcal{T}(\lambda)$ be invertible as an element of $\mathscr{L}(\mathcal{A}^2_\lambda(D_3))$.

Choose a sequence $(A_n)_{n=1}^{\infty}$ of elements of the form

$$A_n = \sum_{p=0}^{m_n} (T_{a_{n,p}}) T_\phi^p,$$

whit each $a_{n,p}$ a (2)-quasi-radial quasi-parabolic function, such that

$$\lim_{n\to\infty} A_n = A.$$

Since the group of invertible elements is open, we can assume that A_n is invertible for all $n \in \mathbb{N}$. Moreover, by the continuity of the inversion we have

$$A^{-1} = \lim_{n\to\infty} A_n^{-1}$$

and thus, A^{-1} will be in $\mathcal{T}(\lambda)$ if each A_n^{-1} is in $\mathcal{T}(\lambda)$.

Fix a A_n and suppose that

$$A_n = \sum_{n=0}^{m} T_{a_p} T_\phi^p.$$

Since A_n is invertible, $\widetilde{A_n}$ is invertible in $\mathscr{L}(l^2(\mathbb{Z}_+^2, L_2(\mathbb{R}_+)))$ and, by Lemma 4.11.1, $\psi(A_n) \neq 0$, for every $\psi \in \mathbb{Z}_+ \times \mathbb{R}_+ \times \{0\}$.

Moreover, let $D = (a_1, \ldots, a_m)$ and let $\mathcal{A}_D$ and $\mathcal{A}_D^*$ as before. Since A_n is invertible in $\mathcal{A}_D$, by Corollary 4.11.7, $\psi(A_n) \neq 0$, for every $\psi \in M_{\infty,\mathbb{R}_+} \times \overline{D}(0, \frac{1}{2})$.

It follows from Theorem 4.10.4 that $\psi(A_n) \neq 0$ for every $\psi \in M(\mathcal{T}(\lambda))$ and consequently, A_n is invertible in $\mathcal{T}(\lambda))$, as we wanted to prove. $\square$

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