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Evolutionary game theory: a general framework to the replicator dynamics

A dissertation presented by

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1 Introduction and technical preliminaries

1.1 Introduction

Evolutionary games form a class of noncooperative games in which the interaction of strategies is studied using evolutionary ideas from two different approaches, static and dynamic. The static approach captures evolutionary concepts through defining and studying equilibrium terms. The dynamic approach, on the other hand, studies the interaction of strategies as a dynamical process determined by a system of differential equations. This manuscript concerns the dynamic approach with a specific dynamical system known as the replicator dynamics. We are particularly interested in the stability of the replicator dynamics for evolutionary games in which the strategy set is a measurable set or, more precisely, a separable metric space.

An evolutionary game is said to be *symmetric* if there are two players only and, furthermore, they have the same strategy sets and the same payoff functions. This type of games models interactions of strategies of a single population, and form part of the so-called population games. On the other hand, *asymmetric* evolutionary games, also known as *multipopulation* games, are games in which there is a finite set of players (or populations) each of which has a different set of strategies and different payoff functions.

Game models with strategies in general measurable spaces are important because they include essentially all the models that appear in theory and applications, from games with finite strategy sets to games with strategies in metric spaces such as some models in oligopoly theory, international trade theory, war of attrition, and public goods, among others. With our proposed model we can introduce an evolutionary dynamics in games where the strategy set is a Borel space (that is, a Borel subset of a complete and separable metric space). We have consequently that the dynamical system lives in a Banach space, which in our case is a space of finite signed measures. In particular, if the strategy set is finite, then the dynamical system is in $\mathbb{R}^m$, where m is the number of strategies of a player for symmetric games, or the total number of strategies of all players for asymmetric games.

The main objective of this work is to present a general, unified framework to study the existence of solutions and the stability of the replicator dynamics for games with metric strategy sets. This means that, first, we establish conditions for the existence of solutions to the replicator dynamics for asymmetric games, and also conditions that ensure the stability of the system in this asymmetric case. Bomze and Potscher [18] suggest an approach (similar to Selten's [92] for the case with finite strategy sets) in which asymmetric games are reduced to the symmetric case; for details see section 3.2 below. This approach, however, has some disadvantages. For instance, the relationship between Nash equilibria and the replicator dynamics is unclear. It is also unclear how to extend stability concepts and results to the asymmetric case. In contrast, with our proposed model is it easy to see the relationship between a Nash equilibrium and the replicator dynamics (see section 3.4) and, in addition, stability concepts have a natural construction from the symmetric case to the asymmetric situation (see Section 3.5).

Second, for symmetric games we study stability criteria in a fairly general context, with respect to different topologies and metrics on a space of measures. We can thus, for instance, relate the Nash equilibria of a certain normal form game with the stability of the replicator dynamics under different metrics (see section 4.4 below), and similarly for strongly uninvadable strategies (section 4.3), a refinement of Nash equilibria. We can also obtain quite general, and at the same time precise results on the approximation of the replicator dynamics by different approximating models, which include finite-dimensional dynamic systems (section 4.5).

Third, we study the replicator dynamic as a limit of a sequence of Markov processes (see chapter 5), and where each Markov process describe a stochastic interaction among the characteristics (genotypes or actions) of the individuals. This stochastic-interaction can be studied by a determinist dynamic under some hypotheses (when the mass-population is enough, for example.)

Concerning some related literature, conditions for the existence of solutions to the replicator dynamics in measure spaces in the symmetric case are given by several authors, including Bomze [17], Oechssler and Riedel [74], and more generally (including dynamics different from the replicator equation) by Cleveland and Ackleh [25]. In section 3.3 we present conditions for the existence of solutions to the replicator dynamics in measure spaces in the asymmetric case and some other important results.

Similarly, conditions for stability have been developed with respect to different topologies, as in for instance Bomze [16], Oechssler and Riedel [74], [75], Eshel and Sansone [36], Veelen and Spreij [98], Cressman and Hofbauer [31].

In section 3.5, below, we present stability results for the replicator dynamics in the asymmetric case. In section 4.3, we present a brief review of stability results in the symmetric case. Also in section 4.3 we establish a result that characterizes the stability of the replicator dynamics with respect to the Wasserstein metric, which is analogous to Theorem 2 of Bomze [16], and is also an approximation to answer the conjecture proposed by Oechssler and Riedel [75].

An important issue in evolutionary games is to study the replicator dynamics as a limit of a sequence of Markov process describing interactions among individuals in a population. These stochastic interactions describe the evolution of the species. There are many references on this issue when the strategy space is finite, for instance, to name a few, Benaim and Weibull [11], Corradi and Sarin [26], Sandholm [91], [90]. However, a more general mathematical structure is needed if the strategy set is a measurable space, which is what we propose in this chapter 5.

On the other hand, in the theory of evolutionary games there are several interesting dynamics, such as, the imitation dynamics, the monotone-selection dynamics, the best-response dynamics, the Brown-von Neumann-Nash dynamics, and so forth (see Hofbauer and Sigmund [50], [51], Sandholm [91], among others). Some of these evolutionary dynamics have been extended to games with strategies in a space of probability measures. For instance, Hofbauer, Oechssler and Riedel [49] extend the Brown-von Neumann-Nash dynamics; Lahkar and Riedel [64] extend the logit dynamics. Moreover, Cheung proposes a general theory for pairwise comparison dynamics [24] and for imitative dynamics [23]. M. Ruijgrok and T. Ruijgrok [87] extend the replicator dynamics with a mutation term. Among all these dynamics, we selected the replicator dynamics partly because it is the most studied for games with strategies in metric spaces, and partly because it has many interesting properties, as can be seen in Cressman [27], Hofbauer and Weibull [52], and many other references. In particular, with the replicator dynamics it is not difficult to construct a proof of the existence of Nash equilibria and, moreover, when the strategy set is finite, we can give a geometric characterization of the set of Nash equilibria; see Harsanyi [44], Hofbauer and Sigmund [50], Ritzberger [83].

Finally, it is noteworthy that today, the evolutionary games have many applications in different areas. For example, genetics and biology [50], modeling cancer [102],[40], [53], spread of epidemics[10], [105], forest management [93], economic development[5], [1], combat money laundering [4], finance [2],[3], among others. We selected the examples in chapter 2 because most of them are classical models in the literature of game theory and, in addition, the corresponding strategy sets are metric spaces. This allows us to relate some of our main theoretical results to interesting particular applications.

1.2 Summary

The remainder of the manuscript is organized as follows. Section 1.3 presents notation and technical requirements.

Chapter 2 introduces a normal form game and presents important related concepts. Also in the section 2.2, we show examples that will be used in rest of this work.

Chapter 3 introduces an evolutionary dynamics for asymmetric games. Section 3.1 shows a heuristic approximation to the replicator dynamics for the asymmetric case. Section 3.2 describes the asymmetric evolutionary game and the replicator dynamics. Section 3.3 establishes conditions for the existence of a solution to the system of differential equations describing the replicator dynamics, and gives some characterizations of the solution. Section 3.4 establishes a relationship between *Nash equilibria* and the replicator dynamics. Section 3.5 introduces conditions to establish the stability of the replicator equations. Section 3.6 proposes examples to illustrate our results. We conclude the chapter in section 3.7 with some general comments on possible extensions.

Chapter 4 introduces an evolutionary dynamics for symmetric games. Section 4.1 describes the replicator dynamics and its relation to evolutionary games. Some important technical issues are also summarized. Section 4.2 establishes the relation between the replicator dynamics and a normal form game using the concepts of *Nash equilibria* and *strongly uninvadable strategies*. Section 4.3 presents a brief review of stability results for the replicator dynamics. Section 4.4 establishes an important relationship between Nash equilibria and the critical points of the replicator dynamics. Section 4.5 proposes approximation schemes for the replicator dynamics in measure spaces, including the approximation by dynamical systems in finite-dimensional spaces. Section 4.6 proposes examples to illustrate our results. Finally, we conclude in section 4.7 with some general comments on possible extensions of our results.

Chapter 5 studies the replicator dynamics as a limit of a sequence of Markov processes. Section 5.1 presents notation and technical requirements. Section 5.2 shows a technique proposed by Kolokoltsov [55],[56] to approximate a sequence of pure jumps models of binary interaction (in a Banach space), through deterministic dynamical system. Section 5.3 use techniques of section 5.2 to establish conditions under which the replicator dynamics is a limit of a sequence of Markov processes.

Chapter 6 presents a summary of contributions and future perspectives.

Finally, Appendix A contain facts on metrics on spaces of probability measures, and the proof of some technical results.

1.3 Technical preliminaries

1.3.1 Spaces of signed measures

Consider a separable metric space A and its Borel σ-algebra $\mathcal{B}(A)$. Let $\mathbb{M}(A)$ be the Banach space of finite signed measures μ on $\mathcal{B}(A)$ endowed with the total variation norm

$$\|\mu\| := \sup_{\|f\|\leq 1} \left| \int_A f(a)\mu(da) \right| = |\mu|(A), \tag{1.1}$$

where $|\mu| = \mu^+ + \mu^-$ denotes the total variation of μ, and μ^+, μ^- stand for the positive and negative parts of μ, respectively. The supremum in (1.1) is taken over functions in the Banach space $\mathbb{B}(A)$ of real-valued bounded measurable functions on A, endowed with the supremum norm

$$\|f\| := \sup_{a\in A} |f(a)|. \tag{1.2}$$

Consider the subset $\mathbb{C}_B(A) \subset \mathbb{B}(A)$ of all real-valued continuous and bounded functions on A. Consider the dual pair $(\mathbb{C}_B(A), \mathbb{M}(A))$ given by the bilinear form $\langle \cdot, \cdot \rangle : \mathbb{C}_B(A) \times \mathbb{M}(A) \to \mathbb{R}$

$$\langle g, \mu \rangle = \int_A g(a)\mu(da). \tag{1.3}$$

We consider the *weak topology* on $\mathbb{M}(A)$ (induced by $\mathbb{C}_B(A)$), i.e., the topology under which all elements of $\mathbb{C}_B(A)$, when regarded as linear functionals $\langle g, \cdot \rangle$ on $\mathbb{M}(A)$ are continuous. In this topology a neighborhood of a point $\mu \in \mathbb{M}(A)$ is of the form

$$\mathcal{V}_\epsilon^{\mathcal{H}}(\mu) := \left\{ \nu \in \mathbb{M}(A) : \quad |\langle g, \nu - \mu \rangle| < \epsilon \quad \forall g \in \mathcal{H} \right\} \tag{1.4}$$

for $\epsilon > 0$ and $\mathcal{H}$ a finite subset of $\mathbb{C}_B(A)$.

Definition 1.1. *A sequence of measures $\mu_n \in \mathbb{M}(A)$ is said to be* weakly convergent *if there exists $\mu \in \mathbb{M}(A)$ such that*

$$\lim_{n\to\infty} \int_A g(a)\mu_n(da) = \int_X g(a)\mu(da) \tag{1.5}$$

for all g in $\mathbb{C}_B(A)$. If $\mathbb{M}(A)$ is the space $\mathbb{P}(A)$ of probability measures on A, sometimes we say that μ_n converges in distribution *to μ.*

1.3.2 Metrics on $\mathbb{P}(A)$

There are many metrics that metrize the weak topology. The following metrics are particularly useful. (For details see, for instance, Shiryaev [94], Billingsley [12] or Villani [101]). This subsection will be used in chapter four, the reader can skip this section and come back later.

Let A be a separable metric space with a metric ϑ, and $\mathbb{P}(A)$ the set of probability measure on A. For any $\mu, \nu \in \mathbb{P}(A)$ we define the following metrics on $\mathbb{P}(A)$.

i) **The Prokhorov metric** r_p, defined as

$$r_p(\mu,\nu) := \inf\{\alpha > 0 : \mu(E) \leq \nu(E_\alpha) + \alpha \ \text{ and } \ \nu(E) \leq \mu(E_\alpha) + \alpha\}, \quad (1.6)$$

where, for $\alpha > 0$, $E_\alpha := \{a \in A : \vartheta(a, E) < \alpha\}$ if $E \neq \phi$. Here ϕ is the empty set, and

$$\vartheta(a, E) := \inf\{\vartheta(a, a') : a' \in E\}.$$

ii) **The bounded Lipschitz metric** r_{bl}, defined as

$$r_{bl}(\mu,\nu) := \sup_{f \in \mathbb{L}_B(A)} \left\{ \int_A f(a)\mu(da) - \int_A f(a)\nu(da) \ : \ \|f\|_{BL} \leq 1 \right\}, \quad (1.7)$$

where $(\mathbb{L}_B(A), \|\cdot\|_{BL})$ is the space of bounded, continuous and real-valued functions on A that satisfy the Lipschitz condition

$$\|f\|_L := \sup \frac{|f(a) - f(b)|}{\vartheta(a,b)} < \infty, \quad (1.8)$$

where the supremum is over all $a \neq b$. For any $f \in \mathbb{L}_B(A)$, the norm $\|f\|_{BL}$ is defined as

$$\|f\|_{BL} := \|f\| + \|f\|_L. \quad (1.9)$$

iii) **The Kantorovich-Rubinstein metric** r_{kr}, defined as

$$r_{kr}(\mu,\nu) := \sup_{f \in \mathbb{L}(A)} \left\{ \int_A f(a)\mu(da) - \int_A f(a)\nu(da) \ : \ \|f\|_L \leq 1 \right\}, \quad (1.10)$$

where $(\mathbb{L}(A), \|\cdot\|_L)$ is the space of continuous real-valued functions on A that satisfy the Lipschitz condition (1.8). Let a_0 a fixed point in A. Then the Kantorovich-Rubinstein metric r_{kr} can be extended as a norm on $\mathbb{M}(A)$ defined as

$$\|\mu\|_{kr} := |\mu(A)| + \sup_{f \in \mathbb{L}(A)} \left\{ \int_A f(a)\mu(da) \ : \ \|f\|_L \leq 1, \ f(a_0) = 0 \right\}, \quad (1.11)$$

for any μ in $\mathbb{M}(A)$ (see Bogachev [15], chapter 8). Note that for any $\mu, \nu \in \mathbb{P}(A)$ $r_{kr}(\mu, \nu) = \|\mu - \nu\|_{kr}$, since

$$\sup_{f \in \mathbb{L}(A)} \left\{ \int_A f(a)\mu(da) - \int_A f(a)\nu(da) \ : \ \|f\|_L \leq 1 \right\}$$

$$= \sup_{f \in \mathbb{L}(A)} \left\{ \int_A \Big(f(a) - f(a_0)\Big)\mu(da) - \int_A \Big(f(a) - f(a_0)\Big)\nu(da) \ : \ \|f\|_L \leq 1 \right\}$$

$$= \sup_{g \in \mathbb{L}(A)} \left\{ \int_A g(a)\mu(da) - \int_A g(a)\nu(da) \ : \ \|g\|_L \leq 1, \ g(a_0) = 0 \right\}$$

iv) Let us suppose that the separable metric space A is also complete (that is, a so-called Polish space), and let a_0 be a fixed point in A. For each p with $1 \leq p < \infty$, we define the space $\mathbb{P}_p(A)$ as

$$\mathbb{P}_p(A) := \left\{ \mu \in \mathbb{P}(A) : \quad \int_A [\vartheta(a, a_0)]^p \mu(da) < \infty \right\}.$$

The L^p**-Wasserstein distance** r_{w_p} between μ and ν in $\mathbb{P}_p(A)$ is defined by

$$r_{w_p}(\mu, \nu) := \left[\inf_{\pi \in \Pi} \int_A \int_A \vartheta(a, b)\pi(da, db) \right]^{\frac{1}{p}}, \tag{1.12}$$

where Π is the set of probability measures on $A \times A$ with marginals μ and ν. In particular, when $p = 1$ we write the L^1**-Wasserstein distance** r_{w_1} as r_w and in addition we have that $r_w = r_{kr}$ on $\mathbb{P}(A)$.

Remark 1.2. *In the rest of this work we will denote by* r_{w^*} **any metric that metrizes the weak topology** *on* $\mathbb{P}(A)$ *(not to be confused with the notation* r_w *of the* L^1*-Wasserstein distance). Moreover, we denote by* r **any metric on** $\mathbb{P}(A)$ *that is either the total variation norm (1.1) or any distance that metrizes the weak topology. An open ball in the metric space* $(\mathbb{P}(A), r)$ *is defined in the classical form*

$$\mathcal{V}_\alpha^r(\mu) := \left\{ \nu \in \mathbb{P}(A) : \quad r(\nu, \mu) < \alpha \right\} \tag{1.13}$$

where $\alpha > 0$.

Remark 1.3. *Let* A *be a separable metric space, and* r_{w^*} *any distance that metrizes the weak topology* τ_{w^*} *in* $\mathbb{P}(A)$*. Let* μ *be any measure in* $\mathbb{P}(A)$*, and consider the family* $\mathcal{V}^{\mathcal{H}}(\mu)$ *of neighborhoods* $\mathcal{V}_\epsilon^{\mathcal{H}}(\mu)$ *of the form (1.4). In addition, consider the family* $\mathcal{V}^{r_{w^*}}(\mu)$ *of the open balls* $\mathcal{V}_\alpha^{r_{w^*}}(\mu)$ *of the form (1.13).*

Both families $\mathcal{V}^{\mathcal{H}}(\mu)$ *and* $\mathcal{V}^{r_{w^*}}(\mu)$ *are* neighborhood basis *for* μ *in the space* $(\mathbb{P}(A), \tau_{w^*})$*. For details see Pedersen [78], chapters I-II.*

Moreover, a neighborhood $\mathcal{V}_{\epsilon}^{\mathcal{H}}(\mu)$ *for* μ *is contained in some* open ball $\mathcal{V}_{\alpha}^{r_{w^*}}(\mu)$ with center μ*. The inverse is also true, i.e., any open ball* $\mathcal{V}_{\alpha}^{r_{w^*}}(\mu)$ *is contained in some neighborhood* $\mathcal{V}_{\epsilon}^{\mathcal{H}}(\mu)$*.*

1.3.3 Product Spaces

Consider two separable metric spaces X and Y with their Borel σ-algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$. We denote by $\sigma[X \times Y]$ the σ-algebra on $X \times Y$ generated by the Cartesian product $\mathcal{B}(X) \times \mathcal{B}(Y)$. For $\mu \in \mathbb{M}(X)$ and $\nu \in \mathbb{M}(Y)$, we denote their product by $\mu \times \nu \in \mathbb{M}(X \times Y)$.

Proposition 1.4. *For* $\mu \in \mathbb{M}(X)$ *and* $\nu \in \mathbb{M}(Y)$*, it holds that*

$$\|\mu \times \nu\| \leq \|\mu\| \|\nu\|. \tag{1.14}$$

As a consequence, $\mu \times \nu$ *is in* $\mathbb{M}(X \times Y)$*.*

Proof. See Heidergott and Leahu [46], Lemma 4.2. □

Now consider a finite family of metric spaces $\{X_i\}_{i=1}^{n}$ and their σ-algebras $\mathcal{B}(X_i)$, as well as the Banach spaces $\mathbb{M}(X_i)$. For $i = 1, ..., n$, let $\mu_i \in \mathbb{M}(X_i)$. Consider the elements $\mu = (\mu_1, \mu_2, ..., \mu_n)$ in the product space $\mathbb{M}(X_1) \times \mathbb{M}(X_2) \times ... \times \mathbb{M}(X_n)$ for which

$$\|\mu\|_{\infty} = \|(\mu_1, ..., \mu_n)\|_{\infty} := \max_{1 \leq i \leq n} \|\mu_i\| < \infty. \tag{1.15}$$

These elements form a Banach space with $\|\cdot\|_{\infty}$ as a norm. We call it the *direct product* of the Banach spaces $\mathbb{M}(X_i)$.

1.3.4 Differentiability

Definition 1.5. *Let* A *be a separable metric space. We say that a mapping* $\mu : [0, \infty) \to \mathbb{M}(A)$ *is* strongly differentiable *if there exists* $\mu'(t) \in \mathbb{M}(A)$ *such that, for every* $t > 0$*,*

$$\lim_{\epsilon \to 0} \left\| \frac{\mu(t+\epsilon) - \mu(t)}{\epsilon} - \mu'(t) \right\| = 0. \tag{1.16}$$

Note that, by (1.1), the left-hand side in (1.16) can be expressed as

$$\lim_{\epsilon \to 0} \sup_{\|g\| \leq 1} \left| \frac{1}{\epsilon} \left[\int_A g(a) \mu(t+\epsilon, da) - \int_A g(a) \mu(t, da) \right] - \int_A g(a) \mu'(t, da) \right|.$$

The *strong derivative* $\mu'(t) \in \mathbb{M}(A)$ is also called a *Fréchet derivative* in the Banach space $\mathbb{M}(A)$. (For weak differentiability, see Remark 4.23.)

1.4 Comments

This chapter presented a general introduction and summary of the manuscript. In addition, some technical preliminaries to be used in the following chapters were presented. The only remaining information to be included are some references that address evolutionary games in an explicit and comprehensive manner. Firths, we mention references for evolutionary games with *finite* strategy spaces. Webb [103] and Weibull [104] are two good introductory books; Hofbauer and Sigmund [50], and ,Sandholm [91], are two books that addresses a larger number of topics for the evolutionary games; Cressman [28] uses techniques based on subgame decompositions of *extensive form games* to analyze convergence results for evolutionary dynamics.

Regarding books that deal with evolutionary game with measurable strategy spaces we only can to mention Bomze[18]. Unfortunately, this book was written in 1989 and does not address several subsequent results that have been developed in this theory. Other references on theoretical advances of this topic are mentioned in the introduction. However, most of them only touch theoretical aspects, and there are few bibliographies about applications, for example oligopoly theory [80], public goods models [80] and preferences economic theory [73], [47].

2 Normal form games

A *game* is a mathematical model that describes the confrontation of a set of decision makers (call players) who choose a strategy to face his opponent and receive a payment as a result of this confrontation. A *normal form game* (also known as a *game in strategic form*) is a *game* which is played one time, and where each player know the strategies and payoffs of her opponents. The solution of a normal form games was proposed by Nash [71].

Section 2.1 introduces a normal form game and presents important related concepts. The following sections show examples that will be used in rest of this work. The most of them are classical models in the literature of game theory and, in addition, the corresponding strategy sets are metric spaces. This allows us to relate some of our main theoretical results (in evolutionary games given in the following chapters) to interesting particular applications.

Section 2.2 shows a linear-quadratic model that can be appliqued in a lot of situations, particularity, in oligopoly theory, international tarted models, or public good games. Section 2.3 show the strategy of commons model which is a classical game used to describe the extraction and use of natural resources. Section 2.4 present the poverty traps model that describe the possible causes of economic underdevelopment of a country. Section 2.5 shows the classical Bertrand game. Sections 2.6 and 2.7 present the Graduate risk model and war attrition game, respectively. These games describe a situation where the players compete for a recurs.

2.1 Normal form games

In this section we introduce normal form games and define the concept of Nash equilibrium as a solution of shuch games.

Consider a set $I := \{1, 2, ..., n\}$ of players. For each player $i \in I$, let A_i be the set of *pure strategies*, which is a separable metric space. Let $\mathcal{B}(A_i)$ be the Borel σ-algebra of A_i, and $\mathbb{P}(A_i)$ the set of probability measures on A_i, also known as the set of *mixed strategies*. For every $i \in I$ and every vector $a := (a_1, ..., a_n)$ in the Cartesian product $A := A_1 \times ... \times A_n$, we write a as (a_i, a_{-i}) where $a_{-i} := (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$ is in $A_{-i} := A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_n$.

Finally, for each player i we assign a payoff function $J_i : \mathbb{P}(A_1)\times...\times\mathbb{P}(A_n) \to \mathbb{R}$ that explains the interrelation with other players, and which is defined as

$$J_i(\mu_1, ..., \mu_n) := \int_{A_1} ... \int_{A_n} U_i(a_1, ..., a_n)\mu_n(da_n)...\mu_1(da_1), \tag{2.1}$$

where $U_i : A_1 \times ... \times A_n \to \mathbb{R}$ is a given measurable function, sometimes we cal U_i a utility or payoff function.

For every $i \in I$ and every vector $\mu := (\mu_1, ..., \mu_n)$ in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$, we sometimes write μ as (μ_i, μ_{-i}), where $\mu_{-i} := (\mu_1, ..., \mu_{i-1}, \mu_{i+1}, ..., \mu_n)$ is in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_{i-1}) \times \mathbb{P}(A_{i+1}) \times ... \times \mathbb{P}(A_n)$.

If $\delta_{\{a_i\}}$ is a Dirac probability measure concentrated at $a_i \in A_i$, the vector $(\delta_{\{a_i\}}, \mu_{-i})$ is written as (a_i, μ_{-i}), and so

$$J_i(\delta_{\{a_i\}}, \mu_{-i}) = J_i(a_i, \mu_{-i}). \tag{2.2}$$

It is convenient to rewrite (2.1) as

$$\mathcal{I}_{(\mu_1,...,\mu_n)}U_i := \int_{A_1} ... \int_{A_n} U_i(a_1, ..., a_n)\mu_n(da_n)...\mu_1(da_1). \tag{2.3}$$

Hence (2.2) becomes

$$\begin{aligned} J_i(a_i, \mu_{-i}) &= \int_{A_{-i}} U_i(a_i, a_{-i})\mu_{-i}(da_{-i}) \\ &= \mathcal{I}_{(\mu_1,...,\mu_{i-1},\mu_{i+1},...,\mu_n)}U_i(a_i). \end{aligned} \tag{2.4}$$

In particular, (2.1) yields

$$J_i(\mu_i, \mu_{-i}) := \int_{A_i} J_i(a_i, \mu_{-i})\mu_i(da_i). \tag{2.5}$$

Finally, a *normal form game* Γ can be described as

$$\Gamma := \left[I,\ \left\{\mathbb{P}(A_i)\right\}_{i\in I},\ \left\{J_i(\cdot)\right\}_{i\in I}\right], \tag{2.6}$$

where

i) $I = \{1, 2, ...n\}$ is the set of players,

ii) for each player $i \in I$ we specify a set of actions (or strategies) $\mathbb{P}(A_i)$ and a payoff function $J_i : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{R}$.

Definition 2.1. *Let* Γ *be a normal form game. A vector* μ^* *in* $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$ *is called* ϵ-equilibrium *(*$\epsilon > 0$*) if for all* $i \in I$

$$J_i(\mu_i^*, \mu_{-i}^*) \geq J_i(\mu_i, \mu_{-i}^*) - \epsilon \qquad \forall \mu_i \in \mathbb{P}(A_i).$$

If the inequality is true when $\epsilon = 0$*, then* μ^* *is called a* Nash equilibrium.

We can obtain from (2.6) a symmetric normal form game with two players when $I = \{1, 2\}$, and the sets of actions and payoff functions are the same for both players, i.e., $\mathbb{P}(A) = \mathbb{P}(A_1) = \mathbb{P}(A_2)$ and $J(\mu_1, \mu_2) = J_1(\mu_1, \mu_2) = J_2(\mu_2, \mu_1)$ for all $\mu_1, \mu_2 \in \mathbb{P}(A)$. Hence, we can describe a *two-players symmetric normal form game* as

$$\Gamma_s := \Big[I = \{1, 2\},\ \mathbb{P}(A),\ J(\cdot)\Big]. \tag{2.7}$$

For symmetric normal form games Γ_s can express a symmetric Nash equilibrium (μ^*, μ^*) in terms of the strategy $\mu^* \in \mathbb{P}(A)$, as follows.

Definition 2.2. *We say that* $\mu^* \in \mathbb{P}(A)$ *is a* Nash equilibrium strategy (NES) *if the pair* (μ^*, μ^*) *is a Nash equilibrium for* Γ_s*. That is,*

$$J(\mu^*, \mu^*) \geq J(\mu, \mu^*) \qquad \forall \mu \in \mathbb{P}(A). \tag{2.8}$$

2.2 A quadratic-linear model

In this subsection we consider games in which we have two players with the following payoff functions:

$$U_1(x, y) = -a_1 x^2 - b_1 xy + c_1 x + d_1 y, \tag{2.9}$$
$$U_2(x, y) = -a_2 y^2 - b_2 yx + c_2 y + d_2 x, \tag{2.10}$$

with $a_1, a_2, b_1, b_2, c_1, c_2 > 0$ and d_1, d_2 any real numbers. Consider the strategy sets $A_1 = [0, M_1]$ and $A_2 = [0, M_2]$ for $M_1, M_2 > 0$ and large enough.

This class of games could represent a Cournot duopoly or models of international trade with linear demand and linear cost (see Bagwell and Wolinsky [6]). It can also represent some models of public good games (see Mas-Colell, Whinston and Green [67]).

If the numbers

$$(2a_2c_1 - b_1c_2),\ (2a_1c_2 - b_2c_1),\ (4a_1a_2 - b_1b_2)$$

are all positive, then we have an interior Nash equilibrium

$$(x^*, y^*) = \left(\frac{2a_2c_1 - b_1c_2}{4a_1a_2 - b_1b_2}, \frac{2a_1c_2 - b_2c_1}{4a_1a_2 - b_1b_2}\right). \tag{2.11}$$

2.3 The tragedy of the commons

The tragedy of commons is a game where the payoff of each player depends of the use of a environmental resource, for example, fish stock in the ocean, tree stocks on a forest area; or any other non-regulated resource, for example, an office refrigerator. The following example comes from Gibbons [41].

Consider a set of farmers $I := \{1, 2, .., n\}$ in a ville and suppose that each farmer $i \in I$ owns x_i goats. Let $\hat{x} := x_1 + ... + x_n$. Each goat needs at least a certain amount of grass to survive, then there is a maximum number of goats that can be grazed on the green $\bar{x}$. The value of each goat is given by a function $v : [0, \bar{x}] \to \mathbb{R}$ such that

i) $v(\hat{x}) > 0$ for $0 \leq \hat{x} < \bar{x}$ and $v(\hat{x}) = 0$ for $\bar{x} \leq \hat{x}$,

ii) $v(\cdot)$ is a concave function with the following proprieties $v'(\hat{x}) < 0$ and $v''(\hat{x}) < 0$ for $\hat{x} \in [0, \bar{x}]$

Let $A_i := [0, \bar{x}]$ be the space of pure strategies of player i. The cost of the farmer i to care a goat is c_i, and the payoff of farmer i is given by

$$U_i(x_i, x_{-i}) = x_i v(x_1 + ... + x_i + ... + x_n) - c_i x_i. \tag{2.12}$$

where $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$.

Note, that $v(\cdot)$ is strictly concave (since $v'' < 0$). Then for each i in I and fixed x'_{-i}, the map $x_i \mapsto v(x'_1 + ... + x_i + ... + x'_n)$ is strictly concave. Consequently, the map $x_i \mapsto U_i(x_i, x'_{-i})$ is also strictly concave. Since for each i in I, A_i is convex and compact, then there exist a unique Nash Equilibrium $(x^*_1, ..., x^*_n)$ for the game (see Rosen [85]).

This Nash equilibrium must maximize $U_i(\cdot, x^*_{-i})$ for each i in I and satisfies the first-order condition

$$v(x^*_1 + ... + x_i + ... + x^*_n) + x_i v'(x^*_1 + ... + x_i + ... + x^*_n) - c_i = 0. \tag{2.13}$$

The social optimum, denoted by $\hat{x}^{**}$, solves

$$\max_{\hat{x} \in [0, \bar{x}]} \{\hat{x} v(\hat{x}) - \hat{x} k\},$$

where $k = \min\{c_1, ..., c_n\}$. Then the social optimum $\hat{x}^{**}$ satisfies the first-order condition

$$v(\hat{x}) + \hat{x} v'(\hat{x}) - k = 0. \tag{2.14}$$

Let $\hat{x}^* = x_1^* + ... + x_n^*$. If $\hat{x}^* \leq \hat{x}^{**}$ and since $v' < 0$ and $v'' < 0$, then $0 < v(\hat{x}^{**}) \leq v(\hat{x}^*)$ and $v'(\hat{x}^{**}) \leq v'(\hat{x}^*) < 0$. Since $x_i^* \leq \hat{x}^{**}$ then the left-hand side of (2.13) is strictly greter than the left-hand side of (2.14), which is a contradiction. Therefore, comparing (2.13) to (2.14) shows that $\hat{x}^* > \hat{x}^{**}$, i.e., the common resource is over-utilized in the Nash equilibrium.

2.4 Poverty traps model

This is an abbreviated version of the model proposed by Accinelli and Sanchez-Carrera [1]. Consider an economy with two populations, workers and firms. Each firm has two possible strategies: to be a modern firm (m) or a traditional firm (τ). A modern firm is a technological company that needs specialist workers to work in optimal conditions.

Similarly, each worker have two possibles strategies: to be a specialist worker (s) or to be an artisan worker (a). A specialist worker has to spend $e > 0$ by concept of education.

The modern company pays $w_m > 0$ by finished product to any type of worker and a premium $p > e$ to specialist workers. On the other hand, a traditional firm pays $w_\tau < w_m$ by finished product. The income of each firm is determined by the workers' productivity. We will denote by $I_{f,w}$, the income of firm type $f \in \{m, \tau\}$ that employs workers type $w \in \{s, a\}$. Assume that $I_{s,m} - I_{s,\tau} > w_m + p - w_\tau$ and $w_m - w_\tau > I_{a,m} - I_{a,\tau}$.

In addition, suppose that each company uses a unique type of worker and each worker is employed in one type of firm only. The payoffs for the game are in the following table.

Worker \ Firm	m	τ
s	$w_m + p - e, \quad I_{s,m} - w_m - p$	$w_\tau - e, \quad I_{s,\tau} - w_\tau$
a	$w_m, \quad I_{a,m} - w_m$	$w_\tau, \quad I_{a,\tau} - w_\tau$

(2.15)

Under the above hypotheses we have two pure Nash equilibria (s, m), (a, τ) and one Nash equilibrium in mixed strategies (μ^*, ν^*) where

$$\mu^*(s) = \frac{e}{p} \tag{2.16}$$

and

$$\nu^*(m) = \frac{(w_m - w_\tau) - (I_{a,\tau} - I_{a,m})}{(I_{s,m} - I_{s,\tau}) - (w_m + p - w_\tau)}. \tag{2.17}$$

2.5 A sales model as a Bertrand game

This example is a Bertrand-duopoly model of sales (proposed by Varian [99]), where each firm (or store) has zero marginal cost and a fixed cost $k > 0$. We will suppose that each consumer desires to purchase, at most, one unit of the homogeneous good produced by the duopoly market and the maximum price that any consumer will pay for the good (consumer's price reservation) is $r > 0$.

We suppose that there are two types of consumers: the *uninformed consumers* which choose any store randomly, and the *informed consumers* which know the whole distribution of prices, i.e., they know the lowest available price. Let I be the number of informed consumers, V the number of uninformed consumers and T the total number of consumers $T = I + V$. We assume that the demand curve facing each firm, is given by

$$q(p,z) = \begin{cases} I + \frac{V}{2}, & \text{if } p < z \\ \frac{V}{2}, & \text{if } z \leq p \end{cases} \tag{2.18}$$

where p is the price of the firm and z is the price of the opponent firm.

Given the demand curve (2.18), each firm maximizes its payoff function

$$U(p,z) = \begin{cases} p\left[I + \frac{V}{2}\right] - k, & \text{if} \quad 0 \leq p < z \leq r \\ p\frac{V}{2} - k, & \text{if} \quad 0 \leq z \leq p \leq r. \end{cases} \tag{2.19}$$

Varian [99], [100] shows that this game has not a Nash equilibrium in pure strategies, and that there exists a symmetric Nash equilibrium in mixed strategies given by

$$\frac{d\mu^*(p)}{dp} = \begin{cases} \left[\frac{rV}{2I}\right] p^{-2}, & \text{if } \bar{p} \leq p \leq r \\ 0, & \text{other case} \end{cases}, \tag{2.20}$$

where $\bar{p} = \frac{rV}{2I+V}$.

2.6 Graduated risk game

The graduated risk game is a symmetric game (proposed by Maynard Smith and Parker [97]), where two players compete for a resource of value $v > 0$. Each player selects the "level of aggression" for the game. This "level of aggression" is captured by a probability distribution $x \in [0,1]$, where x is the probability that neither player is injured, and $\frac{1}{2}(1-x)$ is the probability that player one (or player two) is injured. If the player is injured its payoff is $v - c$ (with $c > 0$), and hence the expected payoff for the player is

$$U(x,y) = \begin{cases} vy + \frac{v-c}{2}(1-y) & \text{if } y > x, \\ \frac{v-c}{2}(1-x) & \text{if } y \leq x, \end{cases} \tag{2.21}$$

where x and y are the "levels of aggression" selected by the player and her opponent, respectively.

If $v < c$, this game has the NES with the density function

$$\frac{d\mu^*(x)}{dx} = \frac{\alpha - 1}{2} x^{\frac{\alpha-3}{2}}, \tag{2.22}$$

where $\alpha = \frac{c}{v}$. Moreover, if $c \leq v$, this games has the NES (see Maynard Smith and Parker [97], and Bishop and Cannings [13])

$$\mu^* = \delta_{\{0\}}. \tag{2.23}$$

2.7 War of attrition game

The war of attrition game was proposed by Maynard Smith [96]. In this two-players symmetric game, each player competes for a reward of value $v > 0$. Each player has a number $m > 0$ of resources for the war, and decides how much resources to spend to win this reward v. If a player is willing to risk more resources than the other player, then he wins the reward v and pays only the resources that the other player spends. Otherwise, the player loses the resources used during the war.

For x, y in the strategy set $A = [0, m]$ (with $v \leq m$), the payoff function is

$$U(x,y) = \begin{cases} v - y & \text{if } y < x, \\ -x & \text{if } y \geq x, \end{cases} \tag{2.24}$$

where x and y are the number of resources spent by the player and her opponent, respectively.

This game has a NES μ^* with the density function

$$\frac{d\mu^*(x)}{dx} = \left[\frac{1}{1 - e^{-m/v}}\right] \frac{1}{v} e^{-x/v}. \tag{2.25}$$

2.8 Comments

This chapter introduced a normal form game and important related concepts. Also, showed examples that will be used in the rest of this work to relate our theoretical results on evolutionary games to these interesting applications. Only remains to give information about references of normal form games. There exists several books that introduce the normal form games, to mention someones Kolokoltsov and Malafeyev [57], Myerson [70], Osborne and Rubinstein [76], Gibbons [41], and Fudenberg and Tirole [38].

Other references on theoretical advances that deal with normal form games with measurable strategy spaces and discontinuous payoff functions we can to mention some classical papers Glicksberg [43], Dasgupta and Maskin [33],[34], Simon [95], Reny [82]. Some recent references on the subject are Carmona and Podczeck[22], Prokopovych and Yannelis [79], Barelli and Meneghe [8],arbonell-Nicolau [20], McLennan, Monteiro and Tourky [68], and Carmona[21].

3 Evolutionary games: the asymmetric case

The theory of evolutionary dynamics in asymmetric games (or of several populations) has been developed for games where the strategy set of each player is finite, as in Balkenborg and Schlag [7], Ritzberger and Weibull [84], Samuelson and Zhang [88], Selten [92]. Nevertheless, there are well-known cases where the sets of strategies are metric spaces, such as oligopoly models and Nash bargaining games (Cressman [30]).

In this chapter we introduce an evolutionary dynamics model for asymmetric games where the strategy sets are measurable spaces (separable metric spaces). Under this hypothesis the replicator dynamics is in a Banach Space. We specify conditions under which the replicator dynamics has a solution. Furthermore, under suitable assumptions, a critical point of the system is stable. Finally, an example illustrates our results.

Section 3.1 shows a heuristic approach to the replicator dynamics in the asymmetric case. Section 3.2 describes the asymmetric evolutionary game and the replicator dynamics. Section 3.3 establishes conditions for the existence of a solution to the system of differential equations (replicator dynamics), and gives some characterizations of the solution (see Theorems 3.5 and 3.6 respectively). Section 3.4 establishes a relationship between the replicator dynamics and a normal form game using the concepts of *Nash equilibrium* and *strongly uninvadable profile* (see Theorems 3.9 and 3.12). Section 3.5 introduces conditions to establish the stability of the replicator dynamics (see Theorem 3.14). Section 3.7 proposes examples to illustrate our results. We conclude the chapter in section 3.8 with some general comments on possible extensions.

3.1 A heuristic approach to the replicator dynamics

Let $I := \{1, 2, ..., n\}$ be the set of different species (or players). Each individual of the species $i \in I$ can choose a single element a_i in a set of characteristics (strategies or actions) A_i, which is a separable metric space. For every $i \in I$

and every vector $a := (a_1, ..., a_n)$ in the Cartesian product $A := A_1 \times ... \times A_n$, we write a as (a_i, a_{-i}) where $a_{-i} := (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$ is in

$$A_{-i} := A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_n.$$

For each $i \in I$, let $\mathcal{B}(A_i)$ be the Borel σ-algebra of A_i, and $\mathbb{P}(A_i)$ the set of probability measures on A_i, also known as the set of *mixed strategies*. For each $i \in I$, let $N_i \in \mathbb{M}$ be a positive measure such that for each E_i in $\mathcal{B}(A_i)$, $N_i(E_i)$ assigns the "number" (or mass) of individuals using pure strategies a_i in E_i. Then the total population of the species i is $N_i(A_i)$ and the proportion of individuals using strategies in E_i is

$$\mu_i(E_i) := \frac{N_i(E_i)}{N_i(A_i)}. \tag{3.1}$$

Indeed, when the set A_i of characteristics of the specie i is not finite, it is convenient to consider the population size not as a "number of individuals" but a measure $N_i \in \mathbb{M}(A_i)$. Then for $i \in I$, we can introduce a probability measure $\mu_i \in \mathbb{P}(A_i)$ as in (3.1) that assigns a population distribution over the action set A_i.

For each species i we assign a payoff function $J_i : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{R}$ that explains the interrelation with the population of other species, and which is defined as in (2.1).

For each i in I, let γ_i^1, γ_i^2 be the background per capita birth and death rates in the population i. The background per capita net birth rate $\gamma_i := \gamma_i^1 - \gamma_i^2$ is modified by the payoff $J_i(a_i, \cdot)$ for using strategy $a_i \in A_i$. The rate of change of the number of individual of the species i for every $E_i \in \mathcal{B}(A_i)$, is

$$N_i'(t, E_i) = \gamma_i N_i(t, E_i) + N_i(t, A_i) \int_{E_i} J_i(a_i, \mu_{-i}(t))\mu_i(t, da_i) \quad \forall E_i \in \mathcal{B}(A_i) \tag{3.2}$$

with some initial positive measure $N_i(0)$ in $\mathbb{M}(A_i)$. The notation $N_i'(t, E_i)$ represents the Fréchet derivative of $N_i(t)$ in the Banach space $\mathbb{M}(A_i)$ (see Definition 1.5) valued at $E_i \in \mathcal{B}(A_i)$ and μ_i is a probability measure defined as in (3.1).

For each t in $[0, \infty)$ and i in I, the term $\int_{E_i} J_i(a_i, \mu_{-i}(t))\mu_i(t, da_i)$ in (3.2) values the efficiency of the strategies in the set E_i of the species i when the other species have a distribution $\mu_{-i}(t)$. Note that if $J_i(\cdot, \cdot) \equiv 0$, the solution of (3.2) is $N_i(t, E_i) = N_i(0, E_i)e^{\gamma_i t}$ for all $E_i \in \mathcal{B}(A_i)$ and $t \geq 0$.

Using (3.1) we have that

$$N_i'(t, E_i) = N_i(t, A_i)\mu_i'(t, E_i) + N_i'(t, A_i)\mu_i(t, E_i)$$

for every $E_i \in \mathcal{B}(A_i)$ and $t \geq 0$. Then for each species i

$$\mu_i'(t, E_i) = \frac{N_i'(t, E_i)}{N_i(t, A_i)} - \frac{N_i'(t, A_i)\mu_i(t, E_i)}{N_i(t, A_i)} \tag{3.3}$$

for evey $E_i \in \mathcal{B}(A_i)$ and $t \geq 0$. Hence, using (3.2) in (3.3), for each i in I, we obtain,

$$\mu_i'(t, E_i) = \int_{E_i} \Big[J_i(a_i, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t))\Big]\mu_i(t, da) \tag{3.4}$$

for each E_i in $\mathcal{B}(A_i)$ and $t \geq 0$. The system of equations (3.4) is known as *the replicator dynamics* for the asymmetric case.

3.2 Asymmetric evolutionary games

In an evolutionary game, the dynamics of the strategies is determined by the solution of a system of differential equations of the form

$$\mu_i'(t) = F_i(\mu_1(t), ..., \mu_n(t)) \qquad \forall \;\; i \in I, \quad t \geq 0, \tag{3.5}$$

with some initial condition $\mu_i(0) = \mu_{i,0}$ for each $i \in I$. The notation $\mu_i'(t)$ represents the Fréchet derivative of $\mu_i(t)$ in the Banach space $\mathbb{M}(A_i)$ (see Definition 1.5). For each $i \in I$, $F_i(\cdot)$ is a mapping

$$F_i : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{M}(A_i).$$

Let

$$F : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n),$$

where $F(\mu) := (F_1(\mu), ..., F_n(\mu))$, and consider the vector

$$\mu'(t) := (\mu_1'(t), ..., \mu_n'(t)).$$

Then the system (3.5) can be expressed as

$$\mu'(t) = F(\mu(t)), \tag{3.6}$$

and we can see that the system lives in the Cartesian product of signed measures

$$\mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n),$$

which is a Banach space with norm as in (1.15), i.e.

$$\|\mu\|_\infty = \|(\mu_1, ..., \mu_n)\|_\infty := \max_{i \in I} \|\mu_i\|.$$

More explicitly, we may write (3.5) as

$$\mu_i'(t, E_i) = F_i(\mu(t), E_i) \quad \forall\, i \in I,\; E_i \in \mathcal{B}(A_i),\; t \geq 0, \tag{3.7}$$

where $\mu_i'(t, E_i)$ and $F_i(\mu(t), E_i)$ are the signed-measures $\mu_i'(t)$ and $F_i(\mu(t))$ valued at $E_i \in \mathcal{B}(A_i)$.

We shall be working with a special class of asymmetric evolutionary games which can be described as

$$\left[I, \left\{\mathbb{P}(A_i)\right\}_{i\in I}, \left\{J_i(\cdot)\right\}_{i\in I}, \left\{\mu_i'(t) = F_i(\mu(t))\right\}_{i\in I}\right], \tag{3.8}$$

where

i) $I = \{1, ..., n\}$ is the finite set of players;

ii) for each player $i \in I$ we have a set of mixed actions $\mathbb{P}(A_i)$ and a payoff function $J_i : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{R}$ (as in (2.1)); and

iii) the replicator dynamics $F_i(\mu(t))$, where

$$F_i(\mu(t), E_i) := \int_{E_i} \Big[J_i(a_i, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t))\Big]\mu_i(t, da_i). \tag{3.9}$$

3.2.1 The symmetric case

We can obtain from (3.8) a symmetric evolutionary game (see Chapter 4) when $I := \{1, 2\}$ and the sets of actions and payoff functions are the same for both players, i.e., $A = A_1 = A_2$ and $U(a, b) = U_1(a, b) = U_2(b, a)$, for all $a, b \in A$. As a consequence, the sets of mixed actions and the expected payoff functions are the same for both players , i.e., $\mathbb{P}(A) = \mathbb{P}(A_1) = \mathbb{P}(A_2)$ and $J(\mu, \nu) = J_1(\mu, \nu) = J_2(\nu, \mu)$, for all $\mu, \nu \in \mathbb{P}(A)$. This kind of model determines the dynamic interaction of strategies of a unique species through the replicator dynamics $\mu'(t) = F(\mu(t))$, where $F : \mathbb{P}(A) \to \mathbb{M}(A)$ is given by

$$F(\mu(t), E) := \int_E \Big[J(a, \mu(t)) - J(\mu(t), \mu(t))\Big]\mu(t, da) \;\; \forall E \in \mathcal{B}(A). \tag{3.10}$$

Finally, as in (3.8), we can describe a symmetric evolutionary games as

$$[I = \{1, 2\}, \;\; \mathbb{P}(A), \;\; J(\cdot), \;\; \mu'(t) = F(\mu(t))]\,. \tag{3.11}$$

3.2.2 Another approximation to asymmetric games

Bomze and Pötscher [18] suggest an approach in which asymmetric games are reduced to symmetric ones. They construct a new strategy set $\bar{A}$ and a new payoff function $J : \bar{A} \times \bar{A} \to \mathbb{R}$. The strategy set $\bar{A}$ decomposes into mutually disjoint sets A_i, that is $\bar{A} := \cup_{i \in I} A_i$, where A_i is the set of strategies of the species $i \in I$. Then any measurable set $E \subset \bar{A}$ may be expressed as a union of mutually disjoint sets E_i, that is $E = \cup_{i \in I} E_i$, where $E_i = E \cap A_i$. Then $\mu(E) = \sum_{i \in I} \mu(E_i) = \sum_{i \in I} \mu_i(E)\mu(A_i)$, where

$$\mu_i(E) := \mu(E|A_i) = \frac{\mu(E \cap A_i)}{\mu(A_i)}. \tag{3.12}$$

The new payoff function is given by

$$J(\mu, \nu) = \sum_{i \in I} \mu(A_i) J_i(\mu_i, \nu_{-i}),$$

where $\nu_{-i} := (\nu_1, ..., \nu_{i-1}, \nu_{i+1}, ..., \nu_n)$ and ν_j as in (3.12) and $J_i(\mu_i, \nu_{-i})$ as in (2.1).

The replicator dynamics is constructed as in the symmetric case (3.10), with

$$F(\mu(t), E) := \sum_{i \in I} \mu(A_i) \int_{E_i} \Big[J_i(a_i, \mu_{-i}(t)) - \mu(A_i) J_i(\mu_i(t), \mu_{-i}(t)) \Big] \mu_i(t, da_i).$$

3.3 Existence

In this section we introduce conditions for the existence and uniqueness of solutions to the differential system (3.4). For this purpose we give conditions under which the operator F in (3.5) (3.6) is Lipschitz, when this operator is defined as in (3.9).

For each $i \in I$ and $t \geq 0$, let

$$\beta_i(a_i|\mu(t)) := J_i(a_i, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t)). \tag{3.13}$$

Hence, by (3.9), $\beta_i(\cdot|\mu(t))$ is the Radon-Nikodym density of $F_i(\mu(t))$ with respect to $\mu_i(t)$, i.e.,

$$F_i(\mu(t), E_i) = \int_{E_i} \beta_i(a_i|\mu(t)) \mu_i(t, da_i) \quad \forall E_i \in \mathcal{B}(A_i). \tag{3.14}$$

Remark 3.1. *i) We will use the usual notation $\mu << \nu$ to indicate that μ is absolutely continuous respect to ν (i.e. for every set $E \in \mathcal{B}(A)$ with $\nu(E) = 0$ we have $\mu(E) = 0$).*

ii) Let A be a separable metric space with Borel σ-algebra $\mathcal{B}(A)$. Suppose that $\nu, \eta \in \mathbb{M}(A)$ and $c_1, c_2 \geq 0$, and let $\mu = c_1\eta + c_2\nu$. If there exists a positive measure $\kappa \in \mathbb{M}(A)$ such that $\nu << \kappa$ and $\eta << \kappa$, then also $\mu << \kappa$. Moreover, the Radon-Nikodym densities

$$\varphi_{\nu\kappa} = \frac{d\nu}{d\kappa} \quad \text{and} \quad \varphi_{\eta\kappa} = \frac{d\eta}{d\kappa},$$

are such that

$$\varphi_{\mu\kappa} = \frac{d\mu}{d\kappa} = c_1\varphi_{\eta\kappa} + c_2\varphi_{\nu\kappa}.$$

Lemma 3.2. *Let ν, η, μ, κ and $\varphi_{\mu\kappa}$ be as in Remark 3.1 Then the total variation norm of μ is given by*

$$\|\mu\| = \int_A |\varphi_{\mu\kappa}(a)|\kappa(da).$$

In particular, the distance between the signed measures ν and η is given by

$$\|\nu - \eta\| = \int_A |(\varphi_{\nu\kappa} - \varphi_{\eta\kappa})(a)|\kappa(da).$$

The following proposition extends to our context some results by Bomze [17](Lemma 1) and Oechssler and Riedel [74](Lemma 3) in the case of symmetric evolutionary games.

Theorem 3.3. *Suppose that, for each $i \in I$, the function $\beta_i(\cdot|\mu)$ in (3.13) satisfies:*

i) there exists $C_i \geq 0$ such that $|\beta_i(a_i|\mu)| \leq C_i$ for each $a_i \in A_i$ and $\|\mu\|_\infty \leq 2$;

ii) there is a constant $D_i > 0$, such that

$$\sup_{a_i \in A_i} |\beta_i(a_i|\eta) - \beta_i(a_i|\nu)| \leq D_i\|\eta - \nu\|_\infty$$

for each ν, η with $\|\eta\|_\infty, \|\nu\|_\infty \leq 2$.

Then there exists a bounded Lipschitz map

$$G : \mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n) \to \mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n),$$

which coincides with F on $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$.

Proof. For each $i \in I$ and ν, η with $\|\eta\|_\infty, \|\nu\|_\infty \leq 2$, let $\mu_i = \frac{|\eta_i|+|\nu_i|}{2}$. Then $\|\mu_i\| \leq 2$, $\eta_i << \mu_i$ and $\nu_i << \mu_i$. Whence there exist the Radon-Nikodym densities $\frac{d\eta_i}{d\mu_i} = \varphi_{\eta_i \mu_i}$ and $\frac{d\nu_i}{d\mu_i} = \varphi_{\nu_i \mu_i}$. Using (3.14) and Lemma 3.2 we have that

$$\begin{aligned}
&\|F_i(\eta) - F_i(\nu)\| \\
&= \int_{A_i} \Big|\beta_i(a_i|\eta)\varphi_{\eta_i\mu_i}(a_i) - \beta_i(a_i|\nu)\varphi_{\nu_i\mu_i}(a_i)\Big| \, \mu_i(da_i) \\
&\leq \int_{A_i} \Big|\beta_i(a_i|\eta) - \beta_i(a_i|\nu)\Big|\Big|\varphi_{\eta_i\mu_i}(a_i)\Big|\mu_i(da_i) \\
&\qquad\qquad + \int_{A_i} \Big|\beta_i(a_i|\nu)\Big|\Big|\varphi_{\eta_i\mu_i}(a_i) - \varphi_{\nu_i\mu_i}(a_i)\Big|\mu_i(da_i) \\
&\leq \int_{A_i} \Big|\beta_i(a_i|\eta) - \beta_i(a_i|\nu)\Big||\eta_i|(da_i) \\
&\qquad\qquad + \int_{A_i} \Big|\beta_i(a_i|\nu)\Big|\Big|\varphi_{\eta_i\mu_i}(a_i) - \varphi_{\nu_i\mu_i}(a_i)\Big|\mu_i(da_i) \\
&\leq 2D_i \max_{j\in I} \|\eta_j - \nu_j\| + C_i\|\eta_i - \nu_i\| \\
&\leq K_i\|\eta - \nu\|_\infty,
\end{aligned}$$

where $K_i := \max\{2D_i, C_i\}$. Therefore

$$\|F(\eta) - F(\nu)\| = \max_{i\in I} \|F_i(\eta) - F_i(\nu)\| \leq K\|\eta - \nu\|_\infty,$$

for all η, ν with $\|\eta\|_\infty, \|\nu\|_\infty \leq 2$, with $K := \max\{K_i : i \in I\}$. Hence, F is Lipschitz continuous on the subset of $\mathbb{M}(A_1)\times ... \times\mathbb{M}(A_n)$ with norm $\|\cdot\|_\infty \leq 2$.

Let us now consider the function

$$G(\mu) := (2 - \|\mu\|_\infty)^+ F(\mu), \tag{3.15}$$

with $(2 - \|\mu\|_\infty)^+ := \max\{0,\ 2 - \|\mu\|_\infty\}$. It is clear that $G(\cdot)$ is bounded and coincides with $F(\cdot)$ on $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$. It remains to show that $G(\cdot)$ is Lipschitz.

Consider η and ν in $\mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n)$. If $\|\eta\|_\infty,\ \|\nu\|_\infty \geq 2$, then $G(\eta) = G(\nu) = 0$ and there is nothing to prove. Now, if $\|\eta\|_\infty > 2 \geq \|\nu\|_\infty$, then

$$\|G(\eta) - G(\nu)\|_\infty = (2 - \|\nu\|_\infty)\|F(\nu)\|_\infty,$$

and

$$\|F(\nu)\|_\infty = \max_{j\in I} \int_{A_j} \Big|\beta_j(a_j|\nu)\Big||\nu_j|(da_j) \leq \max_{j\in I} C_j\|\nu_j\| \leq C\|\nu\|_\infty$$

where $C = \max_{j\in I}\{C_j\}$. Hence

$$\begin{aligned} \|G(\eta) - G(\nu)\|_\infty &\le (2 - \|\nu\|_\infty) C \|\nu\|_\infty \\ &\le 2C(\|\eta\|_\infty - \|\nu\|_\infty) \le 2C\|\eta - \nu\|_\infty. \end{aligned} \tag{3.16}$$

Finally, if $\|\eta\|_\infty, \|\nu\|_\infty \le 2$, then

$$\begin{aligned} &\|G(\eta) - G(\nu)\|_\infty \\ &= \|(2 - \|\eta\|_\infty)F(\eta) - (2 - \|\nu\|_\infty)F(\nu)\|_\infty \\ &\le (2 - \|\eta\|_\infty)\|F(\eta) - F(\nu)\|_\infty + \|F(\nu)\|_\infty \, |\|\nu\|_\infty - \|\eta\|_\infty| \\ &\le 2K\|\eta - \nu\|_\infty + 2C\|\nu - \eta\|_\infty. \end{aligned} \tag{3.17}$$

Using (3.16) and (3.17) we see that, for any $\eta, \nu \in \mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n)$, we have

$$\|G(\eta) - G(\nu)\|_\infty \le 2(K + C)\|\eta - \nu\|_\infty. \qquad \square$$

The following proposition is an extension to our asymmetric games of Lemma 4 of Oechssler and Riedel [74] for symmetric games.

Proposition 3.4. *Let $i \in I$. If the payoff function $U_i(\cdot)$ is bounded, then $\beta_i(\cdot|\mu)$ satisfies the conditions $i)$ and $ii)$ of Theorem 3.3.*

Proof. Suppose that $\|\mu\|_\infty \le 2$ and let $i \in I$. Since $U_i(\cdot)$ is bounded, there exists $C_i' > 0$ such that $|U_i(a)| \le C_i'$ for all $a \in A$. Then, by Proposition 1.4,

$$\begin{aligned} |\beta_i(a_i|\mu)| &= \left| \int_{A_{-i}} U_i(a_i, a_{-i})\mu_{-i}(da_{-i}) - \int_A U_i(a)\mu(da) \right| \\ &\le \left| \int_{A_{-i}} U_i(a_i, a_{-i})\mu_{-i}(da_{-i}) \right| + \left| \int_A U_i(a)\mu(da) \right| \\ &\le C_i'\|\mu_1 \times ... \times \mu_{i-1} \times \mu_{i+1} ... \times \mu_n\| + C_i'\|\mu_1 \times ... \times \mu_n\| \\ &\le 2^{n-1}C_i' + 2^n C_i'. \end{aligned}$$

Letting $C_i := C_i'(2^{n-1} + 2^n)$, the condition $i)$ follows.

To prove the condition $ii)$ in Theorem 3.3, note that for any η and ν with $\|\eta\|_\infty$, $\|\nu\|_\infty \le 2$, using the notation in (2.3), and substracting and adding terms, we obtain, for every $i \in I$,

$$\begin{aligned} &\left| \int_A U_i(a)\eta(da) - \int_A U_i(a)\nu(da) \right| \\ &\qquad \le |\mathcal{I}_{(\eta_1,\eta_2,...,\eta_n)}U_i - \mathcal{I}_{(\nu_1,\eta_2,...,\eta_n)}U_i| \end{aligned}$$

$$
\begin{aligned}
& +|\mathcal{I}_{(\nu_1,\eta_2,\eta_3,\dots,\eta_n)}U_i - \mathcal{I}_{(\nu_1,\nu_2,\eta_3,\dots,\eta_n)}U_i| \\
& + \dots \\
& +|\mathcal{I}_{(\nu_1,\dots,\nu_{n-2},\eta_{n-1},\eta_n)}U_i - \mathcal{I}_{(\nu_1,\dots,\nu_{n-2},\nu_{n-1},\eta_n)}U_i| \\
& +|\mathcal{I}_{(\nu_1,\dots,\nu_{n-1},\eta_n)}U_i - \mathcal{I}_{(\nu_1,\dots,\nu_{n-1},\nu_n)}U_i| \\
\leq \; & \|U_i\|\|\eta_2 \times .... \times \eta_n\|\|\eta_1 - \nu_1\| \\
& +\|U_i\|\|\nu_1 \times \eta_3 \times ... \times \eta_n\|\|\eta_2 - \nu_2\| \\
& + \dots \\
& +\|U_i\|\|\nu_1 \times ... \times \nu_{n-2} \times \eta_n\|\|\eta_{n-1} - \nu_{n-1}\| \\
& +\|U_i\|\|\nu_1 \times .... \times \nu_{n-1}\|\|\eta_n - \nu_n\| \\
\leq \; & 2^{n-1}\|U_i\| \max_{j\in I}\|\eta_j - \nu_j\|. \qquad (3.18)
\end{aligned}
$$

Similarly, for every $i \in I$,

$$
\left|\int_{A_{-i}} U_i(a)\nu_{-i}(da_{-i}) - \int_{A_{-i}} U_i(a)\eta_{-i}(da_{-i})\right| \leq 2^{n-2}\|U_i\| \max_{j\neq i}\|\eta_j - \nu_j\|. \quad (3.19)
$$

Then by (3.18) and (3.19)

$$
\begin{aligned}
|\beta_i(a_i|\eta) - \beta_i(a_i|\nu)| \; = \; & \left|\int_{A_{-i}} U_i(a)\eta_{-i}(da_{-i}) - \int_A U_i(a)\eta(da)\right. \\
& \left. - \int_{A_{-i}} U_i(a)\nu_{-i}(da_{-i}) + \int_A U_i(a)\nu(da)\right| \\
\leq \; & \left|\int_{A_{-i}} U_i(a)\eta_{-i}(da_{-i}) - \int_{A_{-i}} U_i(a)\nu_{-i}(da_{-i})\right| \\
& + \left|\int_A U_i(a)\nu(da) - \int_A U_i(a)\eta(da)\right| \\
\leq \; & 2^n\|U_i\| \max_{j\in I}\|\eta_j - \nu_j\|.
\end{aligned}
$$

To conclude, the latter inequality yields

$$
\sup_{a_i\in A_i}|\beta_i(a_i|\eta) - \beta_i(a_i|\nu)| \leq D_i\|\eta - \nu\|_\infty,
$$

with $D_i = 2^n\|U_i\|$. □

By Theorem 3.3, the differential equation

$$
\mu'(t) = G(\mu(t)), \qquad (3.20)
$$

with G as in (3.15) has a unique solution in the space $\mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n)$ (see Lang [65], chapter IV). If $\mu(t)$ is a solution to (3.20) and

$$\mu(t) \in \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \quad \forall t \geq 0,$$

then $\mu(t)$ is also a solution of the differential equation (3.6), and it is unique since $F(\cdot)$ is Lipschitz in the open ball

$$V_2(0) = \{\mu \in \mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n) : \|\mu\|_\infty < 2\}.$$

Let $\mu(\cdot)$ be a solution of (3.20) (or (3.6)). We say that a set $C \subset \mathbb{M}(A_1) \times ... \times \mathbb{M}(A_n)$ is an *invariant* set for (3.20) (or (3.6)), if $\mu(t)$ is in C for all $t > 0$ when $\mu(0)$ is in C.

The following proposition ensures that the set $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_2)$ is an invariant set for (3.20). Therefore the replicator dynamics has a solution.

Theorem 3.5. *If $\mu(t)$ is a solution to (3.20), with initial condition $\mu(0)$ in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$, then $\mu(t)$ remains in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$ for all $t > 0$. Moreover, $\mu(t)$ is also the unique solution to the replicator dynamics (3.6) with $F(\cdot)$ as in (3.9).*

Proof. First, note that

$$\frac{d\mu_i(t, E_i)}{dt} = \mu'(t, E_i) \qquad \forall i \in I, \ E_i \in \mathbb{B}(A_i), \ t \geq 0. \tag{3.21}$$

Indeed,

$$\begin{aligned}
&\left|\frac{d\mu_i(t, E_i)}{dt} - \mu_i'(t, E_i)\right| \\
&= \lim_{\epsilon \to 0} \left|\frac{\mu_i(t+\epsilon, E_i) - \mu_i(t, E_i)}{\epsilon} - \mu_i'(t, E_i)\right| \\
&= \lim_{\epsilon \to 0} \left|\frac{1}{\epsilon}\left[\int_{A_i} 1_{E_i}(a_i)\mu_i(t+\epsilon, da_i) - \int_{A_i} 1_{E_i}(a_i)\mu_i(t, da_i)\right]\right. \\
&\qquad\qquad \left. - \int_{A_i} 1_{E_i}(a_i)\mu_i'(t, da_i)\right| \\
&\leq \lim_{\epsilon \to 0} \left\|\frac{\mu_i(t+\epsilon) - \mu_i(t)}{\epsilon} - \mu_i'(t)\right\| = 0.
\end{aligned}$$

Now, if $\mu(t)$ is a solution to (3.20), then by (3.21) and (3.15), for each $i \in I$, $E_i \in \mathbb{B}(A_i)$ and $t \geq 0$, we have

$$\frac{d\mu_i(t,E_i)}{dt} = (2-\|\mu(t)\|_\infty)^+\left[\int_{E_i} J_i(a_i,\mu_{-i}(t))\mu_i(t,da_i) \right.$$
$$\left. - J_i(\mu_i(t),\mu_{-i}(t))\mu_i(t,E_i)\right]. \quad (3.22)$$

In particular, for every $i \in I$,

$$\frac{d\mu_i(t,A_i)}{dt} = (2-\|\mu(t)\|_\infty)^+[1-\mu_i(t,A_i)]J_i(\mu_i(t),\mu_{-i}(t)). \quad (3.23)$$

We can express (3.23) as a system of differential equations in $\mathbb{R}^n$, say

$$\frac{d\mu_i(t,A_i)}{dt} = f_i(t,\mu_i(t,A_i)) \quad \text{for} \quad i=1,...,n,$$

where we can see the vector $[f_i(t,\mu_i(t,A_i))]_{i\in I}$ as a function $f:[0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ with

$$f(t,\mu_1(t,A_1),...,\mu_n(t,A_n)) = [f_i(t,\mu_i(t,A_i))]_{i\in I}.$$

The system (3.23) has a critical point if $\mu_i(t,A_i)=1$ for $i=1,...,n.$, (i.e., $f(t,\mu_1(t,A_1),...,\mu_n(t,A_n))=0$). Then if $\mu_i(0,A_i)=1$, we have that $\mu_i(t,A_i)=1$ for all $t\geq 0$ and $i\in I$. Hence the set

$$B := \{\mu \in \mathbb{M}_1\times...\times\mathbb{M}_n : \quad \mu_i(A_i)=1 \ \ \forall i\in I\},$$

is an invariant set for (3.20). Moreover, if $E_i \in \mathbb{B}(A_i)$, $t'\geq 0$ and $\mu_i(t',E_i)=0$, then by (3.22), $\mu_i(t,E_i)=0$ for all $t\geq t'$. In particular for each $E_i\in\mathbb{B}(A_i)$ and $i\in I$,

$$|\mu_i(t,E_i)-\mu_i(s,E_i)| \leq \|\mu_i(t)-\mu_i(s)\| \qquad \forall t,s\geq 0. \quad (3.24)$$

Since for each i in I the map $t\mapsto\mu_i(t)$ is continuous, then by (3.24) so is the map $t\mapsto\mu_i(t,E_i)$ for each $E_i\in\mathbb{B}(A_i)$. Therefore, if $\mu_i(0,E_i)\geq 0$, then we have $\mu_i(t,E_i)\geq 0$ for all $t>0$ and $E_i\in\mathbb{B}(A_i)$. It follows that,

$$\mathbb{P}(A_1)\times...\times\mathbb{P}(A_n)\subset B$$

is an invariant set for the system of differential equations (3.20).

Finally, if $\mu(t)$ is a solution to (3.20) and $\mu(0)$ is in $\mathbb{P}(A_1)\times...\times\mathbb{P}(A_n)$, then $\mu(t)$ is a solution to (3.6), and since F is Lipschitz for all μ with $\|\mu\|_\infty\leq 2$, this solution is unique. □

Theorem 3.6. *Suppose that the conditions i) and ii) of Theorem 3.3 are satisfied. If $\mu(t)$ is a solution to (3.6) with the initial condition $\mu(0)$ in $\mathbb{P}(A_1)\times ...\times\mathbb{P}(A_1)$, then:*

i) for every $i \in I$ and $t > 0$, if μ_i is in $\mathbb{P}(A_i)$, then $\mu_i(0) << \mu_i(t)$ and $\mu_i(t) << \mu_i(0)$, with Radon-Nikodym density

$$\frac{d\mu_i(t)}{d\mu_i(0)}(a_i) = e^{\int_0^t \beta_i(a_i|\mu(s))ds}. \tag{3.25}$$

ii) In particular, for every $i \in I$ and $t > 0$, if ν_i is a probability measure satisfying that $\nu_i << \mu_i(t)$ whenever $\nu_i << \mu_i(0)$, then

$$\log \frac{d\nu_i}{d\mu_i(t)}(a_i) = \log \frac{d\nu_i}{d\mu_i(0)}(a_i) - \int_0^t \beta_i(a_i|\mu(s))ds. \tag{3.26}$$

Proof. The following proof is an adaptation of Ritzberger [83] (lemma 2) and Bomze [17] (lemma 2). Let $\mu(t)$ be the solution to (3.6), with $\mu(0) \in \mathbb{P}(A_i)$ and

$$\varphi_i(t, a_i) := e^{\int_0^t \beta_i(a_i|\mu(s))ds} \geq 0 \qquad \forall i \in I. \tag{3.27}$$

In addition, let

$$\tilde{\mu}_i(t, E_i) := \int_{E_i} \varphi_i(t, a_i)\mu_i(0, da_i) \quad \forall E_i \in \mathcal{B}(A_i),$$

and, by (3.14),

$$F_i(\tilde{\mu}_i(t), E_i) = \int_{E_i} \beta_i(a_i|\mu(t))\tilde{\mu}_i(t, da_i).$$

We will prove that

$$\|\tilde{\mu}'(t) - F(\tilde{\mu}(t))\|_\infty = 0,$$

where $\tilde{\mu}'(t) = (\tilde{\mu}_1'(t), ..., \tilde{\mu}_n'(t))$ and $F(\tilde{\mu}(t)) = (F_1(\tilde{\mu}(t)), ..., F_n(\tilde{\mu}(t)))$.

Let $i \in I$ and fix $t > 0$. Then

$$\begin{aligned}
&\|\tilde{\mu}_i'(t) - F_i(\tilde{\mu}(t))\| \\
&= \lim_{h\to 0} \sup_{\|g\|\leq 1} \left| \frac{1}{h} \int_{A_i} g(a_i)[\varphi_i(t+h, a_i) - \varphi_i(t, a_i)]\mu(0, da_i) \right. \\
&\qquad \left. - \int_{A_i} g(a_i)\beta_i(a_i|\mu(t))\varphi(t, a_i)\mu_i(0, da_i) \right| \\
&\leq \lim_{h\to 0} \int_{A_i} \left| \frac{1}{h}[\varphi_i(t+h, a_i) - \varphi_i(t, a_i)] - \beta_i(a_i|\mu(t))\varphi_i(t, a_i) \right| \mu_i(0, da_i)
\end{aligned}$$

which, by (3.27),

$$\leq \sup_{a_i \in A_i} \left| e^{\int_0^t \beta_i(a_i|\mu(s))ds} \right| \lim_{h\to 0} \int_{A_i} \left| \frac{e^{\int_t^{t+h} \beta_i(a_i|\mu(s))ds} - 1}{h} - \beta_i(a_i|\mu(t)) \right| \mu_i(0, da_i)$$

$$\leq \sup_{a_i \in A_i} \left| e^{tC_i} \right| \int_{A_i} \left| \lim_{h\to 0} \frac{e^{\int_t^{t+h} \beta_i(a_i|\mu(s))ds} - 1}{h} - \beta_i(a_i|\mu(t)) \right| \mu_i(0, da_i) = 0,$$

where the latter equality follows from the conditions *i*) and *ii*) of Theorem 3.3 together with the dominated convergence theorem. To conclude

$$\|\tilde{\mu}'(t) - F(\tilde{\mu}(t))\|_\infty = 0 \qquad \forall t > 0.$$

By the uniqueness in Corollary 1.7, pag. 72 of Lang [65] we thus get (3.25), and therefore

$$\mu_i(t) << \mu_i(0) \quad \forall i \in I.$$

By the condition *ii*) of Theorem 3.3, for each $i \in I$ and $t > 0$, there exists $C_i \geq 0$ such that $-tC_i \leq \int_0^t \beta_i(a_i|\mu(s)) \leq tC_i$. Therefore

$$0 < e^{-tC_i} \leq e^{\int_0^t \beta_i(a_i|\mu(s))} \leq e^{tC_i}.$$

Hence, by (3.25),

$$\int_{E_i} e^{-tC_i} \mu_i(0, da_i) \leq \int_{E_i} \left[e^{\int_0^t \beta_i(a_i|\mu(s))} \right] \mu_i(0, da_i) = \mu_i(t, E_i);$$

thus $\mu_i(0) << \mu_i(t)$.

The assertion *ii*) follows from *i*) and an application of the chain rule for Radon-Nikodym densities (see Bartle [9] chapter 8). □

3.4 Nash equilibrium and the replicator equation

In this section we consider a normal form game Γ as in (2.6), and an asymmetric evolutionary game as in (3.8). We wish to study the relation between a Nash equilibrium of the normal form game Γ and the replicator equation (see Theorem 3.9 below). We also introduce the concept of *strongly uninvadable profile* (Definition 3.10), and its relation with *ϵ-equilibrium* (Definition 2.1).

The following proposition states an important fact about probability measures on separable metric spaces.

Proposition 3.7. *Let A be a separable metric space and μ in $\mathbb{P}(A)$. Then there is a unique closed set $S \subset A$ (called* the support of μ*, in symbols S=Supp(μ)) such that $\mu(A - S) = 0$ and $\mu(O \cap S) > 0$ for every open set O for which $O \cap S \neq \phi$.*

Proof. : See Royden [86], pag. 408. □

Lemma 3.8. *Supposes that $\mu^* = (\mu_1^*, ..., \mu_n^*)$ is a Nash equilibrium of Γ, and let S_i be the support of μ_i^* for some $i \in I$. Then $J_i(a_i, \mu_{-i}^*) = J_i(\mu^*, \mu_{-i}^*)$ for all $a_i \in S_i$, i.e., $J_i(\mu_i^*, \mu_{-i}^*) = J_i(a_i, \mu_{-i}^*)$ μ_i^*-a.s.*

Proof. Using Proposition 3.7 , the proof is similar to the case when the strategy sets are finite (see, e.g., Webb [103]). □

The following theorem gives an important property, namely the relation between a Nash equilibrium of a normal form game and the replicator equation.

Theorem 3.9. *Suppose that $\mu^* = (\mu_1^*, ..., \mu_n^*)$ is a Nash equilibrium of Γ . Then μ^* is a critical point of (3.6), i.e., $F(\mu^*) = 0$, when $F(\cdot)$ is described by the replicator dynamics (3.9).*

Proof. First note that any vector of Dirac measures $\delta_{a'} = (\delta_{a'_1}, ..., \delta_{a'_n})$ (sometimes called a profile of pure strategies) is a critical point of (3.6), since for every $E_i \in \mathcal{B}(A_i)$ and $i \in I$:

$$F_i(\delta'_a, E_i) = \int_{E_i} \left[J_i(a_i, \delta_{a'_{-i}}) - J_i(\delta_{a'_i}, \delta_{a'_{-i}}) \right] \delta_{a'_i}(da_i) = 0.$$

Then if μ^* is a pure Nash equilibrium, i.e., $\mu^* = \delta_{a^*}$, the theorem holds.

Suppose now that the Nash equilibrium μ^* is not pure, and let S_i^* be the support of μ_i^* for $i \in I$. By Lemma 3.8, for all $a_i \in S_i^*$, $J_i(a_i, \mu_{-i}^*) = J_i(\mu_i^*, \mu_{-i}^*)$. Therefore, for any $E_i \in \mathcal{B}(A_i)$,

$$F_i(\mu^*, E_i) = \int_{E_i} \left[J_i(a_i, \mu_{-i}^*) - J(\mu_i^*, \mu_{-i}^*) \right] \mu_i^*(da)$$

$$= \int_{E_i \cap S_i^*} \left[J_i(a_i, \mu_{-i}^*) - J(\mu_i^*, \mu_{-i}^*) \right] \mu_i^*(da) = 0. \qquad \square$$

The following definition is an extended version of strongly uninvadable strategies of symmetric games (for details see Bomze [17]).

Definition 3.10. *A vector $\mu^* \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times ... \times \mathbb{P}(A_n)$ is called a* strong uninvadable profile (SUP) *in a set $\mathcal{C}$ if μ^* is in $\mathcal{C}$ and the followings holds.*

There exists $\epsilon > 0$ *such that for any* $\mu \in \mathcal{C}$ *with* $\|\mu - \mu^*\|_\infty < \epsilon$, *and every* $i \in I$, $J_i(\mu_i^*, \mu_{-i}) > J_i(\mu_i, \mu_{-i})$ *if* $\mu_i \neq \mu_i^*$. *In particular if*

$$\mathcal{C} = \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times ... \times \mathbb{P}(A_n),$$

μ^* *is simply called a* strong uninvadable profile (SUP). *In either case, we call* ϵ *the global invasion barrier.*

Lemma 3.11. *Let* $\mu, \nu \in \mathbb{P}_1(A) \times \cdots \times \mathbb{P}(A_n)$ *and* $\delta > 0$. *Then there exists* α *in* $(0,1)$ *such that* $\|\gamma - \mu\|_\infty \leq \delta$ *if* $\gamma = \alpha\nu + (1-\alpha)\mu$.

Proof. Let $0 < \alpha < \frac{\delta}{\|\nu - \mu\|_\infty}$. Then

$$\|\gamma - \mu\|_\infty = \|\alpha\nu + (1-\alpha)\mu - \mu\|_\infty = \alpha\|\nu - \mu\|_\infty < \delta. \qquad \square$$

As usual, the open neighborhood with center μ^* and radius $\varepsilon > 0$ is defined as

$$V_\varepsilon(\mu^*) := \{\mu \in \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) : \ \|\mu - \mu^*\|_\infty < \varepsilon\}. \tag{3.28}$$

The following theorem gives the relation between an ϵ-equilibrium (or Nash equilibrium) and strong uninvadable profiles.

Theorem 3.12. *Suppose that the payoff function* $U_i(\cdot)$ *in (2.1) is bounded for all* $i \in I$. *Let* μ^* *be a* SUP in a set $\mathcal{C}$ *with global invasion barrier* $\epsilon_1 > 0$. *If the set* $\mathcal{C} \cap V_{\epsilon_1}(\mu^*)$ *has a convex and nonempty interior, then* μ^* *is an* ϵ_2*-equilibrium of* Γ, *where* $\epsilon_2(\cdot) > 0$ *is a function of* ϵ_1 . *Moreover, if* μ^* *is a* SUP, *then* μ^* *is a Nash equilibrium and the boundedness hypothesis on* U_i *is not required.*

Proof. Suppose that μ^* is not an ϵ_2-equilibrium of Γ for any $\epsilon_2 > 0$. Then for $\epsilon_2 > 0$, there exists $i \in I$ and $\nu \in \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$ such that

$$J_i(\nu_i, \mu_{-i}^*) - \epsilon_2 > J(\mu_i^*, \mu_{-i}^*). \tag{3.29}$$

By hypothesis, $\mathcal{C} \cap V_{\epsilon_1}(\mu^*)$ has a convex and nonempty interior. Hence, by Lemma 3.11, there exist $\alpha_1, \alpha_2 \in [0,1]$ such that $\|\eta - \mu^*\| < \epsilon_1$, where $\eta \in \mathcal{C}$ and $\eta := (1-\alpha_1)\mu^* + \alpha_1[(1-\alpha_2)\nu + \alpha_2\kappa]$ for some κ in the interior of $\mathcal{C} \cap V_{\epsilon_1}(\mu^*)$. Since μ^* is a SUP in the set $\mathcal{C}$, $J_i(\mu_i^*, \eta_{-i}) > J_i(\eta_i, \eta_{-i})$, which implies (see Appendix A.1).

$$\begin{aligned}
&(1-\alpha_2)(1-\alpha_1)^{n-1} J_i(\mu_i^*, \mu_{-i}^*) + [(1-\alpha_2)\alpha_1]^{n-1} J_i(\mu_i^*, \nu_{-i}) \\
&\qquad\qquad\qquad\qquad + [\alpha_2\alpha_1]^{n-1} J_i(\mu_i^*, \kappa_{-i})] \\
> \ &(1-\alpha_2)(1-\alpha_1)^{n-1} J_i(\nu_i, \mu_{-i}^*)
\end{aligned}$$

$$-\alpha_2(1-\alpha_1)^{n-1}\left[J_i(\mu_i^*,\mu_{-i}^*)-J_i(\kappa_i,\mu_{-i}^*)\right]+O(\alpha_1). \tag{3.30}$$

Let $\epsilon_2^* = \left(\frac{\alpha_2}{1-\alpha_2}\right)L\epsilon_1$, where $L = 2^{n-1}\max_{i\in I}\|U_i\|$. By (3.18)

$$|J_i(\mu_i^*,\mu_{-i}^*)-J_i(\kappa_i,\mu_{-i}^*)| < L\epsilon_1 \leq \epsilon_2\left(\frac{1-\alpha_2}{\alpha_2}\right) \quad \forall\ \epsilon_2 \geq \epsilon_2^*.$$

Then

$$\begin{aligned}(1-\alpha_2)(1-\alpha_1)^{n-1}&J_i(\mu_i^*,\mu_{-i}^*)\\ &+[(1-\alpha_2)\alpha_1]^{n-1}J_i(\mu_i^*,\nu_{-i})+[\alpha_2\alpha_1]^{n-1}J_i(\mu_i^*,\kappa_{-i})]\\ > \quad &(1-\alpha_2)(1-\alpha_1)^{n-1}[J_i(\nu_i,\mu_{-i}^*)-\epsilon_2]+O(\alpha_1).\end{aligned} \tag{3.31}$$

If (3.29) is true, there exists α_1 in $(0,1)$ sufficiently close to 0, such that the equation (3.31) is violated. So we have that μ^* is an ϵ_2-equilibrium (for $\epsilon_2 \geq \epsilon_2^*$).

Now, suppose that μ^* is a SUP and not a Nash equilibrium of Γ. Then there exists $i \in I$ and $\nu \in \mathbb{P}(A_1)\times\cdots\times\mathbb{P}(A_n)$ such that (3.29) is true with $\epsilon_2 = 0$. By the Lemma 3.11 there exist $\alpha \in [0,1]$ such that $\|\eta-\mu^*\| < \epsilon_1$ where $\eta = (1-\alpha)\mu^*+\alpha\nu$. Since μ^* is a SUP, $J_i(\mu_i^*,\eta_{-i}) > J_i(\eta_i,\eta_{-i})$. Then (see Appendix A.1)

$$\begin{aligned}(1-\alpha)^{n-1}J_i(\mu_i^*,\mu_{-i}^*)&+(\alpha)^{n-1}\left[J_i(\mu_i^*,\nu_{-i})\right]\\ &>(1-\alpha)^{n-1}\left[J_i(\nu_i,\mu_{-i}^*)]\right]+O(\alpha).\end{aligned} \tag{3.32}$$

If μ^* is not a Nash equilibrium, then for α in $(0,1)$ sufficiently small (3.32) is violated. So we have that μ^* is a Nash equilibrium. □

3.5 Stability

In this section we are interested in the stability of the differential system (3.6) (see Definition 3.13). To this end, we establish that uninvadable profiles (Definition 3.10) have some type of stability.

Definition 3.13. *Let μ^* be a critical point of (3.6), i.e., $F(\mu^*) = 0$.*

i) μ^* *is called* Lyapunov stable *if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\|\mu(0)-\mu^*\|_\infty < \delta$, then $\|\mu(t)-\mu^*\|_\infty < \epsilon$ for all $t > 0$.*

ii) μ^* *is called* weakly attracting *if it is Lyapunov stable and, in addition, there exists* $\delta > 0$ *such that if* $\|\mu(0) - \mu^*\|_\infty < \delta$*, then as* $t \to \infty$, $\mu_i(t) \to \mu_i^*$ *weakly for all* $i \in I$.

The following proposition is an extension to asymmetric evolutionary games of Theorem 3 in Oechssler and Riedel [74].

Theorem 3.14. *Suppose that the conditions i) and ii) of Theorem 3.3 hold. Let* $\delta_{a^*} = (\delta_{a_1^*}, ..., \delta_{a_n^*})$ *be a vector of Dirac measures, and* $\mathcal{C}$ *an invariant set for the differential equation (3.6). If* δ_{a^*} *is a* SUP in the set $\mathcal{C}$, *then there exists* $\epsilon > 0$ *such that the set*

$$\mathcal{C} \cap V_\epsilon(\delta_{a^*}),$$

is invariant for (3.6). Moreover, suppose that for all i *in* I*, the map* $\mu \mapsto \beta_i(a_i^*|\mu)$ *is weakly continuous and the set of strategies* A_i *is a compact set. If* $\mathcal{C}$ *is a closed set and* $\mu(0)$ *is in* $\mathcal{C} \cap V_\epsilon(\delta_{a^*})$*, then as* $t \to \infty$, $\mu(t) \to \delta_{a^*}$ *in distribution.*

Proof. First note that the vector of Dirac measures $\delta_{a^*} = (\delta_{a_1^*}, ..., \delta_{a_n^*})$ is a critical point of (3.6) (see the proof of Theorem 3.9). Then if $\mu(0) = \delta_{a^*}$, we have that $\mu(t) = \delta_{a^*}$ for all $t > 0$ and the theorem holds.

Since δ_{a^*} is a SUP in the set $\mathcal{C}$, there exists $\epsilon > 0$ such that for every $\mu \in \mathcal{C}$ with $\|\mu - \delta_{a^*}\|_\infty < \epsilon$ and every $i \in I$, $J_i(\delta_{a_i^*}, \mu_{-i}) > J_i(\mu_i, \mu_{-i})$ *if* $\mu_i \neq \delta_{a_i^*}$.

Suppose that $\mu(0) \neq \delta_{a^*}$ and that $\mu(0)$ is in $\mathcal{C} \cap V_\epsilon(\delta_{a^*})$. By (3.14), for each $i \in I$ and $t \geq 0$,

$$\mu_i'(t, \{a_i^*\}) = \int_{A_i} 1_{\{a^*\}}(a_i)\beta(a_i|\mu(t))\mu_i(t, da_i) = \beta(a_i^*|\mu(t))\mu_i(t, \{a_i^*\}). \quad (3.33)$$

Assume that for each i in I,

$$\mu_i'(0, \{a_i^*\}) = \beta(a_i^*|\mu(0))\mu_i(0, \{a_i^*\}) > 0,$$

and define

$$t_{i,0} := \inf\{t \geq 0 : \mu_i'(t, \{a_i^*\}) = 0\}. \quad (3.34)$$

For each i in I, the function $\beta_i(a_i^*|\mu(t))$ is Lipschitz in $\mu(t)$, and $\mu(t)$ is continuous in t; hence the map $t \to \beta_i(a_i^*|\mu(t))$ is continuous. Also $\mu_i(t, \{a_i^*\})$ is continuous in t. Then by (3.33) the map $t \mapsto \mu_i'(t, \{a_i^*\})$ is continuous. So for each $i \in I$ the set $\{t \geq 0 : \mu_i'(t, \{a_i^*\}) = 0\}$ is closed and $\mu_i'(t_{i,0}, \{a_i^*\}) = 0$. By (3.34), for any i in I

$$\mu_i'(s, \{a_i^*\}) = \beta_i(a_i^*|\mu(s))\mu_i(s, \{a_i^*\}) > 0 \quad \forall \ 0 \leq s < t_0, \quad (3.35)$$

where $t_0 := \min\{t_{1,0}, ..., t_{n,0}\}$. As a consequence of (3.35) we obtain

$$\mu_i(s, \{a_i^*\}) > \mu_i(0, \{a_i^*\}) > 0 \quad \forall\ 0 \le s < t_0,\ i \in I. \tag{3.36}$$

Note that for any $\mu_i \in \mathbb{P}(A_i)$

$$\|\mu_i - \delta_{a_i^*}\| = 2(1 - \mu_i(\{a_i^*\})) \quad \forall\ i \in I. \tag{3.37}$$

If $\|\mu(0) - \delta_{a^*}\|_\infty < \epsilon$, then by (3.36) and (3.37) we have

$$\|\mu(s) - \delta_{a^*}\|_\infty < \epsilon \qquad \forall\ \ 0 \le s < t_0.$$

By continuity of $\mu(t)$ and (3.36) we obtain

$$\mu_i(t_0, \{a_i^*\}) \ge \mu_i(0, \{a_i^*\}) > 0 \quad \forall\ \ 0 \le s < t_0,\ \ i \in I, \tag{3.38}$$

and by (3.37) and (3.38)

$$\|\mu_i(t_0) - \delta_{a_i^*}\|_\infty \le \|\mu(0) - \delta_{a^*}\|_\infty < \epsilon \qquad \forall\ \ i \in I. \tag{3.39}$$

Since $\mathcal{C}$ is an invariant set, by (3.39) we see that $\mu(t_0) \in \mathcal{C} \cap V_\epsilon(\delta_{a^*})$ and so $\beta_i(a_i^*|\mu(t_0)) > 0$ because δ_{a^*} is a SUP in the set $\mathcal{C}$. Then by (3.38)

$$\mu_i'(t_0, \{a_i^*\}) = \beta_i(a_i^*|\mu(t_0))\mu_i(t_0, \{a_i^*\}) > 0 \quad \forall i \in I,$$

so $t \mapsto \mu_i(t, \{a_i^*\})$ is increasing for each i in I and, moreover,

$$\mu(t) \in \mathcal{C} \cap V_\epsilon(\delta_{a^*}) \quad \forall\ \ t \ge 0. \tag{3.40}$$

By hypothesis, A_i is compact for each $i \in I$, so $\mathbb{P}(A_i)$ is compact in the weak topology (see pag. 186, Corollary 5.7.6 in Bobrowski [14]) for all $i \in I$. Then $\mathcal{C} \cap \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$ is compact in the product topology.

On the other hand, δ_{a^*} is a SUP in the set $\mathcal{C}$ and, by (3.40), $\beta_i(a_i^*|\mu(t)) > 0$ for all $t > 0$ and i in I. Moreover, by Theorem 3.6,

$$\mu_i(t, \{a_i^*\}) = \mu_i(0, \{a_i^*\}) e^{\int_0^t \beta_i(a_i^*|\mu(s))ds} \le 1 \quad \forall\ i \in I, \quad t \ge 0;$$

hence

$$\lim_{t\to\infty} \beta_i(a_i^*|\mu(t)) = 0 \quad \forall\ i \in I.$$

Finally, let $v = (v_1, ..., v_n) \in \mathcal{C} \cap \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$ be an accumulation point of the trajectory $\mu(t) = (\mu_1(t), ..., \mu_n(t))$. By (3.40) the distance from

v to δ_{a^*} is at most ϵ. Since δ_{a^*} is a SUP in $\mathcal{C}$ and the map $\mu \mapsto \beta_i(a_i^*|\mu)$ is weakly continuous, if v is such that

$$\beta_i(a_i^*|v) = J_i(a_i^*, v_{-i}) - J_i(v_i, v_{-i}) = 0 \quad \forall\ i \in I,$$

yields that $\delta_{a^*} = v$, which proves that $\mu_i(t) \to \delta_{a_i^*}$ in distribution for all i in I. $\square$

If the vector δ_{a^*} in Theorem 3.14 is a SUP, then we obtain the following corollary, taking $\mathcal{C} = \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$.

Corollary 3.15. *Suppose that the conditions i) and ii) of Theorem 3.3 hold. Let $\delta_{a^*} = (\delta_{a_1^*}, ..., \delta_{a_n^*})$ be a vector of Dirac measures, and suppose that it is a SUP. Then δ_{a^*} is Lyapunov stable for the replicator dynamics. Moreover, if the map $\mu \mapsto \beta_i(a_i^*|\mu)$ is weakly continuous and the set of strategies A_i is compact for all $i \in I$, then δ_{a^*} is weakly attracting.*

Remark 3.16. *Note that if for each i in I the payoff function $U_i(\cdot)$ in (2.1) is continuous, then the map $\mu \mapsto \beta_i(a_i^*|\mu)$ is weakly continuous. This fact is of relevance because many games satisfy that $U_i(\cdot)$ in (2.1) is continuous.*

3.6 Examples

Suppose a game in which it is not assumed that players are "strictly rational" or that they don't have perfect information game (i.e., they do not know the payments and strategy games of his opponents). Then the players could not select the best strategy, and the profile of the game could not be a NE of a Normal form game Γ (2.6).

In evolutionary games (3.8) we assume that the players choose their strategies through an evolutionary dynamics (3.5)-(3.6) which explain the interaction among them. Therefore, the solution of the game is explicated by a trajectory $\mu(t)$ (solution of (3.5)-(3.6)) which depend of a initial profile μ_0. Under some conditions (see Theorem (3.4)), the trajectory $\mu(t)$ is very closed to a "special" NE (solution of Γ). In this "special" NE the strategy of each player satisfies certain condition of dominance, and this NE is call SUP (see Definition 3.10 and Theorem (3.12)). Therefore, for each player the replicator dynamics search and select strategies with certain dominance.

In this section we consider the examples of a quadratic-linear model, the tragedy of commons, and poverty traps model of the sections 2.2, 2.3 and 2.4 repetitively. In each example we prove that the NE of the game is also a SUP. Thus, under the replicator dynamics if the initial profile μ_0 is closed to the NE (which is a SUP), then the players select a profile $\mu(t)$ very closed to the NE for every $t > 0$.

3.6.1 A quadratic-linear model

Consider the game in Section 2.2. We will prove that the Nash equilibrium (2.11) is a SUP for the game. Let $U_1(x, y)$ and $U_2(x, y)$ as in (2.9) and (2.10), respectively. Let

$$\mathcal{C}_1 := \{(\mu, \nu) \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) : \mu(x^*, M_1] = \nu(y^*, M_2] = 0\},$$

$$\mathcal{C}_2 := \{(\mu, \nu) \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) : \mu[0, x^*) = \nu[0, y^*) = 0\},$$

and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. The set $\mathcal{C}$ is invariant for the replicator dynamics (3.6) and $(\delta_{x^*}, \delta_{y^*})$ is in $\mathcal{C}$. On the other hand, let

$$\bar{x}^\mu := \int_{A_1} x\mu(dx), \qquad \bar{y}^\mu := \int_{A_2} y\mu(dy).$$

If (μ, ν) is in $\mathcal{C}_1$, then by Jensen's inequality

$$J_1(\delta_{x^*}, \nu) = \int_{A_2} U_1(x^*, y)\nu(dy) = U_1(x^*, \bar{y}^\nu) > U_1(\bar{x}^\mu, \bar{y}^\nu) \geq J_1(\mu, \nu)$$

$$J_2(\mu, \delta_{y^*}) = \int_{A_1} U_2(x, y^*)\mu(dx) = U_2(\bar{x}^\mu, y^*) > U_2(\bar{x}^\mu, \bar{y}^\nu) \geq J_2(\mu, \nu).$$

This is also true if (μ, ν) is in $\mathcal{C}_2$. Hence, for any $\epsilon > 0$, the vector $\delta_{(x^*, y^*)} = (\delta_{x^*}, \delta_{y^*})$ is a SUP in the set $\mathcal{C}$. Therefore, by Theorem 3.14, for $\epsilon > 0$ the set $\mathcal{C} \cap V_\epsilon(\delta_{(x^*, y^*)})$ is invariant for (3.6). Moreover, since for every i in I, the payoff functions $U_i(\cdot)$ are continuous and the sets of strategies A_i are compact sets, we conclude by Theorem 3.14 and Remark 3.16 that if $\mu(0) \in \mathcal{C} \cap V_\epsilon(\delta_{(x^*, y^*)})$, then $\mu(t) \to \delta_{(x^*, y^*)}$ in distribution.

3.6.2 The tragedy of the commons

In section 2.3, we saw that there is a unique Nash equilibrium $(x_1^*, ..., x_n^*)$ for the "tragedy of the commons". We will prove that it is also a SUP for the game.

For each player i in I, we define the following sets:

$$H_i^1 := \{x_i \in A_i : \; x_i \leq x_i^*\}, \quad H_i^2 := \{x_i \in A_i : \; x_i \geq x_i^*\},$$

$$\mathcal{C}_1 := \{(\mu_1, ...\mu_n) \in \mathbb{P}(A_i) \times ... \times \mathbb{P}(A_n) : \ \mu_i(H_i^1) = 1 \quad \forall i \in I\},$$

$$\mathcal{C}_2 := \{(\mu_1, ...\mu_n) \in \mathbb{P}(A_i) \times ... \times \mathbb{P}(A_n) : \ \mu_i(H_i^2) = 1 \quad \forall i \in I\}.$$

Let $(x_1, ..., x_n)$ be a profile such that $x_i \leq x_i^*$ for all i in I with strictly inequality for some player i. Let $\hat{x} := x_1 + ... + x_n$, and $\hat{x}^* := x_1^* + ... + x_j^* + ... + x_n^*$, then $\hat{x} < \hat{x}^*$.

For all i in I, let U_i be has (2.12) and consider the left-hand side of (2.13). Since $v' < 0$ and $v'' < 0$, then we have $0 < v(\hat{x}^*) < v(\hat{x})$ and $v'(\hat{x}^*) < v'(\hat{x}) < 0$. Therefore for each i in I

$$\frac{\partial U_i(x_i, x_{-i})}{\partial x_i} = v(\hat{x}) + x_i v'(\hat{x}) - c_i > v(\hat{x}^*) + x_i^* v'(\hat{x}^*) - c_i = 0.$$

Thus the map $x_i \mapsto U_i(x_i, x_{-i})$ is increasing in $[0, x_i^*]$, and

$$U_i(x_i^*, x_{-i}) > U_i(x_i, x_{-i}) \quad \forall x_i \in H_i^1, \ x_{-i} \in H_{-i}^1, \tag{3.41}$$

where $H_{-i}^1 = H_1^1 \times ... \times H_{i-1}^1 \times H_{i+1}^1 \times ... \times H_n^1$.

Similarly, if $(x_1, ..., x_n)$ is a profile such that $x_i \geq x_i^*$ for all i in I with strict inequality for son player i. Then the map $x_i \mapsto U_i(x_i, x_{-i})$ is decreasing in $[x_i^*, \bar{x}]$, where $\bar{x}$ is the maximum number of goats that can be in the garden. Hence

$$U_i(x_i^*, x_{-i}) > U_i(x_i, x_{-i}) \quad \forall x_i \in H_i^2, \ x_{-i} \in H_{-i}^2, \tag{3.42}$$

where $H_{-i}^2 = H_1^2 \times ... \times H_{i-1}^2 \times H_{i+1}^2 \times ... \times H_n^2$.

Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. If $\mu \in \mathcal{C}$, then by (3.41) and (3.42)

$$J_i(\delta_{x^*}, \mu_{-i}) > J_i(\mu_i, \mu_{-i}) \quad \forall i \in I.$$

Hence, for any $\epsilon > 0$, the vector $\delta_{x^*} = (\delta_{x_1^*}, ..., \delta_{x_n^*})$ is a SUP in the set $\mathcal{C}$. By Theorem 3.14, the set $\mathcal{C} \cap V_\epsilon(\delta_{x^*})$ is invariant for (3.6). Moreover, since for every i in I, the payoff functions $U_i(\cdot)$ are continuous and the sets of strategies A_i are compact sets, we conclude by Remark 3.16 that if $\mu(0) \in \mathcal{C} \cap V_\epsilon(\delta_{x^*})$, then $\mu(t) \to \delta_{x^*}$ in distribution.

3.6.3 Poverty traps

In section 2.4, we saw that there are three Nash equilibria for the game described by the table (2.15). We will prove that the pure Nash equibribria $\delta_{(s,m)} = (\delta_s, \delta_m)$ and $\delta_{(a,\tau)} = (\delta_a, \delta_\tau)$ are SUPs. To this end, consider the Nash equilibrium (μ^*, ν^*) described by (2.16) and (2.17) respectively.

Let $k_1 = \max\{\mu^*(s), \nu^*(m)\}$ and $k_2 = \max\{1 - \mu^*(s), 1 - \nu^*(m)\}$. Note that the sets of pure strategies for workers and firms are $A_w = \{s, a\}$ and $A_f = \{m, \tau\}$, respectively. Consider the sets

$$\mathcal{C}_1 := \{(\mu, \nu) \in \mathbb{P}(A_w) \times \mathbb{P}(A_f) : \; k_1 < \mu(s), k_1 < \nu(m)\}$$

$$\mathcal{C}_2 := \{(\mu, \nu) \in \mathbb{P}(A_w) \times \mathbb{P}(A_f) : \; k_2 < \mu(a), k_2 < \nu(\tau)\}.$$

Its is easy to check that for any $(\mu, \nu) \in \mathcal{C}_1$,

$$J_w(\delta_s, \nu) > J_w(\mu, \nu) \quad \text{and} \quad J_f(\mu, \delta_m) > J_f(\mu, \nu),$$

where J_w and J_f are the expected payoffs described by (2.1) of the workers and firms respectively. Similarly, if $(\mu, \nu) \in \mathcal{C}_2$, then

$$J_w(\delta_a, \nu) > J_w(\mu, \nu) \quad \text{and} \quad J_f(\mu, \delta_\tau) > J_f(\mu, \nu),$$

Let $\epsilon_1 = 1 - k_1$ and $\epsilon_2 = 1 - k_2$. Note that $\|\delta_w - \mu\| = 1 - \mu(w)$ and $\|\delta_f - \nu\| = 1 - \nu(f)$ for any $w \in A_w$ and $f \in A_f$. Then open balls $V_{\epsilon_1}(\delta_{(s,m)})$ and $V_{\epsilon_2}(\delta_{(s,m)})$ (introduced in 3.28) satisfy that $V_{\epsilon_1}(\delta_{(s,m)}) = \mathcal{C}_1$ and $V_{\epsilon_2}(\delta_{(s,m)}) = \mathcal{C}_2$. This prove that $\delta_{(s,m)}$ and $\delta_{(a,t)}$ are SUPs with barriers ϵ_1 and ϵ_2 respectively.

Hence, the conditions of Corollary 3.15 are satisfied, and so $\delta_{(s,m)}$ and $\delta_{(a,t)}$ are Lyapunov stable for the replicator dynamics. Moreover, since the action spaces A_w and A_f are finite sets, then the replicator dynamics is in $\mathbb{R}^n$ ($n = 4$ in this case) and the maps $(\mu_i, \mu_{-i}) \mapsto J_i(a_i, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t))$ are continuous for $i = w, f$. Therefore, $\delta_{(s,m)}$ and $\delta_{(a,t)}$ are weakly attracting.

3.7 Comments

In this chapter, we introduced a model of asymmetric evolutionary games with strategies in measurable spaces. The model can be reduced, of course, to the particular case of evolutionary games with finite strategy sets. We established conditions under which the replicator dynamics has a solution and we also characterized that solution (Theorem 3.6). Then stability conditions were established, and finally we gave three examples. The first one may be applicable to oligopoly models, theory of international trade, and public good models. The second and third examples deal with the tragedy of commons game and a model of poverty traps.

There are many questions, however, that remain open. For instance, in symmetric evolutionary games with continuous strategy spaces, there are stability conditions with different metrics and topologies. Are these conditions satisfied in the asymmetric case? On the other hand, normal form games with continuous strategies can be approximated by games with discrete strategies. Hence, it would be interesting to investigate if the replicator dynamics with continuous strategies in the asymmetric case can be approximated, in some sense, by games with discrete strategies. (This is true for the symmetric case; see section 4.7.)

4 Evolutionary games: symmetric case

In section 3.2.1 we saw how we can obtain a symmetric evolutionary game (3.8) of a asymmetric evolutionary game (3.5). In this chapter we provide a general framework to study the of the replicator dynamics for symmetric evolutionary games in which the strategy set is a separable metric space. In this case, the replicator dynamics evolves in a space of signed measures. The space of signed measures (particularity the probability spaces) is a very studied mathematical spaces. This allows us to study stability criteria with respect to different topologies and metrics on a space of probability measures, and to establish a relation between symmetric Nash equilibria (of a two-players normal form game (2.7)) and the stability of the replicator dynamics in different metrics.

In two-players normal form game the Symmetric Nash equilibrium can be rewrite in terms of a strategy call NES (see Definition 2.2). In the same form the symmetric SUP can be rewrite in terms of a strategy call *strongly uninvadable strategies* (see Definition 3.4). This particular fact, allows obtain more stability criteria than the asymmetric case.

In this chapter, we also provide conditions to approximate the replicator dynamics on a space of measures by means of a sequence of dynamical systems on finite spaces. Finally, examples illustrate our results.

This chapter is organized as follows. Section 4.1 describes the replicator dynamics and its relation to evolutionary games (compare with sections 3.1-3.2). Some important technical issues are also summarized. Section 4.2 establishes the relation between the replicator dynamics and a normal form game using the concepts of *Nash equilibria* and *strongly uninvadable strategies.* Section 4.3 presents a brief review of results on the stability of the replicator dynamics. Different stability criteria with respect to various metrics and topologies are standardized in the sense that the results (Theorems 4.10, 4.11, 4.12) are expressed in terms of a suitable general metric on a space of probability measures. (For instance, in some cases the metric is required to metrize the weak topology.)

Section 4.5 establishes an important relationship between Nash equilibria and the critical points of the replicator dynamics (Theorem 4.17 and Remarks

4.18 and 4.19). Section 4.6 proposes approximation schemes for the replicator dynamics in measure spaces, including the approximation by dynamical systems in finite-dimensional spaces. Section 4.7 proposes examples to illustrate our results. Finally, we conclude in section 4.8 with some general comments on possible extensions of our results.

In this chapter we use the technical preliminary of chapter 1, particularity section 1.3.2.

4.1 The Model

4.1.1 Symmetric evolutionary games

Consider a population of individuals of a single species. Each individual of this species can choose a single element a in a set of characteristics (the set of pure strategies or pure actions) A, which is a separable metric space. Let $\mathcal{B}(A)$ be the Borel σ-algebra of A, and $\mathbb{P}(A)$ the set of probability measures on A, also known as the set of mixed strategies.

Finally, we also consider a payoff function $J : \mathbb{P}(A) \times \mathbb{P}(A) \to \mathbb{R}$ that explain the interrelation between the population, and which is defined as

$$J(\mu, \nu) := \int_A \int_A U(a, b)\nu(db)\mu(da), \tag{4.1}$$

where $U : A \times A \to \mathbb{R}$ is a given measurable function. If $\delta_{\{a\}}$ is a probability measure concentrated at $a \in A$, the vector $(\delta_{\{a\}}, \mu)$ is written as (a, μ), and then

$$J(\delta_{\{a\}}, \mu) = J(a, \mu)$$

In particular, (4.1) yields

$$J(\mu, \nu) := \int_A J(a, \nu)\mu(da). \tag{4.2}$$

In an evolutionary game, the strategies' dynamics is determined by a differential equation of the form

$$\mu'(t) = F(\mu(t)) \quad t \geq 0, \tag{4.3}$$

with some initial condition $\mu(0) = \mu_0$. The notation $\mu'(t)$ represents the Fréchet derivative of $\mu(t)$ (see Definition 1.5), and $F(\cdot)$ is a mapping $F : \mathbb{P}(A) \to \mathbb{M}(A)$. More explicitly we write (4.3) as

$$\mu'(t, E) = F(\mu(t), E) \quad \forall E \in \mathcal{B}(A), \tag{4.4}$$

where $\mu'(t,E)$ and $F(\mu(t),E)$ are the measures $\mu'(t)$ and $F(\mu(t))$ valued at $E \in \mathcal{B}(A)$.

We shall be working with a special class of so-called symmetric evolutionary games which can be described as a quadruple (compare with (3.11))

$$\Big[I = \{1,2\},\ \mathbb{P}(A),\ J(\cdot),\ \mu'(t) = F(\mu(t))\Big], \tag{4.5}$$

where

i) $I = \{1,2\}$ is the set of players;

ii) for each player $i = 1,2$ we have a set $\mathbb{P}(A)$ of mixed actions and a payoff function $J : \mathbb{P}(A) \times \mathbb{P}(A) \to \mathbb{R}$ (as in (4.1)); and

iii) the dynamics (4.3) is described by the replicator equation (compare with (3.10)), where for each E in $\mathbb{B}(A)$,

$$F(\mu(t),E) := \int_E \Big[J(a,\mu(t)) - J(\mu(t),\mu(t))\Big]\mu(t,da). \tag{4.6}$$

To obtain an heuristic approach to the replicator dynamics (4.3) with $F(\cdot)$ as in (4.6) you can see section 3.2. and deduce the approach to the symmetric case.

4.1.2 Technical issues on the replicator dynamics

For a greater understanding in the reading, the following Theorem summarizes conditions for the existence of a unique solution to the differential equation (4.3) (with $F(\cdot)$ as in (4.6)) and important properties of this solution; see Theorems 3.5 and 3.6, respectively. These technical issues of chapter 3 were rewrote to the symmetric case, and will be use in the rest of the chapter.

For each $t \geq 0$, let

$$\beta(a|\mu(t)) := J(a,\mu(t)) - J(\mu(t),\mu(t)), \tag{4.7}$$

which is the integrand of (4.6). Hence, by (4.7), $\beta(\cdot|\mu(t))$ is the Radon-Nikodym density of $F(\mu(t))$ with respect to $\mu(t)$, i.e.,

$$F(\mu(t),E) = \int_E \beta(a|\mu(t))\mu(t,da) \quad \forall E \in \mathcal{B}(A).$$

Theorem 4.1. *Suppose that the function $\beta(\cdot|\mu)$ in (4.7) satisfies:*

i) there exists $C \geq 0$ such that

$$|\beta(a|\mu)| \leq C \quad \forall a \in A \text{ and } \|\mu\| \leq 2,$$

ii) there is a constant $D > 0$, such that

$$\sup_{a \in A} |\beta(a|\eta) - \beta(a|\nu)| \leq D\|\eta - \nu\| \quad \forall \nu, \eta \text{ with } \|\eta\|, \|\nu\| \leq 2.$$

Then there exists a unique solution to the replicator dynamics (4.3). Moreover, if $\mu(t)$ is a solution of (4.3) with initial condition $\mu(0)$ in $\mathbb{P}(A)$, then $\mu(0) << \mu(t)$ and $\mu(t) << \mu(0)$ for all $t > 0$, with Radon-Nikodym density

$$\frac{d\mu(t)}{d\mu(0)}(a) = e^{\int_0^t \beta(a|\mu(s))ds}. \tag{4.8}$$

In particular, for every $t > 0$, if ν is a probability measure satisfying that $\nu << \mu(t)$ whenever $\nu << \mu(0)$, then

$$\log \frac{d\nu}{d\mu(t)}(a) = \log \frac{d\nu}{d\mu(0)}(a) - \int_0^t \beta(a|\mu(s))ds. \tag{4.9}$$

4.2 The replicator dynamics, NESs and SUSs

In this section we consider symmetric evolutionary games as in (4.5) and compare them with two-players symmetric normal form games (2.7). We wish to study the relation between a Nash equilibrium of a normal form game and the replicator dynamics (Proposition 4.2) . We also define the important concept of *strongly uninvadable strategy* (Definition 4.3), and analyze its relation to a Nash equilibrium (Proposition 4.6).

For the rest of the Chapter, we consider the *two-players symmetric normal form game* Γ_s described as (2.7), and the concept of *Nash equilibrium strategy* (NES) defined in Chapter 2 (see Definition 2.2).

Proposition 4.2. *Let μ^* be a* NES *for Γ_s. Then μ^* is a critical point of (4.3) (i.e., $F(\mu^*) = 0$) when $F(\cdot)$ is described by the replicator dynamics (4.6).*

Proof. See Theorem 3.9. (Also see Mendoza-Palacios and Hernández-Lerma [69], Theorem 5.4.) □

The following definition is a slightly modified version of the strongly uninvadable strategies used in Bomze [17].

Definition 4.3. *Let* r *be a metric on* $\mathbb{P}(A)$ *as in Remark 1.2. A measure* $\mu^* \in \mathbb{P}(A)$ *is called an* r-strongly uninvadable strategy (r-SUS) *if there exists* $\epsilon > 0$ *such that for any* μ *with* $r(\mu, \mu^*) < \epsilon$, *it follows that* $J(\mu^*, \mu) > J(\mu, \mu)$. *We call* ϵ *the global invasion barrier.*

When r is the Prokhorov metric r_p, Oechssler and Riedel [75] name a r_p-SUS as an *evolutionary robust strategy.* If r_{w^*} is any metric that metrizes the weak topology (recall Remark 1.2), Cressman and Hofbauer [31] call a r_{w^*}-SUS a *locally superior strategy.*

We use the notation $\|\cdot\|$-SUS when the metric on $\mathbb{P}(A)$ is given by the total variation norm (1.1).

Proposition 4.4. *Let* r_{w^*} *be a distance that metrizes the weak convergence on* $\mathbb{P}(A)$. *If a measure* $\mu^* \in \mathbb{P}(A)$ *is* r_{w^*}*-SUS, then it is* $\|\cdot\|$*-SUS.*

Proof. Let μ be in the open ball $V_\epsilon^{\|\cdot\|}(\mu^*)$ defined in (1.13). Then there is some open neighborhood $\mathcal{V}_\epsilon^{\mathcal{H}}(\mu^*)$ of the form (1.4) such that $\mu \in \mathcal{V}_\epsilon^{\mathcal{H}}(\mu^*)$ and, by Remark 1.3, there is some open ball $\mathcal{V}_\alpha^{r_{w^*}}(\mu^*)$ such that $\mu \in \mathcal{V}_\alpha^{r_{w^*}}(\mu^*)$. Thus the proposition follows. □

The next lemma is a key fact to provide a general framework to the different stability criteria.

Lemma 4.5. *Let* r_{w^*} *be a distance that metrizes the weak convergence on* $\mathbb{P}(A)$. *For every* $\mu, \nu \in \mathbb{P}(A)$ *and* $\epsilon > 0$, *there exist* α *and* α' *in* $(0,1)$ *and* $\eta, \gamma \in \mathbb{P}(A)$ *such that*

i) $r_{w^*}(\eta, \mu) < \epsilon$ *if* $\eta = \alpha\nu + (1-\alpha)\mu$,

ii) $\|\gamma - \mu\| < \epsilon$ *if* $\gamma = \alpha'\nu + (1-\alpha')\mu$.

Proof. Let α_n be a sequence in $(0,1)$ such that $\alpha_n \to 0$, and let $\eta_n := \alpha_n\nu + (1-\alpha_n)\mu$. If $f \in \mathbb{C}_B(A)$ then

$$\begin{aligned}\lim_{n\to\infty} \int_A f(a)\eta_n(da) &= \lim_{n\to\infty} \alpha_n \int_A f(a)\nu(da) + \lim_{n\to\infty}(1-\alpha_n)\int_A f(a)\mu(da) \\ &= \int_A f(a)\mu(da).\end{aligned}$$

Hence, by Propositions A.1 and A.2 in the Appendix A.2, part *i*) follows.

On the other hand, let $0 < \alpha' < \frac{\epsilon}{\|\nu-\mu\|}$. Then

$$\|\gamma - \mu\| = \|\alpha'\nu + (1-\alpha')\mu - \mu\| = \alpha'\|\nu - \mu\| < \epsilon,$$

and *ii*) holds. □

The following proposition shows that a strongly uninvadable strategy is also a Nash equilibrum strategy. In other words, the concept of SUS is a refinement of NES.

Proposition 4.6. *Let r be a metric on $\mathbb{P}(A)$ as in Remark 1.2. If μ^* is a r-SUS, then μ^* is a NES of Γ_s.*

Proof. Suppose that μ^* is not a NES of Γ_s. Then there exists $\nu \in \mathbb{P}(A)$ such that

$$J(\nu, \mu^*) > J(\mu^*, \mu^*). \tag{4.10}$$

By Lemma 4.5, there exists $\eta := \alpha\nu + (1-\alpha)\mu^*$ for some $\alpha \in (0,1)$, with $r(\eta, \mu^*) < \epsilon$. Since μ^* is r-SUS, $J(\mu^*, \eta) > J(\eta, \eta)$ and so

$$\begin{aligned} \alpha J(\mu^*, \nu) + (1-\alpha)J(\mu^*, \mu^*) \quad > \quad & \alpha\alpha J(\nu, \nu) + (1-\alpha)\alpha J(\nu, \mu^*) \\ & +(1-\alpha)\alpha J(\mu^*, \nu) \\ & +(1-\alpha)(1-\alpha)J(\mu^*, \mu^*). \end{aligned}$$

Hence

$$\alpha J(\mu^*, \nu) + (1-\alpha)J(\mu^*, \mu^*) > \alpha J(\nu, \nu) + (1-\alpha)J(\nu, \mu^*). \tag{4.11}$$

If (4.10) is true, then there exists $\alpha > 0$ sufficiently small such that (4.11) is violated. Thus μ^* is a NES for Γ_s. □

Now, we define the following sets:

i) $\mathcal{N} := \{\mu^* \in \mathbb{P}(A) : \mu^* \text{ is a NES of } \Gamma_s\}$,
$\mathcal{C} := \{\mu^* \in \mathbb{P}(A) : \mu^* \text{ is a critical point of (4.3)}\}$.

ii) If r is a metric on $\mathbb{P}(A)$ as in Remark 1.2,

$$r - \mathcal{SUS} := \{\mu^* \in \mathbb{P}(A) : \mu^* \text{ is } \quad r - \text{SUS } \}.$$

We can summarize Propositions 4.2 and 4.6 as follows:

Corollary 4.7. *Let A be a separable metric space and assume the conditions i) and ii) of Theorem 4.1. If r is a metric on $\mathbb{P}(A)$ as in Remark 1.2, then we have:*

$$r - \mathcal{SUS} \subset \mathcal{N} \subset \mathcal{C}.$$

An improvement of this result is presented in Section 4.4 (see Theorem 4.15).

4.3 Stability of SUSs

This section present a review of results on the stability of a SUS in the replicator dynamics. These results include different stability criteria with respect to various metrics and topologies in the space of probability measures.

4.3.1 The Kullback-Leibler distance

Assume that $\nu << \mu$. We define the cross entropy or Kullback-Leibler distance of ν with respect to μ as

$$K(\mu, \nu) := \int_A \log \left[\frac{d\nu}{d\mu}(a) \right] \nu(da). \tag{4.12}$$

From Jensen's inequality it follows that $0 \leq K(\mu, \nu) \leq \infty$ and $K(\mu, \nu) = 0$ if and only if $\mu = \nu$. The Kullback-Leibler distance is not a metric, since it is not symmetric, i.e., $K(\mu, \nu) \neq K(\nu, \mu)$.

Given $\mu^* \in \mathbb{P}(A)$, $\epsilon > 0$, and a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$, we define the set

$$\mathcal{W}_{\varphi(\epsilon)}(\mu^*) := \Big\{ \mu \in \mathbb{P}(A) : K(\mu, \mu^*) < \varphi(\epsilon) \Big\}. \tag{4.13}$$

Theorem 4.8. *Suppose that A is a separable metric space, and that the conditions i) and ii) of Theorem 3.1 hold. Let μ^* be a $\|\cdot\|$-SUS with global invasion barrier $\epsilon > 0$, and $\mu(\cdot)$ the solution of the replicator dynamics. If $\mu(0) \in \mathcal{W}_{\varphi(\epsilon)}(\mu^*)$, with $\varphi(\epsilon) = \left[\frac{\epsilon}{2}\right]^2$, then:*

i) $\mu(t) \in \mathcal{W}_{\varphi(\epsilon)}(\mu^*)$ *for all* $t \geq 0$*;*

ii) $\|\mu(t) - \mu^*\| < \epsilon$ *for all* $t \geq 0$*;*

iii) for all $t \geq 0$*,* $\mu(t)$ *is in some open ball* $\mathcal{V}_\alpha^{r_{w^*}}(\mu^*)$ *as in (1.13), where* r_{w^*} *is any distance that metrizes the weak topology.*

iv) Moreover, if A is compact and the map $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ is continuous in the weak topology, then $r_{w^*}(\mu(t), \mu^*) \to 0$,.

v) Furthermore, parts i) to iv) are also true with the hypothesis that μ^* is r_{w^*}-SUS.

Proof. Parts $i), ii)$ and $iv)$ are proved in Bomze [16] [1]. Part $iii)$ is a consequence of $ii)$ and Remark 1.3. Finally, $v)$ follows from Proposition 4.4. □.

[1] Bomze [16] proves a more general case for part $iv)$, where any topology τ on $\mathbb{P}(A)$ is considered. He only requires that $\mathbb{P}(A)$ be a τ-compact set and the map $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ be τ-continuous.

4.3.2 The L_1-Wasserstein metric

The following theorem characterizes the stability of the replicator dynamics with respect to the L_1-Wasserstein metric r_w in (1.12). This distance metrizes the weak topology and has important relationships with other distances that also metrize the weak topology (see Proposition A.2). Furthermore, the L_1-Wasserstein is closely related to the variation norm (1.1) and the Kullback-Leibler distance (4.12); see Propositions A.3 and A.4. The following two propositions give better approximations to parts *iii*) and *iv*) of Theorem 4.8.

Theorem 4.9. *Suppose that A is a compact Polish space (with diameter $C > 0$), and the conditions i) and ii) of Theorem 4.1 hold. Let μ^* be a r_w-SUS with global invasion barrier $\epsilon > 0$, and $\mu(\cdot)$ the solution of the replicator dynamics. If $\mu(0) \in \mathcal{W}_{\varphi'(\epsilon)}(\mu^*)$, with $\varphi'(\epsilon) = \left[\frac{\epsilon}{2C}\right]^2$, then*

i) *$\mu(t) \in \mathcal{W}_{\varphi'(\epsilon)}(\mu^*)$ for all $t \geq 0$;*

ii) *$\|\mu(t) - \mu^*\| < \frac{\epsilon}{C}$ for all $t \geq 0$;*

iii) *$r_w(\mu(t), \mu^*) < \epsilon$ for all $t \geq 0$.*

iv) *Moreover, if the map $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ is continuous in the weak topology, then $r_w(\mu(t), \mu^*) \to 0$.*

v) *Furthermore, parts i) to iv) are also true with the hypothesis* that μ^* is $\|\cdot\|$-SUS, with barrier $\frac{\epsilon}{C}$.

Proof. *i*) If $\mu(0)$ is in $\mathcal{W}_{\varphi'(\epsilon)}(\mu^*)$, then by Theorem 4.1 we know that $\mu^* << \mu(t)$ and so $K(\mu(t), \mu^*)$ is well defined for all $t \geq 0$. Using Theorem 4.1 and Fubini's theorem,

$$K(\mu(t), \mu^*) - K(\mu(0), \mu^*) = -\int_A \left[\int_0^t \beta(a|\mu(s))ds\right] \mu^*(da)$$
$$= -\int_0^t J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))ds. \quad (4.14)$$

By the condition *ii*) of Theorem 4.1 there exists $D > 0$ such that, for any $a \in A$ and $\mu, \eta \in \mathbb{P}(A)$

$$|\beta(a|\eta) - \beta(a|\nu)| \leq D\|\eta - \nu\|.$$

So

$$\Big|[J(\mu^*, \eta) - J(\eta, \eta)] - [J(\mu^*, \nu) - J(\nu, \nu)]\Big| = \left|\int_A [\beta(a|\eta) - \beta(a|\nu)]\mu^*(da)\right|$$
$$\leq D\|\eta - \nu\|. \quad (4.15)$$

By (4.15) and since $\mu(s)$ is continuous in s, the map $s \to [J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))]$ is continuous. Therefore, the time derivative of $K(\mu(t), \mu^*)$ exists and since μ^* is a r_w-SUS,

$$\frac{dK(\mu(t), \mu^*)}{dt} = -[J(\mu^*, \mu(t)) - J(\mu(t), \mu(t))] \leq 0. \tag{4.16}$$

Hence $K(\mu(t), \mu^*)$ is nonincreasing in t, and i) holds.

Proof of ii), iii). By Proposition A.3 and (4.14),

$$r_{wl}(\mu(t), \mu^*) \leq C\|\mu(t) - \mu^*\| \leq 2C[K(\mu(0), \mu^*)]^{\frac{1}{2}} < \epsilon. \tag{4.17}$$

Therefore ii) and iii) hold.

iv) Since $K(\mu(t), \mu^*)$ is a nonincreasing function in t and by (4.14), the map

$$t \to \int_0^t \Big[J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))\Big] ds$$

is increasing and

$$\lim_{t\to\infty} \int_0^t \Big[J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))\Big] ds < \infty.$$

Moreover, since the map $s \to [J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))]$ is continuous, we have $\lim_{s\to\infty} [J(\mu^*, \mu(s)) - J(\mu(s), \mu(s))] = 0$.

Since A is compact, the space $\mathbb{P}(A)$ is compact in the weak topology (see Bobrowski [14]), and the distance r_w metrizes this topology (Proposition A.2). Suppose now that $\hat{\mu}$ is an accumulation point of the trajectory $\mu(\cdot)$. By (4.17), the r_w-distance from $\hat{\mu}$ to μ^* is at most ϵ, and since that μ^* is r_w-SUS, $J(\mu^*, \hat{\mu}) > J(\hat{\mu}, \hat{\mu})$ if $\hat{\mu} \neq \mu^*$. By hypothesis, the map $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ is weakly continuous. If $\hat{\mu}$ is such that $J(\mu^*, \hat{\mu}) - J(\hat{\mu}, \hat{\mu}) = 0$, then $\hat{\mu} = \mu^*$, which proves that $r_w(\mu(t), \mu^*) \to 0$.

v) Finally if μ^* is $\|\cdot\|$-SUS with barrier $\frac{\epsilon}{C}$ then, by (4.17), parts i) to iv) follow. □.

4.3.3 Stability of a pure-SUS

The next theorem characterizes the stability of the replicator dynamics for a SUS that is also a Dirac measure.

Theorem 4.10. *Let A be a separable metric space and suppose that the conditions i) and ii) of Theorem 4.1 hold. Let δ_{a^*} be a Dirac measure and r any metric on $\mathbb{P}(A)$ as Remark 1.2. Let us suppose that δ_{a^*} is r-SUS , $\mu(\cdot)$ is a solution of the replicator dynamics, and $\|\mu_0 - \delta_{a^*}\| < \epsilon$ for some small $\epsilon > 0$. Then* *i)* $\|\mu(t) - \delta_{a^*}\| < \epsilon$ *for all* $t \geq 0$*;*

ii) *for all* $t \geq 0$*,* $\mu(t)$ *is in some open ball* $\mathcal{V}_{\alpha}^{r_{w^*}}(\mu^*)$ *as in (1.13), where* r_{w^*} *is any distance that metrizes the weak topology;*

iii) *if A is a compact Polish space (with diameter* $C > 0$*), then for all* $t \geq 0$*,* $r_w(\mu(t), \delta_{a^*}) < C\epsilon$*;*

iv) *if A is a compact metric space (not necessary a Polish space) and the map* $\mu \to J(\delta_{a^*}, \mu) - J(\mu, \mu)$ *is continuous in the weak topology, then* $r_{w^*}(\mu(t), \mu^*) \to 0$*, where* r_{w^*} *is any distance that metrizes the weak topology.*

Proof. Parts *i*) and *iv*) follow from Theorem 3.14 and Corollary 3.15 (See also Mendoza-Palacios and Hernández-Lerma [69] Theorem 6.2). Part *ii*) follows for Proposition 4.4. Finally, Part *iii*) follows from Proposition A.2. □

Theorem 4.10 is also proved by Oechssler and Riedel [74] with slight changes in the definition of $\| \, \|$-SUS.

4.3.4 Related stability results

The following conjecture was proposed by Oechssler and Riedel in [75], when r_{w^*} is a distance that metrizes the weak topology.

Conjecture 4.11. *Let* r *be any metric on* $\mathbb{P}(A)$ *and* r_{w^*} *any distance that metrizes the weak topology. Suppose that A is a separable metric space, and that the conditions i) and ii) of Theorem 4.1 hold. Let* μ^* *be a* r-SUS *and* $\mu(\cdot)$ *the solution of the replicator dynamics. Then*

i) *for* $\epsilon > 0$ *there exist* $\delta > 0$ *such that if* $r(\mu(0), \mu^*) < \delta$*, we have that* $r(\mu(t), \mu^*) < \epsilon$ *for all* $t \geq 0$*;*

ii) *moreover, if part i) is satisfied, and the map* $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ *is continuous in the weak topology and* $\mu^* << \mu(0)$*, then* $r_{w^*}(\mu(t), \mu^*) \to 0$*.*

Remark 4.12. *A double symmetric game (named a potential game by Cressman and Hofbauer [31]) is a game where* $J(\mu, \nu) = J(\nu, \mu)$ *for any* $\mu, \nu \in \mathbb{P}(A)$*. Let* r_{w^*} *be any distance that metrizes the weak topology. Oechssler and Riedel [75] prove that if A is a compact set and* μ^* *is* r_{w^*}-SUS*, then for double symmetric games,* μ^* *satisfies part i) of Conjecture 4.11. Cressman and Hofbauer [31] prove that if part i) is satisfied, then ii) follows for any symmetric game.*

Oechssler and Riedel [75] prove that a r_{w^*}-SUS satisfies other static evolutionary concepts such as *evolutionary stable strategy* (ESS), *continuously stable strategy* (CSS), and *neighborhood invader strategy* (NIS), which characterize dynamic stability in the weak topology for the replicator dynamics. Eshel and Sansone [36], Cressman [29], Cressman, Hofbauer and Riedel [32], use these evolutionary concepts and different hypotheses on the payoff function (4.2) to guarantee dynamic stability. Norman [72] establishes the dynamic stability in terms of strategy sets.

4.4 NESs and stability

In this section we introduce a general definition of dynamic stability for the replicator dynamics (see Definition 4.13), and prove that any *stable* critical point of the replicator dynamics is a NES of Γ_s (see Proposition 4.14). Moreover, in Theorem 4.15 and Remarks 4.16 and 4.17 we relate the stability of the differential equation (4.3), and the static evolutionary concepts NES and SUS.

Consider $\mu, \nu \in \mathbb{P}(A)$. By Propositions A.2, A.3 and A.4 in the Appendix A.2, below, we know that if μ and ν are close with respect to the Kullback-Leibler distance K, then they are close in the total variation norm $\|\cdot\|$, and consequently they are close in the weak topology. This argument is not true in the opposite direction. Hence we say that the Kullback-Leibler distance is "stronger than" the total variation norm, and the total variation norm is "stronger than" any distance that metrizes the weak topology.

Definition 4.13. *Let A be a separable metric space, and r_1 and r_2 the Kullback-Leibler distance or some metric in $\mathbb{P}(A)$ where r_1 is equal to or "stronger than" r_2. A critical point μ^* of the replicator dynamics (4.3) is said to be*

i) $[r_1, r_2]$-stable (in symbols: $[r_1, r_2]$-S) *if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $r_1(\mu(0), \mu^*) < \delta$, then $r_2(\mu(t), \mu^*) < \epsilon$ for all $t > 0$. If $r_1 = r_2 = r^*$ then we only say that μ^** is r^*-stable (in symbols: r^*-S).

ii) $[r_1, r_2]$-asymptotically weakly stable *if it is* $[r_1, r_2]$-stable *and* $\lim_{t\to\infty} \mu(t) = \mu^*$ *in the weak topology.*

Consider the Kullback-Leibler distance K, the total variation norm $\|\cdot\|$, and any distance r_{w^*} that metrizes the weak topology. The following diagram gives the natural implications between the different concepts of stability.

$$\begin{array}{ccccc} K-\mathrm{S} & \Rightarrow & [K,\|\cdot\|]-\mathrm{S} & \Rightarrow & [K,r_{w^*}]-\mathrm{S} \\ & & \Uparrow & & \Uparrow \\ & & \|\cdot\|-\mathrm{S} & \Rightarrow & [\|\cdot\|,r_{w^*}]-\mathrm{S} \\ & & & & \Uparrow \\ & & & & r_{w^*}-\mathrm{S} \end{array} \tag{4.18}$$

These implications are easy to deduce. For example, if the critical point μ^* is $\|\cdot\|$-S, and the initial condition μ_0 satisfies that $K(\mu_0,\mu^*) < \left(\frac{\epsilon}{2}\right)^2$ for a small $\epsilon > 0$, then by Proposition A.3 $\|\mu_0 - \mu^*\| < \epsilon$, hence μ^* is also $[K,\|\cdot\|]$-S. On the other hand, μ^* is $\|\cdot\|$-S, and the initial condition μ_0 is such that $\|\mu(t) - \mu^*\| < \delta$ for all $t > 0$ and some $\delta > 0$, then by Remark 1.3 for any r_{w^*}-metric, $\mu(t) \in \mathcal{V}_\alpha^{r_{w^*}}$ for some small $\alpha > 0$. Hence μ^* is also $[\|\cdot\|, r_{w^*}]$-S.

Van Veelen and Spreij [98] study other relationships among the different concepts of stability in diagram (4.18). They also study relationships between static evolutionary concepts and asymptotic evolutionary stability.

The concept *the support of a probability measure* on a separable metric space is used in the following proposition. See Proposition 3.7 for details.

Proposition 4.14. *Let A be a separable metric space, and r_1, r_2 the Kullback-Leibler distance or some metric in $\mathbb{P}(A)$ where r_1 is equal to or "stronger than" r_2. Suppose that the conditions i) and ii) of Theorem 4.1 are satisfied, and let μ^* be a critical point of (4.3) with $F(\cdot)$ as (4.6). If μ^* is $[r_1,r_2]$-stable, then μ^* is a Nash equilibrium strategy (NES) of Γ_s.*

Proof. If μ^* is a critical point of (4.3) with $F(\cdot)$ as (4.6), then

$$J(a,\mu^*) - J(\mu^*,\mu^*) = 0 \quad \mu^* - \text{a.s.}$$

Suppose that μ^* is not a NES of Γ_s. Then there exist a' in the support of μ^* such that

$$J(a',\mu^*) - J(\mu^*,\mu^*) > \kappa > 0, \tag{4.19}$$

for some κ. By the condition *ii*) of Theorem 4.1 we have that for any $\mu,\eta \in \mathbb{P}(A)$

$$|\beta(a'|\eta) - \beta(a'|\nu)| \leq D\|\eta - \nu\|,$$

and so the map $\mu \to J(a',\mu) - J(\mu,\mu)$ is continuous. Hence, by (4.19), for any $\mu \in \mathbb{P}(A)$ near μ^* in some r_1 distance

$$J(a',\mu) - J(\mu,\mu) > \kappa. \tag{4.20}$$

Let $\epsilon > 0$ and $\mu_0 := \lambda_\epsilon \delta_{a'} + (1-\lambda_\epsilon)\mu^*$ be the initial condition, where $\lambda_\epsilon \in (0,1)$ and $\mu_0 \in \mathcal{W}_{\varphi(\epsilon)}(\mu^*)$, with $\varphi(\epsilon) = \epsilon^2$. The number λ_ϵ indeed exists since

$$K(\mu_0, \mu^*) = \int_{\mathrm{Supp}(\mu^*)} \log\left[\frac{d\mu^*}{d\mu_0}(a)\right]\mu^*(da) = \log\left(\frac{1}{1-\lambda_\epsilon}\right),$$

and the logarithmic function is continuous, and by Propositions A.3 and A.4, μ_0 is near μ^* in the r_1-distance.

By (4.20) and Theorem 4.1 we have

$$\mu(0,\{a'\})e^{\kappa t} \leq \mu(0,\{a'\})e^{\int_0^t \beta(a'|\mu(s))ds} = \mu(t,\{a'\}),$$

for all $t > 0$. Thus $\mu(t,\{a'\})$ is increasing if the initial condition is μ_0 and the trajectory $\mu(t)$ is not close to μ^* in the r_2-distance. So μ^* is not $[r_1, r_2]$-stable. □

Now, let r_1 and r_2 be the Kullback-Leibler distance or some metric on $\mathbb{P}(A)$, where r_1 is equal to or "stronger than" r_2. We define the following set:

$$[r_1, r_2] - \mathcal{S} := \{\mu^* \in \mathbb{P}(A) : \mu^* \text{ is } \quad [r_1, r_2] - \mathrm{S}\}.$$

Theorem 4.15. *Let A be a separable metric space, and consider the conditions i) and ii) of Theorem 4.1. Let r_1 be a metric on $\mathbb{P}(A)$, and let r_2 be the Kullback-Leibler distance or some metric on $\mathbb{P}(A)$ equal to or "stronger than" r_1. Consider the sets $r_1 - \mathcal{SUS}$, $\mathcal{N}$ and $\mathcal{C}$ as in Corollary 4.7. Then we have:*

$$r_1 - \mathcal{SUS} \;\subset\; [K, r_2] - \mathcal{S} \;\subset\; \mathcal{N} \;\subset\; \mathcal{C}.$$

Proof. This is a consequence of Theorem 4.8 and Propositions 4.2 and 4.14. □

Remark 4.16. *Suppose the hypotheses of Theorem 4.15 and let A be a compact Polish space. Then by Theorem 4.9 and Propositions A.2, A.3, we can obtain the implications in Theorem 4.15 with a specific value for the barrier $\epsilon > 0$, for the metrics $\|\cdot\|$, r_p, r_{bl}, r_w, r_{kr}.*

Remark 4.17. *Let r_1 and r_2 be the total variation norm (1.1) or some metric that metrizes the weak topology on $\mathbb{P}(A)$. By Theorem 4.10 and Propositions 4.2, 4.14, we have the following implications if a Dirac measure δ_{a^*} is a r_1-SUS.*

$$\delta_{a^*} \in r_1 - \mathcal{SUS} \;\Rightarrow\; \delta_{a^*} \in [\|\;\|, r_2] - \mathcal{S} \;\Rightarrow\; \delta_{a^*} \in \mathcal{N} \;\Rightarrow\; \delta_{a^*} \in \mathcal{C}.$$

4.5 Finite dimensional approximation

An infinite-dimensional dynamical system, as our proposed model, is not a computable model. To solve this problem, we can introduce some approximation schemes. Oechssler and Riedel [75] propose two approximation theorems. The first theorem establishes the proximity of two paths generated by two different dynamical systems (the original model and a discrete approximation of the model) with the same initial condition. The second theorem establishes the proximity of two paths each with different initial conditions and these paths satisfy the same differential equation (4.3) with $F(\cdot)$ as (4.6).

We propose here two approximation results with hypotheses less restrictive than those by Oechssler and Riedel [75]. Our two approximation theorems extend the results in [75] because we establish the proximity of two paths generated by two different dynamical systems (the original model and a discrete approximation model) with different initial conditions. We will need the following fact.

Lemma 4.18. *Let A be a separable metric space. If the map $\mu : [0,\infty) \to \mathbb{M}(A)$ is* strongly differentiable*, then*

$$\frac{d\|\mu(t)\|}{dt} \leq \|\mu'(t)\|.$$

Proof. see Apendix A.1.1 $\square$.

To obtain a discrete approximation of the infinite-dimensional model (4.3) with $F(\cdot)$ as (4.6) we can apply the following Theorems 4.19 and 4.24 to a discrete approximation of the payoff function U and the initial probability measure μ_0. For some examples of approximation techniques, see Bishop and Cannings [13], Simon [95]. Oechssler and Riedel [75] propose a finite approximation for a game where $A = [0,1]$, and U is a bounded function. They consider the partition $P_k := \{A_i\}_{i=0}^{2^k-1}$, where $A_i := [a_i, a_{i+1})$, $a_i = \frac{i}{2^k}$, for $i = 0, 1, ..., 2^k - 2$ and $A_{2^k-1} := [a_{2^k-1}, 1]$. They propose the discrete approximation to U given by the function

$$U_k(x,y) = U(a_i, a_j), \quad \text{if} \quad (x,y) \in A_i \times A_j, \quad \text{for } i,j = 0,1,...,2^k - 1.$$

Also, we can proximate a probability measure $\mu_0 \in \mathbb{P}(A)$ by a discrete probability distribution μ_k^0 on the partition set P_k. Then we can approximate (4.3) (with $F(\cdot)$ as (4.6)) by a system of differential equation in $\mathbb{R}^{2^k}$:

$$\mu_k'(t, A_i) = \Big[J_k(a_i, \mu_k(t)) - J_k(\mu_k(t), \mu_k(t))\Big]\mu_k(t, A_i), \qquad i = 0, 1, ..., 2^k - 1,$$

with initial condition $\{\mu_k^0(A_i)\}_{i=0}^{2^k-1}$ and where

$$J_k(a_i, \mu_k(t)) = \sum_{j=0}^{2^k-1} U\,(a_i, a_j)\,\mu_k(t, A_j),$$

$$J_k(\mu_k(t), \mu_k(t)) = \sum_{i=0}^{2^k-1} J_k\,(a_i, \mu_k(t))\,\mu_k(t, A_i).$$

Theorem 4.19. *Let A be a separable metric space and let $U, U_\epsilon : A \times A \to \mathbb{R}$ be two bounded functions such that $\|U - U_\epsilon\| < \epsilon$. Consider the replicator dynamics in (4.3) with $F(\cdot)$ as (4.6) induced by U and U_ϵ, i.e.,*

$$\mu'(t, E) = \int_E \Big[J(a, \mu(t)) - J(\mu(t), \mu(t))\Big]\mu(t, da), \tag{4.21}$$

$$\nu'(t, E) = \int_E \Big[J_\epsilon(a, \nu(t)) - J_\epsilon(\nu(t), \nu(t))\Big]\nu(t, da), \tag{4.22}$$

for each E in $\mathcal{B}(A)$ and $t \geq 0$, where $J(a, b) = U(a, b)$, $J_\epsilon(a, b) = U_\epsilon(a, b)$, for all a, b in A. If $\mu(\cdot)$ and $\nu(\cdot)$ are solutions of (4.21) and (4.22), respectively, with initial conditions $\mu(0) = \mu_0$ and $\nu(0) = \nu_0$, then for $T < \infty$

$$\sup_{t\in[0,T]} \|\mu(t) - \nu(t)\| < \|\mu_0 - \nu_0\| e^{QT} + 2\epsilon \left(e^{QT} - \frac{1}{Q}\right). \tag{4.23}$$

where $Q = 5\|U\|$.

Proof. For each $t \geq 0$, let (as in (4.7))

$$\beta(a|\mu) := J(a, \mu) - J(\mu, \mu), \qquad \beta_\epsilon(a|\nu) := J_\epsilon(a, \nu) - J_\epsilon(\nu, \nu),$$

and (as in (4.6))

$$F(\mu, E) := \int_E \beta(a|\mu)\mu(da), \qquad F_\epsilon(\nu, E) := \int_E \beta_\epsilon(a|\nu)\nu(da).$$

Since U is bounded, the conditions $i)$ and $ii)$ of Theorem 4.1 hold, and so there exists $Q > 0$ such that

$$\|F(\nu) - F(\mu)\| \leq Q\|\nu - \mu\| \quad \forall \mu, \nu \in \mathbb{P}(A). \tag{4.24}$$

Actually, $Q = 5\|U\|$; see Mendoza-Palacios and Hernández-Lerma [69] or Oechssler and Riedel [74]. We also have that, for all $\nu \in \mathbb{P}(A)$,

$$\|F_\epsilon(\nu) - F(\nu)\| = \int_A |\beta(a|\nu) - \beta_\epsilon(a|\nu)|\nu(da) \leq 2\|U_\epsilon - U\| < 2\epsilon. \tag{4.25}$$

By Lemma 4.18 and (4.24)-(4.25) we have

$$\begin{aligned}\frac{d\|\nu(t)-\mu(t)\|}{dt} &\leq \|\nu'(t)-\mu'(t)\| \\ &= \|F_\epsilon(\nu(t))-F(\mu(t))\| \\ &\leq \|F_\epsilon(\nu(t))-F(\nu(t))\|+\|F(\nu(t))-F(\mu(t))\| \\ &< 2\epsilon+Q\|\nu(t)-\mu(t)\|\end{aligned}$$

Then

$$\frac{d\|\nu(t)-\mu(t)\|}{dt}-Q\|\nu(t)-\mu(t)\|<2\epsilon.$$

Multiplying by e^{-Qt} we get

$$\frac{d\|\nu(t)-\mu(t)\|e^{-Qt}}{dt}<2\epsilon e^{-Qt},$$

and integrating in the interval $[0,t]$, where $t\leq T$, we get

$$\|\mu(t)-\nu(t)\|e^{-Qt}-\|\mu_0-\nu_0\|e^{-Q0}<2\epsilon\left(\frac{1-e^{-Qt}}{Q}\right).$$

Then for all $t\in[0,T]$

$$\begin{aligned}\|\mu(t)-\nu(t)\| &< \|\mu_0-\nu_0\|e^{Qt}+2\epsilon\left(\frac{e^{Qt}-1}{Q}\right) \\ &\leq \|\mu_0-\nu_0\|e^{QT}+2\epsilon\left(\frac{e^{QT}-1}{Q}\right),\end{aligned}$$

which yields (4.23). □

Corollary 4.20. *Let us assume the hypotheses of Theorem 4.19, and, in addition, suppose that there exist sequences of functions* $\{U_{\epsilon_n}\}_{n=1}^{\infty}$ *and probability measures* $\{\nu^n\}_{n=1}^{\infty}$ *such that* $\|U_{\epsilon_n}-U\|\to 0$ *and* $\|\nu_0^n-\mu_0\|\to 0$*. If* $\mu(\cdot)$ *and* $\nu^n(\cdot)$ *are solutions of (4.21) and (4.22), respectively, with initial conditions* $\mu(0)=\mu_0$ *and* $\nu^n(0)=\nu_0^n$*, then for* $T<\infty$*,*

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\nu_n(t)-\mu(t)\|\to 0.$$

The next approximation theorem establishes the proximity of two paths generated by two different dynamical systems (the original model and a discrete approximation model) with different initial conditions, under the weak topology. To this end we use the Kantorovich-Rubinstein norm $\|\cdot\|_{kr}$ on $\mathbb{M}(A)$ and the L^1-Wasserstein distance r_w on $\mathbb{P}(A)$.

Remark 4.21. *Let A be a separable metric space. We say that a mapping $\mu : [0, \infty) \to \mathbb{M}(A)$ is* weakly differentiable *if there exists $\mu'(t) \in \mathbb{M}(A)$ such that, for every $t > 0$ and $g \in \mathbb{C}_B(A)$*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_A g(a)\mu(t+\epsilon, da) - \int_A g(a)\mu(t, da) \right] = \int_A g(a)\mu'(t, da). \tag{4.26}$$

If $\|\cdot\|_{k,r}$ is the Kantorovich-Rubinstein norm (1.11), then (4.26) can be written as

$$\lim_{\epsilon \to 0} \left\| \frac{\mu(t+\epsilon) - \mu(t)}{\epsilon} - \mu'(t) \right\|_{kr} = 0.$$

Moreover if $\mu'(t)$ is the strong derivative *of $\mu(t)$, then it is also the* weak derivative *of $\mu(t)$. Conversely, if $\mu'(t)$ is the* weak derivative *of $\mu(t)$, and $\mu(t)$ is continuous in t with the norm (1.1), then it is the* strong derivative *of $\mu(t)$. See Heidergott, Hordijk and Leahu [45].*

Lemma 4.22. *Let A be a separable metric space. If the map $\mu : [0, \infty) \to \mathbb{M}(A)$ is* strongly differentiable, *then*

$$\frac{d\|\mu(t)\|_{kr}}{dt} \leq \|\mu'(t)\|_{kr}.$$

Proof. See Appendix A.3.2 □

Let (A, ϑ) be a bounded separable metric space (with diameter $C > 0$), and $\nu \in \mathbb{P}(A)$. If $\|f\|_L \leq 1$ and $f(a_0) = 0$, then for any a in A

$$\frac{f(a)}{C} \leq \sup_{a \in A} \frac{|f(a) - f(a_0)|}{C} \leq \sup_{a,b \in A} \frac{|f(a) - f(b)|}{\vartheta(a, b)} = \|f\|_L = 1.$$

Therefore,

$$\sup_{\substack{\|f\|_L \leq 1 \\ f(a_0)=0}} \int_A f(a)\nu(da) \leq C. \tag{4.27}$$

Lemma 4.23. *Consider a bounded separable metric space (A, ϑ) (with diameter $C > 0$), and the product space $(A \times A, \vartheta^*)$, where*

$$\vartheta^*((a, b), (c, d)) = \max\{\vartheta(a, b), \vartheta(c, d)\}$$

for any a, b, c, d in A. Let $F(\cdot)$ be described as in (4.6). Suppose that the payoff function $U(\cdot)$ in (4.1) is bounded and satisfies that $\|U\|_L < \infty$. Then there exits $Q > 0$ such that

$$\|F(\nu) - F(\mu)\|_{kr} \leq Q\|\nu - \mu\|_{kr} \quad \forall \mu, \nu \in \mathbb{P}(A), \tag{4.28}$$

where $Q := 2\|U\| + 3C\|U\|_L$

Proof. See Appendix A.3.3 □

Theorem 4.24. *Let A be a compact Polish space (with diameter $C > 0$), and let $U, U_\epsilon : A \times A \to \mathbb{R}$ be two bounded functions such that $\|U - U_\epsilon\| < \epsilon$. Suppose that $\|U\|_L < \infty$ and consider the replicator dynamics induced by U and U_ϵ, as in (4.21) and (4.22). If $\mu(\cdot)$ and $\nu(\cdot)$ are solutions of (4.21) and (4.22), respectively, with initial conditions $\mu(0) = \mu_0$ and $\nu(0) = \nu_0$, then for $T < \infty$*

$$\sup_{t\in[0,T]} r_w(\mu(t), \nu(t)) < r_w(\mu_0, \nu_0)e^{QT} + 2C\epsilon\left(\frac{e^{QT} - 1}{Q}\right). \qquad (4.29)$$

with $Q = 2\|U\| + 3C\|U\|_L$ as in (4.28), and r_w is the L^1-Wasserstein distance.

Proof. For each $t \geq 0$, let (as in (4.7))

$$\beta(a|\mu) := J(a, \mu) - J(\mu, \mu), \quad \beta_\epsilon(a|\nu) := J_\epsilon(a, \nu) - J_\epsilon(\nu, \nu),$$

and (as in (4.6))

$$F(\mu, E) := \int_E \beta(a|\mu)\mu(da), \qquad F_\epsilon(\nu, E) := \int_E \beta_\epsilon(a|\nu)\nu(da).$$

Since $\|U\|_L < \infty$, by Lemma 4.23 there exists $Q = 2\|U\| + 3C\|U\|_L > 0$ such that

$$\|F(\nu) - F(\mu)\|_{kr} \leq Q\|\nu - \mu\|_{kr} \quad \forall \mu, \nu \in \mathbb{P}(A). \qquad (4.30)$$

We also have that for each $\nu \in \mathbb{P}(A)$,

$$\begin{aligned} \|F_\epsilon(\nu) - F(\nu)\|_{kr} &= \sup_{\substack{\|f\|_L\leq 1\\ f(a_0)=0}} \int_A f(a)|\beta(a|\nu) - \beta_\epsilon(a|\nu)|\nu(da) \\ &\leq 2\|U_\epsilon - U\| \sup_{\substack{\|f\|_L\leq 1\\ f(a_0)=0}} \int_A f(a)\nu(da) \\ &< 2C\epsilon. \end{aligned} \qquad (4.31)$$

By Lemma 4.22, and (4.30)-(4.31) we have

$$\begin{aligned} \frac{d\|\nu(t) - \mu(t)\|_{kr}}{dt} &\leq \|\nu'(t) - \mu'(t)\|_{kr} \\ &= \|F_\epsilon(\nu(t)) - F(\mu(t))\|_{kr} \\ &\leq \|F_\epsilon(\nu(t)) - F(\nu(t))\|_{kr} + \|F(\nu(t)) - F(\mu(t))\|_{kr} \\ &< 2C\epsilon + Q\|\nu(t) - \mu(t)\|_{kr}. \end{aligned}$$

Then
$$\frac{d\|\nu(t)-\mu(t)\|_{kr}}{dt} - Q\|\nu(t)-\mu(t)\|_{kr} < 2C\epsilon.$$

Multiplying by e^{-Qt} we get
$$\frac{d\|\nu(t)-\mu(t)\|_{kr}e^{-Qt}}{dt} < 2C\epsilon e^{-Qt},$$

and integrating in the interval $[0,t]$, with $t \leq T$, we get
$$\|\mu(t)-\nu(t)\|_{kr}e^{-Qt} - \|\mu_0-\nu_0\|_{kr}e^{-Q0} < 2C\epsilon\left(\frac{1-e^{-Qt}}{Q}\right).$$

Then, by Proposition A.3, for all $t \in [0,T]$
$$\begin{aligned} d_w(\mu(t),\nu(t)) &< d_w(\mu_0,\nu_0)e^{Qt} + 2C\epsilon\left(\frac{e^{Qt}-1}{Q}\right) \\ &\leq d_w(\mu_0,\nu_0)e^{QT} + 2C\epsilon\left(\frac{e^{QT}-1}{Q}\right), \end{aligned}$$

which is equivalent to (4.29). □

Corollary 4.25. *Let us assume the hypotheses of Theorem 4.24, and in addition suppose that there exist sequences of functions* $\{U_{\epsilon_n}\}_{n=1}^{\infty}$ *and probability measures* $\{\nu^n\}_{n=1}^{\infty}$ *such that* $\|U_{\epsilon_n} - U\| \to 0$ *and* $d_w(\nu_0^n, \mu_0) \to 0$*. If* $\mu(\cdot)$ *and* $\nu^n(\cdot)$ *are solutions of (4.21) and (4.22), respectively, with initial conditions* $\mu(0) = \mu_0$ *and* $\nu^n(0) = \nu_0^n$*, then for* $t \in [0,T]$*, with* $T < \infty$*,*
$$\lim_{n\to\infty} \sup_{t\in[0,T]} d_w(\nu_n(t),\mu(t)) \to 0.$$

4.6 Examples

In this section we consider the examples of a quadratic-linear model, a sales model as a Bertrand game, graduate risk game and Ward of attrition game of the sections 2.2, 2.5, 2.6, and 2.7 respectively.

In each example we prove that the NES of the game is also a SUS. Thus, under the replicator dynamics if the initial strategy μ_0 is closed to the NES (which is a SUS), then the player select a strategy $\mu(t)$ very closed to the NES for every $t > 0$. In other words, under the replicator dynamics the player searches and selects strategies that have certain dominance as the SUS.

4.6.1 A linear-quadratic model

In this section we consider the symmetric case game in section 2.2. Then we can rewrite the payoff functions (2.9) and (2.10) as

$$U(x,y) = -ax^2 - bxy + cx + dy,$$

with $a, b, c > 0$ and d any real number.

Let $A = [0, M]$ for $M > 0$, be the strategy set. If $2c(a-b) > 0$ and $4a^2 - b^2 > 0$, then we have an interior Nash equilibrium strategy (NES)

$$x^* = \frac{2c(a-b)}{4a^2 - b^2}.$$

The function $U(x^*, y) - U(y, y)$ has a minimum value of 0 at $y = x^*$ and is strictly concave. So

$$U(x^*, y) - U(y, y) > 0 \qquad \forall y \in A, \quad y \neq x^*,$$

which implies

$$J(x^*, \mu) - J(\mu, \mu) > 0 \qquad \forall \mu \in \mathbb{P}(A), \quad \mu \neq \delta_{x^*}.$$

Then for any metric r on $\mathbb{P}(A)$, the strategy δ_{x^*} is r-SUS. Hence by Theorem 4.10 if $\|\mu_0 - \delta_{x^*}\| = 2(1 - \mu_0(\{x^*\})) < \epsilon$, then

$$\|\mu(t) - \delta_{x^*}\| = 2(1 - \mu(t, \{x^*\})) < \epsilon, \quad r_w(\mu(t), \delta_{x^*}) < M\epsilon \qquad \forall t \geq 0.$$

Moreover, since the payoff function $U(\cdot)$ is continuous and the set of strategies A is compact, we conclude that $\mu(t) \to \delta_{x^*}$ in distribution.

4.6.2 A sales model as a Bertrand game

Consider the sales model in section 2.5. We would prove that the NES (2.20) is a r-SUS for any r-metric in $\mathbb{P}(A)$ (with $A = [0, r]$). Let $U(p, z)$ be the payoff function defined in (2.19) and we suppose that $k > r\frac{\alpha T}{2}$, which implies that the firm has a loss if the opponent firm captures the demand of informed consumer.

Since I is the number of informed consumer, V is the number of uninformed consumer and T is the number of total of total consumer $T = I + V$, we may assume that $I = (1-\alpha)T$ and $V = \alpha T$ for some $\alpha \in (0, 1)$. Then the NES (2.20) is expressed as

$$\frac{d\mu^*(p)}{dp} = \begin{cases} \left[\frac{r\alpha}{2(1-\alpha)}\right] p^{-2}, & \text{if } \bar{p} \leq p \leq r \\ 0, & \text{other case} \end{cases},$$

where $\bar{p} = \frac{r\alpha}{2-\alpha}$. Then we have

$$J(\mu^*, z) - J(z, z) = \begin{cases} k - z\left[\frac{\alpha T}{2}\right], & \text{if } 0 \leq z < \bar{p} \\ \frac{\alpha T}{2}\left[r\left[\log\left(\frac{z}{\bar{p}}\right)\right] + \frac{\alpha r}{1-\alpha}\left[\log\left(\frac{r}{\bar{p}}\right)\right] + \frac{2k}{\alpha T} - z\right], & \text{if } \bar{p} \leq z \leq r \end{cases}.$$

Since $k > r\frac{\alpha T}{2}$, is easy to see that the map $z \mapsto J(\mu^*, z) - J(z, z) > 0$ for all $z \in [0, r]$. Then $J(\mu^*, \mu) - J(\mu, \mu) > 0$, for all $\mu \in \mathbb{P}(A)$. This prove that μ^* is a r-SUS, for any r-metric in $\mathbb{P}(A)$.

Hence, by Theorem 4.9, if $K(\mu_0, \mu^*) < \varphi'(\epsilon) = \left(\frac{\epsilon}{2m}\right)^2$, then

i) $\mu(t) \in \mathcal{W}_{\varphi'(\epsilon)}(\mu^*)$ for all $t \geq 0$;

ii) $\|\mu(t) - \mu^*\| < \frac{\epsilon}{m}$ for all $t \geq 0$;

iii) $r_w(\mu(t), \mu^*) < \epsilon$ for all $t \geq 0$;

Let $\mathcal{C} := \{\mu \in \mathbb{P}(A) : \mu([\bar{p}), r] = 1\}$, then $\mathcal{C}$ is weakly compact and the map $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ is continuous in $\mathcal{C}$ for the weak topology. Hence, $r_w(\mu(t), \mu^*) \to 0$ as $t \to \infty$, if $\mu_0 \in \mathcal{C}$. (See Theorem 3.14.)

4.6.3 Graduated risk game

Consider the game of section 2.6. Bishop and Cannings [13] show that if $v < c$ (in the payoff function (2.21)), then the NES (2.22) satisfies that

$$J(\mu^*, \mu) - J(\mu, \mu) > 0 \quad \forall \mu \in \mathbb{P}(A),$$

i.e., μ^* is r-SUS for any metric r in $\mathbb{P}(A)$.

Hence, by Theorem 4.9, if $K(\mu_0, \mu^*) < \varphi'(\epsilon) = \left(\frac{\epsilon}{2m}\right)^2$, then

i) $\mu(t) \in \mathcal{W}_{\varphi'(\epsilon)}(\mu^*)$ for all $t \geq 0$;

ii) $\|\mu(t) - \mu^*\| < \frac{\epsilon}{m}$ for all $t \geq 0$;

iii) $r_w(\mu(t), \mu^*) < \epsilon$ for all $t \geq 0$.

Since $v < c$ in the payoff function (2.21) and the NES (2.22) we have that the equation

$$J(\mu^*, y) - J(y, y) = \left[\frac{c-v}{2}\right]\left[\frac{\alpha-1}{1+\alpha}\right] + \left[\frac{\alpha v + c}{1+\alpha}\right] y^{\frac{1+\alpha}{2}} - \left[\frac{c-v}{2}\right] y,$$

then the map $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ is continuous in the weak topology. Hence, $r_w(\mu(t), \mu^*) \to 0$ as $t \to \infty$, if $K(\mu_0, \mu^*) < \varphi'(\epsilon)$.

On the other hand, If $c < v$ in (2.21), then the NES (2.23) satisfies that

$$J(\delta_{\{0\}}, y) - J(y, y) = vy > 0 \quad \forall y \in A.$$

Then

$$J(\delta_{\{0\}}, \mu) - J(\mu, \mu) > 0 \quad \forall \mu \in \mathbb{P}(A),$$

i.e., $\delta_{\{0\}}$ also is r-SUS for any metric r in $\mathbb{P}(A)$.

Hence, by Theorem 4.10 if $\|\mu_0 - \delta_0\| < \epsilon$ for some small $\epsilon > 0$, then

i) $\|\mu(t) - \delta_{\{0\}}\| < \epsilon$ for all $t \geq 0$;

ii) $r_w(\mu(t), \delta_{\{0\}}) < \epsilon$ for all $t \geq 0$;

iv) Since the the map $\mu \to J(\delta_{\{0\}}, \mu) - J(\mu, \mu)$ is continuous in the weak topology, $A = [0, 1]$ is compact, then $r_{w^*}(\mu(t), \delta_{\{0\}}) \to 0$,, where r_{w^*} is any distance that metrizes the weak topology.

4.6.4 War of attrition game

Consider the game in section 2.7. Since $v \leq m$ in the payoff function (2.24) and the NES (2.25) we have that the equation

$$J(\mu^*, y) - J(y, y) = \left[\frac{1}{1-e^{-m/v}}\right]\left[2ve^{-y/v} + (y-v)e^{-m/v} - v\right] + y$$

has the positive minimum value

$$\left[\frac{v}{e^{m/v} - 1}\right][e^{m/v}\log(2) - 1] > 0, \qquad \text{at} \;\; y = v\log(2),$$

for $m > \frac{v}{\log(2)}$.

This implies that $J(\mu^*, \mu) - J(\mu, \mu) > 0$ for all μ in $\mathbb{P}(A)$ and $\mu \neq \mu^*$. Then for any metric r on $\mathbb{P}(A)$, the strategy μ^* is r-SUS. Hence, by Theorem 4.9, if $K(\mu_0, \mu^*) < \varphi'(\epsilon) = \left(\frac{\epsilon}{2m}\right)^2$, then *i)* $\mu(t) \in \mathcal{W}_{\varphi'(\epsilon)}(\mu^*)$ for all $t \geq 0$;

ii) $\|\mu(t) - \mu^*\| < \frac{\epsilon}{m}$ for all $t \geq 0$;

iii) $r_w(\mu(t), \mu^*) < \epsilon$ for all $t \geq 0$;

Moreover, since the map $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ is continuous in the weak topology, then $r_w(\mu(t), \mu^*) \to 0$ as $t \to \infty$.

4.7 Comments

In this chapter, we introduced a model of symmetric evolutionary games with strategies in measurable spaces. The model can be reduced, of course, to the particular case of evolutionary games with finite strategy sets. We provide a general framework to the replicator dynamics that allows us to analyze different stability criteria, and establish conditions to approximate the replicator dynamics in a measure space by a sequence of dynamical systems on finite spaces. We also presented three examples. The first one may be applicable to oligopoly models, theory of international trade, and public good models. The second an third examples deal with a graduate risk game and war of attrition game, respectively.

The replicator dynamics has been studied in other general spaces without direct applications to game theory. For instance, Kravvaritis et al. [61], [58], [59] [60], and Papanicolaou and Smyrlis [77] studied conditions for stability and examples for these general cases. These extensions may be applicable in areas such as migration, regional sciences, and spatial economics (see Fujita, Krugman, and Venables [39] chapters 5 and 6).

There are many questions, however, that remain open. For instance, when the set of pure strategies is finite, Cressman [27] shows that under some conditions the stability of monotone selection dynamics is locally determined by the replicator dynamics. Is this true for games with strategies on the space $\mathbb{P}(A)$ of probability measures? Another important issue would be to obtain a stability theorem for several evolutionary dynamics of games with continuous strategies similar to the result by Hofbauer and Sigmund [51] (Theorem 14) for games with a finite strategy set A.

5 The replicator dynamics as a deterministic approximation

In this chapter we see the replicator dynamics as a limit of a sequence of Markov process. There are many references (mentioned in section 1.1) on this issue when the strategy space is finite. However, a more general mathematical structure is needed if the strategy set is a measurable space, which we consider in this chapter. We use a general theorem (Kolokoltsov [55]) in which an infinite-dimensional kinetic equation (a differential equation on a space of measures) is a limit of a sequence of jump Markov process.

Section 5.1 presents notation and technical requirements. Section 5.2 shows a technique proposed by Kolokoltsov [55],[56] to proximate a sequence of pure jumps models of binary interaction (in a Banach space), by means of a deterministic dynamical system. Section 5.3 uses techniques of section 5.2 to establish conditions under which the replicator dynamics is a limit of a sequence of Markov processes. Finally, section 5.4 aggregates comments over futures perspectives.

5.1 Technical Preliminaries

In this section we summarize some facts about the approximation of ordinary differential equations by Markov processes and other topics related. For proofs proofs see e.g. Kallenberg [54], Ethier and Kurtz [37], and and Böttcher, Schilling and Wang [19].

5.1.1 Markov processes

Let F a Banach space, and $L(F)$ the set of all linear bound operators form F into F. A *strongly continuous semigroup* of linear operators on F is a mapping $T : [0, \infty) \to L(F)$ such that

i) $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$, $T(0) = I$ where I is the identity operator.

ii) $\lim_{t\to 0} T(t)x = x$ in the strong topology operator.

Definition 5.1. *The generator G of a strongly continuous semigroup $T(\cdot)$ is defined as follows:*

$$\mathcal{D}(G) := \left\{ f \in F : \lim_{t\to 0+} \left[\frac{T(h) - I}{h} \right] \quad \text{exits} \right\},$$

and for $f \in \mathcal{D}(G)$

$$Gf = \lim_{t\to 0+} \left[\frac{T(h) - I}{h} \right] f.$$

The connection of the linear semigroup theory with the Markov processes is given as follows. Let A be a locally compact metric space. Let $x(\cdot) = \{x(t) : t \geq 0\}$ be a Markov process in A with transition probability $P(s, x, t, E)$, i.e.,

$$P(s, x, t, E) = P(x(t) \in E \quad |x(s) = x).$$

for all $t > s \geq 0,\ x \in A$ and $E \in \mathcal{B}(A)$. Let F the linear real-valued measurable functions f on $[0, \infty) \times A$ such that

$$\int_A |f(s, y)| P(s, x, t, dy) \leq \infty$$

for each s, x, t. For each $t \leq 0$ and $f \in F$, let $T_t f$ be a function on $[0, \infty) \times A$ defined by

$$T_t f(s, x) := \int_A f(s + t, y) P(s, x, s + t, dy).$$

In this case the operator $T_t,\ t \geq 0$, form a semigroup of operators on F.

Let $\mathbb{C}_\infty(A)$ be the set of functions f such that $f \in \mathbb{C}(A)$ and for all $\epsilon > 0$ there exits a compact set $K \subset A$ that satisfies $\sup_{a\in A} |f(a)| \leq \epsilon$. A Markov process is call *homogeneous* if $P(s, x, s + t, E) = P(0, x, t, E)$ for all $s \geq 0$. A (homogeneous) Markov process in a locally compact metric A is call Feller process if for any $f \in \mathbb{C}_\infty(A)$ we have that $T_t f \in \mathbb{C}_\infty$.

5.1.2 Approximation of pure jump process

A jump Markov process describe a stochastic process in continuous time that, intuitively, behaves as follows. Consider that the system starts from a point $x(s) = x \in A$ for some time $s \geq 0$. It stays there a random length of time τ_1 and then "jumps" spontaneously to a new state $y \neq x$. It stays there a random length of time τ_2 (independent of τ_1) and then "jumps" to a new state $z \neq y$, and so on.

By a pure (*homogeneous*)jump Markov process on A we mean a Markov process with a generator of the form

$$Gf(x) = \int_A f(y) - f(x)Q(x, dy).$$

where Q is a transition, that is for every $x \in A$ and $E \in \mathcal{B}(A)$, $Q(\cdot, E)$ is a real-valued measurable function, and $Q(x, \cdot)$ is a signed measure on $\mathcal{B}(A)$. In particular for pure jump processes Q satisfies $Q(x, x) = 0$ and $Q(x, A) < \infty$ for every $x \in A$.

The following theorem see the solution of ordinary differential equations as the limit of a sequence of pure jump Markov processes.

Proposition 5.2. *Let $A \subset \mathbb{R}^m$ (endowed with the euclidean norm $|\cdot|$) and $\{x(t)_n\}$ a sequence of pure jump Markov process in A such that for every $n > 0$, the process $x_n(\cdot)$ have generator*

$$G_n f(x) = \int_A f(y) - f(x)Q_n(x, dy).$$

and $x_n(0) = x_0$. For every $n > 0$ and x in A, consider the functions

$$F_n(x) = \int_A |x - y|Q_n(a, dy) \quad \text{and} \quad H_n(x) = \int_{|x-y|>\epsilon_n} |x - y|Q(x, dy).$$

Where $\{\epsilon_n\}$ is a sequence such that $\lim_{n\to\infty} = 0$. Consider the differential equation

$$x'(t) = G(x(t)), \quad \text{with } x(0) = x_0. \tag{5.1}$$

Where G satisfies a Lipschitz condition. Finally, assume the follow

i) $\sup_n \sup_{x\in A} F_n(x) < \infty$

ii) $\lim_{n\to\infty} \sup_{x\in A} H_n(x) = 0.$

If $G_n(\cdot)$ converges uniformly to $G(\cdot)$, then the sequence of stochastic processes $x_n(t)$ converges weakly to the solution $x(t)$ of the differential equation 5.1. Moreover for every $\epsilon > 0$ and $t > 0$,

$$\lim_{n\to\infty} P(\sup_{s\le t} |x_n(s) - x(s)| > \epsilon) = 0.$$

Proof. See Kurtz[62], [63].

This theorem is can see as a particular case of a theorem of convergence of Feller processes (see Kallenberg [54], chapter 19).

Supposes that Let A be compact metric space, then $\mathbb{M}(A)$ (endowed with the weak topology) is locally compact metric space (see Li [66].) Then under this condition, we can talk about Markov process with sated spaces in $\mathbb{M}(A)$. This Markov processes are call measure-valued Markov processes. For references of this general cases see e.g. Li[66], Dynkin [35] and kolokoltsov [56].

Section 5.2 see the replicator dynamics as a limit of a sequence of measure-valued Markov process.

5.1.3 Notation

Let A be a separable and compact metric space, and consider the Cartesian products $A^j := A \times ... \times A$ (j-times) and $A^\infty := A \times A \times ...$ (infinite-times) with their product topologies. We shall denote by $A^\cup$ the disjoint union of the sets A^j, i.e., $A^\cup = \cup_{j=1}^\infty A^j$, and which is a compact space in A^∞.

For the following Definition we consider the set of the natural numbers $\mathbb{N}$.

Definition 5.3. *A measure μ in $\mathbb{M}(A^\infty)$ is call* symmetric *if for any permutation $\rho : \mathbb{N} \to \mathbb{N}$ that replaces only finitely many elements, we have*

$$\mu(\rho E) = \mu(E) \qquad \forall E \in \mathcal{B}(A^\infty),$$

where

$$\rho E := \Big\{(a_1, a_2, ..., a_j, ...) \in A^\infty : (a_{\rho(1)}, a_{\rho(2)}, ..., a_{\rho(j),...}) \in E\Big\}.$$

The set of symmetric measures *on A^∞ is written as* $\mathbb{M}_S(A^\infty)$.

Similarity, as in Definition 5.3, we can define define a *symmetric measure* on A^j and $A^\cup$. The spaces of symmetric measures on A^j and $A^\cup$ will be denoted by $\mathbb{M}_S(A^j)$ and $\mathbb{M}_S(A^\cup)$, respectively. For more details about symmetric measures see Hewitt and Savage [48], and Bogachev [15] (chapter 10).

Let X be either A^j , A^∞ or $A^\cup$. The spaces of positive measures, and symmetric positive measures on X will be denoted by $\mathbb{M}^+(X)$ and $\mathbb{M}_S^+(X)$ respectively.

A function $f : A^\infty \to \mathbb{R}$ is said to be *symmetric* if for any permutation $\rho : \mathbb{N} \to \mathbb{N}$ and $(a_1, a_2, ..., a_j, ...)$ in A^∞

$$f(a_1, a_2, ..., a_j, ...) = f(a_{\rho(1)}, a_{\rho(2)}, ..., a_{\rho(j)}, ...).$$

Similarity, we can define a real-valued *symmetric function* on A^j and $A^\cup$. Let X be either A^j, A^∞ or $A^\cup$. We shall denoted by $\mathbb{B}_S(X)$ (resp. $\mathbb{C}_S(X)$) the Banach space of symmetric bounded (resp. continuous) real-valued functions on X.

On X we, consider the equivalence relation $\sim$ given by

$$(a_1, a_2, ..., a_j, ...) \sim (b_1, b_2, ..., b_j, ...)$$

if only if there exists a permutation ρ such that $b_i = a_{\rho(i)}$ for all $i = 1, 2, ...$. Let X_S be the quotient space (the space of equivalences cases) with the quotient topology (for details see Pedersen [78] chapter 1). This allows us to identify e.g., $\mathbb{C}_S(X) = \mathbb{C}(X_S)$.

5.2 Pure jump Markov processes for binary interacting individuals

In this section we see how a sequence of general pure jump Markov processes converges weakly to the solution of an infinite-dimensional differential equation (called kinetic equation). For details see Kolokoltsov [55], [56]. In particular we are interested in pure jump Markov processes that emerge from the interaction of two particles, in other words, that originate from binary interacting particles.

In game theory, we are interested in the behavior of individuals, which is why we change the word “particles” (used in physical theory) by “individuals”.

Let A be a separable and compact metric space. The symmetrical laws on A^j (which are uniquely defined by their projections to A^j_S) are called exchangeable systems of j individuals. The elements of $\mathbb{M}^+_S(A^\cup)$ and $\mathbb{C}_S(A^\cup)$ are called, respectively, the states and observables for a Markov process Z_t on $A^\cup$. We shall denote the elements of $A^\cup$ by bold letters, e.g. $\mathbf{a}, \mathbf{b}$. A key observation for the theory of measure-valued limits is the inclusion $A^\cup_S$ to $\mathbb{M}^+(A)$ given by

$$\mathbf{a} = (a_1, ..., a_l) \mapsto h\delta_{a_1} + ... + h\delta_{a_l}, \qquad h > 0 \tag{5.2}$$

which defines a bijection between $A^\cup_S$ and the space $\mathbb{M}^+_{h\delta}(A) \subset \mathbb{M}^+(A)$ of finite linear combinations of δ-measures.

For each $f \in \mathbb{B}_S(A^\cup)$ and $\mathbf{a} = (a_1, ..., a_j) \in A^j \subset A^\cup$, we write $f(\mathbf{a}) = f(a_1, .., a_j)$, $f^+(\mathbf{a}) = f^+(a_1, .., a_j) = f(a_1) + ... + f(a_j)$ and $f^\times(\mathbf{a}) = f^\times(a_1, .., a_j) = f(a_1) \cdot ... \cdot f(a_j)$. For a finite subset of two elements $I = \{i_1, i_2\}$ of a finite set $J = \{1, 2, ..., j\}$ we denote by I^c its complement $I^c = J - I$. Then for $\mathbf{a} = (a_1, ..., a_j) \in A^\cup$, $\mathbf{a}_I = (a_{i_1}, a_{i_2})$ and by $\mathbf{a}_{I^c} = (a_{i^c_1}, ..., a_{i^c_m})$, where $I^c = \{i^c_1, ..., i^c_m)$, and $\mathbf{a} = (\mathbf{a}_I, \mathbf{a}_{I^c})$.

By a pure jump process Z_t on $A^\cup$ that describe the interaction of two individuals, we mean a Markov process with a generator of the form

$$Gf(\mathbf{a}) = \sum_{I \subset \{1,2,..j\}} \int_{A^\cup} f(\mathbf{a}_{I^c}, \mathbf{b}) - f(\mathbf{a}) Q(\mathbf{a}_I, d\mathbf{b}). \tag{5.3}$$

Where the binary interaction transition kernel is such that

$$Q(\mathbf{a}_I) = \int_{A^{\cup}} Q(\mathbf{a}_I, d\mathbf{b}) = \sum_{m=1}^{\infty} \int_{A^m} Q_m(\mathbf{a}_I, db_1 \cdots db_m). \tag{5.4}$$

Changing the state space according to the mapping (5.2) yields the corresponding Markov process Z_t^h on $\mathbb{M}^+_{h\delta}(A)$ The above changing of space leads (5.3) to the generator G_h defined by (for details see Kolokoltsov [55])

$$G_h f(h\delta_{\mathbf{a}})$$

$$= h \sum_{I \subset \{1,2,\dots,j\}} \int_{A^{\cup}} \Big[f(h\delta_{\mathbf{a}} - h\delta_{\mathbf{a}_I} + h\delta_{\mathbf{b}}) - f(h\delta_{\mathbf{a}}) \Big] Q(h\delta_{\mathbf{a}}, \mathbf{a}_I, d\mathbf{b}). \tag{5.5}$$

Using the relation (1.37) in Kolokoltsov [56] (Chapter I) and applying the operator (5.4) over the linear function

$$f_g(\mu) = \langle g, \mu \rangle = \int_A g(a)\mu(da) \qquad g \in \mathbb{C}(A), \tag{5.6}$$

we obtain

$$G_h f_g(h\delta_{\mathbf{a}})$$

$$\begin{aligned} = \; & \frac{1}{2} \int_{A^{\cup}} \int_{A^2} \Big[g^+(\mathbf{b}) - g^+(a_1, a_2) \Big] Q(h\delta_{\mathbf{a}}, (a_1, a_2), d\mathbf{b}) h\delta_{\mathbf{a}}(da_1) h\delta_{\mathbf{a}}(da_2) \\ & - \frac{1}{2} h \int_{A^{\cup}} \int_A \Big[g^+(\mathbf{b}) - g^+(a, a) \Big] Q(h\delta_{\mathbf{a}}, (a, a), d\mathbf{b}) h\delta_{\mathbf{a}}(da) \end{aligned} \tag{5.7}$$

where $g^+(\mathbf{b}) = g^+(b_1, b_2, ..., b_k) = g(b1)+b(2)+...+b(k)$, similarly for $g^+(a_1, a_2)$.

Assume that the value of h is the scale or genetic relevance of each individual. This genetic relevance is decreasing for example with respect to the number of individuals ,i.e., if the population tends to infinity, then $h \to 0$. The genetic relevance of each individual h is high, e.g., in small populations or endangered populations. When the scale or genetic relevance of each individual h is small (e.g. in a huge population) then mass distribution $h\delta_{\mathbf{a}}$ retains this genetic relevance h. It follows that if h tends to 0 and $h\delta_{\mathbf{a}}$ tends to some distribution or probability measure μ, the corresponding generator G_h evaluated in (5.6)-(5.7) tends to

$$Gf_g(\mu)$$
$$= \frac{1}{2}\int_{A^\cup}\int_{A^2}\Big[g^+(\mathbf{b}) - g^+(a_1,a_2)\Big]Q(\mu,(a_1,a_2),d\mathbf{b})\mu(da_1)\mu(da_2). \tag{5.8}$$

Under some hypotheses (see Kolokoltsov [55]) if $\mu(t)$ is solution of the differential equation

$$\frac{d}{dt}\langle g,\mu(t)\rangle = Gf_g(\mu(t)) \qquad \forall g \in \mathbb{C}(A_S), \quad (\text{with } \mu(0)=\mu_0), \tag{5.9}$$

then there exists a subsequence of stochastic process $Z^{h_n}(t)$ (subfamily of $\{Z^h(t)\}_{h>0}$) with generator (5.6), which converges weakly to $\mu(t)$.

5.3 The replicator dynamics as a deterministic approximation

In this section we specify a Markov game which can be approximated by the replicator dynamics. This Markov process models a stochastic interaction between individuals which explain the evolution of the probability distribution of characteristics in a population.

Suppose that in each stage of the game we select a pair of individuals of characteristics $a_1, a_2 \in A$. The agent with characteristic a_1 plays against agent with characteristic a_2 and the transition rate to have $(m-1)$ new agents with characteristic a_1 after this game is given by

$$J_m(a_1, h\delta_{\mathbf{a}}) - J_m(a_2, h\delta_{\mathbf{a}}), \tag{5.10}$$

where

i) $h\delta_{\mathbf{a}}$ is a positive measure on A described by (5.1) and $h>0$ is the scale or the genetic relevance of each individual;

ii) $J_m : \mathbb{M}^+(A)\times\mathbb{M}^+(A) \to \mathbb{R}$ is defined similar (4.1), i.e., $J_m(a_1,a_2) = U_m(a_1,a_2)$ for any $a_1, a_2 \in A$. The function $U_m(\cdot,\cdot)$ can be chosen arbitrarily as long as the average change equals a function $U(\cdot,\cdot)$,i.e., for any $a_1, a_2 \in A$

$$U(a_1,a_2) = \sum_{m=0}^{\infty}(m-1)U_m(a_1,a_2). \tag{5.11}$$

The interaction transition kernels Q_m in (5.3) from the generator (5.7) is of the form

$$Q_m(h\delta_{\mathbf{a}}, (a_1, a_2), db_1 \cdots db_m) = \Big[J_m(a_1, h\delta_{\mathbf{a}}) - J_m(a_2, h\delta_{\mathbf{a}})\Big]\delta_{a_1}(db_1)...\delta_{a_1}(db_m) + \Big[J_m(a_2, h\delta_{\mathbf{a}}) - J_m(a_1, h\delta_{\mathbf{a}})\Big]\delta_{a_2}(db_1)...\delta_{a_2}(db_m). \tag{5.12}$$

Then

$$\int_{A^{\cup}} \Big[g^+(\mathbf{b}) - g^+(a_1, a_2)\Big] Q(h\delta_{\mathbf{a}}, (a_1, a_2), d\mathbf{b})$$
$$= \sum_{m=0}^{\infty}(m-1)\Big[g(a_1)\Big[J_m(a_1, h\delta_{\mathbf{a}}) - J_m(a_2, h\delta_{\mathbf{a}})\Big] + g(a_2)\Big[J_m(a_2, h\delta_{\mathbf{a}}) - J_m(a_1, h\delta_{\mathbf{a}})\Big]\Big]$$
$$= g(a_1)\Big[J(a_1, h\delta_{\mathbf{a}}) - J(a_2, h\delta_{\mathbf{a}})\Big] + g(a_2)\Big[J(a_2, h\delta_{\mathbf{a}}) - J(a_1, h\delta_{\mathbf{a}})\Big] \tag{5.13}$$

Therefore, if h tends to 0 and $h\delta_{\mathbf{a}}$ tends to some probability measure μ, then (by Fubini theorem and (5.13)) the generator G_h in (5.8) has the form

$$\frac{1}{2}\int_A \int_A g(a_1)\Big[J(a_1, \mu) - J(a_2, \mu)\Big]\mu(da_1), \mu(da_2) + \frac{1}{2}\int_A \int_A g(a_2)\Big[J(a_2, \mu) - J(a_1, \mu)\Big]\mu(da_2), \mu(da_1)$$
$$= \int_A \int_A g(a_1)\Big[J(a_1, \mu) - J(a_2, \mu)\Big]\mu(da_1), \mu(da_2)$$
$$= \int_A g(a)\Big[J(a, \mu) - J(\mu, \mu)\Big]\mu(da).$$

Then the kinetic equation (5.8) has the form of the replicator dynamics in the weak topology

$$\frac{d}{dt}\int_A g(s)\mu(t, da) = \int_A g(a)\Big[J(a, \mu(t)) - J(\mu(t), \mu(t))\Big]\mu(t, da), \tag{5.14}$$

for $g \in \mathbb{C}(A)$.

To prove the approximation Theorem 5.4, below, we need the following concepts:

i) Let L a non-negative function in A. We say that the transition kernel is *L-subcritical* if for all $\mathbf{b}$ in $A^{\cup}$ and μ in $\mathbb{M}^{+}(A)$

$$\int_{A^{\cup}} \Big[L^{+}(\mathbf{b}) - L^{+}(a_1, a_2)\Big] Q(\nu, (a_1, a_2), d\mathbf{b}) \leq 0.$$

ii) We say that the transition kernel is L^{+}*-bounded* ($L^{\times}$*-bounded*) if for all (a_1, a_2) in A^2 and μ in $\mathbb{M}^{+}(A)$ and some $c > 0$

$$Q(\mu, (a_1, a_2)) \leq c[L(a_1) + L(a_2)] \qquad \Big(Q(\mu, (a_1, a_2)) \leq cL(a_1) \cdot L(a_2)\Big).$$

iii) We say that the (ND)-condition is satisfied if the number of individuals that can be created by a single act of interaction is uniformly bounded by some number m_0, and P is 1^{+}*-subcritical* (where 1 is a constant function).

Theorem 5.4. *Let A be a compact separable metric space, and let $\{U_m\}_{m=0}^{\infty}$ and U be bounded functions that satisfy (5.11) and*

$$\sum_{m=0}^{\infty} \int_A \int_A |m-1||U_m(a_1, a_2)|\nu(da_1)\nu(da_2) < \infty \qquad \forall \nu \in \mathbb{M}^{+}(A). \tag{5.15}$$

In addition, suppose that the mapping $(\mu, \delta_{(a_1,a_2)}) \to Q(\mu, (a_1, a_2), \cdot)$ is continuous in the weak topology, where Q is defined by (5.4) and (5.13). If the family of initial measures $h\delta_{\mathbf{a}}$ converges weakly to some measure μ (as $h \to 0$), then there exists a subsequence $Z_t^{h_n\delta_{\mathbf{a}}}$ of the family of stochastic process $\{Z_t^{h\delta_{\mathbf{a}}}\}_{h>0}$ defined with generator (5.6) (with transition kernel as (5.4)-(5.13)) that converge weakly to the solution $\mu(\cdot)$ of the replicator dynamics (5.14).

Proof. We will prove that the kernels Q of the generator (5.7) (where Q is defined by (5.4) and (5.13)) satisfy the hypotheses of Theorem 4.2 in Kolokoltsov [55].

Let $L > 0$ be an arbitrary (but fixed) positive number. By (5.4), (5.15), we have

$$\int_{A^{\cup}} \Big[L^{+}(\mathbf{b}) - L^{+}(a_1, a_2)\Big] Q(h\delta_{\mathbf{a}}, (a_1, a_2), d\mathbf{b})$$

$$= L\Big[J(a_1, h\delta_{\mathbf{a}}) - J(a_2, h\delta_{\mathbf{a}})\Big] + L\Big[J(a_2, h\delta_{\mathbf{a}}) - J(a_1, h\delta_{\mathbf{a}})\Big] = 0. \tag{5.16}$$

and

$$\begin{aligned} Q(\mu,(a_1,a_2)) &= \sum_{m=0}^{\infty}\int_{A^m} Q_m(\mu,(a_1,a_2),db_1\cdots db_m) \\ &= \sum_{m=0}^{\infty}\int_{A^m}\Big[J_m(a_1,h\delta_{\mathbf{a}})-J_m(a_2,h\delta_{\mathbf{a}})\Big]\delta_{a_1}(db_1)...\delta_{a_1}(db_m) \\ &\quad+\sum_{m=0}^{\infty}\int_{A^m}\Big[J_m(a_2,h\delta_{\mathbf{a}})-J_m(a_1,h\delta_{\mathbf{a}})\Big]\delta_{a_2}(db_1)...\delta_{a_2}(db_m) \\ &= \sum_{m=0}^{\infty} m\Big[\Big[J_m(a_1,h\delta_{\mathbf{a}})-J_m(a_2,h\delta_{\mathbf{a}})\Big] \\ &\qquad + \Big[J_m(a_2,h\delta_{\mathbf{a}})-J_m(a_1,h\delta_{\mathbf{a}})\Big]\Big] = 0 \qquad (5.17) \end{aligned}$$

Then by (5.16) the kernel $Q(\mu,(a_1,a_2),d\mathbf{b})$ is L-subcritical and 1-sub-critical. By (5.17) the transition kernel is $(1+L^{\alpha})^{+}$-bounded. The (ND) condition is satisfied since the number of individuals that can be created by a single act of interaction is equal to 0. Thus the hypotheses of Theorem 4.2 in Kolokoltsov [55] are satisfied and the assertion is thru. □

Remark 5.5. *Under the conditions of Theorem 5.4, and since the payoff function U is bounded, from Proposition 3.4 and Theorem 4.1 the replicator dynamics in weak form (5.14) is equal to strong form (4.3). Therefore, there exists a subsequence $Z_t^{h_n\delta_{\mathbf{a}}}$ of the family of stochastic process $\{Z_t^{h\delta_{\mathbf{a}}}\}_{h>0}$ defined with generator (5.6) (with transition kernel as (5.4)-(5.13)) that converge weakly to the solution $\mu(\cdot)$ of the replicator dynamics in the to strong form (4.3).*

5.4 Comments

In this chapter we considered the replicator dynamics as a limit of a sequence of measure-valued Markov process. We used a technique proposed by Kolokoltsov [55],[56] to proximate a sequence of pure jumps models of binary interaction (in the space of measure), by means of a deterministic dynamical system.

There are many questions, however, that remain open. For instance, Do we can have numerical approximation for this measure-valued Markov processes? This is an important issue for the application of this theory. When the set of pure strategies is finite, there are other evolutionary dynamics that can be saw as a limit of a sequence of measure-valued Markov process. Is it true for for games whit strategies in the space of measure?. An finally, Do The replicator dynamics in the asymmetric case can be also be approximated by a sequence of stochastic processes?

6 Conclusions and suggestion for future research

This work provides a general framework to study the replicator dynamics for evolutionary games theory in which the strategy set is a separable metric space. We analyze the asymmetric and symmetric case, and included examples to illustrate our results.

For games in the asymmetric case we conclude the follow:

i) Under some conditions there exits the solution for the asymmetric replicator equations(Theorem 3.5) and this solution have special characteristics (Theorem 3.6.) In particular, this conditions are satisfied, when the payoffs of the players are bounded (Proposition 3.4.)

ii) If $\mu^* = (\mu_1^*, ..., \mu_n^*)$ is a Nash equilibrium of a normal form games Γ, then mu^* is a critical point of the replicator dynamics (Theorem 3.9.)

iii) A strong uninvadable profile (SUP) is a Nash equilibrium (Theorem 3.12.) The SUPs are Nash equilibria where the strategy of each player is dominant in a certain subset of her strategies set.

iv) If μ^* is a pure Nash equilibrium and is a SUP, then μ^* is a stable point for the replicator dynamics (Theorem 3.4.)

v) Finally, the symmetric replicator dynamic can be deduced from the asymmetric case (see section 3.2.1.) Therefore, *i*)-to-*iv* are true for the symmetric case.

In two-players normal form game γ_s the Symmetric Nash equilibrium can be rewrite in terms of a strategy call Nash equilibrium strategy (NES). In the same form the symmetric SUP can be rewrite in terms of a strategy call *strongly uninvadable strategies* (SUS). This particular fact, allows obtain more stability criteria than the asymmetric case. In this case, the replicator dynamics evolves in a space of signed measures. This allows us to study stability criteria for the replicator dynamics with respect to different topologies and metrics on a space of probability measures.

The conclusions iv) is valid for pure NES and in terms of the total variation norm $\|\cdot\|$. Let r be any metric on the set of probability measure $\mathbb{P}(A)$, and let r-SUS be a SUS in terms of r (see Definition 4.3.) This is a important point, a SUS is a strategy with dominance in a certain subset of strategy set. The "size" of the subset is determined by the metric r.

For games in the symmetric case we conclude the follow:

vi) For any metric r , if μ^* is a r-SUS, then μ is a NES (Proposition 4.6.)

vii) If μ^* is a stable point for the replicator dynamics, then μ^* is a NES (Proposition 4.14.)

$viii$) For any metric r, if μ^* is a r-SUS, then μ^* is a stable point for the replicator dynamics (Theorems 4.8,4.9,4.10.)

iv) Let $\mathcal{C}$ and $\mathcal{S}$ be the set of the critical and stables points of the replicator dynamics, respectively. Let $\mathcal{N}$ be the set of NESs and $r-\mathcal{SUS}$ the set of SUSs. Then we have the follows contentions (Theorem 4.15)

$$r-\mathcal{SUS} \quad \subset \quad \mathcal{S} \quad \subset \quad \mathcal{N} \quad \subset \quad \mathcal{C}.$$

x) We also analyze the implications between the different concepts of stability in diagram (4.18).

xi The replicator dynamics in a measure space can be to approximated by a sequence of dynamical systems on finite spaces (Theorems 4.19, 4.24.)

xii) The replicator dynamics in a measure space can be to approximated by a sequence of measure-valued Markov processes (Theorem 5.4.)

As future work for evolutionary games we consider several questions.

a) In symmetric evolutionary games with strategies in the space of measures, there are stability conditions with different metrics and topologies. Are these conditions satisfied in the asymmetric case?

b) It would be interesting to investigate if the replicator dynamics with continuous strategies in the asymmetric case can be approximated, in some sense, by games with discrete strategies. (This is true for the symmetric case; see section 4.7.)

c) Sandholm [89] establishes an important relation between the potential games and evolutionari dynamics for games with finite set of strategies. Under some conditions, the potential of a normal form game is a Lyapunov function for the evolutionary dynamics. Cheung [24] make an extension of this results for symmetric games with strategies in the space of measures. This result are true for the asymmetric case?

d) For normal form games with finte set of strategies, with the replicator dynamics we can give a geometric characterization of the set of Nash equilibria; see Harsanyi [44], Hofbauer and Sigmund [50], Ritzberger [83]. Is this geometric characterization true for games with strategies in the space of measure?.

e) When the set of pure strategies is finite, Cressman [27] shows that under some conditions the stability of monotone selection dynamics is locally determined by the replicator dynamics. Is this true for games with strategies on the space $\mathbb{P}(A)$ of probability measures?

f) Another important issue would be to obtain a stability theorem for several evolutionary dynamics of games with continuous strategies and analyze their relation with the replicator dynamics. See Hofbauer and Sigmund [51] (Theorem 14) for games with a finite strategy set A.

g) We considered the replicator dynamics as a limit of a sequence of measure-valued Markov process. Do we can have numerical approximation for this measure-valued Markov processes? This is an important issue for the application of this theory.

h) When the set of pure strategies is finite, there are several evolutionary dynamics that can be saw as a limit of a sequence of measure-valued Markov process. Is it true for for games whit strategies in the space of measure?. An finally, Do The replicator dynamics in the asymmetric case can be also be approximated by a sequence of stochastic processes?

A Appendix

A.1 Technical results for Theorem 3.12

We prove the inequality (3.30) under the hypothesis of Theorem 3.12. Let $\eta^1 = (1-\alpha_1)\mu,\ \eta^2 = \alpha_1(1-\alpha_2)\nu,\ \eta^3 = \alpha_1\alpha_2\kappa$, thus

$$\eta := \eta^1 + \eta^2 + \eta^3.$$

Also note that, for every i in I, $\eta_i^1 = (1-\alpha_1)\mu_i,\ \eta_i^2 = \alpha_1(1-\alpha_2)\nu_i$ and $\eta_i^3 = \alpha_1\alpha_2\kappa_i$. Then

$$\eta = (\eta_1, .., \eta_n) = (\eta_1^1 + \eta_1^2 + \eta_1^3, ..., \eta_n^1 + \eta_n^2 + \eta_n^3).$$

Since μ^* is a SUP in the set $\mathcal{C}$, $J_1(\mu_1^*, \eta_{-1}) > J_1(\eta_1, \eta_{-1})$. Then using the notation in (2.3) we have the following implications

$$\mathcal{I}_{(\mu_1,\eta_2,\eta_3,\ldots,\eta_n)}U_1 > \mathcal{I}_{(\eta_1,\eta_2,\eta_3,\ldots,\eta_n)}U_1$$

$\Rightarrow$

$$\begin{aligned}
\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 \quad + \quad & \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1 \\
> \quad & \mathcal{I}_{(\eta_1^1,\eta_2^1\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1 \\
& + \mathcal{I}_{(\eta_1^2,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^2,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^2,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1 \\
& + \mathcal{I}_{(\eta_1^3,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^3,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^3,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
& \mathcal{I}_{(\mu_1,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1 \\
& \quad > \quad (1-\alpha_1)\Big[\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1\Big] \\
& \quad + \alpha_1(1-\alpha_2)\Big[\mathcal{I}_{(\nu_1,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\nu_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\nu_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1\Big] \\
& \quad + \alpha_1\alpha_2\Big[\mathcal{I}_{(\kappa_1,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\kappa_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\kappa_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1\Big]
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1 \\
&\quad > \quad (1-\alpha_2)\Big[\mathcal{I}_{(\nu_1,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\nu_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\nu_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1\Big] \\
&\qquad + \alpha_2\Big[\mathcal{I}_{(\kappa_1,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\kappa_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\kappa_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1\Big]
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3^1,\ldots,\eta_n^1)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^1,\eta_3^2,\ldots,\eta_n^1)}U_1 + \ldots + \mathcal{I}_{(\mu_1,\eta_2^1,\eta_3^3,\ldots,\eta_n^3)}U_1 + \ldots \\
&+\mathcal{I}_{(\mu_1,\eta_2^2,\eta_3^1,\ldots,\eta_n^1)}U_1 + \ldots + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3^2,\ldots,\eta_n^2)}U_1 + \ldots + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3^3,\ldots,\eta_n^3)}U_1 + \ldots \\
&+\mathcal{I}_{(\mu_1,\eta_2^3,\eta_3^1,\ldots,\eta_n^1)}U_1 + \ldots + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3^3,\ldots,\eta_n^3)}U_1 \\
&\quad > \quad (1-\alpha_2)\Big[\mathcal{I}_{(\nu_1,\eta_2^1,\eta_3^1,\ldots,\eta_n^1)}U_1 + \ldots + \mathcal{I}_{(\nu_1,\eta_2^1,\eta_3^3,\ldots,\eta_3^3)}U_1 \\
&\qquad\qquad + \mathcal{I}_{(\nu_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\nu_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1\Big] \\
&\quad + \alpha_2\Big[\mathcal{I}_{(\kappa_1,\eta_2^1,\eta_3^1,\ldots,\eta_n^1)}U_1 + \ldots + \mathcal{I}_{(\kappa_1,\eta_2^1,\eta_3^3,\ldots,\eta_n^3)}U_1 \\
&\qquad\qquad + \mathcal{I}_{(\kappa_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\kappa_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1\Big]
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3^1,\ldots,\eta_n^1)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3^2,\ldots,\eta_n^2)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3^3,\ldots,\eta_n^3)}U_1 \\
&\qquad > \ (1-\alpha_2)\mathcal{I}_{(\nu_1,\eta_2^1,\eta_3^1,\ldots,\eta_n^1)}U_1 + \alpha_2\mathcal{I}_{(\kappa_1,\eta_2^1,\eta_3^1,\ldots,\eta_n^1)}U_1 + O(\alpha_1)
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&(1-\alpha_1)^{n-1}\mathcal{I}_{(\mu_1,\mu_2,\mu_3,\ldots,\mu_n)}U_1 \\
&\quad + \alpha_2^{n-1}(1-\alpha_1)^{n-1}\mathcal{I}_{(\mu_1,\nu_2,\nu_3,\ldots,\nu_n)}U_1 + \alpha_2^{n-1}\alpha_1^{n-1}\mathcal{I}_{(\mu_1,\kappa_2,\kappa_3,\ldots,\kappa_n^3)}U_1 \\
&\qquad > (1-\alpha_2)(1-\alpha_1)^{n-1}\mathcal{I}_{(\nu_1,\mu_2,\mu_3,\ldots,\mu_n)}U_1 \\
&\qquad\quad + \alpha_2(1-\alpha_1)^{n-1}\mathcal{I}_{(\kappa_1,\mu_2,\mu_3,\ldots,\mu_n)}U_1 + O(\alpha_1)
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&(1-\alpha_2)(1-\alpha_1)^{n-1}\mathcal{I}_{(\mu_1,\mu_2,\mu_3,\ldots,\mu_n)}U_1 \\
&\quad + \alpha_2^{n-1}(1-\alpha_1)^{n-1}\mathcal{I}_{(\mu_1,\nu_2,\nu_3,\ldots,\nu_n)}U_1 + \alpha_2^{n-1}\alpha_1^{n-1}\mathcal{I}_{(\mu_1,\kappa_2,\kappa_3,\ldots,\kappa_n^3)}U_1 \\
&\qquad > (1-\alpha_2)(1-\alpha_1)^{n-1}\mathcal{I}_{(\nu_1,\mu_2,\mu_3,\ldots,\mu_n)}U_1 \\
&\qquad\quad - \alpha_2(1-\alpha_1)^{n-1}\Big[\mathcal{I}_{(\mu_1,\mu_2,\mu_3,\ldots,\mu_n)}U_1 - \mathcal{I}_{(\kappa_1,\mu_2,\mu_3,\ldots,\mu_n)}U_1\Big] + O(\alpha_1),
\end{aligned}$$

and (3.30) follows. The inequality (3.32) is obtain Similarly.

A.2 Technical issues for metrics on $\mathbb{P}(A)$

Proposition A.1. *Let (A, r) be a separable metric space. Then the Prokhorov metric r_p and the bounded Lipschitz metric r_{bl} metrize the weak convergence, i.e., for any sequence $\{\mu_n\} \subset \mathbb{P}(A)$, the following statements are equivalent;*

i) μ_n *converges in the weak topology,*

ii) $r_p(\mu_n, \mu) \to 0$,

iii) $r_{bl}(\mu_n, \mu) \to 0$.

Moreover, for any μ and ν in $\mathbb{P}(A)$

$$\frac{1}{3}[r_p(\mu, \nu)]^2 \leq r_{bl}(\mu, \nu) \leq 2r_p(\mu, \nu).$$

Proof. See Shiryaev [94] chapter 3. □

Proposition A.2. *Let (A, r) be a Polish space and $1 \leq p < \infty$. The L_p-Wasserstein metric r_{w_p} metrizes the weak convergence on $\mathbb{P}_p(A)$, i.e., for any sequence $\{\mu_n\} \subset \mathbb{P}_p(A)$ and $\{\mu\} \subset \mathbb{P}(A)$, the following conditions are equivalents*

i) μ_n *converges in the weak topology,*

ii) $r_{w_p}(\mu_n, \mu) \to 0$.

Moreover, if A is bounded, then the L_p-Wasserstein metric r_{w_p}, the Prokhorov metric r_p , the bounded Lipschitz metric r_{bl} and the Kantorovich-Rubinstein metric r_{kr} all metrize the weak convergence of probability measures in $\mathbb{P}(A)$. Moreover, if $p = 1$ then

$$\frac{1}{3}[r_p(\mu, \nu)]^2 \leq r_{bl}(\mu, \nu) \leq r_{kr}(\mu, \nu) = r_w(\mu, \nu).$$

Proof. See Shiryaev [94] chapter 3, and Givens and Shortt [42]. □

Proposition A.3. *Let A be a separable metric space. Let μ and ν in $\mathbb{P}(A)$, with $\nu << \mu$. Then*

$$\|\mu - \nu\| \leq 2[K(\mu, \nu)]^{\frac{1}{2}}.$$

Moreover, if A is a bounded (with diameter $C > 0$) Polish space, then

$$r_w(\mu, \nu) \leq C\|\mu - \nu\| \leq 2C[K(\mu, \nu)]^{\frac{1}{2}}.$$

Proof. See Reiss [81] chapter 3, and Villani [101] chapter 6. □

Proposition A.4. *Let (A, r) a separable metric space and $1 \leq p < \infty$. If μ and ν in $\mathbb{P}(A)$, then*

$$r_{w_p}(\mu, \nu) \leq 2^{\frac{1}{q}} \left[\int_A [r(a, a_0)]^p |\mu - \nu|(dx) \right]^{\frac{1}{p}}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

In particular, if A is bounded with diameter $C > 0$, then

$$r_w(\mu, \nu) \leq C \|\mu - \nu\|.$$

Proof. Villani [101] chapter 6. □

A.3 Proof of Lemmas 4.20, 4.24, 4.25

A.3.1 Proof of Lemma 4.20

We have the following inequalities

$$\begin{aligned}
\frac{d\|\mu(t)\|}{dt} &= \frac{d}{dt} \sup_{\|f\| \leq 1} \left| \int_A f(a)\mu(t, da) \right| \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\sup_{\|f\| \leq 1} \left| \int_A f(a)\mu(t+\epsilon, da) \right| - \sup_{\|f\| \leq 1} \left| \int_A f(a)\mu(t, da) \right| \right] \\
&\leq \lim_{\epsilon \to 0} \sup_{\|f\| \leq 1} \left| \frac{1}{\epsilon} \left[\int_A f(a)\mu(t+\epsilon, da) - \int_A f(a)\mu(t, da) \right] \right| \\
&= \|\mu'(t)\|.
\end{aligned}$$

A.3.2 Proof of Lemma 4.24

We have the following inequalities

$$\begin{aligned}
\frac{d\|\mu(t)\|_{kr}}{dt} &= \frac{d}{dt}|\mu(t, A)| + \frac{d}{dt} \sup_{\substack{\|f\|_L \leq 1 \\ f(a_0)=0}} \int_A f(a)\mu(t, da) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[|\mu(t+\epsilon, A)| - |\mu(t, A)| \Big] \\
&\quad + \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\sup_{\substack{\|f\|_L \leq 1 \\ f(a_0)=0}} \int_A f(a)\mu(t+\epsilon, da) - \sup_{\substack{\|f\|_L \leq 1 \\ f(a_0)=0}} \int_A f(a)\mu(t, da) \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{\epsilon\to 0}\frac{1}{\epsilon}|\mu(t+\epsilon,A)-\mu(t,A)| \\
&\quad +\lim_{\epsilon\to 0}\sup_{\substack{\|f\|_L\leq 1\\ f(a_0)=0}}\frac{1}{\epsilon}\left[\int_A f(a)\mu(t+\epsilon,da)-\int_A f(a)\mu(t,da)\right] \\
&= \|\mu'(t)\|_{kr} \quad \text{(by Remark 4.23).}
\end{aligned}
$$

A.3.3 Proof of Lemma 4.25

For any a, b in A let

$$
\|U(a,\cdot)\|_L := \sup_{c,d\in A}\frac{|U(a,c)-U(a,d)|}{\vartheta^*((a,c),(a,d))} \leq \|U\|_L, \quad \text{and}
$$

$$
\|U(\cdot,b)\|_L := \sup_{c,d\in A}\frac{|U(c,b)-U(d,b)|}{\vartheta^*((c,b),(d,b))} \leq \|U\|_L.
$$

Then

$$
\begin{aligned}
&|J(\mu,\mu)-J(\nu,\nu)| \\
&\leq \left|\int_A\int_A U(a,b)\mu(da)\mu(db)-\int_A\int_A U(a,b)\nu(da)\mu(db)\right| \\
&\quad +\left|\int_A\int_A U(a,b)\mu(db)\nu(db)-\int_A\int_A U(a,b)\nu(db)\nu(da)\right| \\
&= \left|\|U(a,\cdot)\|_L\int_A\int_A\frac{U(a,b)}{\|U(a,\cdot)\|_L}[\mu-\nu](da)\mu(db)\right| \\
&\quad +\left|\|U(\cdot,b)\|_L\int_A\int_A\frac{U(a,b)}{\|U(\cdot,b)\|_L}[\mu-\nu](db)\nu(da)\right| \\
&\leq 2\|U\|_L\|\mu-\nu\|_{kr}. \qquad \text{(A.1)}
\end{aligned}
$$

Similary

$$
|J(a,\mu)-J(a,\nu)| \leq \|U\|_L\|\mu-\nu\|_{kr}. \qquad \text{(A.2)}
$$

Using (4.31), (A.1) and (A.2) we have

$$
\begin{aligned}
\|F(\mu)-F(\nu)\|_{kr} &= \sup_{\substack{\|f\|_L\leq 1\\ f(a_0)=0}}\int_A f(a)[F(\mu)-F(\nu)](da) \\
&\leq \sup_{\substack{\|f\|_L\leq 1\\ f(a_0)=0}}\int_A f(a)|J(a,\mu)|[\mu-\nu](da)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_A f(a)|J(a,\mu) - J(a,\nu)|\nu(da) \\
&+ \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_A f(a)|J(\mu,\mu)|[\mu-\nu](da) \\
&+ \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_A f(a)|J(\mu,\mu) - J(\nu,\nu)|\nu(da) \\
\le\ & \|U\|\|\mu-\nu\|_{kr} + \|U\|_L\|\mu-\nu\|_{kr} \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_A f(a)\nu(da) \\
&+ \|U\|\|\mu-\nu\|_{kr} + 2\|U\|_L\|\mu-\nu\|_{kr} \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_A f(a)\nu(da) \\
\le\ & \Big[2\|U\| + 3C\|U\|_L\Big]\|\mu-\nu\|_{kr}.
\end{aligned}
$$

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