CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL

Unidad Zacatenco

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Contribuciones al Estudio de las Ecuaciones Pseudo-diferenciales sobre Espacios Ultramétricos

Tesis presentada

Por

Msc. **Víctor Antonio Aguilar Arteaga**

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Director de la Tesis:

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Contributions to the Study of the Pseudodifferential Equations over Ultrametric Spaces

A thesis presented
by

Msc. **Víctor Antonio Aguilar Arteaga**

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Abstract

In this thesis we develop a Cauchy problem for a parabolic type equation on different ultrametric groups which appear as a completion of $\mathbb{Q}$. First, for each finite set S of prime numbers there exists a unique completion $\mathbb{Q}_S$ of $\mathbb{Q}$, which is a second countable, locally compact and totally disconnected topological ring. This topological ring has a natural ultrametric, 'which its associated tree has bounded ramification', that allows to define an additive invariant positive selfadjoint pseudodifferential unbounded operator D^α and to study an abstract heat equation on the Hilbert space $L^2(\mathbb{Q}_S)$. The fundamental solution of this equation is a normal transition function of a Markov process on $\mathbb{Q}_S$.

Later in this work, a class of additive invariant positive selfadjoint pseudodifferential unbounded operators on $L^2(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the ring of finite adéles of the rational numbers, is considered to state a Cauchy problem of parabolic–type equations. These operators come from a set of additive invariant non-Archimedean metrics on $\mathbb{A}_f$. The fundamental solutions of these parabolic equations determines normal transition functions of Markov processes on $\mathbb{A}_f$. Using the fractional Laplacian on the Archimedean place, $\mathbb{R}$, a class of parabolic–type equations on the complete adèle ring, $\mathbb{A}$, is obtained.

The techniques developed here provides a general framework for these kind of problems on different locally compact abelian groups.

This dissertation is based on the the following articles:

- A Heat Equation on Some Adic Completions of $\mathbb{Q}$ and Ultrametric Analysis.
  p–Adic Numbers, Ultrametric Analysis and Applications **9**, no. 3, 165–182, 2017.
- Pseudodifferential Operators and Markov Processes on Adèles.
  p–Adic Numbers, Ultrametric Analysis and Applications **11**, no. 2, 89–113, 2019.

Resumen

En esta tesis se desarrolla el problema de Cauchy para una ecuación de tipo parabólico sobre diferentes grupos ultramétricos que son una completación de $\mathbb{Q}$. Primero, para cada conjunto finito de números primos S existe una única completación $\mathbb{Q}_S$ de $\mathbb{Q}$, que es un anillo topológico segundo contable, localmente compacto y totalmente disconexo. Este anillo topológico tiene una ultramétrica natural, 'asociada a un árbol de ramificación acotada', que permite definir un operador pseudodiferencial no acotado, positivo y autoadjunto D^{α} para estudiar una ecuación abstracta del calor en el espacio de Hilbert $L^2(\mathbb{Q}_S)$. La solución fundamental de esta ecuación es una función de probabilidades de transición de un proceso de Markov en $\mathbb{Q}_S$.

Más adelante, en este trabajo, se considera una clase de operadores pseudodiferenciales no acotados, positivos y autoadjuntos en $L^2(\mathbb{A}_f)$, donde $\mathbb{A}_f$ es el anillo de adeles de los números racionales, para establecer un problema de Cauchy para ecuaciones de tipo parabólico. Estos operadores provienen de un conjunto de métricas no arquimedianas e invariantes por adición definidas en $\mathbb{A}_f$. La soluciones fundamentales de estas ecuaciones parabólicas determinan funciones de probabilidades de transición de procesos de Markov en $\mathbb{A}_f$. Usando el Laplaciano fraccionario en el lugar arquimediano, $\mathbb{R}$, se obtiene una clase de operadores de tipo parabólico en el anillo completo de adeles $\mathbb{A}$.

Las técnicas aquí desarrolladas proporcionan un marco general para este tipo de problemas en diferentes grupos abelianos localmente compactos.

Esta disertación se basa en los siguientes artículos:

- A Heat Equation on Some Adic Completions of $\mathbb{Q}$ and Ultrametric Analysis.
  p–Adic Numbers, Ultrametric Analysis and Applications **9**, no. 3, 165–182, 2017.
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Contents

Overview

Many of the physical phenomena, whether in fluid dynamics, electricity, magnetism, mechanics, optics or heat flow, can be described using partial differential equations and the associated dynamical systems and stochastic processes. We can then argue that such equations are fundamental in Mathematical Physics and that their understanding and computational simulations are of great importance for the development of science and technology.

Through history the mathematical techniques used by physicists have changed according to their needs. In recent years the p–adic analysis has received a lot of attention and the ring of the p–adic integers, $\mathbb{Z}_p$, and the field of the $p-$adic numbers, $\mathbb{Q}_p$, both have been used in some models to the modern physics, giving interesting results and expanding the research on differential equations on $\mathbb{Q}_p^n$ (see e.g., [63], [41] and [73] and the references therein). The field of p-adic numbers, $\mathbb{Q}_p$, where p is a fixed prime number, extends the arithmetic of rational numbers in a different way than real numbers and complex numbers do. This extension is obtained from a different interpretation of the concept of absolute value, $|\cdot|_p$, which allows to define an ultrametric over $\mathbb{Q}$. Intuitively, two integers numbers are close if their difference is divisible by a high power of p, the bigger the power the closer the numbers are.

The investigations of ultrametic spaces in physics have been motivated mainly by two ideas. The first comes from particle physics, in this regard, Roger Penrose mentions (see [22], Ch. 7): "The view of space-time as forming a continuum would imply that a continuous nature would persist, no matter how much a system is magnified. But it is not all clear that continuous description are really appropriate on a scale small enough that continuum phenomena become important. For example, at a scale of 10^{-13} cm (approximately the radius of an elementary particle), the mere attempt at localization of the position of a particle to that accuracy will, as a consequence of the uncertainly principle, imply the probable ocurrence of a very large momentum, with the implication that new particles are created, some of wich may be indistinguishable from the original particle. Thus the concept of "position"for the original particle becomes oscured. More alarming, moreover, is the picture presented if we allow ourselves to discuss phenomena at a dimension of the order of 10^{-33} cm. At such a dimension, the quantum fluctuations in the curvature of space-time (if both present-day quantum theory and gravitation theory can be accurately extrapoled to this degree) would be large enough to produce alterations in topology. Thus, the view of space-time at this dimension would be some kind of chaotic linear superposition on different topologies – a picture in no way resembling a smooth manifold." The first conjecture is due to I. Volovich, he surmised that in smaller distances than Planck's length, spacetime has a non-Archimedean p–adic structure (see e.g., [64], [65] and [66] and the references therein). In 1989 Yuri Manin conjectured that, on the fundamental level our world is neither real, nor p–adic, it is adelic (see [44]). In 1999, M. V. Altaisky and B. G. Sidharth affirmed the following regarding the structure of spacetime (see [6]): "The situation is much like the structure of $\mathbb{Q}_p$, but is not completely identical to it. There is also the question of how distant galaxies of our present Universe could be related to a non-Archimedean p–adic toy model. The answer may be as follows. The present state of the Universe is a result of expansion which has taken place after the Big Bang. Before the Big Bang it might have been only a network of relations between some primary objects emerged from the primary One. Then, due to multiplicative processes the number of the objects (particles) increased greatly, but some of the relations between them were inherited and manifest themselves even in large scale structures. The distance between different objects, even between galaxies, may therefore be measured not only by travelling light waves, but also by the level of their common ancestor in the evolution process." In this work, they propose to abstract the general characteristics of $\mathbb{Q}_p$ to propose a model of spacetime.

The second idea comes from statistical physics, it argues that the not exponential nature of the relaxation processes in glasses, macromolecules and proteins, is a consequence of the hierarchical structure of the state space, which can be connected with p-adic structures (see e.g., [7], [12], [13], [14], [32] y [54] and the references therein).

In addition, ultrametric spaces have been used in finance, particularly in the study of financial markets (see [18], [23], [24], [45], [46], [48], and the references therein), in data analysis (see [49], [50] and the references therein) and in taxonomy (see [42], [9] and the references therein).

One of the subjects that has attracted much of attention is the theory of Markov semigroups and pseudodifferential operators over the n–dimensional p–adic space $\mathbb{Q}_p^n$ or more general local fields, has been deeply explored by several authors (see e.g., [1, 5, 41, 40, 63, 68, 70, 73, 26] and the references therein). In particular, several analogous heat equations on $\mathbb{Q}_p^n$ are now well understood: the fundamental solutions of these equations give rise to transition functions of Markov processes on $\mathbb{Q}_p^n$, which are non–Archimedean counterparts to the classical Brownian motion. Additionally, the theory of stochastic processes and pseudodifferential equations on more general second countable locally compact topological groups has also been studied intensively during the last thirty years (e.g. [10, 31, 51, 56, 69]). In particular, the research of Markov semigroups and pseudodifferential operators over the ring of adèles $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ has also attracted considerably attention (see [20, 36, 38, 40, 44, 61, 62, 68, 71, 72] and the references therein).

This thesis contributes to the study of the pseudodifferential equations and stochastic processes over ultrametric spaces and locally compact abelian groups.

The first part of this work deals with the following problem: given any fixed finite set S of distinct prime numbers, the direct product ring $\mathbb{Q}_S = \prod_{p\in S}\mathbb{Q}_p$ is a second countable, locally compact, totally disconnected, commutative, topological ring which as well is a completion of $\mathbb{Q}$ with respect to non-Archimedean metric. This non-Archimedean metric is not releated with the maximum non–Archimedean metric on $\mathbb{Q}_S$ but it allows to state the hole analytical properties of the polyadic ring $\mathbb{Q}_S$. The ring $\mathbb{Q}_S$ contains, as a maximal compact and open subring, the direct product ring $\mathbb{Z}_S = \prod_{p\in S}\mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p–adic integers. As a topological group, it has a Haar measure $d\mu$ normalised to be a probability measure on $\mathbb{Z}_S$, it is selfdual in the sense of Pontryagin and there exists an additive character $\chi(\cdot)$ such that $x \mapsto \chi(x\cdot)$ gives an explicit isomorphism. Using this identification, the subgroup $\mathbb{Z}_S$ coincides with its own annihilator.

Let us briefly describe the ultrametric of $\mathbb{Q}_S$: let $\psi(n)$ be the second Chebyshev function (see Section 2.1.2). This function determines a "symmetric" filtration of $\mathbb{Q}_S$ by open and closed subgroups:

$$e^{\psi(m)}\mathbb{Z}_S \subset e^{\psi(n)}\mathbb{Z}_S \subset \mathbb{Z}_S \subset e^{\psi(-n)}\mathbb{Z}_S \subset e^{\psi(-m)}\mathbb{Z}_S \quad (m \geq n \geq 1).$$

There is a unique additive invariant ultrametric d on $\mathbb{Q}_S$ such that it has the filtration above as the set of balls centred at zero and the Haar measure of any ball is equal to its radius.

The ultrametric d leads to define a pseudodifferential operator D^α on the Hilbert space $L^2(\mathbb{Q}_S)$. The operator $-D^\alpha$ is a positive selfadjoint unbounded operator and it allows to state an abstract Cauchy problem on $L^2(\mathbb{Q}_S)$ for the classical homogeneous heat equation. This problem is well posed and the normalization property of the ultrametric d allows to give classical bounds of the heat kernel $Z(x,t)$, using properties of the Archimedean Gamma function, and to find and explicit solution of this problem.

The main result on the solution of the Cauchy problem reads as follows (see Section 2.3 for details)

Theorem 2.3.9: If f is a complex valued square integrable function on $\mathbb{Q}_S$, which belongs to the domain of $-D^\alpha$, the Cauchy problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D^\alpha u(x,t) = 0, \; x \in \mathbb{Q}_S, \; t \geq 0, \\ u(x,0) = f(x), \end{cases}$$

has a classical solution $u(x,t)$ determined by the convolution of f with the heat kernel $Z(x,t)$. In addition, $Z(x,t)$ is the transition density of a time and space homogeneous Markov process which is bounded, right–continuous and has no discontinuities other than jumps.

The second part of this manuscript deals with the following problem: the study of a certain class of parabolic-type pseudodifferential equations and their associated Markov stochastic processes on the complete adèle group $\mathbb{A}$ of the rational numbers $\mathbb{Q}$. This locally compact topological ring can be factorised as $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$, where $\mathbb{A}_f$ is the totally disconnected part of $\mathbb{A}$ and $\mathbb{R}$ its connected component at the identity.

Let us briefly describe our construction. The ring $\mathbb{A}_f$ contains, as a maximal compact and open subring, the complete direct product $\prod_{p\in S}\mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p–adic integers and $\mathbb{P}$ denotes the set of all prime numbers. By the Chinese remainder theorem, the product $\prod_{p\in S}\mathbb{Z}_p$ is isomorphic to the profinite completion $\widehat{\mathbb{Z}}$ of the integers. Starting with $\widehat{\mathbb{Z}}$, it is possible to recover $\mathbb{A}_f$ as the inductive limit

$$\mathbb{A}_f = \varinjlim_{n\in\mathbb{N}} \frac{1}{n}\widehat{\mathbb{Z}},$$

which can also be seen to be the ring of fractions of $\widehat{\mathbb{Z}}$ with respect to the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $(e^{\rho(n)})_{n=0}^{\infty}$ be a strictly increasing sequence of natural numbers, beginning with one, totally order by division and cofinal with the natural numbers. In other words $e^{\rho(0)} = 1$, $e^{\rho(n)} < e^{\rho(n+1)}$, $e^{\rho(n+1)}$ is divisible by $e^{\rho(n)}$, and given any positive integer n, there exists an l_n such that $e^{\rho(l_n)}$ is divisible by n. This sequence determines a "symmetric" filtration of $\mathbb{A}_f$ by open and compact subgroups:

$$\{0\} \subset e^{\rho(m)}\widehat{\mathbb{Z}} \subset e^{\rho(n)}\widehat{\mathbb{Z}} \subset \widehat{\mathbb{Z}} \subset e^{-\rho(n)}\widehat{\mathbb{Z}} \subset e^{-\rho(m)}\widehat{\mathbb{Z}} \subset \mathbb{A}_f \quad (m \geq n \geq 1).$$

The intersection of all these subgroups is the trival group and their union is $\mathbb{A}_f$. There is a unique additive invariant ultrametric d_ρ on $\mathbb{A}_f$ such that it has the filtration above as the set of balls centred at zero and such that the Haar measure of any ball is equal to its radius. It follows that d_ρ is an additive invariant ultrametric which is also invariant under multiplications by units of the ring $\widehat{\mathbb{Z}}$. Any of these ultrametrics portrays $\mathbb{A}_f$ as polyadic ring ([3, 26, 34, 37]).

The ultrametric d_ρ leads to define, for each $\alpha > 0$, a pseudodifferential operator D_ρ^α on the Hilbert space $L^2(\mathbb{A}_f)$. The operator $-D_\rho^\alpha$ is a positive selfadjoint unbounded operator and allows us to state an abstract Cauchy problem on $L^2(\mathbb{A}_f)$, analogue to the classical homogeneous heat equation. This problem is well–posed and the values chosen for the ultrametric d_ρ in giving some bounds of the heat kernel $Z_\rho^\alpha(x,t)$ and finding an explicit solution of this problem.

To conclude, given $0 < \beta \leq 2$ and the fractional Laplacian, D_∞^β, in $L^2(\mathbb{R})$, a positive self-adjoint pseudodifferential operator $D_\rho^{\alpha,\beta} = D_\rho^\alpha + D_\infty^\beta$ on $L^2(\mathbb{A})$, is defined. By construction, any of these operators are invariant under translations. The following theorem encloses the results of this writing.

Theorem 3.3.3: If f is any complex valued square integrable function on $\mathrm{Dom}(D_\rho^{\alpha,\beta})$, then the Cauchy problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D_\rho^{\alpha,\beta}u(x,t) = 0, \ x \in \mathbb{A}, \ t > 0, \\ u(x,t) = f(x) \end{cases}$$

has a solution $u(x,t)$ determined by the convolution of f with the heat kernel $Z(x,t)$. Moreover, $Z(x,t)$ is the transition density of a time and space homogeneous Markov process which is bounded, right–continuous and has no discontinuities other than jumps.

The techniques developed here are different from the well known ones for $\mathbb{Q}_p^n$ since they have a more geometrical point of view. For this reason, the method carried out here applies in the general case of a selfdual, second countable, locally compact, Abelian, topological group G with a selfdual filtration $\{H_n\}_{n\in\mathbb{Z}}$ by compact and open subgroups, such that the indices $[H_n : H_{n-1}]$ are uniformly bounded. At the end of this work it is shown how to construct many examples of these groups. (See chapter 4 for details.)

It should be pointed out that numerous descriptions of pseudodifferential equations and Markov processes have been defined and elaborated on other locally compact topological groups related to $\mathbb{Q}$, such as the finite adèle ring $\mathbb{A}_f$ and the complete adèle ring $\mathbb{A}$, as documented in [20], [38], [73] and [72], among others.

In [20], Samuel Estala and Manuel Cruz introduces several rotation and additive invariant ultrametrics on the finite adèle ring $\mathbb{A}_f$ of the rational numbers $\mathbb{Q}$. With these non-Archimedean metrics at hand they define a wide class of rotation and additive invariant Markov processes on $\mathbb{A}_f$. In [72], Markov processes on the ring of adeles are constructed, as the limits of Markov chains on some countable sets consisting of subsets of the direct product of real and p-adic fields. In both articles they developed the technique presented in [4], where Albeverio and Karwowski gave a construction of p–adic valued Markov processes associated to any sequence $a_p(M)_{M\in\mathbb{Z}}$ of non-negative numbers satisfying

$$a_p(M) \geq a_p(M+1) \text{ for every } M \in \mathbb{Z}$$

and

$$\lim_{M\to\infty} a_p(M) = 0.$$

It is performed by constructing Markov chains on the set of p–adic balls of each fixed radius, and take limits of the chains as the radius gets to 0. However, in this thesis, the fundamental solutions of parabolic equations determines normal transition functions of Markov processes on $\mathbb{Q}_S$, $\mathbb{A}_f$ and $\mathbb{A}$.

In [38], A.Y. Khrennikov and Y.V. Radyno define an operator of a multiplication on almost everywhere finite measurable function $|\xi|^\alpha$:

$$M_\alpha : L^2(\mathbb{A}_f) \to L^2(\mathbb{A}_f) : \varphi(\xi) \mapsto |\xi|^\alpha \varphi(\xi),$$

where $\xi = (\xi_2, \xi_3, ..., \xi_p, ...) \in \mathbb{A}_f$, $\alpha = (\alpha_2, \alpha_3, ..., \alpha_p, ...) \in \mathbb{R}^\infty$ and

$$|\xi|^\alpha = \prod_p |\xi_p|_p^{\alpha_p}.$$

In this thesis, for $\alpha > 0$, we consider the pseudodifferential operator

$$D_\rho^\alpha : Dom(D_\rho^\alpha) \subset L^2(\mathbb{A}_f) \to L^2(\mathbb{A}_f)$$

defined by the formula

$$D_\rho^\alpha \phi(x) = \mathcal{F}^{-1}_{\xi\to x}[\|\xi\|_\rho^\alpha \mathcal{F}_{x\to\xi}[f]],$$

every ultrametric (see section 3.1.1)

$$d_\rho(x,y) = e^{-\rho(ord_\rho(x-y))}$$

induces a function given by

$$\|x\|_\rho = e^{-\rho(ord_\rho(x))} \quad (x \in \mathbb{A}_f).$$

In [61], Sergii Torba and Wilson Zuñiga introduce a metric in $\mathbb{A}_f$ as follows: they define a function

$$\|x\| := \begin{cases} \|x\|_0 & \text{if } x \in \prod_p \mathbb{Z}_p \\ \|x\|_1 & \text{if } x \notin \prod_p \mathbb{Z}_p \end{cases}$$

for arbitrary $x \in \mathbb{A}_f$, where

$$\|x\|_0 := \max_p \frac{|x_p|_p}{p}$$

and

$$\|x\|_1 := \max_p |x_p|_p.$$

So the metric in $\mathbb{A}_f$ is the function

$$\rho(x,y) := \|x-y\|, \quad x, y \in \mathbb{A}_f.$$

The range of values of the function ρ coincides with the set $\{0\} \cup \{p^j : p \text{ is prime}, j \in \mathbb{Z} \setminus \{0\}\}$. In their work, the following calculations are obtained

$$B(0,1/2) = \prod_p \mathbb{Z}_p \text{ and } vol\left(\prod_p \mathbb{Z}_p\right) = 1.$$

In this thesis, we use a set of non-Archimedean metrics on $\mathbb{A}_f$, defined in [20], that have the following properties:

- They are invariant under translations and multiplication by units. In particular, the group of translations of $\mathbb{A}_f$ acts transitively on the set of balls of the same radius.
- The maximal compact and open subring $\prod_p \mathbb{Z}_p$ coincides with the unit ball centred at zero.
- If the Haar measure on $\mathbb{A}_f$ is normalized to be a probability measure on $\prod_p \mathbb{Z}_p$, then the diameter of any ball is equal to its Haar measure.

The range of values of these metrics coincides with $(\rho(n))_{n=0}^\infty$, a sequence defined by a strictly increasing sequence of natural numbers $(e^{\rho(n)})_{n=0}^\infty$, which is totally ordered by division and cofinal with the natural numbers, and with $e^{\rho(0)} = 1$.

In both writings, a class of additive invariant positive selfadjoint pseudodifferential unbounded operators on $L^2(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the ring of finite adéles of the rational numbers, is considered to state a Cauchy problem of parabolic–type equations. These operators come from a set of non-Archimedean metrics on $\mathbb{A}_f$. The set $\{0\} \cup \{p^j : p \text{ is prime}, j \in \mathbb{Z} \setminus \{0\}\}$ cannot be defined by one of such sequences $(e^{\rho(n)})_{n=0}^\infty$, so the two investigations are different.

Chapter 1

Preliminaries

In this chapter, we review some basic elements of the theory of p–adic numbers $\mathbb{Q}_p$ as well as the relevant function spaces defined on $\mathbb{Q}_p$ and some aspects of the Fourier analysis on this space. For a comprehensive introduction to these subjects we quote [63].

1.1 The field of p–adic numbers

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers and let $\mathbb{P}$ be the set of prime numbers. Fix a prime number $p \in \mathbb{P}$. If x is any nonzero rational number, it can be written uniquely as $x = p^k \dfrac{a}{b}$, with p not dividing the product ab and $k \in \mathbb{Z}$. The function

$$|x|_p := \begin{cases} p^{-k} & \text{if } x \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

is a non–Archimedean absolute value on $\mathbb{Q}$. The field of p–adic numbers $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the distance induced by $|\cdot|_p$.

Any nonzero p–adic number x has a unique representation of the form

$$p^\gamma \sum_{i=0}^{\infty} a_i p^i,$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $a_i \in \{0, 1, \ldots, p-1\}$ and $a_0 \neq 0$. The value γ, with $\gamma(0) = +\infty$, is called the p–adic order of x. Any series of the above form converges in the topology induced by the p–adic metric.

The fractional part of a p–adic number x is defined by

$$\{x\}_p = \begin{cases} p^\gamma \sum_{i=0}^{-\gamma-1} a_i p^i & \text{if } \gamma < 0, \\ 0 & \text{if } \gamma \geq 0. \end{cases}$$

The field $\mathbb{Q}_p$ is a locally compact topological field. The unit ball

$$\mathbb{Z}_p = \{\, x \in \mathbb{Q}_p \,:\, |x|_p \leq 1 \,\}$$

is the ring of integers and the maximal compact and open subring of $\mathbb{Q}_p$. Denote by dx the Haar measure of the topological Abelian group $(\mathbb{Q}_p, +)$ normalized to be a probability measure on $\mathbb{Z}_p$.

The algebraic and topological properties of the ring $\mathbb{Z}_p$ and the field $\mathbb{Q}_p$ can be expressed, respectively, by an inductive and a projective limit

$$\mathbb{Z}_p = \varprojlim_{l \in \mathbb{N} \cup \{0\}} \mathbb{Z}_p / p^l \mathbb{Z}_p, \qquad \mathbb{Q}_p = \varinjlim_{l \in \mathbb{N} \cup \{0\}} p^{-l} \mathbb{Z}_p. \tag{1.1}$$

In addition to the limits above, to the ring $\mathbb{Z}_p$ and the field $\mathbb{Q}_p$, there correspond, respectively, an infinite rooted tree $\mathcal{T}(\mathbb{Z}_p)$ and an extended tree $\mathcal{T}(\mathbb{Q}_p)$, both trees with constant ramification index p. The endspaces of these trees are $\mathbb{Z}_p$ and $\mathbb{Q}_p \cup \{\infty\}$, respectively.

1.1.1 Bruhat-Schwartz space.

A function $\phi : \mathbb{Q}_p \longrightarrow \mathbb{C}$ is locally constant if for any $x \in \mathbb{Q}_p$, there exists an integer $\ell(x) \in \mathbb{Z}$ such that

$$\phi(x + y) = \phi(x), \quad \text{for all } y \in B_{\ell(x)},$$

where $B_{\ell(x)}$ is the closed ball with centre at zero and radius $p^{\ell(x)}$.

The set of all locally constant functions of compact support on $\mathbb{Q}_p$ forms a $\mathbb{C}$–vector space denoted by $\mathcal{D}(\mathbb{Q}_p)$. The $\mathbb{C}$–vector space $\mathcal{D}(\mathbb{Q}_p)$ is the Bruhat–Schwartz space of $\mathbb{Q}_p$ and an element $\phi \in \mathcal{D}(\mathbb{Q}_p)$ is called a Bruhat–Schwartz function (or simply a test function) on $\mathbb{Q}_p$.

If ϕ belongs to $\mathcal{D}(\mathbb{Q}_p)$ and its not zero everywhere, there exists a largest $\ell = \ell(\phi) \in \mathbb{Z}$ such that, for any $x \in \mathbb{Q}_p$, the following equality holds

$$\phi(x + y) = \phi(x), \text{ for all } y \in B_\ell.$$

This number ℓ is called the parameter of constancy of ϕ.

Denote by $\mathcal{D}_k^\ell(\mathbb{Q}_p)$ the finite dimensional vector space consisting of functions with parameter of constancy is greater or equal than ℓ and whose support is contained in B_k.

A sequence $(f_m)_{m \geq 1}$ in $\mathcal{D}(\mathbb{Q}_p)$ is a Cauchy sequence if there exist $k, \ell \in \mathbb{Z}$ and $M > 0$ such that $f_m \in \mathcal{D}_k^\ell(\mathbb{Q}_p)$ if $m \geq M$ and $(f_m)_{m \geq M}$ is a Cauchy sequence in $\mathcal{D}_k^\ell(\mathbb{Q}_p)$. That is,

$$\mathcal{D}^\ell(\mathbb{Q}_p) = \varinjlim_{k} \mathcal{D}_k^\ell(\mathbb{Q}_p) \quad \text{and} \quad \mathcal{D}(\mathbb{Q}_p) = \varinjlim_{\ell} \mathcal{D}^\ell(\mathbb{Q}_p).$$

With this topology the space $\mathcal{D}(\mathbb{Q}_p)$ is a complete locally convex topological algebra over $\mathbb{C}$. It is also a nuclear space because $\mathcal{D}^\ell(\mathbb{Q}_p)$ is the inductive limit of countable family of finite dimensional algebras and $\mathcal{D}(\mathbb{Q}_p)$ is the inductive limit of countable family of nuclear spaces $\mathcal{D}^\ell(\mathbb{Q}_p)$.

For each compact set $K \subset \mathbb{Q}_p$, let $\mathcal{D}(K) \subset \mathcal{D}(\mathbb{Q}_p)$ be the subspace of test functions whose support is contained in K. The space $\mathcal{D}(K)$ is dense in $\mathrm{C}(K)$, the space of complex–valued continuous functions on K.

1.1.2 Fourier Analysis on $\mathbb{Q}_p$.

An additive character of the field $\mathbb{Q}_p$ is defined as a continuous function $\chi : \mathbb{Q}_p \longrightarrow \mathbb{C}$ such that $\chi(x+y) = \chi(x)\chi(y)$ and $|\chi(x)| = 1$, for all $x, y \in \mathbb{Q}_p$. The function $\chi_p(x) = \exp(2\pi i\{x\}_p)$ defines a canonical additive character of $\mathbb{Q}_p$ which is trivial on $\mathbb{Z}_p$ and not trivial outside $\mathbb{Z}_p$. In fact, all characters of $\mathbb{Q}_p$ are given by $\chi_{p,\xi}(x) = \chi_p(\xi x)$ with $\xi \in \mathbb{Q}_p$.

The Fourier transform of a test function $\phi \in \mathcal{D}(\mathbb{Q}_p)$ is given by the formula

$$\mathcal{F}_p[\phi](\xi) = \widehat{\phi}(\xi) = \int_{\mathbb{Q}_p} \phi(x)\chi_p(\xi x)dx, \qquad (\xi \in \mathbb{Q}_p).$$

The Fourier transform is a continuous linear isomorphism of the space $\mathcal{D}(\mathbb{Q}_p)$ onto itself and the inversion formula holds:

$$\phi(x) = \int_{\mathbb{Q}_p} \widehat{\phi}(\xi)\chi_p(-x\xi)d\xi \qquad (\phi \in \mathcal{D}(\mathbb{Q}_p)).$$

The Parseval – Steklov equality reads as:

$$\int_{\mathbb{Q}_p} \phi(x)\overline{\psi(x)}dx = \int_{\mathbb{Q}_p} \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)}d\xi, \qquad (\phi, \psi \in \mathcal{D}(\mathbb{Q}_p)).$$

Remark 1.1.1. *From expressions (1.1) or the description of $\mathbb{Q}_p$ as collection of endpoints of the tree $\mathcal{T}(\mathbb{Q}_p)$, it follows that the Hilbert space $L^2(\mathbb{Q}_p)$ has a denumerable Hilbert base, which is an analogous of a wavelet base, and therefore it is a separable Hilbert space (see e.g. [5]).*

Remark 1.1.2. *The extended Fourier transform $\mathcal{F} : L^2(\mathbb{Q}_p) \longrightarrow L^2(\mathbb{Q}_p)$ is an isometry of Hilbert spaces and the Parseval – Steklov identity holds on $L^2(\mathbb{Q}_p)$.*

1.2 The finite adèle ring $\mathbb{A}_f$

The prevailing definition of the finite adèle ring $\mathbb{A}_f$ of the rational numbers $\mathbb{Q}$ is given by the restricted direct product of the fields $\mathbb{Q}_p$, with respect to the maximal compact and open subrings $\mathbb{Z}_p$. That is to say,

$$\mathbb{A}_f = \left\{ (x_p)_{p\in\mathbb{P}} \in \prod_{p\in\mathbb{P}} \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for all but finitely many primes } p \in \mathbb{P} \right\}.$$

The restricted direct product topology on $\mathbb{A}_f$ is described as follows. Let $S \subset \mathbb{P}$ be a finite set of prime numbers. The space of S*–adèles* of the rational numbers $\mathbb{Q}$ is the product ring

$$\mathbb{A}_S = \prod_{p\in S} \mathbb{Q}_p \times \prod_{p\notin S} \mathbb{Z}_p.$$

The ring $\mathbb{A}_S$ with the Tychonoff product topology is a second countable locally compact topological ring and contains $\prod_{p\in S} \mathbb{Z}_p$ as a maximal, compact and open subring. For each finite set of primes S, $\mathbb{A}_S$ is a subring of $\mathbb{A}_f$ and $\mathbb{A}_f = \bigcup_S \mathbb{A}_S$, where the union is taken over all finite subsets S of $\mathbb{P}$. The restricted direct product topology on the ring $\mathbb{A}_f$ is the topology of the inductive limit

$$\mathbb{A}_f = \varinjlim_{\substack{S\subset\mathbb{P}\\ |S|<\infty}} \mathbb{A}_S,$$

which essentially states that $\mathbb{A}_S$ is open in $\mathbb{A}_f$. A fundamental system of compact and open neighbourhoods of zero for the restricted direct product topology of $\mathbb{A}_f$ is given by

$$\prod_{p\in S} U_p \times \prod_{p\notin S} \mathbb{Z}_p,$$

where S is a finite subset of $\mathbb{P}$ and U_p is a compact and open subset in $\mathbb{Q}_p$ which contains $0 \in \mathbb{Q}_p$, for all $p \in S$. It follows that a base for the topology of $\mathbb{A}_f$ is the family of compact and open subsets

$$\mathcal{N} = \left\{ x + y \prod_{p\in S} \mathbb{Z}_p \mid x, y \in \mathbb{A}_f \text{ and } y \text{ invertible} \right\}.$$

In summary, $\mathbb{A}_f$, with the restricted direct product topology, is a second countable and totally disconnected locally compact topological ring. The subring $\prod_{p\in S} \mathbb{Z}_p$ is the maximal, compact and open subring of $\mathbb{A}_f$. The Haar measure $d\mu$ of $\mathbb{A}_f$ is usually normalized to be a probability measure on $\prod_{p\in S} \mathbb{Z}_p$, i.e., $d\mu = \prod_{p\in\mathbb{P}} dx_p$. The ring $\mathbb{A}_f$ is the smallest locally compact topological ring which contains each $\mathbb{Q}_p$.

There is an additive character $\chi(x)$ on the totally disconnected Abelian group $\mathbb{A}_f$, which is trivial on $\widehat{\mathbb{Z}}$ and not trivial outside $\widehat{\mathbb{Z}}$, given by

$$\chi(x) = \prod_p \chi_p(x_p), \qquad \big(x = (x_2, \dots, x_p, \dots) \in \mathbb{A}_f\big),$$

where $\chi_p(x_p)$ is the canonical character of $\mathbb{Q}_p$ which is trivial on $\mathbb{Z}_p$ and not trivial outside $\mathbb{Z}_p$. Recall that these characters are given by $\chi_p(x_p) = e^{2\pi i\{x_p\}_p}$, where $\{x_p\}_p$ is the p–adic fractional part of x_p. Since $\mathbb{Q}_p$ is selfdual, it follows that $\mathbb{A}_f$ is also selfdual.

Remark 1.2.1. *It is worth to notice that*

$$\chi(x) = e^{2\pi i\{x\}},$$

where $\{x\} = \sum_{p\in\mathbb{P}} \{x_p\}_p$ *is the unique rational number in* $[0, 1)$ *such that* $x - \{x\} \in \prod_{p\in S} \mathbb{Z}_p$.

Chapter 2

A Heat Equation on some Adic Completions of $\mathbb{Q}$ and Ultrametric Analysis

This chapter introduces the ring $\mathbb{Q}_S$ and an ultrametric on $\mathbb{Q}_S$ invariant under translations by elements of $\mathbb{Q}_S$ and under multiplication by units of $\mathbb{Z}_S$. We will describe $\mathbb{Q}_S$ as topological ring itself, that allows to define an additive invariant positive selfadjoint pseudodifferential unbounded operators D^α and to study an abstract heat equation on the Hilbert space $L^2(\mathbb{Q}_S)$. The fundamental solution of this equation is a normal transition function of a Markov process on $\mathbb{Q}_S$.

2.1 The S–adic ring of $\mathbb{Q}$

2.1.1 The adic ring $\mathbb{Q}_S$

Fix a finite subset $S \subset \mathbb{P}$ and define $\mathbb{Q}_S$ as the direct product

$$\mathbb{Q}_S = \prod_{p\in S} \mathbb{Q}_p.$$

The Tychonoff topology and the componentwise operations provide $\mathbb{Q}_S$ with a structure of a topological ring. The topological ring $\mathbb{Q}_S$ is commutative, second countable, locally compact and totally disconnected. The maximal compact and open subring of $\mathbb{Q}_S$ is the direct product ring $\mathbb{Z}_S = \prod_{p\in S} \mathbb{Z}_p$.

The additive group $(\mathbb{Q}_S, +)$ is a locally compact Abelian group and therefore it has a Haar measure $d\mu$ which can be normalized to be a probability measure on $\mathbb{Z}_S$. The measure $d\mu$ can be expressed in terms of the measures dx_p on the groups $(\mathbb{Q}_p, +)$ as the direct product measure

$$d\mu = \prod_{p\in S} dx_p.$$

With regard to characters, there is a canonical additive character χ on $\mathbb{Q}_S$, which is trivial on $\mathbb{Z}_S$ and not trivial outside $\mathbb{Z}_S$, given by

$$\chi(x) = \prod_{p \in S} \chi_p(x_p), \qquad \big(x = (x_p)_{p\in S} \in \mathbb{Q}_S\big),$$

where $\chi_p(x_p)$ is the canonical character of $\mathbb{Q}_p$. Recall that these characters are given by $\chi_p(x_p) = e^{2\pi i \{x_p\}_p}$, where $\{x_p\}_p$ is the p–adic fractional part of x_p.

For $\xi \in \mathbb{Q}_S$, the application

$$\chi_\xi(x) = \chi(\xi \cdot x) = \prod_{p \in S} \chi_p(\xi_p x_p), \qquad \big(\xi = (\xi_p)_{p\in S} \in \mathbb{Q}_S\big),$$

defines a character on $\mathbb{Q}_S$. Moreover, since $\mathbb{Q}_S$ is a direct product of some p-adic fields, any arbitrary character on $\mathbb{Q}_S$ has the form χ_ξ, for some $\xi \in \mathbb{Q}_S$. Therefore $\mathbb{Q}_S$ is a selfdual group with isomorfism given by $\xi \mapsto \chi_\xi$.

Recall that the annihilator of a compact and open subgroup H of $\mathbb{Q}_S$ is the set of characters that are trivial in H. From the expression of the characters on $\mathbb{Q}_S$, the relation

$$\mathrm{Ann}_{\mathbb{Q}_S}(B_n) = B_{-n},$$

where $\mathrm{Ann}_{\mathbb{Q}_S}(B_n)$ is the annihilator of B_n in $\mathbb{Q}_S$, holds. In particular, $\mathbb{Z}_S$ coincides with its own annihilator.

Remark 2.1.1. *For the scope of this work the relevant properties of $\mathbb{Q}_S$ are that is a topological ring which is second countable, locally compact, and totally disconnected. In addition, as a topological group, $\mathbb{Q}_S$ is selfdual.*

2.1.2 An ultrametric on $\mathbb{Q}_S$

Let us introduce an ultrametric d on $\mathbb{Q}_S$ compatible with its topology, making $(\mathbb{Q}_S, d)$ a complete ultrametric space. The set of rational numbers $\mathbb{Q}$ is diagonally embedded in $\mathbb{Q}_S$ with dense image. In this sense, the ring $\mathbb{Q}_S$ can be thought of as an S–completion of $\mathbb{Q}$ (see [16] for a description of this topology on the ring of integer numbers $\mathbb{Z}$).

We start by defining two arithmetical functions which are related to the set S and similar to the second Chebyshev and von Mangoldt functions (see e.g., [8]). For any natural number n, write

$$\Lambda(n) = \Lambda_S(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some } p \in S \text{ and integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Likewise, let $\psi(n)$ denote the function implicitly given by

$$e^{\psi(n)} = e^{\psi_S(n)} = \mathrm{lcm}\,\big\{\, p^l \leq n : p \in S,\, l \in \mathbb{N} \cup \{0\} \,\big\} \qquad (n \in \mathbb{N}).$$

These arithmetical functions are related by the equations

$$\psi(n) = \sum_{k=1}^{n} \Lambda(k) \quad \text{and} \quad e^{\psi(n)} = \prod_{k=1}^{n} e^{\Lambda(k)} \qquad (n \in \mathbb{N}).$$

For any integer number n, define

$$\psi(n) = \begin{cases} \frac{n}{|n|}\psi(|n|) & \text{if } n \neq 0, \\ 0 & \text{if } n = 0, \end{cases}$$

and

$$\Lambda(n) = \begin{cases} \Lambda(n) & \text{if } n > 0, \\ \Lambda(|n-1|) = \Lambda(|n|+1) & \text{if } n \leq 0. \end{cases}$$

The relations between these functions on the natural numbers extend to the integers in the following way: for any integers $n > m$

$$\psi(n) - \psi(m) = \sum_{k=m+1}^{n} \Lambda(k) \quad \text{and} \quad e^{\psi(n)}/e^{\psi(m)} = \prod_{k=m+1}^{n} e^{\Lambda(k)}.$$

The collection $\{e^{\psi(n)}\mathbb{Z}_S\}_{n\in\mathbb{Z}}$ of compact and open subgroups is a neighbourhood base of zero for the Tychonoff topology on $\mathbb{Q}_S$ and determines a filtration

$$\{0\} \subset \cdots \subset e^{\psi(n)}\mathbb{Z}_S \subset \cdots \subset \mathbb{Z}_S \subset \cdots \subset e^{\psi(m)}\mathbb{Z}_S \subset \cdots \subset \mathbb{Q}_S \qquad \big(n > 0 > m\big),$$

satisfying the properties:

$$\bigcap_{n\in\mathbb{Z}} e^{\psi(n)}\mathbb{Z}_S = \{0\} \quad \text{and} \quad \bigcup_{n\in\mathbb{Z}} e^{\psi(n)}\mathbb{Z}_S = \mathbb{Q}_S.$$

Remark 2.1.2. *From the properties of the above filtration, the topology of the ring $\mathbb{Q}_S$ is expressed by the inductive and projective limits*

$$\mathbb{Q}_S = \varinjlim_{n\in\mathbb{N}} e^{\psi(-n)}\mathbb{Z}_S, \quad \mathbb{Z}_S = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}_S/e^{\psi(n)}\mathbb{Z}_S.$$

In addition, we have the identity

$$e^{\psi(n)}\mathbb{Z}_S = \prod_{p\in S} p^{\mathrm{ord}_p(e^{\psi(n)})}\mathbb{Z}_p,$$

where $\mathrm{ord}_p(\cdot)$ *is the p–adic order function on $\mathbb{Q}_p$.*

For any element $x \in \mathbb{Q}_S$ define the order of x as:

$$\mathrm{ord}(x) := \begin{cases} \max\left\{\, n \,:\, x \in e^{\psi(n)}\mathbb{Z}_S \,\right\} & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

Notice that this order satisfies the following properties:

- $\operatorname{ord}(x) \in \mathbb{Z} \cup \{\infty\}$ and $\operatorname{ord}(x) = \infty$ if and only if $x = 0$,
- $\operatorname{ord}(x+y) \geq \min\{\operatorname{ord}(x), \operatorname{ord}(y)\}$ and
- it takes the values

$$S^{\mathbb{Z}} = \left\{ -p^l \,:\, l \in \mathbb{N} \text{ and } p \in S \right\} \cup \left\{ p^l - 1 \,:\, l \in \mathbb{N} \text{ and } p \in S \right\} \cup \{\infty\}.$$

The nonnegative function $d : \mathbb{Q}_S \times \mathbb{Q}_S \longrightarrow \mathbb{R}^+ \cup \{0\}$ given by

$$d(x,y) = e^{-\psi\left(\operatorname{ord}(x-y)\right)}$$

is an ultrametric on $\mathbb{Q}_S$. This ultrametric d takes values in the set $\{e^{\psi(n)}\}_{n\in\mathbb{Z}} \cup \{0\}$, any ball B_n centred at zero with radius $e^{\psi(n)}$ is precisely the subgroup

$$B_n = B(0, e^{\psi(n)}) = e^{-\psi(n)}\mathbb{Z}_S \qquad (n \in \mathbb{Z}),$$

and any sphere centred at zero and radius $e^{\psi(n)}$ is

$$S_n = S(0, e^{\psi(n)}) = B_n \backslash B_{n-1}.$$

The norm induced by this ultrametric is given by

$$\|x\| = e^{-\psi(\operatorname{ord}(x))}$$

and $\|x\| = e^{\psi(n)}$ if and only if $x \in S_n$.

It is worth to notice that the radius of any ball on $\mathbb{Q}_S$ is equal to its Haar measure:

$$\int_{B_n+y} dx = \int_{B_n} dx = \int_{e^{-\psi(n)}\mathbb{Z}_S} dx = e^{\psi(n)} \qquad (y \in \mathbb{Q}_S,\, n \in \mathbb{Z}).$$

Using this fact, the area of any sphere is given by

$$\int_{S_n+y} dx = \int_{S_n} dx = e^{\psi(n)} - e^{\psi(n-1)} \qquad (y \in \mathbb{Q}_S,\, n \in \mathbb{Z}).$$

Remark 2.1.3. *If $e^{\psi(n)} < e^{\psi(n+1)}$, B_{n+1} is strictly contained in B_n; otherwise, if $e^{\psi(n)} = e^{\psi(n+1)}$, $B_{n+1} = B_n$. This behaviour is controlled by the function $\Lambda(n)$ and there exists a unique increasing bijective function $\rho : \mathbb{Z} \longrightarrow S^{\mathbb{Z}}$, such that $e^{\psi(\rho(n))}$ is a strictly increasing function with $e^{\psi(\rho(0))} = 1$. For this reason, in the sequel, we suppose that $e^{\psi(n)} < e^{\psi(n+1)}$ for any integer number n.*

2.2 Function spaces and pseudodifferential operators on $\mathbb{Q}_S$

The relevant spaces of test functions as well as the basic facts of Fourier analysis on $\mathbb{Q}_S$ are described in this section. It also introduces a pseudodifferential operator D^α on $\mathbb{Q}_S$. The description made here follows closely the account made in [20] for the finite adèle ring of $\mathbb{Q}$. A second point of view is obtained by looking at the product structure of $\mathbb{Q}_S$.

The reader can consult these topics in the excellent books [63], [5], [35]. The theory of general topological vector spaces can be found in [57].

2.2.1 Bruhat–Schwartz test functions on $\mathbb{Q}_S$

The Bruhat–Schwartz space $\mathcal{D}(\mathbb{Q}_S)$ is the space of locally constant functions on $\mathbb{Q}_S$ with compact support. Being $\mathbb{Q}_S$ a totally disconnected space, $\mathcal{D}(\mathbb{Q}_S)$ has a natural topology which can be described by two inductive limits, given any filtration by compact and open sets in $\mathbb{Q}_S$.

Let us describe the topology of $\mathcal{D}(\mathbb{Q}_S)$ using the ultrametric d. If $\phi \in \mathcal{D}(\mathbb{Q}_S)$, there exists a smallest $\ell = \ell_\phi \in \mathbb{Z}$ such that, for every $x \in \mathbb{Q}_S$,

$$\phi(x+y) = \phi(x), \quad \text{for all } \ y \in B_\ell = e^{-\psi(\ell)}\mathbb{Z}_S.$$

This number ℓ is called the parameter of constancy of ϕ. The set of all locally constant functions on $\mathbb{Q}_S$ with common parameter of constancy $\ell \in \mathbb{Z}$ and support in B_k forms a finite dimensional complex vector space of dimension $e^{\psi(\ell)}/e^{\psi(k)}$. Denote this space by $\mathcal{D}^\ell_k(\mathbb{Q}_S)$. The topology of $\mathcal{D}(\mathbb{Q}_S)$ is expressed by the inductive limits

$$\mathcal{D}^\ell(\mathbb{Q}_S) = \varinjlim_k \mathcal{D}^\ell_k(\mathbb{Q}_S) \quad \text{and} \quad \mathcal{D}(\mathbb{Q}_S) = \varinjlim_\ell \mathcal{D}^\ell(\mathbb{Q}_S),$$

which essentially states that every finite dimensional vector space $\mathcal{D}^\ell_k(\mathbb{Q}_S)$ is open in $\mathcal{D}(\mathbb{Q}_S)$. Therefore, $\mathcal{D}(\mathbb{Q}_S)$ is a complete locally convex topological algebra over $\mathbb{C}$ and a nuclear space.

Finally, for each compact subset $K \subset \mathbb{Q}_S$, let $\mathcal{D}(K) \subset \mathcal{D}(\mathbb{Q}_S)$ be the subspace of test functions with support on a fixed compact subset K. The space $\mathcal{D}(K)$ is dense in $\mathrm{C}(K)$, the space of complex valued continuous functions on K. The space $\mathcal{D}(\mathbb{Q}_S)$ is dense in $L^2(\mathbb{Q}_S)$.

Bruhat–Schwartz test functions as a tensor product

Since the Tychonoff topology on $\mathbb{Q}_S$ is the box topology, any function $\phi \in \mathcal{D}(\mathbb{Q}_S)$ can be written as a finite linear combination of elementary functions of the form

$$\phi(x) = \prod_{p \in S} \phi_p(x_p), \qquad \big(x = (x_p)_{p\in S} \in \mathbb{Q}_S\big),$$

where each factor $\phi_p(x_p)$ belongs to the space of test functions $\mathcal{D}(\mathbb{Q}_p)$.

Moreover, from the finite group identification

$$B_l/B_k = \prod_{p\in S} B^p_{\ell_p}/B^p_{k_p},$$

where $\ell_p = \mathrm{ord}_p(e^{\psi(l)})$ and $k_p = \mathrm{ord}_p(e^{\psi(k)})$, the following identity between finite dimensional spaces holds:

$$\mathcal{D}^\ell_k(\mathbb{Q}_S) = \bigotimes_{p\in S} \mathcal{D}^{\ell_p}_{k_p}(\mathbb{Q}_p).$$

Commuting the induced limits with the tensor products, it is possible to show that $\mathcal{D}^\ell(\mathbb{Q}_S)$ corresponds to the algebraic and topological tensor product of nuclear spaces

$\{\mathcal{D}^{\ell}(\mathbb{Q}_p)\}_{p\in S}$, that is to say

$$\mathcal{D}^{\ell}(\mathbb{Q}_S) = \bigotimes_{p\in S} \mathcal{D}^{\ell_p}(\mathbb{Q}_p).$$

Furthermore, $\mathcal{D}(\mathbb{Q}_S)$ corresponds to the algebraic and topological tensor product of nuclear spaces $\{\mathcal{D}(\mathbb{Q}_p)\}_{p\in S}$, that is,

$$\mathcal{D}(\mathbb{Q}_S) \cong \bigotimes_{p\in S} \mathcal{D}(\mathbb{Q}_p).$$

2.2.2 The Fourier transform on $\mathbb{Q}_S$

The Fourier transform of $\phi \in \mathcal{D}(\mathbb{Q}_S)$ is defined by

$$\mathcal{F}[\phi](\xi) = \widehat{\phi}(\xi) = \int_{\mathbb{Q}_S} \phi(x)\chi(\xi x)dx, \qquad (\xi \in \mathbb{Q}_S).$$

In particular, the Fourier transform of any elementary function $\phi = \prod_{p\in S} \phi_p$ is given by

$$\mathcal{F}[\phi](\xi) = \prod_{p\in S} \int_{\mathbb{Q}_p} \phi_p(x_p)\chi_p(\xi_p x_p)dx_p, \qquad \left(\phi(x) = \prod_{p\in S} \phi_p(x_p),\ \xi = (\xi_p)_{p\in S} \in \mathbb{Q}_S\right).$$

The following two integrals are of main importance for the properties of the Fourier transform on the Bruhat-Shwartz space, $\mathcal{D}(\mathbb{Q}_S)$.

Lemma 2.2.1.

$$\int_{B_n} \chi(-\xi x)dx = \begin{cases} e^{\psi(n)} & \text{if } \|\xi\| \le e^{-\psi(n)}, \\ 0 & \text{if } \|\xi\| > e^{-\psi(n)}. \end{cases}$$

Proof. This follows from the equality, $\mathrm{Ann}_{\mathbb{Q}_S}(B_n) = B_{-n}$ and the fact that in any compact topological group, the integral of a nontrivial character over the group is zero. ☐

Lemma 2.2.2. *For any $n \in \mathbb{Z}$,*

$$\int_{S_n} \chi(-\xi x)dx = \begin{cases} e^{\psi(n)} - e^{\psi(n-1)} & \text{if } \|\xi\| \le e^{-\psi(n)}, \\ -e^{\psi(n-1)} & \text{if } \|\xi\| = e^{-\psi(n-1)}, \\ 0 & \text{if } \|\xi\| \ge e^{-\psi(n-2)}. \end{cases}$$

Proof. The integral can be decomposed as

$$\int_{S_n} \chi(-\xi x)dx = \int_{B_n} \chi(-\xi x)dx - \int_{B_{n-1}} \chi(-\xi x)dx,$$

and the last proposition provides the result. ☐

Remark 2.2.3. *From the well known computations of analogous integrals on $\mathbb{Q}_p$, another proof of the last lemmas can be derived directly.*

From lemmas 2.2.1 and 2.2.2, definitions of the spaces $\mathcal{D}_k^\ell(\mathbb{Q}_S)$ and Fourier transform, it follows

$$\mathcal{F} : \mathcal{D}_k^\ell(\mathbb{Q}_S) \longrightarrow \mathcal{D}_{-\ell}^{-k}(\mathbb{Q}_S).$$

Moreover, the Fourier transform $\mathcal{F}$ is a continuous linear isomorphism from $\mathcal{D}(\mathbb{Q}_S)$ into itself. It is worth to notice that this property of the Fourier transform follows directly by construction, because $\mathrm{Ann}_{\mathbb{Q}_S}(B_n) = B_{-n}$.

Remark 2.2.4. *Recall that, in the case of $\mathbb{Q}_p$, the Fourier transform $\mathcal{F}_p$ sends $\mathcal{D}_k^\ell(\mathbb{Q}_p)$ into $\mathcal{D}_{-\ell}^{-k}(\mathbb{Q}_p)$ and the identification of $\mathcal{D}_k^\ell(\mathbb{Q}_S)$ with the tensor product, $\otimes_{p\in S}\mathcal{D}_{k_p}^{\ell_p}(\mathbb{Q}_p)$, of finite dimension spaces. Therefore, $\mathcal{F} : \mathcal{D}_k^\ell(\mathbb{Q}_S) \longrightarrow \mathcal{D}_{-\ell}^{-k}(\mathbb{Q}_S)$ and the Fourier transform gives and isomorphism $\mathcal{D}(\mathbb{Q}_S) \cong \mathcal{D}(\mathbb{Q}_S)$.*

From the description of $\mathbb{Q}_S$ and $\mathbb{Z}_S$, respectively, as an inductive and a projective limit or from the description of $\mathbb{Q}_S \cup \{\infty\}$ as the endspace of a regular infinite tree it follows that the Hilbert space $L^2(\mathbb{Q}_S)$ has a numerable Hilbert base which is a counterpart of a wavelet bases. Therefore, $L^2(\mathbb{Q}_S)$ is a separable Hilbert space. In addition, the Fourier transform

$$\mathcal{F} : L^2(\mathbb{Q}_S) \longrightarrow L^2(\mathbb{Q}_S)$$

is an isometry.

Remark 2.2.5. *Notice that $L^2(\mathbb{Q}_S) \cong \bigotimes_{p\in S} L^2(\mathbb{Q}_p)$, where the tensor denotes the Hilbert tensor product, because each $L^2(\mathbb{Q}_p)$ is a separable Hilbert space, then*

$$\mathcal{F} : L^2(\mathbb{Q}_S) \longrightarrow L^2(\mathbb{Q}_S)$$

is an isometry.

2.2.3 Lizorkin space of test functions on $\mathbb{Q}_S$

Another space of test functions which is useful in the study of the heat equation is the following: Let $\Psi(\mathbb{Q}_S)$ be the space of test functions which vanish at zero

$$\Psi(\mathbb{Q}_S) = \{\, f \in \mathcal{D}(\mathbb{Q}_S) : f(0) = 0 \,\}.$$

This means that for any element $f \in \Psi(\mathbb{Q}_S)$ there exists a ball B_n with centre at zero and radius $e^{\psi(n)}$ such that $f_{|B_n} \equiv 0$. The image of $\Psi(\mathbb{Q}_S)$ under the Fourier transform is the space

$$\Phi(\mathbb{Q}_S) = \{\, g : g = \mathcal{F}[f],\ f \in \Psi(\mathbb{Q}_S) \,\} \subset \mathcal{D}(\mathbb{Q}_S)$$

called the Lizorkin space of test functions of the second kind. The space $\Phi(\mathbb{Q}_S)$ is nontrivial and, as a subspace of $\mathcal{D}(\mathbb{Q}_S)$, it has the subspace topology which makes it a complete topological vector space.

2.2.4 Pseudodifferential operators on $\mathbb{Q}_S$

For any $\alpha > 0$ consider the function $\|\cdot\|^\alpha : \mathbb{Q}_S \longrightarrow \mathbb{R}_{\geq 0}$. The pseudodifferential operator, $D^\alpha : \mathrm{Dom}(D^\alpha) \subset L^2(\mathbb{Q}_S) \longrightarrow L^2(\mathbb{Q}_S)$, defined by the formulae

$$D^\alpha f = \mathcal{F}^{-1}_{\xi\to x}[\|\xi\|^\alpha \mathcal{F}_{x\to\xi}[f]],$$

for any f in the dense domain

$$\mathrm{Dom}(D^\alpha) = \left\{ f \in L^2(\mathbb{Q}_S) : \|\xi\|^\alpha \widehat{f} \in L^2(\mathbb{Q}_S) \right\},$$

is called a pseudodifferential operator with symbol $\|\xi\|^\alpha$.

In other words, if we consider the multiplicative operator $m^\alpha : \mathrm{Dom}(m^\alpha) \subset L^2(\mathbb{Q}_S) \longrightarrow L^2(\mathbb{Q}_S)$ given by

$$m^\alpha(f)(\xi) = \|\xi\|^\alpha f(\xi),$$

with (dense) domain

$$\mathrm{Dom}(m^\alpha) = \left\{ f \in L^2(\mathbb{Q}_S) : \|\xi\|^\alpha f \in L^2(\mathbb{Q}_S) \right\},$$

the unbounded operator D^α with domain $\mathrm{Dom}(D^\alpha)$ is the unique unbounded operator such that the diagram

$$\begin{array}{ccc} L^2(\mathbb{Q}_S) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{Q}_S) \\ {\scriptstyle D^\alpha}\downarrow & & \downarrow{\scriptstyle m^\alpha} \\ L^2(\mathbb{Q}_S) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{Q}_S) \end{array} \tag{2.1}$$

commutes.

Therefore, several properties of D^α can be translated into the multiplicative operator m^α: D^α is a positive selfadjoint unbounded operator. Moreover, the commutative property of the diagram above means that D^α is diagonalized by the unitary Fourier transform $\mathcal{F}$ ($\mathcal{F}$ is surjective and preserves the inner product). Since D^α is a positive and selfadjoint operator, its spectrum is contained in $[0, \infty)$. The characteristic equation

$$D^\alpha f = \lambda f \qquad \big(f \in L^2(\mathbb{Q}_S) \setminus \{0\}\big)$$

is solved by applying it the Fourier transform and solving the resulting equation, $(m^\alpha - \lambda)\widehat{f} = 0$, or

$$(\|\xi\|^\alpha - \lambda)\widehat{f}(\xi) = 0.$$

If $\lambda \in \{e^{\alpha\psi(n)}\}_{n\in\mathbb{Z}}$, the characteristic function of the sphere S_n, Δ_{S_n}, is a solution of the characteristic equation of the multiplicative operator m^α. Otherwise, if $\lambda \notin \{e^{\alpha\psi(n)}\}_{n\in\mathbb{Z}}$, the function $\|\xi\|^\alpha - \lambda$ is bounded from below and λ is in the resolvent set of the multiplicative operator. Since the Fourier transform is unitary, the point spectrum of D^α is the set $\{e^{\alpha\psi(n)}\}_{n\in\mathbb{Z}}$, with corresponding eigenfunctions $\{\mathcal{F}^{-1}(\Delta_{S_n})\}_{n\in\mathbb{Z}}$. Finally, $\{0\}$ forms part of the spectrum as a limit point. Each eigenspace is infinite dimensional and there exist a well defined wavelet base which is also made of eigenfunctions (see [3]) .

2.3 A heat equation on $\mathbb{Q}_S$

This section presents the analysis of the homogeneous heat equation on $\mathbb{Q}_S$ related to the pseudodifferential operator D^α introduced in Section 2.2.4. This study is done by treating the heat equation as an evolution equation on the Hilbert space $L^2(\mathbb{Q}_S, d\mu)$ of square integrable complex valued functions on $\mathbb{Q}_S$. The properties of general evolution equations on Banach spaces can be found in [52] and [30].

For any function $f(x) \in \mathrm{Dom}(D^\alpha)$, the pseudodifferential equation of the form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D^\alpha u(x,t) = 0, \ x \in \mathbb{Q}_S, \ t > 0, \\ u(x,0) = f(x), \end{cases} \tag{2.2}$$

is a non–Archimedean counterpart to the Archimedean homogeneous heat equation.

In the $L^2(\mathbb{Q}_S)$ context, a function $u : \mathbb{Q}_S \times \mathbb{R} \longrightarrow \mathbb{C}$ is called a classical solution of the Cauchy problem if:

a. $u : [0, \infty) \longrightarrow L^2(\mathbb{Q}_S)$ is a continuously differentiable function,

b. $u(x,t) \in \mathrm{Dom}(D^\alpha)$, for all $t \geq 0$, (in particular $f \in \mathrm{Dom}(D^\alpha)$) and,

c. $u(x,t)$ is a solution of the initial value problem.

This problem is called here an abstract Cauchy problem and will be referred as problem (2.2). This problem is well posed and its concept of solution is well understood from the theory of semigroups of linear operators. This solution is described in the following section.

2.3.1 Semigroup of operators

From the Hille–Yoshida Theorem, to the positive selfadjoint operator $-D^\alpha$ there corresponds a strongly continuous contraction semigroup

$$\mathbf{S}(t) = \exp(-tD^\alpha) : L^2(\mathbb{Q}_S) \longrightarrow L^2(\mathbb{Q}_S) \qquad (t \geq 0),$$

with infinitesimal generator $-D^\alpha$.

It follows that $\{\mathbf{S}(t)\}_{t \geq 0}$ has the following properties

- as a function of t, $\mathbf{S}(t)$ is strongly continuous,
- for $t \geq 1$, $\mathbf{S}(t)$ is a bounded operator with operator norm less than one,
- $\mathbf{S}(0)$ is the identity operator in $L^2(\mathbb{Q}_S)$, i.e. $\mathbf{S}(0)(f) = f$, for all $f \in L^2(\mathbb{Q}_S)$,
- it has the semigroup property: $\mathbf{S}(t) \cdot \mathbf{S}(s) = \mathbf{S}(t+s)$,
- if $f \in \mathrm{Dom}(-D^\alpha)$, then $\mathbf{S}(t)f \in \mathrm{Dom}(-D^\alpha)$ for all $t \geq 0$, the L^2 derivative $\frac{d}{dt}\mathbf{S}(t)f$ exists, is continuous for $t \geq 0$, and is given by

$$\frac{d}{dt}\mathbf{S}(t)f\Big|_{t=t_0^+} = -D^\alpha \mathbf{S}(t)f = -\mathbf{S}(t)D^\alpha f \qquad (t_0 \geq 0).$$

All these means that $\mathbf{S}(t)f$ is a classical solution of the heat equation (2.2) with initial condition $f \in \mathrm{Dom}(D^\alpha)$. For any initial data $f \in \mathrm{Dom}(D^\alpha)$, the Fourier transform can be applied to equation (2.2), in the spatial variable x, in order to get the abstract Cauchy problem:

$$\begin{cases} \widehat{u}_t(\xi,t) + \|\xi\|^\alpha \, \widehat{u}(\xi,t) = 0, \quad \xi \in \mathbb{Q}_S, t \geq 0, \\ \widehat{u}(\xi,0) = \widehat{f}(\xi), \quad (\widehat{f} \in \mathrm{Dom}(m^\alpha)). \end{cases} \tag{2.3}$$

The classical solution of this problem is the strongly continuous contraction semigroup $\exp(-tm^\alpha) : L^2(\mathbb{Q}_S) \longrightarrow L^2(\mathbb{Q}_S)$ given by

$$f(\xi) \mapsto f(\xi)\exp(-t\,\|\xi\|^\alpha).$$

Furthermore, from the commutative diagram (2.1), definitions of the infinitesimal generators of $\mathbf{S}(t)$ and $\exp(-tm^\alpha)$ and the abstract Cauchy problems (2.2), (2.3) and the fact that the Fourier transform is unitary, for $t \geq 0$, the diagram

$$\begin{array}{ccc} L^2(\mathbb{Q}_S) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{Q}_S) \\ {\scriptstyle \mathbf{S}(t)}\big\downarrow & & \big\downarrow{\scriptstyle \exp(-tm^\alpha)} \\ L^2(\mathbb{Q}_S) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{Q}_S) \end{array}$$

commutes. That is to say,

$$\mathbf{S}(t) = \mathcal{F}^{-1}\exp(-tm^\alpha)\mathcal{F}.$$

Therefore,

$$\mathcal{F}^{-1}[\exp(-t\,\|\xi\|^\alpha)\mathcal{F}[f](\xi)]$$

is a classical solution of the heat equation (2.2) with initial condition $f \in \mathrm{Dom}(D^\alpha)$.

2.3.2 The heat kernel

The theoretical solution of the heat equation shown above can be found explicitly by introducing heat kernel:

$$Z(x,t) = \mathcal{F}^{-1}\big(\exp(-t\,\|\xi\|^\alpha)\big) = \int_{\mathbb{Q}_S} \chi(-x\xi)\exp(-t\,\|\xi\|^\alpha)d\xi.$$

We estimate the heat kernel using some properties of the Archimedean Gamma function $\Gamma(z)$. Recall that $\Gamma(z)$ is a meromorphic function on the complex plane with simple poles at the nonpositive integers, satisfies the functional equation $\Gamma(z+1) = z\Gamma(z)$ and admits the integral representation

$$\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}ds,$$

in the halfplane $\mathrm{Re}(z) > 0$.

Putting the integral representation of the Archimedean Gamma function into its functional equation, we obtain

$$\Gamma(z+1) = z\int_0^\infty s^{z-1}e^{-s}ds$$
$$= \int_0^\infty e^{-s^{1/z}}ds,$$

which is convergent in the halfplane $\mathrm{Re}(z) > 0$.

The first estimate of the heat kernel is given in the following:

Lemma 2.3.1. *For any $t > 0$ and $\alpha > 0$, the function $\xi \longmapsto \exp(-t\,\|\xi\|^\alpha)$ is integrable over $\mathbb{Q}_S$ and consequently $Z(x,t)$ is well defined for any $t > 0$ and $x \in \mathbb{Q}_S$. Furthermore, for any $t > 0$ and $\alpha > 0$, the heat kernel $Z(x,t)$ satisfies the inequality*

$$|Z(x,t)| \le Ct^{-1/\alpha}, \qquad (x \in \mathbb{Q}_S),$$

where C is a constant depending on α.

Proof. Since the Haar measure of any ball is equal to its radius, we obtain

$$\int_{\mathbb{Q}_S} \exp(-t\,\|\xi\|^\alpha)d\xi = \sum_{n=-\infty}^{\infty}\int_{S_n} \exp(-t\,\|\xi\|^\alpha)d\xi$$
$$= \sum_{n=-\infty}^{\infty} \exp(-te^{\alpha\psi(n)})\big(e^{\psi(n)} - e^{\psi(n-1)}\big)$$
$$< \int_0^\infty \exp(-ts^\alpha)ds$$
$$= t^{-1/\alpha}\Gamma(1/\alpha+1).$$

Since $Z(x,t) = \int_{\mathbb{Q}_S} \chi(-x\xi)\exp(-t\,\|\xi\|^\alpha)d\xi$, this also proves the second assertion with $C = \Gamma(1/\alpha+1)$. ∎

Proposition 2.3.2. *The heat kernel $Z(x,t)$ is a positive function for all x and $t > 0$. In addition*

$$Z(x,t) = \sum_{\substack{n\in\mathbb{Z}\\ e^{\psi(n)}\le\|x\|^{-1}}} e^{\psi(n)}\left\{\exp(-te^{\alpha\psi(n)}) - \exp(-te^{\alpha\psi(n+1)})\right\}.$$

Proof. Using Proposition 2.2.2, if $\|x\| = e^{-\psi(m)}$, then

$$\begin{aligned}
Z(x,t) &= \sum_{n=-\infty}^{\infty} \int_{S_n} \chi(-x\xi) \exp(-t \|\xi\|^{\alpha}) d\xi \\
&= \sum_{n=-\infty}^{\infty} \exp(-te^{\alpha\psi(n)}) \int_{S_n} \chi(-x\xi) d\xi \\
&= \sum_{n=-\infty}^{m+1} \exp(-te^{\alpha\psi(n)}) \int_{S_n} \chi(-x\xi) d\xi \\
&= -\exp(-te^{\alpha\psi(m+1)}) e^{\psi(m)} + \sum_{n=-\infty}^{m} \exp(-te^{\alpha\psi(n)})(e^{\psi(n)} - e^{\psi(n-1)}) \\
&= \sum_{n=-\infty}^{m} e^{\psi(n)} \left\{ \exp(-te^{\alpha\psi(n)}) - \exp(-te^{\alpha\psi(n+1)}) \right\} \\
&= \sum_{\substack{n\in\mathbb{Z} \\ e^{\psi(n)} \leq \|x\|^{-1}}} e^{\psi(n)} \left\{ \exp(-te^{\alpha\psi(n)}) - \exp(-te^{\alpha\psi(n+1)}) \right\}.
\end{aligned}$$

This implies that $Z(x,t)$ is a positive function for all x and $t > 0$. ☐

Remark 2.3.3. *It is important to notice that the expression of the heat kernel in Proposition 2.3.2 does not depend on the algebraic structure of $\mathbb{Q}_S$: it depends only on the values of the second Chebyshev function related to S.*

Lemma 2.3.4. *For any $t > 0$, $\alpha > 0$ and $x \in \mathbb{Q}_S$, the heat kernel $Z(x,t)$ is positive and satisfies the inequality*

$$Z(x,t) \leq Ct \|x\|^{-\alpha-1}, \qquad (x \in \mathbb{Q}_S, t > 0),$$

where C is a constant depending on S and α.

Proof. In order to prove the second assertion we proceed as follows. From the inequality $1 - e^{-s} \leq s$, valid for $s \geq 0$, it follows the inequality

$$\begin{aligned}
Z(x,t) &\leq \|x\|^{-1} \sum_{\substack{n\in\mathbb{Z} \\ e^{\psi(n)} \leq \|x\|^{-1}}} \left\{ \exp(-te^{\alpha\psi(n)}) - \exp(-te^{\alpha\psi(n+1)}) \right\} \\
&\leq \|x\|^{-1} \left(1 - \exp(-te^{\alpha\psi(m+1)})\right) \leq t \|x\|^{-1} e^{\alpha\psi(m+1)} \\
&= te^{\alpha\psi(m)} e^{\alpha\Lambda(m+1)} e^{\psi(m)} \\
&= te^{(\alpha+1)\psi(m)} e^{\alpha\Lambda(m+1)} \\
&= t \|x\|^{-\alpha-1} e^{\alpha\Lambda(m+1)}.
\end{aligned}$$

As a result

$$Z(x,t) \leq Ct \|x\|^{-\alpha-1}, \qquad (x \in \mathbb{Q}_S,\ t > 0),$$

where $C = \max_{p\in S}\{p^{\alpha}\}$. ☐

Remark 2.3.5. *The constant involved in the last result depends on the growing behaviour of the group index $|B_n/B_{n-1}|$ of two consecutive balls centred at zero, which is a uniformly bounded quantity.*

The following proposition exhibits the relevant heat kernel estimate.

Proposition 2.3.6. (Heat kernel estimates) *For each $\alpha > 0$,*

$$Z(x,t) \leq Ct(t^{1/\alpha} + \|x\|)^{-\alpha-1}, \qquad (x \in \mathbb{Q}_S, t > 0).$$

Proof. This is a straight consequence of Lemma 2.3.1 and Lemma 2.3.4. □

Proposition 2.3.7. *The heat kernel satisfies the following properties:*

- *It is the distribution of a probability measure on $\mathbb{Q}_S$, i.e. $Z(x,t) \geq 0$ and*

$$\int_{\mathbb{Q}_S} Z(x,t)dx = 1,$$

for all $t > 0$.

- *It converges to the Dirac distribution as t tends to zero:*

$$\lim_{t\to 0} \int_{\mathbb{Q}_S} Z(x,t)f(x)dx = f(0),$$

for all $f \in \mathcal{D}(\mathbb{Q}_S)$.

- *It has the Markovian property:*

$$Z(x,t+s) = \int_{\mathbb{Q}_S} Z(x-y,t)Z(y,s)dy.$$

Proof. From Proposition 2.3.6, it follows that $Z(x,t)$ is in $L^1(\mathbb{Q}_S)$, for any $t > 0$. Indeed,

$$\begin{aligned}\int_{\mathbb{Q}_S} Z(x,t)dx &= \int_{\mathbb{Z}_S} Z(x,t)dx + \int_{\mathbb{Q}_S\setminus\mathbb{Z}_S} Z(x,t)dx \\ &\leq C_1 + C_2 \int_{\mathbb{Q}_S\setminus\mathbb{Z}_S} \frac{1}{\|x\|^{1+\alpha}}dx \\ &\leq C_1 + C_2 \int_1^{\infty} \frac{1}{s^{1+\alpha}}ds.\end{aligned}$$

Being $\exp(-t\|\xi\|^{\alpha})$ a continuous function on ξ, the Fourier inversion formula implies

$$\int_{\mathbb{Q}_S} Z(x,t)dx = 1.$$

Using this equality, the fact that $f \in \mathcal{D}(\mathbb{Q}_S)$ is a locally constant function of compact support and Proposition 2.3.6, we conclude that

$$\lim_{t\to 0} \int_{\mathbb{Q}_S} Z(x,t)f(x)dx = f(0).$$

Finally,

$$\begin{aligned} Z(x,t+s) &= \mathcal{F}^{-1}\big(\exp(-(t+s)\,\|\xi\|^{\alpha})\big) \\ &= \mathcal{F}^{-1}\big(\exp(-t\,\|\xi\|^{\alpha})\exp(-s\,\|\xi\|^{\alpha})\big) \\ &= \mathcal{F}^{-1}\big(\widehat{Z}(x,t)\widehat{Z}(x,s)\big) \\ &= Z(x,t)*Z(x,s) \\ &= \int_{\mathbb{Q}_S} Z(x-y,t)Z(y,s)dy. \end{aligned}$$

□

Proposition 2.3.8. *For any $t>0$ and $\alpha,\beta>0$, the function*

$$\xi \longmapsto \|\xi\|^{\beta}\exp(-t\,\|\xi\|^{\alpha})$$

is integrable over $\mathbb{Q}_S$. Therefore, $Z(x,t)$ is smooth with respect to t and the derivative

$$\frac{d}{dt}Z(x,t) = -\int_{\mathbb{Q}_S}\|\xi\|^{\alpha}\chi(-x\xi)\exp(-t\,\|\xi\|^{\alpha})d\xi \qquad (t>0)$$

is convergent. Furthermore, $Z(x,t)$ is uniformly continuous on t, i.e. $Z(x,t)\in C\big((0,\infty),C(\mathbb{Q}_S)\big)$.

Proof. The proof is similar to the one in Lemma 2.3.1:

$$\begin{aligned} \int_{\mathbb{Q}_S}\|\xi\|^{\beta}\exp(-t\,\|\xi\|^{\alpha})d\xi &= \int_{\mathbb{Z}_S}\|\xi\|^{\beta}\exp(-t\,\|\xi\|^{\alpha})d\xi + \int_{\mathbb{Q}_S\setminus\mathbb{Z}_S}\|\xi\|^{\beta}\exp(-t\,\|\xi\|^{\alpha})d\xi \\ &< C + \sum_{n=1}^{\infty} e^{\beta\psi(n)}\exp(-te^{\alpha\psi(n)})\big(e^{\psi(n)}-e^{\psi(n-1)}\big) \\ &\le C + \int_0^{\infty} s^{\beta}e^{-ts^{\alpha}}ds \\ &\le C + \frac{\Gamma(\frac{\beta+1}{\alpha})}{\alpha t^{\frac{\beta+1}{\alpha}}}. \end{aligned}$$

For the second part, suppose that $t<t'$. By the mean value theorem

$$\exp(-t'\,\|\xi\|^{\alpha}) - \exp(-t\,\|\xi\|^{\alpha}) = -\|\xi\|^{\alpha}(t'-t)\exp(-t_0\,\|\xi\|^{\alpha}),$$

where $t<t_0<t'$. Then

$$\begin{aligned} \frac{Z(x,t')-Z(x,t)}{t'-t} &= \frac{1}{t'-t}\int_{\mathbb{Q}_S}\chi(-x\xi)[\exp(-t'\,\|\xi\|^{\alpha})-\exp(-t\,\|\xi\|^{\alpha})]d\xi \\ &= -\frac{1}{t'-t}\int_{\mathbb{Q}_S}\chi(-x\xi)[-\|\xi\|^{\alpha}(t'-t)\exp(-t_0\,\|\xi\|^{\alpha})]d\xi \\ &= -\int_{\mathbb{Q}_S}\|\xi\|^{\alpha}\chi(-x\xi)\exp(-t_0\,\|\xi\|^{\alpha})d\xi. \end{aligned}$$

A similar argument shows that these equalities are also valid for $t' < t$. Hence

$$\begin{aligned}
\frac{d}{dt}Z(x,t) &= \lim_{t'\to t}\frac{Z(x,t')-Z(x,t)}{t'-t}\\
&= \lim_{t'\to t} -\int_{\mathbb{Q}_S}\|\xi\|^{\alpha}\chi(-x\xi)\exp(-t_0\|\xi\|^{\alpha})d\xi\\
&= -\int_{\mathbb{Q}_S}\|\xi\|^{\alpha}\chi(-x\xi)\exp(-t\|\xi\|^{\alpha})d\xi.
\end{aligned}$$

Finally, suppose that $t < t'$. By the mean value theorem

$$\exp(-t'\|\xi\|^{\alpha})-\exp(-t\|\xi\|^{\alpha}) = -\|\xi\|^{\alpha}(t'-t)\exp(-t_0\|\xi\|^{\alpha}),$$

where $t < t_0 < t'$. Then

$$\begin{aligned}
|Z(x,t)-Z(x,t')| &= \left|\int_{\mathbb{Q}_S}\chi(-x\xi)[\exp(-t'\|\xi\|^{\alpha})-\exp(-t\|\xi\|^{\alpha})]d\xi\right|\\
&= |t'-t|\left|\int_{\mathbb{Q}_S}\|\xi\|^{\alpha}\chi(-x\xi)\exp(-t_0\|\xi\|^{\alpha})d\xi.\right|\\
&\leq |t'-t|\int_{\mathbb{Q}_S}\|\xi\|^{\alpha}\exp(-t_0\|\xi\|^{\alpha})d\xi\\
&\leq |t'-t|\left(C+\frac{\Gamma(\frac{\alpha+1}{\alpha})}{\alpha t^{\frac{\alpha+1}{\alpha}}}\right).
\end{aligned}$$

Thus, $Z(x,t)$ is Lipschitz continuous. Therefore, $Z(x,t)$ is uniformly continuous on t.

□

2.3.3 The classical solution of the heat equation

Given $t > 0$ define the operator $\mathbf{T}(t) : L^2(\mathbb{Q}_S) \longrightarrow L^2(\mathbb{Q}_S)$ by the convolution with the Heat kernel

$$\mathbf{T}(t)f(x) = Z(x,t) * f(x), \qquad (f \in L^2(\mathbb{Q}_S)),$$

and let $\mathbf{T}(0)$ be the identity operator. From Proposition 2.3.7 and Young's inequality the family of operators $\{\mathbf{T}(t)\}_{t\geq 0}$ is a C_0–semigroup, that is, the family of operators $\{\mathbf{T}(t)\}_{t\geq 0}$ has the following properties

- $\mathbf{T}(0) = I$ (the identity operator),
- $\mathbf{T}(t+s) = \mathbf{T}(t)\mathbf{T}(s)$,
- $\lim_{t\to 0^+}\mathbf{T}(t) = I$.

Let us show that the semigroup $\{\mathbf{T}(t)\}_{t\geq 0}$ gives the solution of the heat equation (2.2) with initial data $f \in \Phi(\mathbb{Q}_S)$. Recall that, for any $f \in \Phi(\mathbb{Q}_S) \subset \mathrm{Dom}(D^{\alpha})$, we have $\widehat{f}(\xi) \in$

$\Psi(\mathbb{Q}_S) \subset \mathrm{Dom}(-m^{\alpha})$. Notice that the abstract cauchy problem (2.3), with initial data $\widehat{f}(\xi) \in \Psi(\mathbb{Q}_S)$, has a unique solution $\widehat{u}(\xi, t)$, such that $\widehat{u}(\cdot, t)$ belongs to $\Psi(\mathbb{Q}_S)$ for any $t \geq 0$, given by

$$\widehat{u}(\xi, t) = \widehat{f}(\xi) \exp(-t \|\xi\|^{\alpha}).$$

Therefore, there is a unique solution to the heat equation (2.2) with initial condition $f \in \Phi(\mathbb{Q}_S)$, such that $u(\cdot, t) \in \Phi(\mathbb{Q}_S)$ for $t \geq 0$, given by

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\big(\widehat{f}(\xi) \exp(-t \|\xi\|^{\alpha})\big) \\ &= \int_{\mathbb{Q}_S} \widehat{f}(\xi) \exp(-t \|\xi\|^{\alpha}) \chi(-x\xi) d\xi \\ &= Z(x,t) * f(x). \end{aligned}$$

The main theorem of the diffusion equation is the following.

Theorem 2.3.9. *Let $\alpha > 0$ and let $\mathbf{S}(t)$ be the C_0–semigroup generated by the operator $-D^{\alpha}$. The operator $\mathbf{S}(t)$ coincides for each $t \geq 0$ with the operator $\mathbf{T}(t)$ given above. In other words, the solution of the abstract Cauchy problem (2.2) is given by $u(x,t) = Z(x,t) * f(x)$, for $t \geq 0$ and $f \in \mathrm{Dom}(D^{\alpha})$. Furthermore, the heat kernel $Z(x,t)$ is the transition density of a time and space homogeneous Markov process which is bounded, right–continuous and has no discontinuities other than jumps.*

Proof. As has been said before, for $f \in \Phi(\mathbb{Q}_S)$, $u(x,t) = Z(x,t) * f(x)$ is a classical solution to the heat equation (2.2). Also, $u(x,t) \in \Phi(\mathbb{Q}_S) \subset \mathrm{Dom}(D^{\alpha})$ for any $t \geq 0$. Since $\Phi(\mathbb{Q}_S)$ is dense in $L^2(\mathbb{Q}_S)$, the operator $\mathbf{S}(t) = \mathbf{T}(t)$ for each $t \geq 0$ and the function $u(x,t) = Z(x,t) * f(x)$ is a solution of the Cauchy problem for any $f \in \mathrm{Dom}(D^{\alpha})$. The last result follows from Lemma 2.3.6 and the fact that $\mathbb{Q}_S$ is a second countable and locally compact ultrametric space (see [29], Theorem 3.6).

□

2.4 Cauchy problem for parabolic type equations on $\mathbb{Q}_S$

In this section two other classical formulations of the Cauchy problem on $\mathbb{Q}_S$ are described.

2.4.1 Homogeneous equation with values in L^2

Recall that a C_0 semigroup $\exp(tL)$, with infinitesimal generator L, is smooth if for any $t > 0$ and any $f \in L^2(\mathbb{Q}_S)$, the element $\exp(tL)f$ of $L^2(\mathbb{Q}_S)$ belongs to the domain of L.

Theorem 2.4.1. *The C_0 semigroup $\mathbf{S}(t)$ is smoothing. In other words, if f is any square integrable function on $\mathbb{Q}_S$, the Cauchy problem*

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D^\alpha u(x,t) = 0, \ x \in \mathbb{Q}_S, \ t > 0, \\ u(x,0) = f(x), \end{cases}$$

has a unique solution

$$u(x,t) = \int_{\mathbb{Q}_S} f(x-y) Z(y,t) dy.$$

Proof. From Proposition 2.3.8, it follows that the C_0 semigroup given by

$$f(\xi) \longmapsto \exp(t \, \|\xi\|^\alpha) f(\xi) \qquad (t \geq 0),$$

is smooth. Since the Fourier transform is unitary, the C_0 semigroup $\mathbf{S}(t)$ is also smooth. □

Chapter 3

Pseudodifferential operators and Markov processes on Adéles

In this chapter a class of Markov processes on the ring of finite adéles of the rational numbers are introduced. A class of non–Archimedean metrics on $\mathbb{A}_f$ are chosen in order to describe this ring as a general polyadic ring and to introduce a family of pseudodifferential operators and parabolic–type equations on $L^2(\mathbb{A}_f)$. The fundamental solutions of these parabolic equations determine transition functions of time and space homogeneous Markov processes on $\mathbb{A}_f$ which are invariantunder multiplication by units. Considering the infinite place $\mathbb{R}$, we extend these results to the complete ring of adèles.

3.1 Ultrametrics on finite adèles

This section portriaits the ring of finite adèles, $\mathbb{A}_f$, as a completion of the rational numbers with respect to certain additive invariant ultrametrics. For a detailed description of these results we quote [3] and [20].

3.1.1 Ultrametrics on $\mathbb{A}_f$

Let $\big(\rho(n)\big)_{n=0}^{\infty}$ be a sequence defined by a strictly increasing sequence of natural numbers $(e^{\rho(n)})_{n=0}^{\infty}$, which is totally ordered by division and cofinal with the natural numbers, and with $e^{\rho(0)} = 1$.

The function $\rho(n) = \log(e^{\rho(n)})$ can be defined to any integer number n, as

$$\rho(n) = \begin{cases} \frac{n}{|n|}\rho(|n|) & \text{if } n \neq 0, \\ 0 & \text{if } n = 0. \end{cases}$$

For any integer n define $\tau(n)$ implicitly by the equation

$$e^{\tau(n)} = \frac{e^{\rho(n)}}{e^{\rho(n-1)}}.$$

Then, for any integers $n > m$, there is a relation

$$\rho(n) - \rho(m) = \sum_{k=m+1}^{n} \tau(k) \quad \text{or equivalently} \quad e^{\rho(n)}/e^{\rho(m)} = \prod_{k=m+1}^{n} e^{\tau(k)}.$$

The collection $\{e^{\rho(n)}\widehat{\mathbb{Z}} \subset \mathbb{A}_f\}_{n\in\mathbb{Z}}$ is a neighborhood base of zero for the restricted product topology on $\mathbb{A}_f$ formed by open and closed subgroups. Additionally, it satisfies the properties:

$$\bigcap_{n\in\mathbb{Z}} e^{\rho(n)}\widehat{\mathbb{Z}} = 0, \quad \text{and} \quad \bigcup_{n\in\mathbb{Z}} e^{\rho(n)}\widehat{\mathbb{Z}} = \mathbb{A}_f.$$

For any element $x \in \mathbb{A}_f$ the ρ–adelic order of x is given by:

$$\mathrm{ord}_\rho(x) := \begin{cases} \max\{\, n \mid x \in e^{\rho(n)}\widehat{\mathbb{Z}} \,\} & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

This function satisfies the following properties:

- $\mathrm{ord}_\rho(x) \in \mathbb{Z} \cup \{\infty\}$ and $\mathrm{ord}_\rho(x) = \infty$ if and only if $x = 0$.
- $\mathrm{ord}_\rho(x + y) \geq \min\{\, \mathrm{ord}_\rho(x), \mathrm{ord}_\rho(y) \,\}$.

The non–Archimedean metric, $d_\rho : \mathbb{A}_f \times \mathbb{A}_f \longrightarrow \mathbb{R}^+ \cup \{0\}$, given by

$$d_\rho(x, y) = e^{-\rho(\mathrm{ord}_\rho(x-y))},$$

defines the restricted product topology and therefore $(\mathbb{A}_f, d_\rho)$ is a complete second countable ultrametric space.

Every non–zero finite adèle $x \in \mathbb{A}_f$ has a unique series representation of the form

$$x = \sum_{l=\gamma}^{\infty} x_l e^{\rho(l)}, \quad (x_l \in \{0, 1, \ldots, e^{\tau(l+1)} - 1\})$$

with $x_\gamma \neq 0$ and $\gamma = \mathrm{ord}_\rho(x) \in \mathbb{Z}$. This series is convergent in the ultrametric of $\mathbb{A}_f$ and the numbers x_l appearing in the representation of x are unique. The value γ, with $\gamma(0) = +\infty$ is the ρ–adelic order of x.

The ring of adelic integers coincides with the unit ball

$$\widehat{\mathbb{Z}} = \{\, x \in \mathbb{A}_f \; : \; \|x\|_\rho \leq 1 \,\}$$

which is the maximal compact and open subring of $\mathbb{A}_f$ as well.

From Remark 1.2.1, the fractional part of a finite adèle $x \in \mathbb{A}_f$ is the unique rational number $\{x\} \in [0, 1)$ such that $x - \{x\} \in \widehat{\mathbb{Z}}$, i.e. $\{x\}$ is equal to x minus its integral part. Then, it is also given by

$$\{x\} := \begin{cases} \displaystyle\sum_{k=\gamma(x)}^{-1} a_k e^{\rho(k)} & \text{if } \gamma(x) < 0, \\ 0 & \text{if } \gamma(x) \geq 0. \end{cases}$$

Denote by dx the Haar measure of the topological Abelian group $(\mathbb{A}_f, +)$ normalised such that the Haar measure of $\widehat{\mathbb{Z}}$ is equal to one.

Note that the ultrametric d_ρ takes values in the set $\{e^{\rho(n)}\}_{n\in\mathbb{Z}} \cup \{0\}$ and the balls centred at zero B_n^ρ are the sets $\{e^{\rho(n)}\widehat{\mathbb{Z}} \subset \mathbb{A}_f\}_{n\in\mathbb{Z}}$, that is,

$$B_n^\rho := B^\rho(0, e^{\rho(n)}) = e^{-\rho(n)}\widehat{\mathbb{Z}}.$$

We denote the sphere centred at zero and radius $e^{\rho(n)}$ as S_n^ρ, i.e.

$$S_n^\rho := S^\rho(0, e^{\rho(n)}) = B_n^\rho \backslash B_{n-1}^\rho.$$

Remark 3.1.1. *Every ultrametric induces a function given by*

$$\|x\|_\rho = e^{-\rho(\mathrm{ord}_\rho(x))} \qquad (x \in \mathbb{A}_f),$$

where $\|x\|_\rho = e^{\rho(n)}$ *if and only if* $x \in S_n^\rho$.

A function $\phi : \mathbb{A}_f \longrightarrow \mathbb{C}$ is locally constant if for any $x \in \mathbb{A}_f$, there exist an open set $U_x \subset \mathbb{A}_f$, such that

$$\phi(y) = \phi(x), \quad \text{for all } y \in U_x.$$

Let $\mathcal{D}(\mathbb{A}_f)$ denote the $\mathbb{C}$–vector space of all locally constant functions with compact support on $\mathbb{A}_f$. The vector space $\mathcal{D}(\mathbb{A}_f)$ is called Bruhat–Schwartz space of $\mathbb{A}_f$ and an element $\phi \in \mathcal{D}(\mathbb{A}_f)$ a Bruhat–Schwartz function (or simply a test function) on $\mathbb{A}_f$ ([17], [3]).

Considering each ultrametric d_ρ, if $\phi : \mathbb{A}_f \longrightarrow \mathbb{C}$ is locally constant, there exists an integer $\ell_\rho(x) \in \mathbb{Z}$ such that

$$\phi(y) = \phi(x), \quad \text{for all } y \in B^\rho_{\ell_\rho(x)}(x),$$

where $B^\rho_{\ell_\rho(x)}(x)$ is the closed ball with centre at x and radius $e^{\rho(\ell_\rho(x))}$.

If ϕ belongs to $\mathcal{D}(\mathbb{A}_f)$ and $\phi(x) \neq 0$ for some $x \in \mathbb{A}_f$, there exists a largest $\ell_\rho = \ell_\rho(\phi) \in \mathbb{Z}$, which is called the parameter of constancy of ϕ with respect to d_ρ, such that, for any $x \in \mathbb{A}_f$, we have

$$\phi(x + y) = \phi(x), \text{ for all } y \in B^\rho_{\ell_\rho}.$$

Denote by $\mathcal{D}^\rho_{\ell,k}(\mathbb{A}_f)$ the finite dimensional vector space consisting of functions whose parameter of constancy is greater than or equal to ℓ and whose support is contained in B_k^ρ.

The topology on $\mathcal{D}(\mathbb{A}_f)$, given by the inductive limit

$$\mathcal{D}(\mathbb{A}_f) = \varinjlim_{\ell\leq k} \mathcal{D}^\rho_{\ell,k}(\mathbb{A}_f),$$

is independent of the choice of d_ρ and with this topology $\mathcal{D}(\mathbb{A}_f)$ is a complete locally convex topological vector space over $\mathbb{C}$. It is also a nuclear space because it is the inductive limit of the countable family of finite dimensional vector spaces $\{\mathcal{D}^\rho_{\ell,k}(\mathbb{A}_f)\}$. A sequence $(f_m)_{m\geq 1}$ in $\mathcal{D}(\mathbb{A}_f)$ is a Cauchy sequence if there exist $k, \ell_\rho \in \mathbb{Z}$ and $M > 0$ such that $f_m \in \mathcal{D}^\rho_{\ell,k}(\mathbb{A}_f)$ if $m \geq M$ and $(f_m)_{m\geq M}$ is a Cauchy sequence in $\mathcal{D}^\rho_{\ell,k}(\mathbb{A}_f)$.

For each compact set $K \subset \mathbb{A}_f$, let $\mathcal{D}(K) \subset \mathcal{D}(\mathbb{A}_f)$ be the subspace of test functions whose support is contained in K. The space $\mathcal{D}(K)$ is dense in $\mathrm{C}(K)$, the space of complex–valued continuous functions on K.

An additive character of the field $\mathbb{A}_f$ is defined as a continuous function $\chi : \mathbb{A}_f \longrightarrow \mathbb{C}$ such that $\chi(x+y) = \chi(x)\chi(y)$ and $|\chi(x)| = 1$, for all $x, y \in \mathbb{A}_f$. The function $\chi(x) = \exp(2\pi i\{x\})$ defines a canonical additive character of $\mathbb{A}_f$, which is trivial on $\widehat{\mathbb{Z}}$ and not trivial outside $\widehat{\mathbb{Z}}$, and all characters of $\mathbb{A}_f$ are given by $\chi_\xi(x) = \chi(\xi x)$, for some $\xi \in \mathbb{A}_f$. The Fourier transform of a test function $\phi \in \mathcal{D}(\mathbb{A}_f)$ is given by the formula

$$\mathcal{F}\phi(\xi) = \widehat{\phi}(\xi) = \int_{\mathbb{A}_f} \phi(x)\chi(\xi x)dx, \qquad (\xi \in \mathbb{A}_f).$$

The Fourier transform is a continuous linear isomorphism of the space $\mathcal{D}(\mathbb{A}_f)$ onto itself and the following inversion formula holds:

$$\phi(x) = \int_{\mathbb{A}_f} \widehat{\phi}(\xi)\chi(-x\xi)d\xi \qquad \big(\phi \in \mathcal{D}(\mathbb{A}_f)\big).$$

Additionally, the Parseval – Steklov equality reads as:

$$\int_{\mathbb{A}_f} \phi(x)\overline{\psi(x)}dx = \int_{\mathbb{A}_f} \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)}d\xi, \qquad \big(\phi, \psi \in \mathcal{D}(\mathbb{A}_f)\big).$$

Last but not least, the Hilbert space $L^2(\mathbb{A}_f)$ is a separable Hilbert space and the extended Fourier transform $\mathcal{F} : L^2(\mathbb{A}_f) \longrightarrow L^2(\mathbb{A}_f)$ is an isometry of Hilbert spaces. Moreover, the Fourier inversion formula and the Parseval – Steklov identity hold on $L^2(\mathbb{A}_f)$.

Remark 3.1.2. *The Haar measure of any ball is equal to its radius:*

$$\int_{y+B_n^\rho} d\xi = \int_{B_n^\rho} d\xi = \int_{e^{-\rho(n)}\widehat{\mathbb{Z}}} d\xi = e^{\rho(n)} \qquad (y \in \mathbb{A}_f,\, n \in \mathbb{Z}),$$

and the area of a sphere is given by

$$\int_{y+S_n^\rho} d\xi = \int_{S_n^\rho} d\xi = e^{\rho(n)} - e^{\rho(n-1)} \qquad (y \in \mathbb{A}_f,\, n \in \mathbb{Z}).$$

Moreover, for any $n \in \mathbb{Z}$ the following formulae hold:

- $$\int_{B_n^\rho} \chi(-\xi x)dx = \begin{cases} e^{\rho(n)} & \text{if } \|\xi\|_\rho \le e^{-\rho(n)}, \\ 0 & \text{if } \|\xi\|_\rho > e^{-\rho(n)}. \end{cases}$$

- $$\int_{S_n^\rho} \chi(-\xi x)dx = \begin{cases} e^{\rho(n)} - e^{\rho(n-1)} & \text{if } \|\xi\|_\rho \le e^{-\rho(n)}, \\ -e^{\rho(n-1)} & \text{if } \|\xi\|_\rho = e^{-\rho(n-1)}, \\ 0 & \text{if } \|\xi\|_\rho \ge e^{-\rho(n-2)}. \end{cases}$$

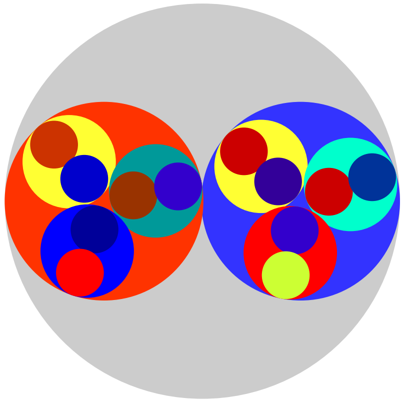

Figure 3.1: The decomposition of $\widehat{\mathbb{Z}}$ by the filtration $\{e^{\psi(n)}\widehat{\mathbb{Z}}\}$

Example 3.1.3. *The sequence* $((n+1)!)_{n=0}^{\infty}$ *is a strictly increasing sequence of natural numbers, beginning with one, totally order by division and cofinal with the natural numbers. The filtration determined by this sequence is given by*

$$\{U_n = \big((|n|+1)!\big)^{n/|n|}\widehat{\mathbb{Z}}\}_{n\in\mathbb{Z}}.$$

Example 3.1.4. *Let* $\psi(n)$ *denote the* second Chebyshev function *(see [8]) defined by the relation*

$$e^{\psi(n)} = \mathrm{lcm}(1, 2, \ldots, n), \qquad (n \in \mathbb{N}),$$

where $\mathrm{lcm}(1, 2, \ldots, n)$ *is the smallest positive integer that is divisible by* $1, 2, 3, \ldots, n$. *Each subgroup* $U_n = e^{\psi(n)}\widehat{\mathbb{Z}} \subset \widehat{\mathbb{Z}}$ *is the intersection of all normal subgroups of* $\widehat{\mathbb{Z}}$ *which has index less than or equal to* n. *In this example, if* $n = p^{\alpha}$ *is a prime power, then* $e^{\psi(p^{\alpha})} = pe^{\psi(p^{\alpha}-1)}$ *and* $U_{p^{\alpha}} = pU_{p^{\alpha}-1}$. *Otherwise, if* n *is not a prime power, then* $e^{\psi(n)} = e^{\psi(n-1)}$. *The elements of the set* $\{e^{\psi(n)}\}_{n=1}^{\infty} \cup \{0\}$ *form an increasing sequence* $(e^{\rho(n)})_{n=0}^{\infty}$ *which is a strictly increasing sequence of natural numbers, beginning with one, totally order by division and cofinal with the natural numbers (see figure 3.1). The filtration determined by this sequence is given by*

$$\{U_n = \big(e^{\psi(|n|+1)}\big)^{n/|n|}\widehat{\mathbb{Z}}\}_{n\in\mathbb{Z}}.$$

3.2 Parabolic–type equations on $\mathbb{A}_f$

This section introduces a positive selfadjoint pseudodifferential unbounded operator D_{ρ}^{α} on $L^2(\mathbb{A}_f)$ related to each ultrametric d_{ρ}, the Hilbert space of square integrable functions on $\mathbb{A}_f$, and solves the abstract Cauchy problem for the homogeneous heat equation on $L^2(\mathbb{A}_f)$ related to D_{ρ}^{α}. The properties of general evolution equations on Banach spaces can be found in [30], [19] and [52]. The reader can consult these and more topics in the excellent books [35], [5], [63] and [73].

3.2.1 Pseudodifferential operators on $\mathbb{A}_f$

For any $\alpha > 0$, consider the pseudodifferential operator $D_\rho^\alpha : \mathrm{Dom}(D_\rho^\alpha) \subset L^2(\mathbb{A}_f) \longrightarrow L^2(\mathbb{A}_f)$ defined by the formula

$$D_\rho^\alpha \phi(x) = \mathcal{F}^{-1}_{\xi \to x}[\|\xi\|_\rho^\alpha \mathcal{F}_{x\to\xi}[f]],$$

for any ϕ in the domain

$$\mathrm{Dom}(D_\rho^\alpha) := \left\{ f \in L^2(\mathbb{A}_f) : \|\xi\|_\rho^\alpha \widehat{f}(\xi) \in L^2(\mathbb{A}_f) \right\}.$$

This operator is a pseudodifferential operator with symbol $\|\xi\|_\rho^\alpha$. It can be seen that the unbounded operator D_ρ^α, with domain $\mathrm{Dom}(D_\rho^\alpha)$, is a positive selfadjoint operator which is diagonalized by the (unitary) Fourier transform. In other words, the following diagram commutes:

$$\begin{array}{ccc} L^2(\mathbb{A}_f) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}_f) \\ {\scriptstyle D_\rho^\alpha}\big\downarrow & & \big\downarrow{\scriptstyle m_\rho^\alpha} \\ L^2(\mathbb{A}_f) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}_f), \end{array} \tag{3.1}$$

where $m_\rho^\alpha : L^2(\mathbb{A}_f) \longrightarrow L^2(\mathbb{A}_f)$ is the multiplicative operator given by $f(\xi) \longmapsto \|\xi\|_\rho^\alpha f(\xi)$, with (dense) domain

$$\mathrm{Dom}(m_\rho^\alpha) := \left\{ f \in L^2(\mathbb{A}_f) \ : \ \|\xi\|_\rho^\alpha f(\xi) \in L^2(\mathbb{A}_f) \right\}.$$

As a result, several properties of D_ρ^α, depending only on the inner product of $L^2(\mathbb{A}_f)$, can be translated into analogue properties of the multiplicative operator m_ρ^α. In particular, the characteristic equation $D_\rho^\alpha f = \lambda f$ with $f \in L^2(\mathbb{A}_f) \setminus \{0\}$ can be solved by applying the Fourier transform. In fact, if $\lambda \in \{e^{\alpha\rho(n)}\}_{n\in\mathbb{Z}}$, the indicator or characteristic function $1_{S_n^\rho}$, of the sphere S_n^ρ, is a solution of the characteristic equation, $(\|\xi\|_\rho^\alpha - \lambda)\widehat{f}(\xi) = 0$, of the multiplicative operator m_ρ^α. Otherwise, if $\lambda \notin \{e^{\alpha\rho(n)}\}_{n\in\mathbb{Z}}$, the function $\|\xi\|_\rho^\alpha - \lambda$ is bounded from below and λ is in the resolvent set of m_ρ^α. Since the Fourier transform is unitary, the point spectrum of D_ρ^α is the set $\{e^{\alpha\rho(n)}\}_{n\in\mathbb{Z}}$, with corresponding eigenfunctions $\{\mathcal{F}^{-1}(1_{S_n^\rho})\}_{n\in\mathbb{Z}}$. It follows that, $\{0\}$ forms part of the spectrum as a limit point. Consequently, $D_\rho^\alpha \neq D_{\rho'}^\alpha$, if the ultrametric d_ρ is not equal to the ultrametric $d_{\rho'}$. Finally, it is worth to mention that each eigenspace is infinite dimensional and there exists a well defined wavelet base which is also made of eigenfunctions.

Remark 3.2.1. *The operator D_ρ^α is derived from the chosen double sequence $\left(e^{\rho(n)}\right)_{n\in\mathbb{Z}}$. Any operator D_ρ^α can be considered a finite adelic analogue of the Vladimirov operator on $\mathbb{Q}_p$.*

3.2.2 A Cauchy problem on $L^2(\mathbb{A}_f)$

For $f(x) \in \mathrm{Dom}(D_\rho^\alpha) \subset L^2(\mathbb{A}_f)$, consider the abstract Cauchy problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D_\rho^\alpha u(x,t) = 0, \ x \in \mathbb{A}_f, \ t \geq 0 \\ u(x,t) = f(x). \end{cases} \tag{3.2}$$

This problem will be pointed as abstract Cauchy problem (3.2). Notice that for each invariant pseudodifferential operator D_ρ^α, the abstract Cauchy problem above is a finite adelic counterpart of the Archimedean abstract Cauchy problem for the homogeneous heat equation.

The abstract Cauchy problem (3.2) is considered in the sense of the Hilbert space $L^2(\mathbb{A}_f)$, that is to say, a function $u : \mathbb{A}_f \times [0,\infty) \longrightarrow \mathbb{C}$ is called a solution if:

a. $u : [0,\infty) \longrightarrow L^2(\mathbb{A}_f)$ is a continuously differentiable function,

b. $u(x,t) \in \mathrm{Dom}(D_\rho^\alpha)$, for all $t \geq 0$ and,

c. $u(x,t)$ satisfies the initial value problem (3.2).

The abstract Cauchy problem (3.2) is well–posed and its solution is well understood from the theory of semigroups of linear operators over Banach spaces. This solution is described in the following section.

3.2.3 Semigroup of operators

From the Hille–Yoshida Theorem, to the positive selfadjoint operator D_ρ^α, there corresponds a strongly continuous contraction semigroup

$$S_\rho^\alpha(t) = \exp(-tD_\rho^\alpha) : L^2(\mathbb{A}_f) \longrightarrow L^2(\mathbb{A}_f) \qquad (t \geq 0),$$

with infinitesimal generator $-D_\rho^\alpha$. It follows that $\{S_\rho^\alpha(t)\}_{t\geq 0}$ has the following properties:

- For any $t \geq 0$, $S_\rho^\alpha(t)$ is a bounded operator with operator norm less or equal to one.
- The application $t \longmapsto S_\rho^\alpha(t)$ is strongly continuous for $t \geq 0$.
- $S_\rho^\alpha(0)$ is the identity operator in $L^2(\mathbb{A}_f)$, i.e. $S_\rho^\alpha(0)(f) = f$, for all $f \in L^2(\mathbb{A}_f)$,
- It has the semigroup property: $S_\rho^\alpha(t) \circ S_\rho^\alpha(s) = S_\rho^\alpha(t+s)$.
- If $f \in \mathrm{Dom}(-D_\rho^\alpha)$, then $S_\rho^\alpha(t)f \in \mathrm{Dom}(-D_\rho^\alpha)$ for all $t \geq 0$, the L^2 derivative $\frac{d}{dt}S_\rho^\alpha(t)f$ exists, it is continuous for $t \geq 0$, and is given by

$$\frac{d}{dt}S_\rho^\alpha(t)f\Big|_{t=t_0^+} = -D_\rho^\alpha S_\rho^\alpha(t_0)f = -S_\rho^\alpha(t_0)D_\rho^\alpha f \qquad (t_0 \geq 0).$$

All this means that $S^{\alpha}_{\rho}(t)f$ is a solution of the Cauchy problem (3.2) with initial condition $f \in \mathrm{Dom}(D^{\alpha}_{\rho})$.

On the other hand, for $f(\xi) \in \mathrm{Dom}(m^{\alpha}_{\rho}) \subset L^2(\mathbb{A}_f)$, consider the abstract Cauchy problem

$$\begin{cases} \frac{\partial u(\xi,t)}{\partial t} + m^{\alpha}_{\rho} u(\xi,t) = 0, \ \xi \in \mathbb{A}_f, \ t \geq 0 \\ u(\xi,t) = f(\xi). \end{cases} \tag{3.3}$$

The solution of this problem is given by the strongly continuous contraction semigroup $\exp(-tm^{\alpha}_{\rho}) : L^2(\mathbb{A}_f) \longrightarrow L^2(\mathbb{A}_f)$ given by

$$f(\xi) \mapsto f(\xi)\exp(-t\,\|\xi\|^{\alpha}_{\rho}),$$

which is the semigroup that corresponds to the positive selfadjoint multiplicative operator m^{α}_{ρ}, under the Hille–Yoshida Theorem and whose infinitesimal generator is equal to $-m^{\alpha}_{\rho}$. In fact, since the function

$$(\xi,t) \mapsto \exp(-t\,\|\xi\|^{\alpha}_{\rho})$$

is uniformly bounded by 1 for any t and $x \in \mathbb{A}_f$, Equation (3.3) gives a C_0-semigroup which coincides with $e^{-tm^{\alpha}}$ in the set of continuous functions of compact support, because the solution of the Heat Equation (3.3) is unique in the set of continuous functions of compact support on $\mathbb{A}_f$. From the fact that the Fourier transform is an isometry on $L^2(\mathbb{A}_f)$ and converts the abstract Cauchy problem (3.2) into (3.3), the commutative diagram (3.1), and corresponding definitions of the infinitesimal generators of $S^{\alpha}_{\rho}(t)$ and $\exp(-tm^{\alpha}_{\rho})$, the following diagram commutes

$$\begin{array}{ccc} L^2(\mathbb{A}_f) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}_f) \\ {\scriptstyle S^{\alpha}_{\rho}(t)}\big\downarrow & & \big\downarrow{\scriptstyle \exp(-tm^{\alpha}_{\rho})} \\ L^2(\mathbb{A}_f) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}_f). \end{array}$$

3.2.4 A heat kernel

In order to describe the theoretical solution given by the Hille–Yosida Theorem we introduce the heat kernel:

$$Z^{\alpha}_{\rho}(x,t) = \mathcal{F}^{-1}\big(\exp(-t\,\|\xi\|^{\alpha}_{\rho})\big) = \int_{\mathbb{A}_f} \chi(-x\xi)\exp(-t\,\|\xi\|^{\alpha}_{\rho})d\xi.$$

The first estimate of the heat kernel is given in the following :

Lemma 3.2.2. *For any $t > 0$, $\alpha > 0$, and $x \in \mathbb{A}_f$, $Z^{\alpha}_{\rho}(x,t)$ is well defined. Furthermore, for any $t > 0$ and $\alpha > 0$, the heat kernel $Z^{\alpha}_{\rho}(x,t)$ satisfies the inequality*

$$\left|Z^{\alpha}_{\rho}(x,t)\right| \leq Ct^{-1/\alpha}, \qquad (x \in \mathbb{A}_f),$$

where C is a constant depending on α.

Proof. Since the Haar measure of any ball is equal to its radius, we obtain

$$\left|Z_\rho^\alpha(x,t)\right| \leq \int_{\mathbb{A}_f} \exp(-t\left\|\xi\right\|_\rho^\alpha)d\xi < \int_0^\infty \exp(-ts^\alpha)ds = t^{-1/\alpha}\Gamma(1/\alpha+1),$$

where Γ denotes the Archimedean gamma function. Therefore, $Z_\rho^\alpha(x,t)$ is well defined and the second assertion holds with $C = \Gamma(1/\alpha+1)$. □

Proposition 3.2.3. *The heat kernel $Z_\rho^\alpha(x,t)$ is a positive function for all x and $t > 0$. In addition*

$$Z_\rho^\alpha(x,t) = \sum_{\substack{n\in\mathbb{Z} \\ e^{\rho(n)}\leq\|x\|_\rho^{-1}}} e^{\rho(n)}\left\{\exp(-te^{\alpha\rho(n)}) - \exp(-te^{\alpha\rho(n+1)})\right\}.$$

Proof. From Remark 3.1.2, if $\|x\|_\rho = e^{-\rho(m)}$, then

$$\begin{aligned}
Z_\rho^\alpha(x,t) &= \sum_{n=-\infty}^{\infty} \int_{S_n^\rho} \chi(-x\xi)\exp(-t\left\|\xi\right\|_\rho^\alpha)d\xi \\
&= \sum_{n=-\infty}^{\infty} \exp(-te^{\alpha\rho(n)}) \int_{S_n^\rho} \chi(-x\xi)d\xi \\
&= \sum_{n=-\infty}^{m+1} \exp(-te^{\alpha\rho(n)}) \int_{S_n^\rho} \chi(-x\xi)d\xi \\
&= -\exp(-te^{\alpha\rho(m+1)})e^{\rho(m)} + \sum_{n=-\infty}^{m} \exp(-te^{\alpha\rho(n)})(e^{\rho(n)} - e^{\rho(n-1)}) \\
&= \sum_{n=-\infty}^{m} e^{\rho(n)}\left\{\exp(-te^{\alpha\rho(n)}) - \exp(-te^{\alpha\rho(n+1)})\right\} \\
&= \sum_{\substack{n\in\mathbb{Z} \\ e^{\rho(n)}\leq\|x\|_\rho^{-1}}} e^{\rho(n)}\left\{\exp(-te^{\alpha\rho(n)}) - \exp(-te^{\alpha\rho(n+1)})\right\}.
\end{aligned}$$

This implies that $Z_\rho^\alpha(x,t)$ is a positive function for all x and $t > 0$.

□

Remark 3.2.4. *It is important to notice that the expression of the heat kernel in Proposition 3.2.3 does not depend on the algebraic structure of $\mathbb{A}_f$. As a matter of fact, $Z_\rho^\alpha(x,t)$ depends only on $\alpha > 0$, the values of the double sequence $(e^{\rho(n)})_{n\in\mathbb{Z}}$ and the ultrametric structure defined by this sequence on $\mathbb{A}_f$.*

Corollary 3.2.5. *The heat kernel is the distribution of a probability measure on $\mathbb{A}_f$, i.e. $Z_\rho^\alpha(x,t) \geq 0$ and*

$$\int_{\mathbb{A}_f} Z_\rho^\alpha(x,t)dx = 1,$$

for all $t > 0$.

Proof. From Proposition 3.2.3 it follows that

$$\begin{aligned}
\int_{\mathbb{A}_f} Z_\rho^\alpha(x,t)dx &= \sum_{l=-\infty}^{\infty} \int_{S_l^\rho} Z_\rho(x,t)dx \\
&= \sum_{l=-\infty}^{\infty} Z_\rho^\alpha(e^{\rho(l)},t)\big(e^{\rho(l)} - e^{\rho(l-1)}\big) \\
&= \sum_{l=-\infty}^{\infty} \left(\sum_{\substack{n\in\mathbb{Z} \\ e^{\rho(n)} \le e^{-\rho(l)}}} e^{\rho(n)} \left\{\exp(-te^{\alpha\rho(n)}) - \exp(-te^{\alpha\rho(n+1)})\right\} \right) \big(e^{\rho(l)} - e^{\rho(l-1)}\big) \\
&= \sum_{n=-\infty}^{\infty} \left(e^{\rho(n)} \left\{\exp(-te^{\alpha\rho(n)}) - \exp(-te^{\alpha\rho(n+1)})\right\}\right) \left(\sum_{l=-\infty}^{-n} \big(e^{\rho(l)} - e^{\rho(l-1)}\big) \right) \\
&= \sum_{n=-\infty}^{\infty} \left\{\exp(-te^{\alpha\rho(n)}) - \exp(-te^{\alpha\rho(n+1)})\right\} \\
&= 1.
\end{aligned}$$

□

Lemma 3.2.6. *For any $t > 0$, $\alpha > 0$ and $x \in \mathbb{A}_f$, the heat kernel $Z_\rho(x,t)$ is positive and satisfies the inequality*

$$Z_\rho^\alpha(x,t) \le \|x\|_\rho^{-1} \big(1 - \exp(-te^{\alpha\rho(m+1)})\big), \tag{3.4}$$

where $\|x\|_\rho = e^{-\rho(m)}$.

Proof. This follows from the inequality

$$\begin{aligned}
Z_\rho^\alpha(x,t) &\le \|x\|_\rho^{-1} \sum_{\substack{n\in\mathbb{Z} \\ e^{\rho(n)} \le \|x\|_\rho^{-1}}} \left\{\exp(-te^{\alpha\rho(n)}) - \exp(-te^{\alpha\rho(n+1)})\right\} \\
&\le \|x\|_\rho^{-1} \big(1 - \exp(-te^{\alpha\rho(m+1)})\big).
\end{aligned}$$

□

Proposition 3.2.7. *The heat kernel satisfies the following properties:*

- *It is the distribution of a probability measure on $\mathbb{A}_f$, i.e. $Z_\rho^\alpha(x,t) \ge 0$ and*

  $$\int_{\mathbb{A}_f} Z_\rho^\alpha(x,t)dx = 1,$$

  for all $t > 0$.

- *It converges to the Dirac distribution as t tends to zero:*

  $$\lim_{t\to 0} \int_{\mathbb{A}_f} Z_\rho^\alpha(x,t)f(x)dx = f(0),$$

  for all $f \in \mathcal{D}(\mathbb{A}_f)$.

- *It has the Markovian property:*

$$Z_\rho^\alpha(x,t+s)=\int_{\mathbb{A}_f} Z_\rho^\alpha(x-y,t)Z_\rho^\alpha(y,s)dy.$$

Proof. From Corollary 3.2.5, $Z_\rho^\alpha(x,t)$ is in $L^1(\mathbb{A}_f)$ for any $t>0$ and

$$\int_{\mathbb{A}_f} Z_\rho^\alpha(x,t)dx=1.$$

Using this equality, the fact that $f\in\mathcal{D}(\mathbb{A}_f)$ is a locally constant function with compact support and Lemma 3.2.6, we conclude that

$$\lim_{t\to 0}\int_{\mathbb{A}_f} Z_\rho^\alpha(x,t)\big(f(x)-f(0)\big)dx=0.$$

The Markovian property follows from the Fourier inversion formula and the related property of the exponential function. □

Remark 3.2.8. *It is worth mentioning that the heat kernel associated to the isotropic Laplacian of the ultrametric space $(\mathbb{A}_f,d_\rho)$, with the Haar measure of $\mathbb{A}_f$ as speed measure, and distribution function $\mathrm{e}^{1/r}$, is equal to (see [15] for the definitions):*

$$\sum_{\substack{n\in\mathbb{Z}\\ e^{\rho(n)}\le\|x\|_\rho^{-1}}} e^{\rho(n)}\left\{\exp(-te^{\alpha\rho(n-1)})-\exp(-te^{\alpha\rho(n)})\right\}.$$

This kernel differs from $Z_\rho^\alpha(x,t)$ only by a term. However this one is very important when considering bounds 3.2.6.

3.2.5 The solution of the heat equation

Given $t>0$ define the operator $T_\rho^\alpha(t):L^2(\mathbb{A}_f)\longrightarrow L^2(\mathbb{A}_f)$ by the convolution with the heat kernel

$$T_\rho^\alpha(t)f(x)=Z_\rho^\alpha(x,t)*f(x),\qquad (f\in L^2(\mathbb{A}_f)),$$

and let $T_\rho^\alpha(0)$ be the identity operator on $L^2(\mathbb{A}_f)$. From Proposition 3.2.7 and Young's inequality the family of operators $\{T_\rho^\alpha(t)\}_{t\ge 0}$ is a strongly continuous contraction semigroup.

The main theorem of the diffusion equation on the ring $\mathbb{A}_f$ is the following.

Theorem 3.2.9. *Let $\alpha>0$ and let $S_\rho^\alpha(t)$ be the C_0–semigroup generated by the operator $-D_\rho^\alpha$. The operators $S_\rho^\alpha(t)$ and $T_\rho^\alpha(t)$ agree for each $t\ge 0$. In other words, for $f\in\mathrm{Dom}(D_\rho^\alpha)$ and for $t>0$ the solution of the abstract Cauchy problem (3.2) is given by the convolution $u(x,t)=Z_\rho^\alpha(x,t)*f(x)$.*

Proof. For $f\in L^1(\mathbb{A}_f)\cap L^2(\mathbb{A}_f)$, the convolution $u(x,t)=Z_\rho^\alpha(x,t)*f(x)$ is in $L^1(\mathbb{A}_f)\cap L^2(\mathbb{A}_f)$ because $Z_\rho^\alpha(x,t)$ is integrable for $t>0$. Then, the Fourier transform $\mathcal{F}_{x\to\xi}u(x,t)$ is equal to

$$\hat{f}(\xi)\exp(-t\,\|\xi\|_\rho^\alpha).$$

From the commutative diagram above and the last equation, $S^{\alpha}_{\rho}(t)(f) = T^{\alpha}_{\rho}(t)(f)$. Since $L^1(\mathbb{A}_f) \cap L^2(\mathbb{A}_f)$ is dense in $L^2(\mathbb{A}_f)$, $S^{\alpha}_{\rho}(t)(f) = T^{\alpha}_{\rho}(t)$, for each $t \geq 0$. As a consequence, the function $u(x,t) = Z^{\alpha}_{\rho}(x,t) * f(x)$ is a solution of the Cauchy problem for any $f \in \mathrm{Dom}(D^{\alpha}_{\rho})$. ☐

3.2.6 Markov processes on $\mathbb{A}_f$

In this section the fundamental solution of the heat equation, $Z^{\alpha}_{\rho}(x,t)$, is shown to be the transition density function of a Markov process on $\mathbb{A}_f$. This family of Markov processes are described for the first time in this writing.

Let $\mathcal{B}$ denote the Borel σ–algebra of $\mathbb{A}_f$ and for $B \in \mathcal{B}$ write 1_B for the characterisctic or indicator function of B. Define

$$\mathbf{p}^{\alpha}_{\rho}(t,x,y) := Z^{\alpha}_{\rho}(x-y,t) \quad (t > 0,\ x, y \in \mathbb{A}_f)$$

and

$$\mathbf{P}^{\alpha}_{\rho}(t,x,B) = \begin{cases} \int_B \mathbf{p}^{\alpha}_{\rho}(t,x,y)dy & \text{if } t > 0, x \in \mathbb{A}_f, B \in \mathcal{B}, \\ 1_B(x) & \text{if } t = 0. \end{cases}$$

From Theorem 3.2.7, it follows that $\mathbf{p}^{\alpha}_{\rho}(t,x,y)$ is a normal transition density and

$\mathbf{P}^{\alpha}_{\rho}(t,x,B)$ is a normal Markov transition function on $\mathbb{A}_f$ which corresponds to a Markov process on $\mathbb{A}_f$ (see [29, Section 2.1], for further detail). Since $\mathbf{p}^{\alpha}_{\rho}(t,x,y)$ depends only on the distance $d_{\rho}(x,y)$, and this distance is additive invariant, it follows that the corresponding Markov process is additive invariant.

In order to portray the properties of the path of the associated Markov process we first state the following:

Lemma 3.2.10. *Let k be any integer, then*

$$\int_{\mathbb{A}_f \setminus B^{\rho}_k} Z^{\alpha}_{\rho}(x,t) \leq 1 - \exp(-te^{\alpha\rho(-k)}).$$

Proof. Similar to Corollary 3.2.5, we have the following

$$\begin{aligned}
\int_{\mathbb{A}_f\setminus B_k^\rho} Z_\rho^\alpha(x,t)dx &= \sum_{l=k+1}^{\infty}\int_{S_l^\rho} Z_\rho^\alpha(x,t)dx \\
&= \sum_{l=k+1}^{\infty} Z_\rho^\alpha(e^{\rho(l)},t)(e^{\rho(l)}-e^{\rho(l-1)}) \\
&= \sum_{l=k+1}^{\infty}\left(\sum_{\substack{n\in\mathbb{Z}\\ e^{\rho(n)}\le e^{-\rho(l)}}} e^{\rho(n)}\left\{\exp(-te^{\alpha\rho(n)})-\exp(-te^{\alpha\rho(n+1)})\right\}\right)(e^{\rho(l)}-e^{\rho(l-1)}) \\
&= \sum_{n=-\infty}^{-(k+1)}\left(e^{\rho(n)}\left\{\exp(-te^{\alpha\rho(n)})-\exp(-te^{\alpha\rho(n+1)})\right\}\right)\left(\sum_{l=k+1}^{-n}\left(e^{\rho(l)}-e^{\rho(l-1)}\right)\right) \\
&\le \sum_{n=-\infty}^{-(k+1)}\left\{\exp(-te^{\alpha\rho(n)})-\exp(-te^{\alpha\rho(n+1)})\right\} \\
&= 1-\exp(-te^{\alpha\rho(-k)}).
\end{aligned}$$

□

Proposition 3.2.11. *The transition function $\mathbf{P}_\rho^\alpha(t,y,B)$ satisfies the following two conditions:*

a. For each $s\ge 0$ and compact subset B of $\mathbb{A}_f$

$$\lim_{x\to\infty}\sup_{t\le s}\mathbf{P}_\rho^\alpha(t,x,B)=0 \qquad (\textit{Condition LB}).$$

b. For each $k>0$ and compact subset B of $\mathbb{A}_f$

$$\lim_{t\to 0+}\sup_{x\in B}\mathbf{P}_\rho^\alpha(t,x,\mathbb{A}_f\setminus B_k(x))=0 \qquad (\textit{Condition MB}).$$

Proof. Let $d(x):=dist(x,B)=e^{\rho(-m_x)}$ where $m_x\in\mathbb{Z}$. From Lemma 3.2.6 it follows that

$$Z_\rho^\alpha(x-y,t)\le[d(x)]^{-1}\left(1-\exp(-se^{\alpha\rho(m_x+1)})\right)$$

for any $y\in B$ and $t\le s$. Since B is compact and α is positive, $d(x)^{-1}\longrightarrow 0$ and $e^{\alpha\rho(m_x+1)}\longrightarrow 0$, as $x\to\infty$. Hence

$$\mathbf{P}_\rho^\alpha(t,x,B)\le[d(x)]^{-1}\left(1-\exp(-se^{\alpha\rho(m_x+1)})\right)\mu(B)\longrightarrow 0$$

as $x\to\infty$. This implies Condition $L(B)$.

Presently, we establish Condition $M(B)$: for $y\in\mathbb{A}_f\setminus B_k^\rho(x)$, we have $\|x-y\|_\rho>e^{\rho(k)}$. Therefore

$$\mathbf{P}_\rho^\alpha(t,x,\mathbb{A}_f\setminus B_k^\rho(x))=\int_{\mathbb{A}_f\setminus B_k^\rho(x)} Z_\rho^\alpha(x-y,t)dy=\int_{\mathbb{A}_f\setminus B_k^\rho(0)} Z_\rho^\alpha(y,t)dy.$$

From Lemma 3.2.10,

$$\int_{\mathbb{A}_f\setminus B_k^{\rho}} Z_\rho^\alpha(y,t)dy \leq 1-\exp(-te^{\alpha\rho(-k)}) \longrightarrow 0,\ t\to 0^+.$$

Since $\mathbf{P}_\rho^\alpha(t,x,B_k^\rho(x))$ is invariant under additive traslations, the last equation implies Condition $M(B)$.

□

Theorem 3.2.12. *The heat kernel $Z_\rho^\alpha(x,t)$ is the transition density of a time and space homogeneous Markov process $W_\rho^\alpha(t)$ on $\mathbb{A}_f$, which is bounded, right–continuous and has no discontinuities other than jumps.*

Proof. The result follows from Proposition 3.2.11 and the fact that $\mathbb{A}_f$ is a second countable and locally compact ultrametric space (see [29, Theorem 3.6]). □

3.3 Cauchy problem for parabolic type equations on $\mathbb{A}$

In this section an abstract Cauchy problem on $L^2(\mathbb{A})$ is presented. First, we recollect several properties of the ring of adèles $\mathbb{A}$. The abstract Cauchy problem on $L^2(\mathbb{A})$ is studied by considering the fractional Laplacian on the Archimedean completion, $\mathbb{R}$, and the pseudodifferential operator on $L^2(\mathbb{A}_f)$, studied in the previous section.

3.3.1 The ring of adèles $\mathbb{A}$

In the present section, the ring of adèles $\mathbb{A}$ of $\mathbb{Q}$ is described as the product of its Archimedean part with its non–Archimedean component. We first consider the locally compact and complete Archimedean field of real numbers $\mathbb{R}$.

The Archimedean place

Recall that the real numbers $\mathbb{R}$ is the unique Archimedean completion of the rational numbers. As a locally compact Abelian group, $\mathbb{R}$, is autodual with pairing function given by $\chi_\infty(\xi_\infty x_\infty)$, where $\chi_\infty(x_\infty)=e^{-2\pi i x_\infty}$ is the canonical character on $\mathbb{R}$. In addition, it is a commutative Lie group. The Schwartz space of $\mathbb{R}$, which we denote here by $\mathcal{D}(\mathbb{R})$, consists of functions $\varphi_\infty:\mathbb{R}\longrightarrow\mathbb{C}$ which are infinitely differentiable and rapidly decreasing. $\mathcal{D}(\mathbb{R})$ has a countable family of seminorms which makes it a nuclear Fréchet space. Let dx_∞ denote the usual Haar measure on $\mathbb{R}$. The Fourier transform

$$\mathcal{F}_\infty[\varphi_\infty](\xi_\infty)=\int_{\mathbb{R}}\varphi_\infty(x_\infty)\chi_\infty(\xi_\infty x_\infty)dx_\infty$$

is an isomorphism from $\mathcal{D}(\mathbb{R})$ onto itself. Moreover, the Fourier inversion formula and the Parseval–Steklov identities hold on $\mathcal{D}(\mathbb{R})$. Furthermore, $L^2(\mathbb{R})$ is a separable Hilbert space, the Fourier transform is an isometry on $L^2(\mathbb{R})$, and the Fourier inversion formula and the Parseval–Steklov identity hold on $L^2(\mathbb{R})$.

Definition 3.3.1. *The* adèle ring $\mathbb{A}$ *of* $\mathbb{Q}$ *is defined as* $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$.

With the product topology, $\mathbb{A}$ is a second countable locally compact Abelian topological ring. If μ_∞ is the Haar (Lebesgue) measure on $\mathbb{R}$ and μ_f denotes the Haar measure on $\mathbb{A}_f$, a Haar measure on $\mathbb{A}$ is given by the product measure $\mu = \mu_\infty \times \mu_f$. Recall that, if χ_∞ and χ_f are the canonical characters on $\mathbb{R}$ and $\mathbb{A}_f$, respectively, then $\chi = (\chi_\infty, \chi_f)$ defines a canonical character on $\mathbb{A}$. $\mathbb{A}$ is a selfdual group in the sense of Pontryagin and we have a paring $\chi_\infty(x_\infty \xi_\infty)$.

Bruhat-Schwartz space

For any $\varphi_\infty \in \mathcal{D}(\mathbb{R})$ and $\varphi_f \in \mathcal{D}(\mathbb{A}_f)$, we have a function φ on $\mathbb{A}$ given by

$$\varphi(x) = \varphi_\infty(x_\infty)\varphi_f(x_f)$$

for any adèle $x = (x_\infty, x_f)$. These functions are continuous on $\mathbb{A}$ and the linear vector space generated by these functions is linearly isomorphic to the algebraic tensor product $\mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{A}_f)$. In the following, we identify these spaces and write $\varphi = \varphi_\infty \otimes \varphi_f$.

Since $\mathbb{A}$ is a locally compact Abelian topological group the Bruhat–Schwartz space $\mathcal{D}(\mathbb{A})$ has natural topology described as follows. First, recall that for any ultrametric d_ρ, $\mathcal{D}^\rho_{\ell,k}(\mathbb{A}_f)$ denotes the set functions with support on $B^\rho_k \subset \mathbb{A}_f$ and parameter of constancy l related to d_ρ. We have the algebraic and topological tensor product of a Fréchet space and finite dimensional space, given by

$$\mathcal{D}(\mathbb{R}) \otimes \mathcal{D}^\rho_{\ell,k}(\mathbb{A}_f)$$

which represents a well defined class of functions on $\mathbb{A}$. These topological vector spaces are nuclear Fréchet, since $\mathcal{D}(\mathbb{R})$ is nuclear Fréchet and $\mathcal{D}^\rho_{\ell,k}(\mathbb{A}_f)$ has finite dimension. We have

$$\mathcal{D}(\mathbb{A}) = \varinjlim_{l \leq k} \mathcal{D}(\mathbb{R}) \otimes \mathcal{D}^\rho_{\ell,k}(\mathbb{A}_f).$$

The space of *Bruhat–Schwartz functions* on $\mathbb{A}$ is the algebraic and topological tensor product of nuclear space vector spaces $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}(\mathbb{A}_f)$, i.e.

$$\mathcal{D}(\mathbb{A}) = \mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{A}_f).$$

The Fourier transform on $\mathbb{A}$

The Fourier transform on $\mathcal{D}(\mathbb{A})$ is defined as

$$\mathcal{F}[\varphi](\xi) = \int_{\mathbb{A}} \varphi(x)\chi(\xi x)dx,$$

for any $\xi \in \mathbb{A}$. It is well–defined on $\mathcal{D}(\mathbb{A})$ and for any function of the form $\varphi = \varphi_\infty \otimes \varphi_f$ it is given by

$$\mathcal{F}[\varphi](\xi) = \mathcal{F}_\infty[\varphi_\infty](\xi_\infty) \otimes \mathcal{F}_f(\varphi_f)(\xi_f) \qquad (\xi = (\xi_\infty, \xi_f) \in \mathbb{A})$$

where $\mathcal{F}_\infty$ and $\mathcal{F}_f$ are the Fourier transforms on $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}(\mathbb{A}_f)$, respectively. In other words, we have $\mathcal{F}_{\mathbb{A}} = \mathcal{F}_\infty \otimes \mathcal{F}_f$.

The Fourier transform $\mathcal{F} : \mathcal{D}(\mathbb{A}) \longrightarrow \mathcal{D}(\mathbb{A})$ is a linear and continuous isomorphism. The inversion formula on $\mathcal{D}(\mathbb{A})$ reads as

$$\mathcal{F}^{-1}[\varphi](\xi) = \int_{\mathbb{A}} \widehat{\varphi}(-\xi)\chi(\xi x)d\xi, \qquad (\xi \in \mathbb{A}),$$

and Parseval–Steklov equality as

$$\int_{\mathbb{A}} \varphi(x)\overline{\psi(x)}dx = \int_{\mathbb{A}} \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}d\xi.$$

The space of square integrable functions $L^2(\mathbb{A})$ on $\mathbb{A}$ is a separable Hilbert space since it is the Hilbert tensor product space $L^2(\mathbb{A}) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{A}_f)$. The Fourier transform $\mathcal{F} : L^2(\mathbb{A}) \longrightarrow L^2(\mathbb{A})$ is an isometry. The Fourier inversion formula and the Parseval-Steklov identity hold.

3.3.2 A Cauchy Problem on $L^2(\mathbb{A})$

In this subsection we consider a class of additive invariant positive selfadjoint pseudodifferential unbounded operators on $L^2(\mathbb{A})$ to state a Cauchy problem for parabolic–type equations.

Archimedean heat kernel

Let us recall the theory of the fractional heat kernel on the real line. For a complete review of this topic the reader may consult [28] and the references therein. For any $0 < \beta \leq 2$, the fractional Laplacian $D^\beta_\infty : \mathrm{Dom}(D^\beta_\infty) \subset L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is given by

$$D^\beta_\infty \phi(x_\infty) = \mathcal{F}^{-1}_{\xi_\infty \to x_\infty}\left[|\xi|^\beta_\infty \mathcal{F}_{x_\infty \to \xi_\infty}[f]\right],$$

for any ϕ in the domain

$$\mathrm{Dom}(D^\beta_\infty) := \left\{ f \in L^2(\mathbb{R}) : |\xi|^\beta_\infty \widehat{f} \in L^2(\mathbb{R}) \right\}.$$

Similar to the case of the finite adèle ring, the operator $D^\beta_\infty \phi(x_\infty)$ is diagonalized by the unitary Fourier transform: if m^β_∞ denotes the multiplicative operator on $L^2(\mathbb{R})$ given by $f(\xi) \longmapsto |\xi|^\beta_\infty f(\xi)$, with domain $\mathrm{Dom}(m^\beta_\infty) := \left\{ f \in L^2(\mathbb{R}) : |\xi|^\beta_\infty \widehat{f}(\xi) \in L^2(\mathbb{R}) \right\}$, then the following diagram commutes:

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\ {\scriptstyle D^\beta_\infty}\big\downarrow & & \big\downarrow{\scriptstyle m^\beta_\infty} \\ L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}). \end{array} \tag{3.5}$$

The pseudodifferential equation

$$\begin{cases} \frac{\partial u(x_\infty,t)}{\partial t} + D^\beta_\infty u(x_\infty,t) = 0, \ x_\infty \in \mathbb{R}, \ t \geq 0; \\ u(x,t) = f(x), \qquad f \in \mathrm{Dom}(D^\beta_\infty) \end{cases} \tag{3.6}$$

is an abstract Cauchy problem whose solution is given by the convolution of f with the *Archimedean heat kernel*:

$$Z_\infty^\beta(x_\infty, t) = \int_{\mathbb{R}} \chi_\infty(\xi_\infty x_\infty) e^{-t|\xi_\infty|^\beta} d\xi_\infty \qquad (t > 0).$$

For $0 < \beta \leq 2$, the following bound holds

$$\left| Z_\infty^\beta(x_\infty, t) \right| \leq \frac{C t^{1/\beta}}{t^{2/\beta} + x_\infty^2} \qquad (\text{for } t > 0,\ x_\infty \in \mathbb{R}).$$

Due to this bound, the Archimedean heat kernel satifies several properties: it is the distribution of a probability measure on $\mathbb{R}$; it converges to the Dirac delta distribution as t tends to zero, and it satisfies the Markovian property. Therefore the Archimedean heat kernel is the transition density of a time and space homogeneous Markov process which is bounded, right–continuous and has no discontinuities other than jumps (see [28, Section 2]).

In addition, the formula

$$S_\infty^\beta(f) = f(x_\infty) * Z_\infty^\beta(x_\infty, t) \qquad (f \in L^2(\mathbb{R}))$$

defines a strongly continuous contraction semigroup with the unbounded operator $\left(D_\infty^\beta, \mathrm{Dom}(D_\infty^\beta)\right)$ as infinitesimal generator. Furthermore, there is a commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\ {\scriptstyle S_\infty^\beta(t)=\exp(-tD_\infty^\beta)}\downarrow & & \downarrow{\scriptstyle \exp(-tm_\infty^\beta)} \\ L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}). \end{array} \tag{3.7}$$

where $\exp(-tm_\infty^\beta)$ is the C_0–semigroup of contractions whose infinitesimal generator corresponds to the operator $-m_\infty^\beta$, under the Hille–Yoshida Theorem.

Tensor product of operators

Let us briefly recall the definition of tensor product of operators on the Hilbert space $L^2(\mathbb{A}) = L^2(\mathbb{A}_f) \otimes L^2(\mathbb{R})$ (see [55, Chapter VIII] for complete detail).

Given two (unbounded) closable operators $(A, \mathrm{Dom}(A))$ and $(B, \mathrm{Dom}(B))$ on $L^2(\mathbb{A}_f)$ and $L^2(\mathbb{R})$, respectively, the algebraic tensor product

$$\mathrm{Dom}(A) \otimes \mathrm{Dom}(B) = \left\{ \sum_{\text{finite}} \lambda_i \phi_f^i \otimes \phi_\infty^i \;:\; \phi_f^i \in \mathrm{Dom}(A),\ \phi_\infty^i \in \mathrm{Dom}(B) \right\} \subset L^2(\mathbb{A})$$

is dense in $L^2(\mathbb{A})$, and the operator $A \otimes B$ given by

$$A \otimes B(\phi_f \otimes \phi_\infty) = A(\phi_f) \otimes B(\phi_\infty),$$

for $\phi_f \otimes \phi_\infty \in \mathrm{Dom}(A) \otimes \mathrm{Dom}(A)$, is closable.

The *tensor product* of A and B is the closure of the operator $A \otimes B$ defined on the algebraic tensor product $\mathrm{Dom}(A) \otimes \mathrm{Dom}(B)$. We denote the closed operator by $A \otimes B$ and its domain by $\mathrm{Dom}(A\otimes B)$. Furthermore, if A and B are selfadjoint, their tensor product $A\otimes B$ is essentially selfadjoint and the spectrum $\sigma(A\otimes B)$ of $A\otimes B$ is the closure in $\mathbb{C}$ of $\sigma(A)\sigma(B)$, where $\sigma(A)$ and $\sigma(B)$ are the corresponding spectrum of A and B.

On the other hand, if A and B are bounded operators, their tensor product $A\otimes B$ is bounded with operator norm

$$\|A\otimes B\|_{L^2(\mathbb{A})}=\|A\|_{L^2(\mathbb{A}_f)}\|B\|_{L^2(\mathbb{R})}.$$

Now, let us recall the definition of the sum of unbounded operators on the Hilbert space $L^2(\mathbb{A})=L^2(\mathbb{A}_f)\otimes L^2(\mathbb{R})$ given by $A+B=A\otimes I+I\otimes B$. Once more, the algebraic tensor product $\mathrm{Dom}(A)\otimes\mathrm{Dom}(B)\subset L^2(\mathbb{A})$ is dense in $L^2(\mathbb{A})$ and the operator $A+B=A\otimes I+I\otimes B$ given by

$$(A+B)(\phi_f\otimes\phi_\infty)=A(\phi_f)\otimes\phi_\infty+\phi_f\otimes B(\phi_\infty),$$

with $\phi_f\otimes\phi_\infty\in\mathrm{Dom}(A)\otimes\mathrm{Dom}(B)$ is essentially selfadjoint. The sum of A and B is the closure of the operator $A+B$ defined on $Dom(A)\otimes Dom(B)$. We denote by $\mathrm{Dom}(A+B)$ the domain of the this closed unbounded operator and with abuse of notation we denote this unbounded operator by $A+B$. The spectrum of $\sigma(A+B)$ of $A+B$ is the closure in $\mathbb{C}$ of $\sigma(A)+\sigma(B)$, where $\sigma(A)$ and $\sigma(B)$ are the corresponding spectrum of A and B, respectively.

Pseudodifferential operators on $\mathbb{A}$

First, notice that the multiplicative operator $\widetilde{m}_\rho^{\alpha,\beta}:L^2(\mathbb{A})\longrightarrow L^2(\mathbb{A})$, given by $f(\xi)\longmapsto(\|\xi_f\|_\rho^\alpha+|\xi_\infty|^\beta)f(\xi)$, with (dense) domain

$$\mathrm{Dom}(\widetilde{m}_\rho^{\alpha,\beta}):=\left\{f\in L^2(\mathbb{A})\,:\,\big(\|\xi_f\|_\rho^\alpha+|\xi_\infty|^\beta\big)\widehat{f}(\xi)\in L^2(\mathbb{A})\right\}$$

is selfadjoint and coincides with $m_\rho^{\alpha,\beta}=m_\rho^\alpha+m_\infty^\beta=m_\rho^\alpha\otimes I+I\otimes m_\infty^\beta$ on the set $\mathrm{Dom}(m_\rho^\alpha)\otimes\mathrm{Dom}(m_\infty^\beta)\subset L^2(\mathbb{A})$. Since $m_\rho^{\alpha,\beta}$ is essentially selfadjoint on the domain $\mathrm{Dom}(m_\rho^\alpha)\otimes\mathrm{Dom}(m_\infty^\beta)$ it follows that $m_\rho^{\alpha,\beta}=\widetilde{m}_\rho^{\alpha,\beta}$.

For any $0<\alpha$ and $0<\beta\le 2$, consider the pseudodifferential operator $\widetilde{D}_\rho^{\alpha,\beta}:\mathrm{Dom}(\widetilde{D}_\rho^{\alpha,\beta})\subset L^2(\mathbb{A})\longrightarrow L^2(\mathbb{A})$ defined by the formula

$$\widetilde{D}_\rho^{\alpha,\beta}\phi(x)=\mathcal{F}^{-1}_{\xi\to x}[m_\rho^{\alpha,\beta}\mathcal{F}_{x\to\xi}[\phi]],$$

for any ϕ in the domain

$$\mathrm{Dom}(\widetilde{D}_\rho^{\alpha,\beta}):=\left\{f\in L^2(\mathbb{A}):\,m_\rho^{\alpha,\beta}(\widehat{\phi})\in L^2(\mathbb{A})\right\}.$$

This unbounded operator is a positive selfadjoint operator which is diagonalized by the (unitary) Fourier transform $\mathcal{F}$, i.e. the following diagram commutes:

$$\begin{array}{ccc} L^2(\mathbb{A}) & \xrightarrow{\ \mathcal{F}\ } & L^2(\mathbb{A}) \\ {\scriptstyle \widetilde{D}_\rho^{\alpha,\beta}}\big\downarrow & & \big\downarrow{\scriptstyle m_\rho^{\alpha,\beta}=\widetilde{m}_\rho^{\alpha,\beta}} \\ L^2(\mathbb{A}) & \xrightarrow{\ \mathcal{F}\ } & L^2(\mathbb{A}). \end{array} \tag{3.8}$$

Therefore, the operator $D_\rho^{\alpha,\beta} = D_\rho^\alpha + D_\infty^\beta = D_\rho^\alpha \otimes I + I \otimes D_\infty^\beta$, which is essentially selfadjoint over the domain $\mathrm{Dom}(D_\rho^\alpha) \otimes \mathrm{Dom}(D_\infty^\beta) \subset L^2(\mathbb{A})$, is equal to the operator $\widetilde{D}_\rho^{\alpha,\beta}$.

A heat equation on $\mathbb{A}$

For $f(x) \in \mathrm{Dom}(D_\rho^{\alpha,\beta}) \subset L^2(\mathbb{A})$, consider the abstract Cauchy problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D_\rho^{\alpha,\beta} u(x,t) = 0, \ x \in \mathbb{A}, \ t \geq 0 \\ u(x,t) = f(x). \end{cases} \tag{3.9}$$

As mentioned above, a function $u : \mathbb{A} \times [0, \infty) \longrightarrow \mathbb{C}$ is called a solution of the abstract Cauchy problem (3.9) in the Hilbert space $L^2(\mathbb{A})$, if:

a. $u : [0, \infty) \longrightarrow L^2(\mathbb{A})$ is a continuously differentiable function on the sense of Hilbert spaces,

b. $u(x,t) \in \mathrm{Dom}(D_\rho^{\alpha,\beta})$, for all $t \geq 0$ and,

c. $u(x,t)$ is a solution of the initial value problem.

Furthermore, this abstract Cauchy problem is well posed and its solution is given by a strongly continuous contraction semigroup. From the Hille–Yoshida theorem, the unbounded operator $-D_\rho^{\alpha,\beta}$ is the infinitesimal generator of a strongly continuous contraction semigroup $S_{\mathbb{A}}^\rho(t) = \exp(-tD_\rho^{\alpha,\beta})$. Additionally, to the unbounded operator $-m_\rho^{\alpha,\beta}$ there corresponds a strongly continuous contraction semigroup $\exp(-tm_\rho^{\alpha,\beta})$ with $m_\rho^{\alpha,\beta}$ as infinitesimal generator.

From an argument as in Section 3.2.3, there is a commutative diagram:

$$\begin{array}{ccc} L^2(\mathbb{A}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}) \\ {\scriptstyle \exp(-tD_\rho^{\alpha,\beta})}\downarrow & & \downarrow{\scriptstyle \exp(-tm_\rho^{\alpha,\beta})} \\ L^2(\mathbb{A}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{A}). \end{array} \tag{3.10}$$

In order to describe the solution of problem (3.9), for fixed $\alpha > 0$ and $0 < \beta \leq 2$, we define the *adelic heat kernel* as

$$Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho) = \int_{\mathbb{A}} \chi(-\xi x) e^{-t(\|\xi_f\|_\rho^\alpha + |\xi_\infty|^\beta)} d\xi \qquad (t > 0, \ x, \xi \in \mathbb{A}),$$

where $\xi = (\xi_f, \xi_\infty)$. That is to say,

$$\begin{aligned} Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho) &= \mathcal{F}_{\mathbb{A}}^{-1}(e^{-t(\|\xi_f\|_\rho^\alpha + |\xi_\infty|^\beta)}) \\ &= \mathcal{F}_\infty^{-1}(e^{-t|\xi_\infty|^\beta}) \mathcal{F}_f^{-1}(e^{-t\|\xi_f\|_\rho^\alpha}) \\ &= Z_f(x_f, t, \rho) \otimes Z_\infty(x_\infty, t). \end{aligned}$$

Proposition 3.3.2. *The adelic heat kernel, $Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho)$, satisfies the following properties:*

- *It is the distribution of a probability measure on $\mathbb{A}$, i.e. $Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho) \geq 0$ and*

$$\int_{\mathbb{A}} Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho)dx = 1,$$

for all $t > 0$.

- *It converges to the Dirac distribution as t tends to zero:*

$$\lim_{t\to 0} \int_{\mathbb{A}} Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho)f(x)dx = f(0),$$

for all $f \in \mathcal{D}(\mathbb{A})$.

- *It has the property:*

$$Z_{\mathbb{A}}(x,t+s,\rho) = \int_{\mathbb{A}} Z_{\mathbb{A}}(x-y,t,\rho)Z_{\mathbb{A}}(y,s,\rho)dy.$$

Proof. From the equality $Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho) = Z_f(x_f,t,\rho) \otimes Z_{\infty}^{\beta}(x_{\infty},t)$ it follows that $Z_{\mathbb{A}}(x,t,\rho)$ is in $L^1(\mathbb{A})$ for any $t > 0$, and also

$$\int_{\mathbb{A}} Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho)dx = 1.$$

Using the corresponding properties of the Archimedean heat kernel and the finite adelic heat kernel, for $f \in \mathcal{D}(\mathbb{A})$, we have

$$\lim_{t\to 0} \int_{\mathbb{A}} Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho)\big(f(x) - f(0)\big)dx = 0.$$

The Markovian property follows from the Fourier inversion formula and the related property of the exponential function. □

Now, for any $f \in L^2(\mathbb{A})$, define

$$T_{\mathbb{A}}^{\rho}(t)(f)(x) = \begin{cases} Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho) * f(x) & t > 0, \\ f(x) & t = 0. \end{cases}$$

From Proposition 3.3.2 and Young's inequality it follows that $\{T_{\mathbb{A}}^{\rho}(t)\}_{t\geq 0}$ is a strongly continuous contraction semigroup. On the other hand, from definition, it follows that

$$S_{\mathbb{A}}^{\rho}(t)(\phi_f \otimes \phi_{\infty}) = \left(Z_f^{\alpha}(x_f,t,\rho) * \phi_f\right) \otimes \left(Z_{\infty}^{\beta}(x_{\infty},t) * \phi_{\infty}\right).$$

Theorem 3.3.3. *If f is any complex valued square integrable function on $\mathrm{Dom}(D_{\rho}^{\alpha,\beta})$, then the Cauchy problem*

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D_{\rho}^{\alpha,\beta}u(x,t) = 0, \ x \in \mathbb{A}, \ t > 0, \\ u(x,t) = f(x) \end{cases}$$

has a classical solution $u(x,t)$ determined by the convolution of f with the heat kernel $Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho)$. Moreover, $Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho)$ is the transition density of a time and space homogeneous Markov process $W_{\rho}^{\alpha,\beta}(t)$ on $\mathbb{A}$, which is bounded, right-continuous and has no discontinuities other than jumps.

Proof. Similar to Section 3.2, for $f \in L^1(\mathbb{A}) \cap L^2(\mathbb{A})$, since the adelic heat kernel is absolute integral, the convolution $Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho) * f(x)$ is in $L^1(\mathbb{A}) \cap L^2(\mathbb{A})$ and

$$\mathcal{F}_{x\to\xi}(Z_{\mathbb{A}}^{\alpha,\beta}(x,t,\rho) * f(x)) = \hat{f}(\xi)\exp(-t\,\|\xi_f\|_\rho^\alpha + |\xi_\infty|^\beta).$$

Therefore $T_{\mathbb{A}}^\rho(t) = S_{\mathbb{A}}^\rho(t)$ coincides on a dense set of $L^2(\mathbb{A})$. The properties of the Markov process follow because, the product of two Markov process which satisfy conditions MB and LB also satisfies those conditions (see [73, Section 4.9]). □

Chapter 4

Final remarks

The theory developed in this work can be applied to more general Abelian topological groups. In detail, let G be a selfdual, second countable, totally disconnected, locally compact, Abelian topological group with a filtration by compact and open subgroups $\{H_n\}_{n\in\mathbb{Z}}$,

$$\{0\} \subset \cdots \subset H_{-n} \subset \cdots \subset H_0 \subset \cdots \subset H_n \subset \cdots \subset G,$$

such that:

1. $H_0 = H$ is a fixed compact and open subgroup of G,
2. the annihilator, $\mathrm{Ann}_G(H_n)$, satisfies

   $$\mathrm{Ann}_G(H_n) = H_{-n},$$

   for all $n \in \mathbb{Z}$, and
3. the following relations are satisfied:

   $$\bigcap_{n\in\mathbb{Z}} H_n = \{0\} \quad \text{and} \quad \bigcup_{n\in\mathbb{Z}} H_n = G.$$

Notice that, since G is autodual, there exist an isomorfism of G with its Pontryagin dual group $\widehat{G}$, i.e. a function $\xi \mapsto \chi_\xi$ which identifies G and $\widehat{G}$ as topological groups. This identification makes expression $\mathrm{Ann}_G(H_n) = H_{-n}$, in property (3) above, meaningful.

Normalize the Haar measure μ on G in such a way that $\mu(H) = 1$. This implies that the measure of any other subgroup H_n of G is given either by the index $[H : H_n]$ or by the index $[H_n : H]$. The group G has a unique G–invariant ultrametric d_G such that the balls centred at zero coincide with the elements of the filtration $\{H_n\}_{n\in\mathbb{Z}}$ and the radius of any ball equals to its Haar measure. Let λ_n be the radius and Haar measure of H_n.

The topological and algebraic properties of G are expressed by the projective and inductive limits:

$$H_0 = \varprojlim_{n\leq 0} H_0/H_n \qquad G = \varinjlim_{n\geq 0} H_n.$$

To the ultrametric spaces, (H, d_H) and (G, d), there correspond, respectively, a tree $\mathcal{T}(H)$ with finite ramification index and endspace H, and an extended tree $\mathcal{T}(G)$ with finite ramification index and endspace $G \cup \{\infty\}$. Consequently, $L^2(G)$ is a separable Hilbert space. In addition, the topology of the locally constant functions of compact support $\mathcal{D}(G)$ is expressed by the inductive limits

$$\mathcal{D}^\ell(G) = \varinjlim_k \mathcal{D}^\ell_k(G) \quad \text{and} \quad \mathcal{D}(G) = \varinjlim_\ell \mathcal{D}^\ell(G)$$

and $\mathcal{D}(G)$ is a locally convex complete topological algebra and a nuclear space.

We have the Fourier transform $\mathcal{F}(f)(\xi) = \int_G f(x)\chi_\xi(x)d\mu(x)$ and due to equality $\mathrm{Ann}_G(H_n) = H_{-n}$, it satisfies $\mathcal{F} : \mathcal{D}^\ell_k(G) \longrightarrow \mathcal{D}^{-k}_{-\ell}(G)$ and gives an isomorfism $\mathcal{D}(G) \cong \mathcal{D}(G)$ of locally convex topological linear space. Furthermore, the Fourier transform $\mathcal{F} : L^2(G) \longrightarrow L^2(G)$ is an isometry of Hilbert spaces.

For any $\alpha > 0$, the function $\|\cdot\|^\alpha_G = d_G(0, \cdot)^\alpha$ defines the pseudodifferential operator $D^\alpha : \mathrm{Dom}(D^\alpha) \subset L^2(G) \longrightarrow L^2(G)$ given by

$$D^\alpha(f) = \mathcal{F}^{-1}_{\xi\to x}[\|\xi\|^\alpha_G \mathcal{F}_{x\to\xi}[f]] \qquad (f \in \mathrm{Dom}(D^\alpha)).$$

The operator D^α is a positive selfadjoint unbounded operator with spectrum $\{0\} \cup \{\lambda^\alpha_n\}_{n\in\mathbb{Z}}$. The heat kernel $Z(x,t) = \mathcal{F}^{-1}_{\xi\to x}\big(\exp(-t\,\|\xi\|^\alpha_G)\big)$ is a well defined positive function, given by a formula similar to the one in Proposition 3.2.3 and satisfies the estimate of Proposition 3.2.6.

As a consequence of all that have been said, the following result holds.

Theorem 4.0.1. *If f belongs to* $\mathrm{Dom}(-D^\alpha) \subset L^2(G)$*, the Cauchy problem*

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + D^\alpha_\rho u(x,t) = 0, \ x \in G, \ t \geq 0, \\ u(x,0) = f(x), \end{cases}$$

has a classical solution $u(x,t)$ determined by the convolution of f with the heat kernel $Z^\alpha(x,t)$. In addition, $Z^\alpha(x,t)$ is the transition density of a time and space homogeneous Markov process $W^\alpha(t)$ on G, which is bounded, right–continuous and has no discontinuities other than jumps.

Examples of these groups can be given as follows:

1. For any fixed positive integer number m, let G be the set of m–adic numbers $\mathbb{Q}_m$ with $H_0 = \mathbb{Z}_m$ the maximal compact and open subring. The filtration given by $H_\ell = m^{\ell^3}\mathbb{Z}_m$, for $\ell \in \mathbb{Z}$ leads to the m–adic analysis of $\mathbb{Q}_m$ (see [26]).

2. Given a sequence $(e^{\rho(n)})^\infty_{n=0}$, which is totally order by division and begins at one, there exists a polyadic completion $\mathbb{Q}(\rho)$ of the rational numbers (see [34, 37]). A filtration can be given as $\{e^{\pm\rho(n)}\mathbb{Z}(\rho)\}$, where $\mathbb{Z}(\rho)$ is the maximal compact and open subring of $\mathbb{Q}(\rho)$ (seee [3]).

3. Set G the restricted direct product of a countable copies of the fixed topological ring $\mathbb{A}_f$ with respect to $\widehat{\mathbb{Z}}$. That is, $G = \prod'_{n\geq 1} \mathbb{A}_f$ and set $H_0 = \prod_{n\geq 1} \widehat{\mathbb{Z}}$. Let $(e^{\rho(n)})_{n=0}^{\infty}$ be a strictly increasing sequence of natural numbers, beginning with one, totally order by division and cofinal with the natural numbers. The members of a filtration can be given as:

   - $H_0 = \prod_{n\geq 1} \widehat{\mathbb{Z}}$,
   - $H_{\pm 1} = e^{\mp\rho(1)}\widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}} \times \cdots$,
   - $H_{\pm 2} = e^{\mp\rho(1)}\widehat{\mathbb{Z}} \times e^{\mp\rho(1)}\widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}} \times \cdots$,
   - $H_{\pm 3} = e^{\mp\rho(2)}\widehat{\mathbb{Z}} \times e^{\mp\rho(1)}\widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}} \times \cdots$,
   - $H_{\pm 4} = e^{\mp\rho(2)}\widehat{\mathbb{Z}} \times e^{\mp\rho(1)}\widehat{\mathbb{Z}} \times e^{\mp\rho(1)}\widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}} \times \cdots$

4. Finite products and restricted direct products of the examples above provides a large class of groups satisfying the requirements.

Bibliography

[1] V.A. Aguilar–Arteaga, M. Cruz–López and S. Estala–Arias. *A Heat Equation on Some Adic Completions of* $\mathbb{Q}$ *and Ultrametric Analysis*. p–Adic Numbers, Ultrametric Analysis and Applications **9**, no. 3, 165–182, 2017.

[2] V.A. Aguilar–Arteaga and S. Estala–Arias. *Pseudodifferential Operators and Markov Processes on Adèles*. p–Adic Numbers, Ultrametric Analysis and Applications **11**, no. 2, 89–113, 2019.

[3] V.A. Aguilar–Arteaga, M. Cruz–López and S. Estala–Arias. *Harmonic Analysis on the Adèle Ring of* $\mathbb{Q}$. Submmited for publishing. arXiv:1803.01964.

[4] Sergio Albeverio and Witold Karwowski. *A random walk on p–adics—the generator and its spectrum* . Stochastic Processes and their Applications, **53**, no. 1, 1–22, 1994.

[5] S. Albeverio, A.Y. Khrennikov, and V.M. Shelkovich. *Theory of p-adic Distributions: Linear and Nonlinear Models*. Number 370. LMS, Lectures Notes Series, Cambridge University Press, 2010.

[6] M. V. Altaisky and B. G. Sidharth. *p–Adic Physics Below and Above Planck Scales*. Chaos, Solitons & Fractals, **1O**, no. 2-3, 167–176, 1999.

[7] A. Ansari, J. Berendzen, S.F. Bowne, H. Frauenfelder, I.E.T. Iben, T.B. Sauke, E. Shyamsunder and R.D. Young, *Protein states and proteinquakes.* Proc. Natl. Acad. Sci. USA, **82**, 5000-5004, 1985.

[8] T.M. Apostol. *Introduction to Analytic Number Theory*. Undergraduate Texts in Mathematics. Springer-Verlag, New York–Heidelberg, 1976.

[9] Alberto Apostolico, Matteo Comin, Andres Dress and Laxmi Parida. *Ultrametric networks: a new tool for phylogenetic analysis*. Algorithms for Molecular Biology, **8**, no. 7, 2013.

[10] D. Applebaum. *Brownian motion and Levy processes on locally compact groups*. Methods of Functional Analysis and Topology **12**, no. 2, 101–112, 2006.

[11] D. Applebaum. *Probabilistic trace and Poisson summation formulae on locally compact abelian groups*. Forum Mathematicum **29**, no. 3, 501–517, 2017.

[12] V.A. Avetisov, A.H. Bikulov, S.V. Kozyrev and V.A. Osipov. *p-adic models of Ultrametric diffusion constrained by hierarchical energy landscapes.* Journal of Physics A: Mathematical and General. **35**, no. 2, 177–189, 2002.

[13] V.A. Avetisov, A.H. Bikulov and V.A. Osipov. *p-adic description of characteristic relaxation in complex systems.* Journal of Physics A: Mathematical and General. **36**, no. 15, 4239–4246, 2003.

[14] V.A. Avetisov, A.H. Bikulov and V.A. Osipov. *p-adic models of ultrametric diffusion in the conformal dynamics of macromolecules.* Proc. Setklov Inst. Math **2452**, no. 1, 48–57, 2004.

[15] A.D. Bendikov, A.A. Grigor'yan, Ch. Pittet and W. Woess. *Isotropic Markov Semigroups on Ultrametric Spaces.* Russian Math. Surveys **69**, no. 4, 589–680, 2014.

[16] Kevin A. Broughan. Adic Topologies for the Rational Integers. *Canadian Journal of Mathematics*, **55**, no. 4, 711–723, 2003.

[17] F. Bruhat. *Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques.* Bulletin de la Société Mathématique de France **89**, 43–75, 1961.

[18] Daniel O. Cajueiro, Thiago R. Serra and Benjamin M. Tabak. *Topological properties of stock market networks: The case of Brazil.* Physica A, **389**, no. 16, 3240–3249, 2010.

[19] T. Cazenave and A. Haraux. *An Introduction to Semilinear Evolution Equations.* Oxford Lecture Series in Mathematics and its Applications **13**. The Clarendon Press, Oxford University Press, New York, 1998.

[20] M. Cruz-López and S. Estala-Arias. Invariant Ultrametrics and Markov Processes on the Finite Adèle Ring of $\mathbb{Q}$. *P-Adic Numbers, Ultrametric Analysis, and Applications*, **8**, no. 2, 89–114, 2016.

[21] M. Del Muto and A. Figà-Talamanca. *Diffusion on locally compact ultrametric spaces.* Expositiones Mathematicae **22**, no. 3, 197 – 211, 2004.

[22] Cecile M. DeWitt and John A. Wheeler. *BATTELLE RECONTRES 1967 Lectures in Mathematics and Physics.* W. A. Benjamin, Inc., New York, USA, 1968.

[23] Bayram Deviren, Mustafa Keskin and Yusuf Kocakaplan. *Topology of the correlation networks among major currencies using hierarchical structure methods.* Physica A, **390**, no. 4, 719–730, 2011.

[24] Bayram Deviren, Mustafa Keskin and Yusuf Kocakaplan. *Hierarchical structures of correlations networks among Turkey's exports and imports by currencies.* Physica A, **391**, no. 24 , 6509–6518, 2012.

[25] J. de Groot. *Non-Archimedean Metrics in Topology.* Proc. Amer. Math. Soc. **7**, no. 5, 948–956, 1956.

[26] M.V. Dolgopolov and A.P. Zubarev. *Some aspects of m-adic analysis and its applications to m-adic stochastic processes.* p–Adic Numbers, Ultrametric Analysis and Applications **3**, no. 1, 39–51, 2011.

[27] B. Dragovich, A. Khrennikov and Ya. Radyno. *Distributions on Adeles.* J. Math. Sci. **142**, no. 3, 2105–2112, 2007.

[28] J. Droniou, T. Gallouet and J. Vovelle. *Global Solution and Smoothing Effect for a Non-local Regularization of a Hyperbolic Equation.* J. Evol. Equ. **3**, no. 3, 499–521, 2003.

[29] E. B. Dynkin. *Markov Processes.* Springer Berlin Heidelberg, 1965.

[30] K.J Engel and R. Nagel. *One–Parameter Semigroups for Linear Evolution Equations.* Springer–Verlag, 2000.

[31] S.N. Evans. *Local properties of Lévy processes on a totally disconnected group.* J. Theoret. Probab. **2**, no. 2, 209–259, 1989.

[32] P.W. Fenimore, H. Frauenfelder and B.H. McMahon. *Myoglobin, the hydrogen atom of biology and paradigm of complexicity.* Proc. Natl. Acad. Sci. USA **100**, no. 15, 8615–8617, 2003.

[33] I.M. Gelfand, M.I. Graev, K.A. Hirsch and I.I. Pyatetskii-Shapiro. (Translator) *Representation Theory and Automorphic Functions*, W. B. Saunders; 1st edition, 1969.

[34] E. Hewitt and K.A. Ross. *Abstract Harmonic Analysis I*, Springer Verlag, Berlin, 1970.

[35] J.I. Igusa. *An Introduction to the Theory of Local Zeta Functions.* Number 14. AMS/IP Studies in Advanced Mathematics, American Mathematical Society, Providence, RI; International Press, Cambridge, MA,, 2000.

[36] W. Karwowski and R. Vilela Mendes. *Hierarchical structures and asymmetric stochastic processes on p-adics and adeles.* J. Math. Phys **35**, no. 9 , 4637–4650, 1994.

[37] V.P. Khavin and N.K. Nikol'skij. (Eds.) *Commutative Harmonic Analysis I*, Encyclopaedia of Mathematical Sciences, Springer–Verlag, Berlin, 1991.

[38] A.Y. Khrennikov and Y.V. Radyno. *On adelic analogue of Laplacian.* Proc. Jangjeon Math. Soc. **6**, no. 1, 1–18, 2003.

[39] A.Y. Khrennikov and S.V. Kozyrev. *Pseudodifferential Operators on Ultrametric Spaces, and Ultrametric Wavelets.* (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **69**, no. 5, 133–148, 2005; translation in Izv. Math. **69**, no. 5, 989–1003, 2005.

[40] A.Y. Khrennikov, A.V. Kosyak and V.M. Shelkovich. *Pseudodifferential Operators on Adèles and Wavelet Bases*. (Russian) Dokl. Akad. Nauk **444**, no. 3, 253–257, 2012; translation in Dokl. Math. **85**, no. 3, 358–362, 2012.

[41] A.N. Kochubei. *Pseudo–differential Equations and Stochastics over Non–Archimedean Fields*. Number 244. Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2001.

[42] Michal Kolar and Frantisek Slanina. *How the quasispecies evolution depends on the topology of the genome space*. Physica A **313**, no. 3–4, 549–568, 2002.

[43] S. Lang. *Algebraic Number Theory. Second edition*. Graduate Texts in Mathematics **110**. Springer–Verlag, New York, 1994.

[44] Y.I. Manin. *Reflections on arithmetical physics*. Conformal invariance and string theory (Poiana Braşov, 1987), 293–303, Perspect. Phys., Academic Press, Boston, MA, 1989.

[45] Rosario N. Mantegna. *Hierarchical structure in financial markets*. The European Physical Journal B **11**, no. 1, 193–197, 1999.

[46] Rosario N. Mantegna and H. Eugene Stanley. *An Introduction to Econophysics. Correlations and Complexity in Finance*. Cambridge University Press, Cambridge, United Kingdom, 1999.

[47] R. B. McFeat. *Geometry of Numbers in Adele Spaces*. Warszawa: Instytut Matematyczny Polskiej Akademi Nauk, 1971.

[48] Brendan J. Moyle, Michael J. Naylor and Lawrence C. Rose. *Topology of foreign exchange markets using hierarchical structure methods*. Physica A **382**, no. 1, 199–208, 2007.

[49] F. Murtagh. *From Data to the p-Adic or Ultrametric Model*. p-Adic Numbers, Ultrametric Analysis and Applications **1**, no. 1, 58–68, 2009.

[50] F. Murtagh. *Symmetry in Data Mining and Analysis: A Unifying View Based on Hierarchy*. Proceedings of the Steklov Institute of Mathematics **265**, no. 1, 177–198, 2009.

[51] K. R. Parthasarathy. *Probability Measures on Metric Spaces*. Probability and Mathematical Statistics A Series of Monographs and Textbooks. Academic Press Inc., New York, 1967.

[52] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer - Verlag, 1983.

[53] D. Ramakrishnan and R. J. Valenza. *Fourier Analysis on Number Fields*. Springer–Verlag, 1999.

[54] R. Rammal, G. Toulouse and M.A. Virasoro. *ultrametricity for physicist.* Rev. Mod. Phys. **58**, no. 3, 765–788, 1986.

[55] M. Reed and B. Simon. *Methods of Modern Mathematical Physics: Functional Analysis I.* Academic, New York (1980).

[56] L. Saloff-Coste. *Opérateurs pseudo-différentiels sur certains groupes totalement discontinus*, Studia Mathematica **83**, no. 3, 205–228, 1986.

[57] H.H. Schaefer and M.P. Wolff. *Topological Vector Spaces.* Springer–Verlag, 1999.

[58] R.A. Struble. *Metrics in Locally Compact Groups.* Compositio Mathematica **28**, no. 3, 217–222, 1974.

[59] K. Taira. *Boundary Value Problems and Markov Processes*, Springer - Verlag, 2009.

[60] J. Tate. *Fourier analysis on algebraic number fields and Hecke zeta functions*, in Algebraic Number Theory, Cambridge University Press, 1967.

[61] S. Torba and W. A. Zúñiga–Galindo. *Parabolic Type Equations and Markov Stochastic Processes on Adeles.* J. Fourier Anal. Appl. **19**, no. 4, 792–835, 2013.

[62] R. Urban. *Markov processes on the adeles and Dedekind's zeta function*, Statistics and Probability Letters **82**, no. 8, 1583–1589, 2012.

[63] V.S. Vladimirov, I.V. Volovich, and E.I. Zelenov. *p–Adic Analysis and Mathematical Physics.* Number 1. Series on Soviet and East European Mathematics, World Scientific Publishing Co., Inc., River Edge, NJ,, 1994.

[64] V.S. Varadarajan. *Reflections on Quanta, Symmetries, and Supersymmetries.* Springer, New York, 2011.

[65] I.V. Volovich. *Number theory as the ultimate physical theory.* p–Adic Numbers, Ultrametric Analysis and Applications **2**, no. 1, 77–87, 2010.

[66] I.V. Volovich. *p-Adic string.* Classical and Quantum Gravity **4**, no. 4, 83–87, 1987.

[67] A. Weil. *Basic Number Theory.* Third edition. Die Grundlehren der Mathematischen Wissenschaften, Band **144**. Springer–Verlag, New York-Berlin, 1974.

[68] K. Yasuda. *Additive processes on local fields.* J. Math. Sci. Univ. Tokyo **3**, no. 3, 629 – 654, 1996.

[69] K. Yasuda. *On infinitely divisible distributions on locally compact groups.* J. Theoret. Probab. **13**, no. 3, 635–657, 2000.

[70] K. Yasuda. *Semi-stable processes on local fields.* Tohoku Math. J. **58**, no. 3, 419 – 431, 2006.

[71] K. Yasuda. *Markov processes on the Adeles and representations of Euler products*. J. Theoret. Probab. **23**, no. 3, 748–769, 2010.

[72] K. Yasuda. *Markov processes on the adeles and Chebyshev function*. Statistics and Probability Letters **83**, no. 1, 238–244, 2013.

[73] W. A. Zúñiga–Galindo. *Pseudodifferential Equations Over Non-Archimedean Spaces*. Lecture Notes in Mathematics. Springer International Publishing, 2016.