CT-695-SS1
Don.- 2012

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Centro de Investigación y de Estudios Avanzados
del Instituto Politécnico Nacional
Unidad Guadalajara

# Sobre el problema de la regulación no lineal utilizando redes neuronales

Tesis que presenta:

**Francisco Javier Lasa Gutiérrez**

para obtener el grado de:

**Maestro en Ciencias**

en la especialidad de:

**Ingeniería Eléctrica**

Directores de Tesis

**Dr. Bernardino Castillo Toledo**
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CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL
BIBLIOTECA CENTRAL

CINVESTAV del IPN Unidad Guadalajara, Guadalajara, Jalisco, Febrero de 2012.

[illegible] CT.-00599
[illegible] CT.-695-SS1
[illegible] 28-08-2012
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ID: 199562

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# On the problem of nonlinear output regulation using neural networks

Presented by:

**Francisco Javier Lasa Gutiérrez**

To obtain the degree in:

**Master in Science**

specialty in:

**Electric Engineering**

Thesis Advisors:

**Dr. Bernardino Castillo Toledo**
**Dr. Stefano Di Gennaro**

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**Tesis de Maestría en Ciencias**
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# Acknowledgments

A special thanks to

My parents, without its support this work could not have been made.

Dr. Bernardino Castillo for his contributions during the development of this work.

Dr. Edgar Sánchez and Dr. Antonio Ramírez for the time they spent reviewing this work.

To CONACyT for granting me the scholarship which allowed me to develop this dissertation.

To my colleagues for their knowledge and advice during these years.

# Abstract

La teoría de control puede ser dividida en dos grandes grupos; el primero de ellos se encarga de estudiar la estabilización de los sistemas, mientras que el segundo se encarga de estudiar cómo hacer que un sistema siga una trayectora de referencia deseada. Mucho esfuerzo se ha hecho en ambos grupos, en éste trabajo se estudia el segundo caso. Mediante redes neuronales dinámicas se identifica un sistema no lineal parcialmente desconocido y utilizando la teoría de regulación se logra que el sistema siga una trayectoria definida.

Un problema común al trabajar con redes neuronales es que las leyes de adaptación de pesos no son contínuas, sin embargo para poder resolver de manera exacta las ecuaciones del regulador es necesario que las leyes sean continuas. En este trabajo se propone una estructura de red neuronal y utilizando una técnica de adaptación de pesos contínua, desarrollada recientemente, se logran resolver las ecuaciones del regulador para la red neuronal.

# Abstract

Control theory can be subdivided into two big categories, the first one studies the stabilization of systems; while the second one studies how to make the system follow a desired trajectory. Great efforts have been made in the research of both groups, in this work the second case is studied. With dynamic neural networks a partially unknown nonlinear system is identified and using the regulation theory trajectory tracking is achieved.

A common problem that arises when working with neural networks is that the adaptation laws are not continuous, however, in order to solve exactly the regulator equations it is necessary to have continuous adaptation laws. In this work a neural network structure is proposed and using a continuous adaptation law of the neural network weight's developed recently, the regulator equations are solved exactly for the neural network.

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# Chapter 1

# Introduction

## 1.1 Preliminaries

Controlling a system to track a desired trajectory and at the same time to reject perturbations has been studied inside the control community for a long time. For the linear case, this problem has been studied by many authors, among whom Smith and Davison (1972), Francis and Wohnam (1975), Francis (1977). In particular, the last work shows that the solution of the Output Regulation problem relies on the solution of two Linear Matrix Equations. In the work of Francis and Wohnam (1975), it has been shown that, for the case of error feedback, any regulator that solves the problem must contain a model of the dynamic system producing the reference and/or disturbance signal. This property is known as the internal model principle. The extension for the non-linear case was considered by Francis and Wohnam, where the output regulation problem was first solved for a class of nonlinear systems where the exogenous signals are constant. Isidori and Byrnes studied the existence of the steady state manifold for the case when the plant is assumed to be known exactly. They used the center manifold theory, and established that it is possible to reduce this problem to a set of mixed nonlinear partial differential and algebraic equations. However, the solution of the regulator equations are, in general, difficult to solve. In the past decades, a lot of effort have been dedicated to obtain good approximations for the solution of this set of equations.

It was not until the last decade that the neural networks became an important tool in this field. In the case of continuous systems, dynamic and recurrent neural networks have been designed in different frameworks to achieve asymptotic tracking of the reference signal and/or disturbance rejection. Zhou and Wang in [40] used a class of radial basis function neural network in order to approximate the solution of the regulator equations. They demonstrated that this class of neural networks can solve the regulator equations, up to a prescribed arbitrarily small error. On this work, it is proved too that the steady-state tracking error for the closed-loop system is bounded. One of the main disadvantages in this result is that the neural network training is made off-line; if a variation in the parameters is made, the solution of the regulator equations will change, and the system will not be able to track the desired trajectory. Also, the neural network used in this work has 181 inputs and 361 centers. The size of the neural network can be made smaller if the learning law is always on.

## 1.2 Motivation

As it can be seen in different works [14] [18] [4], the solution of the Nonlinear Output Regulation Problem is not trivial. It has been shown that the solvability of this problem relies on the solution of a set of partial differential equations[17]. In [35] the author uses a neural network in order to find an approximate solution when the nonlinear system is partially unknown; this solution is then compared with a third order linearization via Taylor serie's expansion. However, the size of the neural network is not known, so they use genetic algorithms to find the right number of neurons, and then train the neural network by means of the descendent gradient algorithm; one of the main disadvantages of this kind of training is that it can get stuck on a local minimum. In [29] a neural network is used in order to solve the trajectory tracking for a nonlinear system; however, the bound of the identification error can get smaller if we use another Lyapunov function. In [5] a black box neural identificator is used in order to approximate the nonlinear system; once the neural identificator has been trained, the nonlinear output regulation problem is solved for this neural network. However, the neural network adaptation law is implemented with non-smooth functions, and the regulator equations cannot be obtained exactly. The major motivation for this dissertation is to look for a structure of neural network in order to obtain the solution of the regulator equations as simple as possible, while the tracking error remains bounded.

## 1.3 Objectives

The main objective of this work is to study the Output Regulation Problem when both the plant and the exosystem are represented by neural network models, this has the advantage that once the neural network is following the plant/exosystem, it will be able to force the system to follow the desired trajectory.

To this end, goals are identified which must be fulfilled in order to complete the main objective. The following list presents the most important topics which need to be studied.

1. Research on the integration of the Output Regulation Theory with Neural Networks in order to find a good neural network structure to work with.

2. Research on the learning algorithms used to train the Neural Network, which can be useful within this framework.

3. Assume parametric variations on the plant and use a neural network to identify the nonlinear system; then, solve the regulator equations for a known exosystem and the neural network.

4. Assume parametric variations on the exosystem and use a neural network to identify the exosystem; then, solve the regulator equations for a known plant and the neural network.

Both problems have a similar framework, a neural network will be adapted on-line in order to deal with the uncertainties.

## 1.4 Thesis structure

This document is organized as follows:

Chapter 2 A brief review on the Output Regulation Problem is presented for both, state feedback and error feedback cases.

Chapter 3 The main algorithms in order to find the exact solution for the Output Regulation Problem are reviewed; a numerical example illustrates how the controller dimension can increase even for simple examples.

Chapter 4 In this section, the main algorithms in order to find an approximate solution for the Output Regulation Problem are reviewed. An approach based on dynamical neural networks is studied too, for the case where the plant has parametric variations as well as where the exosystem is partially unknown.

Chapter 5 Different examples are developed in this section in order to show the approach proposed in chapter 4.

Chapter 6 The conclusions and final comments are stated, as well as future work which can improve this work is suggested.

Appendix A The main mathematical tools used in this work are reviewed. Begining with dynamical systems and ending with the universal approximation theorem of neural networks.

Appendix B The proof for the neural network identification bounds is established here.

# Chapter 2

# The output regulation problem

A common problem in control applications is to design and implement control laws whoch achieve asymptotic tracking and/or disturbance rejection for systems. This is known as the output regulation problem. As first established in Isidori and Byrnes [19], the main condition for the solution of this problem via state-feedback or output-feedback control is the solvability of the so called regulator equations. If this equations are solvable, under some standard assumptions, there exists a state-feedback or output-feedback control law such that the closed-loop system is internally stable, and the tracking error will asymptotically approach to zero for all sufficiently small initial conditions of the plant and sufficiently small reference inputs and/or disturbances. This chapter presents the classical Output Regulation Problem as well as the solution of this problem. The linear output regulation problem is a special case, and was completely solved by the collective efforts of several researchers, including Davison, Francis, and Wohnam, among others.

In order to formulate the Output Regulation Problem formally, consider a system of the form

$$
\begin{aligned}
\dot{x}_t &= f(x_t, \omega_t, u_t) \qquad (2.1)\\
\dot{\omega}_t &= s(\omega_t)\\
e &= h(x_t, \omega_t)
\end{aligned}
$$

with the state $x$ defined in a neighborhood $U$ near the origin in $\mathbb{R}^n$, the input space $\mathbf{R}^m$ and the state $\omega$ defined in a neighborhood $W$ near the origin $\mathbb{R}^q$. Two scenarios can be considered, depending on the available information as follows.

## 2.1 State feedback output regulation

Consider that the plant states $x_t$ and the exosystem states $\omega_t$ are measured; that is, the controller has all the information available. The nonlinear state feedback output regulation is stated as follows.
Given a nonlinear system of the form (2.1), determine, if possible, a control law $u = \alpha(x, \omega)$ such that:

$S_{FI}$ The equilibrium point $x = 0$ of

$$\dot{x}_t = f(x_t, 0, \alpha(\dot{x_t}, 0)) \tag{2.2}$$

is asymptotically stable on the first approximation.

$R_{FI}$ There exists a neighborhood $W \in U \times \Omega$ near $(0,0)$ such that, for every initial condition $(x_0, \omega_0) \in \Omega$ the solution of

$$\begin{aligned} \dot{x}_t &= f(x_t, \omega_t, \alpha(x_t, \omega_t)) \\ \dot{\omega}_t &= s(\omega_t) \end{aligned} \tag{2.3}$$

satisfies

$$\lim_{t\to\infty} e_t = 0 \tag{2.4}$$

The properties of the lineal approxiamtion for the controlled plant play an important role in the solution of the output regulation problem; hence, it is convenient to introduce a notation where the parameters of this approximation appear explicit. Notice that the closed loop system (2.3) can be formulated as:

$$\begin{aligned} \dot{x}_t &= (A + BK)x_t + (P + BL)\omega_t + \varphi(x_t, \omega_t) \\ \dot{\omega}_t &= S\omega_t + \psi(x_t, \omega_t) \end{aligned} \tag{2.5}$$

where $\varphi(x_t, \omega_t)$ and $\psi(x_t, \omega_t)$ vanish in the origin along with their first order derivatives and $A, B, P, K, L, S$ are matrices defined by

$$\begin{aligned} A &= \left[\frac{\partial f}{\partial x}\right]_{0,0,0} & B &= \left[\frac{\partial f}{\partial u}\right]_{0,0,0} \\ P &= \left[\frac{\partial f}{\partial \omega}\right]_{0,0,0} & K &= \left[\frac{\partial \alpha}{\partial x}\right]_{0,0,0} \\ L &= \left[\frac{\partial \alpha}{\partial \omega}\right]_{0,0,0}, & S &= \left[\frac{\partial s}{\partial \omega}\right]_{0,0,0} \end{aligned} \tag{2.6}$$

for every $\omega \subset \Omega_0$.

The necessary and sufficient conditions for the solution of the state feedback output regulator are established in the following theorem.

**Theorem 2.1.1.** *The state feedback output regulation problem has a solution if and only if the pair $(A, B)$ is stabilizable and there exists mappings such that $\pi(\omega_t)$ and $u = c(\omega_t)$, with $\pi(0) = 0$ and $c(0) = 0$, both defined on a neighborhood $\Omega_0 \subset \Omega$, from the origin such that:*

$$\begin{aligned} \frac{\partial \pi}{\partial \omega} s(\omega) &= f(\pi(\omega), \omega, \alpha(\pi(\omega), \omega) \\ 0 &= h(\pi(\omega), \omega) \end{aligned} \tag{2.7}$$

for every $\omega \subset \Omega_0$.

Proof: See [17]

Once $\pi(\omega_t)$ and $c(\omega_t)$ are known from equation (2.7), the control law which solves the output regulation problem is:

$$\alpha(x_t, \omega_t) = c(\omega_t) + K(x_t - \pi(\omega_t)) \tag{2.8}$$

where $K$ is a matrix such that $(A + BK)$ is Hurwitz.

The block diagram for the control law is presented in Figure 2.1.

## 2.2 Error feedback output regulation

For the second scenario, the output error $e_t$ is the only measurement available. The problem can be stated formally as:

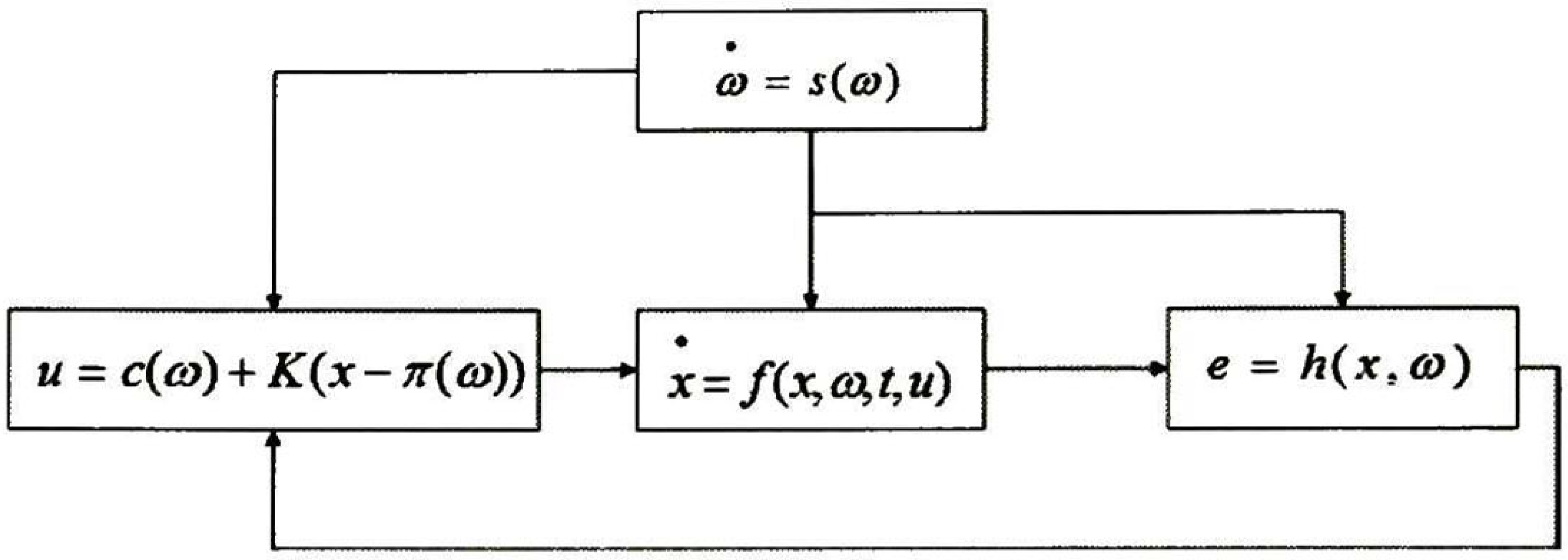


Figure 2.1: Classical nonlinear output regulation problem

Given a nonlinear system of the form (2.1), find, if possible an integer $b$ and two mappings $\theta(\xi_t)$ and $\eta(\xi_t, e_t)$, where $\xi \in \Xi \in \mathbb{R}^b$, such that:

$S_{EF}$ The equilibrium point $(x_t, \xi_t) = (0, 0)$ of

$$\begin{aligned} \dot{x}_t &= f(x_t, 0, \theta(\xi_t)) \\ \dot{\xi}_t &= \eta(\xi_t, h(x, 0)) \end{aligned} \tag{2.9}$$

is asymptotically stable on the first approximation.

$R_{EF}$ There exists a neighborhood $W \in U \times \Xi \times \Omega$ of $(0, 0, 0)$ such that, for every initial condition $(x_0, \xi_0, \omega_0) \in W$ the solution of

$$\begin{aligned} \dot{x}_t &= f(x_t, \omega_t, \theta(\xi_t)) \\ \dot{\xi}_t &= \eta(\xi_t, h(x_t, \omega_t)) \\ \dot{\omega}_t &= s(\omega_t) \end{aligned} \tag{2.10}$$

satisfies

$$\lim_{t \to \infty} e_t = 0 \tag{2.11}$$

As for the case of state feedback, the nonlinear system is expanded in terms of its linear approximation plus a nonlinear term.

$$\begin{aligned} \dot{x}_t &= Ax + BH\xi_t + P\omega_t + \varphi(x_t, \xi_t, \omega_t) \\ \dot{\xi}_t &= F\xi_t + GCx + GQ\omega_t + \chi(x_t, \xi_t, \omega_t) \\ \dot{\omega}_t &= S\omega_t + \psi(\omega_t) \end{aligned} \tag{2.12}$$

where $\varphi(x_t, \xi_t, \omega_t)$, $\chi(x_t, \xi_t, \omega_t)$ and $\psi(\omega_t)$ vanish in the origin along with their first order derivatives, and $C, Q, F, H, G$ are matrices defined by

$$C = \left[\frac{\partial h}{\partial x}\right]_{(0,0)} \qquad Q = \left[\frac{\partial h}{\partial \omega}\right]_{(0,0)}$$
$$F = \left[\frac{\partial \eta}{\partial \xi}\right]_{(0,0)} \qquad G = \left[\frac{\partial \eta}{\partial \omega}\right]_{(0,0)}$$
$$H = \left[\frac{\partial \theta}{\partial \xi}\right]_{(0)}$$

The conditions for the existence of the regulator equations solution is the same as in the case of the full state information problem. In fact if $c(\omega_t)$ is defined as

$$c(\omega_t) = \theta(\rho(\omega_t))$$

then the mapping $x = \pi(\omega_t)$ and $u = c(\omega_t)$ necessarily satisfies (2.7). However, for the case of error feedback, the conditions which guaranteed the solution of the state feedback output regulation problem, does not provide a set of sufficient conditions to the solution of the error feedback output regulation problem. There is an additional condition, which is expressed as a special property of the solution $\pi(\omega_t)$, $c(\omega_t)$.

In order to understand this condition, additional concepts need to be developed. For the case of the full state information, if equations (2.7) are satisfied, the mapping $x = \pi(\omega_t)$ is an invariant manifold for the extended system

$$\begin{aligned} \dot{x}_t &= f(x_t, \omega_t, c(\omega_t)) \\ \dot{\omega}_t &= s(\omega_t) \end{aligned} \tag{2.13}$$

and the error $e_t = h(x_t, \omega_t)$ is zero on every point of that manifold. Then, it is easy to see that for every initial condition $\omega_0$ of the exosystem, that is, for every exogenous input

$$\omega^* = \Phi_t^s(\omega_0) \tag{2.14}$$

if the plant is on the initial condition $x_0 = \pi(\omega_0)$ and the input is

$$u_t^* = c(\omega_t^*) \tag{2.15}$$

then $e_t = 0$ for all $t \geq 0$. Hence, the control law given by the autonomous system

$$\begin{aligned} \dot{\omega}_t &= s(\omega_t) \\ u_t &= c(\omega_t) \end{aligned} \tag{2.16}$$

will force the system to produce a zero error for every exogenous input, if the initial condition of the plant is adequate ($x_0 = \pi(\omega_0)$).

If the equilibrium point is not stable on the first approximation, then, in order to obtain the desired steady state response, the control law must include a stabilization component as in the full state output regulation problem.

Under this control law (2.16), the extended system

$$\begin{aligned} \dot{x}_t &= f(x_t, c(\omega_t) + K(x_t - \pi(\omega_t)) \\ \dot{\omega}_t &= s(\omega_t) \end{aligned} \tag{2.17}$$

will have an invariant manifold of the form $x_t = \pi(\omega_t)$, which will be exponentially attractive.

The following section will establish that the existence for the solution of the error feedback output regulation problem depends (among other things) on a property of the autonomous system (2.16), which could be seen as a function generator of all the inputs that produce zero error. This property requires the notion of system immersion.

**Definition 2.2.1.** *System Immersion: Let the set of smooth functions*

$$\dot{x} = f(x) \quad y = h(x) \tag{2.18}$$

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}) \quad y = \tilde{h}(\tilde{x}) \tag{2.19}$$

*defined on two different state spaces, $X$ and $\tilde{X}$, sharing the same output space $Y \in \mathbb{R}^m$ Suppose that $f(0) = 0$,$h(0) = 0$ and $\tilde{h}(0) = 0$ and denote the systems as $\{X, f, h\}$ and $\{\tilde{X}, \tilde{f}, \tilde{h}\}$ respectively.*
*The system $\{X, f, h\}$ is said to be immersed into the system $\{\tilde{X}, \tilde{f}, \tilde{h}\}$ if there exists a mapping $\tau : X \to \tilde{X}(\tilde{x} = \tau(x))$ that satisfies $\tau(0) = 0$ and*

$$h(x) \neq h(z) \Rightarrow \tilde{h}(\tau(x)) \neq \tilde{h}(\tau(z)) \tag{2.20}$$

*such that*

$$\frac{\partial \tau}{\partial x} f(x) = \tilde{f}(\tau(x)) \tag{2.21}$$

$$h(x) = \tilde{h}(\tau(x)) \tag{2.22}$$

$\forall x \in X$

**Example 2.2.2.** *In this example an immersion will be obtained only to illustrate the concept. The immersion obtained here is not lineal, however, it produces the same output as the original system. Consider the following nonlinear system* $\{X, f, h\}$

$$\begin{aligned} \dot{x} &= \sin(x) * u \\ y &= \sin(x) \end{aligned}$$

*A mapping* $\tau(x) = \begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix}$ *is an immersion of* $\{X, f, h\}$ *into the following system* $\{\tilde{X}, \tilde{f}, \tilde{h}\}$.

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} &= \begin{bmatrix} \tilde{x}_1 \tilde{x}_2 \\ -\tilde{x}_1^2 \end{bmatrix} u \\ y &= \tilde{x}_1 \end{aligned}$$

*The outputs produced by these to sets of differential equations are shown in Figure (2.2). The initial condition must also be mapped, so the immersion need to be observable.*

*This shows that the new state space has increased in dimension but its no longer trigonometric, hence the properties of the new system changed. For the Output Regulation Problem, a linear and observable immersion is needed.*

The above definition can be rephrased as the property that every output generated by $\{X, f, h\}$ can also be generated by $\{\tilde{X}, \tilde{f}, \tilde{h}\}$. This is important because $\{\tilde{X}, \tilde{f}, \tilde{h}\}$ can have properties that $\{X, f, h\}$ does not have.

The following proposition gives the conditions where an immersion into a linear observable space is possible.

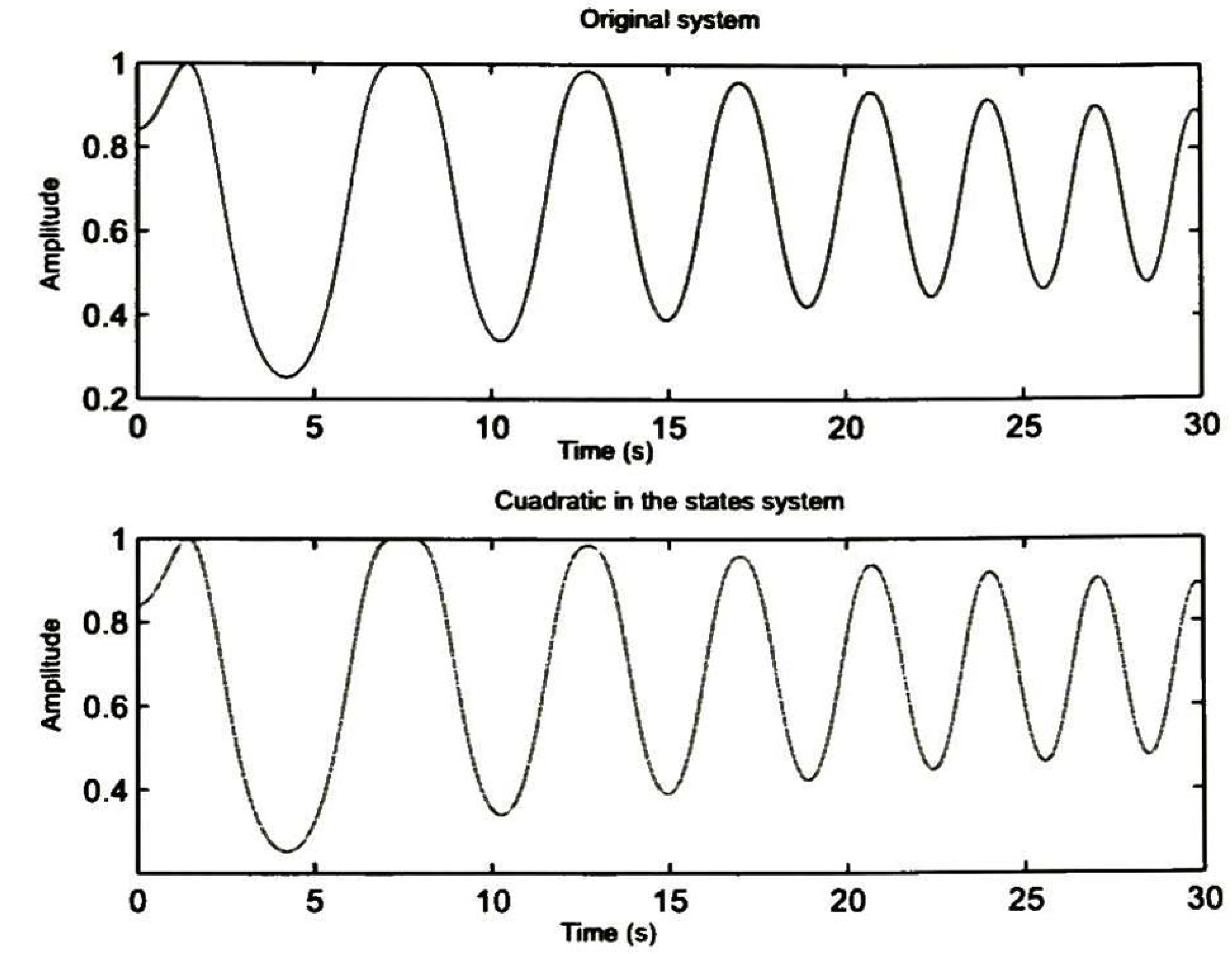


Figure 2.2: Example of an immersion.

**Proposition 2.2.3.** *The following statements are equivalent:*

- $\{X, f, h\}$ *is immersed into a finite dimensional linear observable space.*
- *The observation space* $O$ *of* $\{X, f, h\}$ *has finite dimension over* $\mathbb{R}$.
- *There is an integer* $q$ *and a set of real numbers* $a_0, a_1, ..., a_{q-1}$ *such that:*

$$L_f^q h(x) = a_0 h(x) + a_1 L_f h(x) + ... + a_{q-1} L_f^{q-1} h(x)$$

*Proof: See [17]*

The following result gives the sufficient and necessary conditions for the existence of the nonlinear error feedback output regulation problem solution.

**Theorem 2.2.4.** *The error feedback output regulation problem has a solution if and only if there exist mappings* $x = \pi(\omega)$ *and* $u = c(\omega)$, *with* $\pi(0) = 0$ *and* $c(0) = 0$, *both defined in a neighborhood* $\Omega_0 \subset \Omega$ *near the origin, that satisfies the following conditions*

$$\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega), \omega, c(\omega)) \quad (2.23)$$

$$0 = h(\pi(\omega), \omega) \quad (2.24)$$

*$\forall\Omega_0 \subset \Omega$, and that the autonomous system (2.16) is immersed into*

$$\dot{\xi}_t = \varphi(\xi_t) \quad (2.25)$$

$$u_t = \gamma(\xi_t) \quad (2.26)$$

*defined on a neigborhood $\Xi_0$ near the origin in $\mathbb{R}^\omega$, with $\varphi(0) = 0$ and $\gamma(0) = 0$, and the two matrices*

$$\Phi = \left[\frac{\partial\phi}{\partial\xi}\right]_{\xi=0} \qquad \Gamma = \left[\frac{\partial\gamma}{\partial\xi}\right]_{\xi=0} \quad (2.27)$$

*are such that the pair*

$$\begin{bmatrix} A & 0 \\ NC & \Phi \end{bmatrix} \quad \begin{bmatrix} B \\ 0 \end{bmatrix}$$

*is stabilizable for a selection of N, and the pair*

$$\begin{bmatrix} C & 0 \end{bmatrix}. \begin{bmatrix} A & B\Gamma \\ 0 & \Phi \end{bmatrix}$$

*is detectable.*

*Proof: See [17]*

The block diagram of the nonlinear error output regulation is portrayed in Figure 2.3.

The following example illustrates when an immersion is not possible, and other techniques will be needed in order to solve the Output Regulation Problem.

**Example 2.2.5.** *Consider the following inverted pendulum*

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = g\sin(x_1) - cu \quad (2.28)$$

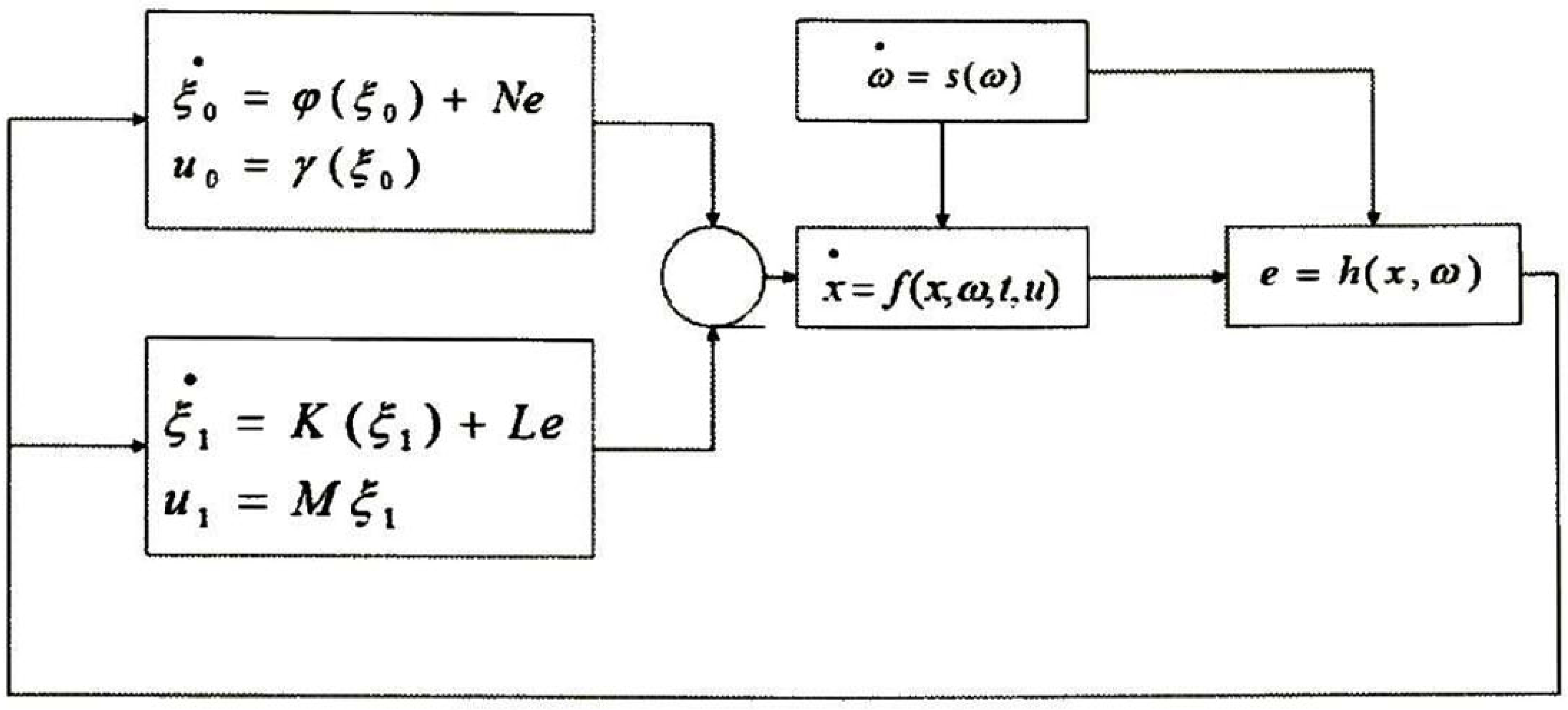


Figure 2.3: Nonlinear error output regulation.

*and the linear oscillator*

$$\begin{aligned} \dot{\omega_1} &= \omega_2 \\ \dot{\omega_2} &= -\alpha^2\omega_1 \end{aligned} \tag{2.29}$$

*The tracking error is defined as* $e = x_1 - \omega_1$. *The parameters g,c are only approximately known. The solution of the regulator equations is*

$$\begin{aligned} \pi_1 &= \omega_1 \\ \pi_2 &= \omega_2 \\ c(\omega, \mu) &= \frac{\alpha^2\omega_1 + g\sin(\omega_1)}{c} \end{aligned} \tag{2.30}$$

*One of the necessary and sufficient conditions to find a linear immersion is that* $c(\omega, \mu)$ *must be polynomial with respect to* $\omega$. *Hence, even for this simple example, a linear immersion does not exists. In Figure 2.4 the tracking error for the inverted pendulum is shown. The controller is able to make the tracking error zero when the parametric variations are known; however, at* $t = 60s$. *the parameter c is changed from its nominal value and the controller is no longer able to make the tracking error zero.*

A lot of effort has been made in order to make this controller robust with respect to uncertainties. In [7] Castillo-Toledo proposed a methodology where the immersion is time-varying, the so called Generalized Immersion, which includes a bigger class of nonlinear systems. In [33] Serrani and Isidori proposed an adaptive scheme where the uncertainties are

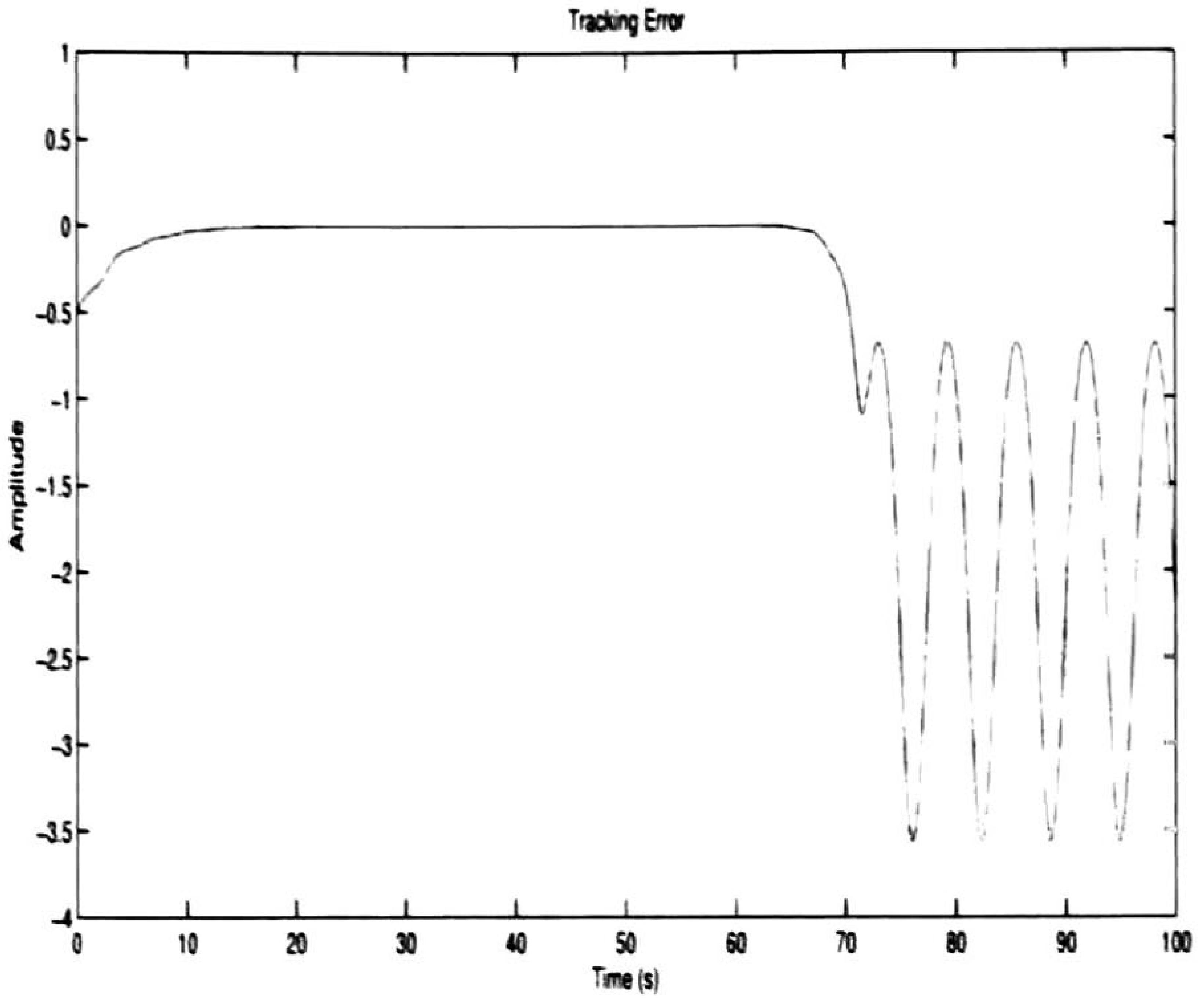


Figure 2.4: Tracking error.

considered constant but unknown, and the adaptive control identifies those parameters; once the error is smaller than a constant, the adaptive law is turned off. Finally, in [24] Obregon has proposed a convergent estimator and has generalized it for $n$ unknown frequencies on the signal.

Chapter 3

# Exact solution of the regulator equations

The Output Regulation Problem has being analysed for a couple decades, by many interesting approaches. They can be classified into two big groups: the first solve the FIB equations exactly, and the second one solve the FIB equations approximately. In this chapter the most significant works on the first group are explained.

## 3.1 Generalized immersion

The concept of Generalized Immersion first appeared in the 90's; the main idea of this theory is to let the immersed system depend on values of the exosystem. This approach is useful because the class of systems which can be immersed under that assumption is bigger than the previous one. In this section this idea will be reviewed in order to compare the proposed solution with this one. As shown in the previous chapter, if it is possible to find the immersion, then the controller is robust with respect to the unknown parameters. It has been established that a linear immersion exists only when the steady state controller is polynomial with respect to the exosystem states. This condition is quite restrictive; hence, a possible way to deal with this situation is to let the immersion depend on the exosystem states.

It is important to notice that the results obtained here guarantee zero tracking error, with the expense of having a high dimensional controller. The main idea can be summarized in the following result.

**Theorem 3.1.1.** *[7] Consider the nonlinear system*

$$\dot{x} = f(x, \omega, u, \mu) \tag{3.1}$$

$$e = h(x, \omega, \mu) \tag{3.2}$$

*The Robust Output Regulation Problem with full exosystem measurement is solvable if and only if there exists mappings $\pi^\alpha(\omega, \mu)$ and $c^\alpha(\omega, \mu)$, with $\pi^\alpha(0,0) = 0$ and $c^\alpha(0,0)$, solving the regulator equations, such that the extended exogenous system with output $c^\alpha(\omega, \mu)$ is immersed into*

$$\frac{d}{dt}\begin{bmatrix}\omega\\ \xi\end{bmatrix} = \begin{bmatrix}s(\omega)\\ \phi(\omega)\xi\end{bmatrix} \tag{3.3}$$

*and the following conditions hold:*

- *The pair*

$$\begin{bmatrix}A & 0\\ NC & \Phi(0)\end{bmatrix}, \begin{bmatrix}B\\ 0\end{bmatrix} \tag{3.4}$$

*is stabilizable for some choice of matrix $N$.*

- *The pair*

$$\begin{bmatrix}C & 0\end{bmatrix}, \begin{bmatrix}A & B\Gamma\\ 0 & \Phi(0)\end{bmatrix}, \Gamma = \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix} \tag{3.5}$$

*is detectable.*

This theorem is similar to the one on which a linear immersion is found, in fact, this is a more general case. The structure of the controller that solves the Robust Output Regulation Problem with exosystem measurement is

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}K & 0\\ 0 & \Phi(\omega)\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} + \begin{bmatrix}L\\ N\end{bmatrix}e \tag{3.6}$$

$$u = M\xi_1 + \Gamma\xi_2 \tag{3.7}$$

For the case of the Error Output Regulation Problem, the following theorem gives sufficient conditions to find a solution for the regulator equations.

**Theorem 3.1.2.** *Consider the nonlinear system*

$$\dot{x} = f(x, \omega, u, \mu) \tag{3.8}$$

$$e = h(x, \omega, \mu) \tag{3.9}$$

*with $s = 1$,$m = 1$ and the exosystem with an additional output $y_\omega = r(\omega)$, that is $s' = 1$. Further, assume that there exists local asymptotic observer for the exosystem state $\omega$ given by*

$$\dot{\hat{\omega}} = g(\hat{\omega}, y_\omega) \tag{3.10}$$

*with the corresponding error dynamics for $\varepsilon = \omega - \hat{\omega}$ as*

$$\dot{\varepsilon} = \phi(\epsilon, \omega) \tag{3.11}$$

*where $\frac{\partial \phi}{\partial \bar{\omega}} = (0, 0)$ is a Hurwitz matrix. Then the Robust Output Regulation Problem with partial exosystem measurement is solvable if and only if the Robust Output Regulation Problem with full exosystem measurement is solvable. Moreover, the corresponding controller has the following form*

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & \Phi(\hat{\omega}) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} L \\ N \end{bmatrix} e \tag{3.12}$$

$$\dot{\hat{\omega}} = g((\ddot{\omega}), y_\omega) \tag{3.13}$$

$$u = M\xi_1 + \Gamma\xi_2 \tag{3.14}$$

The block diagram for the controller that uses the generalized immersion is shown in Figure 3.1.

## 3.2 Simultaneous globally convergent online estimator

This method is a little different to the previous one, it addresses the problem of determining a solution of the regulator equations when the frequency of the exosystem is not known. In fact, multiple frequencies are allowed to be unknown. The dimension of the estimator is $3n$ (where $n$ is the number of unknown frequencies) which is, as far as the author knows, the

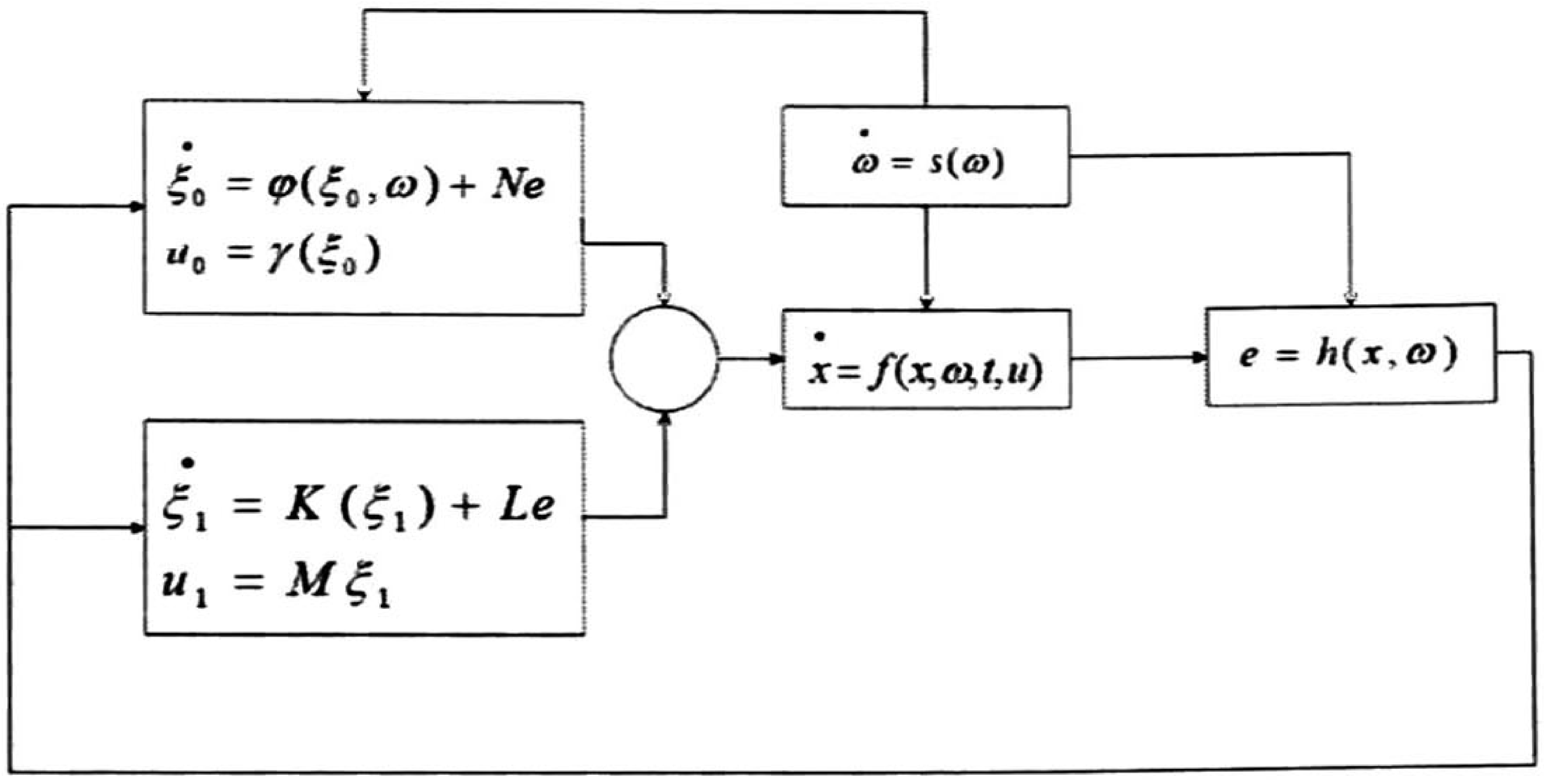


Figure 3.1: Output regulation using generalized immersion.

lower dimensional estimator for the problem[24]. It has the advantage that zero error on the output can be achieved; however, as it is mentioned, the dimension of the controller increases by $3n$ so computational time must be paid in order to achieve the zero error.

Another way to solve this problem is via neural networks, and it will be disscused later that, in spite of not giving zero error, the error is sufficiently small and the controller is lower dimensional.

## 3.3 Conclusions

The problem of nonlinear output regulation has been extensively studied; several methods for determining the exact solution have been studied. However, the solution of this set of partial differential equations by any of this methods result sometimes as difficult as the original problem. For other cases, the controller which solves the Output Regulation Problem increases in dimension. For practical problems, one do not expect the error to be exactly zero; however, one would expect that the error remains bounded, and, if possible, that the bounds can be arbitrarily selected. The following chapter begins with some classical approximations of the Output Regulation Problem and ends with the solution proposed in this work.

Chapter 4

# Approximation of the regulator equations

For the second classification of the regulator solution, one of the most relevant work is the power-series approximation method proposed by Huang and Rugh [16]. These results are based on a kth-order approximation of the plant zero-error manifold, and the control law designed there yield kth-order asymptotic tracking and disturbance rejection properties for the closed-loop system. The error can be made arbitrarily small by increasing the order of the approximation: however, the solution of the regulator equations are more complex, and the controller increases its complexity with each increment in the approximation order.

Another approximation is to use the neural networks. Feedforward neural networks are typically used as approximations of nonlinear systems and/or controllers. They can be classified by the structure of the neural network as well as the algorithms used to adapt the weights. Supervised learning has taken popularity among the control community, mainly because it is capable to adapt the response of the neural network in cases of uncertainties in the parameters of the system. Because neural networks are easily parallelized, they promise to be a viable tool to make complex nonlinear controllers computationally efficient.

## 4.1 Kth-order solution of the state feedback regulator equations

In order to develop the kth-order approximation of the regulator equations solution one more property for the closed loop system is needed.

**Definition 4.1.1.** *Lev $V$ be an open neighborhood of the origin of $\mathbb{R}^q$. A function $o_s^k : V \to \mathbb{R}^s$ is said to be zero up to kth order if it is sufficiently smooth and vanishes at the origin together with all partial derivatives of order less than or equal to $k$. The notation $o^k(x)$ will be used to denote a generic function of $x$ which is zero up to kth order regardless of the dimension of its range space.*

Then, the kth-order nonlinear output regulation problem is defined as:

Determine, if possible, a control law of the form (2.8) or (3.7) such that the closed-loop composite system fulfils the assumption $S_{FI}$ as well as

$R_{KFI}$ For all sufficiently small $x_0$ and $\omega_0$, the solution of

$$\begin{aligned} \dot{x} &= f(x, \omega, u(x, \omega)) \\ \dot{\omega} &= s(\omega) \end{aligned} \tag{4.1}$$

satisfies

$$\lim_{t\to\infty} (e_t - o^k(\omega)) = 0 \tag{4.2}$$

**Theorem 4.1.2.** *The kth-order nonlinear output regulation problem is solvable by a static state feedback controller*

$$u = \alpha(x, \omega) \tag{4.3}$$

*if and only if there exist two sufficiently smooth functions $\pi^{(k)}(\omega)$ and $c^{(k)}(\omega)$ satisfying $\pi^{(k)}(0) = 0$ and $c^{(k)}(0) = 0$ such that*

$$\begin{aligned} \frac{\partial \pi^k}{\partial \omega} s(\omega) &= f(\pi^k(\omega), \omega^k, \alpha(\pi^(k)(\omega), \omega) + o^k(\omega)) \\ o^k(\omega) &= h(\pi^k(\omega), \omega) \end{aligned} \tag{4.4}$$

*Proof: See [15]*

Similar results are obtained for the case of error feedback, see [15] for reference. The block diagram that implements the $kth$ order controller is shown in Figure 4.1.

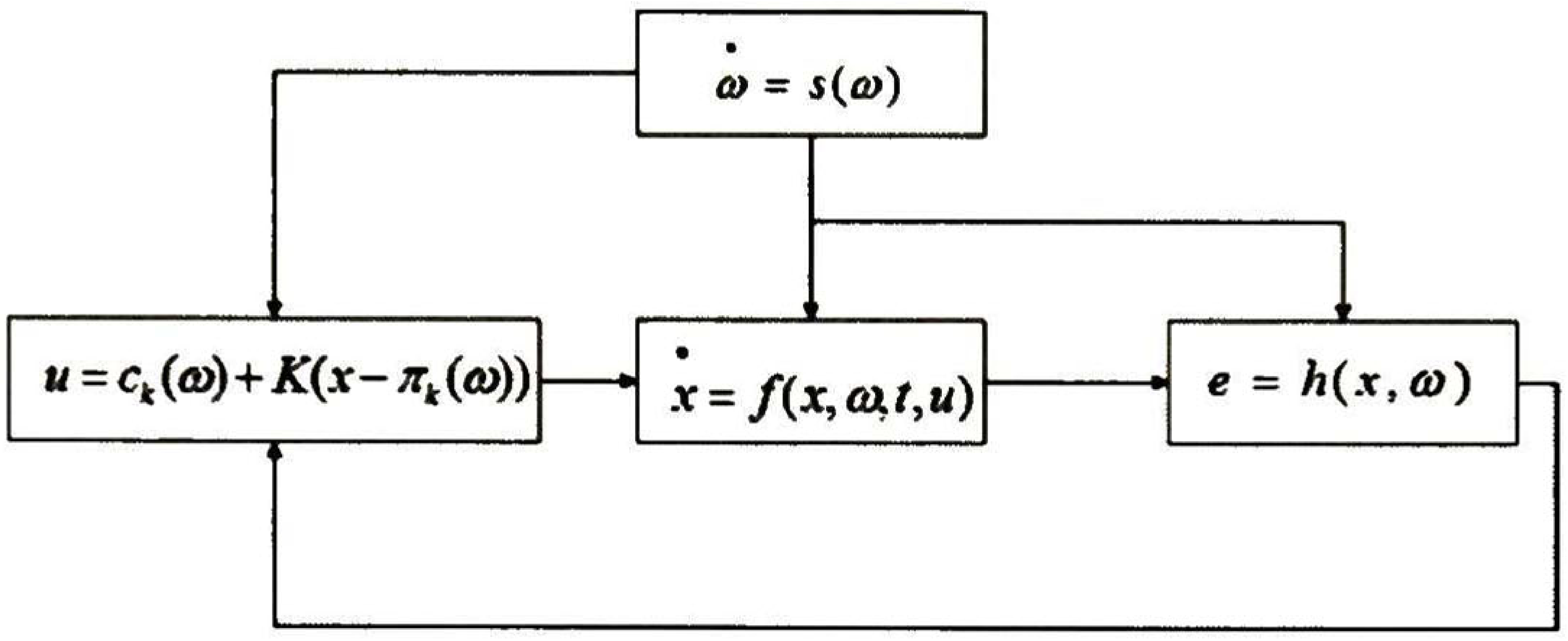


Figure 4.1: K-th order output regulation.

**Example 4.1.3.** *Consider the following nonlinear system*

$$\begin{aligned} \dot{x_1} &= -x_1 + x_2^3 + u \\ \dot{x_2} &= -sin(x_2) + x_1 \\ y &= x_2 \end{aligned} \tag{4.5}$$

*and the following exosystem*

$$\begin{aligned} \dot{\omega_1} &= \omega_2 \\ \dot{\omega_2} &= -\alpha^2\omega_1 \end{aligned} \tag{4.6}$$

*where*

$$e = x_1 - \omega_1 \tag{4.7}$$

*The mappings $\pi(\omega)$ and $c(\omega)$ are obtained as the solution of the FIB equations, the first equation can easily be obtained. Let $e = 0$*

$$\pi_1 = x_1 \tag{4.8}$$

*for the second equation, the FIB equation can be expressed as*

$$\begin{bmatrix} \frac{\partial \pi_2}{\partial \omega_1} & \frac{\partial \pi_2}{\partial \omega_2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = -\sin(\pi_2) + \omega_1 \tag{4.9}$$

*To obtain a closed form solution for this partial differential equation is not trivial; hence a kth order regulator is designed. In this case, a second order approximation is used.*

$$\pi_2(\omega) = \sum_{i=0}^{2} A^i \omega^{[i]} \tag{4.10}$$

*where*

$$A^l = \left\{ \frac{1}{l!} \frac{\partial^l \pi(0)}{\partial \pi_{i1} \cdots \partial \pi_{i2}} \right\} \tag{4.11}$$

$$\omega^{[l]} = \omega \otimes \omega \otimes \cdots \otimes \omega \qquad l - times \tag{4.12}$$

*Expanding equation 4.10*

$$\pi_2(\omega) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} c & d & e \end{bmatrix} \begin{bmatrix} \omega_1^2 \\ \omega_1\omega_2 \\ \omega_2^2 \end{bmatrix} + O(\|\omega\|)^3 \tag{4.13}$$

$$\pi_2(w) = aw_1 + bw_2 + cw_1^2 + dw_1w_2 + ew_2^2 + o\left(\|w\|^3\right)$$

*Taking the partial derivatives with respect to the exosystem states*

$$\begin{aligned} \frac{\partial \pi_2}{\partial w_1} &= a + 2cw_1 + dw_2 \\ \frac{\partial \pi_2}{\partial w_2} &= b + dw_1 + 2ew_2 \end{aligned}$$

*Substituting into the FIB equation*

$$\begin{aligned} (a + 2cw_1 + dw_2)\, w_2 - (b + dw_1 + 2ew_2)\, w_1 &= -\pi_2 + w \\ (a + 2cw_1 + dw_2)\, w_2 - (b + dw_1 + 2ew_2)\, w_1 &= -\left(aw_1 + bw_2 + cw_1^2 + dw_1w_2 + ew_2^2\right) + w \end{aligned}$$

*Finally, solving for each variable of the following equation*

$$w_2\,(a + 2cw_1 + dw_2) - w_1\,(b + 2ew_2 + dw_1) = -cw_1^2 - dw_1w_2 - aw_1 - ew_2^2 - bw_2 + w$$

*The following approximation of $\pi_2$ is used*

$$\begin{aligned} \pi_1 &= w_1 \\ \pi_2 &= \frac{1}{2}w_1 - \frac{1}{2}w_2 \end{aligned}$$

*Now* $\dot{w}_1 = -x_1 + x_2^3 + u$ *is used to obtain* $c(\omega)$,

$$c(\omega) = w_2 + w_1 - \left(\frac{1}{2}w_1 - \frac{1}{2}w_2\right)^3$$

*The kth order output regulation is then solved using the following control law*

$$\begin{aligned} u &= w_2 + w_1 - \left(\frac{1}{2}w_1 - \frac{1}{2}w_2\right)^3 + K(x - \pi(w)) \\ \pi(w) &= \begin{pmatrix} w_1 \\ \frac{1}{2}w_1 - \frac{1}{2}w_2 \end{pmatrix} \end{aligned}$$

*Figure 4.2 shows the output of the system, the nonlinear system is able to track the desired trajectory, finally, Figure 4.3 shows that the error is not zero, however it remains bounded.*

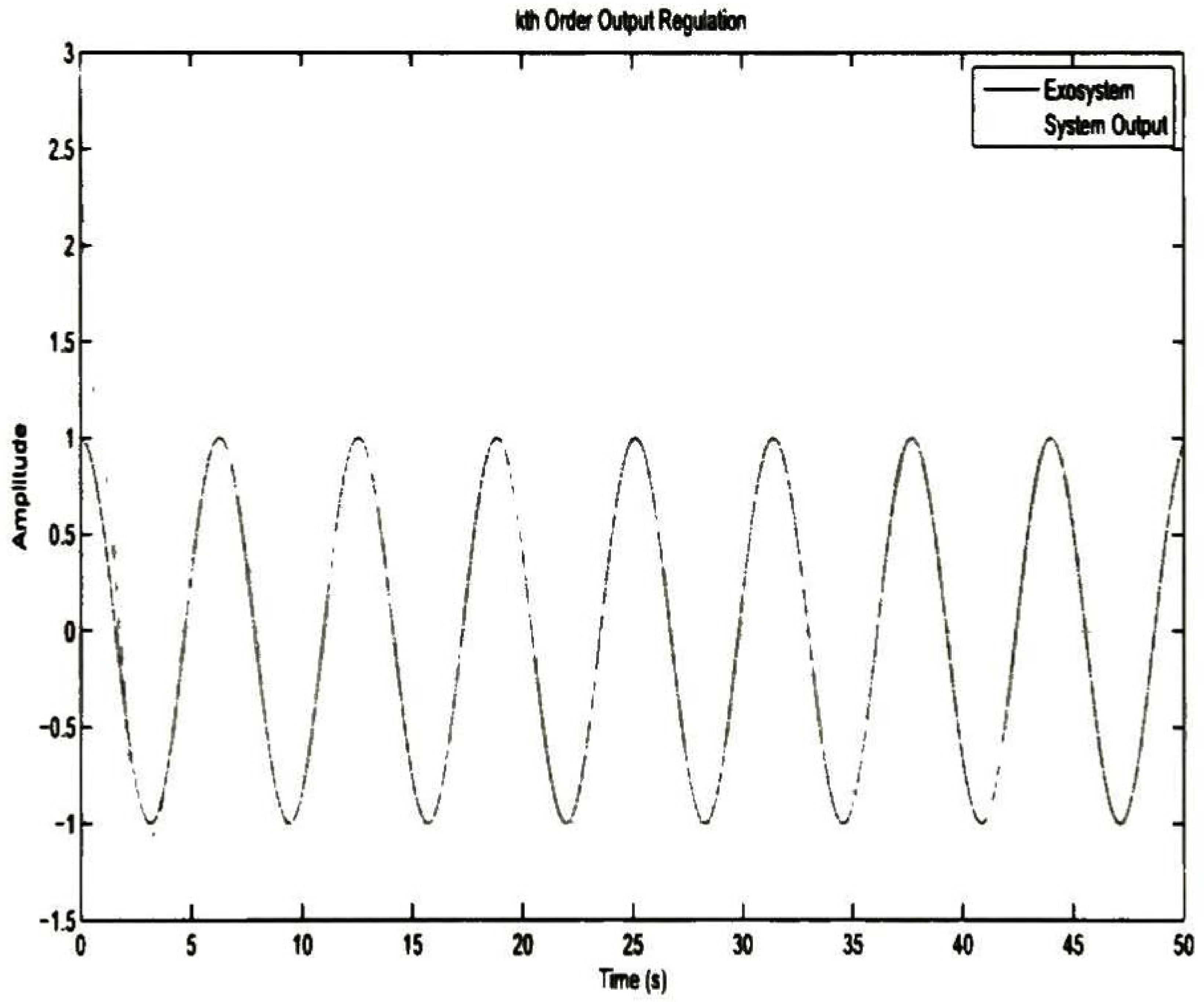


Figure 4.2: K-th order trajectory tracking.

It is important to notice that this controller is not robust with respect to plant uncertainties, however, a linear immersion can be found because a polynomial approximation of the center manifold is being calculated.

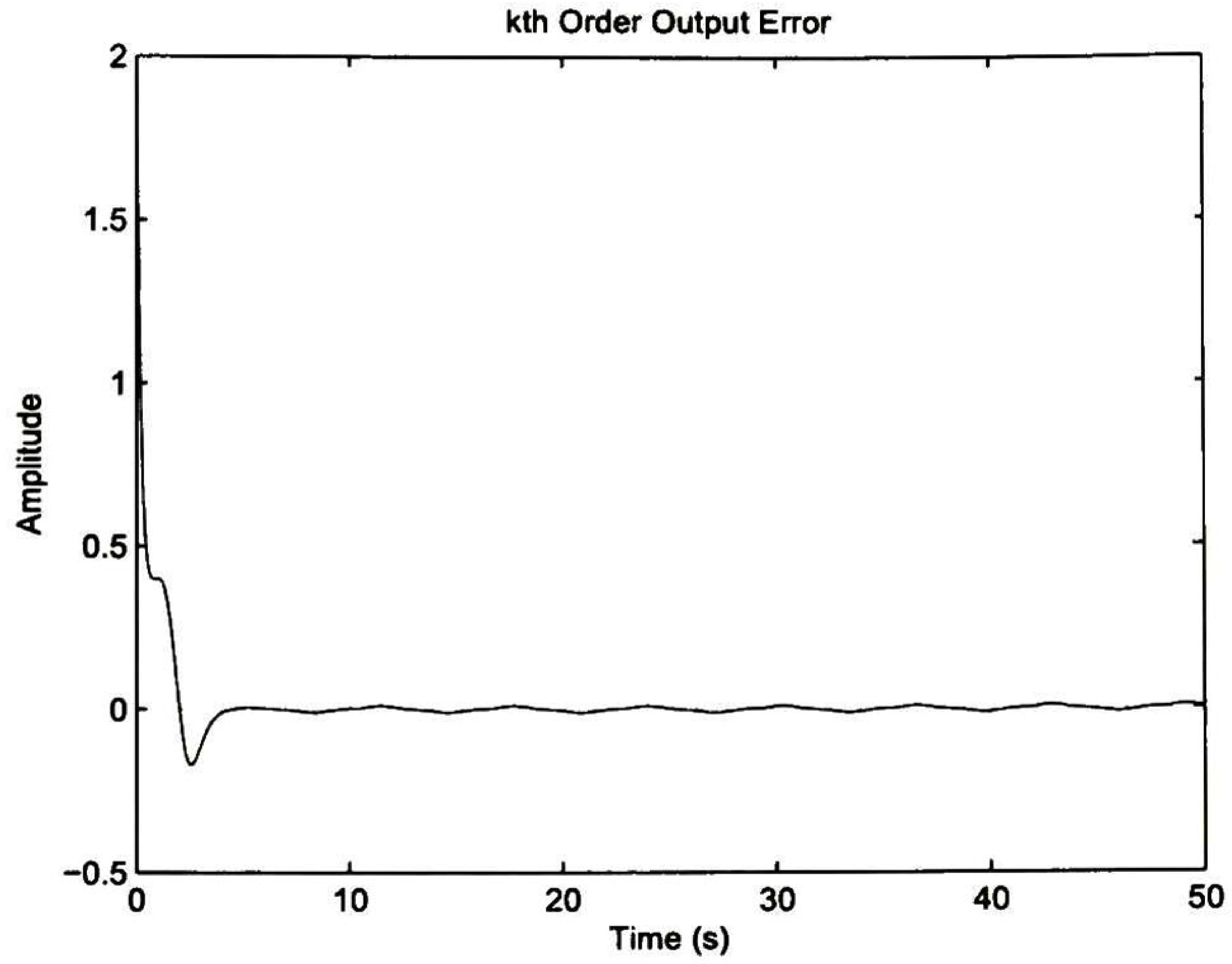


Figure 4.3: K-th order output error.

## 4.2 Neural networks

### 4.2.1 Radial basis function neural networks

In the classical Output Regulation Approach, equation 2.4 can be quite restrictive for practical applications; hence Wang in [35] proposed a way to approximate the solution to the regulator equations by replacing that hypothesis with the following one

$R_{FI}$ For any given $\varepsilon > 0$ design a control law such that for all sufficiently small initial conditions $x_0$ and $\omega_0$, the closed-loop system has a bounded solution for all $t \geq 0$, and

$$\lim_{t\to\infty} \sup \|h_c(x_c, \omega_c)\| \tag{4.14}$$

This problem can also be solved as explained in chapter 3.2.1; however, it has the drawback that the exact knowledge of the plant is needed, and can require tedious computational effort.

To start developing the idea, first the State Feedback Output Regulation Problem is presented, where the control law is of the form:

$$u_t = K(x_t - \pi(\omega_t)) + c(\omega_t) \tag{4.15}$$

where

$$c(\omega_t) = K(x_t - \pi(\omega_t)) - u_t \tag{4.16}$$

Once the feedback gain $K$ is calculated, the feedforward control is a sufficiently smooth function of the exosystem. Using the universal approximation theorem, given any $\varepsilon > 0$, and any compact subset $\Lambda$, there exists a $m$-dimensional vector valued function $\hat{c}(W, \omega)$ such that

$$\max_{\omega \in \Lambda} \|c(\omega) - \hat{c}(W, \omega) \leq \varepsilon \tag{4.17}$$

Replacing $c(\omega_t)$ by $\hat{c}(W, \omega_t)$ leads to the following state feedback neural network control law:

$$u_t = K(x_t - \pi(\omega_t)) + \hat{c}(W, \omega_t) \tag{4.18}$$

In [35] the stability of the closed loop system under this neural network feedback control is proved. Also, a bound for the error is obtained, which depends mainly on the feedback gain $K$ and on the identification error bound, so the training law is critical for this type of controller. An improvement with respect to the linear Output Regulation Feedback control is obtained by training the neural network with genetic algorithms, however the training was made off-line so this type of control is not robust with respect to parametric variations of the plant.

## 4.3 Output regulation for parametric variations using dynamic neural networks

Consider the following nonlinear system

$$\dot{x}_t = f(x_t, \omega_t) + g(x_t)u_t \tag{4.19}$$

$$\dot{\omega}_t = s(\omega_t) \tag{4.20}$$

$$e_t = h(x_t) - r(\omega_t) \tag{4.21}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $h(x) \in \mathbb{R}^m$ the vector function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are partially unknown. It is well known that in order to obtain a solution for the regulator equations it is necessary that the parameters of the plant are known; in some special cases, where an immersion can be found, it is possible to determine a solution even if the parameters change around a neighborhood of the nominal value. However, a linear immersion exists only when the steady state control is polynomial with respect to the exosystem [15], and for the nonlinear case, it is often very difficult, or even impossible to determine such immersion. Advances had been made, and a method to find a generalized immersion has been proposed [7]. This generalized immersion is useful for many nonlinear systems [39]; however the dimension of the controller depends on the structure of $c(\omega)$ and can become large even for simple problems.

In this chapter, a solution for the regulator equations is proposed, based on a neural network identifier which is in the normal form, a continuous adaptation law is used, and based on a Lyapunov-like function, the convergence of the identification error as well as the boundedness of the neural network weights are guaranteed. Normally, the adaptation law use a dead zone function; this is a drawback because in order to improve the identification quality, it is necessary to reduce the size of the dead zone. This can be done by increasing a parameter associated with the Riccati equation. However, this parameter can only be increased up to certain level, beyond which no solution exists. The adaptation law used in this work is a continuous function; as well as its first derivative, this type of adaptation law is useful for the regulation theory because this terms appear in the solution of the regulator equations. The nonlinear function (4.19) can be represented by a known term plus an unknown term in the following way:

$$\dot{x}_t = Ax_t + W_1{}^*\sigma(x_t) + W_2{}^*\phi(x_t)u_t + \Delta f(x_t, u_t) \tag{4.22}$$

A neural network is proposed in order to identify the states, the following structure for the parallel neural network is used.

$$\dot{\hat{x}}_t = A\hat{x}_t + W_1\sigma(\hat{x}_t) + W_2\phi(\hat{x}_t)u_t \tag{4.23}$$

The vector $x \in \mathbb{R}^n$ is the state of the neural network, $u \in \mathbb{R}^m$ is its input. The matrix $A \in \mathbb{R}^{nxn}$ is Hurwitz in order to do the linearisation of the neural network stable. $W_{1,t} \in$

$\mathbb{R}^{nxn}$ and $W_{2,t} \in \mathbb{R}^{nxm}$ are the weights of the neural network describing the output layer connections, and $\sigma(\hat{x}_t) \in \mathbb{R}^n$, $\phi(\hat{x}_t) \in \mathbb{R}^m$ are sigmoidal functions. A parallel neural identifier was used because the solution of the regulator equations of the neural network will only depend on the structure of the identifier and not on that of the plant, it can be seen that if the neural network is able to track the nonlinear plant, then in equation 4.22 the error term $\Delta f(x_t, w_t)$ is zero and, on the contrary, this term will be made arbitrarily small. Both cases are considered, because it is a more common situation when the neural network is not able to follow exactly the nonlinear system.

The proposed control structure is shown in Figure 4.4. Let the estimation error and the

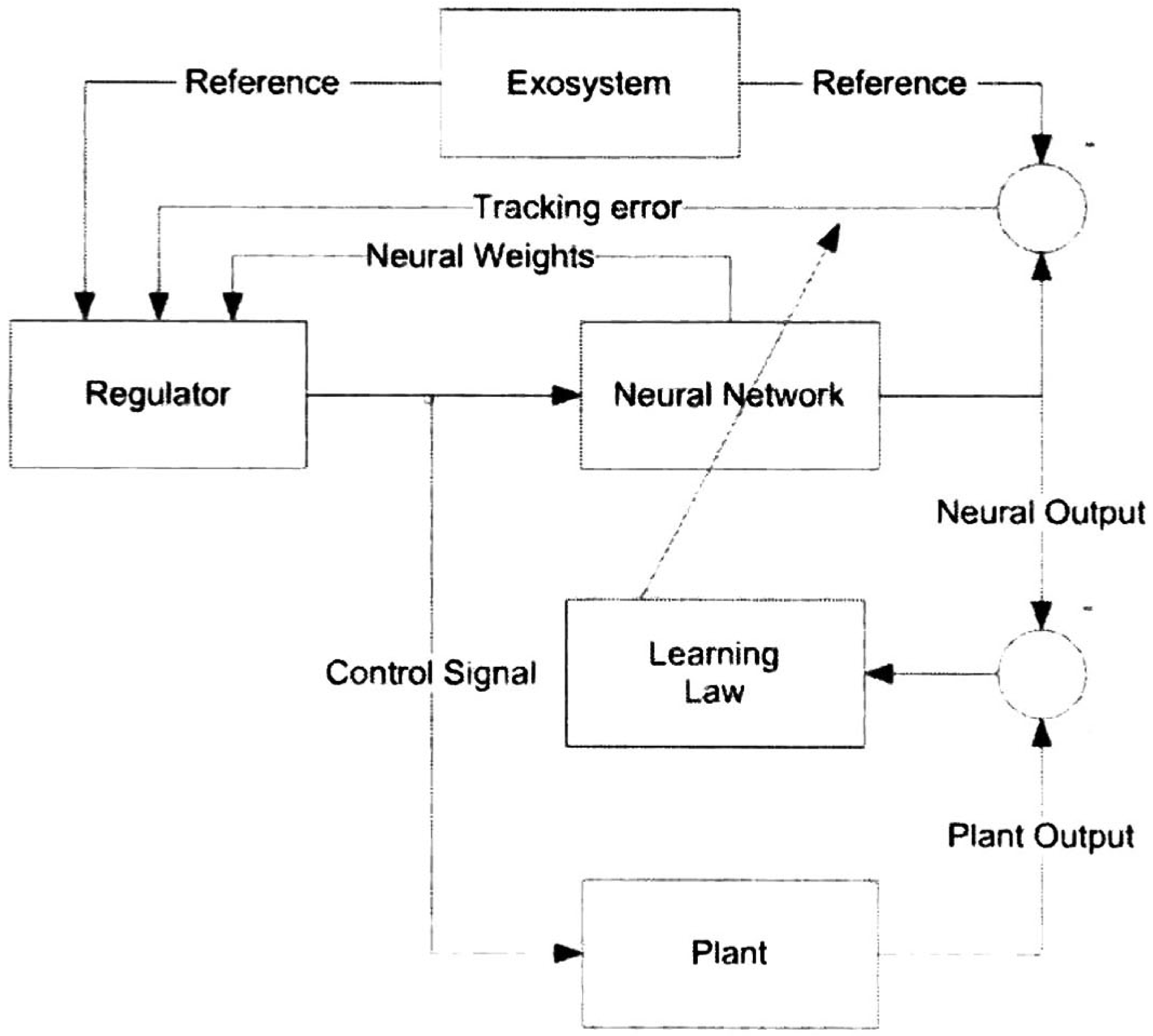


Figure 4.4: Neural identification of the plant.

tracking error be defined, respectively as

$$\Delta_t = \hat{x}_t - x_t$$
$$e_t = \hat{y} - y$$

The following hypotesis are used [29]:

**H.0** The plant satisfies the Lipschitz condition, that is

$$\|f(x,u,t) - f(z,v,t)\| \leq L_1 \|x - z\| + L_2 \|u - v\|$$

where $x, z \in \mathbb{R}^n$, $u, v \in \mathbb{R}^n$, and $L_1, L_2$ are positive constants.

**H.1** The sigmoidal functions satisfy the sector conditions:

$$\begin{aligned} \tilde{\sigma}_t^T \Lambda_\sigma \tilde{\sigma}_t &\leq \Delta_t^T D_\sigma \Delta_t \\ u_t^T \tilde{\phi}_t^T \Lambda_\phi \tilde{\phi}_t u_t &\leq f \Delta_t^T D_\phi \Delta_t \|u_t\|^2 \end{aligned}$$

where

$$\begin{aligned} \tilde{\sigma}_t &:= \sigma(\tilde{x}_t) - \sigma(x_t) \\ \tilde{\phi}_t &:= \phi(\tilde{x}_t) - \phi(x_t) \end{aligned}$$

and $\Lambda_\sigma \in \mathbb{R}^{mxm}$,$D_\sigma \in \mathbb{R}^{nxn}$;$\Lambda_\phi \in \mathbb{R}^{nxn}$,$D_\phi \in \mathbb{R}^{nxn}$ are known constant positive definite matrix.

**H.2** Admissible controls are bounded, that is

$$\|u_t\|^2 \leq u \leq \infty$$

**H.3** Error term is bounded by

$$\|\Delta f(x,u,t)\|_{\Lambda_f}^2 \leq \bar{\eta}$$

where $\Lambda_f \in \mathbb{R}^{nxn}$ is a constant positive definite matrix.

**H.5** The matrices $W_1^*$ and $W_2^*$ are bounded by

$$\begin{aligned} W_1^* \Delta_\sigma^{-1} W_1^{*T} &\leq \bar{W}_1 \\ W_2^* \Lambda_\phi^{-1} W_2^{*T} &\leq \bar{W}_2 \end{aligned}$$

where $W_1^*$ and $W_2^*$ are known positive definite matrices

**H.6** The following Riccati equation

$$A^T P + PA + PRP + Q = 0 \tag{4.24}$$

has a positive solution, such that R and Q are defined as

$$R := \bar{W}_1 + \bar{W}_2 + \Lambda_f^{-1} \tag{4.25}$$

$$Q := D_\sigma + \bar{u} D_\phi + Q_0 \tag{4.26}$$

with the terms $Q_0$ $\bar{W}_1$ and $\bar{W}_2$. Q and R can take almost any value, so the preceding assumptions are realistic, and a solution exists for almost any neural network.

### 4.3.1 Neural network adaptation law

Consider the following neural network

$$\dot{\hat{x}}_t = A\hat{x}_t + W_1 \sigma(\hat{x}_t) + W_2 \phi(\hat{x}_t) u_t \tag{4.27}$$

This neural network is a Hopfield-like one. In order to adjust the weights and minimize the identification error, the following adaptation law is used [26]:

$$\dot{W}_{1,t} = -K_1 P \Delta_t \sigma(\hat{x}_t)^T - \frac{\lambda_{min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}})}{2} \tilde{W}_{1,t} \tag{4.28}$$

$$\dot{W}_{2,t} = -K_2 P \Delta_t \phi(\hat{x}_t)^T - \frac{\lambda_{min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}})}{2} \tilde{W}_{2,t} \tag{4.29}$$

where $K_1$ and $K_2$ are positive definite matrices which can be chosen arbitrarily. $P$ is the solution of the matrix equation given by 4.24 and

$$\tilde{W}_{1,t} := W_{1,t} - W_1^*$$
$$\tilde{W}_{2,t} := W_{2,t} - W_2^*$$

Now the following result is used:

**Theorem 4.3.1.** *If the assumptions H.0 to H.5 are satisfied, and the weight matrices $W_{1,t}$ and $W_{2,t}$ of the neural network (4.27) are adjusted by the differential learning law mentioned above then*

**a)** *Both the identification error and the weights are bounded*

**b)** *The identification error has the following upper bound:*

$$\limsup_{t\to\infty} \Delta_t^T P \Delta_t \leq \frac{\bar{\eta}}{\lambda_{min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}})} \tag{4.30}$$

*Proof: See Appendix B*

### 4.3.2 Regulator equations

Consider the nonlinear system modeled by the neural network as

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, \omega, u) \\ \dot{\omega} &= s(\omega) \\ e &= h(\hat{x}, w) \end{aligned} \tag{4.31}$$

where the first equation describes the dynamics identified by the neural network, whose state $\hat{x}$ is defined in a neighborhood $U$ of the origin in $\mathbb{R}^n$, with control input $u \in \mathbb{R}^m$ and subject to a set of known exogeous input variables $\omega$ defined in a neighborhood $V$ of the origin in $\mathbb{R}^r$ The second equation is known as the exosystem and the third equation defines the error expressed as a function of the neural states and the states of the exosystem.

**Theorem 4.3.2.** *Assume the following assumptions hold*

- *The equilibrium $\omega = 0$ of the exosystem is Lyyapunov stable, and the Jacobian matrix $S = \frac{\partial s(\omega)}{\partial \omega}$ at the equilibrium $\omega = 0$, has all its eigenvalues on the imaginary axis.*

- *There exists a function $k(x)$ such that the Jacobian matrix $A = \frac{\partial f(\hat{x},0,k(\hat{x}))}{\partial \hat{x}}$ evaluated at $\hat{x} = 0$ has all eigenvalues on the open left-half side of the complex plane.*

*Then, the state feedback regulator problem is solvable if there exist $C^r$ $(r \geq 2)$ mappings $x = \pi(\omega)$ and $u = c(\omega)$ with $\pi(0) = 0$ and $c(0) = 0$, both defined in a neighborhood $W^0 \subset W$ of 0, satisfying the conditions:*

$$\frac{\partial \pi(\omega)}{\partial \omega} s(\omega) = f(\pi(\omega), \omega, c(\omega)) \tag{4.32}$$

$$0 = h(\pi(\omega), \omega) \tag{4.33}$$

*In fact, the controller that minimizes the output tracking error is given by*

$$\alpha(\hat{x}, \omega) = K(\hat{x} - \hat{\pi}(\omega)) - c(\omega) \tag{4.34}$$

*Proof.* The proof of this theorem is an immediate consequence of the properties of the neuro identifier, the error can be seen as the sum of the identification error $e_i$ plus the tracking error $e_t$, i.e

$$e_t = e_i + e_t = \{h(x_t) - h(\hat{x}_t)\} + \{h(\hat{x}_t) - r(w_t)\} \tag{4.35}$$

the second term of the error will always be zero independently of the values of the neural network weights, because the regulator equations are being solved for the neural network. The first term of the error will be minimized by the adaptation law. □

## 4.4 Output regulation for a partially unknown exosystem using DNN's

Consider the following nonlinear system

$$\begin{aligned} \dot{x}_t &= f(x, \omega) + g(x)u \\ \dot{\omega} &= s(\omega) \\ e &= h(x) - r(\omega) \end{aligned} \tag{4.36}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $h(x) \in \mathbb{R}^m$ the vector function $s : \mathbb{R}^m \to \mathbb{R}^m$ is partially unknown. This case is quite challenging, it has been recently studied Chen and Huang, Nokiforov, Serrani. Marconi and Isidori and Ye and Huang.. In [24] a globally internal model adaptive sheme is proposed; however, this solution has the disadvantage of having a large dimensional controller.

The identification error is defined as

$$\Delta_t = \hat{\omega}_t - \omega_t$$
$$e_t = \hat{r} - r$$

A parallel neural network is used to identify the exosystem; it is important to notice that the exosystem does not need to be linear, as long as the function is smooth and the neural network is able to track the trajectory, the nonlinear system will be able to follow the exosystem. The parallel neural network used in this example is:

$$\dot{\hat{x}}_t = A\hat{x}_t + W_1\sigma(\hat{x}_t)$$

Notice that the neural network does not depends on the control, that is because generally the exosystem does not depends on the control either. In Figure 4.5 the proposed controller is shown.

### 4.4.1 Neural network adaptation law

In order to adjust the weights and minimize the identification error, the following adaptation law is used:

$$\dot{W}_{1,t} = -K_1 P \Delta_t \sigma(\hat{x}_t)^T - \frac{\lambda_{min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}})}{2} \tilde{W}_{1,t} \tag{4.37}$$

$$\dot{W}_{2,t} = -K_2 P \Delta_t \phi(\hat{x}_t)^T - \frac{\lambda_{min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}})}{2} \tilde{W}_{2,t} \tag{4.38}$$

where $K_1$ and $K_2$ are positive definite matrices which can be chosen arbitrarily, $P$ is the solution of the matrix ecuation given by (4.24) and

$$\tilde{W}_{1,t} := W_{1,t} - W_1^*$$
$$\tilde{W}_{2,t} := W_{2,t} - W_2^*$$

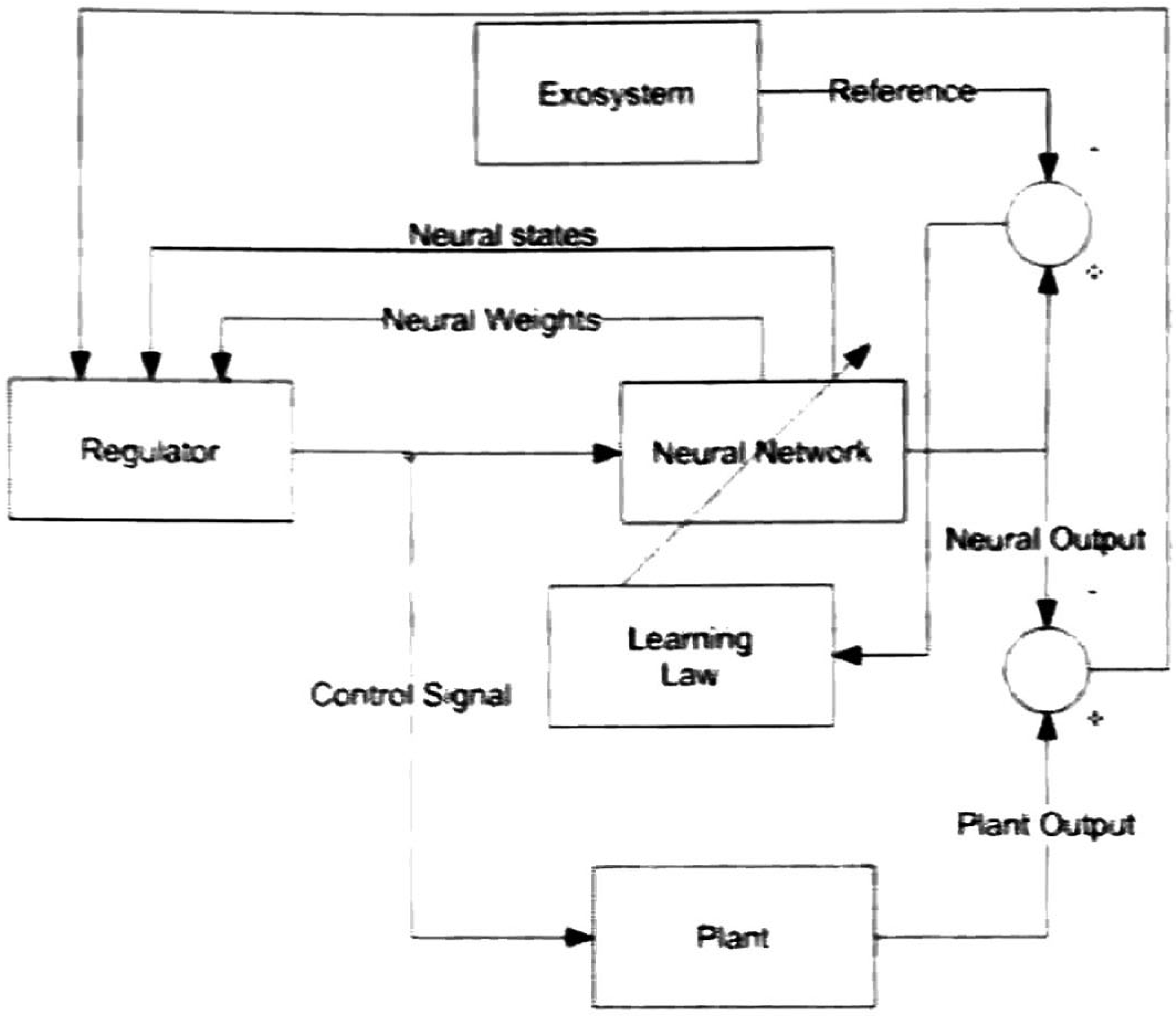


Figure 4.5: Neural identification of the exosystem.

### 4.4.2 Regulator equations

Consider the nonlinear system modeled by the neural network as

$$\begin{aligned} \dot{x} &= f(x, \omega, u) \\ \dot{\hat{\omega}} &= s(\hat{\omega}) \\ e &= h(x, \hat{w}) \end{aligned} \tag{4.39}$$

where the first equation describes the dynamics of the plant, whose state $x$ is defined in a neighborhood $U$ of the origin in $\mathbb{R}^n$ with control input $u \in \mathbb{R}^m$ and subject to a set of known exogeous input variables $\hat{\omega}$ defined in a neighborhood $V$ of the origin in $\mathbb{R}^r$ The second equation is a neural network identifier of the exosystem and the third equation defines the error expressed as a function of the plant states and the neural states of the identifier.

**Theorem 4.4.1.** *Assume the following assumptions hold*

- *The equilibrium* $\hat{\omega} = 0$ *of the neural identifier is Lyapunov stable, and the Jacobian* *matrix* $S = \frac{\partial s(\hat{\omega})}{\partial \hat{\omega}}$ *at the equilibrium* $\hat{\omega} = 0$*, has all its eigenvalues on the imaginary axis.*

- *There exists a function* $k(x)$ *such that the Jacobian matrix* $A = \frac{\partial f(x,0,k(x))}{\partial x}$ *evaluated at* $x = 0$ *has all eigenvalues on the open left-half side of the complex plane.*

*Then, the state feedback regulator problem is solvable if there exist* $C^r$ $(r \geq 2)$ *mappings* $x = \pi(\hat{\omega})$ *and* $u = c(\hat{\omega})$ *with* $\pi(0) = 0$ *and* $c(0) = 0$*, both defined in a neighborhood* $W^0 \subset W$ *of* $0$*, satisfying the conditions:*

$$\frac{\partial \pi(\hat{\omega})}{\partial \hat{\omega}} s(\hat{\omega}) = f(\pi(\hat{\omega}), \hat{\omega}, c(\hat{\omega})) \tag{4.40}$$

$$0 = h(\pi(\hat{\omega}), \hat{\omega}) \tag{4.41}$$

*Proof.* The proof of this theorem is an immediate consequence of the properties of the neuro identifier, the error can be seen as the sum of the identification error $e_i$ plus the tracking error $e_t$, i.e

$$e_t = e_i + e_t = \{\hat{r}(\omega_t) - r(x_t)\} + \{h(x_t) - r(\hat{w}_t)\} \tag{4.42}$$

the second term of the error will always be zero independently of the values of the neural network weights, because the regulator equations are solved for the neural network. The first term of the error will be minimized by the adaptation law. □

## 4.5 Conclusions

A lot of research has been made in done to determine approximate solutions for the regulator equations, often the controller increases in dimension at the expense of removing the nonlinearities, and sometimes the problem is of the same order of complexity as the original one. One method has been proposed here that exploits the advantages of the neural networks in order to do the controller robust with respect to plant uncertainties. Using the same idea, another method has been proposed to manage the case where the exosystem is allowed to vary, or even in some cases to be unknown, in this case, again the neural networks properties are used in order to do the controller robust against exosystem uncertainties.

# Chapter 5

# Illustrative cases

In this chapter a couple of exercises are developed in order to clarify the results given in the last chapter. First the van der poll oscillator is solved considering uncertainties in the plant, then it is solved considering uncertainties in the exosystem. The second example is the inverted pendulum, it is solved first considering uncertainties in the plant, and then it is solved considering uncertainties in the exosystem.

## 5.1 Van der poll oscillator

### 5.1.1 Output regulation for a partially unknown plant

Consider the following nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\mu_1 x_1 + x_1^3 - \mu_2 x_2 + \mu_3 \cos(wt) + u\\ y &= x_1\end{aligned}$$

where the nominal value of $\mu_1$ and $\mu_2$ and $\mu_3$ is 1. The desired trajectory is generated by the following exosystem

$$\begin{aligned} \dot{\omega}_1 &= \omega_2 \\ \dot{\omega}_2 &= -\alpha^2\omega_1 \\ r &= \omega_1 \end{aligned}$$

where $\alpha$ is a known value.

#### 5.1.1.1 Exact solution

The steady state manifold is described by

$$\begin{aligned} \pi_1(\omega) &= \omega_1 \\ \pi_2(\omega) &= \omega_2 \\ c(\omega) &= -\alpha^2\omega_1 + \mu_1\omega_1 + \omega_1^3 + \mu_2\omega_2 + \mu_3\cos(wt) \end{aligned}$$

The solution of the regulator equations rely on the nominal values of the nonlinear system. So the classical output regulation is not able to solve the problem; on Figure 5.1 it can be seen that when a perturbation is applied to the plant, the controller is not able to follow the trajectory. One way to solve this problem is to find a new dynamical system which can reproduce every possible value of $c(\omega)$.

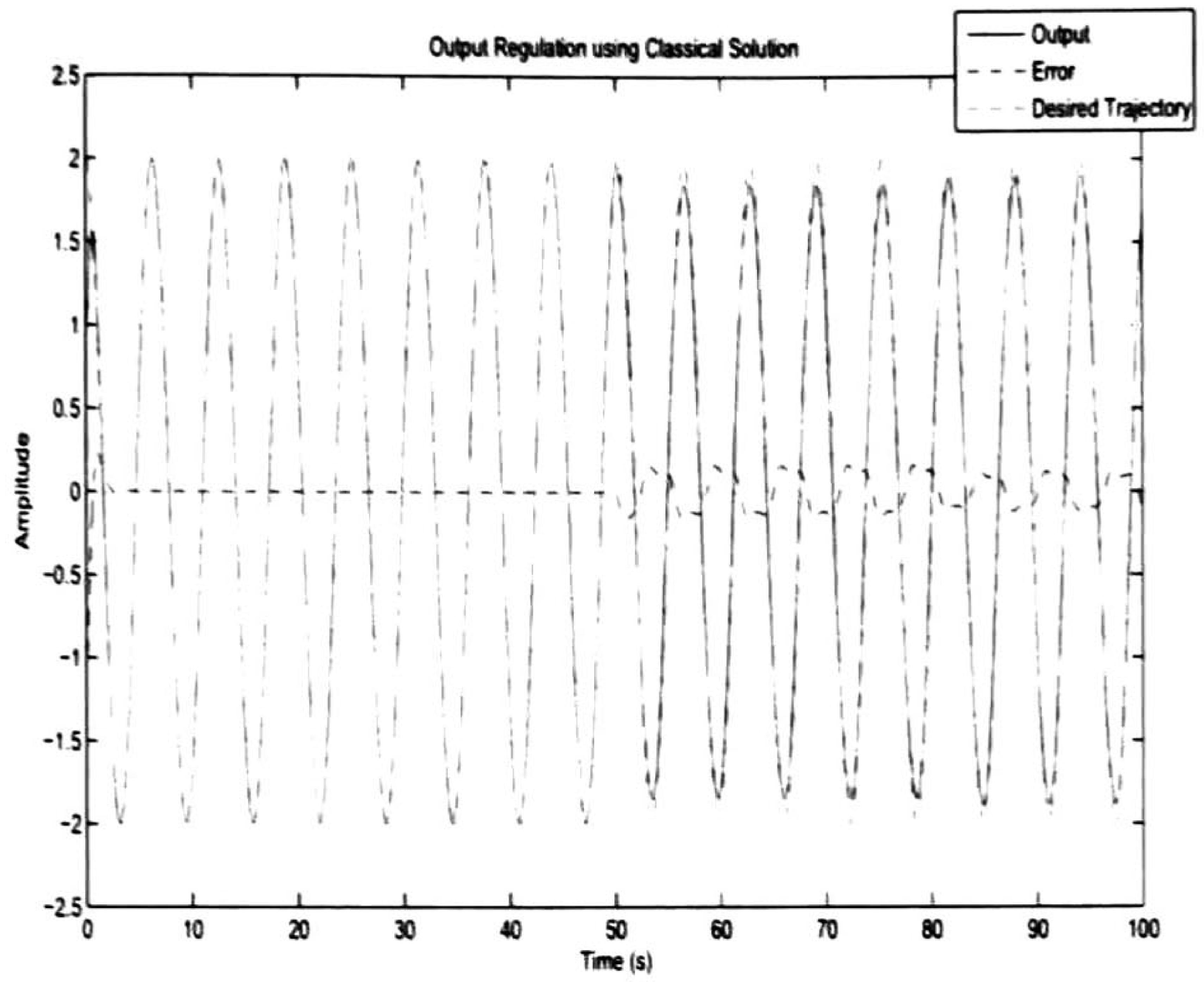


Figure 5.1: System output.

Hence, in order to make the controller robust with respect to $\mu_1$ and $\mu_2$ an immersion of $c(\omega)$ into an observable linear system is going to be found, which will be able to produce all of the $c(\omega)$ trajectories independent of the values of $\mu_1$ and $\mu_2$.

Notice that $c(\omega)$ is almost polynomial with respect to $\omega$, so a linear immersion can be determined for the polynomial part. Consider the space of polynomials of third order or less, that is $\{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9\} = \{\omega_1, \omega_2, \omega_1^2, \omega_1\omega_2, \omega_2^2, \omega_1^3, \omega_1^2\omega_2, \omega_1\omega_2^2, \omega_2^3\}$. Taking the derivative of $z$ a new dynamical system is formed; after some algebraic manipulations, the immersion for this function is given by

$$
\begin{aligned}
\dot{y} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -9 & 0 & -10 & 0 & 0 \end{bmatrix} = \Phi y \\
\Gamma z &= H_0 y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} y
\end{aligned}
$$

Finally, the controller that solves the output regulation problem is given by

$$c(\omega) = K(x - \pi(\omega)) + y + \mu_3 cos(wt) \tag{5.1}$$

The controller is not able to make zero the error because of the parametric variations on cos($wt$),however, Figure 5.2 shows that the controller is able to reject parametric variations; the parameters $\mu_1$ and $\mu_2$ are changed at t=50 s.

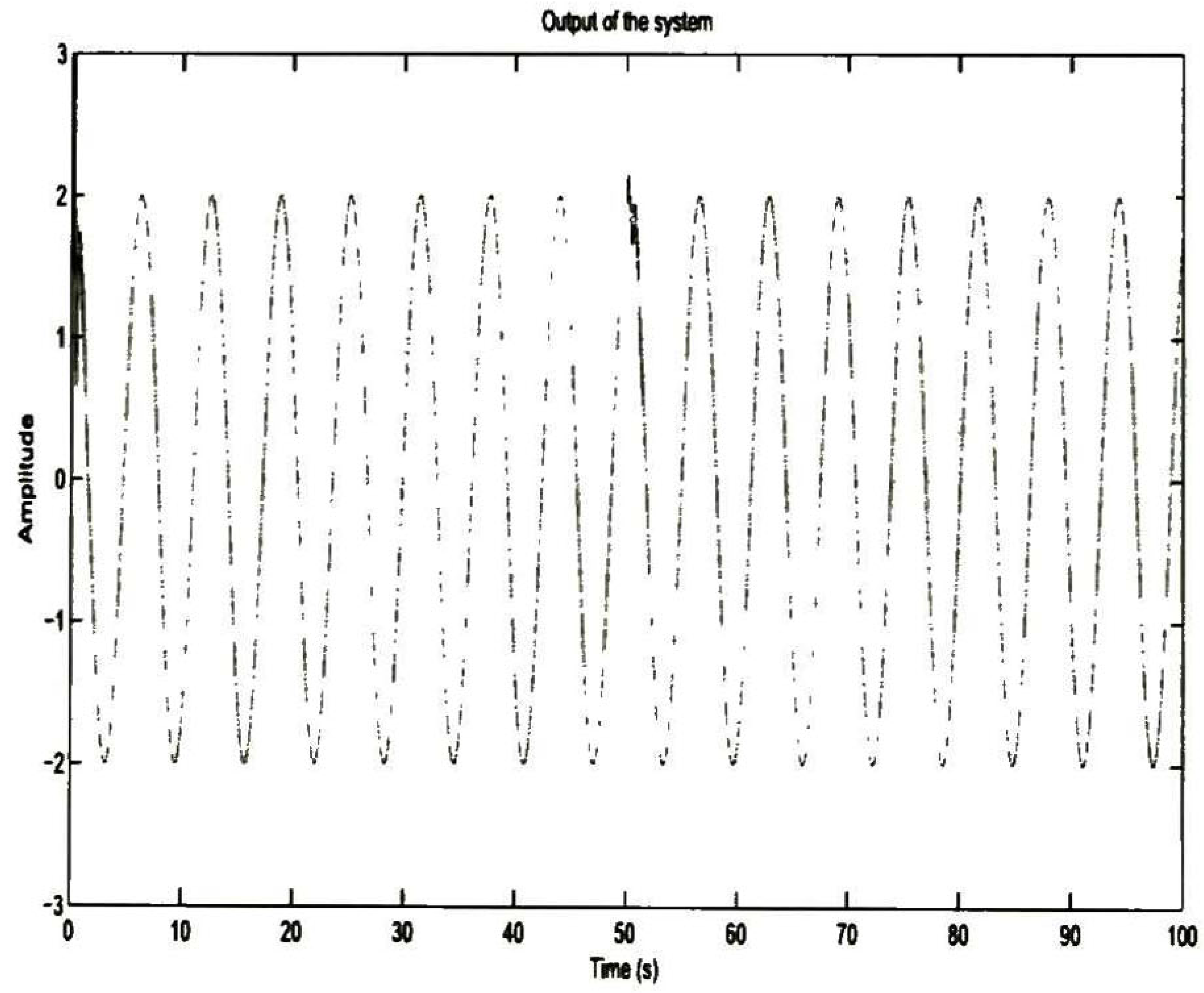


Figure 5.2: Output regulation using immersion.

#### 5.1.1.2 Dynamic neural networks

It is not always possible to obtain the immersion; the method proposed in this research uses the neural network as an identifier of the nonlinear system, and then, the regulator equations are solved for the neural network. The parallel neural network is proposed as

$$\dot{\hat{x}}_t = A\hat{x}_t + bu_t + W_1\sigma(\hat{x}_t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad W_1 = \begin{bmatrix} 0 & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \text{ and } \sigma(\hat{x}) = \begin{bmatrix} \sigma(\hat{x}_1) \\ \sigma(\hat{x}_2) \end{bmatrix}.$$

The adaptation of the weights is done by the following set of differential equations.

$$\dot{W}_{1,t} = -K_1 P \Delta_t \sigma(\hat{x}_t)^T - \frac{\lambda_{min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}})}{2} \tilde{W}_{1,t} \tag{5.2}$$

$$\dot{W}_{2,t} = -K_2 P \Delta_t \phi(\hat{x}_t)^T - \frac{\lambda_{min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}})}{2} \tilde{W}_{2,t} \tag{5.3}$$

where $K_1$ and $K_2$ are positive definite matrices which can be chosen arbitrarily, $P$ is the solution of the matrix ecuation given by (4.24) and

$$\tilde{W}_{1,t} := W_{1,t} - W_1^*$$
$$\tilde{W}_{2,t} := W_{2,t} - W_2^*$$

Using the neural network as the nonlinear plant, and the exosystem, the solution of the regulator equations are

$$\begin{aligned} \pi_1(\omega) &= \omega_1 \\ \pi_2(\omega) &= \omega_2 - \dot{W}_{11}\sigma(\omega_1) \\ c(\omega) &= -2\pi_1 + \pi_1 + \pi_2 - W_{21}\sigma(\pi_1) - W_{22}\sigma(\pi_2) - W_{12}\dot{\sigma}(\pi_1) - \dot{W}_{12}\sigma(pi_1) \end{aligned}$$

Using the proposed adaptation law, and this control the error remains bounded for parametric variations as it is seen in the following graphic. The control law is applied at second 10, a parametric variation is applied at second 50, and then at second 80 another parametric variation is applied.

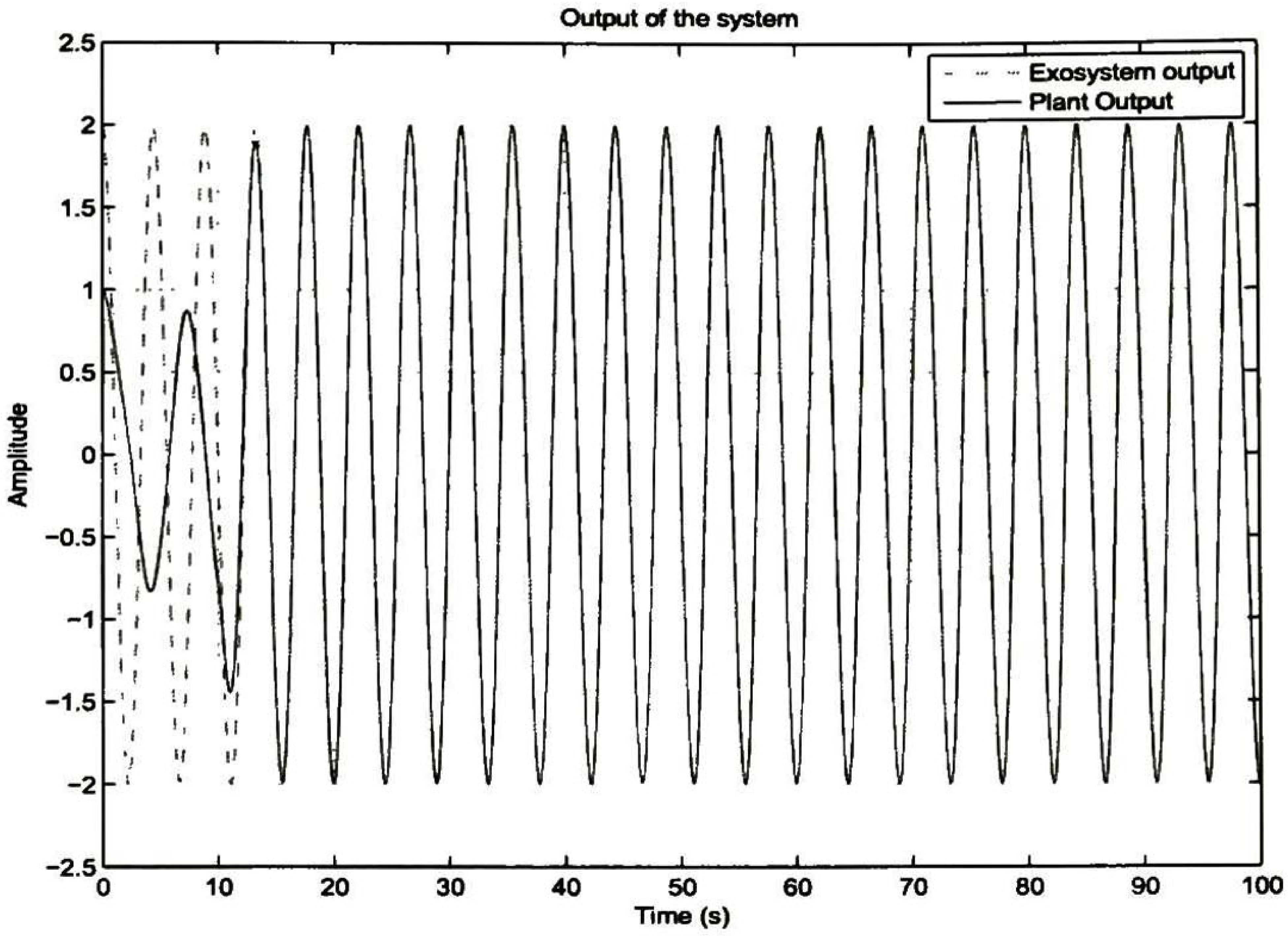


Figure 5.3: System output.

The system is able to track the desired trajectory as long as the identification error remains small, and for small parametric variations the identification error remains bounded; the following graph shows tracking error for the state $x_2$.

In this case, the neural network is able to identify de nonlinear system, and the identification error are in the magnitude of $10^{-3}$ and $10^{-2}$ Figure 5.5 shows the identification error. Finally, Figures 5.6 and 5.7 shows the tracking error as well as the input to the system, respectively.

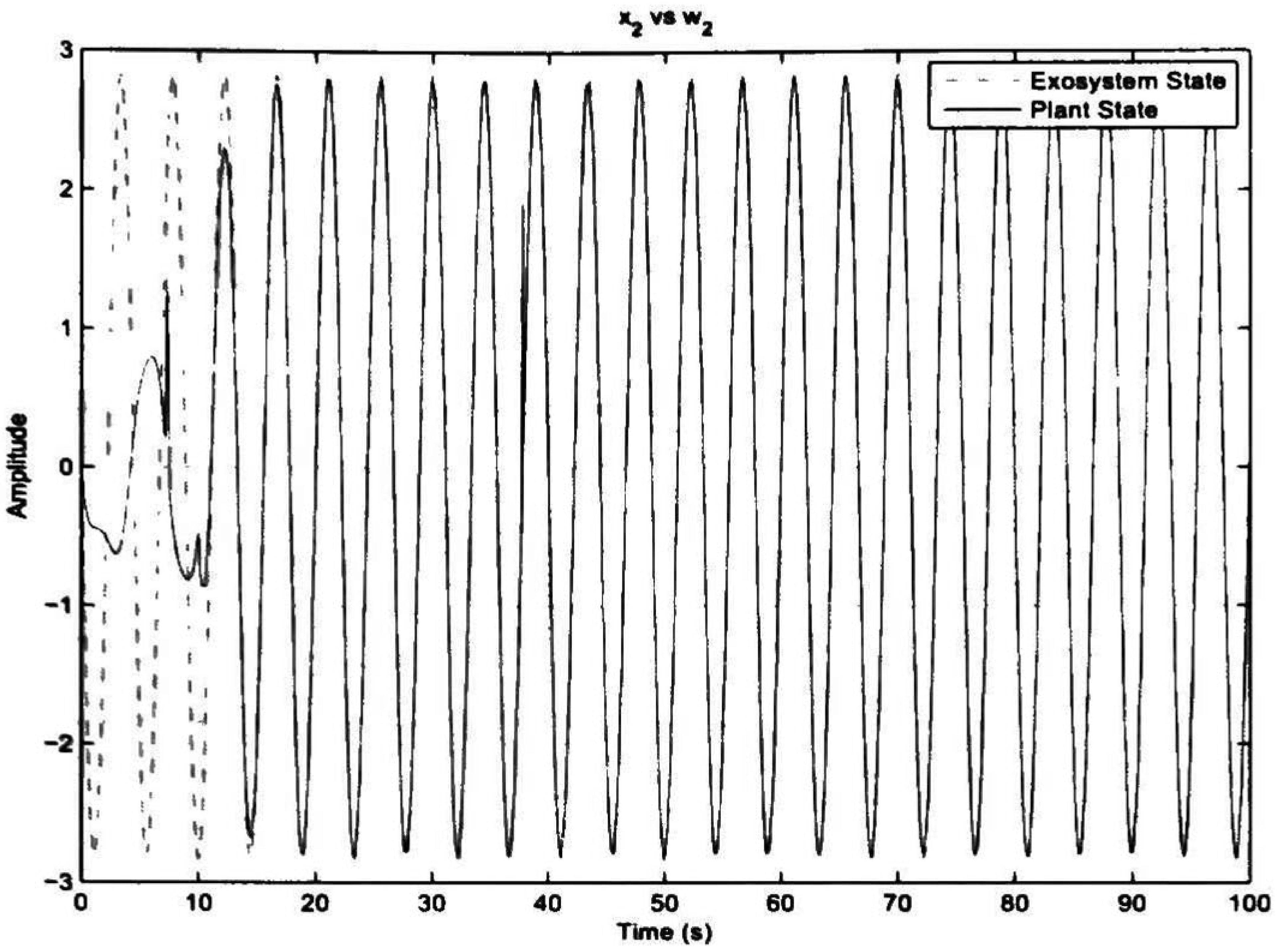


Figure 5.4: Plant state $x_2$.

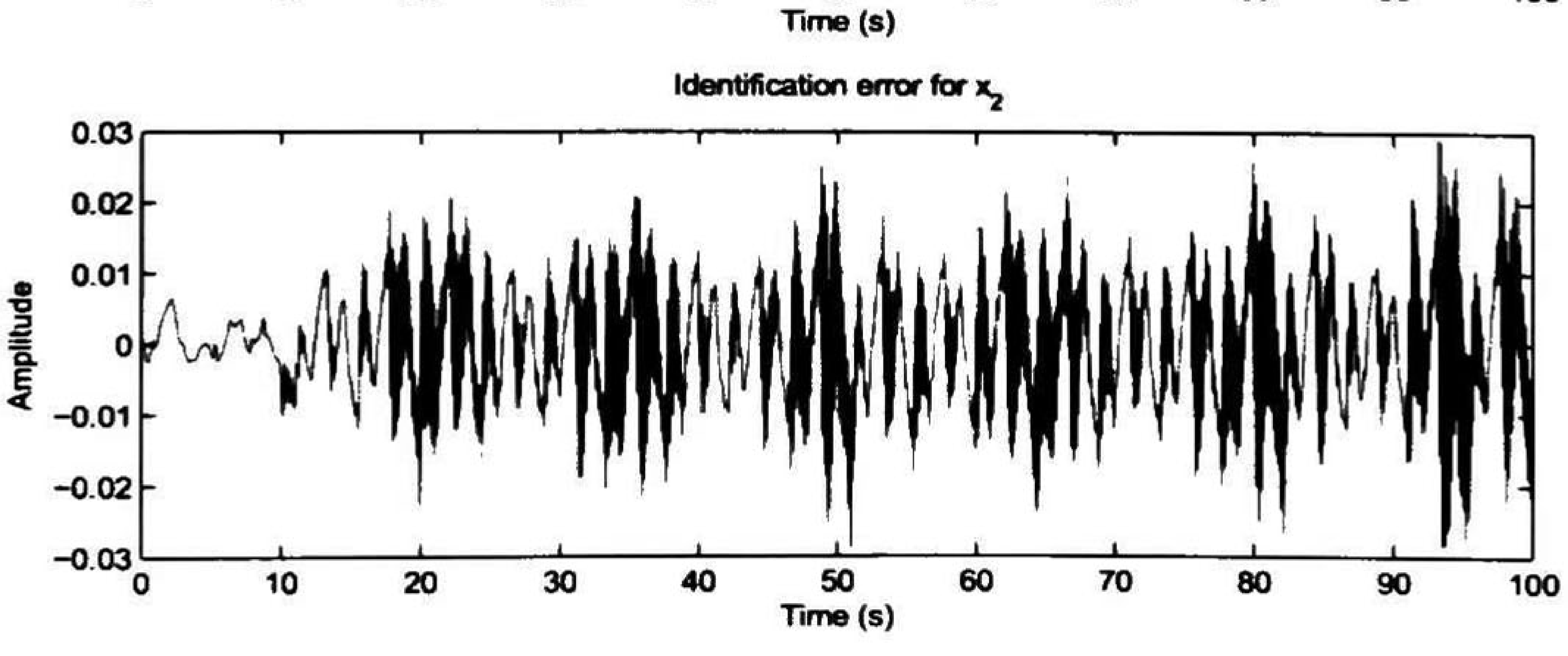


Figure 5.5: Identification error.

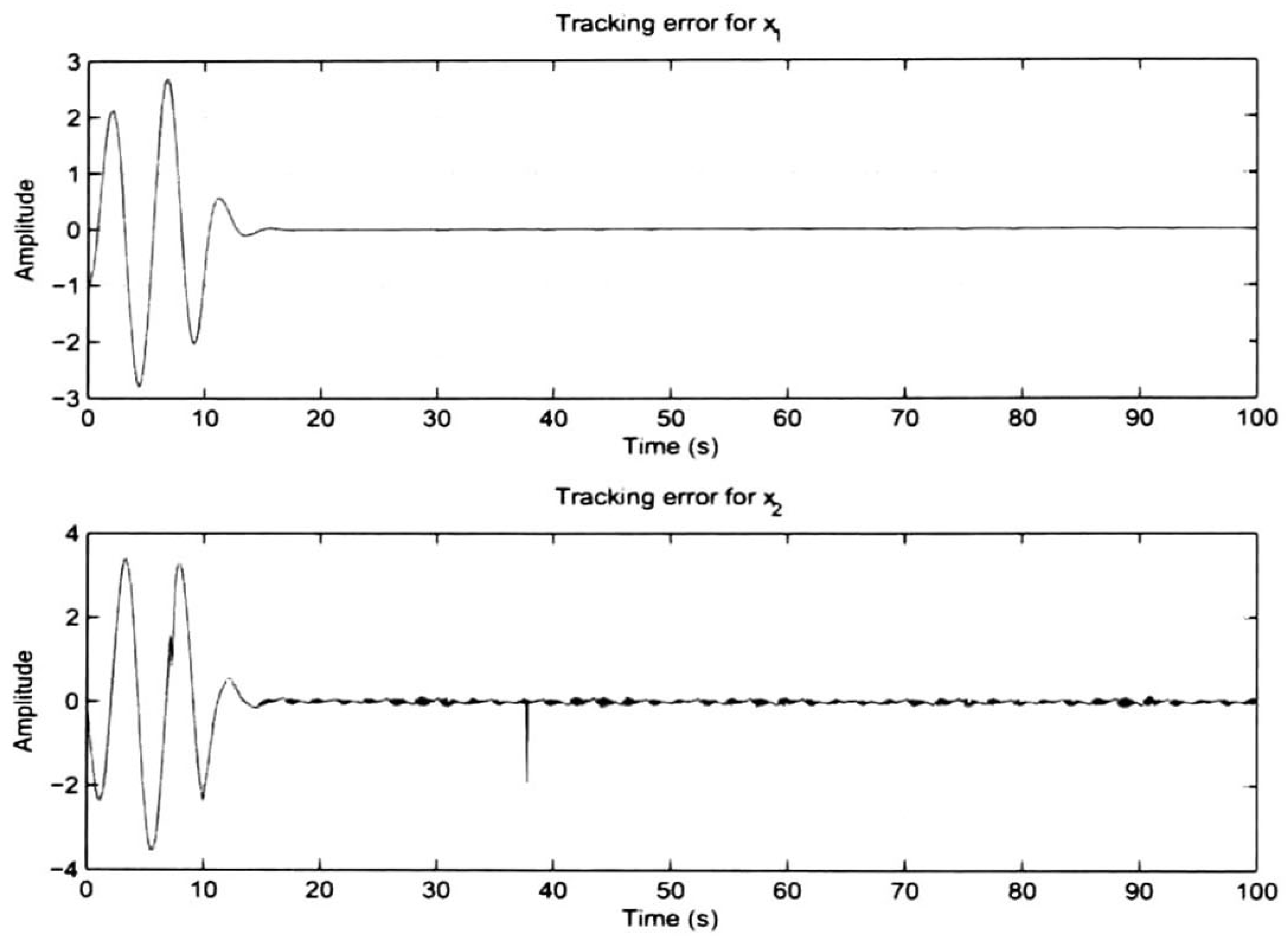


Figure 5.6: Tracking error.

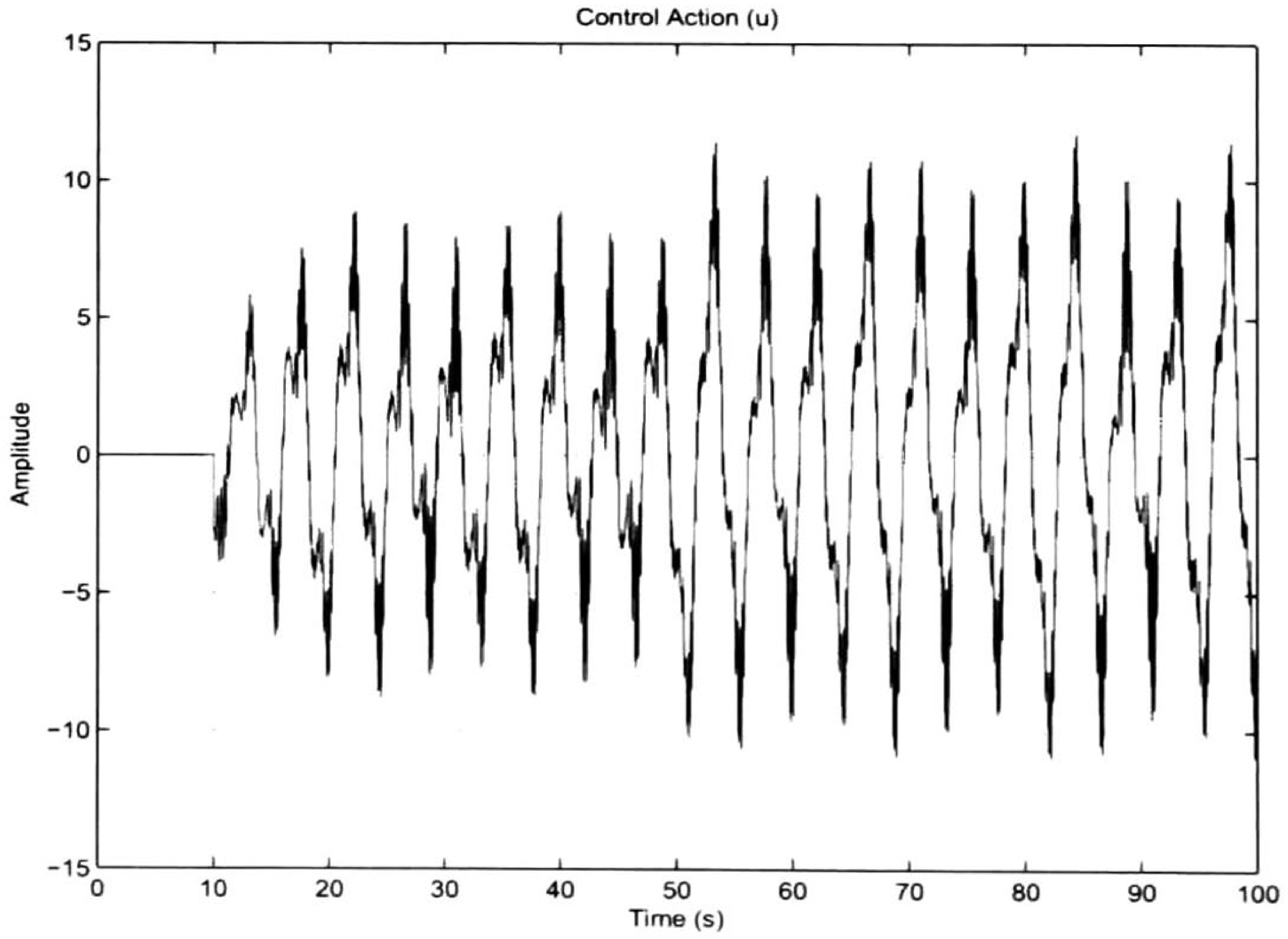


Figure 5.7: Control action $u$.

### 5.1.2 Output regulation for a partially unknown exosystem

Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^3 + x_2 + cos(wt) + u \\ y &= x_1 \end{aligned}$$

The desired trajectory is described by

$$\begin{aligned} \dot{\omega}_1 &= \omega_2 \\ \dot{\omega}_2 &= -\alpha^2\omega_1 \\ r &= \omega_1 \end{aligned}$$

where the parameter $\alpha$ is unknown.

#### 5.1.2.1 Exact Solution

In order to achieve trajectory tracking, one needs to solve the regulator equations

$$\begin{aligned} \pi_1(\omega) &= \omega_1 \\ \pi_2(\omega) &= \omega_2 \\ c(\omega) &= -\alpha^2\omega_1 - \omega_1^3 - \omega_2 - cos(wt) \end{aligned}$$

The solution of the regulator equations depends on the parameter $\alpha$ and, in this case, an immersion is not possible because the parameter $\alpha$ will always appear. Neither the generalized immersion nor the kth-order solution will apply to this problem.

#### 5.1.2.2 Dynamic neural networks

Using the approach described in the previous chapter, a parallel neural network is used to identify the exosystem states. Then, the regulator equations are solved for the nonlinear plant and the neural network. Once the regulator equations are obtained, the plant is going to follow the neural network independent of the values of the exosystem.

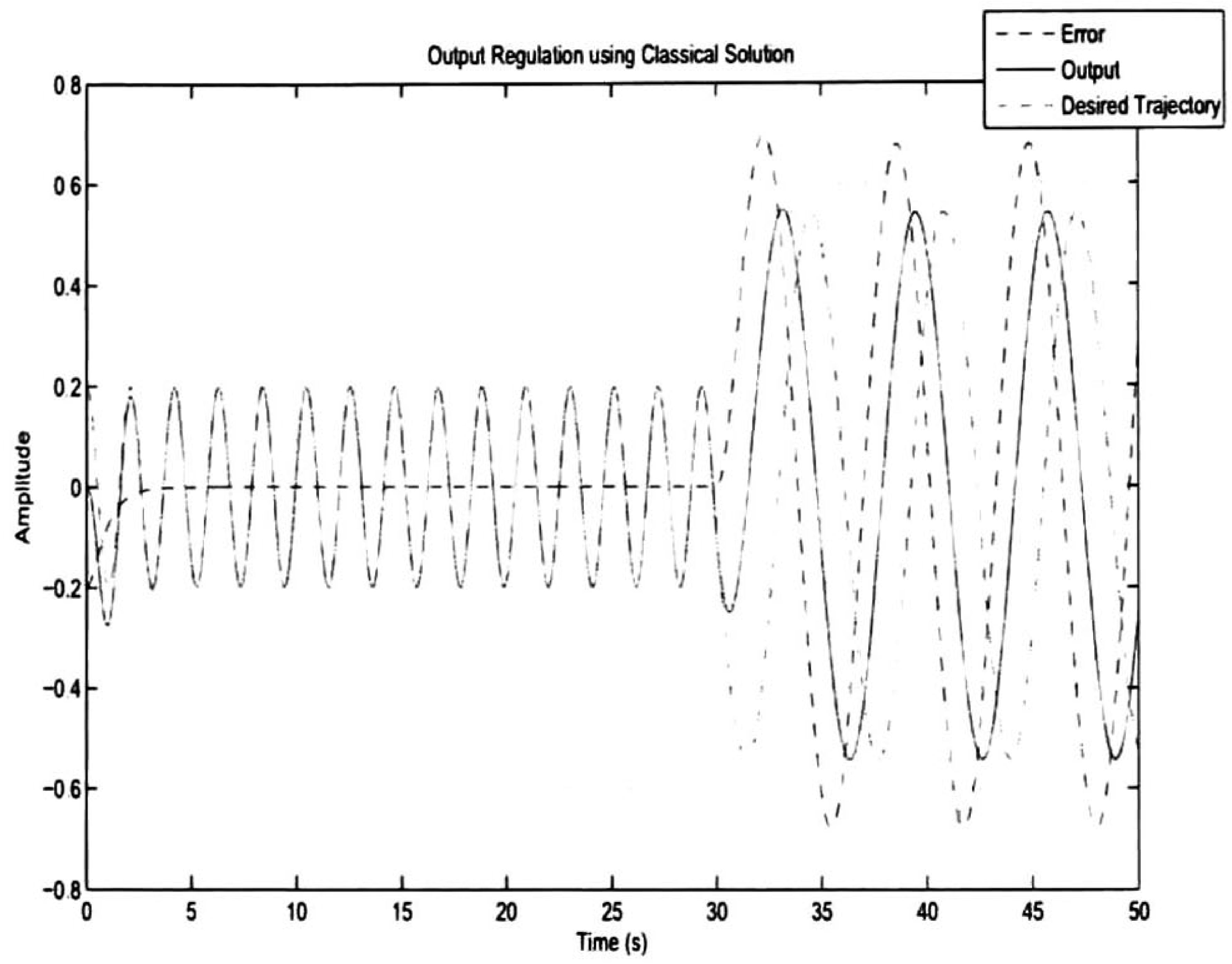


Figure 5.8: Output regulation for an unknown frecuency exosystem.

The structure of the neural network is proposed as follows.

$$\dot{\hat{x}}_t = A\hat{x}_t + W_1\sigma(\hat{x}_t)$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ $W_1 = \begin{bmatrix} 0 & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$ and $\sigma(\hat{x}) = \begin{bmatrix} \sigma(\hat{x}_1) \\ \sigma(\hat{x}_2) \end{bmatrix}$.

The adaptation of the weights is done by the following set of differential equations.

$$\dot{W_{1,t}} = -K_1 P \Delta_t \sigma(\hat{x}_t)^T - \frac{\lambda_{min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}})}{2} \tilde{W_{1,t}} \tag{5.4}$$

$$\dot{W_{2,t}} = -K_2 P \Delta_t \phi(\hat{x}_t)^T - \frac{\lambda_{min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}})}{2} \tilde{W_{2,t}} \tag{5.5}$$

where $K_1$ and $K_2$ are positive definite matrices which can be chosen arbitrarily, $P$ is the solution of the matrix equation given by (4.24) and

$$\tilde{W_{1,t}} := W_{1,t} - W_1^*$$
$$\tilde{W_{2,t}} := W_{2,t} - W_2^*$$

Using the neural network as the exosystem, the solution of the regulator equations are

$$\begin{aligned}
\pi_1(\omega) &= \omega_1 \\
\pi_2(\omega) &= \omega_2 - W_{11}\sigma(\omega_1) \\
c(\omega) &= -\pi_1 + \pi_1^3 + d\pi_2 - g\cos(t) + \hat{x}_1 + \hat{x}_2 - W_{21}\sigma(\hat{x}_1) - W_{22}\sigma(\hat{x}_2) \\
&\quad -W_{12}\dot{\sigma}(\pi_1) - \dot{W}_{12}\sigma(\pi_1)
\end{aligned}$$

The output of the simulation can be seen in Figure 5.9, the first 30 seconds the frequency is three radians per second, after that the frequency is changed to one radian per second. The amplitude of the reference signal also changes, and the neural network is able to follow the trajectory, since the output regulator equations are solved for the neural network, the system is able to track the trajectory as long as the identification error remains bounded. In Figure 5.10 the identification error is shown. It can be seen that the identification error remains bounded. The weights of the neural network evolve as shown in Figure 5.11, it can be seen that the weights are bounded because of the adaptation law. The tracking error is shown in 5.12, it remains bounded and the controller that minimizes the tracking error is shown in 5.13.

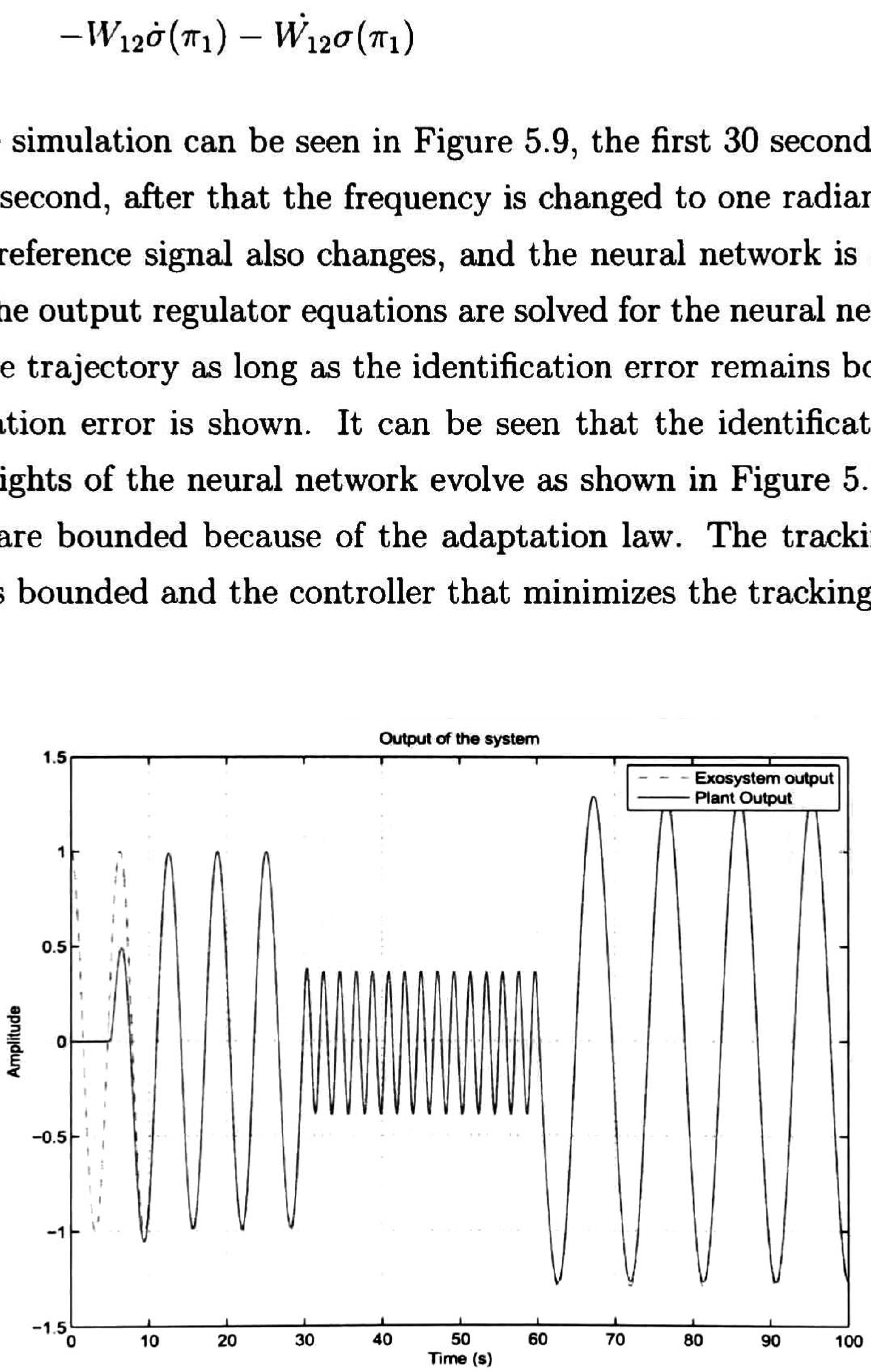


Figure 5.9: Output regulation for an unknown exosystem frecuency using neural networks.

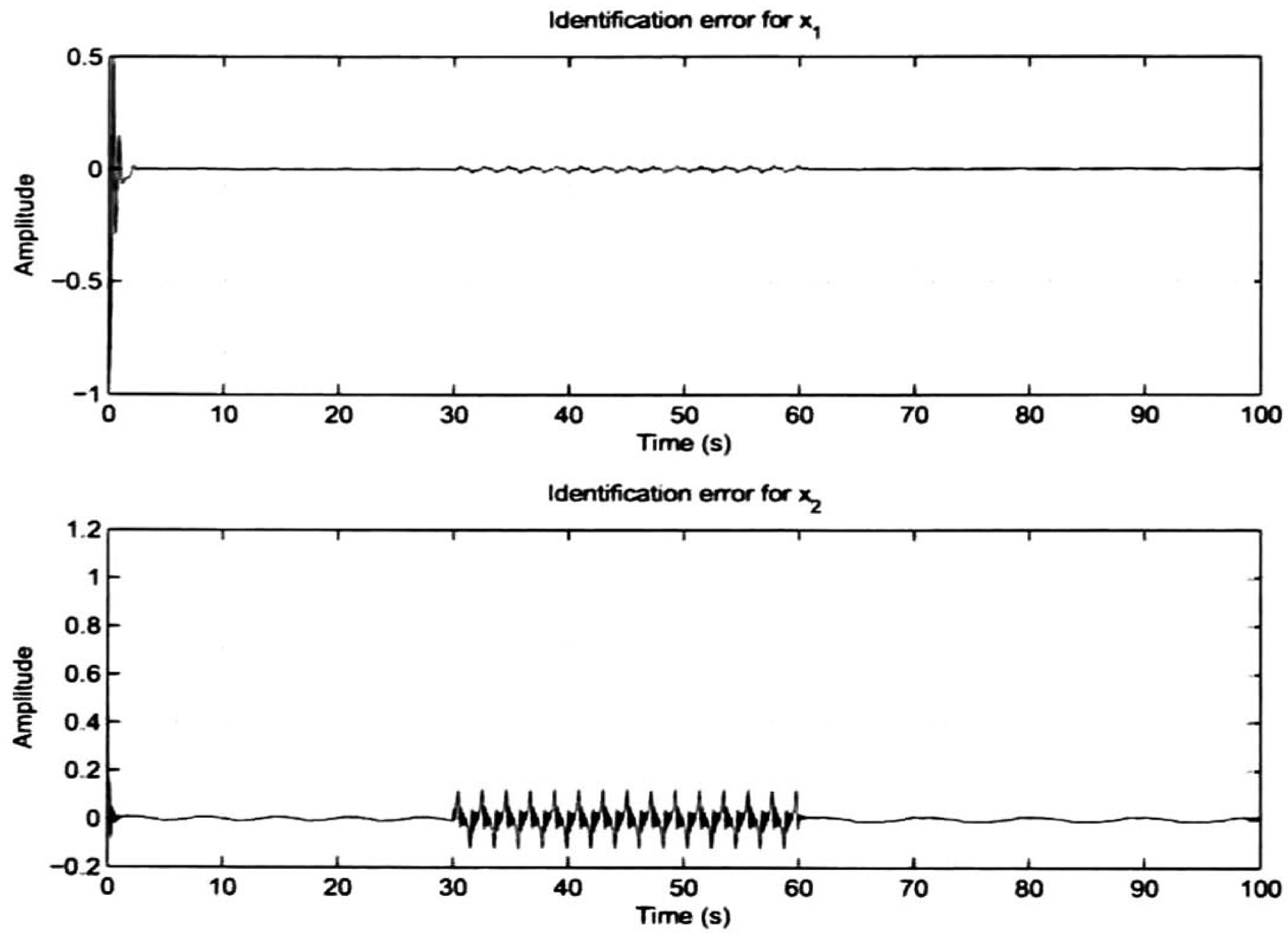


Figure 5.10: Identification error.

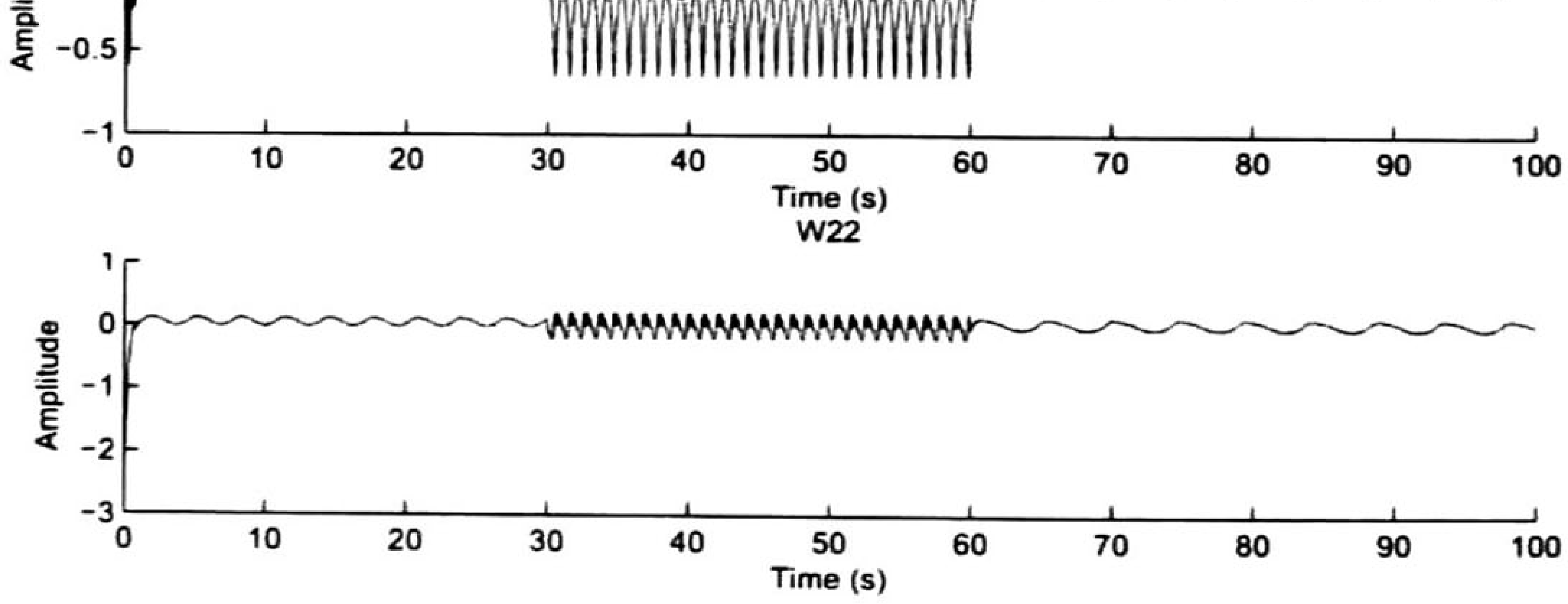


Figure 5.11: Weights of the neural network.

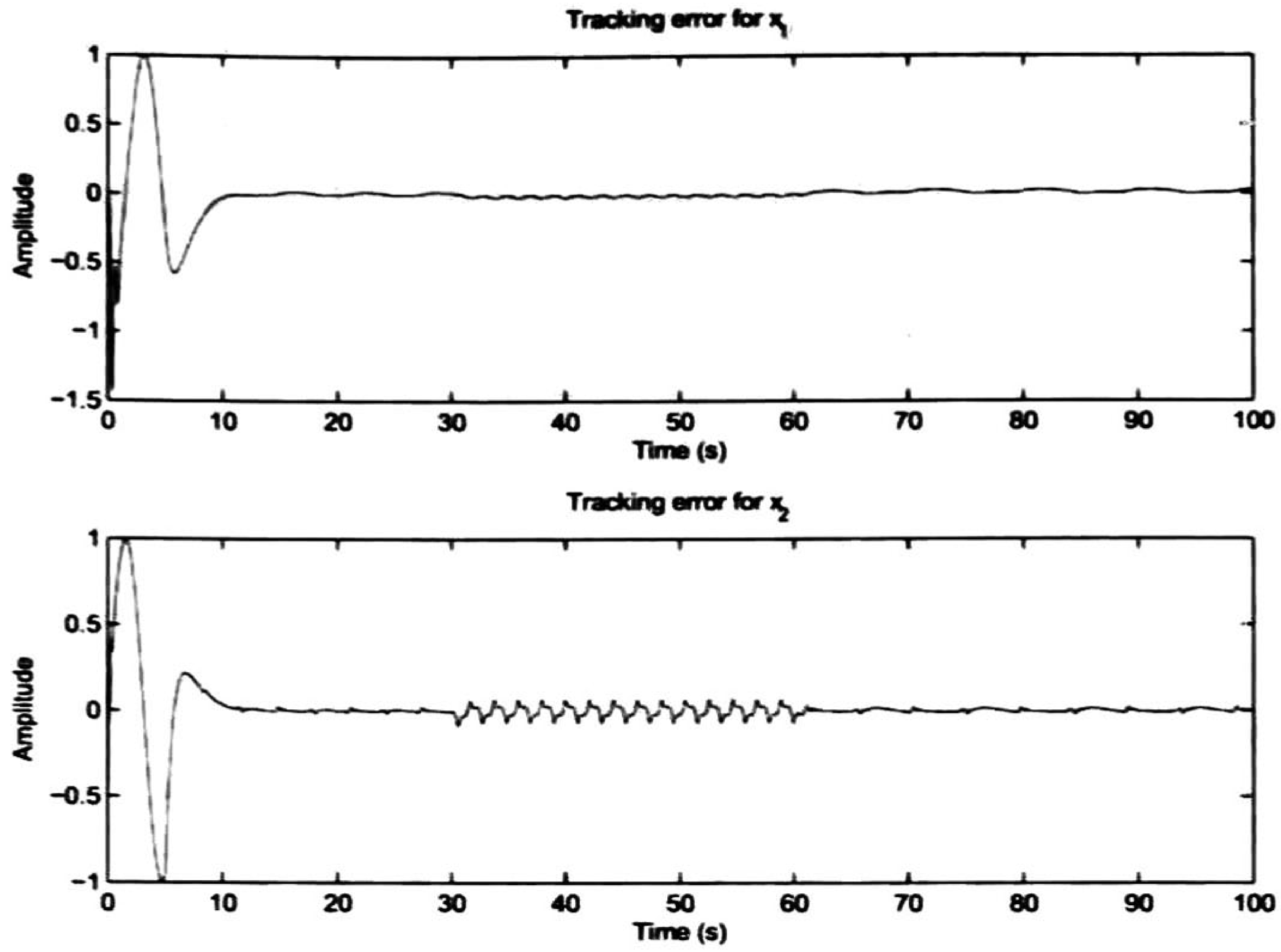


Figure 5.12: Tracking error.

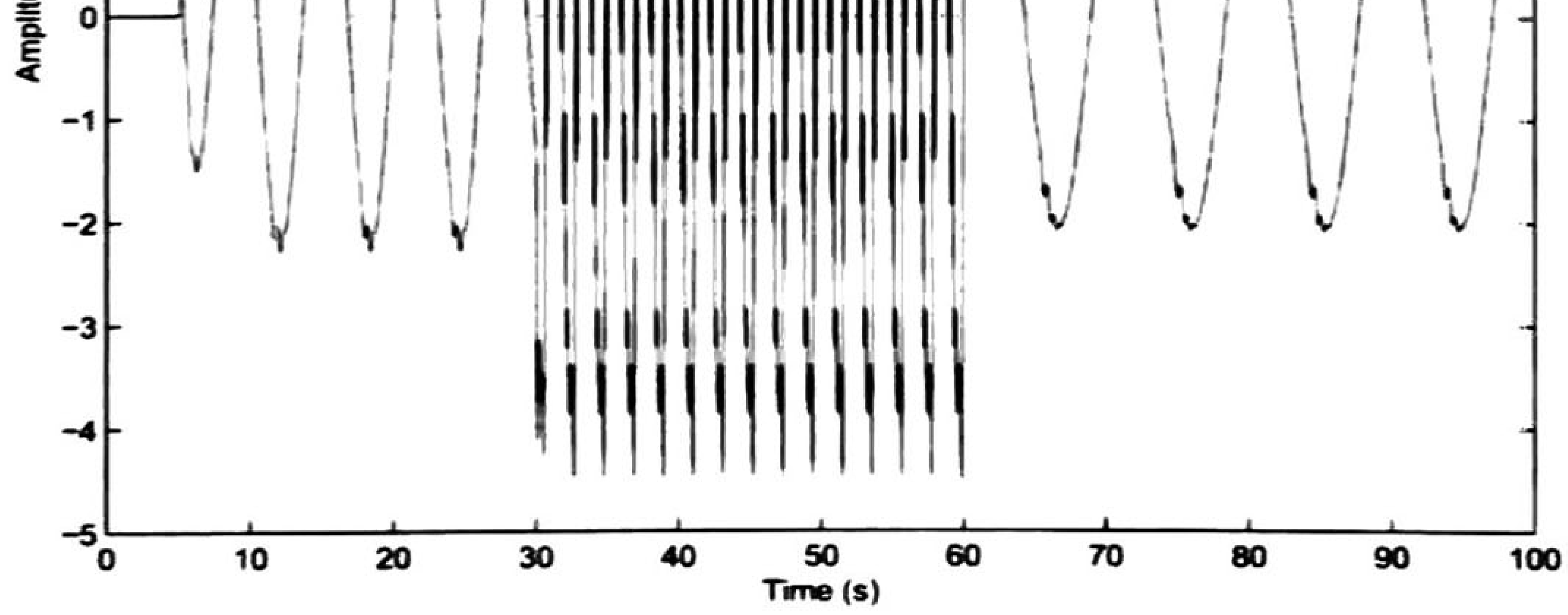


Figure 5.13: Control action $u$.

## 5.2 Inverted pendulum

Using the Euler-Lagrange method, the mathematical model for the inverted pendulum is

$$\tau = (ml_{c2}{}^2 + I_{zz})\ddot{q} + mgl_c sin(q) + \mu_f \dot{q} \tag{5.6}$$

This equation can be represented in state variables if it's solved for $\ddot{q}$.

$$\ddot{q} = -\frac{1}{ml_{c2}^2}(mgl_c sin(q) + \mu_f \dot{q} + \tau) \tag{5.7}$$

Defining the state variables as

$$\begin{aligned} x_1 &= q \\ x_2 &= \dot{q} \\ u &= \tau \\ y &= x_1 \end{aligned} \tag{5.8}$$

the non-linear system is then represented by

$$\dot{x} = f(x) + g(x)u \tag{5.9}$$

$$y = h(x) \tag{5.10}$$

where

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{1}{ml_{c2}^2 + I_{zz}}(mgl_c sin(q) + \mu_f \dot{q} + \tau) \end{bmatrix} \tag{5.11}$$

$$g(x) = \begin{bmatrix} 0 \\ -\frac{1}{ml_{c2}^2 + I_{zz}} \end{bmatrix} \tag{5.12}$$

The following table shows the values of the system

| | | |
|---|---|---|
| $l_c$ | Distance from the joint to the center of gravity | $0.1551\ m$ |
| $m$ | Mass of the link | $0.8293\ Kg$ |
| $I_{zz}$ | Moment of inertia of the link | $0.00595\ kg \cdot m^2$ |
| $\mu_f$ | Viscous friction coefficient | $0.00545\ Kg/s$ |
| $g$ | Gravity coefficient | $9.81\ m/s^2$ |

### 5.2.1 Output regulation for a partially unknown plant

#### 5.2.1.1 Dynamic neural networks

Evaluating the system with the values presented on the above table, the state space representation is

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= 48.7184\sin(x_1) + 0.2104x_2 + 38.61u \\ e &= x_1 - \omega_1 \end{aligned} \tag{5.13}$$

Such that $\omega_1$ is the solution of the following linear exosystem

$$\begin{bmatrix} \dot{\omega_1} \\ \dot{\omega_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \tag{5.14}$$

The steady state manifold for this pair of equations is

$$\begin{aligned} x_{1ss} &= \omega_1 \\ x_{2ss} &= \omega_2 \\ u_{ss} &= -0.259\omega_1 - 1.26\sin(\omega_1) - 0.0054\omega_2 \end{aligned} \tag{5.15}$$

Using the proposed neural network, with the adaptation law described befor, the error remains bounded for parametric variations as it is seen in the following graphic. The control law is applied since the beggining of the simulation, a parametric variation is applied at second 50, and then at second 80 another parametric variation is applied.

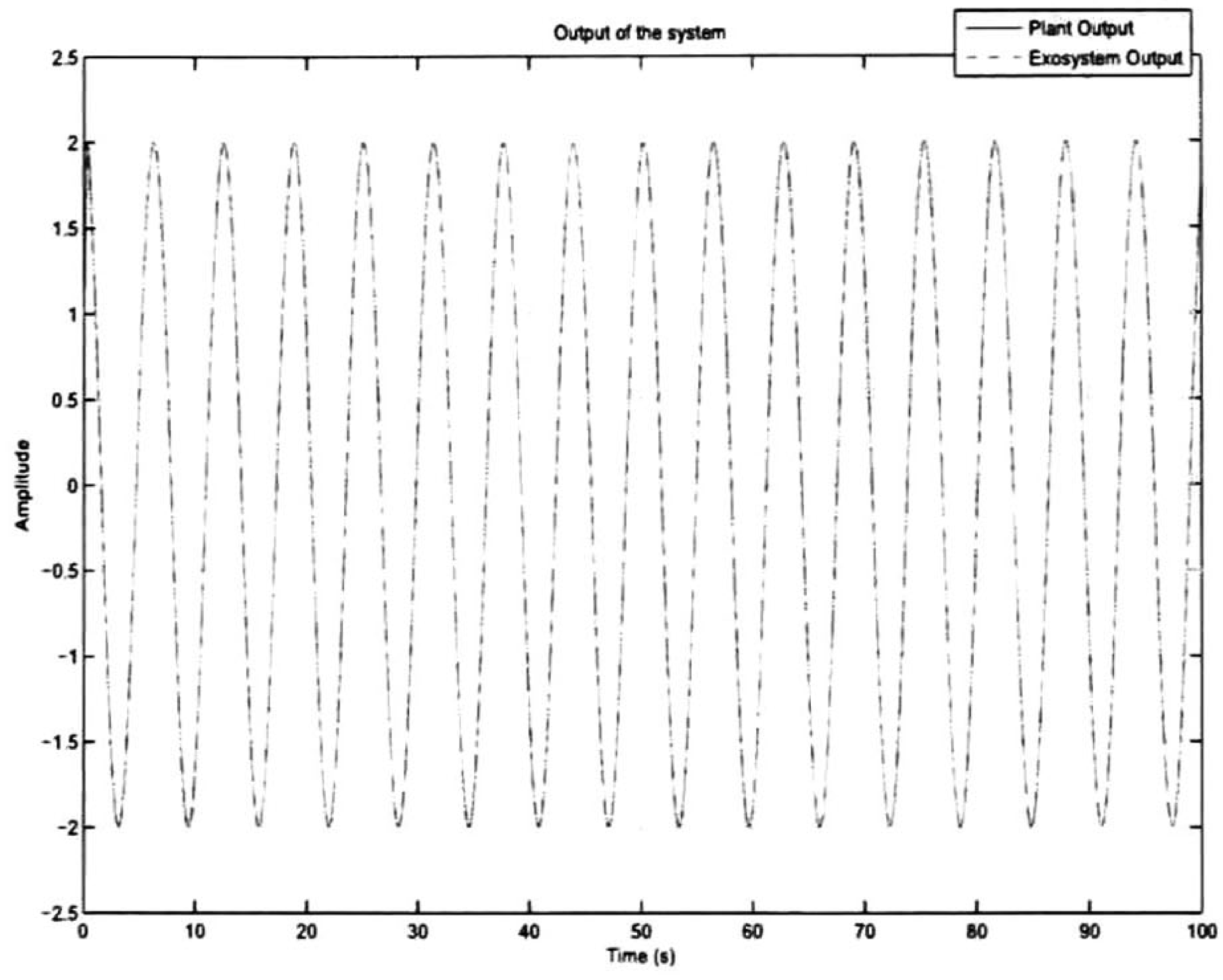


Figure 5.14: System output.

The system is able to track the desired trajectory as long as the identification error remains small, and for small parametric variations the identification error remains bounded, the following graph displays the tracking error for the state $x_2$. In this case, the neural

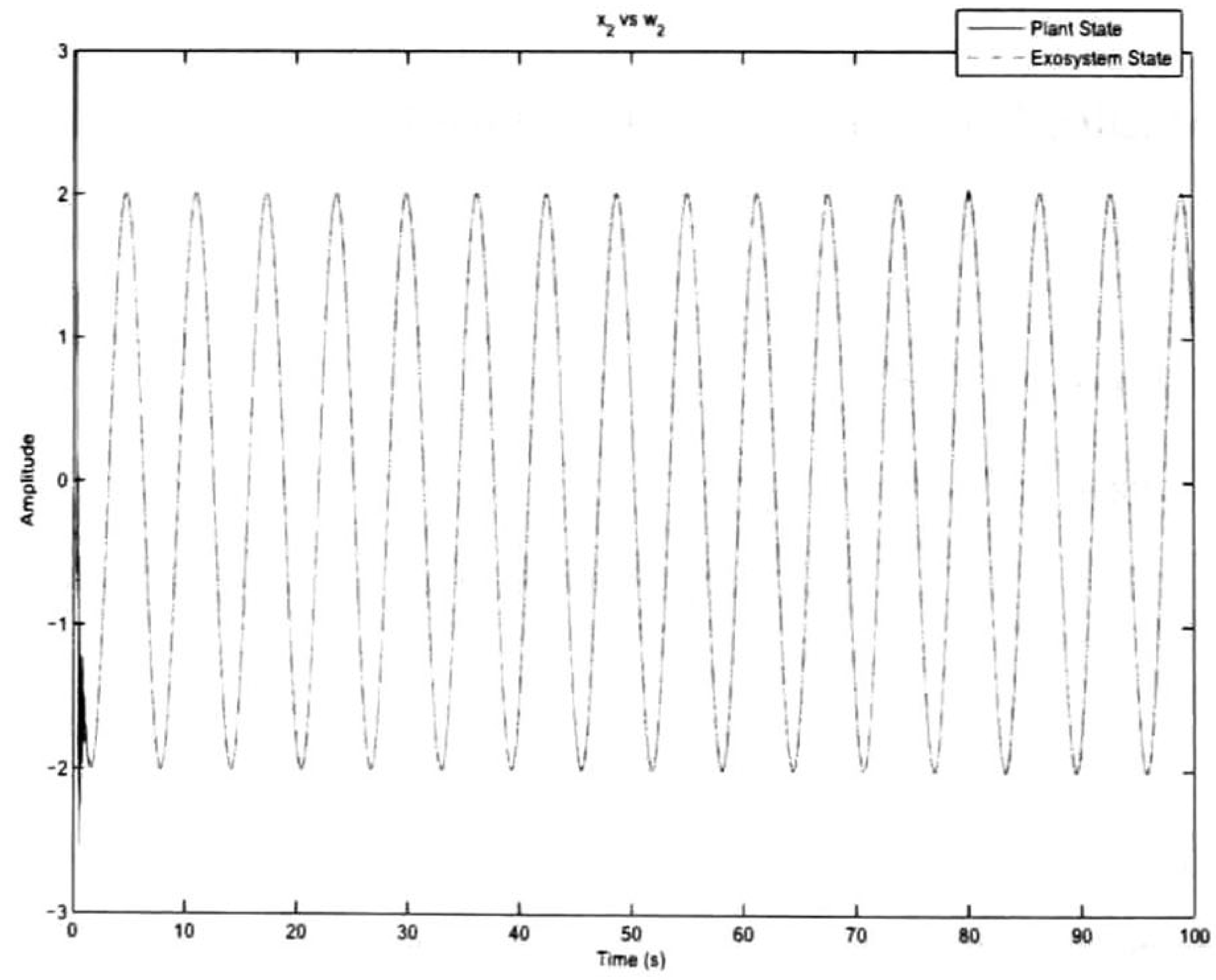


Figure 5.15: Plant state $x_2$.

network is able to identify the nonlinear system, and the identification error are in the magnitude of $10^{-3}$ and $10^{-2}$ Figure 5.16 presents the identification error. Finally, Figure 5.17 portrays the input to the system.

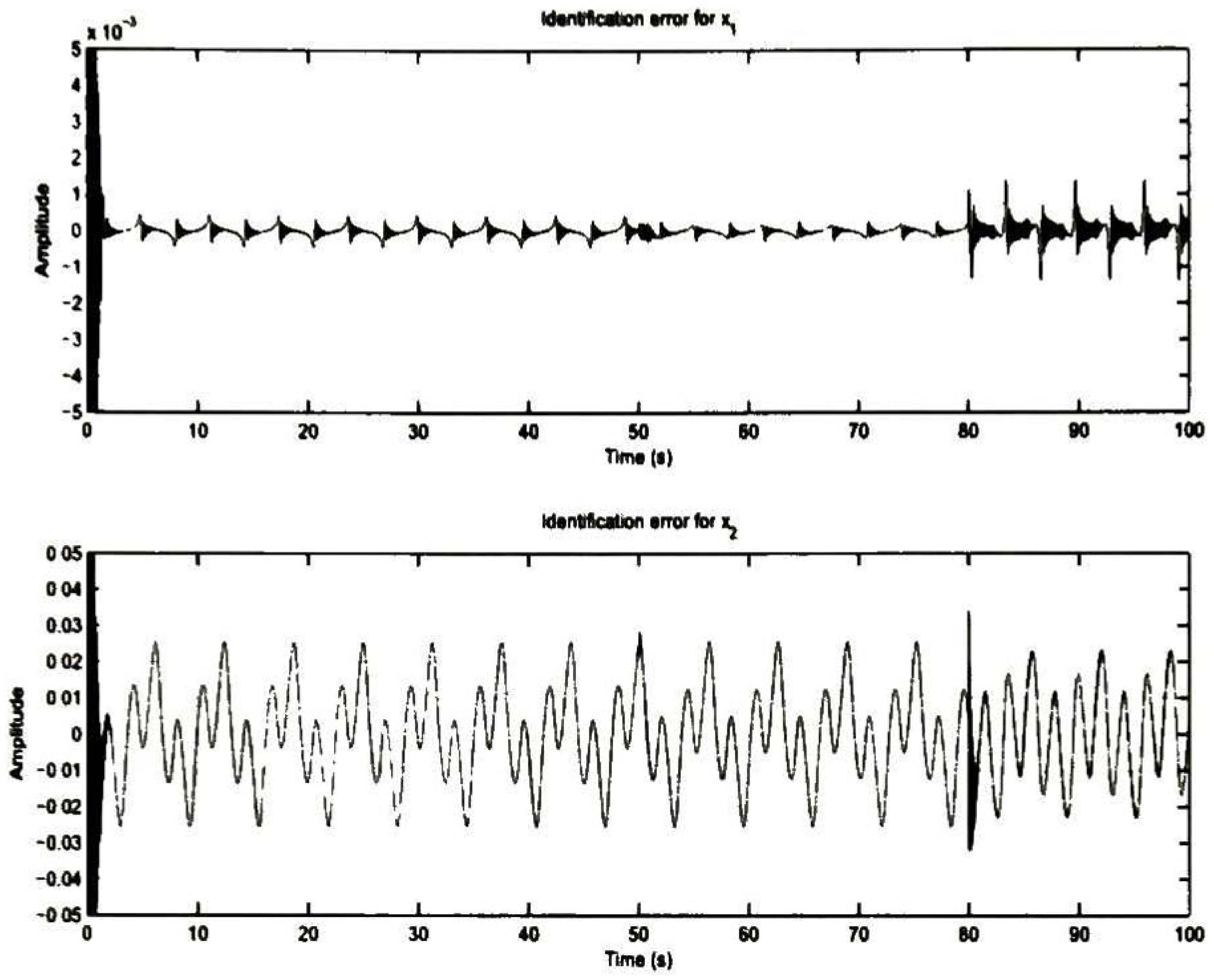


Figure 5.16: Identification error.

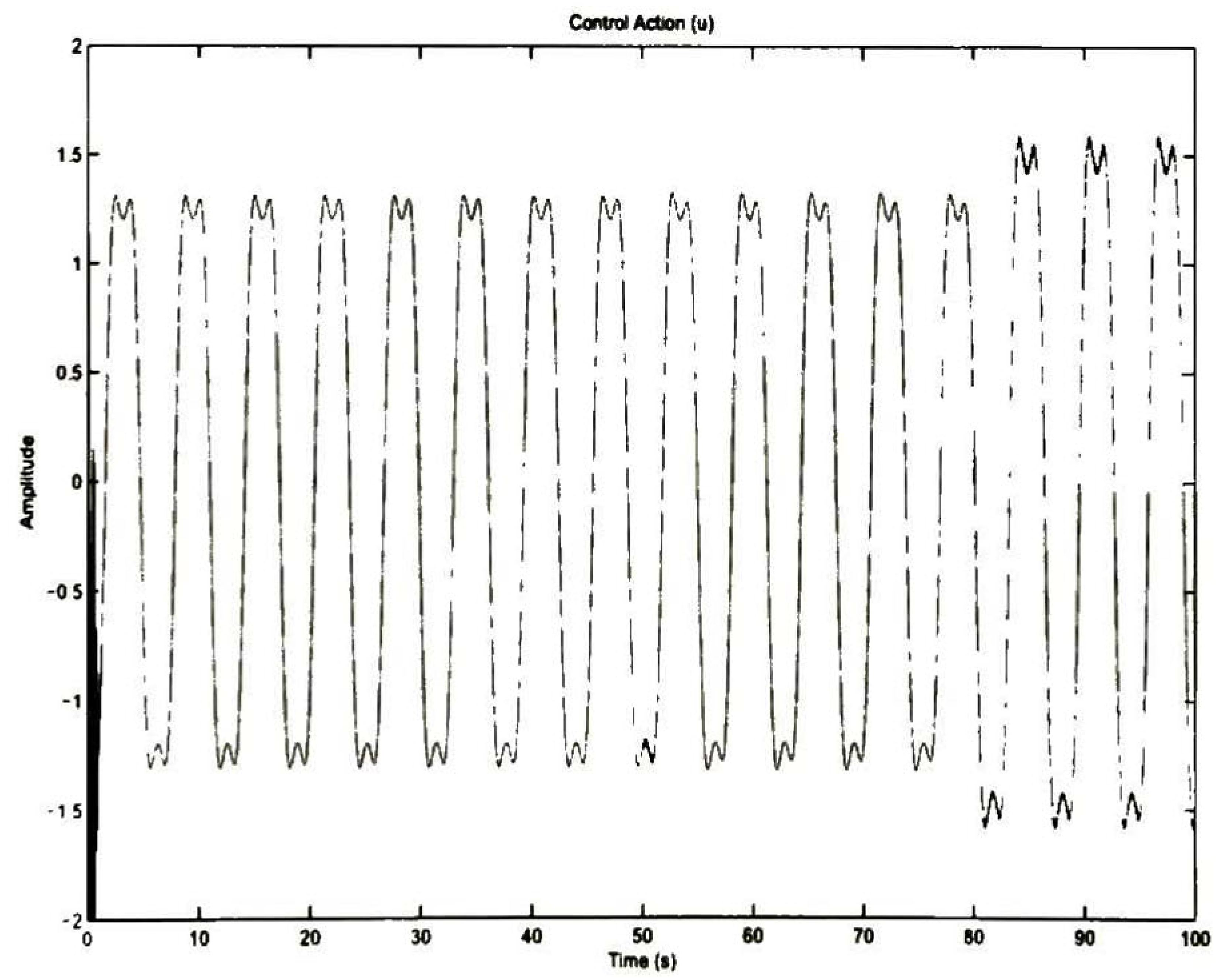


Figure 5.17: Control action $u$.

### 5.2.2 Output regulation for a partially unknown exosystem

#### 5.2.2.1 Dynamic neural networks

Using the approach described in the previous chapter, a parallel neural network is used to identify the exosystem states. Then, the regulator equations are solved for the nonlinear plant and the neural network. Once the regulator equations are obtained, the plant is going to follow the neural network independent exosystem values.

The structure of the neural network proposed is as follows.

$$\dot{\hat{x}}_t = A\hat{x}_t + W_1\sigma(\hat{x}_t)$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $W_1 = \begin{bmatrix} 0 & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$ and $\sigma(\hat{x}) = \begin{bmatrix} \sigma(\hat{x}_1) \\ \sigma(\hat{x}_2) \end{bmatrix}$.

The weight's adaptation is made by the following set of differential equations.

$$\dot{W}_{1,t} = -K_1 P \Delta_t \sigma(\hat{x}_t)^T - \frac{\lambda_{min}(P^{-\frac{1}{2}}Q_0P^{-\frac{1}{2}})}{2}\tilde{W}_{1,t} \tag{5.16}$$

$$\dot{W}_{2,t} = -K_2 P \Delta_t \phi(\hat{x}_t)^T - \frac{\lambda_{min}(P^{-\frac{1}{2}}Q_0P^{-\frac{1}{2}})}{2}\tilde{W}_{2,t} \tag{5.17}$$

where $K_1$ and $K_2$ are positive definite matrices which can be chosen arbitrarily, $P$ is the solution of the matrix equation given by 4.24 and

$$\tilde{W}_{1,t} := W_{1,t} - W_1^*$$
$$\tilde{W}_{2,t} := W_{2,t} - W_2^*$$

Using the neural network as the exosystem, the solution of the regulator equations are

$$\begin{aligned} \pi_1(\omega) &= \omega_1 \\ \pi_2(\omega) &= \omega_2 - W_{11}\sigma(\omega_1) \\ c(\omega) &= \frac{1}{38.61}[-48.7184sin(\pi_1) + 0.2104\pi_2 + \hat{x}_1 + \hat{x}_2 - W_{21}\sigma(\hat{x}_1) - W_{22}\sigma(\hat{x}_2) \\ &-W_{12}\dot{\sigma}(\pi_1) - \dot{W}_{12}\sigma(\pi_1)] \end{aligned}$$

The output of the simulation can be seen in Figure 5.9; the first 30 seconds the frequency is three radians per second, after that the frequency is changed to one radian per second. The amplitude of the reference signal also changes, and the neural network is able to follow the trajectory, since the output regulator equations are solved for the neural network, the system will be able to track the trajectory as long as the identification error remains bounded.
In Figure 5.19 the identification error is presented, it can be seen that the identification error remains bounded. The controller that minimizes the tracking error is portrayed in Figure 5.20.

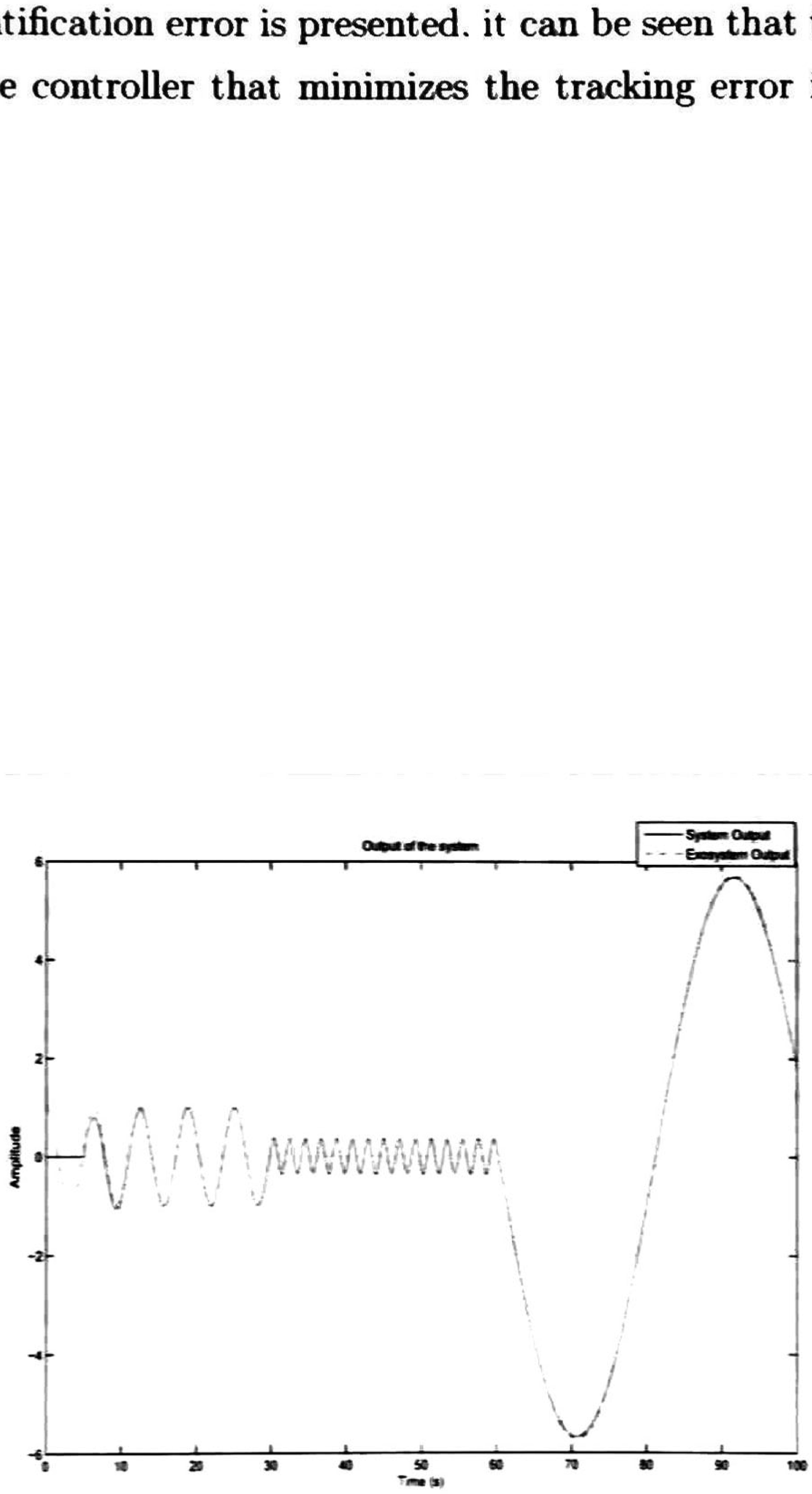


Figure 5.18: Output regulation for an unknown exosystem frecuency using neural networks.

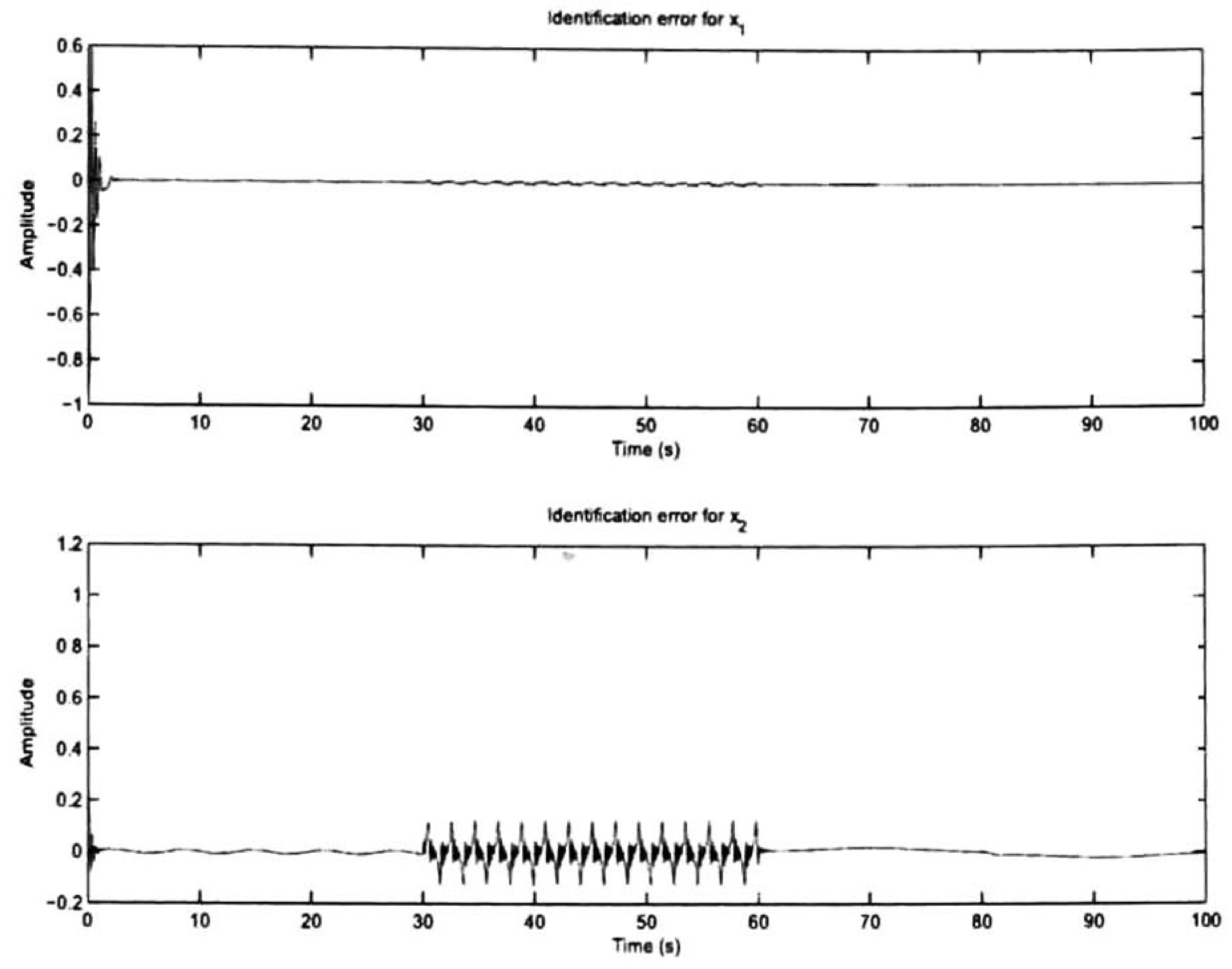


Figure 5.19: Identification error.

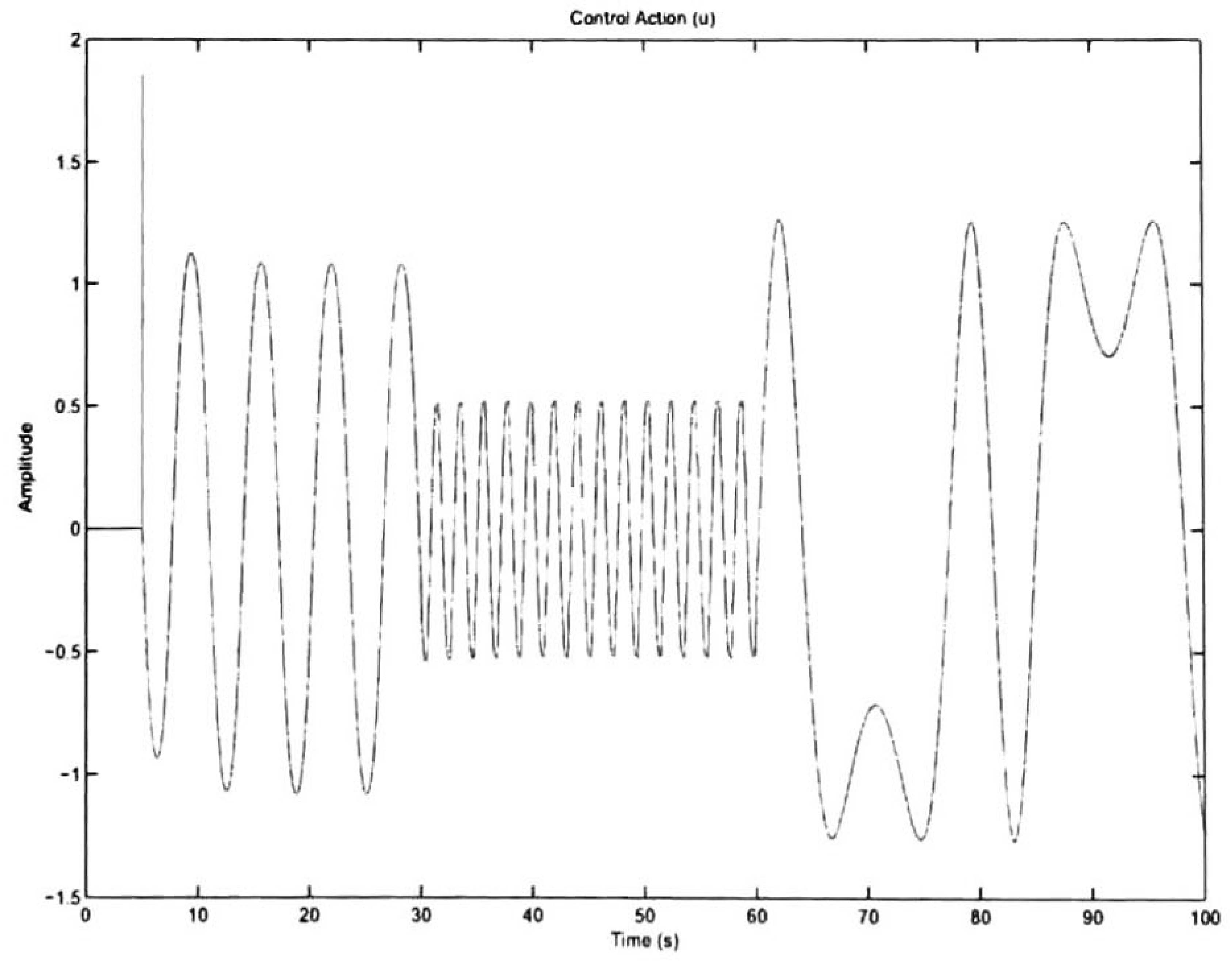


Figure 5.20: Control action $u$.

# Chapter 6

# Conclusions and future work

A lot of research has been done related to the output regulation problem; it is known that the solution of the regulator equations is often impossible to find. There is always something to loose on every method that approximates the solution of these equations.

One of the main disadvantages of the generalized immersion is that the dimension of the controller grows in dimension considerably; using the recently developed adaptive control sheme by Obregon, the controller also grows, but it solves the disadvantage of the generalized immersion in the fact that it does not require to know the states of the exosystem.

In these thesis another method is proposed to approximate the solution of the regulator equations; it has the advantage that the controller does not increase with the complexity of the system, in the case where the neural network is used to identify the plant, as long as the neural network is able to follow the plant the system is able to track the desired trajectory. In the case where the neural network is used to identify the exosystem, as long as the neural network is able to follow the exosystem the system will be able to track the trajectory.

One of the key components in the design of the controller is the adaptation law; it is necessary to determine a continuous adaptation law that does not switch when the error became small, and that also guarantees bounded error and bounded states for the neural network.

Solving the regulator equations where the exosystem is unknown has almost not been studied, there is a lot of open problems in that area. One possible application, which is not studied here, is using this approach to the solution of the output regulation of switched systems; if we use a neural network to identify the plant and the exosystem, then it seems feasible to implement only one controller even if the systems switches.

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## Appendix A

# Foundations

This section deals with mathematical tools that has been used in the development of this work; a general review of non-linear systems is made.

## A.1 Dynamical systems

A nonlinear system can be represented by a set of differential equations of the form

$$\dot{x} = f(x.t.u) \tag{A.1}$$

where $f : D \rightarrow \mathbb{R}^n$ is a locally Lipschitz function. Let $u = g(x(t).t)$, the closed loop system of A.1 can be written as:

$$\dot{x} = f(x.t) \tag{A.2}$$

in the special case where the function $f$ does not depend explicitly on time, the system (A.3) is said to be autonomous.

$$\dot{x} = f(x) \tag{A.3}$$

An important concept when dealing with the state equation is the concept of equilibrium points. For the system (A.3), the equilibrium points are the real roots of the equation

$$f(\bar{x}) = 0 \tag{A.4}$$

Such points can be stable, unstable, or asymptotically stable.

## A.2 Stability of dynamical systems

This concept of stability is very important in the control theory; it is very important that a controller guarantees the stability of the closed loop system, that is, that the output reaches the desired value without drifting to infinify. The following definitions describes the classes into which stability can be classified.

**Definition A.2.1.** *[21] The equilibrium point $x = 0$ of A.3 is*

- *Stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon)$ such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0 \tag{A.5}$$

- *Unstable if not stable*

- *Asymptotically stable if it is stable and $\delta$ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0 \tag{A.6}$$

This concept of stability is usually characterized in the sense of Lyapunov, where an equilibrium point is stable if all solutions starting at nearby point stay nearby; otherwise, it is unstable.

## A.3 Lyapunov stability

The main aspects of the stability for nonlinear systems via the Lyapunov methods will be reviewed here. Extensions for this methods are available in the literature [21]. Lyapunov stability theorems give sufficient conditions for stability, asymptotic stability, and so on, however, they fail to say whether the given conditions are also necessary. Lyapunov stability analysis can be used to establish the boundedness of the solution, even when the system has no equilibrium points. This will be important because the boundedness of the weights of the neural network must be determined.

The next theorem gives the sufficient conditions for a system to be stable.

**Theorem A.3.1.** *Let $x = 0$ be an equilibrium point for (A.3) and $D \subset \mathbb{R}_n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0, V(x) > 0 \quad \forall D - \{0\} \tag{A.7}$$
$$\dot{V}(x) \leq 0 \qquad \forall D$$

*Then $x = 0$ is stable. Moreover, if*

$$\dot{V}(x) < 0 \forall D - \{0\} \tag{A.8}$$

*then $x = 0$ is asymptotically stable.*

*Proof.* See [21] □

## A.4 Steady state

The output regulation problem is used to force a dynamical system to follow a desired trajectory; in order to do so, the controller is able to force the output of the system to converge asymptotically to a desired steady state response.

In order to characterize formally the concept of steady state, consider the following system

$$\dot{x} = f(x, u) \tag{A.9}$$

where $x \in \mathbb{R}^n$ in a neighborhood $U$ close to the origin, and the input $u \in \mathbb{R}^m$ The first assumption is that $f(0,0) = 0$. Let $x(t, x_0)$ be the value of the state $x$ reached at time $t = 0$. Let $u_{ss}$ a specific input and suppose that exists an initial state $x_0^*$ with the property that

$$\lim_{t \to \infty} \|x(t, x_0, u^*(\cdot)) - x(t, x^*, u^*)\| \tag{A.10}$$

for each $x_0$ belonging to a neighborhood $U^*$ of $x_0^*$. If that is the case, then the states

$$x_{ss}(t) = x(t, x_0^*, u_{ss}) \tag{A.11}$$

is called steady state response from (A.9) for a specific input $u_{ss}$.

This definition will be used in the solution of the regulator equations; the controller will consist of two parts, one of the will be a persistent input. Generally, this type of inputs are generated by external systems modelled by differential equations of the form

$$\dot{z} = z(\omega)$$
$$u_{ss} = c(\omega)$$

where the state $\omega \in \mathbb{R}^p$ is defined on a neighborhood $V$ of the origin, and on which $z(0) = 0$,$c(0) = 0$. In order to achieve the bounded input property, it is enough to have that the equilibrium point $\omega = 0$ from $s(\omega)$ be stable in the sense of Lyapunov and to choose the initial condition in $t = 0$ on an appropriate neighborhood $V_0 \subset V$ close to the origin. In order to achieve that the inputs are persistent in time, it is necessary that the equilibrium point $\omega = 0$ be neutrally stable, that is, that the following matrix

$$S = \frac{\partial s}{\partial \omega}|_{\omega=0} \tag{A.12}$$

which characterizes the linear approximation of $s(\omega)$ in $\omega = 0$, has all its eigenvalues on the imaginary axis. [17]

**Proposition A.4.1.** *Consider that A.12 is neutrally stable and that the equilibrium point $x = 0$ of $\dot{x} = f(x, 0)$ is asymptotically stable on the first approximation. Then, a mapping $x = \pi(\omega)$ defined in a neighborhood $V_0 \subset V$ from the origin, with $\pi(0) = 0$, which satisfies*

$$\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega), c(\omega)) \tag{A.13}$$

*for each $v \subset V_0$. In fact, the input $u_{ss}(\omega)$ produce a well defined steady state response*

$$x_{ss}(t) = x(t, \pi(\omega), u_{ss}(\omega)) \tag{A.14}$$

*Proof.* See [17] □

## A.5 Center manifold

Consider the following system

$$\dot{x} = f(x) \tag{A.15}$$

where $f : D \to \mathbb{R}^n$ is continuously differentiable and $D \subset \mathbb{R}^n$ contains the origin $x = 0$. Next, suppose that the origin is an equilibrium point of (A.15). From Khalil [21], it is known that if the linearization of $f$ at the origin, that is, the matrix

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0} \tag{A.16}$$

has all eigenvalues with negative real parts, then the origin is asymptotically stable; if it has some eigenvalues with positive real parts, then the origin is unstable. If A has eigenvalues with zero real parts with the rest of the eigenvalues having negative real parts, then the linearization fails to determine the stability properties of the origin. That is the main reason to study the center manifold theory. A k-dimensional manifold can be seen as the solution of the equation

$$\eta(x) = 0 \tag{A.17}$$

where $\eta : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is sufficiently smooth. For example, the unit circle

$$\{x \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\} \tag{A.18}$$

is a one-dimensional manifold in $\mathbb{R}^2$

The system A.15 can be represented as

$$\dot{x} = Ax + \varphi(x) \tag{A.19}$$

Consider only the systems for which $Re(\lambda_i) \leq 0$ from which

- $m_c$ eigenvalues have zero real parts.
- $m_s$ eigenvalues have negative real parts.

A transformation $T$ always exists such that

$$\begin{bmatrix} y \\ z \end{bmatrix} = Tx, y \in \mathbb{R}^{m_c}, z \in \mathbb{R}^{m_s} \tag{A.20}$$

The system in the new coordinates has the following structure

$$\dot{y} = A_1 y + g_1(y, z) \tag{A.21}$$

$$\dot{z} = A_2 z + g_2(y, z) \tag{A.22}$$

where

- $A_1$ is $m_c \times m_c$ with $Re(eig(A_1)) = 0$
- $A_2$ is $m_s \times m_s$ with $Re(eig(A_2)) < 0$
- the $g_i$ functions are twice continuously differentiable and satisfy

$$g_i(0,0) = 0 \tag{A.23}$$
$$\frac{\partial g_i}{\partial y}(0,0) = 0 \tag{A.24}$$
$$\frac{\partial g_i}{\partial z}(0,0) = 0 \tag{A.25}$$
$$\tag{A.26}$$

**Definition A.5.1.** *A manifold is said to be a local invariant for (A.15) if*

$$\eta(x(0)) = 0 \Rightarrow \eta(x(t)) \equiv 0, \forall t \in [0, t_1) \subset \mathbb{R} \tag{A.27}$$

*Let $t_1 = \infty$ then it is called an invariant manifold.*

**Definition A.5.2.** *If a manifold $z = \pi(y)$ is an invariant manifold for the system and $\pi(\cdot)$ is smooth, then it is called a center manifold if*

$$\pi(0) = 0 \quad , \quad \frac{\partial \pi}{\partial y}(0) = 0 \tag{A.28}$$

With this definitions the following result is given.

**Theorem A.5.3.** *There exist a $\delta > 0$ and a continuously differentiable function $\pi(y)$ defined for all $\|y\| \leq \delta$, such that $z = \pi(y)$ is a center manifold for the system (A.21).*

The motion of the system on the center manifold is determined by the reduced system

$$\dot{y} = A_1 y + g_1(y, z) \tag{A.29}$$

**Lemma A.5.4.** *Suppose $z = \pi(y)$ is a center manifold for the system (A.21) at $(0,0)$. Then, there exist a neighborhood $U^0$ of $(0,0)$ and real numbers $M > 0$ and $K > 0$ such that, if $(y(0), z(0)) \in U^0$, then*

$$\|y(t) - \pi(z(t))\| \leq Me^{-Kt}\|y(0) - \pi(z(0))\| \tag{A.30}$$

That is, the center manifold is locally exponentially attractive. With all the information above, the theorem that is relevant to our work can be stated

**Theorem A.5.5.** *Suppose that the origin $y = 0$ of the reduced system in (A.29) is asymtpotically stable (respectively, stable, unstable). Then, the origin of the full system in (A.21) is asymptotically stable (respectively, stable, unstable).*

## A.6 Neural networks

Artificial Neural Networks (ANNs) are simplified models of biological neural networks. The main purpose of an ANN is to imitate the behaviour of a biological neural network. These neural networks are capable of process information in a parallelized form, making them ideal for real-time applications; however, special hardware must be used in order to take the advantages of a neural network. Field Programmable Gate Arrays look promising to fully implement a neural network that performs parallel computation.

The ANNs does not have the complexity as the human brain, however there are similarities between biological neural networks and artifitial ones: first of all, the construction blocks of both of them are very simple computational elements highly interconnected, and second, the connections between the neurons determine the function of the neural network.

The mathematical model of a neural network will have to include three basic elements.

- A group of synapsis, each one of them characterized by a weight or a synaptic gain. In special, a signal $x_j$ to the input of the synapsis $j$ connected to a neuron $k$ is multiplied by a synaptic weight $\omega_{kj}$. It is important to notice that the first subindex belongs to the neuron, while the second subindex belongs to the input.

- An adder in order to sum the input signals weigthened by its own synaptic weight; the operations described here are linear combinations.

- An activation function in order to limit the amplitude of the neuron output. The activation function is normally normalized between a closed interval $[0, 1]$ or $[-1, 1]$.

Using the above conditions, the neural networks that is used in this work have the following structure

$$\dot{\hat{x}}_t = A\hat{x}_t + bu_t + W_1\sigma(\hat{x}_t) \tag{A.31}$$

where $x \in \mathbb{R}^n$ is the number of neurons that the network will have. and $W_1 \in \mathbb{R}^{n \times n}$ describe the relationship between the hidden layer and the output layer. $\sigma(\cdot)$ belongs to a class of function called sigmoidal functions, that have some properies:

- It is a real-valued differentiable function.
- Its first derivative is bell shaped.
- It has a pair of horizontal asymptotes as $t \to \pm\infty$.

This kind of function is ideal for neural networks because it can implement the activation function, and they make neural networks universal approximators of functions.

### A.6.1 Universal approximation

A neural network can be used as a universal approximator of functions, that is, perform a nonlinear input-output mapping from $\mathbb{R}^n$ (the dimension of the input space) to $\mathbb{R}^l$ (the dimension of the output space). This kind of operator has been studied in ([11]). The first one to demonstrate that a single hidden layer is sufficient to uniformly approximate any continuous function with support in a unit hypercube was Cybenko. It is resumed in the following theorem

**Theorem A.6.1.** *([9])Let $\sigma(\cdot)$ be a stationary, bounded, and monotone increasing function. Let $I_n$ denote the n-dimensional unit hypercube. Let $C(I_n)$ the space of continuous functions on $I_n$. Then for any $f \in C(I_n)$ and $\varepsilon > 0$, there exist an integer $m$ and real constants $\alpha_i, \rho_i$ and $w_{ij}$, with $i = 1, ..., m$ and $j = 1, ..., n$, such that defining $F(u_1, u_2, ..., u_n)$ as*

$$F(u_1, u_2, ..., u_n) = \sum_{i=1}^{m} \alpha_i \sigma(\sum_{j=1}^{n} w_{ij} u_j - \rho_i) \tag{A.32}$$

*it is an approximate realization of $f(\cdot)$, that is,*

$$|F(u_1, u_2, ..., u_n) - f(u_1, u_2, ..., u_n)| < \epsilon, \forall (u_1, u_2, ..., u_n) \in I_n \tag{A.33}$$

# Appendix B

# Neural network adaptation law

Let consider the error dynamics, that is, the derivative of $\Delta_t$

$$\dot{\Delta}_t = \dot{\hat{x}}_t - \dot{x} \tag{B.1}$$

substituting the values

$$\dot{\Delta}_t = A\Delta_t + W_{1,t}\sigma(\hat{x}_t) - W_1^*\sigma(\hat{x}_t) + W_{2,t}\phi(\hat{x}_t)u_t - W_2^*\phi(x_t)u_t - \Delta f \tag{B.2}$$

adding and subtracting the terms $W_1^*\sigma(\hat{x}_t)$ and $W_2^*\phi(\hat{x}_t)u_t$ and taking into account equations B.2 and 2.1, (B.2) can be expressed as:

$$\dot{\Delta}_t = A\Delta_t + \widetilde{W}_{1,t}\sigma(\hat{x}_t) - W_1^*\sigma(\hat{x}_t) + \widetilde{W}_{2,t}\phi(\hat{x}_t)u_t - W_2^*\phi(x_t)u_t - \Delta f \tag{B.3}$$

The following Lyapunov candidate function is used

$$V_t = \Delta_t^T P\Delta_t + tr[\widetilde{W}_{1,t}^T K_1^{-1}\widetilde{W}_{1,t}] + tr[\widetilde{W}_{2,t}^T K_2^{-1}\widetilde{W}_{2,t}] \tag{B.4}$$

where $P$ is the positive solution for the matrix Riccati equation given by (4.24). The first derivative of $V_t$ is

$$\dot{V}_t = \frac{d}{dt}(\Delta_t^T P\Delta_t) + \frac{d}{dt}tr[\widetilde{W}_{1,t}^T K_1^{-1}\widetilde{W}_{1,t}] + \frac{d}{dt}tr[\widetilde{W}_{2,t}^T K_2^{-1}\widetilde{W}_{2,t}] \tag{B.5}$$

so, expanding the equation $\frac{d}{dt}(\Delta_t^T P\Delta_t)$

$$\frac{d}{dt}(\Delta_t^T P\Delta_t) = 2\Delta_t^T P\dot{\Delta}_t \tag{B.6}$$

and substituting (4.23) and (4.22) into (B.6)

$$\begin{aligned}\frac{d}{dt}(\Delta_t^T P \Delta_t) &= 2\Delta_t^T P A \Delta_t \\ &+ 2\Delta_t^T P \widetilde{W}_{1,t}\sigma(\hat{x}_t) \\ &- 2\Delta_t^T P W_1^* \sigma(\hat{x}_t) \\ &+ 2\Delta_t^T P \widetilde{W}_{2,t}\phi(\hat{x}_t)u_t \\ &- 2\Delta_t^T P W_2^* \phi(x_t)u_t \\ &- 2\Delta_t^T P \Delta f\end{aligned}$$

For $\frac{d}{dt}tr[\widetilde{W}_{1,t}^T K_1^{-1}\widetilde{W}_{1,t}]$ we can use some properties of the trace of a matrix to obtain

$$\frac{d}{dt}[tr\widetilde{W}_{1,t}^T K_1^{-1}\widetilde{W}_{1,t}] = tr[\frac{d}{dt}(\widetilde{W}_{1,t}^T K_1^{-1}\widetilde{W}_{1,t})] \quad \text{(B.7)}$$

$$= tr[\dot{\widetilde{W}}_{1,t}^T K_1^{-1}\widetilde{W}_{1,t} + \widetilde{W}_{1,t}^T K_1^{-1}\dot{\widetilde{W}}_{1,t}] \quad \text{(B.8)}$$

$$= tr[\dot{\widetilde{W}}_{1,t}^T K_1^{-1}\widetilde{W}_{1,t}] + tr[\widetilde{W}_{1,t}^T K_1^{-1}\dot{\widetilde{W}}_{1,t}] \quad \text{(B.9)}$$

$$= 2tr[\dot{\widetilde{W}}_{1,t}^T K_1^{-1}\widetilde{W}_{1,t}] \quad \text{(B.10)}$$

Taking the derivative of $\widetilde{W}_{1,t}$

$$\dot{\widetilde{W}}_{1,t} = \dot{W}_{1,t} \quad \text{(B.11)}$$

the term $\dot{W}_{1,t}$ is given by the differential learning law. Thus, substituting (4.28) into the last term of (B.7), $\frac{d}{dt}[tr\widetilde{W}_{1,t}^T K_1^{-1}\widetilde{W}_{1,t}]$ can be computed as

$$\begin{aligned}\frac{d}{dt}[tr\widetilde{W}_{1,t}^T K_1^{-1}\widetilde{W}_{1,t}] &= -2tr[\sigma(\hat{x}_t)\Delta_t^T P K_1 K_1^{-1}\tilde{W}_{1,t}] - \lambda_{\min}(P^{-\frac{1}{2}}Q_0P^{-\frac{1}{2}})tr[\tilde{W}_{1,t}K_1^{-1}\tilde{W}_{1,t}] \\ &= -2\Delta_t^T P\tilde{W}_{1,t}\sigma(\hat{x}_t) - \lambda_{\min}(P^{-\frac{1}{2}}Q_0P^{-\frac{1}{2}})tr[\tilde{W}_{1,t}K_1^{-1}\tilde{W}_{1,t}]\end{aligned}$$

proceeding in a similar way for $tr[\tilde{W}_{2,t}K_2^{-1}\tilde{W}_{2,t}]$

$$\frac{d}{dt}[tr\widetilde{W}_{2,t}^T K_2^{-1}\widetilde{W}_{2,t}] = -2\Delta_t^T P\tilde{W}_{2,t}\phi(\hat{x}_t)u_t - \lambda_{\min}(P^{-\frac{1}{2}}Q_0P^{-\frac{1}{2}})tr[\tilde{W}_{2,t}K_2^{-1}\tilde{W}_{2,t}]$$

finally, substituting into (B.5) $\dot{V}_t$ can be expressed as

$$\begin{aligned} \dot{V}_t &= 2\Delta_t^T PA\Delta_t + 2\Delta_t^T PW_1^* \sigma(\hat{x}_t) \\ &+ 2\Delta_t^T PW_2^* \phi(x_t)u_t \\ &- 2\Delta_t^T P\Delta f \\ &- \lambda_{\min}(P^{-\frac{1}{2}}Q_0P^{-\frac{1}{2}})tr[\tilde{W_{1,t}}K_1^{-1}\tilde{W_{1,t}}] \\ &- \lambda_{\min}(P^{-\frac{1}{2}}Q_0P^{-\frac{1}{2}})tr[\tilde{W_{2,t}}K_2^{-1}\tilde{W_{2,t}}] \end{aligned}$$

Now, an upper bound for $\dot{V}_t$ is determined. To accomplish this task, first the term $2\Delta_t^T PA\Delta_t + 2\Delta_t^T PW_1^* \sigma(\hat{x}_t)$ is considered. Since this term is an scalar, it is possible to express it alternatively as

$$2\Delta_t^T PA\Delta_t + 2\Delta_t^T PW_1^* \sigma(\hat{x}_t) = \Delta_t^T PA\Delta_t + 2\Delta_t^T PW_1^* \sigma(\hat{x}_t) + \sigma(\hat{x}_t)\Delta_t^T PA\Delta_t + 2\Delta_t^T PW_1^* \tag{B.12}$$

using the matrix inequality proved in [29]

$$X^TY + Y^TX \leq X^T\Lambda^{-1}X + Y^T\Lambda Y \tag{B.13}$$

which is valid for any $X, Y \in \mathbb{R}^{n\times k}$ and for any positive definite matrix $0 < \Lambda = \Lambda^T \in \mathbb{R}^{n\times n}$,$2\Delta_t^T PA\Delta_t + 2\Delta_t^T PW_1^* \sigma(\hat{x}_t)$ is bounded by

$$2\Delta_t^T PA\Delta_t + 2\Delta_t^T PW_1^* \sigma(\tilde{x}_t) \leq \Delta_t^T PW_1^* \Lambda_\sigma^{-1} W_1^{*T} P\Delta_t + \tilde{\sigma}_t^T \Lambda_\sigma \tilde{\sigma}_t \tag{B.14}$$

but, from the assumptions A.2 and A.5, we can conclude

$$2\Delta_t^T PA\Delta_t + 2\Delta_t^T PW_1^* \sigma(\tilde{x}_t) \leq \Delta_t^T P\bar{W}_1^* \Delta_t + \Delta_t^T \Lambda_\sigma \Delta_t \tag{B.15}$$

Likewise, using the inequality B.13 in $2\Delta_t^T PW_2^* \tilde{\phi}_t u_t$,

$$\begin{aligned} 2\Delta_t^T PW_2^* \tilde{\phi}_t u_t &= \Delta_t^T PW_2^* \tilde{\phi}_t u_t + u_t^T \tilde{\phi}_t^T W_2^* P\Delta_t \\ &\leq \Delta_t^T PW_2^* \Lambda_\phi^{-1} W_2^{*T} P\Delta_t + u_t^T \tilde{phi}_t^T \Lambda_\phi \tilde{phi}_t u_t \\ &\leq \Delta_t^T P\bar{W}_2 P\Delta_t + \bar{u}_t \Delta_t^T D_\phi \Delta_t \end{aligned}$$

This last inequality is ensured by the assumptions A.2, A.3, and A.5. On the other hand, the following inequality is a corollary from (B.13):

$$-Z^TY - Y^TZ \leq Z^T\Lambda^{-1}Z + Y^T\Lambda Y \tag{B.16}$$

which is valid for any $Z, Y \in \mathbb{R}^{n\times k}$ and for any positive definite matrix $0 < \Lambda = \Lambda^T \in \mathbb{R}^{n\times n}$ Using this result, a bound for $-2\Delta_t^T P\Delta f$ is obtained.

$$-2\Delta_t^T P\Delta f \leq \Delta_t^T P\Lambda_f^{-1} P\Delta + \Delta_f^T \Lambda_f \Delta_f \tag{B.17}$$

In accordance with the assumption A.4

$$-2\Delta_t^T P\Delta f \leq \Delta_t^T P\Lambda_f^{-1} P\Delta + \bar{\eta} \tag{B.18}$$

substituting B.15, B.16, and B.18 into B.12, the following bound for $\dot{V}_t$ can be determined

$$\begin{aligned}
\dot{V}_t \;\leq\; & 2\Delta_t^T PA\Delta_t + \Delta_t^T P\bar{W}_1 P\Delta_t \\
+ \; & \Delta_t^T D_\sigma \Delta_t + \Delta_t^T P\bar{W}_2 P\Delta_t \\
+ \; & \bar{u}\Delta_t^T D_\phi \Delta_t + \Delta_t^T P\Lambda_f^{-1} P\Delta_t + \bar{\eta} \\
- \; & \lambda_{\min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}}) tr[\tilde{W}_{1,t}{}^T K_1^{-1} \tilde{W}_{1,t}] \\
- \; & \lambda_{\min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}}) tr[\tilde{W}_{2,t}{}^T K_2^{-1} \tilde{W}_{2,t}]
\end{aligned}$$

Adding and substracting $\Delta_t Q_0 \Delta t$ into the right-hand side of the last inequality, the expression $A^T P + PA + P(\bar{W}_1 + \bar{W}_w + \Lambda_f^{-1})P + D_\sigma + D_\phi \bar{u} + Q_0$ is formed. However, this expression in accordance with the assumption A.6 is equal to zero. Then

$$\dot{V}_t \leq -\Delta_t^T Q_0 \Delta_t + \bar{\eta} - \lambda_{\min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}}) tr[\tilde{W}_{1,t}{}^T K_1^{-1} \tilde{W}_{1,t}] - \lambda_{\min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}}) tr[\tilde{W}_{2,t}{}^T K_2^{-1} \tilde{W}_{2,t}]$$

Now, consider that

$$\Delta_t^T Q_0 \Delta_t = \Delta_t^T P^{\frac{1}{2}} (P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}}) P^{\frac{1}{2}} \Delta_t \tag{B.19}$$

and using Rayleigh inequality,

$$\lambda_{\min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}}) \Delta_t^T P\Delta_t \leq \Delta_t^T Q_0 \Delta_t \tag{B.20}$$

consequently,

$$\begin{aligned}
\dot{V}_t \;\leq\; & -\lambda_{\min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}}) \Delta_t^T P\Delta_t \\
- \; & \lambda_{\min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}}) tr[\tilde{W}_{1,t}{}^T K_1^{-1} \tilde{W}_{1,t}] \\
- \; & \lambda_{\min}(P^{-\frac{1}{2}} Q_0 P^{-\frac{1}{2}}) tr[\tilde{W}_{2,t}{}^T K_2^{-1} \tilde{W}_{2,t}] + \bar{\eta}
\end{aligned}$$

from the definition of our Lyapunov function candidate, finally $\dot{V}_t$ can be bounded as

$$\dot{V}_t \leq \lambda_{\min}(P^{-\frac{1}{2}}Q_0P^{-\frac{1}{2}})V_t + \bar{\eta} \tag{B.21}$$

which implies that

$$V_t \leq V_0 e^{(-\xi t)} + \frac{\bar{\eta}}{\xi}(1 - e^{(-\xi t)}) \tag{B.22}$$

where $\xi = \lambda_{\min}(P^{-\frac{1}{2}}Q_0P^{-\frac{1}{2}})$. Since $V_t$ is an upperly bounded non-negative function then $\Delta_t, W_{1,t}, W_{w,t} \in L_\infty$ and the first part of the theorem has been proved. On the other hand, from definiton of $V_t$ is evident that

$$\Delta_t^T P \Delta_t \leq \|_t \tag{B.23}$$

but from B.22

$$\Delta_t^T P \Delta_t \leq V_0 e^{(-\xi t)} + \frac{\bar{\eta}}{\xi}(1 - e^{(-\xi t)}) \tag{B.24}$$

finally, taking $\limsup_{t\to\infty}$ for both sides of the last inequality

$$\limsup_{t\to\infty} \Delta_t^T P \Delta_t \leq \frac{\bar{\eta}}{\lambda_{\min}(P^{-\frac{1}{2}}Q_0P^{-\frac{1}{2}})} \tag{B.25}$$

# CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL I.P.N.
## UNIDAD GUADALAJARA

El Jurado designado por la Unidad Guadalajara del Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional aprobó la tesis

On the problem of nonlinear output regulation using neural networks

del (la) C.

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el día 03 de Febrero de 2012.

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