Centro de Investigación y de Estudios Avanzados
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Observabilidad en Sistemas Lineales Conmutados

Tesis que presenta:
Guillermo Ramírez Prado

para obtener el grado de:
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en la especialidad de:
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Directores de Tesis:
Dr. Antonio Ramírez Treviño
Dr. José Javier Ruiz León

CINVESTAV del IPN Unidad Guadalajara, Jalisco, Diciembre 2010

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Obserbavility in Switched Linear Systems

A thesis presented by:
Guillermo Ramírez Prado

to obtain the degree of:
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Thesis Advisors:
Dr. Antonio Ramírez Treviño
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Por:

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Maestro en Ciencias

CINVESTAV Unidad Guadalajara 2001

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Contents

Resumen

En este trabajo se aborda el problema de la observabilidad de Sistemas Híbridos (HS), en particular los Sistemas Lineales Conmutados (SLS), donde la dinámica continua está representada por una familia de sistemas lineales y la discreta por una red de Petri. El problema de la observabilidad consiste en determinar los estados del sistema, tanto continuos como discretos, y el uso de sistemas lineales y redes de Petri facilita el análisis de esta propiedad en los sistemas híbridos. La observabilidad es una propiedad fundamental de los sistemas dinámicos ya que permite diseñar controladores de retroalimentación de estados y diagnosticadores de fallas.

En particular, este trabajo muestra cómo la información obtenida de la familia de sistemas lineales puede ser usada en el análisis de observabilidad de la red de Petri y vice versa, para así obtener una caracterización de la observabilidad en HS. Además se presenta el diseño de observadores asintóticos híbridos. En este trabajo, el estado discreto no se conoce. Observadores de Luenberger junto con identificación de sistemas se utilizan para distinguir entre sistemas lineales. Cada sistema lineal puede ser no observable.

También se busca determinar condiciones necesarias y suficientes para la controlabilidad de SLS. Se presenta al Control Predictivo (MPC) como una técnica de control adecuada para sistemas lineales y para SLS, así como alternativas para el diseño de leyes de control para MPC.

Abstract

This work deals with Observability in finite time of Hybrid Systems (HS), in particular Switched Linear Systems (SLS). It deals with the possibility of recovering the discrete as well as the continuous states of the hybrid system. Observability is a fundamental property of dynamic systems since it allows to implement state feedback controllers or fault diagnosers. Through this work, the continuous part of the systems is represented by a non autonomous Linear System family, and the discrete part is modeled by an interpreted Petri net (IPN).

Based on this model, a novel characterization of the observability in linear hybrid systems, that exploits the information of the input and the output of the IPN as well as the structure of the linear systems, is presented. It is shown that the information of the continuous systems can be used to achieve observability of the discrete system and vice versa. In the present work, the discrete state is unknown, Luenberger observers together with identification are used to distinguish between LS and each LS may not be observable.

Search for necessary and sufficient conditions for controllability of SLS is included on this research. Model Predictive Control (MPC) is presented as an adequate control technique for linear systems and SLS. Different approaches are used for the design of the model predictive control law.

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Chapter 1

Introduction

Hybrid systems (HS) are dynamic systems in which there exists an interaction between a continuous system and a Discrete Event System (DES). This means that in the hybrid system there are continuous and discrete states, in such a way that the behaviour of the continuous part affects the behaviour of the discrete part and viceversa. HS are present in control systems where systems of different structure commute, like power systems, flexible manufacturing systems, air traffic management systems, robots, computer hard drives, etc.

Due to the wide application range and the importance of this kind of systems, HS control is an essential topic in control systems theory. It is important to characterize basic structural properties of HS that will lead to the development and implementation of efficient control squemes.

Among the most important structural properties of any dynamic system there are the observability and controllability notions. These concepts are fundamental to determine system properties and to establish an adecuate control scheme.

Main objectives in this research are the study and characterization of observability properties for HS, particulary for switched linear systems (SLS), where interpreted

Petri nets (IPN) [1] are used to model the discrete part, and a family of linear systems $\mathcal{F}$ to represent the continuous part of the SLS. It is assumed that neither the continuous part nor the discrete part are necessarily autonomous. Objectives also include the study of controllability properties for HS and control.

There exist several works addressing problems in HS, for instance [2] and [3] deal with controllability in HS; [4] and [5] deal with stability. In the later it is proved that the problem of stability in hybrid systems is NP-hard. However, if it is possible to compute a Lyapunov function for all linear systems $LS_i \in \mathcal{F}$ then stability is guarantied. It has also been proved that the controllability in some HS subclasses is an NP-hard problem.

The study of observability [6] in SLS is primordial, since in the SLS exhibiting this property several control architectures can be used, such as feedback pole placement, regulation control, fault diagnosers, etc. When a SLS exhibits the observability property, it is possible to compute the whole system state, i.e. the continuous and discrete states. Unfortunately, the current results characterizing this property for this type of systems are restricted to autonomous systems, leading to computational complex algorithms.

Observability in SLS has been widely studied since it allows to design state feedback controllers, to detect which LS is evolving, to detect and isolate faults, etc. In [7] the observability and controllability of SLS is addressed under the assumption that the discrete events sequence is known. In [2], the study of Piecewise-Affine Hybrid Systems ($PAHS$) is considered. That paper introduces the concept of incremental observability, which is centered on the possibility of reconstructing the hybrid state as the system evolves. The characterization of incremental observability requires the solution of a mixed-integer linear problem, leading to inefficient algorithms. In [8], the observability of Jump Linear Systems (JLS) was characterized. The main results of that paper are the necessary and sufficient conditions for recovering the continuous and discrete states in infinitesimal time, as well as for detecting the commutation times. In [9] the results of [8] were extended to *PAHS*, which is a larger class than *JLS*. That paper provides sufficient conditions for observability, observability after the execution of a single discrete event, observability in infinitesimal time and detection of the commutation times. Although the conditions of [8] and [9] can be analyzed efficiently, those works are constrained to autonomous systems (i.e. systems without inputs).

In [10] the observability of non autonomous *SLS* is addressed. However it restricts its study to the computation of a control law $u(\cdot)$ in such a way that the hybrid state evolution can be recovered from the observed output. Thus, it does not guarantee to compute the state evolution if a closed loop controller scheme is used. In [11], characterizations of observability, reachability and controllability were studied for a class of hybrid systems where the discrete evolution is governed by controllable discrete transitions. The authors derived the conditions to reconstruct the initial continuous state measuring the continuous outputs and inputs for a fixed discrete sequence. Unfortunately, real discrete transitions could be uncontrollable, limiting these results to a small class of hybrid systems.

In [12] and [13] an observability characterization for non autonomous discrete-time *SLS* is presented. The authors presented an observability characterization under arbitrary discrete state sequences. The authors defined the concept of discernability of discrete state sequences to present mode observability, which is similar to the state distinguishability notion presented in [8] and [9]. In [14], the authors extended [12] and [13] to the continuous-time case. In that paper, the characterization of control-discernability (discernability on non autonomous systems) is focused on the existence of a continuous input allowing to distinguish between discrete states by their continuous outputs.

The *HS* observer design problem has been also addressed in several works. In

[15], for instance, it was presented the design of a family of Luenberger observers for continuous systems under the assumption that the discrete state is known. In [16] the design of observers for the discrete part is combined with the design of Luenberger observers. In [17] it was presented the design of a Luenberger-like observer for a class of *SLS*, called detectable switching systems, under the assumption that the discrete state of the switching system is known.

1.1 Objectives and goals

General objectives and goals of the research on observability and controllability of SLS:

Objectives:

1. To characterize observability in SLS.
2. To develop an hybrid observer for SLS.
3. To characterize controllability in SLS.
4. To develop an hybrid controller for SLS.

Goals:

1. The use of geometrical tools to analyze SLS will lead to structural characterization of observability in SLS.
2. Information from the discrete part of the SLS will be used to gain observability in the continuous part of the system, and viceversa.
3. To design observers for a class of observable SLS.
4. To characterize controllability conditions in SLS.

5. To design a controller for a class of SLS.

1.2 Outcomes

Journal publications, book chapters and conference exposure were possible from achieved objectives. Most relevant references are listed next.

1. Rubio-Gómez, L.; Gomez-Gutierrez, D.; Ramirez-Trevio, A.; Ruiz-Leon, J.; Ramirez-Prado, G.; *Diagnosability in Switched Linear Systems*; Book chapter in: "Petri nets Applications"; In-Teh; 2010.

2. Gomez-Gutierrez, D.; Ramirez-Prado, G.; Ramirez-Trevio, A.; Ruiz-Leon, J.; *Observability of Switched Linear Systems*; IEEE Transactions on Industrial Informatics; 6:2, 127-135; 2009.

3. Gomez-Gutierrez, D.; Ramirez-Prado, G.; Ramirez-Trevino, A. & Ruiz-Leon, J.; *Joint State-Mode Observer Design for Switched Linear Systems*; IEEE International Conference on Emerging Technologies and Factory Automation; Hamburg, Germany; 2009.

4. Ramirez-Prado, G.; Gomez-Gutierrez, D.; Ramirez-Trevino, A. & Ruiz-Leon, J.; *Observability of Switched Linear Systems*; Book chapter in: "Recent advances in Control Systems, Robotics and Automation"; Third edition; 2008.

5. Ramirez-Prado, G.; Gomez-Gutierrez, D.; Ramirez-Trevino, A. & Ruiz-Leon, J.; *Observability of Switched Linear Systems*; International Journal of Factory Automation, Robotics and Soft Computing; 2008.

6. Ramírez, Guillermo; Ramirez Treviño, Antonio; Ruiz León, Javier; *Observability of a class of Switched Linear Systems*; IFAC Workshop on Dependable Control of Discrete Systems; Paris, France; 2007.

1.3 Outline

This work is organized as follows.

Chapter 2 presents basic concepts related with linear systems, Petri nets and the observability in these types of systems.

Observability of DES is presented in Chapter 3. It is shown how the continuous information of a SLS can be used, for a DES that is not observable, to be observable. The concept of distinguishability in linear systems is presented and how using this concept, the Interpreted Petri net could become event detectable.

Chapter 4 presents how the discrete system information can be used in the observability of the continuous system. It is shown that when the discrete marking is known, then the structure of the current evolving linear system is known, and from there, the discrete marking sequence and the continuous state can be computed, even in the case when the linear systems are not observable. From these results, joint observability is stated. The main idea is to use information from both the discrete and the continuous part to achieve observability, even when the discrete and the countinuous part of the SLS are unobservable. The design of SLS observers is presented in Chapter 5.

Chapter 6 presents Controllability of SLS. Model Predictive Control (MPC) is presented for periodic discrete-time time-varying linear systems, that may be extended to SLS.

Finally in Chapter 7 conclusions and future work are presented.

Chapter 2

Preliminaries

This chapter presents the definitions of linear systems [6], [18], [19] and Interpreted Petri nets [20], [21], [22] since they conform the structure of hybrid systems [23] [24] studied in this work. The chapter also presents the main results in observability for these types of systems.

2.1 Linear Systems

A linear time-invariant dynamic system (LS) is represented by the dynamic state equation

$$\Sigma : \begin{cases} \dot{x}(t) = Ax\,(t) + Bu\,(t) \\ y(t) = Cx\,(t) \end{cases} \tag{2.1}$$

where A, B, and C are $n \times n$, $n \times p$ and $q \times n$ constants matrices. The state space generated by all possible solutions to $x(t)$ of the LS (2.1) is $\mathcal{X}$.

Remark 2.1 *Through this work the notation $\Sigma\,(A, B, C)$ will be used to denote a particular LS represented by the state equation (2.1).*

The solution to the linear time invariant dynamic equation given by (2.1) is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{2.2}$$

and:

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau. \tag{2.3}$$

Previous solution can also be expressed in the frequency domain. Using the Laplace transform of (2.2) and (2.3)

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s) \tag{2.4}$$

and:

$$Y(s) = C(sI - A)^{-1}x_0 + C(sI - A)^{-1}BU(s) \tag{2.5}$$

respectively, where $U(s)$ and $Y(s)$ are the Laplace transform of $u(t)$ and $y(t)$ respectively.

The rational-function matrix

$$G(s) = C(sI - A)^{-1}B \tag{2.6}$$

is named the transfer function matrix of the dynamic system $\Sigma(A, B, C)$.

2.1.1 Observability of Linear Dynamic Equations

Definition 2.2 *A LS, $\Sigma(A, B, C)$, is said to be observable at t_0 if there exists a finite $t_1 > t_0$ such that for any state x_0 at time t_0, the knowledge of the input $u[t_0, t_1]$ and the output $y[t_0, t_1]$ over the time interval $[t_0, t_1]$ suffices to determine the state x_0. Otherwise, the dynamic equation $\Sigma(A, B, C)$ is said to be unobservable at t_0.*

Theorem 2.3 *Let $\Sigma(A, B, C)$ be a LS. $\Sigma(A, B, C)$ is observable if and only if any of the following conditions is satisfied:*

1. *All columns of Ce^{At} are linearly independent over the time period $[0, \infty]$ and $\mathbb{C}$, the field of complex numbers.*

1' *All columns of $C(sI - A)^{-1}$ are linearly independent over $\mathbb{C}$.*

2. *The observability grammian*

$$W_{ot} = \int_0^t e^{A^*\tau} C^* C e^{A\tau} d\tau \tag{2.7}$$

is nonsingular for any $t > 0$.

3. *The $nq \times n$ observability matrix*

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{2.8}$$

has rank n.

4. *For every eigenvalue λ of A (and consequently for any λ in $\mathbb{C}$) the $(n+q) \times n$ complex matrix*

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \tag{2.9}$$

has rank n, or equivalently $sI - A$ and C are right coprime.

5. *The unobservable subspace $\mathcal{N}$ of (C, A) is the trivial subspace 0, i.e.*

$$\mathcal{N} = \bigcap_{i=1}^{n} \ker\left(CA^{i-1}\right) = 0 \tag{2.10}$$

or equivalently, there exists a nontrivial subspace of states $\mathcal{V}$, such that $\mathcal{V}$ is $A-$invariant ($A\mathcal{V} \subset \mathcal{V}$) and $\mathcal{V} \subset \ker C$.

Proof. *The proofs of statements* $1-4$ *are presented in [6] and [18], the proof of statement* 5 *is presented in [19].* ■

Remark 2.4 *In this work it is assumed that the linear equation in 2.1 is scalar, that is, it has only one input and one output.*

2.2 Interpreted Petri Nets

Interpreted Petri nets are an extension of the Petri nets. Next definitions present the Petri nets and, afterwards the Interpreted Petri nets are presented.

Definition 2.5 *A Petri Net structure* G *is a bipartite digraph represented by the 4-tuple* $G = (P, T, I, O)$ *where:*

- $P = \{p_1, p_2, \ldots, p_r\}$ *is a finite set of vertices called places,*
- $T = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_m\}$ *is a finite set of vertices called transitions,*
- $I : P \times T \longrightarrow \mathbb{Z}^+$ *is a function representing the weighted arcs going from places to transitions,*
- $O : P \times T \longrightarrow \mathbb{Z}^+$ *is a function representing the weighted arcs going from transitions to places, where* $\mathbb{Z}^+$ *is the set of nonnegative integers.*

The symbol $^\bullet\mathbf{t}_j$ denotes the set of all places p_i such that $I(p_i, \mathbf{t}_j) \neq 0$ and $\mathbf{t}_j{}^\bullet$ the set of all places p_i such that $O(p_i, \mathbf{t}_j) \neq 0$. Analogously, $^\bullet p_i$ denotes the set of all transitions $\mathbf{t}_j$ such that $O(p_i, \mathbf{t}_j) \neq 0$ and $p_i{}^\bullet$ the set of all transitions $\mathbf{t}_j$ such that $I(p_i, \mathbf{t}_j) \neq 0$. Pictorially, places are represented by circles, transitions are represented by rectangles, and arcs are depicted as arrows.

The pre-incidence matrix of G is $C^- = [c_{ij}^-]$, where $c_{ij}^- = I(p_i, \mathbf{t}_j)$; the post-incidence matrix of G is $C^+ = [c_{ij}^+]$, where $c_{ij}^+ = O(p_i, \mathbf{t}_j)$; and the incidence matrix of G is $\mathcal{C} = \mathcal{C}^+ - \mathcal{C}^-$ The marking function $M : P \longrightarrow \mathbb{Z}^+$ is a mapping from each place to the nonnegative integers representing the number of tokens (depicted as dots) residing inside each place. The marking of a PN is usually expressed as an $n-$entry vector.

Definition 2.6 *A Petri Net system or Petri Net (PN) is the pair* $N = (G, M_0)$, *where* G *is a* PN *structure and* M_0 *is an initial token distribution.*

In a PN system, a transition $\mathbf{t}_j$ is enabled at marking M_k if $\forall p_i \in P, M_k(p_i) \geqslant I(p_i, \mathbf{t}_j)$; an enabled transition $\mathbf{t}_j$ can be fired reaching a new marking M_{k+1} which can be computed using the Petri net state equation

$$M_{k+1} = M_k + \mathcal{C}v_k, \tag{2.11}$$

where $v_k(i) = 0$, $i \neq j$, $v_k(j) = 1$, this equation is called the PN state equation.

The reachability set of a PN is the set of all possible reachable marking from M_0 firing only enabled transitions; this set is denoted by $R(G, M_0)$.

Definition 2.7 *An Interpreted Petri Net (IPN) is the 4-tuple* $Q = (N, \Sigma, \lambda, \varphi)$ *where:*

- $N = (G, M_0)$ *is a* PN *system.*
- $\Sigma = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ *is the input alphabet of the net, where* α_i *is an input symbol.*
- $\lambda : T \rightarrow \Sigma \cup \{\varepsilon\}$ *is a labelling function of transitions with the following constraint:* $\forall \mathbf{t}_j, \mathbf{t}_k \in T$, $j \neq k$, *if* $\forall p_i$ $I(p_i, \mathbf{t}_j) = I(p_i, \mathbf{t}_k) \neq 0$ *and both* $\lambda(\mathbf{t}_j) \neq \varepsilon$, $\lambda(\mathbf{t}_k) \neq \varepsilon$, *then* $\lambda(\mathbf{t}_j) \neq \lambda(\mathbf{t}_k)$. *In this case* ε *represents an internal system event.*
- $\varphi : R(Q, M_0) \rightarrow (\mathbb{Z}^+)^q$ *is an output function, that associates to each marking in* $R(Q, M_0)$ *an output vector. Here* q *is the number of outputs.*

Remarks:

1. In this work (Q, M_0) will be used instead of $Q = (N, \Sigma, \lambda, \varphi)$ to emphasize the fact that there is an initial marking in an IPN.

2. Function φ is a $q \times r$ matrix, where q is the number of places representing measurable states in the DES and r is the number of places in the model (G, M_0). Each column of this matrix is an elementary or null vector. If the output symbol i is present (turned on) every time that $M(p_j) \geq 1$, then $\varphi(i,j) = 1$, otherwise $\varphi(i,j) = 0$.

3. Equivalent transitions are not allowed, i.e. it is assumed that $\forall \mathbf{t}_i, \mathbf{t}_j$ such that $\mathbf{t}_i \neq \mathbf{t}_j, \lambda(\mathbf{t}_i) = \lambda(\mathbf{t}_j)$, it holds that $\mathcal{C}(\cdot, i) \neq \mathcal{C}(\cdot, j)$. This is not a major constraint because those transitions are redundant.

4. Notice that by definition of λ, IPN are deterministic [25] over labeled transitions, i.e. two transitions with the same associated input symbol (different from symbol ε) cannot have the same input places. However, they can be non deterministic [25] over unlabeled transitions (those t_j such that $\lambda(t_j) = \varepsilon$).

A transition $\mathbf{t}_j \in T$ of an IPN is enabled at marking M_k if $\forall p_i \in P$, $M_k(p_i) \geqslant I(p_i, \mathbf{t}_j)$. If $\lambda(\mathbf{t}_j) = a_i \neq \varepsilon$ is present and $\mathbf{t}_j$ is enabled, then $\mathbf{t}_j$ must fire. If $\lambda(\mathbf{t}_j) = \varepsilon$ and $\mathbf{t}_j$ is enabled then $\mathbf{t}_j$ can be fired. When an enabled transition $\mathbf{t}_j$ is fired in a marking M_k, then a new marking M_{k+1} is reached. This fact is represented as $M_k \xrightarrow{\mathbf{t}_j} M_{k+1}$ and M_{k+1} can be computed using the dynamic part of the IPN state equation:

$$\begin{aligned} M_{k+1} &= M_k + \mathcal{C} v_k \\ y_k &= \varphi(M_k) \end{aligned} \tag{2.12}$$

where C and v_k are defined as in PN and $y_k \in (\mathbb{Z}^+)^q$ is the $k-th$ output vector of the IPN.

According to functions λ and φ, transitions and places of an IPN (Q, M_0) can be classified as follows.

Definition 2.8 *If $\lambda(\mathbf{t}_i) \neq \varepsilon$ the transition $\mathbf{t}_i$ is said to be manipulated. Otherwise it is non manipulated. A place $p_i \in P$ is said to be measurable if the $i-th$ column vector of φ is not null, i.e. $\varphi(\bullet, i) \neq 0$. Otherwise it is non measurable. A place p_i is said to be computable if it is measurable and $\forall j$, $i \neq j$, $\varphi(\bullet, i) \neq \varphi(\bullet, j)$. Otherwise it is non computable.*

Notice that computable places are measurable and the marking of these places can be computed from the output (no other place, when it is marked, generates the same output value of function φ).

2.2.1 Petri nets properties

Definition 2.9 *Let $\sigma = \mathbf{t}_i\mathbf{t}_j\mathbf{t}_k...$ be a firing transition sequence. The Parikh vector $\vec{\sigma} : T \longrightarrow (\mathbb{Z}^+)^m$ of σ maps every transition $\mathbf{t} \in T$ to the number of occurrences of $\mathbf{t}$ in σ.*

Definition 2.10 *A P-semiflow of a Petri Net G is a rational-valued solution of the equation $\mathbf{Y}^T \cdot \mathcal{C} = 0$*

The following proposition presents a fundamental property of P-semiflows

Proposition 2.11 *Let (G, M_0) be a Petri Net system, and let I be a P-semiflow of G then $\forall M \in R(G, M_0)$ $I \quad M = I \cdot M_0$*

Definition 2.12 *A T-semiflow of a Petri Net G is a rational-valued solution of the equation $\mathcal{C} {\cdot} \mathbf{Y} = 0$*

Proposition 2.13 *Let σ be a finite sequence of transitions of a net G which is enabled at a marking M. Then the Parikh vector $\vec{\sigma}$ is a T-semiflow iff $M \xrightarrow{\sigma} M$ (i.e. iff the occurrence of σ reproduces the marking M).*

Definition 2.14 *A sequence of input-output symbols of (Q, M_0) is a sequence $\omega = (\alpha_0, y_0)\ (\alpha_1, y_1) \cdots\ (\alpha_n, y_n)$ where $\alpha_j \in \Sigma \cup \{\varepsilon\}$ and α_{i+1} is the current input of the IPN when the output changes from y_i to y_{i+1}. It is assumed that $\alpha_0 = \varepsilon$, $y_0 = \varphi(M_0)$ and (α_{i+1}, y_{i+1}) belongs to the sequence when:*

- *(α_i, y_i) belongs to the sequence,*
- *$y_{i+1} \neq y_i$, and*
- *there exists no $y_j \neq y_i$, $y_j \neq y_{i+1}$ occurring after the occurrence of y_i and before the occurrence of y_{i+1}.*

Definition 2.15 *Let (Q, M_0) be an IPN. The set $\Lambda(Q, M_0) = \{\omega | \omega$ is a sequence of input-output symbols$\}$ denotes the set of all sequences of input-output symbols of (Q, M_0). The set of all input-output sequences of length greater or equal than k will be denoted by $\Lambda^k(Q, M_0)$, i.e. $\Lambda^k(Q, M_0) = \{\omega \in \Lambda(Q, M_0) \mid |\omega| \geq k\}$.*

Definition 2.16 *If $\omega = (\alpha_0, y_0)(\alpha_1, y_1) \cdots (\alpha_n, y_n)$ is a sequence of input-output symbols, then the firing transition sequence $\sigma \in \pounds(Q, M_0)$ whose firing actually generates ω is denoted by σ_ω. The set of all possible firing transition sequences that could generate the word ω is defined as*

$$\Omega(\omega) = \{\sigma | \sigma \in \pounds(Q, M_0) \ \wedge \ \text{the firing of } \sigma \text{ produces } \omega\}$$

Definition 2.17 *The set of all input-output sequences leading to an ending marking in the IPN (markings enabling no transition or only self-loop transitions) is denoted by $\Lambda_B(Q, M_0)$, i.e., $\Lambda_B(Q, M_0) = \{\omega \in \Lambda(Q, M_0) \mid \exists \sigma \in \Omega(\omega)$ such that $M_0 \xrightarrow{\sigma} M_j$ $\wedge$ if $M_j \xrightarrow{\mathbf{t}_i}$ then $C(\bullet, \mathbf{t}_i) = \vec{0}\}$.*

The prefix of a sequence s is another sequence s' such that there exists a sequence s'' fulfilling that $s = s's''$. The set of all prefixes of s is denoted by $\bar{s}$.

Definition 2.18 *Let $\omega = (\alpha_0, y_0)\ (\alpha_1, y_1)\ \cdots\ (\alpha_n, y_n) \in \Lambda(Q, M_0)$ be a sequence of input-output symbols. The marking sequences set corresponding to ω is defined as $S_\omega = \{M_0\ M_1 \cdots\ M_k |\ M_i \in R(Q, M_0)\ \wedge M_0 \xrightarrow{\mathbf{t}_i} M_1 \xrightarrow{\mathbf{t}_j} \cdots \xrightarrow{\mathbf{t}_m} M_k \wedge\ \sigma_\omega = \mathbf{t}_i\mathbf{t}_j \cdots \mathbf{t}_m \in \Omega(\omega)\}$.*

Definition 2.19 *A Petri Net system is a state machine if $|\mathbf{t}^\bullet| = 1 = |^\bullet\mathbf{t}|$ for every transition $\mathbf{t}$.*

The fundamental property of state machines is that all reachable markings contain exactly the same number of tokens. In other words, the total number of tokens of the system remains invariant under the occurrence of transitions.

Definition 2.20 *A Petri Net systems is a marked Graph if $|p^\bullet| = 1 = |^\bullet p|$ for every place p.*

The fundamental property of marked graphs is that the token counts of circuits remain invariant under the occurrence of transitions.

Definition 2.21 *A Petri Net system (G, M_0) is live if, for every reachable marking M and every transition $\mathbf{t}$ there exists a marking $M' \in R(G, M_0)$ which enables $\mathbf{t}$. If (G, M_0) is a live system, then it is said that M_0 is a live marking of G.*

Definition 2.22 *A Petri Net system (G, M_0) is deadlock-free if every reachable marking enables at least one transition.*

Definition 2.23 *A system is bounded if for every place p there is a natural number b such that $M(b) \leq b$ for every reachable marking M.*

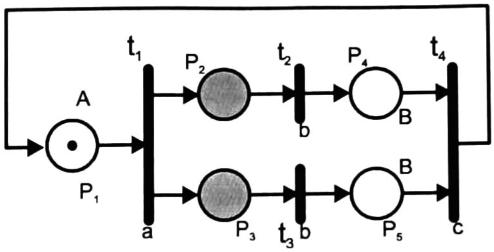

Figure 2.1: A non deterministic IPN.

Definition 2.24 *A marking $M \in R(G, M_0)$ is a home marking if $\forall M_k \in R(G, M_0)$ there is a firing sequence such that $M_k \xrightarrow{\sigma} M$ it is reachable from every reachable marking. In other words, it is a home marking if it is reachable from every reachable marking.*

Definition 2.25 *A Petri Net is cyclic if its initial marking is a home marking.*

Example 2.26 *The IPN of figure 2.1 is composed by five places and four transitions. The output function φ is represented by the matrix:*

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{2.13}$$

Notice that places, p_1, p_4 and p_5 are measurable, however, place p_1 is the only computable place. Since all transitions are labeled, then all of them are manipulated. We obtain the following languages.

$$\pounds(Q, M_0) = \{\mathbf{t}_1, \mathbf{t}_1\mathbf{t}_2, \mathbf{t}_1\mathbf{t}_3, \mathbf{t}_1\mathbf{t}_2\mathbf{t}_3, \mathbf{t}_1\mathbf{t}_3\mathbf{t}_2, \mathbf{t}_1\mathbf{t}_2\mathbf{t}_3\mathbf{t}_4, \mathbf{t}_1\mathbf{t}_3\mathbf{t}_2\mathbf{t}_4, \mathbf{t}_1\mathbf{t}_2\mathbf{t}_3\mathbf{t}_4\mathbf{t}_5, ..\} \tag{2.14}$$

$$\begin{aligned} \Lambda(Q, M_0) = & \left\{ \left(\varepsilon, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \left(\varepsilon, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \left(a, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \right. \\ & \left. \left(\varepsilon, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \left(a, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \left(b, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right), \ldots \right\} \end{aligned} \tag{2.15}$$

$$\Lambda^2(Q, M_0) = \left\{\left(\varepsilon, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\left(a, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)\right. \\ \left.\left(\varepsilon, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\left(a, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)\left(b, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right), \ldots\right\} \tag{2.16}$$

$$\Lambda_B(Q, M_0) = \{\} \tag{2.17}$$

If $\omega = \left(\varepsilon, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\left(a, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)\left(b, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ *then:*

$$\Omega(\omega) = \{\mathbf{t}_1\mathbf{t}_2, \mathbf{t}_1\mathbf{t}_3\} \tag{2.18}$$

and:

$$S_\omega = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\} \tag{2.19}$$

2.2.2 Observability in Interpreted Petri nets

Now, the following concept will be needed to characterize observable IPN and establishes that, even when the precise marking of a place is unknown, it can belong to a conservative marking law. In other words, the location or state of the entities (resources, machines, buffer capacities, etc.) that constitute the DES may be unknown; however the amount of those entities is known. This concept is analogous to that of "macro-markings" used in [26].

Definition 2.27 *Let* (Q, M_0) *be an* IPN *structure and* $M(p_j)$ *be any marking of a place* p_j *in* (Q, M_0). *The set of* s *equations* $CML = \{\sum_{j=1}^{n} \gamma_j^i M(p_j) = k_i | i \in$

$[1, ..., s] \wedge \gamma_j^i \in \mathbb{Z}^+\}$ *form a set of conservative marking laws* (CML) *if* $\forall \gamma_k^i \neq 0$ *it holds that* k_i/γ_k^i *is an integer value and all non computable places* p_n *are contained in at least one equation of the* CML *set. A* CML *is said to be binary* $(BCML)$ *if it holds that* $\forall i, j,\ \gamma_j^i \in \{0, 1\}$ *and* $k_i = 1$*. In addition, the* CML *can be rewritten as:*

$$\Gamma M = K \tag{2.20}$$

where M *is the marking vector,* Γ *is the matrix* $\Gamma[i, j] = \gamma_j^i$ *and* K *is the vector* $K(i) = k_i$.

Remarks:

5. Hereafter $\mathcal{M}_0$ will denote the set of all possible initial markings fulfilling the stated CML, i.e.

$$\mathcal{M}_0 = \{M_0 | \text{such that any } M \in R(Q, M_0) \text{ fulfills the } CML \text{ constraints}\}. \tag{2.21}$$

6. The notation $p_l \in e_i$, where $e_i \in CML$ ($e_i \in BCML$) means that there exists an equation $\sum_{j=1}^{n} \gamma_j^i \cdot M(p_j) = k_i$, named e_i in the CML (in the $BCML$), such that $\gamma_l^i \neq 0$.

Also, $(Q, \mathcal{M}_0)$ will denote an IPN where $M_0 \in \mathcal{M}_0$ and could be unknown. Notation $\mathcal{M}_0^{\mathrm{B}}$ will be used for a $BCML$.

Definition 2.28 *An* IPN *given by* $(Q, \mathcal{M}_0)$ *is event-detectable if any transition firing can be uniquely determined by the knowledge of the input given to* $(Q, \mathcal{M}_0)$ *and output signals that it produces.*

The following lemma provides a structural characterization of the IPN exhibiting event-detectability.

Lemma 2.29 *A live IPN given by $(Q, \mathcal{M}_0)$ is event-detectable if and only if*

1. $\forall \mathbf{t}_i, \mathbf{t}_j \in T$ *such that* $\lambda(\mathbf{t}_i) = \lambda(\mathbf{t}_j)$ *or* $\lambda(\mathbf{t}_i) = \varepsilon$ *it holds that* $\varphi C(\bullet, \mathbf{t}_i) \neq \varphi C(\bullet, \mathbf{t}_j)$ *and*

2. $\forall \mathbf{t}_k \in T$ *it holds that* $\varphi C(\bullet, \mathbf{t}_k) \neq 0$.

Proof. The proof of this result is presented in [27]. ■

As stated previously, a state representation of a dynamic system is said to be observable if the knowledge of its inputs, outputs, and structure suffices to uniquely determine its state, for instance, $\Sigma(A, B, C)$, is said to be observable, at t_0, if there exists a finite time t_1 such that the knowledge of the model structure (A, B, C), the input signal $(u(t))$ and the output signal $(y(t))$ over the interval $t_0 \leq t \leq t_1$ suffices to uniquely determine the initial state $(x(t_0))$. Moreover, since the system is a deterministic one, then $x(t)$ for all $t \geq t_0$, can also be uniquely determined using the knowledge of $x(t_0)$ and $u(t)$, over the interval $t_0 \leq t \leq t_1$.

When the dynamic model is non deterministic (i.e. when the solution of the model is not unique [28]), however, the knowledge of $x(t_0)$ and $u(t)$ over the interval $t_0 \leq t \leq t_1$ does not guarantee the computation of $x(t)$ for all $t \geq t_0$. For instance, even if it is known that the initial marking of the IPN depicted in figure 2.1 is $M_0 = [10000]^T$ and that the input sequence $\sigma = abbcabbcabbc...ab$ is fired, it is not possible to determine if the reached marking is $[01001]^T$ or $[00110]^T$ In this case, there exists no finite transition sequence σ allowing to know the reached marking and all future reached markings of the IPN.

This fact leads to that, in the general case, the observability definition must be changed to ensure that the initial state $x(t_0)$ and all states $x(t)$, $t > t_0$ can be computed in a finite time or length of input firing words. Because of that, in the general case, the following intuitive definition is presented.

A non deterministic dynamic model, for instance an IPN, is observable, at k_0, if there exists a finite integer k_1 such that the knowledge of the model structure (C, λ, φ) and the sequence of input-output symbols ω_k for any $k \geq k_1$ suffices to uniquely determine the state sequence over $k_0 \leq l \leq k$ $(M_{k_0}...M_k)$. The observability definition in IPN can be formally proposed as follows.

Definition 2.30 *An IPN given by (Q, M_0), where M_0 may be unknown, is observable if there exists an integer $k < \infty$ such that $\forall \omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ it holds that the information provided by ω and (Q, M_0) suffices to uniquely determine the initial marking M_0 and the marking M_i reached by the firing of the underlaying firing transition sequence σ_ω.*

Therefore an IPN is observable if for any sequence of input-output signals of length equal or greater than k and for any blocking sequence, the marking sequence reached by the system can be uniquely determined.

Since the set S_ω contains the marking sequences generated by the same input-output sequence $\omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$, then when $|S_w| = 1$ there exists only one marking sequence for the word ω. Thus the initial and the actual marking can be computed from these marking sequence, leading to an observable IPN. This fact is formalized in the following theorem.

Theorem 2.31 *An IPN given by (Q, M_0) is observable if and only if there is an integer $k < \infty$ such that $\forall \omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ it holds that $|S_\omega| = 1$, where S_ω is the marking sequences set corresponding to ω.*

Proof. *(Sufficiency) Assume that there exists an integer $k < \infty$ such that $\forall \omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ it holds that $|S_\omega| = 1$, then a function $\Psi : \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0) \longrightarrow R(Q, M_0) \times R(Q, M_0)$ can be computed, where Ψ fulfills the following: $\forall \omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ it holds that $\Psi(\omega, (Q, M_0)) = (M_0, M_i)$ where M_0 is* *the initial marking and M_i is the marking reached by the firing of the underlaying firing transition sequence σ_ω.*

(Necessity) Suppose that there is no integer $k < \infty$ such that $\forall\omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ it holds that $|S_\omega| = 1$, then for any k there is at least one $\omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ such that $|S_\omega| \neq 1$, therefore $|S_\omega| > 1$. Assume further without lost of generality that $S_\omega = \{\ \gamma_1 = M_i\ M_j \cdots M_k \cdots\ M_n,\ \gamma_2 = M_i'\ M_j' \cdots\ M_k' \cdots\ M_n'\}$.

Since these sequences are different, then there must exist markings M_k, M_k' such that $M_k \neq M_k'$ in γ_1, γ_2 respectively. Notice that when the initial marking of (Q, M_0) is M_k or M_k' then there exist two different values to assign to M_0, or the function Ψ cannot be obtained, a contradiction. ■

2.3 Switched Linear Systems

The definition of a SLS is presented in the following.

Definition 2.32 *Let $\mathcal{F} = \{\Sigma_1, ..., \Sigma_s\}$ be a family of linear systems, all of them of the same dimension, and let (Q, M_0) be an IPN The 2-tupla $\langle\mathcal{F}, (Q, M_0)\rangle$ is a SLS if:*

1. *There exists a function $\Phi : R(Q, M_0) \rightarrow \mathcal{F}$. such that if the current marking of (Q, M_0) is M_k, then the linear system $\Phi(M_k)$ is the only linear system that is evolving.*

2. *There exist bijective functions $\delta_{M_i,M_j} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\delta_{M_i,M_j}(x_f^i) = x_0^j$ indicates how the SLS state is changed when the SLS commutes from Σ_i to Σ_j, $\Sigma_i, \Sigma_j \in \mathcal{F}$*

3. *The elapsed time to change from M_k to M_{k+1} is finite and different from zero.*

4. *The IPN is live, binary and cyclic, an IPN is binary if $M(p_i) \leq 1$ for any place and any reachable marking.*

5. *It is assumed that every $\Sigma_i \in \mathcal{F}$ evolves for a finite $\tau > 0$ before Σ_i commutes to another Σ_j.*

Then, a SLS is described by the following state equation:

$$\begin{aligned} \dot{x}(t) &= A_{M_k}x(t) + B_{M_k}u(t) \\ y_c(t) &= C_{M_k}x(t) \\ M_{k+1} &= M_k + Cv_k \\ y_{d_k} &= \varphi(M_k) \end{aligned} \tag{2.22}$$

where v_k is the firing vector and there exist functions Φ and δ_{M_i,M_j} as described above.

Hereafter continuous and discrete state vectors will be represented as $x(t)$ and $M_k(t)$ respectively.

Let us define the state vector $X(t)$, the output vector $Y(t)$ and the input vector $U(t)$ of the switched linear system as:

$$X(t) = \begin{bmatrix} x(t) \\ M_k(t) \end{bmatrix} \quad Y(t) = \begin{bmatrix} y_c(t) \\ y_{d_k}(t) \end{bmatrix}, \; U(t) = \begin{bmatrix} u(t) \\ v_k(t) \end{bmatrix} \tag{2.23}$$

Condition 2 of previous definition is not restrictive at all, it is fulfilled in all SLS. For instance, some classes of nonlinear systems (NLS) can be approximated by SLS, where each LS is a linearization of the NLS in different operation points. Notice that each LS has its own tangent state space, the functions δ_{M_i,M_j} couple the respective tangent state spaces.

2.3.1 Observability in Switched Linear Systems

Definition 2.33 *The SLS $\Im = \langle \mathcal{F}, (Q, M_0) \rangle$ is said to be observable at t_0 if there exists a finite $t_1 > t_0$ such that for any state X_0 at time t_0, the knowledge of the input*

$U[t_0, t_1]$ and the output $Y[t_0, t_1]$ over the finite time interval $[t_0, t_1]$, and the system structure, suffices to uniquely determine the initial and current state of $\Im$.

It can be observed from previous definitions that if there exist two state trajectories $X_1(t)$, $X_2(t)$, with different initial conditions $X_1(0)$, $X_2(0)$, generating for any time t the same output $Y_1(t) = Y_2(t)$, when the same input $U(t)$ is applied to the system in both trajectories, then the SLS is unobservable. It occurs since the information of $Y_1(t) = Y_2(t) = Y(t)$ and $U(t)$ is not enough to distinguish between initial conditions $X_1(0)$, $X_2(0)$, i.e. there exists no function $\Psi(Y(t), U(t)) = X_1(0), \Psi(Y(t), U(t)) = X_2(0)$.

Definition 2.34 *Let $\mathcal{Q}$ be a dynamic system, then $\mathbf{X}_t(\mathcal{Q})$ denotes the set of finite time t state trajectories that are obtained applying all possible inputs u_i to every possible initial state condition.*

For $s_i \in \mathbf{X}_t(\Im)$, the notation $\boldsymbol{I}(s_i) = u_i[0, t]$ means that the state trajectory s_i was generated applying the input $u_i[0, t]$ to $\Im$ in the time interval $[0, t]$, and $\boldsymbol{O}(s_i) = y_i[0, t]$ denotes the output $y_i[0, t]$ generated by s_i.

Definition 2.35 *Let $\Im$ be a SLS. The state trajectories s_i and $s_j \in \mathbf{X}_\tau(\Im)$ are said to be Input Related, denoted as (IR) $s_i \sim s_j$, if $\boldsymbol{I}(s_i) = \boldsymbol{I}(s_j)$, and they are said to be Output Related, denoted as (OR) $s_i \sim s_j$, if $\boldsymbol{O}(s_i) = \boldsymbol{O}(s_j)$. Notice that both are equivalence relationships.*

The following equivalence classes can be defined in the set $\mathbf{X}_\tau(\Im)$

$$\begin{gathered} C_{u_i}^{\Im} = \{s_p \mid \boldsymbol{I}(s_p) = u_i\}, \\ \text{and} \\ C_{y_i}^{\Im} = \{s_p \mid \boldsymbol{O}(s_p) = y_i\} \end{gathered} \tag{2.24}$$

Up to now, we have grouped the equivalent hybrid state trajectories related to the same input signal into the coset $\mathbf{X}_\tau(\Im)/IR = \Pi^{\Im}_{input} = \{C^{\Im}_{u_i}\}$ and to the same output signal into the coset $\mathbf{X}_\tau(\Im)/OR = \Pi^{\Im}_{output} = \{C^{\Im}_{y_i}\}$. Now, the observability of the SLS will be characterized using these partitions.

Theorem 2.36 *The SLS $\Im$ is observable if and only if there exists a time t such that $\forall i, j$, $\pi^{\Im}_{ij} = C^{\Im}_{u_i} \cap C^{\Im}_{y_j}$, $C^{\Im}_{u_i} \in \Pi^{\Im}_{input}$ and $C^{\Im}_{y_j} \in \Pi^{\Im}_{output}$, it holds that $\left|\pi^{\Im}_{ij}\right| = 1$, where $\left|\pi^{\Im}_{ij}\right|$ denotes the cardinality of the set $\pi^{\Im}_{ij}$.*

Proof. (Sufficiency). Assume that $\forall \pi^{\Im}_{ij} = C^{\Im}_{u_i} \cap C^{\Im}_{y_j}$ it holds that $\left|\pi^{\Im}_{ij}\right| = 1$. Then $s_k \in \pi^{\Im}_{ij}$ is the only sequence in $\pi^{\Im}_{ij}$. Thus a function $\Psi : U \times Y \longrightarrow \mathbf{X}_t(\Im)$ can be found, such that $\Psi(u_i, y_j) = s_k$, where $\boldsymbol{I}(s_k) = u_i$, $\boldsymbol{O}(s_k) = y_j$, and the initial and final states $x_k(0)$, $x_k(\tau)$ of s_k can be computed, i.e. there exists a function $\Lambda(s_k) = (x_k(0), x_k(\tau))$, then $\Lambda \circ \Psi(u_i, y_j) = (x_k(0), x_k(\tau))$.

(Necessity). Assume that there exists $\pi^{\Im}_{ij} = C^{\Im}_{u_i} \cap C^{\Im}_{y_j}$ such that $\left|\pi^{\Im}_{ij}\right| > 1$. Then there exist at least two sequences $s_k, s_l \in \pi^{\Im}_{ij}$, such that $\boldsymbol{I}(s_k) = \boldsymbol{I}(s_l)$ and $\boldsymbol{O}(s_k) = \boldsymbol{O}(s_l)$. Thus for the same u_i, y_j two s_k, s_l (or even more) state sequences are generated. Then, the function $\Psi : U \times Y \longrightarrow \mathbf{X}_t(\Im)$ does not exist. Then, using the input and output sequences is not enough to determine neither the sequence nor the initial and final states of the system. Thus the SLS is unobservable. ■

The previous result characterizes the observability in SLS. This characterization, however, uses all the possible state trajectories leading to complex analysis. Next chapters are devoted to show how to use previous theorem to derive a structural characterization of observability in SLS.

Chapter 3

Mode observability of SLS

This chapter presents two cases when the IPN is not observable. Several places producing the same output and several discrete markings sequences producing the same output information. The first case deals with the case when several marking produce the same output, thus the discrete output information is not enough to determine the discrete state; the second case addresses the problem when several marking sequences produce the same discrete output information, thus the IPN information is not enough to determine which marking sequence was fired (i.e. the continuous systems sequence), thus it is impossible to compute, using the IPN information, which is the discrete marking.

Fortunately, if the continuous information meets some properties, then it will be possible to distinguish the discrete marking.

3.1 Several places producing the same output

Let us consider the IPN of Figure 3.1. This IPN is binary, live, and cyclic. Unfortunately, two markings produce the output B. Thus if symbol B is produced as

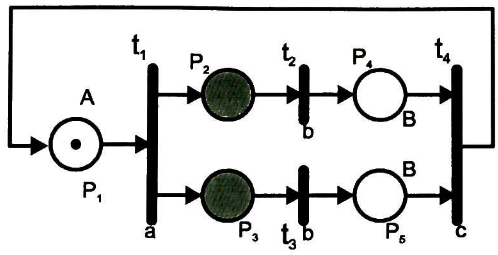

Figure 3.1: Several places producing the same output

output, it is impossible to determine if the system is in marking $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ or $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$

If this IPN is representing the discrete part of a SLS, then the only possibility to distinguish each marking is using the continuous information. In order to do that, the outputs of the continuous systems must contain information to distinguish which linear system is working and afterwards, this information can be used to compute the discrete marking. System identification can be used to distinguish which LS is evolving, however this approach assumes that some white noise can be added to the LS inputs. If this hypothesis cannot be satisfied, then there still exists the possibility to distinguish both continuous systems. The property allowing to distinguish both systems without using an identification scheme is named distinguishability in this work.

Definition 3.1 *The linear systems* $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ *are said to be distinguishable from each other if the knowledge of the input* $u[t_0, t_1]$ *and the output* $y[t_0, t_1]$ *over the finite time interval* $[t_0, t_1]$ *suffices to determine which LS is evolving.*

Similar to the concept of observability two linear systems are indistinguishable from each other using the input and the output iff both systems generate the same output when the same input is applied.

Notation 3.2 *Let $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ be two $SISO$ linear systems, the linear system $\bar{\Sigma}\left\{\bar{A}, \bar{B}, \bar{C}\right\}$ denotes the extended LS formed with the matrices*

$$\begin{aligned}
\bar{A} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \\
\bar{B} &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\
\bar{C} &= \begin{bmatrix} C_1 & -C_2 \end{bmatrix}
\end{aligned} \tag{3.1}$$

Lemma 3.3 *Let $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ be two $SISO$ LS where $A_1 \in \mathbb{R}^n$ and $A_2 \in \mathbb{R}^m$ Then the linear systems $\Sigma_1(A_1, B_1, C_1)$ and $\Sigma_2(A_2, B_2, C_2)$ are distinguishable from each other if and only if the only solution to the equation*

$$\bar{C}e^{\bar{A}t}\left[x_0 + \int_0^t e^{-\bar{A}\tau}\bar{B}u(\tau)d\tau\right] = 0 \tag{3.2}$$

is $x_0 = 0$ and $u(t) = 0$.

Proof. *If the linear systems $\Sigma_1(A_1, B_1, C_1)$ and $\Sigma_2(A_2, B_2, C_2)$ are indistinguishable from each other then there exists an input $u(t)$ such that the same output $y(t)$ is produced by both systems when $u(t)$ is applied, i.e. for two different initial conditions x_0^1, x_0^2 it holds that:*

$$y(t) = C_1e^{A_1t}\left[x_0^1 + \int_0^t e^{-A_1\tau}B_1u(\tau)d\tau\right] \tag{3.3}$$

and

$$y(t) = C_2e^{A_2t}\left[x_0^2 + \int_0^t e^{-A_2\tau}B_2u(\tau)d\tau\right] \tag{3.4}$$

then combining equations (3.3) and (3.4):

$$C_1e^{A_1t}\left[x_0^1 + \int_0^t e^{-A_1\tau}B_1u(\tau)d\tau\right] = C_2e^{A_2t}\left[x_0^2 + \int_0^t e^{-A_2\tau}B_2u(\tau)d\tau\right] \tag{3.5}$$

this equation can be written as

$$C_1e^{A_1t}x_0^1 - C_2e^{A_2t}x_0^2 = \int_0^t \left[-C_1e^{-A_1(t-\tau)}B_1 + C_2e^{-A_2(t-\tau)}B_2\right]u(\tau)d\tau. \tag{3.6}$$

Now, since 3.6 is equivalent to:

$$\begin{array}{c} \begin{bmatrix} C_1 & -C_2 \end{bmatrix} \begin{bmatrix} e^{A_1 t} & 0 \\ 0 & e^{A_2 t} \end{bmatrix} \begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix} = \\ -\int_0^t \begin{bmatrix} C_1 & -C_2 \end{bmatrix} \begin{bmatrix} e^{A_1(t-\tau)} & 0 \\ 0 & e^{A_2(t-\tau)} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(\tau)d\tau \end{array} \tag{3.7}$$

equation (3.7) can be written in terms of the matrices (3.1), with $x_0 = \begin{bmatrix} x_0^1 & x_0^2 \end{bmatrix}^T$ *then*

$$\bar{C}e^{\bar{A}t}\left[x_0 + \int_0^t e^{-\bar{A}\tau}\bar{B}u(\tau)d\tau\right] = 0. \tag{3.8}$$

Since $\Sigma_1(A_1, B_1, C_1)$ *is indistinguishable from* $\Sigma_2(A_2, B_2, C_2)$*, thus there exist solutions* $x_0 \neq 0$ *and* $u(t) \neq 0$ *to equation (3.8). The converse is also true, then* $\Sigma_1(A_1, B_1, C_1)$ *and* $\Sigma_2(A_2, B_2, C_2)$ *are indistinguishable from each other iff the only solution to equation (3.8) is* $x_0 = 0$ *and* $u(t) = 0$. ■

Remark 3.4 *The analysis presented in this work is restricted to inputs that can be transformed into Laplace functions.*

Remark 3.5 *Taking the Laplace transform of equation (3.8) we obtain:*

$$\bar{C}\left(sI - \bar{A}\right)^{-1}x_0 + \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B}U(s) = 0 \tag{3.9}$$

which always has solutions $x_0 = -\alpha\bar{B}$ *and* $U(s) = \alpha$ *for any real constant* α *i.e. when the input is an impulse function. However, impulse functions cannot be implemented and therefore, if the analysis is restricted to inputs that can transformed into Laplace functions, then equation (3.8) has only solution* $x_0 = 0$, $u(t) = 0$ *iff the unique solution space of equation (3.9) is* $\bar{B}$.

Theorem 3.6 *Let* $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ *be two linear systems, where* $A_1 \in \mathbb{R}^{n\times n}$ *and* $A_2 \in \mathbb{R}^{m\times m}$ *Then the linear systems* $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ *are* *distinguishable from each other if and only if the realization* $\bar{\Sigma}\left\{\bar{A},\bar{B},\bar{C}\right\}$ *is minimal and its transfer function does not have transmission zeros (i.e.* $\bar{C}Adj\left(sI-\bar{A}\right)\bar{B}=\alpha$ *for some constant* $\alpha\neq 0$*), where* $\bar{A}$, $\bar{B}$, $\bar{C}$ *are the matrices of (3.1).*

Proof. *(Sufficiency) Assume that the realization* $\bar{\Sigma}\left\{\bar{A},\bar{B},\bar{C}\right\}$ *is minimal and has no transmission zeros, but the LS* $\Sigma_1\left(A_1,B_1,C_1\right)$ *and* $\Sigma_2\left(A_2,B_2,C_2\right)$ *are indistinguishable from each other, then there exist* $x_0\neq 0$ *or* $u(t)\neq 0$ *such that equation (3.8) holds.*

Equation (3.9) can be written as:

$$\frac{\bar{C}Adj\left(sI-\bar{A}\right)x_0}{\det\left(sI-\bar{A}\right)}+\frac{\bar{C}Adj\left(sI-\bar{A}\right)\bar{B}}{\det\left(sI-\bar{A}\right)}U(s)=0, \tag{3.10}$$

since the realization $\left\{\bar{A},\bar{B},\bar{C}\right\}$ *is minimal, there is no cancellation of terms between*

$$\bar{C}Adj\left(sI-\bar{A}\right)\bar{B} \text{ and } \det\left(sI-\bar{A}\right) \tag{3.11}$$

and $U(s)$ *is:*

$$U(s)=-\frac{\bar{C}Adj\left(sI-\bar{A}\right)x_0}{\bar{C}Adj\left(sI-\bar{A}\right)\bar{B}} \tag{3.12}$$

however, since the transfer function $\bar{C}\left(sI-\bar{A}\right)^{-1}\bar{B}$ *has no transmission zeros,*

$$\bar{C}Adj\left(sI-\bar{A}\right)\bar{B} \tag{3.13}$$

does not contain terms in s*. Thus in order to* $U(s)$ *be a proper transfer function,* x_0 *needs to be equal to* $\alpha\bar{B}$ *for any* α*. Therefore* $U(s)=\alpha$*, i.e.* $U(s)$ *is an impulse function or Dirac delta function. Then according to Lemma 3.3 and remark 3.5 the linear systems* $\Sigma_1\left(A_1,B_1,C_1\right)$, $\Sigma_2\left(A_2,B_2,C_2\right)$ *are distinguishable from each other, a contradiction.*

(Necessity) We show that if the realization $\left\{\bar{A},\bar{B},\bar{C}\right\}$ *is not minimal (noncontrollable or nonobservable) or has transmission zeros then the linear systems* $\Sigma_1\left(A_1,B_1,C_1\right)$ *and* $\Sigma_2\left(A_2,B_2,C_2\right)$ *are indistinguishable from each other. The proof is divided in three parts.*

a) *The pair* $\{\bar{A}, \bar{C}\}$ *is nonobservable.*

Then, for input $u(t) = 0$, *the Laplace transform of equation (3.6), becomes:*

$$C_1 (sI - A_1)^{-1} x_0^1 - C_2 (sI - A_2)^{-1} x_0^2 = 0 \tag{3.14}$$

i.e.

$$\bar{C} \left(sI - \bar{A}\right)^{-1} x_0 = 0. \tag{3.15}$$

Since the pair $\{\bar{A}, \bar{C}\}$ *is nonobservable then the columns of* $\bar{C}\left(sI - \bar{A}\right)^{-1}$ *are linearly dependent [6]. Hence, there exists a vector* $x_0 \neq 0$ *such that the equation (3.15) holds. Thus, for input* $u(t) = 0$, *there exist initial conditions* $x_0^1 \neq 0$ *or* $x_0^2 \neq 0$ *such that the LS* $\Sigma_1 (A_1, B_1, C_1)$, $\Sigma_2 (A_2, B_2, C_2)$ *are indistinguishable from each other [8].*

b) *There is a transmission zero* $\lambda \in \mathbb{C}$ *of* $\bar{C}\left(sI - \bar{A}\right)^{-1} \bar{B}$.

Then the output due to the initial state:

$$x_0 = -\left(\bar{A} - \lambda I\right)^{-1} Bk \tag{3.16}$$

and the input

$$u(t) = ke^{\lambda t} \tag{3.17}$$

is identically zero [6]. Hence, equation (3.8) is satisfied, and the LS $\Sigma_1 (A_1, B_1, C_1)$ $\Sigma_2 (A_2, B_2, C_2)$ *are indistinguishable from each other.*

c) *The pair* $\{\bar{A}, \bar{C}\}$ *is noncontrollable.*

Since

$$H(s) = \frac{N(s)}{D(s)} = \bar{C}\left(sI - \bar{A}\right)^{-1} \bar{B} \tag{3.18}$$

and

$$\bar{C}\left(sI - \bar{A}\right)^{-1} \bar{B} = \frac{\bar{C} Adj \left(sI - \bar{A}\right) \bar{B}}{\det\left(sI - \bar{A}\right)} \tag{3.19}$$

then

$$\bar{C}Adj\left(sI-\bar{A}\right)\bar{B}/\det\left(sI-\bar{A}\right) \tag{3.20}$$

are not irreducible and

$$\deg(D\left(s\right)) < \deg(\det\left(sI-\bar{A}\right)), \tag{3.21}$$

because terms in $\bar{C}Adj\left(sI-\bar{A}\right)\bar{B}$ *were cancelled with terms in* $\det\left(sI-\bar{A}\right)$, *thus there exist constants* $\lambda_i \in \mathbb{C}$ *and an input*

$$U(s) = \frac{\alpha}{(s-\lambda_1)\cdots(s-\lambda_p)} \tag{3.22}$$

with $p < \hat{n} = n + m$, *such that*

$$D(s)\left(s-\lambda_1\right)\cdots\left(s-\lambda_p\right) = \det\left(sI-\bar{A}\right) \tag{3.23}$$

where $\lambda_i, i = 1, \ldots, p$ *are the noncontrollable modes, where the modes of* $\bar{A}$ *are the roots of* $\det\left(sI-\bar{A}\right)$. *And therefore,* $H(s)U(s)$ *is:*

$$H(s)U(s) = \bar{C}\left(sI-\bar{A}\right)^{-1}\bar{B}U(s) = \frac{\alpha N(s)}{\det\left(sI-\bar{A}\right)} \tag{3.24}$$

since $H(s)$ *has no transmission zeros, then* $N(s)$ *is constant, and there exists a constant* β *such that* $\beta = \alpha N(s)$. *And therefore, there exist a constant vector* x_0 *such that the equation*

$$\bar{C}Adj\left(sI-\bar{A}\right)x_0 = \beta, \tag{3.25}$$

holds. Since the left hand side of equation (3.25) is [18]:

$$\bar{C}Adj\left(sI-\bar{A}\right) = \bar{C}\left(S_1 s^{\hat{n}-1} + \cdots + S_{\hat{n}}\right) \tag{3.26}$$

where S_i *can be recursively computed as:*

$$\begin{gathered} S_1 = I, \\ S_k = S_{k-1}\bar{A} + a_{k-1}I \text{ for } k = 2, \ldots, \hat{n} \\ 0 = S_{\hat{n}} + a_{\hat{n}}I \end{gathered} \tag{3.27}$$

with a_i, $i = 1, \ldots, \hat{n}$ being the coefficient of the characteristic polynomial of $\bar{A}$, i.e..

$$\det\left(sI - \bar{A}\right) = s^{\hat{n}} + a_1 s^{\hat{n}-1} + \cdots + a_{\hat{n}} \tag{3.28}$$

and

$$\det\left(sI - \bar{A}\right) = \det\left(sI - A_1\right)\det\left(sI - A_2\right). \tag{3.29}$$

Therefore, the equation (3.25) holds, if the following matrix equation is satisfied

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & 0 \\ a_{\hat{n}-1} & a_{\hat{n}-2} & a_{\hat{n}-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \vdots \\ \bar{C}\bar{A}^{\hat{n}-1} \end{bmatrix} x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -\beta \end{bmatrix} \tag{3.30}$$

since the pair $\left\{\bar{A}, \bar{C}\right\}$ is observable $x_0 \neq 0$ can be computed. Thus, there exist $x_0 \neq 0$ and $u(t) \neq 0$ satisfying equation (3.8), thus the linear systems $\Sigma_1(A_1, B_1, C_1)$ and $\Sigma_2(A_2, B_2, C_2)$ are indistinguishable from each other.

■

3.2 Several marking sequences producing the same output

The *IPN* depicted in Figure 3.2 represents the case when several places produce the same output information, as in the previous case, however, the systems may be indistinguishable, thus previous results cannot be used in order to gain observability in the discrete system. Moreover, the sequence of markings M_1M_3 produce the same input-output information that the sequences of markings M_2M_4, thus it is impossible, using the discrete information, to distinguish between both discrete sequences.

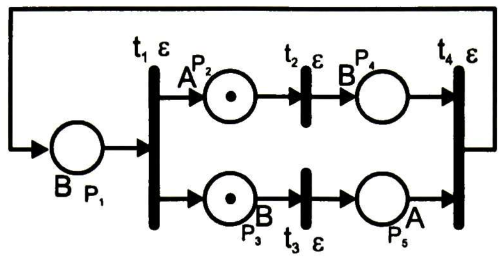

Figure 3.2: *IPN* model of a *DES* where two making sequences produce the same output

However, if some information is obtained from the continuous system commutation, then it is possible to know which discrete sequence was fired. The idea is to use the characterization of indistinguishability subspace for two *LS* and extend this idea to continuous system commutations and use it to distinguish between firing sequences presenting the same discrete output.

Definition 3.7 *The sequence of LS $\Phi(M_i)\cdots\Phi(M_k)$ and $\Phi(M_i')\cdots\Phi(M_k')$ where $M_i\cdots M_k \neq M_i'\cdots M_k'$, are said to be distinguishable from each other, if the information of the input and the output of the LS suffices to determine which sequence of LS evolves.*

It follows from Theorem 2.36 that the sequence of *LS* $\Phi(M_i)\cdots\Phi(M_k)$ and $\Phi(M_i')\cdots\ \Phi(M_k')$ are distinguishable from each other, iff the sequence produce different outputs when the same input is applied.

Notation 3.8 *Hereafter the realization $\{\bar{A}_k, \bar{B}_k, \bar{C}_k\}$ denotes the realization form* *with the matrices*

$$\begin{aligned}\bar{A}_k &= \begin{bmatrix} A_{M_k} & 0 \\ 0 & A_{M'_k} \end{bmatrix} \\ \bar{B}_k &= \begin{bmatrix} B_{M_k} \\ B_{M'_k} \end{bmatrix} \\ \bar{C}_k &= \begin{bmatrix} C_{M_k} & -C_{M'_k} \end{bmatrix}\end{aligned} \tag{3.31}$$

for some markings M_k and M'_k where $\Phi(M_k) = \Sigma_{M_k}(A_{M_k}, B_{M_k}, C_{M_k})$ and $\Phi(M'_k) = \Sigma_{M'_k}\left(A_{M'_k}, B_{M'_k}, C_{M'_k}\right)$.

Lemma 3.9 *Let $\Phi(M_i)\cdots\Phi(M_k)$ and $\Phi(M'_i)\cdots\Phi(M'_k)$ where $M_i\cdots M_k \neq M'_i\cdots M'_k$ be two sequence of LS generating the same discrete input-output information, then $\Phi(M_i)\cdots\ \Phi(M_k)$ and $\Phi(M'_i)\cdots\Phi(M'_k)$ are distinguishable from each other iff the only solution to the equation*

$$\begin{aligned}\bar{C}_i e^{\bar{A}_i(t-t_0)}\left[x(t_0) + \int_{t_0}^{t} e^{-\bar{A}_i(t-t_0-\tau)}\bar{B}_i u(\tau)d\tau\right] &= 0 \text{ for } t \in [t_0, t_1] \\ &\vdots \\ \bar{C}_k e^{\bar{A}_k(t-t_{\hat{k}})}\left[x(t_1) + \int_{t_1}^{t} e^{-\bar{A}_k(t-t_{\hat{k}}-\tau)}\bar{B}_k u(\tau)d\tau\right] &= 0 \text{ for } t \in \left(t_{\hat{k}}, t_{\hat{k}+1}\right)\end{aligned} \tag{3.32}$$

is $x_0 = 0$ and $u(t) = 0$, where $x(t_i) = \bar{\delta}_{i,j} e^{\bar{A}_i(t_1-t_0)} x(t_{i-1})$ with $\bar{\delta}_{i,j} = \begin{bmatrix} \delta_{M_i,M_j} & 0 \\ 0 & \delta_{M'_i,M'_j} \end{bmatrix}$

Proof. *For simplicity the proof is made for sequence of LS of length two.*

(Sufficiency) Assume that there exist two sequences of LS $\Phi(M_i)\Phi(M_j)$ and $\Phi(M'_i)\ \Phi(M'_j)$ with $M_iM_j \neq M'_iM'_j$ such that M_iM_j and $M'_iM'_j$ produce the same discrete input-output sequence w. Now suppose that the corresponding continuous trajectories $\Phi(M_i)\Phi(M_j)$ and $\Phi(M'_i)\Phi(M'_j)$ produce the same continuous output for some input $u(t)$, that is

$$y_1(t) = \begin{cases} C_{M_i} e^{A_{M_i}(t-t_0)} x_1(t_0) + \int_{t_0}^{t} C_{M_i} e^{A_{M_i}(t-\tau)} B_{M_i} u(\tau)d\tau & t \in [t_0, t_1] \\ C_{M_j} e^{A_{M_j}(t-t_1)} x_1(t_1) + \int_{t_1}^{t} C_{M_j} e^{A_{M_j}(t-t_1-\tau)} B_{M_j} u(\tau)d\tau & t \in (t_1, t_2) \end{cases} \tag{3.33}$$

and

$$y_2(t)=\begin{cases} C_{M_i'}e^{A_{M_i'}(t-t_0)}x_2(t_0)+\int_{t_0}^{t}C_{M_i'}e^{A_{M_i'}(t-\tau)}B_{M_i'}u(\tau)d\tau & t\in[t_0,t_1] \\ C_{M_j'}e^{A_{M_j'}(t-t_1)}x_2(t_1)+\int_{t_1}^{t}C_{M_j'}e^{A_{M_j'}(t-t_1-\tau)}B_{M_j'}u(\tau)d\tau & t\in(t_1,t_2) \end{cases} \tag{3.34}$$

with $y_{c_1}=y_{c_2}$*.Now, substracting* $y_{c_2}(t)$ *from* $y_{c_1}(t)$ *and writing the equation in terms of the matrices (3.31):*

$$\begin{gathered} \bar{C}_i e^{\bar{A}_i(t-t_0)}\left[x(t_0)+\int_{t_0}^{t}e^{-\bar{A}_i(t-\tau)}\bar{B}_i u(\tau)d\tau\right]=0 \text{ for } t\in[t_0,t_1] \\ \text{and} \\ \bar{C}_j e^{\bar{A}_j(t-t_1)}\left[x(t_1)+\int_{t_1}^{t}e^{-\bar{A}_j(t-t_1-\tau)}\bar{B}_j u(\tau)d\tau\right]=0 \text{ for } t\in(t_1,t_2) \end{gathered} \tag{3.35}$$

therefore equation (3.35) holds for some $u(t)$ *and* $x(t_0)$, *where* $u(t)\neq 0$ *or* $x(t_0)\neq 0$.

The necessity follows using a similar procedure. ∎

Lemma 3.10 *Let* $\Phi(M_i)\cdots\Phi(M_k)$ *and* $\Phi(M_i')\cdots\Phi(M_k')$ *where* $M_i\cdots M_k\neq M_i'\cdots M_k'$ *be two sequences of LS generating the same discrete input-output information, then* $\Phi(M_i)\cdots\ \Phi(M_k)$ *and* $\Phi(M_i')\cdots\Phi(M_k')$ *are distinguishable from each other if*

$$\begin{gathered} (\bar{\mathcal{W}}_i\cap\bar{\mathcal{W}}_j\bar{\delta}_{i,j}=0)\vee(\bar{\mathcal{W}}_j\cap\bar{\mathcal{W}}_{j+1}\bar{\delta}_{j,j+1}=0). \\ \vee\cdots\vee(\bar{\mathcal{W}}_{k-1}\cap\bar{\mathcal{W}}_k\bar{\delta}_{k-1,k}=0) \end{gathered} \tag{3.36}$$

where $\bar{\mathcal{W}}_i$ *is the indistinguishability subspace of* Σ_{M_i}, $\Sigma_{M_i'}$.

Proof. *For simplicity the proof is made for sequence of LS of length equal to two.*

Assume that $M_iM_j, M_i'M_j'\in S_w$ *where* $M_iM_j\neq M_i'M_j'$ *and* $\bar{\mathcal{W}}_i\cap\bar{\mathcal{W}}_j\bar{\delta}_{i,j}=0$ *then is easy to see that equations*

$$\begin{gathered} \bar{C}_i e^{\bar{A}_i(t-t_0)}\left[x(t_0)+\int_{t_0}^{t}e^{-\bar{A}_i(t-\tau)}\bar{B}_i u(\tau)d\tau\right]=0 \text{ for } t\in[t_0,t_1] \\ \text{and} \\ \bar{C}_j e^{\bar{A}_j(t-t_1)}\left[x(t_1)+\int_{t_1}^{t}e^{-\bar{A}_j(t-t_1-\tau)}\bar{B}_j u(\tau)d\tau\right]=0 \text{ for } t\in(t_1,t_2) \end{gathered} \tag{3.37}$$

do not hold. Notice that, if

$$\bar{C}_i e^{\bar{A}_i(t-t_0)}\left[x(t_0)+\int_{t_0}^{t}e^{-\bar{A}_i\tau}\bar{B}_i u(\tau)d\tau\right]=0 \tag{3.38}$$

for $t \in [t_0, t_1)$, *then the continuous state belongs to the indistinguishability subspace of* $\Sigma_{M_i}(A_{M_i}, B_{M_i}, C_{M_i})$, $\Sigma_{M_i'}(A_{M_i'}, B_{M_i'}, C_{M_i'})$ *i.e.* $x(t) \in \bar{\mathcal{W}}_i$ *for* $t \in [t_0, t_1)$ *because there exists an input* $u(t)$ *that make the LS systems* $\Sigma_{M_i}(A_{M_i}, B_{M_i}, C_{M_i})$ *to be indistinguishable from* $\Sigma_{M_i'}(A_{M_i'}, B_{M_i'}, C_{M_i'})$ *for some initial condition* $x(t_0)$. *In a similar way* $x(t) \in \bar{\mathcal{W}}_j$ *for* $t \in [t_1, t_2)$. *Since* $x(t_1) = \bar{\delta}_{i,j} e^{\bar{A}_1(t_1 - t_0)} x(t_0)$ *then if equation (3.37) holds for* $x_0 \neq 0$ *or* $u(t) \neq 0$ *there exists a vector* $z \neq 0 \in \bar{\mathcal{W}}_i \cap \bar{\mathcal{W}}_j \bar{\delta}_{i,j}$. ■

Chapter 4

Joint Observability of SLS

In this chapter, it is presented the case where the discrete system provides information to the continuous part, in such a way that the value of the continuous variables can be computed, even when every LS of the family is unobservable. This case occurs when the IPN marking sequence can be computed using only the input-output IPN information. Using the IPN marking sequence, the linear system sequence generated by this marking sequence is computed. This chapter also presents that, if the intersection of the unobservable subspaces of the sequence of linear systems is null, then the continuous state can be computed.

4.1 Recovering the initial state of the SLS

Definition 4.1 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS and $\sigma_f = \mathbf{t}_a \mathbf{t}_b ... \mathbf{t}_g$ a sequence in $\pounds(Q, M_0)$ such that $M_0 \xrightarrow{\mathbf{t}_a} M_1 \xrightarrow{\mathbf{t}_b} ... M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$. For each sequence σ_f there exists a linear system sequence $\Phi(M_0)\Phi(M_1)...\Phi(M_{k-1})\Phi(M_k)$ with $k = |\sigma_f|$ Then the observability* *matrix of a sequence associated with σ_f is:*

$$\mathcal{O}_{\sigma_f} = \begin{bmatrix} \mathcal{O}_{M_0} \\ \mathcal{O}_{M_1}\delta_{M_0,M_1}e^{A_{M_0}t_1-t_0} \\ \vdots \\ \mathcal{O}_{M_k}\delta_{M_{k-1},M_k}e^{A_{M_{k-1}}t_k-t_{k-1}}\cdots\delta_{M_0,M_1}e^{A_{M_0}t_1-t_0} \end{bmatrix} \tag{4.1}$$

where $\mathcal{O}_{M_0},...,\mathcal{O}_{M_k}$ are the observability matrices of $\Phi(M_0),...,\Phi(M_k)$ respectively, i.e.

$$\mathcal{O}_{M_i} = \begin{bmatrix} C_{M_i} \\ \vdots \\ C_{M_i}A_{M_i}^{n-1} \end{bmatrix} \tag{4.2}$$

where t_i is the ith switching time.

Theorem 4.2 *Let $\langle\mathcal{F},(Q,M_0)\rangle$ be a SLS, and let $M_0M_1...M_{k-1}M_k$ be a marking sequence evolving in the SLS when the sequence of fired transitions $\sigma = \mathbf{t}_a\mathbf{t}_b...\mathbf{t}_g$ is fired. Assume that the marking sequence can be computed using the input-output information of the IPN, then the continuos initial state can be recovered iff the observability matrix $\mathcal{O}_\sigma$ of the sequence σ has full rank.*

Proof. *Since the marking evolution $M_0 \xrightarrow{\mathbf{t}_a} M_1 \xrightarrow{\mathbf{t}_b} ...M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$ can be computed then the sequence of linear systems $\Phi(M_0)\Phi(M_1)...\Phi(M_k)$ can be also be computed.*

(Sufficiency) Assume that the continuous state cannot be recovered then there exist two sequences generated by the same input producing the same output., i.e.:

$$y_{c_1}=\begin{cases} C_{M_0}(e^{A_{M_0}t-t_0}x_1^0(t_0)+\int_{t_0}^{t}e^{A_{M_0}(t-\tau)}B_{M_0}u(\tau)d\tau) & t\in[t_0,t_1] \\ \vdots \\ C_{M_k}(e^{A_{M_k}t-t_k}x_1^k(t_k)+\int_{t_k}^{t}e^{A_{M_k}(t-t_k-\tau)}B_{M_k}u(\tau)d\tau) & t\in(t_k,t_{k+1}) \end{cases} \tag{4.3}$$

and

$$y_{c_2} = \begin{cases} C_{M_0}(e^{A_{M_0}t-t_0}x_2^0(t_0) + \int_0^t e^{A_{M_0}(t-t_0-\tau)}B_{M_0}u(\tau)d\tau) \ t \in [t_0, t_1] \\ \vdots \\ C_{M_k}(e^{A_{M_k}t-t_k}x_2^k(t_k) + \int_{t_k}^t e^{A_{M_k}(t-t_k-\tau)}B_{M_k}u(\tau)d\tau) \ t \in (t_k, t_{k+1}) \end{cases} \tag{4.4}$$

with $y_{c_1} = y_{c_2}$.

Now, substracting $y_{c_2}(t)$ *from* $y_{c_1}(t)$*:*

$$y_{c_1} - y_{c_2} = 0 = \begin{cases} C_{M_0}e^{A_{M_0}t-t_0}\left(x_1^0(t_0) - x_2^0(t_0)\right) \quad t \in [t_0, t_1] \\ \vdots \\ C_{M_k}e^{A_{M_k}t-t_k}\left(x_1^k(t_k) - x_2^k(t_k)\right) \quad t \in (t_k, t_{k+1}) \end{cases} \tag{4.5}$$

Equation (4.5) implies that $(x_1^i(t_i) - x_2^i(t_i)$ *belongs to the unobservable subspace of* (A_{M_i}, C_{M_i}) *[19]. Now, since:*

$$\begin{aligned} x_k^i(t_i) &= \delta_{M_{i-1},M_i}x_k^{i-1}(t_{i-1}) \\ &= \delta_{M_{i-1},M_i}e^{A_{M_{i-1}}t_k-t_{k-1}}\cdots\delta_{M_0,M_1}e^{A_{M_0}t_1-t_0}x_k^0(t_0) \end{aligned} \tag{4.6}$$

then

$$0 = \begin{cases} C_{M_0}e^{A_{M_0}t-t_0}(x_1^0(t_0) - x_2^0(t_0)) \qquad t \in [t_0, t_1] \\ \vdots \\ C_{M_k}e^{A_{M_k}t-t_k}\delta_{M_{k-1},M_k}e^{A_{M_{k-1}}t_k-t_{k-1}}\cdots \\ \cdots\delta_{M_0,M_1}e^{A_{M_0}t_1-t_0}(x_1^0(t_0) - x_2^0(t_0)) \quad t \in (t_k, t_{k+1}) \end{cases} \tag{4.7}$$

Using Taylor series and Cayley-Hamilton theorem, equation (4.7) can be rewritten as:

$$0 = \begin{cases} C_{M_0}\left(I + A_{M_0}(t-t_0) + \cdots\frac{1}{n-1!}A_{M_0}^{n-1}(t-t_0)^{n-1} + \cdots\right) \\ (x_1^0(t_0) - x_2^0(t_0)), \qquad t \in [t_0, t_1] \\ \vdots \\ C_{M_k}\left(I + A_{M_k}(t-t_k) + \cdots\frac{1}{n-1!}A_{M_k}^{n-1}(t-t_k)^{n-1} + \cdots\right) \\ \delta_{k,k-1}e^{A_{M_{k-1}}t_k-t_{k-1}}\cdots\delta_{M_0,M_1}e^{A_{M_0}t_1-t_0} \\ (x_1^0(t_0) - x_2^0(t_0)), t \in (t_k, t_{k+1}). \end{cases} \tag{4.8}$$

from equation (4.8) the following matrix equation can be established:

$$\begin{bmatrix} \mathcal{O}_{M_0} \\ \mathcal{O}_{M_1}\delta_{M_0,M_1}e^{A_{M_0}t_1-t_0} \\ \vdots \\ \mathcal{O}_{M_k}\delta_{M_{k-1},M_k}e^{A_{M_{k-1}}t_k-t_{k-1}}\dots\delta_{M_0,M_1}e^{A_{M_0}t_1-t_0} \end{bmatrix}\left(x_1^0\left(t_0\right)-x_2^0\left(t_0\right)\right)=0 \tag{4.9}$$

i.e.:

$$\mathcal{O}_\sigma\left(x_1^0\left(t_0\right)-x_2^0\left(t_0\right)\right)=0 \tag{4.10}$$

since $x_1^0\left(t_0\right)-x_2^0\left(t_0\right)\neq 0$ *then the rank of* $\mathcal{O}_\sigma$ *is not full.*

(Necessity) Following a similar procedure. It is easy to show that if the continuous state can be computed then the rank of the observability matrix O_σ *is full.* ■

A sufficient condition for the observability in terms of the unobservable subspace of every linear system can be stated as follows:

Corollary 4.3 *Let* $\langle\mathcal{F},(Q,M_0)\rangle$ *be a SLS, and let* $M_0M_1...M_{k-1}M_k$ *be a marking sequence evolving in the SLS when the sequence of transitions* $\sigma=\mathbf{t}_a\mathbf{t}_b...\mathbf{t}_g$ *is fired. Assume that the marking sequence can be computed using the input-output information of the IPN, then the continuos initial state can be recovered if it holds that*

$$\begin{gathered}\left(\mathcal{N}_{M_0}\cap\mathcal{N}_{M_1}\delta_{M_0,M_1}=0\right)\vee\left(\mathcal{N}_{M_1}\cap\mathcal{N}_{M_2}\delta_{M_1,M_2}=0\right)\\ \vee\cdots\vee\left(\mathcal{N}_{M_{k-1}}\cap\mathcal{N}_{M_k}\delta_{M_{k-1},M_k}=0\right)\end{gathered} \tag{4.11}$$

where $\mathcal{N}_{M_j}=\bigcap_{i=1}^{n}\ker C_{M_j}A_{M_j}^{i-1}$ *is the unobservable subspace of* $\left(C_{M_j},A_{M_j}\right)$

Proof. *Assume that the continuous state cannot be recovered, then acording to Theorem 4.2 the rank of the observability matrix* O_{σ_f} *is not full. And therefore there exist vectors*

$$z_i = \left(x_1^i(t_i)-x_2^i(t_i)\right)\neq 0, \tag{4.12}$$

$$z_{i+1} = \left(x_1^{i+1}(t_{i+1})-x_2^{i+1}(t_{i+1})\right)\neq 0 \tag{4.13}$$

such that

$$\begin{gathered} z_i \in \mathcal{N}_{M_i},\ z_{i+1} \in \mathcal{N}_{M_{i+1}} \\ \text{and} \\ z_{i+1} = \delta_{M_i,M_{i+1}} e^{A_{M_i} t} z_i. \end{gathered} \tag{4.14}$$

Furthermore, the vector $v = e^{A_{M_i}t} z_i \in \mathcal{N}_{M_i}$ *(since* $e^{A_{M_i}t}\mathcal{N}_{M_i} \subseteq \mathcal{N}_{M_i}$ *i.e. the unobservable subspace of* (C_{M_i}, A_{M_i}) *is* $e^{A_{M_i}t}$ *− invariant),* $\delta_{M_i,M_{i+1}} v = z_{i+1} \in \mathcal{N}_{M_{i+1}}$*. Thus* $v \in \mathcal{N}_{M_i} \cap \mathcal{N}_{M_{i+1}} \delta_{M_i,M_{i+1}}$*, hence:*

$$\begin{gathered} (\mathcal{N}_{M_0} \cap \mathcal{N}_{M_1} \delta_{M_0,M_1} \neq 0) \wedge (\mathcal{N}_{M_1} \cap \mathcal{N}_{M_2} \delta_{M_1,M_2} \neq 0) \\ \wedge \cdots \wedge \left(\mathcal{N}_{M_{k-1}} \cap \mathcal{N}_{M_k} \delta_{M_{k-1},M_k} \neq 0\right) \wedge \cdots \end{gathered} \tag{4.15}$$

and thus condition (4.11) is a sufficient condition for the observability of the continuous system after the sequence σ_f *has been fired.* ■

4.2 Joint Observability of SLS

In the joint observability of SLS, the aim is to recover the continuous and discrete states of the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$

Theorem 4.4 *The system* $\langle \mathcal{F}, (Q, M_0) \rangle$ *is observable if*

1. IPN $(Q^\blacklozenge, M_0)$ is observable, and

2. There exists a finite integer k such that every sequence $\sigma_w = \mathbf{t}_a \mathbf{t}_b \cdots \mathbf{t}_g \in \mathcal{L}(Q, M_0)$ of length k such that for $M_i \xrightarrow{\mathbf{t}_a} M_j \xrightarrow{\mathbf{t}_b} \cdots M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$ it holds that:

$$\begin{gathered} \left(\mathcal{N}_{M_i} \cap \mathcal{N}_{M_j} \delta_{M_i,M_j} = 0\right) \vee \left(\mathcal{N}_{M_j} \cap \mathcal{N}_{M_{j+1}} \delta_{M_j,M_{j+1}} = 0\right) \\ \vee \cdots \vee \left(\mathcal{N}_{M_{k-1}} \cap \mathcal{N}_{M_k} \delta_{M_{k-1},M_k} = 0\right) \end{gathered} \tag{4.16}$$

Proof. As the Petri net is observable, when information from the continuous systems is added, the discrete systems sequence can be determined after a finite number of commutations. ■

Notice that condition 2 of Theorem 4.4 is the case when the discrete part provides information to the continuous systems, if this condition holds then it is always possible to recover the continuous and the discrete states of the SLS.

Based on the previous joint observability characterization some sufficient conditions for the observability of particular cases in SLS will be presented. Some of these results can be tested from the structure of the SLS

As stated before, the concept of system distinguishability can be useful in SLS in order to identify the marking of the IPN. Next definition presents how to relabel an IPN using the continuous system information and then a useful proposition is presented.

Definition 4.5 *Let $HS = \langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS and let $\Phi(M_i) = \Sigma_{M_i}(A_{M_i}, B_{M_i}, C_{M_i})$ and $\Phi(M_j) = \Sigma_{M_j}\left(A_{M_j}, B_{M_j}, C_{M_j}\right)$ be two linear systems, where $\Phi(M_i)$, $\Phi(M_j) \in \mathcal{F}$ Then the IPN of HS can be relabeled adding new symbols if the linear systems $\Sigma_{M_i}(A_{M_i}, B_{M_i}, C_{M_i})$ and $\Sigma_{M_j}\left(A_{M_j}, B_{M_j}, C_{M_j}\right)$ are distinguishable from each other. Thus the marking M_i must be labeled different from the marking M_j. The $HS^\blacklozenge = \left\langle \mathcal{F}, (Q^\blacklozenge, M_0) \right\rangle$ denotes the obtained SLS from relabeling $HS = \langle \mathcal{F}, (Q, M_0) \rangle$.*

The obtained SLS after relabeling is the same than the original one, but the information that the continuos systems gives to the discrete part is added explicitly as virtual sensors (labels) attached to IPN marking.

Proposition 4.6 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS. $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable before the first commutation if and only if each LS is observable and the set of labels of every pair of markings of $(Q^\blacklozenge, M_0)$ are different from each other.*

Proof. (Sufficiency) The fact that every pair of markings of $(Q^{\blacklozenge}, M_0)$ are different from each other implies that:

1. If $\varphi(M_i) = \varphi(M_j)$, for some $M_i, M_j \in R(Q, M_0)$ then the linear systems $\Phi(M_i)$ and $\Phi(M_j)$ are distinguishable from each other, thus the initial marking can be computed using the continuous information. Now, as the realization of the extended system is minimal, then the realization of individual systems is also minimal so they are observable. Hence the continuous initial condition can also be computed and the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable.

2. If $\varphi(M_i) \neq \varphi(M_j)$, for some $M_i, M_j \in R(Q, M_0)$ then the initial marking can be computed from the discrete output. Since every LS is observable then because the linear system is known the initial condition of the continuous system can also be computed and the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable.

(Necessity) The proof is divided in two parts

1. If not every LS is observable then the initial condition of the continuous system cannot be computed, thus the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is unobservable

2. If there exists a pair of markings of $(Q^{\blacklozenge}, M_0)$ that are not different from each other, then there exist $M_i, M_j \in R(Q, M_0)$ such that $\varphi(M_i) = \varphi(M_j)$ and for some input $u(t)$ both linear systems produce the same output, that is $\Phi(M_i)$ and $\Phi(M_j)$ are indistinguishable from each other, therefore according to Theorem 2.36 the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is unobservable.

■

Proposition 4.7 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS that is observable in $t \in [t_0, t_1)$ (where*

$t_1$is the time of the first commutation) then the initial condition of the SLS is:

$$M_0 = \left\{ M_k : y(t) - \int_{t_0}^{t} C_{M_i} e^{A_{M_i}(t-\tau)} B_{M_i} u(\tau) d\tau \in \operatorname{Im}\left(C_{M_k} e^{A_{M_k}(t-t_0)} \right) \; t \in [t_0, t_1) \right\} \tag{4.17}$$

and

$$x_0 = \left(C_{M_k} e^{A_{M_k}(t-t_0)} \right)^{-1} \left(y(t) - \int_{t_0}^{t} C_{M_i} e^{A_{M_i}(t-\tau)} B_{M_i} u(\tau) d\tau \right) \tag{4.18}$$

Corollary 4.8 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS where $(Q^{\blacklozenge}, M_0)$ is observable then the system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if and only if there exists an integer $k < \infty$ such that the observability matrix $\mathcal{O}_\sigma$ of every sequence $|\sigma| \geq k$ has full rank.*

A sufficient condition for the observability in terms of the unobservable subspace of every linear system can be stated as follows:

Corollary 4.9 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS where $(Q^{\blacklozenge}, M_0)$ is observable then the system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if there exists an integer k such that every sequence $|\sigma_f| \geq k, \sigma_f = \mathbf{t}_a \mathbf{t}_b ... \mathbf{t}_g \in \pounds(Q, M_0)$, such that $M_0 \xrightarrow{\mathbf{t}_a} M_1 \xrightarrow{\mathbf{t}_b} ... M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$ it holds that:*

$$\begin{aligned} (\mathcal{N}_{M_0} \cap \mathcal{N}_{M_1} \delta_{M_0, M_1} = 0) \vee (\mathcal{N}_{M_1} \cap \mathcal{N}_{M_2} \delta_{M_1, M_2} = 0) \\ \vee \cdots \vee \left(\mathcal{N}_{M_{k-1}} \cap \mathcal{N}_{M_k} \delta_{M_{k-1}, M_k} = 0 \right) \end{aligned} \tag{4.19}$$

where $\mathcal{N}_{M_j} = \bigcap_{i=1}^{n} \ker C_{M_j} A_{M_j}^{i-1}$ is the unobservable subspace of $\left(C_{M_j}, A_{M_j} \right)$

Previous results and concepts will be illustrated next, where even thought nor the linear systems neither the DES are observable, the SLS is observable.

Example 4.10 *Let $\mathcal{H} = \langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS represented as in Definition 2.32, where the IPN model is shown in Fig.4.1, the functions Φ, Π and δ_{M_i, M_j} are given below.*

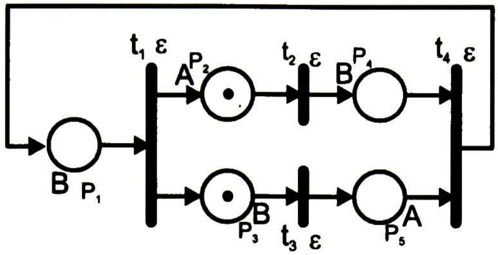

Figure 4.1: IPN model of the DES

The function Φ mapping from IPN markings to the linear systems is

$$\Phi(M_k) = \Sigma_{M_k} : \begin{cases} \dot{x}(\tau) = A_{M_k}x(\tau) + B_{M_k}u(\tau) \\ y = C_{M_k}x(\tau) \end{cases} \tag{4.20}$$

where matrices A_{M_k}, B_{M_k} and C_{M_k} for each marking M_k, are described next

k	$\Phi(M_k) = \Sigma_{M_k}$	A_{M_k}	B_{M_k}	C_{M_k}
0	$\Phi\left(\begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}\right)$	$\begin{bmatrix}-2 & 2 & 2\\0 & -1 & 0\\0 & 1 & -1\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\2\\0\end{bmatrix}^T$
1	$\Phi\left(\begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}\right)$	$\begin{bmatrix}2 & 0 & 0\\2 & 1 & 0\\0 & 1 & -1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}^T$

k	$\Phi(M_k) = \Sigma_{M_k}$	A_{M_k}	B_{M_k}	C_{M_k}
2	$\Phi\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$	$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 0 \\ 2 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$
3	$\Phi\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right)$	$\begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T$
4	$\Phi\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$	$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$

For simplicity the function δ_{M_i,M_j} is the identity matrix.

Since every column in

$$\varphi C = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \tag{4.21}$$

is different from each other, then the IPN is event detectable, and the IPN (Q, M_0) is observable.

Now, notice that every discrete sequence σ starting from any marking of $\mathcal{M}_0$ with $|\sigma| \geq 4$, can be written as $\sigma = xwy$ where $w = \mathbf{t}_2\mathbf{t}_3$ or $w = \mathbf{t}_3\mathbf{t}_2$, the linear systems sequence associated to each w are $\Phi(M_0)\Phi(M_1)\Phi(M_3)$ and $\Phi(M_0)\Phi(M_2)\Phi(M_3)$ respectively.

Notice that, every linear system is unobservable, where the observability matrices are

$$\mathcal{O}_{M_0} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad \mathcal{O}_{M_1} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 6 & 1 & 0 \end{bmatrix} \quad \mathcal{O}_{M_2} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -5 & 0 \end{bmatrix}$$
$$\mathcal{O}_{M_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix} \quad \mathcal{O}_{M_4} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \tag{4.22}$$

However, $rank \begin{bmatrix} \mathcal{O}_{M_1}^T & \mathcal{O}_{M_3}^T \end{bmatrix}^T = 3$ and $rank \begin{bmatrix} \mathcal{O}_{M_2}^T & \mathcal{O}_{M_3}^T \end{bmatrix}^T = 3$, that is $\mathcal{N}_{M_1} \cap \mathcal{N}_{M_3} = 0$ and $\mathcal{N}_{M_2} \cap \mathcal{N}_{M_3} = 0$, thus according to Corollary 4.9, the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable.

Corollary 4.11 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS where $(Q^\blacklozenge, M_0)$ is observable and a binary state machine Petri net then the system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if for each sequence σ generating a minimal $T-semiflow$ of $(Q^\blacklozenge, M_0)$, O_σ has full rank.*

Theorem 4.12 *The system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable in $t \in [t_1, t_2)$ (where t_i is the ith commutation time) if Petri net $(Q^\blacklozenge, M_0)$ is event detectable, and for all $M_i, M_j \in R(Q, M_0)$ such that $M_i \xrightarrow{\mathbf{t}_a} M_j$ for some $\mathbf{t}_a \in Q$,*

$$\mathcal{N}_{M_i} \cap \mathcal{N}_{M_j} \delta_{M_i, M_j} = 0 \tag{4.23}$$

holds.

Chapter 5

Observer design

The design of asymptotic observers for SLS is presented in this chapter. The main idea is to design the observers for the discrete and continuous systems using the asymptotic observers presented in [29] and [6], respectively, taking into account the information provided by the geometric characterization.

The discrete observer is adopted from [29] and [27]. It is depicted in Figure 5.1. In this figure, matrices C^D and C^ϵ represent, respectively, the incidence matrix columns of the manipulated and non manipulated transitions.

Definition 5.1 *Let $\Im$ be an observable SLS described by equation (2.22). Then an observer for the discrete part of the SLS is defined as follows.*

- *Observer structure:*

$$\hat{M}_{k+1} = \hat{M}_k + C\gamma_k + \begin{bmatrix} Id & -Id \end{bmatrix} \begin{bmatrix} \beta_k \\ \delta_k \end{bmatrix} \tag{4.1}$$

- *Initial marking $\hat{M}_0$ is any marking $M \in R(Q, M_0)$ such that $\varphi M = \varphi \hat{M}_0$ where $\hat{M}_0$ is the initial marking of (Q, M_0).*

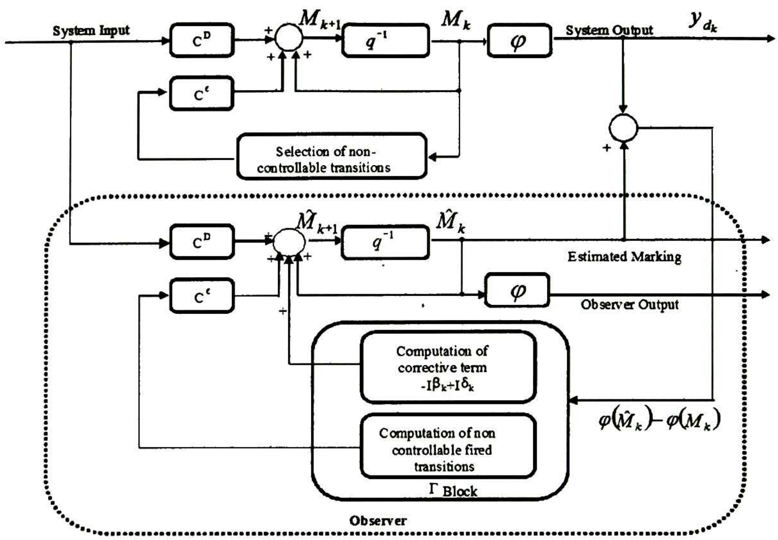

Figure 5.1: Discrete system and observer architecture.

- *Firing rule: when t_j is fired in the SLS then:*

$$\gamma_k = \begin{cases} \overrightarrow{t_j} & \text{if } t_j \text{ is enabled at } \hat{M}_k \\ 0 & \text{otherwise} \end{cases}, \beta_k = \begin{bmatrix} \varpi_1 \\ \vdots \\ \varpi_n \end{bmatrix}, \delta_k = \begin{bmatrix} \upsilon_1 \\ \vdots \\ \upsilon_n \end{bmatrix} \quad \textit{where} \tag{4.2}$$

$$\varpi_i = \begin{cases} 1 & \text{if } (\gamma_k = \overrightarrow{t_j}, \text{ and} \\ & \hat{M}_k(p_i) + C(p_i, \cdot)\gamma_k > 1), \\ & \text{or } (\gamma_k \neq \overrightarrow{t_j}, p_i \in^{\bullet} (t_j), \\ & \text{and } \hat{M}_k(p_i) > 0) \\ 0 & \text{otherwise} \end{cases}, \upsilon_i = \begin{cases} 1 & \text{if } (\gamma_k = \overrightarrow{t_j}, \text{ and} \\ & \hat{M}_k(p_i) + C(p_i, \cdot)\gamma_k < 0), \\ & \text{or } (\gamma_k \neq \overrightarrow{t_j}, p_i \in (t_j)^{\bullet}, \\ & \text{and } \hat{M}_k(p_i) > 0) \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

For the continuous part, the observer structure shown in Figure 5.2 is proposed, it is an adapted architecture from the linear systems state estimator of [6].

Definition 5.2 *Let $\Im$ be a SLS described by equation (2.22), then the continuous*

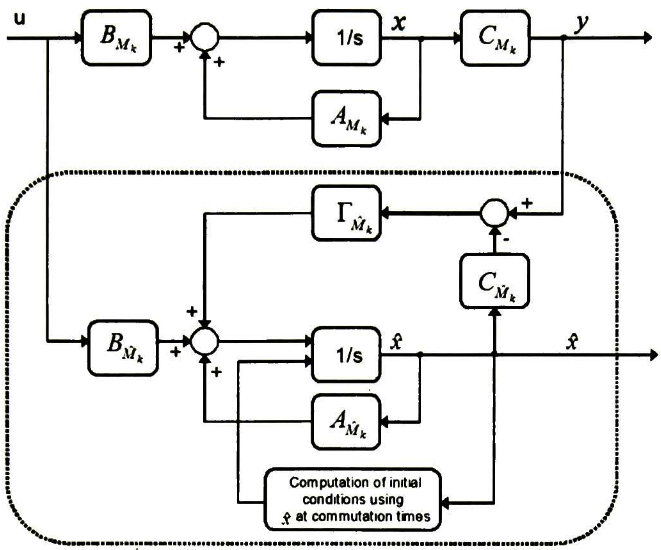

Figure 5.2: Continuous system and observer architecture.

observer structure is the set of observers

$$\begin{aligned}\dot{\hat{x}}(\tau) = \ & (A_{\hat{M}_k} - \Gamma_{\hat{M}_k} C_{\hat{M}_k})\hat{x}(\tau) + B_{\hat{M}_k} u(\tau) \\ & + \Gamma_{\hat{M}_k}(C_{M_k} x(\tau) - C_{\hat{M}_k}\hat{x}(\tau))\end{aligned} \tag{4.4}$$

The parameters Γ_{M_k} *must be computed properly to ensure that* $\lim_{t \longrightarrow \infty} \hat{x} = x$.

Thus the observer is composed by the discrete observer and a family of continuos observers, one for each LS in SLS. The continuous observers are order reduced observers. Thus these observers can only estimate the observable states. As the IPN is event detectable and a binary state machine then after the first transition firing the current marking M_k of the DES is computed by the observer. The observer knows that the LS $\Phi(M_k)$ is currently evolving, thus the reduced order observer for this LS must be also evolving. Then, a gain may be computed to estimate the observable states actual values. A sequence $\sigma_k = t_a t_b ... t_g$. should be executed such that in fulfills conditions of Theorem 4.2. Then there exists a linear system sequence $\Phi(M_k)\Phi(M_k')...\Phi(M_k^{|\sigma_k|'})\Phi(M_k)$,where $M_k \xrightarrow{t_a} M_k' \xrightarrow{t_b} ... \longrightarrow M_k^{|\sigma_k|'} \xrightarrow{t_g} M_k$. The next equation can be stated.

$$\begin{bmatrix} [x] \\ [x'] \\ \vdots \\ \left[x^{|\sigma_k|'}\right] \end{bmatrix} = \begin{bmatrix} O_{M_k} \\ O_{M_{k'}} \\ \vdots \\ O_{M_k^{|\sigma_k|'}} \end{bmatrix} [x_0] \tag{4.5}$$

Where $x^{(b)}$ is the estimated state of system $M_k^{(b)}$. This estimated state is composed by observable and non observable states. However, since this matrix has rank n then it is solvable for x_0. Moreover, once that x_0 is known then vector $x^{|\sigma_f|'}$ can be computed. This value can be given as the initial value of the next observer. From this moment on, the observer has the actual SLS state and the error will be zero. This procedure will be applied every switching time, then the observation error will be zero, even over commutations.

Example 5.3 *Let the continuous part of a SLS be defined by the linear systems family*

$$\Sigma_1 \left\{ A_1 = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \frac{4}{3} & -2 & 1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2 \\ 2/3 \end{bmatrix} \right.$$

$$\Sigma_2 \left\{ A_2 = \begin{bmatrix} 0 & 1 & -1 & \frac{1}{2} \\ 0 & 3 & -1 & \frac{3}{10} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_2 = B_1, \right.$$

with $u(t) = 3\sin(2t)$, *initial conditions* $x_0 = \begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix}^T$ *and initial LS* Σ_1.

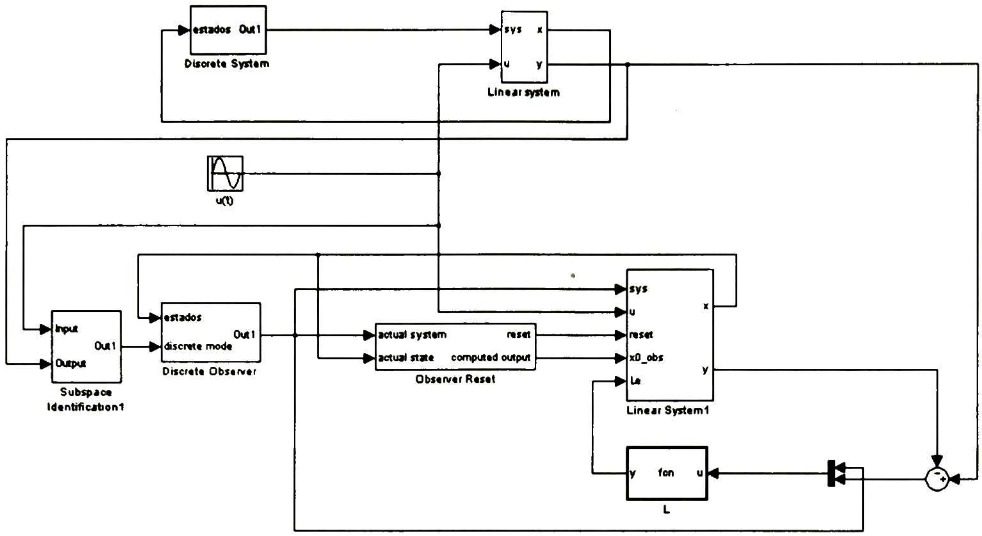

Figure 5.3: Hybrid Observer with Identification

The discrete part of the SLS is defined with the IPN of Figure 5.4, where LS Σ_1 *will evolve when* $M_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ *and LS* Σ_2 *will evolve when* $M_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ *Predicates* $H(t_s)$ *define hyperplanes such that* $H(t_s)x = \varrho_s$.

$$\begin{aligned} H(t_1) &= [\ 0\ \ 0\ \ 1\ \ 0\ ] \text{ with } \varrho_1 = 6 \\ H(t_2) &= [\ 0\ \ 0\ \ 1\ \ 0\ ] \text{ with } \varrho_2 = 6 \end{aligned} \tag{4.6}$$

The predicates define commutation hyperplanes, when the continuous system trajectory hits $H(t_s)$, *then it enables transition* t_s. *Transition* t_s *fires if it is enabled and a new marking in the IPN is obtained. The linear system that corresponds to the new marking activates. For this example, initial conditions for the later system are the final conditions of previous linear system as* $\delta_{M_0,M_1} = \delta_{M_1,M_0} = I$.

First, the mode observer has to identify the LS that is evolving, for this, as the

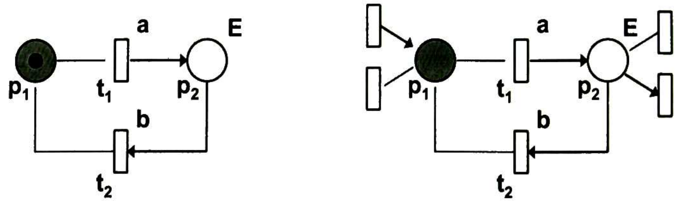

Figure 5.4: Left: IPN model of the discrete event part of SLS. Right: IPN model observer.

IPN is event detectable, the observer shown in Figure 5.4 will converge in at most one commutation. After that the IPN observer state $\hat{M}_k$ is equal to marking M_k of the IPN discrete part model of the SLS.

For the example, the IPN Observer will converge at most at $t_1 = 0.96seg$ which is the first commutation time.

For the continuous part of the SLS, one should make partial order observers for the LS that is evolving. These observers will estimate the state for the LS that the IPN observer says is running. This is a two stage technique, since the IPN observer has to converge for the continuos observer to work properly, and the partial order observers need to know which LS is evolving.

Once the mode observer has converged to the discrete part of the system state. partial observers will give information at each commutation for the state variables that are observable for each LS. One can obtain from the vector state at commutations the values of state variables that do converge as the LS are partially observable.

For the example, at t_1 the vector state is completely unknown, as the discrete observer has just converged and previous states of the IPN observer may point the LS partial observers incorrectly. From time t_1 the continuos observers will converge for the observable variables of current LS. Notice that for the example the observable variables of LS Σ_1 are x_1, x_2 and x_3 and for LS Σ_1 observable variables are x_3 and

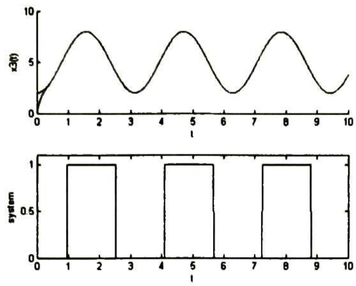

Figure 5.5: State x_3

x_4.

For this example, state variable x_3 is always observable and is used to periodically commutate the SLS, as it can be seen in Figure 5.5. The LS are represented by numbers 0 and 1 for LS Σ_1 and Σ_2 respectively. The actual evolving LS is shown in the same figure.

Using the general solution of linear systems at time $t_2 = 2.53seg$.

$$x(t_2) = e^{A_2(t_2-t_1)}x(t_1) + \int_{t_1}^{t_2} e^{A_2(t_2-\tau)}Bu(\tau)d\tau \tag{4.7}$$

which can be stated as:

$$\begin{bmatrix} x_1^2 \\ x_2^2 \\ 3.9780 \\ -29.9964 \end{bmatrix} = e^{A_2(t_2-t_1)} \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \end{bmatrix} + \begin{bmatrix} -3.7567 \\ -5.8654 \\ -2.0484 \\ -7.0693 \end{bmatrix} \tag{4.8}$$

because partial observers give information on observable states of $x(t_2)$.

At commutation in time $t_3 = 4.1seg$ the next equation can be stated:

$$x(t_3) = e^{A_1(t_3-t_2)}x(t_2) + \int_{t_2}^{t_3} e^{A_1(t_3-\tau)}Bu(\tau)d\tau \tag{4.9}$$

which can be written as:

$$\begin{bmatrix} 76.6307 \\ 81.6285 \\ 6.0175 \\ x_4^3 \end{bmatrix} = e^{A_1(t_3-t_2)} \begin{bmatrix} x_1^2 \\ x_2^2 \\ 3.9780 \\ -29.9964 \end{bmatrix} + \begin{bmatrix} 2.8536 \\ 3.2449 \\ 2.0394 \\ 11.1746 \end{bmatrix} \tag{4.10}$$

State vectors $x(t_1)$, $x(t_2)$ and $x(t_3)$ can be calculated at time t_3 using (4.8) and (4.10) as:

$$x(t_1) = \begin{bmatrix} 1.5741 \\ 2.0520 \\ 6.0264 \\ 4.7755 \end{bmatrix}, x(t_2) = \begin{bmatrix} -2.6842 \\ 19.4578 \\ 3.9780 \\ -29.9964 \end{bmatrix} \text{ and } x(t_3) = \begin{bmatrix} 76.6307 \\ 81.6285 \\ 6.0175 \\ 93.7014 \end{bmatrix} \tag{4.11}$$

At time t_3 the state vector is completely known and is used to update the LS observer.

Using only the partial observers may be enough in some cases, as energy tends to disipate on stable variables on commutations.

It was presented a case where almost every state variable is unstable. Here, the unobservable variables at each commutation is gaining energy as it can be seen in Figure 5.6. Even when the variable error tends to zero when it is observable, once it becomes unobservable on next commutation it gains energy from the previous unobservable variables.

Using the computed state vector $x(t_3)$ on the observer, the error goes to zero, see Figure 5.7, as the LS observer state is the same as the LS system state.

This procedure is repeated at every commutation to avoid numerical errors.

Example 5.4 *Let* $\langle \mathcal{F}, (Q, M_0) \rangle$ *be a SLS represented as in Definition 2.32, where the IPN model is shown in Fig.4.1, the functions* Φ, Π *and* δ_{M_i,M_j} *are given below.*

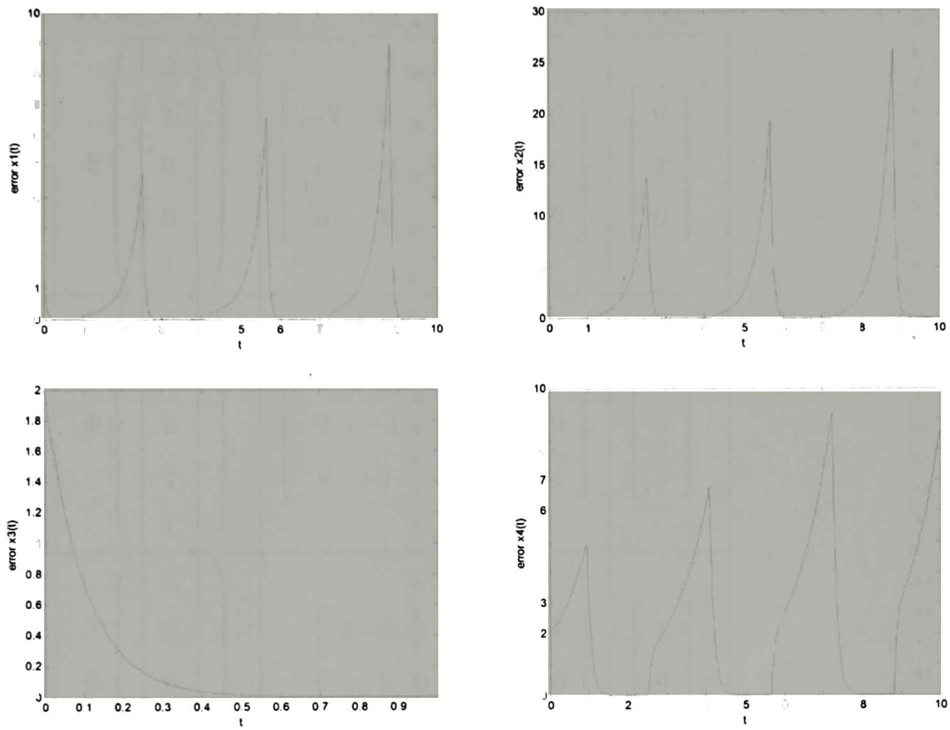

Figure 5.6: Observation error in each state.

The function Φ *mapping from* IPN *markings to the linear systems is:*

$$\Phi\left(M_k\right) = \Sigma_{M_k} \begin{cases} \dot{x}(\tau) = A_{M_k}x(\tau) + B_{M_k}u(\tau) \\ y = C_{M_k}x(\tau) \end{cases}$$

where matrices A_{M_k}, B_{M_k} *and* C_{M_k} *for each marking* M_k, *are described next:*

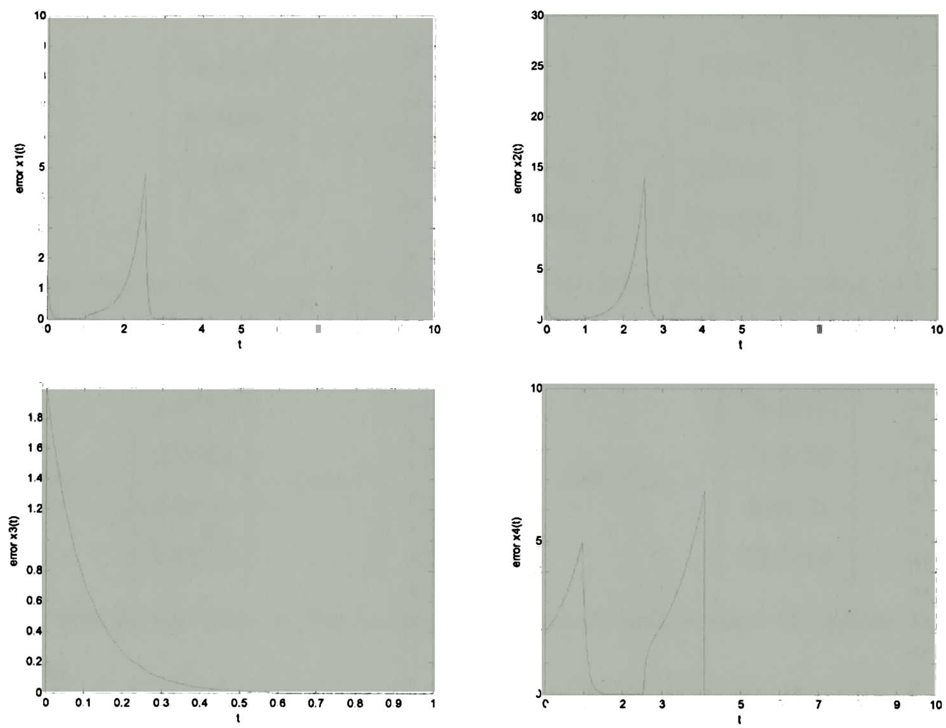

Figure 5.7: Observation error in each state with observer correction.

k	$\Phi\left(M_k\right) = \Sigma_{M_k}$	A_{M_k}	B_{M_k}	C_{M_k}
0	$\Phi\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$	$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 0 \\ 2 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$
1	$\Phi\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$	$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T$

k	$\Phi(M_k) = \Sigma_{M_k}$	A_{M_k}	B_{M_k}	C_{M_k}
2	$\Phi\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$	$\begin{bmatrix} -2 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}^T$
3	$\Phi\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right)$	$\begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T$
4	$\Phi\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$	$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$

Hyperplanes related to markings and transitions are

$$\begin{aligned} H(t_1) &= [\,0 \quad 0 \quad 1\,] \\ H(t_2) &= [\,1 \quad 1 \quad 0\,] \\ H(t_3) &= [\,0 \quad 0 \quad 1\,] \\ H(t_4) &= [\,1 \quad 1 \quad 1\,] \end{aligned} \tag{4.12}$$

Predicates $H(t_s)$ define hyperplanes such that $H(t_s)x = 0$. For simplicity the function δ_{M_i,M_j} is the identity matrix.

The DES is not marking- and sequence-detectable, as the IPN transitions are silent and there are output related sequences for which their initial conditions cannot be determined, For instance, the input-output symbols sequence related to $\alpha_1 = M_0 \xrightarrow{\mathbf{t}_2} M_1 \cdots$ and $\alpha_2 = M_3 \xrightarrow{\mathbf{t}_4} M_4 \cdots$ is the same sequence

$$\omega_{\sigma_1} = \omega_{\sigma_2} = \left(\varepsilon, \begin{bmatrix} 1 & 1 \end{bmatrix}^T\right)\left(\varepsilon, \begin{bmatrix} 0 & 1 \end{bmatrix}^T\right)\cdots \tag{4.13}$$

then the current marking and the firing sequence of the IPN cannot be computed with the input and the output information. However, sequences

$$\bar{\omega}_{\sigma_1} = \left(\varepsilon, \begin{bmatrix} 1 & 1 \end{bmatrix}^T, (-2)\right)\left(\varepsilon, \begin{bmatrix} 0 & 1 \end{bmatrix}^T, (1,2)\right)\cdots \tag{4.14}$$

$$\bar{\omega}_{\sigma_2} = \left(\varepsilon, \begin{bmatrix} 1 & 1 \end{bmatrix}^T, (-2)\right)\left(\varepsilon, \begin{bmatrix} 0 & 1 \end{bmatrix}^T, (3)\right)\cdots \tag{4.15}$$

where $\bar{\omega}_{\sigma_1}, \bar{\omega}_{\sigma_2} \in \bar{\Lambda}\left(\langle \mathcal{F}, (Q, M_0)\rangle\right)$ determined using linear system identification, are different from each other, and α_1 and α_2 are distinguishable. This is true for every output related sequences $|\sigma| \geq 4$ then by introducing the continuous system information the IPN becomes marking- and sequence-detectable.

Notice that every discrete sequence σ starting from any marking of $\mathcal{M}_0$ such that $|\sigma| \geq 4$ can be written as $\sigma = xwy$ where $w = \mathbf{t}_2\mathbf{t}_3$ or $w = \mathbf{t}_3\mathbf{t}_2$, the linear systems sequence associated to each w are $\Phi(M_0)\Phi(M_1)\Phi(M_3)$ and $\Phi(M_0)\Phi(M_2)\Phi(M_3)$ respectively, the observability matrix of each sequence is

$$O_{w_1} = \begin{bmatrix} O_{M_0} \\ H_{M_0,\mathbf{t}_2} \\ O_{M_1} \\ H_{M_1,\mathbf{t}_3} \\ O_{M_3} \end{bmatrix} \text{ and } O_{w_2} = \begin{bmatrix} O_{M_0} \\ H_{M_0,\mathbf{t}_3} \\ O_{M_2} \\ H_{M_2,\mathbf{t}_2} \\ O_{M_3} \end{bmatrix} \tag{4.16}$$

with

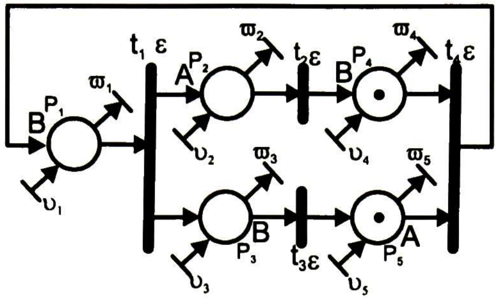

Figure 5.8: Observer of the Discrete State

$$\mathcal{O}_{M_0} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -5 & 0 \end{bmatrix} \quad \mathcal{O}_{M_1} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 6 & 1 & 0 \end{bmatrix} \quad \mathcal{O}_{M_2} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\mathcal{O}_{M_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix} \quad \mathcal{O}_{M_4} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \tag{4.17}$$

Every linear system is not observable, however the observability matrix of w_1 and w_2 has full rank. Therefore the observability matrix of every sequence $|\sigma| \geq 4$ has full rank. Hence by Theorem 4.2 the SLS is observable, and a hybrid observer can be designed.

The Observer is a SLS, the discrete part is designed as in [27] which is shown in Fig. 5.8.

The state observer for the continuous part is the LS

$$\Phi\left(\hat{M}_k\right)=\Sigma_{\hat{M}_k}\left\{\begin{array}{l}\dot{\hat{x}}(\tau)=A_{\hat{M}_k}\hat{x}(\tau)+B_{\hat{M}_k}u(\tau)\\ \qquad +\Gamma_{\hat{M}_k}\left(C_{M_k}x(\tau)-C_{\hat{M}_k}\hat{x}(\tau)\right)\\ \hat{y}(\tau)=C_{\hat{M}_k}\hat{x}(\tau)\end{array}\right. \tag{4.18}$$

where marking $\hat{M}_k$ is named as follows:

$$\begin{aligned}
\hat{M}_0 &= [\,0\ \ 1\ \ 1\ \ 0\ \ 0\,]^T\\
\hat{M}_1 &= [\,0\ \ 0\ \ 1\ \ 1\ \ 0\,]^T\\
\hat{M}_2 &= [\,0\ \ 1\ \ 0\ \ 0\ \ 1\,]^T\\
\hat{M}_3 &= [\,0\ \ 0\ \ 0\ \ 1\ \ 1\,]^T\\
\hat{M}_4 &= [\,1\ \ 0\ \ 0\ \ 0\ \ 0\,]^T
\end{aligned}$$

And LS matrices $(A_{M_k}, B_{M_k}, C_{M_k})$ of the system Σ_{M_k} are the same as the matrices $\left(A_{\hat{M}_k}, B_{\hat{M}_k}, C_{\hat{M}_k}\right)$ of the system $\Sigma_{\hat{M}_k}$. Evolving LS observer $\Sigma_{\hat{M}_k}$ may not be the same as Σ_{M_k} at every time instant.

The observer constant matrices $\Gamma_{\hat{M}_k}$ are computed as

$$\begin{aligned}
\Gamma_{\hat{M}_o} &= [\,85\ \ 1739\ \ 0\,]^T\\
\Gamma_{\hat{M}_1} &= [\,432\ \ 58\ \ 0\,]^T\\
\Gamma_{\hat{M}_2} &= [\,0\ \ 30\ \ -30\,]^T\\
\Gamma_{\hat{M}_3} &= [\,-30\ \ 30\ \ 30\,]^T\\
\Gamma_{\hat{M}_4} &= [\,30\ \ 0\ \ 0\,]^T
\end{aligned}$$

And the initial marking of the IPN observer is $\hat{M}_3$.

According to the firing rule, the IPN marking of the observer is equal to the IPN marking of the system after two commutations, hereafter the continuous observer

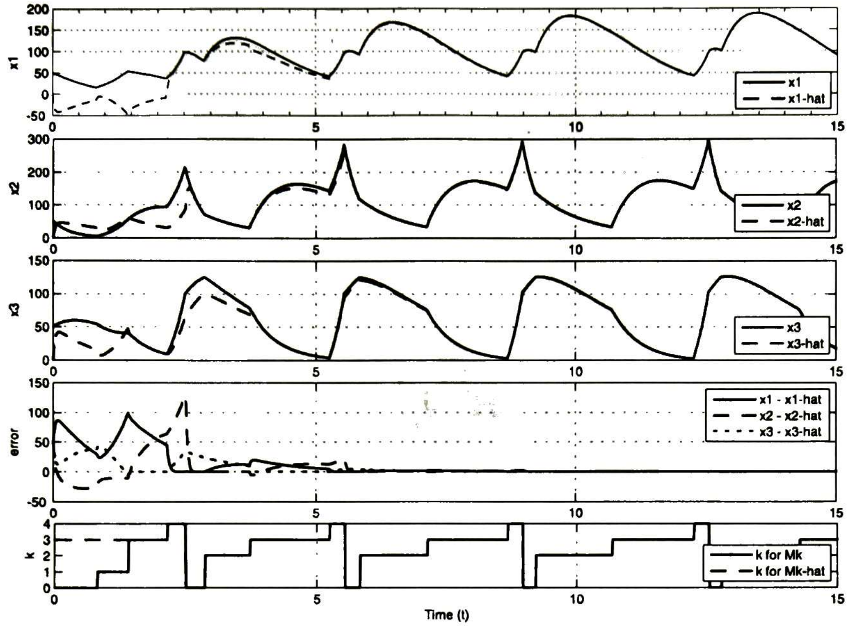

Figure 5.9: SLS states, observed states and observation error..

evolving corresponds to the evolving continuous system, the observable variables of each linear systems are estimated (Fig. 5.9) and the estimation error converges to zero (Fig. 5.9 and 5.10).

The observer herein presented is assimptotic. Once the discrete state has been observed and it has been executed a sequence σ such that the observability matrix O_σ has full rank, then, current state can be calculated at each commutation time. This computed state vector can be used to set the initial condition for the continuous observer for the next commutation.

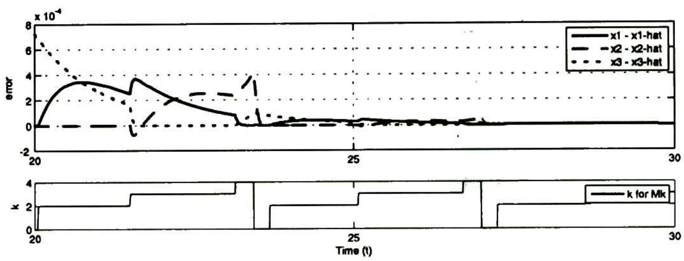

Figure 5.10: SLS Observer error for state vector for interval $t = [20, 30]$

Chapter 6

On Controllability and Control of SLS

Main objectives in this research include study of controllability and control in HS, particularly for switched linear systems (SLS). It is assumed that neither the continuous part nor the discrete part are necessarily autonomous.

Hybrid Systems control may be used for the representation and control of physical systems that modify their operation conditions and for the design of switching controllers like Gain Scheduling [30] or Sliding Modes [31].

Controllability denotes the ability to traslate the state of a the SLS around its entire state space using only admissible manipulations. Controllability is a qualitative property of SLS.

Controllability of DES for SLS is generally based on Supervisory control theory proposed by W.M. Wonham [32], [33]; essentially it states that a DES is controllable if and only if the system behaviour may be restricted to a desired behaviour called specification.

Controllability property on Linear Time Invariant (LTI) systems, states that there is always an input such that any initial state is traslated to any required state. Controllability is presented by Chen in [6]. Wonham presents a geometrical characterization of controllability in [19] which will be used in this chapter.

There exist several works addressing control problems in HS and SLS, for instance [2] and [3] deal with controllability in HS; [4] and [5] deal with stability. In the later it is proved that the problem of stability in hybrid systems is NP-hard. However, if it is possible to compute a Lyapunov function for all linear systems $LS_i \in \mathcal{F}$ then stability is guarantied. It has also been proved that the controllability in some HS subclasses is an NP-hard problem.

Controllability in SLS can be defined using reachability subspaces. It can be stated, from a geometrical point of view, when an SLS is controllable, unfortunately this does not lead to controllability conditions.

Model Predictive Control (MPC) may be used to control LS and HS. MPC is a widely used type of control. It is used in industry, and in applications fields such as automotive and aerospace. Among main advantages of MPC are that it is a predictive control technique, which means MPC estimates the future behaviour of the system and activates the control action based on it, and it is an optimal control technique.

In this chapter, Controllability of SLS is presented. The analysis of this property is stated as Arias and Ruiz did in [34]. Reachable subspaces for SLS, and some conclusions on SLS Controllability are stated.

MPC is presented for periodic discrete-time time-varying linear systems, that may be extended to SLS. Two approaches will be presented: (i) Solving the MPC problem on-line through quadratic programming using fast QP methods, or (ii) Approximating off-line the control law through interpolating or approximating functions, that are evaluated on-line.

6.1 Controllability in SLS

Controllability in LTI systems describes the ability of an external input to traslate the internal state of a system from any initial state to any other final state in a finite time interval.

Definition 6.1 *LTI system $\Sigma(A, B, C)$, represented by equation (2.1), is controllable if for any $x(\tau_0), x(\tau_1)$, there exists an input $u(\tau)$ which is able to transfer state $x(\tau_0)$ into state $x(\tau_1)$ in a finite time τ_1.*

Geometrical characterization of controllability [19] is based in the analysis of its reachable state formed by the solution of the state equation.

The solution to the LTI dynamic equation given in (2.1) is

$$x(t) = e^{A(\tau-\tau_0)}x(\tau_0) + \int_{\tau_0}^{\tau} e^{A(\tau-\varsigma)}Bu(\varsigma)\,d\varsigma \tag{6.1}$$

which is formed by both autonomous and forced responses.

The autonomous dynamic response $e^{A(\tau-\tau_0)}x(\tau_0)$ generates the set of states transformed from the state $x(\tau_0)$ due to state transition $e^{A(\tau-\tau_0)}$.

Forced dynamics of the LTI system may be expresed, using its Taylor expansion, as

$$\int_{\tau_0}^{\tau} e^{A(\tau-\varsigma)}Bu(\varsigma)\,d\varsigma = \sum_{k=0}^{\infty} A^k B \int_{\tau_0}^{\tau} \frac{1}{k!}(\tau-\varsigma)^k u(\varsigma)d\varsigma \tag{6.2}$$

States that are part of forced dynamics may be represented as the linear combination of vectors from matrices $\left\{A^k B \,|k = 0, 1, ..., \infty\right\}$ In general, forced dynamics form an state subspace, which is composed by every state that can be reached from $x(\tau_0)$. In [6], it is shown that controllability of system $\Sigma(A, B, C)$ is determined by the linear independency of the n rows of matrix $e^{A(\tau-\varsigma)}B$. If $\Sigma(A, B, C)$ is controllable, then there exist at least n lineary independent vectors in $\left\{A^k B \,|k = 0, 1, ..., \infty\right\}$, i.e.

the generated space by the forced response of the LTI system is the same as the state space. Which is also a necessary condition.

Reachable subspace R_0 of $\Sigma(A, B, C)$ is the space generated by every state that is part of a solution of $\Sigma(A, B, C)$ when $x(\tau_0) = 0$.

In [19] it is shown that if

$$\tilde{x} = \int_{\tau_0}^{\tau} e^{A(\tau-\varsigma)} Bu(\varsigma)\, d\varsigma, \tau > \tau_0 \tag{6.3}$$

then the reachable states may be represented as a linear combination of vectors $\{A^{r-1}B \,|r = 1, 2, ..., n\}$, with n the size of vector $x(\tau)$. In general, reachable states belong to the addition of subspaces $\operatorname{Im}(B), A\operatorname{Im}(B), A^2\operatorname{Im}(B), ..., A^{n-1}\operatorname{Im}(B)$ i.e.

$$\tilde{x} \in \operatorname{Im}(B) + A\operatorname{Im}(B) + A^2\operatorname{Im}(B) + ... + A^{n-1}\operatorname{Im}(B) \tag{6.4}$$

denoted as

$$\langle A|\operatorname{Im}(B)\rangle = \operatorname{Im}(B) + A\operatorname{Im}(B) + A^2\operatorname{Im}(B) + ... + A^{n-1}\operatorname{Im}(B) \tag{6.5}$$

Subspace $\langle A|\operatorname{Im}(B)\rangle$ is the minimal $A - invariant$ subspace which contains the image of B, i.e.

$$\begin{aligned} \forall x &\in \langle A|\operatorname{Im}(B)\rangle, Ax \in \langle A|\operatorname{Im}(B)\rangle \quad \text{and} \\ \forall u(\tau) &\in \mathsf{R}^p, Bu(\tau) \in \langle A|\operatorname{Im}(B)\rangle \end{aligned} \tag{6.6}$$

Reachable subspace is defined as

$$R_0 = \langle A|\operatorname{Im}(B)\rangle \tag{6.7}$$

Example 6.2 *Let $\Sigma(A, B, C)$ be a LTI system with*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{6.8}$$

Then, $R_0 = \langle A|\operatorname{Im}(B)\rangle = \mathsf{R}^2$

Notice that, if R^n is the state space, $\bar{\mathsf{R}} = \mathsf{R}^n/\mathsf{R}_0$ the quotient space, $P : \mathsf{R}^n \rightarrow \bar{\mathsf{R}}$ the canonical projection, and $\bar{A}$ the mapping induced in R^n by A, then $\bar{x} = Px$, and the system $\Sigma\,(A, B, C)$ represented in the quotient space $\mathsf{R}^n/\mathsf{R}_0$ is $\dot{\bar{x}}(\tau) = \bar{A}\bar{x}(\tau)$. Control law $u(\tau)$ can not affect the states that belong to the set $x \bmod R_0 \neq 0$, and only if $R_0 = \mathsf{R}^n$ then $\Sigma\,(A, B, C)$ is controllable.

Controllability definition used for DES will be the one stated by W.M. Wonham [32], [33]; in his work it is stated that a DES is controllable with respect to a desired behaviour if and only if it is possible to restrict the system behaviour to that which is desired.

In this work, controllability in SLS will be addressed in two ways (as in [34]): First, it will be addessed for SLS where transitions on the DES are manipulable. And later, systems where transitions are non manipulable.

6.1.1 SLS with manipulable transitions

Manipulable transitions allow to control the discrete dynamics. When every transition in a SLS is manipulable, it is posible to forze its commutations until a particular LS is reached, and even stay in this system until it is necessary. For this kind of SLS it is very useful to characterize when it is possible to restrict the discrete dynamics for a particular LS to be activated. It is also important to have the ability of staying in any LS for as long as necessary, i. e. any LS can be activated and it may stay active for as long as needed.

In this type of SLS, any system state may be a commutation state. Then, the discrete dynamics controls the continuous state trajectories.

Next, an example is presented to illustrate this property. Both LS in the example are uncontrollable.

Example 6.3 *Let the continuous part of a SLS be defined by the linear systems*

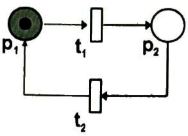

Figure 6.1: Discrete model of SLS

family

$$\Sigma_1 \left\{ A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right.$$
$$\Sigma_2 \left\{ A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right.$$

and the discrete part defined by the Petri net of Figure 6.1.

In this system, each linear system is uncontrollable and the SLS is controllable. Here, Σ_1 is an autonomous system and the maximal controllability subspace of Σ_2 is R_0^2.

$$\begin{aligned} R_0^1 &= \langle A_1 | B_1 \rangle = span \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\ R_0^2 &= \langle A_2 | B_2 \rangle + \langle A_2 | R_0^1 \rangle = span \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \\ R_0^3 &= \langle A_1 | B_1 \rangle + \langle A_1 | R_0^2 \rangle = span \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \end{aligned} \tag{6.9}$$

The reachability space can be expanded in each commutation. For this example, with initial marking $M_0 = [\ 0 \quad 0\ ]^T$ reachable space is R_0^1, after first commutation reachable space is now R_0^2, and after second commutation, the reachable space is R_0^3, which is controllable. The reachable space has been expanded to R^2

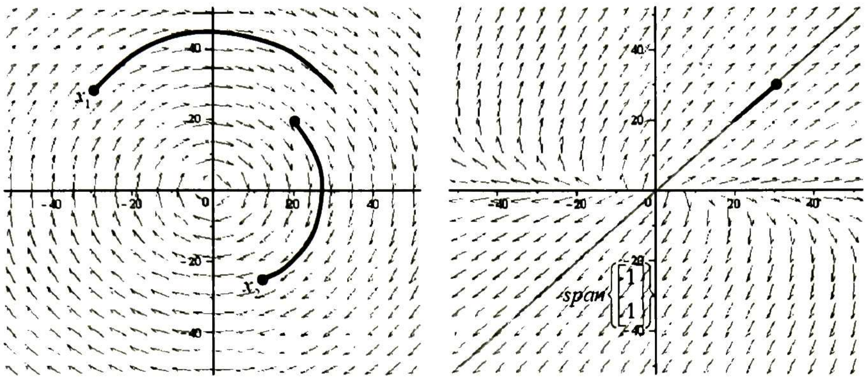

Figure 6.2: Phase portraits of the LS of SLS.

Phase portraits of the system can be seen on Figure 6.2. There, controllability can be appreciated. If it is intended to go from state x_1 to state x_2 then linear system Σ_1 may evolve until system state belongs to the reachability space of Σ_2, switch the continuous dynamics to that of Σ_2 and then move along its invariant until system state is such that if returned to Σ_1 one can move within its controllable subspace and reach x_2.

6.1.2 SLS with non manipulable transitions

When transitions are non manipulable, a HS may be uncontrollable even when each linear system is controllable. In next example it will be shown what happens when system state is restricted by hyperplanes.

Example 6.4 *Let the continuous part of a SLS be defined by*

$$\Sigma_1 \left\{ A_1 = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right.$$

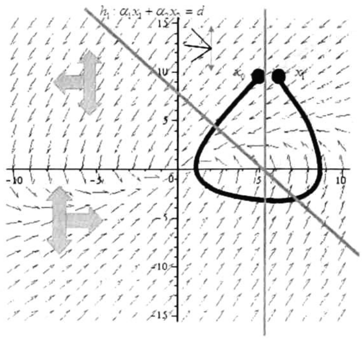

Figure 6.3: Vector field of LS Σ_1 of SLS

$$\Sigma_2 \left\{ A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right.$$

the discrete part defined by the Petri net of Figure 6.1, and hyperplanes on both transitions defined as $h : x_1 + x_2 = 3$.

System Σ_1 *is controllable and one can find an input for the system to go from system state* $\bar{x}_0$ *to* $\bar{x}_1$, *in some finite time* t_1. *But even the states are very close, the trajectory may not be close to them. The vector field of system variable* x_1 *is fixed, so the dynamics depends on* x_2. *If* x_2 *is positive, then* x_1 *decreaces, and if* x_1 *is negative, then* x_2 *grows. As control is not bounded, then it may go in any direction of the vector field. In Figure 6.3 it can be seen that, for system* Σ_1, *for the system state to go from state* $\bar{x}_0$ *to* $\bar{x}_1$ *it should follow a trajectory that may hit an hyperplane, and this is what makes uncontrollable a* SLS.

In Figure 6.4 it is presented the vector fields directions of system variables in different sections or regions of the system state space. There are system states $\bar{x}$ *which cannot reach other states in its same region without hitting an hyperplane, i.e. the trajectory that it should follow, even if it exists, goes through several regions.*

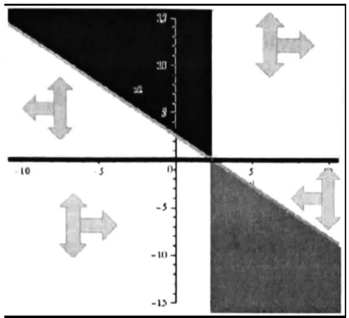

Figure 6.4: Vector fields directions of SLS

In general, in systems of dimension 2, it can be defined two kinds of regions, one where system trajectories translate into those regions will not be able to exit unless the system commutes, and another one where states from the region will not be reached by states outside this region, unless the SLS commute.

In systems with larger dimension, the number of regions explodes and it becomes very difficult to enumerate every class of them, and controllability becomes more complex.

It has been proved ([2], [3], [4], [5]), that the controllability and stability in some HS subclasses are NP-hard problems.

This, and previous section show issues in controlling HS, in particular SLS. Controllability has not been studied further.

6.2 Model Predictive Control

Model Predictive Control (MPC) may be used to control HS. MPC is a widely used type of digital control, it is also known as $MBPC$ (Model Based Predictive Control). It is used in industry, and in applications fields as automotive and aerospace. Among main advantages of MPC are that it is a predictive control technique, which means

MPC estimates the future beaviour of the system and activates the control action based on it, and it is an optimal control technique, i.e. it looks for the optimal solution, taking into account a set of constraints, in order to minimize a cost function.

In literature, there are several different variants of the MPC paradigm, each one with its own special features, but all of them rely on the idea of generating values for process inputs as solution of an on-line optimization problem. That problem is constucted on the basis of a process model and process measurements. Process measurements provide the feedback element in the MPC structure.

Figure 6.5 makes it clear that the behavior of an MPC system can be quite complicated, because the control action is determined as the result of the on-line optimization problem. While engineering intuition may frequently be used in the analysis of the behavior or in the design of MPC system, theory can provide valuable help. Theory can augment human judgement and intuition in the development and implementation of better MPC system that can realize their full potential as "advanced" control system. Some of the benefits of improved MPC system are better control performance, less down time, reduced maintenance requirements, and improved flexibility and agility.

In Model Predictive Control, the control action is obtained by solving an online optimization problem, based on a given prediction model. If the prediction model is linear (possibly time-varying), constraints are linear, and the cost function is quadratic, the resulting optimization problem is a quadratic program (QP). The control variable $u(t)$ is found at each step solving the MPC problem

$$
\begin{array}{ll}
\min & x(t+t_f(t))^T Q_f x(t+t_f(t)) + \sum_{\tau=t,\ \tau\neq t_f(t)}^{t+T-1} x'(t)Qx(t) + u'(t)Ru(t) \\
\text{subj. to} & X_{f_{min}} \leqslant x(t+t_f(t)) \leqslant X_{f_{max}} \\
& X_{min} \leqslant x(\tau) \leqslant X_{max},\ t \neq t_f(t) \\
& U_{min} \leqslant u(\tau) \leqslant U_{max} \\
& x(\tau+1) = Ax(\tau) + Bu(\tau),\ \tau = t, t+1, \dots, t+T-1
\end{array}
\tag{6.10}
$$

where T is the length of the prediction horizon, and with variables $x(t+1), ..., x(t+T)$ and $u(t), ..., u(t+T-1)$. Here, $Q = Q^T \in \mathbf{R}^{n \times n}$, $R = R^T \in \mathbf{R}^{m \times m}$, $Q_f = Q'_f \in \mathbf{R}^n$, X_{min}, X_{max}, $X_{f_{min}} \in \mathbf{R}^n$ and U_{min}, $U_{max} \in \mathbf{R}^m$.

Problem (6.10) is a convex QP Let $u^\star(t), ..., u^\star(t+T-1), x^\star(t+1), ..., x^\star(t+T)$ be optimal for problem (6.10). The MPC policy takes $u(t) = u^\star(t)$. So the input $u(t)$ is a complicated function of the current state $x(t)$, hence the MPC policy has a static state-feedback form $u(t) = \psi_{mpc}(x(t))$. When the quadratic cost term is positive definite [35] it can be shown that $\psi_{mpc}(x(t))$ is a piecewise-affine function.

Several approaches exist to compute the control action. Fast MPC (Fast Model Predictive Control) solver of [36], is based on interior-point methods is a recent and commonly used solver

Fast MPC uses a collection of methods that simplify the structure of the QP and speed up the calculation of the control law. The main strategy is to re-ordering appropriately the variable, such that the interior-point search direction at each step can be found by solving a block tridiagonal system of linear equations. In this way a problem with state dimension n, input dimension m and horizon T takes $O(T(n+m)^3)$ operations per step, instead of $O(T^3(n+m)^3)$ if the variables are not re-ordered. Another technique used in Fast MPC is *warm-starting* [37], [38], in which at each step, the problem is initialized using the predictions made in the previous step. This technique has been proveed to reduce the number of steps required by a factor of 5 or more.

6.3 On model predictive control law

Matrices Q_f, Q, and R are the tuning knobs of the MPC controller. Control law is defined by a set of them such that fullfills (6.10).

Next, it will be presented alternative ways of implementing the model predictive control law using off-line function approximation techniques to get an approximate explicit form of the control law.

In MPC the control, action is given by solving a finite horizon optimal control problem on-line at each time instant. A possible drawback of this method is that the optimization problem may be computationally too expensive, or require excessively complex control code. This problem has usually labeled MPC as a technology for slow processes. The increase in computational performance and memory in microcontroller and computers are progressively changing the concept of "slow" but still solving a QP prevents the application of MPC in many contexts.

To overcome these problems, an alternative approach to evaluate the MPC law was proposed in [35] for linear time-invariant systems with constraints, where most of the computations are performed off-line, and on-line computations reduce to the simple evaluation of a piecewise affine function. The idea is to solve the QP problem *off-line* for all vectors x within a given range, instead of solving it on-line for the current vector $x(t)$, and make the dependence of u on x *explicit.* So, the key idea is to treat the QP problem as a *multiparametric* quadratic programming problem, where $x(t)$ is the vector of parameters.

Exact explicit approach of [35] is not directly applicable to time-varying MPC. Hence, alternative approaches to get explicit solutions that, although approximate, preserve the property of simple on-line operations for function evaluation may be used.

The idea is to generate samples off-line of the optimal control profile using the QP formulation described in the previous sections, and then use function approximation techniques to determine an approximate form of the MPC control law beforehand.

Approximation functions based on either inverse distance weighting (IDW) interpolation [39], or artificial neural network (ANN) [40] may be used. Both function approximation methods have pros and cons: IDW needs no training, but it requires the data points used for approximation to be stored on-line. On the other hand, ANN must be trained off-line using nonlinear optimization procedures, but has the advantage that only the identified ANN parameters need to be stored on-line.

6.3.1 Inverse Distance Weighting

The inverse distance weighting (IDW) method for multivariate interpolation [39] allows the creation of continuous surfaces from a discrete set of data. The method is based on the following interpolation function

$$\hat{u}(x) = \begin{cases} \frac{\sum_{k=0}^{N} w_k(x) u_k}{\sum_{k=0}^{N} w_k(x)} & \text{if } x \neq x_k,\ \forall k = 0, \ldots, N \\ u_k & \text{if } x = x_k \end{cases} \tag{6.11}$$

where

$$w_k(x) = \frac{1}{d(x, x_k)^p} \tag{6.12}$$

is the weighting function, x is an interpolated point, $\{x_k\}_{k=0}^{N}$ are the known grid points, $d(\cdot, \cdot)$ a distance between two points, N is the total number of grid points and p is a positive real power parameter. Greater values of p assign greater influence to values closest to the interpolated point. Clearly, as the distance from x to x_k increases, the weight $w_k(x)$ decreases.

6.3.2 Artificial Neural Network

Artificial neural networks consist in nonlinear and non-parametric models searching relationships between certain input data provided, through the minimization of an error fit functional. The power of ANN's is due to the ability of identifying numerical relations between input data, omitting the natural or formal identification of the model. As any method of processes modeling and system identification, neural networks are only efficient if the input data contain sufficient information. The use of inputs data with high correlation with the output data facilitates statistical processing of the network, providing better results. The most important step in constructing ANNs is the *training phase*, which consists of comparing the input data with the expected known output outcome (the *target*), determining the influence of activated synaptic connections between neurons. If the number of analyzed variables is very high, this phase may require a long time to determine an acceptable condition for convergence.

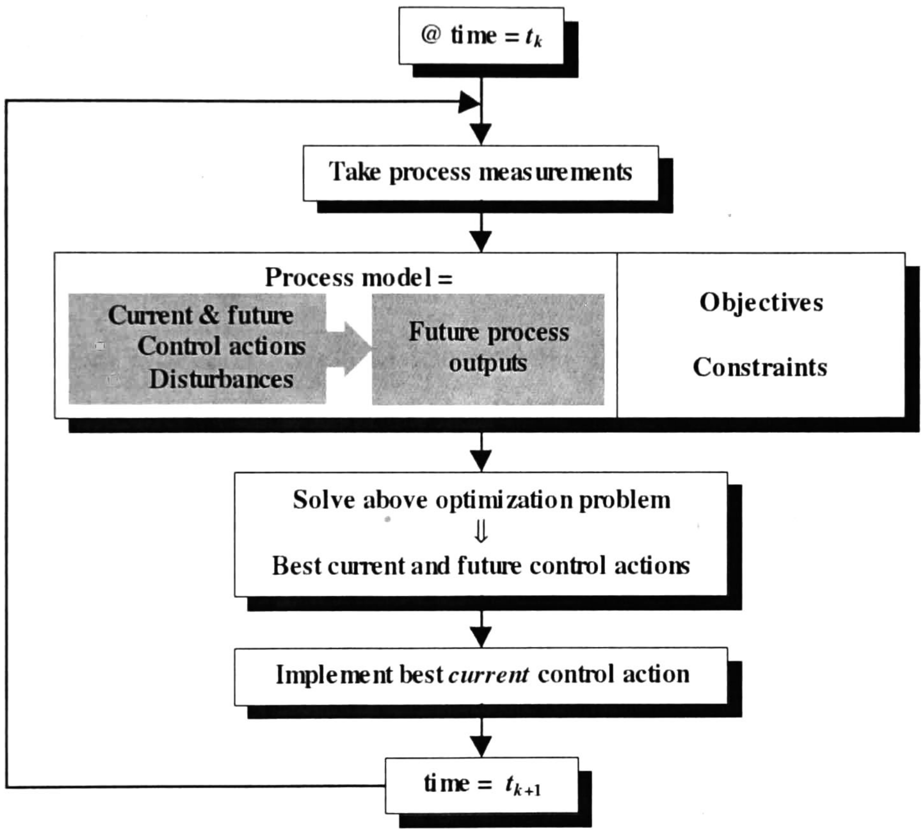

Figure 6.5: Model Predictive Control Scheme

Chapter 7

Conclusions

This work deals with Joint State-Mode Observability in finite time of SLS for autonomous and non autonomous systems, and the design of an asymptotic observer for these systems.

The presented model can be applied to Jump Linear Systems where commutation of the systems does not depend on the continuous variables therefore there is no hyperplane equations, and to Piecewise Affine Systems where hyperplane information is known. It can also be applied to systems where continuous mode identification is possible with or without hyperplane information.

It exploits the input and output information of the SLS (both continuous and discrete information) to compute the discrete and the continuous state in joint mode. In the first one the current marking sequence (discrete state trajectory) is computed and, based on this information in the second stage the continuous state is computed. It is worth noticing that the results herein presented are valid for all possible system inputs, so the results are applied either if the SLS is under a control action or in open loop mode.

Observability property is obtained using information from both, the discrete and the continuous part, using complete or partial information from both. It is important to say that every LS in the continuous part may be unobservable and the Petri net of the discrete part be unobservable also, and the complete SLS be observable. When the discrete part is unobservable, and the information from it is not enough to uniquely determine in which state the DES is, then the joint mode scheme helps to discern between possible discrete states using information or partial information from the continuous part. Also, using identification and information from the discrete part, the continuous observer may reset the variables which, in turn, are now to be unobservable.

Controllability conditions were studied. Also some cases where the SLS is controllable even when its LS are uncontrollable, were studied. It was possible to deffine subregions of the state space where the system still controllable, but it is not possible to state conditions on controllability or controllability subspaces, as the regions where the system remains controllable are not related to its subspaces. SLS composed of linear system of dimenssion 2 and linear hyperplanes were characterized, but it was not extended for larger dimenssions as the problem became undrivable.

Controllability was no further studied as it has been proved ([2], [3], [4], [5]), that controllability and stability properties in some HS subclasses are $NP-hard$ problems.

MPC has been presented for its use on periodic time-varying linear systems, that may be extended to SLS. MPC was succesfully used to solve an aerospace problem. The design of the MPC is rather straightforward once a linear discrete-time time-varying model was available. System constraints are added to the system model for the control formulation. Two approaches were implemented: (i) Solving the MPC problem on-line through quadratic programming using fast QP methods, or (ii) Approximating off-line the control law through interpolating or approximating functions, that are evaluated on-line.

The use of control law off-line approximation gave better results than those using on-line through quadratic programming using fast QP methods. IDW Interpolation results were very similar to those obtained using ANN.

Future work includes the design Hybrid Systems observers using subspace identification of Linear Systems together with Partial Discrete observation, extend study and results to $MIMO$ systems and study of Geometrical Control and MPC for SLS.

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Observabilidad en Sistemas Lineales Conmutados "Observability in Switched Linear Systems"

del (la) C.

Guillermo RAMÍREZ PRADO

el día 14 de Diciembre de 2010.

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