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Control Óptimo Inverso para Sistemas No Lineales en Tiempo Discreto

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Dr. Alexander Georgievich Loukianov
Dr. Edgar Nelson Sánchez Camperos

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Inverse Optimal Control for Discrete-Time Nonlinear Systems

A thesis presented by:
Fernando Ornelas Tellez

to obtain the degree of:
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Thesis Advisors:
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Por:
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Centro de Investigación y de Estudios Avanzados del IPN
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Doctor of Science Thesis
In Electrical Engineering

By·

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Master of Science in Electrical Engineering

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"Because the shape of the whole universe is most perfect and,
in fact, designed by the wisest Creator,
nothing in all of the world will occur
in which no maximum or minimum rule is
somehow shining forth."

Leonhard Euler, 1744.

Resumen

En la teoría de control óptimo se tiene por objetivo la obtención de una acción de control tal que un criterio de costo, el cual pondera el comportamiento de un sistema, sea minimizado. El criterio de costo puede incluir tanto el estado del sistema como las acciones de control. Una solución al problema de control óptimo se puede obtener resolviendo la ecuación de Hamilton-Jacobi-Bellman, lo cual es difícil para sistemas de control no lineales. El problema de control óptimo ha sido resuelto sólo para sistemas lineales, que es conocido como el problema del regulador lineal.

En esta tesis se presenta un enfoque de control óptimo inverso para dar solución al problema de estabilización y seguimiento de trayectorias de sistemas no lineales en tiempo discreto, evitando la solución de la ecuación de Hamilton-Jacobi-Bellman y a la vez minimizando una funcional de costo. La acción de control propuesta es determinada por dos medios: la formulación de una función de control de Lyapunov y en la teoría de Pasividad.

Además, se establece un esquema de control óptimo inverso robusto para evitar dar solución a la ecuación de Hamilton-Jacobi-Isaacs, asociada al problema de control óptimo en sistemas no lineales con perturbaciones.

En problemas prácticos, un controlador basado en el modelo de la planta puede no lograr un desempeño adecuado, esto debido a perturbaciones internas y/o externas, parámetros inciertos, o alguna dinámica no modelada afectando al sistema. Por ello, se establece un controlador neuronal óptimo inverso basado un la combinación de dos técnicas: a) el control óptimo inverso, y b) un identificador neuronal que utiliza una red neuronal recurrente para la obtención de un modelo artificial de un sistema no lineal, el cual se supone desconocido. La red neuronal es entrenada en linea con un filtro de Kalman extendido.

Por otro lado, hay dos ventajas al desarrollar estrategias de control en tiempo discreto: a) la actual tecnología digital es apropiada para implementar controladores digitales en lugar de controladores analógicos; b) un controlador formulado en tiempo discreto puede ser implementado directamente en un procesador digital. En esta tesis se considera una clase de sistemas no lineales conocidos como afines en control. En esta clase se pueden representar una gran variedad de sistemas; muchos de ellos son discretizaciones aproximadas de sistemas de control en tiempo continuo.

Los controladores propuestos se ilustran a través de simulaciones, tanto para estabilización como para seguimiento de trayectorias de sistemas no lineales.

Abstract

Optimal nonlinear control is related to finding a control law for a given system, such that a performance criterion is minimized. This criterion is usually formulated as a cost functional, which is a function of state and control variables. The major drawback for optimal nonlinear control is the need to solve the associated Hamilton-Jacobi-Bellman (HJB) equation. The HJB equation, as far as we are aware, has not been solved for general nonlinear systems. It has been only solved for the linear regulator problem, for which it is particularly well-suited.

This dissertation presents a novel inverse optimal control for stabilization and trajectory tracking of discrete-time nonlinear systems, avoiding the need to solve the associated Hamilton-Jacobi-Bellman equation, and minimizing a meaningful cost function. This stabilizing optimal controller is based on the formulation of a discrete-time control Lyapunov function. Besides, a *robust* inverse optimal control scheme is proposed in order to avoid the associated Hamilton-Jacobi-Isaacs (HJI) equation solution for the case when a disturbance term is affecting the nonlinear systems performance. For realistic situations, a control based on a plant model could not achieve a desired performance, due to internal and external disturbances, uncertain parameters, or unmodelled dynamics. This dissertation establishes a *neural* inverse optimal controller combining two techniques: a) inverse optimal control, and b) an on-line neural identifier, which uses a recurrent neural network, trained with an extended Kalman filter, in order to build a model of an assumed unknown nonlinear system. Two approaches are presented for solving the inverse optimal control problem, one of them is based on passivity theory and the other one is based on the synthesis of a control Lyapunov function.

On the other hand, there are two advantages to work in a discrete-time framework: a) appropriate technology can be used to implement digital controllers rather than analog ones; b) the synthesized controller is directly implemented in a digital processor. In this dissertation, we consider *a class* of nonlinear systems (*affine* nonlinear systems), which represents a great variety of systems, most of them are approximate discretizations of continuous-time systems.

The applicability of the proposed controllers is illustrated, via simulations, by stabilization and trajectory tracking for nonlinear systems.

Notations and Acronyms

Notations

$\forall$	for all
$\in$	belonging to
$\Rightarrow$	implies
$\subset$	contained in
$\subseteq$	contained in or equal to
$\cup$	union
$\cap$	intersection
$:=$	equal by definition
$\lambda_{min}(Q)$	the minimum eigenvalue of matrix Q
$\lambda_{max}(Q)$	the maximum eigenvalue of matrix Q
$P > 0$	a positive definite matrix P
$P \geq 0$	a positive semidefinite matrix P
ΔV	denotes the Lyapunov difference
$\leq$	less that or equal to
$\mathcal{A}$	set or vector space
$\mathcal{K}$	denotes a class $\mathcal{K}$ function
$\mathcal{K}_\infty$	denotes a class $\mathcal{K}_\infty$ function
$\mathcal{KL}$	denotes a class $\mathcal{KL}$ function
$\mathbb{N}$	the set of all natural numbers
$\mathbb{Z}^+$	the set of nonnegative integers
$\mathbb{R}$	the set of all real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{R}_{\geq 0}$	the set of nonnegative real numbers
$\mathbb{R}^n$	n-dimension vector space
$(\cdot)^T$	denotes transpose
$(\cdot)^{-1}$	denotes inverse

$(\cdot)^*$	denotes optimal function
C^ℓ	denotes ℓ-times continuously differentiable function
$\|x\|_n$	the n-norm of vector x
$\|x\|$	the Euclidean norm of vector x
$\alpha_1 \circ \alpha_2$	the composition of two functions, where $\alpha_1(\cdot) \circ \alpha_2(\cdot) = \alpha_1(\alpha_2(\cdot))$

Acronyms

BIBS	Bounded-Input Bounded-State
CLF	Control Lyapunov Function
CT	Continuous-Time
DARE	Discrete-Time Algebraic Riccati Equation
DOF	Degrees of Freedom
DT	Discrete-Time
EKF	Extended Kalman Filter
GS	Globally Stable
GAS	Globally Asymptotically Stable
HJB	Hamilton-Jacobi-Bellman
HJI	Hamilton-Jacobi-Isaacs
ISS	Input-to-State Stable
MI	Matrix Inequality
LTI	Linear Time Invariant
PBC	Passivity-based Control
PID	Proportional Integral Derivative
RHONN	Recurrent High Order Neural Network
RNN	Recurrent Neural Networks
RCLF	Robust Control Lyapunov Function
SG	Speed-Gradient

Contents

Chapter 1

Introduction

This chapter presents a review on the optimal control problem and the conditions in order to obtain its solution. The Hamilton-Jacobi-Bellman (HJB) equation is introduced as a means to solve the optimal control problem; however, the HJB equation solution is an unfeasible task for general nonlinear systems. Then, the inverse optimal control approach is presented as an appropriate alternative methodology to solve the optimal control problem, avoiding the HJB equation solution.

Optimal control is related to finding a control law for a given system such that a performance criterion is minimized. This criterion is usually formulated as a cost functional, which is a function of state and control variables. The optimal control problem can be solved using Pontryagin's maximum principle (a necessary condition) [1], and the method of dynamic programming developed by Bellman [2, 3], which can leads to a nonlinear partial differential equation called the Hamilton-Jacobi-Bellman (HJB) equation (a sufficient condition); nevertheless, solving the HJB equation is not a feasible task [4, 5]. Actually, the HJB equation has so far rarely proved useful except for the linear regulator problem, to which it seems particularly well suited [6].

This dissertation presents a novel inverse optimal control for stabilization and trajectory tracking of discrete-time nonlinear systems, which are affine to the control input, avoiding the need to solve the associated Hamilton-Jacobi-Bellman (HJB) equation, and minimizing a meaningful cost function. The inverse optimal control approach was proposed initially by Kalman [7] for linear systems with quadratic cost functions. We refer the reader to [6, 8, 9] for inverse optimal control of continuous-time linear systems and to [5, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and references therein for the nonlinear continuous-time case. The inverse optimal control problem has been treated for delay systems in [20], and for adaptive control in [21]. To the best of our knowledge, there are few results on discrete-time nonlinear inverse optimal control; see [22], where the proposed control law depends on the knowledge of a Lyapunov function, which is difficult to obtain for general nonlinear systems.

For the inverse approach, a stabilizing feedback control law, based on a priori knowledge of a control Lyapunov function (CLF), is designed first, and then it is established that this control law optimizes a meaningful cost functional. The main characteristic of the inverse approach is that the meaningful cost function is a posteriori determined for the stabilizing feedback control law.

The existence of a CLF implies stabilizability [5] and every CLF can be considered as a meaningful cost function [17, 23, 24]. The CLF approach for control synthesis has been applied successfully to systems for which a CLF can be established such as: feedback linearizable, strict feedback and feed-forward ones [25, 26]. However, systematic techniques for determining CLFs do not exist for general nonlinear systems.

This dissertation presents two approaches for solving the inverse optimal control problem; one of them is based on passivity theory and the other one is based on the synthesis of a control Lyapunov function.

Additionally, in this dissertation a *robust* inverse optimal control scheme is proposed in order to avoid the solution of the Hamilton-Jacobi-Isaacs (HJI) equation associated to the optimal control problem for nonlinear systems which have a disturbance term. Furthermore, it is established a neural inverse optimal controller based on passivity theory and neural networks in order to achieve stabilization and trajectory tracking for uncertain discrete-time nonlinear systems.

In the following, basic fundamentals concerned to inverse optimal control are presented.

1.1 Inverse Optimal Control via Passivity

The concepts of passivity and dissipativity for control systems have received considerable attention lately. These concepts were introduced initially by Popov in the early 1950's [27] and formalized by Willems in the early 1970's [28] with the introduction of the storage and supply rate functions. Dissipative systems present highly desirable properties, which may simplify analysis and controller synthesis. Passivity-based control (PBC) was introduced in [29] to define a controller synthesis methodology, which achieves stabilization by passivation.

Passivity is an alternative approach for stability analysis of feedback systems [30]. One of the passivity advantages is to synthesize stable and robust feedback controllers. Despite the fact that nonlinear passivity for continuous-time has attracted considerable attention, and many results in this direction have been obtained [31, 28, 29, 32, 33, 34, 35] and references therein, there are few ones for discrete-time nonlinear systems [36, 37, 38, 39].

For continuous-time framework, the connection between optimality and passivity was established by Moylan [40] by demonstrating that, as in the linear case, the optimal system has infinite gain margins due to its passivity property with respect to the output.Passivity property for nonlinear systems can be interpreted as a phase property [4, 35], analogous to linear systems, which is introduced to guard against the effects of unmodeled dynamics (fast dynamics) which cause phase delays [4].

For this dissertation, we avoid to solve the associated Hamilton-Jacobi-Bellman equation by proposing a novel inverse optimal controller for discrete-time nonlinear systems based on a quadratic storage function, which is selected as a discrete-time candidate CLF in order to achieve stabilization by means of passivation through the output feedback under detectability conditions. Moreover, a meaningful cost function is minimized. The CLF acts as a Lyapunov function for the closed-loop system. Finally, the CLF is modified in order to achieve asymptotic tracking for given reference trajectories.

1.2 Inverse Optimal Control via CLF

Due to the fact that the optimal control problem solution by Bellman's method is associated with solving a HJB equation, the inverse optimal control via CLF approach is proposed in this dissertation. For this approach, the control law is obtained as a result of solving the Bellman equation. Then, a candidate CLF for the obtained control law is proposed such that it stabilizes the system and a posteriori a meaningful cost functional is minimized.

For this dissertation, a quadratic candidate CLF is used to synthesize the inverse optimal control law. Initially, the candidate CLF depends on a fixed parameter to be selected in order to obtain the solution for the inverse optimal control problem. A posteriori, this parameter is adjusted by means of the speed-gradient (SG) algorithm [41] in order to establish the stabilizing control law and to minimize a cost functional. We refer to this combined approach as the *SG inverse optimal control.* The use of the SG algorithm within the control loop is another novel contribution of this approach. Although the SG has been successfully applied to control synthesis for continuous-time systems, there are very few results of the SG algorithm application for stabilization purposes in the nonlinear discrete-time setting [42].

On the other hand, considering that systems are usually uncertain in their parameters, exposed to disturbances, and that there exist modeling errors, it is desirable to obtain a robust optimal control scheme. Nevertheless, when we deal with the robust optimal control problem, in which a disturbance term is involved in the system, the Hamilton-Jacobi-Isaacs (HJI) partial differential solution is required. A control law as a result of the robust optimal control formulation and the associated HJI solution provides stability, optimality and robustness with respect to disturbances ([17]); however, finding a solution for the HJI equation is the main drawback of the robust optimal control; this solution may not exist or may be extremely difficult to solve in practice.

To overcome the need of the HJI solution, in this dissertation a robust inverse optimal control approach for a class of discrete-time disturbed nonlinear systems is proposed, which does not require solving the HJI equation and guarantees robust stability in the presence of disturbances and a meaningful cost functional is minimized.

1.3 Neural Inverse Optimal Control

For realistic situations, a control based on a plant model could not perform as desired, due to internal and external disturbances, uncertain parameters, or unmodelled dynamics [43]. This fact motivates the need to derive a model based on recurrent high order neural network (RHONN) to identify the dynamics of the plant to be controlled, since this identifier is capable of modeling uncertain nonlinear systems. Also, a RHONN model is easy to implement, relatively simple structure, robustness, it has the capacity to adjust its parameters on-line [44, 45], and it allows incorporating a priori information about the system structure [46]. Three recent books [44, 46, 47] have reviewed the application of recurrent neural networks for nonlinear system neural identification and control. In [46] it is analyzed an adaptive neural identification and control scheme by means of on-line learning, where stability of the closed-loop system is established based on the Lyapunov function method.

Hence, we propose a neural inverse optimal controller for discrete-time uncertain nonlinear systems in order to achieve stabilization and trajectory tracking. For this neural scheme, an assumed uncertain discrete-time nonlinear system is identified by a RHONN model, which is used to synthesize the inverse optimal controller. The neural learning is performed on-line through an extended Kalman filter (EKF) as proposed in [46].

1.4 Motivation and Antecedents

Optimal control laws benefit from adequate stability margins, and the fact that they minimize a meaningful cost functional ensures that control effort is not wasted [48, 49]. Indeed, optimal control theory is introduced in [4] as a synthesis tool which guarantees stability margins. On the other hand, the robustness achieved as a result of the optimality is largely independent of the selected cost functional [4]. Although stability margins do not guarantee robustness, they do characterize basic robustness properties that well designed feedback systems must possess.

In this dissertation, motivated by the favorable stability margins of optimal control systems, we propose an inverse optimal controller, which achieves stabilization and trajectory tracking for discrete-time nonlinear systems, and avoids the HJB equation solution. On the other hand, the inverse optimal control methodology can be applied to uncertain nonlinear systems, which can be modeled by means of a neural identifier, and therefore a robust inverse optimal control scheme is obtained.

To the best of our knowledge, there are only few results on *discrete-time nonlinear inverse optimal control.*

For this dissertation, we consider *a class* of discrete-time nonlinear systems, which are affine in the input. Models of this type describe a great variety of systems. Most of them represent approximate discretizations of continuous-time systems [42]. There are two reasons why one might want to work in a discrete-time (DT) framework. Firstly, appropriate technology can be used to implement and deal with a digital controller rather than an analog one, which is generally more complicated and expensive. Secondly, the synthesized controller is directly implemented in a digital processor.

From the two types of optimality conditions, Pontryagin-type necessary conditions ("Maximum principle") and Bellman-type sufficient conditions ("Dynamic Programming"), the latter is more suitable for feedback design over infinite intervals [4]. Dynamic programming was developed by R. E. Bellman (1920-1984) in the late 1950's [3]. A dynamic programming approach to the problem of optimal control leads to a derivation of the HJB equation. It provides a global optimal control law in the form of state feedback. Unfortunately, it involves the solution of the HJB equation, which is in general analytically intractable. The calculus of variations solution, on the other hand, only requires the solution to a two-point boundary value ordinary differential equation, known as the Euler-Lagrange equation, and while still presenting a challenge, it is tractable when compared with the HJB partial differential equation. However, this solution is not equivalent to the one given by the HJB equation. The Euler-Lagrange equation solve a trajectory optimization problem. That is, they provide an open-loop trajectory corresponding to a specific initial condition. Hence, computational tractability is traded for the lack of a global solution [50]. The existence of a solution of the HJB equation is a sufficient condition for the optimal control problem solution [51].

Optimal nonlinear control solution by dynamic programming requires the HJB partial differential equation solution (first order partial differential equation [6]). For the inverse optimal control, Freeman and Kokotović in [17] exploits the fact that for an optimal problem to be meaningful, it is not necessary to completely specify its cost functional.

To appreciate the importance of this dissertation contribution, one should recall that other methods for selecting the control law, based on the cancellation or domination of nonlinear terms (such as feedback linearization, block control, backstepping technique, and other unmentioned nonlinear feedback designs), do not necessarily possess the desirable properties of optimality and may lead to poor robustness and wasted control effort [49]. Even worse, certain nonlinear terms can represent nonlinear positive feedback, which can have catastrophic effects in the presence of modeling or measurement errors [10]. Other approaches such as variable structure technique may lead to wasted control effort.

1.5 Objectives

General Objective:

To establish discrete-time inverse optimal robust control strategies for a class of nonlinear systems based on passivity and CLF approaches, in order to obtain high performance control strategies.

Specific objectives:

- Discrete-time inverse optimal stabilizing control synthesis for affine systems via passivity and CLF approaches.
- Discrete-time inverse optimal stabilizing control synthesis for affine systems to achieve trajectory tracking.
- Discrete-time inverse optimal stabilizing robust control synthesis for affine systems in presence of disturbances.

1.6 Dissertation Outline

This dissertation is organized as follows.

Chapter 2 Fundamentals

This chapter briefly describes useful results on optimal control theory, Lyapunov stability and passivity theory, required in future chapters, for the inverse optimal control problem solution. Section 2.1 gives a review on optimal control, prior to the introduction of the inverse optimal control problem. Section 2.2 presents general stability analysis, and robust stability results are included in Section 2.3 for disturbed nonlinear systems. Section 2.4 establishes concepts related to passivity theory. Finally, Section 2.5 presents a neural scheme in order to identify uncertain nonlinear systems.

Chapter 3 Inverse Optimal Control: A Passivity Approach

This chapter deals with inverse optimal control via passivity for both stabilization and trajectory tracking problems. In Section 3.1, a stabilizing inverse optimal control is synthesized. In Section 3.2, trajectory tracking is presented by modifying the proposed CLF such that it has a global minimum along the desired trajectory. Examples illustrate the proposed control scheme applicability.

Chapter 4 Inverse Optimal Control: A CLF approach

In this chapter, we establish the inverse optimal control and its solution by proposing a quadratic CLF in Section 4.1; first, the CLF depends on a fixed parameter in order to satisfy stability and optimality condition. A posteriori, the speed gradient algorithm is established in Section 4.2 to compute this CLF parameter and it is used in Section 4.3 to solve the inverse optimal control problem. These results are extended for the inverse optimal control trajectory tracking problem in Section 4.4. Additionally, in Section 4.5 an inverse optimal trajectory tracking for block control form nonlinear systems is proposed. Simulation results illustrate the applicability of the proposed control schemes.

Chapter 5 Neural Inverse Optimal Control

This chapter discusses the combination of Section 2.5, Section 3.1, and Section 3.2 results as presented in Section 5.1 to achieve stabilization and trajectory tracking for uncertain nonlinear systems, by using a RHONN scheme to model uncertain nonlinear systems, and then applying the inverse optimal control methodology. Finally, Section 5.2 establishes a block transformation for the neural model in order to solve the inverse optimal trajectory tracking as a stabilization problem for block control form nonlinear systems. Examples illustrate the applicability of the proposed control techniques.

Chapter 6 Conclusions and Future Work

Conclusions about this dissertation are presented in this final chapter. Also, a future work plan is proposed.

Chapter 2

Fundamentals

This chapter briefly describes useful results on optimal control theory, Lyapunov stability and passivity theory, required in future chapters, for the inverse optimal control problem solution. Section 2.1 gives a review on optimal control, prior to the introduction of the inverse optimal control problem. Section 2.2 presents general stability analysis, and robust stability results are included in Section 2.3 for disturbed nonlinear systems. Section 2.4 establishes concepts related to passivity theory. Finally, Section 2.5 presents a neural scheme in order to identify uncertain nonlinear systems.

2.1 Optimal Control

This section is devoted to briefly discuss the optimal control methodology and their limitations.

Consider the discrete-time affine-in-the-input nonlinear system:

$$x(k+1) = f(x(k)) + g(x(k))\,u(k) \tag{2.1}$$

where $x_k \in \mathbb{R}^n$ is the state of the system, $u_k \in \mathbb{R}^m$ is the control input, $f(x_k) : \mathbb{R}^n \to \mathbb{R}^n$ and $g(x_k) : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth maps, $k \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. We consider that $\overline{x}$ is an isolated equilibrium point of $f(x)+g(x)\overline{u}$ with $\overline{u}$ constant, that is, $f(\overline{x})+g(\overline{x})\overline{u} = \overline{x}$. Without loss of generality, we consider $\overline{x} = 0$ for an $\overline{u}$ constant, $f(0) = 0$ and $rank\{g(x_k)\} = m$ $\forall x_k \neq 0$.

From now on, we will write system (2.1) as:

$$x_{k+1} = f(x_k) + g(x_k)\,u_k, \qquad x_0 = x(0) \tag{2.2}$$

and the subscript $k \in \mathbb{Z}^+$ will stand for the value of the functions and/or variables at the time k.

The following meaningful cost functional is associated with system (2.2):

$$\mathcal{J}(x_k) = \sum_{n=k}^{\infty} \left(l(x_n) + u_n^T R(x_n) u_n \right) \tag{2.3}$$

where $\mathcal{J}(x_k) : \mathbb{R}^n \to \mathbb{R}^+$; $l(x_k) : \mathbb{R}^n \to \mathbb{R}^+$ is a positive semidefinite[1] function and $R(x_k) : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a real symmetric positive definite[2] weighting matrix. The meaningful cost functional (2.3) is a performance measure [52]. The entries of $R(x_k)$ may be functions of the system state in order to vary the weighting on control efforts according to the state value [52]. Considering the state feedback control design approach, we assume that the full state x_k is available.

Equation (2.3) can be rewritten as

$$\begin{aligned} \mathcal{J}(x_k) &= l(x_k) + u_k^T R(x_k) u_k + \sum_{n=k+1}^{\infty} l(x_n) + u_n^T R(x_n) u_n \\ &= l(x_k) + u_k^T R(x_k) u_k + \mathcal{J}(x_{k+1}) \end{aligned} \tag{2.4}$$

where we require the boundary condition $\mathcal{J}(0) = 0$ so that $\mathcal{J}(x_k)$ becomes a Lyapunov function [4, 53]. The value of $\mathcal{J}(x_k)$, if finite, then it is a function of the initial state x_0. When $\mathcal{J}(x_k)$ is at its minimum, which is denoted as $\mathcal{J}^*(x_k)$, it is named the *optimal value function*, and it will be used as a Lyapunov function, i.e., $\mathcal{J}^*(x_k) \triangleq V(x_k)$.

From Bellman's optimality principle [54, 55], it is known that, for the infinite horizon optimization case, the value function $V(x_k)$ becomes time invariant and satisfies the discrete-time (DT) Bellman equation [53, 55, 56]

$$V(x_k) = \min_{u_k} \left\{ l(x_k) + u_k^T R(x_k) u_k + V(x_{k+1}) \right\} \tag{2.5}$$

where $V(x_{k+1})$ depends on both x_k and u_k by means of x_{k+1} in (2.2). Note that the DT Bellman equation is solved backward in time [53].

In order to establish the conditions that the optimal control law must satisfy, we define the discrete-time Hamiltonian $\mathcal{H}(x_k, u_k)$ ([57], pages 830–832) as

$$\mathcal{H}(x_k, u_k) = l(x_k) + u_k^T R(x_k) u_k + V(x_{k+1}) - V(x_k). \tag{2.6}$$

The Hamiltonian is a method to adjoin the constraint (2.2) for the performance index (2.3), and then, solving the optimal control problem by minimizing the Hamiltonian without constraints [54].

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[1] A function $l(z)$ is positive semidefinite (or nonnegative definite) function if for all vectors z, $l(z) \geq 0$. In other words, there are vectors z for which $l(z) = 0$, and for all others z, $l(z) > 0$ [52].

[2] A real symmetric matrix R is positive definite if $z^T R z > 0$ for all $z \neq 0$ [52].

A necessary condition the optimal control law u_k should satisfy is $\frac{\partial \mathcal{H}(x_k,u_k)}{\partial u_k} = 0$ [52], which is equivalent to calculate the gradient of (2.5) right-hand side with respect to u_k, then

$$\begin{aligned} 0 &= 2R(x_k)\,u_k + \frac{\partial V(x_{k+1})}{\partial u_k} \\ &= 2R(x_k)\,u_k + g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}}. \end{aligned} \tag{2.7}$$

Therefore, the optimal control law is formulated as

$$u_k^* = -\frac{1}{2}R^{-1}(x_k)\,g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}} \tag{2.8}$$

with the boundary condition $V(0) = 0$; u_k^* is used when we want to emphasize that u_k is optimal.

Moreover, if $\mathcal{H}(x_k, u_k)$ has a quadratic form in u_k and $R(x_k) > 0$, then

$$\frac{\partial^2 \mathcal{H}(x_k, u_k)}{\partial u_k^2} > 0$$

holds as a sufficient condition such that optimal control law (2.8) (globally [52]) minimizes $\mathcal{H}(x_k, u_k)$ and the performance index (2.3) [54].

Substituting (2.8) into (2.5), we obtain the discrete-time Hamilton-Jacobi-Bellman (HJB) equation described by

$$\begin{aligned} V(x_k) &= l(x_k) + \left(-\frac{1}{2}R^{-1}(x_k)\,g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}}\right)^T \\ &\quad \times R(x_k)\left(-\frac{1}{2}R^{-1}(x_k)\,g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}}\right) + V(x_{k+1}) \\ &= l(x_k) + V(x_{k+1}) + \frac{1}{4}\frac{\partial V^T(x_{k+1})}{\partial x_{k+1}}g(x_k)\,R^{-1}(x_k)g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}} \end{aligned} \tag{2.9}$$

which can be rewritten as

$$l(x_k) + V(x_{k+1}) - V(x_k) + \frac{1}{4}\frac{\partial V^T(x_{k+1})}{\partial x_{k+1}}g(x_k)\,R^{-1}(x_k)g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}} = 0. \tag{2.10}$$

The problem of solving the HJB partial-differential equation (2.10) is not straightforward; this is one of the main disadvantages of discrete-time optimal control for nonlinear systems. To overcome this problem, we propose to solve the inverse optimal control problem.

2.2 Lyapunov Stability

In order to establish stability, let recall the following results.

Definition 2.1 (Radially Unbounded Function [30]) *A function $V(x_k)$ satisfying the condition $V(x_k) \to \infty$ as $\|x_k\| \to \infty$ is said to be radially unbounded.*

Theorem 2.1 (Asymptotic Stability [58]) *The equilibrium $x_k = 0$ of (2.2) is globally asymptotically stable if there is a function $V : \mathbb{R}^n \to \mathbb{R}$ such that (i) V is a positive definite function, radially unbounded, and (ii) $-\Delta V(x_k)$ is a positive definite function, where $\Delta V(x_k) = V(x_{k+1}) - V(x_k)$.*

Theorem 2.2 (Exponential Stability [59]) *Suppose that there exists a positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ and constants c_1, c_2, $c_3 > 0$ and $p > 1$ such that*

$$c_1 \|x\|^p \leq V(x_k) \leq c_2 \|x\|^p \tag{2.11}$$

$$\Delta V(x_k) \leq -c_3 \|x\|^p \qquad \forall k \geq 0, \qquad \forall x \in \mathbb{R}^n. \tag{2.12}$$

Then $x_k = 0$ is an exponentially stable equilibrium for system (2.2).

Clearly, exponential stability implies asymptotic stability. The converse is, however not true.

Due to the fact that the inverse optimal control is based on a Lyapunov function, we establish the following definition.

Definition 2.2 (Control Lyapunov Function [60, 61]) *Let $V(x_k)$ be a radially unbounded function, with $V(x_k) > 0$, $\forall x_k \neq 0$ and $V(0) = 0$. If for any $x_k \in \mathbb{R}^n$, there exist real values u_k such that*

$$\Delta V(x_k, u_k) < 0$$

where the Lyapunov difference $\Delta V(x_k, u_k)$ is defined as $V(x_{k+1}) - V(x_k) = V(f(x_k) + g(x_k)\, u_k) \cdot V(x_k)$. Then $V(\cdot)$ is said to be a "discrete-time control Lyapunov function" (CLF) for system (2.2).

Assumption 2.1 *Let assume that $x = 0$ is an equilibrium point for (2.2), and that there exists a control Lyapunov function $V(x_k)$ such that*

$$\alpha_1(\|x_k\|) \leq V(x_k) \leq \alpha_2(\|x_k\|) \tag{2.13}$$

$$\Delta V(x_k, u_k) \leq -\alpha_3(\|x_k\|) \tag{2.14}$$

where α_1, α_2, and α_3 are class $\mathcal{K}_\infty$ functions[3], and $\|\cdot\|$ denotes the usual Euclidean norm. Then, the origin of the system is an asymptotically stable equilibrium point by means of u_k as input.

[3] α_i, $i = 1, 2, 3$ belong to class $\mathcal{K}_\infty$ functions because later, we will select a radially unbounded function $V(x_k)$.

The existence of this CLF is guaranteed by a converse theorem of the Lyapunov stability theory [62].

As special case, the calculus of class $\mathcal{K}_\infty$– functions in (2.13) simplifies when they take the special form $\alpha_i(r) = \kappa_i r^c$, $\kappa_i > 0$, $c = 2$, and $i = 1, 2$. In particular, for a quadratic positive definite function $V(x_k) = \frac{1}{2} x_k^T P x_k$ with a positive definite and symmetric matrix P, then (2.23) results in

$$\lambda_{min}(P) \left\| x \right\|^2 \leq x_k^T P x_k \leq \lambda_{max}(P) \left\| x \right\|^2 \qquad (2.15)$$

where $\lambda_{min}(P)$ is the minimum eigenvalue of matrix P and $\lambda_{max}(P)$ is the maximum eigenvalue of matrix P.

2.3 Robust Stability Analysis

This section reviews the stability results for disturbed nonlinear systems, for which nonvanishing disturbances are considered. We can no longer study stability of the origin as an equilibrium point, nor should we expect the solution of the disturbed system to approach the origin as $k \to \infty$. The best we can hope for is that if the disturbance is small in some sense, then system solution will be ultimately bounded by a small bound [30]. This brings in the concept of ultimate boundedness.

Definition 2.3 (Ultimate Bound [30, 63]) *The solutions of* (2.2) *with* $u_k = 0$ *are said to be* uniformly ultimately bounded *if there exist positive constants* b *and* c*, and for every* $a \in (0, c)$ *there is a positive constant* $T = T(a)$*, such that*

$$\left\| x_0 \right\| < a \Rightarrow \left\| x_k \right\| \leq b, \qquad \forall k \geq k_0 + T \qquad (2.16)$$

where k_0 *is the initial time instant. They are said to be globally uniformly ultimately bounded if (2.16) holds for arbitrarily large* a*. The constant* b *in (2.16) is known as the* ultimate bound.

The following definition of input-to-state stable (ISS) for the solutions of system (2.2) will be used to study stability properties of a class of disturbed discrete-time nonlinear systems. This ISS property attempts to capture the notion of *bounded-input bounded-state* (BIBS). We say that system (2.2) is *uniformly* BIBS stable if bounded initial states and controls produce uniformly bounded trajectories [64]. The simplest way to introduce the notion of ISS system is as a generalization of global asymptotic stability (GAS) of the trivial solution $x_k = 0$ for (2.2) [65].

Recall that a function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K} - function$ if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a $\mathcal{K}_\infty - function$ if it is a $\mathcal{K}$–function and also $\gamma(s) \to \infty$ as

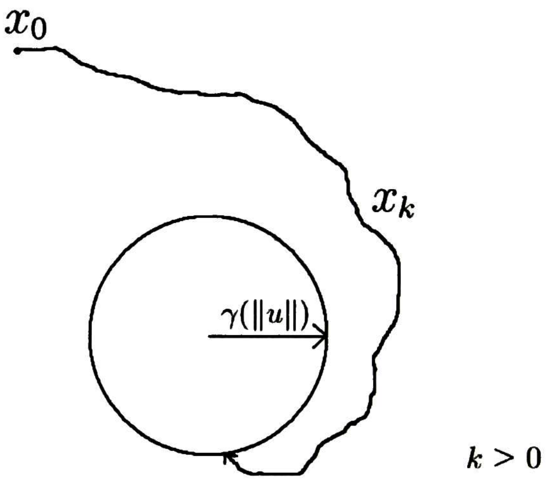

Figure 2.1: System solution trajectories with the ISS property.

$s \to \infty$; and it is a positive definite function if $\gamma(s) > 0$ for all $s > 0$, and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}-function$ if, for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$–function, and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$ [64]. $\mathbb{R}_{\geq 0}$ means nonnegative real numbers.

Definition 2.4 (ISS Property [64, 66]) *System (2.2) is (globally) input-to-state stable with respect to u_k if there exist a $\mathcal{KL}$-funcion β and a $\mathcal{K}$-function γ such that, for each input $u \in l_\infty^m$ and each $x_0 \in \mathbb{R}^n$, it holds that the solution of (2.2) satisfies*

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma\left(\sup_{\tau \in [k_0, \infty)} \|u_\tau\|\right) \tag{2.17}$$

where $\sup_{\tau \in [k_0,\infty)}\{\|u_\tau\| : \tau \in \mathbb{Z}^+\} < \infty$, *which is denoted by* $u \in \ell_\infty^m$.

Thus, system (2.2) is said to be ISS if property (2.17) is satisfied [67].

The interpretation of (2.17) is the following: for a bounded control u, system solution remain in the ball of radius $\beta(\|x_0\|, k) + \gamma\left(\sup_{\tau \in [k_0,\infty)} \|u_\tau\|\right)$ Furthermore, as k increases, all trajectories approach the ball of radius $\gamma\left(\sup_{\tau \in [k_0,\infty)} \|u_\tau\|\right)$ (i.e., all trajectories will be ultimately bounded with ultimate bound γ). Because γ is of class $\mathcal{K}$, this ball is a small neighborhood of the origin whenever $\|u\|$ is small. See Figure 2.1.

Definition 2.5 (Asymptotic Gain Property [64]) *System (2.2) is said to have the $\mathcal{K}$–asymptotic gain if there exists some $\gamma \in \mathcal{K}$ such that*

$$\lim_{k \to \infty} \|x_k(x_0, u)\| \leq \lim_{k \to \infty} \gamma(\|u_k\|) \tag{2.18}$$

for all $x_0 \in \mathbb{R}^n$.

Theorem 2.3 (ISS System [64]) *Consider system (2.2). The following are equivalent:*

(1) It is ISS.

(2) It is BIBS and it admits $\mathcal{K}$*–asymptotic gain.*

Let ℓ_d be the Lipschitz constant such that for all β_1 and β_2 in some bounded neighborhood of (x_k, u_k), the Lyapunov function $V(x_k)$ satisfies the condition ([68])

$$\|V(\beta_1) - V(\beta_2)\| \leq \ell_d \|\beta_1 - \beta_2\|, \qquad \ell_d > 0. \tag{2.19}$$

Definition 2.6 (ISS-Lyapunov Function [64]) *A continuous function* V *on* $\mathbb{R}^n$ *is called an ISS–Lyapunov function for system (2.2) if*

$$\alpha_1(\|x_k\|) \leq V(x_k) \leq \alpha_2(\|x_k\|) \tag{2.20}$$

holds for some $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$, *and*

$$V(f(x_k, u_k)) - V(x_k) \leq -\alpha_3(\|x_k\|) + \sigma(\|u_k\|) \tag{2.21}$$

for some $\alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$. *A smooth ISS-Lyapunov function is one which is smooth.*

Note that if $V(x_k)$ is a ISS–DT Lyapunov function for (2.2), then $V(x_k)$ is a DT Lyapunov function for the 0-input system $x_{k+1} = f(x_k) + g(x_k)\,0$.

Proposition 2.1 *If system (2.2) admits an ISS-Lyapunov function, then it is ISS [64].*

Now, consider the disturbed system

$$x_{k+1} = f(x_k) + g(x_k)\,u_k + d_k, \qquad x_0 = x(0) \tag{2.22}$$

where $x_k \in \mathbb{R}^n$ is the state of the system at time $k \in \mathbb{Z}^+$, $u_k \in \mathbb{R}^m$ is the control, $d_k \in \mathbb{R}^n$ is the disturbance term, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth mappings, $f(0) = 0$. The perturbation term d_k could result from modeling errors, aging, or uncertainties and disturbances which exists for any realistic problem ([30]).

Definition 2.7 (ISS-CLF Function) *A smooth positive definite radially unbounded function* $V\colon \mathbb{R}^n \to \mathbb{R}$ *is said to be an ISS-CLF for system (2.22) if there exists a class* $\mathcal{K}_\infty$ *function* ρ *such that the following implication holds* $\forall x \neq 0$ *and* $\forall d \in \mathbb{R}^n$:

$$\alpha_1(\|x_k\|) \leq V(x_k) \leq \alpha_2(\|x_k\|) \tag{2.23}$$

holds for some $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$, *and*

$$\|x_k\| \geq \rho(\|d_k\|) \Rightarrow \inf_{u_k \in \mathbb{R}^m} \Delta V_d(x_k, d_k) < -\alpha_3(\|x_k\|) \tag{2.24}$$

where $\Delta V_d(x_k, d_k) := V\left(x_{k+1}\right) - V(x_k)$ *and* $\alpha_3 \in \mathcal{K}_\infty$.

Remark 2.1 *The connection between the existence of a Lyapunov function and the input-to-state stability is that, an estimate of the gain function γ in (2.17) is $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$, where $\circ$ means composition[4] of functions with α_1 and α_2 as defined in (2.23) [11].*

Note that if $V(x_k)$ is a ISS–control Lyapunov function for (2.22), then $V(x_k)$ is a control Lyapunov function for the 0-disturbance system $x_{k+1} = f(x_k) + g(x_k)\,u_k$.

Proposition 2.2 (ISS-CLF System) *If system (2.22) admits an ISS–CLF, then it is ISS.*

2.3.1 Optimal Control for Disturbed Systems

For disturbed discrete-time nonlinear system (2.22), the Bellman equation becomes the Isaacs equation described by

$$V(x_k) = \min_{u_k} \left\{ l(x_k) + u_k^T R(x_k)\, u_k + V(x_k, u_k, d_k) \right\} \tag{2.25}$$

and the Hamilton-Jacobi-Isaacs (HJI) equation associated with system (2.22) and cost functional (2.3) is

$$\begin{aligned} 0 &= \inf_{u} \sup_{d \in \mathcal{D}} \left\{ l(x_k) + u_k^T R(x_k)\, u_k + V(x_{k+1}) - V(x_k) \right\} \\ &= \inf_{u} \sup_{d \in \mathcal{D}} \left\{ l(x_k) + u_k^T R(x_k)\, u_k + V(x_k, u_k, d_k) - V(x_k) \right\} \end{aligned} \tag{2.26}$$

where $\mathcal{D}$ is the set of locally bounded functions, and function $V(x_k)$ is unknown. However, finding a solution of HJI equation (2.26) for $V(x_k)$ with (2.8) is the main drawback of the robust optimal control; this solution may not exist or may be pretty difficult to solve [17]. Note that $V(x_{k+1})$ in (2.26) is function of the disturbance term d_k.

2.4 Passivity

Let consider a nonlinear affine system and an output given as

$$x_{k+1} = f(x_k) + g(x_k)\, u_k, \qquad x_0 = x(0) \tag{2.27}$$

$$y_k = h(x_k) + J(x_k)\, u_k \tag{2.28}$$

where $x_k \in \mathbb{R}^n$ is the state of the system at time k, output $y_k \in \mathbb{R}^m$; $h(x_k) : \mathbb{R}^n \to \mathbb{R}^m$, and $J(x_k) : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ are smooth mappings. We assume $h(0) = 0$.

We present definitions, sufficient conditions and key results, which help us to solving the inverse optimal control problem via passivity as follows:

[4] $\alpha_1(\cdot) \circ \alpha_2(\cdot) = \alpha_1(\alpha_2(\cdot))$.

Definition 2.8 (Passivity [69]) *System* (2.27)-(2.28) *is said to be passive if there exists a non-negative function* $V(x_k)$*, called storage function, such that for all* u_k*,*

$$V(x_{k+1}) - V(x_k) \leq y_k^T u_k \tag{2.29}$$

where $(\cdot)^T$ *denotes transpose.*

This storage function may be selected as a candidate CLF if it is a positive definite function [4]. It is worth to note that the output which renders the system passive is not in general the variable we wish to control, and it is used only for control synthesis.

Definition 2.9 (Zero-State Observable System [36]) *A system (2.27)-(2.28) is locally zero-state observable (respectively locally zero-state detectable) if there exists a neighborhood* $\mathcal{Z}$ *of* $x_k = 0$ *in* $\mathbb{R}^n$ *such that for all* $x_0 \in \mathcal{Z}$

$$y_k|_{u_k=0} = h(\phi(k, x_0, 0)) = 0 \quad \forall k \qquad \Longrightarrow x_k = 0 \quad \left(respectively \;\; \lim_{k\to\infty} \phi(k, x_0, 0) = 0\right)$$

where $\phi(k, x_0, 0) = f^k(x_k)$ *is the trajectory of the unforced dynamics* $x_{k+1} = f(x_k)$ *with initial condition* x_0*. If* $\mathcal{Z} = \mathbb{R}^n$*, the system is zero-state observable (respectively zero-state detectable).*

Additionally, in this dissertation, the following definition is introduced.

Definition 2.10 (Feedback Passive System) *System* (2.27)-(2.28) *is said to be feedback passive if there exists a passifying law*

$$u_k = \alpha(x_k) + v_k, \qquad \alpha, v \in \mathbb{R}^m \tag{2.30}$$

with a smooth function $\alpha(x_k)$ *and a storage function* $V(x)$*, such that system* (2.27) *with (2.30), described by*

$$x_{k+1} = \bar{f}(x_k) + g(x_k)\, v_k, \qquad x_0 = x(0) \tag{2.31}$$

and output

$$\bar{y}_k = \bar{h}(x_k) + J(x_k)\, v_k \tag{2.32}$$

satisfies relation (2.29) with v_k *as the new input, where* $\bar{f}(x_k) = f(x_k) + g(x_k)\,\alpha(x_k)$ *and* $\bar{h} : \mathbb{R}^n \to \mathbb{R}^m$ *a smooth mapping, which will be defined latter, with* $\bar{h}(0) = 0$.

To render system (2.27) feedback passive can be summarized as to find a passivation law u_k and an output $\bar{y}_k$, such that relation (2.29) is satisfied with respect to the new input v_k.

2.5 Neural Identification

Analysis of large-scale nonlinear systems requires of a lot of effort, since parameters are difficult to obtain [43]. Hence, to synthesize a controller based on the plant model which have uncertainties is not practical.

For realistic situations, a control based on a plant model can not perform as desired, due to internal and external disturbances, uncertain parameters, or unmodelled dynamics [43]. This fact motivates the need to derive a model based on recurrent high order neural network (RHONN) to identify the dynamics of the plant.

We analyze a general class of systems which are affine in the control with disturbance term as in [70]; the same structure is assumed for the neural network.

2.5.1 Nonlinear Systems

Consider a class of discrete-time disturbed nonlinear system

$$\chi_{k+1} = \bar{f}(\chi_k) + \bar{g}(\chi_k)\, u_k + \Gamma_k \tag{2.33}$$

where $\chi_k \in R^n$ is the system state at time k, $\Gamma_k \in R^n$ is an unknown and bounded perturbation term representing modeling errors, uncertain parameters and disturbances; $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $\bar{g} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth mappings. Without loss of generality, $\chi_k = 0$ is an equilibrium point for (2.33). We assume $\bar{f}(0) = 0$ and $rank\{\bar{g}(\chi_k)\} = m \; \forall \chi_k \neq 0$.

2.5.2 Discrete-Time Recurrent High Order Neural Network

To identify system (2.33), let consider the following discrete-time RHONN proposed in [46]:

$$x_{i,k+1} \;=\; w_{i,k}^T \rho_i(x_k, u_k) \tag{2.34}$$

where $x_k = [x_{1,k}\ x_{2,k}\ \ldots\ x_{n,k}]^T$ x_i is the state of the i-th neuron which identifies the i-th component of state vector χ_k in (2.33), $i = 1, \ldots, n$; w_i is the respective on-line adapted weight vector and $u_k = [u_{1,k}\ u_{2,k}\ \ldots\ u_{m,k}]^T$ is the input vector to the neural network; ρ_i is a L_p dimensional vector defined as

$$\rho_i(x_k, u_k) = \begin{bmatrix} \rho_{i_1} \\ \rho_{i_2} \\ \vdots \\ \rho_{i_{L_p}} \end{bmatrix} = \begin{bmatrix} \prod_{\ell \in I_1} Z_{i_\ell}^{d_{i_\ell}(1)} \\ \prod_{\ell \in I_2} Z_{i_\ell}^{d_{i_\ell}(2)} \\ \vdots \\ \prod_{\ell \in I_{L_p}} Z_{i_\ell}^{d_{i_\ell}(L_p)} \end{bmatrix} \tag{2.35}$$

where d_{i_ℓ} are nonnegative integers, L_p is the respective number p of high-order connections, $\{I_1, I_2, \ldots, I_{L_p}\}$ is a collection of non-ordered subsets of $\{1, 2, \ldots, n+m\}$. Z_i is a vector defined as

$$Z_i = \begin{bmatrix} Z_{i_1} \\ \vdots \\ Z_{i_n} \\ Z_{i_{n+1}} \\ \vdots \\ Z_{i_{n+m}} \end{bmatrix} = \begin{bmatrix} S(x_{1,k}) \\ \vdots \\ S(x_{n,k}) \\ u_{1,k} \\ \vdots \\ u_{m,k} \end{bmatrix}$$

where the sigmoid function $S(\cdot)$ is defined by

$$S(x) = \frac{\alpha_i}{1 + e^{-\beta_i x}} - \gamma_i$$

with $S(\cdot) \in [-\gamma_i, \alpha_i - \gamma_i]$; α_i, β_i and γ_i are positive constants.

We propose the following modification of the discrete-time RHONN (2.34) for the system described by (2.33) [71]:

- Neural weights associated to the control inputs could be fixed ($w_i^{'}$) to ensure controllability of the identifier.

Based on this modification and using the structure of the system (2.33), we propose the following neural network model:

$$x_{i,k+1} = w_{i,k}^T \rho_i(x_k) + w_i^{'T} \psi_i(x_k, u_k) \tag{2.36}$$

in order to identify (2.33), where x_i is the i-th neuron state; $w_{i,k}$ is the on-line adjustable weight vector and $w_i^{'}$ is the fixed weight vector; ψ denotes a function of x or u corresponding to the plant structure (2.33) or external inputs to the network, respectively. Vector ρ_i in (2.36) is like (2.35), however Z_i is redefined as

$$Z_i = \begin{bmatrix} Z_{i_1} \\ \vdots \\ Z_{i_n} \end{bmatrix} = \begin{bmatrix} S(x_{1,k}) \\ \vdots \\ S(x_{n,k}) \end{bmatrix}$$

The on-line adjustable weight vector $w_{i,k}$ is defined as

$$w_{i,k} = \begin{bmatrix} w_{i\,1,k} & \cdots & w_{i\,L_p,k} \end{bmatrix}^T$$

Remark 2.2 *It is worth to notice that* (2.36) *does not consider the disturbance term* (Γ_k) *due the RHONN weights are adjusted on-line, and hence the RHONN identifies the dynamics of the nonlinear system, which includes the disturbance effects.*

RHONN Models

From results presented in [44], we can assume that there exists a RHONN which models (2.33); thereby, plant model (2.33) can be described by

$$\chi_{k+1} = W_k^* \rho(\chi_k) + W'^* \psi(\chi_k, u_k) + v_k \tag{2.37}$$

where $W_k^* = [w_{1,k}^{*T} \; w_{2,k}^{*T} \; \ldots \; w_{n,k}^{*T}]^T$ and $W'^* = [w_1^{*'T} \; w_2^{*'T} \; \ldots \; w_n^{*'T}]^T$ are the optimal unknown weight matrices, and the modelling error v_k is given by

$$v_k = \bar{f}(\chi_k) + \bar{g}(\chi_k)\, u_k + \Gamma_k - W_k^* \rho(\chi_k) - W'^* \psi(\chi_k, u_k).$$

The modelling error term v_k can be render arbitrarily small selecting appropriately the number L_p of high-order connections [44]. The ideal weight matrices W_k^* and W'^* are artificial quantities required for analytical purpose. In general, it is assumed that this vector exists and is constant but unknown. Optimal unknown weight vectors $w_{i,k}^*$ will be approximate by the on-line adjustable weight vectors $w_{i,k}$ [46].

For neural identification of (2.33), two possible models for (2.36) can be used

- Parallel model

$$x_{i,k+1} = w_{i,k}^T \rho_i(x_k) + w_i'^T \psi_i(x_k, u_k) \tag{2.38}$$

- Series–parallel model

$$x_{i,k+1} = w_{i,k}^T \rho_i(\chi_k) + w_i'^T \psi_i(\chi_k, u_k). \tag{2.39}$$

On-line Learning Law

For the RHONN weights on-line learning, we use an EKF [72]. The weights become the states to be estimated; the main objective of the EKF is to find the optimal values for the weight vector $w_{i,k}$ such that the prediction error is minimized. The EKF solution to the training problem is given by the following recursion:

$$\begin{aligned} M_{i,k} &= [R_{i,k} + H_{i,k}^T P_{i,k} H_{i,k}]^{-1} \\ K_{i,k} &= P_{i,k} H_{i,k} M_{i,k} \\ w_{i,k+1} &= w_{i,k} + \eta_i K_{i,k} e_{i,k} \\ P_{i,k+1} &= P_{i,k} - K_{i,k} H_{i,k}^T P_{i,k} + Q_{i,k} \end{aligned} \tag{2.40}$$

where vector $w_{i,k}$ represents the estimate of the i-th weight (state) of the i-th neuron at update step k. This estimate is a function of the Kalman gain K_i and the neural identification error $e_{i,k} = \chi_{i,k} - x_{i,k}$, where χ_i is the plant state and x_i is the RHONN state. The Kalman gain is a function of the approximate error covariance matrix P_i, a matrix of derivatives of the network's outputs with respect to all trainable weight parameters H_i as follow

$$H_{i,k} = \left[\frac{\partial x_{i,k}}{\partial w_{i,k}}\right]^T \tag{2.41}$$

and a global scaling matrix M_i. Here, Q_i is the covariance matrix of the process noise and R_i is the measurement noise covariance matrix. As additional parameter we introduce the rate learning η_i such that $0 \leq \eta_i \leq 1$. Usually P_i, Q_i and R_i are initialized as diagonal matrices, with entries $P_i(0)$, $Q_i(0)$ and $R_i(0)$ respectively. We set to Q_i and R_i fixed. During training, the values of H_i, K_i and P_i are ensured to be bounded [46].

Theorem 2.4 (Neural Identification [46]) *The RHONN (2.34) trained with the EKF based algorithm (2.40) to identify the nonlinear plant (2.33), ensures that the neural identification error* $e_{i,k}$*, is semiglobally uniformly ultimately bounded; moreover, the RHONN weights remain bounded.*

Chapter 3

Inverse Optimal Control: A Passivity Approach

This chapter deals with inverse optimal control via passivity for both stabilization and trajectory tracking problems. In Section 3.1, a stabilizing inverse optimal control is synthesized. In Section 3.2, trajectory tracking is presented by modifying the proposed CLF such that it has a global minimum along the desired trajectory. Examples illustrate the proposed control scheme applicability.

3.1 Inverse Optimal Control via Passivity

In this section, we proceed to develop an inverse optimal control law for system (2.2), which can be globally asymptotically stabilized by the output feedback $u_k = -y_k$. It is worth to mention that, the output with respect to which the system is rendered passive could not be the variable which we wish to control. The passive output will only be a preliminary step for control synthesis; additionally, we need to ensure that the output variables, which we want to control, behaves as desired.

Let us state the conditions to achieve inverse optimality via passivation in the following theorem.

Theorem 3.1 *Assume an affine discrete-time nonlinear system* (2.27) *with input* (2.30) *and output* (2.32)*, which is zero-state detectable. Consider passivity condition* (2.29) *with:*

(1) *a candidate CLF as*

$$V(x_k) = \frac{1}{2} x_k^T P \, x_k, \qquad P = P^T > 0 \tag{3.1}$$

(2) *a control input* (2.30) *with* $\alpha(x_k)$ *defined as*

$$\alpha(x_k) = -\left(I_m + J(x_k)\right)^{-1} h(x_k) \tag{3.2}$$

and v_k as new input, where I_m is the $m \times m$ identity matrix, $(\cdot)^{-1}$ denotes inverse; $h(x_k)$ and $J(x_k)$ are defined as

$$h(x_k) = g^T(x_k)\, P\, f(x_k) \tag{3.3}$$

and

$$J(x_k) = \frac{1}{2}\, g^T(x_k)\, P\, g(x_k). \tag{3.4}$$

If there exists P such that

$$\left(f(x_k) + g(x_k)\,\alpha(x_k)\right)^T P\left(f(x_k) + g(x_k)\,\alpha(x_k)\right) - x_k^T\, P\, x_k \leq 0, \tag{3.5}$$

*then, **(a)** system (2.27) with (2.30) and (2.32) is feedback passive in accordance with Definition 2.10; **(b)** system (2.27) with (2.30) is globally asymptotically stabilized at the equilibrium point $x_k = 0$ by the output feedback $v_k = -\bar{y}_k$ with output $\bar{y}_k = \bar{h}(x_k) + J(x_k)\, v_k$, $\bar{h}(x_k) = g^T(x_k)\, P\, \bar{f}(x_k)$ and $\bar{f}(x_k) = f(x_k) + g(x_k)\,\alpha(x_k)$; **(c)** moreover, with $V(x_k)$ as a CLF, control law (3.2) is inverse optimal in the sense that it minimizes the meaningful functional given as*

$$\mathcal{J}(x_k) = \sum_{k=0}^{\infty} L(x_k, \alpha(x_k)) \tag{3.6}$$

where $L(x_k, \alpha(x_k)) = l(x_k) + \alpha^T(x_k)\,\alpha(x_k)$, $l(x_k) = -\frac{\bar{f}^T(x_k)\, P\, \bar{f}(x_k) - x_k^T\, P\, x_k}{2} \geq 0$, and optimal value function $\mathcal{J}^*(x_0) = V(x_0)$.

Proof. (a) Passivation: System (2.27) with input (2.30) and output (2.32) must be rendered passive, such that the inequality $V(x_{k+1}) - V(x_k) \leq \bar{y}_k^T\, v_k$ is fulfilled with v_k as new input and with the output $\bar{y}_k$ defined as in (2.32). Therefore, from (2.29) we have

$$\begin{aligned}\frac{\left(f(x_k) + g(x_k)\,\alpha(x_k)\right)^T(x_k)\, P\left(f(x_k) + g(x_k)\,\alpha(x_k)\right) - x_k^T\, P\, x_k}{2} \\ + \frac{2\bar{f}^T(x_k)\, P\, g(x_k)\, v_k + v_k^T\, g^T(x_k)\, P\, g(x_k)\, v_k}{2} \\ \leq \bar{h}^T(x_k)\, v_k + v_k^T\, J^T(x_k)\, v_k\end{aligned} \tag{3.7}$$

with $\alpha(x_k)$ as defined in (3.2), $\bar{h}(x_k) = g^T(x_k)\, P\, \bar{f}(x_k)$ and $\bar{f}(x_k) = f(x_k) + g(x_k)\,\alpha(x_k)$. We rewrite inequality (3.7) as follows:

- From the first term of (3.7), we have

$$\left(f(x_k) + g(x_k)\,\alpha(x_k)\right)^T P\left(f(x_k) + g(x_k)\,\alpha(x_k)\right) - x_k^T\, P\, x_k \leq 0. \tag{3.8}$$

- $2\,\bar{f}^{\,T}(x_k)\,P\,g(x_k)\,v_k = 2\,\bar{h}^T(x_k)\,v_k$, then

$$\bar{h}(x_k) = g^T(x_k)\,P\,\bar{f}(x_k). \tag{3.9}$$

- $v^T\,g^T(x_k)\,P\,g(x_k)\,v_k = 2\,v_k^T\,J^T(x_k)\,v_k$, then

$$J(x_k) = \frac{1}{2}\,g^T(x_k)\,P\,g(x_k). \tag{3.10}$$

Assume that system (2.27) with input (2.30) and output (2.32) fulfills the zero-state detectability property; from (3.8)-(3.10), we deduce that, if there exists P such that (3.8) is satisfied, then system (2.27) with (2.30) and (2.32) is feedback passive, with $V(x_k)$ as storage function, and $\bar{h}(x_k)$ and $J(x_k)$ as defined in (3.9) and (3.10), respectively.

(b) Stability: In order to show the asymptotic stability of closed-loop system (2.27), (2.30) with output feedback [73]

$$\begin{aligned} v_k &= -\bar{y}_k \\ &= -(I_m + J(x_k))^{-1}\,\bar{h}(x_k), \end{aligned} \tag{3.11}$$

let consider

$$\begin{aligned} \Delta V(x_k) &:= V(x_{k+1}) - V(x_k) \\ &= \frac{[\bar{f}(x_k) + g(x_k)\,v_k]^T\,P\,[\bar{f}(x_k) + g(x_k)\,v_k] - x_k^T\,P\,x_k}{2} \\ &= \frac{\bar{f}^T(x_k)\,P\,\bar{f}(x_k) - x_k^T\,P\,x_k}{2} - \bar{f}^T(x_k)\,P\,g(x_k)\,v_k + v_k^T\,\frac{1}{2}\,g^T(x_k)\,P\,g(x_k)\,v_k \\ &= \frac{\bar{f}^T(x_k)\,P\,\bar{f}(x_k) - x_k^T\,P\,x_k}{2} + \bar{h}^T(x_k)\,(I_m + J(x_k))^{-1}\,\bar{h}(x_k) \\ &\quad + \bar{h}^T(x_k)\,(I_m + J(x_k))^{-1}\,\frac{1}{2}\,g^T(x_k)\,P\,g(x_k)\,(I_m + J(x_k))^{-1}\,\bar{h}(x_k) \end{aligned} \tag{3.12}$$

and noting that $\frac{1}{2}\,g^T(x_k)\,P\,g(x_k) = \left(I_m + \frac{1}{2}\,g^T(x_k)\,P\,g(x_k)\right) - I_m = (I_m + J(x_k)) - I_m$, we obtain

$$\begin{aligned} \Delta V(x_k) &= \frac{\bar{f}^T(x_k)\,P\,\bar{f}(x_k) - x_k^T\,P\,x_k}{2} + \bar{h}^T(x_k)\,(I_m + J(x_k))^{-1}\,\bar{h}(x_k) \\ &\quad + \bar{h}^T(x_k)\,(I_m + J(x_k))^{-1}\,[(I_m + J(x_k)) - I_m]\,(I_m + J(x_k))^{-1}\,\bar{h}(x_k) \\ &= \frac{\bar{f}^T(x_k)\,P\,\bar{f}(x_k) - x_k^T\,P\,x_k}{2} - \bar{h}^T(x_k)\,(I_m + J(x_k))^{-2}\,\bar{h}(x_k). \end{aligned} \tag{3.13}$$

From (3.11), we have $v_k = -\,(I_m + J(x_k))^{-1}\,\bar{h}(x_k)$, and under condition (3.8), (3.13) results in

$$\begin{aligned} \Delta V(x_k) &= \frac{\bar{f}^T(x_k)\,P\,\bar{f}(x_k) - x_k^T\,P\,x_k}{2} - \|v_k\|^2 \\ &< 0. \end{aligned} \tag{3.14}$$

Since $V(x_k) \to \infty$ as $\|x_k\| \to \infty$, then $V(x_k)$ is a radially unbounded function; therefore, the solution $x_k = 0$ of the closed-loop system (2.27) with (2.30) and output feedback (2.32), is globally asymptotically stable.

(c) Optimality: Control law (3.2) is established to be an inverse optimal law since, (i), it stabilizes system (2.27) according to **(b)**; and (ii), it minimizes the cost functional (3.6).

In order to show minimization of (3.6) by means of $\alpha(x_k)$, we firstly obtain control law (3.2) as calculating the gradient of Hamiltonian (2.6) with respect to $\alpha(x_k)$ such that $\frac{\partial \mathcal{H}(x_k,\alpha(x_k))}{\partial \alpha(x_k)} = 0$, with $L(x_k, \alpha(x_k))$ as defined in (3.6), and then solving this last equality for $\alpha(x_k)$. So, using (2.6) we have

$$\begin{aligned} 0 &= \min_{\alpha(x_k)} \{L(x_k, \alpha(x_k)) + V(x_{k+1}) - V(x_k)\} \\ &= \min_{\alpha(x_k)} \{l(x_k) - y_k^T\, \alpha(x_k) + V(x_{k+1}) - V(x_k)\} \\ &= -h^T(x_k) - \alpha^T(x_k)\,(J(x_k) + J^T(x_k)) + (f^T(x_k) - y_k^T\, g^T(x_k))\, P\, g(x_k) \\ &= -h^T(x_k) - \alpha^T(x_k)\,(J(x_k) + J^T(x_k)) + f^T(x_k)\, P\, g(x_k) - y_k^T\, g^T(x_k)\, P\, g(x_k). \end{aligned} \tag{3.15}$$

Considering $h^T(x_k) = f^T(x_k)\, P\, g(x_k)$ (3.3), we obtain

$$0 = -\alpha^T(x_k)\, J(x_k) - h^T(x_k)\, J(x_k) - \alpha^T(x_k)\, J^T(x_k)\, J(x_k)$$

then

$$\alpha^T(x_k)\,(J(x_k) + J^T(x_k)\, J(x_k)) = -h^T(x_k)\, J(x_k)$$

and finally

$$(J(x_k) + J^2(x_k))\, \alpha(x_k) = -J(x_k)\, h(x_k). \tag{3.16}$$

From (3.16), it immediately follows that the inverse optimal control law is

$$\alpha(x_k) \;=\; -\,(I_m + J(x_k))^{-1}\, h(x_k)$$

as established in (3.2). In order to obtain the optimal value function ($\mathcal{J}^*$) for the meaningful cost functional (3.6), we solve (2.6) for $L(x_k, \alpha(x_k))$, and adding over $[0, N]$, where $N \in \mathbb{Z}^+$, yields

$$\sum_{k=0}^{N} L(x_k, \alpha(x_k)) = -V(x_N) + V(x_0) + \sum_{k=0}^{N} H(x_k, \alpha(x_k)).$$

Letting $N \to \infty$ and noting that $V(x_N) \to 0$ for any x_0, and $H(x_k, \alpha(x_k)) = 0$ for inverse optimal control $\alpha(x_k)$, then $\mathcal{J}^*(x_0) = V(x_0)$, which is named as the optimal value function. Then, inverse optimality for (3.6) is established. ■

Corollary 3.1 *If system* (2.27) *with input* (2.30) *and output* (2.32) *is single input single output, the control law is given by*

$$\alpha(x_k) = -(1 + J(x_k))^{-1} h(x_k) \tag{3.17}$$

where $h(x_k)$ and $J(x_k)$ are defined as in Theorem 3.1.

Corollary 3.2 *Consider that function $g(x_k)$ in (2.27) is entry-wise bounded, i.e., $|g_{ij}(x_k)| \leq b_{ij} > 0$, $\forall x \in \mathbb{R}^n$, $i = 1, \ldots, n$, $j = 1, \ldots, m$*

$$g(x_k) = \begin{bmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nm} \end{bmatrix} \leq \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} = B_1. \tag{3.18}$$

Then, solution of inequality (3.5) *for $P = P^T > 0$ reduces to the solution of the following matrix inequality (MI):*

$$\begin{bmatrix} P - 2\,P\,B_1\left(I_m + \frac{1}{2}\,B_1^T\,P\,B_1\right)^{-2}\,B_1^T\,P & 0 \\ 0 & -P \end{bmatrix} < 0 \tag{3.19}$$

for P

Proof. Using (3.18) and $\alpha(x_k)$ in (3.2) with $\alpha(x_k) = -(I_m + \frac{1}{2}\,B_1^T\,P\,B_1)^{-1}\,B_1^T\,P\,f(x_k)$, the left-hand side of (3.5) reduces to

$$\begin{aligned}
&\left(f(x_k) + g(x_k)\,\alpha(x_k)\right)^T P\left(f(x_k) + g(x_k)\,\alpha(x_k)\right) - x_k^T\,P\,x_k \\
&\quad = f^T(x_k)\,P\,f(x_k) + 2\,f^T(x_k)\,P\,B_1\,\alpha(x_k) + \alpha^T(x_k)\,B_1^T\,P\,B_1\,\alpha(x_k) - x_k^T\,P\,x_k \\
&\quad = f^T(x_k)\,P\,f(x_k) - x_k^T\,P\,x_k - 2\,h^T(x_k)\,\left(I_m + J(x_k)\right)^{-1}\,h(x_k) \\
&\qquad + h^T(x_k)\,\left(I_m + J(x_k)\right)^{-1}\,B_1^T\,P\,B_1\,\left(I_m + J(x_k)\right)^{-1}\,h(x_k).
\end{aligned} \tag{3.20}$$

Note that $h(x_k) = B_1^T\,P\,f(x_k)$, $J(x_k) = \frac{1}{2}\,B_1^T\,P\,B_1$ and $B_1^T\,P\,B_1 = 2(I_m + J(x_k)) - 2\,I_m$; then, (3.5) is satisfied by finding P for (3.20) such that the following inequality holds:

$$\begin{aligned}
&f^T(x_k)\,P\,f(x_k) - x_k^T\,P\,x_k - 2\,h^T(x_k)\,\left(I_m + J(x_k)\right)^{-1}\,h(x_k) \\
&+ h^T(x_k)\,\left(I_m + J(x_k)\right)^{-1}\,\left[2(I_m + J(x_k)) - 2\,I_m\right]\,\left(I_m + J(x_k)\right)^{-1}\,h(x_k) \\
&\quad = \quad f^T(x_k)\,P\,f(x_k) - x_k^T\,P\,x_k - 2\,h^T(x_k)\,\left(I_m + J(x_k)\right)^{-2}\,h(x_k) \\
&\quad = \quad f^T(x_k)\,P\,f(x_k) - 2\,f^T(x_k)\,P\,B_1\left(I_m + \frac{1}{2}\,B_1^T\,P\,B_1\right)^{-2}\,B_1^T\,P\,f(x_k) - x_k^T\,P\,x_k \\
&\quad = \quad f^T(x_k)\left[P - 2\,P\,B_1\left(I_m + \frac{1}{2}\,B_1^T\,P\,B_1\right)^{-2}\,B_1^T\,P\right]\,f(x_k) - x_k^T\,P\,x_k \\
&\quad = \quad \begin{bmatrix} f(x_k) & x_k \end{bmatrix}^T \begin{bmatrix} P - 2\,P\,B_1\left(I_m + \frac{1}{2}\,B_1^T\,P\,B_1\right)^{-2}\,B_1^T\,P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} f(x_k) \\ x_k \end{bmatrix} \\
&\quad < \quad 0
\end{aligned} \tag{3.21}$$

which is guaranteed by the solution of (3.19). ■

Remark 3.1 *It is worth to note that full state measurement is not been considered. Instead of, detectability property is required.*

3.1.1 A Class of Nonlinear Systems

Let consider a special class of systems for (2.27), as:

$$x_{k+1} = A\,x_k + \varphi(x_k) + B\,u_k, \qquad x_0 = x(0). \tag{3.22}$$

where $A = \frac{\partial f}{\partial x_k}\Big|_{x_k=0} \in \mathbb{R}^{n\times n}$, $B = g(0)$ and $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a norm-bounded function inside a sufficiently small neighborhood Ω of $x_k = 0$ in $\mathbb{R}^n$, which represents the higher-order terms with $\|\varphi(x_k)\| \leq b_0\,\|x_k\|$ and $b_0 > 0$.

Hence, we establish the following result.

Theorem 3.2 *Consider system* (3.22) *with input* $u_k = \alpha(x_k) + v_k$, $\alpha(x_k)$ *defined as in* (3.2), *and the output defined by* (2.32) *to be zero-state detectable. There always exists a quadratic control Lyapunov function* $V(x_k) = x_k^T\,P\,x_k$, $P = P^T > 0$ *such that* (3.22) *with* $u_k = \alpha(x_k) + v_k$ *and* (2.32) *is feedback passive with* v_k *as new input for all* $x_k \in \Omega$ *and* $c_3 > c_4$, *where* c_3 *is a positive constant and* c_4 *is a constant. Furthermore, the equilibrium point* $x_k = 0$ *of* (3.22) *is exponentially stabilized by the output feedback* $v_k = -\bar{y}_k$. *If* $\Omega = \mathbb{R}^n$, *global exponential stability is achieved.*

Moreover, with $V(x_k)$ *as a CLF, control law* (3.2) *is inverse optimal in the sense that minimizes the meaningful functional given as in* (4.6).

Proof. Along the lines of the analysis for Theorem 3.1, this proof can be reduced to only demonstrate that condition (3.8) is upper bounded as $\bar{f}^{\,T}(x_k)\,P\,\bar{f}(x_k) - x_k^T\,P\,x_k \leq -(c_3 - c_4)\,\|x_k\|^2$ with $\bar{f}(x_k) = A\,x_k + \varphi(x_k) + B\,\alpha(x_k)$. Considering control law (3.2) with $h(x_k) = B^T\,P\,f(x_k)$, $f(x_k) = A\,x_k + \varphi(x_k)$ and $J(x_k) = \frac{1}{2}\,B^T\,P\,B$, and taking into account (2.11)-(2.12) with $p = 2$, $c_1 = \lambda_{min}(P)$ and $c_2 = \lambda_{max}(P)$, where $\lambda_{min(max)}(P)$ denotes the minimum (maximum) eigenvalue of matrix P, then the left-hand side of (3.5) becomes

$$\begin{aligned}
f^T(x_k)Pf(x_k) - x_k^T P x_k - 2\,f^T(x_k)\,P\,B(x_k)\left(I_m + \frac{1}{2}\,g^T(x_k)P\,g(x_k)\right)^{-2} g^T(x_k)\,P\,f(x_k) \\
= [A\,x_k + \varphi(x_k)]^T\,P\,[A\,x_k + \varphi(x_k)] - x_k^T\,P\,x_k - 2\,[A\,x_k + \varphi(x_k)]^T\,P\,B \\
\times\left(I_m + \frac{1}{2}\,B^T P\,B\right)^{-2} B^T\,P\,[A\,x_k + \varphi(x_k)]
\end{aligned}$$

$$
\begin{aligned}
&= x_k^T A^T P A x_k + 2\varphi^T(x_k) P A x_k + \varphi^T(x_k) P \varphi(x_k) - x_k^T P x_k \\
&\quad -2 x_k^T A^T P B \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T P A x_k - 4\varphi^T(x_k) P B \\
&\quad \times \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T P A x_k - 2\varphi^T(x_k) P B \\
&\quad \times \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T P \varphi^T(x_k) \\
&\leq x_k^T \left[A^T P A - P - 2 A^T P B \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T P A\right] x_k \\
&\quad +2 b_0 \lambda_{max}(P) \|A\| \|x_k\|^2 + b_0^2 \lambda_{max}(P) \|x_k\|^2 - 4 b_0 \lambda_{min}^2(P) \|A\| \Bigg\| B \\
&\quad \times \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T \Bigg\| \|x_k\|^2 - 2 b_0^2 \lambda_{min}^2(P) \\
&\quad \times \Bigg\| B \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T \Bigg\| \|x_k\|^2 \\
&\leq x_k^T \left[A^T \left(I_n - 2 B P \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T\right) P A - P\right] x_k \\
&\quad +b_0\Big(2 b_0 \|A\| + 1\Big)\lambda_{max}(P) \|x_k\|^2 - 2 b_0 \Big(2 \|A\| + b_0\Big) \lambda_{min}^2(P) \Bigg\| B \\
&\quad \times \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T \Bigg\| \|x_k\|^2 \\
&\leq x_k^T \left[\varsigma^2 A^T P A - P\right] x_k + b_0\Big(2 b_0 \|A\| + 1\Big)\lambda_{max}(P) \|x_k\|^2 \\
&\quad -2 b_0 \Big(2 \|A\| + b_0\Big) \lambda_{min}^2(P) \Bigg\| B \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T \Bigg\| \|x_k\|^2 \qquad (3.23)
\end{aligned}
$$

where I_n is the $n \times n$ identity matrix and ς^2 is a scalar defined as

$$
\varsigma^2 = \Bigg\| I_n - 2 B P \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T \Bigg\|.
$$

For $0 < \|P\| < \infty$, it will yields $\varsigma^2 \in (0, 1)$. Hence, (3.23) reduces to

$$
\leq x_k^T \left[(\varsigma A)^T P (\varsigma A) - P\right] x_k + c_4 \|x_k\|^2 \qquad (3.24)
$$

where

$$
c_4 = b_0 \Big(2 b_0 \|A\| + 1\Big)\lambda_{max}(P) - 2 b_0 \Big(2 \|A\| + b_0\Big) \lambda_{min}^2(P) \Bigg\| B \left(I_m + \frac{1}{2} B^T P B\right)^{-2} B^T \Bigg\|.
$$

Due to the fact that $\varsigma \in (0,1)$, then for an appropriate value of $P = P^T > 0$, the eigenvalues of (ςA) will be placed inside the unit circle. It ensures the existence of P such that $x_k^T \left[(\varsigma A)^T P (\varsigma A) - P\right] x_k + c_4 \|x_k\|^2 \leq 0$ is guaranteed for all $x_k \in \Omega$. As result, system (3.22) with input $u_k = \alpha(x_k) + v_k$ and output (2.32) with $\bar{h}(x_k) = B^T P (A x_k + \varphi(x_k) + B \alpha(x_k))$, is feedback passive.

Furthermore, by means of P, we can achieve a desired negativity amount [26] for

$$x_k^T \left[(\varsigma A)^T P (\varsigma A) - P\right] x_k$$

in (3.24). This negativity amount can be assigned using a positive definite matrix Q as follows:

$$\begin{aligned} x_k^T \left[(\varsigma A)^T P (\varsigma A) - P\right] x_k &\leq -x_k^T Q x_k \\ &\leq -\lambda_{min}(Q) \|x_k\|^2 \\ &= -c_3 \|x_k\|^2 \qquad c_3 = \lambda_{min}(Q). \end{aligned} \tag{3.25}$$

Hence, from (3.24)-(3.25), we obtain

$$\bar{f}^{\,T}(x_k) P \bar{f}(x_k) - x_k^T P x_k \leq -(c_3 - c_4) \|x_k\|^2$$

Finally, exponential stability will be achieved for $c_3 > c_4$ by output feedback $v_k = -\bar{y}_k$, in which condition (3.14) results in

$$\begin{aligned} \Delta V(x_k) &= -\frac{c_3 - c_4}{2} \|x_k\|^2 - \|v_k\|^2 \\ &< 0. \end{aligned} \tag{3.26}$$

Moreover, if all the assumptions hold globally $\forall x_k \in \Omega$ and $\Omega = \mathbb{R}^n$, then the origin is globally exponentially stable.

The minimization of meaningful cost follows along the same lines as that of Theorem 3.1.

■

3.2 Trajectory Tracking

In this section, we modify the CLF (3.1) such that the new storage function (energy function) has a global minimum along the desired trajectory $x_{\delta,k}$.

To achieve tracking, first we redefine the CLF (3.1) as

$$V(x_k, x_{\delta,k}) = \frac{1}{2}(x_k - x_{\delta,k})^T K^T P K (x_k - x_{\delta,k}) \tag{3.27}$$

where $x_{\delta,k}$ is the desired trajectory and K is an additional gain matrix introduced to modify the convergence rate of the tracking error.

Theorem 3.3 *Assume an affine discrete-time nonlinear system (2.27), and define an output as*

$$y_k = h(x_k, x_{\delta,k+1}) + J(x_k)\, u_k \tag{3.28}$$

which is zero-state detectable. Consider a candidate CLF defined by (3.27) in order to satisfy the modified passivity condition

$$V(x_{k+1}, x_{\delta,k+1}) - V(x_k, x_{\delta,k}) \leq y_k^T\, u_k. \tag{3.29}$$

If there exists $\overline{P} = \overline{P}^T > 0$*, such that*

$$f^T\,\overline{P}\,f + x_{\delta,k+1}^T\,\overline{P}\,x_{\delta,k+1} - 2\,f^T\,\overline{P}\,x_{\delta,k+1} - (x_k - x_{\delta,k})^T\,\overline{P}\,(x_k - x_{\delta,k}) \leq 0 \tag{3.30}$$

where $\overline{P} = K^T\,P\,K$*; then, system (2.27) with output (3.28), is passive and the system solution is globally asymptotically stabilized along the desired trajectory* $(x_{\delta,k})$*, by the output feedback*

$$\begin{aligned} u_k &= -y_k \\ &= -\left(I_m + J(x_k)\right)^{-1} h(x_k, x_{\delta,k+1}) \end{aligned} \tag{3.31}$$

with

$$h(x_k, x_{\delta,k+1}) = g^T(x_k)\,\overline{P}\,(f(x_k) - x_{\delta,k+1})$$

and

$$J(x_k) = \frac{1}{2}\,g^T(x_k)\,\overline{P}\,g(x_k).$$

Moreover, with (3.27) as a CLF, this control law is inverse optimal in the sense that minimizes the meaningful functional given as

$$\mathcal{J}(x_k, x_{\delta,k}) = \sum_{k=0}^{\infty} L(x_k, x_{\delta,k}, u_k) \tag{3.32}$$

where $L(x_k, x_{\delta,k}, u_k)$ *is a non-negative function.*

Proof. Let (3.27) be a candidate CLF. System (2.27) with output (3.28), must be rendered passive, such that the inequality (3.29) is fulfilled. Thus, from (3.29), and considering one step ahead for $x_{\delta,k}$, we have

$$\frac{(x_{k+1} - x_{\delta,k+1})^T\,K^T\,P\,K\,(x_{k+1} - x_{\delta,k+1})}{2} - \frac{(x_k - x_{\delta,k})^T\,K^T\,P\,K\,(x_k - x_{\delta,k})}{2} \leq h^T(x_k, x_{\delta,k})u_k + u_k^T J^T(x_k) u_k. \tag{3.33}$$

Defining $\overline{P} = K^T\,P\,K$ and substituting (2.27) in (3.33), we obtain

$$\frac{(f + g\,u_k - x_{\delta,k+1})^T \overline{P}\,(f + g\,u_k - x_{\delta,k+1})}{2} - \frac{(x_k - x_{\delta,k})^T \overline{P}\,(x_k - x_{\delta,k})}{2} \leq h^T u_k + u_k^T J^T u_k. \quad (3.34)$$

Then, (3.34) becomes

$$f^T \overline{P}\, f + x_{\delta,k+1}^T \overline{P}\, x_{\delta,k+1} - 2\, f^T \overline{P}\, x_{\delta,k+1} - (x_k - x_{\delta,k})^T \overline{P}\,(x_k - x_{\delta,k}) + (2 f^T \overline{P}\, g - 2 x_{\delta,k+1}^T \overline{P}\, g) u_k + u_k^T g^T \overline{P}\, g\, u_k \leq 2\, h^T u_k + 2\, u_k^T J^T u_k. \quad (3.35)$$

From (3.35), passivity is achieved if:

- From the first term of (3.35), we can find $\overline{P} = \overline{P}^T > 0$ such that

$$f^T \overline{P}\, f + x_{\delta,k+1}^T \overline{P}\, x_{\delta,k+1} - 2\, f^T \overline{P}\, x_{\delta,k+1} - (x_k - x_{\delta,k})^T \overline{P}\,(x_k - x_{\delta,k}) \leq 0. \quad (3.36)$$

- $(2 f^T \overline{P} g - 2 x_{\delta,k+1}^T \overline{P}\, g)\, u_k = 2\, h^T u_k$, then

$$h(x_k, x_{\delta,k+1}) = g^T(x_k)\, \overline{P}\,(f(x_k) - x_{\delta,k+1}). \quad (3.37)$$

- $u^T g^T \overline{P}\, g\, u_k = 2\, u_k^T J^T u_k$, then

$$J(x_k) = \frac{1}{2}\, g^T(x_k)\, \overline{P}\, g(x_k). \quad (3.38)$$

If system (2.27) with output (3.28) fulfill the zero-state detectability property, from (3.36)-(3.38) we deduce that, if there exist a $\overline{P}$, such that (3.36) is satisfied, then system (2.27) with output (3.28) is passive, where $h(x_k, x_{\delta,k+1})$ and $J(x_k)$ are defined as (3.37) and (3.38), respectively. To guarantee asymptotic trajectory tracking, we choose $u_k = -y_k$ and then condition (3.29) defined as $V(x_{k+1}, x_{\delta,k+1}) - V(x_k, x_{\delta,k}) \leq -y_k^T y_k < 0\ \forall x_k \neq 0$ is satisfied. Due that $V(x_k, x_{\delta,k})$ is a radially unbounded function, the solution of the closed-loop system (2.27) with (3.31) as input, is globally asymptotically stable along the desired trajectory $(x_{\delta,k})$.

The minimization of meaningful cost functional is established similarly as in Theorem 3.1, and hence it is omitted. ∎

Comment 3.1 *If passivity condition* (3.29) *is not satisfied, then a passivation procedure, as proposed in Theorem 3.1, must be established, with* (3.27) *as CLF.*

Comment 3.2 *Theorem 3.3 guarantees that system* (2.27) *indeed tracks the desired trajectory* $x_{\delta,k}$.

3.3 Examples

In this section, for the proposed inverse optimal control scheme based on passivity, we illustrate the applicability of the obtained results by means of two examples.

3.3.1 Unstable Nonlinear System

Stabilization

We synthesize an inverse optimal control law for a discrete-time second order nonlinear system (unstable for $u_k = 0$) of the form (2.27) with:

$$f(x_k) = \begin{bmatrix} 2.2\, sin(0.5\, x_{1,k}) + 0.1\, x_{2,k} \\ 0.1\, x_{1,k}^2 + 1.8\, x_{2,k} \end{bmatrix} \tag{3.39}$$

and

$$g(x_k) = \begin{bmatrix} 0 \\ 2 + 0.1\, cos(x_{2,k}) \end{bmatrix} \tag{3.40}$$

According to (3.2), the inverse optimal control law is formulated as

$$\alpha(x_k) = -\left(1 + \frac{1}{2}\, g^T(x_k)\, P\, g(x_k)\right)^{-1} g^T(x_k)\, P\, f\,(x_k) \tag{3.41}$$

where we propose a positive definite matrix P as

$$P = \begin{bmatrix} 0.20 & 0.15 \\ 0.15 & 0.20 \end{bmatrix}$$

The phase portrait for this open-loop ($u_k = 0$) unstable system with initial conditions $x_0 = [2 \quad -2]^T$ is displayed in Figure 3.1.

Figure 3.2 presents the stabilization time response for x_k of this system with initial conditions $x_0 = [2 \quad -2]^T$; this figure also includes the applied inverse optimal control law (3.41), which achieves asymptotic stability; the respective phase portrait is displayed in Figure 3.3.

Trajectory Tracking

In accordance with Theorem 3.3, the control law for trajectory tracking is given in (3.31) as $u_k = -y_k$, in which $y_k = h(x_k, x_{\delta,k+1}) + J(x_k)\, u_k$, where

$$h(x_k, x_{\delta,k+1}) = g^T(x_k)\, \overline{P}\, (f(x_k) - x_{\delta,k+1})$$

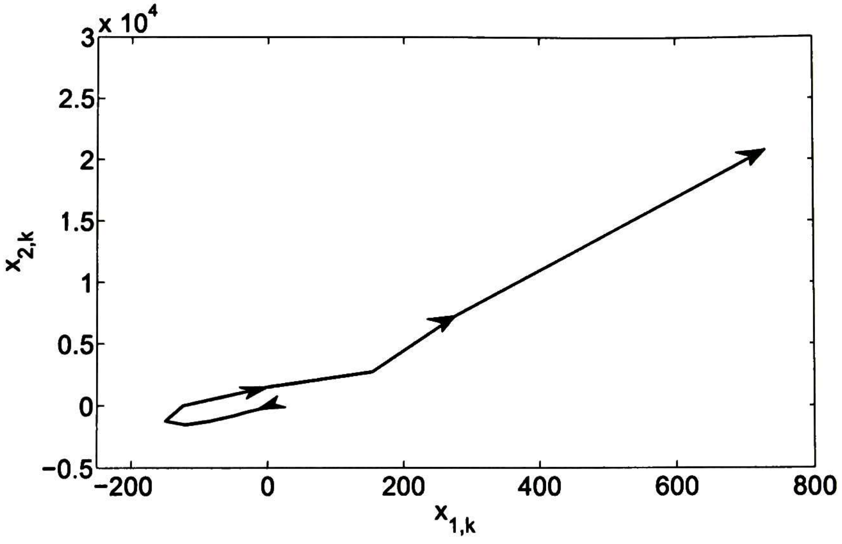

Figure 3.1: Unstable phase portrait.

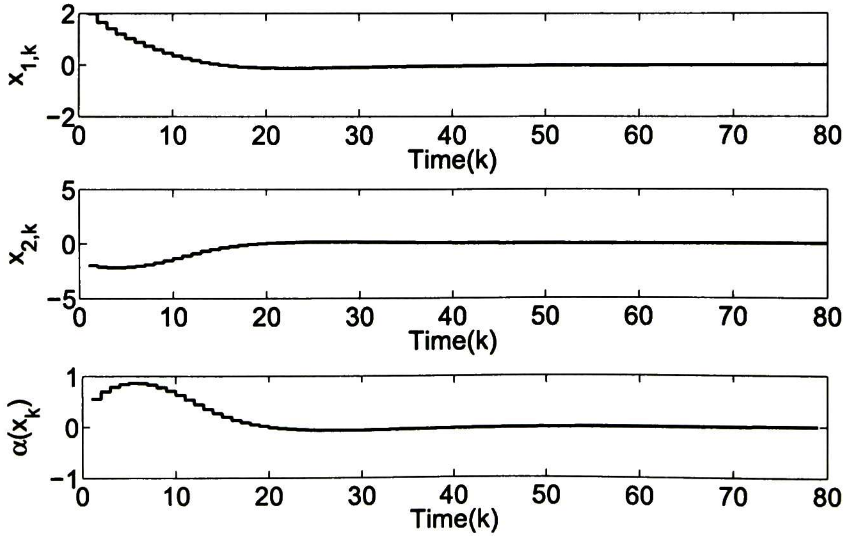

Figure 3.2: Stability time response for (2.27) with (3.39)-(3.40) and (3.41).

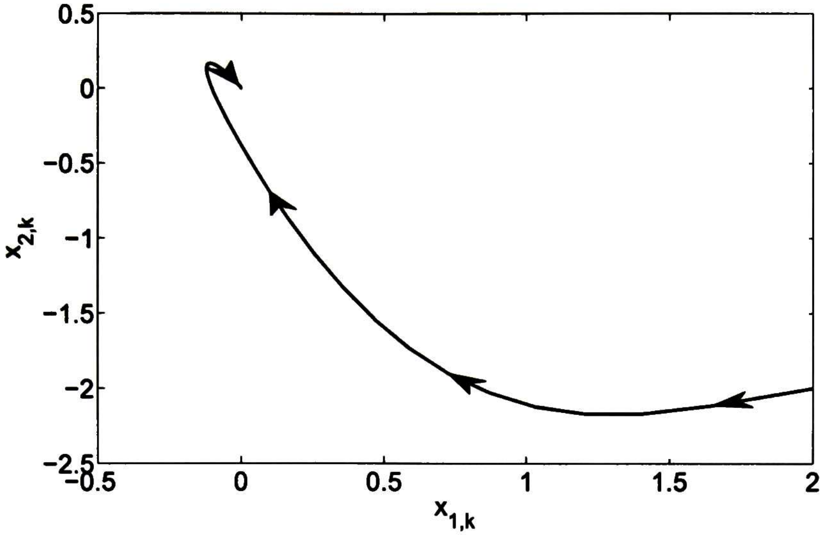

Figure 3.3: Stabilized phase portrait for (2.27) with (3.39)-(3.40) and (3.41).

and

$$J(x_k) = \frac{1}{2} g^T(x_k) \overline{P} g(x_k)$$

with $f(x_k)$ and $g(x_k)$ as defined in (3.39) and (3.40), respectively. Hence, we adjust gain matrix $\overline{P} = K^T P K$ for (3.31) in order to achieve trajectory tracking for $x_k = [x_{1,k}\ x_{2,k}]^T$ The signal reference for $x_{2,k}$ is:

$$x_{2\delta,k} = 1.5\, sin(0.12\, k)\ rad.$$

and reference $x_{1\delta,k}$ is defined accordingly.

Figure 3.4 presents the trajectory tracking for x_k with

$$P = \begin{bmatrix} 0.00340 & 0.00272 \\ 0.00272 & 0.00240 \end{bmatrix}; \quad K = \begin{bmatrix} 0.10 & 0 \\ 0 & 12.0 \end{bmatrix}$$

3.3.2 Planar Robot

We apply Section 3.2 results to synthesize position trajectory tracking control for a two DOF planar rigid robot. The robot model is described in Appendix A.

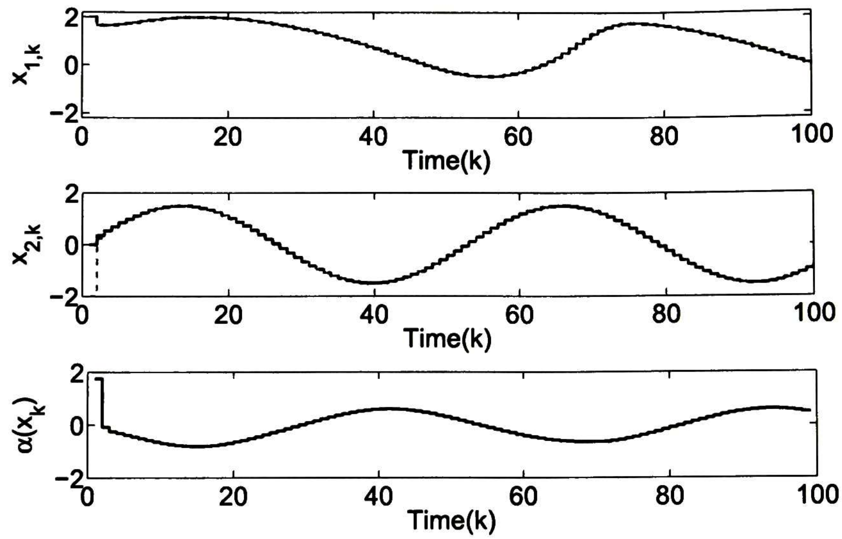

Figure 3.4: Tracking performance of x_k. Solid line ($x_{\delta,k}$) is the reference signal and dashed line is the evolution of x_k. Control signal is also displayed.

Robot as an Affine System

In order to easy the controller synthesis, we rewrite (A.1) in a block structure form as

$$\begin{aligned} x_{12,k+1} &= f_1(x_k) \\ x_{34,k+1} &= f_2(x_k) + g(x_k)\, u(x_k), \qquad x_0 = x(0) \end{aligned} \tag{3.42}$$

where $x_k = [x_{12,k}^T \;\; x_{34,k}^T]^T$ being $x_{12,k} = [x_{1,k} \;\; x_{2,k}]^T$ the position variables, and $x_{34,k} = [x_{3,k} \;\; x_{4,k}]^T$ the velocity variables, for link 1 and link 2 respectively;

$$f_1(x_k) = \begin{bmatrix} x_{1,k} + x_{3,k}T \\ x_{2,k} + x_{4,k}T \end{bmatrix}$$

$$f_2(x_k) = \begin{bmatrix} x_{3,k} + c(-D_{22}(V_1+F_1) + D_{12}(V_2+F_2)) \\ x_{4,k} + c(D_{12}(V_1+F_1) - D_{11}(V_2+F_2)) \end{bmatrix}$$

$$g(x_k) = \left[\begin{array}{c|c} D_{22} & -D_{12} \\ \hline -D_{12} & D_{11} \end{array}\right]$$

with $c = T/(D_{11}D_{22} - D_{12}^2)$.

Control Synthesis

For trajectory tracking, we propose the desired storage function as

$$V(x_k, x_{\delta,k}) = \frac{1}{2}(x_k - x_{\delta,k})^T K^T P K (x_k - x_{\delta,k})$$

where $x_{\delta,k}$ are the reference trajectories; K is an additional gain to modify the convergence rate and P is synthesized to achieve passivity, according to Section 3.2, which can be written, respectively, with a block structure as:

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

and

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

where K_2 is chosen as the 2×2 identity matrix.

Thus, from passivity condition (3.29) for system (3.42), and according to (3.37)-(3.38), the output is established as

$$y_k = h(x_k, x_{\delta,k+1}) + J(x_k)u_k$$

where

$$h(x_k, x_{\delta,k+1}) = g^T(x_k)\left(P_{22}\, f_2(x_k) - K_1^T\, P_{12}\, x_{12\delta,k+1} + K_1^T\, P_{12}\, f_1(x_k)\right)$$

and

$$J(x_k) = \frac{1}{2} g^T(x_k)\, P_{22}\, g(x_k).$$

Global asymptotic convergence to state reference trajectory is guaranteed with (3.31), if we can find a positive definite matrix $\overline{P}$ satisfying (3.30).

Simulation Results

The reference signals are

$$x_{1\delta,k} = 2.0\, sin(1.0\, k\, T)\ rad$$

$$x_{2\delta,k} = 1.5\, sin(1.2\, k\, T)\ rad.$$

References $x_{3\delta,k}$ and $x_{4\delta,k}$, are defined accordingly.

These signals are selected to illustrate the ability of the proposed algorithm to track nonlinear trajectories.

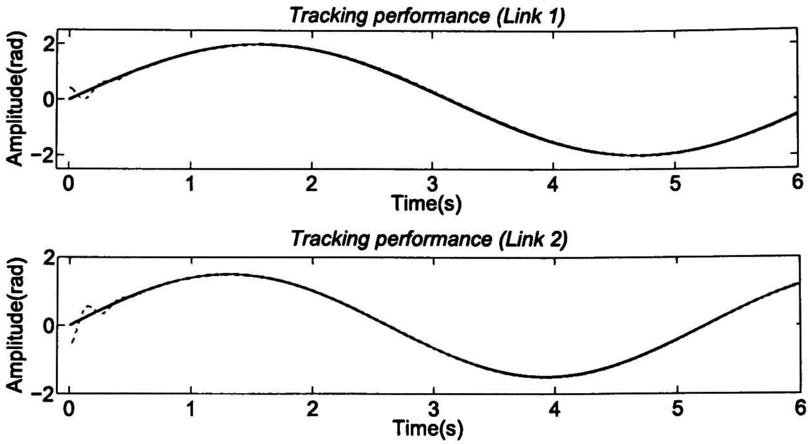

Figure 3.5: Tracking performance of Link 1 and Link 2, respectively. $x_{12\delta}$ (*solid line*) are the reference signal and x_{12} (*dashed line*) are the Link positions

Equation (3.30) is satisfied with

$$P_{11} = 100 * \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}; \quad P_{12} = 100 * \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$P_{21} = P_{12}^T; \quad P_{22} = P_{11}^T; \quad K_1 = \begin{bmatrix} 170 & 0 \\ 0 & 110 \end{bmatrix}$$

The tracking performance for both, link 1 and link 2 position, are shown in the Figure 3.5, with initial conditions $x_{1,k} = 0.4\ rad; x_{2,k} = -0.5\ rad; x_{3,k} = 0\ rad/s; x_{4,k} = 0$; and $T = 0.001$.

The control signals u_1 and u_2 are displayed in Figure 3.6.

3.4 Conclusions

This chapter has presented a novel discrete-time inverse optimal control scheme, which achieve stabilization and trajectory tracking for nonlinear system and is inverse optimal in the sense that, a posteriori, minimizes a meaningful cost functional. The controller synthesis is based on the selection of a storage function used as candidate CLF and a passifying law to render passive the system. The applicability of the proposed methods is illustrated by means of two examples. The fist one is an unstable nonlinear system in which stabilization and trajectory tracking are achieved, and for the second one (a planar robot) trajectory tracking is accomplished. Research will continue to apply the proposed control law in real-time.

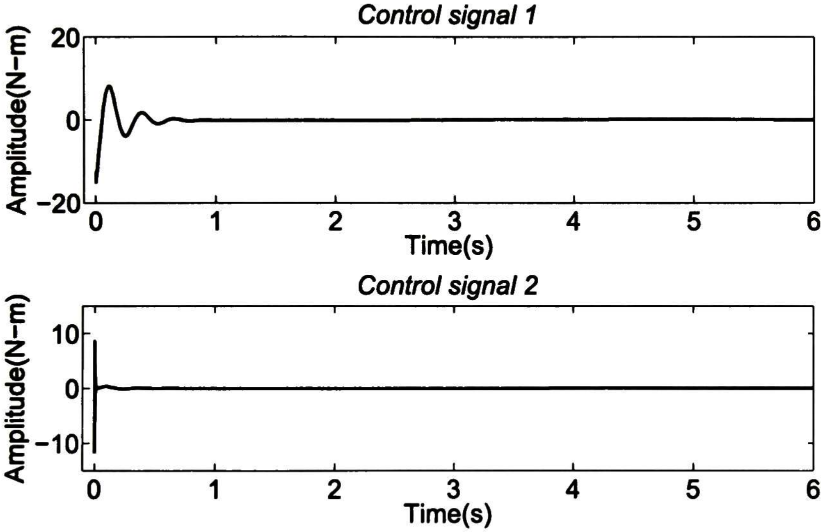

Figure 3.6: Control signal time responses.

Chapter 4

Inverse Optimal Control: A CLF Approach

In this chapter, we establish the inverse optimal control and its solution by proposing a quadratic CLF in Section 4.1; first, the CLF depends on a fixed parameter in order to satisfy stability and optimality condition. A posteriori, the speed gradient algorithm is established in Section 4.2 to compute this CLF parameter and it is used in Section 4.3 to solve the inverse optimal control problem. These results are extended for the inverse optimal control trajectory tracking problem in Section 4.4. Additionally, in Section 4.5 an inverse optimal trajectory tracking for block control form nonlinear systems is proposed. Simulation results illustrate the applicability of the proposed control schemes.

4.1 Inverse Optimal Control via CLF

Motivated by the favorable stability margins of optimal control systems, we synthesize a stabilizing feedback control law, which will be optimal with respect to a meaningful cost functional. At the same time, we want to avoid the difficult task of solving the HJB partial differential equation. In the inverse optimal control problem, a candidate CLF is used to construct an optimal control law directly without solving the associated HJB equation [17]. We focus on inverse optimality because it avoids to solve the HJB partial differential equations and still allows to obtain Kalman-type stability margins [5].

In contrast with the inverse optimal control via passivity approach, in which a storage function is used as a candidate CLF and the inverse optimal control law is selected as the output feedback, for the inverse optimal control via CLF, the control law is obtained as a result of solving the Bellman equation. Then, a candidate CLF for the obtained control law is proposed such that it stabilizes the system and a posteriori a meaningful cost functional is minimized.

For this dissertation, a quadratic candidate CLF is used to synthesize the inverse optimal control law. We establish the following assumptions and definitions which allow the inverse optimal control solution via the CLF approach.

Assumption 4.1 *The full state of system (2.2) is measurable.*

Definition 4.1 (Inverse Optimal Control Law) *Let define the control law*

$$u_k^* = -\frac{1}{2}R^{-1}(x_k)g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}} \tag{4.1}$$

to be inverse optimal (globally) stabilizing if:

(i) *It achieves (global) asymptotic stability of $x = 0$ for system (2.2);*

(ii) *$V(x_k)$ is (radially unbounded) positive definite function such that inequality*

$$\overline{V} := V(x_{k+1}) - V(x_k) + u_k^{*T} R(x_k) u_k^* \leq 0 \tag{4.2}$$

is satisfied.

When we select $l(x_k) := -\overline{V} \geq 0$, then $V(x_k)$ is a solution for the HJB equation

$$l(x_k) + V(x_{k+1}) - V(x_k) + \frac{1}{4}\frac{\partial V^T(x_{k+1})}{\partial x_{k+1}}g(x_k)\, R^{-1}(x_k)\, g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}} = 0.$$

We can establish the main conceptual differences between optimal control and inverse optimal control as follows:

- For optimal control, the meaningful cost indexes $l(x_k) \geq 0$ and $R(x_k) > 0$ are given a priori; then, they are used to calculate $u(x_k)$ and $V(x_k)$ by means of HJB equation solution.

- For inverse optimal control, a candidate CLF ($V(x_k)$) and the meaningful cost index $R(x_k)$ are given a priori, and then these functions are used to calculate the inverse control law $u^*(x_k)$ and the meaningful cost index $l(x_k)$, defined as $l(x) := -\overline{V}$

As established in Definition 4.1, the inverse optimal control problem is based on the knowledge of $V(x_k)$. Thus, we propose a CLF $V(x_k)$, such that (i) and (ii) are guaranteed. That is, instead of solving (2.10) for $V(x_k)$, we propose a control Lyapunov function $V(x_k)$ with the form:

$$V(x_k) = \frac{1}{2}x_k^T P x_k, \qquad P = P^T > 0 \tag{4.3}$$

for control law (4.1) in order to ensure stability of the equilibrium point $x_k = 0$ of system (2.2), which will be achieved by defining an appropriate matrix P. Moreover, it will be established that control law (4.1) with (4.3), which is referred to as the *inverse optimal* control law, optimizes a meaningful cost functional of the form (2.3).

Consequently, by considering $V(x_k)$ as in (4.3), control law (4.1) takes the following form:

$$\begin{aligned} \alpha(x_k) &:= u_k^* \\ &= -\frac{1}{2}\left(R(x_k) + P_2(x_k)\right)^{-1} P_1(x_k) \end{aligned} \tag{4.4}$$

where $P_1(x_k) = g^T(x_k)\, P\, f(x_k)$ and $P_2(x_k) = \frac{1}{2}\, g^T(x_k)\, P\, g(x_k)$. It is worth to point out that P and $R(x_k)$ are positive definite and symmetric matrices; thus, the existence of the inverse in (4.4) is ensured.

Once we have proposed a CLF for solving the inverse optimal control in accordance with Definition 4.1, the respective solution is presented, for which P is considered a fixed matrix.

Theorem 4.1 *Consider the affine discrete-time nonlinear system* (2.2). *If there exists a matrix* $P = P^T > 0$ *such that the following inequality holds:*

$$V_f(x_k) - \frac{1}{4} P_1^T(x_k)\left(R(x_k) + P_2(x_k)\right)^{-1} P_1(x_k) \le -\zeta_Q\, \|x_k\|^2 \tag{4.5}$$

where $V_f(x_k) = \frac{1}{2}\left[V(f(x_k)) - V(x_k)\right]$, *with* $V(f(x_k)) = f^T(x_k)\, P\, f(x_k)$ *and* $\zeta_Q > 0$; $P_1(x_k)$ *and* $P_2(x_k)$ *as defined in* (4.4); *then, the equilibrium point* $x_k = 0$ *of system* (2.2) *is globally exponentially stabilized by the control law* (4.4), *with CLF* (4.3).

Moreover, with (4.3) as a CLF, this control law is inverse optimal in the sense that it minimizes the meaningful functional given by

$$\mathcal{J}(x_k) = \sum_{k=0}^{\infty}\left(l(x_k) + u_k^T\, R(x_k)\, u_k\right) \tag{4.6}$$

with

$$l(x_k) = -\overline{V}\big|_{u_k^* = \alpha(x_k)} \tag{4.7}$$

and optimal value function $\mathcal{J}^*(x_0) = V(x_0)$.

Proof. First, we analyze stability. Global stability for the equilibrium point $x_k = 0$ of system (2.2) with (4.4) as input, is achieved if function $\overline{V}$ in (4.2), is satisfied. In order to achieve (4.2), then

$$\begin{aligned}
\overline{V} &= V(x_{k+1}) - V(x_k) + \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\
&= \frac{f^T(x_k)\, P\, f(x_k) + 2 f^T(x_k)\, P\, g(x_k)\, \alpha(x_k)}{2} \\
&\quad + \frac{\alpha^T(x_k)\, g^T(x_k)\, P\, g(x_k)\, \alpha(x_k) - x_k^T\, P\, x_k}{2} + \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\
&= V_f(x_k) - \frac{1}{2} P_1^T(x_k)\, (R(x_k) + P_2(x_k))^{-1}\, P_1(x_k) + \frac{1}{4} P_1^T(x_k)\, (R(x_k) + P_2(x_k))^{-1}\, P_1(x_k) \\
&= V_f(x_k) - \frac{1}{4} P_1^T(x_k)\, (R(x_k) + P_2(x_k))^{-1}\, P_1(x_k). \qquad (4.8)
\end{aligned}$$

Selecting P such that $\overline{V} \leq 0$, stability of $x_k = 0$ is guaranteed. Furthermore, by means of P, we can achieve a desired negativity amount [26] for the closed-loop function $\overline{V}$ in (4.8). This negativity amount can be bounded using a positive definite matrix Q as follows:

$$\begin{aligned}
\overline{V} &= V_f(x_k) - \frac{1}{4} P_1^T(x_k)\, (R(x_k) + P_2(x_k))^{-1}\, P_1(x_k) \\
&\leq -x_k^T\, Q\, x_k \\
&\leq -\lambda_{min}(Q)\, \|x_k\|^2 \\
&= -\zeta_Q\, \|x_k\|^2, \qquad \zeta_Q = \lambda_{min}(Q) \qquad (4.9)
\end{aligned}$$

where $\|\cdot\|$ stands for the Euclidean norm and $\zeta_Q > 0$ denotes the minimum eigenvalue of matrix Q ($\lambda_{min}(Q)$). Thus, from (4.9) follows condition (4.5).

Considering (4.8)-(4.9), if $\overline{V} = V(x_{k+1}) - V(x_k) + \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \leq -\zeta_Q \|x_k\|^2$, then $\Delta V = V(x_{k+1}) - V(x_k) \leq -\zeta_Q \|x_k\|^2$ Moreover, as $V(x_k)$ is a radially unbounded function, then the solution $x_k = 0$ of the closed-loop system (2.2) with (4.4) as input, is globally exponentially stable according to Theorem 2.2.

When function $-l(x_k)$ is set to be the (4.2) right-hand side, i.e., $l(x_k) = -\overline{V}\big|_{u_k^* = \alpha(x_k)} \geq 0$, then $V(x_k)$ is a solution of the HJB equation (2.10) as established in Definition 4.1.

In order to establish optimality, considering that (4.4) stabilizes (2.2), and substituting $l(x_k)$ in (4.6), we obtain

$$\begin{aligned}
\mathcal{J}(x_k) &= \sum_{k=0}^{\infty} \left(l(x_k) + u_k^T\, R(x_k)\, u_k \right) \\
&= \sum_{k=0}^{\infty} \left(-\overline{V} + u_k^T\, R(x_k)\, u_k \right) \\
&= -\sum_{k=0}^{\infty} \left[V_f(x_k) - \frac{1}{4} P_1^T(x_k)\, (R(x_k) + P_2(x_k))^{-1}\, P_1(x_k) \right] + \sum_{k=0}^{\infty} u_k^T\, R(x_k)\, u_k \qquad (4.10)
\end{aligned}$$

Now, factorizing (4.10) and then adding the identity matrix $I_m \in \mathbb{R}^{m\times m}$ presented as $I_m = (R(x_k) + P_2(x_k))\,(R(x_k) + P_2(x_k))^{-1}$, we obtain

$$
\begin{aligned}
\mathcal{J}(x_k) &= -\sum_{k=0}^{\infty}\Big[V_f(x_k) - \frac{1}{2}P_1^T(x_k)\,(R(x_k)+P_2(x_k))^{-1}\,P_1(x_k) \\
&\quad +\frac{1}{4}P_1^T(x_k)\,(R(x_k)+P_2(x_k))^{-1}\,P_2(x_k)\,(R(x_k)+P_2(x_k))^{-1}\,P_1(x_k) \\
&\quad +\frac{1}{4}P_1^T(x_k)\,(R(x_k)+P_2(x_k))^{-1}\,R(x_k)\,(R(x_k)+P_2(x_k))^{-1}\,P_1(x_k)\Big] \\
&\quad +\sum_{k=0}^{\infty} u_k^T\,R(x_k)\,u_k. \qquad (4.11)
\end{aligned}
$$

Being $\alpha(x_k) = -\frac{1}{2}\,(R(x_k)+P_2(x_k))^{-1}\,P_1(x_k)$, then (4.11) becomes

$$
\begin{aligned}
\mathcal{J}(x_k) &= -\sum_{k=0}^{\infty}\Big[V_f(x_k) + P_1^T(x_k)\,\alpha(x_k) + \alpha^T(x_k)\,P_2(x_k)\,\alpha(x_k)\Big] + \sum_{k=0}^{\infty}\Big[u_k^T\,R(x_k)\,u_k \\
&\quad -\alpha^T(x_k)R(x_k)\alpha(x_k)\Big] \\
&= -\sum_{k=0}^{\infty}\Big[V(x_{k+1}) - V(x_k)\Big] + \sum_{k=0}^{\infty}\Big[u_k^T\,R(x_k)\,u_k - \alpha^T(x_k)R(x_k)\alpha(x_k)\Big]. \qquad (4.12)
\end{aligned}
$$

After evaluating the summation for $k = 0$, then (4.12) can be written as

$$
\begin{aligned}
\mathcal{J}(x_k) &= -\sum_{k=1}^{\infty}\Big[V(x_{k+1}) - V(x_k)\Big] - V(x_1) + V(x_0) + \sum_{k=0}^{\infty}\Big[u_k^T\,R(x_k)\,u_k - \alpha^T(x_k)R(x_k)\alpha(x_k)\Big] \\
&= -\sum_{k=2}^{\infty}\Big[V(x_{k+1}) - V(x_k)\Big] - V(x_2) + V(x_1) - V(x_1) + V(x_0) + \sum_{k=0}^{\infty}\Big[u_k^T\,R(x_k)\,u_k \\
&\quad -\alpha^T(x_k)R(x_k)\alpha(x_k)\Big]. \qquad (4.13)
\end{aligned}
$$

For notation convenience in (4.13), the upper limit ∞ will treated as $N \to \infty$, and thus

$$
\begin{aligned}
\mathcal{J}(x_k) &= -V(x_N) + V(x_{N-1}) - V(x_{N-1}) + V(x_0) \\
&\quad + \lim_{N\to\infty}\sum_{k=0}^{N}\Big[u_k^T\,R(x_k)\,u_k - \alpha^T(x_k)R(x_k)\alpha(x_k)\Big] \\
&= -V(x_N) + V(x_0) + \lim_{N\to\infty}\sum_{k=0}^{N}\Big[u_k^T\,R(x_k)\,u_k - \alpha^T(x_k)R(x_k)\alpha(x_k)\Big].
\end{aligned}
$$

Letting $N \to \infty$ and noting that $V(x_N) \to 0$ for all x_0, then

$$\mathcal{J}(x_k) = V(x_0) + \sum_{k=0}^{\infty} \left[u_k^T R(x_k) u_k - \alpha^T(x_k) R(x_k) \alpha(x_k)\right] \tag{4.14}$$

Thus, the minimum value of (4.14) is reached with $u_k = \alpha(x_k)$. Hence, the control law (4.4) minimizes the cost functional (4.6). The optimal value function of (4.6) is $\mathcal{J}^*(x_0) = V(x_0)$ for all x_0. ∎

Remark 4.1 *It is worth to note the state measurements are required for this controller. Work is progressing to complete the design with observer schemes.*

Optimal control will be in general of the form (4.1) and the minimum value of the performance index will be function of the initial state x_0. If system (2.2) and the control law (4.1) are given, we shall say that the pair $\{V(x_k), l(x_k)\}$ is a solution to the *inverse optimal control problem* if the performance index (2.3) is minimized by (4.1), and the minimum value being $V(x_0)$ [40].

As proposed in Corollary 3.2, a MI is established to solve inequality (4.5) as follows.

Corollary 4.1 *Consider function $g(x_k)$ in (2.2) to be entry-wise bounded as proposed in (3.18) and $R(x_k)$ in (4.6) to be a fixed positive definite matrix denoted by R. Then, solution of inequality (4.5) for $P = P^T > 0$ reduces to the solution of the following MI:*

$$\begin{bmatrix} P - \frac{1}{2} P B_1 \left(R + \frac{1}{2} B_1^T P B_1\right)^{-1} B_1^T P & 0 \\ 0 & -(P - 2Q) \end{bmatrix} < 0 \tag{4.15}$$

for P.

Proof. From (4.9), we have

$$V_f(x_k) - \frac{1}{4} P_1^T(x_k) \left(R(x_k) + P_2(x_k)\right)^{-1} P_1(x_k) \leq -x_k^T Q x_k$$

hence,

$$\frac{1}{2}(f^T(x_k) P f(x_k) - x_k^T P x_k) - \frac{1}{4} f^T(x_k) P B_1 \left(R + \frac{1}{2} B_1^T P B_1\right)^{-1} B_1^T P f(x_k) \leq x_k^T Q x_k$$

Then, (4.5) is satisfied by finding P such that the following inequality holds:

$$\begin{aligned} &\frac{1}{2} f^T(x_k) \left[P - \frac{1}{2} P B_1 \left(R + \frac{1}{2} B_1^T P B_1\right)^{-1} B_1^T P\right] f(x_k) - \frac{1}{2} x_k^T (P - 2Q) x_k \\ &= \begin{bmatrix} f(x_k) & x_k \end{bmatrix}^T \begin{bmatrix} P - \frac{1}{2} P B_1 \left(R + \frac{1}{2} B_1^T P B_1\right)^{-1} B_1^T P & 0 \\ 0 & -(P - 2Q) \end{bmatrix} \begin{bmatrix} f(x_k) \\ x_k \end{bmatrix} \\ &< 0 \end{aligned} \tag{4.16}$$

which is guaranteed by solving (4.15). ■

As a special case, the discrete-time inverse optimal control for linear systems is described in Appendix B.

4.1.1 Example

The applicability of the developed method is illustrated as follows.

Stabilization of an Unstable Nonlinear System

Consider the discrete-time second order unstable nonlinear system of the form (2.2) with

$$f(x_k) = \begin{bmatrix} x_{1,k}\, x_{2,k} - 0.8\, x_{2,k} \\ x_{1,k}^2 + 1.8\, x_{2,k} \end{bmatrix} \tag{4.17}$$

and

$$g(x_k) = \begin{bmatrix} 0 \\ -2 + cos(x_{2,k}) \end{bmatrix} \tag{4.18}$$

According to (4.4), the stabilizing optimal control law is formulated as

$$u_k^* = -\frac{1}{2}\left(R(x_k) + \frac{1}{2}\, g^T(x_k)\, P\, g(x_k)\right)^{-1} g^T(x_k)\, P\, f\,(x_k)$$

where the positive definite matrix P is selected as

$$P = 10 * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $R(x_k)$ is a constant matrix

$$R(x_k) = 1.$$

The state penalty term $l(x_k)$ in (4.6) is calculated according to (4.7).

Figure 4.1 shows the stabilization of this system with initial conditions $x_0 = [2 \quad -2]^T$, and Figure 4.2 displays the evaluation of the cost functional $\mathcal{J}(x_k)$.

Remark 4.2 *By using a CLF, system (4.17)-(4.18), is not required to be stable for* $u_k = 0$.

Remark 4.3 *In this example, according to Theorem 4.1, the optimal value function is calculated as* $\mathcal{J}^*(x_0) = V(x_0) = \frac{1}{2}\, x_0^T\, P\, x_0 = 40$, *which is reached as shown in Figure 4.2.*

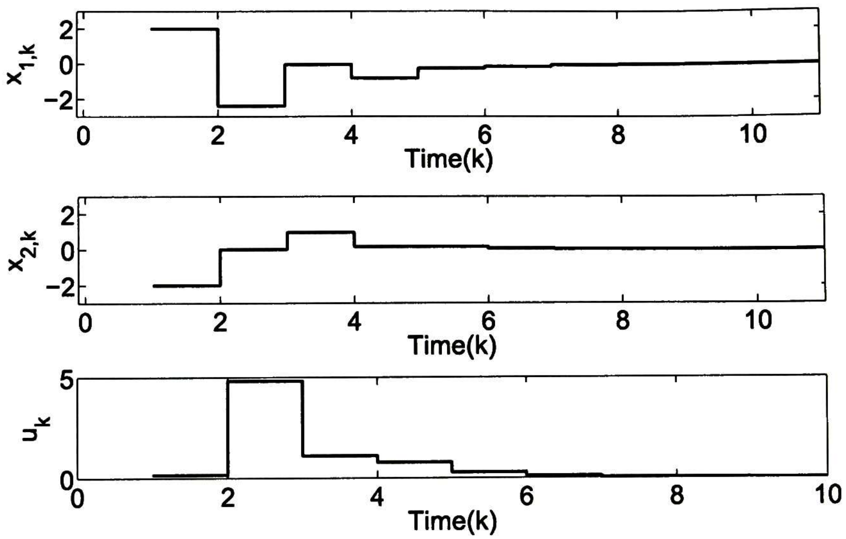

Figure 4.1: Stabilization of a nonlinear system.

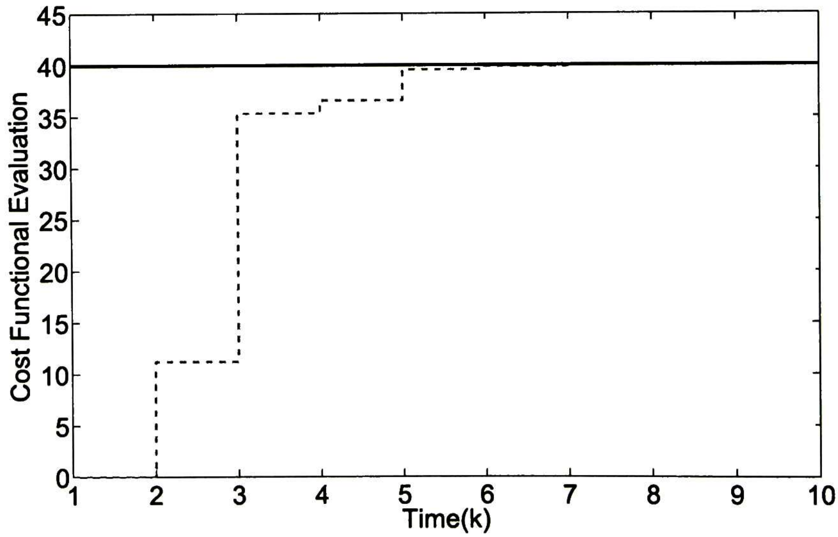

Figure 4.2: Solid line shows the optimal value for the cost functional $(\mathcal{J}^*(x_0))$. Dashed line shows the evaluation for the cost functional $\mathcal{J}(x_k)$ at the k-step.

4.1.2 Robust Inverse Optimal Control

Optimal controllers are known to be robust with respect to certain plant parameter variations, disturbances and unmodelled dynamics as provided by stability margins, which implies that the Lyapunov difference $\Delta V < 0$ for optimal control schemes might still holds even for internal and/or external disturbances in the plant, and therefore stability will be maintained [74].

In this section, we establish a robust inverse optimal controller to achieve disturbance attenuation for a disturbed discrete-time nonlinear system. At the same time, this controller is optimal with respect to a meaningful cost functional, and we avoid to solve the Hamilton-Jacobi-Isaacs (HJI) partial differential equation [17].

Let us consider the disturbed discrete-time nonlinear system

$$x_{k+1} = f(x_k) + g(x_k)\,u_k + d_k, \qquad x_0 = x(0) \tag{4.19}$$

where $d_k \in \mathbb{R}^n$ is a disturbance, which is bounded by

$$\|d_k\| \leq \ell_k' + \alpha_4(\|x_k\|) \tag{4.20}$$

with $\ell_k' \leq \ell$; ℓ is a positive constant and $\alpha_4(\|x_k\|)$ is a $\mathcal{K}_\infty$–function, and suppose that $\alpha_4(\|x_k\|)$ in (4.20) is of the same order as the $\mathcal{K}_\infty$–function $\alpha_3(\|x_k\|)$, i.e.

$$\alpha_4(\|x_k\|) = \delta\,\alpha_3(\|x_k\|), \qquad \delta > 0. \tag{4.21}$$

In the next definition, we establish the discrete-time *robust inverse optimal* control problem

Definition 4.2 *The control law*

$$u_k^* = \alpha(x_k) = -\frac{1}{2}R^{-1}(x_k)g^T(x_k)\frac{\partial V(x_{k+1})}{\partial x_{k+1}} \tag{4.22}$$

is robust inverse optimal (globally) stabilizing if:

(i) *It achieves (global) ISS for system* (4.19)*;*

(ii) $V(x_k)$ *is (radially unbounded) positive definite such that the inequality*

$$\begin{aligned} \overline{V}_d(x_k, d_k) : &= V(x_{k+1}) - V(x_k) + u_k^T\,R(x_k)\,u_k \\ &\leq -\sigma(x_k) + \ell_d\,\|d_k\| \end{aligned} \tag{4.23}$$

is satisfied, where $\sigma(x_k)$ is a positive definite function and ℓ_d is a positive constant. The value of function $\sigma(x_k)$ represents a desired amount of negativity [26] of the closed-loop Lyapunov difference $\overline{V}_d(x_k, d_k)$.

For the robust inverse optimal control solution, let consider the continuous state feedback control law (4.22), with (4.3) as a candidate CLF, where $P \in \mathbb{R}^{n \times n}$ is assumed to be positive definite and symmetric matrix. Taking one step ahead for (4.3), then control law (4.22) results in (4.4).

Hence, a robust inverse optimal controller is stated as follows.

Theorem 4.2 *Consider a disturbed affine discrete-time nonlinear system (4.19). If there exists a matrix $P = P^T > 0$ such that the following inequality holds*

$$V_f(x_k) - \frac{1}{4} P_1^T(x_k) \left(R(x_k) + P_2(x_k)\right)^{-1} P_1(x_k) \leq -\zeta \alpha_3(\|x_k\|), \quad \forall \|x_k\| \geq \rho(\|d_k\|) \tag{4.24}$$

with δ in (4.21) satisfying

$$\delta < \frac{\eta}{\ell_d} \tag{4.25}$$

where $V_f(x_k) = \frac{1}{2}[V(f(x_k)) - V(x_k)]$, and with $P_1(x_k) = g^T(x_k) P f(x_k)$ and $P_2(x_k) = \frac{1}{2} g^T(x_k) P g(x_k)$; ζ, $\ell_d > 0$, $\eta = (1-\theta)\zeta > 0$, $0 < \theta < 1$, and with ρ a $\mathcal{K}_\infty$- function; then, the solution of the closed-loop system (4.19) and (4.4) is ISS with the ultimate bound γ (i.e., $\|x_k\| \leq \gamma$, $\forall k \geq k_0 + T$) and (4.3) as a ISS-CLF in (2.23)-(2.24). The ultimate bound γ in (2.17) becomes $\gamma = \alpha_3^{-1}\left(\frac{\ell_d \ell}{\theta_1 \zeta}\right)\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$.

Moreover, with (4.3) as a ISS-CLF, control law (4.4) *is inverse optimal in the sense that minimizes the meaningful functional given as*

$$\mathcal{J} = \sup_{d \in \mathcal{D}} \left\{ \lim_{\tau \to \infty} \left[V(x_\tau) + \sum_{k=0}^{\tau} \left(l_d(x_k) + u_k^T R(x_k) u_k + \ell_d \|d_k\| \right) \right] \right\} \tag{4.26}$$

where $\mathcal{D}$ is the set of locally bounded functions, and

$$l_d(x_k) := -V_d(x_k, d_k)$$

with $V_d(x_k, d_k)$ a negative definite function.

Proof. First, we analyze stability for system (4.19) with nonvanishing disturbance d_k. It is worth to note that asymptotic stability of $x = 0$ is not reached anymore [30]; ISS property for solution of system (4.19) can be only ensured if stabilizability is assumed. Stability analysis for disturbed system can be treated by two terms; we propose a Lyapunov difference for the nominal system (i.e., $x_{k+1} = f(x_k) + g(x_k) u_k$), denoted by ΔV. and additionally, a difference for disturbed system (4.19) denoted by ΛV. The Lyapunov difference for the disturbed system is defined as

$$\Delta V_d(x_k, d_k) = V(x_{k+1}) - V(x_k). \tag{4.27}$$

Let first define the function $V_{nom}(x_{k+1})$ as the $k+1$–step using the Lyapunov function $V(x_k)$ for the nominal system (2.2). Then, adding and subtracting $V_{nom}(x_{k+1})$ in (4.27)

$$\Delta V_d(x_k, d_k) = \underbrace{V(x_{k+1}) - V_{nom}(x_{k+1})}_{\Lambda V:=} + \underbrace{V_{nom}(x_{k+1}) - V(x_k)}_{\Delta V:=} \tag{4.28}$$

From (4.2) with $\sigma(x_k) = \zeta\alpha_3(\|x_k\|)$, $\zeta > 0$ and the control law (4.4), we obtain

$$\begin{aligned} \Delta V &= V_f(x_k) - \frac{1}{4}P_1^T(x_k)\left(R(x_k) + P_2(x_k)\right)^{-1} P_1(x_k) \\ &\le -\zeta\alpha_3(\|x_k\|) \end{aligned}$$

in (4.28), which is ensured by means of $P = P^T > 0$. On the other hand, since $V(x_k)$ is a C^1 (indeed it is C^2 differentiable)function in x_k for all k, then ΛV satisfies condition (2.19) as

$$\begin{aligned} |\Lambda V| &\le \ell_d \left\|f(x_k) + g(x_k)\,u_k(x_k) + d_k - f(x_k) - g(x_k)\,u_k(x_k)\right\| \\ &= \ell_d \|d_k\| \\ &\le \ell_d\,\ell + \ell_d\,\alpha_4(\|x_k\|) \end{aligned}$$

where ℓ and ℓ_d are positive constants. Hence, the Lyapunov difference $\Delta V_d(x_k, d_k)$ for the disturbed system is determined as

$$\begin{aligned} \Delta V_d(x_k, d_k) &= \Lambda V + \Delta V \\ &\le |\Lambda V| + \Delta V \\ &\le \ell_d\,\alpha_4(\|x_k\|) + \ell_d\,\ell + V_f(x_k) - \frac{1}{4}P_1^T(x_k)\left(R(x_k) + P_2(x_k)\right)^{-1} P_1(x_k) \\ &\le -\zeta\alpha_3(\|x_k\|) + \ell_d\,\alpha_4(\|x_k\|) + \ell_d\,\ell \qquad (4.29) \\ &= -\zeta\alpha_3(\|x_k\|) + \theta\zeta\alpha_3(\|x_k\|) - \theta\zeta\alpha_3(\|x_k\|) + \ell_d\,\alpha_4(\|x_k\|) + \ell_d\,\ell \\ &= -(1-\theta)\zeta\alpha_3(\|x_k\|) - \theta\zeta\alpha_3(\|x_k\|) + \ell_d\,\alpha_4(\|x_k\|) + \ell_d\,\ell \\ &= -(1-\theta)\zeta\alpha_3(\|x_k\|) + \ell_d\,\alpha_4(\|x_k\|), \qquad \forall\,\|x_k\| \ge \alpha_3^{-1}\left(\frac{\ell_d\ell}{\theta\zeta}\right) \end{aligned}$$

where $0 < \theta < 1$. In particular, using condition (4.21) in the previous expression, we obtain

$$\begin{aligned} \Delta V_d(x_k, d_k) &\le -(1-\theta)\zeta\alpha_3(\|x_k\|) + \ell_d\,\alpha_4(\|x_k\|) \\ &= -\eta\alpha_3(\|x_k\|) + \ell_d\,\delta\alpha_3(\|x_k\|) \qquad (4.30) \\ &= -(\eta - \ell_d\,\delta)\alpha_3(\|x_k\|), \qquad \eta = (1-\theta)\zeta > 0 \end{aligned}$$

which is negative definite if condition $\delta < \frac{\eta}{\ell_d}$ (4.25) is satisfied. Therefore, if condition (4.25) holds and considering $V(x_k)$ (4.3) as a radially unbounded ISS-CLF, then by the Proposition 2.2, the closed-loop system (4.19) and (4.4) is ISS, which implies BIBS stability and $\mathcal{K}$–asymptotic gain according to Theorem 2.3.

Now, by Definition 2.7 and Remark 2.1, the solution of the closed-loop system (4.19) and (4.4) is ultimately bounded with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$, which results in, $\gamma = \alpha_3^{-1}\left(\frac{\ell_d \ell}{\theta \zeta}\right)\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$. Hence, according to Definition 2.3, the solution is ultimately bounded with ultimate bound $b = \gamma$.

In order to establish inverse optimality, considering that the control (4.4) achieves ISS for the system (4.19), and substituting $l_d(x_k)$ in (4.26), it follows that

$$\begin{aligned}
\mathcal{J} &= \sup_{d\in\mathcal{D}}\left\{\lim_{\tau\to\infty}\left[V(x_\tau) + \sum_{k=0}^{\tau}\Big(l_d(x_k) + u_k^T R(x_k)\, u_k + \ell_d \left\|d_k\right\|\Big)\right]\right\} \\
&= \sup_{d\in\mathcal{D}}\left\{\lim_{\tau\to\infty}\left[V(x_\tau) + \sum_{k=0}^{\tau}\Big(-\Lambda V - \Delta V + u_k^T R(x_k)\, u_k + \ell_d \left\|d_k\right\|\Big)\right]\right\} \\
&= \sup_{d\in\mathcal{D}}\left\{\lim_{\tau\to\infty}\left[V(x_\tau) - \sum_{k=0}^{\tau}\Big(V_f(x_k) - \frac{1}{4}P_1^T(x_k)\,(R(x_k) + P_2(x_k))^{-1}\, P_1(x_k)\right.\right. \\
&\quad \left.\left. + \ell_d\, \ell + \ell_d \delta \alpha_3(\|x_k\|)\Big) + \sum_{k=0}^{\tau} u_k^T R(x_k)\, u_k + \sum_{k=0}^{\tau}\ell_d \left\|d_k\right\|\right]\right\} \\
&= \lim_{\tau\to\infty}\left[V(x_\tau) - \sum_{k=0}^{\tau}\Big(V_f(x_k) - \frac{1}{4}P_1^T(x_k)\,(R(x_k) + P_2(x_k))^{-1} P_1(x_k)\Big)\right. \\
&\quad \left. + \sum_{k=0}^{\tau} u_k^T R(x_k)\, u_k + \sup_{d\in\mathcal{D}}\left\{\sum_{k=0}^{\tau}\left(\ell_d \left\|d_k\right\| - \ell_d\, \ell - \ell_d \delta \alpha_3(\|x_k\|)\right)\right\}\right]
\end{aligned} \tag{4.31}$$

Adding the term

$$\frac{1}{4}P_1^T(x_k)\,(R(x_k) + P_2(x_k))^{-1}\, R(x_k)\,(R(x_k) + P_2(x_k))^{-1} P_1(x_k)$$

at the first addition term of (4.31) and subtracting at the second addition term of (4.31), yields

$$\begin{aligned}
\mathcal{J} &= \lim_{\tau\to\infty}\left[V(x_\tau) - \sum_{k=0}^{\tau}(V(x_{k+1}) - V(x_k)) + \sum_{k=0}^{\tau}\Big(u_k^T R(x_k)\, u_k - \frac{1}{4}P_1^T(x_k)\,(R(x_k)\right. \\
&\quad + P_2(x_k))^{-1} R(x_k)\,(R(x_k) + P_2(x_k))^{-1}\, P_1(x_k)\Big) \\
&\quad \left. + \sup_{d\in\mathcal{D}}\left\{\sum_{k=0}^{\tau}\left(\ell_d \left\|d_k\right\| - \ell_d\, \ell - \ell_d \delta \alpha_3(\|x_k\|)\right)\right\}\right]
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\tau\to\infty}\Bigg[V(x_\tau) - \sum_{k=0}^{\tau}(V(x_{k+1}) - V(x_k)) + \sum_{k=0}^{\tau}\left[u_k^T R(x_k)\, u_k - \alpha^T(x_k)R(x_k)\alpha(x_k)\right] \\
&\quad + \sup_{d\in\mathcal{D}}\left\{\sum_{k=0}^{\tau}(\ell_d\,\|d_k\| - \ell_d\,\ell - \ell_d\delta\alpha_3(\|x_k\|))\right\}\Bigg] \\
&= \lim_{\tau\to\infty}\Bigg[V(x_\tau) - V(x_\tau) + V(x_0) + \sum_{k=0}^{\tau}\left[u_k^T R(x_k)\, u_k - \alpha^T(x_k)R(x_k)\alpha(x_k)\right] \\
&\quad + \sum_{k=0}^{\tau}\left(\sup_{d\in\mathcal{D}}\{\ell_d\,\|d_k\|\} - \ell_d\,\ell - \ell_d\delta\alpha_3(\|x_k\|)\right)\Bigg] \qquad (4.32)
\end{aligned}$$

If $\sup_{d\in\mathcal{D}}\{\ell_d\,\|d_k\|\}$ is taken as the worst case by considering the equality for (4.20), we obtain

$$\begin{aligned}
\sup_{d\in\mathcal{D}}\{\ell_d\,\|d_k\|\} &= \ell_d \sup_{d\in\mathcal{D}}\{\|d_k\|\} \\
&= \ell_d\,\ell + \ell_d\,\delta\alpha_3(\|x_k\|). \qquad (4.33)
\end{aligned}$$

Therefore

$$\sum_{k=0}^{\tau}\left(\sup_{d\in\mathcal{D}}\{\ell_d\,\|d_k\|\} - \ell_d\,\ell - \ell_d\delta\alpha_3\,(\|x_k\|)\right) = 0. \qquad (4.34)$$

Thus, the minimum value of (4.32) is reached with $u_k = \alpha(x_k)$. Hence, the control law (4.4) minimizes the cost functional (4.26). The optimal value function of (4.26) is $\mathcal{J}^*(x_0) = V(x_0)$. ∎

Remark 4.4 *It is worth to note that, in the inverse optimality analysis proof, equality for* (4.21) *is considered in order to optimize with respect to the worst case for the disturbance.*

As special case, the manipulation of class $\mathcal{K}_\infty-$ functions in Definition 2.6 is simplified when the class $\mathcal{K}_\infty-$ functions takes the special form $\alpha_i(r) = \kappa_i r^c$, $\kappa_i > 0$, $c > 1$, and $i = 1,2,3$. In this case, exponential stability is achieved [30]. Let us assume that the disturbance term d_k in (4.19) satisfies the bound

$$\|d_k\| \le \ell + \delta\,\|x_k\|^2 \qquad (4.35)$$

where ℓ and δ are positive constants.

Corollary 4.2 *Consider the disturbed affine discrete-time nonlinear system* (4.19) *with* (4.35). *If there exists a matrix* $P = P^T > 0$ *such that the following inequality holds*

$$V_f(x_k) - \frac{1}{4}P_1^T(x_k)\,(R(x_k) + P_2(x_k))^{-1}\,P_1(x_k) \le -\zeta_Q\,\|x_k\|^2 \qquad \forall\,\|x_k\| \ge \rho(\|d_k\|) \qquad (4.36)$$

where $\zeta_Q > 0$ denotes the minimum eigenvalue of matrix Q as established in (4.9), and δ in (4.35) satisfies

$$\delta < \frac{\zeta_Q}{\ell_d} \tag{4.37}$$

then, the solution of closed-loop for system (4.19), (4.4) is ISS, with (4.3) as a ISS-CLF. The ultimate bound γ in (2.17) becomes $\gamma = \sqrt{\frac{\ell_d \ell}{\theta\eta}} \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$ with $0 < \theta < 1$ and $\eta > 0$. This bound is reached exponentially.

Moreover, with (4.3) as a ISS-CLF, this control law is inverse optimal in the sense that minimizes the meaningful functional given as

$$\mathcal{J} = \sup_{d \in \mathcal{D}} \left\{ \lim_{\tau \to \infty} \left[V(x_\tau) + \sum_{k=0}^{\tau} \left(l_d(x_k) + u_k^T R(x_k) u_k + \ell_d \|d_k\| \right) \right] \right\} \tag{4.38}$$

where $\mathcal{D}$ is the set of locally bounded functions, and

$$l_d(x_k) = -\Lambda V - \Delta V.$$

Proof. Stability is analyzed similar to the proof of Theorem 4.2, where Lyapunov difference is treated by means of two terms as in (4.28). For the first one, disturbance term is considered, and for the second one, Lyapunov difference in order to achieve exponential stability for undisturbed system is analyzed. For the latter, Lyapunov difference ΔV is considered from (4.9), hence the Lyapunov difference ΔV becomes $\Delta V \leq -\zeta_Q \|x_k\|^2$ with a positive constant ζ_Q, and since $V(x_k)$ is a C^1 function in x_k for all k, then ΛV satisfies the bound condition (2.19) as

$$\begin{aligned} |\Lambda V| &\leq \ell_d \|f(x_k) + g(x_k)\, u_k + d_k - f(x_k) + g(x_k)\, u_k\| \\ &= \ell_d \|d_k\| \\ &\leq \ell_d \ell + \ell_d \delta \|x_k\|^2 \end{aligned} \tag{4.39}$$

where ℓ_d and δ are positive constants, and the bound disturbance (4.35) is regarded. Hence, from (4.9) and (4.39) the Lyapunov difference $\Delta V_d(x_k, d_k)$ for disturbed system (4.19) is established as

$$\begin{aligned}
\Delta V_d(x_k, d_k) &= \Lambda V + \Delta V \\
&\leq |\Lambda V| + \Delta V \\
&\leq \ell_d \, \delta \, \|x_k\|^2 + \ell_d \, \ell + V_f(x_k) - \frac{1}{4} P_1^T(x_k) \, (R(x_k) + P_2(x_k))^{-1} \, P_1(x_k) \\
&\leq -x_k^T \, Q \, x_k + \ell_d \, \delta \, \|x_k\|^2 + \ell_d \, \ell \\
&\leq -\zeta_Q \, \|x_k\|^2 + \ell_d \, \delta \, \|x_k\|^2 + \ell_d \, \ell \\
&= -(\zeta_Q - \ell_d \delta) \, \|x_k\|^2 + \ell_d \, \ell \\
&= -\eta \, \|x_k\|^2 + \ell_d \, \ell, \qquad \eta = \zeta_Q - \ell_d \, \delta > 0 \\
&= -\eta \, \|x_k\|^2 + \theta\eta \, \|x_k\|^2 - \theta\eta \, \|x_k\|^2 + \ell_d \, \ell, \qquad 0 < \theta < 1 \\
&= -(1 - \theta)\eta \, \|x_k\|^2 \qquad \|x_k\| > \sqrt{\frac{\ell_d \, \ell}{\theta\eta}}.
\end{aligned} \tag{4.40}$$

At this point, it must be ensured that η in (4.40) is positive, and thus $\delta < \frac{\zeta_Q}{\ell_d}$, i.e., the condition (4.37).

To this end, as $V(x_k)$ is a radially unbounded function ISS-CLF, then, by Proposition 2.2, the solution of the closed-loop system (4.4), (4.19) is ISS with exponential convergence to the ultimate bound γ, which results in $\gamma = \sqrt{\frac{\ell_d \ell}{\theta\eta}} \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$

In order to establish inverse optimality, considering that (4.4) achieve ISS for (4.19), and substituting $l_d(x_k)$ in (4.38), it follows that

$$\begin{aligned}
\mathcal{J} &= \sup_{d \in \mathcal{D}} \left\{ \lim_{\tau \to \infty} \left[V(x_\tau) + \sum_{k=0}^{\tau} \left(l_d(x_k) + u_k^T \, R(x_k) \, u_k + \ell_d \, \|d_k\| \right) \right] \right\} \\
&= \sup_{d \in \mathcal{D}} \left\{ \lim_{\tau \to \infty} \left[V(x_\tau) + \sum_{k=0}^{\tau} \left(-\Lambda V - \Delta V + u_k^T \, R(x_k) \, u_k + \ell_d \, \|d_k\| \right) \right] \right\} \\
&= \sup_{d \in \mathcal{D}} \left\{ \lim_{\tau \to \infty} \left[V(x_\tau) - \sum_{k=0}^{\tau} \left(V_f(x_k) - \frac{1}{4} P_1^T(x_k) \, (R(x_k) + P_2(x_k))^{-1} \, P_1(x_k) \right. \right. \right. \\
&\qquad \left. \left. \left. + \ell_d \, \ell + \ell_d \delta \, \|x_k\|^2 \right) + \sum_{k=0}^{\tau} u_k^T \, R(x_k) \, u_k + \sum_{k=0}^{\tau} \ell_d \, \|d_k\| \right] \right\} \\
&= \lim_{\tau \to \infty} \left[V(x_\tau) - \sum_{k=0}^{\tau} \left(V_f(x_k) - \frac{1}{4} P_1^T(x_k) \, (R(x_k) + P_2(x_k))^{-1} \, P_1(x_k) \right) \right. \\
&\qquad \left. + \sum_{k=0}^{\tau} u_k^T \, R(x_k) \, u_k + \sup_{d \in \mathcal{D}} \left\{ \sum_{k=0}^{\tau} \left(\ell_d \, \|d_k\| - \ell_d \, \ell - \ell_d \delta \, \|x_k\|^2 \right) \right\} \right]
\end{aligned}$$

Adding the term $\frac{1}{4}P_1^T(x_k)\left(R(x_k)+P_2(x_k)\right)^{-1}R(x_k)\left(R(x_k)+P_2(x_k)\right)^{-1}P_1(x_k)$ at the first addition term and subtracting at second addition term, yields

$$\begin{aligned}
\mathcal{J} &= \lim_{\tau\to\infty}\Bigg[V(x_\tau)-\sum_{k=0}^{\tau}\left(V(x_{k+1})-V(x_k)\right)+\sum_{k=0}^{\tau}\Big(u_k^T R(x_k)\,u_k \\
&\quad -\frac{1}{4}P_1^T(x_k)\left(R(x_k)+P_2(x_k)\right)^{-1}R(x_k)\left(R(x_k)+P_2(x_k)\right)^{-1}P_1(x_k)\Big) \\
&\quad +\sup_{d\in\mathcal{D}}\left\{\sum_{k=0}^{\tau}\left(\ell_d\left\|d_k\right\|-\ell_d\,\ell-\ell_d\delta\left\|x_k\right\|^2\right)\right\}\Bigg] \\
&= \lim_{\tau\to\infty}\Bigg[V(x_\tau)-\sum_{k=0}^{\tau}\left(V(x_{k+1})-V(x_k)\right)+\sum_{k=0}^{\tau}\left[u_k^T R(x_k)\,u_k-\alpha^T(x_k)R(x_k)\alpha(x_k)\right] \\
&\quad +\sup_{d\in\mathcal{D}}\left\{\sum_{k=0}^{\tau}\left(\ell_d\left\|d_k\right\|-\ell_d\,\ell-\ell_d\delta\left\|x_k\right\|^2\right)\right\}\Bigg] \\
&= \lim_{\tau\to\infty}\Bigg[V(x_\tau)-V(x_\tau)+V(x_0)+\sum_{k=0}^{\tau}\left[u_k^T R(x_k)\,u_k-\alpha^T(x_k)R(x_k)\alpha(x_k)\right] \\
&\quad +\sum_{k=0}^{\tau}\left(\sup_{d\in\mathcal{D}}\left\{\ell_d\left\|d_k\right\|\right\}-\ell_d\,\ell-\ell_d\delta\left\|x_k\right\|^2\right)\Bigg]
\end{aligned}\tag{4.41}$$

If $\sup_{d\in\mathcal{D}}\left\{\ell_d\left\|d_k\right\|\right\}$ is taken as the worst case by considering the equality for (4.35), we obtain

$$\begin{aligned}
\sup_{d\in\mathcal{D}}\left\{\ell_d\left\|d_k\right\|\right\} &= \ell_d\sup_{d\in\mathcal{D}}\left\{\left\|d_k\right\|\right\} \\
&= \ell_d\,\ell+\ell_d\delta\left\|x_k\right\|^2
\end{aligned}\tag{4.42}$$

Therefore

$$\sum_{k=0}^{\tau}\left(\sup_{d\in\mathcal{D}}\left\{\ell_d\left\|d_k\right\|\right\}-\ell_d\,\ell-\ell_d\delta\left\|x_k\right\|^2\right)=0.\tag{4.43}$$

Thus, the minimum value of (4.41) is reached with $u_k=\alpha(x_k)$, and the control law (4.4) minimizes the cost functional (4.38). The optimal value function of (4.38) is $\mathcal{J}^*(x_0)=V(x_0)$. ∎

Remark 4.5 *Terminal penalty $V(x_\tau)$ in (4.26) and (4.38) is a necessary function to avoid imposing the assumption $x_\tau\to 0$ as $\tau\to\infty$. Hence, inverse optimality is guaranteed only outside of ball, which is bounded by function γ as defined in* (2.17).

4.2 Speed-Gradient Algorithm for the Inverse Optimal Control

In Section 4.1, a candidate CLF $V(x_k) = \frac{1}{2}x_k^T P x_k$ is proposed in order to solve the inverse optimal control problem as established in Definition 4.1, for which an adequate selection of the fixed matrix P must be done such that condition (4.5) is fulfilled. We propose to use the speed-gradient algorithm to calculate this matrix P in a recursive way to ensure the fulfillment of condition (4.5). Then, a candidate CLF $V(x_k)$ described by

$$V(x_k) = \frac{1}{2}x_k^T P_k x_k, \qquad P_k = P_k^T > 0 \tag{4.44}$$

is proposed for control law (4.1) in order to guarantee stability for the equilibrium point $x_k = 0$ of system (2.2). The stability will be achieved by defining an appropriate matrix P_k. Moreover, it will be established that the control law (4.1) based on (4.44) optimizes the meaningful cost functional (2.3).

Consequently, by considering $V(x_k)$ as in (4.44), the control law (4.1) becomes

$$u_k^* = -\frac{1}{2}\left(R(x_k) + \frac{1}{2}g^T(x_k)\, P_k\, g(x_k)\right)^{-1} g^T(x_k)\, P_k\, f(x_k). \tag{4.45}$$

It is worth to point out that P_k and $R(x_k)$ are positive definite and symmetric matrices; thus, the existence of the inverse in (4.45) is ensured.

To compute a time variant value for P_k, which ensures stability for system (2.2) with (4.45), we will use the speed-gradient (SG) algorithm as follows.

4.2.1 Speed-Gradient Algorithm

In [41], a discrete-time application of the SG algorithm is formulated to find a control law u_k which ensures the control goal:

$$\mathcal{Q}(x_{k+1}) \leq \Delta, \qquad \text{for } k \geq k^*, \tag{4.46}$$

where $\mathcal{Q}$ is a control goal function, a constant $\Delta > 0$, and $k^* \in \mathbb{Z}^+$ is the time at which the control goal is achieved. $\mathcal{Q}$ ensures stability if it is a positive definite function.

Based on the SG application proposed in [41], we consider the control law given by (4.45), with Δ in (4.46) a state dependent function $\Delta(x_k)$.

Control law (4.45) at every time depends on the matrix P_k. Let define the matrix P_k at every time k as:

$$P_k = p_k\, P'$$

where $P' = P'^T > 0$ is a given constant matrix and p_k is a scalar parameter to be adjusted by the SG algorithm. Then, (4.45) is transformed into:

$$u_k^* = -\frac{p_k}{2}\left(R(x_k) + \frac{p_k}{2}\, g^T(x_k)\, P'\, g(x_k)\right)^{-1} g^T(x_k)\, P'\, f(x_k). \tag{4.47}$$

The SG algorithm is now reformulated for the inverse optimal control problem.

Definition 4.3 (SG Goal Function) *Consider a time-varying parameter $p_k \in \mathcal{P} \subset \mathbb{R}^+$, with $p_k > 0$ for all k, and $\mathcal{P}$ is the set of admissible values for p_k. A nonnegative C^1 function $\mathcal{Q} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the form*

$$\mathcal{Q}(x_k, p_k) = V_{sg}(x_{k+1}), \tag{4.48}$$

where $V_{sg}(x_{k+1}) = \frac{1}{2}\, x_{k+1}^T\, P'\, x_{k+1}$ with x_{k+1} as defined in (2.2), is referred to as SG goal function for system (2.2). We define $\mathcal{Q}_k(p) := \mathcal{Q}(x_k, p_k)$.

Definition 4.4 (SG Control Goal) *Consider a constant $p^* \in \mathcal{P}$. The SG control goal for system (2.2) with (4.47) is defined as finding p_k so that the SG goal function $\mathcal{Q}_k(p)$, as in (4.48), fulfills:*

$$\mathcal{Q}_k(p) \leq \Delta(x_k), \qquad \text{for} \quad k \geq k^* \tag{4.49}$$

where

$$\Delta(x_k) = V_{sg}(x_k) - \frac{1}{p_k}\, u_k^T\, R(x_k)\, u_k \tag{4.50}$$

with $V_{sg}(x_k) = \frac{1}{2}\, x_k^T\, P'\, x_k$ and u_k as defined in (4.47); $k^* \in \mathbb{Z}^+$ is the time at which the SG control goal is achieved.

Remark 4.6 *Solution p_k must guarantees that $V_{sg}(x_k) > \frac{1}{p_k}\, u_k^T R(x_k)\, u_k$ in order to obtain a positive definite function $\Delta(x_k)$.*

To conclude, the SG algorithm is used to calculate p_k in order to achieve the SG control goal defined above.

Proposition 4.1 *Consider a discrete-time nonlinear system of the form (2.2) with (4.47) as input. Let $\mathcal{Q}$ be a SG goal function as defined in (4.48), and denoted by $\mathcal{Q}_k(p)$. Let $\bar{p}, p^* \in \mathcal{P}$ be positive constant values and $\Delta(x_k)$ be a positive definite function with $\Delta(0) = 0$ and ϵ^* be a sufficiently small positive constant. Assume that:*

- *A1. There exist p^* and ϵ^* such that*

$$\mathcal{Q}_k(p^*) \leq \epsilon^* \ll \Delta(x_k) \quad \text{and} \quad 1 - \epsilon^*/\Delta(x_k) \approx 1. \tag{4.51}$$

- *A2. For all $p_k \in \mathcal{P}$:*

$$(p^* - p_k)^T \nabla_p \mathcal{Q}_k(p) \leq \epsilon^* - \Delta(x_k) < 0 \tag{4.52}$$

where $\nabla_p \mathcal{Q}_k(p)$ denotes the gradient of $\mathcal{Q}_k(p)$ with respect to p_k.

Then, for any initial condition $p_0 > 0$, there exists a $k^* \in \mathbb{Z}^+$ such that the SG Control Goal (4.49) is achieved by means of the following dynamic variation of parameter p_k:

$$p_{k+1} = p_k - \gamma_{d,k} \nabla_p \mathcal{Q}_k(p), \tag{4.53}$$

with

$$\gamma_{d,k} = \gamma_c\, \delta_k\, |\nabla_p \mathcal{Q}_k(p)|^{-2} \qquad 0 < \gamma_c \leq 2\,\Delta(x_k)$$

and

$$\delta_k = \begin{cases} 1 & for \quad Q(p_k) > \Delta(x_k) \\ 0 & \quad otherwise. \end{cases} \tag{4.54}$$

Finally, for $k \geq k^*$, p_k becomes a constant value denoted by $\bar{p}$ and the SG algorithm is completed.

Proof. We follow similar arguments as the ones given for the SG discrete-time version [41]. Let us consider the positive definite Lyapunov function as $V_p(p_k) = |p_k - p^*|^2$. Then, the Lyapunov difference is given as

$$\begin{aligned}
\Delta V_p(p_k) &= |p_{k+1} - p^*|^2 - |p_k - p^*|^2 \\
&= (p_{k+1} - p_k)^T [(p_{k+1} - p_k) + 2(p_k - p^*)] \\
&= -\gamma_{d,k}\, \nabla_p \mathcal{Q}_k(p)\, [-\gamma_{d,k}\, \nabla_p \mathcal{Q}_k(p) + 2(p_k - p^*)] \\
&\leq -2\,\gamma_{d,k}\, (\Delta(x_k) - \epsilon^*) + \gamma_{d,k}^2\, |\nabla_p \mathcal{Q}_k(p)|^2 \\
&\leq -2\,\gamma_c\, \delta_k\, (\Delta(x_k) - \epsilon^*)\, |\nabla_p \mathcal{Q}_k(p)|^{-2} + \gamma_c^2\, \delta_k^2\, |\nabla_p \mathcal{Q}_k(p)|^{-4}\, |\nabla_p \mathcal{Q}_k(p)|^2 \\
&= -\frac{\gamma_c\, [2\,\Delta(x_k)\, (1 - \epsilon^*/\Delta(x_k)) - \gamma_c]}{|\nabla_p \mathcal{Q}_k(p)|^2} \\
&\approx -\frac{\gamma_c\, [2\,\Delta(x_k) - \gamma_c]}{|\nabla_p \mathcal{Q}_k(p)|^2}
\end{aligned} \tag{4.55}$$

for $\mathcal{Q}_k(p) > \Delta(x_k)$, $\delta_k = 1$ and boundness of p_k is guaranteed if $0 < \gamma_c \leq 2\,\Delta(x_k)$. Finally, when $k \geq k^*$, then $\delta_k = 0$, which means the algorithm concludes; hence, $\mathcal{Q}_k(p) \leq \Delta(x_k)$ and p_k becomes a constant value denoted by $\bar{p}$ ($p_k = \bar{p}$). ■

Since the parameter p_k is a scalar value, the gradient $\nabla_p \mathcal{Q}_k(p)$ in (4.53) is reduced to be the partial derivative of $\mathcal{Q}_k(p)$ with respect to p_k, i.e., $\frac{\partial}{\partial p_k}\, \mathcal{Q}_k(p)$.

Remark 4.7 *Parameter γ_c in (4.53) is selected such that solution p_k ensures the requirement $V_{sg}(x_k) > \frac{1}{p_k}\, u_k^T R(x_k)\, u_k$ in Remark 4.6. Then, we have a positive definite function $\Delta(x_k)$.*

Remark 4.8 *With $\mathcal{Q}(x_k, p_k)$ as defined in* (4.48)*, the dynamic variation of parameter p_k in* (4.53) *results in*

$$p_{k+1} = p_k + 8\,\gamma_{d,k}\,\frac{f^T(x_k)\,P'\,g(x_k)\,R(x_k)^2\,g^T(x_k)\,f(x_k)}{\left(2\,R(x_k) + p_k\,g^T(x_k)\,P'\,g(x_k)\right)^3}$$

which is positive for all time k if $p_0 > 0$. Therefore positiveness for p_k is ensured and requirement $P_k = P_k^T > 0$ for (4.44) *is guaranteed.*

When SG Control Goal (4.49) is achieved, then $p_k = \bar{p}$ for $k \geq k^*$ Thus, matrix P_k in (4.45) is considered constant and $P_k = P$ where P is computed as $P = \bar{p}\,P'$, with P' a *design* positive definite matrix. Under these constraints, we obtain:

$$\begin{aligned} \alpha(x_k) &:= u_k^* \\ &= -\frac{1}{2}\left(R(x_k) + \frac{1}{2}\,g^T(x_k)\,P\,g(x_k)\right)^{-1} g^T(x_k)\,P\,f(x_k). \end{aligned} \tag{4.56}$$

Summary of the proposed SG algorithm to calculate parameter p_k

Considering the closed-loop system (2.2) with (4.47) as input, we obtain

$$x_{k+1} = f(x_k) - \frac{p_k}{2} g(x_k)\left(R(x_k) + \frac{p_k}{2}\,g^T(x_k)\,P'\,g(x_k)\right)^{-1} g^T(x_k)\,P'\,f\,(x_k).$$

Then, we propose the SG Goal Function

$$\mathcal{Q}_k(p_k) = x_{k+1}^T\,x_{k+1}.$$

The dynamic variation of parameter p_k is establish as

$$p_{k+1} = p_k - \gamma \nabla_p\,\mathcal{Q}_k(p_k), \qquad p_0 = p(0).$$

Finally, when condition (4.49) is fulfilled, the SG algorithm finishes.

4.3 SG Inverse Optimal Control

Once the control law (4.56) has been established, we demonstrate that it ensures stability and optimality for (2.2) without solving the HJB equation (2.10). Thus, the main contribution of this chapter is stated as the following theorem.

Theorem 4.3 *Consider that system* (2.2) *with* (4.47) *has achieved the SG control goal* (4.49) *by means of* (4.53). *Let* $V(x_k) = \frac{1}{2}x_k^T P x_k$ *be a candidate Lyapunov function with* $P = P^T > 0$. *Then, inverse optimal control law* (4.56) *renders the equilibrium point* $x_k = 0$ *of system* (2.2) *to be globally asymptotically stable. Moreover, with* $V(x_k) = \frac{1}{2}x_k^T P x_k$ *as CLF and* $P = \bar{p} P'$ *this control law* (4.56) *is inverse optimal in the sense that it minimizes the meaningful cost functional given by*

$$\mathcal{J}(x_k) = \sum_{k=0}^{\infty} \left(l(x_k) + u_k^T R(x_k)\, u_k \right) \tag{4.57}$$

where

$$l(x_k) := -\overline{V} \tag{4.58}$$

with $\overline{V}$ *defined as*

$$\overline{V} = V(x_{k+1}) - V(x_k) + \alpha^T(x_k)\, R(x_k)\, \alpha(x_k)$$

and optimal value function $\mathcal{J}^*(x_0) = V(x_0)$.

Proof. Considering that system (2.2), (4.47) and (4.53) has achieved the SG Control Goal (4.49) for $k \geq k^*$, then (4.49) can be rewritten as:

$$\begin{aligned} V_{sg}(x_{k+1}) - V_{sg}(x_k) + \frac{1}{\bar{p}}\alpha^T(x_k)\, R(x_k)\, \alpha(x_k) &= \frac{1}{2}x_{k+1}^T P' x_{k+1} - \frac{1}{2}x_k^T P' x_k \\ &\quad + \frac{1}{\bar{p}}\alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\ &\leq 0. \end{aligned} \tag{4.59}$$

Multiplying (4.59) by the positive constant $\bar{p}$, we obtain

$$\begin{aligned} \overline{V} &:= \frac{\bar{p}}{2}x_{k+1}^T P' x_{k+1} - \frac{\bar{p}}{2}x_k^T P' x_k + \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\ &= \frac{1}{2}x_{k+1}^T P x_{k+1} - \frac{1}{2}x_k^T P x_k + \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\ &= V(x_{k+1}) - V(x_k) + \alpha^T(x_k)\, R(x_k)\, \alpha(x_k) \\ &\leq 0 \end{aligned} \tag{4.60}$$

and condition (4.2) is fulfilled.

From (4.60), obviously $V(x_{k+1}) - V(x_k) \leq 0$ for all $x_k \neq 0$ and therefore global asymptotic stability is achieved in accordance with Theorem 2.1.

When function $-l(x_k)$ is set to be the (4.60) right-hand side, i.e., $l(x_k) = -\overline{V} \geq 0$, then with $V(x_k) = \frac{1}{2}x_k^T P x_k$ as CLF is a solution of the HJB equation (2.10) for $k \geq k^*$

In order to obtain the optimal value function for the meaningful cost functional (4.57), we proceed as in Theorem 4.1. ■

4.3.1 Example

The proposed methodology is illustrated by an example. We synthesize an inverse optimal control law for a discrete-time second order nonlinear system (unstable for $u_k = 0$) of the form (2.2) with:

$$f(x_k) = \begin{bmatrix} x_{1,k}\, x_{2,k} - 0.8\, x_{2,k} \\ x_{1,k}^2 + 1.8\, x_{2,k} \end{bmatrix} \tag{4.61}$$

and

$$g(x_k) = \begin{bmatrix} 0 \\ -2 + cos(x_{2,k}) \end{bmatrix} \tag{4.62}$$

According to (4.56), the inverse optimal control law is formulated as

$$u_k^* = -\frac{1}{2}\left(R(x_k) + \frac{1}{2}\, g^T(x_k)\, P_k\, g(x_k)\right)^{-1} g^T(x_k)\, P_k\, f\,(x_k)$$

where the positive definite matrix $P_k = p_k\, P'$ is calculated by the SG algorithm with P' as the identity matrix, that is

$$\begin{aligned} P_k &= p_k\, P' \\ &= p_k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

and $R(x_k)$ is a constant matrix

$$R(x_k) = 0.5.$$

The state penalty term $l(x_k)$ in (4.57) is calculated according to (4.58).

Figure 4.3 shows the solution x_k of this system with initial conditions $x_0 = [2 \quad -2]^T$; this figure also includes the applied inverse optimal control law, which achieves asymptotic stability.

Figure 4.4 displays both the SG algorithm solution p_k with initial condition $p_0 = 2.5$ and final value $\bar{p} = 3.4613$. Evaluation of the cost functional $\mathcal{J}(x_k)$ is also displayed in this figure.

Notice that the open-loop system (4.61), has an unstable equilibrium point for $u_k = 0$. In this example, according to Theorem 4.3, the optimal value function is calculated as $\mathcal{J}^*(x_0) = V(x_0) = \frac{1}{2}\, x_0^T\, P\, x_0 = 13.8452$, which is reached as shown in Figure 4.4.

Remark 4.9 *It is worth to note that, full state measurement is required. Work is progressing to complete the design with observer schemes.*

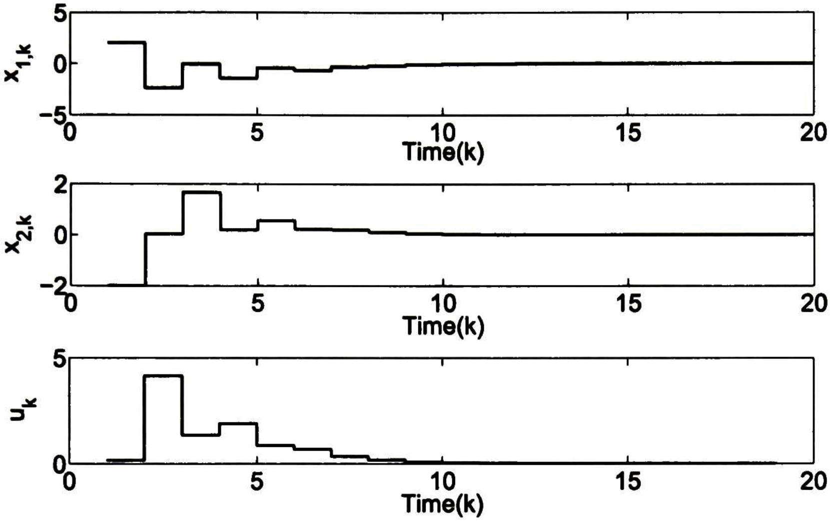

Figure 4.3: Stabilization of a nonlinear system.

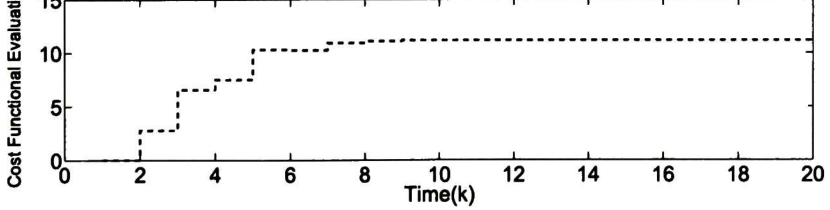

Figure 4.4: p_k and $\mathcal{J}(x_k)$ time evolution.

4.4 SG Trajectory Tracking Inverse Optimal Control

This section deals with the inverse optimal control methodology for trajectory tracking in combination with the speed-gradient algorithm discussed in Section 4.2.

Consider the affine discrete-time nonlinear system (2.2). The following meaningful cost functional is associated with the trajectory tracking problem for system (2.2):

$$\mathcal{J}(z_k) = \sum_{n=k}^{\infty} \left(l(z_n) + u_n^T R(z_n) u_n \right) \tag{4.63}$$

where $z_k = x_k - x_{\delta,k}$ with $x_{\delta,k}$ as the desired trajectory for x_k; $z_k \in \mathbb{R}^n$; $\mathcal{J}(z_k) : \mathbb{R}^n \to \mathbb{R}^+$; $l(z_k) : \mathbb{R}^n \to \mathbb{R}^+$ is a positive semidefinite function and $R(z_k) : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a real symmetric positive definite weighting matrix. The meaningful cost functional (4.63) is a performance measure [52]. The entries of $R(z_k)$ can be fixed or functions of the system state in order to vary the weighting on control efforts according to the state value [52]. Considering the state feedback control design problem, we assume that the full state x_k is available. Using the optimal value function $\mathcal{J}^*(x_k)$ for (4.63) as Lyapunov function $V(x_k)$, equation (4.63) can be rewritten as

$$\begin{aligned} V(z_k) &= l(z_k) + u_k^T R(z_k) u_k + \sum_{n=k+1}^{\infty} l(z_n) + u_n^T R(z_n) u_n \\ &= l(z_k) + u_k^T R(z_k) u_k + V(z_{k+1}) \end{aligned} \tag{4.64}$$

where we require the boundary condition $V(0) = 0$ so that $V(z_k)$ becomes a Lyapunov function.

Similar to Section 2.1 procedure, we establish the conditions that the optimal control law must satisfy. We define the discrete-time Hamiltonian $\mathcal{H}(z_k, u_k)$ as

$$\mathcal{H}(z_k, u_k) = l(z_k) + u_k^T R(z_k) u_k + V(z_{k+1}) - V(z_k). \tag{4.65}$$

A necessary condition that the optimal control law should satisfy is $\frac{\partial \mathcal{H}(z_k,u_k)}{\partial u_k} = 0$, then

$$\begin{aligned} 0 &= 2R(z_k) u_k + \frac{\partial V(z_{k+1})}{\partial u_k} \\ &= 2R(z_k) u_k + \frac{\partial z_{k+1}}{\partial u_k} \frac{\partial V(z_{k+1})}{\partial z_{k+1}} \\ &= 2R(z_k) u_k + g^T(x_k) \frac{\partial V(z_{k+1})}{\partial z_{k+1}} \end{aligned} \tag{4.66}$$

Therefore, the optimal control law to achieve trajectory tracking is formulated as

$$u_k^* = -\frac{1}{2} R^{-1}(z_k) \, g^T(x_k) \frac{\partial V(z_{k+1})}{\partial z_{k+1}} \tag{4.67}$$

with the boundary condition $V(0) = 0$. For solving the trajectory tracking inverse optimal control problem, it is necessary to solve the following HJB equation:

$$l(z_k) + V(z_{k+1}) - V(z_k) + \frac{1}{4}\frac{\partial V^T(z_{k+1})}{\partial z_{k+1}} g(x_k)\, R^{-1}(z_k)\, g^T(x_k)\, \frac{\partial V(z_{k+1})}{\partial z_{k+1}} = 0 \tag{4.68}$$

which is a challenging task. To overcome this problem, we propose to solve the inverse optimal control problem. The main characteristic of the inverse problem is that a stabilizing feedback control law is designed first, and then it is established that this law optimizes the meaningful cost functional (4.63).

Definition 4.5 (Trajectory Tracking Inverse Optimal Control) *Consider the tracking error as $z_k = x_k - x_{\delta,k}$, being $x_{\delta,k}$ the desired trajectory for x_k. Let define the control law*

$$u_k^* = -\frac{1}{2}R^{-1}(z_k)g^T(x_k)\frac{\partial V(z_{k+1})}{\partial z_{k+1}} \tag{4.69}$$

will be inverse optimal (globally) stabilizing along the desired trajectory $x_{\delta,k}$ if:

(i) *It achieves (global) asymptotic stability of $x_k = 0$ for system* (2.2) *along reference $x_{\delta,k}$;*

(ii) *$V(z_k)$ is (radially unbounded) positive definite function such that inequality*

$$\overline{V} := V(z_{k+1}) - V(z_k) + u_k^{*T}\, R(z_k)\, u_k^* \leq 0 \tag{4.70}$$

is satisfied.

When we select $l(z_k) := -\overline{V}$. then $V(z_k)$ is a solution for (4.68), and cost functional (4.63) *is minimized.*

As established in Definition 4.5, the inverse optimal control law for trajectory tracking problem is based on the knowledge of $V(z_k)$. Then, we propose a CLF, $V(z_k)$, such that (i) and (ii) are guaranteed. Hence, instead of solving (4.68) for $V(z_k)$, a quadratic candidate CLF $V(z_k)$ for (4.69) is proposed with the form:

$$V(z_k) = \frac{1}{2}z_k^T\, P_k\, z_k, \qquad P_k = P_k^T > 0 \tag{4.71}$$

in order to ensure stability of the tracking error z_k, where

$$\begin{aligned} z_k &= x_k - x_{\delta,k} \\ &= \begin{bmatrix} (x_{1,k} - x_{1\delta,k}) \\ (x_{2,k} - x_{2\delta,k}) \\ \vdots \\ (x_{n,k} - x_{n\delta,k}) \end{bmatrix} \end{aligned} \tag{4.72}$$

Moreover, it will be established that the control law (4.69) with (4.71), which is referred to as the *inverse optimal* control law, optimizes a meaningful cost functional of the form (4.63).

Consequently, by considering $V(x_k)$ as in (4.71), control law (4.69) takes the following form:

$$\begin{aligned} u_k^* &= -\frac{1}{4}R^{-1}(z_k)g^T(x_k)\frac{\partial z_{k+1}^T P_k z_{k+1}}{\partial z_{k+1}} \\ &= -\frac{1}{2}R^{-1}(z_k)g^T(x_k)P_k z_{k+1} \\ &= -\frac{1}{2}\left(R(z_k)+\frac{1}{2}g^T(x_k)P_k g(x_k)\right)^{-1} g^T(x_k)P_k(f(x_k)-x_{\delta,k+1}). \end{aligned} \tag{4.73}$$

It is worth to point out that P_k and $R(z_k)$ are positive definite and symmetric matrices; thus, the existence of the inverse in (4.73) is ensured.

To compute P_k, which ensures trajectory tracking of x_k for system (2.2) with (4.73) along the desired trajectory $x_{\delta,k}$, we will use the speed-gradient (SG) algorithm.

4.4.1 Speed-Gradient Algorithm for Trajectory Tracking

As proceed in 4.2, the control goal function is established as

$$\mathcal{Q}(z_{k+1}) \leq \Delta, \qquad \text{for } k \geq k^* \tag{4.74}$$

where $\mathcal{Q}$ is a control goal function, a constant $\Delta > 0$, and $k^* \in \mathbb{Z}^+$ is the time at which the control goal is achieved.

Control law (4.73) at every time depends on the matrix P_k. Let define the matrix P_k at every time k as:

$$P_k = p_k \overline{P}$$

where p_k is a scalar parameter to be adjusted by the SG algorithm, $\overline{P} = K^T P' K$ with K an additional diagonal gain matrix of appropriate dimension introduced to modify the convergence rate of the tracking error and $P' = P'^T > 0$ a *design* constant matrix of appropriate dimension. Then, (4.73) becomes:

$$u_k = -\frac{p_k}{2}\left(R(z_k)+\frac{p_k}{2}g^T(x_k)\overline{P}g(x_k)\right)^{-1} g^T(x_k)\overline{P}(f(x_k)-x_{\delta,k+1}). \tag{4.75}$$

The SG algorithm is now reformulated for the trajectory tracking inverse optimal control problem.

Definition 4.6 (SG Goal Function for Trajectory Tracking) *Consider a time-varying parameter $p_k \in \mathcal{P} \subset \mathbb{R}^+$, with $p_k > 0$ for all k, and $\mathcal{P}$ is the set of admissible values for p_k. A nonnegative C^1 function $\mathcal{Q} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the form*

$$\mathcal{Q}(z_k, p_k) = V_{sg}(z_{k+1}), \tag{4.76}$$

where $V_{sg}(z_{k+1}) = \frac{1}{2} z_{k+1}^T P' z_{k+1}$ *is referred to as SG goal function for system* (2.2), *with* $z_{k+1} = x_{k+1} - x_{\delta,k+1}$, x_{k+1} *as defined in* (2.2), *control law* (4.75) *and desired reference* $x_{\delta,k+1}$. *We define* $\mathcal{Q}_k(p) := \mathcal{Q}(z_k, p_k)$.

Definition 4.7 (SG Control Goal for Trajectory Tracking) *Consider a constant* $p^* \in \mathcal{P}$. *The SG control goal for system* (2.2) *with* (4.75) *is defined as finding* p_k *so that the SG goal function* $\mathcal{Q}_k(p)$ *as defined in* (4.76) *fulfills:*

$$\mathcal{Q}_k(p) \leq \Delta(z_k), \qquad for \quad k \geq k^*, \tag{4.77}$$

where

$$\Delta(z_k) = V_{sg}(z_k) - \frac{1}{p_k} u_k^T R(z_k) u_k \tag{4.78}$$

with $V_{sg}(z_k) = \frac{1}{2} z_k^T P' z_k$ *and* u_k *as defined in* (4.75); $k^* \in \mathbb{Z}^+$ *is the time at which the SG control goal is achieved.*

Remark 4.10 *Solution* p_k *must guarantees that* $V_{sg}(z_k) > \frac{1}{p_k} u_k^T R(z_k) u_k$ *in order to obtain a positive definite function* $\Delta(z_k)$.

The SG algorithm is used to compute p_k in order to achieve the SG control goal defined above.

Proposition 4.2 *Consider a discrete-time nonlinear system of the form* (2.2) *with* (4.75) *as input. Let* $\mathcal{Q}$ *be a SG goal function as defined in* (4.76), *and denoted by* $\mathcal{Q}_k(p)$. *Let* $\bar{p}, p^* \in \mathcal{P}$ *be positive constant values and* $\Delta(z_k)$ *be a positive definite function with* $\Delta(0) = 0$ *and* ϵ^* *be a sufficiently small positive constant. Assume that:*

- *A1. There exist* p^* *and* ϵ^* *such that*

$$\mathcal{Q}_k(p^*) \leq \epsilon^* \ll \Delta(z_k) \quad and \quad 1 - \epsilon^*/\Delta(z_k) \approx 1. \tag{4.79}$$

- *A2. For all* $p_k \in \mathcal{P}$:

$$(p^* - p_k)^T \nabla_p \mathcal{Q}_k(p) \leq \epsilon^* - \Delta(z_k) < 0 \tag{4.80}$$

  where $\nabla_p \mathcal{Q}_k(p)$ *denotes the gradient of* $\mathcal{Q}_k(p)$ *with respect to* p_k.

Then, for any initial condition $p_0 > 0$, *there exists a* $k^* \in \mathbb{Z}^+$ *such that the SG Control Goal* (4.77) *is achieved by means of the following dynamic variation of parameter* p_k:

$$p_{k+1} = p_k - \gamma_{d,k} \nabla_p \mathcal{Q}_k(p), \tag{4.81}$$

with

$$\gamma_{d,k} = \gamma_c \, \delta_k \, |\nabla_p \mathcal{Q}_k(p)|^{-2} \qquad 0 < \gamma_c \leq 2\,\Delta(z_k)$$

and

$$\delta_k = \begin{cases} 1 & for \quad Q(p_k) > \Delta(z_k) \\ 0 & otherwise. \end{cases} \tag{4.82}$$

Finally, for $k \geq k^*$, p_k becomes a constant value denoted by $\bar{p}$ and the SG algorithm is completed.

Proof. It follows closely the one of Proposition 4.1. ■

Remark 4.11 *Parameter γ_c in (4.81) is selected such that solution p_k ensures the requirement $V_{sg}(z_k) > \frac{1}{p_k} u_k^T R(z_k) u_k$ in Remark 4.10. Then, we have a positive definite function $\Delta(z_k)$.*

When SG Control Goal (4.77) is achieved, then $p_k = \bar{p}$ for $k \geq k^*$. Thus, matrix P_k in (4.73) is considered constant, that is $P_k = P$, where P is computed as $P = \bar{p} K P' K$, with P' a *design* positive definite matrix. Under these constraints, we obtain:

$$\begin{aligned} \alpha(z) &:= u_k \\ &= -\frac{1}{2}\left(R(z_k) + \frac{1}{2} g^T(x_k) P g(x_k)\right)^{-1} g^T(x_k) P (f(x_k) - x_{\delta,k+1}). \end{aligned} \tag{4.83}$$

The following theorem establishes the trajectory tracking via inverse optimal control.

Theorem 4.4 *Consider that system (2.2) with (4.75) has achieved the SG control goal (4.77) by means of (4.81). Let $V(z_k) = \frac{1}{2} z_k^T P z_k$ be a candidate Lyapunov function with $P = P^T > 0$. Then, trajectory tracking inverse optimal control law (4.83) renders solution x_k of system (2.2) to be globally asymptotically stable along the desired trajectory $x_{\delta,k}$. Moreover, with $V(x_k) = \frac{1}{2} z_k^T P z_k$ as CLF and $P = \bar{p} P'$, this control law (4.83) is inverse optimal in the sense that it minimizes the meaningful cost functional given by*

$$\mathcal{J}(z_k) = \sum_{k=0}^{\infty} \left(l(z_k) + u_k^T R(z_k) u_k\right) \tag{4.84}$$

where

$$l(z_k) := -\overline{V} \tag{4.85}$$

with $\overline{V}$ defined as

$$\overline{V} = V(z_{k+1}) - V(z_k) + \alpha^T(z) R(z_k) \alpha(z)$$

and $\alpha(z)$ as defined in (4.83).

Proof. It follows closely the given in Theorem 4.3 and hence it is omitted. ■

4.4.2 Example

To illustrate the applicability of the proposed methodology, we synthesize a trajectory tracking inverse optimal control law in order to achieve trajectory tracking for a discrete-time second order nonlinear system (unstable for $u_k = 0$) of the form (2.2) with:

$$f(x_k) = \begin{bmatrix} 2\,x_{1,k}\,sin(0.5\,x_{1,k}) + 0.1\,x_{2,k}^2 \\ 0.1\,x_{1,k}^2 + 1.8\,x_{2,k} \end{bmatrix} \tag{4.86}$$

and

$$g(x_k) = \begin{bmatrix} 0 \\ 2 + 0.1\,cos(x_{2,k}) \end{bmatrix} \tag{4.87}$$

According to (4.83), the trajectory tracking inverse optimal control law is formulated as

$$u_k = -\frac{1}{2}\left(R(z_k) + \frac{1}{2}\,g^T(x_k)\,P\,g(x_k)\right)^{-1} g^T(x_k)\,P\,(f(x_k) - x_{\delta,k+1})$$

where the positive definite matrix $P_k = p_k\,P'$ is calculated by the SG algorithm with P' as the identity matrix, that is

$$\begin{aligned} P_k &= p_k\,P' \\ &= p_k \begin{bmatrix} 0.020 & 0.016 \\ 0.016 & 0.020 \end{bmatrix} \end{aligned}$$

and $R(x_k)$ is a constant matrix

$$R(x_k) = 0.5.$$

The reference for $x_{2,k}$ is:

$$x_{2\delta,k} = 1.5\,sin(0.12\,k)\;rad.$$

and reference $x_{1\delta,k}$ is defined accordingly.

Figure 4.5 presents the trajectory tracking for x_k with initial condition $p_0 = 2.5$ for the SG algorithm. Control law $\alpha(z_k)$ is also displayed.

4.5 Trajectory Tracking for Block Control Form Systems

Consider system (2.2) to be (globally) stabilized by inverse optimal control law $u_k = \alpha(x_k)$ as proposed in (4.56). Let consider system (2.2) can be presented (possibly after a nonlinear

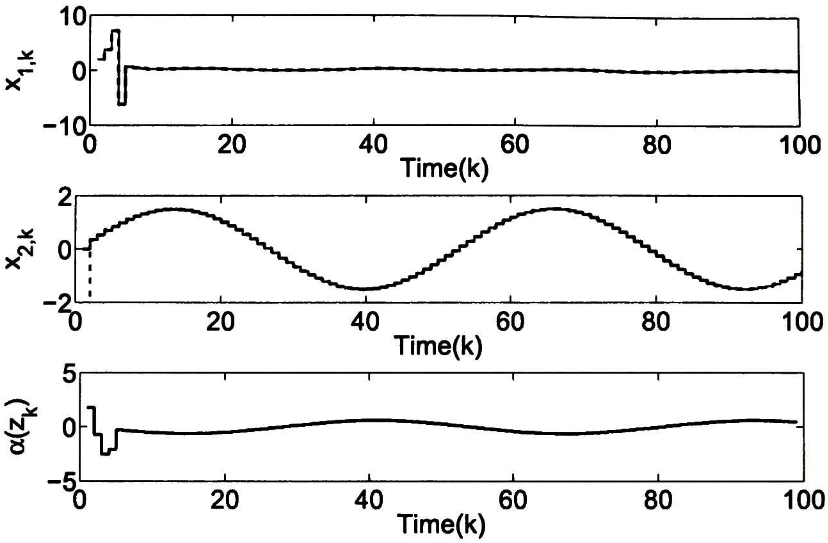

Figure 4.5: Tracking performance of x_k. Solid line ($x_{\delta,k}$) is the reference signal and dashed line is the evolution of x_k. Control signal is also displayed.

transformation) in the nonlinear block control form [75] consisting of r blocks as

$$\begin{aligned} x_{k+1}^1 &= f^1(x_k^1) + B^1(x_k^1)\,x_k^2 \\ &\vdots \\ x_{k+1}^{r-1} &= f^{r-1}(x_k^1, x_k^2, \ldots, x_k^{r-1}) + B^{r-1}(x_k^1, x_k^2, \ldots, x_k^{r-1})\,x_k^r \\ x_{k+1}^r &= f^r(x_k) + B^r(x_k)\,\alpha(x_k) \end{aligned} \tag{4.88}$$

where $x_k \in \mathbb{R}^n$, $x_k = \left[x_k^{1T}\ x_k^{2T}\ \ldots\ x_k^{rT}\right]^T$; $x^j \in \mathbb{R}^{n_j}$; $j = 1, \ldots, r$; n_j denotes the order of each r-th block; $x^j = \left[x^{j_1}\ x^{j_2}\ \ldots\ x^{j_{n_j}}\right]^T$; input $\alpha(x_k) \in R^m$; $f^j : \mathbb{R}^n \to \mathbb{R}^n$ and $B^j : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are smooth mappings. Without loss of generality, $x_k = 0$ is an equilibrium point for (4.88). We assume $f^j(0) = 0$, $rank\{B^j(x_k)\} = m_j\ \forall x_k \neq 0$ and $n = \sum_{j=1}^{r} n_j$.

For trajectory tracking of first block in (4.88), let define the tracking error as

$$z_k^1 = x_k^1 - x_{\delta,k}^1 \tag{4.89}$$

where $x_{\delta,k}^j$ is the desired trajectory signal.

Once defined the first new variable (4.89), we take one step ahead

$$z_{k+1}^1 = f^1(x_k^1) + B^1(x_k^1)\,x_k^2 - x_{\delta,k+1}^1. \tag{4.90}$$

Equation (4.90) is viewed as a block with state z_k^1 and the state x_k^2 is considered as a pseudo-control input, where desired dynamics can be imposed, which can be solved with the anticipation of the desired dynamics for (4.90) as follows:

$$\begin{aligned} z^1_{i,k+1} &= f^1\left(x^1_k\right) + B^1\left(x^1_k\right) x^2_k - x^1_{\delta,k+1} \\ &= f^1\left(z^1_k\right) + B^1\left(x^1_k\right) z^2_k \end{aligned} \tag{4.91}$$

Then, x^2_k is calculated as

$$x^2_{\delta,k} = \left(B^1\left(x^1_k\right)\right)^{-1}\left(x^1_{\delta,k+1} - f^1\left(x^1_k\right) + f^1\left(z^1_k\right) + B^1\left(x^1_k\right) z^2_k\right). \tag{4.92}$$

Note that the calculated value of state $x^2_{\delta,k}$ in (4.92) is not the real value of such state; instead of, it represents the desired behavior for x^2_k. Hence, to avoid confusions this desired value of x^2_k is referred as $x^2_{\delta,k}$ in (4.92).

Proceeding in the same way as for the first block, a second variable in the new coordinates is defined as

$$z^2_k = x^2_k - x^2_{\delta,k}.$$

Taking one step ahead in z^2_k yields

$$\begin{aligned} z^2_{k+1} &= x^2_{k+1} - x^2_{\delta,k+1} \\ &= f^2\left(x^1_k, x^2_k\right) + B^2\left(x^1_k, x^2_k\right) x^3_k - x^2_{\delta,k+1}. \end{aligned}$$

The desired dynamics for this block is imposed as

$$\begin{aligned} z^2_{k+1} &= f^2\left(x^1_k, x^2_k\right) + B^2\left(x^1_k, x^2_k\right) x^3_k - x^2_{\delta,k+1} \\ &= f^1\left(z^1_k\right) + B^2\left(x^1_k, x^2_k\right) z^2_k \end{aligned} \tag{4.93}$$

These steps are taken iteratively. At the last step, the known desired variable is $x^r_{\delta,k}$, and the last new variable is defined as

$$z^r_k = x^r_k - x^r_{\delta,k}.$$

As usually, taking one step ahead yields

$$z^r_{k+1} = f^r\left(x_k\right) + B^r\left(x_k\right)\alpha(x_k) - x^r_{\delta,k+1} \tag{4.94}$$

and the desired dynamics for this block is imposed by means of

$$\alpha(x_k) = \left(B^r\left(x_k\right)\right)^{-1}\left(x^r_{\delta,k+1} - f^r\left(x_k\right) + f^r\left(z_k\right) + B^r\left(z_k\right)\alpha(z_k)\right). \tag{4.95}$$

Hence, system (4.88) can be presented in the new variables $z = \left[z^{1T} z^{2T} \cdots z^{rT}\right]$ of the form

$$\begin{aligned} z^1_{k+1} &= f^1\left(z^1_k\right) + B^1\left(x^1_k\right) z^2_k \\ &\vdots \\ z^{r-1}_{k+1} &= f^{r-1}\left(z^1_k, z^2_k, \ldots, z^{r-1}_k\right) + B^{r-1}\left(z^1_k, z^2_k, \ldots, z^{r-1}_k\right) z^r_k \\ z^r_{k+1} &= f^r\left(z_k\right) + B^r\left(z_k\right)\alpha(z_k) \end{aligned} \tag{4.96}$$

which in a general form can be described by

$$z_{k+1} = f(z_k) + g(z_k)\,\alpha(z_k). \tag{4.97}$$

System (4.97) can be decomposed for x_{k+1} as $x_{k+1} - x_{\delta,k+1} = f(z_k) + g(z_k)\,\alpha(z_k)$, and thus

$$x_{k+1} = f(z_k) + g(z_k)\,\alpha(z_k) + x_{\delta,k+1}. \tag{4.98}$$

Theorem 4.5 *Consider the equilibrium point $x_k = 0$ of system (4.88) to be (globally) asymptotically stabilized by inverse optimal control law $\alpha(x_k)$ (4.56), and therefore the Lyapunov difference becomes $V(x_{k+1}) - V(x_k) < 0$. Then, solution x_k of (4.88) with (4.95) as input is (globally) asymptotically stabilized along the desired trajectory $x_{\delta,k}$. Moreover, control law (4.95) minimizes the following cost functional:*

$$\mathcal{J}(z_k) = \sum_{k=0}^{\infty} \left(l(z_k) + \alpha(z_k)^T\, R(z_k)\, \alpha(z_k) \right) \tag{4.99}$$

with

$$l(z_k) = -\overline{V}(z_k) \geq 0. \tag{4.100}$$

Proof. Let system (4.88) to be described in a general form by (2.2) with $u_k = \alpha(x_k)$ (4.56), and system (4.96) to be described by (4.97). Consider a candidate Lyapunov function as $V(z_k)$. Then, Lyapunov difference becomes

$$\begin{aligned} \Delta V :&= V(z_{k+1}) - V(z_k) \\ &= V(x_{k+1} - x_{\delta,k+1}) - V(z_k). \end{aligned} \tag{4.101}$$

Substituting (4.98) in (4.101) we have

$$\begin{aligned} \Delta V &= V(x_{k+1} - x_{\delta,k+1}) - V(z_k) \\ &= V(f(z_k) + g(z_k)\,\alpha(z_k)) - V(z_k). \end{aligned} \tag{4.102}$$

Due to the fact $V(x_{k+1}) - V(x_k) = V(f(x_k) + g(x_k)\,\alpha(x_k)) - V(x_k) < 0$ for (2.2), then the Lyapunov difference for transformed system is $V(z_{k+1}) - V(z_k) = V(f(z_k) + g(z_k)\,\alpha(z_k)) - V(z_k) < 0$, and (global) asymptotic stability is guaranteed for transformed system (4.97).

The minimization of a meaningful cost functional is established similarly as in Theorem 3.1, and hence it is omitted. ■

4.5.1 Example

In this section, we illustrate the applicability of the obtained results by means of an unstable example for solving the trajectory tracking problem. By illustration easily and space limitation, we synthesize an trajectory tracking inverse optimal control law for a discrete-time

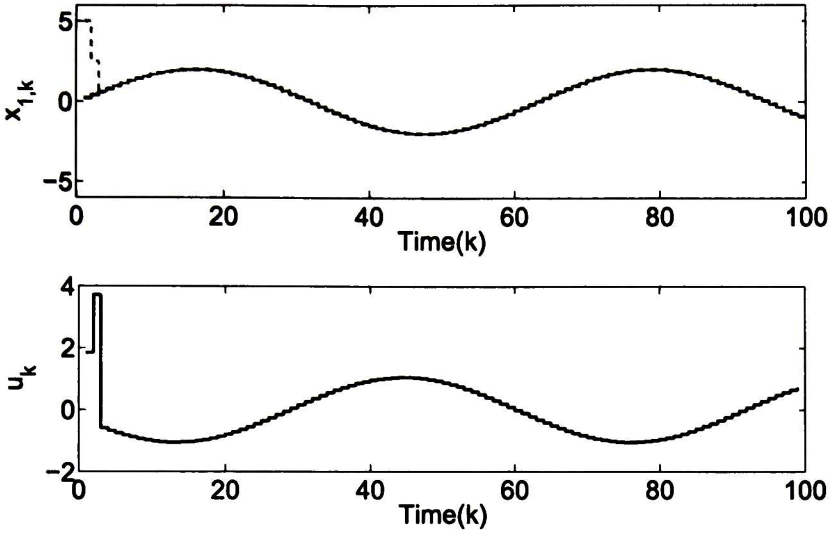

Figure 4.6: Tracking performance of x_k. Solid line ($x_{\delta,k}$) is the reference signal and dashed line is the evolution of $x_{1,k}$. Control signal is also displayed.

second order system (unstable for $u_k = 0$) of the form (2.2) with:

$$f(x_k) = \begin{bmatrix} 1.5\,x_{1,k} + x_{2,k} \\ x_{1,k} + 2\,x_{2,k} \end{bmatrix} \tag{4.103}$$

and

$$g(\chi_k) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.104}$$

In accordance with Section 4.5, control law (4.95) becomes

$$\begin{aligned} \alpha(x_k) \;&=\; x^1_{\delta,k+2} - 1.5(1.5\,x_{1,k} + x_{2,k}) + 1.5(1.5\,z_{1,k} + z_{2,k}) + 2\,(1.5\,z_{1,k} \\ &\quad +2\,z_{2,k}) + 2\alpha(z_k) - x_{1,k} - 2\,z_{2,k}. \end{aligned} \tag{4.105}$$

Figure 4.6 presents the trajectory tracking for $x_{1,k}$ with

$$P = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

4.6 Conclusions

This chapter has established the inverse optimal control problem for a class of discrete-time nonlinear systems. To avoid the solution of the Hamilton-Jacobi-Bellman equation, we propose a discrete-time control Lyapunov function (CLF) in a quadratic form, which depends first on a fixed parameter and finally this parameter is adjusted by means of the speed-gradient algorithm. Based on this CLF, the inverse optimal control strategy is synthesized. Stability and the corresponding conditions for the inverse optimal control solution are established. Then, these results are extended to establish an inverse optimal trajectory tracking. Additionally, an inverse optimal control scheme was presented to achieve trajectory tracking for block control form nonlinear systems. Simulation results illustrate that the proposed controller ensures stabilization and trajectory tracking for nonlinear system, and this control law minimizes a meaningful cost functional.

Chapter 5

Neural Inverse Optimal Control

This chapter discusses the combination of Section 2.5, Section 3.1, and Section 3.2 results as presented in Section 5.1 to achieve stabilization and trajectory tracking for uncertain nonlinear systems, by using a RHONN scheme to model uncertain nonlinear systems, and then applying the inverse optimal control methodology. Finally, Section 5.2 establishes a block transformation for the neural model in order to solve the inverse optimal trajectory tracking as a stabilization problem for block control form nonlinear systems. Examples illustrate the applicability of the proposed control techniques.

Stabilization and trajectory tracking results can be applied to disturbed nonlinear systems, which can be modeled by means of a neural identifier presented in Section 2.5, and obtaining a robust inverse optimal controller. Two procedures to achieve robust trajectory tracking with the neural model are presented. First, based on passivity approach, we propose a neural inverse optimal controller which uses a CLF with a global minimum on the desired trajectory. Second, a block transformation for a neural identifier is applied in order to obtain an error system on the desired reference, and then, a neural inverse optimal stabilization control law for the error resulting system is synthesized.

5.1 Neural Inverse Optimal Control Scheme

First, the stabilization problem for discrete-time nonlinear systems is discussed.

5.1.1 Stabilization

As described in Section 2.5, for neural identification of (2.33) a series-parallel neural model (2.39) can be used. Then, for this neural model, the stabilization results established in

Section 3 are applied as follows.

Model (2.39) can be represented as a system of the form (2.27)

$$x_{k+1} = f(x_k) + g(x_k)\, u_k$$

and if there exists $P = P^T > 0$ satisfying condition (3.5), this system can be asymptotically stabilized by the inverse optimal control law

$$\alpha(x_k) = -\left(I_m + \frac{1}{2}\, g^T(x_k)\, P\, g(x_k)\right)^{-1} g^T(x_k)\, P\, f(x_k)$$

in accordance with Theorem 3.1.

An example illustrates the previously mentioned.

Example

We synthesize a neural inverse optimal control law for a discrete-time second order nonlinear system (unstable for $u_k = 0$) of the form (2.27) with:

$$f(\chi_k) = \begin{bmatrix} 0.5\, \chi_{1,k}\, sin(0.5\, \chi_{1,k}) + 0.2\, \chi_{2,k}^2 \\ 0.1\, \chi_{1,k}^2 + 1.8\, \chi_{2,k} \end{bmatrix} \tag{5.1}$$

and

$$g(\chi_k) = \begin{bmatrix} 0 \\ 2 + 0.1\, cos(\chi_{2,k}) \end{bmatrix} \tag{5.2}$$

Neural Identifier for (5.1)-(5.2)*:* Assume system (5.1)-(5.2) to be unknown. In order to identify this uncertain system, from (2.36) and (2.39), we propose the following series–parallel neural network

$$\begin{aligned} x_{1,k+1} &= w_{11,k}\, S(\chi_1) + w_{12,k}\, S(\chi_2)^2 \\ x_{1,k+1} &= w_{21,k}\, S(\chi_1)^2 + w_{22,k}\, S(\chi_2) + w_2'\, u_k \end{aligned} \tag{5.3}$$

which can be rewritten as $x_{k+1} = f(x_k) + g(x_k)\, u_k$, where

$$f(x_k) = \begin{bmatrix} w_{11,k}\, S(\chi_1) + w_{12,k}\, S(\chi_2)^2 \\ w_{21,k}\, S(\chi_1)^2 + w_{22,k}\, S(\chi_2) \end{bmatrix} \tag{5.4}$$

and

$$g(x_k) = \begin{bmatrix} 0 \\ w_2' \end{bmatrix} \tag{5.5}$$

with $w_2' = 0.8$. Initial adjustable weights are given in a random way; $\eta_1 = \eta_2 = 0.99$, $P_1 = P_2 = 10\, I_2$, where I_2 is the 2×2 identity matrix; $Q_1 = Q_2 = 1300\, I_2$, $R_1 = 1000$ and $R_2 = 4500$.

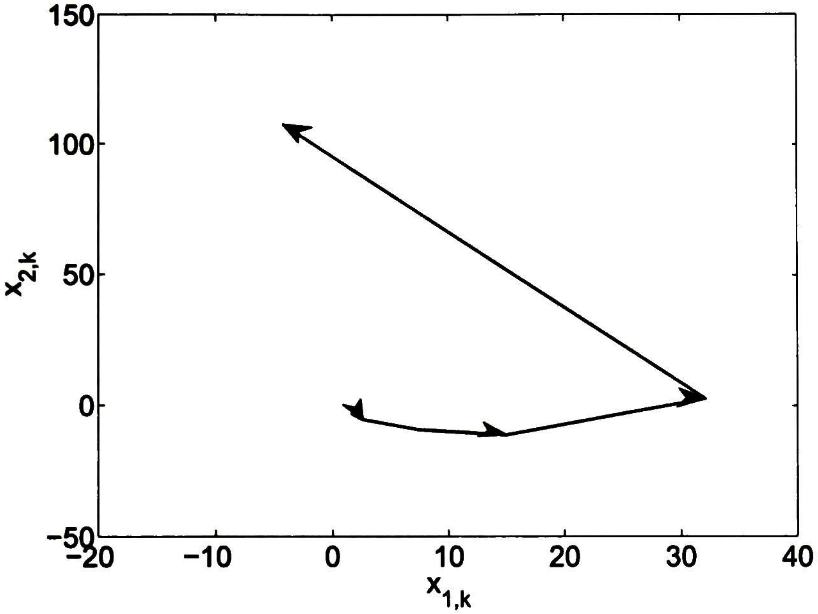

Figure 5.1: Unstable phase portrait.

Control Synthesis

According to (3.2), the inverse optimal control law is formulated as

$$\alpha(x_k) = -\left(1 + \frac{1}{2}\, g^T(x_k)\, P\, g(x_k)\right)^{-1} g^T(x_k)\, P\, f\,(x_k) \tag{5.6}$$

where we propose a positive definite matrix P as

$$P = \begin{bmatrix} 0.0005 & 0.0319 \\ 0.0319 & 3.2942 \end{bmatrix}$$

Simulation Results

The phase portrait for this unstable open-loop ($u_k = 0$) system with initial conditions $\chi_0 = [2 \quad -2]^T$ is displayed in Figure 5.1.

Figure 5.2 presents the stabilization time response for x_k of this system with initial conditions $\chi_0 = [2 \quad -2]^T$. Initial conditions for RHONN are $x_0 = [0.5 \quad 2]^T$; this figure also includes the applied inverse optimal control law (5.6), which achieves asymptotic stability.

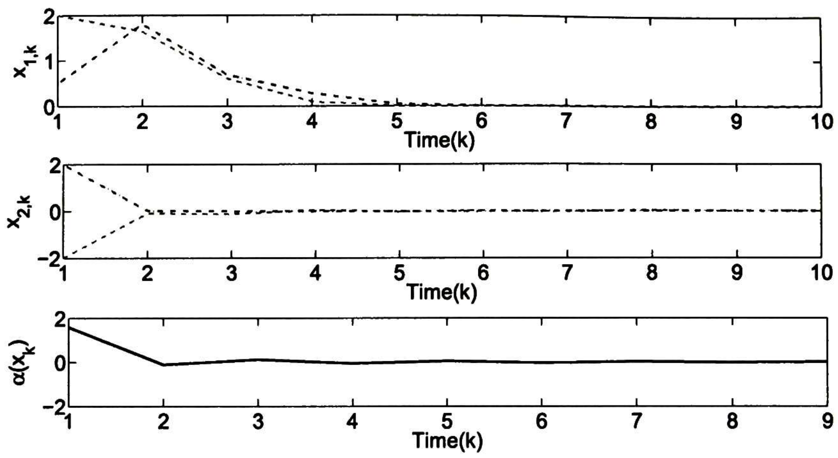

Figure 5.2: Stability time response for (5.1)-(5.2) and (5.6) as input. Dashed line is the evolution of χ_k and dash-dot line is x_k. Control signal is also displayed.

5.1.2 Trajectory Tracking

The tracking of a desired trajectory, defined in terms of the plant state χ_i formulated as (2.33), can be established as the following inequality:

$$\|\chi_{i\delta} - \chi_i\| \leq \|\chi_i - x_i\| + \|\chi_{i\delta} - x_i\| \tag{5.7}$$

where $\|\cdot\|$ stands for the Euclidean norm, $\chi_{i\delta}$ is the desired trajectory signal, which is assumed smooth and bounded. Inequality (5.7) is valid considering the separation principle for discrete-time nonlinear systems [76], and based on (5.7), the tracking of a desired trajectory can be divided into the following two requirements:

Requirement 5.1

$$\lim_{k\to\infty} \|\chi_i - x_i\| \leq \zeta_i \tag{5.8}$$

with ζ_i *a small positive constant.*

Requirement 5.2

$$\lim_{k\to\infty} \|\chi_{i\delta} - x_i\| = 0. \tag{5.9}$$

In order to fulfill Requirement 5.1, an on-line neural identifier based on (2.36) is proposed to ensure (5.8) [46], while (5.9), in Requirement 5.2, is guaranteed by a discrete-time controller developed using inverse optimal control technique, as presented in this dissertation. A general control scheme is shown in Figure 5.3.

Now, trajectory tracking is illustrated as established in Section 3.2 for the neural scheme as follows.

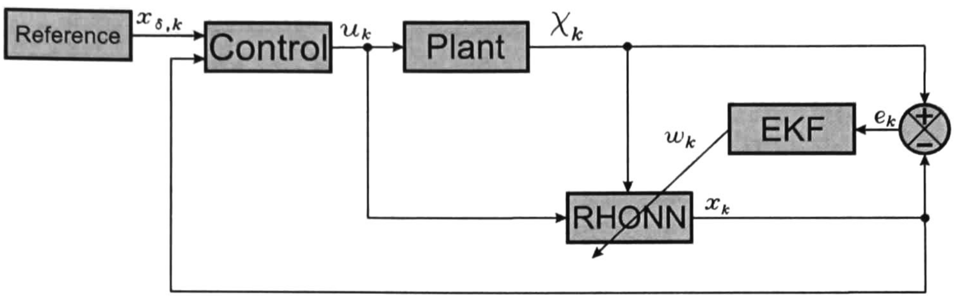

Figure 5.3: Control scheme.

Example

In accordance with Theorem 3.3, the control law to achieve trajectory tracking for system (2.27) is given in (3.31) as $u_k = -y_k$, for which $y_k = h(x_k, x_{\delta,k+1}) + J(x_k)\,u_k$, where

$$h(x_k, x_{\delta,k+1}) = g^T(x_k)\,\overline{P}\,(f(x_k) - x_{\delta,k+1})$$

and

$$J(x_k) = \frac{1}{2}\,g^T(x_k)\,\overline{P}\,g(x_k)$$

with $f(x_k)$ and $g(x_k)$ as defined in (5.1) and (5.2), respectively. Hence, we adjust gain matrix $\overline{P} = K^T\,P\,K$ for (3.31) in order to achieve trajectory tracking for $x_k = [x_{1,k}\ x_{2,k}]^T$. The reference for $x_{2,k}$ is:

$$x_{2\delta,k} = 2\,sin(0.075\,k)\ rad.$$

and reference $x_{1\delta,k}$ is defined accordingly.

Figure 5.4 presents the trajectory tracking for x_k with

$$P = \begin{bmatrix} 0.0484 & 0.0387 \\ 0.0387 & 0.0484 \end{bmatrix}; \quad K = \begin{bmatrix} 0.100 & 0.00 \\ 0.00 & 8.25 \end{bmatrix}$$

5.2 Block Control Form: A Nonlinear Systems Particular Class

In this section, we develop a neural inverse optimal control scheme for systems which have a special state representation referred as the block control (BC) form [75]. For these systems, the trajectory tracking problem is solved as a stabilization problem after a block transformation.

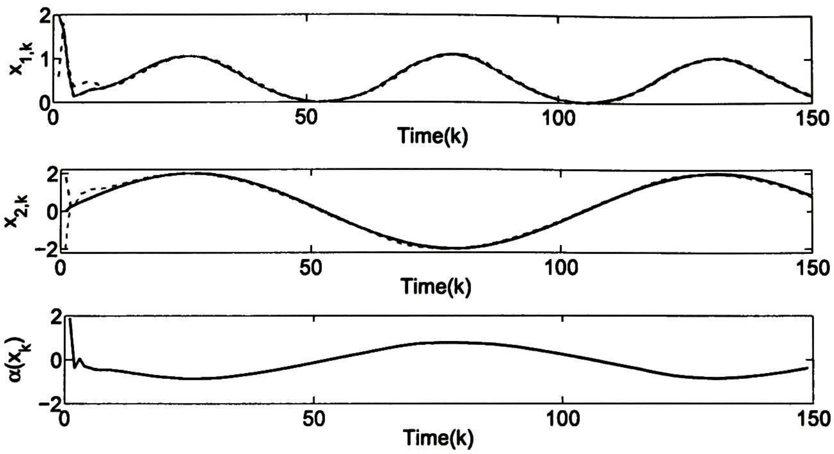

Figure 5.4: Tracking performance of χ_k. Solid line ($\chi_{\delta,k}$) is the reference signal, dashed line is the evolution of χ_k and dash-dot line is x_k. Control signal is also displayed.

5.2.1 Block Transformation

Let consider that system (2.33), under an appropriate nonsingular transformation, can be rewritten as the following BC form with r blocks:

$$\begin{aligned}
\chi_{i_1,k+1} &= f_{i_1}(\chi_{i_1,k}) + B_{i_1}(\chi_{i_1,k})\chi_{i_1,k} + \Gamma_{i_1,k} \\
\chi_{i_2,k+1} &= f_{i_2}(\chi_{i_1,k},\chi_{i_2,k}) + B_{i_2}(\chi_{i_1,k},\chi_{i_2,k})\chi_{i_3,k} + \Gamma_{i_2,k} \\
&\vdots \\
\chi_{i_r,k+1} &= f_{i_r}(\chi_k) + B_{i_r}(\chi_k)\,u_k + \Gamma_{i_r,k}
\end{aligned} \tag{5.10}$$

where $\chi_k \in \mathbb{R}^n$ is the system state, $\chi_k = \left[\begin{array}{cccc} \chi_{i1,k}^T & \chi_{i2,k}^T & \cdots & \chi_{i2,k}^T \end{array}\right]^T$; $i = 1, \ldots, n_r$; $u_k \in \mathbb{R}^{m_r}$ is the input vector. We assume that f_{i_j}, B_{i_j} and Γ_{i_j} are smooth functions, $j = 1, \ldots, r$, $f_{i_j}(0) = 0$ and $rank\{B_{i_j}(\chi_k)\} = m_j \; \forall \chi_k \neq 0$. The unmatched and matched disturbance terms are represented by $\Gamma_{i_r,k}$. The whole system order is $n = \sum\limits_{j=1}^{r} n_j$.

To identify (5.10), we propose a neural identifier with the same BC structure.

The essential feature of the proposed method is the conversion of the system (2.36), using the series–parallel model (2.39) to identify (5.10), to the BC form consisting of r blocks as:

$$\begin{aligned}
x_{i_1,k+1} &= W_{i_1,k}\,\rho_{i_1}(\chi_{i_1,k}) + W_{i_1}^{'}\,\chi_{i_2,k} \\
x_{i_2,k+1} &= W_{i_2,k}\,\rho_{i_2}(\chi_{i_1,k},\chi_{i_2,k}) + W_{i_2}^{'}\,\chi_{i_3,k} \\
&\vdots \\
x_{i_r,k+1} &= W_{i_r,k}\,\rho_{i_r}(\chi_{i_1,k},\ldots,\chi_{i_r,k}) + W_{i_r}^{'}\,u_k
\end{aligned} \tag{5.11}$$

where $x_k = \left[x_{i_1}^T x_{i_2}^T \cdots x_{i_r}^T\right]^T \in \mathbb{R}^n$, $x_{i_r} \in \mathbb{R}^{n_r}$ denotes the i-th neuron system state corresponding to the r-th block; $i = 1, \ldots, n_r$; $W_{i_1,k} = [w_{1_1,k}^T \; w_{2_1,k}^T \; \ldots \; w_{n_{r1},k}^T]^T$ is the on-line adjustable weight matrix and $W_{i_r}' = [w_{1_1}'^T \; w_{2_1}'^T \; \ldots \; w_{n_{r1}}'^T]^T$ is the fixed weight matrix; n_r denotes the order for each r-th block, and the whole system order becomes $n = \sum_{j=1}^{r} n_j$.

First, we define the tracking error as

$$z_{i_1,k} = x_{i_1,k} - \chi_{i_1\delta,k} \tag{5.12}$$

where $\chi_{i_1\delta,k}$ is the desired trajectory signal.

Once defined the first new variable (5.12), one step ahead is taken as

$$z_{i_1,k+1} = W_{i_1,k}\,\rho_{i_1}(\chi_{i_1,k}) + W_{i_1}'\,\chi_{i_2,k} - \chi_{i_1\delta,k+1}. \tag{5.13}$$

Equation (5.13) is viewed as a block with state $z_{i_1,k}$ and the state $\chi_{i_2,k}$ is considered as a pseudo-control input, where desired dynamics can be imposed. This can be solved with the anticipation of the desired dynamics for this block as follows:

$$z_{i_1,k+1} = W_{i_1,k}\,\rho_{i_1}(\chi_{i_1,k}) + W_{i_1}'\,\chi_{i_2,k} - \chi_{i_1\delta,k+1} = K_{i_1}\,z_{i_1,k} \tag{5.14}$$

where $K_{i_1} = diag\{k_{11}, \ldots, k_{n_1 1}\}$ with $|k_{q1}| < 1$, $q = 1, \ldots, n_1$, in order to assure stability for block (5.14). From (5.14), $\chi_{i_2,k}$ is calculated as

$$\chi_{i_2\delta,k} \;=\; \left(W_{i_1}'\right)^{-1}\left(-W_{i_1,k}\,\rho_{i_1}\left(\chi_{i_1,k}\right) + \chi_{i_1\delta,k+1} + K_{i_1}\,z_{i_1,k}\right). \tag{5.15}$$

Note that the calculated value of state $\chi_{i_2\delta,k}$ in (5.15) is not the real value of such state; instead, it represents the desired behavior for $\chi_{i_2,k}$. So, to avoid confusions this desired value of $\chi_{i_2,k}$ is referred as $\chi_{i_2\delta,k}$ in (5.15).

Proceeding in the same way as for the first block, a second variable in the new coordinates is defined as

$$z_{i_2,k} = x_{i_2,k} - \chi_{i_2\delta,k}.$$

Taking one step ahead in $z_{i_2,k}$ yields

$$\begin{aligned} z_{i_2,k+1} &= x_{i_2,k+1} - x_{i_2\delta,k+1} \\ &= W_{i_2,k}\,\rho_{i_2}(\chi_{i_1,k}, \chi_{i_2,k}) + W_{i_2}'\,\chi_{i_3,k} - \chi_{i_2\delta,k+1}. \end{aligned}$$

The desired dynamics for this block is imposed as

$$z_{i_2,k+1} = W_{i_2,k}\,\rho_{i_2}(\chi_{i_1,k}, \chi_{i_2,k}) + W_{i_2}'\chi_{i_3,k} - \chi_{i_2\delta,k+1} = K_{i_2} z_{i_2,k} \tag{5.16}$$

where $K_{i_2} = diag\{k_{12}, \ldots, k_{n_2 2}\}$ with $|k_{q2}| < 1$, $q = 1, \ldots, n_2$.

These steps are taken iteratively. At the last step, the known desired variable is $\chi^{r}_{r\delta,k}$, and the last new variable is defined as

$$z_{i_r,k} = x_{i_r,k} - \chi_{i_r\delta,k}.$$

As usually, taking one step ahead yields

$$z_{i_r,k+1} = W_{i_r,k}\,\rho_{i_r}(\chi_{i_1,k},\ldots,\chi_{i_r,k}) + W'_{i_r}\,u_k - \chi_{i_r\delta,k+1}. \tag{5.17}$$

System (5.11) can be represented in the new variables of the form

$$\begin{aligned} z_{i_1,k+1} &= K_1\,z_{i_1,k} + W'_{i_1}\,z_{i_2,k} \\ z_{i_2,k+1} &= K_{i_2}\,z_{i_2,k} + W'_{i_2}\,z_{i_3,k} \\ &\vdots \\ z_{i_r,k+1} &= W_{i_r,k}\,\rho_{i_r}(\chi_{i_1,k},\ldots,\chi_{i_r,k}) - \chi_{i_r\delta,k+1} + W'_{i_r}\,u_k. \end{aligned} \tag{5.18}$$

5.2.2 Block Inverse Optimal Control

Now, the problem is to stabilize the transformed system (5.18) at the origin to achieve trajectory tracking along $\chi_{\delta,k}$. System (5.18) can be presented in a general form as

$$z_{k+1} = f(z_k) + g(z_k)\,u_k. \tag{5.19}$$

where $z_k = \left[z^T_{i_1,k} z^T_{i_2,k} \cdots z^T_{i_r,k}\right]$.

Then, for this system, we apply the inverse optimal control law (3.2) as

$$\alpha(z_k) = -\,\left[I_m + J(z_k)\right]^{-1}\,h(z_k)$$

with

$$h(z_k) = g(z_k)^T\,P\,f(z_k) \tag{5.20}$$

and

$$J(z_k) = \frac{1}{2}g(z_k)^T\,P\,g(z_k) \tag{5.21}$$

in order to achieve stabilization of the error system (5.19).

5.2.3 Planar Robot Example

As application, the transformation methodology for a two DOF planar rigid robot is developed to establish an error system as described in the previous section. Then, Section 3.1 and Section 2.5 results are used to synthesize a neural inverse optimal controller in order to achieve global stabilization of the error system, and therefore, position trajectory tracking is ensured for the planar robot.

Robot Model Description

After an Euler discretization of the robot dynamics, the discrete-time robot model is described by (A.1), which has the BC form.

Remark 5.1 *The structure of system (A.1) is used to design the neural network identifier. The parameters of the system (A.1) are assumed to be unknown for the control synthesis.*

Remark 5.2 *For system* (A.1), $n = 4$, $r = 2$, $n_1 = 2$, $n_2 = 2$. *To identify this system, the series-parallel model* (2.39) *is used.*

Neural Identifier for the Planar Robot

To identify the uncertainty robot model, from (2.36), (2.39) and (5.11), we propose the following series–parallel neural network according to Remark 5.1 and Remark 5.2

$$\begin{aligned}
\begin{bmatrix} x_{1_1,k+1} \\ x_{2_1,k+1} \end{bmatrix} &= \begin{bmatrix} w_{1_11,k}\, S(\chi_{1_1,k}) + w'_{1_1}\, \chi_{1_2,k} \\ w_{2_11,k}\, S(\chi_{2_1,k}) + w'_{2_1}\, \chi_{2_2,k} \end{bmatrix} \\
\begin{bmatrix} x_{1_2,k+1} \\ x_{2_2,k+1} \end{bmatrix} &= \left[\begin{matrix} w_{1_21,k}\, S(\chi_{1_1,k}) + w_{1_22,k}\, S(\chi_{2_1,k}) \cdots \\ w_{2_21,k}\, S(\chi_{1_1,k}) + w_{2_22,k}\, S(\chi_{2_1,k}) \cdots \end{matrix} \right. \\
&\qquad \begin{matrix} +w_{1_23,k}\, S(\chi_{1_2,k}) + w_{1_24,k}\, S(\chi_{2_2,k}) \cdots \\ +w_{2_23,k}\, S(\chi_{1_2,k}) + w_{2_24,k}\, S(\chi_{2_2,k}) \cdots \end{matrix} \\
&\qquad \left. \begin{matrix} +w'_{1_21}\, u_{1,k} + w'_{1_22}\, u_{2,k} \\ +w'_{2_21}\, u_{1,k} + w'_{2_22}\, u_{2,k} \end{matrix} \right]
\end{aligned} \tag{5.22}$$

where $x_{j_1,k}$ identifies to $\chi_{j_1,k}$ and $x_{j_2,k}$ identifies to $\chi_{j_2,k}$; $j = 1, 2$; w_{j_rp} are the adjustable weights, p is the corresponding number of adjustable weights; w'_{j_rp} are fixed parameters.

To update the weights, the adaptation algorithm (2.40) is implemented.

Control Synthesis

Let define system (5.22) in a r-block control form as

$$x_{1,k+1} := \begin{bmatrix} x_{1_1,k+1} \\ x_{2_1,k+1} \end{bmatrix} = W_{1,k}\, \rho_1(\chi_{1,k}) + W'_1\, \chi_{2,k} \tag{5.23}$$

$$x_{2,k+1} := \begin{bmatrix} x_{1_2,k+1} \\ x_{2_2,k+1} \end{bmatrix} = W_{2,k}\, \rho_2(\chi_{1,k}, \chi_{2,k}) + W'_2\, u_k$$

with $\chi_{1,k}, \chi_{2,k}, \rho_1, \rho_2, W_{1,k}, W_{2,k}, W'_1$ and W'_2 of appropriated dimension according to (5.22).

The goal is to force the angle position $x_{1,k}$ to track a desired reference signal $\chi_{1\delta,k}$. This is achieved by designing a control law as described in Section 5.2.1. First the tracking error is defined as

$$z_{1,k} = x_{1,k} - \chi_{1\delta,k}.$$

Then using (5.22) and introducing the desired dynamics for $z_{1,k}$ results in

$$\begin{aligned} z_{1,k+1} &= W_{1,k}\,\rho_1(\chi_{1,k}) + W_1'\,\chi_{2,k} - \chi_{1\delta,k+1} \\ &= K_1\,z_{1,k}. \end{aligned} \tag{5.24}$$

where $K_1 = diag\{k_{1_1}, k_{2_1}\}$ with $|k_{1_1}|, |k_{2_1}| < 1$.

The desired value $\chi_{2\delta,k}$ for the pseudo-control input $\chi_{2,k}$ is calculated from (5.24) as

$$\chi_{2\delta,k} = \left(W_1'\right)^{-1}\left(-W_{1,k}\,\rho_1(\chi_{1,k}) + \chi_{1\delta,k+1} + K_1\,z_{1,k}\right). \tag{5.25}$$

At the second step, we introduce a new variable as

$$z_{2,k} = x_{2,k} - \chi_{2\delta,k}.$$

Taking one step ahead, we have

$$z_{2,k+1} = W_{2,k}\,\rho_2(\chi_{1,k}, \chi_{2,k}) - \chi_{2\delta,k+1} + W_2'\,u_k \tag{5.26}$$

Now, the system (5.22) in the new variables $z_{1,k}$ and $z_{2,k}$ is represented of the following form:

$$\begin{aligned} z_{1,k+1} &= K_1\,z_{1,k} + W_1'\,z_{2,k} \\ z_{2,k+1} &= W_{2,k}\,\rho_2(\chi_{1,k}, \chi_{2,k}) - \chi_{2\delta,k+1} + W_2'\,u_k. \end{aligned} \tag{5.27}$$

If we rewrite the system (5.27) in the form (5.18)-(5.19), from Section 5.2.2, the proposed control law is given as

$$\begin{aligned} u_k &= \alpha(z_k) \\ &= -(1 + J(z_k))^{-1}\,h(z_k) \end{aligned} \tag{5.28}$$

where $h(z_k)$ and $J(z_k)$ are defined as in (5.20) and (5.21), respectively.

Simulation Results

The initial conditions are given in the Table 5.1; the identifier (5.22) and controller parameters are shown in Table 5.2, where I_4 is the 4×4 identity matrix. The sample time is $T = 0.001\ s$ and the adjustable weights are initialized in a random way.

Table 5.1: Initial conditions

$\chi_{1_1,0}$	0.5 rad	$\chi_{2_1,0}$	-0.5 rad
$\chi_{1_2,0}$	0 rad/s	$\chi_{2_2,0}$	0 rad/s
$P_{1_1,0}$	10	$P_{2_1,0}$	10
$P_{1_2,0}$	$10\, I_4$	$P_{2_1,0}$	$10\, I_4$
$x_{1_1,0}$	0 rad	$x_{2_1,0}$	0 rad
$x_{1_2,0}$	0 rad/s	$x_{2_2,0}$	0 rad/s

Table 5.2: Identifier and controller parameters

PARAMETER	VALUE	PARAMETER	VALUE
Q_{1_1}	1000	Q_{2_1}	1000
Q_{1_2}	$1000\, I_4$	Q_{2_1}	$1000\, I_4$
R_{1_1}	10000	R_{2_1}	10000
R_{1_2}	10000	R_{2_1}	10000
w'_{1_1}	0.001	w'_{2_1}	0.001
$w'_{1_2 1}$	0.900	$w'_{1_2 2}$	0.010
$w'_{2_2 1}$	0.010	$w'_{2_2 2}$	0.700
k_{1_1}	0.960	k_{2_1}	0.970
η_{1_1}	0.900	η_{1_2}	0.900
η_{2_1}	0.900	η_{2_2}	0.900

The value of matrix P is

$$P = \begin{bmatrix} 1.1520 & 0.1512 & 0.2880 & 0.0151 \\ 0.1512 & 0.0794 & 0.1512 & 0.0198 \\ 0.2880 & 0.1512 & 1.1520 & 0.1512 \\ 0.0151 & 0.0198 & 0.1512 & 0.0794 \end{bmatrix}$$

The reference signals are

$$\chi_{1_1\delta,k} = 2.0\, sin(1.0\, k\, T)\ rad$$

$$\chi_{2_1\delta,k} = 1.5\, sin(1.2\, k\, T)\ rad.$$

References $\chi_{1_2\delta,k}$ and $\chi_{2_2\delta,k}$, are defined accordingly.

The reference signals are selected to illustrate the ability of the proposed algorithm to track nonlinear trajectories.

The tracking performance of the link 1 and link 2 position are shown in the Fig. 5.5. The control signals $u_{1,k}$ and $u_{2,k}$ responses are displayed in Fig. 5.6.

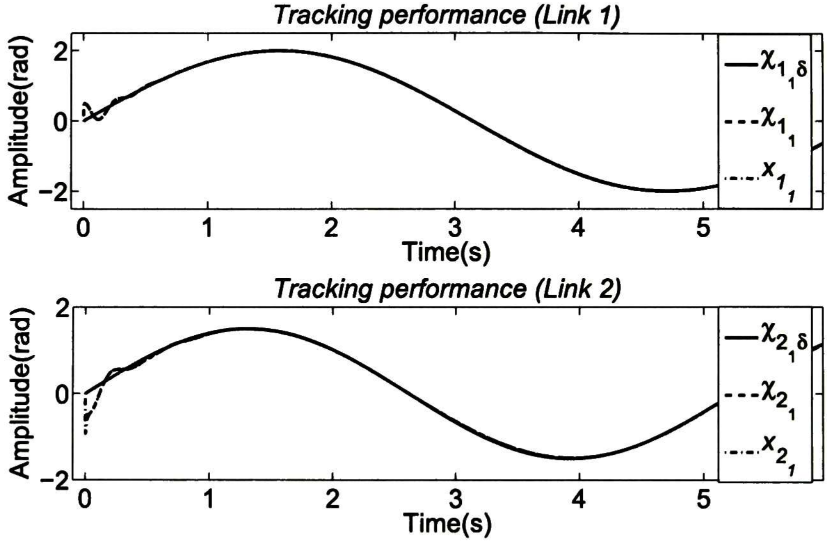

Figure 5.5: Planar robot tracking performance.

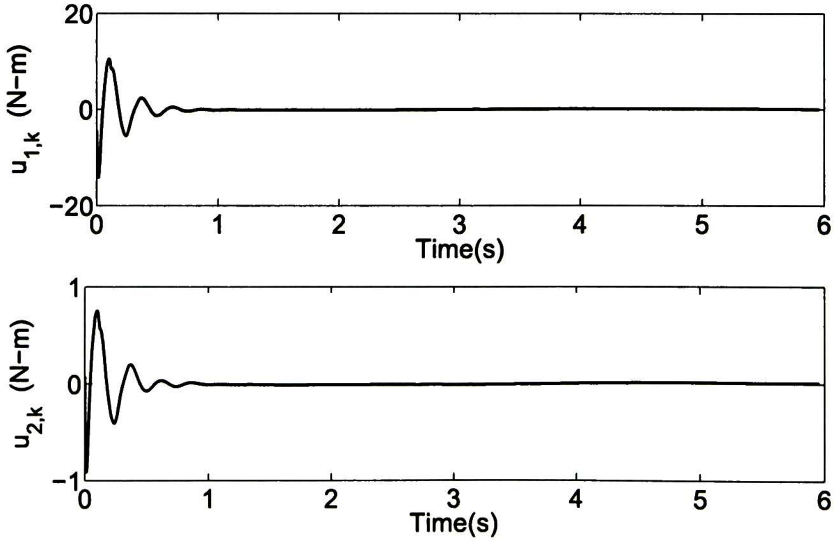

Figure 5.6: Control signal time responses.

5.3 Conclusions

This chapter has presented the application of inverse optimal control to a neural identifier in order to achieve a robust stabilization and robust trajectory tracking scheme for a class of nonlinear systems. We begin analyzing the robust stabilization and robust trajectory tracking via passivity. A posteriori, a transformation on the desired trajectory is established, then, stabilization on this transformed system is performed to achieve trajectory tracking. We use discrete-time recurrent neural networks to model an uncertain nonlinear systems; thus, an explicit knowledge of the plant is not necessary. The proposed approach is successfully applied to implement a robust control based on recurrent high order neural network, passivity and inverse optimality by means of examples. In the simulation results, it can be seen that the required goal is achieved, i.e., the designed controller maintains stability or trajectory tracking of the unknown system. The training of the neural network is performed on-line using an extended Kalman filter.

Chapter 6

Conclusions and Future Work

Conclusions about this dissertation are presented in this final chapter. Also, a future work plan is proposed.

6.1 Conclusions

This dissertation has presented a novel discrete-time inverse optimal control scheme, which achieves stabilization and trajectory tracking for nonlinear systems and is inverse optimal in the sense that, a posteriori, control law minimizes a meaningful cost functional. To avoid the Hamilton-Jacobi-Bellman equation solution, we proposed a discrete-time quadratic control Lyapunov function (CLF). The controller synthesis is based on two approaches: *a*) inverse optimal control based on passivity, in which the storage function is used as CLF, and *b*) an inverse optimal control based on the selection of a CLF. Furthermore, a robust inverse optimal control was established in order to guarantee stability for nonlinear systems, which are affected by internal and/or external disturbances.

We use discrete-time recurrent neural networks to model uncertain nonlinear systems; thus, an explicit knowledge of the plant is not necessary. The proposed approach is successfully applied to implement a robust controller based on recurrent high order neural network and inverse optimality via passivity. By means of simulations, it can be seen that the required goal is achieved, i.e., the designed controller maintains stability of the plant with unknown parameters. For neural network training, an on-line extended Kalman filter is performed.

The applicability of the proposed controllers is illustrated, via simulations, by stabilization and trajectory tracking for an unstable nonlinear system and a two degrees of freedom (DOF) planar robot.

The contributions of this dissertation are:

- To synthesize an *inverse optimal discrete-time controller for nonlinear systems via passivity (Chapter 3)*. The controller synthesis is based on the selection of a storage function, which is used as candidate CLF, and a passifying law to render passive the system. This controller achieves stabilization and trajectory tracking for discrete-time nonlinear systems and is inverse optimal in the sense that, a posteriori, minimizes a meaningful cost functional.
- To synthesize an *inverse optimal discrete-time controller for nonlinear systems via CLF (Chapter 4)*. The control law is obtained as a result of solving the Bellman equation. Then, a candidate CLF for the obtained control law is proposed such that stabilization and trajectory tracking for discrete-time nonlinear systems are achieved; a posteriori, a meaningful cost functional is minimized. The CLF depends on a fixed parameter in order to satisfy stability and optimality condition. A posteriori, the speed gradient algorithm is established to compute this CLF parameter.
- To establish a *neural inverse optimal discrete-time control scheme for uncertain nonlinear systems (Chapter 5)*. For this neural scheme, an assumed uncertain discrete-time nonlinear system is identified by a RHONN model, which is used to synthesize the inverse optimal controller in order to achieve stabilization and trajectory tracking. The neural learning is performed on-line through an extended Kalman filter.

6.2 Future Work

The following topics are suggested as continuation of this dissertation.

- To establish a robust inverse optimal controller for trajectory tracking.
- To synthesize an inverse optimal controller based on observer schemes when the state is not fully measurable.
- To generalize the inverse optimal control scheme for general CLF (a non-necessarily quadratic CLF).
- To determine an appropriate selection for matrix P and control weight matrix $R(x_k)$ for the cost functional such that these matrices reflect desired performance attributes, such as convergence speed, control effort and inputs and/or state constraints.
- To extend the speed-gradient algorithm for entry-wise calculation of matrix P.
- To combine inverse optimal control and sliding modes in a discrete-time framework.

Appendix A

Robot Model

After discretizing by means of the Euler approximation the robot dynamics, the discrete-time planar robot model is written as:

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + x_{3,k}T \\ x_{2,k+1} &= x_{2,k} + x_{4,k}T \\ x_{3,k+1} &= x_{3,k} + \left(\frac{-D_{22}(V_1+F_1) + D_{12}(V_2+F_2)}{D_{11}D_{22} - D_{12}^2} + \frac{D_{22}\,u_{1,k} - D_{12}\,u_{2,k}}{D_{11}D_{22} - D_{12}^2} \right) T \\ x_{4,k+1} &= x_{4,k} + \left(\frac{D_{12}(V_1+F_1) - D_{11}(V_2+F_2)}{D_{11}D_{22} - D_{12}^2} + \frac{-D_{12}\,u_{1,k} + D_{11}\,u_{2,k}}{D_{11}D_{22} - D_{12}^2} \right) T \end{aligned} \tag{A.1}$$

where T is the sampling time, $u_{1,k}$ and $u_{2,k}$ are the applied torques; $x_1 = \theta_1$, $x_2 = \theta_2$ are the positions; $x_3 = \dot{\theta}_1$, $x_4 = \dot{\theta}_2$ are the velocities; $s_2 = sin(x_2)$, $c_2 = cos(x_2)$ and, with entries in (A.1) as

$D_{11}(\Theta) = m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2 + 2l_1 l_{c2} c_2) + I_{zz_1} + I_{zz_2}$

$D_{12}(\Theta) = m_2 l_{c2}^2 + m_2 l_1 l_{c2} c_2 + I_{zz_2}$

$D_{22}(\Theta) = m_2 l_{c2}^2 + I_{zz_2}$

$V_1(\Theta,\dot{\Theta}) = -m_2 l_1 l_{c2} s_2 (\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 - m_2 l_1 l_{c2} \dot{\theta}_1 \dot{\theta}_2 s_2$

$V_2(\Theta,\dot{\Theta}) = m_2 l_1 l_{c2} s_2 (\dot{\theta}_1)^2$

$F_1(\Theta,\dot{\Theta}) = \mu_1 \dot{\theta}_1$

$F_2(\Theta,\dot{\Theta}) = \mu_2 \dot{\theta}_2$.

The parameters of the planar robot used for simulation (MATLAB[1]) are given in Table A.1.

[1] It is a trademark of the MathWorks Inc.

Table A.1: Model parameters

PARAMETER	VALUE	DESCRIPTION
l_1	$0.3\ m$	Length of the link 1
l_{c1}	$0.2\ m$	Mean length of the link 1
l_2	$0.25\ m$	Length of the link 2
l_{c2}	$0.1\ m$	Mean length of the link 2
m_1	$1\ Kg$	Mass of the link 1
m_2	$0.3\ Kg$	Mass of the link 2
I_{zz1}	$0.05\ Kg-m^2$	Moment of inertia 1
I_{zz2}	$0.004\ Kg-m^2$	Moment of inertia 2
μ_1	$0.005\ Kg/s$	Friction coefficient 1
μ_2	$0.0047\ Kg/s$	Friction coefficient 2

Appendix B

Inverse Optimal Control for Linear Systems

For the special case of linear systems, it can be established that inverse optimal control is an alternative way to solve the DT algebraic Riccati equation (DARE) [53] (HJB equation in DT nonlinear systems). Particularly, for the DT linear system

$$x_{k+1} = A\,x_k + B\,u_k, \qquad x_0 = x(0). \tag{B.1}$$

According to Theorem 4.1, the inverse control law provides

$$\begin{aligned} u_k^* &= -\frac{1}{2}(R + P_2(x_k))^{-1} P_1(x_k) \\ &= -\frac{1}{2}(R + \frac{1}{2} B^T\,P\,B)^{-1} B^T\,P\,A\,x_k \end{aligned} \tag{B.2}$$

where $P_1(x_k)$ and $P_2(x_k)$ are defined as

$$P_1(x_k) = B^T\,P\,A\,x_k, \tag{B.3}$$

and

$$P_2(x_k) = \frac{1}{2} B^T\,P\,B. \tag{B.4}$$

If we can find P satisfying (4.5), then the closed-loop system (B.1) with the inverse optimal control law (B.2) is globally asymptotically stable.

Selecting $R = \frac{1}{2}\overline{R} > 0$ in (B.2) yields

$$\begin{aligned} u_k^* &= -\frac{1}{2}(\frac{1}{2}\overline{R} + \frac{1}{2} B^T\,P\,B)^{-1} B^T\,P\,A\,x_k \\ &= -(\overline{R} + B^T\,P\,B)^{-1} B^T\,P\,A\,x_k \end{aligned} \tag{B.5}$$

Moreover, choosing $Q = \frac{1}{2}\overline{Q} > 0$ and the inverse optimal control law (B.5), the HJB (2.5) with $l(x_k) = x_k^T\, Q\, x_k$ becomes

$$\begin{aligned} V(x_k) &= l(x_k) + u_k^{*T} R(x_k)\, u_k^* + V(x_{k+1}) \\ &= x_k^T\, Q\, x_k + u_k^{*T} R(x_k)\, u_k^* + V(f(x_k)) \\ &\quad + P_1^T(x_k)\, u_k^* + u_k^{*T}\, P_2(x_k)\, u_k^*. \end{aligned}$$

Then

$$\begin{aligned} \frac{x_k^T\, P\, x_k}{2} &= \frac{x_k^T\, \overline{Q}\, x_k}{2} + \frac{u_k^{*T} \overline{R}(x_k)\, u_k^*}{2} \\ &\quad + \frac{x_k^T\, A^T\, P\, A\, x_k}{2} \\ &\quad - x_k^T\, A^T\, P\, B(\overline{R} + B^T\, P\, B)^{-1} B^T\, P\, A\, x_k \\ &\quad + \frac{u_k^{*T}\, B^T\, P\, B\, u_k^*}{2} \end{aligned}$$

which can be rewritten as

$$\begin{aligned} x_k^T\, P\, x_k &= x_k^T\, \overline{Q}\, x_k + x_k^T\, A^T\, P\, A\, x_k \\ &\quad - 2\, x_k^T\, A^T\, P\, B(\overline{R} + B^T\, P\, B)^{-1} B^T\, P\, A\, x_k \\ &\quad + x_k^T\, A^T\, P\, B(\overline{R} + B^T\, P\, B)^{-1} \times \\ &\quad (\overline{R} + B^T\, P\, B)(\overline{R} + B^T\, P\, B)^{-1} B^T\, P\, A\, x_k. \end{aligned} \tag{B.6}$$

Finally, from (B.6) the DT algebraic Riccati equation

$$P = \overline{Q} + A^T\, P\, A - A^T\, P\, B(\overline{R} + B^T\, P\, B)^{-1} B^T\, P\, A \tag{B.7}$$

is obtained.

Stabilization of an Unstable Linear System

In order to illustrate the applicability of the result in Theorem 4.1 for a linear example, let consider an unstable linear system described by (B.1), where

$$A = \begin{bmatrix} 0.3 & -0.8 \\ 0.8 & 1.8 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

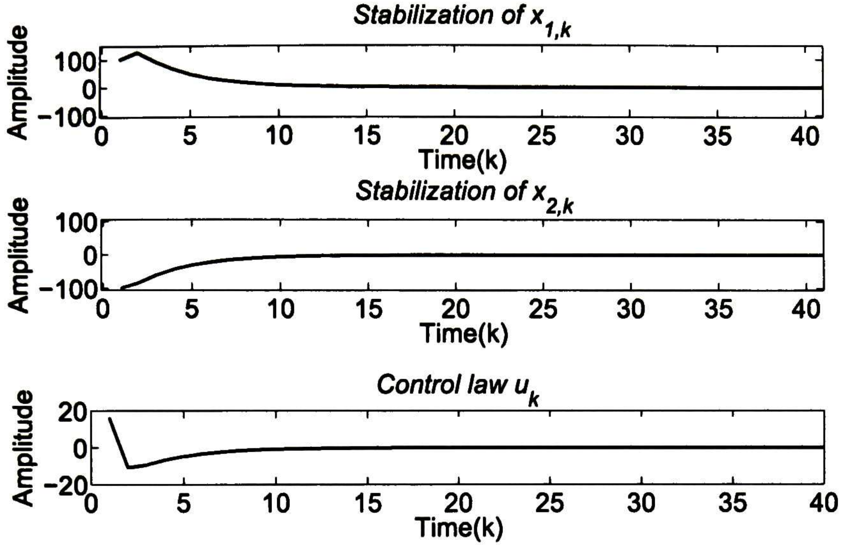

Figure B.1: Stabilization of a linear system.

Eigenvalues of this system are $\lambda_{1,2} = [1.05, \pm 0.2784]$ for $u_k = 0$. For this example, the inverse optimal control law (B.5) is used, and the definite positive matrix P is selected as

$$P = \begin{bmatrix} 0.5441 & 0.3197 \\ 0.3197 & 1.0010 \end{bmatrix}$$

and

$$\overline{R} = 0.1.$$

The new eigenvalues for the closed-loop system are $\lambda_1 = 0.724$ and $\lambda_2 = 0.7136$. Figure B.1 displays the stabilization of this system with initial conditions $x_0 = [100 \quad -100]^T$.

Appendix C

Publications

C.1 Journals Publications

- Fernando Ornelas, Edgar N. Sanchez, Alexander G. Loukianov, "Inverse optimal control for discrete-time nonlinear systems via passivation", (Submitted) *Optimal Control Applications and Methods*, 2011.

- Fernando Ornelas, Edgar N. Sanchez, Alexander G. Loukianov, "Discrete-time neural inverse optimal control for uncertain nonlinear systems via passivation", (Submitted) *IEEE Trans. on Neural Networks*, 2011.

- Blanca S. Leon, Alma Y. Alanis, Edgar N. Sanchez, Eduardo Ruiz-Velazquez, Fernando Ornelas, "Inverse Optimal Neural Control for Discrete Time Nonlinear Positive Systems", (Submitted) *International Journal of Adaptive Control and Signal Processing*, 2011.

- Blanca S. Leon, Alma Y. Alanis, Edgar N. Sanchez, Eduardo Ruiz-Velazquez, Fernando Ornelas, "Blood glucose level inverse optimal neural control for type 1 diabetes mellitus patients", (Submitted) *Information Sciences (Elsevier)*, 2011.

C.2 International Congress Publications

- Fernando Ornelas, Edgar N. Sanchez, Alexander G. Loukianov, Eva M. Navarro-López, "Speed-gradient inverse optimal control for discrete-time nonlinear systems", *IEEE Conference on Decision and Control (CDC 2011)*, Orlando, FL, USA, December, 2011.

- Blanca Leon, Alma Y. Alanis, Edgar N. Sanchez, Fernando Ornelas, Eduardo Ruiz-Velazquez, "Inverse optimal trajectory tracking for discrete-time nonlinear positive systems", *IEEE Conference on Decision and Control (CDC 2011)*, Orlando, FL, USA, December, 2011.

- Fernando Ornelas, Alexander G. Loukianov and Edgar N. Sanchez, "Discrete-time robust inverse optimal control for a class of nonlinear systems" *18-th IFAC World Congress*, Milano, Italy, August 28–September 2, 2011.

- Fernando Ornelas, Alexander G. Loukianov, Edgar N. Sanchez, "Discrete-time nonlinear systems inverse optimal control: A control Lyapunov approach", *IEEE Multiconference on Systems and Control (MSC 2011)*, Denver, CO, USA, September 28-30, 2011.

- Fernando Ornelas, Edgar N. Sanchez and Alexander G. Loukianov, "Discrete-time Inverse Optimal Control for Nonlinear Systems Trajectory Tracking". *Conference on Decision and Control (CDC 2010)*, Atlanta, Georgia, USA, December 15–17, 2010.

- Fernando Ornelas, Edgar N. Sanchez and Alexander G. Loukianov, "Planar Robot trajectory-following using inverse optimal neural control", *World Automation Congress (WAC 2010)*, Japan, September 19–23, 2010.

- Fernando Ornelas, Edgar N. Sanchez and Alexander G. Loukianov, "Real-time of discrete-time inverse optimal control", *IEEE Multi-Conference on Systems and Control (MSC 2010)*, Yokohama, Japan, September 8-10, 2010.

- Fernando Ornelas, Edgar N. Sanchez and Alexander G. Loukianov, "Decentralized Inverse Optimal Neural Control for Trajectory Tracking", *Asociación de México de Control Automático (AMCA 2010)*, Puerto Vallarta, Jalisco, México, October 6–8, 2010.

- Fernando Ornelas, Edgar N. Sanchez and Alexander G. Loukianov, "Discrete-time inverse optimal neural control: application for a planar Robot", 6*th International Conference on Electrical Engineering, Computing Science and Automatic Control (ICEEE 2009)*, Toluca, México, November 10-13, 2009.

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CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL I.P.N.
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Control Óptimo Inverso para Sistemas No Lineales en Tiempo Discreto / Inverse Optimal Control for Discrete-Time Nonlinear Systems

del (la) C.

Fernando ORNELAS TELLEZ

el día 22 de Agosto de 2011.

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