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Aplicación de Técnicas LMI para el Problema de Regulación de la Salida para Sistemas No Lineales

Tesis que presenta:
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LMI-based Techniques for Output Regulation Problems for Nonlinear Systems

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LMI-based Techniques for Output Regulation Problems for Nonlinear Systems

Master of Science Thesis
In Electrical Engineering

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“ *No debería uno contar nunca nada, ni dar datos ni aportar historias ni hacer que la gente recuerde a seres que jamás han existido ni pisado la tierra o cruzado el mundo ...*

Javier Marías

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Resumen

El seguimiento de trayectoria es uno de los principales problemas en la teoría de control; una herramienta muy útil para resolver dicho problema es la teoría de regulación de la salida, que permite hacer seguimiento asintótico y rechazo de perturbaciones. En general, el problema de regulación de la salida resulta difícil de resolver porque depende de un conjunto de ecuaciones diferenciales parciales en el caso no lineal. Los modelos Takagi-Sugeno han sido utilizados con el fin de resolver las ecuaciones mencionadas anteriormente de una manera sistemática y sencilla. El presente trabajo de tesis muestra un enfoque sistemático basado en modelos Takagi-Sugeno y desigualdades matriciales lineales que soluciona el problema de regulación no lineal de la salida tomando ventaja de las representaciones convexas de los mapeos no lineales y las derivadas de las funciones de membresía, una de las principales contribuciones que lo hacen distinto a los trabajos anteriores sobre este tema. Esta solución está basada en desigualdades matriciales lineales elemento por elemento que pueden ser resueltas numéricamente por métodos de punto interior de programación convexa.

Abstract

Tracking a signal is one of the most important issues in control theory and has been thoroughly studied by the output regulation theory. In general, output regulation theory is difficult to solve in the nonlinear case because the solution depends on a set of partial differential equations appears in the nonlinear case. Takagi-Sugeno models have been introduced in order to solve the aforementioned equations in a more systematic and numerically convenient procedure. This thesis develops a novel approach based on Takagi-Sugeno models and linear matrix inequalities to solve the nonlinear output regulation problem by taking advantage of convex representations of the involved nonlinear mappings and the time-derivatives of the membership functions, being the latter one of the main original contributions of this work that makes it different from previous results. In addition, the results are fully based on element-wise linear matrix inequalities which are numerically solvable by convex optimization methods.

Chapter 1

Introduction

1.1 Overview

An important task in control theory is tracking a desired signal while rejecting perturbations; that is why in the literature there exist different methods to achieve these goals. In the 70's, regulation theory appeared to provide an answer to the previous problem in the framework of differential geometry; Francis [13] showed that the solution for the linear case relies on solving two linear matrix equations, called the Francis Equations. For nonlinear systems, in the early 90's, Isidori and Byrnes provided a first set of mathematical conditions to solve the nonlinear output regulation problem (NORP); basically, they proved that the problem is solvable if and only if a solution to a set of nonlinear partial differential equation, named Francis-Isidori-Byrnes equations (FIB) exists [19]. This result was based on the centre manifold theory [8]. In the following years many researchers worked in the nonlinear output regulation area [7, 17, 18].

Generally speaking, the output regulation problem consists in finding a control law such that the equilibrium point of the closed-loop system without external signals is exponentially stable, and the tracking error tends asymptotically to zero under the influence of reference signals and/or perturbations given by an external dynamical system named exosystem. Two main cases are considered in regulation theory: 1) when full information of all the states and the exosystem is available and used it is called Nonlinear Output Regulation Problem via State Feedback (NORPSF); and 2) when only the tracking error is used (for instance, because it is the only data available) it is named the Nonlinear Output Regulation Problem via Error Feedback (NORPEF).

Nevertheless, finding the solution of the differential equations (FIB equations) is, in most cases, very difficult or even impossible. For this reason, linear approximations are very common in nonlinear systems when solving the output regulation problem described above [5, 7, 17, 27]. In recent years, a good path has been established to tackle the NORP through the use of Takagi-Sugeno (TS) models and linear matrix inequalities (LMIs) [11, 23]. Due to the fact that TS models, via sector nonlinearity approach [32], can exactly represent a wide variety of nonlinear models in a compact set of the state space by capturing the nonlinearities in membership functions (MFs) sharing the convex sum property, the use of the Lyapunov direct method allows many control problems to be solved in terms of LMI conditions [32]. Expressing control problems as LMIs is a very convenient feature for systematically express and search solutions since they can be efficiently solved through optimization convex techniques which are readily available in commercial technical software [6, 29]. Because of the aforementioned reasons, researchers have been encouraged to recast traditional nonlinear problems in terms of LMIs [9, 11, 25].

The basic idea of a TS-LMI approach for solving the NORP (also known as the fuzzy regulation problem) is to systematically search via LMIs the nonlinear mappings involved in the solution of the FIB equations by assuming a TS structure of the nonlinear system [11, 23, 25]. This idea is straightforward to implement for the discrete-time case since no time-derivatives of the MFs are involved [10]. As for the continuous-time case, the partial derivative in the FIB equations leads to the time-derivatives of the MFs which breaks the convex structure of the problem and makes it hard to handle it conveniently [25]. A similar problem appeared independently when nonquadratic Lyapunov functions were employed for continuous-time TS models: the first solutions offered no more than bounding a priori these time-derivatives, which is not always possible and presents some technical troubles [4, 26, 30].

In recent years, a way to escape from the aforementioned time-derivative problem has been developed by taking into account the local character of the TS models [3, 14]: this work reformulates the solution of the NORP by taking into account these improvements. The results here provided face some difficulties shown in [11, 25], thus generalizing those previous results. Moreover, in contrast with former approaches, the whole set of results are expressed in terms of LMIs, including the solution to the FIB equations, thus eliminating the need of solving equations and inequalities separately.

1.2 Objectives

The goals of the current work are:

- To obtain a systematic solution for nonlinear output regulation problems in terms of LMIs. To do that, FIB equations need to be represented in a convex-sum form (TS models).

- To achieve zero tracking error; when this goal is not possible, to provide a bound of the steady-state tracking error for any initial condition inside the modelling region.

- To provide full-information conditions to solve the NORP by the inclusion of the time-derivative of MFs.

- To synthesize a complete fuzzy controller that allows trajectory tracking via regulation theory.

1.3 Thesis Structure

Chapter 2 covers the foundations on TS models, LMIs, relaxation lemmas and some facts on synthesis of controllers for TS models: its first section presents the procedure to construct a TS model from nonlinear systems; the second section summarizes concepts on LMI, since it is the principal mathematical tool used in this work; the third section provides some useful relaxation lemmas in the analysis of TS models; the final section presents facts concerning analysis of TS models and different parallel distributed compensation (PDC) control laws used in this work.

Chapter 3 provides new developments for linear output regulation theory via LMI and convex optimization techniques: the first section shows the developments leading to LMI conditions that solve the Linear Output Regulation Problem via State Feedback (LORPSF); the second section provides the solution of Linear Output Regulation Problem via Error Feedback (LORPEF) through LMIs; in the last part two examples are included in order to test the presented results.

Chapter 4 shows the main result of the thesis: conditions for solving the nonlinear output regulation problem via Takagi-Sugeno mappings and LMIs are stated. Firstly, the NORPSF and the NORPEF are solved via LMIs, then the NORPSF is solved through a complete LMI approach with full-information even for the bounds of the time-derivatives of the MFs.

Chapter 5 draws some the conclusions and suggests future work on this field.

Chapter 2

Nonlinear Control Based on Convex Models

The focus of this chapter is on TS fuzzy models (which are equivalent to the quasi-linear parameter varying models) and LMI techniques. It also covers stability analysis and controller design (stabilization).

2.1 Takagi-Sugeno Models

Nonlinear plants have any number of exact representations in a TS form via sector-nonlinearity approach [32]. A brief introduction about this approach is presented in the following.

Consider the following affine in-control nonlinear system

$$\dot{x}(t) = f(x) + g(x)u(t), \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $f(x)$ and $g(x)$ are nonlinear and smooth functions.

To that end, suppose that each of the p nonlinearities in (2.1) is bounded, i.e., $nl_j(\cdot) \in [\underline{nl_j}, \overline{nl_j}]$, $j \in \{1, \ldots, p\}$ in a compact region $\Delta \supset \mathbf{0}$ of the state-space. Let $z(\cdot) \in \mathbb{R}^p$ be the premise vector; nonlinearities are expressed in terms of the components of this vector. The above described nonlinearities are captured in the following weighting functions:

$$\omega_0^j(\cdot) = \frac{nl_j(\cdot) - \underline{nl_j}}{\overline{nl_j} - \underline{nl_j}}, \quad \omega_1^j(\cdot) = 1 - \omega_0^j(\cdot), \quad j \in \{1, \ldots, p\}, \tag{2.2}$$

from which the following membership functions (MFs) arise:

$$h_i = h_{1+i_1+i_2\times 2+\cdots+i_p\times 2^{p-1}} = \prod_{j=1}^{p} \omega_{i_j}^j(z_j), \tag{2.3}$$

with $i \in \{1, \ldots, 2^p\}$, $i_j \in \{0, 1\}$. The MFs (2.3) hold the convex-sum property in Δ: $\sum_{i=1}^{r} h_i(\cdot) = 1$, $h_i(\cdot) \geq 0$. For simplicity, the sums will be written as $\Upsilon_z = \sum_{i=1}^{r} h_i(\cdot)\Upsilon_i$ and their inverse as $\Upsilon_z^{-1} = \left(\sum_{i=1}^{r} h_i(\cdot)\Upsilon_i\right)^{-1}$ Other sums arising in the sequel will be denoted as $\dot{\Upsilon}_z = \sum_{i=1}^{r} \dot{h}_i(\cdot)\Upsilon_i$ and the double convex-sum $\Upsilon_{zz} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\cdot)h_j(\cdot)\Upsilon_{ij}$.

Then, an exact representation of (2.1) in Δ is:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\left(A_i x(t) + B_i u(t)\right) = A_z x(t) + B_z u(t), \tag{2.4}$$

with A_i, B_i, $i = 1, \ldots, r$ being matrices of proper dimensions determined according to the specific combination of minima and maxima of the involved nonlinearities, and $h_i(\cdot)$, $i = 1, \ldots, r$, being MFs defined as in (2.3), where $r = 2^p \in \mathbb{N}$ is the number of rules. This model is in the TS form [33] and it is an exact rewriting of the nonlinear model (2.1).

Example 2.1. Consider the following nonlinear system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + 1 & -\dfrac{\sin(x_2)}{x_2} \\ -1 & -2x_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.5}$$

where $x_1 \in [-1, 1]$ and $x_2 \in [-3, 3]$. Nonlinearities can be chosen in a variety of ways; a natural choice is: $nl_1 = x_2 + 1$, $nl_2 = -\dfrac{\sin(x_2)}{x_2}$, and $nl_3 = -2x_2^2$. Then the maximum and minimum value of each nonlinearity is $nl_1 \in [-2, 4]$, $nl_2 \in [-1, -0.05]$ and $nl_3 \in [-18, 0]$. the weights are: $\omega_0^1 = \dfrac{x_2 + 3}{6}$, $\omega_1^1 = 1 - \omega_0^1$, $\omega_0^2 = \dfrac{-\sin(x_2)/x_2}{0.95}$, $\omega_1^2 = 1 - \omega_0^2$, $\omega_0^3 = \dfrac{-x_2^2}{9}$, and $\omega_1^3 = 1 - \omega_0^3$,

The arising MFs are:

$$h_1 = \omega_0^1 \omega_0^2 \omega_0^3$$

$$h_2 = \omega_0^1 \omega_0^2 \omega_1^3$$

$$h_3 = \omega_0^1 \omega_1^2 \omega_0^3$$

$$h_4 = \omega_0^1 \omega_1^2 \omega_1^3$$

$$h_5 = \omega_1^1 \omega_0^2 \omega_0^3$$

$$h_6 = \omega_1^1 \omega_0^2 \omega_1^3$$

$$h_7 = \omega_1^1 \omega_1^2 \omega_0^3$$

$$h_8 = \omega_1^1 \omega_1^2 \omega_1^3$$

The following TS model is equivalent to (2.5)

$$\dot{x}(t) = \sum_{i=1}^{8} h_i(z(t)) A_i x(t), \tag{2.6}$$

with $A_1 = \begin{bmatrix} 4 & -0.05 \\ -1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & -0.05 \\ -1 & -18 \end{bmatrix}$, $A_3 = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 4 & -1 \\ -1 & -18 \end{bmatrix}$, $A_5 = \begin{bmatrix} 2 & -0.05 \\ -1 & 0 \end{bmatrix}$, $A_6 = \begin{bmatrix} 2 & -0.05 \\ -1 & -18 \end{bmatrix}$, $A_7 = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$, and $A_8 = \begin{bmatrix} 2 & -1 \\ -1 & -18 \end{bmatrix}$.

2.2 Linear Matrix Inequalities

This section introduces linear matrix inequalities which are a fundamental tool for analysis and synthesis of controllers for TS models and can be easily implemented with convex optimizations techniques.

Maybe the first use of LMIs in control theory appeared around 1890, published by Aleksandr Lyapunov, who demonstrated that the differential equation

$$\frac{d}{dt}x(t) = Ax(t),$$

is stable if and only if there exist $P = P^T > 0$ such that

$$A^T P + PA < 0, \tag{2.7}$$

which is an LMI on the constraint P

Definition 2.1. A linear matrix inequality has the following form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{2.8}$$

where $x \in \mathbb{R}^m$ is the vector of the decision variables and $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $i = 0, \ldots, m$, are given symmetric matrices. The symbol $>$ means that $F(x)$ is positive definite while non strict LMIs can appear as $F(x) \geq 0$. In addition, $<$ will stand for negative definiteness in matrix expressions [6, 29, 32].

Very often in the LMIs the variables are matrices, for example, the Lyapunov inequality (2.7) where A is given and P is the variable. In this case the LMI will not be written explicitly in the form $F(x) > 0$. Of course, the inequality (2.7) can be readily put in the form (2.8):

take $F_0 = 0$, $F_i = -A^T - A$, and x_i, $i = 1, \ldots, m$ as each unknown entry of $P \in \mathbb{R}^{n \times n}$, finally $m = n(n+1)/2$.

Some useful LMIs properties are presented [6, 12, 22, 29]:

Proposition 2.1 (System of LMIs). *A set of LMIs $F_1(x) > 0, F_2(x) > 0, \ldots, F_n(x) > 0$ can be expressed as the following single LMI:*

$$F(x) = \begin{bmatrix} F_1(x) & 0 & \ldots & 0 \\ 0 & F_2(x) & \ldots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & \ldots & 0 & F_n(x) \end{bmatrix} > 0. \tag{2.9}$$

The following property is important because it is used to transform nonlinear inequalities into linear ones.

Proposition 2.2 (Schur complement). *The following inequalities are equivalent:*

$$Q(x) - S(x)R(x)^{-1}S(x)^T > 0, \quad R(x) > 0, \tag{2.10}$$

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0, \tag{2.11}$$

where $Q(x) = Q(x)^T$ $R(x) = R(x)^T$ and $S(x)$ depend affinely on x.

Note that inequality (2.10) is a nonlinear one that become LMI via Schur complement.

Proposition 2.3 (Congruence). *Let $P(x) > 0$ and $Q(x)$ being matrices that depend affinely on x. If $Q(x)$ is full column rank, the expression $Q(x)P(x)Q(x)^T$ is also positive definite.*

Proposition 2.4 (S-procedure). *Let $F_0 < 0$ being a matrix that depends affinely on x and*

$$F_1(x) \leq 0, \cdots, F_r(x) \leq 0, \tag{2.12}$$

there exist positive real scalars $s_1, \ldots, s_r$ such that

$$F_0(x) - \sum_{i=1}^{r} s_i F_i(x) < 0. \tag{2.13}$$

Proposition 2.5 (Finsler's Lemma). *Let $x \in \mathbb{R}^n$, $Q = Q^T \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times n}$ such that $rank(R) < n$; the following expressions are equivalent:*

- $x^T Q x < 0$, $\forall x \in \{x \in \mathbb{R}^n : x \neq 0, Rx = 0\}$.
- $\exists X \in \mathbb{R}^{n\times m} : Q + XR + R^T X^T < 0$.

In the following results, an asterisk $(*)$ inside a matrix is equivalent to the transpose of its corresponding symmetric element; in a sum it is the transpose of the terms on its left-hand side. The symbol $\prec$ ($\succ$) stands for element-wise lower-than (greater-than) relation in matrix expressions. Then, the inequality (2.11) can be rewritten as:

$$\begin{bmatrix} Q(x) & (*) \\ S(x)^T & R(x) \end{bmatrix} > 0. \tag{2.14}$$

Example 2.2. Consider the quadratic matrix inequality

$$A^T P + PA + PBR^{-1}B^T P + Q < 0, \tag{2.15}$$

where A, B, $Q = Q^T$ $R = R^T > 0$ are given matrices of appropriate dimensions, and $P = P^T$ is the variable. Note that this is a quadratic matrix inequality in the variable P. With the help of the Schur complement, it can be expressed as the following linear matrix inequality:

$$\begin{bmatrix} -A^T P - PA - Q & (*) \\ B^T P & R \end{bmatrix} > 0. \tag{2.16}$$

There are three problems related to LMI:

1. *Feasibility*: to determine if there exist elements $x \in X$ such that $F(x) < 0$. The LMI $F(x) < 0$ is called feasible if such x exists, otherwise it is said to be infeasible.

2. *Eigenvalue problem (EVP)*: to minimize the maximum eigenvalue of a matrix that depends affinely on a variable, subject to an LMI constraint

$$\begin{aligned} &\text{minimize} \quad \lambda \\ &\text{subject to} \quad \lambda I - A(x) > 0, \qquad B(x) > 0, \end{aligned}$$

   where A, B, are symmetric matrices that depend affinely on the optimization variable x.

3. *Generalized Eigenvalue Problem (GEVP)*:to minimize eigenvalues of a pair of matrices which depend affinely on a variable, subject to a set of LMI-constraints or to determine that the problem is infeasible.

$$\begin{aligned} &\text{minimize} \quad \lambda \\ &\text{subject to} \quad \lambda B(x) - A(x) > 0, \qquad B(x) > 0, \qquad C(x) > 0 \end{aligned}$$

   where A, B, C, are symmetric matrices that are affine functions of x.

2.3 Relaxation Lemmas

As will be showed later, Lyapunov direct method leads to expressions containing convex sums when applied to TS models; the MFs contained in these sums cannot be included in the LMI expressions and there are several ways to drop them. The authors usually referred to this as sum relaxation lemmas. Relaxations help fill the gap introduced by MFs into the inequalities used for analysis and control design of TS models, since LMIs methodology does not include the shape of the MFs [1]. Some relaxation lemmas follow:

Lemma 2.1. [31] *Let Υ_{ij}, $i, j \in \{1, \ldots, r\}$ being a collection of matrices of proper size. Then the double convex-sum:*

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right) h_j\left(z(t)\right) \Upsilon_{ij} < 0, \tag{2.17}$$

is verified if:

$$\begin{aligned} &\Upsilon_{ii} < 0, \quad \forall i \in \{1, \ldots, r\}, \\ &\Upsilon_{ij} + \Upsilon_{ji} < 0, \quad \forall (i,j) \in \{1, \ldots, r\}^2, \quad i < j. \end{aligned} \tag{2.18}$$

Lemma 2.2. [34] *Let Υ_{ij}, $i, j \in \{1, \ldots, r\}$ being a collection of matrices of proper size. The inequality (2.17) is verified if the following conditions hold:*

$$\begin{aligned} &\Upsilon_{ii} < 0, \quad \forall i \in \{1, \ldots, r\}, \\ &\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad \forall (i,j) \in \{1, \ldots, r\}^2, \quad i \neq j. \end{aligned} \tag{2.19}$$

There exist other relaxations with different degrees of conservatism and/or complexity [21, 24, 28]. The relaxations here presented are considered more convenient since they do not involve new slack variables while making a good compromise between numerical complexity and quality of solutions.

2.4 Stabilization of Takagi-Sugeno Models

This section shows different ways for analysis of Takagi-Sugeno models and controller design.

First of all, consider an autonomous fuzzy system in the TS form:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) A_i x(t) = A_z x(t). \tag{2.20}$$

Sufficient conditions for the stability of (2.20) are given in the sense of Lyapunov:

Theorem 2.1. [32] *The equilibrium point of the continuous fuzzy system (2.20) is globally asymptotically stable if there exist a common positive define matrix $P = P^T > 0$ such that:*

$$A_i^T P + P A_i < 0 \quad i = 1, \ldots, r, \tag{2.21}$$

that is, a common P has to exist for all subsystems.

Proof. Stability conditions are immediately derived using a quadratic Lyapunov function $V(x) = x(t)^T P x(t)$ is positive and radially unbounded if $P = P^T > 0$ and its time-derivative satisfies:

$$\dot{V}(x) = \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) < 0. \tag{2.22}$$

Substituting (2.20) in (2.22):

$$\begin{aligned} \dot{V}(x) &= (A_z x(t))^T P x(t) + x(t)^T P A_z x(t) \\ &= x(t)^T (A_z^T P + P A_z) x(t). \end{aligned} \tag{2.23}$$

Since MFs in A_z hold the convex-sum property a sufficient condition to guarantee $\dot{V}(x) < 0$ is (2.21). □

Note that the previous theorem reduces to the Lyapunov stability theorem for linear systems when $r = 1$. Moreover the stability condition showed in Theorem 2.1 is expressed in LMIs. This a feasibility problem, which determines if there exists a common symmetric positive-definite matrix P and can be solved efficiently via convex optimization techniques [6, 29]. This result is for continuous-time systems and can be extended to the discrete-time case with a similar procedure.

The design of a feedback controller consists in finding gains for the TS system to stabilize it. Three aproaches are presented: 1) common gain K for all the subsystems, i.e., $u = Kx$; 2) different K for each subsystem, i.e., $u = K_z x$, called the parallel distributed compensation (PDC) and 3) the non-quadratic fuzzy control with $u = K_z P_z^{-1} x$, one of several non-PDC control laws.

In the following theorems, continuous-time TS models are considered:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) \left(A_i x(t) + B_i u(t)\right). \tag{2.24}$$

Simple State Feedback. For this approach the control law is $u(t) = Kx(t)$, then the closed-loop system is:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) \left(A_i + B_i K\right) x(t). \tag{2.25}$$

Theorem 2.2. *The equilibrium point of the continuous closed-loop fuzzy system (2.25) is globally asymptotically stabilizable if* $\exists X = X^T > 0$ *and* M *such that:*

$$A_i X + X A_i^T + B_i M + M^T B_i < 0, \quad i = 1, \ldots, r. \tag{2.26}$$

hold for (2.25). Then the feedback gain is $K = MX^{-1}$ *The common quadratic Lyapunov function is* $V(x(t)) = x(t)^T P x(t)$ *where* $P = X^{-1}$

Proof. Consider a quadratic Lyapunov function candidate $V(x) = x^T P x$, $V(0) = 0$ and $V(x) > 0$, $\forall x \neq 0$ with $P = P^T > 0$, then,

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T (A_z + B_z K)^T P x + x^T P (A_z + B_z K) x < 0. \end{aligned} \tag{2.27}$$

Sufficient conditions to guarantee $\dot{V}(x) < 0$ are

$$A_z P + K^T B_z^T P + P A_z + P B_z K < 0. \tag{2.28}$$

The expression (2.28) is not an LMI. In order to obtain a LMI problem it is necessary to apply the congruence property: pre- and post-multiplying (2.28) by X, where $X = P^{-1}$, plus a change of variables $M = KX$. □

Parallel Distributed Compensator (PDC). This design is also named as fuzzy controller and consists in designing a control law for each subsystem of the TS fuzzy model. The designed controller shares the same MFs as the TS model. Then the control law is

$$u(t) = \sum_{i=1}^{r} h_i(z(t)) K_i x(t) = K_z x(t). \tag{2.29}$$

The closed-loop system of (2.24) with (2.29) is:

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \left(A_i + B_i K_j\right) x(t) = (A_z + B_z K_z) x(t). \tag{2.30}$$

Theorem 2.3. [32] *The equilibrium point of the continuous closed-loop fuzzy system (2.30) is globally asymptotically stabilizable if* $\exists X = X^T > 0$ *and* M_i, $i = 1, \ldots, r$ *such that:*

$$A_z X + X A_z^T + B_z M_z + M_z^T B_z^T < 0, \tag{2.31}$$

hold for (2.30). Then the feedback gains are $K_z = M_z X^{-1}$ *The common Lyapunov function is* $V(x(t)) = x(t)^T P x(t)$ *where* $P = X^{-1}$

Proof. The proof follows the same lines as for the previous theorem [32]. □

Remark 2.1. The inequality (2.31) needs a relaxation lemma provided in previous section in order to drop off the double sums implied by z.

For example, if the Relaxation Lemma 2.1 is applied, the inequality (2.32) turns into the following LMIs:

$$A_iX + B_iM_i + (*) < 0, \quad \forall i,$$
$$A_iX + A_jX + B_iM_j + B_jM_i + (*) < 0, \quad i < j.$$

Remark 2.2. Despite the fact that the fuzzy controller (2.29) is constructed using a local design structure, feedback gains K_i should be determined using global design conditions in order to guarantee global stability.

Remark 2.3. Since the TS model (2.24) is obtained from the sector nonlinearity approach, the control feedback (2.29) ensures the global stability for the original nonlinear system in the modelled compact of the state variables.

Non-Quadratic Fuzzy Control. This approach is one among several non-PDC schemes and consists in designing a control law for each subsystem of the TS fuzzy model with a fuzzy Lyapunov function (FLF) candidate which shares the same MFs of the TS fuzzy model.

The non-quadratic fuzzy Lyapunov function used in this approach is

$$V(x) = x^T \left(\sum_{j=1}^{r} h_j(z) P_j \right)^{-1} x = x^T P_z^{-1} x, \tag{2.32}$$

with $P_z = P_z^T > 0$. Then, the control law used is

$$u(t) = \sum_{i=1}^{r} h_i(z(t)) K_i \left(\sum_{j=1}^{r} h_j(z(t)) P_j \right)^{-1} x(t) = K_z P_z^{-1} x(t). \tag{2.33}$$

The closed-loop system of (2.24) with the control law (2.33) arises:

$$\dot{x}(t) = (A_z + B_z K_z P_z^{-1}) x(t). \tag{2.34}$$

Theorem 2.4. [2] *The equilibrium point of the continuous closed-loop fuzzy system (2.34) is locally asymptotically stabilizable if* $\exists X = X^T > 0$ *and* M_i, $i = 1, \ldots, r$ *such that:*

$$A_z P_z + P_z A_z^T + B_z K_z + K_z^T B_z^T - \sum_{k=1}^{p} (-1)^{d_k^{\alpha}} \beta_k \left(P_{g_1(z,k)} - P_{g_2(z,k)} \right) < 0, \tag{2.35}$$

hold for (2.34). With $g_1(j,k) = \lfloor (j-1)/2^{p+1-2} \rfloor \times 2^{2+1-k} + 1 + (j-1) \mod 2^{p-2}$, $g_2(j,k) = g_1(j,k) + 2^{p-2}$ *and* d_k^{α} *calculated from* $\alpha - 1 = d_p^{\alpha} + d_{p-1}^{\alpha} \times 2 + \ldots + d_1^{\alpha} \times 2^{p-1}$, $\alpha \in \{1, \ldots, 2^p\}$.

Proof. Consider a non-quadratic fuzzy Lyapunov function (2.32) and its time-derivative

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P_z^{-1} x + x^T P_z^{-1} \dot{x} + x^T \dot{P}_z^{-1} x \\ &= x^T \left(\left(A_z + B_z K_z P_z^{-1} \right)^T P_z^{-1} + P_z^{-1} \left(A_z + B_z K_z P_z^{-1} \right) + \dot{P}_z^{-1} \right) x. \end{aligned} \tag{2.36}$$

A sufficient condition to guarantee $\dot{V}(x) < 0$:

$$A_z^T P_z^{-1} + P_z^{-1} K_z^T B_z^T P_z^{-1} + P_z^{-1} A_z + P_z^{-1} B_z K_z P_z^{-1} + \dot{P}_z^{-1} < 0. \tag{2.37}$$

Pre- and post-multiplying by P_z (congruence property) the above expression, and considering the fact that $P_z \dot{P}_z^{-1} P_z = -\dot{P}_z$, yields:

$$P_z A_z^T + K_z^T B_z^T + A_z P_z + B_z K_z - \dot{P}_z < 0. \tag{2.38}$$

In [14] has been shown that

$$\dot{P}_z = \sum_{k=1}^{p} \frac{\partial \omega_0^k}{\partial z_k} \dot{z}_k \left(P_{g_1(z,k)} - P_{g_2(z,k)} \right). \tag{2.39}$$

Consider the bound $\beta_k > 0$ for

$$\left|\frac{\partial \omega_0^k}{\partial z_k}\dot{z}_k\right| \leq \beta_k, \tag{2.40}$$

by the property $Y \pm \beta_k \times Z < 0 \Rightarrow Y + \frac{\partial \omega_0^k}{\partial z_k}\dot{z}_k Z < 0$ the sign combinations captured by $(-1)^{d_k^\alpha}$ complete the proof. □

Remark 2.4. In [15] and [16] different ways to guarantee the bounds β_k and to find an estimation of the region of attraction are proved based-on the fact that the premise vector $z(t)$ as well as the weights ω_o^k depend on the state vector $x(t)$. Although this procedure adds more LMIs, they can be solved simultaneously.

Remark 2.5. When $P_{g_1(z,k)} - P_{g_2(z,k)} = 0$ the problem turns into a quadratic case, i.e., this approach includes the quadratic case.

Remark 2.6. The inequality (2.35) needs a relaxation lemma as those provided in the previous section in order to drop off the double sums implied by z.

Chapter 3

Linear Output Regulation Problems: an LMI Approach

This chapter presents a new LMI approach for Linear Output Regulation Problems (LORP), since they are interesting in the sense that nonlinear ones can be approximated via linearization. The basis of regulation theory are reviewed in Appendix A.

In order to solve the LORP trough LMIs the following result is stated:

Theorem 3.1. *Let L, $R : \mathbb{R}^p \to \mathbb{R}^{m\times n}$ being linear matrix continuously differentiable functions of the decision vector $x \in \mathbb{R}^p$ If a unique solution $\bar{x} \in \mathbb{R}^p$ exists to the problem $L(x) = R(x)$, this solution is approximated with arbitrary accuracy by the element-wise LMI minimization problem*

$$\min \varepsilon > 0 : -\varepsilon \prec L(x) - R(x) \prec \varepsilon. \tag{3.1}$$

Proof. From $L(x) = R(x)$ it follows that $L(\bar{x}) - R(\bar{x}) = 0$, with $0 \in \mathbb{R}^{n\times m}$. By continuity and uniqueness of solution $\bar{x}$, this implies that $\forall\varepsilon > 0$ arbitrarily small such that $\exists\delta > 0 :$ $|x - \bar{x}| \Rightarrow |L(x) - R(x)| \prec \varepsilon$ with ε arbitrarily small. This leads straightforwardly to 3.1. $\square$

Remark 3.1. Note that the LMI minimization problem (3.1) is twofold, i.e.:

$$\min \varepsilon > 0 \quad : -\varepsilon \prec L(x) - R(x) \prec \varepsilon$$
$$\Leftrightarrow \qquad \min \varepsilon > 0 : \begin{cases} L(x) - R(x) - \varepsilon \prec 0 \\ L(x) - R(x) + \varepsilon \succ 0 \end{cases}$$

3.1 LORP via State Feedback

The previous theorem allows the LORP to be restated in terms of LMIs. This rewriting significantly improves the regulator design, since it accomplishes this task under a single framework which can be easily implemented via convex optimization techniques already available in commercial LMI toolboxes.

Recall the regulation theory stated in Appendix A; the following theorem provides an LMI solution to LORP via state feedback (LORPSF).

Theorem 3.2. *Assume A1 and A2 in (A.1).Then, the LORPSF has a solution if and only if* $\exists X = X^T > 0$, $M \in \mathbb{R}^{m\times n}$, $\Pi \in \mathbb{R}^{n\times q}$, $\Gamma \in \mathbb{R}^{m\times q}$ *and an arbitrarily small* $\varepsilon > 0$ *such that*

$$X < 0, \quad AX + BM + (*) < 0,$$
$$\min \varepsilon > 0 : -\varepsilon \prec \begin{bmatrix} A\Pi + B\Gamma + E - \Pi S & 0 \\ 0 & C\Pi + Q \end{bmatrix} \prec \varepsilon. \tag{3.2}$$

The control law is given by (A.2) with $K = MX^{-1}$ *and* $L = \Gamma - K\Pi$.

Proof. In the Appendix A sufficient and necessary conditions for solving the LORPSF can be checked. The first of these conditions is to guarantee $A + BK$ to be Hurwitz, which can be obtained from the Lyapunov function candidate $V = x^T P x$, $P = P^T > 0$ through the following development with no influence of the exosystem ($w = 0$, then $u = Kx$):

$$\begin{aligned}
\dot{V} &= \dot{x}^T P x + x^T P \dot{x} = x^T \left(P\left(A + BK\right) + \left(A^T + K^T B^T\right) P\right) x < 0 \\
&\Leftrightarrow\ P\left(A + BK\right) + \left(A^T + K^T B^T\right) P < 0 \\
&\Leftrightarrow\ AX + BM + XA^T + MB^T < 0,
\end{aligned} \tag{3.3}$$

with $X = P^{-1}$ and $M = KP^{-1}$; this gives the first set of LMIs in (3.2). The second one -corresponding to the minimization LMI problem- follows directly from Francis equations (A.3) and Theorem 3.1. □

LORPSF is applied when full-information is available; however, for most of the real situations some states (from the plant $x(t)$ or exosystem $w(t)$) can be unmeasurable, then an error feedback is suitable. This approach is shown in the next section.

3.2 LORP via Error Feedback

When only the measure of the tracking error is available, the LORP is solved by constructing an observer for states $x(t)$ and/or $w(t)$, then the tracking error is asymptotically driven to zero via error feedback. The next theorem provides conditions to solve LORP via error feedback (LORPEF).

Theorem 3.3. *Assume A1, A2 and A3 for the linear system (A.1). Then, the LORPEF has a solution if and only if* $\exists X_1 = X_1^T > 0$, $X_2 = X_2^T > 0$, $M_1 \in \mathbb{R}^{m\times n}$, $M_2 \in \mathbb{R}^{(n+q)\times m}$,

$\Pi \in \mathbb{R}^{n\times q}$, $\Gamma \in \mathbb{R}^{m\times q}$, *and an arbitrarily small* $\varepsilon > 0$ *such that*

$$\begin{aligned} &X_1 < 0, \quad AX_1 + BM_1 + (*) < 0,\\ &X_2 < 0, \quad \bar{A}X_2 + \bar{B}M_2 + (*) < 0,\\ &\min \varepsilon > 0 : -\varepsilon \prec \begin{bmatrix} A\Pi + B\Gamma + E - \Pi S & 0 \\ 0 & C\Pi + Q \end{bmatrix} \prec \varepsilon, \end{aligned} \tag{3.4}$$

with $\bar{A} = \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}^T$ $\bar{B} = \begin{bmatrix} C & Q \end{bmatrix}^T$ *The control law is given by (A.4) with* $K = M_1X_1^{-1}$ *and the observer gains* $G = \begin{bmatrix} G_0 \\ G_1 \end{bmatrix} = \left(M_2X_2^{-1}\right)^T$, $F = \begin{bmatrix} A + G_0C + BK & P + G_0Q + B(\Gamma - K\Pi) \\ G_1C & S + G_1Q \end{bmatrix}$ *and* $H = \begin{bmatrix} K & \Gamma - K\Pi \end{bmatrix}$.

Proof. As in the proof of Theorem 3.2, the Lyapunov direct method can be used to stabilize the pairs (A, B) and $(\bar{A}, \bar{B})$ thus producing the first two LMI conditions in (3.4). The last set of LMIs in (3.4) corresponds to a minimization LMI problem, and it arises from applying Theorem 3.1 to conditions (A.5). Gains K and G are obtained from the first set of LMIs, Π and Γ from the third one, while F and H are calculated for the control law (A.4). □

3.3 Examples

In this section, two very well-known nonlinear models are considered to illustrate the LMI output regulation techniques: the first one shows the LORPSF applied to the ball-and-beam system while the second one solves the LORPEF for an underactuated two-link PENDUBOT system. These nonlinear plants are hard-to-drive, but by LMI-based regulation on their linearized models proves to be fruitful.

Example 3.1. Consider the nonlinear model of the ball-and-beam system [32]:

$$\dot{x}(t) = \begin{bmatrix} x_2 \\ B\left(x_1 x_4^2 - g\sin\left(x_3\right)\right) \\ x_4 \\ u \end{bmatrix} x(t), \tag{3.5}$$

with x_1 corresponding to the ball distance to the origin, x_3 the angle between the right-hand side of the bar and the horizontal axis, x_2 and x_4 the linear and angular velocities respectively, with $B = 0.7134$, $g = 9.81$.

A linear representation of this model at the origin gives

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -Bg & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t). \tag{3.6}$$

Note that $E = 0$, i.e., there is no direct influence of the exosystem.

Let $y = x_1$ be the output of interest to be driven to follow w_1 from the exosystem model

$$\dot{w}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w(t), \quad Q = \begin{bmatrix} -1 & 0 \end{bmatrix} \tag{3.7}$$

which generates a sinusoidal reference.

Through LMI conditions (3.2) in Theorem 3.2, the following gains have been obtained via convex optimization techniques:

$$K = \begin{bmatrix} 2.4317 & 2.8124 & -21.7505 & -3.4033 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -0.1429 & 0 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1429 & 0 \\ 0 & 0.1429 \end{bmatrix} \quad \text{and} \quad \varepsilon = 1.9233 \times 10^{-16}$$

In Figure 3.1 a simulation has been run from initial conditions $x(0) = 0$ and $w(0) = \begin{bmatrix} 0.5 & 0 \end{bmatrix}^T$ to show the trajectory tracking due to the gains calculated via Theorem 3.2. The tracking error is presented in Figure 3.2. The corresponding control law is showed in Figure 3.3. It is important to stress that the performance shown in these figures corresponds to the control laws being applied from the output regulation theorems to the original nonlinear plant.

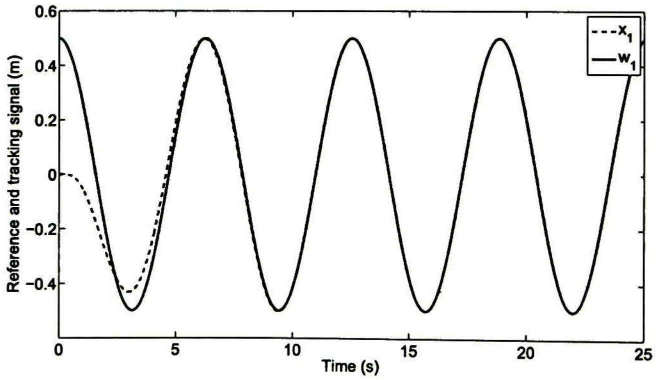

Figure 3.1: *Trajectory tracking of reference w_1 by x_1 in Example 3.1.*

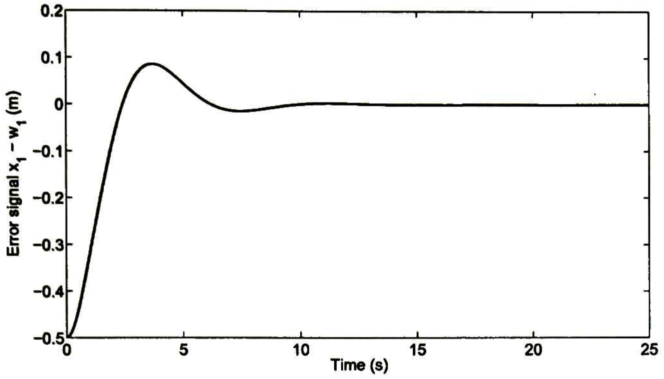

Figure 3.2: *Error signal $x_1 - w_1$ in Example 3.1.*

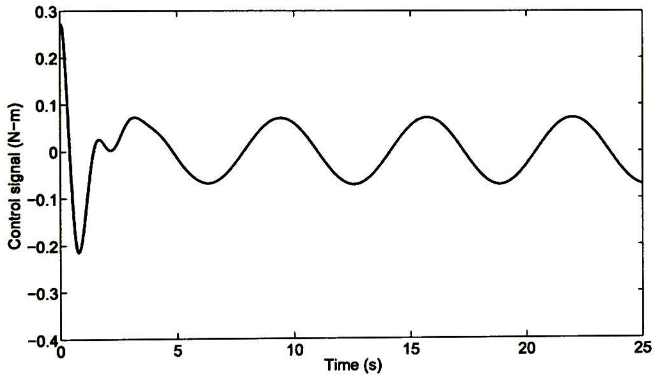

Figure 3.3: *Control signal $u(t)$ in Example 3.1.*

Example 3.2. Consider the nonlinear system of an underactuated two-link PENDUBOT system [27]:

$$\dot{x}(t) = \begin{bmatrix} x_3 \\ x_4 \\ b_3(x_2)p_1(x) \\ b_4(x_2)p_2(x) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_3(x_2) \\ b_4(x_2) \end{bmatrix} u(t), \tag{3.8}$$

with x_1 being the base link angle with respect to the horizontal, x_2 the angle between the two links, x_3 and x_4 their respective angular velocities. The functions $b_3 = \frac{D_{22}}{D_{11}D_{22}-D_{22}^2}$, $b_4 = \frac{-D_{12}}{D_{11}D_{22}-D_{12}^2}$, $p_1 = \frac{D_{12}}{D_{22}}\left(\bar{C}_2+\bar{G}_2-\bar{F}_2\right)-\bar{C}_1-\bar{G}_1-\bar{F}_1$, $p_2 = \frac{D_{11}}{D_{12}}\left(\bar{C}_2+\bar{G}_2+\bar{F}_2\right)-\bar{C}_2-\bar{G}_1-\bar{F}_1$, $D_{11} = m_1 l_{c2}^2 + m_2\left(l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos(x_2)\right) + I_1 + I_2$, $D_{12} = m_2\left(l_{c2}^2 + 2l_1 l_{c2}\cos(x_2)\right) + I_2$, $D_{22} = m_2 l_{c2}^2 + I_2$, $\bar{C}_1 = -2m_2 l_1 l_{c2} x_3 x_4 \sin(x_2) - m_2 l_1 l_{c2} x_4 \sin(x_2)$, $\bar{C}_2 = m_2 l_1 l_{c2} x_3^2 \sin(x_2)$, $\bar{G}_1 = m_1 g l_{c1}\cos(x_1) + m_2 g l_1 \cos(x_1) + m_2 g l_{c2}\cos(x_1+x_2)$, $\bar{G}_2 = m_2 g l_{c2}\cos(x_1+x_2)$, $\bar{F}_1 = \mu_1 x_3$, and $\bar{F}_2 = \mu_2 x_4$. The parameters $l_1 = 0.26987$, $l_{c1} = 0.13494$, $l_2 = 0.38417$, $l_{c2} = 0.19208$, $m_1 = 0.5289$, $m_2 = 0.3346$, $I_1 = 1.3863 \times 10^{-2}$, $I_2 = 1.6749 \times 10^{-2}$, $\mu_1 = 0.00545$, $\mu_2 = 0.00047$, and $g = 9.81$.

Maintaining the system in the upright position is usually the control objective on the PENDUBOT with model (3.8); a linearization around $x(t) = \begin{bmatrix} 90^\circ & 0 & 0 & 0 \end{bmatrix}^T$ comes thus at hand:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 32.2499 & -10.0172 & 0 & 1 \\ -29.8052 & 37.6597 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 26.6507 \\ -42.5387 \end{bmatrix} u(t), \tag{3.9}$$

with $y = x_1$ as the desired output. The exosystem is given by

$$\dot{w}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w(t), \quad Q = \begin{bmatrix} -1 & 0 \end{bmatrix} \tag{3.10}$$

as to follow a sinusoidal reference which corresponds to the PENDUBOT system swinging from one side to the other around the upright position.

Usually not all the states are available, so the LORPEF suits this case. Using conditions (3.4) in Theorem 3.3 the following gains have been obtained:

$$K = \begin{bmatrix} 2.3734 & 3.8478 & 0.8677 & 0.7584 \end{bmatrix},\ \Gamma = \begin{bmatrix} -1.6288 & 0.0653 \end{bmatrix},\ \Pi = \begin{bmatrix} 1 & 0 \\ -1.0212 & 0.0719 \\ 0 & 1 \\ -0.0719 & -5.2742 \end{bmatrix},$$

$$G = 1 \times 10^3 \begin{bmatrix} 0.0095 & -0.8531 & 0.4194 & -5.2742 & -0.0094 & 0.0012 \end{bmatrix}^T \text{ and } \varepsilon = 8.1797 \times 10^{-15}.$$

Figure 3.4 draws the trajectory tracking under initial conditions $x(0) = 0$, $w(0) = \begin{bmatrix} 0.3 & 0 \end{bmatrix}^T$ and $\xi(0) = \begin{bmatrix} 0.1 & 0.2 & 0 & 0.2 & 0.3 & 0 \end{bmatrix}^T$ Figure 3.5 shows the error evolution along the time when the nonlinear model (3.8) has been fed with the control law (A.4). Figure 3.6 illustrates the control signal corresponding to this simulation.

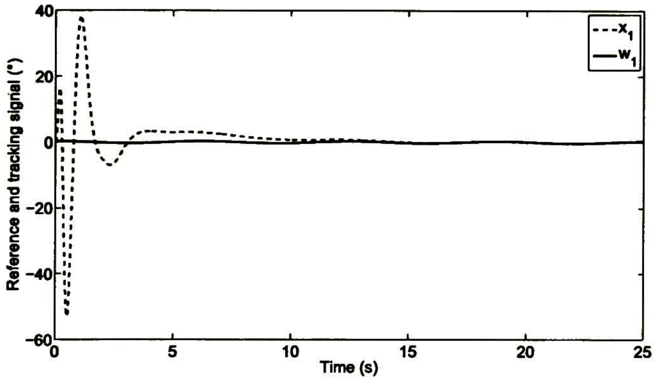

Figure 3.4: *Trajectory tracking of reference w_1 by x_1 in Example 3.2.*

The advantages of expressing these results in such a way have been to express and solve output regulation conditions via convex optimization techniques which are easily implemented in commercially available software, thus providing a numerically better approach for systematize output regulation of nonlinear plants.

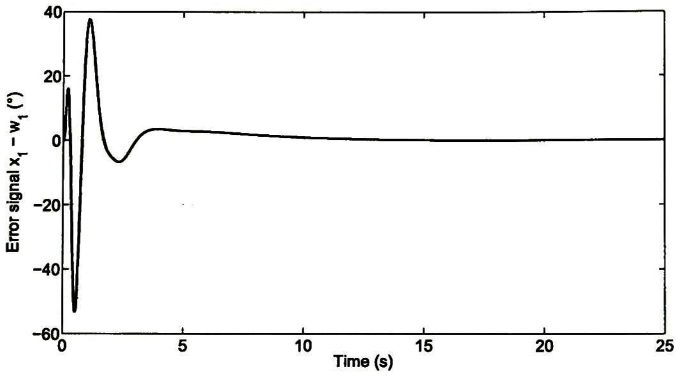

Figure 3.5: *Error signal $x_1 - w_1$ in Example 3.2.*

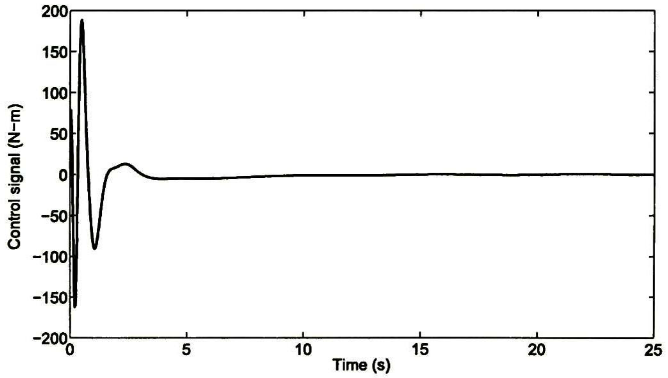

Figure 3.6: *Control signal $u(t)$ in Example 3.2.*

The next chapter extends this approach by incorporating the element-wise LMI approach hereby presented as well as the most recent novelties in convex modeling to the nonlinear output regulator problem.

Chapter 4

Nonlinear Output Regulation via TS Models and LMIs

As was mentioned in the introduction, finding a solution to the NORP is difficult because complicated nonlinear partial differential equations have to be solved. Since TS models and LMIs have successfully deal with complicated nonlinear problems, several researchers have tried to solve the NORP under this framework. The principal motivation of a TS-LMI approach for solving the NORP (so-called Fuzzy Regulation Problem) is to systematically search via LMIs the nonlinear mappings from the FIB equations; to do that, it is assumed a TS structure for them [11, 25, 35]. The FIB equations combined with the nonlinear mappings via TS models lead to the time-derivatives of the MFs. This chapter shows how to deal with an old problem when TS models are used: the time-derivatives of the MFs [25]. To that end ideas are taken from [3, 14] where it is shown a way to avoid from this problem.

4.1 NORP based on Takagi-Sugeno Models

NORP is stated in Appendix A, where two classical NORP are briefly described. This section provides a new approach based on TS models and LMIs to solve the NORP a) via state feedback and b) via error feedback.

The nonlinear system (A.6) has the following TS model [33]:

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r}(z(t))\left(A_i x(t) + B_i u(t) + E_i w(t)\right) = A_z x(t) + B_z x(t) + E_z w(t) \\
\dot{w}(t) &= \sum_{i=1}^{r}(z(t)) S_i w(t) = S_z w(t) \\
e(t) &= \sum_{i=1}^{r}(z(t))\left(C_i x(t) - Q_i w(t)\right) = C_z x(t) - Q_z w(t),
\end{aligned}
\tag{4.1}
$$

with $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{o\times n}$, $E_i \in \mathbb{R}^{n\times q}$, $Q_i \in \mathbb{R}^{o\times q}$, $S_i \in \mathbb{R}^{q\times q}$. $i \in \{1,\ldots,r\}$, and $h_i(\cdot)$, $i = 1,\ldots,r$ being MFs satisfying the convex-sum property: $\sum_{i=1}^{r} h_i(\cdot) = 1$, $h_i(\cdot) \geq 0$ and $r = 2^p \in \mathbb{N}$ being the number of rules.

The model (4.1) is an exact representation of the nonlinear system (A.6) in the compact region Δ; i.e., it is not an approximation.

Using the TS model defined above, the FIB equations (A.8) yields:

$$
\begin{aligned}
\begin{bmatrix} \dot{\pi}(w) \\ 0 \end{bmatrix} &= \sum_{i=1}^{r} h_i\left(z(t)\right)\left(\begin{bmatrix} A_i & E_i \\ C_i & -Q_i \end{bmatrix}\begin{bmatrix} \pi(w) \\ w \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix}\gamma(w)\right) \\
&= \begin{bmatrix} A_z & E_z \\ C_z & -Q_z \end{bmatrix}\begin{bmatrix} \pi(w) \\ w \end{bmatrix} + \begin{bmatrix} B_z \\ 0 \end{bmatrix}\gamma(w),
\end{aligned}
\tag{4.2}
$$

As in [11, 25] the nonlinear mappings $x = \pi(w(t))$ and $u = \gamma(w(t)$ will be assumed to share the same convex structure as TS model (4.1) in order to find them via LMIs, i.e.:

$$\begin{aligned}\pi(w(t)) &= \sum_{j=1}^{r} h_j(z(t))\,\Pi_j w(t) = \Pi_z w(t),\\ \gamma(w(t)) &= \sum_{j=1}^{r} h_j(z(t))\,\Gamma_j w(t) = \Gamma_z w(t),\end{aligned} \tag{4.3}$$

with $\Pi_j \in \mathbb{R}^{n\times q}$, $\Gamma_j \in \mathbb{R}^{m\times q}$, $j \in \{1,\ldots,r\}$.

Thus, substituting the fuzzy mappings (4.3) in (4.2) yields:

$$\begin{bmatrix} \Pi_z\dot{w} + \dot{\Pi}_z w \\ 0 \end{bmatrix} = \begin{bmatrix} A_z & E_z \\ C_z & -Q_z \end{bmatrix}\begin{bmatrix} \Pi_z w \\ w \end{bmatrix} + \begin{bmatrix} B_z \\ 0 \end{bmatrix}\Gamma_z w, \tag{4.4}$$

from which, after substituting $\dot{w}$ and dropping out the common w arises:

$$\begin{bmatrix} \Pi_z S_z + \dot{\Pi}_z \\ 0 \end{bmatrix} = \begin{bmatrix} A_z\Pi_z + E_z \\ C_z\Pi_z - Q_z \end{bmatrix} + \begin{bmatrix} B_z\Gamma_z \\ 0 \end{bmatrix} = \begin{bmatrix} A_z\Pi_z + E_z + B_z\Gamma_z \\ C_z\Pi_z - Q_z \end{bmatrix} \tag{4.5}$$

Moreover, the nonlinear control law for **NORPSF** (A.7) can be rewritten as:

$$u(t) = \alpha(x,w) = \Gamma_z w + K_z(x - \Pi_z w). \tag{4.6}$$

On the other hand, for the **NORPEF**, the control law (A.9):

$$\begin{aligned}\dot{\xi}(t) &= \eta(\xi,e) = F_z\xi + G_z e\\ u(t) &= \theta(\xi) = H_z\xi,\end{aligned} \tag{4.7}$$

with $F_z \in \mathbb{R}^{(n+q)\times(n+q)}$, $G_z \in \mathbb{R}^{(n+q)\times p}$, $H_z \in \mathbb{R}^{m\times(n+q)}$ defined as follows:
$F_z = \begin{bmatrix} A_z - G_z^0C_z + B_zK_z & E_z + G_z^0Q_z + B_z(\Gamma_z - K_z\Pi_z) \\ -G_z^1C_z & S_z + G_z^1Q_z \end{bmatrix}$, $G_z = \begin{bmatrix} G_z^0 \\ G_z^1 \end{bmatrix}$ and $H_z = \begin{bmatrix} K_z & \Gamma_z - K_z\Pi_z \end{bmatrix}$ has the form

$$u(t) = \theta(\xi) = \Gamma_z\xi_1 + K_z(\xi_0 - \Pi_z\xi_1). \tag{4.8}$$

Generally, finding an exact tracking is difficult due to: 1) regularly, there is no inverse for B_z, then the nonlinear mapping $\gamma(w)$ is approximated; 2) $\dot{\Pi}_z \neq 0$, where time-derivative of the MFs arises in (4.5). Results in [11, 25] stress in some these difficulties by considering particular cases ($\dot{\Pi}_z = 0$ from $\Pi_j = \Pi$, $\forall j \in \{1,\ldots,r\}$), or bounds which are not known a priori. In [3, 14] a way to fully take advantage of the information available in the time-derivatives of the MFs was presented in a different context: it keeps an LMI form so it can be efficiently solved by convex optimization techniques.

The following results incorporate these novelties; they have two sets of LMIs to be solved concurrently: a first one focused on stabilizing the TS model (4.1); a second one intended to solve the FIB equations (4.5).

Theorem 4.1. *Assume H1 and H2. The NORPSF has a solution if* $\exists X = X^T \in \mathbb{R}^{n\times n}$*,* $M_j \in \mathbb{R}^{m\times n}$*,* $\Pi_j \in \mathbb{R}^{n\times q}$*,* $\Gamma_j \in \mathbb{R}^{m\times q}$*,* $j \in \{1,\ldots,r\}$*,* $\beta_k > 0$*,* $k \in \{1,\ldots,r\}$*, and an arbitrarily small* $\varepsilon > 0$ *such that:*

$$X > 0, \quad A_z X + B_z M_z + (*) < 0, \tag{4.9}$$

$$-\varepsilon \prec \begin{bmatrix} A_z\Pi_z + B_z\Gamma_z + E_z - \Pi_z S_z - \sum_{k=1}^{p}(-1)^{d_k^\alpha}\beta_k\left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)}\right) \\ C_z\Pi_z - Q_z \end{bmatrix} \prec \varepsilon, \tag{4.10}$$

$$\left|\dot{\omega}_0^k\right| \leq \beta_k, \tag{4.11}$$

with $g_1(j,k) = \lfloor (j-1)/2^{p+1-k} \rfloor \times 2^{p+1-k} + 1 + (j-1) \mod 2^{p-k}$*,* $g_2(j,k) = g_1(j,k) + 2^{p-2}$*,* $\lfloor\cdot\rfloor$ *the floor function and* d_k^α *obtained from* $\alpha - 1 = d_p^\alpha + d_{p-1}^\alpha \times 2 + \ldots + d_1^\alpha \times 2^{p-1}$ *The control law is given by (4.6), in this case with* $K^* = K_z = M_z X^{-1}$

Proof. The first set of conditions (4.9) corresponds to the stabilization of the TS model (4.1) when $w(t) = 0$, using a quadratic Lyapunov function $V(x) = x^T Px$ and the PDC control law $u = K_z x$; the procedure is direct as shown in the previous sections.

In [14] it has been proved that:

$$\dot{\Pi}_z = \sum_{k=1}^{p} \dot{\omega}_0^k \left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)}\right), \tag{4.12}$$

with $g_1(j,k)$ and $g_2(j,k)$ defined as above. Substituting (4.12) in (4.5) gives

$$\begin{bmatrix} A_z\Pi_z + E_z + B_z\Gamma_z - \Pi_z S_z - \sum_{k=1}^{p} \dot{\omega}_0^k \left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)}\right) \\ C_z\Pi_z - Q_z \end{bmatrix} = 0. \tag{4.13}$$

Theorem 3.1 provides sufficient conditions for matrix equality (4.13) to hold if an arbitrarily small $\varepsilon > 0$ exists such that the following hold:

$$-\varepsilon \prec \begin{bmatrix} A_z\Pi_z + B_z\Gamma_z + E_z - \Pi_z S_z - \sum_{k=1}^{p} \dot{\omega}_0^k \left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)}\right) \\ C_z\Pi_z - Q_z \end{bmatrix} \prec \varepsilon. \tag{4.14}$$

If $\left|\dot{\omega}_0^k\right| \leq \beta_k$, $\beta_k > 0$, then (4.10) follows from (4.14) by the property $Y \pm \beta_k \times Z < 0 \Leftrightarrow Y + \dot{\omega}_0^k Z < 0$ since all the sign combinations of the sum involving $\dot{\omega}_0^k$ are considered in (4.10). □

Theorem 4.2. *Assume H1, H2, and H3. The NORPEF has a solution if* $\exists X_1 = X_1^T \in \mathbb{R}^{n\times n}$, $X_2 = X_2^T \in \mathbb{R}^{(n+q)\times(n+q)}$, $M_{1j} \in \mathbb{R}^{m\times n}$, $M_{2j} \in \mathbb{R}^{m\times(n+q)}$, $\Pi_j \in \mathbb{R}^{n\times q}$, $\Gamma_j \in \mathbb{R}^{m\times q}$, $j \in \{1,\ldots,r\}$, $\beta_k > 0$, $k \in \{1,\ldots,r\}$, *and an arbitrarily small* $\varepsilon > 0$ *such that:*

$$X_1 > 0, \quad A_z X_1 + B_z M_{1z} + (*) < 0, \tag{4.15}$$

$$X_2 > 0, \quad \bar{A}_z X_2 + \bar{B}_z M_{2z} + (*) < 0, \tag{4.16}$$

$$-\varepsilon \prec \begin{bmatrix} A_z\Pi_z + B_z\Gamma_z + E_z - \Pi_z S_z - \sum_{k=1}^{p}(-1)^{d_k^\alpha}\beta_k\left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)}\right) \\ C_z\Pi_z - Q_z \end{bmatrix} \prec \varepsilon, \qquad (4.17)$$

$$\left|\dot{\omega}_0^k\right| \le \beta_k, \qquad (4.18)$$

with $g_1(j,k)$, $g_2(j,k)$ and d_k^α defined as in the previous theorem. The control law is given by (4.8),in this case with $K^* = K_z = M_{1z}X_1^{-1}$ and $G_z = (M_{2z}X_2^{-1})^T$

Proof. As in the previous theorem, a quadratic candidate Lyapunov function is used to stabilize the pairs (A_z, B_z) and $(\bar{A}_z, \bar{B}_z)$ with $\bar{A}_z = \begin{bmatrix} A_z & E_z \\ 0 & S_z \end{bmatrix}^T$ and $\bar{B}_z = \begin{bmatrix} C_z & -Q_z \end{bmatrix}^T$
The remaining part of the procedure is identical to the Theorem 4.1. □

Remark 4.1. Conditions in Theorem 4.1 and Theorem 4.2 can be solved simultaneously given a systematic approach by convex optimization techniques.

Remark 4.2. A relaxation lemma needs to be applied in order to drop off the double sums implied by z.

Remark 4.3. The LMIs obtained from Theorem 4.1 and Theorem 4.2 can be solved as a convex minimization problem over ε; furthermore, this represents a bound to each entry of the steady-state error $e_{ss} = C_z\Pi_z - Q_z$ if the mapping $x = \Pi_z$ is accurate.

Remark 4.4. The bound β_k can be verified a posteriori by running a simulation of the closed-loop system.

4.1.1 Examples

Two examples are presented to test the effectiveness of the new approach. The first example solves the NORP when full information is available, resulting $\dot{\Pi}_z = 0$. The second one, employs an error feedback since not all the states are accessible, where $\dot{\Pi}_z \neq 0$ and the selection of $\left|\dot{\omega}_0^k\right| \leq \beta_k$ proves to be crucial.

Example 4.1. Consider the following nonlinear system:

$$\dot{x}(t) = \begin{bmatrix} -1 & x_2^2 \\ 0.76 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0.5x_1^2 \\ 0 & 0 \end{bmatrix} w(t), \tag{4.19}$$

with $x(t)$ the state vector and $w(t)$ the exosystem vector which generates the reference signal and perturbations:

$$\dot{w}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w(t). \tag{4.20}$$

From the sector nonlinearity approach with premise variables are $z_1 = x_1^2$, $z_2 = x_2^2$, the following TS model exactly represents (4.19) and (4.20) in the compact region $\Delta = \{|x_1| \leq 1, |w_1| \leq 1\} \supset \mathbf{0}$:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{4} h_i(z(t)) \left(A_i x(t) + B_i u(t) + E_i w(t)\right), \\ \dot{w}(t) &= \sum_{i=1}^{4} h_i(z(t)) S_i w(t), \end{aligned} \tag{4.21}$$

with $A_1 = A_2 = \begin{bmatrix} -1 & 1 \\ 0.76 & 0.5 \end{bmatrix}$, $A_3 = A_4 = \begin{bmatrix} -1 & 0 \\ 0.76 & 0.5 \end{bmatrix}$, $B_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $S_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $i = 1, \ldots, 4$, $E_1 = E_3 = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix}$, $E_2 = E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\omega_0^1 = x_1^2$, $\omega_0^2 = x_2^2$, $\omega_1^1 = 1 - \omega_0^1$, $\omega_1^2 = 1 - \omega_0^2$, $h_1 = \omega_0^1\omega_0^2$, $h_2 = \omega_0^1\omega_1^2$, $h_3 = \omega_1^1\omega_0^2$, and $h_4 = \omega_1^1\omega_1^2$.

The task is to make the output $y = x_1$ follow the reference w_1; it therefore requires to define $C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $Q_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $i = 1, \dots, 4$. Applying Theorem 4.1 and the relaxation lemma 2.2 over the double sums, the resulting stabilizing gains are:

$$K_1 = K_2 = \begin{bmatrix} -1.4259 & -5.6353 \end{bmatrix} \quad \text{and} \quad K_3 = K_4 = \begin{bmatrix} -1.4259 & -4.6353 \end{bmatrix}$$

The bounds $\beta_k = 1 \times 10^8$, $k = 1, 2$ are verified a posteriori and elected to satisfy $\left|\dot{\omega}_0^k\right| \leq \beta_k$. The TS mappings obtained:

$$\Pi_i = \begin{bmatrix} 1 & 0 \\ -0.304 & -0.608 \end{bmatrix} \quad i = 1, \dots, 4, \quad \Gamma_1 = \begin{bmatrix} 1.304 & 1.108 \end{bmatrix},$$
$$\Gamma_2 = \begin{bmatrix} 1.304 & 1.608 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \quad \text{and} \quad \Gamma_4 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

In this case, $\dot{\Pi}_z = 0$, then, the election of β_k is not relevant, since it is clearly that the term $\sum_{k=1}^{p}(-1)^{d_k^\alpha}\beta_k\left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)}\right)$ is zero.

The exact tracking is shown in Figure 4.1. The simulations was run with initial conditions $x(0) = \begin{bmatrix} 0.4 & 0.1 \end{bmatrix}^T$ and $w(0) = \begin{bmatrix} -0.8 & 0 \end{bmatrix}^T$ In Figure 4.2 the tracking error is presented.

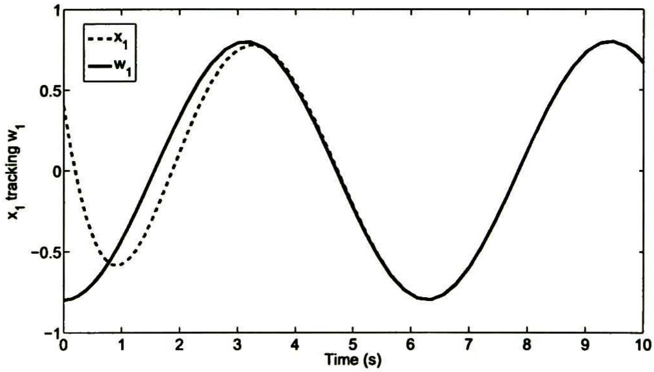

Figure 4.1: *x_1 tracking w_1 in Example 4.1.*

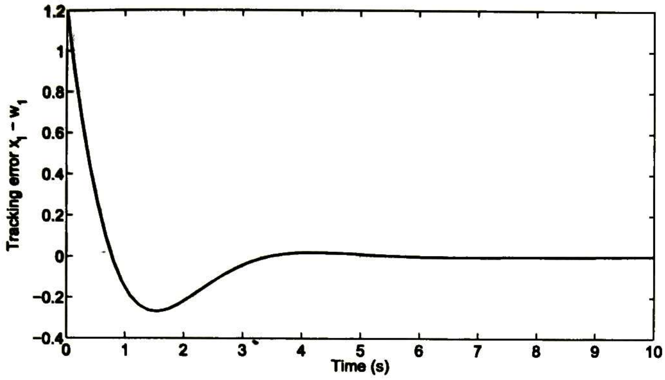

Figure 4.2: *Tracking error $x_1 - w_1$ in Example 4.1.*

Example 4.2. Consider the nonlinear system described by:

$$\dot{x}(t) = \begin{bmatrix} 0.3 + 0.027x_1^2 & 0.9 \\ 0.3 & 0.4x_2^2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0.3x_2^2 \end{bmatrix} w(t), \tag{4.22}$$

with $x(t)$ the state vector and $w(t)$ the exosystem vector which generates the references signal (sinusoidal in this case) and the perturbations to be rejected, described by:

$$\dot{w}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w(t). \tag{4.23}$$

The following TS model is rewritten from the nonlinear models (4.22) and (4.23) in the region $\Delta = \{|x_1| \leq 1, |w_1| \leq 1\} \supset \mathbf{0}$, with the premise variables $z_1 = x_1^2$ and $z_2 = x_2^2$,

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{4} h_i(z(t)) \left(A_i x(t) + B_i u(t) + E_i w(t)\right), \\ \dot{w}(t) &= \sum_{i=1}^{4} h_i(z(t)) S_i w(t), \end{aligned} \tag{4.24}$$

4.1 NORP based on Takagi-Sugeno Models

NORP is stated in Appendix A, where two classical NORP are briefly described. This section provides a new approach based on TS models and LMIs to solve the NORP a) via state feedback and b) via error feedback.

The nonlinear system (A.6) has the following TS model [33]:

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r}(z(t))\left(A_i x(t) + B_i u(t) + E_i w(t)\right) = A_z x(t) + B_z x(t) + E_z w(t) \\
\dot{w}(t) &= \sum_{i=1}^{r}(z(t)) S_i w(t) = S_z w(t) \\
e(t) &= \sum_{i=1}^{r}(z(t))\left(C_i x(t) - Q_i w(t)\right) = C_z x(t) - Q_z w(t),
\end{aligned} \tag{4.1}$$

with $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{o\times n}$, $E_i \in \mathbb{R}^{n\times q}$, $Q_i \in \mathbb{R}^{o\times q}$, $S_i \in \mathbb{R}^{q\times q}$, $i \in \{1,\ldots,r\}$, and $h_i(\cdot)$, $i = 1,\ldots,r$ being MFs satisfying the convex-sum property: $\sum_{i=1}^{r} h_i(\cdot) = 1$, $h_i(\cdot) \geq 0$ and $r = 2^p \in \mathbb{N}$ being the number of rules.

The model (4.1) is an exact representation of the nonlinear system (A.6) in the compact region Δ; i.e., it is not an approximation.

Using the TS model defined above, the FIB equations (A.8) yields:

$$\begin{aligned}
\begin{bmatrix} \dot{\pi}(w) \\ 0 \end{bmatrix} &= \sum_{i=1}^{r} h_i\left(z(t)\right)\left(\begin{bmatrix} A_i & E_i \\ C_i & -Q_i \end{bmatrix}\begin{bmatrix} \pi(w) \\ w \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix}\gamma(w)\right) \\
&= \begin{bmatrix} A_z & E_z \\ C_z & -Q_z \end{bmatrix}\begin{bmatrix} \pi(w) \\ w \end{bmatrix} + \begin{bmatrix} B_z \\ 0 \end{bmatrix}\gamma(w),
\end{aligned} \tag{4.2}$$

As in [11, 25] the nonlinear mappings $x = \pi(w(t))$ and $u = \gamma(w(t)$ will be assumed to share the same convex structure as TS model (4.1) in order to find them via LMIs, i.e.:

$$\begin{aligned} \pi(w(t)) &= \sum_{j=1}^{r} h_j\left(z(t)\right)\Pi_j w(t) = \Pi_z w(t), \\ \gamma(w(t)) &= \sum_{j=1}^{r} h_j\left(z(t)\right)\Gamma_j w(t) = \Gamma_z w(t), \end{aligned} \tag{4.3}$$

with $\Pi_j \in \mathbb{R}^{n\times q}$, $\Gamma_j \in \mathbb{R}^{m\times q}$, $j \in \{1,\ldots,r\}$.

Thus, substituting the fuzzy mappings (4.3) in (4.2) yields:

$$\begin{bmatrix} \Pi_z\dot{w} + \dot{\Pi}_z w \\ 0 \end{bmatrix} = \begin{bmatrix} A_z & E_z \\ C_z & -Q_z \end{bmatrix} \begin{bmatrix} \Pi_z w \\ w \end{bmatrix} + \begin{bmatrix} B_z \\ 0 \end{bmatrix} \Gamma_z w, \tag{4.4}$$

from which, after substituting $\dot{w}$ and dropping out the common w arises:

$$\begin{bmatrix} \Pi_z S_z + \dot{\Pi}_z \\ 0 \end{bmatrix} = \begin{bmatrix} A_z\Pi_z + E_z \\ C_z\Pi_z - Q_z \end{bmatrix} + \begin{bmatrix} B_z\Gamma_z \\ 0 \end{bmatrix} = \begin{bmatrix} A_z\Pi_z + E_z + B_z\Gamma_z \\ C_z\Pi_z - Q_z \end{bmatrix} \tag{4.5}$$

Moreover, the nonlinear control law for **NORPSF** (A.7) can be rewritten as:

$$u(t) = \alpha(x,w) = \Gamma_z w + K_z(x - \Pi_z w). \tag{4.6}$$

On the other hand, for the **NORPEF**, the control law (A.9):

$$\begin{aligned} \dot{\xi}(t) &= \eta(\xi,e) = F_z\xi + G_z e \\ u(t) &= \theta(\xi) = H_z\xi, \end{aligned} \tag{4.7}$$

with $F_z \in \mathbb{R}^{(n+q)\times(n+q)}$, $G_z \in \mathbb{R}^{(n+q)\times p}$, $H_z \in \mathbb{R}^{m\times(n+q)}$ defined as follows: $F_z = \begin{bmatrix} A_z - G_z^0 C_z + B_z K_z & E_z + G_z^0 Q_z + B_z(\Gamma_z - K_z\Pi_z) \\ -G_z^1 C_z & S_z + G_z^1 Q_z \end{bmatrix}$, $G_z = \begin{bmatrix} G_z^0 \\ G_z^1 \end{bmatrix}$ and $H_z = \begin{bmatrix} K_z & \Gamma_z - K_z\Pi_z \end{bmatrix}$ has the form

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Generally, finding an exact tracking is difficult due to: 1) regularly, there is no inverse for B_z, then the nonlinear mapping $\gamma(w)$ is approximated; 2) $\dot{\Pi}_z \neq 0$, where time-derivative of the MFs arises in (4.5). Results in [11, 25] stress in some these difficulties by considering particular cases ($\dot{\Pi}_z = 0$ from $\Pi_j = \Pi$, $\forall j \in \{1, \ldots, r\}$), or bounds which are not known a priori. In [3, 14] a way to fully take advantage of the information available in the time-derivatives of the MFs was presented in a different context: it keeps an LMI form so it can be efficiently solved by convex optimization techniques.

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$$X > 0, \quad A_z X + B_z M_z + (*) < 0, \tag{4.9}$$

$$-\varepsilon \prec \begin{bmatrix} A_z\Pi_z + B_z\Gamma_z + E_z - \Pi_z S_z - \sum_{k=1}^{p}(-1)^{d_k^\alpha}\beta_k\left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)}\right) \\ C_z\Pi_z - Q_z \end{bmatrix} \prec \varepsilon, \tag{4.10}$$

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with $g_1(j,k) = \lfloor (j-1)/2^{p+1-k} \rfloor \times 2^{p+1-k} + 1 + (j-1) \mod 2^{p-k}$, $g_2(j,k) = g_1(j,k) + 2^{p-2}$, $\lfloor\cdot\rfloor$ *the floor function and* d_k^α *obtained from* $\alpha - 1 = d_p^\alpha + d_{p-1}^\alpha \times 2 + \ldots + d_1^\alpha \times 2^{p-1}$ *The control law is given by (4.6), in this case with* $K^* = K_z = M_z X^{-1}$

Proof. The first set of conditions (4.9) corresponds to the stabilization of the TS model (4.1) when $w(t) = 0$, using a quadratic Lyapunov function $V(x) = x^T P x$ and the PDC control law $u = K_z x$; the procedure is direct as shown in the previous sections.

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Two examples are presented to test the effectiveness of the new approach. The first example solves the NORP when full information is available, resulting $\dot{\Pi}_z = 0$. The second one, employs an error feedback since not all the states are accessible, where $\dot{\Pi}_z \neq 0$ and the selection of $\left|\dot{\omega}_0^k\right| \leq \beta_k$ proves to be crucial.

Example 4.1. Consider the following nonlinear system:

$$\dot{x}(t) = \begin{bmatrix} -1 & x_2^2 \\ 0.76 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0.5x_1^2 \\ 0 & 0 \end{bmatrix} w(t), \tag{4.19}$$

with $x(t)$ the state vector and $w(t)$ the exosystem vector which generates the reference signal and perturbations:

$$\dot{w}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w(t). \tag{4.20}$$

From the sector nonlinearity approach with premise variables are $z_1 = x_1^2$, $z_2 = x_2^2$, the following TS model exactly represents (4.19) and (4.20) in the compact region $\Delta = \{|x_1| \leq 1, |w_1| \leq 1\} \supset \mathbf{0}$:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{4} h_i(z(t)) \left(A_i x(t) + B_i u(t) + E_i w(t)\right), \\ \dot{w}(t) &= \sum_{i=1}^{4} h_i(z(t)) S_i w(t), \end{aligned} \tag{4.21}$$

with $A_1 = A_2 = \begin{bmatrix} -1 & 1 \\ 0.76 & 0.5 \end{bmatrix}$, $A_3 = A_4 = \begin{bmatrix} -1 & 0 \\ 0.76 & 0.5 \end{bmatrix}$, $B_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $S_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $i = 1, \ldots, 4$, $E_1 = E_3 = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix}$, $E_2 = E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\omega_0^1 = x_1^2$, $\omega_0^2 = x_2^2$, $\omega_1^1 = 1 - \omega_0^1$, $\omega_1^2 = 1 - \omega_0^2$, $h_1 = \omega_0^1 \omega_0^2$, $h_2 = \omega_0^1 \omega_1^2$, $h_3 = \omega_1^1 \omega_0^2$, and $h_4 = \omega_1^1 \omega_1^2$.

The task is to make the output $y = x_1$ follow the reference w_1; it therefore requires to define $C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $Q_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $i = 1, \dots, 4$. Applying Theorem 4.1 and the relaxation lemma 2.2 over the double sums, the resulting stabilizing gains are:

$$K_1 = K_2 = \begin{bmatrix} -1.4259 & -5.6353 \end{bmatrix} \quad \text{and} \quad K_3 = K_4 = \begin{bmatrix} -1.4259 & -4.6353 \end{bmatrix}$$

The bounds $\beta_k = 1 \times 10^8$, $k = 1, 2$ are verified a posteriori and elected to satisfy $\left|\dot{\omega}_0^k\right| \leq \beta_k$. The TS mappings obtained:

$$\Pi_i = \begin{bmatrix} 1 & 0 \\ -0.304 & -0.608 \end{bmatrix} \quad i = 1, \dots, 4, \quad \Gamma_1 = \begin{bmatrix} 1.304 & 1.108 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} 1.304 & 1.608 \end{bmatrix} \quad \Gamma_3 = \begin{bmatrix} 1 & 0.5 \end{bmatrix}, \quad \text{and} \quad \Gamma_4 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

In this case, $\dot{\Pi}_z = 0$, then, the election of β_k is not relevant, since it is clearly that the term $\sum_{k=1}^{p} (-1)^{d_k^\alpha} \beta_k \left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)}\right)$ is zero.

The exact tracking is shown in Figure 4.1. The simulations was run with initial conditions $x(0) = \begin{bmatrix} 0.4 & 0.1 \end{bmatrix}^T$ and $w(0) = \begin{bmatrix} -0.8 & 0 \end{bmatrix}^T$ In Figure 4.2 the tracking error is presented.

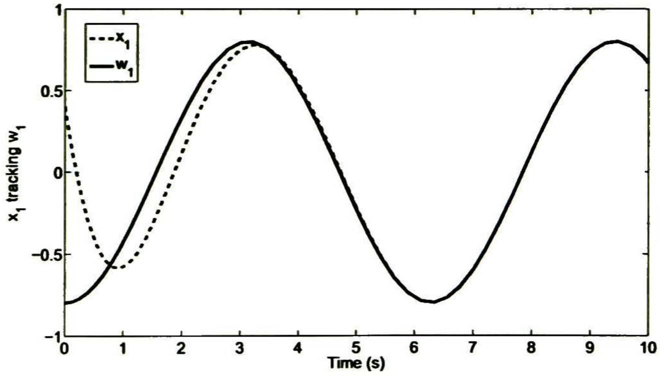

Figure 4.1: *x_1 tracking w_1 in Example 4.1.*

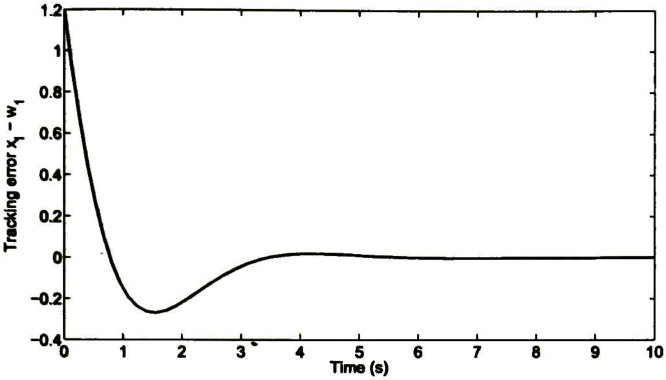

Figure 4.2: *Tracking error $x_1 - w_1$ in Example 4.1.*

Example 4.2. Consider the nonlinear system described by:

$$\dot{x}(t) = \begin{bmatrix} 0.3 + 0.027x_1^2 & 0.9 \\ 0.3 & 0.4x_2^2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0.3x_2^2 \end{bmatrix} w(t), \tag{4.22}$$

with $x(t)$ the state vector and $w(t)$ the exosystem vector which generates the references signal (sinusoidal in this case) and the perturbations to be rejected, described by:

$$\dot{w}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w(t). \tag{4.23}$$

The following TS model is rewritten from the nonlinear models (4.22) and (4.23) in the region $\Delta = \{|x_1| \leq 1, |w_1| \leq 1\} \supset \mathbf{0}$, with the premise variables $z_1 = x_1^2$ and $z_2 = x_2^2$,

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{4} h_i(z(t)) \left(A_i x(t) + B_i u(t) + E_i w(t) \right), \\ \dot{w}(t) &= \sum_{i=1}^{4} h_i(z(t)) S_i w(t), \end{aligned} \tag{4.24}$$

with $A_1 = \begin{bmatrix} 0.327 & 0.9 \\ 0.3 & 0.4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.327 & 0.9 \\ 0.3 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0.3 & 0.9 \\ 0.3 & 0.4 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0.3 & 0.9 \\ 0.3 & 0 \end{bmatrix}$,
$B_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $S_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $i = 1, \ldots, 4$, $E_1 = E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.3 \end{bmatrix}$, $E_2 = E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,
$\omega_0^1 = x_1^2$, $\omega_0^2 = x_2^2$, $\omega_1^1 = 1 - \omega_0^1$, $\omega_1^2 = 1 - \omega_0^2$, $h_1 = \omega_0^1\omega_0^2$, $h_2 = \omega_0^1\omega_1^2$, $h_3 = \omega_1^1\omega_0^2$, and $h_4 = \omega_1^1\omega_1^2$.

If $y = x_1$ is asked to follow a sinusoidal reference w_1, then $C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $Q_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $i = 1, \ldots, 4$. Using Theorem 4.2 and the relaxation lemma 2.2 over the double sums, results in following stabilization gains:

$$K_1 = \begin{bmatrix} -2.3578 & -1.6417 \end{bmatrix} \quad K_2 = \begin{bmatrix} -2.3578 & -1.2417 \end{bmatrix}$$
$$K_3 = \begin{bmatrix} -2.3893 & -1.6570 \end{bmatrix}, \quad K_4 = \begin{bmatrix} -2.3893 & -1.2570 \end{bmatrix}$$
$$G_1 = \begin{bmatrix} 10.9143 & 13.3953 & 5.0924 & -1.3855 \end{bmatrix}^T \quad G_2 = \begin{bmatrix} 10.8429 & 13.2989 & 5.0923 & -1.3832 \end{bmatrix}^T,$$
$$G_3 = \begin{bmatrix} 10.7302 & 13.2045 & 5.0082 & -1.3699 \end{bmatrix}^T, \quad G_4 = \begin{bmatrix} 10.6588 & 13.1081 & 4.9720 & -1.3676 \end{bmatrix}^T$$

The proposed bounds $\beta_1 = 1.5$ and $\beta_2 = 5$ are verified in simulation. Moreover, as a result of the computation a minimum $\varepsilon = 0.0083$ is found.

The fuzzy mappings are:

$$\Pi_1 = \Pi_2 = \begin{bmatrix} 0.9937 & -0.0026 \\ -0.3489 & 1.1050 \end{bmatrix} \quad \Pi_3 = \Pi_4 = \begin{bmatrix} 0.9937 & 0.0029 \\ -0.3486 & 1.1031 \end{bmatrix} \quad \Gamma_1 = \begin{bmatrix} -1.264 \\ -1.090 \end{bmatrix}^T$$
$$\Gamma_2 = \begin{bmatrix} -1.4027 \\ -0.3483 \end{bmatrix}^T \quad \Gamma_3 = \begin{bmatrix} -1.2634 \\ -1.0859 \end{bmatrix}^T \quad \text{and} \quad \Gamma_4 = \begin{bmatrix} -1.4016 \\ -0.3444 \end{bmatrix}^T$$

Note that, in this case $\dot{\Pi}_z \neq 0$, and $\varepsilon = 0.0083$ which corresponds to the bound on the steady state tracking error e_{ss}. Figure 4.3 shows the time evolution of the tracking error for several initial conditions. In addition, in Figure 4.4 the bound β_k is verified for the initial conditions $x(0) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^T$ $w(0) = \begin{bmatrix} -0.6 & 0 \end{bmatrix}^T$, and $\xi(0) = \begin{bmatrix} 0.4 & 0.1 & -0.3 & 0 \end{bmatrix}^T$

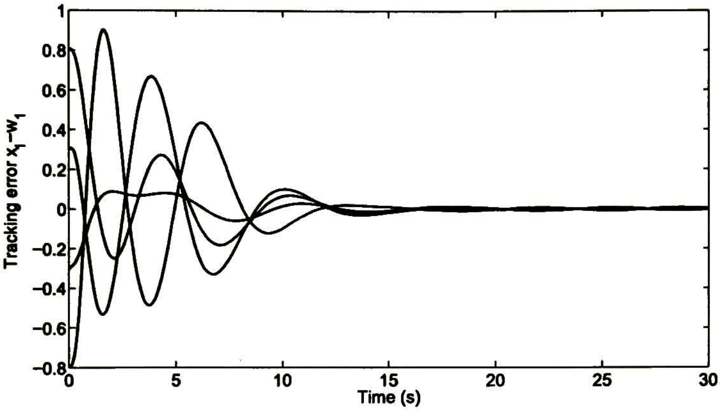

Figure 4.3: *Tracking error $x_1 - w_1$ in Example 4.2.*

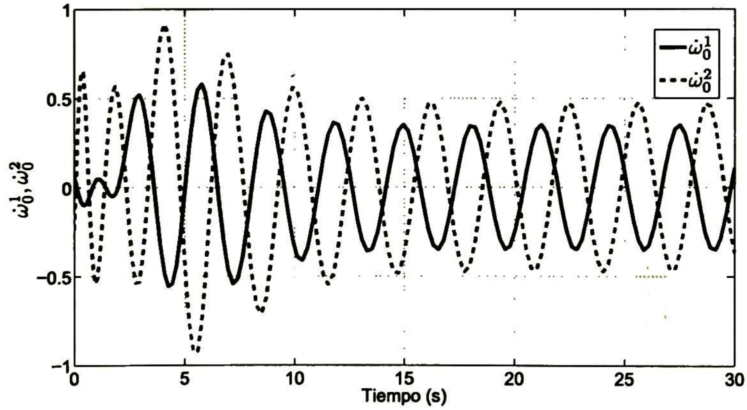

Figure 4.4: *Bounds β_k in Example 4.2.*

Conditions have been presented to solve the fuzzy regulation problem. This is a new approach completely based on LMIs and TS models.

Theorems 4.1 and 4.2 present one disadvantage: the bounds β_k need to be checked a posteriori via simulation, since they are not guaranteed via LMIs. In the next section this issue is faced.

4.2 NORP via TS Fuzzy Mappings: a Full-Information LMI Approach

As was shown in the previous section, bounds β_k need to be guaranteed. This can be achieved since the MFs depends ultimately on the state vector $x(t)$ and/or the exosystem vector $w(t)$. The following developments include a new procedure that ensures the bounds over $\left|\dot{\omega}_0^k\right|$. Here, it is convenient to introduce a new arrangement of the TS model (4.1) as follows:

$$\begin{aligned}
\dot{\bar{x}}(t) &\equiv \begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \sum_{i=1}^{r} h_i\left(z(t)\right)\left(\begin{bmatrix} A_i & E_i \\ 0 & S_i \end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u(t)\right) \\
&= \sum_{i=1}^{r} (z(t))\left(\bar{A}_i\bar{x}(t) + \bar{B}_i u(t)\right) = \bar{A}_z\bar{x}(t) + \bar{B}_z u(t), \\
e(t) &= \sum_{i=1}^{r} (z(t))\left(\begin{bmatrix} C_i & -Q_i \end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}\right) \\
&= \sum_{i=1}^{r} (z(t))\bar{C}_i\bar{x}(t) = \bar{C}_z\bar{x}(t),
\end{aligned} \tag{4.25}$$

with $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{o\times n}$, $E_i \in \mathbb{R}^{n\times q}$, $Q_i \in \mathbb{R}^{o\times q}$, $S_i \in \mathbb{R}^{q\times q}$, $i \in \{1,\ldots,r\}$, being matrices of proper size derived form the sector nonlinearity approach with $r = 2^p \in \mathbb{N}$ the number of rules, $\bar{x}(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$, $\bar{A}_i = \begin{bmatrix} A_i & E_i \\ 0 & S_i \end{bmatrix}$, $\bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}$, and $\bar{C}_i = \begin{bmatrix} C_i & -Q_i \end{bmatrix}$, $i = 1,\ldots,r$ being shorthand matrix notations. This model is in the Takagi-Sugeno form [33].

Using this new shorthand notation, the control law (4.6) is rewritten as

$$\begin{aligned} u(t) &= \alpha(x,w) = \Gamma_z w + K^*(x - \Pi_z w) \\ &= \begin{bmatrix} K^* & \Gamma_z - K^*\Pi_z \end{bmatrix} \bar{x} = \bar{K}_z \bar{x}, \end{aligned} \tag{4.26}$$

with $\bar{K}_z = \begin{bmatrix} K^* & \Gamma_z - K^*\Pi_z \end{bmatrix}$. The stabilizing part of the control law $\alpha(x,0) = K^*x$ is normally designed a priori in regulation theory; it will be therefore assumed to be known in the sequel. Several options for this term was discussed before.

Substituting (4.26) in (4.25), the following closed-loop TS model is obtained:

$$\begin{aligned} \dot{\bar{x}}(t) &= \left(\bar{A}_z + \bar{B}_z \bar{K}_z\right) \bar{x}(t), \\ e(t) &= \bar{C}_z \bar{x}(t). \end{aligned} \tag{4.27}$$

We search to express the FIB equations in its convex form (4.5) so in the next theorem LMI conditions are found to solve them through convex optimization techniques. Former results in [11, 25] will be straightforwardly generalized by the new approach.

Theorem 4.3. *Assume H1 and H2. The NORPSF has a solution if* $\exists L_{ij} = L_{ij}^T > 0$, $\Pi_j \in \mathbb{R}^{n\times q}$, $\Gamma_j \in \mathbb{R}^{m\times q}$, $i,j \in \{1,\dots,r\}$, $\beta_k > 0$, $k \in \{1,\dots,r\}$, *and an arbitrarily small* $\varepsilon > 0$ *such that:*

$$-\varepsilon \prec \begin{bmatrix} A_z\Pi_z + B_z\Gamma_z + E_z - \Pi_z S_z - \sum_{k=1}^{p} (-1)^{d_k^\alpha} \beta_k \left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)}\right) \\ C_z\Pi_z - Q_z \end{bmatrix} \prec \varepsilon, \tag{4.28}$$

$$\begin{bmatrix} \frac{2\beta_k}{(\lambda_\omega^k + \lambda_x)^2} I - L_{zz} & 0 & 0 \\ 0 & \frac{2\beta_k}{(\lambda_\omega^k + \lambda_x)^2} I & (*) \\ 0 & \bar{A}_z + \bar{B}_z\bar{K}_z & L_{zz} \end{bmatrix} \geq 0, \tag{4.29}$$

with $g_1(j,k) = \lfloor (j-1)/2^{p+1-k} \rfloor \times 2^{p+1-k} + 1 + (j-1) \mod 2^{p-k}$, $g_2(j,k) = g_1(j,k) + 2^{p-2}$, $\lfloor \cdot \rfloor$ *the floor function,* λ_x, λ_ω^k *being bounds such that* $|\bar{x}| \leq \lambda_x$, $\left| \frac{\partial \omega_0^k}{\partial \bar{x}} \right| \leq \lambda_\omega^k$ *and* d_k^α *obtained from* $\alpha - 1 = d_p^\alpha + d_{p-1}^\alpha \times 2 + \ldots + d_1^\alpha \times 2^{p-1}$ *The control law is given by (4.26).*

Proof. In [14] has been shown that:

$$\dot{\Pi}_z = \sum_{k=1}^{p} \dot{\omega}_0^k \left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)} \right), \tag{4.30}$$

with $g_1(j,k)$ and $g_2(j,k)$ defined as above. Substituting (4.30) in (4.5) gives

$$\begin{bmatrix} A_z \Pi_z + E_z + B_z \Gamma_z - \Pi_z S_z - \sum_{k=1}^{p} \dot{\omega}_0^k \left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)} \right) \\ C_z \Pi_z - Q_z \end{bmatrix} = 0. \tag{4.31}$$

Theorem 3.1 provides sufficient conditions for matrix equality (4.31) to hold if an arbitrarily small $\varepsilon > 0$ exists such that the following hold:

$$-\varepsilon \prec \begin{bmatrix} A_z \Pi_z + B_z \Gamma_z + E_z - \Pi_z S_z - \sum_{k=1}^{p} \dot{\omega}_0^k \left(\Pi_{g_1(z,k)} - \Pi_{g_2(z,k)} \right) \\ C_z \Pi_z - Q_z \end{bmatrix} \prec \varepsilon. \tag{4.32}$$

If $\left| \dot{\omega}_0^k \right| \leq \beta_k$, $\beta_k > 0$, then (4.28) follows from (4.32) by the property $Y \pm \beta_k \times Z < 0 \Leftrightarrow Y + \dot{\omega}_0^k Z < 0$ since all the sign combinations of the sum involving $\dot{\omega}_0^k$ are considered in (4.28). On the other hand, condition $\left| \dot{\omega}_0^k \right| \leq \beta_k$ needs to be guaranteed. Partially inspired in [16] and taking into account that $\dot{\omega}_0^k = \left(\frac{\partial \omega_0^k}{\partial \bar{x}} \right)^T \dot{\bar{x}} = \left(\frac{\partial \omega_0^k}{\partial \bar{x}} \right)^T \left(\bar{A}_z + \bar{B}_z \bar{K}_z \right) \bar{x}$, this inequality can be rewritten as follows:

$$\begin{aligned} &\left| \left(\frac{\partial \omega_0^k}{\partial \bar{x}} \right)^T \left(\bar{A}_z + \bar{B}_z \bar{K}_z \right) \bar{x} \right| \leq \beta_k \\ \Leftrightarrow &\left| \left(\frac{\partial \omega_0^k}{\partial \bar{x}} \right)^T \left(\bar{A}_z + \bar{B}_z \bar{K}_z \right) \bar{x} \right| + \left| \bar{x}^T \left(\bar{A}_z + \bar{B}_z \bar{K}_z \right)^T \left(\frac{\partial \omega_0^k}{\partial \bar{x}} \right) \right| \leq 2\beta_k. \end{aligned} \tag{4.33}$$

Moreover, since $2\left|u^T v\right| \leq u^T L u + v^T L^{-1} v$ with $L = L^T > 0$, the previous condition is satisfied if

$$\left(\frac{\partial \omega_0^k}{\partial \bar{x}}\right)^T L \left(\frac{\partial \omega_0^k}{\partial \bar{x}}\right) + \bar{x}^T \left(\bar{A}_z + \bar{B}_z \bar{K}_z\right)^T L^{-1} \left(\bar{A}_z + \bar{B}_z \bar{K}_z\right) \bar{x}$$
$$= \begin{bmatrix} \frac{\partial \omega_0^k}{\partial \bar{x}} \\ \bar{x} \end{bmatrix}^T \begin{bmatrix} L & 0 \\ 0 & \left(\bar{A}_z + \bar{B}_z \bar{K}_z\right)^T L^{-1} \left(\bar{A}_z + \bar{B}_z \bar{K}_z\right) \end{bmatrix} \begin{bmatrix} \frac{\partial \omega_0^k}{\partial \bar{x}} \\ \bar{x} \end{bmatrix} \leq 2\beta_k. \quad (4.34)$$

Recall that $|\bar{x}| \leq \lambda_x$ and $\left|\frac{\partial \omega_0^k}{\partial \bar{x}}\right| \leq \lambda_\omega^k$. It is clear that $\left\|\begin{bmatrix} \frac{\partial \omega_0^k}{\partial \bar{x}} & \bar{x} \end{bmatrix}\right\|^2 \leq \left(\lambda_\omega^k + \lambda_x\right)^2$, so (4.34) is implied by

$$\begin{bmatrix} \frac{\partial \omega_0^k}{\partial \bar{x}} \\ \bar{x} \end{bmatrix}^T \begin{bmatrix} L & 0 \\ 0 & \left(\bar{A}_z + \bar{B}_z \bar{K}_z\right)^T L^{-1} \left(\bar{A}_z + \bar{B}_z \bar{K}_z\right) \end{bmatrix} \begin{bmatrix} \frac{\partial \omega_0^k}{\partial \bar{x}} \\ \bar{x} \end{bmatrix}$$
$$\leq \begin{bmatrix} \frac{\partial \omega_0^k}{\partial \bar{x}} \\ \bar{x} \end{bmatrix}^T \frac{2\beta_k}{(\lambda_\omega^k + \lambda_x)^2} \begin{bmatrix} \frac{\partial \omega_0^k}{\partial \bar{x}} \\ \bar{x} \end{bmatrix},$$

which is implied by the following inequality:

$$\begin{bmatrix} \frac{2\beta_k}{(\lambda_\omega^k + \lambda_x)^2} I - L_{zz} & 0 \\ 0 & \frac{2\beta_k}{(\lambda_\omega^k + \lambda_x)^2} I - L_{zz} - \left(\bar{A}_z + \bar{B}_z \bar{K}_z\right)^T L^{-1} \left(\bar{A}_z + \bar{B}_z \bar{K}_z\right) \end{bmatrix} \geq 0. \quad (4.35)$$

Using Schur complement on the previous expression as well as taking $L = L_{zz} > 0$ lead the desired result in (4.29), thus concluding the proof. □

Remark 4.5. In contrast with approaches bounding the time-derivatives of the MFs, the theorem above relies only on a priori known bounds, since λ_x and λ_ω^k depend entirely on the state $\bar{x}$ whose bounds are known a priori from the compact set Δ on which the TS model (4.25) has been defined.

Remark 4.6. Via the relaxation lemma in (2.1) and (2.2), and a proper definition of Υ_{zz}, inequalities (4.28) and (4.29) turn into LMIs by removing the double sums implied by the subscript notation. Other relaxations with a different degree of conservatism can be equally used [21, 28].

Remark 4.7. As mentioned before, the stabilizing part of the control law $\alpha(x, 0) = K^* x$ is normally designed beforehand. The simplest choice is the state feedback through a constant gain, i.e., $K^* = K$ to render stable the linear approximation of the pair $(f(x), g(x))$ in $x = 0$. Since the first attempts to solve the NORPSF via TS models of the nonlinear mappings, authors were inclined to use fuzzy control laws along with these solutions: a) Parallel Distributed Compensation (PDC) [11, 32], where $K^* = K_z$ and a common quadratic Lyapunov function $V = x^T P x$ are used; b) Nonquadratic fuzzy control [25, 30] where $K^* = K_z P_z^{-1}$ and a fuzzy Lyapunov function $V = x^T P_z^{-1} x$ should be employed. Theorem 4.3 is directly compatible with any of these approaches.

4.2.1 Examples

The following examples perform trajectory tracking by solving the NORPSF using Theorem 4.3. They employ single state feedback gains for the stabilizing part of (4.26), i.e., $K^* = K$. The first one naturally produces the solution $\Pi_j = \Pi$, $\forall j \in \{1, \ldots, r\}$, which corresponds to a case that has been previously reported [25]; the second one holds for different Π_j, Γ_j, thus fully exploiting the nonlinear nature of mappings $\pi(w(t))$ and $\gamma(w(t))$.

Example 4.3. Consider the following nonlinear model corresponding to a simple pendulum

[20]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\frac{\sin(x_1)}{x_1} & -\frac{k}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{4.36}$$

with $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ being the state vector, x_1 the angle between the upright position and the pendulum bar, x_2 the angular velocity, and $g = 9.82$ m/s^2, $l = 0.8$ m, $k = 0.1$, $m = 0.1$ kg being the model parameters. Consider as well the following Van der Pol oscillator as the exosystem providing references to (4.36) with $w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ as its state:

$$\dot{w}(t) = \begin{bmatrix} 0 & 1 \\ -1 & (1 - w_1^2) \end{bmatrix} w(t). \tag{4.37}$$

By sector nonlinearity approach with premise variables $z_1(t) = \sin(x_1)/x_1$ and $z_2(t) = w_1^2$, the following exact representation of (4.36) and (4.37) in the TS form is obtained for the region $\Delta = \{|x_i| \le 4, |w_i| \le 2\} \supset \mathbf{0}$:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{4} h_i(z(t)) \left(A_i x(t) + B_i u(t)\right), \\ \dot{w}(t) &= \sum_{i=1}^{4} h_i(z(t)) S_i w(t), \end{aligned} \tag{4.38}$$

with $A_1 = A_2 \begin{bmatrix} 0 & 1 \\ -g/l & -k/m \end{bmatrix}$, $A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ 0.1892g/l & -k/m \end{bmatrix}$, $B_i = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $i = 1, \ldots, 4$, $S_1 = S_3 = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$, $S_2 = S_4 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, $\omega_0^1 = \dfrac{\sin(x_1)/x_1 + 0.1892}{1.18292}$, $\omega_0^2 = \dfrac{w_1^2}{4}$, $\omega_1^1 = 1 - \omega_0^1$, $\omega_1^2 = 1 - \omega_0^2$, $h_1 = \omega_0^1\omega_0^2$, $h_2 = \omega_0^1\omega_1^2$, $h_3 = \omega_1^1\omega_0^2$, and $h_4 = \omega_1^1\omega_1^2$.

Let x_1 be the output of interest to be driven to follow w_1, i.e., $C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $Q_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $i = 1, \ldots, 4$. Note that there is no direct transfer between w and x, i.e., $E_i = 0$.

In order to use Theorem 4.3, an ordinary state feedback is first designed to stabilize (4.38) under $w(t) = 0$: using a quadratic Lyapunov function $V = x^T P x$ the gain $K^* = K = \begin{bmatrix} -163.514 & -9.404 \end{bmatrix}$ has been obtained. Bounds $\lambda_x = 6.3246$, $\lambda_\omega^1 = 0.3688$, and $\lambda_\omega^2 = 1$ are deduced from Δ. Bounds $\beta_k = 1 \times 10^6 : \left|\dot{\omega}_0^k\right| \leq \beta_k$, $k = 1, 2$ are fixed as large as possible.

Applying the relaxation lemma (2.2) to conditions in Theorem 4.3 as pointed out by Remark 4.6, the following gains have been obtained from the corresponding LMIs:

$$\Pi_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad j \in \{1, \ldots, r\}, \quad \Gamma_1 = \begin{bmatrix} 11.3 \\ -2 \end{bmatrix}^T$$

$$\Gamma_2 = \begin{bmatrix} 11.3 \\ 2 \end{bmatrix}^T \quad \Gamma_3 = \begin{bmatrix} -3.32 \\ -2 \end{bmatrix}^T \quad \text{and} \quad \Gamma_4 = \begin{bmatrix} -3.32 \\ 2 \end{bmatrix}^T$$

Note that since matrices Π_j are all the same, $\Pi_z = 0$, which corresponds to simpler situations [25]. In Figure 4.5 a simulation has been run from initial conditions $x(0) = \begin{bmatrix} 3 & 0 \end{bmatrix}^T$ and $w(0) = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$ to show the trajectory tracking due to the gains calculated via Theorem 4.3. Figure 4.6 presents the tracking error $x_1 - w_1$.

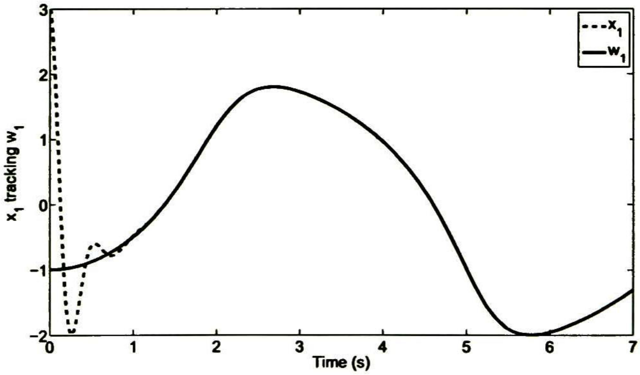

Figure 4.5: *Trajectory tracking of reference signal w_1 by x_1 in Example 4.3.*

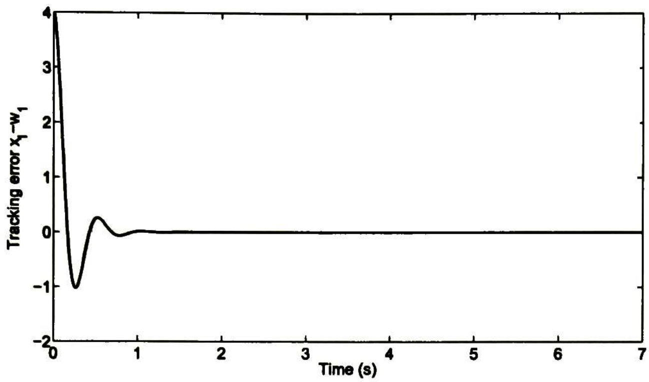

Figure 4.6: *Tracking error $x_1 - w_1$ in Example 4.3.*

Example 4.4. Consider the following nonlinear model:

$$\dot{x}(t) = \begin{bmatrix} 0.2 + 0.0137x_1^2 & 1 \\ 0.4 & 0.3x_2^2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0.1x_2^2 \end{bmatrix} w(t), \tag{4.39}$$

with $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ being the state vector, and $w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ the exosystem vector corresponding to a sinusoidal reference generated by:

$$\dot{w}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w(t). \tag{4.40}$$

Choosing the premise variables $z_1(t) = x_1^2$ and $z_2(t) = x_2^2$, the following TS model exactly representing (4.4) and (4.40) in $\Delta = \{|x_i| \leq 1, |w_i| \leq 1\} \supset \mathbf{0}$ is obtained:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{4} h_i(z(t)) \left(A_i x(t) + B_i u(t) + E_i w(t)\right), \\ \dot{w}(t) &= \sum_{i=1}^{4} h_i(z(t)) S_i w(t), \end{aligned} \tag{4.41}$$

with $A_1 = \begin{bmatrix} 0.2137 & 1 \\ 0.4 & 0.3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.2137 & 1 \\ 0.4 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0.2 & 1 \\ 0.4 & 0.3 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0.2 & 1 \\ 0.4 & 0 \end{bmatrix}$, $B_i = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $S_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $i = 1, \dots, 4$, $E_1 = E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}$, $E_2 = E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\omega_0^1 = x_1^2$, $\omega_0^2 = x_2^2$, $\omega_1^1 = 1 - \omega_0^1$, $\omega_1^2 = 1 - \omega_0^2$, $h_1 = \omega_0^1\omega_0^2$, $h_2 = \omega_0^1\omega_1^2$, $h_3 = \omega_1^1\omega_0^2$, and $h_4 = \omega_1^1\omega_1^2$.

Consider that the output signal $y = x_1$ is asked to follow w_1, i.e., $C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $Q_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $i = 1, \dots, 4$. Theorem 4.3 comes at hand provided an ordinary state feedback is first designed to stabilize (4.4) under $w(t) = 0$. As in the previous example, a quadratic approach is used to calculate the gain $K^* = K = \begin{bmatrix} -3.9926 & -4.5594 \end{bmatrix}$ for the stabilizing part of (4.26). Bounds $\lambda_x = \lambda_\omega^k = 2$, $k = 1, 2$ are inferred from Δ, while bounds $\beta_1 = 60$, $\beta_2 = 220$: $\left|\dot{\omega}_0^k\right| \leq \beta_k$, $k = 1, 2$ are fixed as large as possible. The LMIs resulting from applying the Relaxation Lemma (2.2) to conditions in Theorem 4.3 yield the following gains when the minimum value of ε has been found to be $\varepsilon = 0.0067$:

$$\Pi_1 = \Pi_2 = \begin{bmatrix} 0.9939 & -0.0001 \\ -0.2056 & 0.9940 \end{bmatrix} \quad \Pi_3 = \Pi_4 = \begin{bmatrix} 0.9939 & 0.0001 \\ -0.2055 & 0.9940 \end{bmatrix} \quad \Gamma_1 = \begin{bmatrix} -1.3299 \\ -0.6038 \end{bmatrix}^T$$

$$\Gamma_2 = \begin{bmatrix} -1.3915 \\ -0.2059 \end{bmatrix}^T \quad \Gamma_3 = \begin{bmatrix} -1.3299 \\ -0.6037 \end{bmatrix}^T \quad \text{and} \quad \Gamma_4 = \begin{bmatrix} -1.3916 \\ -0.2059 \end{bmatrix}^T$$

Note that in this case $\Pi_z \neq 0$, while finding the minimum value of ε; moreover, since this is a bound on the steady state tracking error as pointed out in Remark 4.3, the effects on tracking can be seen in Figures 4.7 and 4.8 for several initial conditions of the error: in all cases the bound $e_{ss} \leq \varepsilon = 0.0067$ holds.

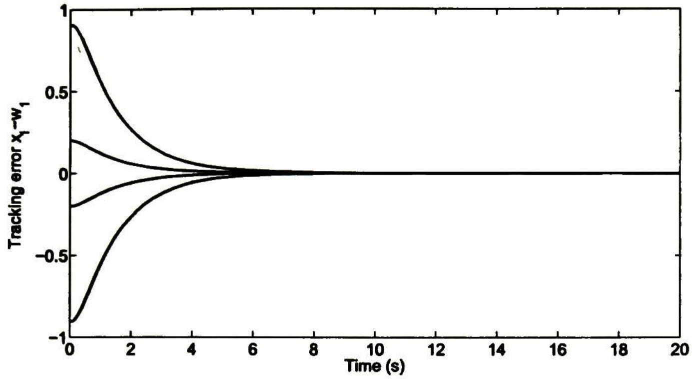

Figure 4.7: *Tracking error $x_1 - w_1$ for several initial conditions in Example 4.4.*

Figure 4.8 shows a close view on the steady state tracking error. The aim of this image is to show that the steady-state error bound is respected. In dashed lines the $\varepsilon < 0.0067$ is plotted, in continuous lines different tracking errors for several initial conditions are plotted.

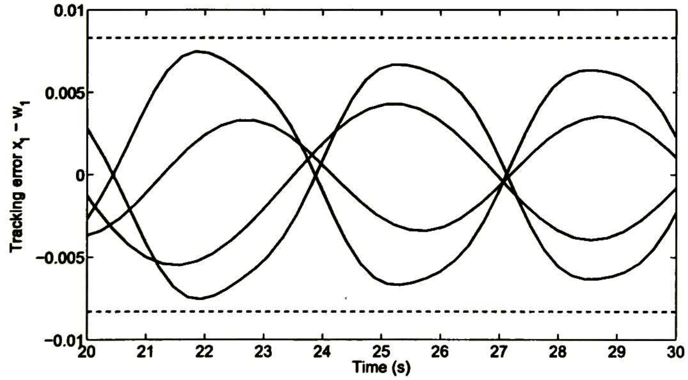

Figure 4.8: *Zoom-in on tracking error and the steady state error bound e_{ss} in Example 4.4.*

Finally, the next example is intended to show the importance of the time-derivatives of

MFs due to $\dot{\Pi}_z$. This is achieved by considering $\dot{\Pi}_z = 0$ as in many previous works. Then, the expression (4.28) provided in Theorem 4.3 yields:

$$-\varepsilon \prec \begin{bmatrix} A_z\Pi_z + B_z\Gamma_z + E_z - \Pi_z S_z \\ C_z\Pi_z - Q_z \end{bmatrix} \prec \varepsilon, \tag{4.42}$$

No time-derivatives of the MFs appear, then (4.29) is useless because there is no $\dot{\omega}_0^k$ to be bounded.

Example 4.5. Consider the following TS model valid in the compact region $\Delta = \{|x_i| \leq 1\} \supset \mathbf{0}$:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{2} h_i(z(t))\left(A_i x(t) + B_i u(t)\right), \\ \dot{w}(t) &= \sum_{i=1}^{2} h_i(z(t)) S_i w(t), \end{aligned} \tag{4.43}$$

with $A_1 = \begin{bmatrix} 0.02 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_i = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$, $S_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $i = 1, 2$, $\omega_0^1 = x_1^2$, $\omega_1^1 = 1 - \omega_0^1$, $h_1 = \omega_0^1$, and $h_2 = \omega_1^1$.

Applying conditions derived in Theorem 4.3 and a Relaxation Lemma 2.2, with bound $\lambda_x = 2$ and $\lambda_\omega^1 = 2$ calculated from Δ, the bound $\beta_1 = 20$ is chosen such that $|\dot{\omega}_0^1| \leq \beta_1$; the following gains result:

$K^* = K = \begin{bmatrix} -3.1178 & -2.4130 \end{bmatrix}$, $\Pi_1 = \begin{bmatrix} 0.9918 & -0.0002 \\ -0.0102 & 0.9919 \end{bmatrix}$, $\Pi_2 = \begin{bmatrix} 0.9918 & 0.0002 \\ -0.0097 & 0.9919 \end{bmatrix}$,

$\Gamma_1 = \begin{bmatrix} -1.9837 \\ -0.0247 \end{bmatrix}^T$ and $\Gamma_2 = \begin{bmatrix} -1.9838 \\ -0.0232 \end{bmatrix}^T$ The bound for the steady-state error is $e_{ss} \leq \varepsilon = 0.0095$.

Tracking error is shown in Figure 4.9. In order to see if the bound over the steady-state error is accomplished Figure 4.10 gives a zoom-in on the steady-state error. In dashed lines the bound ε is plotted and in continuous lines the steady-state error is illustrated.

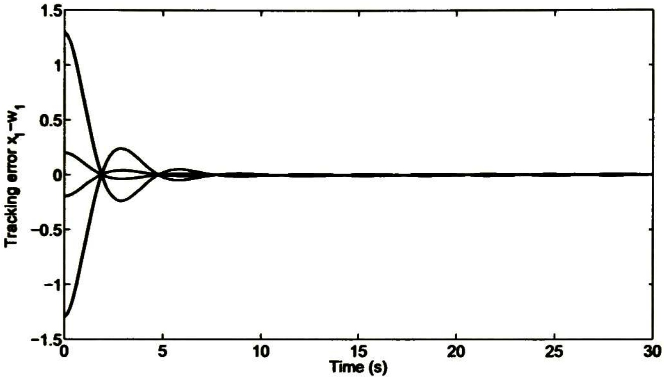

Figure 4.9: *Tracking error $x_1 - w_1$ for several initial conditions in Example 4.5.*

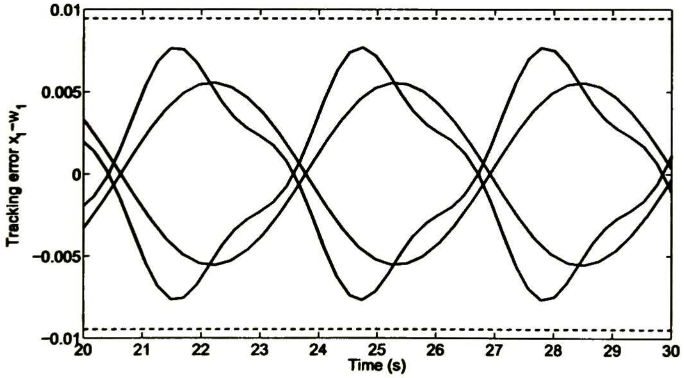

Figure 4.10: *Zoom-in on tracking error and the steady state error bound e_{ss} in Example 4.5.*

On the other hand, if the term $\dot{\Pi}_z$ is removed, then the condition (4.42) provides the following gains:

$$K^* = K = \begin{bmatrix} -3.1178 & -2.4130 \end{bmatrix}, \Pi_1 = \begin{bmatrix} 1 & 0 \\ -0.02 & 1 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} -2 \\ -0.04 \end{bmatrix}^T,$$

and $\Gamma_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}^T$ The bound for the steady-state error is $e_{ss} \leq \varepsilon = 5.4778 \times 10^{-11}$

Results are illustrated by figures 4.11 and 4.12.

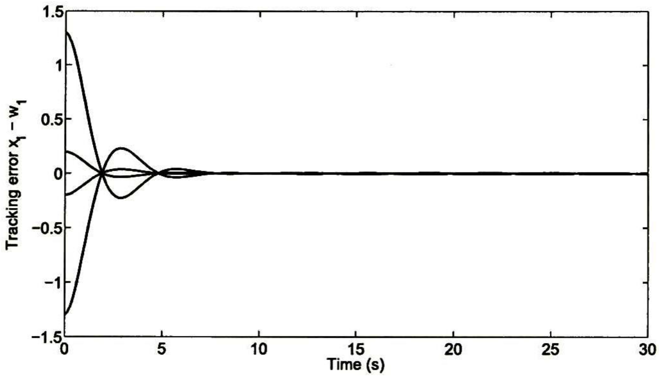

Figure 4.11: *Tracking error $x_1 - w_1$ for several initial conditions in Example 4.5.*

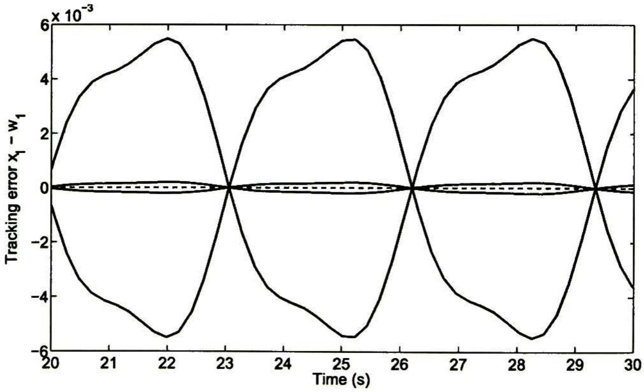

Figure 4.12: *Zoom-in on tracking error and the steady state error bound e_{ss} in Example 4.5.*

A comparison between Figure 4.10 and Figure 4.12 yields that: when Π_z is despised, the bounds over the steady-state error are not respected for all the initial conditions, i.e., the tracking is guaranteed in a small neighborhood. This shows the importance of the time-derivative of the MFs: they are included, the bound of the steady-state error is respected.

Chapter 5

Conclusions

This work has presented a novel approach to solve two classical output regulation problems under the LMI framework: state feedback (full information) and error feedback. The advantages of expressing these results in such a way have been to express and solve output regulation conditions via convex optimization techniques which are easily implemented in commercially available software, thus providing a numerically better approach for systematize output regulation of linear and nonlinear plants.

Concerning the implementation of the linear case a simple solution has been developed since there is no time-derivative of the membership functions. On the other hand, for nonlinear systems the Takagi-Sugeno models are introduced in order to solve the NORP in terms of LMIs; this representation is extended to the nonlinear mappings assuming the same TS structure as the plant and the exosystem. This procedure obliges to consider the time-derivative of the MFs, which is the principal problem solved by this research using recent novelties in stabilization of TS models.

The solutions thus offered have generalized previous results on the subject since they handle the time-derivatives of the nonlinear mappings in the FIB equations, which are represented via TS models. In this way, the TS-LMI framework provides a full-information systematic methodology for solve the NORP. Moreover, LMI conditions thus found can be efficiently solved via convex optimization techniques.

Several examples were presented in order to illustrate the effectiveness of the approaches and to stress the advantages as well as the drawbacks of considering the time-derivatives of the MFs into the regulator design.

Future work should concentrate in relaxing the bound-dependency of the current approach as well as to go beyond the modeling limitations of the TS current representation of nonlinear mappings in the FIB equations. The latter is quite important and the main drawback of the present approach, since nonlinear mappings may not share the same nonlinearities as the plant and the exosystem.

Appendix A

Regulation Theory

This section provides a brief review of the basis of the well-known output regulation problem. Essentially, the regulation problem consists in finding a state or error feedback control law such that the equilibrium point of the closed-loop system in absence of external signals is exponentially stable; and the tracking error goes to zero when the system is under the influence of the exosystem [17, 18].

A.1 Linear Output Regulation

This section provides conditions to solve the output regulation for the linear case when state or error feedback applies.

Consider the linear system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) \\ \dot{w}(t) &= Sw(t) \\ e(t) &= Cx(t) + Qw(t), \end{aligned} \tag{A.1}$$

where $x(t) \in X \subset \mathbb{R}^{n\times 1}$ is the state vector to be driven, $w(t) \in W \subset \mathbb{R}^{q\times 1}$ is the exosystem that provides the references/perturbations, $e(t) \in \mathbb{R}^{o\times 1}$ is the tracking error and the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{o\times n}$, $E \in \mathbb{R}^{n\times q}$, $Q \in \mathbb{R}^{o\times q}$, $S \in \mathbb{R}^{q\times q}$.

Linear Output Regulation Problem via State Feedback (LORPSF). Consider the matrices $\{A, B, C, E, Q, S\}$ given in the linear system (A.1), then the LORPSF consists in finding gains $K \in \mathbb{R}^{m\times n}$, $L \in \mathbb{R}^{m\times q}$ such that with

$$u(t) = Kx(t) + Lw(t). \tag{A.2}$$

- $A + BK$ is Hurwitz.
- $\forall\,(x(0), w(0)) = (x^0, w^0) \to \lim_{t\to\infty} e(t) = 0.$

The following assumptions are well-known conditions from the regulation theory to solve the LORPSF [13, 27]:

A1. $Re\{\sigma(S)\} \geq 0.$

A2. (A, B) is stabilizable.

Then, if A1 and A2 are satisfied, the LORPSF has a solution if and only if $\exists \Pi \in \mathbb{R}^{n\times q}$, $\Gamma \in \mathbb{R}^{m\times q}$, and $K \in \mathbb{R}^{m\times n}$ such that:

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + E \\ 0 &= C\Pi + Q, \end{aligned} \tag{A.3}$$

is satisfied. The control law is given by (A.2) with K stabilizing (A, B) by any method, and $L = \Gamma - K\Pi$.

The equations (A.3) are called the Francis equations.

A common situation is when only the components of the error signal can be measured. In this case the controller needs to be provided with error feedback.

Linear Output Regulation Problem via Error Feedback (LORPEF). Consider the matrices $\{A, B, C, E, Q, S\}$ given in the linear system (A.1), then the LORPEF consists in finding $F \in \mathbb{R}^{(n+q)\times(n+q)}$, $G \in \mathbb{R}^{(n+q)\times o}$, $H \in \mathbb{R}^{m\times(n+q)}$ such that with

$$\begin{aligned} \dot{\xi}(t) &= F\xi(t) - Ge(t) \\ u(t) &= Kx(t) + Lw(t), \end{aligned} \tag{A.4}$$

- $\begin{bmatrix} A & BH \\ GC & F \end{bmatrix}$ is Hurwitz.
- $\forall\,(x(0), \xi(0), w(0)) = (x^0, \xi^0, w^0) \rightarrow \lim\limits_{t\to\infty} e(t) = 0.$

In order to solve the LORPEF consider A1 and A2 as assumptions plus the following:

A3. $\left(\begin{bmatrix} A & E \\ 0 & S \end{bmatrix}, \begin{bmatrix} C & Q \end{bmatrix}\right)$ is detectable.

Then, if the assumptions A1, A2, and A3 are fulfilled the LORPEF has a solution if and only if $\exists \Pi \in \mathbb{R}^{n\times q}$, $\Gamma \in \mathbb{R}^{m\times q}$, and $K \in \mathbb{R}^{m\times n}$ such that:

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + E \\ 0 &= C\Pi + Q, \end{aligned} \tag{A.5}$$

hold. The control law is given by (A.4) with $G = \begin{bmatrix} G_0 \\ G_1 \end{bmatrix}$ being the observer gain of the pair

$\left(\begin{bmatrix} A & E \\ 0 & S \end{bmatrix}, \begin{bmatrix} C & Q \end{bmatrix}\right)$, from which $F = \begin{bmatrix} A - G_0C + BK & E - G_0Q + B(\Gamma - K\Pi) \\ -G_1C & S - G_1Q \end{bmatrix}$, and $H = \begin{bmatrix} K & \Gamma - K\Pi \end{bmatrix}$.

The importance of the results in the linear case is because nonlinear plants can be handled by linearization in an operation point resulting [27].

A.2 Nonlinear Output Regulation

In this section the NORP is going to be presented for two well-known as the corresponding nonlinear versions of the previous section: the state feedback and the error feedback cases.

Consider the nonlinear system:

$$\begin{aligned} \dot{x}(t) &= f(x) + g(x)u + p(x)w \\ \dot{w}(t) &= s(w) \\ e(t) &= h(x) + q(w), \end{aligned} \tag{A.6}$$

where $x(t) \in X \subset \mathbb{R}^{n\times 1}$ is the state vector to be driven, $u(t) \in \mathbb{R}^{m\times 1}$ is the input vector, $w(t) \in W \subset \mathbb{R}^{q\times 1}$ is the exosystem that provides the references/perturbations, and $e(t) \in \mathbb{R}^{o\times 1}$ the tracking error.

Nonlinear Output Regulation Problem via State Feedback (NORPSF). Consider the nonlinear system (A.6), then the NORPSF consists in finding a control law

$$u = \alpha(x, w), \tag{A.7}$$

such that:

- $\dot{x}(t) = f(x) + g(x)\alpha(x,0)$ has an exponentially stable equilibrium point in $x = 0$ with $\alpha(x,0) = Kx$, $K \in \mathbb{R}^{n\times m}$.

- $\exists U \subset X \times W \supset (0,0) : \forall\,(x(0), w(0)) \in U \Rightarrow \lim_{t\to\infty} e(t) = 0$.

The following assumptions are considered [19]:

H1. $w = 0$ is a stable equilibrium point of $\dot{w}(t) = s(w)$ and $\exists \tilde{W} \subset W \supset 0 : \forall w(0) \in \tilde{W}$ is Poisson-stable.

H2. $(f(x), g(x))$ has a stabilizable linear approximation in $x = 0$ via gain K.

If H1 and H2 are satisfied, the NORPSF has a solution if and only if $\exists \pi(w), \gamma(w) : \pi(0) = 0, \gamma(0) = 0$ as mappings in $W^0 \subset W \supset 0$ such that [18]:

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f\left(\pi(w)\right) + g\left(\pi(w)\right)\gamma(w) + p\left(\pi(w)\right) w \\ 0 &= h\left(\pi(w)\right) + q(w), \end{aligned} \tag{A.8}$$

is satisfied. The control law is given by (A.7) with $\alpha(x,w) = \gamma(w) + K\left(x - \pi(w)\right)$.

Nonlinear Output Regulation Problem via Error Feedback (NORPEF). Consider the nonlinear system (A.6) then the NORPEF consists in finding a control law

$$\begin{aligned} \dot{\xi}(t) &= \eta(\xi, e) \\ u(t) &= \theta(\xi), \end{aligned} \tag{A.9}$$

where $\xi(t) \in \Xi \subset \mathbb{R}^{(n+q)\times 1}$, $\xi(t) \doteq \begin{bmatrix} \xi_0 & \xi_1 \end{bmatrix}^T$ such that:

- $\dot{x}(t) = f(x) + g(x)\theta(\xi)$ and $\dot{\xi}(t) = \eta(\xi, h(x))$ has an exponentially stable equilibrium point in $(x,\xi) = (0,0)$.

- $\exists U \subset X \times \Xi \times W \supset (0,0,0) : \forall\, (x(0), \xi(0), w(0)) \in U$ such that the closed-loop system:

$$\begin{aligned} \dot{x}(t) &= f(x) + g(x)\theta(\xi) + p(x)w \\ \dot{\xi}(t) &= \eta(\xi, h(x) + q(w)) \\ \dot{w}(t) &= s(w), \end{aligned} \tag{A.10}$$

satisfies $\lim_{t\to\infty} e(t) = 0$.

In this case, the previous assumptions H1 and H2 are considered, but a third one needs to be taken into account:

H3. $\left(\begin{bmatrix} f(x) + p(x)w \\ s(x) \end{bmatrix}, h(x) + q(w) \right)$ is a detectable pair at the linear approximation in $(x, w) = (0, 0)$.

If H1, H2, and H3 are satisfied, the NORPEF has a solution if and only if $\exists \pi(w), \gamma(w) :$ $\pi(0) = 0, \gamma(0) = 0$ as mappings in $W^0 \subset W \supset 0$ such that [18]:

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f\left(\pi(w)\right) + g\left(\pi(w)\right)\gamma(w) + p\left(\pi(w)\right)w \\ 0 &= h\left(\pi(w)\right) + q(w), \end{aligned} \tag{A.11}$$

is satisfied. The control law is given by (A.9) with $\theta(\xi) = \gamma(\xi_1) + K(\xi_0 - \pi(\xi_1))$.

Appendix B

Publications

Published articles for proceedings in international conferences:

- M. Bernal. R. Márquez, V. Estrada-Manzo, B. Castillo-Toledo, *An Element-Wise Linear Matrix Inequality Approach for Output Regulation Problems*, World Automotion Congress 2012, Pto. Vallarta, México. In press.

- M. Bernal. R. Márquez, V. Estrada-Manzo, B. Castillo-Toledo, *Nonlinear Output Regulation via Takagi-Sugeno Fuzzy Mappings: a Full-Information LMI Approach*, IEEE World Congress on Computational Intelligence 2012, Brisbane, Australia. In press.

Accepted articles for journals:

- V. Estrada-Manzo, B. Castillo-Toledo, M. Bernal, *Regulación No Lineal de la Salida Basada en Modelos Takagi-Sugeno*, RIEE&C Journal. August 2012. In press.

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Aplicación de técnicas LMI para el problema de regulación de la salida para sistemas no lineales / LMI-based Techniques for Output Regulation Problem for Nonlinear Systems

del (la) C.

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