Centro de Investigación y de Estudios Avanzados
del Instituto Politécnico Nacional
Unidad Guadalajara

Formación y seguimiento de trayectoria de sistemas multi-agentes

Tesis que presenta:
Alejandro Cervantes Herrera

para obtener el grado de:
Doctor en Ciencias

en la especialidad de:
Ingeniería Eléctrica

Director de Tesis
Dr. José Javier Ruíz León

CINVESTAV del IPN Unidad Guadalajara, Guadalajara, Jalisco, Diciembre de 2012.

Centro de Investigación y de Estudios Avanzados

del Instituto Politécnico Nacional

Unidad Guadalajara

Formation and trajectory tracking of multi-agent systems

A thesis presented by:
Alejandro Cervantes Herrera

to obtain the degree of:
Doctor in Science

in the subject of:
Electrical Engineering

Thesis Advisors:
Dr. José Javier Ruiz León

CINVESTAV del IPN Unidad Guadalajara, Guadalajara, Jalisco, December, 2012

Formación y seguimiento de trayectoria de sistemas multi-agentes

Tesis de Doctorado en Ciencias
Ingeniería Eléctrica

Por:
Alejandro Cervantes Herrera
Maestro en Ciencias con Especialidad en Ingeniería Eléctrica
Centro de Investigación y de Estudios Avanzados del I. P. N.
2005-2007

Becario de Conacyt, expediente no. 172539

Director de Tesis
Dr. José Javier Ruíz León

CINVESTAV del IPN Unidad Guadalajara, Diciembre de 2012.

Formation and trajectory tracking of multi-agent systems

Doctor of Science Thesis
In Electrical Engineering

By:

Alejandro Cervantes Herrera

Master of Science Thesis In Electrical Engineering
Centro de Investigación y de Estudios Avanzados del I. P. N.
2005-2007

Scholarship granted by CONACYT, No. 172539

Thesis Advisors:

Dr. José Javier Ruiz León

CINVESTAV del IPN Unidad Guadalajara, December, 2012.

Resumen

Los seres vivos en la naturaleza se reúnen en comunidades de individuos con intereses o características en común. Tales comunidades usualmente reaccionan como entidades individuales. El comportamiento de un individuo de la comunidad afecta el comportamiento de algunos de los miembros mas cercanos a él y a través de ellos puede llegar a afectar el comportamiento de todo el grupo.

El estudio de sistemas multiagentes puede ser concebido como un intento de replicar el comportamiento de las comunidades en la naturaleza. En particular, el consenso se ocupa del acuerdo entre los diferentes agentes, en el cual, el estado de los agentes alcanza un valor de estado estable no especificado y que depende del algoritmo de consenso y de la comunicación entre los agentes.

Cuando los problemas de consenso y seguimiento de trayectoria se abordan de manera conjunta, las perturbaciones en la trayectoria seguida por un agente afectarán las trayectorias que sus vecinos seguirán, de manera similar como se observa en bancos de peces y rebaños.

En este trabajo, se presenta el diseño de leyes de control distribuidas para el consenso y regulación de la salida de sistemas multiagentes. Se considera que solo un subconjunto de los agentes tiene acceso a la referencia a seguir, que es tomada como la salida de un agente virtual o exosistema. Se considera de igual forma que ninguno de los agentes tiene acceso al estado del exosistema y que todos los agentes tienen el conocimiento de las salidas de sus vecinos.

Se presentan tres casos diferentes. En el primero, se resolvió el problema del consenso y regulación de la salida para sistemas multiagentes compuestos de sistemas lineales multivariables con diferentes dimensiones, manteniendo una formación del sistema multiagente y considerando una red de comunicación fija entre sus miembros.

En el segundo y tercer caso, se resolvió el problema de consenso y regulación de la salida para sistemas multiagentes compuestos por una clase de sistemas lineales conmutados (SLS) de diferente dimensión. En el segundo caso se da una solución similar al primero, pero en el tercer caso, se resuelve el problema de consenso y regulación de la salida para una clase de sistemas SLS multiagentes considerando redes de comunicación cambiantes en el tiempo, pero la formación del sistemas multiagente no es considerada.

Abstract

Living creatures in nature gather together in groups of individuals with common characteristics or interests. Such groups of individuals often react as single entities. The behavior of one individual of the group affects the behavior of some of its members, and through them it may affect the behavior of the whole group.

The study of multi-agent systems can be thought of as an attempt to replicate the behavior of communities of individuals in nature. In particular, consensus deals with the problem of the agreement of agents, in which the state of the agents reaches a steady state value which is unspecified and is dependent on the communication topology and on the employed consensus algorithm.

When consensus and trajectory tracking are addressed together, the disturbances on the trajectory followed by one agent will affect the trajectories followed by its neighbors, similar to what happens in shoals and flocks.

In this work, the design of distributed control laws for the output consensus and output regulation of multi-agents systems is presented. It is considered that only a subset of the agents has access to the reference, which is taken as the output of a virtual agent (exosystem). It is also considered that none of the agents has access to the exosystem state and that every agent has access to the output of all its neighbors.

Three different approaches are presented. In the first one, the problem of output consensus and output regulation is solved for multi-agent systems composed by multi-input multi-output linear systems of different dimensions maintaining the multi-agent system in formation under a fixed communication topology.

In the second and third case, the problem of output consensus and output regulation is solved for a class of switched linear multi-agent systems of different dimension. The second approach is similar to the first one, but the third addresses the problem of output consensus and output regulation of a class of switched linear multi-agent systems under switching communication topologies although multi-agent formation is not considered.

The proposed control laws guarantee multi-agent system output consensus and output regulation by means of the stability of the observer-regulation error dynamics.

Agradecimientos

Quiero dedicar esta tesis:

A mi familia que siempre ha estado conmigo y sin cuyo apoyo jamás habría podido lograr esta y muchas otras metas.

También quiero agradecer muy especialmente:

A mi asesor el Doctor José Javier Ruiz León quien me dio su apoyo y guía para lograr este objetivo.

Al Doctor Antonio Ramírez Treviño quien colaboró de cerca con este trabajo y me ayudó con sus conocimientos y consejos.

A todos mis profesores que con su empeño han ayudado a formarme como profesional.

A Lorena del Carmen Ramírez Sandoval por su compañía y cariño todo este tiempo.

A mis tías y tíos y a la congregación de las hijas mínimas de María inmaculada en especial a la madre la madre Eloisa quien me ha dado su compresión y paciencia.

A mis compañeros que me brindaron su apoyo y amistad e hicieron mas ligeros los minutos de cansancio y presión.

A todos mis amigos que de alguna forma u otra siempre aportan algo nuevo y divertido a mi vida.

Y al CONACYT por otorgarme los recursos necesarios para realizar mis estudios.

Contents

Chapter 1

Introduction

A multi-agent system can be described as a group of dynamical systems which interact with each other by means of the state or the outputs of their neighbors, in order to achieve a common goal. Multi-agent systems execute common tasks with distributed control actions and local information. In recent years, multi-agent systems have attracted the attention of the scientific community for their capacity to perform cooperative and coordinated tasks, under an individual control paradigm and their multiple applications in the fields of power systems, autonomous vehicles, transportation systems, and military applications. Many topics such as consensus, formation, trajectory tracking, and flocking have been widely studied, and approaches as self organization, goal achievement with robustness to component failures, and network evolution are becoming the objective of the research.

Output consensus deals with the problem of the agreement of the outputs of the agents composing a multi-agent system. The restriction of consensus only to the output of the systems opens the possibility of multi-agent interaction to wide different systems, including linear, nonlinear, systems of different dimensions, etc. in a common multi-agent system.

The output regulation deals with the problem of trajectory tracking of a signal generated by an exosystem. The multi-agent interaction allows that a group of systems succeed in trajectory tracking even when not all the agents in the group know the reference to track.

Addressing output consensus and output regulation at the same time allows the interaction of different system in a coordinated trajectory tracking with restricted information. Such trajectories can be thought of as actual trajectories, like in the displacement of a group of military vehicles or as a totally different concept like the tension and frequency of the energy on a power system generated by different generators.

The problem of output consensus and output regulation of multi-agent systems is the topic this thesis is focus on. This problem is addressed from three different points of view. First, the problem of output consensus and output regulation allowing formation is addressed for multi-agent systems, composed by multi-input multi-output linear systems, under fixed communication topologies. Then, the same problem is addressed for multi-agent systems composed by agents of class of switched linear systems (*SLS*). Finally, the problem of output consensus and output regulation is addressed for the same class of switched multi-agent systems but considering switching communication topologies. This final approach can be applied to linear multivariable multi-agent systems as a particular case, but formation is not allowed.

In the following section some works related to the topic of this thesis are presented and briefly described.

1.1 State of the art

Previous results related to consensus problems and a theoretical framework for the analysis of consensus algorithms in multi-agent systems is presented in [30], where an overview of methods for convergence and analysis of consensus algorithms is presented.

In [31] and [27] consensus algorithms are introduced, for both linear and nonlinear systems, and a Lyapunov function is used to analyze the convergence of the consensus algorithms.

A general formulation of multi-agent formation is presented in [10], where the authors also address some problems such as how the information flow topology affects the system coordination stability and performance.

In [17], necessary and sufficient conditions for an appropriate decentralized linear stabilizing feedback are established and a relationship between the rate of convergence to the formation and the eigenvalues of the (directed) Laplacian matrix is established.

Necessary and sufficient conditions for a distributed consensus controller to stabilize a set of multi-agent systems, and consensus of multi-agent systems under switching interaction topologies are studied in [41].

Formation control laws based on artificial potential fields and consensus algorithms for a group of nonholonomic vehicles are proposed in [23].

In [32], the problem of "flocking" with obstacles is addressed, where flocking is defined as achieving both structural and navigational stability. A flock is allowed to divide into multiple flocks to avoid obstacles and continue its trajectory. The systems considered are restricted to have integrator dynamics, and no stability results are presented.

A feedback control strategy which achieves convergence of a multi-agent system, for single-integrator dynamics with a desired formation and avoiding collisions is proposed in [5]. It is shown that under certain assumptions, when the control law forces the multi-agent system to attempt to reach an unfeasible formation, it drives the agents velocity vectors to a common value at steady state, which establishes some connection between formation infeasibility and a sort of flocking.

Consensus of multiple autonomous vehicles is addressed in [18], by using virtual leaders and artificial potential fields among neighboring vehicles. Virtual leaders are used to manipulate formation topology and direct the motion of the formation. Results are also restricted to vehicles with integrator dynamics.

A decentralized dynamic controller is presented in [9], which deals with the problem of cooperation among a collection of vehicles performing a shared task using inter-vehicle communication, considering linear dynamics. The authors establish the conditions for the stability of formations related to the topology of communication network, and discuss some topics on network information flow.

An approach to regulation of multi-agent linear systems is presented in [36], where the authors demonstrate zero tracking error under switching interaction topologies using observers for the exosystem. However, each agent observer depends on the exosystem output and/or on the full state of the observers of its neighbors.

In [42], the regulation of multi-agent linear systems considering uncertainties on the matrices of the system description is addressed. The results presented in that paper assure zero tracking error even when only a subset of the systems has access to the reference signal based on an internal model approach. But this approach does not allow the multi-agent system to achieve a formation.

The robust regulation of multi-agent linear systems considering uncertainties was also addressed in [40]. The results obtained provide sufficient conditions for multi-agent zero tracking error under switched connection topologies based on a canonical internal model approach. Similar to the previous work, the results presented do not allow the multi-agent system to achieve a formation.

Although, many works have appeared addressing topics related to multi-agent consensus, most of them are restricted to multi-agent systems with relatively simple dynamics like integrator dynamics or linear systems with identical dynamics. However, there are still few results on output consensus and trajectory tracking. The work developed in this thesis addresses the problem from a different perspective, and considers complex dynamics like the switched linear systems, for which the application to linear multivariable systems is a particular case. In the following section the objectives of this thesis are stated.

1.2 Objectives

The main objectives of this work are stated as follows:

- To design a distributed control law for the output consensus and output regulation of linear multivariable multi-agent systems, under a fixed communication topology, when only a subset of the agents has access to the reference (exosystem output) and none of them to the state of the exosystem.

- To achieve and maintain multi-agent formation while the linear multi-agent system achieves regulation.

- To design a distributed control law for the output consensus and output regulation of a class of switched linear multi-agent systems, under a fixed communication topology, when only a subset of the agents has access to the reference and none of them to the state of the exosystem.
- To achieve and maintain multi-agent formation while the switched linear multi-agent system achieves regulation.
- To design a distributed control law for the output consensus and output regulation of a class of switched linear multi-agent systems, under a switching communication topology, when a possibly varying subset of the agents has access to the reference and none of them to the state of the exosystem.

1.3 Contributions

In this work, the design of distributed control laws for the output consensus and output regulation of multi-agents systems is presented. It is considered that only a subset of the agents has access to the reference, which is taken as the output of a virtual agent (exosystem). It is also considered that none of the agents has access to the exosystem state and that every agent has access to the output of all its neighbors.

The communication among the agents is restricted such that it can be represented by graphs which contain a spanning tree and do not contain any loops.

In order to reach consensus, it is also considered that each agent has to follow the same reference provided by the exosystem, whose structure is known.

Under such assumptions, distributed observers for the state of the exosystem are designed depending on the outputs of the neighbors of each agent.

Three different approaches are presented. In the first one, the problem of output consensus and output regulation is solved for multi-agent systems composed by multi-input multi-output linear systems of different dimensions maintaining the multi-agent system in formation under a fixed communication topology. This approach is similar to the one presented in [36], but in the case of the work herein presented, the observer depends only on the outputs of the neighbors of each agent.

In the second and third case, the problem of output consensus and output regulation is solved for switched linear multi-agent systems of different dimension. The second approach is similar to the first one, but the third addresses the problem of output consensus and output regulation of *SLS* systems under switching interaction topologies although multi-agent formation is not allowed. The third approach includes the problem of output consensus and output regulation of linear multivariable multi-agent systems as a particular case.

The proposed control laws guarantee multi-agent system output consensus and output regulation by means of the stability of the observer-regulation error dynamics.

1.4 List of publications

1. A. Cervantes-Herrera, C. López-Limón, A. Ramírez-Treviño, and J. Ruiz-León, "Seguimiento de trayectoria de multiagentes exhibiendo consenso," Congreso Anual de la Asociación de México de Control Automático AMCA-2010, Puerto Vallarta México, 2010.
2. C. López-Limón, A. Cervantes-Herrera, J. Ruiz-León and A. Ramírez-Treviño, "Formation and trajectory tracking of a class of nonlinear systems with super twisting control," IEEE International Conference on Emerging Technologies and Factory Automation ETFA-2011, Toulouse, France, 2011.
3. C. López-Limón, A. Cervantes-Herrera, J. Ruiz-León, and A. Ramírez-Treviño, "Trajectory tracking and consensus of SISO linear multi-agent systems with formation changes," 8th International Conference on Electrical Engineering, Computing Science and Automatic Control CCE-2011, pp. 391-396, Mérida, Yucatán, México, 2011.
4. A. Cervantes-Herrera, C. López-Limón, A. Ramírez-Treviño, and J. Ruíz-León, "Feedback stabilization of switched linear systems," IEEE Challenges and Opportunities for the Engineering of the Future CESA-2012, pp. 189-193, Santiago, Chile, 2012.
5. A. Cervantes-Herrera, J. Ruiz-León, C. López-Limón, and A. Ramírez-Treviño, "A distributed control design for the output regulation and output consensus of a class of switched linear multi-agent systems," 17th IEEE International Conference on Emerging Technologies and Factory Automation ETFA-2012, Krakow, Poland, 2012
6. C. López-Limón, J. Ruiz-León, A. Cervantes-Herrera, and A. Ramírez-Treviño, "Consensus and trajectory tracking of SISO linear multi-agent systems under switching communication topologies and formation changes," Submitted to Kybernetika, 2012.
7. A. Cervantes-Herrera, J. Ruiz-León, C. López-Limón, and A. Ramírez-Treviño, "A distributed control design for the output regulation and output consensus of a class of switched linear multi-agent systems, under switching communication topologies," To be submitted.

1.5 Organization of the document

This document is organized as follows: The second chapter presents concepts and previous results on consensus and trajectory tracking of multi-agent systems. The contributions of this work appear in the third and the fourth chapter. The third chapter shows a result on multi-input multi-output linear multi-agent systems distributed output regulation and output consensus. The fourth chapter presents the design of distributed control laws for the output consensus and output regulation of switched linear multi-agent systems considering the cases of fixed and switching communication topologies. Finally, some conclusions and future research goals are stated in chapter five.

Chapter 2

Preliminaries

Consensus and trajectory tracking have been very active research topics in the field of multi-agent systems, and many works regarding diverse topics, such as, different consensus algorithms, convergence rate, performance, robustness and stability have appeared, some of them will be presented in this chapter as a basis for the results presented in chapters three and four.

A multi-agent system can be described as a network of dynamical systems called agents. In general, these agents interact with each other by means of the knowledge of the state or the output of their neighbors. This knowledge is achieved by means of a communication framework which is usually described by a graph as detailed in the following section.

2.1 Graph theory fundaments

The relations among a system of agents are commonly described by a graph $\mathcal{G} = (\vartheta, \Phi, \mathcal{A})$, where ϑ is a set of nodes (agents), $\Phi \in \vartheta \times \vartheta$ is a set of edges that connect a node to another (self edges are not allowed), and $\mathcal{A} = [\alpha_{i,j}] \in \mathbb{R}^{N \times N}$ is its adjacency matrix containing positive weights describing the relationships among nodes. An edge $(\nu_i, \nu_j) \in \Phi$ means that node ν_j can get information from node ν_i. If an edge (ν_i, ν_j) is contained in Φ, this implies that the term $\alpha_{j,i}$ of the adjacency matrix is different from zero and vice versa, nevertheless, it does not imply that the edge (ν_j, ν_i) is also contained in Φ. The set of neighbors of node i will be denoted by $\Theta_i = \{\nu_j : (\nu_j, \nu_i) \in \Phi, j = 1, \ldots, N\}$ and by $\rho_i = |\Theta_i|$ its cardinality.

The Laplacian $\mathcal{L}$ will be defined as $\mathcal{L} = \mathcal{A} - diag\left\{\Sigma_{j=1}^{N}\alpha_{1j} \cdots \Sigma_{j=1}^{N}\alpha_{Nj}\right\}$. By definition, every row sum of the Laplacian matrix is zero. Therefore, the Laplacian matrix always has a zero eigenvalue corresponding to a right eigenvector with identical nonzero elements.

A spanning tree is a graph in which every node can get information from only one node, except for one called root. The root node does not receive information from any node. A graph $\mathcal{G}$ is said to have a spanning tree if every one of its nodes and a subset of its edges form a spanning tree, which means that at least one of the nodes has a communication path to every other node.

If any two distinct nodes of a graph $\mathcal{G}$ can be connected via a path that follows the direction of its edges, then it is called strongly connected.

When the relations among the agents change in time, a switching graph becomes necessary. A switching graph will be defined as $\mathcal{G}_{\varrho t} = (\vartheta, \Phi_{\varrho t}, \mathcal{A}_{\varrho t})$, where $\varrho_t :. [t_0, \infty) \rightarrow \{1, \ldots, \gamma\}$ is the switching signal that determines the communication topology at time t. The set of neighbors of node i at time t, $\Theta_{i_{\varrho t}}$, will be defined analogously as $\Theta_{i_{\varrho t}} = \{\nu_j : (\nu_j, \nu_i) \in \Phi_{\varrho t}, j = 1, \ldots, N\}$ and $\rho_{i_{\varrho t}} = |\Theta_{i_{\varrho t}}|$ its cardinality.

In this work, for the case of multi-agent systems, it will be considered that the condition $(\nu_i, \nu_j) \in \Phi \Leftrightarrow (\nu_j, \nu_i) \notin \Phi$ is met.

In the case of switching interaction topologies, it will be considered that the set of nodes ϑ is constant, and also, that the condition $(\nu_i, \nu_j) \in \Phi_{\varrho t} \Leftrightarrow (\nu_j, \nu_i) \notin \Phi_{\varrho t}$ is met.

2.2 Consensus

In networks of dynamic systems consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all the agents [30].

The most general definition of consensus of multi-agents implies that the state of every system converge to the same value. Such a value will depend on the initial values of the states of the agents, the communication topology and the employed consensus algorithm.

A consensus algorithm is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network [30].

The most basic approach to multi-agent consensus is when only agents with integrator dynamics are considered.

Consider a multi-agent system composed of N agents with integrator dynamics

$$\dot{x}_i = u_i \qquad i = \{1, 2, \cdots, N\} \tag{2.1}$$

and a consensus algorithm

$$u_i(t) = \sum_{j \in \Theta_i} (x_j(t) - x_i(t)) \tag{2.2}$$

hence, the multi-agent dynamics are

$$\dot{x}(t) = -\mathcal{L}x(t)$$

where $\mathcal{L}$ is the Laplacian corresponding to the graph $\mathcal{G}$ of the communication topology, and $x(t)^T = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_N(t) \end{bmatrix}$.

Corollary 2.1. *(Corollary 1 of [33]) Consider a multi-agent system as (2.1) where each node applies control (2.2). Assume its communication topology is such that $\mathcal{G}$ is strongly connected. Then, control (2.2) globally asymptotically solves a consensus problem.*

The equilibrium $x^* = a\mathbf{1}^T$, $a \in \mathbb{R}$ of the multi-agent system is a right eigenvector of $\mathcal{L}$ associated with the eigenvalue $\lambda = 0$, and $\mathbf{1} \in \mathbb{R}^N$ is a row vector full of ones.

Different consensus algorithms, convergence analysis, equilibrium points, and topics related, for systems with integrator and double integrator dynamics, are found in [30], [33], [32], [35], [8], [6], [18], [13], [34], and references therein.

In [41] a controller parametrization for the consensus of linear multi-agent systems with identical dynamics is presented.

Consider a linear multi-agent system composed by N identical linear dynamics

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \qquad i = 1, \ldots, N, \tag{2.3}$$

a communication topology according to a graph $\mathcal{G}$, a consensus error

$$\zeta_i(t) = \sum_{j=1}^{N} \alpha_{i,j} \left[(x_j(t) - x_i(t))\right] \tag{2.4}$$

and a consensus error state feedback

$$u_i(t) = K\zeta_i(t). \tag{2.5}$$

$PD(n)$ is the set of $n \times n$ positive definite matrices, $Skew(n)$ is the set of $n \times n$ skew symmetric matrices, and $B^\dagger \in \mathbb{R}^{m \times n}$ is a generalized inverse matrix of B. Therefore,

- Both $BB^\dagger$ and $I - BB^\dagger$ are symmetric matrices. Furthermore $BB^\dagger B = B$, $B^\dagger BB^\dagger = B^\dagger$, $B^T BB^\dagger = B^T$

– $BB^\dagger$ is an orthogonal projection matrix to ImB.

Lemma 2.2. *(Lemma 1 of [41]) For given $Q_i \in PD(n)$, $i = 1, \cdots, N-1$, if there is K to satisfy*

$$(A + \lambda_i BK)Pi + Pi(A + \lambda_i BK)^T + Q_i = 0, \qquad P_i \in PD(n), \tag{2.6}$$

where $\lambda_i = 1, \cdots, N-1$ are the nonzero eigenvalues of the Laplacian matrix associated with the graph $\mathcal{G}$ having a spanning tree, then P_i, $i = 1, \cdots, N-1$, satisfy

$$(I - BB^\dagger)(AP_i + P_iA^T + Q_i)(I - BB^\dagger) = 0, \qquad i = 1, \cdots, N-1, \tag{2.7}$$

$$\begin{aligned}\lambda_j \left[B^\dagger(AP_i + P_iA^T + Q_i)(I - \frac{1}{2}BB^\dagger)P_i^{-1} + \lambda_i BWP_i^{-1}\right] = \\ \lambda_i \left[B^\dagger(AP_j + P_jA^T + Q_i)(I - \frac{1}{2}BB^\dagger)P_j^{-1} + \lambda_j^{-1}\right]\end{aligned} \tag{2.8}$$

and K is taken the form of

$$K = -\frac{1}{\lambda_i}\left[B^\dagger(AP_i + P_iA^T + Q_i)(I - \frac{1}{2}BB^\dagger)P_i^{-1}\right] - B^\dagger WP_i^{-1}, \tag{2.9}$$

with $\lambda_i = 0$, $W = BB^\dagger W \in Skew(n)$, $W = BB^\dagger WBB^\dagger \in Skew(n)$.

Hence, from the previous feedback parametrization the achievement of consensus is guaranteed.

Theorem 2.3. *(Theorem 1 of [41]) Assume that the interconnection digraph of the multi-agent system (2.3) has a spanning tree. Then the following conditions are equivalent:*

1. *The consensus of system (2.3) is achieved, under control (2.5).*
2. *Any feedback controller K makes $A + \lambda_i BK$ stable (Hurwitz), where $i = 1, \cdots, N-1$ are the nonzero eigenvalues of Laplacian matrix $\mathcal{L}$.*
3. *For given matrices $Q_i \in PD(n)$, $i = 1, \cdots, N-1$,, the feedback controller K satisfies (2.6).*
4. *The matrices $P_i \in PD(n)$, $i = 1, \cdots, N-1$ satisfy (2.7) and (2.8).*
5. *The feedback controller K can be written in the form of (2.9) with $P_i \in PD(n)$, $i = 1, \cdots, N-1$ satisfying (2.7), (2.8).*

Further information about consensus of linear systems can be found in [31], [27], [2], [4], [19], [26], and references therein.

Output consensus

The output consensus will be defined as in [16]. Consider a multi-agent system composed by N linear dynamics

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) \qquad i = 1, \ldots, N, \tag{2.10}$$

with outputs

$$y_i(t) = C_i x_i(t) \tag{2.11}$$

The goal of the controller $u_i(t)$ is to guarantee that $\lim\limits_{t\to\infty}(y_j(t) - y_i(t)) = 0$ for all $i, j \in \{1, 2, \cdots, N\}$, and for any initial conditions $x_i(0)$.

This implies that a certain signal $y_0(t)$ exists such that $\lim\limits_{t\to\infty}(y_0(t) - y_i(t)) = 0$ for all $i \in \{1, 2, \cdots, N\}$, and $y_0(t)$ is the outcome of the on-line consensus mechanism among the agents. Since the overall system is linear, the synchronized signal $y_0(t)$ needs to be the output of a certain linear system, namely, exosystem.

It should be noted that the $i-th$ controller $u_i(t)$ uses only the output information of agent i and the outputs of its neighbors.

Different results on output consensus are found in [44], [45], and [39], and [7].

2.3 Output regulation

Consider a linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned} \tag{2.12}$$

The problem of output regulation consist on designing a control law

$$u(t) = Kx(t) + \Gamma\omega(t), \tag{2.13}$$

such that,

$$\lim_{t\to\infty}(y(t) - y_0(t)) = 0 \tag{2.14}$$

where

$$y_0(t) = R\omega(t) \tag{2.15}$$

is a reference to follow for the output of system (2.12), and $\omega(t)$ is the state of an exosystem

$$\dot{\omega}(t) = S\omega(t). \tag{2.16}$$

Consider also the following hypotheses

Hypothesis 2.1. *($\mathcal{H}2.1$)* $\sigma(S) \subset \{\lambda \in \mathcal{C} \mid \mathbf{Re}[\lambda] \geq 0\}$

where $\sigma(S)$ is the spectrum of S.

Hypothesis 2.2. *($\mathcal{H}2.2$) The pair (A, B) is stabilizable.*

Proposition 2.4. *(Proposition 1 of [15]) Suppose $\mathcal{H}2.1$ and $\mathcal{H}2.2$ hold. Then, the linear state feedback regulator problem is solvable if and only if there exist matrices Π and Γ which solve the linear matrix equations*

$$\Pi S = A\Pi + B\Gamma \tag{2.17}$$

$$C\Pi - R = 0. \tag{2.18}$$

For topics on regulation, please refer to [11], [15], [12], and [14].

2.4 Distributed output regulation of linear multi-agent systems

Consider a linear multi-agent system composed by a set of N agents with different linear dynamics described by

$$\begin{array}{ll} \dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) & \quad i = 1, \dots, N. \\ y_i(t) = C_i x_i(t) & \end{array} \tag{2.19}$$

where, $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^q$, and $u_i \in \mathbb{R}^m$

The problem of distributed output regulation is defined as the problem of designing a local control strategy for the output regulation of a multi-agent systems.

2.4.1 Internal model approach

In [42], the problem of distributed output regulation of linear multi-agent systems is addressed. A linear multi-agent system as (2.19), with systems of the same dimension ($n_i = n$), is considered, and parameter variations in the matrices are allowed.

Define the output consensus error as

$$\zeta_i(t) = \frac{1}{\mid \Theta_i \mid} \sum_{j \in \Theta_i} (y_j(t) - y_i(t)) \tag{2.20}$$

and consider the control law

$$\begin{aligned} u_i(t) &= -K_i x_i(t) + K_{zi} z_i(t) \\ \dot{z}_i(t) &= G_1 z_i(t) + G_2 v_i(t) \end{aligned} \tag{2.21}$$

where

$$v_i(t) = \begin{cases} \zeta_i(t) & (\nu_0, \nu_i) \notin \Phi \\ y_0(t) - y_i(t) & (\nu_0, \nu_i) \in \Phi, \end{cases} \tag{2.22}$$

and the condition $(\nu_0, \nu_i) \in \Phi$ indicates that agent i knows the state of the exosystem.

Hypothesis 2.3. *($\mathcal{H}2.3$)*

$$rank \begin{pmatrix} A_i - \lambda I & B_i \\ C_i & 0 \end{pmatrix} = n + q, \qquad \lambda \in \sigma(S), \qquad i \in \{1, 2, \cdots, N\}$$

Hypothesis 2.4. *($\mathcal{H}2.4$) The graph $\mathcal{G}$ describing the communication topology of the multi-agent system does not contain loops, and incorporates a spanning tree with root on ν_0 (reference agent).*

Hypothesis 2.5. *($\mathcal{H}2.5$) The pair of matrices (G_1, G_2) incorporates a p-copy internal model of matrix S.*

Theorem 2.5. *(Theorem 3.1 of [42]) Under hypotheses $\mathcal{H}2.1$–$\mathcal{H}2.5$, the distributed output regulation of system (2.19) can be solved by a dynamic state feedback control of the form (2.21).*

The matrices K_i and K_{zi} in control (2.21) must be calculated such that the matrices

$$\begin{pmatrix} A_i - B_i K_i & B_i K_{zi} \\ G_2 C_i & G_1 \end{pmatrix}$$

are Hurwitz for $i \in \{1, 2, \cdots, N\}$.

2.4.2 Observer approach

In [36], the problem of distributed output regulation of linear multi-agent systems is addressed. A linear multi-agent system as (2.19) is considered. The communication topology is considered to be switching, and only a subset of the agents is supposed to receive the exosystem state at time t.

In order to compute the state of the exosystem the following observers are proposed

$$\dot{\hat{\omega}}_i(t) = S\hat{\omega}_i(t) + l \left[\sum_{j=1}^{N} \alpha_{i,j} \left(\hat{\omega}_j(t) - \hat{\omega}_i(t) \right) + \alpha_{i,0} \left(\omega(t) - \hat{\omega}_i(t) \right) \right] \tag{2.23}$$

The observer of the agent i depends on the exosystem observed state of its neighbors.

Consider the control law

$$u_i(t) = -K_i x_i(t) + \Gamma_i \hat{\omega}_i(t) \tag{2.24}$$

and, the hypothesis

Hypothesis 2.6. *There exists a subsequence $\{i_k\}$ with $t_{i_{k+1}} - t_{i_k} < \tau$ for some positive τ such that every node is reachable from the node ν_0 in the union graph $\bigcup\limits_{\sigma_t \in \{i_k\}} \mathcal{G}_{\sigma_t}$.*

Theorem 2.6. *(Theorem 1 of [36]) Assume that there exists a solution for the equation system (2.17), (2.18) and that Hypotheses $\mathcal{H}2.1$, $\mathcal{H}2.2$, and $\mathcal{H}2.6$ are met. Then, the distributed output regulation problem of system (2.19) can be solved by the distributed dynamic state feedback control law (2.24) where K_i, $i = 1, \cdots, N$, are such that $A_i - B_i K_i$ are Hurwitz, and l is any positive constant.*

Chapter 3

Output consensus and output regulation of MIMO linear multi-agent systems

Multi-agent consensus has been widely studied in the last decade, and many works have been presented attempting to solve consensus problems with obstacles and control stability.

In this chapter, a first approach to the problem of output consensus and output regulation of a multi-agent system with linear dynamics is presented. The proposed control scheme attains output consensus and output regulation of the multi-agent system, so that the N agents in formation track a desired reference by means of a virtual leader agent that has communication to a subset of the agents. The virtual leader dynamics are described as a function of the state of an exosystem. Given that, not all the agents receive the reference and none of them receives the state of the exosystem. Observers, dependent on the outputs of the neighbors of the agents, are designed in order to compute the state of the exosystem and solve regulation.

Different approaches have appeared addressing the problem of robust output consensus an output regulation of linear multi-agent systems. In [42] and [40] the problem is addressed, for fixed and switching communication topologies respectively, based on the internal model principle [12]. But none of these works allow the multi-agent system formation. A different approach is the presented in [36]. In this paper the problem is addressed in a similar way to the developed here, but the exosystem observer of a given agent depends on the reference and on the state of the observers of its neighbors. Nevertheless, the result in [36] allows the consideration of switching communication topologies.

3.1 Problem statement

Consider a set of N agents with different linear dynamics described by

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) \\ y_i(t) &= C_i x_i(t) \end{aligned} \qquad i = 1, \dots, N, \tag{3.1}$$

where, $x_i(t) \in \mathbb{R}^{n_i}$ is the state, $u_i(t) \in \mathbb{R}^m$ the control, and $y_i(t) \in \mathbb{R}^q$ the output vector of the i-th agent. The matrices $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m}$ and $C_i \in \mathbb{R}^{q \times n_i}$ are supposed to be such that $rank B_i \geq q$, and it will be considered that every agent is stabilizable.

Consider also an exosystem

$$\dot{\omega}(t) = S\omega(t), \tag{3.2}$$

and its output

$$y_0(t) = \begin{bmatrix} R & 0 \end{bmatrix} \omega(t) \tag{3.3}$$

where, $\omega(t) \in \mathbb{R}^p$, $S \in \mathbb{R}^{p \times p}$, $R \in \mathbb{R}^{q \times p-1}$, and the last entry of vector ω is a constant $\omega_p(t) = 1$.

Regard $y_i^d = d_i \omega_p(t) \in \mathbb{R}^q$ as the desired relative position of the output of the i-th agent with respect to the virtual agent output $y_0(t)$ (multi-agent system reference) and let it also depend on the exosystem state $\omega(t)$. Then define the i-th agent output consensus and regulation error as

$$\begin{aligned} \zeta_i(t) = \sum_{j=1}^{N} \alpha_{i,j} & \left[(y_j(t) - y_i(t)) - (d_j - d_i)\omega_p(t) \right] \\ & + \alpha_{i,0} \left[(y_0 - y_i(t)) + d_i \omega_p(t) \right]. \end{aligned} \tag{3.4}$$

Note that this definition is similar to the consensus error (2.4). The weight $\alpha_{i,0}$ corresponds to the difference between the i-th agent output and the reference y_0 (virtual agent output).

The problem of output consensus and regulation of multi-agent systems consists on obtaining a control law such that the output consensus and regulation error $\lim_{t \to \infty} \zeta_i(t) = 0$ for each agent $i = \{1, 2, \cdots, N\}$.

3.2 Distributed output consensus and regulation control

Hypothesis 3.1. *($\mathcal{H}3.1$) The graph $\mathcal{G}$ describing the communication topology of the multi-agent system does not contain loops, and incorporates a spanning tree with root on ν_0 (reference agent).*

Given that it is considered that each agent has only the information of the output of its neighbors including the exosystem, none of the agents has access to its state. Thus, order to obtain the state of the exosystem, the following observers are proposed

$$\dot{\hat{\omega}}_i(t) = \bar{S}\hat{\omega}_i(t) + \kappa_i(\sum_j y_j(t) - \rho_i l\hat{\omega}_i(t)) \quad \forall j \in \Theta_i \tag{3.5}$$

where $L = \begin{bmatrix} R & 0 & \mathbf{I}_q \end{bmatrix}$,

$$\bar{S} = \begin{bmatrix} S & 0 \\ 0 & 0_q \end{bmatrix} \tag{3.6}$$

0_q is a $q \times q$ zero matrix. The matrix κ_i should be computed such that the matrix $(\bar{S} - \rho_i\kappa_i L)$ is Hurwitz in order to guarantee the stability of the error of observation as shown later.

Remark 3.1. *Given that the agents are allowed to follow the reference in formation, the outputs of each agent may have an offset from the actual reference value, hence the matrix $\bar{S}$ is used as an extended exosystem matrix to filter the unknown d_j's from the exosystem states.*

Hypothesis 3.2. *($\mathcal{H}3.2$) a) The linear systems (3.1) under the state feedback controls $u_i(t) = -Kx_i(t)$ are Hurwitz.*

b) There exists solutions Π_i to the following equations

$$\begin{aligned} \Pi_i S &= (A_i - B_i K_i)\Pi_i + (P_i + B_i\Gamma_i) \\ 0 &= C_i\Pi_i - R \qquad\qquad i = 1, 2, \cdots, N. \end{aligned}$$

Now, set the distributed output regulation control law as

$$u_i(t) = -K_i x_i(t) + \begin{bmatrix} \Gamma_i & 0_q \end{bmatrix}\hat{\omega}_i(t). \tag{3.7}$$

Note that control (3.7) is of the form (2.13) and depends on the exosystem observer state $\hat{\omega}_i(t)$.

Define the i-th observer error as

$$\tilde{\omega}_i(t) = \bar{\omega}_i(t) - \hat{\omega}_i(t), \tag{3.8}$$

where $\bar{\omega}_i(t) = \begin{bmatrix} \omega(t) \\ \delta_i \end{bmatrix}$, and $\delta_i = \sum_j d_j/\rho_i$, $\forall j \in \Theta_i$.

Note that an extended exosystem observer $\hat{\omega}_i(t)$ is defined in order to filtrate from the reference the offsets from the outputs of the neighbors of agent i. The vector δ_i is the media of the separation of the neighbors of agent i from the reference.

It will be demonstrated later that the extended variables of the observer will converge to this value.

The dynamics of the observer dynamics will be described by

$$\dot{\hat{\omega}}_i(t) = \bar{S}\tilde{\omega}_i(t) - \kappa_i(\sum_j y_j(t) - \rho_i L\hat{\omega}_i(t)). \tag{3.9}$$

Consider that $\bar{\Pi}_i = \begin{bmatrix} \Pi_i & 0_q \end{bmatrix}$, and that Hypothesis ($\mathcal{H}3.2$) is met, then the i-th agent regulation error can be defined as

$$e_i(t) = x_i(t) - \bar{\Pi}_i\hat{\omega}_i(t) \tag{3.10}$$

and its dynamics is given by

$$\dot{e}_i(t) = (A_i - B_iK_i)e_i(t) - \bar{\Pi}_i\kappa_i(\sum_j y_j(t) - \rho_i L\hat{\omega}_i(t)). \tag{3.11}$$

From (3.8) and (3.10), it can be inferred that

$$\begin{aligned} y_i(t) &= C_i x_i(t) \\ &= C_i\left(e_i(t) + \bar{\Pi}_i\left(\bar{\omega}_i(t) - \tilde{\omega}_i(t)\right)\right) \\ &= C_i e_i(t) + \bar{R}_i\left(\bar{\omega}_i(t) - \tilde{\omega}_i(t)\right) \end{aligned} \tag{3.12}$$

where, $\bar{R}_i = \begin{bmatrix} R & d_i & 0_q \end{bmatrix}$.

Take the second term in (3.9) and substitute y_j as in (3.12)

$$\left(\sum_j y_j(t) - \rho_i L\hat{\omega}_i(t)\right) = -\rho_i L\hat{\omega}_i(t) + \sum_j\left\{\bar{R}_j\left(\bar{\omega}_j(t) - \tilde{\omega}_j(t)\right) + C_je_j(t)\right\}. \tag{3.13}$$

Given that $\omega_p(t) = 1$, it can be seen that

$$\sum_j \bar{R}_j\bar{\omega}_j(t) = \rho_i L\bar{\omega}_i(t), \tag{3.14}$$

thus,

$$\left(\sum_j \bar{R}_j\bar{\omega}_j(t) - \rho_i L\hat{\omega}_i(t)\right) = \rho_i L\tilde{\omega}_i(t). \tag{3.15}$$

Remark 3.2. *The observed variables $\hat{\omega}_{i,(p+\alpha)}$ where $\alpha = 1, 2, \cdots, q$, will tend asymptotically to $\delta_i = \sum_j d_j/\rho_i$, $\forall j \in \Theta_i$.*

Using (3.13) and (3.15), the i-th observer and regulation error dynamics (3.9), (3.11) can be restated as

$$\dot{\tilde{\omega}}_i(t) = \kappa_i\left(\sum_j\left(\bar{R}_j\tilde{\omega}_j(t) - C_je_j(t)\right)\right) + \left(\bar{S} - \kappa_i\rho_i L\right)\tilde{\omega}_i(t) \tag{3.16}$$

$$\dot{e}_i(t) = \bar{\Pi}_i \kappa_i \sum_j \left(\bar{R}_j \tilde{\omega}_j(t) - C_j e_j(t)\right) + (A_i - B_i K_i) e_i(t) - \bar{\Pi}_i \kappa_i \rho_i L \tilde{\omega}_i(t). \tag{3.17}$$

Define the i-th agent observer-regulation error as

$$\xi_i(t) = \begin{bmatrix} \tilde{\omega}_i(t) \\ e_i(t) \end{bmatrix} \tag{3.18}$$

and multi-agent observer-regulation error as

$$\xi(t) = \begin{bmatrix} \xi_1^T(t) & \xi_2^T(t) & \cdots & \xi_N^T(t) \end{bmatrix}^T \tag{3.19}$$

Under Hypothesis ($\mathcal{H}3.1$), the Laplacian $\mathcal{L}$ is a lower triangular matrix. Thus, the multi-agent observer-regulation error dynamics will have a block lower triangular form

$$\dot{\xi}(t) = \begin{bmatrix} \Omega_1 & 0 & \cdots & 0 & 0 \\ \varrho_{2,1} & \Omega_2 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ \varrho_{N-1,1} & \varrho_{N-1,2} & & \Omega_{N-1} & 0 \\ \varrho_{N,1} & \varrho_{N,2} & \cdots & \varrho_{N,N-1} & \Omega_N \end{bmatrix} \xi(t) \tag{3.20}$$

with matrix blocks

$$\Omega_i = \begin{bmatrix} \bar{S} - \kappa_i \rho_i L & 0 \\ -\bar{\Pi}_i \kappa_i \rho_i L & A_i - B_i K_i \end{bmatrix} \tag{3.21}$$

on its diagonal. The terms ϱ_{i,j_i} are considered constant.

Now, the main result of this chapter can be stated.

Theorem 3.1. *Consider that Hypothesis ($\mathcal{H}2.1$), ($\mathcal{H}2.2$), ($\mathcal{H}3.1$), and ($\mathcal{H}3.2$) are met, then the distributed output regulation and output consensus of the linear multi-agent system (3.1) can be solved by a control law of the form (3.7).*

Proof. Consider a linear multi-agent system with agents of the form (3.1), and control law (3.7).

Then, under Hypothesis ($\mathcal{H}3.1$) the corresponding multi-agent observer-regulation error will have the form (3.20). Given the block triangular form of the observer-regulation error dynamics and the stability of the blocks (3.21), the stability of the observer-regulation error (3.20) is guaranteed.

Finally, given the stability of the observer-regulation error, in steady state, the following holds

$$\hat{\omega}_i(t) = \bar{\omega}_i(t) \tag{3.22}$$

$$x_i(t) = \Pi_i \omega(t) \tag{3.23}$$

$$y_i(t) = \begin{bmatrix} R & d_i \end{bmatrix} \omega(t) \tag{3.24}$$

thus, the i-th output consensus and regulation error $\zeta_i(t) = 0$ for $i = 1, 2, \cdots, N$. □

The following example is presented in order to illustrate the previous result.

3.3 Example

Consider 3 agents with the following dynamics

$$\dot{x}_1(t) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & -3 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u_1(t),$$

$$y_1(t) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} x_1(t)$$

$$\dot{x}_2(t) = \begin{bmatrix} 0 & 1 & 0 & 5 & 4 \\ -2 & -1 & 3 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 \\ 4 & 0 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_2(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} u_2(t),$$

$$y_2(t) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} x_2(t)$$

$$\dot{x}_3(t) = \begin{bmatrix} -3 & 0 & 0 & 5 \\ 3 & 2 & 0 & 3 \\ 1 & 4 & -2 & 0 \\ 0 & -2 & -2 & -1 \end{bmatrix} x_3(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} u_3(t),$$

$$y_3(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \end{bmatrix} x_3(t)$$

with initial conditions

$$x_1(0) = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad x_2(0) = \begin{bmatrix} 5 \\ -6 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \quad x_3(0) = \begin{bmatrix} 3 \\ 7 \\ -9 \\ 2 \end{bmatrix}$$

The multi-agent system including the above systems is connected according to the following Laplacian matrix

$$\mathcal{L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

in this way, only the second system receives the reference to track. The graph corresponding to this communication topology is shown in Fig. 3.1.

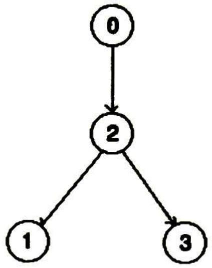

Figure 3.1: Communication topology graph

The desired relative positions of each output with respect to the reference are considered constant, and are described by the next constant vectors

$$d_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad d_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad d_3 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

Let the reference exosystem be

$$\dot{\omega}(t) = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega(t)$$

where the zero row accounts for a constant $\omega_3(t)$,and consider the initial conditions

$$\omega(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

where the last initial condition must be one in order to respect the relative positions defined by the d_i's.

Take the reference matrix as

$$R = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then, the exosystem observers are described by

$$\dot{\hat{\omega}}_1(t) = \bar{S}\hat{\omega}_1(t) + \begin{bmatrix} 4.58 & 5.54 \\ -5.20 & 7.76 \\ 0 & 0 \\ 0.84 & -14.91 \\ 6.02 & 0.21 \end{bmatrix} \left(y_2(t) - \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \hat{\omega}_1(t) \right)$$

$$\dot{\hat{\omega}}_2(t) = \bar{S}\hat{\omega}_2(t) + \begin{bmatrix} 4.58 & 5.54 \\ -5.20 & 7.76 \\ 0 & 0 \\ 0.84 & -14.91 \\ 6.02 & 0.21 \end{bmatrix} \left(y_0(t) - \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \hat{\omega}_2(t) \right)$$

$$\dot{\hat{\omega}}_3(t) = \bar{S}\hat{\omega}_3(t) + \begin{bmatrix} 4.58 & 5.54 \\ -5.20 & 7.76 \\ 0 & 0 \\ 0.84 & -14.91 \\ 6.02 & 0.21 \end{bmatrix} \left(y_2(t) - \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \hat{\omega}_3(t) \right)$$

where

$$\bar{S} = \begin{bmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The initial conditions of the observers were taken as

$$\hat{\omega}_1(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \hat{\omega}_2(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \hat{\omega}_3(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The third initial condition is restricted to be the one, in order to obtain the desired separations y_i^d's.

Finally, according to the previous the distributed control laws are computed as

$$u_1(t) = \begin{bmatrix} 5.39 & 2.25 & 4.36 \\ 3.91 & 3.63 & 6.44 \end{bmatrix} x_1(t) + \begin{bmatrix} 1.69 & 4.77 & 15.94 & 0 & 0 \\ -5.57 & 14.36 & -24.04 & 0 & 0 \end{bmatrix} \hat{\omega}_1(t)$$

$$u_2(t) = \begin{bmatrix} 8.84 & 0.07 & 3.81 & 8.34 & 6.74 \\ 0.32 & -0.67 & 9.50 & 1.53 & 0.89 \end{bmatrix} x_2(t) + \begin{bmatrix} 4.16 & 8.36 & 0 & 0 & 0 \\ 2.52 & -7.31 & 0 & 0 & 0 \end{bmatrix} \hat{\omega}_2(t)$$

$$u_3(t) = \begin{bmatrix} 11.05 & 22.59 & 2.80 & 12.38 \\ -22.54 & -50 & -9.85 & -19.83 \end{bmatrix} x_3(t) + \begin{bmatrix} 4.73 & 5.92 & 89.42 & 0 & 0 \\ -16.87 & -5.96 & -178.79 & 0 & 0 \end{bmatrix} \hat{\omega}_3(t)$$

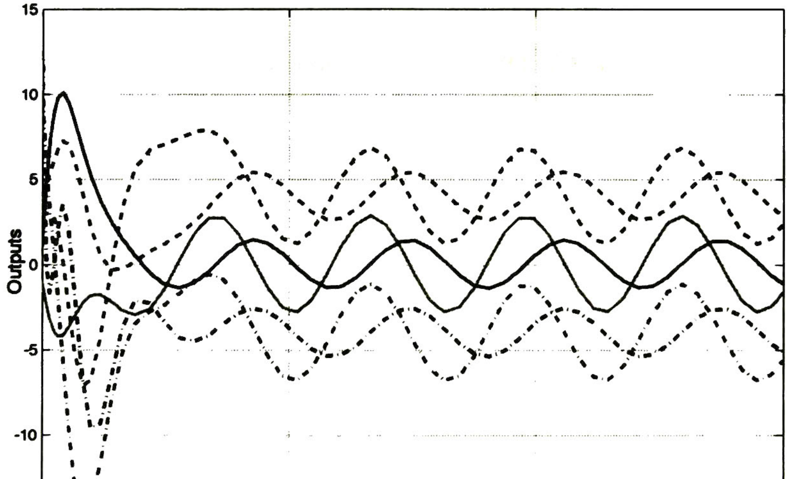

Figure 3.2: Outputs achieving consensus and regulation

The trajectories of the outputs of each system obtained using this technique in simulation are shown in Fig. 3.2. The outputs of the same agent share the same line style. The exosystem $\omega(t)$ is composed by a sine function $\omega_1(t)$, a cosine $\omega_2(t)$ and a constant $\omega_3(t) = 1$. It can be seen in this figure that the value of the reference is attained by every output.

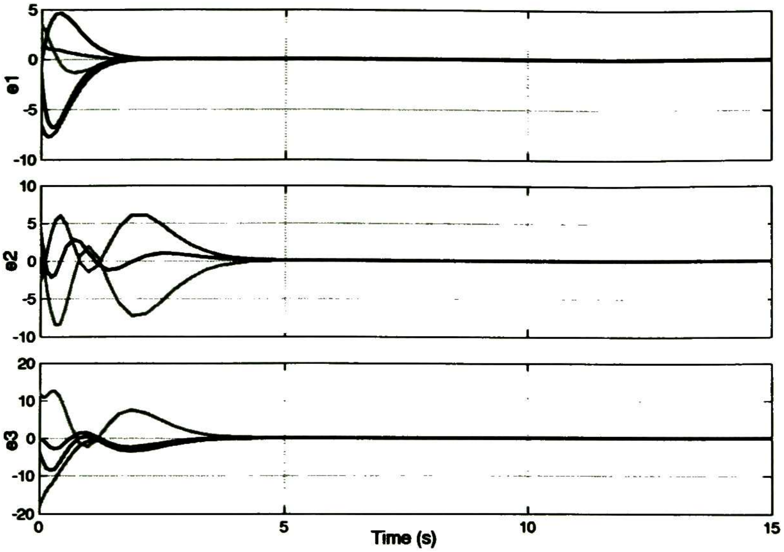

Figure 3.3: Observer-regulation errors (3.19)

Fig. 3.3 displays the behavior of the observer-regulation errors (3.19), and the regulation errors $error_i = x_i - \Pi_i \omega$ are shown in Fig. 3.4. It can be seen that both errors are stable.

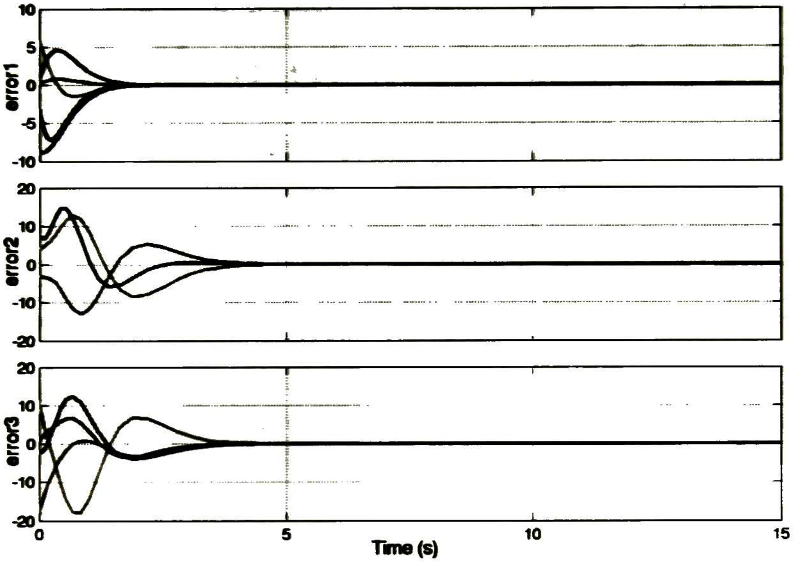

Figure 3.4: Regulation errors

Chapter 4

Output consensus and output regulation of a class of *SLS* multi-agent systems

The design of distributed control laws for the output consensus and output regulation of switched linear multi-agent systems is addressed in this chapter, where each agent receives the measurement of the outputs of its neighbors. The communication among the agents is described as a graph, and it is restricted to be such that the corresponding graph has a spanning tree.

The cases of fixed and switching communication topologies are undertaken. In the first case, it is considered that a subset of the agents has access to the output of the reference (exosystem) and none of them has access to the state of the exosystem.

In the case of switching communication topologies, it is considered that the set of agents (nodes) remain constant under any communication topology. Similar to the case of fixed topologies, it is required that each of the graphs representing each of the communication topologies contains a spanning tree. It is also considered that only a subset of the agents has access to the output of the reference and none of them has access to the state of the exosystem. This subset is allowed to be different in any communication topology.

The subset of the agents with access to the references is, in both cases, restricted to be different from the empty set.

4.1 Switched linear systems

A switched linear system (*SLS*) $\mathcal{V} = \langle \mathcal{F}, \sigma_t \rangle$ is a hybrid dynamical system where $\mathcal{F} = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_k\}$ is a collection of linear systems of the form

$$\Sigma_{\sigma_t} \begin{cases} \dot{x}(t) = A_{\sigma_t} x(t) + B_{\sigma_t} u(t), & x(t_0) = x_0 \\ y(t) = Cx(t) \end{cases} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^q$, are the state, input and output variables respectively and A_i, B_i and C, $i \in \{1, \ldots, k\}$ are constant matrices of appropriate dimensions, and $\sigma_t : [t_0, \infty) \to \{1, \ldots, k\}$ is the switching signal that determines the evolving linear dynamics,.

Remark 4.1. *In systems as (4.1), it is often taken that* $y(t) = C_i x(t)$, *but for the object of this work only systems with* $C_i = C$ *for all* $i \in \{1, \ldots, k\}$ *will be considered.*

The last restriction is required in order to fulfill the *SLS* systems regulation conditions to be introduced later in this chapter.

This model presents a state $x(t)$ that evolves according to k different dynamics. The switching between any of these dynamics can be triggered by many different events: the change of state on an automaton, the elapsed of a certain amount of time, the reach of a given region of space by the state, direct manipulation of the controller, etc.

4.1.1 Switched linear systems stability

In this section, results on *SLS* stability are presented. Several new results have been presented to this respect, the readers interested in related topics are referred to [21], [1], [3], [20], [46], [29], [38], [43], [37], [22], and [24].

Lyapunov stability

The following definitions and results on Lyapunov stability were taken from [21].

The switched linear system (4.1) under a state feedback control

$$u(t) = -K_i x(t) \qquad \sigma_t = i \tag{4.2}$$

is *uniformly asymptotically stable* if there exist a positive constant g and a class $\mathcal{KL}$ function f such that for all switching signals σ_t the solutions of (4.1) with $|x(0)| \leq g$ satisfy the inequality

$$|x(t)| \leq f(|x(0)|, t) \qquad \forall t \geq 0. \tag{4.3}$$

If the inequality (4.3) is valid for all switching signals and all initial conditions, the system (4.1) under a state feedback control(4.2) is *globally uniformly asymptotically stable (GUAS).*

A positive definite continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is a common Lyapunov function for the family of systems $\mathcal{F}$ if there exists a positive definite continuous function $Q : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(A_i - B_i K_i)\, x(t) \leq -Q(x(t)) \qquad \forall x(t),\, \forall i \tag{4.4}$$

Then, the following result on stability can be stated.

Theorem 4.1. *(Theorem 2.1 [21]) If all systems in the family $\mathcal{F}$ with control (4.2) share a radially unbounded common Lyapunov function $V(x(t))$, then the switched system (4.1) is globally uniformly asymptotically stable (GUAS).*

This result assures the stability of the *SLS* system under any switching sequence. It is important to mention that if a common Lyapunov function for the *SLS* is not found, it is not enough proof to assert that a *SLS* is unstable.

In ([28]) the following result is stated

Lemma 4.2. *If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and matrices $Z_i \in \mathbb{R}^{m_i \times n}$, $i = \{1, \cdots, N\}$, such that*

$$A_i P + P A_i^T + B_i Z_i + Z_i^T B_i^T < 0$$

then the switched control law (4.2) with

$$K_i = Z_i P^{-1}$$

assures the closed-loop stability of the switched system (4.1).

Now, it is possible to state the problem of finding a common Lyapunov function for a *SLS* system as a LMI program to compute a switched state feedback.

Phase plane stability condition for *SLS* systems

In the following, a sufficient condition for *SLS* systems stability is presented. This condition is related to the eigenvectors of the subsystems of the *SLS* system.

An eigenvector associated to a linear system $\dot{x}(t) = Ax(t)$ is a vector $v \neq 0$, such that $Av = \lambda v$, where λ is an eigenvalue of the system.

Consider a switched linear system

$$\dot{x}(t) = A_i x(t) + B_i u(t) \tag{4.5}$$

where the pairs (A_i, B_i) are controllable and a switched state feedback (4.2).

Theorem 4.3. *If the switched state feedback (4.2) is such that, the matrices* $(A_i - B_iK_i)$ *are Hurwitz, and its eigenvectors are the same for all* i*, then the SLS system will be stable.*

If a *SLS* system is stabilized using this result, then the phase planes of its composing linear dynamics will align and the *SLS* system will have a phase plane similar to a linear system.

For the proof and control design related to the previous result, refer to the appendix.

4.1.2 Output regulation of switched linear systems

Let

$$\dot{\omega}(t) = S\omega(t) \tag{4.6}$$

where $\omega(t) \in \mathbb{R}^p$ and $S \in \mathbb{R}^{p\times p}$, be an exosystem from which a reference $R\omega(t)$ will be defined for a *SLS* system and consider the input

$$u(t) = -K_ix(t) + \Gamma_i\omega(t) \tag{4.7}$$

then, from [25], the problem of Output Regulation via Full Information for *SLS* (*ORFI*) consists on, given A_i, B_i, C, P_i, R, S, and having full access to $x(t)$, find a feedback law of the form (4.7) such that

1) system (4.1) with control (4.2) is asymptotically stable under arbitrary switching laws, and

2) for each initial condition x_0, the solution $x(t)$ of system (4.1) with (4.7) is such that $Cx(t) - R\omega(t)$ tends to zero uniformly under arbitrary switching laws.

Consider the following

Hypothesis 4.1. *($\mathcal{H}4.1$) System (4.6) is antistable, i.e. all the eigenvalues of S have nonnegative real part.*

Hypothesis 4.2. *($\mathcal{H}4.2$) a) system (4.1) with (4.2) has a common Lyapunov function.*

b) there exists a solution Π_i *to the following equations*

$$\begin{aligned} \Pi_iS &= (A_i - B_iK_i)\Pi_i + (P_i + B_i\Gamma_i) \\ 0 &= C\Pi_i - R \qquad\qquad i = 1, 2, \cdots, k. \end{aligned}$$

Hypothesis 4.3. *($\mathcal{H}4.3$) It will be considered that the evolving dynamics of system (4.1) and its switching instants are always known.*

Theorem 4.4. *(Theorem 1 [25]) Assume ($\mathcal{H}4.1$) and ($\mathcal{H}4.2$). Then the ORFI of switched system (1) is solvable only if for $\forall i, j$, $i, j = 1, 2, \cdots, k$, the following condition holds*

$$C(A_l - B_l K_l)^m (\Pi_i - \Pi_j) = 0, \; m = 0, 1, \cdots, s \; l = i, j,$$

where s is the index of the admissible pair (C, $A_i - B_i K_i$).

A more restrictive condition is given if ($A_i - B_i K_i$, C) is observable, as in the following result.

Theorem 4.5. *(Theorem 2 [25]) Assume ($\mathcal{H}4.1$), ($\mathcal{H}4.2$), and that ($A_i - B_i K_i$, C), $i = 1, 2, \cdots, k$ is observable. Then the ORFI of (1) is solvable if and only if $\Pi_i = \Pi_j$, for $i, j = 1, 2, \cdots, k$.*

From Theorem 4.4, it is clear that a sufficient condition for the problem of *ORFI* to be solvable is: if a state of a subsystem $\Sigma_i \in \mathcal{F}$ is observable, its respective elements of Π_i have to be equal to those of Π_j for $i, j = 1, 2, \cdots, k$.

A class of *SLS* systems that meets the conditions of Theorem 4.4 is the one of systems of the form

$$\dot{x}(t) = \begin{bmatrix} A_1 & A_2 & 0 \\ A_{3,i} & A_{4,i} & 0 \\ A_{5,i} & A_{6,i} & A_{7,i} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_{1,i} \\ B_{2,i} \end{bmatrix} u(t) \tag{4.8}$$

$$y(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x^\eta(t) \\ x_3(t) \end{bmatrix} \tag{4.9}$$

for $i = 1, 2, \cdots, k$, where $x_1(t) \in \mathbb{R}^{\eta - q}$, $x_2(t) \in \mathbb{R}^q$, $x_3(t) \in \mathbb{R}^{n-\eta}$, $u(t) \in \mathbb{R}^m$, $q \leq m$, $B_{1,i}$ is a full row rank matrix, $A_{7,i}$ is Hurwitz and the subsystem

$$\dot{x}^\eta(t) = A_i^\eta x^\eta(t) + B_i^\eta u(t), \quad x^\eta(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{4.10}$$

is controllable. Matrices A_i^η and B_i^η are defined as follows

$$A_i^\eta = \begin{bmatrix} A_1 & A_2 \\ A_{3,i} & A_{4,i} \end{bmatrix}, \quad B_i^\eta = \begin{bmatrix} 0 \\ B_{1,i} \end{bmatrix}$$

In particular, subsystem (4.10) with output $y^\eta(t) = C_1 x^\eta(t)$ also fulfills the conditions of Theorem 4.5.

4.2 Design of a distributed control law for the output consensus and output regulation of *SLS* multi-agent systems

In the present section, the problem of output consensus and output regulation of *SLS* multi-agent systems is addressed.

The proposed approach starts by computing a local state feedback control law for each linear dynamics of a *SLS* agent. Such local feedback control is computed to ensure the stability of the *SLS* system. Under the assumptions made on communication topologies, every agent is capable of getting the exosystem state by means of an observer to be regulated, and the stability of the *SLS* agents guarantee the stability of the switched observer-regulation error. Therefore, the outputs of the *SLS* system achieve consensus and, regulation on a given formation.

4.2.1 Problem statement

Consider a system of N agents with different *SLS* dynamics of class (4.8), (4.9) described by

$$\begin{array}{ll} \dot{x}_i(t) = A_{i,j_i}x_i(t) + B_{i,j_i}u_i(t) & i = 1,\dots,N \\ y_i(t) = C_i x_i(t) & j_i = 1,\dots,k_i \end{array} \tag{4.11}$$

where, $x_i(t) \in \mathbb{R}^{n_i}$ is the state, $u_i(t) \in \mathbb{R}^{m_i}$ the control, and $y_i(t) \in \mathbb{R}^q$ the output of the i-th agent, and a virtual reference agent whose dynamics is represented by the exosystem (4.6) with output

$$y_0(t) = \begin{bmatrix} R & 0 \end{bmatrix} \omega(t) \tag{4.12}$$

where $R \in \mathbb{R}^{q \times p-1}$ and the last entry of vector ω is a constant $\omega_p(t) = 1$.

Remark 4.2. *It is considered that rank* $\{C_i\} = q$ *for* $i = 1,\dots,N$.

Let $\mathcal{G}$ be a graph representing the communication between the *SLS* agents of system (4.11) together with the exosystem (4.6), then the following hypothesis is assumed.

Hypothesis 4.4. *($\mathcal{H}4.4$) The graph $\mathcal{G}$ does not contain loops, and incorporates a spanning tree with root on ν_0 (reference agent).*

Consider $y_i^d = d_i\omega_p(t) \in \mathbb{R}^q$ as the constant desired relative position of the output of the i-th agent with respect to the virtual agent output (reference). Then define the i-th agent output consensus and regulation error as

$$\begin{aligned} \zeta_i(t) = \sum_{j=1}^{N} \alpha_{i,j} \left[(y_j(t) - y_i(t)) - (d_j - d_i)\omega_p(t) \right] \\ + \alpha_{i,0} \left[(y_0(t) - y_i(t)) + d_i\omega_p(t) \right]. \end{aligned} \tag{4.13}$$

Note that the weight $\alpha_{i,0}$ corresponds to the difference between the i-th agent output and the virtual agent (reference) output (4.12).

Thus, the problem of distributed output regulation and output consensus of *SLS* multi-agent systems consists on obtaining a distributed control law such that every agent output consensus and regulation error $\zeta_i(t)$ tends to zero asymptotically.

Next, the problem of distributed output regulation and output consensus of *SLS* multi-agent systems, is addressed for agents with different dimension and dynamics.

4.2.2 Distributed output consensus and regulation control

For ease of notation and without loss of generality, consider the controllable part (4.10) of N *SLS* agents of class (4.8), with output (4.9) as in (4.11).

Given that only a subset of the *SLS* agents will have access to the reference (4.12) and none of them to the exosystem state (4.6), the following observers are proposed

$$\dot{\hat{\omega}}_i(t) = \bar{S}\hat{\omega}_i(t) + \kappa_i(\sum_j y_j(t) - \rho_i L\hat{\omega}_i(t)) \quad \forall j \in \Theta_i \tag{4.14}$$

where $L = \begin{bmatrix} R & 0 & \mathbf{I}_q \end{bmatrix}$, and κ_i is such that the matrix $\left(\bar{S} - \rho_i\kappa_i L\right)$ is Hurwitz,

$$\bar{S} = \begin{bmatrix} S & 0 \\ 0 & 0_q \end{bmatrix} \tag{4.15}$$

and 0_q is a $q \times q$ zero matrix.

Remark 4.3. *The matrix $\bar{S}$ is used as an extended exosystem matrix to filter the unknown d_j's from the exosystem states.*

Now, set the distributed output regulation control law as

$$u_i(t) = -K_{i,j_i}x_i(t) + \begin{bmatrix} \Gamma_{i,j_i} & 0_q \end{bmatrix}\hat{\omega}_i(t), \tag{4.16}$$

for $j_i \in \{1, \cdots k_i\}$. Note that control (4.16) is of the form (4.7) and depends on the exosystem observer state $\hat{\omega}_i(t)$.

Define the i-th observer error as

$$\tilde{\omega}_i(t) = \bar{\omega}_i(t) - \hat{\omega}_i(t), \tag{4.17}$$

where $\bar{\omega}_i(t) = \begin{bmatrix} \omega(t) \\ \delta_i \end{bmatrix}$, and $\delta_i = \sum_j d_j/\rho_i$, $\forall j \in \Theta_i$. Hence, its dynamics will be described by

$$\dot{\tilde{\omega}}_i(t) = \bar{S}\tilde{\omega}_i(t) - \kappa_i(\sum_j y_j(t) - \rho_i L\hat{\omega}_i(t)). \tag{4.18}$$

Consider that $\bar{\Pi}_i = \begin{bmatrix} \Pi_i & 0_q \end{bmatrix}$, and that Hypothesis ($\mathcal{H}4.2$) is met, then the i-th agent regulation error can be defined as

$$e_i(t) = x_i(t) - \bar{\Pi}_i \hat{\omega}_i(t) \tag{4.19}$$

and its dynamics is given by

$$\dot{e}_i(t) = (A_{i,j_i} - B_{i,j_i} K_{i,j_i}) e_i(t) - \bar{\Pi}_i \kappa_i (\sum_j y_j(t) - \rho_i L \hat{\omega}_i(t)). \tag{4.20}$$

From (4.17) and (4.19), it can be inferred that

$$\begin{aligned} y_i(t) &= C_i x_i(t) \\ &= C_i \left(e_i(t) + \bar{\Pi}_i \left(\bar{\omega}_i(t) - \tilde{\omega}_i(t) \right) \right) \\ &= C_i e_i(t) + \bar{R}_i \left(\bar{\omega}_i(t) - \tilde{\omega}_i(t) \right) \end{aligned} \tag{4.21}$$

where, $\bar{R}_i = \begin{bmatrix} R & d_i & 0_q \end{bmatrix}$.

Take the second term in (4.18) and substitute y_j as in (4.21)

$$\left(\sum_j y_j(t) - \rho_i L \hat{\omega}_i(t) \right) = -\rho_i L \hat{\omega}_i(t) + \sum_j \left\{ \bar{R}_j \left(\bar{\omega}_j(t) - \tilde{\omega}_j(t) \right) + C_j e_j(t) \right\}. \tag{4.22}$$

Given that $\omega_p(t) = 1$, it can be seen that

$$\sum_j \bar{R}_j \bar{\omega}_j(t) = \rho_i L \bar{\omega}_i(t), \tag{4.23}$$

thus,

$$\left(\sum_j \bar{R}_j \bar{\omega}_j(t) - \rho_i L \hat{\omega}_i(t) \right) = \rho_i L \tilde{\omega}_i(t). \tag{4.24}$$

Remark 4.4. *The observed variables $\hat{\omega}_{i,(p+\alpha)}$ where $\alpha = 1, 2, \cdots, q$, will tend asymptotically to $\delta_i = \sum_j d_j / \rho_i$, $\forall j \in \Theta_i$.*

Using (4.22) and (4.24),z the i-th observer and regulation error dynamics (4.18), (4.20) can be restated as

$$\dot{\tilde{\omega}}_i(t) = \kappa_i \left(\sum_j \left(\bar{R}_j \tilde{\omega}_j(t) - C_j e_j(t) \right) \right) + \left(\bar{S} - \kappa_i \rho_i L \right) \tilde{\omega}_i(t) \tag{4.25}$$

$$\begin{aligned} \dot{e}_i(t) = \bar{\Pi}_i \kappa_i \sum_j \left(\bar{R}_j \tilde{\omega}_j(t) - C_j e_j(t) \right) \\ + (A_{i,j_i} - B_{i,j_i} K_{i,j_i}) e_i(t) - \bar{\Pi}_i \kappa_i \rho_i L \tilde{\omega}_i(t). \end{aligned} \tag{4.26}$$

Define the i-th agent observer-regulation error as

$$\xi_i(t) = \begin{bmatrix} \tilde{\omega}_i(t) \\ e_i(t) \end{bmatrix} \tag{4.27}$$

and *SLS* multi-agent observer-regulation error as

$$\xi(t) = \left[\ \xi_1^T(t)\ \ \xi_2^T(t)\ \ \cdots\ \ \xi_N^T(t)\ \right]^T \tag{4.28}$$

Under Hypothesis ($\mathcal{H}$4.4), the Laplacian $\mathcal{L}$ is a lower triangular matrix. Thus, the *SLS* multi-agent observer-regulation error dynamics will have a block lower triangular form

$$\dot{\xi}(t) = \begin{bmatrix} \Omega_{1,j_1} & 0 & \cdots & 0 & 0 \\ \varrho_{2,1} & \Omega_{2,j_2} & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ \varrho_{N-1,1} & \varrho_{N-1,2} & & \Omega_{N-1,j_{N-1}} & 0 \\ \varrho_{N,1} & \varrho_{N,2} & \cdots & \varrho_{N,N-1} & \Omega_{N,j_N} \end{bmatrix} \xi(t) \tag{4.29}$$

with switched matrix blocks

$$\Omega_{i,j_i} = \begin{bmatrix} \bar{S} - \kappa_i \rho_i L & 0 \\ \\ -\bar{\Pi}_i \kappa_i \rho_i L & A_{i,j_i} - B_{i,j_i} K_{i,j_i} \end{bmatrix} \tag{4.30}$$

on its diagonal. The terms ϱ_{i,j_i} are considered constant.

Now, the main result of this section can be stated.

Theorem 4.6. *Consider that Hypothesis ($\mathcal{H}$4.1)-($\mathcal{H}$4.4) are met, then the distributed output regulation and output consensus of the SLS multi-agent system with agents of the form (4.8) with output (4.9) can be solved by a control law of the form (4.16).*

Proof. Consider a *SLS* multi-agent system with agents of the form (4.8) with output (4.9), an exosystem (4.6) with output (4.12), and control law (4.16).

Then, under Hypothesis ($\mathcal{H}$4.4) the corresponding *SLS* multi-agent observer-regulation error will have the form (4.29). Given Hypothesis ($\mathcal{H}$4.1), the switched block matrices Ω_{i,j_i} are such that there exist positive definite matrices P_i and Q_{i,j_i}, for which the equations

$$\Omega_{i,j_i}^T P_i - P_i \Omega_{i,j_i} = -Q_{i,j_i} \tag{4.31}$$

are met.

Set

$$V(t) = \sum_i V_i(t) \tag{4.32}$$

as a common Lyapunov function for system (4.29) where

$$V_i(t) = \xi_i(t)^T P_i \xi_i(t). \tag{4.33}$$

Consider $V_1(t)$, whose derivative

$$\dot{V}_1(t) = -\xi_1(t)^T Q_{1,j_1} \xi_1(t) \tag{4.34}$$

is clearly negative, independent from the value of $\xi_1(t)$. Thus, the observer-regulation error $\xi_1(t)$ is stable. For $V_2(t)$ its derivative is

$$\dot{V}_2(t) = -\xi_2(t)^T Q_{2,j_2} \xi_2(t) + \xi_1(t)^T \varrho_{2,1}^T P_2 \xi_2(t) + \xi_2(t)^T P_2 \varrho_{2,1} \xi_1(t) \tag{4.35}$$

hence, given the stability of $\xi_1(t)$ the terms with $\varrho_{2,1}$ will become zero and the derivative of $V_2(t)$ will be negative, therefore, the observer-regulation error $\xi_2(t)$ is stable. The same reasoning can be applied to the rest of the $V_i(t)$'s. In this way, It is concluded that the derivative of the common Lyapunov function (4.32) is negative, and as a result, the observer-regulation error (4.29) is stable.

Finally, given the stability of the observer-regulation error, in steady state, the following holds

$$\hat{\omega}_i(t) = \bar{\omega}_i(t) \tag{4.36}$$

$$x_i(t) = \Pi_i \omega(t) \tag{4.37}$$

$$y_i(t) = \begin{bmatrix} R & d_i \end{bmatrix} \omega(t) \tag{4.38}$$

thus, the i-th output consensus and regulation error $\zeta_i(t) = 0$ for $i = 1, 2, \cdots, N$. □

4.2.3 Example

Consider 3 agents with the following switched linear dynamics

$$\dot{x}_1(t) = \begin{cases} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 9 & -2 & 4 & 0 \\ 0 & 0 & 1 & -3 \\ 7 & 4 & -1 & -1 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 7 & 0 \end{bmatrix} u_1(t) & (x_1(t) > 0 \; and \; x_2(t) > 0) \\ \begin{bmatrix} 1 & 2 & 3 & 6 \\ 9 & -2 & 4 & 0 \\ 4 & -2 & 5 & 1 \\ 6 & 2 & 5 & -3 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix} u_1(t) & not(x_1(t) > 0 \; and \; x_2(t) > 0) \end{cases}$$

$$y_1(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix} x_1(t)$$

$$\dot{x}_2(t) = \begin{cases} \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & 0 \\ 4 & 7 & -7 \end{bmatrix} x_2(t) + \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} u_2(t) & (x_1(t) > 0 \; and \; x_2(t) > 0) \\ \begin{bmatrix} 10 & -1 & 0 \\ -3 & 1 & 0 \\ 9 & -3 & -1 \end{bmatrix} x_2(t) + \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} u_2(t) & (x_1(t) < 0 \; and \; x_2(t) < 0) \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix} x_2(t) + \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} u_2(t) & (x_1(t) \cdot x_2(t) \leq 0) \end{cases}$$

$$y_2(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} x_2(t)$$

$$\dot{x}_3(t) = \begin{cases} \begin{bmatrix} -3 & 0 & 0 & 5 \\ 3 & 2 & 0 & 3 \\ 1 & 4 & -2 & 0 \\ 0 & -2 & -2 & -1 \end{bmatrix} x_3(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} u_3(t) & (x_1(t) > 0 \; and \; x_2(t) > 0) \\ \begin{bmatrix} -3 & 0 & 0 & 5 \\ 3 & 2 & 0 & 3 \\ 1 & 4 & -2 & 0 \\ 0 & -2 & -2 & -1 \end{bmatrix} x_3(t) + \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u_3(t) & not(x_1(t) > 0 \; and \; x_2(t) > 0) \end{cases}$$

$$y_3(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \end{bmatrix} x_3(t)$$

with initial conditions

$$x_1(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix} \quad x_2(0) = \begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix} \quad x_3(0) = \begin{bmatrix} 3 \\ 7 \\ -9 \\ 2 \end{bmatrix}$$

The multi-agent system including the above systems is connected according to the following Laplacian matrix

$$\mathcal{L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

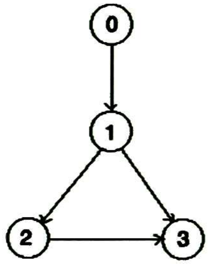

Figure 4.1: Communication topology graph

so that only the first agent receives the reference to track. The graph corresponding to this communication topology is shown in Fig. 4.1.

The desired relative positions of each output with respect to the reference are described by the next constant vectors

$$d_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad d_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad d_3 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

Let the reference exosystem be

$$\dot{\omega}(t) = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega(t)$$

where the zero row accounts for a constant $\omega_3(t)$, consider the initial conditions

$$\omega(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and take the reference matrix as

$$R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, the exosystem observers are described by

$$\dot{\hat{\omega}}_1(t) = \bar{S}\hat{\omega}_1(t) + \begin{bmatrix} 4.58 & 5.54 \\ -5.20 & 7.76 \\ 0 & 0 \\ 0.84 & -14.91 \\ 6.02 & 0.21 \end{bmatrix} \left(y_0(t) - \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \hat{\omega}_1(t) \right)$$

$$\dot{\hat{\omega}}_2(t) = \bar{S}\hat{\omega}_2(t) + \begin{bmatrix} 4.58 & 5.54 \\ -5.20 & 7.76 \\ 0 & 0 \\ 0.84 & -14.91 \\ 6.02 & 0.21 \end{bmatrix} \left(y_1(t) - \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \hat{\omega}_2(t) \right)$$

$$\dot{\hat{\omega}}_3(t) = \bar{S}\hat{\omega}_3(t) + \begin{bmatrix} 4.58 & 5.54 \\ -5.20 & 7.76 \\ 0 & 0 \\ 0.84 & -14.91 \\ 6.02 & 0.21 \end{bmatrix} \left(y_1(t) + y_2(t) - 2\begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \hat{\omega}_3(t) \right)$$

where

$$\bar{S} = \begin{bmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The initial conditions of the observers were taken as

$$\hat{\omega}_1(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \hat{\omega}_2(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \hat{\omega}_3(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The third initial condition is restricted to be the one, in order to obtain the desired separations y_i^d's.

Finally, according to the previous, the switched distributed control laws are computed as

$$u_1(t) = \begin{cases} \begin{bmatrix} 1.14 & 0.63 & -0.05 & 0.63 \\ 6.31 & 1.91 & 6.31 & 3.29 \end{bmatrix} x_1(t) + \begin{bmatrix} 1.40 & -0.56 & 2.89 & 0 & 0 \\ -3.77 & 1.83 & -6.67 & 0 & 0 \end{bmatrix} \hat{\omega}_1(t) \\ \begin{bmatrix} 4.47 & 0.62 & 4.15 & 2.81 \\ 6.89 & 2.43 & 5.53 & 2.37 \end{bmatrix} x_1(t) + \begin{bmatrix} -1.54 & 0.78 & -2.61 & 0 & 0 \\ 9.83 & -3.96 & 20.14 & 0 & 0 \end{bmatrix} \hat{\omega}_1(t) \end{cases}$$

$$u_2(t) = \begin{cases} \begin{bmatrix} -4 & 2.33 & 0 \\ 1 & 0.33 & 0 \end{bmatrix} x_2(t) + \begin{bmatrix} -2.33 & 5.66 & 0 & 0 & 0 \\ 1.66 & -1.83 & 0 & 0 & 0 \end{bmatrix} \hat{\omega}_2(t) \\ \begin{bmatrix} -11.66 & 5.66 & 0 \\ 4.33 & -0.33 & 0 \end{bmatrix} x_2(t) + \begin{bmatrix} -2.33 & 5.66 & 0 & 0 & 0 \\ 1.66 & -1.83 & 0 & 0 & 0 \end{bmatrix} \hat{\omega}_2(t) \\ \begin{bmatrix} -1.66 & 4 & 0 \\ 1.33 & 0 & 0 \end{bmatrix} x_2(t) + \begin{bmatrix} -2.33 & 5.66 & 0 & 0 & 0 \\ 1.66 & -1.83 & 0 & 0 & 0 \end{bmatrix} \hat{\omega}_2(t) \end{cases}$$

$$u_3(t) = \begin{cases} \begin{bmatrix} 11.05 & 22.59 & 2.80 & 12.38 \\ -22.54 & -50 & -9.85 & -19.83 \end{bmatrix} x_3(t) + \begin{bmatrix} 4.73 & 5.92 & 89.42 & 0 & 0 \\ -16.87 & -5.96 & -178.79 & 0 & 0 \end{bmatrix} \hat{\omega}_3(t) \\ \begin{bmatrix} 12.10 & 32.63 & 11.27 & 2.68 \\ -0.49 & -4.94 & -4.23 & 4.88 \end{bmatrix} x_3(t) + \begin{bmatrix} 19.31 & -5.56 & 91.79 & 0 & 0 \\ -7.32 & 5.79 & -0.69 & 0 & 0 \end{bmatrix} \hat{\omega}_3(t). \end{cases}$$

The order of the switching controls is the same as the one of the *SLS* dynamics to which they correspond.

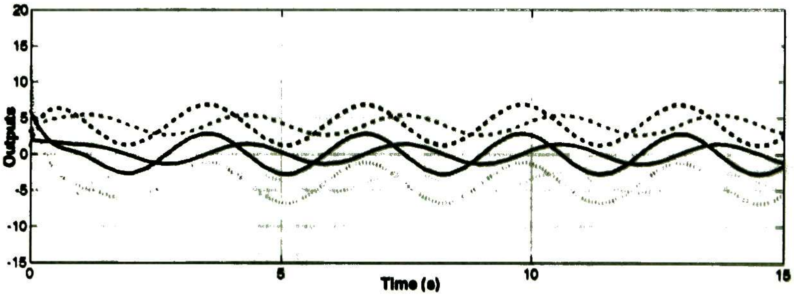

Figure 4.2: Output reaching formation and tracking.

The trajectories of the outputs of the multi-agent system obtained in simulation are shown in Fig. 4.2. The outputs, $y_{i,1}$ and $y_{i,2}$ follow the same trajectory respectively. The outputs of the same system share the same line style. The exosystem ω is in this case composed by a sine function ω_1, a cosine ω_2 and a constant $\omega_2 = 1$. It can be seen in this figure that the value of the reference is attained by every output.

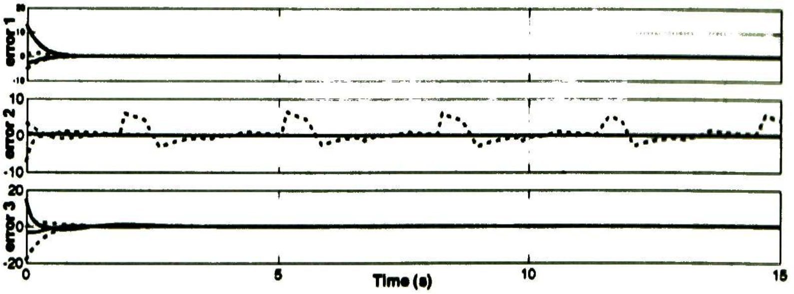

Figure 4.3: Output regulation and consensus error $e_i = x_i - \Pi_{i,j}\hat{\omega}$.

Fig. 4.3 displays the behavior of the regulation error (4.20), while Fig. 4.4 displays the regulation error with respect to the state of the exosystem $error_i = x_i - \Pi_{i,j}\omega$.

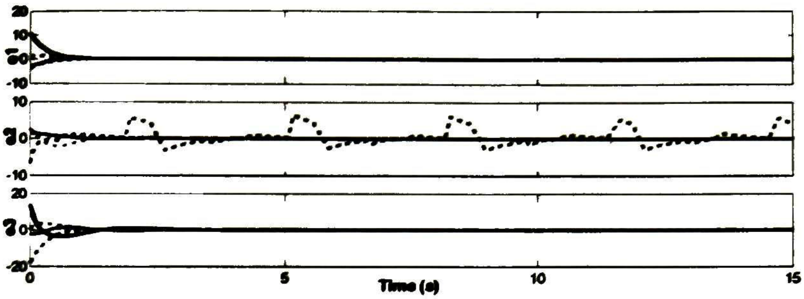

Figure 4.4: Output regulation and consensus error $error_i = x_i - \Pi_{i,j}\omega$.

From these figures, it can be seen that both errors are very similar and stable except by the dashed lines on errors of system two.

The dashed lines on errors of system two never decay to zero because the corresponding state does not share the same terms on the corresponding Π, but this error does not affect the regulation, given that such a state is not observable.

4.3 A distributed control design for the output regulation and output consensus of a class of switched linear multi-agent systems under switching communication topologies

In the present section, the problem of output consensus and output regulation of *SLS* multi-agent systems under switching communication topologies is addressed.

In the proposed approach, a local state feedback control law is computed for each linear dynamics of a *SLS* agent, such that, each of the linear dynamics of *SLS* system share a common Lyapunov function, thus the *SLS* agent dynamics is stabilized, and as in the previous approach the exosystem state is obtained by means of an observer.

Under the assumptions made on communication topologies, the existence of Lyapunov functions for each of the agents guarantee the existence of a common Lyapunov function for the *SLS* system with a negative derivative under any communication topology meeting the restrictions. Hence, the stability of the switched regulation and consensus error is guaranteed under switching communication topologies, and the outputs of the *SLS* multi-agent system achieve consensus and regulation.

4.3.1 Problem statement

Consider a system of N agents with different *SLS* dynamics of class (4.8), (4.9) described by

$$\begin{array}{ll} \dot{x}_i(t) = A_{i,j_i}x_i(t) + B_{i,j_i}u_i(t) & i = 1,\dots,N \\ y_i(t) = C_i x_i(t) & j_i = 1,\dots,k_i \end{array} \tag{4.39}$$

where, $x_i(t) \in \mathbb{R}^{n_i}$ is the state, $u_i(t) \in \mathbb{R}^{m_i}$ the control, and $y_i(t) \in \mathbb{R}^q$ the output of the i-th agent, and a virtual reference agent whose dynamics is represented by the exosystem (4.6) with output

$$y_0(t) = R\omega(t) \tag{4.40}$$

where $R \in \mathbb{R}^{q \times p-1}$

Remark 4.5. *It is considered that* $\text{rank}\{C_i\} = q$ *for* $i = 1,\dots,N$.

Let $\mathcal{G}_{\varrho_t}$ be a switching graph representing communication among the *SLS* agents of system (4.39) together with the exosystem (4.6), then the following hypothesis is assumed.

Hypothesis 4.5. *($\mathcal{H}4.5$) The graph $\mathcal{G}_{\varrho_t}$ does not contain loops, and incorporates a spanning tree with root on ν_0 (reference agent) for any ϱ_t.*

Define the i-th agent output consensus and regulation error as

$$\zeta_{i_{\varrho_t}}(t) = \sum_{j=1}^{N} \alpha_{i,j_{\varrho_t}} \left[(y_j(t) - y_i(t))\right] + \alpha_{i,0_{\varrho_t}} \left[(y_0(t) - y_i(t))\right]. \tag{4.41}$$

Note that the weight $\alpha_{i,0_{\varrho_t}}$ corresponds to the difference between the i-th agent output and the virtual agent (reference) output (4.40).

Thus, the problem of distributed output regulation and output consensus of *SLS* multi-agent systems under switching interaction topologies consists on obtaining a distributed control law such that every agent output consensus and regulation error $\zeta_{i_{\varrho_t}}(t)$ tends to zero asymptotically under any switching sequence of ϱ_t.

Next, the problem of distributed output regulation and output consensus of *SLS* multi-agent systems under switching interaction topologies, is addressed for agents with different dimension and dynamics.

4.3.2 Distributed output consensus and regulation control

For ease of notation and without loss of generality, consider the controllable part (4.10) of N *SLS* agents of class (4.8), with output (4.9) as in (4.39).

Given that only a subset of the *SLS* agents will have access to the reference (4.40) and none of them to the exosystem state (4.6), the following observers are proposed

$$\dot{\hat{\omega}}_i(t) = S\hat{\omega}_i(t) + \frac{\kappa_i}{\rho_{i_{et}}}(\sum_j y_j(t) - \rho_{i_{et}} R\hat{\omega}_i(t)) \quad \forall j \in \Theta_{i_{et}} \tag{4.42}$$

where κ_i is such that the matrix $(S - \kappa_i R)$ is Hurwitz. Now, set the distributed output regulation control law as

$$u_i(t) = -K_{i,j_i} x_i(t) + \Gamma_{i,j_i}\hat{\omega}_i(t), \tag{4.43}$$

for $j_i \in \{1, \cdots k_i\}$. Note that control (4.43) is of the form (4.7) and depends on the exosystem observer state $\hat{\omega}_i(t)$.

Define the i-th observer error as

$$\tilde{\omega}_i(t) = \omega(t) - \hat{\omega}_i(t), \tag{4.44}$$

hence, its dynamics will be described by

$$\dot{\tilde{\omega}}_i(t) = S\tilde{\omega}_i(t) - \frac{\kappa_i}{\rho_{i_{et}}}(\sum_j y_j(t) - \rho_{i_{et}} R\hat{\omega}_i(t)). \tag{4.45}$$

Consider that Hypothesis ($\mathcal{H}$4.2) is met, then the i-th agent regulation error can be defined as

$$e_i(t) = x_i(t) - \Pi_i\hat{\omega}_i(t) \tag{4.46}$$

and its dynamics is given by

$$\dot{e}_i(t) = (A_{i,j_i} - B_{i,j_i}K_{i,j_i})e_i(t) - \Pi_i\frac{\kappa_i}{\rho_{i_{et}}}(\sum_j y_j(t) - \rho_{i_{et}} R\hat{\omega}_i(t)). \tag{4.47}$$

From (4.44) and (4.46), it can be inferred that

$$\begin{aligned} y_i(t) &= C_i x_i(t) \\ &= C_i\left(e_i(t) + \Pi_i\left(\omega(t) - \tilde{\omega}_i(t)\right)\right) \\ &= C_i e_i(t) + R\left(\omega(t) - \tilde{\omega}_i(t)\right). \end{aligned} \tag{4.48}$$

Take the second term in (4.45) and substitute y_j as in (4.48)

$$\left(\sum_j y_j(t) - \rho_{i_{et}} R\hat{\omega}_i(t)\right) = \sum_j \left\{R\left(\omega(t) - \tilde{\omega}_j(t)\right) + C_j e_j(t)\right\} - \rho_{i_{et}} R\hat{\omega}_i(t). \tag{4.49}$$

It can be seen that

$$\sum_j R\omega(t) = \rho_{i_{et}} R\omega(t), \tag{4.50}$$

thus,

$$\left(\sum_j R\omega(t) - \rho_{i_{\varrho t}} R\hat{\omega}_i(t)\right) = \rho_{i_{\varrho t}} R\tilde{\omega}_i(t). \tag{4.51}$$

Using (4.49) and (4.51), the i-th observer and regulation error dynamics (4.45), (4.47) can be restated as

$$\dot{\tilde{\omega}}_i(t) = \frac{\kappa_i}{\rho_{i_{\varrho t}}}\left(\sum_j (r\tilde{\omega}_j(t) - C_j e_j(t))\right) + (S - \kappa_i r)\,\tilde{\omega}_i(t) \tag{4.52}$$

$$\begin{aligned}\dot{e}_i(t) = {} & \Pi_i \frac{\kappa_i}{\rho_{i_{\varrho t}}} \sum_j (R\tilde{\omega}_j(t) - C_j e_j(t)) \\ & + (A_{i,j_i} - B_{i,j_i} K_{i,j_i}) e_i(t) - \Pi_i \kappa_i R\tilde{\omega}_i(t).\end{aligned} \tag{4.53}$$

Define the i-th agent observer-regulation error as

$$\xi_i(t) = \begin{bmatrix} \tilde{\omega}_i(t) \\ e_i(t) \end{bmatrix} \tag{4.54}$$

and SLS multi-agent observer-regulation error as

$$\xi(t) = \begin{bmatrix} \xi_1^T(t) & \xi_2^T(t) & \cdots & \xi_N^T(t) \end{bmatrix}^T \tag{4.55}$$

Consider that Hypothesis ($\mathcal{H}$4.5) is met, then, given any communication topology $\mathcal{G}_{\gamma_0}$, there exist an order O_{γ_0}, such that, the Laplacian $\mathcal{L}_{\gamma_0}$ is a lower triangular matrix. Thus, the reordered SLS multi-agent observer-regulation error dynamics will have a block lower triangular form

$$\dot{\xi}_{\gamma_0}(t) = \bar{A}_{\gamma_0} \xi_{\gamma_0}(t) \tag{4.56}$$

where

$$\bar{A}_{\gamma_0} = \begin{bmatrix} \Omega_{1,j_1} & 0 & \cdots & 0 & 0 \\ \beta_{2,1} & \Omega_{2,j_2} & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ \beta_{N-1,1} & \beta_{N-1,2} & & \Omega_{N-1,j_{N-1}} & 0 \\ \beta_{N,1} & \beta_{N,2} & \cdots & \beta_{N,N-1} & \Omega_{N,j_N} \end{bmatrix}$$

with indexes according to the order O_{γ_0} and switched matrix blocks

$$\Omega_{i,j_i} = \begin{bmatrix} S - \kappa_i R & 0 \\ -\Pi_i \kappa_i R & A_{i,j_i} - B_{i,j_i} K_{i,j_i} \end{bmatrix} \tag{4.57}$$

on its diagonal, where $\xi_{\varrho t}(t) = \mathcal{T}_{\varrho t}\xi(t)$ for $\varrho_t = \gamma_0$ and $\mathcal{T}_{\varrho t}\xi(t) \in \mathbb{R}^{\Sigma n_i}$ is a transformation matrix, which changes the order of the agents.

Hence, it is clear that the *SLS* multi-agent observer-regulation error will have switched linear dynamics

$$\dot{\xi}(t) = \tilde{A}_{\varrho_t}\xi(t) \tag{4.58}$$

where $\tilde{A}_{\varrho_t} \in \mathbb{R}^{\Sigma n_i \times \Sigma n_i}$.

Now, the main result of this section can be stated.

Theorem 4.7. *Consider that Hypothesis ($\mathcal{H}$4.1)-($\mathcal{H}$4.3), and ($\mathcal{H}$4.5) are met, then the distributed output regulation and output consensus of the SLS multi-agent system under switched interaction topologies with agents of the form (4.8) with output (4.9) can be solved by a control law of the form (4.43).*

Proof. Consider a *SLS* multi-agent system with agents of the form (4.8) with output (4.9), an exosystem (4.6) with output (4.40), and control law (4.43).

Then, under Hypothesis ($\mathcal{H}$4.5) and a given order O_{γ_0} the corresponding *SLS* multi-agent observer-regulation error will have the form (4.56). Given Hypothesis ($\mathcal{H}$4.1), the switched block matrices Ω_{i,j_i} are such that there exist positive definite matrices P_i and Q_{i,j_i}, for which the equations

$$\Omega_{i,j_i}^T P_i - P_i\Omega_{i,j_i} = -Q_{i,j_i} \tag{4.59}$$

are met.

Set

$$V(t) = \sum_i V_i(t) \tag{4.60}$$

as a common Lyapunov function for system (4.56) where

$$V_i(t) = \xi_i(t)^T P_i \xi_i(t). \tag{4.61}$$

Consider $V_1(t)$, whose derivative

$$\dot{V}_1(t) = -\xi_1(t)^T Q_{1,j_1}\xi_1(t) \tag{4.62}$$

is clearly negative, independent from the value of $\xi_1(t)$. Thus, the observer-regulation error $\xi_1(t)$ is stable. For $V_2(t)$ its derivative is

$$\dot{V}_2(t) = -\xi_2(t)^T Q_{2,j_2}\xi_2(t) + \xi_1(t)^T \varrho_{2,1}^T P_2\xi_2(t) + \xi_2(t)^T P_2\varrho_{2,1}\xi_1(t) \tag{4.63}$$

hence, given the stability of $\xi_1(t)$ the terms with $\varrho_{2,1}$ will become zero and the derivative of $V_2(t)$ will be negative, therefore, the observer-regulation error $\xi_2(t)$ is stable. The same reasoning can be applied to the rest of the $V_i(t)$'s. In this way, it is concluded that the derivative of the common Lyapunov function (4.60) is negative, and as a result, the observer-regulation error (4.56) is stable.

and the Λ_i's are diagonal matrices containing the absolute values of the eigenvalues of the matrices A_i.

Define a common Lyapunov function for the N linear subsystems

$$V(x) = x(t)^T P x(t) \qquad i = 1, 2, \cdots, N \tag{A.3}$$

where

$$P = (EE^T)^{-1} \tag{A.4}$$

Taking the derivative of $V(x(t))$

$$\dot{V}(x(t)) = x(t)^T (PA_i + A_i^T P) x(t) \tag{A.5}$$

is obtained, clearly in order to guarantee that $\dot{V}(x(t))$ is negative and consequently the *SLS* system is stable, it is needed that $L = PA_i + A_i^T P < 0$.

Let L be

$$L = (EE^T)^{-1} A_i + A_i^T (EE^T)^{-1} = -(EE^T)^{-1} W E^{-1} \tag{A.6}$$

solving for W from (A.6)

$$\begin{aligned} A_i + (EE^T) A_i^T (EE^T)^{-1} &= -WE^{-1} \\ A_i E + (EE^T) A_i^T (EE^T)^{-1} E &= -W \\ A_i E + EE^T A_i^T E^{-T} E^{-1} E &= -W \\ A_i E + EE^T A_i^T E^{-T} &= -W \\ -2E\Lambda_i &= -W. \end{aligned}$$

Substituting W in equation (A.6)

$$\begin{aligned} L &= -(EE^T)^{-1} (2E\Lambda_i) E^{-1} \\ L &= -2E^{-T} E^{-1} E \Lambda_i E^{-1} \end{aligned}$$

hence

$$L = -2E^{-T} \Lambda_i E^{-1} \tag{A.7}$$

which is negative defined for all Λ_i. This demonstrates that there exists a common Lyapunov function with negative derivative for all the linear subsystems, therefore, the *SLS* system is stable. □

A.2 Design of a stabilizing control

Consider a switched linear system

$$\dot{x}(t) = A_i x(t) + B_i u(t) \tag{A.8}$$

where the pars (A_i, B_i) are controllable In order to stabilize the switched linear system, a switched state feedback will be designed to move the eigenvectors of its composing linear subsystems.

This way the objective is to design an algorithm to find a set of eigenvectors to be shared by all the subsystems of system (A.8), which at the same time guarantee the stability of every subsystem.

Let

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) \tag{A.9}$$

be a controllable linear system which belongs to a switched linear system like (A.8), consider a state feedback input $u_1(t) = -K_1 x_1(t)$, then

$$\dot{x}_1(t) = (A_1 - B_1 K_1) x_1(t). \tag{A.10}$$

It is clear that the eigenvectors of system (A.10) are in the kernel of matrix $A_1 - \lambda_i I - B_1 K_1$

$$(A_1 - \lambda_i I - B_1 K_1) v_i = 0 \tag{A.11}$$

hence, calculating $ker\{A_1 - \lambda_i I - B_1 K_1\}$, the eigenvectors of system (A.10) related to eigenvalue λ_i will be found, but K_1 is unknown.

To overcome this problem define a set of new vectors

$$q_i = K_1 v_i \tag{A.12}$$

now the augmented vector $\begin{bmatrix} v_i^T & q_i^T \end{bmatrix}^T$ satisfies the following condition

$$\begin{bmatrix} A_1 - \lambda_i I & B_1 \end{bmatrix} \begin{bmatrix} v_i \\ q_i \end{bmatrix} = 0. \tag{A.13}$$

Condition (A.13) enables us to calculate the $v_i = ker\{A_1 - \lambda_i I - B_1 K_1\}$ as a function of λ_i

$$v_i = \vartheta(\lambda_i) \tag{A.14}$$

without using an explicit value for K_1. Unfortunately, in general, these functions are nonlinear.

Having these expressions for the subsystems of a *SLS*, a relation between the eigenvalues of the subsystems while guaranteeing the same eigenvectors, will be sought.

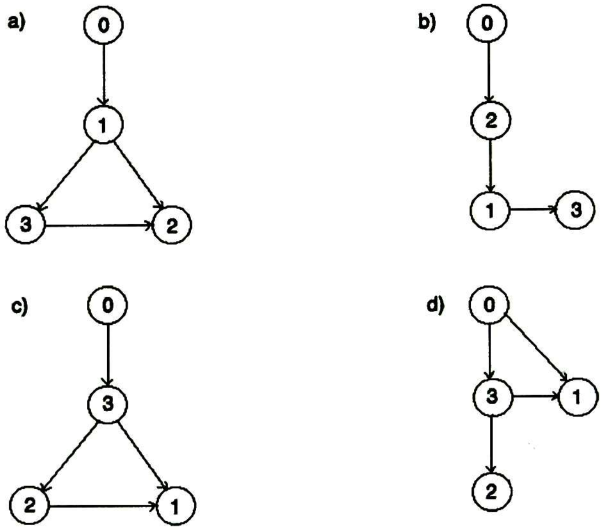

Figure 4.5: Graphs of the corresponding communication topologies

consider the initial conditions

$$\omega(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and take the reference matrix as

$$R = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then, the exosystem observers are described by

$$\dot{\hat{\omega}}_i(t) = S\hat{\omega}_i(t) + \begin{bmatrix} 2.03 & -1.84 \\ -5.53 & 7.84 \\ 5.41 & -2.34 \end{bmatrix} \mathcal{Y}_{i_{et}}$$

$$\mathcal{Y}_{1_{et}} = \begin{cases} \left(y_0(t) - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \hat{\omega}_1(t)\right) & \mu.0s \leq t < \mu.250s \\ \left(y_2(t) - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \hat{\omega}_1(t)\right) & \mu.250s \leq t < \mu.500s \\ \left(\frac{y_2(t) + y_3(t)}{2} - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \hat{\omega}_1(t)\right) & \mu.500s \leq t < \mu.750s \\ \left(\frac{y_0(t) + y_3(t)}{2} - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \hat{\omega}_1(t)\right) & \mu.750s \leq t < \mu + 1.0s \end{cases}$$

$$\mathcal{Y}_{2_{et}} = \begin{cases} \left(\frac{y_1(t) + y_3(t)}{2} - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \hat{\omega}_2(t)\right) & \mu.0s \leq t < \mu.250s \\ \left(y_0(t) - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \hat{\omega}_2(t)\right) & \mu.250s \leq t < \mu.500s \\ \left(y_3(t) - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \hat{\omega}_2(t)\right) & \mu.500s \leq t < \mu.750s \\ \left(y_1(t) - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \hat{\omega}_2(t)\right) & \mu.750s \leq t < \mu + 1.0s \end{cases}$$

$$\mathcal{Y}_{3_{et}} = \begin{cases} \left(y_1(t) - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \hat{\omega}_3(t)\right) & \mu.0s \leq t < \mu.500s \\ \left(y_0(t) - \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \hat{\omega}_3(t)\right) & \mu.500s \leq t < \mu + 1.0s \end{cases}$$

where $\mu = \{1, 2, \cdots 14\}$. The initial conditions of the observers were taken as

$$\hat{\omega}_1(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \hat{\omega}_2(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \hat{\omega}_3(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Finally, according to the previous the switched distributed control laws are computed as

$$u_1(t) = \begin{cases} \begin{bmatrix} 1.14 & 0.63 & -0.05 & 0.63 \\ 6.31 & 1.91 & 6.31 & 3.29 \end{bmatrix} x_1(t) + \begin{bmatrix} 1.41 & -0.56 & 0.72 \\ -3.77 & 1.83 & -1.66 \end{bmatrix} \hat{\omega}_1(t) \\ \begin{bmatrix} 4.47 & 0.62 & 4.15 & 2.81 \\ 6.89 & 2.43 & 5.53 & 2.37 \end{bmatrix} x_1(t) + \begin{bmatrix} -1.54 & 0.78 & -0.65 \\ 9.83 & -3.96 & 5.03 \end{bmatrix} \hat{\omega}_1(t) \end{cases}$$

$$u_2(t) = \begin{cases} \begin{bmatrix} -4 & 2.33 & 0 \\ 1 & 0.33 & 0 \end{bmatrix} x_2(t) + \begin{bmatrix} -2.33 & 5.66 & 1 \\ 1.66 & -1.83 & 0.5 \end{bmatrix} \hat{\omega}_2(t) \\ \begin{bmatrix} -11.66 & 5.66 & 0 \\ 4.33 & -0.33 & 0 \end{bmatrix} x_2(t) + \begin{bmatrix} -2.33 & 5.66 & 1 \\ 1.66 & -1.83 & 0.5 \end{bmatrix} \hat{\omega}_2(t) \\ \begin{bmatrix} -1.66 & 4 & 0 \\ 1.33 & 0 & 0 \end{bmatrix} x_2(t) + \begin{bmatrix} -2.33 & 5.66 & 1 \\ 1.66 & -1.83 & 0.5 \end{bmatrix} \hat{\omega}_2(t) \end{cases}$$

$$u_3(t) = \begin{cases} \begin{bmatrix} 11.05 & 22.59 & 2.80 & 12.38 \\ -22.54 & -50 & -9.85 & -19.83 \end{bmatrix} x_3(t) + \begin{bmatrix} 4.73 & 5.92 & -22.35 \\ -16.87 & -5.96 & 44.69 \end{bmatrix} \hat{\omega}_3(t) \\ \begin{bmatrix} 12.10 & 32.63 & 11.27 & 2.68 \\ -0.49 & -4.94 & -4.23 & 4.88 \end{bmatrix} x_3(t) + \begin{bmatrix} 19.31 & -5.56 & -22.94 \\ -7.32 & 5.79 & 0.17 \end{bmatrix} \hat{\omega}_3(t). \end{cases}$$

The order of the switching controls is the same as the one of the *SLS* dynamics to which they correspond. The references and the trajectories of the outputs of the multi-agent system obtained in simulation are shown in Fig. 4.6. The outputs, $y_{i,1}$ and $y_{i,2}$ follow the same trajectory respectively. The exosystem ω is in this case composed by a sine function ω_1, a cosine ω_2 and a constant $\omega_3 = 1$. It can be seen in this figure that the value of the reference is attained by every output.

Fig. 4.7 displays the behavior of the regulation error (4.47), while Fig. 4.8 displays the regulation error with respect to the state of the exosystem $error_i = x_i - \Pi_{i,j}\omega$. From these figures, it can be seen that both errors are stable except by the dashed lines on errors of system two.

The dashed lines on errors of system two never decay to zero because the corresponding state does not share the same terms on the corresponding Π. But, this error does not affect the regulation given that such a state is not observable.

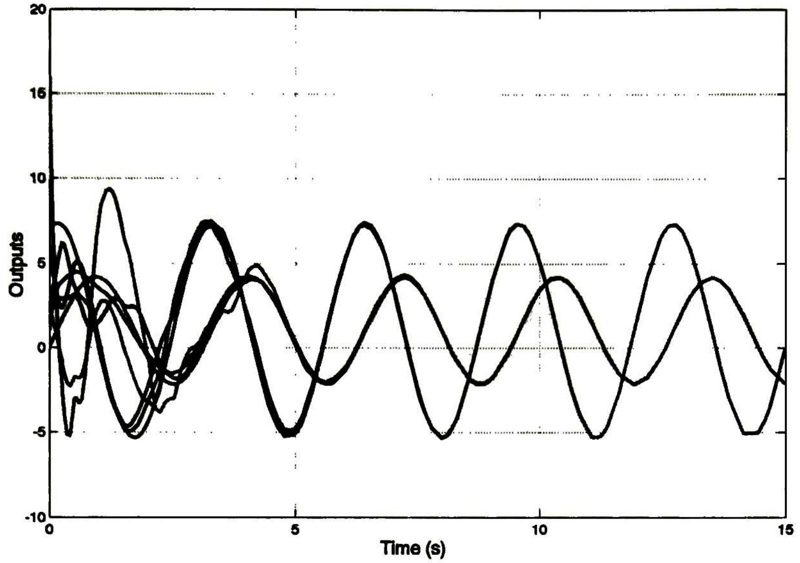

Figure 4.6: Outputs reaching consensus and tracking.

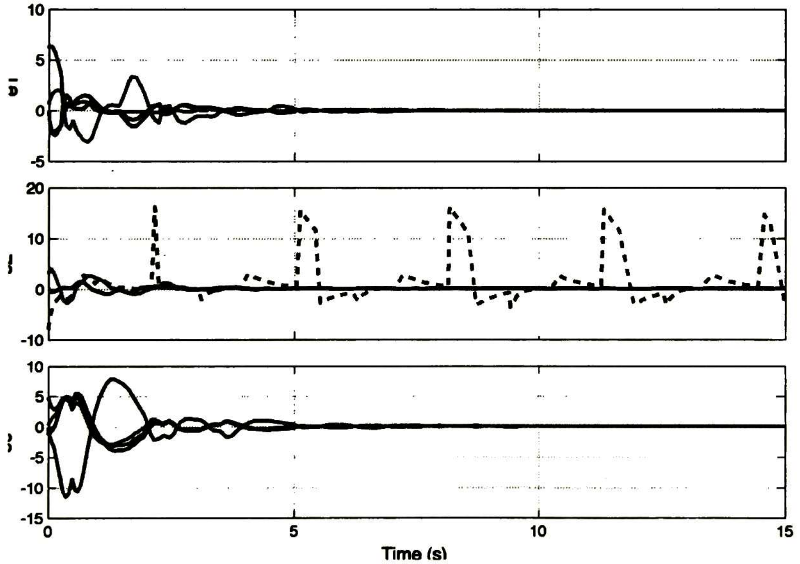

Figure 4.7: Output regulation and consensus error $e_i = x_i - \Pi_{i,j}\hat{\omega}$.

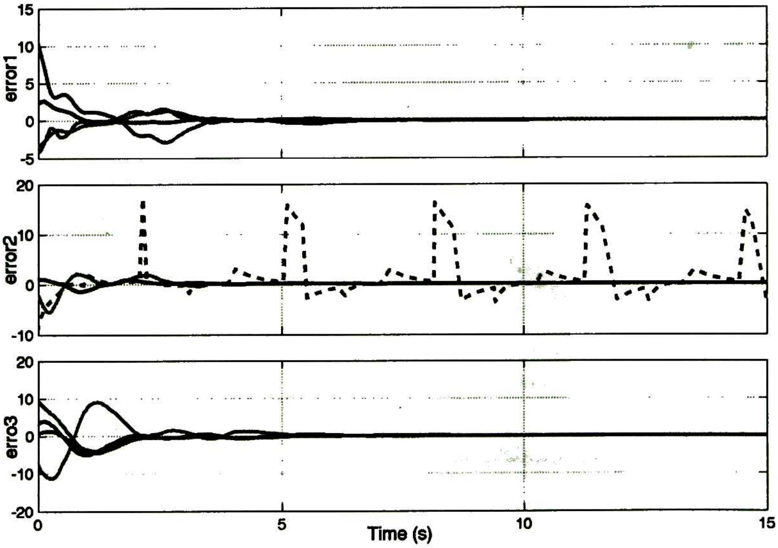

Figure 4.8: Output regulation and consensus error $error_i = x_i - \Pi_{i,j}\omega$.

Chapter 5

Conclusions and future work

5.1 Conclusions

In Chapter 3, it is presented a control strategy for the output consensus and output regulation of multi-agent systems with agents described by linear multi-input and multi-output systems with different dimensions, and a reference produced by an exosystem considered a virtual agent. The number of outputs of each agent to be controlled is restricted to be the same.

It was demonstrated that the proposed control law guarantees that output consensus and output regulation is attained by the group of agents in formation, when only a subset of the agents has access to the reference, and none of them has access to the state of the exosystem.

The agents make use of an observer, with the outputs of its neighbors as inputs, to get the state of the exosystem.

The output consensus and output regulation error stability is guaranteed by properly selecting a stabilizing state feedback for each of the agents.

Recent works addressing the same problem are [42], [40], and [36]. The three of them solve the problem of output consensus and output regulation of linear multi-agent systems. In [42] and [40] control laws are designed based on an internal model approach, such that the control is robust to parameter variations in the systems. In addition, the work in [40] takes into account systems with different dimensions and switching communication topologies. But none of these works allow the multi-agent system formation. In [40], an approach similar to the presented in this thesis is used, with the difference that the observer of one agent depends on the state of the observers of its neighbors, while here, only the knowledge of the outputs of its neighbors is considered. Another difference is that such characteristic of the observers allow the consideration of switched interaction topologies.

In Chapter 4 two different approaches to the output consensus and output regulation of *SLS* multi-agent systems are presented. The first of them addresses the design of a control law for the distributed output consensus and output regulation of a group of *SLS* agents of class (4.8) in formation.

It was demonstrated that the proposed strategy achieves the output regulation and consensus of the *SLS* multi-agent system preserving a desired formation even when only a subset of the agents has access to the reference, and none of them have access to the state of the exosystem.

The agents obtain the exosystem state by means of distributed observers based in the outputs of their neighbors.

The observer-regulation error results with a lower block triangular form, and its stability is guaranteed by properly selecting a switched state feedback to assure the stability of the agents.

Finally, the second approach in Chapter 4 addresses the design of a distributed control law for output consensus and output regulation for a group of *SLS* agents of class (4.8) under switched communication topologies.

It was demonstrated that the proposed control law achieves the output regulation and consensus of the *SLS* multi-agent system, under switching communication topologies, even when only a subset of the agents has access to the reference, and none of them have access to the state of the exosystem.

The observer-regulation error results in a switched linear system. If the state feedback is selected such that each of the agents has a common Lyapunov function, then the existence of a common Lyapunov function for the dynamics of the observer-regulation error for a given communication topology is guaranteed. Under the assumptions made on communication topologies, the common Lyapunov function for the dynamics of the observer-regulation error for a given communication topology is a valid common Lyapunov function for the observer-regulation error under any topology. Hence, the switched dynamics of the observer-regulation error share a common Lyapunov function, consequently, the observer-regulation error is stable.

5.2 Future work

Among the future work appears the following:

1. Improve the control laws adding robustness to parameter variations.

2. Consider the case of delays in the communication between agents.
3. Make use of an observer to compute the states of the agents.
4. Design of a control law for *SLS* output consensus and regulation avoiding agents collision.

Bibliography

[1] M. S. Branicky. Multiple lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on automatic control*, 43(4):475–482, 1998.

[2] T. Chen, J. Zhu, and Y. Shi. Consensus of linear time-invariant multi-agent systems based on dynamic output feedback. *Proceedings of the 21st annual international conference on Chinese control and decision conference*, pages 4237–4241, 2009.

[3] D. Cheng, L. Guo, and J. Huang. On quadratic lyapunov functions. *IEEE transactions on automatic control*, 48(5):885–890, 2003.

[4] G. A. de Castro and F. Paganini. Convex synthesis of controllers for consensus. In *Proceeding of the 2004 American Control Conference*, pages 4933–4938.

[5] D. V. Dimarogonas and K. J. Kyriakopoulos. Formation control and collision avoidance for multi-agent systems and a connection between formation infeasibility and flocking behavior. *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 84–89, 2005.

[6] D. V. Dimarogonas, S. G. Loizou, K. J. Kyriakopoulos, and M. M. Zavlanos. A feedback stabilization and collision avoidance scheme for multiple independent non-point agents. *Automatica*, pages 229–243, 2006.

[7] V. B. et al., editor. *Recent Advances in Learning and Control*, chapter chapter, pages 51–64. Springer-Verlag Berlin Heidelberg, 2008.

[8] L. Fang, P. J. Antsaklis, and A. Tzimas. Asynchronous consensus protocols: preliminary results, simulations and open questions. In *Proceedings of the 44th IEEE Conference on decision and control, and the european control conference*, pages 2194–2199, 2005.

[9] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IFAC World Congress. Barcelona*, 2002.

[10] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9):1465–1476, 2004.

[11] B. A. Francis. The linear multivariable regulator problem. In *Proceedings of the IEEE Conference on decsicion and control*, volume 15, pages 873–878, 1976.

[12] B. A. Francis and W. M. Wonham. The internal model principle of control theory. *Automatica*, 12:457–465, 1976.

[13] Y. Hong, L. Gao, D. Cheng, and J. Hu. Lyapunov-based approach to multiagent systems with switching jointly connected interconnection. *IEEE Transactions on automatic control*, 52(5), 2007.

[14] A. Isidori. *Non Linear Control Systems.* Springer, 1995.

[15] A. Isidori and C. I. Byrnes. Output regulation of nonlinear systems. *IEEE Transactions on automatic control*, 35(2):131–140, 1990.

[16] H. Kim, H. Shim, and J. H. Seo. Output consensus of heterogeneous uncertain linear multi-agent systems. *IEEE Transactions on automatic control*, 2011.

[17] G. Lafferriere, A. Williams, J. Caughman, and J. Veerman. Decentralized control of vehicle formations. *Systems and control letters*, 54(9):899–910, 2005.

[18] N. E. Leonard and E. Fiorelli. Virtual leaders, artificial potencials and coordinated control of groups. In *Proceedings of the 40th Conference on Decision and Control*, pages 2969–2973.

[19] Z. Li, Z. Duan, and L. Huang. Leader-follower consensus of multi-agent systems. *American Control Conference*, pages 3256–3261, 2009.

[20] Z. G. Li, C. Y. Wen, and Y. C. Soh. Observer-based stabilization of switching linear systems. *Automatica*, 39:517–524, 2003.

[21] D. Liberzon. *Switching in systems and control.* Birkhäuser, 2003.

[22] H. Lin and P. J. Antsaklis. Stability and stabilizability of switched linear systems: a survey of recent results. *IEEE transactions on automatic control*, 54(2):308–322, 2009.

[23] K. Listmann, M. Masalawala, and J. Adamy. Consensus for formation control of nonholonomic mobile robots. *Proceedings of the IEEE International Conference on Robotics and Automation*, 2009.

[24] X. Liu. Stabilization of switched linear systems with mode-dependent time-varying delays. *Applied Mathematics and Computation*, pages 2581–2586, 2010.

[25] Y. Liu and J. Zhao. Output regulation of a class of switched linear systems with disturbances. In *Proceedings of the American Control Conference*, pages 882–883, 2001.

[26] C.-Q. Ma and J.-F. Zhang. Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Transactions on Automatic Control*, 2009.

[27] Z. Meng, W. Ren, Y. Cao, and Z. You. Leaderless and leader-following consensus with communication and input delays under a directed network topology. *IEEE Transactions on systems, man, and cybernetics, Part B*, pages 75–87, 2011.

[28] V. F. Montagner, V. J. S. Leite, R. C. L. F. Oliveira, and P. L. D. Peres. State feedback control of switched linear systems: An lmi approach. *Journal of computational and applied mathematics*, 2006.

[29] I. Necoara, B. D. Schutte, T. van den Boom, and J. Hellendoorn. Robustly stabilizing mpc for perturbed pwl systems. In *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference*, pages 3759–3764, year.

[30] R. Olfati-Saber, J. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *IEEE Transactions on Automatic Control*, 95(1):215–233, 2007.

[31] R. Olfati-Saber and R. M. Murray. Consensus protocols for networks of dynamic agents. In *Proceedings of the American Control Conference*, pages 951–956, 2003.

[32] R. Olfati-Saber and R. M. Murray. Flocking with obstacle avoidance: Cooperation with limited information in mobile networks. *Proceedings. 42nd IEEE Conference on Decision and Control*, pages 2022 – 2028, 2003.

[33] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time delays. *IEEE Transactions on automatic control*, 2004.

[34] A. Olshevsky and J. N. Tsitsiklis. On the nonexistence of quadratic lyapunov functions for consensus algorithms. *IEEE Transactions on automatic control*, 53(11):2642–2645, 2008.

[35] J. S. Sharma, editor. *Cooperative control of distributed multi-agent systems*. John Wiley and Sons., 2007.

[36] Y. Su and J. Huang. Cooperative output regulation with application to multi-agent consensus under switching network. *IEEE Transactions on systems, man, and cybernetics—Part B: Cybernetics*, 2012.

[37] Z. Sun. Stabilization and optimization of switched linear systems. *Automatica*, 42:783–788, 2006.

[38] Z. Sun and S. S. Ge. Analysis and synthesis of switched linear control systems. *Automatica*, 41:181–195, 2005.

[39] J. Thunberg and X. Hu. Optimal output consensus control and outlier detection. In *Proceedings of the 19th International symposium on mathematical theory of networks and systems*, pages 2247–2249, 2010.

[40] X. Wang and F. Han. Robust coordination control of switching multi-agent systems via output regulation approach. *Kybernetika*, 47(5):755–772, 2011.

[41] X. Wang and Y. Hong. Parametrization and geometric analysis of coordination controllers for multi-agent systems. *Kybernetica*, 45(5):785–800, 2009.

[42] X. Wang, Y. Hong, J. Huang, and Z.-P. Jiang. A distributed control approach to a robust output regulation problem for multi-agent linear systems. *IEEE Transactions on automatic control*, 55(12):2891–2895, 2010.

[43] G. Xie and L. Wang. Periodical stabilization of switched linear systems. *Journal of computational and applied mathematics*, 81:176–187, 2005.

[44] T. Yang, A. Saberi, A. A. Stoorvogel, and H. F. Grip. Output consensus for networks of non-identical introspective agents. In *50th IEEE Conference on decision and control and european control conference*, pages 1286–1292, 2011.

[45] X.-R. Yang and G.-P. Liu. Output consensus of linear multi-agent systems. In *UKACC International Conference on Control*, pages 1094–1099, 2012.

[46] L. Zhang, Y. Chen, and P. Cui. Stabilization for a class of second-order switched systems. *Nonlinear Analysis*, 62:1527–1535, 2005.

Appendix A

Phase plane aligning stability condition for SLS systems: proof and control design

This appendix presents the proof and control design related to the Theorem 4.3 in Chapter 4

Consider a switched linear system

$$\dot{x}(t) = A_i x(t) + B_i u(t)$$

where the pairs (A_i, B_i) are controllable and a switched state feedback (4.2).

Theorem 4.3: *If the switched state feedback (4.2) is such that, the matrices $(A_i - B_i K_i)$ are Hurwitz, and its eigenvectors are the same for all i, then the SLS system will be stable.*

A.1 Proof

Proof. Let

$$\dot{x}(t) = A_i x(t) \qquad i = 1, 2, \ldots, N \tag{A.1}$$

be a SLS system composed by N stable linear subsystems sharing all their eigenvectors, where $x(t) \in \mathbb{R}^n$ and let $E \in \mathbb{R}^{n \times n}$ be a matrix formed by those eigenvectors

$$A_i E = -E \Lambda_i \tag{A.2}$$

and the Λ_i's are diagonal matrices containing the absolute values of the eigenvalues of the matrices A_i.

Define a common Lyapunov function for the N linear subsystems

$$V(x) = x(t)^T P x(t) \qquad i = 1, 2, \cdots, N \tag{A.3}$$

where

$$P = (EE^T)^{-1} \tag{A.4}$$

Taking the derivative of $V(x(t))$

$$\dot{V}(x(t)) = x(t)^T (PA_i + A_i^T P) x(t) \tag{A.5}$$

is obtained, clearly in order to guarantee that $\dot{V}(x(t))$ is negative and consequently the *SLS* system is stable, it is needed that $L = PA_i + A_i^T P < 0$.

Let L be

$$L = (EE^T)^{-1} A_i + A_i^T (EE^T)^{-1} = -(EE^T)^{-1} W E^{-1} \tag{A.6}$$

solving for W from (A.6)

$$\begin{aligned} A_i + (EE^T) A_i^T (EE^T)^{-1} &= -WE^{-1} \\ A_i E + (EE^T) A_i^T (EE^T)^{-1} E &= -W \\ A_i E + EE^T A_i^T E^{-T} E^{-1} E &= -W \\ A_i E + EE^T A_i^T E^{-T} &= -W \\ -2E\Lambda_i &= -W. \end{aligned}$$

Substituting W in equation (A.6)

$$\begin{aligned} L &= -(EE^T)^{-1} (2E\Lambda_i) E^{-1} \\ L &= -2E^{-T} E^{-1} E \Lambda_i E^{-1} \end{aligned}$$

hence

$$L = -2E^{-T} \Lambda_i E^{-1} \tag{A.7}$$

which is negative defined for all Λ_i. This demonstrates that there exists a common Lyapunov function with negative derivative for all the linear subsystems, therefore, the *SLS* system is stable. □

A.2 Design of a stabilizing control

Consider a switched linear system

$$\dot{x}(t) = A_i x(t) + B_i u(t) \tag{A.8}$$

where the pars (A_i, B_i) are controllable In order to stabilize the switched linear system, a switched state feedback will be designed to move the eigenvectors of its composing linear subsystems.

This way the objective is to design an algorithm to find a set of eigenvectors to be shared by all the subsystems of system (A.8), which at the same time guarantee the stability of every subsystem.

Let

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) \tag{A.9}$$

be a controllable linear system which belongs to a switched linear system like (A.8), consider a state feedback input $u_1(t) = -K_1 x_1(t)$, then

$$\dot{x}_1(t) = (A_1 - B_1 K_1) x_1(t). \tag{A.10}$$

It is clear that the eigenvectors of system (A.10) are in the kernel of matrix $A_1 - \lambda_i I - B_1 K_1$

$$(A_1 - \lambda_i I - B_1 K_1) v_i = 0 \tag{A.11}$$

hence, calculating $ker\{A_1 - \lambda_i I - B_1 K_1\}$, the eigenvectors of system (A.10) related to eigenvalue λ_i will be found, but K_1 is unknown.

To overcome this problem define a set of new vectors

$$q_i = K_1 v_i \tag{A.12}$$

now the augmented vector $\begin{bmatrix} v_i^T & q_i^T \end{bmatrix}^T$ satisfies the following condition

$$\begin{bmatrix} A_1 - \lambda_i I & B_1 \end{bmatrix} \begin{bmatrix} v_i \\ q_i \end{bmatrix} = 0. \tag{A.13}$$

Condition (A.13) enables us to calculate the $v_i = ker\{A_1 - \lambda_i I - B_1 K_1\}$ as a function of λ_i

$$v_i = \vartheta(\lambda_i) \tag{A.14}$$

without using an explicit value for K_1. Unfortunately, in general, these functions are nonlinear.

Having these expressions for the subsystems of a *SLS*, a relation between the eigenvalues of the subsystems while guaranteeing the same eigenvectors, will be sought.

Consider a *SLS* composed of N different subsystems of dimension n, and let $v_{i,j}$ be the j-th eigenvector of the i-th subsystem. Then, given that it is wanted that all the subsystems have the same eigenvectors, it is set

$$v_{i,j} = \vartheta(\lambda_{i,j}) = \vartheta(\lambda_{i+1,j}) = v_{i+1,j} \tag{A.15}$$

which results in a set of n nonlinear simultaneous equations, therefore, no solution can be found (for the general case).

Nevertheless, for the case of subsystems of dimension two, such a solution can always be found, furthermore, the expressions for the eigenvectors can be expressed in a linear form. For subsystems of dimension three with one input, only three solutions can be found for any pair of subsystems, in other words, in general, for two systems of dimension three with one input, only there exist three different eigenvectors to be shared.

These results make this approach hardly useful for systems of dimensions bigger than two.

Consider a controllable linear system of dimension two part of a *SLS*

$$\dot{x}_1(t) = A_1 x_1(t) + b_1 u_1(t), \tag{A.16}$$

with input $u = -K_1 x_1(t)$ and let v_1 and v_2 be the eigenvectors required to be common for the system, such that

$$(A_1 - b_1 K_1)v_1 = \lambda_{1,1} v_1 \tag{A.17}$$

and

$$(A_1 - b_1 K_1)v_2 = \lambda_{1,2} v_2. \tag{A.18}$$

Rearranging equations (A.17) and (A.18) leaving on the right side only the terms independent from $\lambda_{1,1}$, $\lambda_{1,2}$ y K_1

$$\begin{aligned} \lambda_{1,1} v_1 + b_1 K_1 v_1 &= A_1 v_1 \\ \lambda_{1,2} v_2 + b_1 K_1 v_2 &= A_1 v_2 \end{aligned}$$

define $\lambda_1 = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} \end{bmatrix}^T$

$$\begin{aligned} \begin{bmatrix} v_1 & 0 \end{bmatrix} \lambda_1 + b_1 v_1^T K_1^T &= A_1 v_1 \\ \begin{bmatrix} 0 & v_2 \end{bmatrix} \lambda_1 + b_1 v_2^T K_1^T &= A_1 v_2 \end{aligned}$$

In this way the equations can be set on a matrix form

$$\begin{bmatrix} v_1 & 0 & b_1 v_1^T \\ 0 & v_2 & b_1 v_2^T \end{bmatrix} \begin{bmatrix} \lambda_1 \\ K_1^T \end{bmatrix} = \begin{bmatrix} A_1 v_1 \\ A_1 v_2 \end{bmatrix} \tag{A.19}$$

solving for λ_1 and K_1^T, the state feedback that makes v_1 and v_2 the eigenvectors of system (A.16) can be calculated

$$\begin{bmatrix} \lambda_1 \\ K_1^T \end{bmatrix} = \begin{bmatrix} v_1 & 0 & b_1 v_1^T \\ 0 & v_2 & b_1 v_2^T \end{bmatrix}^{-1} \begin{bmatrix} A_1 v_1 \\ A_1 v_2 \end{bmatrix} \tag{A.20}$$

At this point, the eigenvectors of the linear subsystems can be assigned, but given that by the time of moving the eigenvectors to a set value, the eigenvalues move on a random way, thus a new problem arises: how to find a set of eigenvectors that at the same time of being common for the linear subsystems of a *SLS* system, also be associated to eigenvalues with negative real part?

Consider a similarity transformation to take the system to its controller form

$$\dot{x}_1(t) = A_1 x_1(t) + b_1 u(t) \quad \Longrightarrow \quad \dot{\tilde{x}}_1(t) = T_1^{-1} A_1 T_1 \tilde{x}_1(t) + T_1^{-1} b_1 u(t)$$

where for an $x_1(t) \in \mathbb{R}^n$

$$T_1^{-1} A_1 T_1 = \begin{bmatrix} 0 & I_{n-1} \\ \alpha_1 & \alpha_2 \end{bmatrix}$$

the terms α_i are the coefficients of the characteristic polynomial of A_1, and

$$T_1^{-1} b_1 = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix}$$

Applying the previous procedure to move the eigenvectors of $\dot{\tilde{x}}_1(t)$

$$\begin{bmatrix} \tilde{v_{1,1}} & 0 & 0 & 0 \\ \tilde{v_{1,2}} & 0 & \tilde{v_{1,1}} & \tilde{v_{1,2}} \\ 0 & \tilde{v_{2,1}} & 0 & 0 \\ 0 & \tilde{v_{2,2}} & \tilde{v_{2,1}} & \tilde{v_{2,2}} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \tilde{K_1}^T \end{bmatrix} = \begin{bmatrix} T_1^{-1} A_1 T_1 \tilde{v_1} \\ T_1^{-1} A_1 T_1 \tilde{v_2} \end{bmatrix} \tag{A.21}$$

where $\tilde{v_1}$ and $\tilde{v_2}$ are the eigenvectors of the matrix $(T_1^{-1} A_1 T_1 - T_1^{-1} b_1 \tilde{K_1})$ and $v_1 = T_1 \tilde{v_1}$ and $v_2 = T_1 \tilde{v_2}$.

Rewriting (A.21) it is obtained

$$\begin{bmatrix} \tilde{v_{1,1}} & 0 & 0 & 0 \\ \tilde{v_{1,2}} & 0 & \tilde{v_{1,1}} & \tilde{v_{1,2}} \\ 0 & \tilde{v_{2,1}} & 0 & 0 \\ 0 & \tilde{v_{2,2}} & \tilde{v_{2,1}} & \tilde{v_{2,2}} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \tilde{K_1}^T \end{bmatrix} = \begin{bmatrix} \tilde{v_{1,2}} \\ \alpha_1 \tilde{v_{1,1}} + \alpha_2 \tilde{v_{1,2}} \\ \tilde{v_{2,2}} \\ \alpha_1 \tilde{v_{2,1}} + \alpha_2 \tilde{v_{2,2}} \end{bmatrix} \tag{A.22}$$

where it can be clearly observed the next relationships between the elements of the eigenvectors $\tilde{v_1}$ and $\tilde{v_2}$

$$\begin{aligned} \tilde{v_{1,1}}\lambda_{1,1} &= \tilde{v_{1,2}} \\ \tilde{v_{2,1}}\lambda_{1,2} &= \tilde{v_{2,2}}. \end{aligned} \tag{A.23}$$

The relationships shown in (A.23) can be written in the form of a vector equation

$$\begin{aligned} \begin{bmatrix} \lambda_{1,1} & -1 \end{bmatrix} \tilde{v_1} &= 0 \\ \begin{bmatrix} \lambda_{1,2} & -1 \end{bmatrix} \tilde{v_2} &= 0. \end{aligned} \tag{A.24}$$

Substituting the vectors $\tilde{v_1}$ and $\tilde{v_2}$, a relationship between eigenvalues and eigenvectors is found

$$\begin{aligned} \begin{bmatrix} \lambda_{1,1} & -1 \end{bmatrix} T_1^{-1} v_1 &= 0 \\ \begin{bmatrix} \lambda_{1,2} & -1 \end{bmatrix} T_1^{-1} v_2 &= 0. \end{aligned} \tag{A.25}$$

Simplifying the equations in (A.25) in a matrix form

$$\begin{bmatrix} \lambda_{1,1} & -1 \\ \lambda_{1,2} & -1 \end{bmatrix} T_1^{-1} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = 0. \tag{A.26}$$

Consider the result in equation (A.26) for two different linear subsystems of dimension two, then, it can be seen that

$$\begin{bmatrix} \lambda_{1,1} & -1 \\ \lambda_{1,2} & -1 \end{bmatrix} T_1^{-1} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \lambda_{2,1} & -1 \\ \lambda_{2,2} & -1 \end{bmatrix} T_2^{-1} \begin{bmatrix} v_1 & v_2 \end{bmatrix}. \tag{A.27}$$

Given that the eigenvector matrix is invertible, it can be eliminated from equation (A.27)

$$\begin{bmatrix} \lambda_{1,1} & -1 \\ \lambda_{1,2} & -1 \end{bmatrix} T_1^{-1} = \begin{bmatrix} \lambda_{2,1} & -1 \\ \lambda_{2,2} & -1 \end{bmatrix} T_2^{-1} \tag{A.28}$$

Solving for the eigenvalues of the second subsystem, a direct relation between the eigenvalues of two different linear systems of dimension two which share eigenvectors is found

$$\begin{bmatrix} \lambda_{2,1} & -1 \\ \lambda_{2,2} & -1 \end{bmatrix} = \begin{bmatrix} \lambda_{1,1} & -1 \\ \lambda_{1,2} & -1 \end{bmatrix} T_1^{-1} T_2. \tag{A.29}$$

Repeating the previous process for all the subsystems of the *SLS* system and solving in every case for the same eigenvalues, the problem of finding the combination of eigenvalues that stabilizes the *SLS* system reduces to solve an LMI program to find the values of the dependent eigenvalues

$$\begin{bmatrix} \lambda_{i,1} & -1 \\ \lambda_{i,2} & -1 \end{bmatrix} = \begin{bmatrix} \lambda_{1,1} & -1 \\ \lambda_{1,2} & -1 \end{bmatrix} T_1^{-1} T_i \tag{A.30}$$

including the relations with the other eigenvalues as restrictions to the program

$$\begin{bmatrix} \lambda_{1,1} & -1 \\ \lambda_{1,2} & -1 \end{bmatrix} T_1^{-1} T_i < 0. \qquad \text{(A.31)}$$

In this way, if a solution exists for the LMI, it guarantees the negativity of the eigenvalues of all the linear dynamics that composes the *SLS* system.

Finally, having calculated the stabilizing eigenvalues, the *SLS* stabilizing feedback can be calculated through the well known methods.

A.3 Example

Consider a piecewise linear system of dimension two,

$$\dot{x}(t) = A_i x(t) + bu(t) \qquad i \in \{1, 2, 3\} \qquad \text{(A.32)}$$

composed of three different linear subsystems with matrices

$$A_1 = \begin{bmatrix} 10 & 22 \\ 10 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 1 \\ -21 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} -7 & 8 \\ 9 & 2 \end{bmatrix}$$

each of them with unstable dynamics, therefore, each of the subsystems has at least one eigenvalue with positive real part.

The subsystems are distributed in the state space as displayed on Fig. A.1. Employing the proposed stabilizing procedure, the desired eigenvalues are found

$$\lambda_1 = \begin{bmatrix} -78 \\ -79 \end{bmatrix} \quad \lambda_2 = \begin{bmatrix} -1 \\ -1.0455 \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} -39 \\ -39.3636 \end{bmatrix}$$

consequently, the required feedback gains

$$K_1 = \begin{bmatrix} 366 & 167 \end{bmatrix} \quad K_2 = \begin{bmatrix} -4.81 & 8.04 \end{bmatrix} \quad K_3 = \begin{bmatrix} 138.45 & 73.36 \end{bmatrix}$$

will produce the following common eigenvectors

$$v = \begin{bmatrix} 0.2425 & -0.2400 \\ -0.9701 & 0.9708 \end{bmatrix}$$

The phase plane of the autonomous *SLS* system is shown in Fig. A.2.

Applying the feedback gains obtained previously, the resulting phase plane of the piecewise linear system is shown in Fig. A.3, where all the system trajectories are stable and phase plane is similar to that of a linear system with eigenvectors v.

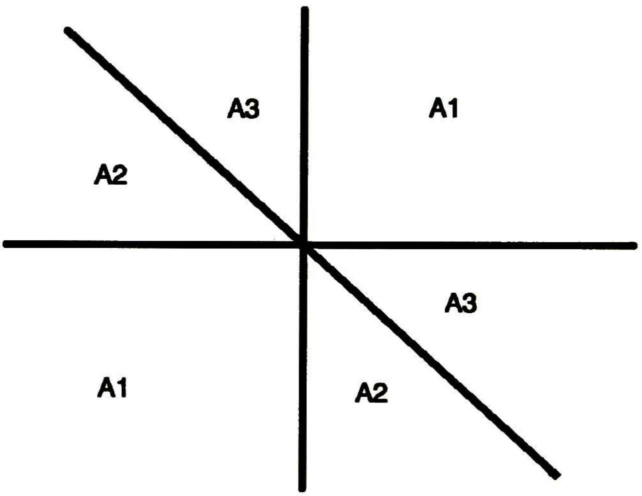

Figure A.1: Phase plane of the autonomous PWL system

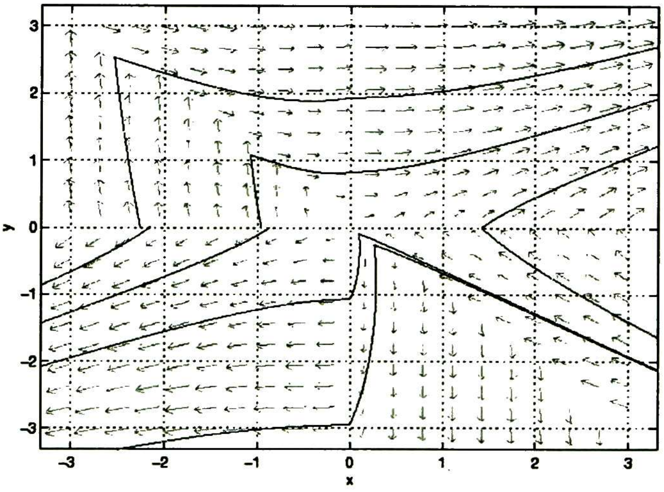

Figure A.2: Phase plane of the autonomous PWL system

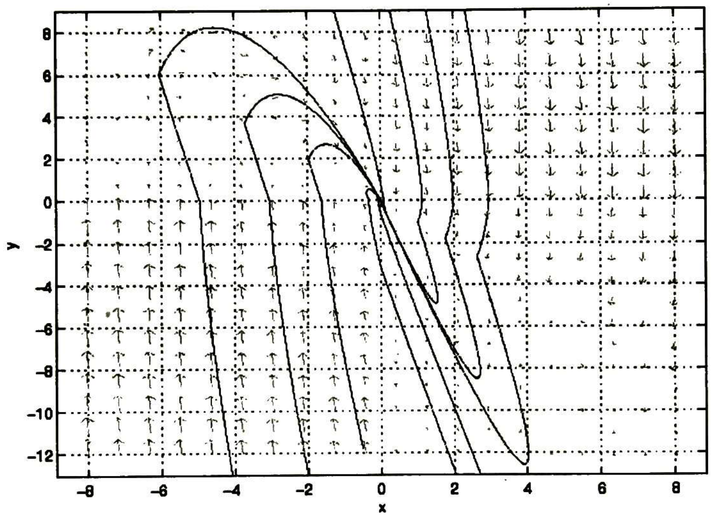

Figure A.3: Phase plane of the stabilized PWL system

CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL I.P.N.
UNIDAD GUADALAJARA

El Jurado designado por la Unidad Guadalajara del Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional aprobó la tesis

Formación y seguimiento de trayectoria de sistemas multi-agentes/
Formation and trajectory tracking of multi-agent systems

del (la) C.

Alejandro CERVANTES HERRERA

el día 14 de Diciembre de 2012.

Dra. Ofelia Begovich Mendoza
Investigador CINVESTAV 3B
CINVESTAV Unidad Guadalajara

Dr. Deni Librado Torres Román
Investigador CINVESTAV 3B
CINVESTAV Unidad Guadalajara

Dr. José Javier Ruíz León
Investigador CINVESTAV 3B
CINVESTAV Unidad Guadalajara

Dr. Antonio Ramírez Treviño
Investigador CINVESTAV 3A
CINVESTAV Unidad Guadalajara

Dr. Guillermo Ramírez Prado
Profesor Investigador
Universidad Autónoma de
Aguascalientes