Centro de Investigación y de Estudios Avanzados
del
Instituto Politécnico Nacional

DEPARTAMENTO DE FÍSICA

Parámetros de Michel en presencia de neutrinos de Majorana

Tesis que presenta

Juan Manuel Márquez Morales

para obtener el Grado de

Maestro en Ciencias

en la Especialidad de

Física

Director de tesis: **Dr. Pablo Roig Garcés**

Ciudad de México Agosto, 2021

CENTRO DE INVESTIGACION Y DE ESTUDIOS AVANZADOS
DEL INSTITUTO POLITECNICO NACIONAL

PHYSICS DEPARTMENT

"Michel parameters in the presence of Majorana neutrinos"

Thesis submitted by

Juan Manuel Márquez Morales

In order to obtain the

Master of Science

degree, speciality in

Physics

Supervisor: **Dr. Pablo Roig Garcés**

Mexico City August, 2021.

Michel parameters in the presence of Majorana neutrinos.

A thesis submitted for the degree of

Master of Science

Juan Manuel Márquez Morales

Advisor: Dr. Pablo Roig Garcés

Centro de Investigación y de Estudios Avanzados

del Instituto Politécnico Nacional

August 2021

Acknowledgments

I wish to express my sincere appreciation to my advisor, Dr. Pablo Roig Garcés, for his patience, motivation and enthusiasm. His approachable personality and deep knowledge helped me a lot in all this research. I would also like to thank Dr. Gabriel López Castro and Dr. Eduard de la Cruz Burelo for their helpful comments and questions.

I thank my friends and colleagues for their support, invaluable discussions as well as the fun moments and distractions that made these two years such a great experience.

Finally, I wish to express my deepest gratitude to my parents, Paulina and Juan Manuel, for their love and support throughout my life. This thesis is dedicated to them.

Abstract

Leptonic decays provide a clean laboratory where many high-precision measurements can be made in order to test the internal consistency of the SM and reveal the signature of possible new physics.

The leptonic decay of a charged lepton $l^- \longrightarrow l'^- \bar{\nu}_{l'} \nu_l$ is described within the model-independent approach with the help of *Michel parameters*.

We analyse the effects of Majorana neutrinos on leptonic decays using the most general four-lepton effective interaction Hamiltonian of dimension six. We calculate the specific energy and angular distribution of the final charged lepton, complemented with the decaying-lepton polarization. We discuss the new generalized *Michel parameters* and focus on the effects of the heavy Majorana masses that would lead to sizable contributions on scenarios where the new sterile Majorana neutrinos have non-negligible mixing.

Keywords – Michel parameters; Majorana neutrinos; new physics; leptonic decay.

Resumen

Los decaimientos leptónicos proveen un laboratorio limpio, dónde se pueden realizar muchas medidas de precisión para poner a prueba la consistencia del modelo estándar y revelar la presencia de posible nueva física.

El decaimiento leptónico de un leptón cargado $l^- \longrightarrow l'^- \bar{\nu}_{l'} \nu_l$ está descrito, dentro de la aproximación independiente de modelo, por los llamados parámetros de Michel.

En este trabajo, analizamos los efectos de neutrinos de Majorana en decaimientos leptónicos usando el Hamiltoniano efectivo de dimensión seis más general para una interacción de cuatro leptones. Calculamos la distribución angular y energética del leptón cargado final, complementado con la polarización de la partícula que decae. Discutimos los nuevos parámetros de Michel generalizados y nos enfocamos en los efectos de masas de Majorana pesadas, que darían lugar a contribuciones medibles en escenarios donde los neutrinos estériles de Majorana tuvieran mezclas no despreciables.

Palabras clave – Parámetros de Michel; neutrinos de Majorana; nueva física; decaimiento leptónico.

Contents

Chapter 1

Introduction

The Standard Model of Elementary Particles (SM) is the most successful theoretical model to describe nature. It has been tested so many times and remains consistent with the experimental data [1]. Nevertheless, the SM does not explain everything, such as gravity interaction, neutrino masses, dark matter, baryon asymmetry, etc. These are some physical phenomena that motivate the research of new physics (NP) that is beyond the SM [2].

Nowadays, the Dirac or Majorana nature of neutrinos as well as the mechanism from which they acquire mass are one of the unsolved puzzles that lie beyond SM. There are many minimal extensions of the SM in order to account for nonzero masses and mixings for the active neutrinos; adding new gauge singlet fields, such as right-handed neutrinos. Some of them are the well-known νSM [3, 4] and the seesaw-mechanisms [5, 6, 7, 8, 9] (see Appendix A).

Also, the Dirac or Majorana nature of neutrinos is crucial for understanding the origin of their masses and some physical processes. If neutrinos were Majorana particles, then lepton number is not conserved and the neutrinos would be their own antiparticles, i.e, there is no conserved quantum number that allows to distinguish between neutrino or anti-neutrino (see Appendix B). Thus, making interesting from the phenomenological point of view to test the Dirac or Majorana nature of the neutrinos through lepton number violating processes, specially in the heavy sector, where the Majorana contribution could lead to measurable changes [10].

Much theoretical and experimental researches are being carried out to discover how the Majorana nature of neutrinos and the specific origin of their masses can affect physical processes, leading to small deviation from the well-known SM results and new decay modes. By doing this, we will have a better understanding of nature and we will get closer to a more fundamental theory, see e.g. [5, 11, 12, 13, 14].

One of the experimental strategies in the search of a fundamental description of nature which goes beyond the SM is to perform high precision measurements where an observed discrepancy with the SM would reveal the signature of NP. For this purpose, muon and tau leptons are specially important, where many precision measurements can be made, see e.g. [15, 16, 17].

For example, studies of muon decays both determine the overall strength and establish the Lorentz structure of weak interactions, as well as setting extraordinary limits on charged lepton flavor violating processes (LFV) [18]. Measurements of the muon's anomalous magnetic moment offer singular sensitivity to the completeness of the standard model and the predictions of many speculative theories. In general, due to the incredible high-precision and high-sensitivity experiments with muons, this lepton is a perfect candidate for precision SM tests.

On the other hand, the tau lepton experiments are not as accurate as those made with muons. However, the tau lepton has some unique features that make it as important as the muon in the search for NP. Nowadays, the detailed study of higher-order electroweak corrections and QCD contributions has promoted the physics of the tau lepton to the level of precision tests. The pure leptonic or semileptonic character of tau decays provides a clean laboratory to test the structure of the weak currents and the universality of their couplings to the gauge bosons. Moreover, the tau is the only known lepton massive enough to decay into hadrons; its semileptonic decays are then an ideal tool for studying strong interaction effects in very clean conditions [15].

Finally, since one naively expects the fermions to be sensitive to the possible NP proportional to their masses, the large tau mass allows one to investigate NP contributions, through a broad range of kinematically-allowed decay modes, complementing the high-precision searches performed in muon decay.

In this work we analyse the leptonic decays $l^- \longrightarrow l'^- \bar{\nu}_{l'} \nu_l$, where the lepton pair (l,l') may be (μ,e), (τ,e) or (τ,μ), using the most general four-lepton effective interaction Hamiltonian to test the structure of the weak currents and the Majorana nature of neutrinos in the framework of low-scale seesaw scenarios, where the new sterile Majorana neutrinos have non-negligible mixings and some of them require masses low enough to be produced on-shell, where the genuine effects of the heavy Majorana masses could be measurable.

In Ch. 2 and Ch. 3 we briefly introduce the main concepts about the SM formulation and the effective theories framework. In Ch. 4 we reproduce in detail the well known results of lepton decays, including the analysis of *Michel parameters* and we present the current experimental limits on the effective couplings. Finally, in Ch. 5 we provide detailed expressions for the amplitude and decay rate of the process under consideration, once considering the effects of Majorana neutrinos and we estimate its suppression using the experimental constraints on the heavy-light mixing angles as a function of the mass of one heavy state. Our conclusions are given in Ch. 6.

Chapter 2

Standard Model

The Standard Model of Elementary Particles (SM) is the most successful theory we've ever had to describe nature. Its predictions have been tested in a huge number of experiments and they had an incredible agreement. Nevertheless, the SM does not describe all the physics phenomena and is far away from the so called *Theory of everything*.
Nowadays, the precision tests of the SM play an important role to verify its predictions and give insights of the possible new physics that is beyond.
In this chapter, we give a brief introduction to the SM and its components.

2.1 Introduction to the SM

The SM is a quantum field theory (QFT) that is used to describe fundamental particles and their interactions. Its Lagrangian possesses a gauge invariance under the group $SU(3)_C \times SU(2)_L \times U(1)_Y$ together with Poincare and CPT invariance. Also, the SM exhibits additional symmetries, not postulated at the outset of its construction, collectively denoted *accidental symmetries*, such as baryon number, lepton number, custodial symmetry, etc.
The SM includes the electromagnetic, strong and weak forces and all their carrier particles, and explains well how these forces act on all of the matter particles. The corresponding force results from the exchange of the force-carrier particles, which belong to a group called *bosons*. The particle content is described in a more qualitative way in the next section.
In the QFT context, the objects described by the SM are *quantum fields* which are defined at all points in spacetime. These fields are the fermion fields, Ψ, which account for matter particles; the electroweak boson fields W_1, W_2, W_3 and B; the gluon field, G_a and the Higgs field, ϕ.
The SM Lagrangian Density $\mathcal{L}_{SM}$ describes the behavior of these fields, including their corresponding kinetic terms and their mutual interactions, via *coupling* terms. Upon

writing the most general Lagrangian with massless neutrinos, one finds that the dynamics depends on 19 parameters, whose numerical values are established by experiment. The explicit form of this Lagrangian can be found in [19].

All the theoretical aspects of this construction are out of this brief introduction, but can be found in many research articles and textbooks, which have been done specially to give a formal and understandable undergraduate and graduate level introduction to the SM, see e.g. [20, 21, 22].

2.2 Particle Content

In the current view, all matter is made out of three kinds of elementary particles: leptons, quarks and mediators, also called *gauge bosons*, see Fig. 2.1.

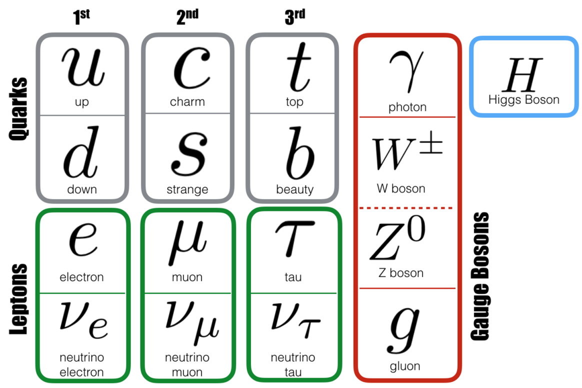

Figure 2.1: Particle content of the SM [23].

Leptons are fermion particles which have spin 1/2 and do not interact via the strong force. There are six leptons in the SM, the electron (e), muon (μ) and tau (τ) together with their corresponding neutrinos (ν_e, ν_μ, ν_τ). Neutrinos do not carry electric charge, so their motion is directly influenced only by the weak nuclear force, which makes them notoriously difficult to detect. By contrast the electron, muon and tau all interact electromagnetically.

Quarks just like leptons are spin 1/2 particles, but they do interact with the strong force and thus have an associated *color* charge. Similarly, there are six *flavours* of quarks: up (u), down (d), charm (c), strange (s), top (t) and bottom (b).

Quarks and leptons fall naturally into three families or generations. Each member of a generation has greater mass than the corresponding particles of lower generations but the same quantum numbers, such as charge and spin. The first generation charged particles do not decay (except the d quark in free neutron decay), hence all ordinary matter is made of such particles.

Every interaction has its mediators; the photon for the electromagnetic force, two W's and a Z for the weak force and eight gluons for the strong force.

Finally, to explain why some particles have mass, it was proposed the so called Higgs mechanism [24], that required that a spinless particle should exist. This particle was called the Higgs boson and it was discovered in 2012 by the ATLAS and CMS experiments at the Large Hadron Collider (LHC) at CERN [25, 26].

Together with their corresponding antiparticles and the fact that each quark and antiquark comes in three colors, there are 12 leptons, 36 quarks, 12 mediators and 1 Higgs boson, all told. So we have a minimum of 61 particles to contend with, see Fig.2.2, but as we just discussed, they are tightly interrelated. The eight gluons, for example, are identical except for their colors, and the second and third generations mimic the first.

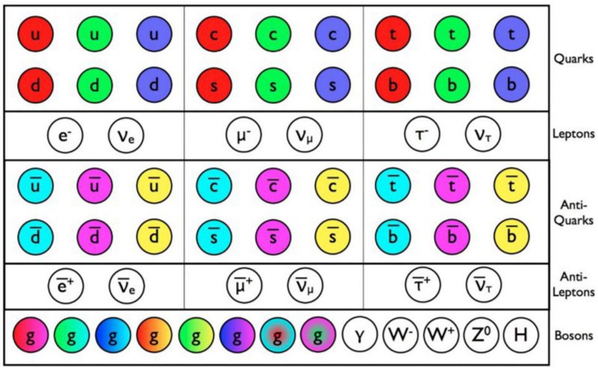

Figure 2.2: Particle content of the SM [27].

2.3 Gauge Fields and the Electroweak Sector

For this thesis, the electroweak sector is specially important since our results are obtained within the framework of the effective field theory that corresponds to the low-energy limit of the renormalizable $SU(2)_L \times U(1)_Y$ electroweak sector.

The electroweak sector interacts with the symmetry group $SU(2)_L \times U(1)_Y$, where the subscript L indicates coupling only to left handed fermions, as we shall see explicitly in the lepton decay formulation in further chapters.

The electroweak Lagrangian is [28]

$$\mathcal{L}_{EW} = \sum_{\Psi} \bar{\Psi}\gamma^{\mu}\left(i\partial_{\mu} - g'\frac{1}{2}Y_W B_{\mu} - g\frac{1}{2}\sigma_{La}W_{\mu}^{a}\right)\Psi - \frac{1}{4}W_a^{\mu\nu}W_{\mu\nu}^{a} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu}, \tag{2.1}$$

where B_{μ} is the $U(1)$ gauge field, Y_W is the weak hypercharge,the generator of the $U(1)$ group. W_{μ}^{a} is the 3-component $SU(2)$ gauge field (a=1,2,3); σ_{La} are the Pauli matrices, the generators of the $SU(2)$ group and the subscript L has the same meaning as before. Finally g' and g are the $U(1)$ and $SU(2)$ couplings respectively and $W_a^{\mu\nu}$ together with $B^{\mu\nu}$ are the field strength tensors for the weak isospin and weak hypercharge fields.

Due to the Higgs mechanism, the electroweak boson fields W_1, W_2, W_3 and B mix to create the states which are physically observable. To retain gauge invariance, the underlying fields must be massless, but the observable states can gain masses in the process [24].

The states are:

The massive neutral Z boson: $Z = cos\theta_W W_3 - sin\theta_W B$.

The massless neutral A boson: $A = sin\theta_W W_3 + cos\theta_W B$.

The massive charged W bosons: $W^{\pm} = \frac{1}{\sqrt{2}}\left(W_1 \mp iW_2\right)$

where θ_W is the weak mixing angle.

The A field is the photon, which corresponds to the well-known electromagnetic four-potential. The $W^{\pm}$ fields are precisely the fields involved in the lepton decay, that enter explicitly in the charged currents interactions, as we will discuss later.

2.4 Beyond the Standard Model

As we introduced before, the SM is a really successful and predictive theory. But it does not explain everything. There are some physical phenomena that motivate the research of new physics that is beyond the SM.

Here we summarize some of the most important unsolved problems in physics that lie beyond the SM. Some questions, that the SM do not answer are the following

- How to unify all the forces, including gravity, in a consistent theory?
- Why does its 19 parameters have the values that we measure?
- Why are there three generations of particles?

- Why is there more matter than antimatter in the universe?
- Where does dark matter fit into the SM? Does it even consist of one or more new particles?
- What is the mechanism from which neutrinos acquire mass? Are they Dirac or Majorana fermions?

The search for a fundamental description of nature which goes beyond the SM is driven by two complementary experimental strategies. The first is to build increasingly energetic colliders, such as the Large Hadron Collider (LHC) at CERN, to excite matter into a new form.

The second approach is to perform high precision measurements where an observed discrepancy with the SM would reveal the signature of new physics [16].

The high accuracy achieved by the most recent experiments allows to make stringent tests of the SM structure at the level of quantum corrections. Confronting these measurements with the theoretical predictions, one can check the internal consistency of the SM framework and determine its parameters.

In this thesis, we implement the second strategy, where the test of the Lorentz structure and Majorana neutrinos effects in the $l^- \longrightarrow l'^- \bar{\nu}_{l'} \nu_l$ transition amplitudes are discussed.

Chapter 3

Effective Theories

Effective field theories are one of the most important tools that provide a comprehensive and easier description of nature at certain physical scales. Although the concept of effective theory is in some sense intuitive, implementing them in a mathematically consistent way in an interacting quantum field theory is not so obvious.
In this chapter, we summarize the basic principles of effective theories, based on [29, 30, 31], as well as one of its most famous examples: The Fermi Theory of Weak Interactions.

3.1 Introduction to EFT

The idea behind effective field theories (EFT) is that you can calculate without knowing the exact theory. This allows you to compute an experimentally measurable quantity with some finite uncertainty. That is why engineers are able to design and build bridges without any knowledge of strong interactions or quantum gravity; you can even calculate the hydrogen atom energy levels without using the fact that the proton is made up of quarks, the existence of weak interactions or any detailed input from QED or QCD; just taking into account an electron of mass m_e interacting via a Coulomb potential with a proton treated as an infinitely heavy point particle with charge $+1$.
In order to analyze a particular physical system in a simpler way, it is necessary to isolate the most relevant ingredients from the rest, obtaining a good approximation without having to understand everything, in a way that we are able to study the low-energy dynamics of a physics problem that involves widely separated energy scales,independently of the details of the high-energy interactions.
Thus, effective field theories are the appropriate theoretical tool to describe low-energy physics, where low is defined with respect to some energy scale Λ. The approximation to the specific problem can always be improved by taking into account the corrections induced by the neglected energy scales as small perturbations.
In field theories this implies taking explicitly into account the relevant degrees of freedom,

i.e. those states with $m << \Lambda$, while the heavier excitations with $M >> \Lambda$ are integrated out from the action.[29]

There are many examples of successful EFT used in physics, such as Fermi theory of weak interactions, heavy quark effective theory (HQET), non-relativistic quantum chromodynamics (NRQCD), chiral perturbation theory (χPT), soft-collinear effective theory (SCET), etc. [30]

3.2 The EFT Lagrangian

The EFT is characterized by an effective Lagrangian of the form:*

$$\mathcal{L} = \sum_i c_i O_i \tag{3.1}$$

Where O_i are operators constructed with the light fields, and c_i its couplings.

We have said that high energy dynamics can be ignored in the study of processes at low energies, but actually the high energy or heavy degrees of freedom are taken into account in the couplings c_i. We will verify this explicitly in the next section.

For the action to be dimensionless, the Lagrangian density needs to have dimension 4. Thus, if we define the dimension of the operators O_i to be d_i, then the coefficients dimension are fixed as follows:

$$[O_i] = d_i \quad \longrightarrow \quad c_i \sim \frac{1}{\Lambda^{d_i - 4}} \tag{3.2}$$

with Λ some characteristic short-distance scale of the system.

The EFT Lagrangian can be seen as an expansion of a more fundamental theory with a small expansion parameter δ, known as the power counting parameter. Thus, calculations are done in an expansion to some order n in δ, so that the error is of order δ^{n+1}.

Usually, the expansion parameter is the ratio of a low-energy scale E (The energy at which the observables will be computed) such as the external momentum p or particle mass m, and the short-distance scale Λ, $\delta = m/\Lambda$.

It is important to remember that the EFT is useful in its predictions only when $E < \Lambda$; otherwise, when $E > \Lambda$, higher powers contribute more than lower ones and the calculations

*From now on we will use natural units.

are no longer valid.
Another aspect to emphasize is that the behavior of the different operators is determined by their dimension. Actually, we can distinguish three types of operators [29]:

- Relevant ($d_i < 4$)
- Marginal ($d_i = 4$)
- Irrelevant ($d_i > 4$)

In the low-energy regime ($E << \Lambda$), the *Relevant* operators are enhanced by $(\Lambda/E)^{4-d_i}$ becoming more important at lower energies, while the $Irrelevant$ operators are suppressed by $(E/\Lambda)^{d_i-4}$ being weak, but of course,this does not mean that they are not important. Finally, the $Marginal$ operator give raise to dimensionless coefficients and are equally important at all energy scales, although quantum effects could modify their scaling behaviour on either side.
So, explicitly, the EFT Lagrangian is given by an infinite series of terms of increasing operator dimension that has to be treated as an expansion in powers of (E/Λ),

$$\mathcal{L} = \mathcal{L}_{d\leq 4} + \mathcal{L}_5(E/\Lambda) + \mathcal{L}_6(E/\Lambda)^2 + ... \tag{3.3}$$

Finally, it is known that EFT are generally not renormalizable, but that is not a big deal, since it can be shown [29, 30], using well-known renormalization schemes together with dimensional analysis, that the EFT behaves for all practical purposes like a renormalizable field theory if one works to some fixed order in $1/\Lambda$. This is because there are only a finite number of terms in the effective Lagrangian $\mathcal{L}$ that are allowed to a given order in $1/\Lambda$. Terms of higher order in $1/\Lambda$ can be safely neglected because they can never be multiplied by positive powers of Λ to produce effects comparable to lower order terms.
Another way to say it, is that a non-renormalizable theory is just as good as a renormalizable theory for computations, provided one is satisfied with a finite accuracy. The reason why QED is so successful to describe the low–energy scattering of electrons with positrons is not renormalizability, but rather the fact that M_Z is very heavy and the leading non-renormalizable contributions are suppressed by $(E/M_Z)^2$.

To summarize this ideas, we can build a set of general principles for EFT [29]:

1. Dynamics at low energies (large distances) does not depend on details of dynamics at high energies (short distances).

2. Choose the appropriate description of the important physics at the considered scale. If there are large energy gaps, put to zero (infinity) the light (heavy) scales, i.e.

$$0 \longleftarrow m << E << \Lambda \longrightarrow \infty$$

   Finite corrections induced by these scales can be incorporated as perturbations.

3. The EFT describes the low–energy physics, to a given accuracy ϵ, in terms of a finite set of parameters:

$$\epsilon \leq (E/\Lambda)^{(d_i-4)}$$

4. The EFT has the same infra-red (but different ultra-violet) behaviour than the underlying fundamental theory.

5. The only remnants of the high–energy dynamics are in the low–energy couplings and in the symmetries of the EFT.

3.3 Fermi Theory of Weak Interactions

Historically, Fermi created his theory as an attempt to explain beta decay even before the Standard Model (SM) existed; he based his intuition on electromagnetism which involves a vector current. Nowadays, as we know, Fermi's vector form was replaced by the $V-A$ interaction form which agrees with the experimental results such as parity violation [32]. In this section, we will discuss how to obtain the Fermi theory as the low-energy limit of the renormalizable $SU(2) \times U(1)$ electroweak theory at tree level.

In the Standard Model, weak decays proceed at lowest order through the exchange of a $W^{\pm}$ boson between two fermionic left–handed currents (except for the heavy quark top which decays into a real W). The charged current Lagrangian that describes this process is given in the SM by:

$$\mathcal{L} = -\frac{g}{2\sqrt{2}} W_\mu^\dagger \left\{ \sum_l \bar{\nu}_l \gamma^\mu (1-\gamma^5) l + \sum_{ij} \bar{u}_i \gamma^\mu (1-\gamma^5) V_{ij} d_j \right\} + h.c \tag{3.4}$$

Where $g/\sqrt{2}$ is the W coupling constant, u_i are the up-type quarks, d_j are down type quarks and V_{ij} is the CKM mixing matrix, l is a lepton and ν_l the corresponding neutrino. Let us now focus on the leptonic sector, the derivation of the quark sector will be almost identical.

The tree-level amplitude for the leptonic $l \longrightarrow l' + \bar{\nu}_{l'} + \nu_l$ decay from (3.4) is:

$$\mathcal{M} = \left(\frac{-ig}{2\sqrt{2}}\right)^2 [\bar{u}_4\gamma^\mu(1-\gamma^5)v_2] \left(\frac{-ig_{\mu\nu}}{p^2 - M_W^2}\right) [\bar{u}_3\gamma^\nu(1-\gamma^5)u_1], \tag{3.5}$$

where the W boson propagator is in the 't Hooft-Feynman gauge, p is the momentum transferred by the W, and u_1,v_2,u_3 and u_4 are the spinors corresponding to l,$\bar{\nu}_{l'}$,ν_l and l' respectively.

The amplitude (3.5) produces a non-local four-fermion interaction, because of the factor of $p^2 - M_W^2$ in the denominator. However , if the momentum transfer p is small compared with M_W, i.e, $p << M_W$, the non-local interaction can be approximated by a local interaction using the Taylor series expansion

$$\frac{1}{p^2 - M_W^2} = -\frac{1}{M_W^2}\left(1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + ...\right) \tag{3.6}$$

Thus, to lowest order, the amplitude is

$$\mathcal{M} = \left(\frac{i}{M_W^2}\right)\left(\frac{ig}{2\sqrt{2}}\right)^2 [\bar{u}_4\gamma^\mu(1-\gamma^5)v_2][\bar{u}_3\gamma_\mu(1-\gamma^5)u_1] + \mathcal{O}\left(\frac{1}{M_W^4}\right), \tag{3.7}$$

which is the same amplitude as that produced by the local Lagrangian

$$\mathcal{L} = -\frac{g^2}{8M_W^2}[\bar{l}'\gamma^\mu(1-\gamma^5)\nu_{l'}][\bar{\nu}_l\gamma^\nu(1-\gamma^5)l] + \mathcal{O}\left(\frac{1}{M_W^4}\right). \tag{3.8}$$

Equation (3.8) is the lowest order Lagrangian for leptonic decay in the EFT ($p << M_W$). It is usually written, for historical reasons, in terms of the so-called Fermi coupling constant G_F

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}}[\bar{l}'\gamma^\mu(1-\gamma^5)\nu_{l'}][\bar{\nu}_l\gamma^\nu(1-\gamma^5)l] + \mathcal{O}\left(\frac{1}{M_W^4}\right). \tag{3.9}$$

Where the relation between G_F and the W coupling g is

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}. \tag{3.10}$$

These results can be generalized to the quark sector, so weak decays can then be described through an effective local four-fermion interaction

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}}\mathcal{J}_\mu \mathcal{J}^{\mu\dagger}, \tag{3.11}$$

where

$$\mathcal{J}^\mu = \sum_l \bar{\nu}_l \gamma^\mu (1-\gamma^5) l + \sum_{ij} \bar{u}_i \gamma^\mu (1-\gamma^5) V_{ij} d_j \tag{3.12}$$

Equation (3.11) is what we called the Fermi theory of weak interactions. Fermi theory is only valid in the regime ($p << M_W$); which describes to a really good accuracy the weak decays where p is of the order of the decaying particle mass such as m_b, m_c and m_l; since $m_b, m_c, m_l << M_W$. It contains operators of dimension 6 and, therefore, a coupling with dimension -2. Thus, the *Irrelevant* ($d > 4$) property, given in the previous section, of the operators give us an answer of why this type of interactions are *weak* at low energies ($E << M_W$).

Equation (3.10) is technically called a *matching condition*; and as we described before, shows explicitly the relation between the effective coupling constant and the underlying electroweak theory parameters (g,M_W).

Again, the Fermi Lagrangian, as any other EFT, has the usual expansion form, with the expansion parameter $\delta = p/M_W$. The higher order corrections $\mathcal{O}(1/M_W^4)$ can be neglected, provided we are satisfied with an accuracy not better than (p^2/M_W^2).

To summarize; at low energies ($p << M_W$), there is not enough energy to produce a physical W boson. Therefore, the vector-boson propagator shrinks to a point and can be well approximated through a local four-fermion interaction (Fig. 3.1), i.e, for weak decays the electroweak theory can be replaced by the Fermi effective theory up to corrections of order $\mathcal{O}(1/M_W^4)$.

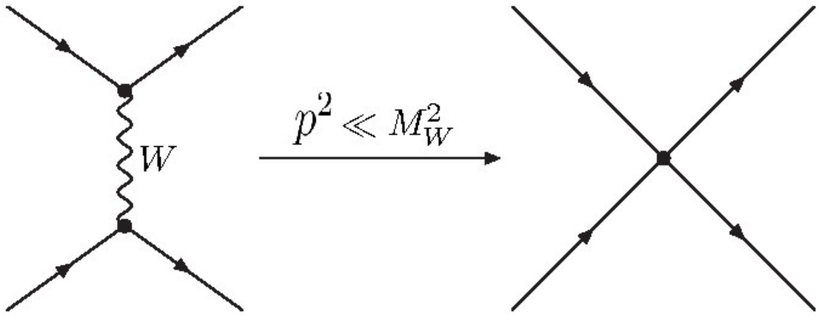

Figure 3.1: Left: Non-local weak interaction by the exchange of a W boson. Right:Local weak interaction described by Fermi's theory.

Fermi theory of weak interactions include the effect of W exchange via dimension-six four-fermion operators and the higher energy parameters are *hidden* in the effective coupling G_F.

Also notice that the EFT Lagrangian (3.11) has the same chiral symmetry as the fundamental theory, coupling just left-handed particles (right-handed antiparticles) via the V-A Lorentz structure.

This might seem trivial, since we obtained the EFT explicitly as a limit of the electroweak fundamental theory. But we often do not know the underlying theory that gives rise to the EFT; in that case the EFT is constructed as the most general analysis consistent with the symmetries and the known light particles. Nevertheless the properties of the unknown high-energy theory will be reflected as non-trivial symmetries in the EFT as we shall see explicitly at the end of the next chapter.

Chapter 4

Lepton Decays

Muon and tau leptonic decays are specially important processes in the SM since they provide a clean laboratory to test its prediction, such as the structure of the weak currents and the universality of their couplings to the gauge bosons.

All experimental results obtained so far confirm the SM scenario. Although, the increased sensitivities of the most recent experiments result in interesting limits on possible new physics contributions to the lepton decay amplitudes.

In this chapter, we summarize the well known results of lepton decays, and introduced the so called *Michel parameters* that characterize the energy and angular distributions of the final charged lepton in the case of the most general four-lepton interaction Hamiltonian.

4.1 Unpolarized Charged Lepton Decays in the SM

Let us consider the leptonic decays $l^- \longrightarrow l'^- \bar{\nu}_{l'} \nu_l$ Fig.4.1, where the lepton pair (l,l') may be (μ,e), (τ,e) or (τ,μ).

Since $m_l << M_W$ we can safely use the Fermi theory of weak interactions to describe the charged lepton decays.

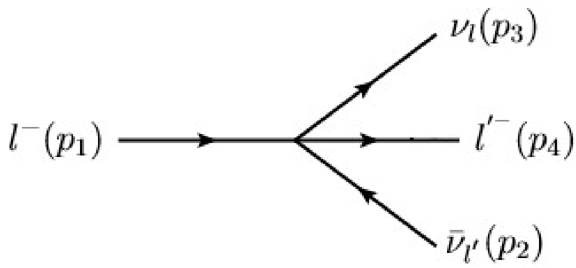

Figure 4.1: Lepton decay in Fermi's theory.

Thus, the decay is characterized by the local four-fermion interaction Lagrangian

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}}[\bar{l}'\gamma^\mu(1-\gamma^5)\nu_{l'}][\bar{\nu}_l\gamma^\nu(1-\gamma^5)l]. \tag{4.1}$$

Which corresponds to a matrix element

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}[\bar{u}_4\gamma^\mu(1-\gamma^5)v_2][\bar{u}_3\gamma_\mu(1-\gamma^5)u_1], \tag{4.2}$$

where the spinors subscripts agree with the corresponding particle momentum, as shown in Fig.4.1.

The differential decay rate is

$$d\Gamma = \frac{(2\pi)^4\delta^4(p_1-p_2-p_3-p_4)}{2m_1}\frac{d^3p_2d^3p_3d^3p_4|\mathcal{M}|^2}{(2\pi)^3 2E_2(2\pi)^3 2E_3(2\pi)^3 2E_4} \tag{4.3}$$

$|\mathcal{M}|^2$ can be calculated using the standard trace techniques.

Since we are not dealing with polarized leptons, the decay rate can be obtained by averaging over the initial $l(p_1)$ spins and summing over all final spins.

$$|\mathcal{M}|^2 \longrightarrow |\overline{\mathcal{M}}|^2 \equiv \frac{1}{2}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 \tag{4.4}$$

This, together with the spinors completeness relations leads to the following trace expression [20]

$$|\overline{\mathcal{M}}|^2 = \frac{G_F^2}{4}Tr[(\not{p}_4+m_4)\gamma^\mu(1-\gamma^5)(\not{p}_2+m_2)\gamma^\nu(1-\gamma^5)]Tr[(\not{p}_3+m_3)\gamma_\mu(1-\gamma^5)(\not{p}_1+m_1)\gamma_\nu(1-\gamma^5)], \tag{4.5}$$

where $\not{p}$ is the usual Dirac slash-notation $\not{p} \equiv \gamma^\mu p_\mu$.

Working with massless neutrinos ($m_2 = m_3 = 0$) and using the well-known trace identities; we obtain

$$\begin{aligned}|\overline{\mathcal{M}}|^2 = 16G_F^2[p_2^\mu p_4^\nu + p_2^\nu p_4^\mu - g^{\mu\nu}p_2\cdot p_4 + i\epsilon^{\mu\nu\rho\sigma}p_{2\rho}p_{4\sigma}]\\ [p_{1\mu}p_{3\nu} + p_{1\nu}p_{3\mu} - g_{\mu\nu}p_1\cdot p_3 + i\epsilon_{\mu\nu\tau\omega}p_1^\tau p_3^\omega]\end{aligned} \tag{4.6}$$

Using the symmetry properties of the above products and the identity $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\tau\omega} = -2(g_\tau^\rho g_\omega^\sigma - g_\omega^\rho g_\tau^\sigma)$ it is straightforward to evaluate the equation (4.6),

$$|\overline{\mathcal{M}}|^2 = 64G_F^2(p_1\cdot p_2)(p_3\cdot p_4), \tag{4.7}$$

thus the unpolarized differential decay rate (4.3) is

$$d\bar{\Gamma} = \frac{(2\pi)^4\delta^4(p_1 - p_2 - p_3 - p_4)}{2m_1}\frac{d^3p_2 d^3p_3 d^3p_4 64G_F^2(p_1\cdot p_2)(p_3\cdot p_4)}{(2\pi)^3 2E_2(2\pi)^3 2E_3(2\pi)^3 2E_4}. \tag{4.8}$$

Assuming that the neutrinos are not detected, we can integrate over their momenta

$$d\bar{\Gamma} = \frac{4G_F^2}{(2\pi)^5}\frac{d^3p_4 p_1^\mu p_4^\nu}{m_1 E_4}\int\frac{d^3p_2 d^3p_3}{E_2E_3}\delta^4(p_1 - p_2 - p_3 - p_4)p_{2\mu}p_{3\nu}. \tag{4.9}$$

The integral we need to evaluate is of the form

$$I_{\mu\nu} \equiv \int\frac{d^3p_2 d^3p_3}{E_2E_3}\delta^4(p_2 + p_3 - q)p_{2\mu}p_{3\nu}. \tag{4.10}$$

with $q \equiv p_1 - p_4$. We know that (4.10) is a second rank tensor which can only depend on q. Therefore, using the covariance properties of this tensor integral, it must be of the form

$$I_{\mu\nu} = Aq^2 g_{\mu\nu} + Bq_\mu q_\nu \tag{4.11}$$

Also, we can build the following scalars

$$\begin{aligned} g^{\mu\nu}I_{\mu\nu} &= (4A + B)q^2 \\ q^\mu q^\nu I_{\mu\nu} &= (A + B)q^4 \end{aligned} \tag{4.12}$$

The explicit calculation of (4.12) allows us to obtain the values of the constants A and B, leading to $B = 2A = \pi/3$ [33]. Thus,

$$I_{\mu\nu} = \frac{\pi}{6}[q^2 g_{\mu\nu} + 2q_\mu q_\nu] \tag{4.13}$$

$$d\bar{\Gamma} = \frac{4G_F^2}{(2\pi)^5}\frac{d^3p_4 p_1^\mu p_4^\nu}{m_1 E_4}\frac{\pi}{6}[q^2 g_{\mu\nu} + 2q_\mu q_\nu], \tag{4.14}$$

which reduces to

$$d\bar{\Gamma} = \frac{2\pi G_F^2}{3(2\pi)^5}\frac{d^3p_4}{m_1 E_4}[q^2(p_1\cdot p_4) + 2(p_1\cdot q)(p_4\cdot q)], \tag{4.15}$$

In the initial lepton rest frame, we can easily compute the following scalar products:

$$\begin{aligned} &p_1 = (m_1, \vec{0}), \quad p_4 = (E_4, \vec{p_4}), \quad p_1^2 = m_1^2 \quad p_4^2 = m_4^2, \\ &p_1 \cdot p_4 = m_1 E_4, \quad p_1 \cdot q = m_1^2 - m_1 E_4, \quad p_4 \cdot q = m_1 E_4 - m_4^2, \end{aligned} \tag{4.16}$$

yielding

$$d\bar{\Gamma} = \frac{2\pi G_F^2}{3(2\pi)^5}\frac{d^3p_4}{m_1 E_4}[(m_1^2 + m_4^2 - 2m_1E_4)(m_1E_4) + 2(m_1^2 - m_1E_4)(m_1E_4 - m_4^2)], \tag{4.17}$$

Using, $d^3p_4 = 2\pi|\vec{p_4}|^2 d\cos\theta d|\vec{p_4}|$ and $E_4^2 = |\vec{p_4}|^2 + m_4^2 \longrightarrow d|\vec{p_4}| = \frac{E_4}{|\vec{p_4}|}dE_4$, equation (4.17) is nicely written as follow,

$$d\bar{\Gamma} = \frac{4\pi^2 G_F^2}{3(2\pi)^5}\sqrt{E_4^2 - m_4^2} \quad [3m_1^2E_4 + 3m_4^2E_4 - 4m_1E_4^2 - 2m_1m_4^2]dE_4 d\cos\theta, \tag{4.18}$$

Equation (4.18) shows the explicit energy and (isotropic) angular distribution of the final charged lepton, when specific polarizations are not measured.

Finally, from the kinematics analysis of $1 \longrightarrow 3$ processes; we know that [21]

$$\begin{aligned} &E_4 = \frac{m_1^2 + m_4^2 - s_{23}}{2m_1}, \quad \textit{with} \quad s_{23} = (p_2 + p_3)^2 = (p_1 - p_4)^2 \\ &\therefore (m_2 + m_3)^2 \le s_{23} \le (m_1 - m_4)^2, \end{aligned} \tag{4.19}$$

considering massless neutrinos ($m_2 = m_3 = 0$), the energy limits of the final charged lepton are

$$m_4 \le E_4 \le \frac{m_1^2 + m_4^2}{2m_1} \tag{4.20}$$

Thus, we can obtain the final leptonic decay width, integrating over $\cos\theta$ and E_4 as follows

$$\Gamma = \frac{4\pi^2 G_F^2}{3(2\pi)^5}\int_{-1}^{1} d\cos\theta \int_{m_4}^{\frac{m_1^2+m_4^2}{2m_1}} \sqrt{E_4^2 - m_4^2} \quad [3m_1^2E_4 + 3m_4^2E_4 - 4m_1E_4^2 - 2m_1m_4^2]dE_4, \tag{4.21}$$

which leads us to the final result for the total decay width

$$\Gamma = \frac{G_F^2 m_1^5}{192\pi^3} f\left(\frac{m_4^2}{m_1^2}\right), \tag{4.22}$$

where $f(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 log(x)$.

For the precision electroweak tests one must take into account radiative QED corrections, higher-order electroweak corrections and the non-local structure of the W propagator. With these considerations, the leptonic decay widths are given by [20, 15]

$$\Gamma_{l\longrightarrow l'} \equiv \Gamma[l^- \longrightarrow l'^- \bar{\nu}_{l'}\nu_l(\gamma)] = \frac{G_{ll'}^2 m_1^5}{192\pi^3} f\left(\frac{m_{l'}^2}{m_l^2}\right)\left(1+\delta_{RC}^{ll'}\right), \tag{4.23}$$

where $m_l = m_1$, $m_{l'} = m_4$, (γ) represents additional photons or lepton pairs which have been included inclusively in $\delta_{RC}^{ll'}$ and [34, 35, 36]

$$\delta_{RC}^{ll'} = \frac{\alpha}{2\pi}\left[\frac{25}{4} - \pi^2 + \mathcal{O}\left(\frac{m_{l'}^2}{m_l^2}\right)\right] + ... \tag{4.24}$$

$$G_{ll'}^2 = \left[\frac{g^2}{4\sqrt{2}M_W^2}(1+\Delta r)\right]^2 \left[1 + \frac{3}{5}\frac{m_l^2}{M_W^2} + \frac{9}{5}\frac{m_{l'}^2}{M_W^2} + \mathcal{O}\left(\frac{m_{l'}^4}{m_l^2 M_W^2}\right)\right] \tag{4.25}$$

Equation (4.23) can be used to obtain the Fermi constant G_F from the observed muon lifetime τ_μ; since $\tau_\mu^{-1} = \Gamma[\mu^- \longrightarrow e^-\bar{\nu}_e\nu_\mu(\gamma)]$.

Nowadays, τ_μ= 2.1969811(22) $\times$ 10^{-6}s, is the most accurate particle lifetime ever measured [37]; combined with the electron and muon masses implies

$$G_F = (1.1663787 \pm 0.0000006) \times 10^{-5} \quad GeV^{-2} \tag{4.26}$$

4.2 Polarized Charged Lepton Decays in the SM

The expression in (4.2) can be generalized to allow for a polarized lepton. The only difference is that now we should not sum over both spin orientations for the initial lepton, rather we should take the spinor corresponding to that value of spin.

Still, the spin sum and trace identities process are an excellent machinery that we would like to continue using. For this purpose, an alternative is to sum over both spins, but introduce a spin projection matrix in the amplitude which will select out only one spin orientation [21].

Thus, for an initial polarized lepton, equation (4.2) is now written as follows

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}[\bar{u_4}\gamma^\mu(1-\gamma^5)v_2][\bar{u_3}\gamma_\mu(1-\gamma^5)\frac{1}{2}(1+\gamma^5 \not{s})u_1], \tag{4.27}$$

where $\frac{1}{2}(1+\gamma^5 \not{s})$ is the spin projector, s^μ is the polarization four-vector which has the components

$$s^\mu = (0, \hat{s}) \tag{4.28}$$

in the rest frame of the decaying particle, $\hat{s}$ being a three-vector along the direction in which the leptons spins have been align, and satisfies the properties $p_1 \cdot s = 0$ and $s^2 = -1$. With these considerations, the calculation is done in the same way as in the previous section, giving the following squared amplitude

$$|\mathcal{M}|^2 = 64G_F^2(p_1 \cdot p_2 - m_1(p_2 \cdot s))(p_3 \cdot p_4), \tag{4.29}$$

Considering the polarization of the decaying particle does not change the dependence of the squared amplitude upon the neutrino four-momenta, as compared to the unpolarized case. Thus, the neutrinos phase-space integral (4.10) is going to be the same, leading to

$$\begin{aligned} d\Gamma = \frac{4\pi^2 G_F^2}{3(2\pi)^5}\sqrt{E_4^2 - m_4^2} & \left\{[3m_1^2E_4 + 3m_4^2E_4 - 4m_1E_4^2 - 2m_1m_4^2]\right. \\ + \mathcal{P}\sqrt{E_4^2 - m_4^2} & \left.[m_1^2 + 3m_4^2 - 4E_4m_1]\cos\theta\right\} \quad dE_4 d\cos\theta, \end{aligned} \tag{4.30}$$

where $\mathcal{P}$ is the initial lepton net polarization and θ is the angle between the l^- spin and the final charged lepton momentum.

Equation (4.30) shows the explicit energy and angular distribution of the final lepton when the initial lepton polarization is known. Furthermore, we can integrate over the final lepton energy, where the integration limits have been discussed in the previous section. Integration over E_4 gives

$$\frac{d\Gamma}{d\cos\theta} = \frac{G_F^2 m_1^5}{384\pi^3}\left\{f\left(\frac{m_4^2}{m_1^2}\right) - \frac{1}{3}\cos\theta\left(1 - \frac{m_4}{m_1}\right)h\left(\frac{m_4}{m_1}\right)\right\}, \tag{4.31}$$

with $h(x) = 1 + 5x + 15x^2 + 3x^3$. We see, first of all, that the $\cos\theta$ term makes the differential decay rate non-uniform, this kind of terms signal parity violation. Furthermore, if we integrate the angular configuration, the $\cos\theta$ term gives no contribution. In fact, the total rate obtained in this case is equal to the total rate for unpolarized leptons that was obtained in (4.22).

Study of polarized lepton decay played a crucial role in determining the $(V - A)$ nature of the interaction, and in the general case to study the effects of new physics described by new operators and couplings of the effective weak Hamiltonian.

4.3 Lorentz Structure of the Charged Current

In the previous section we described the SM predictions for the three-body decay of polarized and unpolarized leptons. Here, we will generalize these results using the most general four-lepton effective interaction Hamiltonian, in such a way that, with high statistics, we will be able to investigate the Lorentz structure of the decay amplitudes through the analysis of the energy and angular distribution of the final charged lepton, complemented with polarization information whenever available.

The most general, local, derivative-free, lepton-number conserving, four-lepton interaction Hamiltonian, consistent with locality and Lorentz invariance is [15, 38]

$$\mathcal{H} = 4\frac{G_{ll'}}{\sqrt{2}} \sum_{n,\epsilon,\omega} g^n_{\epsilon\omega} \left[\bar{l'}_\epsilon \Gamma^n (\nu_{l'})_\sigma\right] \left[(\bar{\nu}_l)_\lambda \Gamma_n l_\omega\right]. \tag{4.32}$$

The subindices $\epsilon, \omega, \sigma, \lambda$ label the chiralities (L, R) of the corresponding fermions, and $n = S, V, T$ the type of interaction: scalar $(\Gamma^S = I)$,vector $(\Gamma^V = \gamma^\mu)$ and tensor $(\Gamma^T = \sigma^{\mu\nu}/\sqrt{2})$.

Since tensor interactions can contribute only for opposite chiralities of the charged leptons, as we will prove it later, this leads to the existence of 10 complex coupling constants, related to 4 scalar, 4 vector and 2 tensors interactions.

Once an unphysical global phase is removed, it leaves 19 real numbers to be determined by the experiment. Furthermore the global factor $G_{ll'}$, which is determined by the total decay rate, leads to the following normalization of the coupling constants [39]

$$\begin{aligned} 1 = &\frac{1}{4}(|g^S_{RR}|^2 + |g^S_{RL}|^2 + |g^S_{LR}|^2 + |g^S_{LL}|^2) + 3(|g^T_{RL}|^2 + |g^T_{RL}|^2) \\ &+ (|g^V_{RR}|^2 + |g^V_{RL}|^2 + |g^V_{LR}|^2 + |g^V_{LL}|^2). \end{aligned} \tag{4.33}$$

Thus, $|g^S_{\epsilon\omega}| \leq 2$,$|g^V_{\epsilon\omega}| \leq 1$ and $|g^T_{\epsilon\omega}| \leq 1/\sqrt{3}$.

The Standard Model predicts $|g^V_{LL}| = 1$ and all others couplings being zero. In the search for new physics it is important to calculate the final charged-lepton distribution using the

Hamiltonian (4.32).

The following calculation was done considering an initial lepton polarization and was computed with the help of the *FeynCalc* Mathematica package [40, 41, 42].

Expanding the Hamiltonian (4.32), where due to the chiral projectors properties, for a given ϵ, ω, n, the neutrino chiralities σ and λ are uniquely determined; we obtain

$$\begin{aligned}\mathcal{H} = 4\frac{G_{ll'}}{\sqrt{2}}\Big\{ & g^S_{LL}\left[\bar{l}'_L \nu_{l'R}\right]\left[\bar{\nu}_{lR} l_L\right] + g^V_{LL}\left[\bar{l}'_L \gamma^\mu \nu_{l'L}\right]\left[\bar{\nu}_{lL}\gamma_\mu l_L\right] + g^S_{RR}\left[\bar{l}'_R \nu_{l'L}\right]\left[\bar{\nu}_{lL} l_R\right] \\ & + g^V_{RR}\left[\bar{l}'_R \gamma^\mu \nu_{l'R}\right]\left[\bar{\nu}_{lR}\gamma_\mu l_R\right] + g^S_{LR}\left[\bar{l}'_L \nu_{l'R}\right]\left[\bar{\nu}_{lL} l_R\right] + g^V_{LR}\left[\bar{l}'_L \gamma^\mu \nu_{l'L}\right]\left[\bar{\nu}_{lR}\gamma_\mu l_R\right] \\ & + g^T_{LR}\left[\bar{l}'_L \frac{\sigma^{\mu\nu}}{\sqrt{2}} \nu_{l'R}\right]\left[\bar{\nu}_{lL}\frac{\sigma_{\mu\nu}}{\sqrt{2}} l_R\right] + g^S_{RL}\left[\bar{l}'_R \nu_{l'L}\right]\left[\bar{\nu}_{lR} l_L\right] + g^V_{RL}\left[\bar{l}'_R \gamma^\mu \nu_{l'R}\right]\left[\bar{\nu}_{lL}\gamma_\mu l_L\right] \\ & + g^T_{RL}\left[\bar{l}'_R \frac{\sigma^{\mu\nu}}{\sqrt{2}} \nu_{l'L}\right]\left[\bar{\nu}_{lR}\frac{\sigma_{\mu\nu}}{\sqrt{2}} l_L\right]\Big\}\end{aligned} \tag{4.34}$$

The tensor couplings g^T_{LL} and g^T_{RR} terms are identically zero,

$$\mathcal{H}^T_{LL} \equiv g^T_{LL}\left[\bar{l}'_L \frac{\sigma^{\mu\nu}}{\sqrt{2}} \nu_{l'R}\right]\left[\bar{\nu}_{lR}\frac{\sigma_{\mu\nu}}{\sqrt{2}} l_L\right] = 0 \tag{4.35}$$

$$\mathcal{H}^T_{RR} \equiv g^T_{RR}\left[\bar{l}'_R \frac{\sigma^{\mu\nu}}{\sqrt{2}} \nu_{l'L}\right]\left[\bar{\nu}_{lL}\frac{\sigma_{\mu\nu}}{\sqrt{2}} l_R\right] = 0 \tag{4.36}$$

This is due to the Fierz transformations identities [21], that ensure

$$\begin{aligned}\frac{\mathcal{H}^T_{LL}}{g^T_{LL}} &= \left[\bar{l}'_L \frac{\sigma^{\mu\nu}}{\sqrt{2}} \nu_{l'R}\right]\left[\bar{\nu}_{lR}\frac{\sigma_{\mu\nu}}{\sqrt{2}} l_L\right] \\ &= 3\left[\bar{l}'_L l_L\right]\left[\bar{\nu}_{lR}\nu_{l'R}\right] - \frac{1}{2}\left[\bar{l}'_L \frac{\sigma^{\mu\nu}}{\sqrt{2}} l_L\right]\left[\bar{\nu}_{lR}\frac{\sigma_{\mu\nu}}{\sqrt{2}}\nu_{l'R}\right] + 3\left[\bar{l}'_L\gamma^5 l_L\right]\left[\bar{\nu}_{lR}\gamma^5\nu_{l'R}\right] \\ &= 3\left[\bar{l}' P_R P_L l\right]\left[\bar{\nu}_l P_L P_R \nu_{l'}\right] - \frac{1}{4}\left[\bar{l}' P_R \sigma^{\mu\nu} P_L l\right]\left[\bar{\nu}_l P_L \sigma^{\mu\nu} P_R \nu_{l'}\right] \\ &\quad + 3\left[\bar{l}' P_R \gamma^5 P_L l\right]\left[\bar{\nu}_l P_L \gamma^5 P_R \nu_{l'}\right] \\ &= -\frac{1}{4}\left[\bar{l}'\sigma^{\mu\nu} P_R P_L l\right]\left[\bar{\nu}_l \sigma^{\mu\nu} P_L P_R \nu_{l'}\right] + 3\left[\bar{l}'\gamma^5 P_R P_L l\right]\left[\bar{\nu}_l\gamma^5 P_L P_R \nu_{l'}\right] \\ &= 0,\end{aligned} \tag{4.37}$$

where $P_{R,L} \equiv \frac{1}{2}(1 \pm \gamma^5)$ are the chirality projectors and we have used the properties $\overline{\Psi}_{R,L} = \overline{\Psi} P_{L,R}$, $\Psi_{R,L} = P_{R,L}\Psi$, $P_L P_R = P_R P_L = 0$, $[P_{R,L}, \gamma^5] = [P_{R,L}, \sigma^{\mu\nu}] = 0$.

The same transformations can be applied to $\mathcal{H}^T_{RR}$ verifying that the result is zero too.

The Hamiltonian (4.34) can be written in terms of the chirality operators. Using all the properties described before, it takes the form

$$\begin{aligned}\mathcal{H} = \frac{G_{ll'}}{\sqrt{2}}\Big\{ & g^S_{LL}\left[\bar{l}'(1+\gamma^5)\nu_{l'}\right]\left[\bar{\nu}_l(1-\gamma^5)l\right] + g^V_{LL}\left[\bar{l}'\gamma^\mu(1-\gamma^5)\nu_{l'}\right]\left[\bar{\nu}_l\gamma_\mu(1-\gamma^5)l\right] \\ & + g^S_{RR}\left[\bar{l}'(1-\gamma^5)\nu_{l'}\right]\left[\bar{\nu}_l(1+\gamma^5)l\right] + g^V_{RR}\left[\bar{l}'\gamma^\mu(1+\gamma^5)\nu_{l'}\right]\left[\bar{\nu}_l\gamma_\mu(1+\gamma^5)l\right] \\ & + g^S_{LR}\left[\bar{l}'(1+\gamma^5)\nu_{l'}\right]\left[\bar{\nu}_l(1+\gamma^5)l\right] + g^V_{LR}\left[\bar{l}'\gamma^\mu(1-\gamma^5)\nu_{l'}\right]\left[\bar{\nu}_l\gamma_\mu(1+\gamma^5)l\right] \\ & + g^T_{LR}\left[\bar{l}'\frac{\sigma^{\mu\nu}}{\sqrt{2}}(1+\gamma^5)\nu_{l'}\right]\left[\bar{\nu}_l\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1+\gamma^5)l\right] + g^S_{RL}\left[\bar{l}'(1-\gamma^5)\nu_{l'}\right]\left[\bar{\nu}_l(1-\gamma^5)l\right] \\ & + g^V_{RL}\left[\bar{l}'\gamma^\mu(1+\gamma^5)\nu_{l'}\right]\left[\bar{\nu}_l\gamma_\mu(1-\gamma^5)l\right] + g^T_{RL}\left[\bar{l}'\frac{\sigma^{\mu\nu}}{\sqrt{2}}(1-\gamma^5)\nu_{l'}\right]\left[\bar{\nu}_l\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1-\gamma^5)l\right]\Big\}\end{aligned} \tag{4.38}$$

Also, if we take into account the decaying lepton polarization, as discussed in the last section, we need to make the change $l \longrightarrow \frac{1}{2}(1+\gamma^5\not{s})l$.
Thus, the Hamiltonian (4.38) together with the spin projector leads the following amplitude, for the process shown in Fig.4.1

$$\begin{aligned}\mathcal{M} = -\,i\frac{G_{ll'}}{2\sqrt{2}}\Big\{ & g^S_{LL}\left[\bar{u}_4(1+\gamma^5)v_2\right]\left[\bar{u}_3(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{LL}\left[\bar{u}_4\gamma^\mu(1-\gamma^5)v_2\right] \\ & \left[\bar{u}_3\gamma_\mu(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{RR}\left[\bar{u}_4(1-\gamma^5)v_2\right]\left[\bar{u}_3(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] \\ & + g^V_{RR}\left[\bar{u}_4\gamma^\mu(1+\gamma^5)v_2\right]\left[\bar{u}_3\gamma_\mu(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{LR}\left[\bar{u}_4(1+\gamma^5)v_2\right] \\ & \left[\bar{u}_3(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{LR}\left[\bar{u}_4\gamma^\mu(1-\gamma^5)v_2\right]\left[\bar{u}_3\gamma_\mu(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] \\ & + g^T_{LR}\left[\bar{u}_4\frac{\sigma^{\mu\nu}}{\sqrt{2}}(1+\gamma^5)v_2\right]\left[\bar{u}_3\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{RL}\left[\bar{u}_4(1-\gamma^5)v_2\right] \\ & \left[\bar{u}_3(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{RL}\left[\bar{u}_4\gamma^\mu(1+\gamma^5)v_2\right]\left[\bar{u}_3\gamma_\mu(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] \\ & + g^T_{RL}\left[\bar{u}_4\frac{\sigma^{\mu\nu}}{\sqrt{2}}(1-\gamma^5)v_2\right]\left[\bar{u}_3\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1-\gamma^5)(1+\gamma^5\not{s})u_1\right]\Big\}\end{aligned} \tag{4.39}$$

There are 10 pairs of leptonic currents, so in order to find the squared amplitude we need to calculate the following terms

$$\mathcal{M} \equiv \mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3+\mathcal{M}_4+\mathcal{M}_5+\mathcal{M}_6+\mathcal{M}_7+\mathcal{M}_8+\mathcal{M}_9+\mathcal{M}_{10} \tag{4.40}$$

$$|\mathcal{M}|^2 = \sum_{i=1}^{10} |\mathcal{M}_i|^2 + 2\,\mathrm{Re}\Big[\sum_{i=2}^{10} \mathcal{M}_1\mathcal{M}_i^* + \sum_{i=3}^{10} \mathcal{M}_2\mathcal{M}_i^* + \sum_{i=4}^{10} \mathcal{M}_3\mathcal{M}_i^* + \sum_{i=5}^{10} \mathcal{M}_4\mathcal{M}_i^* + \sum_{i=6}^{10} \mathcal{M}_5\mathcal{M}_i^* + \sum_{i=7}^{10} \mathcal{M}_6\mathcal{M}_i^* + \sum_{i=8}^{10} \mathcal{M}_7\mathcal{M}_i^* + \sum_{i=9}^{10} \mathcal{M}_8\mathcal{M}_i^* + \mathcal{M}_9\mathcal{M}_{10}^*\Big], \tag{4.41}$$

these are a total of 55 terms (10 squared and 45 crossed terms); fortunately only 8 crossed terms are non-zero due to the trace and projectors properties.
The *FeynCalc* computation give us the following polarized squared amplitude

$$\begin{aligned}
|\mathcal{M}|^2 = 16G_{ll'}^2 &(|g_{LL}^S|^2 + |g_{LR}^S|^2 - 4|g_{LR}^T|^2 + |g_{RL}^S|^2 - 4|g_{RL}^T|^2 + |g_{RR}^S|^2)(p_2\cdot p_4)(p_1\cdot p_3)\\
&+ m_1(-|g_{LL}^S|^2 + |g_{LR}^S|^2 - 4|g_{LR}^T|^2 - |g_{RL}^S|^2 + 4|g_{RL}^T|^2 + |g_{RR}^S|^2)(p_2\cdot p_4)(p_3\cdot s)\\
&+ (4|g_{LL}^V|^2 + 8|g_{LR}^T|^2 + 8|g_{RL}^T|^2 + 4|g_{RR}^V|^2)(p_3\cdot p_4)(p_1\cdot p_2)\\
&+ m_1(-4|g_{LL}^V|^2 + 8|g_{LR}^T|^2 - 8|g_{RL}^T|^2 + 4|g_{RR}^V|^2)(p_3\cdot p_4)(p_2\cdot s)\\
&+ (8|g_{LR}^T|^2 + 4|g_{LR}^V|^2 + 8|g_{RL}^T|^2 + 4|g_{RL}^V|^2)(p_1\cdot p_4)(p_2\cdot p_3)\\
&+ m_1(8|g_{LR}^T|^2 + 4|g_{LR}^V|^2 - 8|g_{RL}^T|^2 - 4|g_{RL}^V|^2)(p_2\cdot p_3)(p_4\cdot s)\\
&+ 2\,\mathrm{Re}\Big[m_4 m_1(p_2\cdot p_3)(g_{LL}^S g_{RR}^{V*} + g_{LL}^V g_{RR}^{S*} + g_{LR}^S g_{RL}^{V*} + 6g_{LR}^T g_{RL}^{V*} + g_{LR}^V g_{RL}^{S*} + 6g_{LR}^V g_{RL}^{T*})\\
&+ (im_4\epsilon^{p_1p_2p_3s})(g_{LL}^S g_{RR}^{V*} - g_{LL}^V g_{RR}^{S*} + g_{LR}^S g_{RL}^{V*} - 2g_{LR}^T g_{RL}^{V*} - g_{LR}^V g_{RL}^{S*} + 2g_{LR}^V g_{RL}^{T*})\\
&+ m_4(p_1\cdot p_3)(p_2\cdot s)(g_{LL}^S g_{RR}^{V*} - g_{LL}^V g_{RR}^{S*} - g_{LR}^S g_{RL}^{V*} + 2g_{LR}^T g_{RL}^{V*} + g_{LR}^V g_{RL}^{S*} - 2g_{LR}^V g_{RL}^{T*})\\
&+ m_4(p_1\cdot p_2)(p_3\cdot s)(-g_{LL}^S g_{RR}^{V*} + g_{LL}^V g_{RR}^{S*} + g_{LR}^S g_{RL}^{V*} - 2g_{LR}^T g_{RL}^{V*} - g_{LR}^V g_{RL}^{S*} + 2g_{LR}^V g_{RL}^{T*})\\
&+ 2g_{LR}^S g_{LR}^{T*}(-m_1(p_2\cdot s)(p_3\cdot p_4) + m_1(p_2\cdot p_3)(p_4\cdot s) + im_1\epsilon^{p_2p_3p_4s} - i\epsilon^{p_1p_2p_3p_4}\\
&+ (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_2)(p_3\cdot p_4)) + 2g_{RL}^S g_{RL}^{T*}(m_1(p_2\cdot s)(p_3\cdot p_4) - m_1(p_2\cdot p_3)(p_4\cdot s)\\
&+ im_1\epsilon^{p_2p_3p_4s} + i\epsilon^{p_1p_2p_3p_4} + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_2)(p_3\cdot p_4))\Big].
\end{aligned} \tag{4.42}$$

A quick way, at least for the terms with the g_{LL}^V coefficient, to verify the result is by making $|g_{LL}^V| = 1$ and all the other couplings zero, thus the amplitude (4.42) reduces to the SM prediction obtained in (4.29).
The neutrinos phase space integral does not change and therefore has the same value as (4.13). Thus, following the same steps of the first section using the amplitude (4.42) we

obtain from *FeynCalc*, in the decaying-lepton rest frame, the differential decay rate:

$$\begin{aligned}
\frac{d\Gamma_{l\to l'}}{dxd\Omega} = \frac{m_1 w^4 G_{ll'}^2}{24\pi^4}\sqrt{x^2-x_0^2}\Big\{&(4x^2-3x-x_0^2)\Big[\frac{1}{4}(|g_{LL}^S|^2+|g_{LR}^S|^2+|g_{RL}^S|^2+|g_{RR}^S|^2)\\
&+|g_{LL}^V|^2+|g_{RR}^V|^2+|g_{LR}^T|^2+|g_{RL}^T|^2\Big]+6x(1-x)\Big[\frac{1}{4}(|g_{LL}^S|^2+|g_{LR}^S|^2+|g_{RL}^S|^2\\
&+|g_{RR}^S|^2)+|g_{LL}^V|^2+|g_{RR}^V|^2+|g_{LR}^V|^2+|g_{RL}^V|^2+3|g_{LR}^T|^2+3|g_{RL}^T|^2\Big]\\
&+\mathrm{Re}\Big[3x_0(1-x)\big(g_{LL}^V g_{RR}^{S*}+g_{RR}^V g_{LL}^{S*}+g_{LR}^V(g_{RL}^{S*}+6g_{RL}^{T*})+g_{RL}^V(g_{LR}^{S*}+6g_{LR}^{T*})\big)\\
&-(4x^2-3x-x_0^2)(g_{LR}^S g_{LR}^{T*}+g_{RL}^S g_{RL}^{T*})\Big]\\
&+\mathcal{P}\sqrt{x^2-x_0^2}\cos\theta\Big[\frac{1}{4}(1-x)(24|g_{RL}^V|^2-48|g_{LR}^T|^2-24|g_{LR}^V|^2+48|g_{RL}^T|^2)\\
&+\frac{1}{4}\big(2-2x-\sqrt{1-x_0^2}\big)(|g_{LL}^S|^2-|g_{RR}^S|^2+|g_{RL}^S|^2-|g_{LR}^S|^2+4(|g_{LL}^V|^2-|g_{RR}^V|^2\\
&+|g_{RL}^T|^2-|g_{LR}^T|^2))+\mathrm{Re}\Big[(4x-4-\sqrt{1-x_0^2})(g_{LR}^S g_{LR}^{T*}-g_{RL}^S g_{RL}^{T*})\Big]\Big]\Big\}
\end{aligned} \tag{4.43}$$

where $\mathcal{P}$ is the initial lepton net polarization, θ is the angle between the l^- spin and the final charged-lepton momentum, $w \equiv (m_1^2+m_4^2)/2m_1$, $x \equiv E_4/\omega$ is the reduced energy and $x_0 \equiv m_4/\omega$.

The differential decay rate (4.43) is usually written in terms of the so called *Michel parameters* and the isotropic and anisotropic functions $F(x)$, $A(x)$ [43, 44].

In terms of the $g_{\epsilon\omega}^n$ couplings, the *Michel parameters* are

$$\begin{aligned}
\rho &= \frac{3}{4}(\beta^+ + \beta^-) + (\gamma^+ + \gamma^-),\\
\xi &= 3(\alpha^- - \alpha^+) + (\beta^- - \beta^+) + \frac{7}{3}(\gamma^+ - \gamma^-),\\
\xi\delta &= \frac{3}{4}(\beta^- - \beta^+) + (\gamma^+ - \gamma^-),\\
\eta &= \frac{1}{2}\mathrm{Re}[g_{LL}^V g_{RR}^{S*} + g_{RR}^V g_{LL}^{S*} + g_{LR}^V(g_{RL}^{S*}+6g_{RL}^{T*}) + g_{RL}^V(g_{LR}^{S*}+6g_{LR}^{T*})],
\end{aligned} \tag{4.44}$$

where

$$\begin{aligned}
\alpha^+ &\equiv |g_{RL}^V|^2 + \frac{1}{16}|g_{RL}^S + 6g_{RL}^T|^2, & \alpha^- &\equiv |g_{LR}^V|^2 + \frac{1}{16}|g_{LR}^S + 6g_{LR}^T|^2,\\
\beta^+ &\equiv |g_{RR}^V|^2 + \frac{1}{4}|g_{RR}^S|^2, & \beta^- &\equiv |g_{LL}^V|^2 + \frac{1}{4}|g_{LL}^S|^2,\\
\gamma^+ &\equiv \frac{3}{16}|g_{RL}^S - 2g_{RL}^T|^2, & \gamma^- &\equiv \frac{3}{16}|g_{LR}^S - 2g_{LR}^T|^2,
\end{aligned} \tag{4.45}$$

are positive-definite combinations of the decay constants, corresponding to a final right-handed $(\alpha^+, \beta^+, \gamma^+)$ or left-handed $(\alpha^-, \beta^-, \gamma^-)$ lepton.

The functions $F(x)$ and $A(x)$ in terms of the *Michel parameters* are

$$\begin{aligned} F(x) &= x(1-x) + \frac{2}{9}\rho\left(4x^2 - 3x - x_0^2\right) + \eta x_0(1-x), \\ A(x) &= 1 - x + \frac{2}{3}\delta\left(4x - 4 + \sqrt{1-x_0^2}\right). \end{aligned} \tag{4.46}$$

Finally, it is useful to write explicitly the following *Michel parameters* as

$$\begin{aligned} \frac{4}{3}\rho = &\frac{1}{4}(|g^S_{LL}|^2 + |g^S_{LR}|^2 + |g^S_{RL}|^2 + |g^S_{RR}|^2) + |g^V_{LL}|^2 + |g^V_{RR}|^2 + |g^T_{LR}|^2 + |g^T_{RL}|^2 \\ &- \mathrm{Re}\left[g^S_{LR} g^{T*}_{LR} + g^S_{RL} g^{T*}_{RL}\right], \end{aligned} \tag{4.47}$$

$$\begin{aligned} 4\xi = &|g^S_{LL}|^2 - |g^S_{LR}|^2 + |g^S_{RL}|^2 - |g^S_{RR}|^2 + 4(|g^V_{LL}|^2 - |g^V_{RR}|^2) + 12(|g^V_{LR}|^2 - |g^V_{RL}|^2) \\ &+ 20(|g^T_{LR}|^2 - |g^T_{RL}|^2) + 16\,\mathrm{Re}\left[g^S_{LR} g^{T*}_{LR} - g^S_{RL} g^{T*}_{RL}\right], \end{aligned} \tag{4.48}$$

$$\begin{aligned} 2\xi - \frac{16}{3}\xi\delta = &-\frac{1}{2}(|g^S_{LL}|^2 - |g^S_{LR}|^2 + |g^S_{RL}|^2 - |g^S_{RR}|^2) - 2(|g^V_{LL}|^2 - |g^V_{RR}|^2) \\ &+ 6(|g^V_{LR}|^2 - |g^V_{RL}|^2) + 14(|g^T_{LR}|^2 - |g^T_{RL}|^2) + 4\,\mathrm{Re}\left[g^S_{LR} g^{T*}_{LR} - g^S_{RL} g^{T*}_{RL}\right], \end{aligned} \tag{4.49}$$

with this parametrization and the normalization condition it is straightforward from (4.43) to obtain the reduced expression of the final charged-lepton distribution

$$\frac{d\Gamma_{l\to l'}}{dx d\cos\theta} = \frac{m_1 w^4}{2\pi^3} G^2_{ll'} \sqrt{x^2 - x_0^2}\left\{F(x) - \frac{\xi}{3}\mathcal{P}\sqrt{x^2 - x_0^2}\cos\theta A(x)\right\} \tag{4.50}$$

Thus, for an unpolarized lepton ($\mathcal{P} = 0$), the distribution is characterized by the parameters ρ and η. Two more parameters, ξ and δ, can be determined when the initial lepton polarization is known. If the polarization of the final charged lepton is also measured, 5 additional independent parameters $(\xi', \xi'', \eta'', \alpha', \beta')$ appear [1]. In the SM, $\rho = \delta = 3/4$, $\eta = \eta'' = \alpha' = \beta' = 0$ and $\xi = \xi' = \xi'' = 1$.

We can obtain the total decay rate integrating (4.50) over all energy and angular configurations. As discussed in the last section, the $\cos\theta$ term gives no contribution, so we only care about the isotropic part.

The integral limits in terms of the new variables are:

$$m_4 \le E_4 \le \frac{m_1^2 + m_4^2}{2m_1} \longrightarrow \frac{m_4}{\omega} \le x \le \frac{m_1^2 + m_4^2}{2m_1\omega} \longrightarrow x_0 \le x \le 1. \tag{4.51}$$

Thus

$$\Gamma_{l\to l'} = \frac{m_1 w^4}{\pi^3} G_{ll'}^2 \int_{x_0}^{1} \sqrt{x^2 - x_0^2} F(x) dx$$
$$= \frac{m_1 w^4}{\pi^3} G_{ll'}^2 \int_{x_0}^{1} \sqrt{x^2 - x_0^2} \left[x(1-x) + \frac{2}{9}\rho \left(4x^2 - 3x - x_0^2\right) + \eta x_0 (1-x) \right] dx, \tag{4.52}$$

leading to

$$\Gamma_{l\to l'} = \frac{G_{ll'}^2}{192\pi^3 m_1^3} \Big\{ m_1^8 - m_4^8 - 8m_4^2 m_1^6 + 8m_4^6 m_1^2 - 24 m_4^4 m_1^4 log\left(\frac{m_4}{m_1}\right)$$
$$+ \eta \Big[4m_4 m_1^7 - 4m_4^7 m_1 - 36 m_4^5 m_1^3 + 36 m_4^3 m_1^5 + 48(m_4^5 m_1^3 + m_4^3 m_1^5) \tag{4.53}$$
$$log\left(\frac{m_4}{m_1}\right) \Big] \Big\},$$

which can be rewritten as follows

$$\Gamma_{l\to l'} = \frac{\hat{G}_{ll'}^2 m_1^5}{192\pi^3} f(m_4^2/m_1^2) \left(1 + \delta_{RC}^{ll'}\right), \tag{4.54}$$

where

$$\hat{G}_{ll'} \equiv G_{ll'} \sqrt{1 + 4\eta \frac{m_4}{m_1} \frac{g(m_4^2/m_1^2)}{f(m_4^2/m_1^2)}} \tag{4.55}$$

$g(x) = 1 + 9x - 9x^2 - x^3 + 6x(1+x)log(x)$, and the SM radiative correction $\delta_{RC}^{ll'}$ has been included.

The normalization $G_{e\mu}$ corresponds to the Fermi coupling G_F, measured in the μ decay. Unlike all other *Michel parameters*, the dependence of (4.54) on η is left but it is suppressed by a factor of (m_4/m_1).

Thus, it may be a negligible contribution in the case of $\mu^- \to e^- \nu_\mu \bar{\nu}_e$ or $\tau^- \to e^- \nu_\tau \bar{\nu}_e$ but not for $\tau^- \to \mu^- \nu_\tau \bar{\nu}_\mu$, where it can change the partial width decay by up to 18% [45]. It is this difference that allows one to determine η by comparing the branching ratios of the two τ leptonic decays.

If e/μ universality is assumed[†], the leptonic decay ratio B_μ/B_e implies:

$$\eta = 0.016 \pm 0.013. \tag{4.56}$$

[†]Assuming lepton universality leads to a more restrictive value for η. We show the value of η without assuming lepton universality in table 4.1

A non-zero value of η would show that there are at least two different couplings with opposite chiralities for the charged leptons. Assuming the V-A coupling g^V_{LL} to be dominant, then the term $\frac{1}{2}\,\mathrm{Re}[g^V_{LL} g^{S*}_{RR}]$ is the only one linear in non-standard couplings in the whole spectrum, thus the measurement of η is of particular interest for the determination of new couplings, specially the scalar coupling g^S_{RR}, that usually appears in many extensions of the standard model.
The experimental status on the τ-decay Michel parameters [46, 47, 48, 49, 50, 51, 52] together with the more accurate values measured in μ decay [53, 54, 55, 56] are shown in table 4.1.

Table 4.1: Michel parameters

	$\mu^- \to e^- \nu_\mu \bar{\nu}_e$	$\tau^- \to e^- \nu_\tau \bar{\nu}_e$	$\tau^- \to \mu^- \nu_\tau \bar{\nu}_\mu$
ρ	0.74979 ± 0.00026	0.747 ± 0.010	0.763 ± 0.020
η	0.057 ± 0.034	—	0.094 ± 0.073
ξ	$1.0009^{+0.0016}_{-0.0007}$	0.994 ± 0.040	1.030 ± 0.059
$\xi\delta$	$0.7511^{+0.0012}_{-0.0006}$	0.734 ± 0.028	0.778 ± 0.037
ξ'	1.00 ± 0.04	—	—
ξ''	0.65 ± 0.36	—	—

The polarization of the charged lepton emitted in the τ decay, related to ξ' and ξ'', has never been measured. In principle, this could be done using the radiative decays , since the distribution of the photons emitted by the daughter lepton is sensitive to the lepton polarization [57].

In order to established bounds on the couplings, is convenient to introduce the probabilities [39]

$$Q_{\epsilon\omega} = \frac{1}{4}|g^S_{\epsilon\omega}|^2 + |g^V_{\epsilon\omega}|^2 + 3(1-\delta_{\epsilon\omega})|g^T_{\epsilon\omega}|^2 \tag{4.57}$$

for the decay of an ω-handed l^- into an ϵ-handed daughter lepton.

The probabilities $Q_{\epsilon\omega}$ can be extracted from the measurable shape parameters, as follows:

$$\begin{aligned} Q_{LL} = \beta^- &= \frac{1}{4}\left(-3 + \frac{16}{3}\rho - \frac{1}{3}\xi + \frac{16}{9}\xi\delta + \xi' + \xi''\right) \\ Q_{RR} = \beta^+ &= \frac{1}{4}\left(-3 + \frac{16}{3}\rho + \frac{1}{3}\xi - \frac{16}{9}\xi\delta - \xi' + \xi''\right) \\ Q_{LR} = \alpha^- + \gamma^- &= \frac{1}{4}\left(5 - \frac{16}{3}\rho + \frac{1}{3}\xi - \frac{16}{9}\xi\delta + \xi' - \xi''\right) \\ Q_{RL} = \alpha^+ + \gamma^+ &= \frac{1}{4}\left(5 - \frac{16}{3}\rho - \frac{1}{3}\xi + \frac{16}{9}\xi\delta - \xi' - \xi''\right) \end{aligned} \tag{4.58}$$

Upper bounds on any of these probabilities translate into corresponding limits for all couplings with the given chiralities.

The present 90% CL bounds on the μ-decay couplings and 95% CL bounds on the τ-decay couplings are given in table 4.2 and 4.3 respectively [1].

Table 4.2: 90% CL experimental bounds for the $\mu^- \to e^-\nu_\mu\bar{\nu}_e$ couplings

$\mu^- \to e^-\nu_\mu\bar{\nu}_e$			
$\lvert g^S_{RR}\rvert < 0.035$	$\lvert g^S_{LR}\rvert < 0.050$	$\lvert g^S_{RL}\rvert < 0.412$	$\lvert g^S_{LL}\rvert < 0.550$
$\lvert g^V_{RR}\rvert < 0.017$	$\lvert g^V_{LR}\rvert < 0.023$	$\lvert g^V_{RL}\rvert < 0.104$	$\lvert g^V_{LL}\rvert > 0.960$
$\lvert g^T_{RR}\rvert \equiv 0$	$\lvert g^T_{LR}\rvert < 0.015$	$\lvert g^T_{RL}\rvert < 0.103$	$\lvert g^T_{LL}\rvert \equiv 0$
$\lvert g^S_{LR} + 6g^T_{LR}\rvert < 0.143$	$\lvert g^S_{LR} + 2g^T_{LR}\rvert < 0.108$	$\lvert g^S_{LR} - 2g^T_{LR}\rvert < 0.070$	
$\lvert g^S_{RL} + 6g^T_{RL}\rvert < 0.418$	$\lvert g^S_{RL} + 2g^T_{RL}\rvert < 0.417$	$\lvert g^S_{RL} - 2g^T_{RL}\rvert < 0.418$	

Table 4.3: 95% CL experimental bounds for the leptonic τ-decay couplings

$\tau^- \to e^-\nu_\tau\bar{\nu}_e$			
$\lvert g^S_{RR}\rvert < 0.70$	$\lvert g^S_{LR}\rvert < 0.99$	$\lvert g^S_{RL}\rvert \leq 2$	$\lvert g^S_{LL}\rvert \leq 2$
$\lvert g^V_{RR}\rvert < 0.17$	$\lvert g^V_{LR}\rvert < 0.13$	$\lvert g^V_{RL}\rvert < 0.52$	$\lvert g^V_{LL}\rvert \leq 1$
$\lvert g^T_{RR}\rvert \equiv 0$	$\lvert g^T_{LR}\rvert < 0.082$	$\lvert g^T_{RL}\rvert < 0.51$	$\lvert g^T_{LL}\rvert \equiv 0$
$\tau^- \to \mu^-\nu_\tau\bar{\nu}_\mu$			
$\lvert g^S_{RR}\rvert < 0.72$	$\lvert g^S_{LR}\rvert < 0.95$	$\lvert g^S_{RL}\rvert \leq 2$	$\lvert g^S_{LL}\rvert \leq 2$
$\lvert g^V_{RR}\rvert < 0.18$	$\lvert g^V_{LR}\rvert < 0.12$	$\lvert g^V_{RL}\rvert < 0.52$	$\lvert g^V_{LL}\rvert \leq 1$
$\lvert g^T_{RR}\rvert \equiv 0$	$\lvert g^T_{LR}\rvert < 0.079$	$\lvert g^T_{RL}\rvert < 0.51$	$\lvert g^T_{LL}\rvert \equiv 0$

Note that the most accurate experimental values agree, within the uncertainty, with $|g^V_{LL}| = 1$ and all others couplings being zero. Thus if some couplings of the same order in the EFT expansion are much more important than others, there has to be an underlying symmetry that explains it. This non-trivial symmetry is precisely a remnant of the high–energy dynamics as we discussed in the last section, showing that the SM provides the dominant contribution to the decay rate, being $|g^V_{LL}| \approx 1$ due to the V-A structure of the electroweak theory.

Finally, processes such as radiative muons and taus decays as well as five-body leptonic decays of muons and tau leptons can be also used to obtain independent constraints and get an additional information about the structure of weak interactions, see e.g.[58, 59, 60].

Chapter 5

Lepton Decays with Majorana Neutrinos

The effective analysis of lepton decay allows us to investigate the Lorentz structure of the decay amplitudes through the analysis of the energy and angular distribution of the final charged lepton, as we just discussed. Besides that, another possibility in these scenarios is to test the Dirac or Majorana nature of neutrinos.

In this chapter, we analyse the decay amplitude coming from the most general four-lepton interaction Hamiltonian, including the effects due to the Majorana nature of neutrinos, that could be measured in the most recent and future experiments.

All these studies are based on scenarios where the new sterile Majorana neutrinos have non-negligible mixings and some of them require masses low enough to be produced on-shell.

All the basic information about Majorana fermions, including its Feynman rules, together with neutrinos masses and mixings models are given in appendices A and B.

5.1 The Effective Amplitude for Majorana Neutrinos

In this framework, the charged weak current interaction is written in the basis of mass eigenstates of the charged leptons l and the neutrino N_j after diagonalizing the charged lepton and neutrino mass matrices.

Working in the basis where the charged leptons are already diagonalized, the current neutrino ($\nu_{L,R}$) is assumed to be the superposition of the mass-eigenstate neutrinos (N_j) with the mass m_j, that is,

$$\nu_{lL} = \sum_j U_{lj} N_{jL}, \quad \nu_{lR} = \sum_j V_{lj} N_{jR}, \tag{5.1}$$

where $j = \{1, 2, ..., n\}$ with n the number of mass-eigenstate neutrinos.

As shown by Langacker and London [61], explicit lepton-number nonconservation still leads to a matrix element equivalent to (4.32). Thus, in the mass basis, the effective

Hamiltonian (4.34) for the $l^- \longrightarrow l'^- \overline{N}_j N_k$ process is written as:

$$\begin{aligned}\mathcal{H} = 4\frac{G_{ll'}}{\sqrt{2}}\sum_{j,k}\Big\{ & g^S_{LL}\left[\bar{l}'_L V_{l'j} N_{jR}\right]\left[\overline{N}_{kR} V^*_{lk} l_L\right] + g^V_{LL}\left[\bar{l}'_L \gamma^\mu U_{l'j} N_{jL}\right]\left[\overline{N}_{kL} U^*_{lk}\gamma_\mu l_L\right] \\ & + g^S_{RR}\left[\bar{l}'_R U_{l'j} N_{jL}\right]\left[\overline{N}_{kL} U^*_{lk} l_R\right] + g^V_{RR}\left[\bar{l}'_R \gamma^\mu V_{l'j} N_{jR}\right]\left[\overline{N}_{kR} V^*_{lk}\gamma_\mu l_R\right] \\ & + g^S_{LR}\left[\bar{l}'_L V_{l'j} N_{jR}\right]\left[\overline{N}_{kL} U^*_{lk} l_R\right] + g^V_{LR}\left[\bar{l}'_L \gamma^\mu U_{l'j} N_{jL}\right]\left[\overline{N}_{kR} V^*_{lk}\gamma_\mu l_R\right] \\ & + g^T_{LR}\left[\bar{l}'_L \frac{\sigma^{\mu\nu}}{\sqrt{2}} V_{l'j} N_{jR}\right]\left[\overline{N}_{kL} U^*_{lk}\frac{\sigma_{\mu\nu}}{\sqrt{2}} l_R\right] + g^S_{RL}\left[\bar{l}'_R U_{l'j} N_{jL}\right]\left[\overline{N}_{kR} V^*_{lk} l_L\right] \\ & + g^V_{RL}\left[\bar{l}'_R \gamma^\mu V_{l'j} N_{jR}\right]\left[\overline{N}_{kL} U^*_{lk}\gamma_\mu l_L\right] + g^T_{RL}\left[\bar{l}'_R \frac{\sigma^{\mu\nu}}{\sqrt{2}} U_{l'j} N_{jL}\right]\left[\overline{N}_{kR} V^*_{lk}\frac{\sigma_{\mu\nu}}{\sqrt{2}} l_L\right]\Big\}.\end{aligned} \tag{5.2}$$

Note that $\overline{N}$ represents an antineutrino for the Dirac neutrino case, but should be identified with N for the Majorana neutrino case ($N{=}N^c{=}C\overline{N}^T$), as discussed in appendix B. Then, the Hamiltonian (5.2) for the case of Majorana neutrinos is

$$\begin{aligned}\mathcal{H} = 4\frac{G_{ll'}}{\sqrt{2}}\sum_{j,k}\Big\{ & g^S_{LL} V_{l'j} V^*_{lk}\left[\bar{l}'_L N_{jR}\right]\left[N_{kR} l_L\right] + g^V_{LL} U_{l'j} U^*_{lk}\left[\bar{l}'_L \gamma^\mu N_{jL}\right]\left[N_{kL}\gamma_\mu l_L\right] \\ & + g^S_{RR} U_{l'j} U^*_{lk}\left[\bar{l}'_R N_{jL}\right]\left[N_{kL} l_R\right] + g^V_{RR} V_{l'j} V^*_{lk}\left[\bar{l}'_R \gamma^\mu N_{jR}\right]\left[N_{kR}\gamma_\mu l_R\right] \\ & + g^S_{LR} V_{l'j} U^*_{lk}\left[\bar{l}'_L N_{jR}\right]\left[N_{kL} l_R\right] + g^V_{LR} U_{l'j} V^*_{lk}\left[\bar{l}'_L \gamma^\mu N_{jL}\right]\left[N_{kR}\gamma_\mu l_R\right] \\ & + g^T_{LR} V_{l'j} U^*_{lk}\left[\bar{l}'_L \frac{\sigma^{\mu\nu}}{\sqrt{2}} N_{jR}\right]\left[N_{kL}\frac{\sigma_{\mu\nu}}{\sqrt{2}} l_R\right] + g^S_{RL} U_{l'j} V^*_{lk}\left[\bar{l}'_R N_{jL}\right]\left[N_{kR} l_L\right] \\ & + g^V_{RL} V_{l'j} U^*_{lk}\left[\bar{l}'_R \gamma^\mu N_{jR}\right]\left[N_{kL}\gamma_\mu l_L\right] + g^T_{RL} U_{l'j} V^*_{lk}\left[\bar{l}'_R \frac{\sigma^{\mu\nu}}{\sqrt{2}} N_{jL}\right]\left[N_{kR}\frac{\sigma_{\mu\nu}}{\sqrt{2}} l_L\right]\Big\}\end{aligned} \tag{5.3}$$

and in terms of the chirality operators

$$\begin{aligned}\mathcal{H} = \frac{G_{ll'}}{\sqrt{2}}\sum_{j,k}\Big\{ & g^S_{LL} V_{l'j} V^*_{lk}\left[\bar{l}'(1+\gamma^5)N_j\right]\left[N_k(1-\gamma^5)l\right] + g^V_{LL} U_{l'j} U^*_{lk}\left[\bar{l}'\gamma^\mu(1-\gamma^5)N_j\right] \\ & \left[N_k\gamma_\mu(1-\gamma^5)l\right] + g^S_{RR} U_{l'j} U^*_{lk}\left[\bar{l}'(1-\gamma^5)N_j\right]\left[N_k(1+\gamma^5)l\right] + g^V_{RR} V_{l'j} V^*_{lk}\left[\bar{l}'\gamma^\mu(1+\gamma^5)N_j\right] \\ & \left[N_k\gamma_\mu(1+\gamma^5)l\right] + g^S_{LR} V_{l'j} U^*_{lk}\left[\bar{l}'(1+\gamma^5)N_j\right]\left[N_k(1+\gamma^5)l\right] + g^V_{LR} U_{l'j} V^*_{lk}\left[\bar{l}'\gamma^\mu(1-\gamma^5)N_j\right] \\ & \left[N_k\gamma_\mu(1+\gamma^5)l\right] + g^T_{LR} V_{l'j} U^*_{lk}\left[\bar{l}'\frac{\sigma^{\mu\nu}}{\sqrt{2}}(1+\gamma^5)N_j\right]\left[N_k\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1+\gamma^5)l\right] + g^S_{RL} U_{l'j} V^*_{lk}\left[\bar{l}'(1-\gamma^5)N_j\right] \\ & \left[N_k(1-\gamma^5)l\right] + g^V_{RL} V_{l'j} U^*_{lk}\left[\bar{l}'\gamma^\mu(1+\gamma^5)N_j\right]\left[N_k\gamma_\mu(1-\gamma^5)l\right] + g^T_{RL} U_{l'j} V^*_{lk}\left[\bar{l}'\frac{\sigma^{\mu\nu}}{\sqrt{2}}(1-\gamma^5)N_j\right] \\ & \left[N_k\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1-\gamma^5)l\right]\Big\}.\end{aligned} \tag{5.4}$$

Also, if we take into account the decaying lepton polarization, as discussed in the last chapter, we need to make the change $l \longrightarrow \frac{1}{2}(1+\gamma^5 \not{s})l$.

We now have to calculate the amplitude for the process. Unlike the Dirac case, the indistinguishable properties of the Majorana neutrinos have strong consequences in the amplitude. This time, the possible first order Feynman diagrams for the $l^- \longrightarrow l'^- N_j N_k$ decay are:

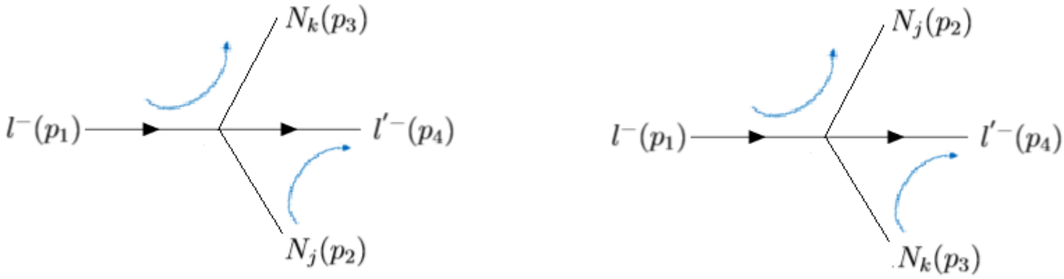

Figure 5.1: Feynman diagrams for the $l^- \longrightarrow l'^- N_j N_k$ decay

Where the second diagram is only possible in the Majorana neutrino case and we already defined the orientation for each fermion chain (blue arrows).

Thus, applying the Feynman rules for Majorana fermions seen in appendix B together with the Hamiltonian (5.4) and the spin projector leads the following amplitude:

$$
\begin{aligned}
\mathcal{M}_{jk} = &- i\frac{G_{ll'}}{2\sqrt{2}}\Big\{ g^S_{LL} V_{l'j} V^*_{lk} \left[\bar{u}_4(1+\gamma^5)v_2\right]\left[\bar{u}_3(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{LL} U_{l'j} U^*_{lk}\left[\bar{u}_4\gamma^\mu(1-\gamma^5)v_2\right] \\
&\left[\bar{u}_3\gamma_\mu(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{RR} U_{l'j} U^*_{lk}\left[\bar{u}_4(1-\gamma^5)v_2\right]\left[\bar{u}_3(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] \\
&+ g^V_{RR} V_{l'j} V^*_{lk}\left[\bar{u}_4\gamma^\mu(1+\gamma^5)v_2\right]\left[\bar{u}_3\gamma_\mu(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{LR} V_{l'j} U^*_{lk}\left[\bar{u}_4(1+\gamma^5)v_2\right] \\
&\left[\bar{u}_3(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{LR} U_{l'j} V^*_{lk}\left[\bar{u}_4\gamma^\mu(1-\gamma^5)v_2\right]\left[\bar{u}_3\gamma_\mu(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] \\
&+ g^T_{LR} V_{l'j} U^*_{lk}\left[\bar{u}_4\frac{\sigma^{\mu\nu}}{\sqrt{2}}(1+\gamma^5)v_2\right]\left[\bar{u}_3\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{RL} U_{l'j} V^*_{lk}\left[\bar{u}_4(1-\gamma^5)v_2\right] \\
&\left[\bar{u}_3(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{RL} V_{l'j} U^*_{lk}\left[\bar{u}_4\gamma^\mu(1+\gamma^5)v_2\right]\left[\bar{u}_3\gamma_\mu(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] \\
&+ g^T_{RL} U_{l'j} V^*_{lk}\left[\bar{u}_4\frac{\sigma^{\mu\nu}}{\sqrt{2}}(1-\gamma^5)v_2\right]\left[\bar{u}_3\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1-\gamma^5)(1+\gamma^5\not{s})u_1\right]\Big\} \\
&+ i\frac{G_{ll'}}{2\sqrt{2}}\Big\{ g^S_{LL} V_{l'k} V^*_{lj}\left[\bar{u}_4(1+\gamma^5)v_3\right]\left[\bar{u}_2(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{LL} U_{l'k} U^*_{lj}\left[\bar{u}_4\gamma^\mu(1-\gamma^5)v_3\right] \\
&\left[\bar{u}_2\gamma_\mu(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{RR} U_{l'k} U^*_{lj}\left[\bar{u}_4(1-\gamma^5)v_3\right]\left[\bar{u}_2(1+\gamma^5)(1+\gamma^5\not{s})u_1\right]
\end{aligned}
$$

$$+ g^V_{RR} V_{l'k} V^*_{lj} \left[\bar{u}_4 \gamma^\mu (1+\gamma^5) v_3\right] \left[\bar{u}_2 \gamma_\mu (1+\gamma^5)(1+\gamma^5 \not{s}) u_1\right] + g^S_{LR} V_{l'k} U^*_{lj} \left[\bar{u}_4 (1+\gamma^5) v_3\right]$$
$$\left[\bar{u}_2 (1+\gamma^5)(1+\gamma^5 \not{s}) u_1\right] + g^V_{LR} U_{l'k} V^*_{lj} \left[\bar{u}_4 \gamma^\mu (1-\gamma^5) v_3\right] \left[\bar{u}_2 \gamma_\mu (1+\gamma^5)(1+\gamma^5 \not{s}) u_1\right]$$
$$+ g^T_{LR} V_{l'k} U^*_{lj} \left[\bar{u}_4 \frac{\sigma^{\mu\nu}}{\sqrt{2}} (1+\gamma^5) v_3\right] \left[\bar{u}_2 \frac{\sigma_{\mu\nu}}{\sqrt{2}} (1+\gamma^5)(1+\gamma^5 \not{s}) u_1\right] + g^S_{RL} U_{l'k} V^*_{lj} \left[\bar{u}_4 (1-\gamma^5) v_3\right],$$
$$\left[\bar{u}_2 (1-\gamma^5)(1+\gamma^5 \not{s}) u_1\right] + g^V_{RL} V_{l'k} U^*_{lj} \left[\bar{u}_4 \gamma^\mu (1+\gamma^5) v_3\right] \left[\bar{u}_2 \gamma_\mu (1-\gamma^5)(1+\gamma^5 \not{s}) u_1\right]$$
$$+ g^T_{RL} U_{l'k} V^*_{lj} \left[\bar{u}_4 \frac{\sigma^{\mu\nu}}{\sqrt{2}} (1-\gamma^5) v_3\right] \left[\bar{u}_2 \frac{\sigma_{\mu\nu}}{\sqrt{2}} (1-\gamma^5)(1+\gamma^5 \not{s}) u_1\right] \Bigg\} \tag{5.5}$$

The amplitude (5.5) contains twenty pairs of lepton currents. The first ten pairs of lepton currents, between red brackets, come from the first Feynman diagram. The last contribution, between blue brackets, come from the second Feynman diagram and is only possible in the Majorana neutrino case.

Note also that the Majorana contribution to the amplitude is obtained from the first one (red brackets) by the exchange $p_2 \leftrightarrow p_3$ and $j \leftrightarrow k$.

A first inspection of (5.5) explicitly shows a difficulty for the amplitude squared calculation. Specifically the interference terms between the first and second (Majorana) contribution will lead to an expression of the form:

$$\mathcal{M}_{int} \approx [\bar{u}_4 \Gamma v_2][\bar{u}_3 \Gamma' u_1] \cdot [\bar{v}_3 \Gamma'' u_4][\bar{u}_1 \Gamma''' u_2]. \tag{5.6}$$

Thus, once we sum over polarizations, this interference term will have the contributions

$$\sum_{s2} v_2 u_2 \quad \text{and} \quad \sum_{s3} \bar{u}_3 \bar{v}_3, \tag{5.7}$$

where the spinors completeness relation cannot be directly applied, so the trace mechanism discussed in the last chapter needs to be modified.

If we want to recover this mechanism we should modify our lepton currents in order to have the specific form of spinors sum, so that we can apply the usual completeness relation.

One way to do this is by making Fierz transformations to the Majorana contribution currents and using the relations seen in appendix B to transform the u and v spinors in such a way that we can use the completeness relation in all the spinors polarization sum. Another way, much easier than the last one, is to define another orientation (blue arrows) for the fermion chains in the Feynman diagrams. This change won't affect the observables, as discussed in appendix B.

Defining the fermion chain orientation as follows

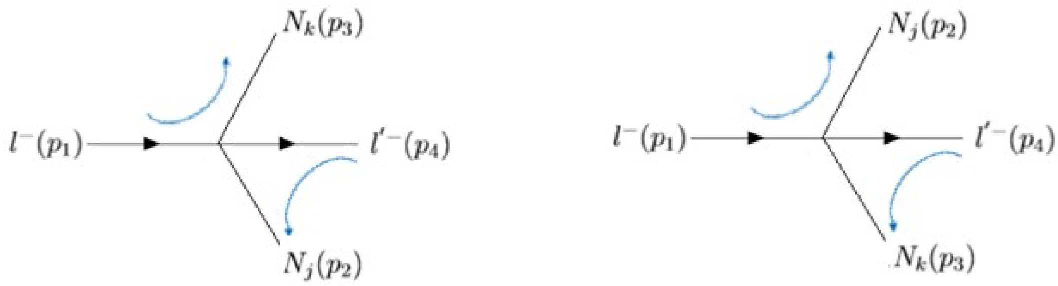

Figure 5.2: Feynman diagrams for the $l^- \longrightarrow l'^- N_j N_k$ decay

the amplitude, following the Feynman rules for Majorana fermions, is now:

$$
\begin{aligned}
\mathcal{M}_{jk} = & -i\frac{G_{ll'}}{2\sqrt{2}}\Big\{ g^S_{LL} V^*_{l'j} V^*_{lk} \left[\bar{u}_2(1+\gamma^5)v_4\right] \left[\bar{u}_3(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{LL} U^*_{l'j} U^*_{lk} \left[\bar{u}_2\gamma^\mu(-1-\gamma^5)v_4\right] \\
& \left[\bar{u}_3\gamma_\mu(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{RR} U^*_{l'j} U^*_{lk} \left[\bar{u}_2(1-\gamma^5)v_4\right] \left[\bar{u}_3(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] \\
& + g^V_{RR} V^*_{l'j} V^*_{lk} \left[\bar{u}_2\gamma^\mu(-1+\gamma^5)v_4\right] \left[\bar{u}_3\gamma_\mu(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{LR} V^*_{l'j} U^*_{lk} \left[\bar{u}_2(1+\gamma^5)v_4\right] \\
& \left[\bar{u}_3(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{LR} U^*_{l'j} V^*_{lk} \left[\bar{u}_2\gamma^\mu(-1-\gamma^5)v_4\right] \left[\bar{u}_3\gamma_\mu(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] \\
& + g^T_{LR} V^*_{l'j} U^*_{lk} \left[\bar{u}_2\frac{\sigma^{\mu\nu}}{\sqrt{2}}(-1-\gamma^5)v_4\right] \left[\bar{u}_3\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{RL} U^*_{l'j} V^*_{lk} \left[\bar{u}_2(1-\gamma^5)v_4\right] \\
& \left[\bar{u}_3(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{RL} V^*_{l'j} U^*_{lk} \left[\bar{u}_2\gamma^\mu(-1+\gamma^5)v_4\right] \left[\bar{u}_3\gamma_\mu(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] \\
& + g^T_{RL} U^*_{l'j} V^*_{lk} \left[\bar{u}_2\frac{\sigma^{\mu\nu}}{\sqrt{2}}(-1+\gamma^5)v_4\right] \left[\bar{u}_3\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1-\gamma^5)(1+\gamma^5\not{s})u_1\right]\Big\} \\
& + i\frac{G_{ll'}}{2\sqrt{2}}\Big\{ g^S_{LL} V^*_{l'k} V^*_{lj} \left[\bar{u}_3(1+\gamma^5)v_4\right] \left[\bar{u}_2(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{LL} U^*_{l'k} U^*_{lj} \left[\bar{u}_3\gamma^\mu(-1-\gamma^5)v_4\right] \\
& \left[\bar{u}_2\gamma_\mu(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{RR} U^*_{l'k} U^*_{lj} \left[\bar{u}_3(1-\gamma^5)v_4\right] \left[\bar{u}_2(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] \\
& + g^V_{RR} V^*_{l'k} V^*_{lj} \left[\bar{u}_3\gamma^\mu(-1+\gamma^5)v_4\right] \left[\bar{u}_2\gamma_\mu(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{LR} V^*_{l'k} U^*_{lj} \left[\bar{u}_3(1+\gamma^5)v_4\right] \\
& \left[\bar{u}_2(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{LR} U^*_{l'k} V^*_{lj} \left[\bar{u}_3\gamma^\mu(-1-\gamma^5)v_4\right] \left[\bar{u}_2\gamma_\mu(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] \\
& + g^T_{LR} V^*_{l'k} U^*_{lj} \left[\bar{u}_3\frac{\sigma^{\mu\nu}}{\sqrt{2}}(-1-\gamma^5)v_4\right] \left[\bar{u}_2\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1+\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^S_{RL} U^*_{l'k} V^*_{lj} \left[\bar{u}_3(1-\gamma^5)v_4\right] \\
& \left[\bar{u}_2(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] + g^V_{RL} V^*_{l'k} U^*_{lj} \left[\bar{u}_3\gamma^\mu(-1+\gamma^5)v_4\right] \left[\bar{u}_2\gamma_\mu(1-\gamma^5)(1+\gamma^5\not{s})u_1\right] \\
& + g^T_{RL} U^*_{l'k} V^*_{lj} \left[\bar{u}_3\frac{\sigma^{\mu\nu}}{\sqrt{2}}(-1+\gamma^5)v_4\right] \left[\bar{u}_2\frac{\sigma_{\mu\nu}}{\sqrt{2}}(1-\gamma^5)(1+\gamma^5\not{s})u_1\right]\Big\}.
\end{aligned}
\tag{5.8}
$$

The new amplitude (5.8) will lead to interference terms of the form:

$$\mathcal{M}_{int} \approx [\bar{u}_2 \Gamma v_4][\bar{u}_3 \Gamma^{'} u_1] \cdot [\bar{v}_4 \Gamma^{''} u_3][\bar{u}_1 \Gamma^{'''} u_2]. \tag{5.9}$$

Thus, once we sum over polarizations, the spinors completeness relation can be directly applied and all the trace mechanism can be used as usual.

From (5.8), there are now 20 pairs of leptonic currents, so in order to find the squared amplitude we need to calculate the following terms

$$\begin{aligned}\mathcal{M}_{jk} \equiv& \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 + \mathcal{M}_7 + \mathcal{M}_8 + \mathcal{M}_9 + \mathcal{M}_{10} + \mathcal{M}_{11} \\ &+ \mathcal{M}_{12} + \mathcal{M}_{13} + \mathcal{M}_{14} + \mathcal{M}_{15} + \mathcal{M}_{16} + \mathcal{M}_{17} + \mathcal{M}_{18} + \mathcal{M}_{19} + \mathcal{M}_{20}\end{aligned} \tag{5.10}$$

$$\begin{aligned}|\mathcal{M}_{jk}|^2 = \sum_{i=1}^{20} |\mathcal{M}_i|^2 + 2\,\mathrm{Re}\Big[&\sum_{i=2}^{20} \mathcal{M}_1 \mathcal{M}_i^* + \sum_{i=3}^{20} \mathcal{M}_2 \mathcal{M}_i^* + \sum_{i=4}^{20} \mathcal{M}_3 \mathcal{M}_i^* + \sum_{i=5}^{20} \mathcal{M}_4 \mathcal{M}_i^* \\ &+ \sum_{i=6}^{20} \mathcal{M}_5 \mathcal{M}_i^* + \sum_{i=7}^{20} \mathcal{M}_6 \mathcal{M}_i^* + \sum_{i=8}^{20} \mathcal{M}_7 \mathcal{M}_i^* + \sum_{i=9}^{20} \mathcal{M}_8 \mathcal{M}_i^* + \sum_{i=10}^{20} \mathcal{M}_9 \mathcal{M}_i^* \\ &+ \sum_{i=11}^{20} \mathcal{M}_{10} \mathcal{M}_i^* + \sum_{i=12}^{20} \mathcal{M}_{11} \mathcal{M}_i^* + \sum_{i=13}^{20} \mathcal{M}_{12} \mathcal{M}_i^* + \sum_{i=14}^{20} \mathcal{M}_{13} \mathcal{M}_i^* + \sum_{i=15}^{20} \mathcal{M}_{14} \mathcal{M}_i^* \\ &+ \sum_{i=16}^{20} \mathcal{M}_{15} \mathcal{M}_i^* + \sum_{i=17}^{20} \mathcal{M}_{16} \mathcal{M}_i^* + \sum_{i=18}^{20} \mathcal{M}_{17} \mathcal{M}_i^* + \sum_{i=19}^{20} \mathcal{M}_{18} \mathcal{M}_i^* + \mathcal{M}_{19} \mathcal{M}_{20}^* \Big],\end{aligned} \tag{5.11}$$

these are a total of 210 terms (20 squared and 190 crossed terms) that need to be computed. This time we are considering finite neutrino masses, so almost all the contributions will be different from zero, leading to an expression difficult to handle.

After some simplifications and considering all energetically allowed neutrino pairs, the *FeynCalc* computation give us the following polarized squared amplitude:

$$\begin{aligned}\sum_{j,k} |\mathcal{M}|^2_{jk} = 16 G^2_{ll'} \Big\{ &\Big[m_1 A^-(p_2 \cdot s) + A^+(p_1 \cdot p_2) \Big] (p_3 \cdot p_4) + \frac{1}{2} \Big[m_1 B^-(p_4 \cdot s) + B^+(p_1 \cdot p_4) \Big] (p_2 \cdot p_3) \\ &+ \mathrm{Re} \Big\{ - i m_j D_L^+ (\epsilon^{p_1 p_3 p_4 s} + i m_1 (p_3 \cdot p_4)) - m_j (p_1 \cdot p_3) \Big[D_L^- (p_4 \cdot s) + m_4 D_R^+ \Big] \\ &+ m_j (p_3 \cdot s) \Big[D_L^- (p_1 \cdot p_4) + m_1 m_4 D_R^- \Big] + \frac{m_j m_k}{2} \Big[- E^+ (p_1 \cdot p_4) + m_1 E^- (p_4 \cdot s) \Big] \\ &+ \frac{1}{2} (p_2 \cdot p_3) \Big[F^+ (p_1 \cdot p_4) + m_1 F^- (p_4 \cdot s) \Big] + G^+ (p_1 \cdot p_2)(p_3 \cdot p_4) \\ &+ m_1 G^- (p_2 \cdot s)(p_3 \cdot p_4) - m_1 m_2 m_3 m_4 H^+ + m_1 m_4 J^+ (p_2 \cdot p_3) \Big\} \Big\} + \binom{j \leftrightarrow k}{2 \leftrightarrow 3}\end{aligned}$$

$$+ 16G_{ll'}^2 \,\mathrm{Re}\Big\{i\Big[m_1 C^+ \epsilon^{p_2p_3p_4s} - C^- \epsilon^{p_1p_2p_3p_4}\Big] - iQ\Big[m_j \epsilon^{p_1p_3p_4s} + m_k \epsilon^{p_1p_2p_4s}\Big]\Big\}, \tag{5.12}$$

where we defined the parameters as:

$$A^\pm \equiv \sum_{j,k} \Big\{|[f_{RR}^S]_{jk}|^2 + |[f_{LR}^S]_{jk}|^2 + 4(|[f_{RR}^V]_{jk}|^2 + |[f_{LR}^T]_{jk}|^2) \pm (R \leftrightarrow L)\Big\}$$

$$B^\pm \equiv \sum_{j,k} \Big\{8[|[f_{LR}^V]_{jk}|^2 + 2|[f_{LR}^T]_{jk}|^2 \pm (R \leftrightarrow L)]\Big\}$$

$$C^\pm \equiv \sum_{j,k} \Big\{[f_{LR}^S]_{jk}[f_{LR}^S]_{kj}^* - 12[f_{LR}^T]_{jk}[f_{LR}^T]_{kj}^* \pm (R \leftrightarrow L)\Big\}$$

$$D_L^+ \equiv \sum_{j,k} \Big\{([f_{LL}^S]_{kj} + 2[f_{LL}^V]_{jk})((2[f_{LR}^S]_{kj}^* - [f_{LR}^S]_{jk}^*) + 6[f_{LR}^T]_{jk}^* - 2([f_{LR}^V]_{jk}^* + [f_{LR}^V]_{kj}^*))$$
$$+ (R \leftrightarrow L)\Big\}$$

$$D_L^- \equiv -\sum_{j,k} \Big\{([f_{LL}^S]_{kj} + 2[f_{LL}^V]_{jk})([f_{LR}^S]_{jk}^* + 2(2[f_{LR}^T]_{kj}^* + [f_{LR}^T]_{jk}^*) - 2([f_{LR}^V]_{jk}^* + [f_{LR}^V]_{kj}^*))$$
$$- (R \leftrightarrow L)\Big\}$$

$$D_R^\pm \equiv \pm\sum_{j,k} \Big\{([f_{RR}^S]_{kj} + 2[f_{RR}^V]_{jk})((2[f_{LR}^S]_{kj}^* - [f_{LR}^S]_{jk}^*) + 6[f_{LR}^T]_{jk}^* - 2([f_{LR}^V]_{jk}^* + [f_{LR}^V]_{kj}^*))$$
$$\pm (R \leftrightarrow L)\Big\}$$

$$E^\pm \equiv \sum_{j,k} \Big\{[f_{LL}^S]_{jk}[f_{LL}^S]_{kj}^* + 4[f_{LL}^V]_{jk}[f_{LL}^V]_{kj}^* + 4[f_{LL}^S]_{jk}[f_{LL}^V]_{jk}^* + 4[f_{RL}^S]_{jk}([f_{RL}^V]_{jk}^* + [f_{RL}^V]_{kj}^*)$$
$$+ 24[f_{RL}^V]_{jk}([f_{RL}^T]_{jk}^* + [f_{RL}^T]_{kj}^*) \pm (R \leftrightarrow L)\Big\}$$

$$F^\pm \equiv \sum_{j,k} \Big\{[f_{LR}^S]_{jk}[f_{LR}^S]_{kj}^* + 8[f_{LR}^V]_{jk}[f_{LR}^V]_{kj}^* + 20[f_{LR}^T]_{jk}[f_{LR}^T]_{kj}^* + 4[f_{LR}^S]_{jk}(2[f_{LR}^T]_{kj}^* + [f_{LR}^T]_{jk}^*)$$
$$\pm (R \leftrightarrow L)\Big\}$$

$$G^\pm \equiv \sum_{j,k} \Big\{4[f_{RR}^S]_{kj}[f_{RR}^V]_{jk}^* - 4[f_{LR}^T]_{kj}[f_{LR}^T]_{jk}^* - [f_{LR}^S]_{kj}[f_{LR}^S]_{jk}^* + 4[f_{LR}^T]_{jk}^*([f_{LR}^S]_{kj} - [f_{LR}^S]_{jk})$$
$$\pm (R \leftrightarrow L)\Big\}$$

$$H^+ \equiv \sum_{j,k} \Big\{[f_{LR}^S]_{kj}(2[f_{RL}^S]_{kj}^* - [f_{RL}^S]_{jk}^* + 6[f_{RL}^T]_{jk}^*) + 6[f_{LR}^T]_{jk}([f_{RL}^S]_{jk}^* + 2(2[f_{RL}^T]_{kj}^* + [f_{RL}^T]_{jk}^*))$$
$$+ 2([f_{LL}^S]_{kj} + 2[f_{LL}^V]_{jk})([f_{RR}^S]_{kj}^* + 2[f_{RR}^V]_{jk}^*) + 8[f_{LR}^V]_{jk}([f_{RL}^V]_{jk}^* + [f_{RL}^V]_{kj}^*)\Big\}$$

$$J^+ \equiv \sum_{j,k} \Big\{2([f_{LR}^V]_{jk} + [f_{LR}^V]_{kj})([f_{RL}^S]_{kj}^* + 6[f_{RL}^T]_{kj}^*) + 2([f_{RL}^V]_{jk}^* + [f_{RL}^V]_{kj}^*)([f_{LR}^S]_{kj} + 6[f_{LR}^T]_{kj})$$
$$+ ([f_{LL}^S]_{kj} + 2[f_{LL}^V]_{jk})([f_{RR}^S]_{jk}^* + 2[f_{RR}^V]_{kj}^*)\Big\}$$

$$Q \equiv \sum_{j,k} 4([f^S_{LL}]_{kj} + 2[f^V_{LL}]_{jk})([f^T_{LR}]^*_{kj} - [f^T_{LR}]^*_{jk}) \tag{5.13}$$

with the constants $[f^n_{lm}]_{jk}$:

$$\begin{aligned}
[f^S_{LL}]_{jk} &\equiv g^S_{LL} V^*_{l'j} V^*_{lk}, & [f^S_{RR}]_{jk} &\equiv g^S_{RR} U^*_{l'j} U^*_{lk}, \\
[f^V_{LL}]_{jk} &\equiv g^V_{LL} U^*_{l'j} U^*_{lk}, & [f^V_{RR}]_{jk} &\equiv g^V_{RR} V^*_{l'j} V^*_{lk}, \\
[f^S_{LR}]_{jk} &\equiv g^S_{LR} V^*_{l'j} U^*_{lk}, & [f^S_{RL}]_{jk} &\equiv g^S_{RL} U^*_{l'j} V^*_{lk}, \\
[f^V_{LR}]_{jk} &\equiv g^V_{LR} U^*_{l'j} V^*_{lk}, & [f^V_{RL}]_{jk} &\equiv g^V_{RL} V^*_{l'j} U^*_{lk}, \\
[f^T_{LR}]_{jk} &\equiv g^T_{LR} V^*_{l'j} U^*_{lk}, & [f^T_{RL}]_{jk} &\equiv g^T_{RL} U^*_{l'j} V^*_{lk}.
\end{aligned} \tag{5.14}$$

Finally, we can simplify even more with the following parameter combination:

$$\begin{aligned}
K^\pm \equiv A^\pm + \operatorname{Re} G^\pm = \sum_{j,k} \Big\{ & |[f^S_{RR}]_{jk}|^2 + |[f^S_{LR}]_{jk}|^2 + 4(|[f^V_{RR}]_{jk}|^2 + |[f^T_{LR}]_{jk}|^2) \\
& + \operatorname{Re}\Big[4[f^S_{RR}]_{kj}[f^V_{RR}]^*_{jk} - 4[f^T_{LR}]_{kj}[f^T_{LR}]^*_{jk} - [f^S_{LR}]_{kj}[f^S_{LR}]^*_{jk} \\
& + 4[f^T_{LR}]^*_{jk}([f^S_{LR}]_{kj} - [f^S_{LR}]_{jk})\Big] \pm (R \leftrightarrow L)\Big\} \\
L^\pm \equiv B^\pm + \operatorname{Re} F^\pm = \sum_{j,k} \Big\{ & 8[|[f^V_{LR}]_{jk}|^2 + 2|[f^T_{LR}]_{jk}|^2] + \operatorname{Re}\Big[[f^S_{LR}]_{jk}[f^S_{LR}]^*_{kj} + 8[f^V_{LR}]_{jk}[f^V_{LR}]^*_{kj} \\
& + 20[f^T_{LR}]_{jk}[f^T_{LR}]^*_{kj} + 4[f^S_{LR}]_{jk}(2[f^T_{LR}]^*_{kj} + [f^T_{LR}]^*_{jk})\Big] \pm (R \leftrightarrow L)\Big\}
\end{aligned} \tag{5.15}$$

Thus, the completely reduced form of the amplitude is:

$$\begin{aligned}
\sum_{j,k} |\mathcal{M}|^2_{jk} = 16 G^2_{ll'} \Big\{ & \Big[m_1 K^-(p_2 \cdot s) + K^+(p_1 \cdot p_2)\Big](p_3 \cdot p_4) + \frac{1}{2}\Big[m_1 L^-(p_4 \cdot s) + L^+(p_1 \cdot p_4) \\
& + 2 m_1 m_4 \operatorname{Re} J^+\Big](p_2 \cdot p_3) + \operatorname{Re}\Big\{ m_j \Big[D^+_L\Big(m_1(p_3 \cdot p_4) - i\epsilon^{p_1 p_3 p_4 s}\Big) - \Big(D^-_L(p_4 \cdot s) \\
& + m_4 D^+_R\Big)(p_1 \cdot p_3) + \Big(D^-_L(p_1 \cdot p_4) + m_1 m_4 D^-_R\Big)(p_3 \cdot s)\Big] + \frac{m_j m_k}{2}\Big[m_1 E^-(p_4 \cdot s) \\
& - E^+(p_1 \cdot p_4) - 2 m_1 m_4 H^+\Big]\Big\}\Big\} + \binom{j \leftrightarrow k}{2 \leftrightarrow 3} \\
& + 16 G^2_{ll'} \operatorname{Re}\Big\{ i\Big[m_1 C^+ \epsilon^{p_2 p_3 p_4 s} - C^- \epsilon^{p_1 p_2 p_3 p_4}\Big] - iQ\Big[m_j \epsilon^{p_1 p_3 p_4 s} + m_k \epsilon^{p_1 p_2 p_4 s}\Big]\Big\}.
\end{aligned} \tag{5.16}$$

5.2 The Effective Decay Rate for Majorana Neutrinos

The differential decay rate for the process $l^- \longrightarrow l^{'-} N_j N_k$ is

$$d\Gamma = \frac{1}{2!} \sum_{j,k} d\Gamma^{jk}. \tag{5.17}$$

Explicitly:

$$d\Gamma = \frac{1}{2!} \frac{(2\pi)^4 \delta^4(p_1 - p_2 - p_3 - p_4)}{2m_1} \frac{d^3p_2 d^3p_3 d^3p_4}{(2\pi)^3 2E_2 (2\pi)^3 2E_3 (2\pi)^3 2E_4} \sum_{j,k} |\mathcal{M}|^2_{jk}, \tag{5.18}$$

where we are taking into account all the possible neutrino mass final states and the sum extends over all energetically allowed neutrino pairs. Also, the factor 1/2! is taken into account due to the indistinguishable properties of the final neutrinos mass eigenstates.
Assuming that the neutrinos are not detected, we can integrate over their momenta. Due to the explicit dependence of the amplitude (5.16) in the neutrinos momenta, we will have to compute three kinds of phase space integrals, of the form:

$$\begin{aligned} I_{\mu\nu} &\equiv \int \frac{d^3p_2 d^3p_3}{E_2 E_3} \delta^4(p_2 + p_3 - q) p_{2\mu} p_{3\nu}, \\ I_\mu &\equiv \int \frac{d^3p_2 d^3p_3}{E_2 E_3} \delta^4(p_2 + p_3 - q) p_{(2,3)\mu}, \\ I &\equiv \int \frac{d^3p_2 d^3p_3}{E_2 E_3} \delta^4(p_2 + p_3 - q), \end{aligned} \tag{5.19}$$

with $q \equiv p_1 - p_4$.
Again, we can use the covariance properties of these tensor integrals and the well known three body space phase results to compute them, leading to:

$$\begin{aligned} I_{\mu\nu} &= \frac{\pi}{6}[q^2 g_{\mu\nu} + 2q_\mu q_\nu], \\ I_\mu &= \pi q_\mu, \\ I &= 2\pi. \end{aligned} \tag{5.20}$$

It is important to emphasize that in the phase space integration we are neglecting neutrino masses, in good agreement with the known final lepton energy distribution. If more precision is needed, the calculation must take into account non-negligible effects due to neutrino masses. For this thesis, we are dealing with neutrino masses effects up to the matrix element calculation.

Thus, following the same steps of the last chapter, using the amplitude (5.12) we obtain from *FeynCalc*, in the decaying-lepton rest frame, the differential decay rate:

$$\begin{aligned}\frac{d\Gamma_{l\to l'}}{dx d\cos\theta} = \frac{m_1 w^4}{2\pi^3} G_{ll'}^2 \sqrt{x^2 - x_0^2}\Big\{ & x(1-x)(\epsilon^+ + 2\zeta^+) + \frac{\epsilon^+}{6}\left(4x^2 - 3x - x_0^2\right) + x_0(1-x) \\ & \left(\frac{\lambda}{2}\right) + \mathcal{P}\sqrt{x^2 - x_0^2}\cos\theta\Big[(1-x)\left(\frac{\epsilon^-}{3} - 2\zeta^-\right) + \frac{\epsilon^-}{6}\left(4x - 4 + \sqrt{1 - x_0^2}\right)\Big] \\ & + \frac{1}{16}\frac{(m_j + m_k)}{m_1}\Big\{\kappa_L^+\big(x(1 + \sqrt{1 - x_0^2}) - x_0^2\big) - \kappa_R^+\big(x_0(1-x) + x_0\sqrt{1 - x_0^2}\big) \\ & + \mathcal{P}\sqrt{x^2 - x_0^2}\cos\theta\Big[\kappa_R^- x_0 + \kappa_L^-\big(1 + \sqrt{1 - x_0^2}\big)\Big]\Big\} - \frac{1}{8}\frac{m_j m_k}{m_1^2}\big(1 + \sqrt{1 - x_0^2}\big) \\ & \Big\{x\sigma^+ + 2x_0(\phi^+) + \mathcal{P}\sqrt{x^2 - x_0^2}\cos\theta(\sigma^-)\Big\}\Big\},\end{aligned} \tag{5.21}$$

where $\mathcal{P}$ is the initial lepton net polarization, θ is the angle between the l^- spin and the final charged-lepton momentum, $w \equiv (m_1^2 + m_4^2)/2m_1$, $x \equiv E_4/\omega$ is the reduced energy and $x_0 \equiv m_4/\omega$.

We have explicitly separated the contribution with linear neutrino mass dependence (blue brackets) and with quadratic neutrino mass dependence (red brackets) to better distinguish the neutrino mass effect on the differential decay rate.

Note that each neutrino mass term is suppressed by the decaying particle mass m_1 and thus, these contributions will be negligible in the case of light active neutrinos.

Finally, the real parameters appearing in (5.21), were defined as:

$$\begin{aligned}\lambda \equiv \frac{1}{2}\operatorname{Re} J^+ = \operatorname{Re}\sum_{j,k}\Big\{ & ([f_{LR}^V]_{jk} + [f_{LR}^V]_{kj})([f_{RL}^S]_{kj}^* + 6[f_{RL}^T]_{kj}^*) + ([f_{RL}^V]_{jk}^* + [f_{RL}^V]_{kj}^*) \\ & ([f_{LR}^S]_{kj} + 6[f_{LR}^T]_{kj}) + \frac{1}{2}([f_{LL}^S]_{kj} + 2[f_{LL}^V]_{jk})([f_{RR}^S]_{jk}^* + 2[f_{RR}^V]_{kj}^*)\Big\}\end{aligned}$$

$$\begin{aligned}\epsilon^\pm \equiv \frac{1}{4}K^\pm = \sum_{j,k}\Big\{ & \frac{1}{4}(|[f_{RR}^S]_{jk}|^2 + |[f_{LR}^S]_{jk}|^2) + (|[f_{RR}^V]_{jk}|^2 + |[f_{LR}^T]_{jk}|^2) + \operatorname{Re}\Big[[f_{RR}^S]_{kj}[f_{RR}^V]_{jk}^* \\ & - [f_{LR}^T]_{kj}[f_{LR}^T]_{jk}^* - \frac{1}{4}[f_{LR}^S]_{kj}[f_{LR}^S]_{jk}^* + [f_{LR}^T]_{jk}^*([f_{LR}^S]_{kj} - [f_{LR}^S]_{jk})\Big] \pm (R \leftrightarrow L)\Big\}\end{aligned}$$

$$\begin{aligned}\zeta^\pm \equiv \frac{1}{16}L^\pm = \sum_{j,k}\Big\{ & \frac{1}{2}[|[f_{LR}^V]_{jk}|^2 + 2|[f_{LR}^T]_{jk}|^2] + \frac{1}{16}\operatorname{Re}\Big[[f_{LR}^S]_{jk}[f_{LR}^S]_{kj}^* + 8[f_{LR}^V]_{jk}[f_{LR}^V]_{kj}^* \\ & + 20[f_{LR}^T]_{jk}[f_{LR}^T]_{kj}^* + 4[f_{LR}^S]_{jk}(2[f_{LR}^T]_{kj}^* + [f_{LR}^T]_{jk}^*)\Big] \pm (R \leftrightarrow L)\Big\}\end{aligned}$$

$$\kappa_L^+ \equiv \operatorname{Re} D_L^+ = \operatorname{Re}\sum_{j,k}\Big\{([f^S_{LL}]_{kj} + 2[f^V_{LL}]_{jk})((2[f^S_{LR}]^*_{kj} - [f^S_{LR}]^*_{jk}) + 6[f^T_{LR}]^*_{jk} - 2([f^V_{LR}]^*_{jk} + [f^V_{LR}]^*_{kj})) + (R \leftrightarrow L)\Big\}$$

$$\kappa_L^- \equiv \operatorname{Re} D_L^- = -\operatorname{Re}\sum_{j,k}\Big\{([f^S_{LL}]_{kj} + 2[f^V_{LL}]_{jk})([f^S_{LR}]^*_{jk} + 2(2[f^T_{LR}]^*_{kj} + [f^T_{LR}]^*_{jk}) - 2([f^V_{LR}]^*_{jk} + [f^V_{LR}]^*_{kj})) - (R \leftrightarrow L)\Big\}$$

$$\kappa_R^\pm \equiv \operatorname{Re} D_R^\pm = \pm\operatorname{Re}\sum_{j,k}\Big\{([f^S_{LL}]_{kj} + 2[f^V_{LL}]_{jk})((2[f^S_{LR}]^*_{kj} - [f^S_{LR}]^*_{jk}) + 6[f^T_{LR}]^*_{jk} - 2([f^V_{LR}]^*_{jk} + [f^V_{LR}]^*_{kj})) \pm (R \leftrightarrow L)\Big\}$$

$$\sigma^\pm \equiv \operatorname{Re} E^\pm = \operatorname{Re}\sum_{j,k}\Big\{[f^S_{LL}]_{jk}[f^S_{LL}]^*_{kj} + 4[f^V_{LL}]_{jk}[f^V_{LL}]^*_{kj} + 4[f^S_{LL}]_{jk}[f^V_{LL}]^*_{jk} + 4[f^S_{RL}]_{jk}([f^V_{RL}]^*_{jk} + [f^V_{RL}]^*_{kj}) + 24[f^V_{RL}]_{jk}([f^T_{RL}]^*_{jk} + [f^T_{RL}]^*_{kj}) \pm (R \leftrightarrow L)\Big\}$$

$$\phi^+ \equiv \operatorname{Re} H^+ = \operatorname{Re}\sum_{j,k}\Big\{[f^S_{LR}]_{kj}(2[f^S_{RL}]^*_{kj} - [f^S_{RL}]^*_{jk} + 6[f^T_{RL}]^*_{jk}) + 6[f^T_{LR}]_{jk}([f^S_{RL}]^*_{jk} + 2(2[f^T_{RL}]^*_{kj} + [f^T_{RL}]^*_{jk})) + 2([f^S_{LL}]_{kj} + 2[f^V_{LL}]_{jk})([f^S_{RR}]^*_{kj} + 2[f^V_{RR}]^*_{jk}) + 8[f^V_{LR}]_{jk}([f^V_{RL}]^*_{jk} + [f^V_{RL}]^*_{kj})\Big\} \tag{5.22}$$

5.3 Majorana Contributions (Rough Estimation)

From (5.21) we can highlight several things. First of all, we notice that the new parameters were defined in such a way that the full isotropic distribution is described by all the parameters with "+" upper-index, while the complete anisotropic part is described by all the parameters with "−" upper-index. Thus, we can give a useful interpretation of the parameters upper-indexes, as the ones describing the isotropic (+) or anisotropic (−) dependence.

It is now important to discuss the new features about the Majorana distribution. Specially, the possible measurable contribution of the Majorana mass terms and the deviations of the standard *Michel parameters* due to this Majorana neutrino nature.

We will first talk about the contribution with no neutrino mass dependence. Comparing this first contribution of (5.21) with the differential decay rate for the standard Michel case, we notice that they are related just by the parameter exchange:

$$\begin{aligned} \eta &\to \frac{\lambda}{2}, & \rho &\to \frac{3}{4}\epsilon^+ \\ \xi &\to 6\zeta^- - \epsilon^-, & \xi\delta &\to -\frac{3}{4}\epsilon^- \end{aligned} \tag{5.23}$$

and $x(1-x) \to x(1-x)(\epsilon^+ + 2\zeta^+)$.

In the SM limit ($|f_{LL}^V| = 1$) we obtain $\lambda = 0$, $\epsilon^\pm = \pm 1$ and $\zeta^\pm = 0$. Leading to $\eta = 0$, $\rho = 3/4$, $\xi = 1$ and $\delta = 3/4$ in agreement with the well-known results.

The other two contributions of (5.21) have explicit neutrino mass dependence and they contain completely new parameters due to the Majorana properties of neutrinos. The $\kappa_N^\pm$ parameters fully characterize the linear mass term, and thus, are less suppressed, as we shall see later, than the $\sigma^\pm$ and ϕ^+ parameters, which describe the quadratic neutrino mass term.

Furthermore, we can emphasize the importance of $\kappa_R^\pm$ and ϕ^+ as new genuine low-energy parameters (proportional to x_0), just as η in the standard Michel distribution. That is, the dependence of the total decay rate on $\kappa_R^\pm$ and ϕ^+ is left but it will be suppressed by a factor of (m_4/m_1), as in equation (4.55) for the η case, being more relevant in tau decays, where these NP contributions could be measurable.

In the SM limit ($|f_{LL}^V| = 1$) these parameters have the values $\kappa_N^\pm = 0$, $\sigma^\pm = 4$ and $\phi^+ = 0$. If precise measurements are made, then, the coupling constants bounds could be improved with this new data. Also, the measurement of this kind of contributions, that lead to small deviations from the standard Michel distribution, would offer information about the Majorana nature of neutrinos.

We will now try to roughly estimate the size of these new Majorana contributions and the deviations from the standard *Michel parameters* based on some of the best direct experimental constraints on heavy neutrino mixing [5, 62, 63, 64, 65, 66, 67]. We shall focus on the suppression due only to the Majorana masses and mixings (considering just one extra heavy neutrino for simplicity). Of course there will be other suppression factors such as the x_0^2 dependence and the explicit form of the new Majorana parameters, but this won't be the main contribution, so we are not taking them into account in the analysis.

If the heavy neutrino is forbidden by kinematics, then only the light neutrinos will be produced as final states and thus the suppression will be really high. Considering the light neutrino masses to be of order $\mathcal{O}(\text{eV})$ and the decaying particle mass of order $\mathcal{O}(10^9\text{eV})$, then the new Majorana contributions will be suppressed by a factor of $\sim 10^{-9}$ for the linear mass terms and $\sim 10^{-18}$ for the quadratic one. Both of them out of the scope of near-future experiments.

In addition, the absence of heavy neutrinos would have an impact on the non-unitarity of the mixing matrix, leading to small deviations from unity once the squared mixing matrix elements are summed over all the energetically allowed neutrinos.

In contrast, if the heavy neutrinos are kinematically accessible, then the suppression of the Majorana terms will change, depending on the heavy neutrino mass and its mixing with the active and sterile sector.

Following our notation, the mixing matrix elements U_{lj} and V_{lj} are suppressed depending on the neutrino mass, as follows:

Neutrino	U_{lj}	V_{lj}
Light ($j \leq 3$)	Not suppressed	Suppressed
Heavy ($j \geq 4$)	Suppressed	Not suppressed

Table 5.1: Suppression of the mixing matrix elements.

Considering the experimental constraints on an invisible heavy neutrino, obtained from different experimental sources and some reasonable phenomenological assumptions (v_4 decays and its lifetime), see e.g. [62, 67]. We have:

Neutrino	Mass (MeV)	Mixing $\|U_{l4}\|^2$
Heavy ($l = e$)	0.001 - 0.45	10^{-3}
	10 - 55	10^{-8}
	135 - 350	10^{-6}
Heavy ($l = \mu$)	10 - 30	10^{-4}
	70 - 300	10^{-5}
	175 - 300	10^{-8}
Heavy ($l = \tau$)	100 - 1.2×10^3	$10^{-7} - 10^{-3}$
	$1\times10^3 - 60\times10^3$	$10^{-5} - 10^{-3}$

Table 5.2: Experimental constraints on heavy neutrino mixing.

Thus, it is possible to have one or two heavy neutrinos in the final states and each term will be suppressed by some of the mixing matrix elements, depending on the specific form of the $[f^n_{lm}]_{jk}$ coupling constants.

For this rough estimation, we will consider both cases (one and two final heavy neutrinos) and the mixing suppression as $|U_{l4}|^2$ for one final heavy neutrino and $|U_{l4}|^2|U_{l'4}|^2$ for two heavy final-state neutrinos.

With these considerations we can summarize the results of the final Majorana terms suppression in the following table:

Neutrino	Mass (MeV)	Mixing Suppression	Linear Term Suppression (m_ν)	Quadratic Term Suppression (m_ν^2)
Light (2)	1×10^{-6}	—	10^{-9}	10^{-18}
Heavy (1) ($l = e$)	0.001 - 0.45	10^{-3}	$10^{-9} - 10^{-7}$	$10^{-18} - 10^{-16}$
	10 - 55	10^{-8}	10^{-10}	10^{-19}
	135 - 350	10^{-6}	10^{-7}	10^{-16}
Heavy (1) ($l = \mu$)	10 - 30	10^{-4}	10^{-6}	10^{-15}
	70 - 300	10^{-5}	$10^{-7} - 10^{-6}$	$10^{-16} - 10^{-15}$
	175 - 300	10^{-8}	10^{-9}	10^{-18}
Heavy (1) ($l = \tau$)	100 - 1.2×10^3	$10^{-7} - 10^{-3}$	$10^{-8} - 10^{-3}$	$10^{-18} - 10^{-12}$
	$1\times10^3 - 60\times10^3$	$10^{-5} - 10^{-3}$	$10^{-5} - 10^{-3}$	$10^{-14} - 10^{-12}$
Heavy (2) ($\mu \to eNN$)	10 - 30	10^{-12}	10^{-14}	10^{-16}
	175 - 300	$10^{-14} - 10^{-11}$	$10^{-15} - 10^{-12}$	$10^{-16} - 10^{-13}$
Heavy (2) ($\tau \to eNN$)	135 - 350	$10^{-13} - 10^{-9}$	$10^{-14} - 10^{-10}$	$10^{-14} - 10^{-10}$
Heavy (2) ($\tau \to \mu NN$)	100 - 300	$10^{-12} - 10^{-8}$	$10^{-13} - 10^{-9}$	$10^{-14} - 10^{-10}$
	175 - 350	$10^{-15} - 10^{-11}$	$10^{-16} - 10^{-12}$	$10^{-16} - 10^{-12}$

Table 5.3: Majorana decay rate suppression effects.

The mean life of the muon and tau has been measured to a precision of order 10^{-6} and 10^{-3} respectively [1]. In order to make new precision tests, the most recent and future experiments are working hard to increased this sensitivity.

Thus, from table 5.3, sadly, almost all of the new Majorana contributions are really suppressed and out of the scope of the near future experiments. But a few of them are just in the current precision limit and could be measurable in current and forthcoming experiments.

Specifically, in the case of one final heavy neutrino with a mass around $10^2 - 10^3 MeV$ the linear term suppression could be low enough to make sizeable distortions in the differential decay rate, specially in tau decays. Thus, making this new Majorana contribution a potential and exciting deviation waiting to be measured.

Finally, we try to estimate the possible measurable changes between the well-known *Michel parameters* (η, ρ, ξ and $\xi\delta$) and the new redefined parameters ($\lambda, \epsilon^{\pm}$ and $\zeta^{\pm}$) as shown in equation (5.23).

In order to make a rough estimation, we will consider $[f^n_{lm}]_{jk} = [f^n_{lm}]_{kj} \equiv f^n_{lm}$, focusing on the new information that the Majorana parameters would have.

From their specific definition, we have the following differences:

$$
\begin{aligned}
\eta - \frac{\lambda}{2} &= \frac{1}{2}\,\mathrm{Re}\Big[-\frac{1}{2} f^S_{LL} f^{S*}_{RR} - 2 f^V_{LL} f^{V*}_{RR} - f^V_{LR}(f^{S*}_{RL} + 6 f^{T*}_{RL}) - f^V_{RL}(f^{S*}_{LR} + 6 f^{T*}_{LR})\Big] \\
\rho - \frac{3}{4}\epsilon^+ &= \frac{3}{16}|f^S_{RL} - 2f^T_{RL}|^2 + \frac{3}{16}|f^S_{LR} - 2f^T_{LR}|^2 - \frac{3}{4}\,\mathrm{Re}[f^S_{RR} f^{V*}_{RR} + f^S_{LL} f^{V*}_{LL}] \\
\xi - (6\zeta^- - \epsilon^-) &= -3\Big[|f^V_{LR}|^2 + \frac{1}{16}|f^S_{LR} + 6f^T_{LR}|^2 - |f^V_{RL}|^2 - \frac{1}{16}|f^S_{RL} + 6f^T_{RL}|^2\Big] \\
&\quad + \frac{7}{16}|f^S_{RL} - 2f^T_{RL}|^2 - \frac{7}{16}|f^S_{LR} - 2f^T_{LR}|^2 - \mathrm{Re}[f^S_{LL} f^{V*}_{LL} - f^S_{RR} f^{V*}_{RR}] \\
\xi\delta - (-\frac{3}{4}\epsilon^-) &= \frac{3}{16}|f^S_{RL} - 2f^T_{RL}|^2 - \frac{3}{16}|f^S_{LR} - 2f^T_{LR}|^2 + \frac{3}{4}\,\mathrm{Re}[f^S_{RR} f^{V*}_{RR} - f^S_{LL} f^{V*}_{LL}]
\end{aligned}
\tag{5.24}
$$

Or, in a more compact form:

$$
\begin{aligned}
\eta - \frac{\lambda}{2} &= A', & \rho - \frac{3}{4}\epsilon^+ &= B'^+ \\
\xi - (6\zeta^- - \epsilon^-) &= C', & \xi\delta - (-\frac{3}{4}\epsilon^-) &= B'^-
\end{aligned}
\tag{5.25}
$$

where

$$
\begin{aligned}
A' &= -\frac{1}{2}\,\mathrm{Re}\Big[\frac{1}{4} f^S_{LL} f^{S*}_{RR} + f^V_{LL} f^{V*}_{RR} + f^V_{LR}(f^{S*}_{RL} + 6 f^{T*}_{RL})\Big] + (R \leftrightarrow L) \\
B'^{\pm} &= \frac{3}{16}|f^S_{RL} - 2f^T_{RL}|^2 - \frac{3}{4}\,\mathrm{Re}[f^S_{LL} f^{V*}_{LL}] \pm (R \leftrightarrow L) \\
C' &= 3|f^V_{RL}|^2 + \frac{3}{16}|f^S_{RL} + 6f^T_{RL}|^2 + \frac{7}{16}|f^S_{RL} - 2f^T_{RL}|^2 + \mathrm{Re}[f^S_{RR} f^{V*}_{RR}] - (R \leftrightarrow L)
\end{aligned}
\tag{5.26}
$$

Furthermore, using the current experimental values of the standard *Michel parameters* from table 4.1, we can find the lower and upper limits of the new generalized *Michel parameters* for the muon and tau decay. We emphasize that these limits are valid only if we consider that the reported experimental values are the *Michel parameters* (η, ρ, ξ and $\xi\delta$). If neutrinos are Majorana particles, then the experimental values will correspond to the generalized *Michel parameters* and these limits are no longer valid.

Using this data, we obtain the following bounds:

For $\mu^- \to e^- \overline{N}_j N_k$:

$$\begin{aligned} &0.023 - A' \leq \frac{\lambda}{2} \leq 0.091 - A', && 0.74719 - B'^{+} \leq \frac{3}{4}\epsilon^{+} \leq 0.75239 - B'^{+} \\ &1.0002 - C' \leq (6\zeta^{-} - \epsilon^{-}) \leq 1.0025 - C', && 0.7505 - B'^{-} \leq (-\frac{3}{4}\epsilon^{-}) \leq 0.7523 - B'^{-} \end{aligned} \tag{5.27}$$

For $\tau^- \to e^- \overline{N}_j N_k$:

$$\begin{aligned} &-- \leq \frac{\lambda}{2} \leq --, && 0.737 - B'^{+} \leq \frac{3}{4}\epsilon^{+} \leq 0.757 - B'^{+} \\ &0.954 - C' \leq (6\zeta^{-} - \epsilon^{-}) \leq 1.034 - C', && 0.706 - B'^{-} \leq (-\frac{3}{4}\epsilon^{-}) \leq 0.762 - B'^{-} \end{aligned} \tag{5.28}$$

For $\tau^- \to \mu^- \overline{N}_j N_k$:

$$\begin{aligned} &0.021 - A' \leq \frac{\lambda}{2} \leq 0.167 - A', && 0.743 - B'^{+} \leq \frac{3}{4}\epsilon^{+} \leq 0.783 - B'^{+} \\ &0.971 - C' \leq (6\zeta^{-} - \epsilon^{-}) \leq 1.089 - C', && 0.741 - B'^{-} \leq (-\frac{3}{4}\epsilon^{-}) \leq 0.815 - B'^{-} \end{aligned} \tag{5.29}$$

Back to the equation (5.24), we see that these differences are the changes between the standard *Michel parameters* and the generalized ones. For the SM case it is straightforward that all these differences are identically zero, as it should be. But if other couplings are not zero, then it will be some tiny changes, depending on the specific Lorentz structure of the underlying theory.

In order to estimate these differences, we shall focus on the main contribution. Assuming the $V - A$ coupling f^V_{LL} to be dominant, then the linear terms in non-standard couplings are the most important.

For example, for a non-zero f^V_{LL} and f^V_{RR} couplings, we have:

$$\begin{aligned} \eta - \frac{\lambda}{2} &= -\operatorname{Re}[f^V_{LL} f^{V*}_{RR}], \qquad \rho - \frac{3}{4}\epsilon^+ = 0 \\ \xi - (6\zeta^- - \epsilon^-) &= 0, \qquad \xi\delta - (-\frac{3}{4}\epsilon^-) = 0 \end{aligned} \tag{5.30}$$

And, for a non-zero f^V_{LL} and f^S_{LL} couplings:

$$\begin{aligned} \eta - \frac{\lambda}{2} &= 0, \qquad \rho - \frac{3}{4}\epsilon^+ = -\frac{3}{4}\operatorname{Re}[f^S_{LL} f^{V*}_{LL}] \\ \xi - (6\zeta^- - \epsilon^-) &= -\operatorname{Re}[f^S_{LL} f^{V*}_{LL}], \quad \xi\delta - (-\frac{3}{4}\epsilon^-) = -\frac{3}{4}\operatorname{Re}[f^S_{LL} f^{V*}_{LL}] \end{aligned} \tag{5.31}$$

Using the explicit form of the coupling constants, $f^S_{LL} = g^S_{LL} V^*_{l'j} V^*_{lk}$, $f^V_{RR} = g^V_{RR} V^*_{l'j} V^*_{lk}$, $f^V_{LL} = g^V_{LL} U^*_{l'j} U^*_{lk}$ and the suppression effects discussed in table 5.1, we see that in any case (0,1 or 2 final heavy neutrinos), the suppression of $f^V_{LL} f^{V*}_{RR}$ and $f^S_{LL} f^{V*}_{LL}$ will be of the order of the heavy neutrino mixing.

From table 5.2 this suppression will be of order 10^{-8}–10^{-3} depending on the lepton involved in the decay process. For all the other terms that are not linear in non-standard couplings, we can follow the same process, and ensure that they lead to negligible contributions due to the high suppression.

Thus, the generalized *Michel parameters* will have really small deviations from the previous ones, that could be measurable if the suppression is low enough to be in the precision regime of the near-future experiments, modifying the coupling constants upper-bounds seen in the last chapter.

Furthermore, from (5.30) and (5.31) is explicitly shown how the measurement of the discrepancy between a generalized *Michel parameter* and a classical one, could characterize the underlying Lorentz structure of a theory. For example, a theory with non-zero f^V_{LL} and f^V_{RR} couplings, will be characterized by the measurement of one anomalous parameter, as in equation (5.30); while a theory with non-zero f^V_{LL} and f^S_{LL} couplings will imply the existence of three anomalous parameters, as in equation (5.31).

Of course, the presence of other non-zero couplings will make the characterization not as immediate as the last example. So, in a general case, the complete expression (5.24) must be taken into account in order to analyse the Lorentz structure of the theory.

Finally, since the maximum measurable effect of a neutrino mass or the presence of a generalized *Michel parameter* in the decay rate would be of order 10^{-3} for τ-decay and

10^{-6} for μ-decay , as shown in the last discussion and in table 5.3, it is important for the precision electroweak tests to consider all radiative corrections and their subleading effects.

As discussed in the last chapter, these effects take into account radiative QED corrections, higher-order electroweak corrections and the non-local structure of the W propagator, all of these within the SM framework. Leading to [34, 35, 36, 20]:

$$\Gamma_{l\longrightarrow l'} \equiv \Gamma[l^- \longrightarrow l'^- \bar{\nu}_{l'}\nu_l(\gamma)] = \frac{\hat{G}^2_{ll'} m_1^5}{192\pi^3} f\left(\frac{m_4^2}{m_1^2}\right)\left(1+\delta^{ll'}_{RC}\right), \tag{5.32}$$

where $\hat{G}_{ll'} \equiv G_{ll'}F(MP)$ with $F(MP)$ a function that depends on the $Michel\ parameters$ (MP), specifically the low energy ones, as seen in equation (4.55) for the standard case with the η parameter.

The explicit form of these SM radiative corrections are:

$$\delta^{ll'}_{RC} = \frac{\alpha}{2\pi}\left[\frac{25}{4} - \pi^2 + \mathcal{O}\left(\frac{m_4^2}{m_1^2}\right)\right] + \ldots \tag{5.33}$$

$$G^2_{ll'} = \left[\frac{g^2}{4\sqrt{2}M_W^2}(1+\Delta r)\right]^2 \left[1 + \frac{3}{5}\frac{m_1^2}{M_W^2} + \frac{9}{5}\frac{m_4^2}{M_W^2} + \mathcal{O}\left(\frac{m_4^4}{m_1^2 M_W^2}\right)\right] \tag{5.34}$$

These corrections can also be analyzed at the level of differential decay rate. Specifically, the most recent corrections induced by the W-boson propagator to the differential rates of the leptonic decay of a polarized muon and tau lepton and the numerical effect of these corrections are discussed in [68].

It is important to emphasize that we can safely employ these radiative corrections (calculated in the SM limit $|f^V_{LL}| = 1$) in order to measure with high precision the $Michel\ Parameters$, since, to a high degree of precision, the current experimental information is consistent with a $V - A$ structure, so possible deviations are expected to be very small and can therefore be treated at the tree level, making the SM radiative corrections the main higher-order contributions. You can see [69] for a helpful discussion.

We can summarize the main numerical contributions, including hadronic corrections, in the following table:

Radiative Corrections	Numerical Effect (μ-decay)	Numerical Effect (τ-decay)
Electroweak	$(3/5)(m_\mu^2/M_W^2) \sim 1.0 \times 10^{-6}$	$(3/5)(m_\tau^2/M_W^2) \sim 2.9 \times 10^{-4}$
QED	$\mathcal{O}(\alpha) \sim 10^{-3}$	$\mathcal{O}(\alpha) \sim 10^{-3}$
Hadronic	$\mathcal{O}(\alpha^2/\pi^2) \sim 10^{-5}$	$\mathcal{O}(\alpha^2/\pi^2) \sim 10^{-5}$

Table 5.4: Main numerical effects of radiative corrections.

The subleading contributions of these corrections are of order $\mathcal{O}(10^{-11} - 10^{-7})$, which are out of experimental reach in the foreseeable future.

From table 5.4, it is evident that the three main radiative corrections must be taken into account in the muon decay analysis, since the smallest one of them is of the same magnitude as the present experimental relative uncertainty of the muon decay rate.

In principle, for tau decays, due to the current precision achieved, the QED correction is the most important. For the Majorana effects, the electroweak and hadronic corrections are not needed, since they would imply a correction up to 1-10% to something that has not yet been measured.

Putting all together, the possible Majorana nature of neutrinos will affect the leptonic decay rate in two ways, with the generalization of the well-known *Michel parameters* and the contribution of terms proportional to the Majorana masses, characterized with completely new parameters. In both cases, the suppression effects are really high, but under certain circumstances, these contributions could be measurable in near and future experiments if mixings of heavy neutrinos remain at their current values.

Chapter 6

Conclusions

There are some physical phenomena that motivate the research of NP that is beyond the SM. Leptonic decays provide a clean laboratory where many high-precision measurements can be made in order to test the internal consistency of the SM and reveal the signature of possible NP.

In particular, we do not know whether the observed neutrinos are Dirac or Majorana particles or if the lepton sector includes additional fermion singlets (sterile neutrinos). The Majorana nature of neutrinos as well as the existence of sterile neutrinos, would affect the leptonic decays in a non-trivial way.

In this work we have studied the lepton decay $l^- \longrightarrow l'^- N_j N_k$, where N_j and N_k are Majorana mass-eigenstate neutrinos. We have constructed its matrix element by using the most general four-lepton effective interaction Hamiltonian and obtained the specific energy and angular distribution of the final charged lepton, complemented with the decaying-lepton polarization.

We have introduced the generalized *Michel parameters*, that arise due to the neutrinos Majorana properties, for the term without neutrino mass dependence and defined completely new Majorana parameters for the terms with explicit neutrino mass dependence. In order to estimate the size of the genuine Majorana contributions and the deviation of the generalized *Michel parameters* from the classical ones, we have used some of the best experimental constraints on an invisible heavy neutrino. Once the suppression of the terms has been estimated, almost all of the possible results are many orders of magnitude lower than the experimental precision of current and forthcoming experiments. Nevertheless, there are some heavy masses range together with a non-negligible heavy-light mixing that lead to contributions right at the current experimental precision limit. Specifically, for the case of τ-decay with one heavy final-state neutrino with a mass around $10^2 - 10^3 MeV$ the linear term suppression could be of order 10^{-3}, low enough to be measured in current and forthcoming experiments.

Thus, the measure of a sizeable distortion in the differential decay rate would be a clear signal of the Majorana neutrino nature as well as possible NP.

Finally, there is a lot of work that can be done to complement these results. It would also be interesting to consider the final lepton polarization, where new parameters and Majorana effects could be measured, as well as analyze other type of leptonic decays, such as radiative muon and tau decay with Majorana neutrinos.

References

[1] Particle Data Group . Phys. Rev. D 86, (2020). http://pdg.lbl.gov/.

[2] Kai Schmidt-Hoberg. *Phenomenology of Physics beyond the Standard Model*, (2015).

[3] B. Pontecorvo. Zh. Eksp. Teor. Fiz. 34, 247, (1957).

[4] Z. Maki, M. Nakagawa, and S. Sakata. Prog. Theor. Phys.28, 870, (1962).

[5] Anupama Atre, Tao Han, Silvia Pascoli, Bin Zhang . *The Search for Heavy Majorana Neutrinos*, (2009). http://arxiv.org/abs/0901.3589v2.

[6] R. N. Mohapatra and J. W. F. Valle. Phys. Rev. D 34, 1642, (1986).

[7] J. Bernabéu, A. Santamaría, J. Vidal, A. Méndez, and J. W. F. Valle. Phys. Lett. B 187, 303, (1987).

[8] M. Malinsky, J.C. Romao, and J. W.F. Valle. , Phys. Rev. Lett. 95, 161801, (2005).

[9] A. G. Dias , C. A. de S. Pires , P. S. Rodrigues da Silva, A. Sampieri . *A Simple Realization of the Inverse Seesaw Mechanism*, (2012). http://arxiv.org/abs/1206.2590v2.

[10] G. Hernández-Tomé, J.I. Illana , M. Masip, G. López Castro and P. Roig. *Effects of heavy Majorana neutrinos on lepton flavor violating processes*, (2020). https://doi.org/10.1103/PhysRevD.101.075020.

[11] Takeshi Fukuyama, Joji Tsumura . *Detecting Majorana nature of neutrinos in muon and tau decay*, (2008). http://arxiv.org/abs/0809.5221v2.

[12] Masaru Doi, Tsuneyuki Kotani, Hiroyuki Nishiura. *New Parametrization in Muon Decay and the Type of Emitted Neutrino*, (2005). http://arxiv.org/abs/hep-ph/0502136v2.

[13] Duarte, L., Peressutti, J., Romero, I. et al. *Majorana neutrinos with effective interactions in B decays*, (2019). https://doi.org/10.1140/epjc/s10052-019-7104-x.

[14] S.N. Gninenko. *The MiniBooNE anomaly and heavy neutrino decay*, (2009). http://arxiv.org/abs/0902.3802v5.

[15] A.Pich. *Precision Tau Physics*, (2013). https://arxiv.org/pdf/1310.7922.pdf.

[16] A.Pich. *Precision tests of the Standard Model*, (1997). http://arxiv.org/abs/hep-ph/9711279v1.

[17] T.P. Gorringe, D.W. Hertzog. *Precision Muon Physics*, (2015). http://arxiv.org/abs/1506.01465v1.

[18] Y. Amhis et al. (HFLAV Collaboration). Eur. Phys. J. C 77, 895, (2017).

[19] W. N. Cottingham and D. A. Greenwood. *An Introduction to the Standard Model of Particle Physics*. Cambridge University Press, (2007). http://einstein-schrodinger.com/Standard_Model.pdf.

[20] P. Langacker. *The Standard Model and Beyond*. CRC Press, (2010).

[21] Palash B. Pal. *An Introductory course of particle physics.* CRC Press, (2015).

[22] David Griffiths. *Introduction to Elementary Particles.* John Wiley Sons, Inc., (1987).

[23] University of Zurich. *Standard Model*, (2018). https://www.physik.uzh.ch/en/researcharea/lhcb/outreach/StandardModel.html.

[24] Abdelhak Djouadi. *The Higgs Mechanism and the Origin of Mass*, (2011). https://www.researchgate.net/publication/225267856_The_Higgs_Mechanism_and_the_Origin_of_Mass.

[25] CMS Collaboration. (2013). http://arxiv.org/abs/1207.7235v2.

[26] ATLAS Collaboration. (2013). http://arxiv.org/abs/1207.7214v2.

[27] Ethan Siegel. *Beyond the galaxy.* World Scientific Publishing, (2015).

[28] A.Pich. *The Standard Model of Electroweak Interactions*, (2005). http://arxiv.org/abs/hep-ph/0502010v1.

[29] A. Pich. *Effective Field Theory*, (1998). http://arXiv.org/abs/hep-ph/9806303v1.

[30] Aneesh V. Manohar. *Effective Field Theories*, (1996). http://arXiv.org/abs/hep-ph/9606222v1.

[31] Aneesh V. Manohar. *Introduction to Effective Field Theories*, (2017). https://arxiv.org/pdf/1804.05863.

[32] G. Rajasekaran. *Fermi and the Theory of Weak Interactions*, (2014). http://arxiv.org/abs/1403.3309v1.

[33] M. Maggiore. *A Modern Introduction to Quantum Field Theory.* Oxford University Press, (2005).

[34] R. E. Behrends, R. J. Finkelstein, and A. Sirlin. Radiative corrections to decay processes. *Phys. Rev.*, 101:866–873, Jan (1956).

[35] S. M. Berman. Radiative corrections to muon and neutron decay. *Phys. Rev.*, 112:267–270, Oct (1958).

[36] Toichiro Kinoshita and Alberto Sirlin. Radiative corrections to fermi interactions. *Phys. Rev.*, 113:1652–1660, Mar (1959).

[37] D. M. Webber et al. (MuLan Collaboration). Phys. Rev. Lett 106, (2011).

[38] L. Michel. Proc. Phys. Soc. A 63, 514, 1371, (1950).

[39] W. Fetscher, H.J. Gerber and K.F. Johnson. Phys. Lett., B173 102, (1986).

[40] V. Shtabovenko, R. Mertig and F. Orellana. *FeynCalc 9.3: New features and improvements.* Comput. Phys. Commun., (2020). https://arxiv.org/abs/2001.04407.

[41] V. Shtabovenko, R. Mertig and F. Orellana. *New Developments in FeynCalc 9.0.* Comput. Phys. Commun, (2016). http://arxiv.org/abs/1601.01167.

[42] R. Mertig, M. Böhm, and A. Denner. *Feyn Calc - Computer-algebraic calculation of Feynman amplitudes.* Comput. Phys. Commun, (1991).

[43] C. Bouchiat and L. Michel. Phys. Rev., 106 170, (1957).

[44] T. Kinoshita and A. Sirlin. Phys. Rev., 107 593, (1957).

[45] Stahl,A. The michel parameter η in τ decays. *Phys. Lett.*, 324:121–124, (1994).

[46] SLD Collaboration . Phys. Rev. Lett. 78 4691, (1997).

[47] ALEPH Collaboration . Eur. Phys. J. C 22 217, (2001).

[48] DELPHI Collaboration . Eur. Phys. J. C 16 229, (2000).

[49] OPAL Collaboration . Eur. Phys. J. C 8 3, (1999).

[50] L3 Collaboration . Phys. Lett. B 438 405, (1998).

[51] ARGUS Collaboration . Phys. Lett. B 431 179, (1998).

[52] CLEO Collaboration . Phys. Lett. D 56 5320, (1997).

[53] TWIST Collaboration . Phys. Rev. D 85 092013, (2012).

[54] N. Danneberg et al. . Phys. Rev. Lett. 94 021802, (2005).

[55] B. Balke et al. . Phys. Rev. D 37 587, (1988).

[56] I. Beltrami et al. . Phys. Lett. B 194 326, (1987).

[57] A. Stahl and H. Voss . Z. Phys. C 74 73, (1997).

[58] A. Flores-Tlalpa, G. Lopez Castro and P. Roig . *Five-body leptonic decays of muon and tau leptons*. JHEP. 185, (2016). https://doi.org/10.1007/JHEP04(2016)185.

[59] A.B. Arbuzov, T.V. Kopylova . *Michel parameters in radiative muon decay*. JHEP. 109, (2016). https://doi.org/10.1007/JHEP09(2016)109.

[60] Sasak Junya and Belle Collaboration . *Study of five-body leptonic decays of tau at Belle experiment*. J. Phys. Conf. Ser. 912 012002, (2017).

[61] P. Langacker and D. London. Phys. Rev.D39, 266, (1989).

[62] André de Gouvea, Andrew Kobach. *Global Constraints on a Heavy Neutrino*, (2016). http://arxiv.org/abs/1511.00683v3.

[63] Enrique Fernandez-Martinez, Josu Hernandez-Garcia, Jacobo Lopez-Pavon . *Global constraints on heavy neutrino mixing*, (2016). http://arxiv.org/abs/1605.08774v2.

[64] Mattias Blennow, Pilar Coloma, Enrique Fernandez-Martinez, Josu Hernandez-Garcia, Jacobo Lopez-Pavon . *Non-Unitarity, sterile neutrinos, and Non-Standard neutrino Interactions*, (2017). http://arxiv.org/abs/1609.08637v3.

[65] Marco Drewes, Bjorn Garbrech. *Combining Experimental and Cosmological Constraints on Heavy Neutrinos*, (2017). http://arxiv.org/abs/1502.00477v3.

[66] Frank F. Deppisch, P. S. Bhupal Dev, Apostolos Pilaftsis. *Neutrinos and Collider Physics*, (2015). http://arxiv.org/abs/1502.06541v3.

[67] Andrew Kobach and Sean Dobbs. *Heavy Neutrinos and the Kinematics of Tau Decays*, (2015). http://arxiv.org/abs/1412.4785v2.

[68] M. Fae, L. Mercolli and M. Passera. *W-propagator corrections to μ and τ leptonic decays*, (2013). http://arxiv.org/abs/1310.1081v2.

[69] Alberto Sirlin and Andrea Ferroglia. *Radiative Corrections in Precision Electroweak Physics: Historical Perspective*, (2012). http://arxiv.org/abs/1210.5296v1.

[70] Y. Fukuda et al. *(Super-Kamiokande Collaboration)*. Phys. Rev. Lett. 81, 1562, (1998).

[71] R. N. Mohapatra and P.B. Pal. World Sci. Lect. Notes Phys.60, 1, (1998).

[72] R. N. Mohapatra and Palash B. Pal. *Massive Neutrinos in Physics and Astrophysics*. World Scientific Publishing, (2004).

[73] Ansgar Denner, H. Eck, O. Hahn, and J. Kublbeck. . *Feynman rules for fermion number violating interactions.* Nucl. Phys. B 387: 467-481, (1992).

Appendices

Massive Neutrinos

A1 Neutrinos in the SM

In the SM neutrinos come in three flavours, corresponding to the charged lepton associated. They belong to $SU(2)$ doublets

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L , \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L , \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \tag{A.1}$$

and alike (e_R, μ_R, τ_R), there are no SU(2) neutrino singlets.

Thus, in the SM, neutrinos are left handed and antineutrinos right handed. These are the only chiralities that participate in weak interactions as we can see explicitly from the following interaction Lagrangians. Neutrinos interact only through the weak force via the charged current (CC):

$$\begin{aligned} W^- &\longrightarrow l_\alpha^- + \overline{\nu}_\alpha \\ W^+ &\longrightarrow l_\alpha^+ + \nu_\alpha, \end{aligned} \tag{A.2}$$

with the interaction Lagrangian

$$\mathcal{L}_{int}^{CC} = -\frac{g}{2\sqrt{2}} \left(\sum_\alpha \overline{\nu}_\alpha \gamma_\mu (1 - \gamma_5) l_\alpha W^\mu + h.c. \right) \tag{A.3}$$

And via the neutral current (NC):

$$Z^0 \longrightarrow \nu_\alpha + \overline{\nu}_\alpha, \tag{A.4}$$

with the interaction Lagrangian

$$\mathcal{L}_{int}^{NC} = -\frac{g}{4\cos\theta_W}\left(\sum_{\alpha}\overline{\nu}_\alpha\gamma_\mu(1-\gamma_5)\nu_\alpha Z^\mu + h.c.\right), \tag{A.5}$$

where $\alpha = e, \mu, \tau$.

In the SM, fermion masses appears in the Lagrangian as a Dirac mass term of the form $m\overline{\Psi}\Psi$.

Thus, decomposing into its chiral states ($\Psi = \Psi_L + \Psi_R$):

$$\begin{aligned}\mathcal{L}_{m_D} &= m_D\overline{\Psi}\Psi = m_D(\overline{\Psi}_L + \overline{\Psi}_R)(\Psi_L + \Psi_R)\\ &= m_D(\overline{\Psi}_L\Psi_L + \overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L + \overline{\Psi}_R\Psi_R),\end{aligned} \tag{A.6}$$

and using the properties of the chiral projectors: $\Psi_{L,R} = P_{L,R}\Psi$, $\overline{\Psi}_{L,R} = \overline{\Psi}P_{R,L}$ and $P_{L,R}P_{R,L} = 0$ we obtain, for the Dirac mass term

$$\mathcal{L}_{m_D} = m_D(\overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L). \tag{A.7}$$

Thus, the mass term couples L and R chiral states of a particle.

But since SM neutrinos have only a L-chiral state, i.e. , there is no R-chiral states for neutrinos (N_R), this mass term is not possible, so neutrinos cannot have a Dirac mass in the SM.

The only option left, is to try to make a mass term from ν_L alone. This kind of term, known as Majorana mass term, is of the form:

$$\mathcal{L}_{m_M} = \frac{1}{2}m_M(\overline{\nu_L^C}\nu_L + \overline{\nu}_L\nu_L^C), \tag{A.8}$$

where ν^C is the charge conjugate field, defined as $\nu^C = C\overline{\nu}^T$.

This kind of term breaks all charges in two units, but as we shall see in Appendix B, neutrinos can be Majorana fermions, so this term is not forbidden. However, this mass term is not invariant under weak isospin, so it cannot appear in the SM Lagrangian.

In conclusion, the fact that no right handed neutrinos N_R appear in the SM and that the $\overline{\nu_L^C}\nu_L$ term is forbidden by weak isospin, make the neutrinos massless within the SM.

Nowadays, we know that neutrinos do have mass [70]. So we need to introduced neutrino masses beyond the SM.

A2 Massive Neutrinos

In order to account for neutrino masses, the SM framework needs to be extended.
There are many extensions of the SM that try to explain the experimental results of neutrino masses, these new models usually incorporate right-handed neutrino fields and extend even more the particle content with the existence of a new Higgs triplet or other lepton $SU(2)$ singlets, etc.
Every model has a characteristic energy scale for this new physics and in principle, some of them could lead to observable deviations from the SM predictions in the most recent experiments, due to the high accuracy achieved.
A minimal extension is the so-called νSM [71], which adds right-handed components for the three neutrinos families. The neutrino masses appear as a Dirac mass term in the Lagrangian via Yukawa couplings with the Higgs doublet, just like for all other fermions. Nevertheless, the νSM requires extremely tiny Yukawa couplings ($Y_\nu \simeq 10^{-13}$) in order to explain the observed neutrino masses. As a consequence of the mass mechanism, there is a mixing between the mass eigenstates and the flavour eigenstates, analogous to the quark sector. This mixing is described by the well-known Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [3, 4].
In the νSM the effects of Majorana neutrinos are suppressed by a factor of $(m_\nu/E)^2$, being E the energy scale of the process and m_ν the corresponding neutrino mass ($m_\nu \sim eV$). Thus, in this model, any experimental observation of these effects is out of scope, even with the accuracy achieved by the recent experiments.
Other models include a heavy neutrino sector (N) with Majorana masses, where the processes can also be mediated by those heavy neutrinos, but the heavy-light mixing implies then a suppression of order of $(E/m_N)^2$.
In the simplest scenarios, the heavy neutrino mass can be as large as $m_N \simeq 10^{14}\ GeV$ so $(E/m_N)^2$ is really small. Even so, there are scenarios, usually known as low-scale seesaw models, that allow for arbitrary masses in the heavy neutrino sector and then unsuppressed heavy-light mixings, where precise measurements of the possible new physics could be made, see e.g., [14, 6, 7, 8, 11, 12].
In the next sections we are going to briefly describe two neutrino mass models, the type

I seesaw model, where the tiny neutrino masses appear for a very large value of the Majorana mass of the right singlet. And a low-scale seesaw model, called inverse seesaw model, a well-motivated variant with an unsuppressed heavy-light mixing.

A3 Type I Seesaw Model

Here we summarize the main properties of the Type I Seesaw Model, based on [5].
Once the right-handed neutrino field (N_R) is included, no SM symmetry forbids the Majorana mass term to appear. Thus, if, in addition to the Dirac mass term, the right-handed singlets have Majorana masses, the full mass Lagrangian becomes:

$$\mathcal{L}_{D+M} = \overline{\nu}_L m_D N_R + \frac{1}{2}\overline{N_R^C} m_R N_R + h.c. \tag{A.9}$$

Where ν_L and N_R are, respectively, the weak eigenstates of the left-handed and right-handed neutrinos. Explicitly, we write $\nu_L^T = (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L})$ and $N_R^T = (N_{eR}, N_{\mu R}, N_{\tau R})$. Actually, the leptonic content in the theory can include n right-handed SM singlets (n≥2 for at least two massive neutrinos), but here we choose only three, that correspond to the right-handed parts of the weak eigenstates, just to show the general idea of the type I seesaw mechanism.
Now, using the identity $\overline{\nu}_L m_D N_R = \overline{(N_R)^C} m_D^T (\nu_L)^C$, the mass term (A.9) can be written as:

$$\mathcal{L}_{D+M} = \frac{1}{2}\left[\overline{(\nu_L)} m_D (N_R) + \overline{(N_R)^C} m_D^T (\nu_L)^C + \overline{(N_R)^C} m_R (N_R)\right] + h.c. \tag{A.10}$$

$$\mathcal{L}_{D+M} = \frac{1}{2}\begin{pmatrix}\overline{(\nu_L)} & \overline{(N_R)^C}\end{pmatrix} \mathcal{M} \begin{pmatrix}(\nu_L)^C \\ (N_R)\end{pmatrix}, \tag{A.11}$$

where $\mathcal{M}$ is a symmetric 6×6 matrix defined by

$$\mathcal{M} = \begin{pmatrix} 0 & m_D \\ m_D^T & m_R \end{pmatrix}. \tag{A.12}$$

The $\mathcal{M}$ matrix can be diagonalized by some orthogonal matrix in principle to determine the neutrino masses. However, we use the 6×6 unitary matrix $\mathcal{U}$ in order to obtain positive values for masses [72]:

$$\mathcal{U}^\dagger \mathcal{M} \mathcal{U}^* = \mathcal{U}^\dagger \begin{pmatrix} 0 & m_D \\ m_D^T & m_R \end{pmatrix} \mathcal{U}^* = \begin{pmatrix} m_\nu & 0 \\ 0 & m_N \end{pmatrix} \equiv \mathcal{D} \tag{A.13}$$

$$\mathcal{M} = \mathcal{U}\mathcal{D}\mathcal{U}^T \tag{A.14}$$

Here, $\mathcal{D}$ is a diagonal matrix whose elements represent the masses of the Majorana-type neutrinos and the mass eigenvalues are of the order

$$m_\nu \approx \frac{m_D^2}{m_R}, \quad m_N \approx m_R \tag{A.15}$$

where $m_R >> m_D$ and m_D is of the order of the mass of a charged lepton or quark. Therefore, substituting (A.14) in (A.11) we obtain:

$$\mathcal{L}_{D+M} = \frac{1}{2} \left(\overline{(\nu_L)} \quad \overline{(N_R)^C} \right) \mathcal{U}\mathcal{D}\mathcal{U}^T \begin{pmatrix} (\nu_L)^C \\ (N_R) \end{pmatrix} = \frac{1}{2} \left(\overline{(\nu_L^m)} \quad \overline{(N_R^m)^C} \right) \mathcal{D} \begin{pmatrix} (\nu_L^m)^C \\ (N_R^m) \end{pmatrix}, \tag{A.16}$$

with the mass eigenstates ν^m and N^m.

Thus, in this scenario, the small masses of the left-handed Majorana-type neutrinos (ν^m) are naturally explained by the seesaw mechanism under the assumption that the right-handed Majorana neutrinos (N^m) have large masses.

Suppose the Dirac mass of the neutrino is around $m_D \approx 10\ GeV$. Then if the mass of the heavy partner is around $m_N \approx 10^{14}\ GeV$, the mass of the light neutrino (m_ν) will be in the meV range, as we now know it is.

Furthermore, the weak eigenstates of neutrinos are expressed as superpositions of the mass eigenstates Majorana neutrinos as follows:

$$\begin{pmatrix} (\nu_L) \\ (N_R)^C \end{pmatrix} = \mathcal{U} \begin{pmatrix} (\nu_L^m) \\ (N_R^m)^C \end{pmatrix}. \tag{A.17}$$

The unitary matrix can be parameterized as

$$\mathcal{U} = \begin{pmatrix} U_{3\times3} & V_{3\times3} \\ X_{3\times3} & Y_{3\times3}, \end{pmatrix} \tag{A.18}$$

then

$$\nu_{aL} = \sum_{m=1}^{3} U_{am}\nu_L^m + \sum_{m=4}^{6} V_{am}(N_R^m)^C, \tag{A.19}$$

$$N_{bR} = \sum_{m=1}^{3} X_{Bm}^{*}(\nu_L^m)^C + \sum_{m=4}^{6} Y_{bm}^{*}N_R^m. \tag{A.20}$$

Parametrically, U$U^\dagger$ and $Y^\dagger$ Y $\sim \mathcal{O}(1)$, V$V^\dagger$ and $X^\dagger$ X$\sim \mathcal{O}(m_\nu/m_N)$.
From the neutrinos mixing (A.19) and (A.20) we see that our familiar left-handed neutrinos are mostly the light Majorana neutrinos, with small masses $m_\nu \approx m_D^2/m_N$ and the right-handed neutrinos are mostly the heavy Majorana neutrinos with large masses m_N.
In conclusion,in this model, the sterile right-handed Majorana neutrinos need to have really large masses in order for m_ν to be in the meV range. With these conditions the negligible heavy-light mixing implies then a high suppression for the Majorana effects. Finally, the N^m Majorana neutrinos have masses really high to be produced on-shell.
So, within this model, the suppression for Majorana neutrinos effects in leptonic decays will lead to SM deviations that are out of scope for the future experiments.

A4 Inverse Seesaw Model

As we discussed in the type I seesaw model, in order to have small neutrino masses the typical scale of the extra particles (such as right-handed neutrinos) is in general very high, potentially very close to the gauge coupling unification (GUT) scale, thus implying that direct experimental tests of the seesaw hypothesis might be impossible. In contrast, low-scale seesaw mechanisms, in which sterile fermions are added to the SM particle content with masses around the electroweak scale or even lower, are very attractive from a phenomenological point of view since the new states can be produced in colliders or low-energy experiments, and their contributions to physical processes can be sizeable.
In this section we briefly summarize the main properties of the inverse seesaw model (ISS), based on [6, 9], which is a low-scale seesaw mechanism.
The ISS requires the addition of right-handed neutrinos (N_R) and extra sterile fermions (S).

$$(\nu, N^C, S) \quad \text{with} \quad L = (+1, -1, +1), \tag{A.21}$$

where N and S are $SU(2)$ singlets.

With this lepton content is possible to build the general mass Lagrangian, as we did for the type I seesaw model case, and obtain the $\mathcal{M}$ matrix, analogue to (A.12), defined by

$$\mathcal{L}_{m_\nu} = \frac{1}{2} n_L^T C \mathcal{M} n_L + h.c., \tag{A.22}$$

where $n_L = (\nu_L, (N_R)^C, S)^T$ and

$$\mathcal{M} = \begin{pmatrix} 0 & m_D & 0 \\ m_D^T & 0 & M \\ 0 & M^T & \mu \end{pmatrix}. \tag{A.23}$$

Once $\mathcal{M}$ is diagonalized, the light mass-eigenstate neutrinos (ν^m) acquire the mass eigenvalues

$$m_\nu = m_D (M^T)^{-1} \mu M^{-1} m_D^T. \tag{A.24}$$

The distinctive feature of the ISS is that an additional dimensionful parameter (μ) allows to accommodate the smallness of the active neutrino masses for a low seesaw scale. In turn, this allows for sizeable mixings between the active and the additional sterile states. Such features are in clear contrast with, for instance, the canonical type I seesaw.

It happens that standard neutrinos with mass at sub-eV scale are obtained for m_D at electroweak scale, M at TeV scale and μ at keV scale.Thus, m_ν can be very light even if M is far below GUT scale.

In this case all the heavy neutrinos may develop masses around TeV scale and their mixing with the standard neutrinos is modulated by the ratio m_D/M, not as supressed as the one obtained from the type I seesaw model.

In conclusion, this kind of low-scale seesaw mechanism leads to non-negligible mixings and masses for the new heavy neutrinos not as high as the type I seesaw model, so their possible contributions to physical processes can be sizeable in the most recent and future experiments.

Majorana Fermions

B1 Majorana Fermions

For a Dirac fermion we already have an invariant that is quadratic in the fermion fields, the well known invariant $\overline{\Psi}\Psi$ that is identified as the mass term for such fermion field Ψ. Actually, there can be other kinds of quadratic invariants involving fermion fields. Especially we can ask whether we can make any invariant of the form $\Psi_a A_{ab} \Psi_b$, where a,b are spinor indexes and A is a constant matrix. The spinor indexes are contracted in order to obtain a Lorentz invariant and thus, this new term can be written as $\Psi^T A \Psi$ in matrix notation.
Under Lorentz transformations, a fermion field transforms as follows:

$$\Psi(x) \longrightarrow \Psi^{'}(x') = exp\left(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right)\Psi(x), \tag{B.1}$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right], \tag{B.2}$$

and $w^{\mu\nu}$'s are the parameters of the transformation.
Thus, in order for $\Psi^T A \Psi$ to be Lorentz invariant, it needs to satisfy:

$$\Psi^{'T}(x')A\Psi^{'}(x') = \Psi^T(x)A\Psi(x). \tag{B.3}$$

Using (B.1) in the left hand side of (B.3) we have

$$\Psi^{'T}(x')A\Psi^{'}(x') = \Psi^T(x)\ exp\left(-\frac{i}{4}\omega^{\mu\nu T}\sigma_{\mu\nu}\right) A\ exp\left(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right)\Psi(x), \tag{B.4}$$

or, examining the first order terms in the transformation parameters $w^{\mu\nu}$

$$\begin{aligned}\Psi^{'T}(x')A\Psi^{'}(x') &= \Psi^T(x)\left(1 - \frac{i}{4}\omega^{\mu\nu}\sigma^T_{\mu\nu}\right) A \left(1 - \frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right)\Psi(x) \\ &= \Psi^T(x)\left(A - \frac{i}{4}\omega^{\mu\nu}(\sigma^T_{\mu\nu}A + A\sigma_{\mu\nu})\right)\Psi(x).\end{aligned} \tag{B.5}$$

Equation (B.5) together with (B.3) shows that invariance will be achieved if the matrix A satisfies the condition

$$\sigma_{\mu\nu}^T A + A\sigma_{\mu\nu} = 0, \tag{B.6}$$

i.e., if

$$\sigma_{\mu\nu}^T = -A\sigma_{\mu\nu}A^{-1}. \tag{B.7}$$

Finally, it is well known that, under charge conjugation C, a fermion field transforms as

$$\Psi(x) \longrightarrow \Psi^c(x) = \gamma_0 C\Psi^*(x), \tag{B.8}$$

such that the Lorentz transformation properties of $\Psi^c(x)$ and $\Psi(x)$ are identical.
This implies, from (B.1), that

$$\Psi'^c(x') = \gamma_0 C\Psi'^*(x') = \gamma_0\ C\ exp\left(\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}^*\right)\Psi(x)^*, \tag{B.9}$$

needs to be equal to

$$\Psi'^c(x') = exp\left(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right)\Psi(x)^c = exp\left(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right)\gamma_0 C\Psi^*(x). \tag{B.10}$$

Then, comparing these two equations to first order, we obtain the relation

$$\gamma_0 C\sigma_{\mu\nu}^* = -\sigma_{\mu\nu}\gamma_0 C, \tag{B.11}$$

or

$$\sigma_{\mu\nu}^* = -C^{-1}\gamma_0\sigma_{\mu\nu}\gamma_0 C = -C^{-1}\sigma_{\mu\nu}^\dagger C, \tag{B.12}$$

where we have used $\gamma_0\gamma_0 = 1$ and $\sigma_{\mu\nu}^\dagger = \gamma_0\sigma_{\mu\nu}\gamma_0$.
Thus, taking the hermitian conjugate and using the unitary nature of the C matrix that ensures $C^\dagger = C^{-1}$, we obtain

$$\sigma_{\mu\nu}^T = -C^{-1}\sigma_{\mu\nu}C. \tag{B.13}$$

Comparing (B.13) with (B.7), we find that we can choose

$$A = C^{-1}. \tag{B.14}$$

Thus, $\Psi^T C^{-1}\Psi$ is a Lorentz invariant quadratic in the fermion field.

This kind of invariant was not considered before in the SM formulation, because it can annihilate two units of *charge* and would therefore defy any *charge* conservation. Where we refer to *charge* as any internal quantum number that carry the corresponding fermion state, such as electric charge or lepton number.

It is immediate that if we deal with electrically charged fermions, we cannot use this kind of invariant just because of electric charge conservation. Meanwhile neutrinos do not have electric charge and only carry lepton number; even so, this new invariant would not conserve lepton number, but as we discuss in the second chapter, lepton number is just an accidental symmetry and can easily cease to be so if there are extra fields in the model. Thus it is worthwhile to consider the possibility of this kind of term in the neutrino case.

Now that we have constructed two kinds of mass terms $\overline{\Psi}\Psi$ and $\Psi^T C^{-1}\Psi$, the equality of both requires

$$\overline{\Psi}\Psi = e^{i\alpha}\Psi^T C^{-1}\Psi, \tag{B.15}$$

where we have introduce a phase $e^{i\alpha}$ in order for $\Psi^T C^{-1}\Psi$ to be hermitian.

Thus

$$\overline{\Psi} = e^{i\alpha}\Psi^T C^{-1}, \tag{B.16}$$

and taking the transpose

$$\gamma_0^T \Psi^* = e^{i\alpha}\left(C^{-1}\right)^T \Psi \tag{B.17}$$

$$\gamma_0^T \Psi^* = -e^{i\alpha} C^{-1}\Psi \;\longrightarrow\; C\gamma_0^T \Psi^* = -e^{i\alpha}\Psi \;\longrightarrow\; \gamma_0 C\Psi^* = e^{i\alpha}\Psi, \tag{B.18}$$

where we have used the identities $C^T = -C$ and $C\gamma_\mu^T = -\gamma_\mu C$.

So

$$\Psi^c(x) = \gamma_0 C\Psi^*(x) = e^{i\alpha}\Psi(x). \tag{B.19}$$

A fermion field that satisfies (B.19) is called a Majorana field and from the definition of the conjugate field, it implies that a Majorana fermion is its own antiparticle.

Imposing this Majorana condition on the plane wave expansion of a Dirac field, we find that it is of the form

$$\Psi(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(d_s(\mathbf{p})u_s(\mathbf{p})e^{-ip\cdot x} + e^{-i\alpha} d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{ip\cdot x}\right). \tag{B.20}$$

Thus the plane wave expansion (B.20) is the explicit form of a Majorana field and can annihilate as well as create a Majorana particle. Where the u-spinors and the v-spinors satisfy the relations

$$v_s = \gamma_0 C u_s^*, \qquad u_s = \gamma_0 C v_s^*. \tag{B.21}$$

This formalism, as we can note from the Majorana condition (B.19) would lead to expressions involving the C matrix, which make it representation dependent, as we shall see in the Z decay example at the end of this appendix.

B2 Feynman Rules for Majorana Fermions

In order to work with C-independent expressions, we shall change to a more suitable approach.

Here, we summarize the Feynman rules for Majorana fermions described in [73]. These rules do not involve explicit charge conjugation matrices and thus lead us to representation independent calculations.

We consider a typical coupling term $\mathcal{L}_I = \overline{\chi}\Gamma\chi$ where each χ can be a Dirac or a Majorana fermion and Γ denotes a generic fermionic interaction including Dirac matrices, coupling constants h^i_{abc} and boson fields:

$$\overline{\chi}\Gamma\chi = h^i_{abc}\overline{\chi}_a\Gamma_i\chi_b\Phi_c, \tag{B.22}$$

where the field Φ summarizes scalar and vector fields and $\Gamma_i = 1, i\gamma_5, \gamma_\mu\gamma_5, \gamma_\mu, \sigma_{\mu\nu}$.

Finally, the charge conjugated Γ' is given by $\Gamma' = C\Gamma^T C^{-1}$ and

$$\Gamma_i' = C\Gamma_i^T C^{-1} = \eta_i\Gamma_i, \tag{B.23}$$

with

$$\eta_i = \begin{cases} 1 & \text{for } \Gamma_i = 1, i\gamma_5, \gamma_\mu\gamma_5 \\ -1 & \text{for } \Gamma_i = \gamma_\mu, \sigma_{\mu\nu} \end{cases} \tag{B.24}$$

In the case of a pure Majorana fermion vertex we obtained $\Gamma = \Gamma'$.

Let Φ be a scalar or vector field and λ, Ψ Majorana and Dirac fermions respectively.

Due to the Goldstone equivalence theorem, these rules also define how the fermion chain interacts with the massive gauge boson.

In our Feynman diagrams, fermions are denoted by solid lines. For Dirac fermions (Ψ), each line carries an arrow which indicates the fermion number flow. Majorana fermions (λ) lines do not carry arrows. The Dirac propagator is denoted by $S(p)$.

The Feynman rules are as follows:

- Draw all possible Feynman diagrams for the process.
- Fix an arbitrary orientation (fermion flow) for each fermion chain. This is shown as blue arrows.
- Start at an external leg (for closed loops at some arbitrary propagator) and write down the Dirac matrices proceeding opposite to the chosen orientation (blue arrows) through the chain.
- For each internal propagator, external line and vertex insert the appropriate analytic expression as given in Figs. B2.1, B2.2 and B2.3 corresponding to the chosen fermion flow.
- Multiply by a factor (-1) for every closed loop.
- Multiply by the permutation parity of the spinors in the obtained analytical expression with respect to some reference order.
- As far as the determination of the combinatorial factor is concerned, Majorana fermions behave exactly like real scalar or vector fields.

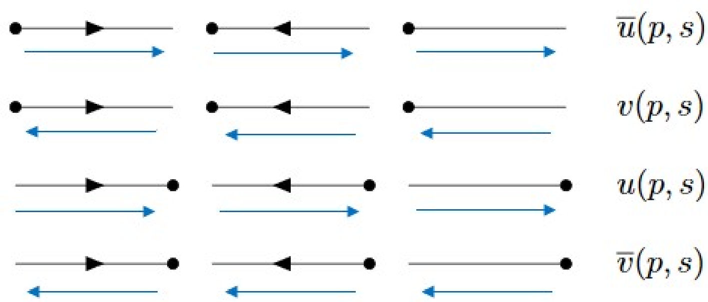

Figure B2.1: Feynman rules for external fermion lines with orientation (blue arrows). The momentum p flows from left to right.

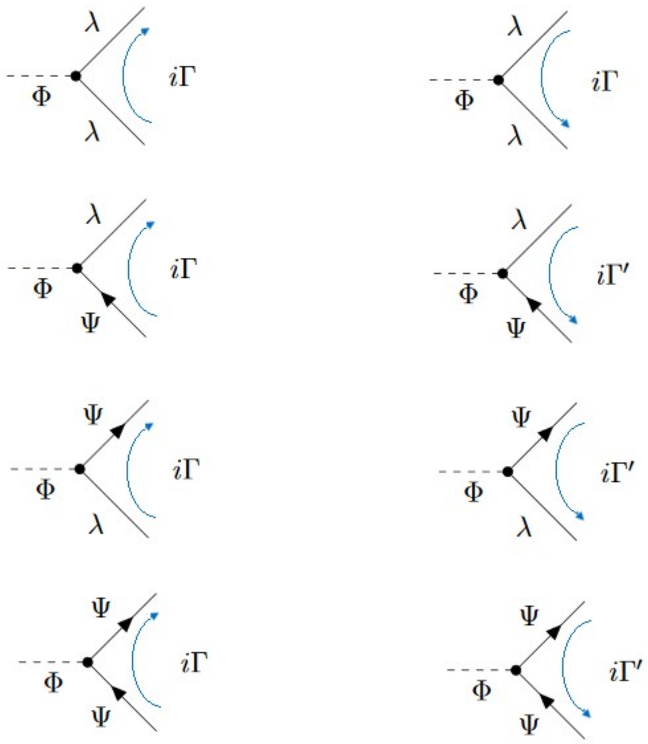

Figure B2.2: Feynman rules for fermionic vertices with orientation (blue arrows).

Figure B2.3: Feynman rules for fermion propagators with orientation (blue arrows). The momentum p flows from left to right.

Finally, it is important to note that the analytical expressions are independent of the chosen orientation (fermion flow) as shown explicitly in [73].

B3 Example: Z Decay into two Majorana Neutrinos

Let's discuss first the decay of a general boson into two Majorana fermions with momenta p_1 and p_2 of the form $\phi \longrightarrow \lambda(p_1) + \lambda(p_2)$. We assume the interaction Lagrangian contains the bilinear $\overline{\Psi} F \Psi$, where F is some numerical matrix. We conclude from the explicit form of the Majorana field operator (B.20), that the amplitude will be

$$\mathcal{M} = e^{-i\alpha} \left[\overline{u}(p_1) F v(p_2) - \overline{v}(p_1) F u(p_2)\right] \mathcal{M}_0, \tag{B.25}$$

where $\mathcal{M}_0$ is a factor that comes from the field operator of the initial boson state.
The amplitude (B.25) shows explicitly the Majorana nature of the final states, since both, Ψ and $\overline{\Psi}$, can create a Majorana particle, it is possible that $\lambda(p_1)$ comes from the action of the Ψ field operator, and $\lambda(p_2)$ from $\overline{\Psi}$; but it can also be the other way around. So we have two contributions in (B.25) because of these two possibilities. If we were dealing with Dirac particles, only one of these two terms would be in the amplitude.
Now, using (B.21) in (B.25) we obtain

$$\begin{aligned} \mathcal{M} &= e^{-i\alpha} \left[\overline{u}(p_1) F v(p_2) - u^T(p_1) C^\dagger \gamma_0^\dagger \gamma_0 F \gamma_0 C v^*(p_2)\right] \mathcal{M}_0 \\ &= e^{-i\alpha} \left[\overline{u}(p_1) F v(p_2) - u^T(p_1) C^{-1} F \gamma_0 C v^*(p_2)\right] \mathcal{M}_0, \end{aligned} \tag{B.26}$$

where we have used all the properties of the C matrix described before.
Finally, since each term is ultimately a number, we can write the second term as the transpose of the matrix expression. Thus, the amplitude (B.25) can be expressed as follows

$$\mathcal{M} = e^{-i\alpha}\, \overline{u}(p_1) \left[F + C F^T C^{-1}\right] v(p_2) \mathcal{M}_0. \tag{B.27}$$

We note the explicit matrix C in (B.27), which makes it representation dependent, as we discussed before.
For the Lorentz bilinears, we have [21]

F	1	γ_μ	$\sigma_{\mu\nu}$	$\gamma_\mu\gamma_5$	γ^5
C F$^T C^{-1}$	1	$-\gamma_\mu$	$-\sigma_{\mu\nu}$	$\gamma_\mu\gamma_5$	γ^5

If ϕ corresponds to the Z boson, then $F = \gamma^\mu(a - b\gamma^5)$ and $\mathcal{M}_0 = \epsilon_\mu(k)$, where $\epsilon_\mu(k)$ denotes the polarization vector for the Z boson.
Thus, the amplitude (B.27) for the Z decay into Majorana fermions is written as

$$\mathcal{M} = e^{-i\alpha}\, \overline{u}(p_1) \left[\gamma^\mu(a - b\gamma^5) + C\ (\gamma^\mu(a - b\gamma^5))^T C^{-1}\right] v(p_2) \epsilon_\mu(k). \tag{B.28}$$

But

$$C\ (\gamma^\mu(a - b\gamma^5))^T C^{-1} = aC\gamma^{\mu T} C^{-1} - bC(\gamma^\mu \gamma^5)^T C^{-1} \tag{B.29}$$

and as we just discussed

$$\begin{aligned} C\gamma^{\mu T} C^{-1} &= -\gamma^\mu \\ C(\gamma^\mu \gamma^5)^T C^{-1} &= \gamma^\mu \gamma^5. \end{aligned} \tag{B.30}$$

Thus, amplitude (B.28) is reduced to

$$\mathcal{M} = e^{-i\alpha}\ \overline{u}(p_1) \left[\gamma^\mu(a - b\gamma^5) + \gamma^\mu(-a - b\gamma^5)\right] v(p_2)\epsilon_\mu(k). \tag{B.31}$$

$$\mathcal{M} = e^{-i\alpha}\ \overline{u}(p_1)\gamma^\mu(-2b\gamma^5)v(p_2)\epsilon_\mu(k). \tag{B.32}$$

Then, the polar vector term does not contribute to the Feynman amplitude and the axial vector term gets a contribution two times bigger.

Actually, for the standard Z decay into Dirac fermions, it is well known that [21]

$$\mathcal{M}_Z = \overline{u}(p_1)\gamma^\mu(a - b\gamma^5)v(p_2)\epsilon_\mu(k), \tag{B.33}$$

that leads to the following total decay rate

$$\Gamma_{Z_D} = \frac{M_Z}{12\pi}(a^2 + b^2), \tag{B.34}$$

where the fermion masses have been neglected.

As we can see, the amplitude for the Majorana case (B.32) is exactly the same, up to a non-physical global phase, as the one for the Dirac case (B.33), just by the change $a \to 0$ and $b \to 2b$. So the total decay rate to Majorana fermions can be obtain directly from (B.34) taking into account the above considerations and multiplying it by a factor (1/2!) due to the indistinguishable property of the final Majorana states.

Thus, for the Z decay into Majorana fermions, the total decay rate is given by

$$\Gamma_{Z_M} = \frac{1}{2}\frac{M_Z}{12\pi}(-2b)^2, \tag{B.35}$$

$$\Gamma_{Z_M} = \frac{M_Z}{6\pi}b^2, \tag{B.36}$$

Finally, if we consider the final fermion states as neutrinos, we know, from the SM, that its couplings to the Z boson is purely left-chiral, i.e., $a = b$, so the total decay rate for the Z boson into two Dirac neutrinos (B.34) reduces to

$$\Gamma_{Z_D} = \frac{M_Z}{6\pi} b^2. \tag{B.37}$$

Comparing (B.36) with (B.37) it is immediate that

$$\Gamma_{Z_D} = \Gamma_{Z_M} \tag{B.38}$$

Thus, the Dirac or Majorana nature of the neutrino does not make any difference in the Z boson decay rate. Actually, for non-zero neutrino masses, there will be corrections that depend on the Dirac or Majorana nature of the neutrino. However, for practical purposes this difference is useless since the corrections would contain the factor m_ν/M_Z, which is really tiny.

It is important to note that this result is consequence of the pure left-chiral coupling $(V - A)$ of the neutrino. Maybe, if there were any other type of coupling or new physics involved in the process, the Majorana effect would lead to a measurable difference in the total decay rate, as discussed in appendix A.

We therefore have to look for other kind of signatures in order to distinguish between the Dirac or Majorana nature of neutrinos. A way to do that, specially important for this thesis, is the search for Majorana neutrino effects in the muon and tau decay, due to new non-standard couplings and the existence of new high energy physics that may be detectable in near future experiments, see e.g. [11, 12].

Finally, let us now obtain the amplitude of the Z decay process using the Feynman rules for Majorana fermions described before, in order to verify the consistency of both methods. The possible tree level Feynman diagrams for this process are shown in Fig.B3.1,

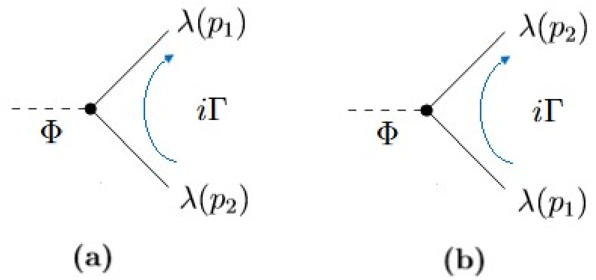

Figure B3.1: Possible Feynman diagrams for the $\phi \longrightarrow \lambda(p_1) + \lambda(p_2)$ process with a fix orientation (blue arrows)

where we have two distinct contributions due to the Majorana nature of the final particles. Now, to construct the amplitude, we just insert the appropriate analytic expression as given in Figs. B2.1, B2.2 and B2.3 corresponding to the chosen fermion flow. Thus

$$i\mathcal{M}' = i\overline{u}(p_1)\Gamma v(p_2) - i\overline{u}(p_2)\Gamma v(p_1), \tag{B.39}$$

where the minus sign is due to the permutation parity of the spinors and the reference order of the external fermions has been chosen as $(p_1; p_2)$.
If we choose the other possible orientation, as shown in Fig. B3.2 , we will have

$$i\mathcal{M}'' = -i\overline{u}(p_2)\Gamma v(p_1) + i\overline{u}(p_1)\Gamma v(p_2). \tag{B.40}$$

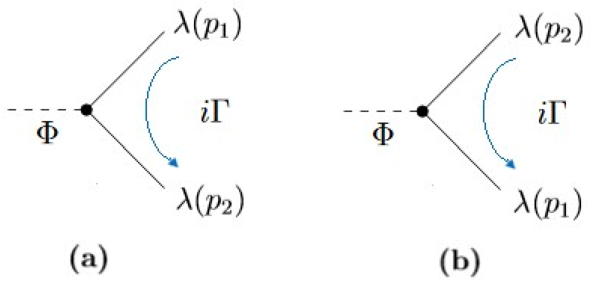

Figure B3.2: Possible Feynman diagrams for the $\phi \longrightarrow \lambda(p_1) + \lambda(p_2)$ process with a fix orientation (blue arrows).

So, as we can see immediately, $\mathcal{M}' = \mathcal{M}''$. Thus, as we discussed before, the amplitude is independent of the chosen orientation of the fermion chain.

Taking $\mathcal{M}'$ with Γ defined as (B.22), we have

$$\mathcal{M}' = \overline{u}(p_1)h_i^c\Gamma^i\phi_c v(p_2) - \overline{u}(p_2)h_i^c\Gamma^i\phi_c v(p_1), \tag{B.41}$$

where for the Z decay $h_i^c\Gamma^i\phi_c = \gamma^\mu(a - b\gamma^5)\epsilon_\mu(k)$, which in terms of the notation used at the beginning of this subsection, $h_i^c\Gamma^i\phi_c = F\mathcal{M}_0$. This leads to the following amplitude

$$\mathcal{M}' = [\overline{u}(p_1)Fv(p_2) - \overline{u}(p_2)Fv(p_1)]\mathcal{M}_0, \tag{B.42}$$

Thus

$$\begin{aligned}\mathcal{M}' &= \left[\overline{u}(p_1)Fv(p_2) - v^T(p_2)C^\dagger\gamma_0^\dagger\gamma_0 F\gamma_0 Cu^*(p_1)\right]\mathcal{M}_0 \\ &= \left[\overline{u}(p_1)Fv(p_2) - v^T(p_2)C^{-1}F\gamma_0 Cu^*(p_1)\right]\mathcal{M}_0 \\ &= \left[\overline{u}(p_1)Fv(p_2) - u^\dagger(p_1)C^T\gamma_0^T F^T(C^{-1})^T v(p_2)\right]\mathcal{M}_0 \\ &= \left[\overline{u}(p_1)Fv(p_2) + \overline{u}(p_1)CF^TC^{-1}v(p_2)\right]\mathcal{M}_0,\end{aligned} \tag{B.43}$$

where in the third line we used the fact that each term is ultimately a number, so we can write it as the transpose of the matrix expression and also we have used all the properties of the C matrix described before.

Finally,

$$\mathcal{M}' = \overline{u}(p_1)\left[F + CF^TC^{-1}\right]v(p_2)\mathcal{M}_0. \tag{B.44}$$

Comparing (B.44) with (B.27), we see that both amplitudes are the same, up to a non-physical global phase. So both of them would lead to the same observables; ensuring the consistency of both methods.