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Hamiltonian first-order gravity in terms of manifestly Lorentz-covariant phase-space variables

by

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Gravedad de primer orden Hamiltoniana en términos de variables de espacio fase manifiestamente covariantes de Lorentz

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THESIS

“Hamiltonian first-order gravity in terms of manifestly Lorentz-covariant phase-space variables”

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To my friends,
to whom I owe a lot.

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Abstract

Loop quantum gravity is one of the main proposals in the search for a quantum theory of gravity. Its starting point is the Hamiltonian formulation of general relativity encompassed by the Ashtekar-Barbero variables. This formulation describes gravity as an $SU(2)$ [or $SO(3)$] invariant theory, which translates into a quantum description without the Lorentz invariance. The absence of the Lorentz symmetry is a direct consequence of a partial gauge fixing implemented during the classical Hamiltonian analysis.

In this work we explore the Hamiltonian formulation of two different actions, the Holst action and a BF-type action with the Barbero-Immirzi parameter. Both actions describe general relativity in the first-order formalism. During their usual Hamiltonian analysis, we found the presence of second-class constraints which we explicitly solve. We do it without resorting to any gauge fixing and in a manifestly Lorentz-covariant fashion. Later, thanks to the use of canonical transformations, we obtain different Hamiltonian formulations for general relativity, all of them exposing their Lorentz-covariant nature explicitly. With the Lorentz symmetry intact, we explore two different gauge fixings, one that allows us to land at the usual Ashtekar-Barbero formulation and one that leads us to a new description invariant under $SU(1,1)$ [$SO(2,1)$] transformations. Finally, we present a new method that bypasses the appearance of second-class constraints from the very beginning, which simplifies the Hamiltonian analysis considerably.

Resumen

La gravedad cuántica de lazos es una de las principales propuestas en busca de una teoría cuántica de la gravedad. Su punto de partida es la formulación Hamiltoniana de la relatividad general descrita por las variables de Ashtekar-Barbero. Esta formulación describe la gravedad como una teoría invariante ante transformaciones locales del grupo $SU(2)$ [ó $SO(3)$], lo cual se traduce en una descripción cuántica sin la invarianza de Lorentz. La ausencia de la simetría de Lorentz es una consecuencia directa de una fijación de norma parcial que se implementa a nivel clásico durante el análisis Hamiltoniano.

En este trabajo exploramos la formulación Hamiltoniana de dos acciones diferentes, la acción de Holst y una acción tipo BF con el parámetro de Barbero-Immirzi. Ambas acciones describen la relatividad general en el formalismo de primer orden. Durante su análisis Hamiltoniano usual aparecen constricciones de segunda clase las cuales resolvemos explícitamente. Además, lo hacemos sin recurrir a ninguna fijación de norma y de una manera manifiestamente covariante de Lorentz. Luego, gracias al uso de transformaciones canónicas, obtenemos diferentes formulaciones Hamiltonianas para la relatividad general, todas ellas mostrando su naturaleza covariante de Lorentz de manera explícita. Con la simetría de Lorentz intacta, exploramos dos fijaciones de norma diferentes, una que nos permite llegar a la formulación usual de Ashtekar-Barbero y otra que nos lleva a una nueva descripción invariante ante transformaciones del grupo $SU(1,1)$ [ó $SO(2,1)$]. Finalmente, presentamos un nuevo método que evita, desde el principio del análisis, la aparición de constricciones de segunda clase, lo cual simplifica el análisis Hamiltoniano considerablemente.

Contents

Chapter 1

Introduction

1.1 Context

Until now, the general theory of the relativity of motion (general relativity for short) gives the best-known description of gravity. It was formulated by Albert Einstein in 1915 [1, 2], and it describes gravity as a spacetime deformation rather than an action-at-a-distance force. The theory has surpassed every experimental test so far [3], from the bending of light as it moves through the spacetime to the discovery of gravitational waves [4]. Nonetheless, this description of gravity is incomplete since it breaks down at the singularity points, such as the center of a black hole or the big bang. The incompleteness of the theory embarks us on the mission to find a more general perspective that embodies the complete nature of gravity. Here it is where the quest for a quantum theory of gravity begins.

Among the different proposals that attempt to describe the quantum nature of gravity,[1] we found those that try to implement the canonical quantization program. The first efforts towards this approach began when Arnowitt, Deser, and Misner provided the first Hamiltonian formalism for general relativity [6] (commonly referred to as the ADM formulation). They decomposed the spacetime manifold into spacelike leaves to foliate it along a timelike direction. The foliation is characterized by four fields, the lapse function and the shift vector, which act as Lagrange multipliers in the Hamiltonian framework. Therefore, they impose restrictions on the phase-space variables for general relativity. The restrictions are categorized—according to Dirac's criteria [7, 8]—as first-class constraints, and are the gauge generators responsible for the spacetime diffeomorphism symmetry. Despite being widely used in numerical general relativity, the development of a quantum theory along the ADM road ended due to theoretical and technical complications [9].

The canonical path resurged in the mid-80s when Ashtekar performed a complex canonical transformation from the $SO(3)$-ADM variables to the now known Ashtekar

[1] See Ref. [5] for a discussion of the different approaches.

variables [10, 11]. The Ashtekar formulation offers several advantages over its predecessor. First, we have the geometric nature of the canonical variables. In particular, the configuration variable is an $SO(3,\mathbb{C})$-valued connection, and thus, it motivates the use of loop variables [12, 13]. Second, the first-class constraints in the Ashtekar formalism are much more manageable than the former case leading to a more appealing formulation. Third, the Ashtekar formalism incorporates local Lorentz transformations as part of the gauge symmetries of the theory. Although the Ashtekar formulation exposed new insights into the quantum character of gravity, due to the complex nature of the phase-space variables, it is necessary to introduce reality conditions to recover a real description. When the conditions are implemented at the classical level we lose the advantages of the Ashtekar complex variables [14], whereas at the quantum domain the conditions are too challenging to handle. Thus, this road was abandoned as well.

As the use of connection variables appeared to be fruitful for the canonical quantization program, the attention then focused on first-order formulations of gravity. Here, we can employ an orthonormal tetrad field and an internal Lorentz connection to describe the dynamics of general relativity with first-order equations. We can derive such formulation from the Palatini (or Einstein-Cartan) action, and incorporate it into the canonical quantization program. However, it is during the Hamiltonian analysis that other restrictions among the phase-space variables appear, they are known—again, in Dirac's terminology [7, 8]—as second-class constraints, and, unlike the first-class constraints, we need to get rid of them to continue with the canonical quantization.

We can deal with the second-class constraints using a partial gauge fixing that reduces the internal Lorentz group into its compact subgroup $SO(3)$ [15]. As a consequence, the second-class constraints become easier to solve, and the ensuing formulation results invariant under $SO(3)$ rotations. This description is precisely the $SO(3)$-ADM formulation and uses $SO(3)$ vectors as its canonical variables [16]. Therefore, the techniques of the Ashtekar approach are unfitting for this description.

Finally, in 1995, following the steps and ideas of Ashtekar, Barbero implemented a real canonical transformation from the $SO(3)$-ADM formalism. He obtained a formulation characterized by a real $SO(3)$ [or $SU(2)$] connection as a configuration variable and a densitized triad field as its associated momentum [17]. The clear geometrical meaning of the canonical variables allowed the use of the quantization techniques of Ashtekar complex formalism. This Hamiltonian description received the name of the Ashtekar-Barbero formulation, and it became the starting point into what is known as *loop quantum gravity*

[18–21]. Nevertheless, in the Ashtekar-Barbero description, the first-class constraints are not as simple as they are in their complex counterpart, and the canonical conjugated variables are $SO(3)$ $[SU(2)]$ covariant fields as opposed to the Lorentz-covariant fields of the Ashtekar formalism.

Loop quantum gravity has unraveled essential results about the quantum nature of gravity, like the discreteness of the spacetime [22] and the occurrence of a "big bounce" that avoids the big bang singularity [23–26]. Also, it has been possible to derive the entropy associated with a black hole [27–31]. However, despite the promising results of loop quantum gravity, the theory is not complete yet. One of the main concerns is the absence of the Lorentz symmetry in the quantum domain. If it is one of the fundamental symmetries of nature, we must find a way to incorporate it back to the formalism.

1.2 Motivation

One way to promote the Lorentz symmetry into the quantum realm is to avoid any gauge fixing at the classical level. Hence, we must deal with the second-class constraints without spoiling the Lorenz invariance. The task was accomplished in a manifestly covariant fashion for the 4-dimensional Palatini action [32]. It resulted in a Hamiltonian description formed solely by first-class constraints with Lorentz vectors as their canonical conjugated variables. Nevertheless, no further results have arisen from here.

In the next year after Barbero's work, Holst presented a different path to derive the Ashtekar-Barbero formulation [33]. He proposed a new action principle from which the Ashtekar-Barbero formalism emerges after a gauge fixing. The Holst action (as it was later coined) is a first-order description that, at the Lagrangian standpoint, renders the same dynamics as the Palatini action. Thus, both actions are equivalent, at least from the classical viewpoint.

Holst action leads to a new and alternative Hamiltonian framework for general relativity. Despite not being exempt from the appearance of second-class constraints, some progress has been made in this direction. Barros e Sá dealt with the second-class constraints of the Holst action by explicitly solving them [34]. He addressed the problem without resorting to any gauge fixing. However, to simplify the solution of the second-class constraints, he split the internal symmetry group. Although it does not break the Lorentz invariance, the lack of manifest Lorentz covariance makes the ensuing formulation quite cumbersome to manage.

On the other hand, Alexandrov and collaborators faced the problem of the second-class constraints from a different, but equivalent, perspective. They introduced the so-called Dirac bracket and ended up with a description with manifestly Lorentz-covariant variables [35–37]. Nonetheless, some of the variables that label the phase space in their Hamiltonian formulation do not commute with each other. Therefore, the implementation of this description into the quantization program might be troublesome.

A few years later, Cianfrani and Montani attempted to promote the Lorentz invariance into the quantum regime with a different solution for the second-class constraints [38]. Their approach followed similar ideas to those of Barros e Sá. However, their solution turned out to be incomplete. Thus, it misleads to an incorrect Hamiltonian description.

Due to the difficulties of these three approaches (the one from Barros e Sá, the one from Alexandrov and collaborators, and the one from Crianfrani and Montani), we have been unable to implement the complete (4-dimensional) Lorentz symmetry into the quantum realm. Nevertheless, if we consider an alternative gauge fixing, we can derive a Hamiltonian formulation invariant under the 3-dimensional Lorentz group [39]. Thus, in principle, we can explore some of the Lorentz symmetry into the quantum domain. Although some interesting results have been exposed in this direction by Ref. [39], they did not present the complete Hamiltonian description. In particular, the form of the Hamiltonian constraint is still missing.

1.3 Outline

This dissertation deals with the second-class constraints of the Holst action in a manifestly Lorentz-covariant fashion. Along the way, we tackle some of the issues enlisted above and disclose the complete Hamiltonian picture for Holst action.

We begin our discussion in Chapter 2, where we start with the Hamiltonian analysis of the Holst action. Here, we sketch the key features of Dirac's method for constrained systems and classify the constraints that arise during the formalism. We also solve the second-class constraints, but in a way similar to Cianfrani and Montani. With the correct solution, we fix their mistake and obtain a formulation described by a noncanonical symplectic structure. Then, with the suitable Darboux map, we connect our formulation with the one found by Barros e Sá. We finish this chapter showing how the noncanonical and canonical descriptions are reduced to the Ashtekar-Barbero formulation.

Next, in Chapter 3, we explicitly solve the second-class constraints in a manifestly Lorentz-covariant fashion and, employing canonical transformations, we derive three alternative formulations that maintain their explicit Lorentz covariance. At the end, we conclude this chapter by exposing a gauge fixing in all the previous formulations. We observe that they collapse either to the Ashtekar-Barbero formalism or to the $SO(3)$-ADM description.

In Chapter 4, we explore a different gauge fixing, one that reduces the Lorentz group to its subgroup $SU(1,1)$ [or $SO(2,1)$]. The remnant formulation comes straightforwardly thanks to the explicit covariant nature of the variables involved. Remarkably, the description invariant under local $SU(1,1)$ [or $SO(2,1)$] transformations resembles the formulation of Ashtekar-Barbero. The form of the Hamiltonian constraints is the same as the $SO(3)$ Ashtekar-Barbero formulation.

Another type of classical formulations of interest for the quantization program of gravity are the BF formulations for general relativity. They are the starting point in the covariant version of loop quantum gravity known as the spin foam models [40–42]. Moreover, they are known to be related to the Ashtekar original variables in the Hamiltonian framework [43]. Thus, in Chapter 5, we describe the Hamiltonian analysis for a real BF-type action that, at the Lagrangian level, is equivalent to the Holst action. Hence, we make contact between the BF descriptions for gravity and the Ashtekar-Barbero formalism.

In Chapter 6, we use what we learned in the previous chapters about the structure of the phase space, and we develop a method to bypass the introduction of second-class constraints from the very beginning. We do it while maintaining the complete Lorentz-covariant nature of the variables. Furthermore, we generalize the canonical transformations enlisted in Chapter 3 with a two-parameter family of canonical transformations.

Finally, in Chapter 7, we conclude with some final remarks and discuss future implications for the obtained results. In addition, we devote Appendix A to define the notation used throughout the entire document. On the other hand, in Appendix B, we enlist the contributions of this dissertation.

Chapter 2

Revisiting the Hamiltonian formalism of Holst action

In this chapter we introduce the reader to the Hamiltonian formulation of general relativity. We start with the Holst action with a cosmological constant term and use Dirac's method for constrained systems [7, 8]. Along the way, we encounter the presence of second-class constraints, which we solve without resorting to any gauge fixing. The solution leads us to a noncanonical symplectic structure. Then, with the proper Darboux map, we construct a Hamiltonian description labeled with canonical conjugated variables. Finally, at the end of this chapter, we impose the gauge fixing known as "*time gauge*" and arrive at the Ashtekar-Barbero formulation for general relativity.

The analysis and results of this chapter were published in Ref. [44].

2.1 The Holst action

In the first-order formalism, gravity is described by a set of four orthonormal 1-forms (cotetrads) and an internal connection. Let M represent the spacetime manifold. Then, at each point of it, we denote the cotetrad field by e^I and the connection with $\omega^I{}_J$. The indices I, $J, \dots$ take the values $\{0, 1, 2, 3\}$. They are group indices and are lowered and raised with the internal metric $(\eta_{IJ}) = \text{diag}\,(\sigma, 1, 1, 1)$, being $\sigma = \pm 1$. We work with the internal group $SO(3, 1)$ when $\sigma = -1$ and $SO(4)$ if $\sigma = 1$. They are the Lorentz and Euclidean groups, respectively; $\omega^I{}_J$ is the group-valued connection compatible with the metric, $d\eta_{IJ} - \omega^K{}_I \eta_{KJ} - \omega^K{}_J \eta_{IK} = 0$, and thus, $\omega_{IJ} = -\omega_{JI}$.

After the conventions mentioned above, we consider the action

$$S[e, \omega] = \frac{1}{\kappa} \int_M \left[* \left(e^I \wedge e^J \right) \wedge F_{IJ} + \frac{\sigma}{\gamma} e^I \wedge e^J \wedge F_{IJ} - 2\Lambda\rho \right], \tag{2.1}$$

where $\kappa = 16\pi G$ (G as Newton's gravitational constant). The first term inside the square brackets represents de Palatini action, with the asterisk being the internal dual [see (A.2)] and $F^I{}_J$ denotes the curvature of the connection $\omega^I{}_J$

$$F^I{}_J := d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J. \tag{2.2}$$

The second term in (2.1) is the Holst term [33] coupled through the Barbero-Immirzi parameter γ [45]. Also, the last term in (2.1) involves the cosmological constant Λ and the volume form $\rho = (1/4!)\epsilon_{IJKL}e^I \wedge e^J \wedge e^K \wedge e^L$, with ϵ_{IJKL} the totally antisymmetric $SO(3,1)$ [or $SO(4)$] tensor ($\epsilon_{0123} = 1$).

The action in Eq. (2.1) describes general relativity with a cosmological constant Λ. The Barbero-Immirzi parameter γ in (2.1) drops out from the equations of motion. In fact, the Holst term is said to be of topological nature since it does not propagate any physical degree of freedom [46]. Although the Barbero-Immirzi parameter drops out classically; at the quantum regime its significance is unclear because it appears in the spectra of the area and volume operators, and on the formula of the black hole entropy [18, 47].

Furthermore, the action (2.1) is invariant under spacetime diffeomorphisms and local Lorentz (Euclidean) transformations. They constitute the distinctive symmetries of general relativity in the first-order formalism.[1]

2.2 Classification of the constraints

Before we begin with the Hamiltonian description, we introduce the γ-hat notation defined in Eq. (A.4). Using this notation, action (2.1) acquires the form

$$S[e,\omega] = \frac{1}{\kappa}\int_M \left[* \left(e^I \wedge e^J \right) \wedge \overset{(\gamma)}{F}_{IJ} - 2\Lambda\rho \right]. \tag{2.3}$$

Next, we define the notion of evolution and choose a coordinate with respect to which the system evolves. For this reason, we assume that the spacetime manifold M is diffeomorphic to $\mathbb{R} \times \Sigma$, with Σ a 3-dimensional spacelike manifold without boundary, $\partial\Sigma = 0$. Then, we foliate the spacetime with spacelike surfaces Σ_t for every $t \in \mathbb{R}$. Each Σ_t is diffeomorphic to Σ, and every point $p \in M$ is labeled with the coordinates $\{x^\alpha\} = \{t, x^a\}$, where $\{x^a\}$ label the points on Σ_t. In this adapted coordinates, the differential forms are

$$e^I = e_\mu{}^I dx^\mu = e_t{}^I dt + e_a{}^I dx^a, \tag{2.4a}$$

[1] For a new perspective on the symmetries of first-order general relativity, see Refs. [48–50].

$$\omega^I{}_J = \omega_\mu{}^I{}_J dx^\mu = \omega_t{}^I{}_J dt + \omega_a{}^I{}_J dx^a, \tag{2.4b}$$
$$F^I{}_J = \frac{1}{2} F_{\mu\nu}{}^I{}_J dx^\mu \wedge dx^\nu = F_{ta}{}^I{}_J dt \wedge dx^a + \frac{1}{2} F_{ab}{}^I{}_J dx^a \wedge dx^b. \tag{2.4c}$$

The splitting of the spacetime indices into "space" indices, $a = \{1, 2, 3\}$, and a "time" direction does not break any of the general relativity symmetries since the splitting of the indices is arbitrary.

Also, to describe the foliation, it is convenient to define an internal vector orthogonal to $e_a{}^I$. We denote it by n_I and demand it to be normalized to σ, so it is a timelike vector in the Lorentzian case. The two properties: $n_I e_a{}^I = 0$ and $n_I n^I = \sigma$, are enough to determine the explicit form of n_I

$$n_I = \frac{1}{6\sqrt{q}} \epsilon_{IJKL} \tilde{\eta}^{abc} e_a{}^J e_b{}^K e_c{}^L, \tag{2.5}$$

with $\tilde{\eta}^{abc}$ being the totally antisymmetric tensor density ($\tilde{\eta}^{123} = +1$) and $q = \det(q_{ab}) > 0$ (of weight +2) being the determinant of the induced metric, $q_{ab} := \eta_{IJ} e_a{}^I e_b{}^J$, on Σ_t.

After the splitting of the local indices, the action acquires the form

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \Big[\tilde{\eta}^{abc} e_a{}^I e_b{}^J * \overset{(\gamma)}{F}_{tcIJ} + \tilde{\eta}^{abc} e_t{}^I e_a{}^J * \overset{(\gamma)}{F}_{bcIJ} - 2\sqrt{q} e_t{}^I n_I \Big], \tag{2.6}$$

where we omitted the wedge product between dt and $d^3x := dx^1 \wedge dx^2 \wedge dx^3$. Also, from Eqs. (2.2), (2.4b), and (2.4c), we have

$$F_{\mu\nu}{}^I{}_J = \partial_\mu \omega_\nu{}^I{}_J - \partial_\nu \omega_\mu{}^I{}_J + \omega_\mu{}^I{}_K \omega_\nu{}^K{}_J - \omega_\nu{}^I{}_K \omega_\mu{}^K{}_J. \tag{2.7}$$

Next, we reparametrize the four fields $e_t{}^I$ with the usual lapse function N and the shift vector N^a of the ADM formalism

$$e_t{}^I = N n^I + N^a e_a{}^I. \tag{2.8}$$

The reparametrization of the cotetrad $e_\mu{}^I$ and an integration by parts lead us to the action

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[\tilde{\Pi}^{aIJ} \partial_t \overset{(\gamma)}{\omega}_{aIJ} - \tilde{H} + \partial_a \left(\tilde{\Pi}^{aIJ} \overset{(\gamma)}{\omega}_{tIJ} \right) \right], \tag{2.9}$$

where we identified $\overset{(\gamma)}{\omega}_{aIJ}$ as the configuration variable and defined its associated momentum as

$$\tilde{\Pi}^{aIJ} := \frac{1}{2}\tilde{\eta}^{abc}\epsilon^{IJ}{}_{KL}e_b{}^K e_c{}^L. \tag{2.10}$$

The boundary term in the action (2.9) is a direct consequence of the integration by parts. However, we will neglect it since $\partial\Sigma = 0$. Also, in the action (2.9) we have the Hamiltonian density

$$\tilde{H} := -\omega_{tIJ}\tilde{\mathcal{G}}^{IJ} + N^a\tilde{\mathcal{V}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}}, \tag{2.11}$$

with

$$\begin{aligned} \tilde{\mathcal{G}}^{IJ} &:= D_a \overset{(\gamma)}{\tilde{\Pi}}{}^{aIJ} = \partial_a \overset{(\gamma)}{\tilde{\Pi}}{}^{aIJ} + 2\omega_a{}^{[I|}{}_K \overset{(\gamma)}{\tilde{\Pi}}{}^{aK|J]}, & \text{(2.12a)}\\ \tilde{\mathcal{V}}_a &:= \tilde{\Pi}^{bIJ} \overset{(\gamma)}{F}{}_{abIJ}, & \text{(2.12b)}\\ \tilde{\tilde{\mathcal{S}}} &:= \tilde{\Pi}^{aIK}\tilde{\Pi}^b{}_K{}^J \overset{(\gamma)}{F}{}_{abIJ} + 2\sigma q\Lambda. & \text{(2.12c)} \end{aligned}$$

Notice that we used the antisymmetrizer notation defined in (A.1b). Since ω_{tIJ}, N^a, and $\underset{\sim}{N} := q^{-1/2}N$ appear linearly in the action, they act as Lagrange multipliers. Therefore, they impose the constraints: $\tilde{\mathcal{G}}^{IJ} \approx 0$, $\tilde{\mathcal{V}}_a \approx 0$, and $\tilde{\tilde{\mathcal{S}}} \approx 0$. Here the symbol "$\approx$" stands for a weak equality, it means that the equality is valid only on the constraint surface (see Ref. [7,8]).

Thanks to the properties enlisted in (A.7a), we see from (2.9) that to work with the canonical pair $(\overset{(\gamma)}{\omega}{}_{aIJ}, \tilde{\Pi}^{aIJ})$ or with $(\omega_{aIJ}, \overset{(\gamma)}{\tilde{\Pi}}{}^{aIJ})$ is equivalent to each other.[2] Although we should express the Hamiltonian in terms of any of these pairs, we mixed the notation to show the constraints in its simplest form, thanks to the fact that the only difference between these variables is the internal projector $P^{IJ}{}_{KL}$. However, to have an appropriate Hamiltonian description, we need to relate q with the canonical variables. Using (2.10) we derive the relation

$$qq^{ab} = \frac{\sigma}{2}\tilde{\Pi}^{aIJ}\tilde{\Pi}^b{}_{IJ}, \tag{2.13}$$

where q^{ab} stands for inverse of q_{ab} ($q_{ac}q^{cb} = \delta_a^b$). At this point, the entire action (2.9) is ultimately described by the canonical pair $(\overset{(\gamma)}{\omega}{}_{aIJ}, \tilde{\Pi}^{aIJ})$, and it obeys the fundamental Poisson bracket

$$\left\{\overset{(\gamma)}{\omega}{}_{aIJ}(t,x), \tilde{\Pi}^{bKL}(t,y)\right\} = \kappa\delta_a^b\delta_{[I}^K\delta_{J]}^L\delta^3(x,y), \tag{2.14}$$

with $\delta^3(x,y)$ being the 3-dimensional Dirac delta for the points $x, y \in \Sigma_t$.

[2] This apparent ambiguity disappears once we solve the second-class constraints. Also, if we did not define the momentum $\tilde{\Pi}^{aIJ}$ the ambiguity does not show up, see Chapter 6.

Our description is not complete yet, the definition of the canonical momenta in Eq. (2.10) defines 18 variables $\tilde{\Pi}^{aIJ}$ constructed out of the 12 components of $e_a{}^I$. This mismatch imply the existence of the six primary constraints:

$$\tilde{\tilde{\Phi}}^{ab} := *\tilde{\Pi}^{aIJ}\tilde{\Pi}^{b}{}_{IJ} \approx 0. \tag{2.15}$$

Consequently, following Dirac's method [7, 8],[3] we must preserve the constraint (2.15) under time evolution. Thus, we impose $\partial_t\tilde{\tilde{\Phi}}^{ab} \approx 0$. Computing $\partial_t\tilde{\tilde{\Phi}}^{ab} \approx 0$, using either the equations of motion or the Poisson Bracket, results in

$$\partial_t\tilde{\tilde{\Phi}}^{ab} = \left\{\tilde{\tilde{\Phi}}^{ab}, \tilde{H}\right\} \approx 2\underset{\sim}{N}\Psi^{ab} \approx 0, \tag{2.16}$$

where Ψ^{ab} is a tensor density of weight $+3$ given by

$$\Psi^{ab} := \epsilon_{IJKL}\tilde{\Pi}^{(a|IM}\tilde{\Pi}^{c}{}_{M}{}^{J}D_c\tilde{\Pi}^{|b)KL}. \tag{2.17}$$

Therefore, to maintain the evolution of the constraint $\tilde{\tilde{\Phi}}^{ab}$, either $\underset{\sim}{N} \approx 0$ or $\Psi^{ab} \approx 0$. Because the former case imply a degenerate spacetime metric (a case outside of our current scope, see Ref. [52] for a lower dimension example) we take the latter case. Thereby, (2.16) implies that Ψ^{ab} is a secondary constraint. The evolution of Ψ^{ab} fixes one of the Lagrange multipliers, and thus, no tertiary restrictions appear. These are all of the constraints of the theory. Notice that the Poisson bracket among $\tilde{\tilde{\Phi}}^{ab}$ and Ψ^{ab} does not vanish on the constraint surface, and we will deal with it in the next section.

After all of the constraints are taken into account, the action now reads

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \Big[\tilde{\Pi}^{aIJ}\partial_t \overset{(\gamma)}{\omega}{}_{aIJ} - \tilde{H}_T\Big], \tag{2.18}$$

with $\tilde{H}_T$ being the total Hamiltonian density

$$\tilde{H}_T := -\omega_{tIJ}\tilde{\mathcal{G}}^{IJ} + N^a\tilde{\mathcal{V}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}} + \underset{\sim}{\phi}_{ab}\tilde{\tilde{\Phi}}^{ab} + \psi_{ab}\Psi^{ab}, \tag{2.19}$$

which incorporates the new constraints $\tilde{\tilde{\Phi}}^{ab}$ and Ψ^{ab} together with their corresponding Lagrange multipliers $\underset{\sim}{\phi}_{ab}$ and ψ_{ab} (of weight -2).

Continuing with the Hamiltonian analysis, we compute the Poisson bracket among the constraints, and we classify them according to Dirac's criteria [7, 8]. The constraints $\tilde{\mathcal{G}}^{IJ}$, $\tilde{\mathcal{V}}_a$, and $\tilde{\tilde{\mathcal{S}}}$, are first class and are known, respectively, as the Gauss, vector, and scalar (or

[3] See also Ref. [51] for more details on the method.

Hamiltonian) constraint. They generate the gauge symmetries of the theory. The Gauss constraint generates local Lorentz (Euclidean) transformations, whereas the vector and scalar constraints are responsible for generating spacetime diffeomorphisms. On the other hand, $\tilde{\tilde{\Phi}}^{ab}$ and Ψ^{ab} are second-class constraints; it means that the Poisson bracket among them does not vanish on the constraint surface. Therefore, second-class constraints must be suitably handled.

The classification of constraints also provides a way to count the number of degrees of freedom (d.o.f.) of the theory. Using the formula [53]:

$$\text{d.o.f.} = \frac{1}{2}\left(\begin{array}{c} \#\ \text{Phase-space} \\ \text{variables} \end{array} - 2\times \begin{array}{c} \#\ \text{First-class} \\ \text{constraints} \end{array} - \begin{array}{c} \#\ \text{Second-class} \\ \text{constraints} \end{array} \right), \quad (2.20)$$

we see that the theory possesses $(1/2)(2\times 18 - 2\times 10 - 12) = 2$ d.o.f. per space point, which is what one expects in general relativity. When we eliminate the second-class constraints from the formalism, the number of degrees of freedom is not altered. Thus, getting rid of the second-class constraints implies a reduction of the number of phase-space variables, and thus, we end up with a smaller phase space.

There are two equivalent ways to deal with the second-class constraints: one is to work with a modified Poisson bracket—the Dirac bracket—that incorporates the second-class constraints in its definition; the other consists in explicitly solving the second-class constraints. During this work, we focus on the second alternative, and we show different ways of solving the second-class constraints for the Holst action. The approach that uses the Dirac bracket is reported in Refs. [35–37].

2.3 Solution of the second-class constraints: noncanonical phase-space variables

Here we get rid of the second-class constraints following the guidance of Refs. [34] and [38]. Thus, we split the internal indices into their *electric* and *magnetic* components. Although the solution reported by Cianfrani and Montani in Ref. [38] is incomplete, we mend their mistake and provide the correct solution. We have already published the results of the upcoming sections; they are found in Ref. [44].

Let us begin by splitting the internal indices into their 0-component and i-components [$i = (1, 2, 3)$]. Then, we notice that the 18 components of $\tilde{\Pi}^{aIJ}$ are divided into the $9 + 9$

variables $\tilde{\Pi}^{ai0}$ and $\tilde{\Pi}^{aij}$, of which, according to (2.15), only 12 of them are independent. Solving the constraint $\tilde{\tilde{\Phi}}^{ab} = 0$ results in

$$\begin{aligned} \tilde{\Pi}^{ai0} &=: \tilde{\Pi}^{ai}, & (2.21a) \\ \tilde{\Pi}^{aij} &= -2\tilde{\Pi}^{a[i}\chi^{j]}. & (2.21b) \end{aligned}$$

The nine components of the tensor density $\tilde{\Pi}^{ai}$ plus the three components of the internal vector χ_i represent the 12 independent variables contained in $\tilde{\Pi}^{aIJ}$; they are going to partially label the coordinates of the points of the phase space. Furthermore, using Eq. (2.13), we can give some geometrical meaning to these variables. Let $\det(\tilde{\Pi}^{ai}) \neq 0$, then, we denote the inverse of $\tilde{\Pi}^{ai}$ with $\underset{\sim}{\Pi}_{ai}$ ($\underset{\sim}{\Pi}_{ai}\tilde{\Pi}^{aj} = \delta_i^j$, $\underset{\sim}{\Pi}_{ai}\tilde{\Pi}^{bi} = \delta_a^b$) and use (2.13) to obtain the relation

$$q_{ab} = \varepsilon |\det(\tilde{\Pi}^{ai})| \Theta^{ij} \underset{\sim}{\Pi}_{ai} \underset{\sim}{\Pi}_{bj}, \tag{2.22}$$

with $\varepsilon := \text{sgn}(1 + \sigma\chi_i\chi^i)$ and

$$\Theta^i{}_j := \delta^i_j + \sigma\chi^i\chi_j. \tag{2.23}$$

Therefore, $\tilde{\Pi}^{ai}$ is a nonorthonormal densitized basis for Σ_t, and χ_i is the deviation that prevents $\tilde{\Pi}^{ai}$ from becoming a densitized orthonormal triad.

Although $\Theta^i{}_j$ is an internal metric for the nonorthogonal basis, we are not using it to lower or raise the internal indices. Instead, we use δ^i_j to deal with the internal indices. Also, to simplify future expressions, it is convenient to employ the internal matrix

$$\vartheta^i{}_j := \left(1 + \sigma\chi_k\chi^k\right)\delta^i_j - \sigma\chi^i\chi_j, \tag{2.24}$$

which is related to $\Theta^i{}_j$ through $\Theta^i{}_j = (1 + \sigma\chi_k\chi^k)\left(\vartheta^{-1}\right)^i{}_j$.

Continuing with the solution of the remaining second-class constraint, $\Psi^{ab} = 0$, a direct substitution of (2.21a) and (2.21b) into the constraint (2.17) leads us to

$$\begin{aligned} \Psi^{ab} = &-4\sigma\epsilon_{ijk}\tilde{\Pi}^{(a|i}\tilde{\Pi}^{ck}\Big[\left(1 + \sigma\chi_l\chi^l\right)\partial_c\tilde{\Pi}^{|b)j} + \vartheta^l{}_m\tilde{\Pi}^{|b)m}\left(\omega_c{}^j{}_l + \sigma\omega_{c0l}\chi^j\right) \\ &-\sigma\tilde{\Pi}^{|b)l}\chi_l\left(\omega_{c0}{}^j + \partial_c\chi^j\right)\Big] = 0. \end{aligned} \tag{2.25}$$

This equality represents a set of six linear equations for the 18 unknowns ω_{a0i} and ω_{aij}. Hence, the solution ought to be parametrized by 12 free variables. Let the electric components of the connection ω_{a0i} be nine of these variables while we introduce $\tilde{Y}^i$ to account for the remaining three. Then, the solution of (2.25) is

$$\begin{aligned}
\omega_{a0i} &= \omega_{a0i}, &\quad (2.26a)\\
\omega_{aij} &= \Omega_{aij} + 2\sigma\omega_{a0[i}\chi_{j]} - 2\underset{\sim}{\Pi}_{ak}\Theta^{k}{}_{[i}\tilde{Y}_{j]}, &\quad (2.26b)
\end{aligned}$$

where Ω_{aij} stands for the particular solution

$$\begin{aligned}
\Omega_{aij} \;=\;& \Theta_{[i|k}\tilde{\Pi}^{b}{}_{|j]}\left(\partial_b\underset{\sim}{\Pi}_a{}^{k} - \partial_a\underset{\sim}{\Pi}_b{}^{k}\right) - \underset{\sim}{\Pi}_a{}^{k}\tilde{\Pi}^{b}{}_{[i|}\partial_b\Theta_{|j]k} - \sigma\chi_{[i|}\partial_a\chi_{|j]}\\
&-\Theta_{[i|k}\tilde{\Pi}^{b}{}_{|j]}\underset{\sim}{\Pi}_a{}^{k}\tilde{\Pi}^{cl}\partial_b\underset{\sim}{\Pi}_{cl} + \Theta^{kl}\underset{\sim}{\Pi}_{ak}\tilde{\Pi}^{b}{}_{[i}\tilde{\Pi}^{c}{}_{j]}\partial_c\underset{\sim}{\Pi}_{bl}. \qquad (2.27)
\end{aligned}$$

Cianfrani and Montani's approach did not consider the existence of the variables $\tilde{Y}^i$, meaning that they provided a particular solution of (2.25) only. Thus, their approach is incorrect simply because their solution is not ultimately equivalent to the constraint $\Psi^{ab} = 0$.

Now, we substitute (2.21a), (2.21b), (2.26a), (2.26b), and (2.27) into (2.18), after some algebra we have

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left(\mu_{ai}\dot{\tilde{\Pi}}^{ai} + \tilde{\nu}_i\dot{\chi}^i + \tilde{\alpha}^{ai}\dot{\omega}_{a0i} + \beta_i\dot{\tilde{Y}}^i - \tilde{H}' + \partial_a\tilde{B}^a\right). \qquad (2.28)$$

This action is composed by several terms. Let us dissect each of the parts that make up the action. First, we have the kinetic terms

$$\mu_{ai}\dot{\tilde{\Pi}}^{ai} + \tilde{\nu}_i\dot{\chi}^i + \tilde{\alpha}^{ai}\dot{\omega}_{a0i} + \beta_i\dot{\tilde{Y}}^i, \qquad (2.29)$$

where μ_{ai}, $\tilde{\nu}_i$, $\tilde{\alpha}^{ai}$, and β_i are functions of the variables ω_{a0i}, $\tilde{\Pi}^{ai}$, χ_i, and $\tilde{Y}^i$ only; they are explicitly given by

$$\begin{aligned}
\mu_{ai} \;:=\;& \vartheta_{ij}\partial_a\chi^j + \partial_a\chi_i + 2\left(1+\sigma\chi^k\chi_k\right)\underset{\sim}{\Pi}_{aj}\chi^j\tilde{Y}_i - 2\underset{\sim}{\Pi}_{aj}\Theta^{j}{}_{i}\chi^k\tilde{Y}_k\\
&-\tilde{\Pi}^{bl}\chi_l\Big[2\Theta^{k}{}_{i}\partial_{[a}\underset{\sim}{\Pi}_{b]k} - 2\Theta^{jk}\underset{\sim}{\Pi}_{aj}\tilde{\Pi}^{c}{}_{i}\partial_{[b}\underset{\sim}{\Pi}_{c]k} + \Theta^{k}{}_{i}\underset{\sim}{\Pi}_{ak}\tilde{\Pi}^{cm}\partial_b\underset{\sim}{\Pi}_{cm}\\
&-2\sigma\underset{\sim}{\Pi}_a{}^{j}\chi_{(j|}\partial_b\chi_{|i)}\Big] - \tilde{\Pi}^{b}{}_{i}\chi_k\Big[2\Theta^{jk}\partial_{[b}\underset{\sim}{\Pi}_{a]j} - \Theta^{jk}\underset{\sim}{\Pi}_{aj}\tilde{\Pi}^{cl}\partial_b\underset{\sim}{\Pi}_{cl}\\
&+2\sigma\underset{\sim}{\Pi}_{aj}\chi^{(j}\partial_b\chi^{k)}\Big] - \frac{2}{\gamma}\epsilon_{ijk}\underset{\sim}{\Pi}_{al}\Theta^{jl}\tilde{Y}^k, \qquad (2.30a)\\
\tilde{\nu}_i \;:=\;& 4\sigma\tilde{\Pi}^{a}{}_{i}\tilde{\Pi}^{bj}\chi_j\chi^k\partial_{[a}\underset{\sim}{\Pi}_{b]k} - 2\sigma\tilde{\Pi}^{a}{}_{[i}\chi_{j]}\chi^j\tilde{\Pi}^{bk}\partial_a\underset{\sim}{\Pi}_{bk} + 4\sigma\tilde{\Pi}^{a}{}_{[i}\chi_{j]}\omega_{a0}{}^{j}\\
&+4\sigma\tilde{\Pi}^{a}{}_{[i|}\chi^j\partial_a\chi_{|j]} + 4\sigma\chi^j\chi_{[i}\tilde{Y}_{j]} - \frac{2\sigma}{\gamma}\epsilon_{ijk}\tilde{\Pi}^{aj}\Big[\tilde{\Pi}^{bk}\partial_a(\chi^l\underset{\sim}{\Pi}_{bl}) + \omega_{a0}{}^{k}
\end{aligned}$$

$$
-\frac{1}{2}\chi^k\tilde{\Pi}^{bl}\partial_a \underset{\sim}{\Pi}_{bl}\Bigg], \tag{2.30b}
$$

$$
\tilde{\alpha}^{ai} := -2\vartheta^i{}_j\tilde{\Pi}^{aj}, \tag{2.30c}
$$

$$
\beta_i := 4\chi_i. \tag{2.30d}
$$

The second part that constitutes the action (2.28) is the first-class Hamiltonian $\tilde{H}'$; it is formed by first-class constraints only

$$
\tilde{H}' := \epsilon_{ijk}\omega_t{}^{jk}\tilde{\mathcal{G}}^i_{\text{rot}} - 2\omega_{ti0}\tilde{\mathcal{G}}^i_{\text{boost}} + N^a\tilde{\mathcal{V}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}}, \tag{2.31}
$$

where the constraints are given by

$$
\begin{aligned}
\tilde{\mathcal{G}}^i_{\text{boost}} &:= \tilde{\mathcal{G}}^{i0} = \partial_a\left(P^i{}_j\tilde{\Pi}^{aj}\right) + \Omega_a{}^i{}_j P^j{}_l\tilde{\Pi}^{al} + 2\sigma\tilde{\Pi}^{a[j}\omega_{a0}{}^{i]}\chi_j \\
&\quad -\frac{\sigma}{\gamma}\epsilon^{ijk}\omega_{a0j}\tilde{\Pi}^a{}_k - \frac{1}{\gamma}\epsilon_{jkl}\omega_{a0}{}^j\tilde{\Pi}^{ak}\chi^l\chi^i + \left(\vartheta^i{}_j + P^i{}_j\right)\tilde{Y}^j, \qquad (2.32\text{a}) \\
\tilde{\mathcal{G}}^i_{\text{rot}} &:= -\frac{1}{2}\epsilon^{ijk}\tilde{\mathcal{G}}_{jk} = \partial_a\left(Q^i{}_j\tilde{\Pi}^{aj}\right) + \Omega_a{}^i{}_j Q^j{}_l\tilde{\Pi}^{al} + 2\frac{\sigma}{\gamma}\tilde{\Pi}^{a[j}\omega_{a0}{}^{i]}\chi_j \\
&\quad -\epsilon^{ijk}\omega_{a0j}\tilde{\Pi}^a{}_k - \sigma\epsilon_{jkl}\omega_{a0}{}^j\tilde{\Pi}^{ak}\chi^l\chi^i + \left(\frac{1}{\gamma}\vartheta^i{}_j + Q^i{}_j\right)\tilde{Y}^j, \qquad (2.32\text{b}) \\
\tilde{\mathcal{V}}_a &= 2\omega_{a0i}\partial_b\left(P^i{}_j\tilde{\Pi}^{bj}\right) + 2\Upsilon_{ai}\partial_b\left(Q^i{}_j\tilde{\Pi}^{bj}\right) - 4P^i{}_j\tilde{\Pi}^{bj}\partial_{[a}\omega_{b]0i} \\
&\quad -4Q^i{}_j\tilde{\Pi}^{bj}\partial_{[a}\Upsilon_{b]i} + 2\omega_{a0i}\mathcal{G}^i_{\text{boost}} + 2\Upsilon_{ai}\mathcal{G}^i_{\text{rot}}, \qquad (2.32\text{c}) \\
\tilde{\tilde{\mathcal{S}}} &= -2\tilde{\Pi}^{ai}\chi_i\tilde{\mathcal{V}}_a - 2\sigma\left(1+\sigma\chi_n\chi^n\right)\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}\Bigg[\frac{\sigma}{\gamma}\partial_a\omega_{b0}{}^k + \partial_a\Upsilon_b{}^k \\
&\quad -\frac{1}{2}\epsilon^{klm}\left(2\frac{\sigma}{\gamma}\omega_{a0l}\Upsilon_{bm} + \sigma\omega_{a0l}\omega_{b0m} + \Upsilon_{al}\Upsilon_{bm}\right)\Bigg] \\
&\quad +2\sigma\Lambda\left|1+\sigma\chi_i\chi^i\right||\det(\tilde{\Pi}^{ai})|. \qquad (2.32\text{d})
\end{aligned}
$$

For the sake of simplicity, we introduced the internal matrices

$$
P^i{}_j := \delta^i_j + \frac{\sigma}{\gamma}\epsilon^i{}_{jk}\chi^k, \tag{2.33}
$$

$$
Q^i{}_j := \frac{1}{\gamma}\delta^i_j + \epsilon^i{}_{jk}\chi^k, \tag{2.34}
$$

and also we defined

$$
\Upsilon_{ai} := \frac{1}{2}\epsilon_{ijk}\left(\Omega_a{}^{jk} + 2\sigma\omega_{a0}{}^j\chi^k - 2\underset{\sim}{\Pi}_{al}\Theta^{lj}\tilde{Y}^k\right). \tag{2.35}
$$

Finally, in the action we find the boundary term

$$\tilde{B}^a := -2\dot{\tilde{\Pi}}^{ai}\chi_i + \frac{1}{\gamma}\epsilon_{ijk}\left(\Theta^{il}\underset{\sim}{\Pi}_{bl}\tilde{\Pi}^{aj}\dot{\tilde{\Pi}}^{bk} - \sigma\dot{\chi}^i\chi^j\tilde{\Pi}^{ak}\right), \tag{2.36}$$

which might be neglected just like the others. Nonetheless, we will keep track of it since it can be reabsorbed in the Darboux map given below.

Notice how the splitting of the indices back in (2.21) force us to split the Gauss constraint into two parts, the part that generates boost transformations $\tilde{\mathcal{G}}^i_{\text{boost}}$ and the one that generates the $SO(3)$ rotations $\tilde{\mathcal{G}}^i_{\text{rot}}$.[4] Since both generators are present, the theory is still invariant under the complete Lorentz (Euclidean) group.

The action (2.28) depends on the Lagrange multipliers ω_{tij}, ω_{ti0}, N^a, and $\underset{\sim}{N}$ as well as on the variables ω_{a0i}, $\tilde{\Pi}^{ai}$, χ_i, and $\tilde{Y}^i$. Therefore, the variables label the coordinates in our phase space, and the quantities μ_{ai}, $\tilde{\nu}_i$, $\tilde{\alpha}^{ai}$, and β_i are the components of a noncanonical symplectic potential. A quantum theory developed from our noncanonical variables might be troublesome because of the lack of canonical variables and by the complicated form of the first-class constraints. Nevertheless, at the classical level, this description is completely equivalent to Einstein's theory.

In spite of having noncanonical variables, this formulation also possesses $(1/2)(24-2\times 10) = 2$ d.o.f per space point [see (2.20)]. Thus, our description is correct. On the other hand, notice that neglecting the variables $\tilde{Y}^i$ yields to an incorrect count in the number of d.o.f. since, to begin with, the number of phase-space variables is odd. We need the variables $\tilde{Y}^i$ to correctly label each point of the phase space; the incompleteness of the solution reported in Ref. [38] leads to an incorrect parametrization of the phase space for general relativity.

2.4 Description with canonical conjugated variables through a Darboux map

For a given noncanonical symplectic structure, Darboux's theorem states that it is always possible to find a set of canonical pairs to label the points of our phase space. Thereby, given the 24 noncanonical variables $\left(\omega_{a0i}, \tilde{\Pi}^{ai}, \chi_i, \tilde{Y}^i\right)$ we can find 12 canonical

[4] Their names are only valid with the Lorentzian signature. In the Euclidean case, it is a linear combination of both constraints the generator of $SO(4)$ rotations.

pairs to render an equivalent description. In fact, we make contact with Barros e Sá's description if we consider the Darboux map

$$
\begin{aligned}
\tilde{\Pi}^{ai} &= \tilde{\Pi}^{ai}, & (2.37a)\\
A_{ai} &= -\gamma \overset{(\gamma)}{\omega}_{a0i} - \gamma\epsilon_{ijk}\chi^{j}\left(\Upsilon_{a}{}^{k} + \frac{\sigma}{\gamma}\omega_{a0}{}^{k}\right), & (2.37b)\\
\chi_i &= \chi_i, & (2.37c)\\
\tilde{\zeta}_i &= -\gamma \overset{(\gamma)}{\omega}_{aij}\,\tilde{\Pi}^{aj}. & (2.37d)
\end{aligned}
$$

The new variables A_{ai} and $\tilde{\zeta}^i$ replace ω_{a0i} and $\tilde{Y}^i$, and they become the new configuration variables for the now canonical momenta $\tilde{\Pi}^{ai}$ and χ_i, correspondingly. To implement the Darboux map into the previous description, we invert (2.37b) and (2.37d)

$$
\begin{aligned}
\omega_{a0i} &= -\left(\vartheta^{-1}\right)_{i}{}^{j}\left\{\frac{1}{\gamma}A_{aj} + \frac{1}{2}\epsilon^{klm}Q_{kj}\Omega_{alm} + \frac{\gamma^2}{2\left(\gamma^2-\sigma\right)}M_{jkl}\underset{\sim}{\Pi}_{a}{}^{k}\Theta^{l}{}_{m}\left[\frac{1}{\gamma}\tilde{\zeta}^{m}\right.\right.\\
&\quad \left.\left. -\tilde{\Pi}^{b}{}_{n}\left(\frac{1}{\gamma}S^{mnp}A_{bp} - T^{mnpq}\Omega_{bpq}\right)\right]\right\}, & (2.38a)\\
\tilde{Y}^{i} &= -\frac{\gamma^2}{2\left(\gamma^2-\sigma\right)}\Theta^{i}{}_{j}\left[\tilde{\Pi}^{a}{}_{k}\left(T^{jklm}\Omega_{alm} - \frac{1}{\gamma}S^{jkl}A_{al}\right) + \frac{1}{\gamma}\tilde{\zeta}^{j}\right], & (2.38b)
\end{aligned}
$$

where we introduced the following internal quantities:

$$
\begin{aligned}
M_{ijk} &:= \delta_{ij}\chi_k - \vartheta_{ik}\chi_j + \frac{1}{\gamma}\epsilon_{ijk} - \frac{\sigma}{\gamma}\epsilon_{ikl}\chi^{l}\chi_j, & (2.39a)\\
S_{ijk} &:= \sigma\epsilon_{ijl}Q^{lm}\left(\vartheta^{-1}\right)_{mk}, & (2.39b)\\
T^{ij}{}_{kl} &:= \delta^{i}_{[k}\delta^{j}_{l]} - \frac{\sigma}{2}\epsilon^{ijm}\epsilon_{klq}\left(\vartheta^{-1}\right)^{np}Q_{mn}Q^{q}{}_{p}. & (2.39c)
\end{aligned}
$$

Substituting (2.38a), (2.38b) together with Eqs. (2.39a)–(2.39c) into all terms that form the action (2.28), we find

$$
S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left(\frac{2}{\gamma}\dot{A}_{ai}\tilde{\Pi}^{ai} + \frac{2}{\gamma}\dot{\chi}_i\tilde{\zeta}^{i} - \tilde{H}'\right). \tag{2.40}
$$

Therefore, it is easy to see that the pairs $(A_{ai}, \tilde{\Pi}^{ai})$ and $(\chi_i, \tilde{\zeta}^i)$ are indeed canonical variables because they obey the fundamental Poisson brackets

$$
\begin{aligned}
\left\{A_{ai}(t,x),\, \tilde{\Pi}^{bj}(t,y)\right\} &= \frac{\kappa\gamma}{2}\delta^{b}_{a}\delta^{j}_{i}\delta^{3}(x,y), & (2.41)\\
\left\{\chi_i(t,x),\, \tilde{\zeta}^{j}(t,y)\right\} &= \frac{\kappa\gamma}{2}\delta^{j}_{i}\delta^{3}(x,y). & (2.42)
\end{aligned}
$$

On the other hand, the first-class Hamiltonian is the same as that given in (2.31), but now the constraints read

$$\begin{aligned}
\tilde{\mathcal{G}}^i_{\text{boost}} &= \partial_a\left(P^i{}_j\tilde{\Pi}^{aj}\right) + \frac{2\sigma}{\gamma}A_{aj}\tilde{\Pi}^{a[i}\chi^{j]} - \frac{\sigma}{\gamma}\tilde{\zeta}_j\chi^j\chi^i - \frac{1}{\gamma}\tilde{\zeta}^i, && \text{(2.43a)}\\
\tilde{\mathcal{G}}^i_{\text{rot}} &= \partial_a\left(Q^i{}_j\tilde{\Pi}^{aj}\right) + \frac{1}{\gamma}\epsilon^i{}_{jk}\left(A_a{}^j\tilde{\Pi}^{ak} - \tilde{\zeta}^j\chi^k\right), && \text{(2.43b)}\\
\tilde{\mathcal{V}}_a &= \frac{4}{\gamma}\tilde{\Pi}^{bi}\partial_{[a}A_{b]i} + \frac{2}{\gamma}\tilde{\zeta}_i\partial_a\chi^i - \frac{2\gamma^2}{\gamma^2-\sigma}\left[\frac{1}{\gamma^2}A_{ai}\left(\tilde{\zeta}^i + \sigma\tilde{\zeta}_j\chi^j\chi^i\right)\right.\\
&\quad -\frac{2\sigma}{\gamma^2}\tilde{\Pi}^{b[i}\chi^{j]}A_{ai}A_{bj} - \frac{\sigma}{\gamma^3}\epsilon_{ijk}\left(\tilde{\Pi}^{bi}A_b{}^j + \tilde{\zeta}^i\chi^j\right)A_a{}^k\\
&\quad \left.+\left(Q^i{}_j\mathcal{G}^j_{\text{boost}} - P^i{}_j\mathcal{G}^j_{\text{rot}}\right)J_{ai}\right], && \text{(2.43c)}\\
\tilde{\tilde{\mathcal{S}}} &= -2\tilde{\Pi}^{ai}\chi_i\tilde{\mathcal{V}}_a - 2\sigma\left(1+\sigma\chi_p\chi^p\right)\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}\left\{\partial_a J_b{}^k - \frac{2}{\gamma}\left(A_{al}+J_{al}\right)J_b{}^k\chi^l\right.\\
&\quad -\frac{\sigma\gamma^2}{2(\gamma^2-\sigma)}\left[\epsilon^{klm}\left(\frac{1}{\gamma^2}A_{al}A_{bm} + \sigma J_{al}J_{bm} + \frac{2}{\gamma^2}A_{al}J_{bm}\right)\right.\\
&\quad \left.\left.+\frac{2}{\gamma}A_{al}J_b{}^l\chi^k + \epsilon^{lmn}J_{al}J_{bm}\chi_n\chi^k\right]\right\} + 2\sigma\Lambda\left|1+\sigma\chi_i\chi^i\right|\,|\det(\tilde{\Pi}^{ai})|, && \text{(2.43d)}
\end{aligned}$$

where we defined

$$J_{ai} := \frac{1}{2}\left(\underset{\sim}{\Pi}_{aj}\tilde{M}^j{}_i + \epsilon_{ijk}\underset{\sim}{\Pi}_a{}^j\tilde{\zeta}^k\right), \tag{2.44}$$

with

$$\begin{aligned}
\tilde{M}_{ij} &:= \frac{2}{\left(1+\sigma\chi_r\chi^r\right)^2}\left[\delta^k_i\delta^l_j - \frac{1}{4}\left(\vartheta^{-1}\right)_{ij}\vartheta^{kl}\right]\epsilon_{kmp}\epsilon_{lnq}\vartheta^{mn}\tilde{f}^{(pq)}, && \text{(2.45)}\\
\tilde{f}^i{}_j &:= \left(1+\sigma\chi_m\chi^m\right)\left\{\epsilon^i{}_{kl}\tilde{\Pi}^{ak}\left[\left(1-\frac{\sigma}{\gamma^2}\right)\underset{\sim}{\Pi}_{bj}\partial_a\tilde{\Pi}^{bl} - \frac{\sigma}{\gamma}\chi^l A_{aj}\right]\right.\\
&\quad \left.+\frac{\sigma}{\gamma^2}\left(\tilde{\Pi}^{ak}A_{ak}\delta^i_j - A_a{}^i\tilde{\Pi}^a{}_j + \tilde{\zeta}^i\chi_j\right)\right\} + \sigma\left(P^i{}_k\tilde{\mathcal{G}}^k_{\text{rot}} + Q^j{}_k\tilde{\mathcal{G}}^k_{\text{boost}}\right)\chi_j. && \text{(2.46)}
\end{aligned}$$

This is almost the same formulation presented by Barros e Sá in Ref. [34]. The main difference between our description and that of Ref. [34] (besides the rescaled phase-space coordinates),[5] is that we have not neglected any of the terms proportional to the Gauss constraint as it was done in such a reference. Furthermore, here we introduced the quantity J_{ai} which allows us to write the Hamiltonian constraint in a simple way. Finally, we remark that the Darboux map given in Eqs. (2.37a)-(2.37d) reincorporates the boundary term $\tilde{B}^a$ into the canonical variables. Thus, this description does not neglect the boundary term

[5] To compare with Barros e Sá's description, we have to make the changes: $\tilde{\Pi}^{ai} \to -\tilde{\Pi}^{ai}$, $A_{ai} \to -\gamma A_{ai}$, and $\zeta_i \to \gamma\zeta_i$. We rescaled the variables in order to make a clear contact with the Ashtekar-Barbero formulation (employing the usual conventions).

$\tilde{B}^a$, although it still neglects the boundary term of the action (2.9).

Sometimes it is customary to work with the diffeomorphism constraint $\tilde{\mathcal{D}}_a$ instead of the vector constraint. This constraint is also first class and generates the spatial diffeomorphisms tangent to Σ_t. To derive the diffeomorphism constraint we use the identity

$$\begin{aligned}\mathcal{G}^i_{\text{boost}} - \frac{\sigma}{\gamma}\mathcal{G}^i_{\text{rot}} \quad = \quad & \left(1 - \frac{\sigma}{\gamma^2}\right)\partial_a\tilde{\Pi}^{ai} + \frac{2\sigma}{\gamma}A_{aj}\tilde{\Pi}^{a[i}\chi^{j]} - \frac{\sigma}{\gamma}\tilde{\zeta}_j\chi^j\chi^i - \frac{1}{\gamma}\tilde{\zeta}^i \\ & -\frac{\sigma}{\gamma^2}\epsilon^i{}_{jk}\left(A_a{}^j\tilde{\Pi}^{ak} - \tilde{\zeta}^j\chi^k\right). \end{aligned} \tag{2.47}$$

Then, we rewrite (2.43c) as

$$\tilde{\mathcal{V}}_a \quad = \quad 2\tilde{\mathcal{D}}_a + \frac{2\gamma^2}{\gamma^2 - \sigma}\left[\left(\frac{1}{\gamma}A_{ai} - J_{aj}Q^j{}_i\right)\mathcal{G}^i_{\text{boost}} + \left(J_{aj}P^j{}_i - \frac{\sigma}{\gamma^2}A_{ai}\right)\mathcal{G}^i_{\text{rot}}\right], \tag{2.48}$$

where we identified the diffeomorphism constraint as

$$\tilde{\mathcal{D}}_a := \frac{2}{\gamma}\tilde{\Pi}^{ai}\partial_{[a}A_{b]i} - \frac{1}{\gamma}A_{ai}\partial_b\tilde{\Pi}^{bi} + \frac{1}{\gamma}\tilde{\zeta}^i\partial_a\chi_i. \tag{2.49}$$

Thus, redefining the Lagrange multipliers that enforce the Gauss constraints

$$\lambda_i \quad := \quad \frac{1}{2}\epsilon_{ijk}\omega_t{}^{jk} + \frac{\gamma^2}{\gamma^2 - \sigma}N^a\left(J_{aj}P^j{}_i - \frac{\sigma}{\gamma^2}A_{ai}\right), \tag{2.50}$$

$$\rho_i \quad := \quad -\omega_{ti0} + \frac{\gamma^2}{\gamma^2 - \sigma}N^a\left(\frac{1}{\gamma}A_{ai} - J_{aj}Q^j{}_i\right), \tag{2.51}$$

our theory is described by the action

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[\frac{2}{\gamma}\tilde{\Pi}^{ai}\dot{A}_{ai} + \frac{2}{\gamma}\tilde{\zeta}^i\dot{\chi}_i - \left(2\lambda_i\tilde{\mathcal{G}}^i_{\text{rot}} + 2\rho_i\tilde{\mathcal{G}}^i_{\text{boost}} + 2N^a\tilde{\mathcal{D}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}}\right)\right]. \tag{2.52}$$

The implementation of this formulation into the quantization program has not been attempted due to the complicated form of the constraints, particularly the scalar constraint. However, it generalizes the Ashtekar-Barbero formulation because it is invariant under Lorentz (Euclidean) transformations. In the next section, we obtain the Ashtekar-Barbero formulation from the current Hamiltonian approach.

2.5 Gauge fixing: time gauge

The formulations enlisted above—either with canonical variables or not—are fully invariant under $SO(3,1)$ $[SO(4)]$ transformations. If we want to make contact with the Ashtekar-Barbero formulation, we need to break this symmetry group down to its compact subgroup $SO(3)$. To do it, we have to eliminate the boost freedom of the theory. We accomplish it when we consider the gauge condition

$$\chi_i = 0, \tag{2.53}$$

since it does not Poisson-commute with the constraint $\tilde{\mathcal{G}}^i_{\text{boost}}$,[6]

$$\left\{\chi_i(t,x), \tilde{\mathcal{G}}^j_{\text{boost}}(t,y)\right\}\Big|_{\chi_i=0} = \frac{1}{\gamma}\delta_i^j \delta^3(x,y). \tag{2.54}$$

Therefore, given that δ_i^j is nonsingular, the imposed-by-hand constraint $\chi_i = 0$ and the constraint $\tilde{\mathcal{G}}^i_{\text{boost}} = 0$ are now second class. Thus, we must get rid of them.

When the condition (2.53) is taken into account, the spatial metric acquires the form [see Eq. (2.22)]

$$q_{ab} = |\det(\tilde{\Pi}^{ai})| \underset{\sim}{\Pi}_{ai} \underset{\sim}{\Pi}_b{}^i. \tag{2.55}$$

Hence, $\tilde{\Pi}^{ai}$ becomes a densitized triad for the spacelike submanifold Σ_t. Furthermore, imposing $\chi_i = 0$ aligns the local time direction with a vector normal to Σ_t [54, 55]. This is why the gauge condition (2.53) receives the name "*time gauge*".

On the other hand, the remaining Gauss constraint obeys the algebra

$$\left\{\tilde{\mathcal{G}}^i_{\text{rot}}(t,x), \tilde{\mathcal{G}}^j_{\text{rot}}(t,y)\right\} = \frac{\kappa\gamma}{2}\epsilon^{ij}{}_k \tilde{\mathcal{G}}^k_{\text{rot}} \delta^3(x,y), \tag{2.56}$$

which is the Lie algebra of the $SO(3)$ [or equivalently $SU(2)$] group. Thus, after solving the second-class constraints (2.53) and $\tilde{\mathcal{G}}^i_{\text{boost}} = 0$, we arrive at a formulation invariant under $SO(3)$ rotations.

To continue, we introduce the covariant derivative compatible with $\tilde{\Pi}^{ai}$

$$\nabla_a \tilde{\Pi}^{bi} := \partial_a \tilde{\Pi}^{bi} + \Gamma^b{}_{ca}\tilde{\Pi}^{ci} - \Gamma^c{}_{ca}\tilde{\Pi}^{bi} + \epsilon^i{}_{jk}\Gamma_a{}^j \tilde{\Pi}^{bk} = 0. \tag{2.57}$$

This definition involves 27 equations for $18+9$ unknowns, which are $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$ and Γ_{ai},

[6] The condition Poisson-commutes with all the other first-class constraints.

respectively. Solving (2.57) results in

$$\Gamma^a{}_{bc} = \frac{1}{2}q^{ad}\left(\partial_b q_{dc} + \partial_c q_{bd} - \partial_d q_{bc}\right), \tag{2.58}$$

$$\Gamma_{ai} = \epsilon_{ijk}\tilde{\Pi}^{bj}\left(\partial_{[b}\underset{\sim}{\Pi}_{a]}{}^k + \underset{\sim}{\Pi}_a{}^{[l|}\tilde{\Pi}^{c|k]}\partial_b\underset{\sim}{\Pi}_{cl}\right). \tag{2.59}$$

This means that $\Gamma^a{}_{bc}$ are the Christoffel symbols for the space metric q_{ab} (2.55), while Γ_{ai} is the spin connection. The field strength of Γ_{ai} is defined as

$$R_{abi} := \partial_a\Gamma_{bi} - \partial_b\Gamma_{ai} + \epsilon_{ijk}\Gamma_a{}^j\Gamma_b{}^k. \tag{2.60}$$

At this point, the analysis bifurcates depending on whether or we consider canonical variables or not to describe the phase space. Let us to analyze each case separately.

Noncanonical variables

In the time gauge, the variable Ω_{aij}, introduced as the particular solution of (2.25) [see Eq. (2.27)], and the spin connection of Eq. (2.59) are related by

$$\Omega_{aij} = -\epsilon_{ijk}\Gamma_a{}^k. \tag{2.61}$$

Then, using (2.32a), (2.53), and (2.57), we solve $\tilde{\mathcal{G}}^i_{\text{boost}} = 0$ and obtain

$$\tilde{Y}_i = \frac{\sigma}{2\gamma}\epsilon_{ijk}\omega_{a0}{}^j\tilde{\Pi}^{ak}. \tag{2.62}$$

Next, we substitute (2.53) and (2.62) into the action (2.28), then we get

$$S = \int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left(\mu_{ai}\dot{\tilde{\Pi}}^{ai} + \tilde{\alpha}^{ai}\dot{\omega}_{a0i} - \tilde{H}' + \partial_a\tilde{B}^a\right). \tag{2.63}$$

Therefore, the theory is described by the phase-space variables ω_{a0i} and $\tilde{\Pi}^{ai}$ only, where the symplectic potential, derived from Eqs. (2.30a) and (2.30c), is

$$\mu_{ai} = -\frac{2\sigma}{\gamma^2}\omega_{b0[i}\tilde{\Pi}^b{}_{j]}\underset{\sim}{\Pi}_a{}^j, \tag{2.64}$$

$$\tilde{\alpha}^{ai} = -2\tilde{\Pi}^{ai}. \tag{2.65}$$

Also, the boundary term of (2.36) is given by

$$\tilde{B}^a = \frac{1}{\gamma}\epsilon_{ijk}\underset{\sim}{\Pi}_b{}^i\tilde{\Pi}^{aj}\dot{\tilde{\Pi}}^{bk}, \tag{2.66}$$

and the constraints that make up the first-class Hamiltonian (2.32b)-(2.32d) are given by

$$\tilde{\mathcal{G}}^i = \left(1-\frac{\sigma}{\gamma^2}\right)\epsilon^i{}_{jk}\tilde{\Pi}^{aj}\omega_{a0}{}^k, \tag{2.67a}$$

$$\tilde{\mathcal{V}}_a = 4\nabla_{[b}\left(\omega_{a]0i}\tilde{\Pi}^{bi}\right) - \frac{\sigma}{\gamma^2-\sigma}\epsilon_{ijk}\tilde{\Pi}^{bi}\underset{\sim}{\Pi}_a{}^j\nabla_b\tilde{\mathcal{G}}^k, \tag{2.67b}$$

$$\begin{aligned}\tilde{\tilde{\mathcal{S}}} &= \sigma\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}R_{ab}{}^k + 2\tilde{\Pi}^{a[i|}\tilde{\Pi}^{b|j]}\omega_{a0i}\omega_{b0j} \\ &\quad -\frac{\sigma\gamma^2}{2\left(\gamma^2-\sigma\right)^2}\tilde{\mathcal{G}}^i\tilde{\mathcal{G}}_i + 2\sigma\Lambda|\det(\tilde{\Pi}^{ai})|.\end{aligned} \tag{2.67c}$$

We omitted the label "rot" in the rotational Gauss constraint because such a distinction is no longer necessary.

The formulation is still described by the noncanonical variables ω_{a0i} and $\tilde{\Pi}^{ai}$. Nevertheless, we can rearrange the kinetic terms as

$$\mu_{ai}\dot{\tilde{\Pi}}^{ai} + \tilde{\alpha}^{ai}\dot{\omega}_{a0i} + \partial_a\tilde{B}^a = \frac{2}{\gamma}\tilde{\Pi}^{ai}\partial_t\left(-\gamma\omega_{a0i} + \Gamma_{ai} - \frac{\sigma}{\gamma}\underset{\sim}{\Pi}_a{}^j\tilde{\Pi}^b{}_{[i|}\omega_{b0|j]}\right). \tag{2.68}$$

From here, it is straightforward to identify the configuration variable

$$A_{ai} := -\gamma\omega_{a0i} + \Gamma_{ai} - \frac{\sigma}{\gamma}\underset{\sim}{\Pi}_a{}^j\tilde{\Pi}^b{}_{[i|}\omega_{b0|j]}. \tag{2.69}$$

If it were not for the last term in the right-hand side of the last equation, the definition of A_{ai} would take the exact form of Barbero's canonical transformation [17]. However, this is not the case because Eq. (2.69) is not a canonical transformation, but rather it is the Darboux map from our noncanonical approach to the canonical pair (A_{ai}, $\tilde{\Pi}^{ai}$). Although we obtained the same connection A_{ai} as Barbero did, we have derived it from a different perspective. Furthermore, imposing the time gauge in the Darboux map of Eq. (2.37b) results in Eq. (2.69). Therefore, the time gauge helps us to identify the Darboux map directly from the action.

Continuing with the analysis, we invert (2.69)

$$\omega_{a0i} = \frac{1}{2\gamma}\left[\frac{2\gamma^2-\sigma}{\gamma^2-\sigma}\delta_a^b\delta_i^j - \frac{\sigma}{\gamma^2-\sigma}\underset{\sim}{\Pi}_a{}^j\tilde{\Pi}^b{}_i\right]\left(\Gamma_{bj} - A_{bj}\right), \tag{2.70}$$

and we substitute ω_{a0i} back into the action (2.63) to get

$$S = \int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[\frac{2}{\gamma}\tilde{\Pi}^{ai}\dot{A}_{ai} - \left(\epsilon_{ijk}\omega_t{}^{jk}\tilde{\mathcal{G}}^i + N^a\tilde{\mathcal{V}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}}\right)\right]. \tag{2.71}$$

Notice that the constraints (2.67a)-(2.67c) acquire the form

$$\tilde{\mathcal{G}}^i = \frac{1}{\gamma}\left(\partial_a \tilde{\Pi}^{ai} + \epsilon^{ijk} A_{aj} \tilde{\Pi}^a{}_k\right), \tag{2.72a}$$

$$\tilde{\mathcal{V}}_a = \frac{2}{\gamma}\tilde{\Pi}^{bi} F_{abi} + 2\left(\Gamma_{ai} - A_{ai}\right)\tilde{\mathcal{G}}^i, \tag{2.72b}$$

$$\begin{aligned}\tilde{\tilde{\mathcal{S}}} = & \frac{1}{\gamma^2}\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}\left[F_{ab}{}^k + \left(\sigma\gamma^2 - 1\right) R_{ab}{}^k\right] + 2\sigma\Lambda|\det(\tilde{\Pi}^{ai})| \\ & -\frac{2}{\gamma}\tilde{\Pi}^a{}_i \nabla_a \tilde{\mathcal{G}}^i + \frac{\sigma}{2\left(\gamma^2 - \sigma\right)}\tilde{\mathcal{G}}^i \tilde{\mathcal{G}}_i,\end{aligned} \tag{2.72c}$$

with

$$F_{abi} := \partial_a A_{bi} - \partial_b A_{ai} + \epsilon_{ijk} A_a{}^j A_b{}^k \tag{2.73}$$

being the field strength of the connection A_{ai}. Also, to obtain the form of the scalar constraint (2.72c), we used the identity

$$2\nabla_{[a}\left(A_{b]i} - \Gamma_{b]i}\right) = F_{abi} - R_{abi} - \epsilon_{ijk}(A_a{}^j - \Gamma_a{}^j)(A_b{}^k - \Gamma_b{}^k). \tag{2.74}$$

To continue, we collect all the terms proportional to the Gauss constraint. Thus, we integrate by parts the term involving the covariant derivative in (2.72c) and redefine the Lagrange multiplier that imposes the Gauss constraint as

$$\mu_i := \epsilon_{ijk}\omega_t{}^{jk} + 2N^a(\Gamma_{ai} - A_{ai}) + \frac{\sigma \underset{\sim}{N}}{2(\gamma^2 - \sigma)}\tilde{\mathcal{G}}_i + \frac{1}{\gamma}\tilde{\Pi}^a{}_i \nabla_a \underset{\sim}{N}. \tag{2.75}$$

Then, the Hamiltonian description is given by the action

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[\frac{2}{\gamma}\dot{A}_{ai}\tilde{\Pi}^{ai} - \left(2\mu_i \tilde{\mathcal{G}}^i + 2N^a \tilde{\mathcal{C}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{C}}}\right) + \partial_a\left(\frac{2}{\gamma}\underset{\sim}{N}\tilde{\Pi}^{ai}\tilde{\mathcal{G}}_i\right)\right], \tag{2.76}$$

where the boundary term is a consequence of an integration by parts, and the vector and scalar constraints are

$$\tilde{\mathcal{C}}_a := \frac{1}{\gamma}\tilde{\Pi}^{bi} F_{abi}, \tag{2.77a}$$

$$\tilde{\tilde{\mathcal{C}}} := \frac{1}{\gamma^2}\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}\left[F_{ab}{}^k + \left(\sigma\gamma^2 - 1\right) R_{ab}{}^k\right] + 2\sigma\Lambda|\det(\tilde{\Pi}^{ai})|. \tag{2.77b}$$

This is the Ashtekar-Barbero formulation with cosmological constant, and we have derived it from a Hamiltonian description with a noncanonical symplectic structure. Here, the phase space is labeled with the canonical conjugated variables (A_{ai}, $\tilde{\Pi}^{ai}$), which are an internal $SO(3)$ [or $SU(2)$] connection and a densitized triad field, respectively.

Canonical variables

Given our description of Sec. 2.4, the time gauge in the canonical variables approach is straightforward. We begin solving $\tilde{\mathcal{G}}^i_{\text{boost}} = 0$; its solution reads

$$\tilde{\zeta}_i = \gamma \partial_a \tilde{\Pi}^a{}_i = -\gamma \epsilon_{ijk} \Gamma_a{}^j \tilde{\Pi}^{ak}, \tag{2.78}$$

where to get the second equality we used (2.57). Also, in the time gauge, J_{ai} is

$$J_{ai} \quad = \quad -\frac{\sigma}{2\gamma^2}\left(\delta^b_a \delta^j_i + \tilde{\Pi}^b{}_i \underset{\sim}{\Pi}_a{}^j\right)(A_{bj} - \Gamma_{bj}) - \Gamma_{ai}. \tag{2.79}$$

Then, substituting Eqs. (2.53), (2.78), and (2.79) back into the action (2.52) yields

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[\frac{2}{\gamma}\tilde{\Pi}^{ai}\dot{A}_{ai} - \left(2\lambda_i \tilde{\mathcal{G}}^i + 2N^a \tilde{\mathcal{D}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}}\right)\right], \tag{2.80}$$

where the constraints (2.43b), (2.49), and (2.43d) are

$$\begin{aligned}
\tilde{\mathcal{G}}^i \quad &= \quad \frac{1}{\gamma}\left(\partial_a \tilde{\Pi}^{ai} + \epsilon^{ijk} A_{aj} \tilde{\Pi}^a{}_k\right), && (2.81\text{a})\\
\tilde{\mathcal{D}}_a \quad &= \quad \frac{2}{\gamma}\tilde{\Pi}^{bi}\partial_{[a}A_{b]i} - \frac{1}{\gamma}A_{ai}\partial_b \tilde{\Pi}^{bi}, && (2.81\text{b})\\
\tilde{\tilde{\mathcal{S}}} \quad &= \quad \frac{1}{\gamma^2}\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}\left[F_{ab}{}^k + \left(\sigma\gamma^2 - 1\right)R_{ab}{}^k\right] + 2\sigma\Lambda|\det(\tilde{\Pi}^{ai})| \\
&\quad\quad -\frac{2}{\gamma}\tilde{\Pi}^{ai}\nabla_a \tilde{\mathcal{G}}_i + \frac{\sigma}{2\left(\gamma^2 - \sigma\right)}\tilde{\mathcal{G}}^i \tilde{\mathcal{G}}_i, && (2.81\text{c})
\end{aligned}$$

with F_{abi} being the field strength of the connection A_{ai} [see Eq. (2.73)].

As before, we work with the vector constraint rather than the diffeomorphism constraint

$$\tilde{\mathcal{C}}_a := \tilde{\mathcal{D}}_a + A_{ai}\tilde{\mathcal{G}}^i = \frac{1}{\gamma}\tilde{\Pi}^{bi}F_{abi}. \tag{2.82}$$

Then, we collect all the terms proportional to the Gauss constraint, so the action (2.80) becomes

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[\frac{2}{\gamma}\dot{A}_{ai}\tilde{\Pi}^{ai} - \left(2\mu_i \tilde{\mathcal{G}}^i + 2N^a \tilde{\mathcal{C}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{C}}}\right) + \partial_a\left(\frac{2}{\gamma}\underset{\sim}{N}\tilde{\Pi}^{ai}\tilde{\mathcal{G}}_i\right)\right], \tag{2.83}$$

where we integrated by parts the term with the covariant derivative in (2.81c) and redefined the Lagrange multiplier

$$\mu_i := \lambda_i - N^a A_{ai} + \frac{\sigma \underset{\sim}{N}}{2(\gamma^2 - \sigma)}\tilde{\mathcal{G}}_i + \frac{1}{\gamma}\tilde{\Pi}^a{}_i \nabla_a \underset{\sim}{N}. \tag{2.84}$$

Also, the scalar constraint is

$$\tilde{\tilde{\mathcal{C}}} := \frac{1}{\gamma^2}\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}\left[F_{ab}{}^{k} + \left(\sigma\gamma^2 - 1\right) R_{ab}{}^{k}\right] + 2\sigma\Lambda|\det(\tilde{\Pi}^{ai})|. \tag{2.85}$$

Therefore, we reach the Ashtekar-Barbero formulation with cosmological constant once again, which is the starting point of loop quantum gravity. The canonical conjugated variables, the connection A_{ai} and the densitized triad field $\tilde{\Pi}^{ai}$, are used to construct the loop variables involved in the quantum theory. Our analysis ends here since the quantum description is out of the scope of the present work.

2.6 Comments

We finish this chapter with some final remarks about the nonmanifestly Lorentz-covariant solution of the second-class constraints. Although a Hamiltonian formulation of this type was already reported by Barros e Sá, we found an equivalent way to describe the phase space with a noncanonical symplectic structure. Furthermore, it is illustrative how both descriptions are connected through a Darboux map. In fact, we developed our Hamiltonian description when we tried to make contact between the two known works at that time, the one from Barros e Sá and the one from Cianfrani and Montani. Since we could not find the relation between them because of the lack of variables, we completed the analysis and found the link that was missing.

Among the results presented in this chapter, our main contributions are:

- The solution of the second-class constraints and the ensuing Hamiltonian formulation (Sec. 2.3).
- The Darboux map that leads us to a canonical description (Sec. 2.4).

Also, it is worth to mention that Barros e Sá made contact with the Ashtekar-Barbero formulation by fixing the gauge before solving the second-class constraints; he did not exposed the method we presented in Sec. 2.5. Our results are found in Ref. [44].

Chapter 3

Manifestly Lorentz-covariant formulation through the solution of the second-class constraints of Holst action

In this chapter we present a manifestly Lorentz-covariant Hamiltonian formulation for the Holst action. We accomplish it by solving the second-class constraints in terms of canonical conjugated variables that explicitly exhibit their Lorentz covariance. Subsequently, we derive different Hamiltonian formulations related to each other via canonical transformations; the ensuing formulations are also manifestly Lorentz covariant. Moreover, two of these canonical transformations allow us to connect the Hamiltonian formalisms of Holst and Palatini actions. Finally, at the end of the chapter, we explore the time gauge in all the Hamiltonian descriptions previously found, and show they either become the Ashtekar-Barbero formulation or the $SO(3)$-ADM description.

The analysis and results of this chapter were published in Ref. [56].

3.1 Hamiltonian action

We begin our analysis right after Sec. 2.2, where we showed that Holst action with a cosmological constant Λ

$$S[e,\omega] = \frac{1}{\kappa}\int_M \left\{ \left[*\left(e^I \wedge e^J\right) + \frac{\sigma}{\gamma} e^I \wedge e^J \right] \wedge F_{IJ} - 2\Lambda\rho \right\}, \tag{3.1}$$

is equivalent—up to a neglected boundary term, see Eq. (2.9)—[1]to

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \Big[\tilde{\Pi}^{aIJ} \partial_t \overset{(\gamma)}{\omega}{}_{aIJ} - \tilde{H}_T \Big]. \tag{3.2}$$

[1]Throughout this chapter, we will exhibit the boundary terms that appear during the Hamiltonian analysis. However, we only display them in the first equation they appear; after that, we neglect them.

Here, $(\tilde{\Pi}^{aIJ}, \overset{(\gamma)}{\omega}{}_{aIJ})$ are the canonical conjugated variables and the total Hamiltonian is given by

$$\tilde{H}_T := -\omega_{tIJ}\tilde{\mathcal{G}}^{IJ} + N^a\tilde{\mathcal{V}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}} + \underset{\sim}{\phi}_{ab}\tilde{\tilde{\Phi}}^{ab} + \psi_{ab}\Psi^{ab}, \tag{3.3}$$

where ω_{tIJ}, N^a, $\underset{\sim}{N}$, $\underset{\sim}{\phi}_{ab}$, ψ_{ab} are Lagrange multipliers that impose the constraints

$$\tilde{\mathcal{G}}^{IJ} := D_a \overset{(\gamma)}{\tilde{\Pi}}{}^{aIJ} = \partial_a \overset{(\gamma)}{\tilde{\Pi}}{}^{aIJ} + 2 \overset{(\gamma)}{\omega}{}_a{}^{[I|}{}_K\tilde{\Pi}^{aK|J]} \approx 0, \tag{3.4a}$$

$$\tilde{\mathcal{V}}_a := \tilde{\Pi}^{bIJ} \overset{(\gamma)}{F}{}_{abIJ} \approx 0, \tag{3.4b}$$

$$\tilde{\tilde{\mathcal{S}}} := \tilde{\Pi}^{aIK}\tilde{\Pi}^b{}_K{}^J \overset{(\gamma)}{F}{}_{abIJ} + 2\sigma q\Lambda \approx 0, \tag{3.4c}$$

$$\tilde{\tilde{\Phi}}^{ab} := *\tilde{\Pi}^{aIJ}\tilde{\Pi}^b{}_{IJ} \approx 0, \tag{3.4d}$$

$$\Psi^{ab} := \epsilon_{IJKL}\tilde{\Pi}^{(a|IM}\tilde{\Pi}^c{}_M{}^J D_c\tilde{\Pi}^{|b)KL} \approx 0. \tag{3.4e}$$

Furthermore, we remind the reader that q_{ab} is the induced metric on Σ_t with q^{ab} being its inverse, and $q = \det(q_{ab})$. They fulfill the relation

$$qq^{ab} = \frac{\sigma}{2}\tilde{\Pi}^{aIJ}\tilde{\Pi}^b{}_{IJ}. \tag{3.5}$$

Also, from (2.7), the spatial components of the curvature are

$$F_{ab}{}^I{}_J = \partial_a\omega_b{}^I{}_J - \partial_b\omega_a{}^I{}_J + \omega_a{}^I{}_K\omega_b{}^K{}_J - \omega_b{}^I{}_K\omega_a{}^K{}_J. \tag{3.6}$$

The constraints $\tilde{\mathcal{G}}^{IJ}$, $\tilde{\mathcal{V}}_a$, and $\tilde{\tilde{\mathcal{S}}}$ are first class and are associated with the gauge symmetries of the theory. On the other hand, $\tilde{\tilde{\Phi}}^{ab}$ and Ψ^{ab} are the second-class constraints that must be handled somehow in the formalism.

3.2 Solution of the second-class constraints: manifestly Lorentz-covariant phase-space variables

We start with the second-class constraint $\tilde{\tilde{\Phi}}^{ab} = 0$. Equation (3.4d) defines a set of six quadratic equations for the 18 components in $\tilde{\Pi}^{aIJ}$; it means that $\tilde{\Pi}^{aIJ}$ has 12 independent variables that will label the coordinates in our smaller phase space. We denote the independent variables as $\tilde{\Pi}^{aI}$, so the solution of $\tilde{\tilde{\Phi}}^{ab} = 0$ is [15, 32]

$$\tilde{\Pi}^{aIJ} = 2\epsilon\tilde{\Pi}^{a[I}m^{J]}, \tag{3.7}$$

where $\epsilon = \pm 1$ is a sign ambiguity since the constraint is quadratic in the momenta and m_I is an arbitrary internal vector that depends only on $\tilde{\Pi}^{aI}$. Exploiting its arbitrariness, we demand m_I to fulfill a pair of properties: to be orthogonal to $\tilde{\Pi}^{aI}$ ($m_I\tilde{\Pi}^{aI} = 0$) and to be a normalized timelike vector in the Lorentzian signature ($m_I m^I = \sigma$). This vector might remind us of the internal vector n_I introduced in Sec 2.2. Although they are indeed related, for the moment we only consider m_I as the vector that solves the constraint (3.4d) and that satisfy the two properties enlisted above.[2] These two properties are enough to determine the explicit form of m_I [*cf.* Eq. (2.5)]

$$m_I = \frac{1}{6\sqrt{|h|}}\epsilon_{IJKL}\underset{\sim}{\eta}_{abc}\tilde{\Pi}^{aJ}\tilde{\Pi}^{bK}\tilde{\Pi}^{cL}, \tag{3.8}$$

where $h := \det(\tilde{\tilde{h}}^{ab})$ (of weight +4) with $\tilde{\tilde{h}}^{ab} := \eta_{IJ}\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ}$.

Moreover, h is related to the determinant of the spatial metric q. From (3.5) and (3.7) we find

$$qq^{ab} = \tilde{\tilde{h}}^{ab}. \tag{3.9}$$

Thereby, $\underset{\approx}{h}_{ab}$ (the inverse of $\tilde{\tilde{h}}^{ab}$, $\underset{\approx}{h}_{ac}\tilde{\tilde{h}}^{cb} = \delta^b_a$) is the densitized metric for the submanifold Σ_t. Likewise, the previous relation implies

$$q^2 = h. \tag{3.10}$$

Therefore, $h > 0$, and we can safely remove the absolute value bars in (3.8). The relation between q and h clearly suggest that the spatial part of the tetrad field ${e_a}^I$ and the new phase-space variables $\tilde{\Pi}^{aI}$ are related. For the moment, let us ignore this fact. We will elaborate on this relation in Chapter 6.

To simplify future expressions, it is convenient to introduce two quantities. The first one is an internal projector that we can derive from Eq. (3.8)

$$q^I{}_J := \underset{\approx}{h}_{ab}\tilde{\Pi}^{aI}\tilde{\Pi}^b{}_J = \delta^I_J - \sigma m^I m_J, \tag{3.11}$$

which projects onto the orthogonal plane to m_I. The second one is the covariant derivative ∇_a compatible with $\tilde{\Pi}^{aI}$, i.e., it is the one that satisfies

$$\nabla_a\tilde{\Pi}^{bI} := \partial_a\tilde{\Pi}^{bI} + \Gamma^b{}_{ac}\tilde{\Pi}^{cI} - \Gamma^c{}_{ac}\tilde{\Pi}^{bI} + \Gamma_a{}^I{}_J\tilde{\Pi}^{bJ} = 0. \tag{3.12}$$

The components $\Gamma_{aIJ} = -\Gamma_{aJI}$ and $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$ are 36 unknowns that we can determinate

[2]The relation between both vectors will become clear in Chapter 6.

from the 36 equations defined in (3.12). Solving for these variables we find

$$\begin{aligned}\Gamma_{aIJ} &= \underset{\approx}{h}_{ab}\tilde{\Pi}^{c}{}_{[I|}\partial_c\tilde{\Pi}^{b}{}_{|J]} + \underset{\approx}{h}_{bc}\tilde{\Pi}^{b}{}_{[I|}\partial_a\tilde{\Pi}^{c}{}_{|J]} - \sigma\underset{\approx}{h}_{ab}\tilde{\Pi}^{c}{}_{[I}m_{J]}m_K\partial_c\tilde{\Pi}^{bK} \\ &\quad +\sigma\underset{\approx}{h}_{bc}\tilde{\Pi}^{b}{}_{[I}m_{J]}m_K\partial_a\tilde{\Pi}^{cK} + \underset{\approx}{h}_{ab}\underset{\approx}{h}_{cd}\tilde{\Pi}^{c}{}_{K}\tilde{\Pi}^{b}{}_{[I}\tilde{\Pi}^{f}{}_{J]}\partial_f\tilde{\Pi}^{dK} \\ &\quad -\underset{\approx}{h}_{ab}\underset{\approx}{h}_{cd}\tilde{\Pi}^{b}{}_{K}\tilde{\Pi}^{c}{}_{[I}\tilde{\Pi}^{f}{}_{J]}\partial_f\tilde{\Pi}^{dK}, \qquad (3.13)\\ \Gamma^{a}{}_{bc} &= \frac{1}{2}q^{ad}\left(\partial_b q_{dc} + \partial_c q_{bd} - \partial_d q_{bc}\right). \qquad (3.14)\end{aligned}$$

Thus, $\Gamma^{a}{}_{bc}$ are just the Christoffel symbols for the metric q_{ab}. In the meantime, we define the curvature for the internal connection Γ_{aIJ} as

$$R_{ab}{}^{I}{}_{J} = \partial_a\Gamma_b{}^{I}{}_{J} - \partial_b\Gamma_a{}^{I}{}_{J} + \Gamma_a{}^{I}{}_{K}\Gamma_b{}^{K}{}_{J} - \Gamma_b{}^{I}{}_{K}\Gamma_a{}^{K}{}_{J}. \qquad (3.15)$$

Now we face the remaining constraint $\Psi^{ab} = 0$. After substituting (3.7) back into (3.4e) and using (3.12), the constraint acquires the form

$$\Psi^{ab} = -2\sigma\epsilon\epsilon_{IJKL}\tilde{\Pi}^{(a|I}\tilde{\Pi}^{|b)M}\tilde{\Pi}^{cJ}m^K\left(\Gamma_c{}^{L}{}_{M} - \omega_c{}^{L}{}_{M}\right) = 0. \qquad (3.16)$$

This is a system of six linear equations for the 18 unknowns contained in ω_{aIJ}. Thus, solving for ω_{aIJ} (or equivalently for $\overset{(\gamma)}{\omega}_{aIJ}$) implies the existence of 12 free variables in the general solution. These free variables will play the role of phase-space coordinates, and most likely will not form a set of canonical variables—just as in the case exposed in Sec. 2.3—. Therefore, we need to find the appropriate Darboux map that leads us to a description with canonical coordinates. However, we simplify part of the process when we use (3.7) and manipulate the kinetic term of the action (3.2)

$$\begin{aligned}\tilde{\Pi}^{aIJ}\partial_t \overset{(\gamma)}{\omega}_{aIJ} &= 2\epsilon\tilde{\Pi}^{aI}m^J\partial_t \overset{(\gamma)}{\omega}_{aIJ}, \qquad (3.17)\\ &= 2\tilde{\Pi}^{aI}\partial_t\left(\epsilon \overset{(\gamma)}{\omega}_{aIJ} m^J + \epsilon m_I \overset{(\gamma)}{\omega}_{bJK} \underset{\approx}{h}_{ac}\tilde{\Pi}^{cJ}\tilde{\Pi}^{bK}\right). \qquad (3.18)\end{aligned}$$

Thereby, it is natural to define the 12 variables

$$C_{aI} := \epsilon\left(\overset{(\gamma)}{\omega}_{aIJ}m^J + m_I \overset{(\gamma)}{\omega}_{bJK}\underset{\approx}{h}_{ac}\tilde{\Pi}^{cJ}\tilde{\Pi}^{bK}\right), \qquad (3.19)$$

which will act as the canonical configuration variables. Thus, Eqs. (3.16) and (3.19) are the complete set of 18 equations that solve the constraint $\Psi^{ab} = 0$, and, at the same time, give us the canonical variables (C_{aI}, $\tilde{\Pi}^{aI}$). The solution for both, (3.16) and (3.19), is

$$\overset{(\gamma)}{\omega}_{aIJ} = M_a{}^{b}{}_{IJK}C_b{}^{K} + \tilde{N}^{b}{}_{IJ}\underset{\sim}{\lambda}_{ab}, \qquad (3.20)$$

with

$$
\begin{aligned}
M_a{}^b{}_{IJK} &:= \epsilon\sigma\Big[-\delta_a^b m_{[I}\eta_{J]K}+\delta_a^b P_{IJKL}m^L+\underset{\sim}{h}_{ac}\tilde{\Pi}^b{}_{[I}\tilde{\Pi}^c{}_{J]}m_K \\
&\quad +\frac{1}{2\gamma}\epsilon_{IJLM}\underset{\sim}{h}_{ac}\tilde{\Pi}^c{}_K\tilde{\Pi}^{bL}m^M\Big], \qquad (3.21a)\\
\tilde{N}^a{}_{IJ} &:= \epsilon_{IJKL}\tilde{\Pi}^{aK}m^L, \qquad (3.21b)\\
\underset{\sim}{\lambda}_{ab} &:= \frac{1}{2}\epsilon_{IJKL}\left(\underset{\sim}{h}_{ab}\underset{\sim}{h}_{cd}-2\underset{\sim}{h}_{c(a}\underset{\sim}{h}_{b)d}\right)\tilde{\Pi}^{cI}\tilde{\Pi}^{fJ}\tilde{\Pi}^{dM}m^L\Gamma_f{}^K{}_M. \qquad (3.21c)
\end{aligned}
$$

We write $\overset{(\gamma)}{\omega}_{aIJ}$ as in Eq. (3.20) to highlight the $12+6$ variables C_{aI} and $\underset{\sim}{\lambda}_{ab}=\underset{\sim}{\lambda}_{ba}$, respectively; they account for the 18 original variables that compose $\overset{(\gamma)}{\omega}_{aIJ}$. Furthermore, we can interpret C_{aI} as the 12 dynamical variables contained in $\overset{(\gamma)}{\omega}_{aIJ}$ [defined in (3.19)], whereas $\underset{\sim}{\lambda}_{ab}$ are six nondynamical variables determined by (3.16).

With the solutions for both second-class constraints (3.4d) and (3.4e), we substitute (3.7) and (3.20) [together with (3.21a), (3.21b), and (3.21c)] back into the action (3.2), then, we obtain

$$
S=\frac{1}{\kappa}\int_{\mathbb{R}}dt\int_{\Sigma_t}d^3x\left[2\tilde{\Pi}^{aI}\dot{C}_{aI}-\left(-\omega_{tIJ}\tilde{\mathcal{G}}^{IJ}+N^a\tilde{\mathcal{V}}_a+\underset{\sim}{N}\tilde{\tilde{\mathcal{S}}}\right)\right]. \qquad (3.22)
$$

The term inside the parenthesis is the first-class Hamiltonian, which is formed by the first-class constraints

$$
\begin{aligned}
\tilde{\mathcal{G}}^{IJ} &= 2\tilde{\Pi}^{a[I}C_a{}^{J]}+4\epsilon P^{IJ}{}_{KL}\tilde{\Pi}^{a[M}m^{K]}\Gamma_a{}^L{}_M\approx 0, \qquad (3.23a)\\
\tilde{\mathcal{V}}_a &= 4\nabla_{[a}\left(C_{b]I}\tilde{\Pi}^{bI}\right)-4\epsilon\tilde{\Pi}^{b[I}m^{J]}\overset{(\gamma)}{\Gamma}_{aIK}\Gamma_b{}^K{}_J+\epsilon\sigma\tilde{\mathcal{G}}^{IJ}m_J\Big[2C_{aI}\\
&\quad -2\epsilon m^K\overset{(\gamma)}{\Gamma}_{aIK}+\underset{\sim}{h}_{ab}\tilde{\Pi}^{bK}\tilde{\mathcal{G}}_{IK}\Big]\approx 0, \qquad (3.23b)\\
\tilde{\tilde{\mathcal{S}}} &= -\sigma\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ}R_{abIJ}+2\tilde{\Pi}^{a[I|}\tilde{\Pi}^{b|J]}\Big[C_{aI}C_{bJ}-2\epsilon C_{aI}m^K\overset{(\gamma)}{\Gamma}_{bJK}\\
&\quad +\left(\Gamma_{aIL}+\frac{2}{\gamma}*\Gamma_{aIL}\right)\Gamma_{bJK}m^Km^L+\frac{1}{\gamma^2}q^{KL}\Gamma_{aIK}\Gamma_{bJL}\Big]\\
&\quad +\tilde{\mathcal{G}}^{IJ}\Big[-\frac{1}{4}\tilde{\mathcal{G}}_{IJ}+\frac{1}{4}\left(P^{-1}\right)_{IJKL}\tilde{\mathcal{G}}^{KL}-\frac{\sigma}{2}m_Im^K\tilde{\mathcal{G}}_{JK}\Big]\\
&\quad -2\epsilon\tilde{\Pi}^{aI}m^J\nabla_a\tilde{\mathcal{G}}_{IJ}+2\sigma\sqrt{h}\Lambda\approx 0. \qquad (3.23c)
\end{aligned}
$$

We can simplify our formulation if we collect all the terms proportional to the Gauss constraint, but first, we notice that the vector constraint can be rewritten as

$$\tilde{\mathcal{V}}_a = 2\tilde{\mathcal{D}}_a + \tilde{\mathcal{G}}^{IJ}\left(\Gamma_{aIJ} + 2\epsilon\sigma C_{aI}m_J + 2\sigma m_I m^K \overset{(\gamma)}{\Gamma}_{aJK} - \epsilon\sigma \underset{\sim}{h}_{ab}\tilde{\Pi}^{bK} m_I \tilde{\mathcal{G}}_{JK}\right), \tag{3.24}$$

where $\tilde{\mathcal{D}}_a$ is the diffeomorphism constraint given by

$$\tilde{\mathcal{D}}_a := 2\tilde{\Pi}^{bI}\partial_{[a}C_{b]I} - C_{aI}\partial_b\tilde{\Pi}^{bI}. \tag{3.25}$$

Now, we integrate by parts the term containing the covariant derivative in (3.23c) and collect all the terms proportional to the Gauss constraint. Doing this, the action becomes

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[2\tilde{\Pi}^{aI}\dot{C}_{aI} - \left(\lambda_{IJ}\tilde{\mathcal{G}}^{IJ} + 2N^a\tilde{\mathcal{D}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{H}}}\right) + \partial_a\left(2\epsilon\underset{\sim}{N}\tilde{\Pi}^{aI}m^J\tilde{\mathcal{G}}_{IJ}\right)\right], \tag{3.26}$$

where we defined the Lagrange multiplier as

$$\begin{aligned}\lambda_{IJ} \quad := \quad & -\omega_{tIJ} + N^a\left(\Gamma_{aIJ} + 2\epsilon\sigma C_{a[I}m_{J]} + 2\sigma m_{[I|}m^K \overset{(\gamma)}{\Gamma}_{a|J]K} - \epsilon\sigma\underset{\sim}{h}_{ab}\tilde{\Pi}^{bK}m_{[I}\tilde{\mathcal{G}}_{J]K}\right) \\ & + \underset{\sim}{N}\left[-\frac{1}{4}\tilde{\mathcal{G}}_{IJ} + \frac{1}{4}\left(P^{-1}\right)_{IJKL}\tilde{\mathcal{G}}^{KL} - \frac{\sigma}{2}m_{[I}m^K\tilde{\mathcal{G}}_{J]K}\right] + 2\epsilon\tilde{\Pi}^a{}_{[I}m_{J]}\nabla_a\underset{\sim}{N},\end{aligned} \tag{3.27}$$

and the new Hamiltonian constraint is

$$\begin{aligned}\tilde{\tilde{\mathcal{H}}} \quad = \quad & -\sigma\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ}R_{abIJ} + 2\tilde{\Pi}^{a[I|}\tilde{\Pi}^{b|J]}\Bigg[C_{aI}C_{bJ} - 2\epsilon C_{aI}m^K \overset{(\gamma)}{\Gamma}_{bJK} \\ & + \left(\Gamma_{aIL} + \frac{2}{\gamma} * \Gamma_{aIL}\right)\Gamma_{bJK}m^K m^L + \frac{1}{\gamma^2}q^{KL}\Gamma_{aIK}\Gamma_{bJL}\Bigg] + 2\sigma\sqrt{h}\Lambda.\end{aligned} \tag{3.28}$$

Our formulation is then described by the action (3.26) formed exclusively by the first-class constraints (3.23a), (3.25), and (3.28). It is written with the manifestly Lorentz-(Euclidean)-covariant variables (C_{aI}, $\tilde{\Pi}^{aI}$) that obey the fundamental Poisson bracket

$$\left\{C_{aI}(t,x), \tilde{\Pi}^{bJ}(t,y)\right\} = 2\kappa\delta^b_a\delta^J_I\delta^3(x,y). \tag{3.29}$$

Thanks to the explicit covariant nature of the variables, the previous description is much more appealing than those exposed in (2.28) or (2.52). However, the geometrical meaning of the canonical variables is not as clear as in the Ashtekar-Barbero formulation. Although the canonical momentum has a clear interpretation, since $\tilde{\Pi}^{aI}$ transforms as a densitized vector under spatial diffeomorphisms and as an $SO(3,1)$ [$SO(4)$] internal vector under Lorentz (Euclidean) transformations, the configuration variable C_{aI} behaves as a 1-form under spatial diffeomorphisms. However, its transformation law under Lorentz (Euclidean) transformations is quite challenging to interpret. Nevertheless, our description, as opposed to the Ashtekar-Barbero formulation, is invariant under the complete symmetry group $SO(3,1)$ [$SO(4)$].

3.3 Alternative parametrizations through canonical transformations

In this section, we exploit the use of canonical transformations to derive alternative Hamiltonian descriptions for general relativity. All of the following formulations are described by manifestly Lorentz (Euclidean) covariant variables.

1. The first canonical transformation we consider is

$$K_{aI} = C_{aI} - \epsilon\left(\Gamma_{aIJ}m^J + \underset{\sim}{h}_{ab}\tilde{\Pi}^{bJ}\tilde{\Pi}^{cK}\Gamma_{cJK}m_I\right), \tag{3.30a}$$

$$\tilde{\Pi}^{aI} = \tilde{\Pi}^{aI}. \tag{3.30b}$$

Under it, the theory is now described by the action

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[2\tilde{\Pi}^{aI}\dot{K}_{aI} - \left(\lambda_{IJ}\tilde{\mathcal{G}}^{IJ} + 2N^a\tilde{\mathcal{D}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{H}}}\right) + \partial_a\left(2\epsilon\dot{\tilde{\Pi}}^{aI}m_I\right)\right], \tag{3.31}$$

where the pair $(K_{aI}, \tilde{\Pi}^{aI})$ are the new canonical variables. The boundary term appears from the substitution of (3.30) in the kinetic term of (3.26)

$$2\tilde{\Pi}^{aI}\dot{C}_{aI} = 2\tilde{\Pi}^{aI}\dot{K}_{aI} + \partial_a\left(2\epsilon\dot{\tilde{\Pi}}^{aI}m_I\right), \tag{3.32}$$

but, since Σ_t does not possess a boundary, the transformation is canonical. In fact, the fundamental Poisson bracket is

$$\left\{K_{aI}(t,x), \tilde{\Pi}^{bJ}(t,y)\right\} = \frac{\kappa}{2}\delta^b_a\delta^J_I\delta^3(x,y), \tag{3.33}$$

and the constraints are

$$\tilde{\mathcal{G}}^{IJ} = 2\tilde{\Pi}^{a[I}K_a{}^{J]} + \frac{2\epsilon}{\gamma}\epsilon^{IJ}{}_{KL}\tilde{\Pi}^{a[M}m^{K]}\Gamma_a{}^L{}_M \approx 0, \tag{3.34a}$$

$$\tilde{\mathcal{D}}_a = 2\tilde{\Pi}^{bI}\partial_{[a}K_{b]I} - K_{aI}\partial_b\tilde{\Pi}^{bI} \approx 0, \tag{3.34b}$$

$$\tilde{\tilde{\mathcal{H}}} = -\sigma\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ}R_{abIJ} + 2\tilde{\Pi}^{a[I|}\tilde{\Pi}^{b|J]}\Big[K_{aI}K_{bJ} - \frac{2\epsilon}{\gamma}K_{aI}m^K * \Gamma_{bJK}$$

$$
\left. + \frac{1}{\gamma^2} q^{KL} \Gamma_{aIK} \Gamma_{bJL} \right] + 2\sigma\sqrt{h}\Lambda \approx 0. \tag{3.34c}
$$

In the formulation with the variables $(K_{aI},\, \tilde{\Pi}^{aI})$, the diffeomorphism constraint maintains the same structure, whereas the other two constraints take a more compact form. Also, the presence of the Barbero-Immirzi parameter is still noticeable. Thus, the transformation (3.30) connects two formulations inherent to the Holst action.

2. For the next canonical transformation we go back to the formulation with the variables C_{aI} and $\tilde{\Pi}^{aI}$, and we consider the transformation

$$
\begin{aligned}
Q_{aI} &= C_{aI} - \epsilon \left(\overset{(\gamma)}{\Gamma}{}_{aIJ}\, m^J + \underset{\sim}{h}_{ab} \tilde{\Pi}^{bJ} \tilde{\Pi}^{cK}\, \overset{(\gamma)}{\Gamma}{}_{cJK}\, m_I \right), & (3.35\text{a})\\
\tilde{\Pi}^{aI} &= \tilde{\Pi}^{aI}, & (3.35\text{b})
\end{aligned}
$$

which leads the action (3.26) (neglecting the boundary term) to the form

$$
S = \frac{1}{\kappa} \int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[2\tilde{\Pi}^{aI} \dot{Q}_{aI} - \left(\lambda_{IJ} \tilde{\mathcal{G}}^{IJ} + 2N^a \tilde{\mathcal{D}}_a + \underset{\sim}{N} \tilde{\tilde{\mathcal{H}}} \right) \right]. \tag{3.36}
$$

The neglected boundary term comes from

$$
2\tilde{\Pi}^{aI} \dot{C}_{aI} = 2\tilde{\Pi}^{aI} \dot{Q}_{aI} + \partial_a \left(2\epsilon \dot{\tilde{\Pi}}^{aI} m_I - \frac{\epsilon\sigma}{\gamma} \sqrt{h} \tilde{\eta}^{abc} \underset{\approx}{h}_{bd} \underset{\approx}{h}_{ce} \dot{\tilde{\Pi}}^{dI} \tilde{\Pi}^{e}{}_I \right). \tag{3.37}
$$

Therefore, the pair $(Q_{aI},\, \tilde{\Pi}^{aI})$ obeys the Poisson Bracket

$$
\left\{ Q_{aI}(t,x),\, \tilde{\Pi}^{bJ}(t,y) \right\} = \frac{\kappa}{2} \delta^b_a \delta^J_I \delta^3(x,y), \tag{3.38}
$$

and the first-class constraints in terms of $(Q_{aI},\, \tilde{\Pi}^{aI})$ are

$$
\begin{aligned}
\tilde{\mathcal{G}}^{IJ} &= 2\tilde{\Pi}^{a[I} Q_a{}^{J]} \approx 0, & (3.39\text{a})\\
\tilde{\mathcal{D}}_a &= 2\tilde{\Pi}^{bI} \partial_{[a} Q_{b]I} - Q_{aI} \partial_b \tilde{\Pi}^{bI} \approx 0, & (3.39\text{b})\\
\tilde{\tilde{\mathcal{H}}} &= -\sigma \tilde{\Pi}^{aI} \tilde{\Pi}^{bJ} R_{abIJ} + 2\tilde{\Pi}^{a[I|} \tilde{\Pi}^{b|J]} Q_{aI} Q_{bJ} + 2\sigma\sqrt{h}\Lambda \approx 0. & (3.39\text{c})
\end{aligned}
$$

Again, the canonical transformation does not modify the diffeomorphism constraint, but it reduces the form of the other two constraints. Moreover, the remarkable aspect of the Hamiltonian formulation (3.36) is the absence of the Barbero-Immirzi parameter. In fact, this is the same formulation that arises during the Hamiltonian analysis of the Palatini action [32,57]. Thus, the inverse of the transformation (3.35) is a Lorentz-covariant version of Barbero's canonical transformation [17].

3. Finally, we present one more description. Although the next transformation was not originally presented in Ref. [56], it is already reported in Refs. [57] and [58]. This time we consider the canonical transformation

$$\begin{aligned} \mathcal{Q}_{aI} &= C_{aI} - \frac{\epsilon}{\gamma}\left(*\Gamma_{aIJ}m^J + \underset{\sim}{h}_{ab}\tilde{\Pi}^{bJ}\tilde{\Pi}^{cK} * \Gamma_{cJK}m_I \right), && (3.40a) \\ \tilde{\Pi}^{aI} &= \tilde{\Pi}^{aI}. && (3.40b) \end{aligned}$$

Using this canonical transformation, the action (3.26) becomes

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[2\tilde{\Pi}^{aI}\dot{\mathcal{Q}}_{aI} - \left(\lambda_{IJ}\tilde{\mathcal{G}}^{IJ} + 2N^a\tilde{\mathcal{D}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{H}}} \right) \right], \quad (3.41)$$

where, again, we neglected the boundary term that emerges from

$$2\tilde{\Pi}^{aI}\dot{C}_{aI} = 2\tilde{\Pi}^{aI}\dot{\mathcal{Q}}_{aI} - \partial_a\left(\frac{\epsilon\sigma}{\gamma}\sqrt{h}\tilde{\eta}^{abc}\underset{\sim}{h}_{bd}\underset{\sim}{h}_{ce}\dot{\tilde{\Pi}}^{dI}\tilde{\Pi}^e{}_I \right). \quad (3.42)$$

It is clear from (3.41) that the new variables obey the Poisson bracket

$$\left\{ \mathcal{Q}_{aI}(t,x), \tilde{\Pi}^{bJ}(t,y) \right\} = \frac{\kappa}{2}\delta^b_a\delta^J_I\delta^3(x,y), \quad (3.43)$$

and the first-class constraints are given by

$$\begin{aligned} \tilde{\mathcal{G}}^{IJ} &= 2\tilde{\Pi}^{a[I}\mathcal{Q}_a{}^{J]} + 2\epsilon\tilde{\Pi}^{aM}m^{[I}\Gamma_a{}^{J]}{}_M - 2\epsilon\tilde{\Pi}^{a[I|}m^M\Gamma_a{}^{|J]}{}_M \approx 0, && (3.44a) \\ \tilde{\mathcal{D}}_a &= 2\tilde{\Pi}^{bI}\partial_{[a}\mathcal{Q}_{b]I} - \mathcal{Q}_{aI}\partial_b\tilde{\Pi}^{bI} \approx 0, && (3.44b) \\ \tilde{\tilde{\mathcal{H}}} &= -\sigma\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ}R_{abIJ} + 2\tilde{\Pi}^{a[I|}\tilde{\Pi}^{b|J]}\Big[\mathcal{Q}_{aI}\mathcal{Q}_{bJ} - 2\epsilon\mathcal{Q}_{aI}m^K\Gamma_{bJK} \\ &\quad + \Gamma_{aIL}\Gamma_{bJK}m^K m^L \Big] + 2\sigma\sqrt{h}\Lambda \approx 0. && (3.44c) \end{aligned}$$

Instinctively, the lack of the Barbero-Immirzi parameter suggests that this description is inherent to the Palatini action. This statement is correct since the last formulation emerges during the Hamiltonian analysis of the Palatini action [57]. Thus, the inverse of the transformation (3.40) is also a Lorentz-covariant version of Barbero's canonical transformation.

A few remarks about the canonical transformations. Regardless of the canonical variables, the diffeomorphism constraints are the same in terms of them. Therefore, all the configuration variables transform as 1-forms under spatial diffeomorphisms, and the momentum transforms as a vector of weight +1. Furthermore, under Lorentz (Euclidean) transformations, $\tilde{\Pi}^{aI}$ transforms as a Lorentz (Euclidean) vector. In contrast, the transformation law for the majority of the configurational variables are quite complicated, just the variable Q_{aI} has a clear geometrical meaning since it transforms as an $SO(3,1)$ [$SO(4)$] vector.

We notice some interesting facts about the descriptions enlisted above. When we consider the limit $\gamma \to \infty$ in the formulation with the variables $\left(C_{aI},\, \tilde{\Pi}^{aI}\right)$, the formulation becomes the one described by the variables $\left(\mathcal{Q}_{aI},\, \tilde{\Pi}^{aI}\right)$. Also, if the same limit is applied in the $\left(K_{aI},\, \tilde{\Pi}^{aI}\right)$ formulation, we end up with the description of the $\left(Q_{aI},\, \tilde{\Pi}^{aI}\right)$ variables. Explicitly:

$$\left(C_{aI},\, \tilde{\Pi}^{aI}\right) \overset{\gamma\to\infty}{\longrightarrow} \left(\mathcal{Q}_{aI},\, \tilde{\Pi}^{aI}\right), \tag{3.45a}$$

$$\left(K_{aI},\, \tilde{\Pi}^{aI}\right) \overset{\gamma\to\infty}{\longrightarrow} \left(Q_{aI},\, \tilde{\Pi}^{aI}\right). \tag{3.45b}$$

Furthermore, when we combine Eqs. (3.30a), (3.35a), and (3.40a), we found:

$$C_{aI} + Q_{aI} = K_{aI} + \mathcal{Q}_{aI}. \tag{3.46}$$

We do not know if this relationship has some implications, but it is an intriguing relation.

3.4 Gauge fixing: time gauge

Let us explore the previous manifestly Lorentz-covariant formulations under a gauge fixing. We consider the gauge that reduces the internal symmetry group, $SO(3,1)$ or $SO(4)$, to its compact subgroup $SO(3)$. During the first part of this section, we keep the analysis quite general so that it is valid for the formulations of the last two sections. However, as the analysis progresses, we are obligated to consider each case separately.

Consider the gauge condition

$$\tilde{\Pi}^{a0} = 0, \tag{3.47}$$

which, regardless of the canonical pair considered, Poisson-commutes with almost every first-class constraint. The only nontrivial Poisson bracket is

$$\{\tilde{\Pi}^{a0}(t,x),\, \tilde{\mathcal{G}}^{i0}(t,y)\} = -\frac{\sigma\kappa}{2}\tilde{\Pi}^{ai}\delta^3(x,y). \tag{3.48}$$

Therefore, since we consider $\det(\tilde{\Pi}^{ai}) \neq 0$, the condition (3.47) is a second-class constraint that must be solved together with the now second-class constraint $\tilde{\mathcal{G}}^{i0} = 0$. Consequently, we lose some of the generators of the Lorentz (Euclidean) group, the remnant subgroup will be generated by the constraints $\tilde{\mathcal{G}}^{ij}$.

Defining

$$\tilde{\mathcal{G}}_i := -\frac{1}{2}\epsilon_{ijk}\tilde{\mathcal{G}}^{jk}, \tag{3.49}$$

we notice that the generators fulfill the algebra

$$\left\{\tilde{\mathcal{G}}_i(t,x), \tilde{\mathcal{G}}_j(t,x)\right\} = \frac{\kappa}{2}\epsilon_{ijk}\tilde{\mathcal{G}}^k\delta^3(x,y), \tag{3.50}$$

with $\epsilon_{ijk} := \epsilon_{0ijk}$. Thereby, $\tilde{\mathcal{G}}_i$ obeys the Lie algebra corresponding to the $SO(3)$ [or $SU(2)$] group. Hence, when we impose (3.47) as gauge condition, we break down the Lorentz (Euclidean) symmetry and leave behind a theory invariant under $SO(3)$ [$SU(2)$] rotations.

We have removed the boost freedom from the theory, as in Sec. 2.5. Thus, this gauge fixing also receives the name "*time gauge*". Also, from equation (3.8), we deduce that (3.47) implies $m_i = 0$ and $m_0 = \text{sgn}[\det(\Pi^{ai})]$. Henceforth, for the Lorentzian signature, m_I is a timelike vector aligned with the internal time direction. Furthermore, let $\underset{\sim}{\Pi}_{ai}$ denote the inverse of $\tilde{\Pi}^{ai}$ ($\underset{\sim}{\Pi}_{ai}\tilde{\Pi}^{aj} = \delta_i^j$ and $\underset{\sim}{\Pi}_{ai}\tilde{\Pi}^{bi} = \delta_a^b$), then, from Eq. (3.9), we obtain

$$q_{ab} = |\det(\tilde{\Pi}^{ai})|\underset{\sim}{\Pi}_{ai}\underset{\sim}{\Pi}_b{}^i. \tag{3.51}$$

Thus, $\tilde{\Pi}^{ai}$ is an orthonormal densitized basis for Σ_t.

Another important consequence of the time gauge happens to the internal connection Γ_{aIJ}, it discomposes into two parts: the connection along the time direction Γ_{a0i} and the connection tangent to Σ_t, rewritten as $\Gamma_{ai} := -(1/2)\epsilon_{ijk}\Gamma_a{}^{jk}$. Both parts are directly computed from (3.13), the former is identically zero whereas the latter becomes

$$\Gamma_{ai} = \epsilon_{ijk}\left(\partial_{[b}\underset{\sim}{\Pi}_{a]}{}^j + \underset{\sim}{\Pi}_a{}^{[l|}\tilde{\Pi}^{c|j]}\partial_b\underset{\sim}{\Pi}_{cl}\right)\tilde{\Pi}^{bk}, \tag{3.52}$$

and, from (3.12), it fulfills the equation

$$\nabla_a \tilde{\Pi}^{bi} = \partial_a \tilde{\Pi}^{bi} + \Gamma^b{}_{ac} \tilde{\Pi}^{ci} - \Gamma^c{}_{ac} \tilde{\Pi}^{bi} + \epsilon^i{}_{jk} \Gamma_a{}^j \tilde{\Pi}^{bk} = 0. \tag{3.53}$$

Therefore, Γ_{ai} is the spin connection compatible with $\tilde{\Pi}^{ai}$, and its field strength is

$$R_{abi} := -\frac{1}{2} \epsilon_{ijk} R_{ab}{}^{jk} = \partial_a \Gamma_{bi} - \partial_b \Gamma_{ai} + \epsilon_{ijk} \Gamma_a{}^j \Gamma_b{}^k. \tag{3.54}$$

All the discussion until now is valid for every formulation of the previous sections since most of the results are related to the canonical momenta, which are the same in all cases. Next, we need to solve the constraint $\tilde{\mathcal{G}}^{i0} = 0$, depending on the case we are considering; it will fix either C_{a0}, K_{a0}, Q_{a0}, or $\mathcal{Q}_{a0}$. Thus, in principle, we need to separate our analysis for each canonical pair. However, imposing (3.47) in (3.30a) leads us to $C_{ai} = K_{ai}$, and, from (3.46), we conclude $Q_{ai} = \mathcal{Q}_{ai}$. Thence, we need to bifurcate our analysis into two cases only, one for the variables C_{ai} (or K_{ai}) and one for the variables Q_{ai} (or $\mathcal{Q}_{ai}$). Let us first explore the former case, since it is the one that arises naturally after solving the second-class constraints.

Time Gauge for the variables ($C_{ai} = K_{ai}$, $\tilde{\Pi}^{ai}$)

From (3.23a) and (3.34a) the solution of $\tilde{\mathcal{G}}^{i0} = 0$ reads

$$\begin{aligned} C_{a0} &= -\sigma \epsilon m^0 \underset{\sim}{\Pi}_{ai} \partial_b \tilde{\Pi}^{bi}, && (3.55) \\ K_{a0} &= 0. && (3.56) \end{aligned}$$

Regardless of which one of the formulation we consider, either (C_{aI}, $\tilde{\Pi}^{aI}$) or (K_{aI}, $\tilde{\Pi}^{aI}$), we will arrive at a description of general relativity under $SO(3)$ [$SU(2)$] transformations described by the canonical pair ($C_{ai} = K_{ai}$, $\tilde{\Pi}^{ai}$). This formulation is given by the action

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[2\tilde{\Pi}^{ai} \dot{C}_{ai} - \left(2\lambda_i \tilde{\mathcal{G}}^i + 2N^a \tilde{\mathcal{D}}_a + \underset{\sim}{N} \tilde{\tilde{\mathcal{H}}} \right) \right], \tag{3.57}$$

where $\lambda_i := -(1/2)\epsilon_{ijk}\lambda^{jk}$ and

$$\begin{aligned} \tilde{\mathcal{G}}^i &= \epsilon \frac{m^0}{\gamma} \left[\partial_a \tilde{\Pi}^{ai} + \epsilon^i{}_{jk} (\epsilon m^0 \gamma C_a{}^j) \tilde{\Pi}^{ak} \right], && (3.58a) \\ \tilde{\mathcal{D}}_a &= 2\tilde{\Pi}^{bi} \partial_{[a} C_{b]i} - C_{ai} \partial_b \tilde{\Pi}^{bi}, && (3.58b) \\ \tilde{\tilde{\mathcal{H}}} &= \sigma \epsilon_{ijk} \tilde{\Pi}^{ai} \tilde{\Pi}^{bj} R_{ab}{}^k + 2\tilde{\Pi}^{a[i|} \tilde{\Pi}^{b|j]} \left(C_{ai} - \frac{\epsilon m^0}{\gamma} \Gamma_{ai} \right) \left(C_{bj} - \frac{\epsilon m^0}{\gamma} \Gamma_{bj} \right) \\ &\quad + 2\sigma\Lambda |\det(\tilde{\Pi}^{ai})|. && (3.58c) \end{aligned}$$

Taking a closer look at the Gauss constraint in (3.58a), we identify the connection

$$A_{ai} := \epsilon m^0 \gamma C_{ai}. \tag{3.59}$$

Using A_{ai} as our configuration variable, we rewrite the action and obtain

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[\frac{2\epsilon m^0}{\gamma}\tilde{\Pi}^{ai}\dot{A}_{ai} - \left(2\lambda_i \tilde{\mathcal{G}}^i + 2N^a \tilde{\mathcal{D}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{H}}}\right)\right]. \tag{3.60}$$

The theory now obeys the fundamental Poisson bracket

$$\left\{A_{ai}(t,x), \tilde{\Pi}^{bj}(t,y)\right\} = \frac{\kappa\gamma\epsilon m^0}{2}\delta_a^b \delta_i^j \delta^3(x,y), \tag{3.61}$$

and the constraints are

$$\tilde{\mathcal{G}}^i = \epsilon \frac{m^0}{\gamma}\left[\partial_a \tilde{\Pi}^{ai} + \epsilon^i{}_{jk} A_a{}^j \tilde{\Pi}^{ak}\right], \tag{3.62a}$$

$$\tilde{\mathcal{D}}_a = \epsilon \frac{m^0}{\gamma}\left(2\tilde{\Pi}^{bi}\partial_{[a}A_{b]i} - A_{ai}\partial_b \tilde{\Pi}^{bi}\right), \tag{3.62b}$$

$$\begin{aligned}\tilde{\tilde{\mathcal{H}}} = {} & \frac{1}{\gamma^2}\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}\left[F_{ab}{}^k + \left(\sigma\gamma^2 - 1\right) R_{ab}{}^k\right] + 2\epsilon \frac{m^0}{\gamma}\tilde{\Pi}^{ai}\nabla_a \tilde{\mathcal{G}}_i \\ & + 2\sigma\Lambda |\det(\tilde{\Pi}^{ai})|,\end{aligned} \tag{3.62c}$$

where

$$F_{abi} := \partial_a A_{bi} - \partial_b A_{ai} + \epsilon_{ijk} A_a{}^j A_b{}^k \tag{3.63}$$

is the field strength of the connection A_{ai}. To get (3.62c) we used the identity

$$2\nabla_{[a}\left(A_{b]i} - \Gamma_{b]i}\right) = F_{abi} - R_{abi} - \epsilon_{ijk}(A_a{}^j - \Gamma_a{}^j)(A_b{}^k - \Gamma_b{}^k). \tag{3.64}$$

To arrive at the usual Ashtekar-Barbero formulation we need to do two things. First, we use the vector constraint defined in Eq. (2.82) instead of the diffeomorphism constraint of (3.62b). Second, we collect all the terms proportional to the Gauss constraint; to do it, we integrate by parts the term with covariant derivate in Eq. (3.62c). Then, neglecting the boundary term, we end up with the action

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[\frac{2\epsilon m^0}{\gamma}\tilde{\Pi}^{ai}\dot{A}_{ai} - \left(2\nu_i \tilde{\mathcal{G}}^i + 2N^a \tilde{\mathcal{C}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{C}}}\right)\right], \tag{3.65}$$

with $\nu_i := \lambda_i - N^a A_{ai} + (1/\gamma)\tilde{\Pi}^a{}_i \nabla_a \underset{\sim}{N}$ and

$$\tilde{\mathcal{G}}^i = \epsilon \frac{m^0}{\gamma} \left[\partial_a \tilde{\Pi}^{ai} + \epsilon^i{}_{jk} A_a{}^j \tilde{\Pi}^{ak}\right], \tag{3.66a}$$

$$\tilde{\mathcal{C}}_a = \epsilon \frac{m^0}{\gamma} \tilde{\Pi}^{bi} F_{abi}, \tag{3.66b}$$

$$\tilde{\tilde{\mathcal{C}}} = \frac{1}{\gamma^2} \epsilon_{ijk} \tilde{\Pi}^{ai} \tilde{\Pi}^{bj} \left[F_{ab}{}^k + \left(\sigma\gamma^2 - 1\right) R_{ab}{}^k\right] + 2\sigma\Lambda |\det(\tilde{\Pi}^{ai})|. \tag{3.66c}$$

The sign ϵm^0—which comes from the ambiguity of solving the quadratic constraint (3.4d) and from the sign of the determinant of $\tilde{\Pi}^{ai}$—plays no important role, because it could be reabsorbed into the Lagrange multipliers that accompany the Gauss and diffeomorphism constraints. Apart from that, this is the Ashtekar-Barbero formulation [17] and is neatly derived from our description with manifestly Lorentz (Euclidean) covariant phase-space variables. Therefore, our formulation is indeed a generalization of the Ashtekar-Barbero description when the symmetry group remains intact.

Time Gauge for the variables $(Q_{ai} = \mathcal{Q}_{ai}, \tilde{\Pi}^{ai})$

Now, we fix the gauge for the pairs of variables $(Q_{aI}, \tilde{\Pi}^{aI})$ or $(\mathcal{Q}_{aI}, \tilde{\Pi}^{aI})$. We solve $\tilde{\mathcal{G}}^{i0} = 0$ from (3.39a) and (3.44a), and get

$$Q_{a0} = 0, \tag{3.67}$$

$$\mathcal{Q}_{a0} = -\sigma\epsilon m^0 \underset{\sim}{\Pi}_{ai} \partial_b \tilde{\Pi}^{bi}. \tag{3.68}$$

Thus, substituting (3.47) and the correspondent solution of $\tilde{\mathcal{G}}^{i0} = 0$ into (3.36) or (3.41), yields

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[2\tilde{\Pi}^{ai} \dot{Q}_{ai} - \left(2\lambda_i \tilde{\mathcal{G}}^i + 2N^a \tilde{\mathcal{D}}_a + \underset{\sim}{N} \tilde{\tilde{\mathcal{H}}}\right)\right], \tag{3.69}$$

where the constraints, derived either from (3.39) or (3.44), are

$$\tilde{\mathcal{G}}^i = \epsilon^i{}_{jk} Q_a{}^j \tilde{\Pi}^{ak} \approx 0, \tag{3.70a}$$

$$\tilde{\mathcal{D}}_a = 2\tilde{\Pi}^{bi} \partial_{[a} Q_{b]i} - U_{ai} \partial_b \tilde{\Pi}^{bi} \approx 0, \tag{3.70b}$$

$$\tilde{\tilde{\mathcal{H}}} = \sigma \epsilon_{ijk} \tilde{\Pi}^{ai} \tilde{\Pi}^{bj} R_{ab}{}^k + 2\tilde{\Pi}^{a[i|} \tilde{\Pi}^{b|j]} Q_{ai} Q_{bj} + 2\sigma\Lambda |\det(\tilde{\Pi}^{ai})|. \tag{3.70c}$$

This is precisely the $SO(3)$-ADM formulation [15, 16], which one obtains after performing the Hamiltonian formulation of the Palatini action plus the time gauge [32,57]. Thus, in the manifestly Lorentz-covariant formulations where the Barbero-Immirzi parameter is absent, they all collapse to the $SO(3)$-ADM formulation once the time gauge is taken into account.

Finally, notice that in the time gauge the inverse of the canonical transformations, either (3.35a) or (3.40a), become

$$\epsilon m^0 \gamma Q_{ai} = A_{ai} - \Gamma_{ai}, \tag{3.71}$$

where we defined A_{ai} as in (3.59). This canonical transformation is, up to the sign ϵm^0, the inverse of Barbero's canonical transformation [17]. Hence, (3.35a) and (3.40a) are indeed Lorentz-covariant versions of the inverse of the Barbero's canonical transformation.

3.5 Comments

We end the discussion remarking three main results of this chapter:

(i) We have solved, in a manifestly Lorentz-covariant fashion, the second-class constraints that arise during the Hamiltonian analysis of Holst action. From the constraint (3.4d), we identified the 12 independent variables $\tilde{\Pi}^{aI}$ that compose the original momentum $\tilde{\Pi}^{aIJ}$. Additionally, we split the 18 fields of the internal connection ω_{aIJ} into the 12 canonical variables C_{aI} and the six nondynamical variables $\underset{\sim}{\lambda}_{ab}$, the latter are fixed by (3.4e).

(ii) We exposed different sets of Hamiltonian formulations, all of them made of first-class constraints only and described by canonical conjugated variables that are explicitly Lorentz covariant. Furthermore, they relate to each other by canonical transformations, and two of them, namely the transformations of Eqs. (3.35) and (3.40), link the Holst to the Palatini action.

(iii) Finally, in the time gauge, we notice that the previous Lorentz-covariant formulations either collapse to the Ashtekar-Barbero formalism or to the $SO(3)$-ADM description; it depends on whether or not the Barbero-Immirzi parameter is present in the formulation.

All of the results presented during this chapter are new, and they were published in Ref. [56].

Chapter 4

$SU(1,1)$ phase-space variables from manifestly Lorentz-covariant phase-space variables for Holst action

In this chapter we explore an alternative gauge fixing; one that reduces the original Lorentz group into its subgroup $SU(1,1)$ [or $SO(2,1)$]. To accomplish it, we need to reformulate our description. Thus, we foliate the spacetime manifold with timelike leaves along a spacelike direction. From there, we continue with the usual Hamiltonian analysis and found the presence of second-class constraints. Using the techniques developed in the previous chapter, we solve the second-class constraints in a manifestly covariant fashion. Then, we impose the gauge condition known as "*space gauge*" and arrive at a formulation for general relativity described by an $SU(1,1)$ connection and a densitized triad.

The upcoming description is already published in Ref. [59].

4.1 Unusual Hamiltonian description

Since we want to arrive at a formulation invariant under $SU(1,1)$ [or $SO(2,1)$] transformations, we need to change some of the usual assumptions in order to have the correct physical interpretation. First, we consider a spacetime manifold M diffeomorphic to $\Omega \times \mathbb{R}$, where Ω is a 3-dimensional timelike submanifold that might have a boundary. Then, without loss of generality, we foliate the spacetime along the spacelike direction x^3, where each surface $x^3 = \texttt{constant}$ is diffeomorphic to Ω. The study of timelike foliations is not estrange in the context of loop quantum gravity or in the spin foams approach, see for instance Refs. [39, 60–62].

We start again from the Holst action with cosmological constant Λ. Using the γ-hat notation of (A.4), Holst action has the form

$$S[e,\omega] = \frac{1}{\kappa}\int_M \left[* \left(e^I \wedge e^J\right) \wedge \overset{(\gamma)}{F}{}_{IJ} - 2\Lambda\rho \right], \tag{4.1}$$

where the internal dual was defined in (A.2), $\rho = (1/4!)\epsilon_{IJKL}e^I \wedge e^J \wedge e^K \wedge e^L$ is the volume form, ϵ_{IJKL} being the totally antisymmetric Lorentz tensor ($\epsilon_{0123} = 1$), and $F^I{}_J$ is the curvature of the connection $\omega^I{}_J$ [see (2.2)]. Also, to deal with the internal indices, we restrict ourselves exclusively to the Minkowski metric $(\eta_{IJ}) = \mathrm{diag}(-1, 1, 1, 1)$.

The Hamiltonian analysis is as straightforward as the one presented in Sec. 2.2, we only need to make a few changes. We begin by defining the notion of "evolution" along the spacelike direction x^3. Then, we express the differential forms as

$$e^I = e_\mu{}^I dx^\mu = e_{\bar{a}}{}^I dx^{\bar{a}} + e_3{}^I dx^3, \tag{4.2a}$$

$$\omega^I{}_J = \omega_\mu{}^I{}_J dx^\mu = \omega_{\bar{a}}{}^I{}_J dx^{\bar{a}} + \omega_3{}^I{}_J dx^3. \tag{4.2b}$$

The bar over the indices indicate that they take the values $\bar{a} = \{0, 1, 2\}$. Now, we parametrize $e_3{}^I$ with the four fields N and $N^{\bar{a}}$ (analogous to the lapse function and shift vector of the usual $3+1$ decomposition)

$$e_3{}^I = Nn^I + N^{\bar{a}} e_{\bar{a}}{}^I, \tag{4.3}$$

where n^I is an internal vector satisfying $n^I n_I = 1$ and $n_I e_{\bar{a}}{}^I = 0$. The induced metric on Ω is $q_{\bar{a}\bar{b}} = \eta_{IJ} e_{\bar{a}}{}^I e_{\bar{b}}{}^J$, and its determinant $q = \det(q_{\bar{a}\bar{b}}) < 0$ since Ω is a timelike surface.

All of the previous considerations lead us to the action

$$S = \frac{1}{\kappa}\int_\Omega dV \int_{\mathbb{R}} dx^3 \left[\tilde{\Pi}^{\bar{a}IJ}\partial_3 \overset{(\gamma)}{\omega}{}_{\bar{a}IJ} - \tilde{H} + \partial_{\bar{a}}\left(\tilde{\Pi}^{\bar{a}IJ}\omega_{3IJ}\right)\right], \tag{4.4}$$

with $dV := dx^0 \wedge dx^1 \wedge dx^2$. Also, we took $\overset{(\gamma)}{\omega}{}_{\bar{a}IJ}$ as the configuration variable and identified its conjugated momentum as

$$\tilde{\Pi}^{\bar{a}IJ} := -\frac{1}{2}\tilde{\eta}^{\bar{a}\bar{b}\bar{c}}\epsilon^{IJ}{}_{KL} e_{\bar{b}}{}^K e_{\bar{c}}{}^L, \tag{4.5}$$

where $\tilde{\eta}^{\bar{a}\bar{b}\bar{c}} := \tilde{\eta}^{\bar{a}\bar{b}\bar{c}3}$ is a totally antisymmetric tensor density ($\tilde{\eta}^{012} = +1$). Therefore, The action is described by the variables $(\overset{(\gamma)}{\omega}{}_{\bar{a}IJ}, \tilde{\Pi}^{\bar{a}IJ})$ [or equivalently $(\omega_{\bar{a}IJ}, \overset{(\gamma)}{\tilde{\Pi}}{}^{\bar{a}IJ})$], and they satisfy the commutation relation

$$\left\{\overset{(\gamma)}{\omega}{}_{\bar{a}IJ}(x, x^3), \tilde{\Pi}^{\bar{b}KL}(y, x^3)\right\} = \kappa\delta^{\bar{b}}_{\bar{a}}\delta^K_{[I}\delta^L_{J]}\delta^3(x, y), \tag{4.6}$$

where x and y are points in Ω. Also, since the 3-dimensional manifold might have a boundary, we will maintain the boundary terms throughout the entire analysis. Continuing with the analysis, $\tilde{H}$ is the Hamiltonian density given by

$$\tilde{H} := -\omega_{3IJ}\tilde{\mathcal{G}}^{IJ} + N^{\bar{a}}\tilde{\mathcal{V}}_{\bar{a}} + \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}}, \tag{4.7}$$

where ω_{3IJ}, $N^{\bar{a}}$, and $\underset{\sim}{N} := |q|^{-1}N$ are Lagrange multipliers imposing the constraints

$$\begin{aligned}
\tilde{\mathcal{G}}^{IJ} &:= D_{\bar{a}}\overset{(\gamma)}{\tilde{\Pi}}{}^{\bar{a}IJ} = \partial_{\bar{a}}\overset{(\gamma)}{\tilde{\Pi}}{}^{\bar{a}IJ} + 2\omega_{\bar{a}}{}^{[I|}{}_{K}\overset{(\gamma)}{\tilde{\Pi}}{}^{\bar{a}K|J]} \approx 0, && (4.8a)\\
\tilde{\mathcal{V}}_{\bar{a}} &:= \overset{(\gamma)}{\tilde{\Pi}}{}^{\bar{b}IJ}F_{\bar{a}\bar{b}IJ} \approx 0, && (4.8b)\\
\tilde{\tilde{\mathcal{S}}} &:= \tilde{\Pi}^{\bar{a}IK}\tilde{\Pi}^{\bar{b}}{}_{K}{}^{J}\overset{(\gamma)}{F}_{\bar{a}\bar{b}IJ} - 2q\Lambda \approx 0, && (4.8c)
\end{aligned}$$

where $F_{\bar{a}\bar{b}IJ} := \partial_{\bar{a}}\omega_{\bar{b}IJ} - \partial_{\bar{b}}\omega_{\bar{a}IJ} + \omega_{\bar{a}IK}\omega_{\bar{b}}{}^{K}{}_{J} - \omega_{\bar{b}IK}\omega_{\bar{a}}{}^{K}{}_{J}$, and q is related to $\tilde{\Pi}^{\bar{a}IJ}$ through

$$qq^{\bar{a}\bar{b}} = -\frac{1}{2}\tilde{\Pi}^{\bar{a}IJ}\tilde{\Pi}^{\bar{b}}{}_{IJ}. \tag{4.9}$$

Furthermore, the definition of momentum $\tilde{\Pi}^{\bar{a}IJ}$ introduces six new constraints whose evolution leads us to add another six secondary constraints. Thus, the theory is in fact described by the action

$$S = \frac{1}{\kappa}\int_{\Omega} dV \int_{\mathbb{R}} dx^3 \left[\overset{(\gamma)}{\tilde{\Pi}}{}^{\bar{a}IJ}\partial_3\omega_{\bar{a}IJ} - \tilde{H}_T + \partial_{\bar{a}}\left(\tilde{\Pi}^{\bar{a}IJ}\omega_{3IJ}\right)\right]. \tag{4.10}$$

The total Hamiltonian,

$$\tilde{H}_T := -\omega_{3IJ}\tilde{\mathcal{G}}^{IJ} + N^{\bar{a}}\tilde{\mathcal{V}}_{\bar{a}} + \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}} + \underset{\sim}{\phi}_{\bar{a}\bar{b}}\tilde{\tilde{\Phi}}^{\bar{a}\bar{b}} + \psi_{\bar{a}\bar{b}}\Psi^{\bar{a}\bar{b}}, \tag{4.11}$$

is formed by the linear combination of constraints (4.8a), (4.8b), (4.8c), and

$$\begin{aligned}
\tilde{\tilde{\Phi}}^{\bar{a}\bar{b}} &:= *\tilde{\Pi}^{\bar{a}}{}_{IJ}\tilde{\Pi}^{\bar{b}IJ} \approx 0, && (4.12a)\\
\Psi^{\bar{a}\bar{b}} &:= \epsilon_{IJKL}\tilde{\Pi}^{(\bar{a}|IM}\tilde{\Pi}^{\bar{c}}{}_{M}{}^{J}D_{\bar{c}}\tilde{\Pi}^{|\bar{b})KL} \approx 0. && (4.12b)
\end{aligned}$$

$\underset{\sim}{\phi}_{\bar{a}\bar{b}}$, and $\psi_{\bar{a}\bar{b}}$ (of weight -2, so that $\Psi^{\bar{a}\bar{b}}$ has weight +3) are also Lagrange multipliers. As in the usual case, $\tilde{\mathcal{G}}^{IJ}$, $\tilde{\mathcal{V}}_{\bar{a}}$, and $\tilde{\tilde{\mathcal{S}}}$ are the first-class constraints that generate the gauge symmetries of the theory (local Lorentz transformations and spacetime diffeomorphisms) whereas $\tilde{\tilde{\Phi}}^{\bar{a}\bar{b}}$ and $\Psi^{\bar{a}\bar{b}}$ are second class. We deal with them in next section.

Notice that the Hamiltonian description given here is the same as the one in the previous chapter (see Sec. 3.1, keep in mind that here $\sigma = -1$). Therefore, the form of the constraints does not depend on the foliation considered, they maintain the same functional form. This is a direct manifestation of the diffeomorphism invariance of general relativity.

4.2 Solution of the second-class constraints: manifestly Lorentz-covariant phase-space variables

The next step is to get rid of the second-class constraints. Since we follow the same method described in the previous chapter, we present only the main results. Thus, the solution to $\tilde{\tilde{\Phi}}^{\bar{a}\bar{b}} = 0$ is

$$\tilde{\Pi}^{\bar{a}IJ} = 2\epsilon\tilde{\Pi}^{\bar{a}[I}m^{J]}, \tag{4.13}$$

where $\epsilon = \pm 1$ and m^I is an arbitrary vector. We choose m_I such that it fulfills the two properties: $m_I m^I = 1$ and $m_I\tilde{\Pi}^{\bar{a}I} = 0$. Explicitly, it is

$$m_I := \frac{1}{6\sqrt{|h|}}\epsilon_{IJKL}\underset{\sim}{\eta}_{\bar{a}\bar{b}\bar{c}}\tilde{\Pi}^{\bar{a}J}\tilde{\Pi}^{\bar{b}K}\tilde{\Pi}^{\bar{c}L}, \tag{4.14}$$

where $h := \det(\tilde{\tilde{h}}^{\bar{a}\bar{b}}) < 0$ and $\tilde{\tilde{h}}^{\bar{a}\bar{b}} := \tilde{\Pi}^{\bar{a}I}\tilde{\Pi}^{\bar{b}}{}_I$. Notice that we use a different normalization factor for m_I, which is required to have consistency with the gauge fixing we are going to consider in next section.

As in the previous chapter, we introduce two important quantities. The first one is the projector onto the plane orthogonal to m_I

$$q^I{}_J := \underset{\approx}{h}_{\bar{a}\bar{b}}\tilde{\Pi}^{\bar{a}I}\tilde{\Pi}^{\bar{b}}{}_J = \delta^I_J - m^I m_J, \tag{4.15}$$

where $\underset{\approx}{h}_{\bar{a}\bar{b}}$ is the inverse of $\tilde{\tilde{h}}^{\bar{a}\bar{b}}$ ($\underset{\approx}{h}_{\bar{a}\bar{c}}\tilde{\tilde{h}}^{\bar{c}\bar{b}} = \delta^{\bar{b}}_{\bar{a}}$). The second one is the covariant derivative compatible with $\tilde{\Pi}^{\bar{a}I}$

$$\nabla_{\bar{a}}\tilde{\Pi}^{\bar{b}I} := \partial_{\bar{a}}\tilde{\Pi}^{\bar{b}I} + \Gamma_{\bar{a}}{}^I{}_J\tilde{\Pi}^{\bar{b}J} + \Gamma^{\bar{b}}{}_{\bar{a}\bar{c}}\tilde{\Pi}^{\bar{c}I} - \Gamma^{\bar{c}}{}_{\bar{a}\bar{c}}\tilde{\Pi}^{\bar{b}I} = 0, \tag{4.16}$$

which is a system of 36 equations for the 18 + 18 components of $\Gamma_{\bar{a}IJ} = -\Gamma_{\bar{a}JI}$ and $\Gamma^{\bar{a}}{}_{\bar{b}\bar{c}} = \Gamma^{\bar{a}}{}_{\bar{c}\bar{b}}$. Their explicit form is similar to the expressions found in (3.13) and (3.14), respectively, we only need add a bar over the lowercase indices. Furthermore, the curvature of the internal connection $\Gamma_{\bar{a}IJ}$ is

$$R_{\bar{a}\bar{b}}{}^I{}_J = \partial_{\bar{a}}\Gamma_{\bar{b}}{}^I{}_J - \partial_{\bar{b}}\Gamma_{\bar{a}}{}^I{}_J + \Gamma_{\bar{a}}{}^I{}_K\Gamma_{\bar{b}}{}^K{}_J - \Gamma_{\bar{b}}{}^I{}_K\Gamma_{\bar{a}}{}^K{}_J. \tag{4.17}$$

On the other hand, before we solve the remaining constraint, we complement the six constraints in (4.12b) with the definition of $C_{\bar{a}I}$ that involves 12 equations

$$C_{\bar{a}I} := \epsilon \left(\overset{(\gamma)}{\omega}_{\bar{a}IJ}\, m^J + m_I\, \overset{(\gamma)}{\omega}_{\bar{b}JK}\, \underset{\approx}{h}_{\bar{a}\bar{c}} \tilde{\Pi}^{\bar{c}J} \tilde{\Pi}^{\bar{b}K} \right). \tag{4.18}$$

Thus, solving (4.12b) and (4.18) jointly, we arrive at the solution for $\overset{(\gamma)}{\omega}_{\bar{a}IJ}$

$$\overset{(\gamma)}{\omega}_{\bar{a}IJ} = M_{\bar{a}}{}^{\bar{b}}{}_{IJK} C_{\bar{b}}{}^K + \underset{\sim}{\lambda}_{\bar{a}\bar{b}} \tilde{N}^{\bar{b}}{}_{IJ}, \tag{4.19}$$

where

$$\begin{aligned} M_{\bar{a}}{}^{\bar{b}}{}_{IJK} &:= \epsilon \Big[-\delta^{\bar{b}}_{\bar{a}} m_{[I}\eta_{J]K} + \delta^{\bar{b}}_{\bar{a}} P_{IJKL} m^L + \underset{\approx}{h}_{\bar{a}\bar{c}} \tilde{\Pi}^{\bar{b}}{}_{[I} \tilde{\Pi}^{\bar{c}}{}_{J]} m_K \\ &\quad + \frac{1}{2\gamma} \epsilon_{IJLM} \underset{\approx}{h}_{\bar{a}\bar{c}} \tilde{\Pi}^{\bar{c}}{}_K \tilde{\Pi}^{\bar{b}L} m^M \Big], \qquad (4.20) \\ \tilde{N}^{\bar{a}}{}_{IJ} &:= \epsilon_{IJKL} \tilde{\Pi}^{\bar{a}K} m^L, \qquad (4.21) \\ \underset{\sim}{\lambda}_{\bar{a}\bar{b}} &:= -\frac{1}{2} \epsilon_{IJKL} \left(\underset{\approx}{h}_{\bar{a}\bar{b}} \underset{\approx}{h}_{\bar{c}\bar{d}} - 2 \underset{\approx}{h}_{\bar{c}(\bar{a}} \underset{\approx}{h}_{\bar{b})\bar{d}} \right) \tilde{\Pi}^{\bar{c}I} \tilde{\Pi}^{\bar{f}J} \tilde{\Pi}^{\bar{d}M} m^L \Gamma_{\bar{f}}{}^K{}_M. \qquad (4.22) \end{aligned}$$

The solutions (4.13) and (4.19) reduce the action (4.10) to

$$S = \frac{1}{\kappa} \int_{\Omega} dV \int_{\mathbb{R}} dx^3 \left[2\tilde{\Pi}^{\bar{a}I} \partial_3 C_{\bar{a}I} - \left(-\omega_{3IJ} \tilde{\mathcal{G}}^{IJ} + N^{\bar{a}} \tilde{\mathcal{V}}_{\bar{a}} + \underset{\sim}{N} \tilde{\tilde{\mathcal{S}}} \right) + \partial_{\bar{a}} \left(\tilde{\Pi}^{\bar{a}IJ} \omega_{3IJ} \right) \right], \tag{4.23}$$

which is formed entirely by the first-class constraints

$$\begin{aligned} \tilde{\mathcal{G}}^{IJ} &= 2\tilde{\Pi}^{\bar{a}[I} C_{\bar{a}}{}^{J]} + 4\epsilon P^{IJ}{}_{KL} \tilde{\Pi}^{\bar{a}[M} m^{K]} \Gamma_{\bar{a}}{}^L{}_M \approx 0, \qquad (4.24a) \\ \tilde{\mathcal{V}}_{\bar{a}} &= 4\nabla_{[\bar{a}} \left(C_{\bar{b}]I} \tilde{\Pi}^{\bar{b}I} \right) - 4\epsilon \tilde{\Pi}^{\bar{b}[I} m^{J]} \overset{(\gamma)}{\Gamma}_{\bar{a}IK} \Gamma_{\bar{b}}{}^K{}_J + \epsilon \tilde{\mathcal{G}}^{IJ} m_J \Big[2C_{\bar{a}I} \\ &\quad -2\epsilon m^K \overset{(\gamma)}{\Gamma}_{\bar{a}IK} + \underset{\approx}{h}_{\bar{a}\bar{b}} \tilde{\Pi}^{\bar{b}K} \tilde{\mathcal{G}}_{IK} \Big] \approx 0, \qquad (4.24b) \\ \tilde{\tilde{\mathcal{S}}} &= -\tilde{\Pi}^{\bar{a}I} \tilde{\Pi}^{\bar{b}J} R_{\bar{a}\bar{b}IJ} + 2\tilde{\Pi}^{\bar{a}[I|} \tilde{\Pi}^{\bar{b}|J]} \Big[C_{\bar{a}I} C_{\bar{b}J} - 2\epsilon C_{\bar{a}I} m^K \overset{(\gamma)}{\Gamma}_{\bar{b}JK} \\ &\quad + \left(\Gamma_{\bar{a}IL} + \frac{2}{\gamma} * \Gamma_{\bar{a}IL} \right) \Gamma_{\bar{b}JK} m^K m^L - \frac{1}{\gamma^2} q^{KL} \Gamma_{\bar{a}IK} \Gamma_{\bar{b}JL} \Big] \\ &\quad + \tilde{\mathcal{G}}^{IJ} \Big[-\frac{1}{4} \tilde{\mathcal{G}}_{IJ} + \frac{1}{4} \left(P^{-1} \right)_{IJKL} \tilde{\mathcal{G}}^{KL} - \frac{1}{2} m_I m^K \tilde{\mathcal{G}}_{JK} \Big] \\ &\quad -2\epsilon \tilde{\Pi}^{\bar{a}I} m^J \nabla_{\bar{a}} \tilde{\mathcal{G}}_{IJ} + 2\sqrt{|h|} \Lambda \approx 0. \qquad (4.24c) \end{aligned}$$

Before we collect all the terms proportional to the Gauss constraint, we rewrite the vector constraint as

$$\tilde{\mathcal{V}}_{\bar{a}} = 2\tilde{\mathcal{D}}_{\bar{a}} + \tilde{\mathcal{G}}^{IJ}\left(\Gamma_{\bar{a}IJ} + 2\epsilon C_{\bar{a}I}m_J + 2m_I m^K \overset{(\gamma)}{\Gamma}{}_{\bar{a}JK} - \epsilon \underset{\approx}{h}_{\bar{a}\bar{b}}\tilde{\Pi}^{\bar{b}K}m_I\tilde{\mathcal{G}}_{JK}\right), \tag{4.25}$$

where $\tilde{\mathcal{D}}_a$ is the diffeomorphism constraint given by

$$\tilde{\mathcal{D}}_{\bar{a}} := 2\tilde{\Pi}^{\bar{b}I}\partial_{[\bar{a}}C_{\bar{b}]I} - C_{\bar{a}I}\partial_{\bar{b}}\tilde{\Pi}^{\bar{b}I}. \tag{4.26}$$

Next, we integrate by parts the term containing the derivative of $\tilde{\mathcal{G}}^{IJ}$ in (4.24c) and redefine the Lagrange multiplier as

$$\begin{aligned}\lambda_{IJ} \quad := \quad & -\omega_{3IJ} + N^{\bar{a}}\left(\Gamma_{\bar{a}IJ} + 2\epsilon C_{\bar{a}[I}m_{J]} + 2m_{[I|}m^K \overset{(\gamma)}{\Gamma}{}_{\bar{a}|J]K} - \epsilon \underset{\approx}{h}_{\bar{a}\bar{b}}\tilde{\Pi}^{\bar{b}K}m_{[I}\tilde{\mathcal{G}}_{J]K}\right)\\ & + \underset{\sim}{N}\left[-\frac{1}{4}\tilde{\mathcal{G}}_{IJ} + \frac{1}{4}\left(P^{-1}\right)_{IJKL}\tilde{\mathcal{G}}^{KL} - \frac{1}{2}m_{[I}m^K\tilde{\mathcal{G}}_{J]K}\right] + 2\epsilon\tilde{\Pi}^{\bar{a}}{}_{[I}m_{J]}\nabla_{\bar{a}}\underset{\sim}{N}.\end{aligned} \tag{4.27}$$

Thus, the action becomes

$$\begin{aligned}S \quad = \quad & \frac{1}{\kappa}\int_{\Omega} dV \int_{\mathbb{R}} dx^3 \left[2\tilde{\Pi}^{\bar{a}I}\partial_3 C_{\bar{a}I} - \left(\lambda_{IJ}\tilde{\mathcal{G}}^{IJ} + 2N^{\bar{a}}\tilde{\mathcal{D}}_{\bar{a}} + \underset{\sim}{N}\tilde{\tilde{\mathcal{H}}}\right)\right.\\ & \left. + \partial_{\bar{a}}\left(2\epsilon\tilde{\Pi}^{\bar{a}I}m^J\omega_{3IJ} + 2\epsilon\underset{\sim}{N}\tilde{\Pi}^{\bar{a}I}m^J\tilde{\mathcal{G}}_{IJ}\right)\right],\end{aligned} \tag{4.28}$$

where the Gauss and diffeomorphism constraints are given in (4.24a) and (4.26), respectively, and the Hamiltonian constraint is given by

$$\begin{aligned}\tilde{\tilde{\mathcal{H}}} \quad = \quad & -\tilde{\Pi}^{\bar{a}I}\tilde{\Pi}^{\bar{b}J}R_{\bar{a}\bar{b}IJ} + 2\tilde{\Pi}^{\bar{a}[I|}\tilde{\Pi}^{\bar{b}|J]}\left[C_{\bar{a}I}C_{\bar{b}J} - 2\epsilon C_{\bar{a}I}m^K \overset{(\gamma)}{\Gamma}{}_{\bar{b}JK}\right.\\ & \left. + \left(\Gamma_{\bar{a}IL} + \frac{2}{\gamma} * \Gamma_{\bar{a}IL}\right)\Gamma_{\bar{b}JK}m^K m^L - \frac{1}{\gamma^2}q^{KL}\Gamma_{\bar{a}IK}\Gamma_{\bar{b}JL}\right] + 2\sqrt{|h|}\Lambda.\end{aligned} \tag{4.29}$$

At this point, we are tempted to explore the different sets of canonical formulations presented in Sec. 3.3. However, there is no much insight we can gain from displaying them here. We just remind the reader that under those transformations, we must keep track of the boundary terms that arise during the canonical transformations.

4.3 Gauge fixing: space gauge

The group $SU(1,1)$ [or $SO(2,1)$] is one of the subgroups belonging to Lorentz group, whose generators are two boost transformations and one rotation. We can derive a Hamiltonian description invariant under local $SU(1,1)$ transformations if we consider the gauge condition

$$\tilde{\Pi}^{\bar{a}3} = 0. \tag{4.30}$$

Using Eq. (4.14), the former condition implies $m^{\bar{i}} = 0$ and $m^3 = -\text{sgn}[\det(\tilde{\Pi}^{\bar{a}\bar{i}})]$, for $\bar{i} = \{0,\, 1,\, 2, \}$. Therefore, the only nonzero component of m_I is along an internal spatial direction, so we shall name this gauge condition the "*space gauge*". Moreover, from the normalization of m_I, we have

$$m_I m^I = m_3 m^3 = \left(m^3\right)^2 = 1. \tag{4.31}$$

Thus, in order to keep a real description, m_I must be a spacelike vector. This is the reason why we have assumed it since the very beginning.

Continuing with the analysis, the Poisson bracket of the condition (4.30) with $\tilde{\mathcal{G}}^{\bar{i}3}$ is not zero; it is

$$\left\{\tilde{\Pi}^{\bar{a}3}(x,x^3),\, \tilde{\mathcal{G}}^{\bar{i}3}(y,x^3)\right\} = -\frac{\kappa}{2}\tilde{\Pi}^{\bar{a}\bar{i}}\delta^3(x,y). \tag{4.32}$$

As a result, since $\det(\tilde{\Pi}^{\bar{a}\bar{i}}) \neq 0$, $\tilde{\mathcal{G}}^{\bar{i}3} = 0$ and Eq. (4.30) are second-class constraints. Fixing $\tilde{\mathcal{G}}^{\bar{i}3}$ drops out the freedom to perform boost transformations along the x^3 axis and the rotations around axes x^1 and x^2. On the other hand, we define

$$\tilde{\mathcal{G}}_{\bar{i}} := -\frac{1}{2}\epsilon_{\bar{i}\bar{j}\bar{k}}\tilde{\mathcal{G}}^{\bar{j}\bar{k}}, \tag{4.33}$$

where $\epsilon_{\bar{i}\bar{j}\bar{k}} := \epsilon_{\bar{i}\bar{j}\bar{k}3}$, and then, the remaining Gauss constraint $\tilde{\mathcal{G}}_{\bar{i}}$ obeys the algebra

$$\left\{\tilde{\mathcal{G}}_{\bar{i}}\,(x,x^3),\, \tilde{\mathcal{G}}_{\bar{j}}\,(y,x^3)\right\} = \frac{\kappa}{2}\epsilon_{\bar{i}\bar{j}}{}^{\bar{k}}\tilde{\mathcal{G}}_{\bar{k}}\delta^3(x,y), \tag{4.34}$$

which corresponds to the Lie algebra of the group $SU(1,1)$ [or $SO(2,1)$]. The constraint $\tilde{\mathcal{G}}_{\bar{i}}$ generates rotations around the x^3 axis and boost transformations along the axes x^1 and x^2.

Under the space gauge, the internal connection $\Gamma_{\bar{a}IJ}$ splits into two parts: $\Gamma_{\bar{a}3\bar{i}} = 0$ and $\Gamma_{\bar{a}\bar{i}} := -(1/2)\epsilon_{\bar{i}\bar{j}\bar{k}}\Gamma_{\bar{a}}{}^{\bar{j}\bar{k}}$; the latter being the spin connection compatible with $\tilde{\Pi}^{\bar{a}\bar{i}}$ since, from (4.16), it satisfies

$$\nabla_{\bar{a}}\tilde{\Pi}^{\bar{b}\bar{i}} = \partial_{\bar{a}}\tilde{\Pi}^{\bar{b}\bar{i}} + \Gamma^{\bar{b}}{}_{\bar{a}\bar{c}}\tilde{\Pi}^{\bar{c}\bar{i}} - \Gamma^{\bar{c}}{}_{\bar{a}\bar{c}}\tilde{\Pi}^{\bar{b}\bar{i}} + \epsilon^{\bar{i}}{}_{\bar{j}\bar{k}}\Gamma_{\bar{a}}{}^{\bar{j}}\tilde{\Pi}^{\bar{b}\bar{k}} = 0. \tag{4.35}$$

Explicitly it is

$$\Gamma_{\bar{a}\bar{i}} = \epsilon_{\bar{i}\bar{j}\bar{k}}\left(\partial_{[\bar{b}}\underset{\sim}{\Pi}_{\bar{a}]}{}^{\bar{j}} + \underset{\sim}{\Pi}_{\bar{a}}{}^{[\bar{l}|}\tilde{\Pi}^{\bar{c}|\bar{j}]}\partial_{\bar{b}}\underset{\sim}{\Pi}_{\bar{c}\bar{l}}\right)\tilde{\Pi}^{\bar{b}\bar{k}}, \tag{4.36}$$

where $\underset{\sim}{\Pi}_{\bar{a}\bar{i}}$ is the inverse of $\tilde{\Pi}^{\bar{a}\bar{i}}$ ($\underset{\sim}{\Pi}_{\bar{a}\bar{i}}\tilde{\Pi}^{\bar{b}\bar{i}} = \delta^{\bar{b}}_{\bar{a}}$ and $\underset{\sim}{\Pi}_{\bar{a}\bar{i}}\tilde{\Pi}^{\bar{a}\bar{j}} = \delta^{\bar{j}}_{\bar{i}}$). Furthermore, the field strength of $\Gamma_{\bar{a}\bar{i}}$ is

$$R_{\bar{a}\bar{b}\bar{i}} := -\frac{1}{2}\epsilon_{\bar{i}\bar{j}\bar{k}}R_{\bar{a}\bar{b}}{}^{\bar{j}\bar{k}} = \partial_{\bar{a}}\Gamma_{\bar{b}\bar{i}} - \partial_{\bar{b}}\Gamma_{\bar{a}\bar{i}} + \epsilon_{\bar{i}\bar{j}\bar{k}}\Gamma_{\bar{a}}{}^{\bar{j}}\Gamma_{\bar{b}}{}^{\bar{k}}. \tag{4.37}$$

Moving forward, we solve $\tilde{\mathcal{G}}^{i3} = 0$ and obtain

$$C_{\bar{a}3} = -\epsilon m^3\underset{\sim}{\Pi}_{\bar{a}\bar{i}}\partial_{\bar{b}}\tilde{\Pi}^{\bar{b}\bar{i}}. \tag{4.38}$$

Substituting (4.30) and (4.38) into the action (4.28), it acquires the form

$$S = \frac{1}{\kappa}\int_\Omega dV\int_{\mathbb{R}} dx^3\left[2\tilde{\Pi}^{\bar{a}i}\partial_3 C_{\bar{a}i} - \left(2\lambda_{\bar{i}}\tilde{\mathcal{G}}^{\bar{i}} + 2N^{\bar{a}}\tilde{\mathcal{D}}_{\bar{a}} + \underset{\sim}{N}\tilde{\tilde{\mathcal{H}}}\right) + \partial_{\bar{a}}\left(2\epsilon\tilde{\Pi}^{\bar{a}\bar{i}}m^3\omega_{3\bar{i}3}\right)\right], \tag{4.39}$$

where $\lambda_{\bar{i}} := -(1/2)\epsilon_{\bar{i}\bar{j}\bar{k}}\lambda^{\bar{j}\bar{k}}$ and the constraints are

$$\begin{aligned}
\tilde{\mathcal{G}}^{\bar{i}} &= \epsilon\frac{m^3}{\gamma}\left[\partial_{\bar{a}}\tilde{\Pi}^{\bar{a}\bar{i}} + \epsilon^{\bar{i}}{}_{\bar{j}\bar{k}}(\epsilon m^3\gamma C_{\bar{a}}{}^{\bar{j}})\tilde{\Pi}^{\bar{a}\bar{k}}\right], &(4.40\text{a})\\
\tilde{\mathcal{D}}_{\bar{a}} &= 2\tilde{\Pi}^{\bar{a}\bar{i}}\partial_{[\bar{a}}C_{\bar{b}]\bar{i}} - C_{\bar{a}\bar{i}}\partial_{\bar{b}}\tilde{\Pi}^{\bar{a}\bar{i}}, &(4.40\text{b})\\
\tilde{\tilde{\mathcal{H}}} &= \epsilon_{\bar{i}\bar{j}\bar{k}}\tilde{\Pi}^{\bar{a}\bar{i}}\tilde{\Pi}^{\bar{b}\bar{j}}R_{\bar{a}\bar{b}}{}^{\bar{k}} + 2\tilde{\Pi}^{\bar{a}[\bar{i}|}\tilde{\Pi}^{\bar{b}|\bar{j}]}\left(C_{\bar{a}\bar{i}} - \frac{\epsilon m^3}{\gamma}\Gamma_{\bar{a}\bar{i}}\right)\left(C_{\bar{b}\bar{j}} - \frac{\epsilon m^3}{\gamma}\Gamma_{\bar{b}\bar{j}}\right)\\
&\quad +2\Lambda|\det(\tilde{\Pi}^{\bar{a}\bar{i}})|. &(4.40\text{c})
\end{aligned}$$

From Eq. (4.40a) we identify the connection

$$A_{\bar{a}\bar{i}} := \epsilon\gamma m^3 C_{\bar{a}\bar{i}}. \tag{4.41}$$

Thus, using $A_{\bar{a}\bar{i}}$ as our configuration variable, the action (4.39) is rewritten as

$$\begin{aligned}
S &= \frac{1}{\kappa}\int_\Omega dV\int_{\mathbb{R}} dx^3\left[\frac{2\epsilon m^3}{\gamma}\tilde{\Pi}^{\bar{a}i}\partial_3 A_{\bar{a}i} - \left(2\lambda_{\bar{i}}\tilde{\mathcal{G}}^{\bar{i}} + 2N^{\bar{a}}\tilde{\mathcal{D}}_{\bar{a}} + \underset{\sim}{N}\tilde{\tilde{\mathcal{H}}}\right)\right.\\
&\quad \left.+\partial_{\bar{a}}\left(2\epsilon\tilde{\Pi}^{\bar{a}\bar{i}}m^3\omega_{3\bar{i}3}\right)\right], \qquad (4.42)
\end{aligned}$$

and the constraints are given by

$$\tilde{\mathcal{G}}^{\bar{i}} = \epsilon\frac{m^3}{\gamma}\left[\partial_{\bar{a}}\tilde{\Pi}^{\bar{a}\bar{i}} + \epsilon^{\bar{i}}{}_{\bar{j}\bar{k}}A_{\bar{a}}{}^{\bar{j}}\tilde{\Pi}^{\bar{a}\bar{k}}\right], \tag{4.43a}$$

$$\tilde{\mathcal{D}}_{\bar{a}} = \epsilon\frac{m^3}{\gamma}\left(2\tilde{\Pi}^{\bar{b}\bar{i}}\partial_{[\bar{a}}A_{\bar{b}]\bar{i}} - A_{\bar{a}\bar{i}}\partial_{\bar{b}}\tilde{\Pi}^{\bar{b}\bar{i}}\right), \tag{4.43b}$$

$$\begin{aligned}\tilde{\tilde{\mathcal{H}}} &= \frac{1}{\gamma^2}\epsilon_{\bar{i}\bar{j}\bar{k}}\tilde{\Pi}^{\bar{a}\bar{i}}\tilde{\Pi}^{\bar{b}\bar{j}}\left[F_{\bar{a}\bar{b}}{}^{\bar{k}} - \left(\gamma^2+1\right)R_{\bar{a}\bar{b}}{}^{\bar{k}}\right] + 2\epsilon\frac{m^3}{\gamma}\tilde{\Pi}^{\bar{a}\bar{i}}\nabla_{\bar{a}}\tilde{\mathcal{G}}_i \\ &\quad +2\Lambda|\det(\tilde{\Pi}^{\bar{a}\bar{i}})|,\end{aligned} \tag{4.43c}$$

where we used the identity

$$2\nabla_{[\bar{a}}\left(A_{\bar{b}]\bar{i}} - \Gamma_{\bar{b}]\bar{i}}\right) = F_{\bar{a}\bar{b}\bar{i}} - R_{\bar{a}\bar{b}\bar{i}} - \epsilon_{\bar{i}\bar{j}\bar{k}}(A_{\bar{a}}{}^{\bar{j}} - \Gamma_{\bar{a}}{}^{\bar{j}})(A_{\bar{b}}{}^{\bar{k}} - \Gamma_{\bar{b}}{}^{\bar{k}}). \tag{4.44}$$

Alternatively, as in Sec. 2.5, we can use the vector constraint instead of the diffeomorphism constraint

$$\tilde{\mathcal{C}}_{\bar{a}} := \tilde{\mathcal{D}}_{\bar{a}} + A_{\bar{a}\bar{i}}\tilde{\mathcal{G}}^{\bar{i}} = \epsilon\frac{m^3}{\gamma}\tilde{\Pi}^{\bar{b}\bar{i}}F_{\bar{a}\bar{b}\bar{i}}, \tag{4.45}$$

which requires a redefinition of the Lagrange multiplier

$$\mu_{\bar{i}} := \lambda_{\bar{i}} - N^{\bar{a}}A_{\bar{a}\bar{i}} - \frac{1}{\gamma}\tilde{\Pi}^{\bar{a}}{}_{\bar{i}}\nabla_{\bar{a}}\underset{\sim}{N}. \tag{4.46}$$

Then, we integrate by parts the term with the covariant derivative in (4.43c) and, with the redefinition of the Lagrange multiplier $\mu_{\bar{i}}$, we arrive at

$$\begin{aligned}S &= \frac{1}{\kappa}\int_\Omega dV\int_{\mathbb{R}} dx^3\left[\frac{2\epsilon m^3}{\gamma}\tilde{\Pi}^{\bar{a}i}\partial_3 A_{\bar{a}i} - \left(2\mu_{\bar{i}}\tilde{\mathcal{G}}^{\bar{i}} + 2N^{\bar{a}}\tilde{\mathcal{C}}_{\bar{a}} + \underset{\sim}{N}\tilde{\tilde{\mathcal{C}}}\right)\right. \\ &\quad \left. +\partial_{\bar{a}}\left(2\epsilon\tilde{\Pi}^{\bar{a}\bar{i}}m^3\omega_{3\bar{i}3} - 2\epsilon\frac{m^3}{\gamma}\underset{\sim}{N}\tilde{\Pi}^{\bar{a}\bar{i}}\tilde{\mathcal{G}}_{\bar{i}}\right)\right],\end{aligned} \tag{4.47}$$

where the scalar constraint $\tilde{\tilde{\mathcal{C}}}$ is given by

$$\tilde{\tilde{\mathcal{C}}} = \frac{1}{\gamma^2}\epsilon_{\bar{i}\bar{j}\bar{k}}\tilde{\Pi}^{\bar{a}\bar{i}}\tilde{\Pi}^{\bar{b}\bar{j}}\left[F_{\bar{a}\bar{b}}{}^{\bar{k}} - \left(\gamma^2+1\right)R_{\bar{a}\bar{b}}{}^{\bar{k}}\right] + 2\Lambda|\det(\tilde{\Pi}^{\bar{a}\bar{i}})|. \tag{4.48}$$

The Hamiltonian formulation is encompassed by the action (4.47) and the constraints (4.43a), (4.45), and (4.48). It is what we call the Ashtekar-Barbero-like formulation. It resembles the original formulation reported in Ref. [17], but it is constructed with $SU(1,1)$-[$SO(2,1)$]-covariant objects instead of the $SU(2)$ [$SO(3)$] fields of the original approach. Although the group $SU(1,1)$ is not compact, it might be possible to implement this description into the canonical quantization program (see Ref. [63]).

Finally, for completeness purposes, if we had applied the space gauge in a formulation described by the variables $(Q_{\bar{a}I}, \tilde{\Pi}^{\bar{a}I})$ or $(\mathcal{Q}_{\bar{a}I}, \tilde{\Pi}^{\bar{a}I})$, we would have obtained the formulation characterized by the constraints

$$\tilde{\mathcal{G}}^{\bar{i}} = \epsilon^{\bar{i}}{}_{\bar{j}\bar{k}} Q_{\bar{a}}{}^{\bar{j}} \tilde{\Pi}^{\bar{a}\bar{k}} \approx 0, \tag{4.49a}$$

$$\tilde{\mathcal{D}}_{\bar{a}} = 2\tilde{\Pi}^{\bar{b}\bar{i}} \partial_{[\bar{a}} Q_{\bar{b}]\bar{i}} - Q_{\bar{a}\bar{i}} \partial_{\bar{b}} \tilde{\Pi}^{\bar{b}\bar{i}} \approx 0, \tag{4.49b}$$

$$\tilde{\tilde{\mathcal{H}}} = \epsilon_{\bar{i}\bar{j}\bar{k}} \tilde{\Pi}^{\bar{a}\bar{i}} \tilde{\Pi}^{\bar{b}\bar{j}} R_{\bar{a}\bar{b}}{}^{\bar{k}} + 2\tilde{\Pi}^{\bar{a}[\bar{i}|} \tilde{\Pi}^{\bar{b}|\bar{j}]} Q_{\bar{a}\bar{i}} Q_{\bar{b}\bar{j}} + 2\Lambda |\det(\tilde{\Pi}^{ai})|. \tag{4.49c}$$

This formulation is similar to the $SO(3)$-ADM description, but it is invariant under the $SO(2,1)$ group. Thus, we might call it the $SO(2,1)$-ADM description.

4.4 Comments

We finish this chapter summarizing the results we presented here. First, we foliated the spacetime manifold with timelike leaves and found a Hamiltonian description with first- and second-class constraints. Then, we explicitly solve the second-class constraints in the same manifestly Lorentz-covariant fashion of the previous chapter. At the end, using the space gauge, we found a description for general relativity invariant under $SU(1,1)$ [or $SO(2,1)$] transformations. Remarkably, this new formulation has the same structure as the Ashtekar-Barbero formulation, and it is also constructed with a densitized triad and an internal connection.

We also want to point out that a similar description was already reported in Ref. [39]. However, the procedure exposed in that work was not as neat or clear as ours, since they used the nonmanifestly formulation of Barros e Sá. Furthermore, their Hamiltonian description is incomplete since they did not present the scalar constraint, which has a complicated form in the Barros e Sá description.

Finally, we remark that the results exposed in this chapter are found in Ref. [59], and are published under the terms of the *Creative Commons Attribution 4.0* International license. We, the authors, own the rights for the article distribution.

Chapter 5

Hamiltonian analysis for a BF-type action for general relativity with the Barbero-Immirzi parameter

BF theories are first-order formulations where the fundamental variables are an internal connection and a B field. In four dimensions, the B field is a 2-form. Although pure BF theories are topological—in the sense that they do not propagate physical degrees of freedom—, with the addition of constraints on the B field we can break its topological nature and describe physical theories such as general relativity.[1] Plebański presented the first BF-type action that describes general relativity [65]. It was later shown that the Hamiltonian analysis of the Plebański action leads to the Ashtekar complex formulation [43].

In this chapter, we use a BF-type action that is equivalent, at the Lagrangian level, to the Holst action. From there, we perform its Hamiltonian analysis and classify its constraints. Once the solution of the second-class constraints is done, either with manifestly Lorentz-covariant variables or not, we obtain the same Hamiltonian description of Chapters 2 and 3. Thus, we can connect the Hamiltonian formalism of BF gravity with the Ashtekar-Barbero formulation.

Some of the results of this chapter were published in Ref. [66].

5.1 Classification of the constraints

The action we consider is [67, 68]

$$
\begin{aligned}
S[B,\omega,\phi,\mu] \quad = \quad & \frac{1}{\kappa}\int_M \Bigg[\left(B^{IJ} + \frac{1}{\gamma} * B^{IJ} \right) \wedge F_{IJ} - \phi_{IJKL} B^{IJ} \wedge B^{KL} \\
& - \mu \phi_{IJKL} \epsilon^{IJKL} + \mu\lambda + l_1 B_{IJ} \wedge B^{IJ} + l_2 B_{IJ} \wedge * B^{IJ} \Bigg], \qquad (5.1)
\end{aligned}
$$

[1] For an extensive review in BF formulations for gravity see Ref. [64].

where M is a 4-dimensional manifold, $\kappa = 16\pi G$, γ represents the Barbero-Immirzi parameter, and the asterisk stands for the internal dual [see Eq. (A.2)]. Note that the action depends on the field B^{IJ}, the internal Lorentz (Euclidean) connection $\omega^I{}_J$ (with $F^I{}_J = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J$ being its curvature), the internal tensor $\phi_{IJKL} = \phi_{KLIJ} = -\phi_{JIKL} = -\phi_{IJLK}$, and in the 4-form μ. The constants λ, l_1, and l_2 are related to the cosmological constant Λ.

To begin with the Hamiltonian analysis, we consider the same assumptions we done in Chapters 2 and 3 (see Appendix A). Furthermore, along this section we follow Ref. [69] where this analysis was first reported. Continuing with the analysis, we decompose the differential forms as

$$B^{IJ} = \frac{1}{2}B_{\mu\nu}{}^{IJ}dx^\mu \wedge dx^\nu = B_{ta}{}^{IJ}dt \wedge dx^a + \frac{1}{2}B_{ab}{}^{IJ}dx^a \wedge dx^b, \tag{5.2a}$$

$$\omega^I{}_J = \omega_\mu{}^I{}_J dx^\mu = \omega_t{}^I{}_J dt + \omega_a{}^I{}_J dx^a, \tag{5.2b}$$

$$\mu = \tilde{\mu}_0 d^4x, \tag{5.2c}$$

and rewrite the action (5.1) as

$$\begin{aligned} S = {} & \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \Big[\tilde{\Pi}^{aIJ}\partial_t \overset{(\gamma)}{\omega}{}_{aIJ} + \omega_{tIJ} D_a \overset{(\gamma)}{\tilde{\Pi}}{}^{aIJ} + \frac{1}{2}\tilde{\eta}^{abc} \overset{(\gamma)}{F}{}_{abIJ}\, B_{tc}{}^{IJ} + \tilde{\mu}_0\lambda \\ & - \left(2B_{ta}{}^{IJ}\tilde{\Pi}^{aKL} + \tilde{\mu}_0\epsilon^{IJKL}\right)\phi_{IJKL} + 2l_1\tilde{\Pi}^{aIJ}B_{taIJ} + 2l_2 * \tilde{\Pi}^{aIJ}B_{taIJ}\Big], \end{aligned} \tag{5.3}$$

where

$$\tilde{\Pi}^{aIJ} := \frac{1}{2}\tilde{\eta}^{abc}B_{bc}{}^{IJ}. \tag{5.4}$$

Here, D_a stands for the covariant derivative associated with $\omega_a{}^I{}_J$; explicitly

$$D_a\tilde{\Pi}^{aIJ} = \partial_a\tilde{\Pi}^{aIJ} + 2\omega_a{}^{[I|}{}_K\tilde{\Pi}^{aK|J]}. \tag{5.5}$$

To simplify the number of variables involved in the analysis, we use the equation of motion for ϕ_{IJKL}

$$\tilde{\Pi}^{aIJ}B_{ta}{}^{KL} + \tilde{\Pi}^{aKL}B_{ta}{}^{IJ} + \tilde{\mu}_0\epsilon^{IJKL} = 0. \tag{5.6}$$

Given the symmetries of ϕ_{IJKL}, Eq. (5.6) is a system of 21 independent equations. It can be shown that the solution of Eq. (5.6) is [69, 70]

$$\tilde{\mu}_0 = -\frac{\sigma}{12}\epsilon_{IJKL}B_{ta}{}^{IJ}\tilde{\Pi}^{aKL}, \tag{5.7}$$

$$B_{ta}{}^{IJ} = \frac{\sigma}{2}\underset{\sim}{N}q_{ab} * \tilde{\Pi}^{aIJ} - \underset{\sim}{\eta}_{abc}N^b\tilde{\Pi}^{cIJ}, \tag{5.8}$$

together with the constraint

$$\tilde{\tilde{\Phi}}^{ab} := -\sigma * \tilde{\Pi}^{aIJ}\tilde{\Pi}^{b}{}_{IJ} = 0, \tag{5.9}$$

where q_{ab} is the inverse of q^{ab}, which fulfills the relation

$$qq^{ab} := \frac{\sigma}{2}\tilde{\Pi}^{aIJ}\tilde{\Pi}^{b}{}_{IJ}, \tag{5.10}$$

with $q := \det(q_{ab})$. Moreover, note that we have introduced four arbitrary fields $\underset{\sim}{N}$ and N^a.

Using (5.7) and (5.8), the action (5.3) becomes

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \Big[\tilde{\Pi}^{aIJ}\partial_t \overset{(\gamma)}{\omega}{}_{aIJ} + \omega_{tIJ}\tilde{\mathcal{G}}^{IJ} - N^a\tilde{\mathcal{V}}_a - \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}} - \underset{\sim}{\phi}_{ab}\tilde{\tilde{\Phi}}^{ab}\Big], \tag{5.11}$$

we also added the multiplier $\underset{\sim}{\phi}_{ab}$ to impose the constraint (5.9). Here, we observe that ω_{tIJ}, N^a, and $\underset{\sim}{N}$, play the role of Lagrange multipliers imposing the constraints

$$\begin{aligned}
\tilde{\mathcal{G}}^{IJ} &:= D_a \overset{(\gamma)}{\tilde{\Pi}}{}^{aIJ} \approx 0, && (5.12a)\\
\tilde{\mathcal{V}}_a &:= \tilde{\Pi}^{bIJ} \overset{(\gamma)}{F}{}_{abIJ} \approx 0, && (5.12b)\\
\tilde{\tilde{\mathcal{S}}} &:= \frac{\sigma}{2}\tilde{\eta}^{abc}q_{ad} * \tilde{\Pi}^{dIJ} \overset{(\gamma)}{F}{}_{bcIJ} + q\,(12\sigma l_2 - \lambda) \approx 0. && (5.12c)
\end{aligned}$$

The next step in the Hamiltonian analysis is to compute the Poisson algebra among the constraints (5.9) and (5.12a)-(5.12c). The algebra does not close because of the Poisson bracket between $\tilde{\tilde{\Phi}}^{ab}$ and $\tilde{\tilde{\mathcal{S}}}$. Therefore, the evolution of $\tilde{\tilde{\Phi}}^{ab}$ leads to the secondary constraint [69]

$$\Psi^{ab} := 2\tilde{\eta}^{(a|cd}q_{cf}\tilde{\Pi}^{f}{}_{IJ}D_d\tilde{\Pi}^{|b)IJ} \approx 0. \tag{5.13}$$

We incorporate the previous constraint into the action, then, it reads

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \Big[\tilde{\Pi}^{aIJ}\partial_t \overset{(\gamma)}{\omega}{}_{aIJ} + \omega_{tIJ}\tilde{\mathcal{G}}^{IJ} - N^a\tilde{\mathcal{V}}_a - \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}} - \underset{\sim}{\phi}_{ab}\tilde{\tilde{\Phi}}^{ab} - \psi_{ab}\Psi^{ab}\Big], \tag{5.14}$$

where the Lagrange multiplier ψ_{ab} (of weight -2) imposes (5.13). According to Dirac's classification of constraints, $\tilde{\mathcal{G}}^{IJ}$, $\tilde{\mathcal{V}}_a$, and $\tilde{\tilde{\mathcal{S}}}$ are first class; they generate local Lorentz (Euclidean) transformations and spacetime diffeomorphisms. On the other hand, $\tilde{\tilde{\Phi}}^{ab}$ and Ψ^{ab} (of weight +3) are second class, and we deal with them in the next section. All together, they account for the $(1/2)(2 \times 18 - 2 \times 10 - 12) = 2$ local d.o.f. of general relativity. Note that the formulation (5.14) is the same one that arises from Holst action when second-class constraints are involved.

5.2 Solutions for the second-class constraints

At this point, we want to study the solution of the second-class constraints with nonmanifestly Lorentz-covariant variables and with manifestly Lorentz-covariant variables. Thus, we bifurcate the analysis for the remaining of this chapter.

5.2.1 Nonmanifestly Lorentz-covariant solution

We begin with the solution in terms of nonmanifestly Lorentz-covariant variables. Here, instead of the path we followed in Sec. 2.3, we follow an approach closely related to the one in Ref. [34]. Thus, we can introduce canonical variables to simplify the solution of the second-class constraints, so we can avoid the formulation with a noncanonical symplectic structure. It is worth to mention that we have already reported the upcoming results; they are found in Ref. [66].

First, we solve the constraint (5.9), its solution is

$$\tilde{\Pi}^{ai0} =: \tilde{\Pi}^{ai}, \tag{5.15a}$$

$$\tilde{\Pi}^{aij} = -2\tilde{\Pi}^{a[i}\chi^{j]}. \tag{5.15b}$$

Therefore, the 12 independent variables that constitute $\tilde{\Pi}^{aIJ}$ are $\tilde{\Pi}^{ai}$ and χ_i.

Before we continue with the next constraint, we notice that we can rearrange the kinetic term of the action (5.14). Hence, using (5.15a) and (5.15b), we have [34]

$$\tilde{\Pi}^{aIJ}\partial_t \overset{(\gamma)}{\omega}{}_{aIJ} = \frac{2}{\gamma}\tilde{\Pi}^{ai}\dot{A}_{ai} + \frac{2}{\gamma}\tilde{\zeta}^i\dot{\chi}_i, \tag{5.16}$$

where we made the definitions

$$A_{ai} := -\gamma \overset{(\gamma)}{\omega}{}_{a0i} - \gamma \overset{(\gamma)}{\omega}{}_{aij}\, \chi^j, \tag{5.17a}$$

$$\tilde{\zeta}^i := -\gamma \overset{(\gamma)}{\omega}{}_a{}^i{}_j \tilde{\Pi}^{aj}. \tag{5.17b}$$

Thus, the Hamiltonian formulation is described by the canonical pairs $(A_{ai}, \tilde{\Pi}^{ai})$ and

$(\chi_i,\, \tilde{\zeta}^i)$ since the only nonvanishing commutation relations are

$$\begin{aligned}\left\{A_{ai}(t,x),\, \tilde{\Pi}^{bj}(t,y)\right\} &= \frac{\kappa\gamma}{2}\delta^b_a\delta^j_i\delta^3(x,y), &(5.18a)\\ \left\{\chi_i(t,x),\, \tilde{\zeta}^j(t,y)\right\} &= \frac{\kappa\gamma}{2}\delta^j_i\delta^3(x,y). &(5.18b)\end{aligned}$$

The next step is to parametrize $\overset{(\gamma)}{\omega}{}_{a0i}$ and $\overset{(\gamma)}{\omega}{}_{aij}$ with the canonical variables $\tilde{\Pi}^{ai}$, A_{ai}, χ_i, and $\tilde{\zeta}^i$. However, since (5.17a) and (5.17b) are 12 equations for the 18 components of $\overset{(\gamma)}{\omega}{}_{a0i}$ and $\overset{(\gamma)}{\omega}{}_{aij}$, we need to introduce the six free variables $\tilde{M}_{ij} = \tilde{M}_{ji}$. Then, we invert (5.17a) and (5.17b) and get

$$\begin{aligned}\overset{(\gamma)}{\omega}{}_{a0i} &= -\frac{1}{\gamma}A_{ai} - \frac{1}{2}\epsilon_{ijk}\chi^j \underset{\sim}{\Pi}_{al}\tilde{M}^{kl} - \underset{\sim}{\Pi}_{a[i}\tilde{\zeta}_{j]}\chi^j, &(5.19a)\\ \overset{(\gamma)}{\omega}{}_{aij} &= \frac{1}{2}\epsilon_{ijk}\underset{\sim}{\Pi}_{al}\tilde{M}^{kl} + \underset{\sim}{\Pi}_{a[i}\tilde{\zeta}_{j]}. &(5.19b)\end{aligned}$$

The next step is to solve the constraint $\Psi^{ab}=0$ given in (5.13). Substituting (5.15a), (5.15b), (5.19a), and (5.19b) into (5.13) implies

$$\frac{1}{4}\left(1-\frac{\sigma}{\gamma^2}\right)\underset{\sim}{\Pi}_{ai}\underset{\sim}{\Pi}_{bj}\Psi^{ab} = 2\tilde{f}_{(ij)} - (1+\sigma\chi^p\chi_p)\,\epsilon_{ikm}\epsilon_{jln}\Theta^{mn}\tilde{M}^{kl} = 0. \qquad (5.20)$$

The quantities $\Theta^i{}_j$ and $\tilde{f}^i{}_j$ are the ones already defined in (2.23) and (2.46), respectively. Therefore, the constraint (5.13) allows us to fix the variables $\tilde{M}_{ij}$. The solution of (5.20) is

$$\begin{aligned}\tilde{M}_{ij} &= \frac{2}{(1+\sigma\chi_r\chi^r)^2}\left[\delta^k_i\delta^l_j - \frac{1}{4}\left(\vartheta^{-1}\right)_{ij}\vartheta^{kl}\right]\epsilon_{kmp}\epsilon_{lnq}\vartheta^{mn}\tilde{f}^{(pq)}, &(5.21)\\ &= \frac{1}{(1+\sigma\chi_m\chi^m)^2}\left[\left(\tilde{f}^k{}_k + \sigma\tilde{f}_{kl}\chi^k\chi^l\right)\delta_{ij} + \left(\sigma\tilde{f}^k{}_k - \tilde{f}_{kl}\chi^k\chi^l\right)\chi_i\chi_j\right.\\ &\quad \left. -2\tilde{f}_{(ij)} - 2\,\sigma\left(\chi_{(i}\tilde{f}_{j)k} + \tilde{f}_{k(i}\chi_{j)}\right)\chi^k\right], &(5.22)\end{aligned}$$

with $\vartheta^i{}_j$ defined in (2.24).

We have successfully solved the second-class constraints (5.9) and (5.13), and we have derived a Hamiltonian formulation described by the canonical variables $(A_{ai},\, \tilde{\Pi}^{ai})$ and $(\chi_i,\, \tilde{\zeta}^i)$. Then, we substitute (5.15a), (5.15b), (5.19a), (5.19b), and (5.22) into the action (5.14) and in the constraints (5.12a)-(5.12c), and get

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\Sigma_t} d^3x \left[\frac{2}{\gamma}\tilde{\Pi}^{ai}\dot{A}_{ai} + \frac{2}{\gamma}\tilde{\zeta}^i\dot{\chi}_i - \epsilon_{ijk}\omega_t{}^{jk}\tilde{\mathcal{G}}^i_{\text{rot}} + 2\omega_{ti0}\tilde{\mathcal{G}}^i_{\text{boost}} - N^a\tilde{\mathcal{V}}_a - \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}}\right], \quad (5.23)$$

where the constraints are given by

$$
\begin{aligned}
\tilde{\mathcal{G}}^i_{\text{boost}} &= \partial_a\left(P^i{}_j\tilde{\Pi}^{aj}\right)+\frac{2\sigma}{\gamma}A_{aj}\tilde{\Pi}^{a[i}\chi^{j]}-\frac{\sigma}{\gamma}\tilde{\zeta}_j\chi^j\chi^i-\frac{1}{\gamma}\tilde{\zeta}^i, &(5.24\text{a})\\
\tilde{\mathcal{G}}^i_{\text{rot}} &= \partial_a\left(Q^i{}_j\tilde{\Pi}^{aj}\right)+\frac{1}{\gamma}\epsilon^i{}_{jk}\left(A_a{}^j\tilde{\Pi}^{ak}-\tilde{\zeta}^j\chi^k\right), &(5.24\text{b})\\
\tilde{\mathcal{V}}_a &= \frac{4}{\gamma}\tilde{\Pi}^{bi}\partial_{[a}A_{b]i}+\frac{2}{\gamma}\tilde{\zeta}_i\partial_a\chi^i-\frac{2\gamma^2}{\gamma^2-\sigma}\Bigg[\frac{1}{\gamma^2}A_{ai}\left(\tilde{\zeta}^i+\sigma\tilde{\zeta}_j\chi^j\chi^i\right)\\
&\quad -\frac{2\sigma}{\gamma^2}\tilde{\Pi}^{b[i}\chi^{j]}A_{ai}A_{bj}-\frac{\sigma}{\gamma^3}\epsilon_{ijk}\left(\tilde{\Pi}^{bi}A_b{}^j+\tilde{\zeta}^i\chi^j\right)A_a{}^k\\
&\quad +\left(Q^i{}_j\mathcal{G}^j_{\text{boost}}-P^i{}_j\mathcal{G}^j_{\text{rot}}\right)J_{ai}\Bigg], &(5.24\text{c})\\
\tilde{\tilde{\mathcal{S}}} &= -2\tilde{\Pi}^{ai}\chi_i\tilde{\mathcal{V}}_a-2\sigma\left(1+\sigma\chi_p\chi^p\right)\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}\Bigg\{\partial_aJ_b{}^k-\frac{2}{\gamma}\left(A_{al}+J_{al}\right)J_b{}^k\chi^l\\
&\quad -\frac{\sigma\gamma^2}{2(\gamma^2-\sigma)}\Bigg[\epsilon^{klm}\left(\frac{1}{\gamma^2}A_{al}A_{bm}+\sigma J_{al}J_{bm}+\frac{2}{\gamma^2}A_{al}J_{bm}\right)\\
&\quad +\frac{2}{\gamma}A_{al}J_b{}^l\chi^k+\epsilon^{lmn}J_{al}J_{bm}\chi_n\chi^k\Bigg]\Bigg\}+2\sigma\Lambda\left|1+\sigma\chi_i\chi^i\right|\left|\det(\tilde{\Pi}^{ai})\right|, &(5.24\text{d})
\end{aligned}
$$

where $\Lambda := (6l_2-\sigma\lambda/2)$, while $P^i{}_j$, $Q^i{}_j$, and J_{ai} are defined in (2.33), (2.34), and (2.44), respectively. This is the same description we encountered in Sec. 2.4. Therefore, the Hamiltonian formulation of the BF action (5.1) is utterly equivalent to the Hamiltonian formulation of the Holst action. From this point we can easily make contact with the Ashtekar-Barbero formulation (see Sec. 2.5). Thus, it is possible to derive the Ashtekar-Barbero formalism from the BF-type action (5.1).

5.2.2 Manifestly Lorentz-covariant solution

Let us move on to the manifestly Lorentz-covariant formalism. The solution of (5.9) is

$$\tilde{\Pi}^{aIJ}=2\epsilon\tilde{\Pi}^{a[I}m^{J]}, \tag{5.25}$$

where $\epsilon=\pm1$ and m_I is an arbitrary internal vector. We choose m_I so that it satisfies: $m_I\tilde{\Pi}^{aI}=0$ and $m_Im^I=\sigma$, its explicit form is given in Eq. (3.8). Moreover, let us to define the covariant derivative as

$$\nabla_a\tilde{\Pi}^{bI}:=\partial_a\tilde{\Pi}^{bI}+\Gamma^b{}_{ac}\tilde{\Pi}^{cI}-\Gamma^c{}_{ac}\tilde{\Pi}^{bI}+\Gamma_a{}^I{}_J\tilde{\Pi}^{bJ}=0, \tag{5.26}$$

where $\Gamma_{aIJ}=-\Gamma_{aJI}$ and $\Gamma^a{}_{bc}=\Gamma^a{}_{cb}$. The solution for Γ_{aIJ} and $\Gamma^a{}_{bc}$ is that given in (3.13) and (3.14), correspondingly. Also, from the definition of m_I in Eq. (3.8), we get the identity

$$\sqrt{h}\tilde{\eta}^{abc}\underset{\approx}{h}_{cd}\tilde{\Pi}^{d}{}_{I} = -\sigma\epsilon_{IJKL}m^{J}\tilde{\Pi}^{aK}\tilde{\Pi}^{bL}. \tag{5.27}$$

Next, we substitute (5.25) into the second-class constraint (5.13), and we use Eqs. (5.26) and (5.27) to simplify the expression. Then, the constraint $\Psi^{ab} = 0$ of Eq. (5.13) reads

$$\Psi^{ab} = 4\epsilon_{IJKL}\tilde{\Pi}^{(a|I}\tilde{\Pi}^{|b)M}\tilde{\Pi}^{cJ}m^{K}\left(\Gamma_{c}{}^{L}{}_{M} - \omega_{c}{}^{L}{}_{M}\right) = 0. \tag{5.28}$$

After a quick comparison with Eq. (3.16), we notice that both expressions are different by a global factor of $(-2\sigma\epsilon)$ which does not alter the result. Thus, we can use the same solution given in Eq. (3.20).

If we compare the action (5.14) and constraints (5.12a)-(5.12c) with those of (3.2) and (3.4a)-(3.4c), we found that they only differ in the scalar constraint $\tilde{\tilde{\mathcal{S}}}$. Hence, since the solution of the second-class constraints is the same in both cases, we only need to compute the scalar constraint to see if both formulations are equivalent. Thus, substituting (5.25) and (3.20) into (5.12c) yields

$$\begin{aligned}
\tilde{\tilde{\mathcal{S}}} \;=\; & -\epsilon\Bigg\{ -\sigma\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ}R_{abIJ} + 2\tilde{\Pi}^{a[I|}\tilde{\Pi}^{b|J]}\Bigg[C_{aI}C_{bJ} - 2\epsilon C_{aI}m^{K}\overset{(\gamma)}{\Gamma}{}_{bJK} \\
& + \left(\Gamma_{aIL} + \frac{2}{\gamma} * \Gamma_{aIL}\right)\Gamma_{bJK}m^{K}m^{L} + \frac{1}{\gamma^{2}}q^{KL}\Gamma_{aIK}\Gamma_{bJL}\Bigg] \\
& + \tilde{\mathcal{G}}^{IJ}\Bigg[-\frac{1}{4}\tilde{\mathcal{G}}_{IJ} + \frac{1}{4}\left(P^{-1}\right)_{IJKL}\tilde{\mathcal{G}}^{KL} - \frac{\sigma}{2}m_{I}m^{K}\tilde{\mathcal{G}}_{JK}\Bigg] \\
& - 2\epsilon\tilde{\Pi}^{aI}m^{J}\nabla_{a}\tilde{\mathcal{G}}_{IJ} + 2\sigma\sqrt{h}\Lambda\Bigg\} \approx 0.
\end{aligned} \tag{5.29}$$

with $\Lambda := -\epsilon(6l_2 - \sigma\lambda/2)$, whereas $q^{I}{}_{J}$ and $R_{ab}{}^{I}{}_{J}$ are given by (3.11) and (3.15), respectively.

The only difference between (5.29) and (3.23c) is the global factor $(-\epsilon)$, which can be reabsorbed into the Lagrange multiplier that impose the scalar constraint; we just need to make the change $\underset{\sim}{N} \to -\epsilon\underset{\sim}{N}$. Thus, from this departing point we can also derive the Ashtekar-Barbero formulation as we did in Sec. 3.4.

5.3 Comments

We finish the chapter with a few remarks. The first part of the analysis was carried out in Ref. [69]. There, the authors considered the action (5.1) and classified the constraints that emerge during its Hamiltonian analysis. Following that point of view, we took their results and solved the second-class constraints in a nonmanifestly covariant fashion. We already reported that solution; it is found in Ref. [66], where we also presented the complete path from the BF-type action to the Ashtekar-Barbero formulation. On the other hand, for the manifestly covariant solution, we followed the procedure of Chapter 3, and our results agree with those of Chapter 3.

Chapter 6

Manifestly Lorentz-covariant Hamiltonian analysis for Holst action without introducing second-class constraints

As we have seen, second-class constraints usually emerge in the real first-order Hamiltonian formalism of general relativity. They are introduced due to the mismatch between the number of independent variables that compose the tetrad (or B field) and the internal connection. However, if we can identify the canonical variables from the very beginning, we can avoid to introduce second-class constraints. We develop this idea throughout this chapter for the Holst action, and we find the suitable parametrization for the fundamental variables of the action that do this job. The spatial part of the tetrad field is directly related to the canonical momenta, and we decompose the spatial part of the internal connection into two parts, one associated with the configuration variables and one composed by auxiliary fields. After we integrate out the auxiliary fields from the formalism, we end up with a Hamiltonian formulation described by first-class constraints only. Furthermore, to complement this chapter, we generalize the canonical transformations enlisted in Sec. 3.3, and we explore the consequences of the time gauge in this general description.

The analysis and results of this chapter are published in Ref. [58].

6.1 The parametrization of the Lagrangian variables

We begin our analysis with the Holst action with cosmological constant Λ

$$S[e,\omega] = \frac{1}{\kappa}\int_M \left[* \left(e^I \wedge e^J\right) \wedge \overset{(\gamma)}{F}{}_{IJ} - 2\Lambda\rho \right], \tag{6.1}$$

where the internal dual and the γ-hat notation are defined in (A.2) and (A.4), respectively; and $\rho = (1/4!)\epsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L$ is the volume form. This action depends on the tetrad field e^I and in the 1-form connection $\omega^I{}_J$ through its curvature

$$F^I{}_J = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J. \tag{6.2}$$

Along this chapter, we want to keep the analysis as general as possible. Thus, we assume that the spacetime manifold M is diffeomorphic to $\mathbb{R}\times\Xi$, with Ξ as a 3-dimensional submanifold that can be either a spacelike or a timelike surface. Under this consideration, the internal indices are raised or lowered with metric $(\eta_{IJ}) = \mathrm{diag}(\sigma,1,1,1)$ if Ξ is a spacelike surface, whereas we use the $(\eta_{IJ}) = \mathrm{diag}(1,-1,1,1)$ when Ξ is a timelike surface. Also, we will omit the boundary terms that appear during the Hamiltonian analysis. If the reader is interested in the boundary terms, they are found during the analyses developed in Chapters 3 and 4.

Next, we choose the x^0 direction to define the notion of evolution, whereas x^a denotes the coordinates that label the points of Ξ, with a, b, c, ... $=1,2,3$. Then, we express the differential forms as

$$\begin{aligned} e^I &= e_\mu{}^I dx^\mu = e_0{}^I dx^0 + e_a{}^I dx^a, && (6.3a)\\ \omega^I{}_J &= \omega_\mu{}^I{}_J dx^\mu = \omega_0{}^I{}_J dx^0 + \omega_a{}^I{}_J dx^a, && (6.3b)\\ F^I{}_J &= \frac{1}{2}F_{\mu\nu}{}^I{}_J dx^\mu\wedge dx^\nu = F_{0a}{}^I{}_J dx^0\wedge dx^a + \frac{1}{2}F_{ab}{}^I{}_J dx^a\wedge dx^b, && (6.3c)\end{aligned}$$

with $F_{\mu\nu}{}^I{}_J = \partial_\mu\omega_\nu{}^I{}_J - \partial_\nu\omega_\mu{}^I{}_J + \omega_\mu{}^I{}_K\omega_\nu{}^K{}_J - \omega_\nu{}^I{}_K\omega_\mu{}^K{}_J$.

We denote the induced metric on Ξ as $q_{ab} := e_{aI}e_b{}^I$, and we define internal vector n_I orthogonal to Ξ. Thus, n_I fulfills the two properties: $n_I e_a{}^I = 0$ and $n_I n^I = \tau$, for a fixed value of τ. Furthermore, let $q = \det(q_{ab})$, if Ξ is a timelike surface, then $q<0$ and n_I is a spacelike vector with norm $\tau = 1$. On the other hand, if Ξ is a spacelike surface, then $q>0$ and n_I (in the Lorentzian signature, $\sigma=-1$) is a timelike vector, so $\tau=\sigma$. Explicitly, n_I is

$$n_I := \frac{1}{6\sqrt{|q|}}\epsilon_{IJKL}\eta^{abc}e_a{}^J e_b{}^K e_c{}^L. \tag{6.4}$$

The case for a null foliation ($\tau=0$) is out of the scope of this analysis, see Ref. [71] for a Hamiltonian description on the light front for the Palatini action.

After $3+1$ decomposition of the fields is made and the considerations mentioned above are done, the action (6.1) takes the form

$$\begin{aligned} S &= \frac{1}{\kappa}\int_{\mathbb{R}} dx^0 \int_\Xi d^3x \Big\{ -2\tilde{\Pi}^{aI}n^J\partial_0 \overset{(\gamma)}{\omega}_{aIJ} + \omega_{0IJ}\tilde{\mathcal{G}}^{IJ} + |q|^{-1/2}e_0{}^I\Big[2\tilde{\Pi}^a{}_I\tilde{\Pi}^{bJ}n^K \overset{(\gamma)}{F}_{abJK} \\ &\quad + n_I\Big(\tilde{\Pi}^{aJ}\tilde{\Pi}^{bK}\overset{(\gamma)}{F}_{abJK} - 2\Lambda|q|\Big)\Big]\Big\}, \end{aligned} \tag{6.5}$$

where we performed an integration by parts and defined the quantities

$$\begin{aligned}\tilde{\Pi}^{aI} &:= \sqrt{|q|}q^{ab}e_b{}^I, &(6.6)\\ \tilde{\mathcal{G}}^{IJ} &:= -2P^{IJ}{}_{KL}\left[\partial_a\left(\tilde{\Pi}^{aK}n^L\right)+2\omega_a{}^K{}_M\tilde{\Pi}^{a[M}n^{L]}\right]. &(6.7)\end{aligned}$$

In addition, we define the densitized metric $\underset{\approx}{h}_{ab} := |q|^{-1}q_{ab}$, whose inverse is $\tilde{\tilde{h}}^{ab} = \tilde{\Pi}^{aI}\tilde{\Pi}^b{}_I$, which allows us to invert (6.6) and obtain

$$e_a{}^I = |h|^{1/4}\underset{\approx}{h}_{ab}\tilde{\Pi}^{bI}, \tag{6.8}$$

where $h = \det(\tilde{\tilde{h}}^{ab})$. Thus, we have mapped the 12 components of $e_a{}^I$ into the 12 variables in $\tilde{\Pi}^{aI}$. Moreover, if we substitute Eq. (6.8) into Eq. (6.4) we get

$$n_I = \frac{\sigma\tau}{6\sqrt{|h|}}\epsilon_{IJKL}\underset{\sim}{\eta}_{abc}\tilde{\Pi}^{aJ}\tilde{\Pi}^{bK}\tilde{\Pi}^{cL}, \tag{6.9}$$

so we can express the internal vector n_I in terms of $\tilde{\Pi}^{aI}$ only. Also, notice that, from (6.6) or (6.8), n_I satisfies $n_I\tilde{\Pi}^{aI} = 0$.

On the other hand, the remaining four components of the tetrad $e_0{}^I$ are parametrized in terms of the lapse function N and the shift vector N^a as follows

$$e_0{}^I = Nn^I + N^a e_a{}^I = Nn^I + N^a|h|^{1/4}\underset{\approx}{h}_{ab}\tilde{\Pi}^{bI}. \tag{6.10}$$

Notice that the last equality is completely function of $\tilde{\Pi}^{aI}$, N, and N^a if we consider n_I as given by (6.9). Therefore, Eqs. (6.8) and (6.10) define a map (N, N^a, $\tilde{\Pi}^{aI}$) $\mapsto$ ($e_\mu{}^I$), whose inverse is given by (6.6) and

$$\begin{aligned}N &= \tau e_0{}^I n_I, &(6.11)\\ N^a &= q^{ab}e_0{}^I e_{bI}, &(6.12)\end{aligned}$$

where n_I is taken as in (6.4) and q^{ab} is the inverse of q_{ab} ($q^{ac}q_{cb} = \delta^a_b$).

After substituting the new parameterization given by (6.8) and (6.10) into the action (6.5), it acquires the form

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dx^0\int_{\Xi} d^3x\left[-2\tilde{\Pi}^{aI}n^J\partial_0 \overset{(\gamma)}{\omega}{}_{aIJ} + \omega_{0IJ}\tilde{\mathcal{G}}^{IJ} - N^a\tilde{\mathcal{V}}_a - \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}}\right], \tag{6.13}$$

where we defined $\underset{\sim}{N} := |h|^{-1/4}N$ and

$$
\begin{aligned}
\tilde{\mathcal{V}}_a &:= -2\tilde{\Pi}^{bI}n^J \overset{(\gamma)}{F}_{abIJ}, && \text{(6.14a)}\\
\tilde{\tilde{\mathcal{S}}} &:= -\tau\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ} \overset{(\gamma)}{F}_{abIJ} + 2\tau\sqrt{|h|}\Lambda. && \text{(6.14b)}
\end{aligned}
$$

Instead of defining the canonical momentum conjugated to $\overset{(\gamma)}{\omega}_a{}^I{}_J$ (which leads to the introduction of second-class constraints, as we saw in Chapter 2), we rearrange the first term in (6.13) as

$$
-2\tilde{\Pi}^{aI}n^J\partial_0 \overset{(\gamma)}{\omega}_{aIJ} = 2\tilde{\Pi}^{aI}\dot{C}_{aI}, \tag{6.15}
$$

where

$$
C_{aI} := W_a{}^b{}_{IJK} \overset{(\gamma)}{\omega}_b{}^{JK} \tag{6.16}
$$

and $W_a{}^b{}_{IJK} = -W_a{}^b{}_{IKJ}$ is

$$
W_a{}^b{}_{IJK} := -\left(\delta_a^b\eta_{I[J}n_{K]} + n_I\underset{\approx}{h}_{ac}\tilde{\Pi}^c{}_{[J}\tilde{\Pi}^b{}_{K]}\right). \tag{6.17}
$$

We can interpret $W_a{}^b{}_{IJK}$ as an operator that projects 12 dynamical variables contained in $\overset{(\gamma)}{\omega}_a{}^I{}_J$. Furthermore, the null vectors that compose the presymplectic structure of (6.13) are in the kernel of $W_a{}^b{}_{IJK}$. Therefore, we can solve the 12 equations in (6.16) to express $\overset{(\gamma)}{\omega}_a{}^I{}_J$ in terms of C_{aI} plus the six free variables $\underset{\sim}{\lambda}_{ab} = \underset{\sim}{\lambda}_{ba}$ that drop out of the presymplectic structure. The solution for (6.16) is

$$
\overset{(\gamma)}{\omega}_{aIJ} = M_a{}^b{}_{IJK}C_b{}^K + \tilde{N}^b{}_{IJ}\underset{\sim}{\lambda}_{ab}, \tag{6.18}
$$

with the expressions for $M_a{}^b{}_{IJK}$ and $\tilde{N}^a{}_{IJ}$ given by

$$
\begin{aligned}
M_a{}^b{}_{IJK} &:= \tau\delta_a^b n_{[I}\eta_{J]K} - \tau\delta_a^b P_{IJKL}n^L - \tau\underset{\approx}{h}_{ac}\tilde{\Pi}^b{}_{[I}\tilde{\Pi}^c{}_{J]}n_K \\
&\quad -\frac{\tau}{2\gamma}\epsilon_{IJLM}\underset{\approx}{h}_{ac}\tilde{\Pi}^c{}_K\tilde{\Pi}^{bL}n^M, && \text{(6.19a)}\\
\tilde{N}^a{}_{IJ} &:= \epsilon_{IJKL}\tilde{\Pi}^{aK}n^L. && \text{(6.19b)}
\end{aligned}
$$

In addition to the objects (6.17), (6.19a), and (6.19b), we introduce the tensor density $\underset{\sim}{U}_{ab}{}^{cIJ} = \underset{\sim}{U}_{ba}{}^{cIJ} = -\underset{\sim}{U}_{ab}{}^{cJI}$ defined by

$$
\underset{\sim}{U}_{ab}{}^{cIJ} := \sigma\tau\left(1 - \frac{\sigma}{\gamma^2}\right) * (P^{-1})^{IJKL}\delta^c{}_{(a}\underset{\approx}{h}_{b)e}\tilde{\Pi}^e{}_K n_L. \tag{6.20}
$$

They all together satisfy the following orthogonality relations:

$$
\begin{aligned}
W_a{}^{cIKL} M_c{}^b{}_{KLJ} &= \delta^b_a \delta^I_J, &(6.21a)\\
\underset{\sim}{U}_{ab}{}^{cIJ} \tilde{N}^d{}_{IJ} &= \delta^c_{(a}\delta^d_{b)}, &(6.21b)\\
W_a{}^{(b}{}_{IJK} \tilde{N}^{c)JK} &= 0, &(6.21c)\\
\underset{\sim}{U}_{ab}{}^{cIJ} M_c{}^d{}_{IJK} &= 0, &(6.21d)
\end{aligned}
$$

as well as the completeness relation

$$
M_a{}^c{}_{IJM} W_c{}^{bMKL} + \tilde{N}^c{}_{IJ} \underset{\sim}{U}_{ac}{}^{bKL} = \delta^b_a \delta^K_{[I}\delta^L_{J]}. \tag{6.22}
$$

Therefore, $W_a{}^b{}_{IJK}$ and $\underset{\sim}{U}_{ab}{}^{cIJ}$ are the orthogonal projectors that decompose $\overset{(\gamma)}{\omega}{}_a{}^I{}_J$ into the $12+6$ variables C_{aI} and $\underset{\sim}{\lambda}_{ab}$. Thus, the map $(\overset{(\gamma)}{\omega}{}_a{}^I{}_J) \mapsto (C_{aI}, \underset{\sim}{\lambda}_{ab})$, is given by (6.16) and

$$
\underset{\sim}{\lambda}_{ab} = \underset{\sim}{U}_{ab}{}^{cIJ} \overset{(\gamma)}{\omega}{}_{cIJ} \ . \tag{6.23}
$$

Going back to the action (6.13), we substitute the new parametrization for $\overset{(\gamma)}{\omega}{}_a{}^I{}_J$ given in (6.18) and obtain

$$
S = \frac{1}{\kappa}\int_{\mathbb{R}} dx^0 \int_{\Xi} d^3x \left[2\tilde{\Pi}^{aI}\dot{C}_{aI} + \omega_{0IJ}\tilde{\mathcal{G}}^{IJ} - N^a\tilde{\mathcal{V}}_a - \underset{\sim}{N}\tilde{\tilde{\mathcal{S}}}\right], \tag{6.24}
$$

with

$$
\begin{aligned}
\tilde{\mathcal{G}}^{IJ} &= 2\tilde{\Pi}^{a[I}C_a{}^{J]} + 4P^{IJ}{}_{KL}\tilde{\Pi}^{a[K}n^{M]}\Gamma_a{}^L{}_M, &(6.25a)\\
\tilde{\mathcal{V}}_a &= 2\left(2\tilde{\Pi}^{bI}\partial_{[a}C_{b]I} - C_{aI}\partial_b\tilde{\Pi}^{bI}\right) + (P^{-1})_{IJKL}\tilde{\mathcal{G}}^{IJ}\left(M_a{}^{bKLM}C_{bM}\right.\\
&\quad \left. + \underset{\sim}{\lambda}_{ab}\tilde{N}^{bKL}\right), &(6.25b)\\
\tilde{\tilde{\mathcal{S}}} &= -\tau\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ}R_{abIJ} + 2\tilde{\Pi}^{a[I|}\tilde{\Pi}^{b|J]}\left[C_{aI}C_{bJ} + 2C_{aI}\overset{(\gamma)}{\Gamma}{}_{bJK}n^K\right.\\
&\quad \left. + \frac{\sigma\tau}{\gamma^2}q^{KL}\Gamma_{aIK}\Gamma_{bJL} + \left(\Gamma_{aIL} + \frac{2}{\gamma} * \Gamma_{aIL}\right)\Gamma_{bJK}n^K n^L\right] + 2\tau\Lambda\sqrt{|h|}\\
&\quad + 2\tilde{\Pi}^{aI}n^J\nabla_a\tilde{\mathcal{G}}_{IJ} - \frac{1}{4}\left[\tilde{\mathcal{G}}^{IJ} - (P^{-1})^{IJKL}\tilde{\mathcal{G}}_{KL} + 2\tau n^I\tilde{\mathcal{G}}^J{}_K n^K\right]\tilde{\mathcal{G}}_{IJ}\\
&\quad + \frac{\sigma\gamma^2}{\gamma^2-\sigma}G^{abcd}\big(\underset{\sim}{\lambda}_{ab} - \underset{\sim}{U}_{ab}{}^{eIJ}\overset{(\gamma)}{\Gamma}{}_{eIJ}\big)\big(\underset{\sim}{\lambda}_{cd} - \underset{\sim}{U}_{cd}{}^{fKL}\overset{(\gamma)}{\Gamma}{}_{fKL}\big). &(6.25c)
\end{aligned}
$$

The quantities $q^I{}_J$ and Γ_{aIJ} are given, respectively, by (3.11) and (3.13) just replacing $\sigma \to \tau$. Also, the curvature $R_{ab}{}^I{}_J$ is the same as that given by (3.15), and we have defined

$G^{abcd} := \tilde{\tilde{h}}^{ab}\tilde{\tilde{h}}^{cd} - \tilde{\tilde{h}}^{(a|c}\tilde{\tilde{h}}^{|b)d}$, which is a tensor density of weight $+4$.

To simplify our analysis, we collect all the terms proportional to the Gauss constraint, so we redefine the Lagrange multiplier that imposes it as

$$\begin{aligned}
\lambda_{IJ} \;&:=\; -\omega_{0IJ} + N^a\omega_{aIJ} - 2\tilde{\Pi}^a{}_{[I}n_{J]}\nabla_a \underset{\sim}{N} \\
&\quad -\frac{1}{4}\underset{\sim}{N}\left[\tilde{\mathcal{G}}_{IJ} - (P^{-1})_{IJKL}\tilde{\mathcal{G}}^{KL} + 2\tau n_{[I}\tilde{\mathcal{G}}_{J]K}n^K\right].
\end{aligned} \tag{6.26}$$

Thus, after integrating by parts the term containing the covariant derivative of the Gauss constraint in (6.25c), we have

$$S \;=\; \frac{1}{\kappa}\int_{\mathbb{R}} dx^0 \int_{\Xi} d^3x \left[2\tilde{\Pi}^{aI}\dot{C}_{aI} - \left(\lambda_{IJ}\tilde{\mathcal{G}}^{IJ} + 2N^a\tilde{\mathcal{D}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{H}}}\right)\right], \tag{6.27}$$

with

$$\tilde{\mathcal{D}}_a \;:=\; 2\tilde{\Pi}^{bI}\partial_{[a}C_{b]I} - C_{aI}\partial_b\tilde{\Pi}^{bI}, \tag{6.28a}$$

$$\begin{aligned}
\tilde{\tilde{\mathcal{H}}} \;&:=\; -\tau\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ}R_{abIJ} + 2\tilde{\Pi}^{a[I|}\tilde{\Pi}^{b|J]}\Bigg[C_{aI}C_{bJ} + 2C_{aI}\overset{(\gamma)}{\Gamma}{}_{bJK}\,n^K \\
&\quad + \frac{\sigma\tau}{\gamma^2}q^{KL}\Gamma_{aIK}\Gamma_{bJL} + \left(\Gamma_{aIL} + \frac{2}{\gamma}*\Gamma_{aIL}\right)\Gamma_{bJK}n^K n^L\Bigg] + 2\tau\Lambda\sqrt{|h|} \\
&\quad + \frac{\sigma\gamma^2}{\gamma^2-\sigma}G^{abcd}\big(\underset{\sim}{\lambda}_{ab} - \underset{\sim}{U}_{ab}{}^{eIJ}\overset{(\gamma)}{\Gamma}{}_{eIJ}\big)\big(\underset{\sim}{\lambda}_{cd} - \underset{\sim}{U}_{cd}{}^{fKL}\overset{(\gamma)}{\Gamma}{}_{fKL}\big).
\end{aligned} \tag{6.28b}$$

At this point, we have parametrized the original 24 variables that constitute the internal connection $\overset{(\gamma)}{\omega}{}_\mu{}^I{}_J$ with the $12+6+6$ variables in C_{aI}, $\underset{\sim}{\lambda}_{ab}$, and λ_{IJ}. The map $(\overset{(\gamma)}{\omega}{}_\mu{}^I{}_J) \mapsto (C_{aI}, \underset{\sim}{\lambda}_{ab}, \lambda_{IJ})$ is given by Eqs. (6.16), (6.23), and (6.26). On the other hand, its inverse map is given by Eqs. (6.18) and

$$\begin{aligned}
\overset{(\gamma)}{\omega}{}_{0IJ} \;&=\; -\overset{(\gamma)}{\lambda}{}_{IJ} - 2P_{IJKL}\tilde{\Pi}^{aK}n^L\nabla_a\underset{\sim}{N} + N^a\left(M_a{}^b{}_{IJK}C_b{}^K + \underset{\sim}{\lambda}_{ab}N^b{}_{IJ}\right) \\
&\quad -\frac{1}{4}\underset{\sim}{N}\left(\overset{(\gamma)}{\tilde{\mathcal{G}}}{}_{IJ} - \tilde{\mathcal{G}}_{IJ} + 2\tau P_{IJKL}n^K\tilde{\mathcal{G}}^{LM}n_M\right).
\end{aligned} \tag{6.29}$$

To complete the Hamiltonian analysis, we must deal with the variables $\underset{\sim}{\lambda}_{ab}$. Since they appear quadratically in the action, they are auxiliary fields [53]. Thus, we can integrate them out using their equation of motion. From the action (6.27) with the constraints (6.25a), (6.28a), and (6.28b), we compute the equation of motion for $\underset{\sim}{\lambda}_{ab}$

$$2\underset{\sim}{N}G^{abcd}\big(\underset{\sim}{\lambda}_{cd} - \underset{\sim}{U}_{cd}{}^{fIJ}\overset{(\gamma)}{\Gamma}{}_{fIJ}\big) = 0. \tag{6.30}$$

Given that $\underset{\sim}{N} \neq 0$ and the G^{abcd} is invertible, $\left(G^{-1}\right)_{abcd} = (1/2)(\underset{\approx}{h}_{ab}\underset{\approx}{h}_{cd} - 2\underset{\approx}{h}_{(a|c}\underset{\approx}{h}_{|b)d})$, the solution for $\underset{\sim}{\lambda}_{ab}$ is

$$\underset{\sim}{\lambda}_{ab} = \underset{\sim}{U}_{ab}{}^{cIJ} \overset{(\gamma)}{\Gamma}{}_{cIJ} \,. \tag{6.31}$$

Substituting $\underset{\sim}{\lambda}_{ab}$ back into the action (6.27), we have the Hamiltonian description for the Holst action given by the action (6.27) and the first-class constraints (6.25a), (6.28a), and

$$\begin{aligned} \tilde{\tilde{\mathcal{H}}} \;=\; & -\tau\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ}R_{abIJ} + 2\tilde{\Pi}^{a[I|}\tilde{\Pi}^{b|J]}\Bigg[C_{aI}C_{bJ} + 2C_{aI}\overset{(\gamma)}{\Gamma}{}_{bJK}\, n^K \\ & + \frac{\sigma\tau}{\gamma^2}q^{KL}\Gamma_{aIK}\Gamma_{bJL} + \left(\Gamma_{aIL} + \frac{2}{\gamma} * \Gamma_{aIL}\right)\Gamma_{bJK}n^K n^L\Bigg] + 2\tau\Lambda\sqrt{|h|}. \end{aligned} \tag{6.32}$$

Notice that the Hamiltonian formulation described by the action (6.27) and the constraints (6.25a), (6.28a), and (6.32); is the same we found in Secs. 3.2 ($\tau = \sigma$) and 4.2 ($\tau = 1$) where we explicitly solved the second-class constraints. Thus, the method presented in this chapter rendered the same results while we avoided the introduction of second-class constraints. Furthermore, in this description the sign ambiguity ϵ does not appear because we did not involve any second-class constraint. Also, the internal vector m_I, that arises from the solution of the second-class constraints, is directly associated with the internal vector that characterizes the spacetime foliation n_I. In fact, from (6.9) and (3.8), we conclude that they differ at most by a sign

$$n_I = \sigma\tau m_I. \tag{6.33}$$

6.2 Canonical transformation

With the aid of the projector $W_a{}^b{}_{IJK}$, defined in Eq. (6.17), we realize that the canonical transformations enlisted in Sec. 3.3 belong to the family of transformations

$$\begin{aligned} X_{aI} &= C_{aI} - W_a{}^b{}_{IJK}\left(\alpha\Gamma_b{}^{JK} + \frac{\beta}{\gamma} * \Gamma_b{}^{JK}\right), && (6.34\text{a}) \\ \tilde{\Pi}^{aI} &= \tilde{\Pi}^{aI}, && (6.34\text{b}) \end{aligned}$$

where α and β are real parameters.

The substitution of (6.34) into the kinetic term of the action (6.27) results in

$$2\tilde{\Pi}^{aI}\dot{C}_{aI} = 2\tilde{\Pi}^{aI}\dot{X}_{aI} + \partial_a\left(-2\alpha n_I\dot{\tilde{\Pi}}^{aI} + \frac{\tau\beta}{\gamma}\sqrt{|h|}\tilde{\eta}^{abc}\underset{\approx}{h}_{bd}\underset{\approx}{h}_{cf}\dot{\tilde{\Pi}}^{dI}\tilde{\Pi}^f{}_I\right). \tag{6.35}$$

Therefore, the Hamiltonian formulation is described by the canonical variables $(X_{aI},\, \tilde{\Pi}^{aI})$, and the action

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dx^0 \int_{\Xi} d^3x \left[2\tilde{\Pi}^{aI}\dot{X}_{aI} - \left(\lambda_{IJ}\tilde{\mathcal{G}}^{IJ} + 2N^a\tilde{\mathcal{D}}_a + \underset{\sim}{N}\tilde{\tilde{\mathcal{H}}}\right)\right], \tag{6.36}$$

with the constraints given by

$$\tilde{\mathcal{G}}^{IJ} = 2\tilde{\Pi}^{a[I}X_a{}^{J]} + 4\left[(1-\alpha)\,\delta^I_{[K}\delta^J_{L]} + \frac{1}{2}\left(\frac{1-\beta}{\gamma}\right)\epsilon^{IJ}{}_{KL}\right]\tilde{\Pi}^{a[K}n^{M]}\Gamma_a{}^L{}_M \approx 0, \tag{6.37a}$$

$$\tilde{\mathcal{D}}_a = 2\tilde{\Pi}^{bI}\partial_{[a}X_{b]I} - X_{aI}\partial_b\tilde{\Pi}^{bI} \approx 0, \tag{6.37b}$$

$$\begin{aligned}\tilde{\tilde{\mathcal{H}}} = {} & -\tau\tilde{\Pi}^{aI}\tilde{\Pi}^{bJ}R_{abIJ} + 2\tilde{\Pi}^{a[I|}\tilde{\Pi}^{b|J]}\left\{X_{aI}X_{bJ} + \sigma\tau\left(\frac{1-\beta}{\gamma}\right)^2 q^{KL}\Gamma_{aIK}\Gamma_{bJL}\right. \\ & +2\left[(1-\alpha)\,\delta^L_{[J}\delta^M_{K]} + \frac{1}{2}\left(\frac{1-\beta}{\gamma}\right)\epsilon_{JK}{}^{LM}\right]X_{aI}n^K\Gamma_{bLM} \\ & \left.+(1-\alpha)\left[(1-\alpha)\,\Gamma_{aIL} + 2\left(\frac{1-\beta}{\gamma}\right) * \Gamma_{aIL}\right]\Gamma_{bJK}n^K n^L\right\} \\ & +2\tau\sqrt{|h|}\Lambda \approx 0.\end{aligned} \tag{6.37c}$$

For particular values of α and β, we recover the cases previously discussed:

- Variables $\left(C_{aI},\, \tilde{\Pi}^{aI}\right)$: $\alpha = 0$, $\beta = 0$ (Identity transformation).
- Variables $\left(K_{aI},\, \tilde{\Pi}^{aI}\right)$: $\alpha = 1$, $\beta = 0$.
- Variables $\left(Q_{aI},\, \tilde{\Pi}^{aI}\right)$: $\alpha = 1$, $\beta = 1$.
- Variables $\left(\mathcal{Q}_{aI},\, \tilde{\Pi}^{aI}\right)$: $\alpha = 0$, $\beta = 1$.

Furthermore, from the structure of the constraints, if $\beta = 1$ the Barbero-Immirzi parameter disappears from the formalism. Thus, the remaining formulation can be thought as naturally associated with the Palatini action. On the other hand, if $\alpha = 0$ and $\beta \neq 1$, we obtain a description similar to one rendered by the pair of $\left(C_{aI},\, \tilde{\Pi}^{aI}\right)$ with a rescaled Barbero-Immirzi parameter $\gamma/(1-\beta)$. Finally, regardless of the values of α and β, the diffeomorphism constraints have the same form. Therefore, under spatial diffeomorphisms, the configuration variables X_{aI} always transforms as a 1-form, and the canonical momentum Π^{aI} always transforms as a vector of weight $+1$.

6.3 Gauge fixing

The Hamiltonian description given in this chapter allows us to consider either the time gauge or the space gauge, we only need to keep in mind the different conventions. For the time gauge, Ξ is a spacelike surface with $q > 0$ ($h > 0$) and the norm of n_I is $\tau = \sigma$, while the internal metric is $(\eta_{IJ}) = \text{diag}(\sigma, 1, 1, 1)$. On the other hand, for the space gauge, Ξ is a timelike surface with $q < 0$ ($h < 0$) and $\tau = 1$, and the internal metric is $(\eta_{IJ}) = \text{diag}(1, -1, 1, 1)$. The metrics are different so that in both cases the gauge condition, $\tilde{\Pi}^{a0} = 0$, is the same.

However, in this section we only display the case for the time gauge. Hence, we consider $h > 0$, $\tau = \sigma$, and $(\eta_{IJ}) = \text{diag}(\sigma, 1, 1, 1)$. Then, we impose the gauge the condition

$$\tilde{\Pi}^{a0} = 0, \tag{6.38}$$

which is equivalent to $n_i = 0$, with i, j, k ... running from 1 to 3. Also, from (6.9) we conclude that $n_0 = \text{sgn}[\det(\tilde{\Pi}^{ai})]$. Next, we see that the condition (6.38) does not commutes with $\tilde{\mathcal{G}}^{i0}$

$$\{\tilde{\Pi}^{a0}, \tilde{\mathcal{G}}^{i0}\} = -\frac{\kappa\sigma}{2}\tilde{\Pi}^{ai}. \tag{6.39}$$

Therefore, we solve (6.38) and $\tilde{\mathcal{G}}^{i0} = 0$ simultaneously, and get

$$X_{a0} = \sigma n^0 (1 - \alpha) \underset{\sim}{\Pi}_{ai} \partial_b \tilde{\Pi}^{bi}. \tag{6.40}$$

In the time gauge, the action (6.36) is

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} dx^0 \int_{\Xi} d^3x \left[2\tilde{\Pi}^{ai} \dot{X}_{ai} - \left(2\lambda_i \tilde{\mathcal{G}}^i + 2N^a \tilde{\mathcal{D}}_a + \underset{\sim}{N} \tilde{\tilde{\mathcal{H}}} \right) \right], \tag{6.41}$$

where we defined $\lambda_i := -(1/2)\epsilon_{ijk}\lambda^{jk}$ and $\tilde{\mathcal{G}}_i := -(1/2)\epsilon_{ijk}\tilde{\mathcal{G}}^{jk}$, and the constraints acquire the form

$$\begin{aligned} \tilde{\mathcal{G}}^i &= -\frac{n^0}{\gamma}(1 - \beta)\partial_a \tilde{\Pi}^{ai} + \epsilon^i{}_{jk} X_a{}^j \tilde{\Pi}^{ak}, & (6.42a) \\ \tilde{\mathcal{D}}_a &= 2\tilde{\Pi}^{bi} \partial_{[a} X_{b]i} - X_{ai} \partial_b \tilde{\Pi}^{bi}, & (6.42b) \\ \tilde{\tilde{\mathcal{H}}} &= \sigma \epsilon_{ijk} \tilde{\Pi}^{ai} \tilde{\Pi}^{bj} R_{ab}{}^k + 2\sigma\Lambda |\det(\tilde{\Pi}^{ai})| \\ &\quad + 2\tilde{\Pi}^{a[i|}\tilde{\Pi}^{b|j]} \left[X_{ai} + \frac{n^0}{\gamma}(1 - \beta)\Gamma_{ai} \right] \left[X_{bj} + \frac{n^0}{\gamma}(1 - \beta)\Gamma_{bj} \right]. & (6.42c) \end{aligned}$$

The first thing we notice is the absence of the parameter α; it becomes irrelevant in the gauge fixing. In contrast, the value of parameter β (either $\beta = 1$ or $\beta \neq 1$) dictates the theory under consideration. We analyze more carefully this in what follows:

a) Case $\beta = 1$

In this case, the analysis is straightforward. From (6.42a)-(6.42c) we obtain

$$\begin{aligned}
\tilde{\mathcal{G}}^i &= \epsilon^i{}_{jk} X_a{}^j \tilde{\Pi}^{ak}, && (6.43a)\\
\tilde{\mathcal{D}}_a &= 2\tilde{\Pi}^{bi}\partial_{[a}X_{b]i} - X_{ai}\partial_b \tilde{\Pi}^{bi}, && (6.43b)\\
\tilde{\tilde{\mathcal{H}}} &= \sigma\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}R_{ab}{}^k + 2\tilde{\Pi}^{a[i|}\tilde{\Pi}^{b|j]}X_{ai}X_{bj} + 2\sigma\Lambda|\det(\tilde{\Pi}^{ai})|. && (6.43c)
\end{aligned}$$

Thus, $\beta = 1$ implies the absence of the Barbero-Immirzi parameter and, under the time gauge, the Hamiltonian description becomes the $SO(3)$-ADM formulation.

b) Case $\beta \neq 1$

In this case we begin by rewriting the Gauss constraint as

$$\tilde{\mathcal{G}}^i = -\frac{n^0}{\gamma}(1-\beta)\left[\partial_a\tilde{\Pi}^{ai} + \epsilon^i{}_{jk}\left(-n^0\frac{\gamma}{1-\beta}X_a{}^j\right)\tilde{\Pi}^{ak}\right], \tag{6.44}$$

from which we identify the internal connection as

$$\mathcal{A}_{ai} := -n^0\frac{\gamma}{1-\beta}X_{ai}, \tag{6.45}$$

with its corresponding field strength given by

$$\mathcal{F}_{abi} := \partial_a\mathcal{A}_{bi} - \partial_b\mathcal{A}_{ai} + \epsilon_{ijk}\mathcal{A}_a{}^j\mathcal{A}_b{}^k. \tag{6.46}$$

Moreover, using (6.46) and (3.54), we derive the identity

$$\epsilon_{ijk}(\mathcal{A}_a{}^j - \Gamma_a{}^j)(\mathcal{A}_b{}^k - \Gamma_b{}^k) = \mathcal{F}_{abi} - R_{abi} - 2\nabla_{[a}(\mathcal{A}_{b]i} - \Gamma_{b]i}). \tag{6.47}$$

With $\mathcal{A}_{ai}$ as our new configuration variable and, using the identity (6.47), we rewrite the action (6.41) and get

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dx^0 \int_{\Xi} d^3x \left[-\frac{2}{\gamma}n^0(1-\beta)\tilde{\Pi}^{ai}\dot{\mathcal{A}}_{ai} - 2\lambda_i\tilde{\mathcal{G}}^i - 2N^a\tilde{\mathcal{D}}_a - \underset{\sim}{N}\tilde{\tilde{\mathcal{H}}}\right], \tag{6.48}$$

where the constraints are given by

$$\tilde{\mathcal{G}}^i = -\frac{n^0}{\gamma}(1-\beta)\left(\partial_a\tilde{\Pi}^{ai} + \epsilon^i{}_{jk}\mathcal{A}_a{}^j\tilde{\Pi}^{ak}\right), \tag{6.49a}$$

$$
\begin{aligned}
\tilde{\mathcal{D}}_a &= -\frac{n^0}{\gamma}(1-\beta)\left(2\tilde{\Pi}^{bi}\partial_{[a}\mathcal{A}_{b]i} - \mathcal{A}_{ai}\partial_b\tilde{\Pi}^{bi}\right), && (6.49b) \\
\tilde{\tilde{\mathcal{H}}} &= \frac{(1-\beta)^2}{\gamma^2}\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}\left\{\mathcal{F}_{ab}{}^k + \left[\frac{\sigma\gamma^2}{(1-\beta)^2} - 1\right]R_{ab}{}^k\right\} \\
&\quad +2\sigma n^0\Lambda\det(\tilde{\Pi}^{ai}) - 2\frac{n^0}{\gamma}(1-\beta)\tilde{\Pi}^a{}_i\nabla_a\tilde{\mathcal{G}}^i. && (6.49c)
\end{aligned}
$$

Furthermore, integrating by parts the last term of (6.49c), and defining

$$
\tilde{\mathcal{C}}_a := \tilde{\mathcal{D}}_a + \mathcal{A}_{ai}\tilde{\mathcal{G}}^i = -\frac{n^0}{\gamma}(1-\beta)\mathcal{F}_{abi}, \tag{6.50}
$$

we arrive at

$$
S = \frac{1}{\kappa}\int_{\mathbb{R}} dx^0 \int_{\Xi} d^3x \left[-\frac{2}{\gamma}n^0(1-\beta)\tilde{\Pi}^{ai}\dot{\mathcal{A}}_{ai} - 2\mu_i\tilde{\mathcal{G}}^i - 2N^a\tilde{\mathcal{C}}_a - \underset{\sim}{N}\tilde{\tilde{\mathcal{C}}}\right], \tag{6.51}
$$

with $\mu_i := \lambda_i - \mathcal{A}_{ai}\tilde{\mathcal{G}}^i + [n^0(1-\beta)/\gamma]\tilde{\Pi}^a{}_i\nabla_a\underset{\sim}{N}$ and

$$
\tilde{\tilde{\mathcal{C}}} = \frac{(1-\beta)^2}{\gamma^2}\epsilon_{ijk}\tilde{\Pi}^{ai}\tilde{\Pi}^{bj}\left\{\mathcal{F}_{ab}{}^k + \left[\frac{\sigma\gamma^2}{(1-\beta)^2} - 1\right]R_{ab}{}^k\right\} + 2\sigma\Lambda|\det(\tilde{\Pi}^{ai})|. \tag{6.52}
$$

This is the Ashtekar-Barbero formulation with a rescaled Barbero-Immirzi parameter $\gamma/(1-\beta)$ and internal connection given by $\mathcal{A}_{ai}$. Both connections, namely $\mathcal{A}_{ai}$ and the original Ashtekar-Barbero connection A_{ai}, are related to each other via

$$
\mathcal{A}_{ai} = \left(\frac{1}{1-\beta}\right)(A_{ai} - \Gamma_{ai}) + \Gamma_{ai}. \tag{6.53}
$$

It is remarkable how the complete family of two-parameter canonical transformations given in (6.34) either collapse to the $SO(3)$-ADM description or to the Ashtekar-Barbero formalism. Furthermore, in the time gauge, only one of the two parameters becomes relevant and it is precisely this parameter which dictates the nature of the Hamiltonian formalism.

6.4 Comments

To conclude, we want to remark the distinctive aspects of this chapter. We avoided the introduction of second-class constraints when performing the Hamiltonian analysis of the Holst action. Thus, they are not mandatory in the real first-order formalism, as it is often believed. We accomplish such feat with the adequate parametrization of fundamental variables that describe the Holst action. The 16 components of the tetrad field become the

12 canonical momenta of the phase space $\tilde{\Pi}^{aI}$ [Eq. (6.6)] plus the four Lagrange multipliers N and N^a [Eqs. (6.11) and (6.12)]. On the other hand, the 24 fields that compose the internal Lorentz connection are map into the 12 configuration variables C_{aI} [Eq. (6.16)], the six Lagrange multipliers λ_{IJ} [Eq. (6.26)], and the six auxiliary fields $\underset{\sim}{\lambda}_{ab}$ [Eq. (6.23)]. Then, we integrate out the auxiliary fields $\underset{\sim}{\lambda}_{ab}$ and obtain a Hamiltonian description formed exclusively by first-class constraints.

Furthermore, we generalized the canonical transformations introduced in Sec. 3.3 with a two-parameter family of canonical transformations, whose values determine the presence or absence of the Barbero-Immirzi parameter. At the end, we explored the time gauge for the generalized variables $(X_{aI}, \tilde{\Pi}^{aI})$, and we found that the only relevant parameter is a rescaled version of Barbero-Immirzi parameter $\gamma/(1-\beta)$. Its presence or absence dictates whether we are working in the Ashtekar-Barbero formalism or in the $SO(3)$-ADM description. All of the results of this chapter are published in Ref. [58] under the terms of the *Creative Commons Attribution 4.0* International license. We, the authors, own the rights for the article distribution.

The results of this chapter open a new avenue that allows us to study different formulations of gravity. For instance, in the n-dimensional Palatini action, the second-class constraints are reducible when $n > 4$ [72]. Thus, the treatment to solve them is troublesome. However, with the use of the same techniques enlisted above, the Hamiltonian analysis is simple and direct. We published these results recently in Ref. [57]. Also, the method reported here has served as an inspiration to avoid the second-class constraints in BF gravity, see Ref. [73].

Moreover, the new parameterization could allow us to couple fermions into the Hamiltonian formalism of gravity. So far, this has been achieved only when the time gauge is considered [74, 75]. It might also be interesting to explore the case when we add more topological terms to the Palatini action [76–78]. Some of this work is currently under development.

Chapter 7

Conclusions

In this work we explored the Hamiltonian framework surrounding the Holst action. Along the way, we exposed different methods to solve the second-class constraints that arose during the Hamiltonian analysis. The first type of solution we presented was in a nonmanifestly Lorentz-covariant fashion, and it served to illustrate the former state of the Hamiltonian formalism for the Holst action. Here, we followed the approach of Cianfrani and Montani [38], and, even though their original solution was incomplete, we were able to mend their mistake and find a suitable Hamiltonian formulation (see Sec. 2.3). Moreover, although noncanonical variables described the resulting phase space, we made contact with the Barros e Sá formulation through the Darboux map exhibited in Sec. 2.4.

Next, in Sec. 3.2, we derived a manifestly Lorentz-covariant solution for the second-class constraints of the Holst action. The ensuing Hamiltonian formulation is formed by first-class constraints only and explicitly shows its Lorentz covariance. Furthermore, once we take the time gauge into account, our description neatly collapses to the Ashtekar-Barbero formulation (see Sec. 3.4).

With the manifestly Lorentz-covariant formulation at hand, we devoted Chapter 4 to explore an alternative gauge fixing. The new gauge is known as the space gauge, and it preserves some of the boost freedom of the Lorentz group. Thanks to the explicit covariant nature of the variables involved, we straightforwardly arrived at a complete formulation for general relativity invariant under local $SU(1,1)$ transformations. Remarkably, the new formulation resembles the Ashtekar-Barbero description.

Regarding the BF approaches, we also studied the Hamiltonian formalism of a BF-type action for general relativity that, at the Lagrangian level, reduces to the Holst's case. In Chapter 5, we showed that once we get rid of the second-class constraints, the BF action defined in Eq. (5.1) is entirely equivalent to the Holst action at the Hamiltonian level. Thus, we can also derive the Ashtekar-Barbero formulation from a BF-type action for general relativity.

Another important result was presented in Sec. 3.3. We obtained alternative Hamiltonian formulations through the use of canonical transformations. They are descriptions formed solely by first-class constraints, and they are manifestly Lorentz covariant too. Furthermore, some of the canonical transformations directly relate the Hamiltonian descriptions of the Holst and Palatini actions, specifically, the maps are: $\left(C_{aI}, \tilde{\Pi}^{aI}\right) \mapsto \left(Q_{aI}, \tilde{\Pi}^{aI}\right)$ or $\left(C_{aI}, \tilde{\Pi}^{aI}\right) \mapsto \left(\mathcal{Q}_{aI}, \tilde{\Pi}^{aI}\right)$ of Eqs. (3.35) and (3.40), respectively.[1] Therefore, we showed the equivalence of both actions at the Hamiltonian level with the complete symmetry group $SO(3,1)$ [or $SO(4)$].

In Sec. 6.2, we generalized the canonical transformations mentioned above. We did it with the map $\left(C_{aI}, \tilde{\Pi}^{aI}\right) \mapsto \left(X_{aI}, \tilde{\Pi}^{aI}\right)$, defined in Eq. (6.34). This map depends on two real parameters α and β, and, depending on their value, we can describe any of the formulations enlisted in Sec. 3.3. Moreover, in the time gauge, we noticed that the parameter α becomes irrelevant. In contrast, the value of β determines if the formulation is the $SO(3)$-ADM description ($\beta = 1$), or if it is the Ashtekar-Barbero formulation ($\beta \neq 1$). Thus, in the time gauge, only one parameter is important, and it is precisely the Barbero-Immirzi parameter (up to a rescaled version of it).

The last result we presented was exposed in Chapter 6 of this thesis. There, we demonstrated that we could avoid the introduction of the second-class constraints from the beginning. We achieved it by noticing that the canonical momenta of the smaller phase space $\tilde{\Pi}^{aI}$ is related to the spatial part of the tetrad field $e_a{}^I$. Therefore, the 16 independent fields that compose $e_\mu{}^I$ are translated into four Lagrange multipliers N and N^a plus 12 canonical momenta $\tilde{\Pi}^{aI}$. Furthermore, and similarly to the previous case, the Lorentz connection $\omega_\mu{}^I{}_J$ is decomposed into 12 dynamical variables C_{aI} (which play the role of the configuration variables), plus six Lagrange multipliers λ_{IJ}, and six auxiliary fields $\underset{\sim}{\lambda}_{ab}$. Once we integrate out the auxiliary fields $\underset{\sim}{\lambda}_{ab}$ through their own equation of motion, we land at the same Hamiltonian description of Chapter 3. Thus, this method simplifies the Hamiltonian analysis significantly.

The method of Chapter 6 is not restricted to the Holst action, it is a generic procedure that can be applied to avoid the introduction of second-class constraints in other contexts. For instance, in the Hamiltonian analysis for the n-dimensional Palatini action, there are second-class constraints that are reducible so further treatment is required [72, 79]. However, employing the method described in Chapter 6, we can avoid the troublesome

[1] The maps $\left(K_{aI}, \tilde{\Pi}^{aI}\right) \mapsto \left(Q_{aI}, \tilde{\Pi}^{aI}\right)$ or $\left(K_{aI}, \tilde{\Pi}^{aI}\right) \mapsto \left(\mathcal{Q}_{aI}, \tilde{\Pi}^{aI}\right)$ also prove the equivalence of both actions at the Hamiltonian level. They can be derived from the expressions (3.30), (3.35), and (3.40).

procedure of dealing with the second-class constraints and directly arrive at a formulation constructed with first-class constraints only [57]. Furthermore, the method can be implemented into the BF approaches, as it is treated in Ref. [73]. Also, without second-class constraints, it will be easier to couple fermionic matter into the Hamiltonian description of gravity. Although some attempts have been made in the regime of the time gauge [75, 80], the explicitly covariant description with fermions fields is currently under development. We hope to report it soon.

As a final remark, we have found several Hamiltonian descriptions for general relativity using the Holst action. In all of these formulations, the canonical variables lack the characteristic geometrical meaning of the Ashtekar-Barbero description. Although the momentum variable always transforms as an internal Lorentz vector, the configuration variable does not transform as an $SO(3,1)$ connection. Therefore, in all of our descriptions, we can not implement the techniques developed in loop quantum gravity. However, our formulations describe the phase space of general relativity with first-class constraints only, and with canonical conjugated variables that explicitly exhibit their Lorentz covariance. Perhaps this could motivate the use of alternative strategies that will lead us to a quantum description of gravity without losing the Lorentz invariance.

Appendix

A

Conventions and notation

Throughout this work we use a 4-dimensional spacetime manifold M, in general we will assume that M is diffeomorphic to $\mathbb{R}\times\Sigma$, with Σ as a spacelike 3-dimensional submanifold without boundary ($\partial\Sigma = 0$). However, in Chapter 4, we consider that M is diffeomorphic to $\Omega\times\mathbb{R}$, with Ω a 3-dimensional timelike submanifold that might have a boundary. Also, be aware that Chapter 6 has its own conventions.

Independently of the topology of M, we define an orthonormal 1-form basis e^I at each point $p\in M$. The latin capital letters beginning in the middle of the alphabet (I, J, $K, ...$) are group indices. They take values $\{0,\, 1,\, 2,\, 3\}$ and are lowered or raised with the internal metric $(\eta_{IJ}) = \mathrm{diag}\,(\sigma,\, 1,\, 1,\, 1)$, being $\sigma = \pm 1$, where $\sigma = -1$ ($\sigma = +1$) indicates a Lorentzian (Euclidean) signature. These indices represent quantities valued in the group $SO(3,1)$ for $\sigma = -1$ or $SO(4)$ for $\sigma = 1$. In general, we maintain both signatures, except in Chapter 4 when we strictly stick to the Lorentz group. Sometimes we will split the internal indices and use lowercase latin letters (i, j, k, ...) to denote that they take the values $\{1,\, 2,\, 3\}$. On the other hand, for the spacetime indices, we use greek letters (α, β, $\mu, ...$), so we label the local coordinates as $\{x^\mu\} = \{t, x^a\}$, with a, b, $c, ... = 1,\, 2,\, 3$. In Chapter 4, we use a bar over the lowercase indices, either internal ($\bar{i}$, $\bar{j}$, $\bar{k}, ...$) or spacetime ($\bar{a}$, $\bar{b}$, $\bar{c}, ...$), to indicate that they take the values $\{0,\, 1,\, 2\}$.

Regardless of the set of indices, we define the symmetrizer and the antisymmetrizer, respectively, as

$$A_{(xy)} := \frac{1}{2}\left(A_{xy} + A_{yx}\right), \tag{A.1a}$$

$$A_{[xy]} := \frac{1}{2}\left(A_{xy} - A_{yx}\right). \tag{A.1b}$$

Also, for any antisymmetric internal quantity $U_{IJ} = -U_{JI}$, we define its correspondent internal dual by

$$*\, U_{IJ} := \frac{1}{2}\epsilon_{IJKL}U^{KL}, \tag{A.2}$$

where ϵ_{IJKL} is the totally antisymmetric $SO(3,1)$ [or $SO(4)$] tensor and $\epsilon_{0123}=1$. The internal dual satisfies the properties

$$
\begin{aligned}
*U_{IJ}V^{IJ} &= U_{IJ}*V^{IJ}, && \text{(A.3a)}\\
*\left(U_{[I|K}V^{K}{}_{|J]}\right) &= *U_{[I|K}V^{K}{}_{|J]} = U_{[I|K}*V^{K}{}_{|J]}. && \text{(A.3b)}
\end{aligned}
$$

Similarly, let γ be a real number, then, it is convenient to define the γ-hat notation as

$$\overset{(\gamma)}{U}_{IJ} := \left(\delta^K_{[I}\delta^L_{J]} + \frac{1}{2\gamma}\epsilon_{IJ}{}^{KL}\right)U_{KL} = P_{IJ}{}^{KL}U_{KL}, \tag{A.4}$$

where $P_{IJ}{}^{KL}$ defines the internal projector

$$P_{IJ}{}^{KL} := \delta^K_{[I}\delta^L_{J]} + \frac{1}{2\gamma}\epsilon_{IJ}{}^{KL}, \tag{A.5}$$

with $\left(P^{-1}\right)^{IJ}{}_{KL}$ being its inverse as long as $\gamma^2 \neq \sigma$. Explicitly $\left(P^{-1}\right)^{IJ}{}_{KL}$ is given by

$$\left(P^{-1}\right)^{IJ}{}_{KL} = \frac{\gamma^2}{\gamma^2-\sigma}\left(\delta^I_{[K}\delta^J_{L]} - \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL}\right). \tag{A.6}$$

Notice that they fulfill $P_{IJ}{}^{MN}\left(P^{-1}\right)_{MN}{}^{KL} = \delta^K_{[I}\delta^L_{J]}$. From (A.3) it is straightforward to prove the identities

$$
\begin{aligned}
\overset{(\gamma)}{U}_{IJ}\,V^{IJ} &= U_{IJ}\,\overset{(\gamma)}{V}{}^{IJ}, && \text{(A.7a)}\\
\overset{(\gamma)}{\left(U_{[I|K}V^{K}{}_{|J]}\right)} &= \overset{(\gamma)}{U}_{[I|K}\,V^{K}{}_{|J]} = U_{[I|K}\,\overset{(\gamma)}{V}{}^{K}{}_{|J]}. && \text{(A.7b)}
\end{aligned}
$$

Finally, when working with tensor densities, we will use tildes above (below) the correspondent variable; the tildes above (below) indicate a positive (negative) weight equivalent to the number of tildes. For convenience, sometimes we omit the use of tildes, but its weight is mentioned somewhere in the text. Two of the most common tensor density we use are $\tilde{\eta}^{\alpha\beta\mu\nu}$ and $\underset{\sim}{\eta}_{\alpha\beta\mu\nu}$, they are totally antisymmetric tensor densities and satisfy $\tilde{\eta}^{0123}=1$ and $\underset{\sim}{\eta}_{0123}=1$, respectively. From these tensor densities, we sometimes use $\tilde{\eta}^{abc} := \tilde{\eta}^{0abc}$ and $\tilde{\eta}^{\bar{a}\bar{b}\bar{c}} := \tilde{\eta}^{\bar{a}\bar{b}\bar{c}3}$, or $\underset{\sim}{\eta}_{abc} := \underset{\sim}{\eta}_{0abc}$ and $\underset{\sim}{\eta}_{\bar{a}\bar{b}\bar{c}} := \underset{\sim}{\eta}_{\bar{a}\bar{b}\bar{c}3}$.

Appendix B

List of publications

Throughout this thesis, I exposed the work I developed during the last few years, in which I dedicated time to understand the Hamiltonian formulations of first-order general relativity. My Ph.D. thesis work generated the following publications:

- M. Celada, M. Montesinos, and **J. Romero**. "Barbero's formulation from a BF-type action with the Immirzi parameter". Classical and Quantum Gravity **33** 115014 (2016).

- M. Montesinos, **J. Romero**, M. Celada. "Manifestly Lorentz-covariant variables for the phase space of general relativity". Phys. Rev. D **97** 024014 (2018).

- M. Montesinos, **J. Romero**, R. Escobedo, M. and Celada. "$SU(1,1)$ Barbero-like variables derived from Holst action". Phys. Rev. D **98** 124002 (2018).[1]

- M. Montesinos, **J. Romero**, M. Celada. "Revisiting the solution of the second-class constraints of the Holst action". Phys. Rev. D **99** 064029 (2019).

- M. Montesinos, **J. Romero**, M. Celada. "Canonical analysis of Holst action without second-class constraints". Phys. Rev. D **101** 084003 (2020).[1]

- M. Montesinos, R. Escobedo, **J. Romero**, and M.Celada. "Canonical analysis of n-dimensional Palatini action without second-class constraints". Phys. Rev. D **101** 024042 (2020).[1]

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