CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS
DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO
DEPARTAMENTO DE FÍSICA

“Nudos, relaciones tipo estrella-triángulo y teoría cuántica de campos”

Tesis que presenta

José de la Cruz Moreno

para obtener el Grado de

Doctor en Ciencias

en la Especialidad de

Física

Director de tesis: Dr. Héctor Hugo García Compeán

Ciudad de México Enero, 2021

CENTER FOR RESEARCH AND ADVANCED STUDIES OF THE NATIONAL POLYTECHNIC INSTITUTE

PHYSICS DEPARTMENT

"Knots, star-triangle type relations and quantum field theory"

by

José de la Cruz Moreno

In order to obtain the

Doctor of Science

degree, speciality in

Physics

Advisor: Ph. D. Héctor Hugo García Compeán

Mexico City January, 2021

Acknowledgements

Dedicated to my parents María Florida and José
who has always been the most beautiful and charming support for me,
they have believed in me since I was a child, I really love them.
Father, mother, I can truly say from inside my heart that this achievement is ours,
I am very grateful for the peaceful and amorous life you give me,
thank you so much for all your love!

Dedicated to my close friends because real friendship is invaluable,
I will never forget all those days when we supported each other,
all those talks and games to renovate our spirit, and
to keep smiling against the problems.

I am very grateful to my thesis supervisor Dr. Héctor Hugo García Compeán
who inspired me to work in very interesting top-notch subjects,
he is a friendly and an admirable person,
I really appreciate all the support and discussions we had
either in academic issues or in casual conversations,
he was always well-disposed to give me advices and suggestions,
and to work hard when necessary.
Thank you so much for all these years professor!

I also thank Eduardo López González, a friend of mine and a co-author in one of the research projects that is part of this thesis, my committee members Dr. Roberto Alejandro Santos Silva, Dr. Abdel Pérez Lorenzana, Dr. Miguel A. Xicoténcatl Merino, Dr. Riccardo Capovilla Chiariglione and Dr. Josué De Santiago Sanabria, for drawing my attention to some points that led to this improved version of the thesis, and CONACYT for the financial support given during my doctoral stay at CINVESTAV-IPN.

Resumen

A lo largo de los años se han estudiado invariantes topológicos, dualidades y correspondencias que involucran teorías cuánticas de campos. El presente trabajo es un avance en esta dirección y concierne por un lado la obtención de invariantes topológicos de nudos a partir de la teoría *gauge* cuantizada de Chern-Simons y por otro la correspondencia *gauge*/YBE (ecuación de Yang-Baxter por sus siglas en inglés) para dualidades de teorías $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ *quiver gauge* supersimétricas. Esto se detalla a continuación.

Es bien sabido que la teoría *gauge* de Chern-Simons permite construir invariantes topológicos de nudos tanto en el régimen perturbativo como en el no perturbativo, a saber, invariantes de Vassiliev (también llamados invariantes de tipo finito) e invariantes polinomiales, respectivamente. El interés de este trabajo está en los primeros, donde la superposición de amplitudes específicas a cierto orden en teoría de perturbaciones es un invariante de Vassiliev de ese orden. Por otro lado, dichas amplitudes pueden ser reescritas en un contexto geométrico y topológico como integrales de Bott-Taubes en espacios de configuración. Este dicccionario nos permite escribir los invariantes de Vassiliev como integrales de Bott-Taubes y esto se realiza hasta orden tres en la constante de acoplamiento, lo que extiende un trabajo previo de Thurston. Una vez que se tienen los invariantes de Vassiliev en espacios de configuración es más fácil incorporar información de un campo vectorial suave y sin divergencias en la variedad diferenciable donde la teoría de Chern-Simons está definida para obtener invariantes más finos. Dicha incoporación se realiza al reemplazar los nudos con ciclos asintóticos de Schwartzman. La primera parte de esta tesis está dedicada a la construcción explícita de los llamados invariantes de nudos de Vassiliev promedio asintóticos de orden superior.

La correspondencia *gauge*/YBE es una correspondencia entre dualidades de teorías *quiver gauge* supersimétricas y modelos integrables en mecánica estadística. El diccionario aún está bajo construcción y es esto lo que motiva la segunda parte de la tesis, donde se obtienen relaciones tipo estrella-triángulo a partir de algunas dualidades de teorías $2d$ $\mathcal{N} = (0,2)$ *quiver gauge* supersimétricas, esto es, se determinan completamente tanto los pesos de Boltzmann como los factores de interacción y normalización. El primer análisis se realiza para la dualidad $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ Intriligator-Pouliot, proveniente de una reducción dimensional de la dualidad $4d$ $\mathcal{N} = 1$ $USp(2N)$ Intriligator-Pouliot, para distintos valores de N. Para $N = 1$ se obtiene una identidad triángulo, para $N = 2, 5$ (y la generalización $N = 3k + 2$ con k un entero no negativo) el resultado es una variante de la relación tipo estrella-triángulo y para $N = 3, 4$ se deduce una forma similar de la relación tipo estrella-triángulo asimétrica. El segundo análisis se realiza para la dualidad $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ para teorías con materia en la representación tensorial antisimétrica que proviene de una reducción dimensional de la dualidad $4d$ $\mathcal{N} = 1$ $USp(2N)$ Csáki-Skiba-Schmaltz. En este análisis se obtiene una identidad tipo triángulo para cada valor de N.

Finalmente, se plantean sugerencias de trabajo futuro para ambas líneas de investigación.

Abstract

Along the years, topological invariants, dualities and correspondences involving quantum field theories have been studied. The present work is a development in this framework that concerns on the one hand topological knot invariants from quantized Chern-Simons gauge theory and on the other hand gauge/YBE correspondence (YBE stands for Yang-Baxter equation from now on) for $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ supersymmetric quiver gauge theory dualities. This is detailed as follows.

It is well known that Chern-Simons gauge theory leads to topological knot invariants in both the perturbative and the non-perturbative regimen, namely, finite type or Vassiliev invariants and polynomial invariants, respectively. We are interested in the former ones where the superposition of specific amplitudes at any order in perturbation theory is a Vassiliev invariant of that order. On the other hand, the amplitudes can be written in a geometrical and topological framework as Bott-Taubes integrals in configuration spaces. This dictionary let us write Vassiliev invariants as Bott-Taubes integrals and this is performed up to order three in the coupling constant, extending a previous work of Thurston. Once we have Vassiliev invariants in configuration spaces it is easier to incorporate information of a smooth divergenceless vector field on the manifold where Chern-Simons theory is defined in order to obtain richer invariants. The incorporation is done via the asymptotic cycles of Schwartzman that replace the knots. The first part of the thesis is devoted to this instance, explicit construction of the so called higher-order average asymptotic Vassiliev invariants for knots.

Gauge/YBE correspondence is a correspondence between supersymmetric quiver gauge theory dualities and integrable models in statistical mechanics. The dictionary is still under construction and this motivated the second part of the thesis where star-triangle type relations are derived from some $2d$ $\mathcal{N} = (0,2)$ supersymmetric quiver gauge theory dualities, that is, Boltzmann weights, interaction and normalization factors were completely determined. The first worked duality is $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ Intriligator-Pouliot duality coming from dimensional reduction of $4d$ $\mathcal{N} = 1$ $USp(2N)$ Intriligator-Pouliot duality. The description was performed for many values of N. For $N = 1$ the result was a triangle identity, for $N = 2, 5$ (and the generalization $N = 3k + 2$ with k a non-negative integer) a slight variation of the star-triangle type relation was found and for $N = 3, 4$ a similar form of the asymmetric star-triangle type relation was obtained. The second worked duality is $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ duality for theories with matter in the antisymmetric tensor representation that comes from dimensional reduction of $4d$ $\mathcal{N} = 1$ $USp(2N)$ Csáki-Skiba-Schmaltz duality. In this case a triangle type identity was obtained for any value of N.

Finally, some suggestions for future work are given for both research lines.

Contents

Introduction

The main goal of this thesis is to deepen into the impact of quantum field theories in different areas of mathematics and physics. Historically, these two sciences had influenced each other in a very powerfull way and this work is an example of this great symbiosis when expressing relations and correspondences between knot theory, dynamical systems, integrable models and diverse quantum field theories. It is pretty interesting for me the way seemingly distant topics can be related as a consistent physical mathematical structure, it really impress me the amount of dualities that exist in theoretical physics as well as the depth of their implications. There were several academic motivations for writting this thesis; some of them were seminal papers [1] and [2] that I knew thanks to my thesis supervisor, the first one builds a chain complex for a manifold M by using supersymmetric quantum mechanics to study its so called Morse homology while in the second one Chern-Simons gauge theory is shown to be the natural framework where knot invariants have physical realization as correlation functions of Wilson loop operators; motivation also came when reading the research articles my thesis supervisor have written, all of them combine physics and mathematics in a systematic and creative way. All in all, the main content of this thesis is adapted from articles that are either published or submitted to publication concerning research work of my doctoral stay at CINVESTAV-IPN.

As a Ph.D. student my research interest is focused in mathematical aspects of string theory and quantum field theories, specifically, the subjects I am interested in are gauge/YBE correspondence, Vassiliev invariants and their generalizations, gauge/string theory duality, Khovanov homology from a physical point of view, topological sigma models, entanglement entropy via knot theory and knot invariants via Kapustin-Witten equations. All these topics share two principal features, namely, they involve topological invariants and quantum field theories. This was precisely the reason I started to work on two projects. The first project is an explicit construction of topological knot invariants, that are also dynamical system invariants, via Chern-Simons gauge theory. Due to the existence of gauge/string theory duality and gauge/YBE correspondence the natural question is then if those invariants have a counterpart either in string theory or in integrable models. This second direction was one of the motivations to start working on gauge/YBE correspondence and it led to the second project where star-triangle type relations associated with some $2d$ $\mathcal{N} = (0,2)$ supersymmetric quiver gauge theory dualities were found.

The first project is described in chapter 1 as a more detailed version of reference [3]. An appropriate introduction to the topic is given as follows. It is well known that Chern-Simons gauge theory is the appropriate physical framework to describe topological invariants of 3-manifolds such as the Ray-Singer torsion [4, 5]. In particular, knot and link invariants were described in this context through a non-perturbative treatment, which gives rise to the Jones polynomial [2]. The perturbative analysis of Chern-Simons action also leads to invariants of knots and links [6, 7] (for a review see, for instance, Refs. [8, 9, 10, 11]), in particular, the so called *finite type invariants* from which Vassiliev invariants are an example [12, 13, 14, 15] (for an introduction to Vassiliev invariants see, for instance, [16]). In the context of quantum field theory many developments were given in more physical terms in Refs. [7, 17, 18]. To be precise, in Ref. [7] it is proven that at any order in perturbation theory the resulting expression is a Vassiliev invariant of that order, *i.e.*, a Vassiliev invariant is a sum of the amplitudes of certain Feynman diagrams.

On the other hand, in the study of dynamical systems, quantum field theory also has produced a significant contribution. In Ref. [19], Verjovsky and Vila-Freyer used the Chern-Simons theory and the idea of asymptotic homology cycles of foliations previously proposed by Schwartzman [20] for determining topological invariants of certain dynamical systems. Some further developments on the interface between dynamical systems and algebraic topology were discussed, for instance, in Refs. [21, 22, 23, 24]. To be more precise, one of the goals of the article [19] was to construct topological invariants of triplets $(M, \mathcal{F}, \mu)$, where M is the underlying 3-manifold, $\mathcal{F}$ is the foliation in M generated by a non-singular global volume-preserving vector field X and μ is the transverse measure invariant under such a flow. In Ref. [19], using the Abelian Chern-Simons action on $\mathbb{R}^3$ or $\mathbf{S}^3$ with a volume-preserving vector field X, it was shown that the exact evaluation of Chern-Simons integral functional in Witten's theory [2] leads to the link invariant for a pair of orbits of one non-singular vector field X. The result is precisely the *helicity* invariant or *asymptotic Hopf invariant*, obtained previously by a series of authors in different contexts of (astro)physics and mathematics [25, 26, 27, 28, 29, 30]. This is called the average asymptotic linking number and it can be regarded to be a topological invariant of the dynamical system defined by the triplet $(M, \mathcal{F}, \mu)$. For a review on these topics see, for instance, [31, 32, 33]. Thus, in this context it would be possible that the Jones-Witten invariants of manifolds M_1 and M_2 are equivalent but the invariants of the triplets $(M_1, \mathcal{F}_1, \mu_1)$ and $(M_2, \mathcal{F}_2, \mu_2)$ are inequivalent as invariants of dynamical systems.

Moreover in [19] it was also discussed the non-Abelian case. This is quite more complicated than the Abelian one. In this reference it is also found the formal definition of average asymptotic Jones-Witten invariant in terms of the average asymptotic Wilson loop functional. In the process the definition requires to consider the holonomy of the connection and the Ergodic theorem. This is a suitable definition, however, it makes very hard the possibility to make explicit computations.

Helicity or linking numbers can be extended in different directions, one of them is the generalization to higher dimensions. In Ref. [34] starting from a BF theory in n dimensions on a homologically trivial manifold, it was obtained a generalization of the helicity (or Jones-Witten invariant) found in [19]. Moreover in Ref. [35], with the use of results from [21, 22, 23], it was possible to find invariants for triplets $(M_4, \mathcal{F}, \mu)$ in the cases of the Donaldson-Witten and Seiberg-Witten invariants. These invariants of four-dimensional dynamical systems involve the use of non-Abelian groups, however, as it was discussed in Ref. [35], the observables are Lie algebra valued functionals. Consequently, the mentioned complication arising from non-Abelian features does not appear there. As we mentioned above, in the perturbative Chern-Simons theory the relevant invariants are the Vassiliev ones. These objects can be obtained from the expansion of the Wilson loops and therefore they are also Lie algebra valued. Thus similarly to the situation of Ref. [35], the complication does not appear in this case.

On the perturbative theory, where Vassiliev invariants are defined, the work has not been so extensive. Configuration spaces were introduced into this context in Refs. [6, 12, 13, 14, 15, 36, 37] to compute Feynman diagrams in Chern-Simons theory. Another important development is the proposal of integration in the configuration space, known as Bott-Taubes integrals. These integrals were introduced in the seminal paper [38] in order to study the Feynman diagrams with 3 points on the knot and 1 point outside of it (for a recent overview on this subject see Ref. [39]). Later Thurston [40] generalized the work of Ref. [38] to the case of integration on the configuration space constructed from Feynman diagrams with p points lying on the knot and q points lying outside of it. Moreover, Thurston's work also provides a guide to translate Bott-Taubes integrals into Chern-Simons expressions. One of the advantages of the configuration space formalism is that the Feynman amplitudes can be expressed as integrals of differential forms in configuration space. As a consequence of this fact, in Ref. [41], Komendarczyk and Volić introduced a volume-preserving vector field X into this context with the aim of proposing a manner to obtain *average asymptotic Vassiliev invariants*. Average symptotic Vassiliev invariants were studied also in the context of Kontsevich's integrals in Ref. [42]. In the present chapter we extend the work done in Refs. [40, 41], which uses Bott-Taubes integration. We use systematically the perturbative expansion of Chern-Simons theory to find Bott-Taubes integrals associated to higher-order terms of the perturbative expansion of Chern-Simons theory. The Vassiliev invariants are computed explicitly up to third order in the coupling constant. Furthermore, the results obtained here are used to find their corresponding average asymptotic Vassiliev invariants. In order to do all this work, we compile information from various authors into mathematical diagrams. This is highly convenient because some results and constructions are spread out in the literature.

This chapter is organized as follows. In section 1.1 we briefly overview the emergence of Vassiliev invariants from the perturbative Chern-Simons theory in the Lorentzian signature. Section 1.2 is devoted to review the Bott-Taubes integrals and the evaluation on a vector field X. Moreover we overview the general construction to define average asymptotic invariants introduced in Ref. [41]. In section 1.3 we obtain the Bott-Taubes integrals for perturbative diagrams of first, second and third orders. Finally, in section 1.4 we introduce vector fields in the description found in section 1.3 to obtain the average asymptotic Vassiliev invariants corresponding to first, second and third order Feynman amplitudes.

The second project is described in chapter 2 as an extended version of reference [43] whose appropriate introduction is given as follows. Classical and quantum integrable systems defined through the Yang-Baxter equation (YBE) have been studied from diverse view points in the literature. Important work has been summarized in many compendia reported at early stages, see for instance, [44, 45, 46, 47, 48, 49]. Recently, a surprising relation between quiver gauge theories in various dimensions with diverse degrees of supersymmetry and integrable models in statistical mechanics has starting to be explored by many authors, for an overview see [50] and references therein. This relation is termed in the literature as the *gauge/YBE correspondence*. In this correspondence the underlying spin lattice in the integrable model is identified to the quiver diagram of the quiver gauge theory. Moreover the self-interaction and nearest-neighbour interaction of spins correspond to the gauge vector supermultiplets in the adjoint representation of the gauge group and the chiral multiplet in the bifundamental representation of the gauge group, respectively. As a result of the work on this subject a dictionary of this correspondence has been established between the structure and features of the integrable models and the quiver field theory. For instance, the spin variables can be identified with the gauge holonomies along non-trivial homology 1-cycles, the rapidity line can be identified with the zig-zag path, the spectral parameter with the R-charge, the statistical partition function to the field theory partition function, the star-star relation to the Seiberg(-like) duality, the Yang-Baxter equation with the Yang-Baxter duality, etc.

There are plenty of integrable models that have been obtained from supersymmetric dualities via the gauge/YBE correspondence [50]. This have been done for different dimensions, amounts of supersymmetry, gauge groups and diverse curved manifolds. To state some examples, there are integrable models associated to $2d$ $\mathcal{N} = (2,2)$ theories (see, for instance, [51, 52]), $3d$ $\mathcal{N} = 2$ theories (see, for instance, [53, 54]) and $4d$ $\mathcal{N} = 1$ theories (see, for instance, [55, 56]). A large list of more dualities is given in [57]. Despite the rich zoo of new integrable models obtained from the gauge/YBE correspondence and as far as we know, there are no explicit integrable models associated with $2d$ $\mathcal{N} = (0,2)$ theories and we have found very few literature about this topic (see, for example, [58] for the context of brane constructions).

The above considerations have motivated the authors to study $2d$ supersymmetric $\mathcal{N} = (0,2)$ field theories. It would be interesting to investigate whether this family of theories can be incorporated to the context of the mentioned correspondence and to check if there are integrable models that can be associated to these models. However, as a first step in this direction we will concentrate in the present work in studying what kind of star-triangle relations can be associated to some of the dualities obeyed for these supersymmetric theories. Thus the aim of the present chapter is to study what kind of star-triangle relations (or some of their variants as the star-tringle type relation, STR type, or the triangle identity) arises from some dualities in supersymmetric quiver gauge theories. In this direction, we first analyse the Intriligator-Pouliot duality in $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ theories coming from dimensional reduction (see [59] for description of this reduction) of $4d$ $\mathcal{N} = 1$ $USp(2N)$ confining Intriligator-Pouliot theory originally studied in [60]. The analysis is carried out for different values of N. We also study a new duality for $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ theories with matter in the antisymmetric tensor representation found in [61] that arises from a dimensional reduction of $4d$ $\mathcal{N} = 1$ $USp(2N)$ Csáki-Skiba-Schmaltz duality first discussed in [62]. The expressions obtained in this work share many features with the standard star-triangle relation, such as the general distribution and dependence on the spin variables and spectral parameters, although they have not exactly the same form. In certain cases we found some similarity with star-triangle type relations discussed in the literature of Yang–Baxter/$3D$-consistency correspondence [63, 64, 65, 66]. It must be remarked that despite all those similarities the integrability properties of the STR type expressions we found in this work are still unclear and they probably do not lead to integrable systems.

Intriligator-Pouliot and Csáki-Skiba-Schmaltz dualities are important in the context of Seiberg-like duality [67, 68] while $2d$ $\mathcal{N} = (0,2)$ theories have special interest since the discovery of trialities among them [69, 70]. In the remarkable paper [69] the authors studied the space of $2d$ $\mathcal{N} = (0,2)$ supersymmetric quiver gauge theories and there it was found a *triality* among them. There it is also speculated the possibility that the triality would be associated with the tetrahedron equation of statistical mechanics (see [71], for instance, for some work on this equation) in a similar way that Seiberg's duality is related with Yang-Baxter equation.

It is worth mentioning that we focused into these $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ dualities as a first step to study simple models but it would be interesting to extend the analysis for other gauge groups. Also it must be remarked that in both cases the dualities are given between a gauge theory and a Landau-Ginzburg model (where there is no gauge symmetry) and it actually helped us to obtain either STR type expressions or triangle identities. To be more precise, if there is no gauge symmetry then there is no integration in the corresponding index of the theory and this highly resembles the structural form of both the STR type expressions and the triangle identities. This situation however is by far not general for $2d$ $\mathcal{N} = (0,2)$ theories as it can be regarded in the triality studied in [69]; this is maybe one of the reasons why the authors in that reference speculated about the introduction of the tetrahedron equation.

This chapter is organized as follows: in Section 2.1 a brief overview of the gauge/YBE correspondence is given. In Section 2.2 we review the $4d$ Csáki-Skiba-Schmaltz duality and its associated STR type expression. Finally, section 2.3 is devoted to obtain some slight variants of the star-tringle type relations from $2d$ theories with supersymmetry $\mathcal{N} = (0,2)$ and $USp(2N)$ gauge group. For $2d$ Intriligator-Pouliot duality this is carried out for the first five values of N and a general case with $N = 3k + 2$. It is also shown that the relation associated with $2d$ Csáki-Skiba-Schmaltz duality for any value of N is a triangle type identity.

I have just started to study the many dualities and correspondences in theoretical physics, my interest is to explore some of them to find more relations among the seemingly different areas involved. In the last part of the thesis there are some conclusions about the influence of the described research articles and perspectives for future works in those directions, also, appendices A.1, A.2, A.3 and B.1 are included to give some mathematical technicalities and calculations needed in the bulk of the thesis.

CHAPTER 1

Vassiliev Invariants for Flows Via Chern-Simons Perturbation Theory

This chapter is basically an extended version of the research article "Vassiliev Invariants for Flows Via Chern-Simons Perturbation Theory" (see reference [3]) done in collaboration with my thesis supervisor and with another author, and submitted to International Journal of Modern Physics A on November 2020. This work explicitly builds topological knot invariants, that are also dynamical system invariants, from perturbative analysis of quantum Chern-Simons gauge theory and inclusion of a vector field in its three manifold. It is well known that both non-perturbative and perturbative analysis of this theory lead to topological knot and link invariants, the former are called quantum invariants while the latter are known as finite type invariants or Vassiliev invariants. To be precise, at any order in perturbation theory the superposition of specific amplitudes is a knot invariant of that order; Bott-Taubes integrals on configuration spaces are introduced to study them in a geometrical and algebraic-topological framework. One of the consequences of this formalism is that the resulting amplitudes are rewritten in cohomological terms in configuration space and so a traduction from Bott-Taubes integrals to Chern-Simons perturbative amplitudes can be done. The authors calculate explicit expressions for this program up to third order in the coupling constant, expanding some previous work done by Thurston, and incorporate a smooth divergenceless vector field on the manifold where Chern-Simons theory is defined in order to make contact with previous articles about dynamical system invariants such as the Hopf invariant. This way, by using the obtained Bott-Taubes integrals, some examples of higher-order average asymptotic Vassiliev invariants, where roughly speaking the knot is replaced by asymptotic cycles generated by the orbits of the vector field, are explicitly built extending the work of Komendarczyk and Volić.

1.1 Vassiliev invariants from perturbative Chern-Simons theory

In this section we briefly overview the perturbative expansion of Chern-Simons theory. We are not intending to be exhaustive but just to give the background material to introduce the notation and conventions we will follow in further sections. This section is written following Refs. [12, 13, 17].

The Chern-Simons action (or functional) is written as

$$I_{CS}(A) = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right), \tag{1.1}$$

where A is a $\mathcal{G}$-valued connection on a trivial G-principal bundle over a 3-dimensional manifold M which we will take from now on as $\mathbb{R}^3$ or $\mathbf{S}^3$. Here G is any compact and semi-simple

Lie group and $\mathcal{G} = Lie(G)$ is its associated Lie algebra. Moreover Tr: $\mathcal{G} \times \mathcal{G} \to \mathbb{R}$ is the Killing quadratic form on $\mathcal{G}$. In Eq. (1.1) k is an integer number i.e. $k \in \mathbb{Z}$ and it represents the inverse of the coupling constant of the theory. In this theory the unnormalized correlation functions are given by

$$\langle W_R^K(A) \rangle = \int DAe^{iI_{CS}(A)} W_R^K(A), \tag{1.2}$$

where

$$W_R^K(A) = \mathrm{Tr}_R \left[\mathrm{P} \exp \left(\oint_K A_\mu^a(x) t_a dx^\mu \right) \right] \tag{1.3}$$

is the Wilson loop operator and Tr_R is the Killing form in the representation R of G, t_a are the generators of the Lie algebra at representation R and $K : \mathbf{S}^1 \to M$ is the knot, *i.e.*, a smooth embedding. A nonperturbative analysis [2] of correlation functions (1.2) reveals that these functions coincide with the unnormalized Jones polynomial

$$J(q, K) = \langle W_R^K(A) \rangle. \tag{1.4}$$

These objects are polynomials in the variable $q = \exp\left(\frac{2\pi i}{k + h^\vee}\right)$, where $h^\vee$ stands for the dual Coxeter number of G (for $SU(N)$ it is N). It depends on the knot K, the Lie group G and its representation R. For example, $SU(N)$ in the fundamental representation gives the HOMFLY-PT polynomial, $SO(N)$ in the fundamental representation gives the Kauffman polynomial and $SU(2)$ in the $(n+1)$-dimensional representation[1] gives the n-colored Jones polynomial[2]. The Jones polynomial can be written as finite q-series

$$J(q, K) = \sum_n a_n q^n, \tag{1.5}$$

where a_n are integer numbers.

Chern-Simons gauge theory can be quantized via BRST method and the resulting quantum action in components in $\mathbb{R}^3$ looks like [13]

$$I = \frac{k}{4\pi} \int_{\mathbb{R}^3} \mathrm{Tr} \left(\varepsilon^{ijk} A_i \partial_j A_k + 2\bar{c}\partial_i \partial^i c + 2\phi \partial^i A_i + \frac{1}{3} \varepsilon^{ijk} A_i [A_j, A_k] + 2\bar{c}\partial_i [A^i, c] \right), \tag{1.6}$$

where c and $\bar{c}$ are the ghost and anti-ghost fields, which are Grassmann and $\mathcal{G}$-valued scalar fields coupled to the gauge fields, and ϕ is a $\mathcal{G}$-valued scalar field. This action is composed by the following three parts

$$I = I_0 + I_g + I', \tag{1.7}$$

where I_0 is the kinetic (or free) part of I,

$$I_0 = \frac{k}{4\pi} \int_{\mathbb{R}^3} \mathrm{Tr} \left(\varepsilon^{ijk} A_i \partial_j A_k + 2\bar{c}\partial_i \partial^i c \right), \tag{1.8}$$

I_g is the gauge fixing action,

$$I_g = \frac{k}{4\pi} \int_{\mathbb{R}^3} \mathrm{Tr} \left(2\phi \partial^i A_i \right), \tag{1.9}$$

[1] In physics literature this is called the j spin representation. Here $2j = n$, *i.e.*, its dimension is $2j + 1$.

[2] The case $n = 1$ (or $j = 1/2$) is the famous Jones polynomial.

and I' is the interaction action,

$$I' = \frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{Tr}\left(\frac{1}{3}\varepsilon^{ijk}A_i[A_j, A_k] + 2\bar{c}\partial_i[A^i, c]\right). \tag{1.10}$$

The correlation function (1.2) is then replaced by

$$\langle W_R^K(A)\rangle = \int DAD\phi DcD\bar{c}\; e^{iI} W_R^K(A), \tag{1.11}$$

where the Wilson loop operator $W_R^K(A)$ is given as in (1.3). The integral corresponding to I_g is a constraint in the space of connections such that integration should be performed on a submanifold $\mathcal{U}$ of the space of all connections $\mathcal{A}$. All additional fields such as the ghost fields are introduced in a gauge invariant way in $\mathcal{U}$. Thus the above vacuum expectation value can be written as

$$\langle W_R^K(A)\rangle = \int DAD\phi DcD\bar{c}\; e^{i(I_0 + I_g)} e^{iI'} W_R^K(A). \tag{1.12}$$

From now on the expression $\langle W_R^K(A)\rangle$ stands for this vacuum expectation value. Of course for the case of n-component links $K = K_1 \cup \cdots \cup K_n$ the required expression is

$$\langle W_{R_1}^{K_1}(A)\cdots W_{R_n}^{K_n}(A)\rangle = \int DAD\phi DcD\bar{c}\; e^{i(I_0 + I_g)} e^{iI'} W_{R_1}^{K_1}(A)\cdots W_{R_n}^{K_n}(A), \tag{1.13}$$

where the component K_i has a representation R_i of the gauge group.

The perturbative analysis is performed over the coupling constant $1/k$ of the interacting terms in I' and in the Wilson loop functional $W_R^K(A)$, while that corresponding to the free part of the action will remain the same. The perturbative expression for a Wilson loop in the fundamental representation $\mathbf{R}$ at order two in $1/k$ is

$$\begin{aligned}
W_R^K(A) = {} & \mathrm{Tr}_{\mathbf{R}}\Bigg[1 + \oint_K ds A_i^a(K(s))\dot{K}^i(s)t_a \\
& \quad + \iint_{s_1<s_2} ds_1 ds_2 A_{i_1}^{a_1}(K(s_1))A_{i_2}^{a_2}(K(s_2))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)t_{a_1}t_{a_2} + \cdots\Bigg] \\
= {} & \mathrm{Tr}\Bigg[I + \oint_K ds A_i^a(K(s))\dot{K}^i(s)\mathbf{R}(t_a) \\
& \quad + \iint_{s_1<s_2} ds_1 ds_2 A_{i_1}^{a_1}(K(s_1))A_{i_2}^{a_2}(K(s_2))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\mathbf{R}(t_{a_1})\mathbf{R}(t_{a_2}) + \cdots\Bigg] \\
= {} & \dim(\mathbf{R}) \\
& + \oint_K ds A_i^a(K(s))\dot{K}^i(s)[\mathbf{R}(t_a)]_{\alpha\alpha} \\
& + \iint_{s_1<s_2} ds_1 ds_2 A_{i_1}^{a_1}(K(s_1))A_{i_2}^{a_2}(K(s_2))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)[\mathbf{R}(t_{a_1})]_{\alpha_1\alpha_2}[\mathbf{R}(t_{a_2})]_{\alpha_2\alpha_1} \\
& + \cdots.
\end{aligned} \tag{1.14}$$

The interaction part at the same order is given by

$$
\begin{aligned}
e^{iI'} = 1 &+ \frac{1}{1!}\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3}\mathrm{Tr}\left(\frac{1}{3}\varepsilon^{ijk}A_i[A_j,A_k]+2\bar{c}\partial_i[A^i,c]\right)\right] \\
&+ \frac{1}{2!}\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3}\mathrm{Tr}\left(\frac{1}{3}\varepsilon^{ijk}A_i[A_j,A_k]+2\bar{c}\partial_i[A^i,c]\right)\right]^2+\cdots.
\end{aligned}
\tag{1.15}
$$

The diagrams for a knot to be analysed in this work can be built from Chern-Simons perturbation theory by considering the pertubative expansion of its Wilson loop (1.14) at order four (the information relative to the group is not written explicitly), *i.e.*,

$$
\begin{aligned}
W_R^K(A) \approx 1 &+ \left(\frac{1}{\sqrt{k}}\right)\oint_{\mathbf{S}^1} ds A_i(K(s))\dot{K}^i(s) \\
&+ \left(\frac{1}{\sqrt{k}}\right)^2 \iint_{s_1<s_2} ds_1 ds_2 A_{i_1}(K(s_1))A_{i_2}(K(s_2))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2) \\
&+ \left(\frac{1}{\sqrt{k}}\right)^3 \iiint_{s_1<s_2<s_3} ds_1 ds_2 ds_3 \Big[A_{i_1}(K(s_1))A_{i_2}(K(s_2))A_{i_3}(K(s_3)) \\
&\qquad\qquad \times\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\Big] \\
&+ \left(\frac{1}{\sqrt{k}}\right)^4 \iiiint_{s_1<s_2<s_3<s_4} ds_1 ds_2 ds_3 ds_4 \Big[A_{i_1}(K(s_1))A_{i_2}(K(s_2))A_{i_3}(K(s_3))A_{i_4}(K(s_4)) \\
&\qquad\qquad \times\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\dot{K}^{i_4}(s_4)\Big],
\end{aligned}
\tag{1.16}
$$

where $K:\mathbf{S}^1\to\mathbf{S}^3$ is the knot embedding, and the interaction term (1.15) at first order, *i.e.*,

$$
\begin{aligned}
e^{iI'} &\approx 1+\left(\frac{1}{\sqrt{k}}\right)^3\frac{ik}{4\pi}\int_{\mathbb{R}^3}\mathrm{Tr}\left(\frac{1}{3}\varepsilon^{ijk}A_i[A_j,A_k]+2\bar{c}\partial_i[A^i,c]\right) \\
&= 1+\left(\frac{1}{\sqrt{k}}\right)\frac{i}{4\pi}\int_{\mathbb{R}^3}\mathrm{Tr}\left(\frac{1}{3}\varepsilon^{ijk}A_i[A_j,A_k]+2\bar{c}\partial_i[A^i,c]\right).
\end{aligned}
\tag{1.17}
$$

In the procedure to get the previous equations we have imposed the following redefinitions

$$
A\longmapsto\frac{A}{\sqrt{k}},\qquad c\longmapsto\frac{c}{\sqrt{k}},\qquad \bar{c}\longmapsto\frac{\bar{c}}{\sqrt{k}},
\tag{1.18}
$$

for the gauge, ghost and anti-ghost fields, respectively. Thus the vacuum expectation value (1.12) gives

$$
\begin{aligned}
&\int DAD\phi DcD\bar{c}e^{i(I_0+I_g)}e^{iI'}W_R^K(A) \\
&\approx\int DAD\phi DcD\bar{c}e^{i(I_0+I_g)}\left[1+\left(\frac{1}{\sqrt{k}}\right)\frac{i}{4\pi}\int_{\mathbb{R}^3}\mathrm{Tr}\left(\frac{1}{3}\varepsilon^{ijk}A_i[A_j,A_k]+2\bar{c}\partial_i[A^i,c]\right)\right]
\end{aligned}
$$

$$\times \Bigg[1 + \left(\frac{1}{\sqrt{k}}\right) \oint_{\mathbf{S}^1} ds A_i(K(s)) \dot{K}^i(s)$$
$$+ \left(\frac{1}{\sqrt{k}}\right)^2 \iint\limits_{s_1<s_2} ds_1 ds_2 A_{i_1}(K(s_1)) A_{i_2}(K(s_2)) \dot{K}^{i_1}(s_1) \dot{K}^{i_2}(s_2)$$
$$+ \left(\frac{1}{\sqrt{k}}\right)^3 \iiint\limits_{s_1<s_2<s_3} ds_1 ds_2 ds_3 A_{i_1}(K(s_1)) A_{i_2}(K(s_2)) A_{i_3}(K(s_3)) \dot{K}^{i_1}(s_1) \dot{K}^{i_2}(s_2) \dot{K}^{i_3}(s_3)$$
$$+ \left(\frac{1}{\sqrt{k}}\right)^4 \iiiint\limits_{s_1<s_2<s_3<s_4} ds_1 ds_2 ds_3 ds_4 \Big\{ A_{i_1}(K(s_1)) A_{i_2}(K(s_2)) A_{i_3}(K(s_3)) A_{i_4}(K(s_4))$$
$$\times \dot{K}^{i_1}(s_1) \dot{K}^{i_2}(s_2) \dot{K}^{i_3}(s_3) \dot{K}^{i_4}(s_4) \Big\} \Bigg]. \tag{1.19}$$

The interest of this work is focused on some normalized terms coming from Eq. (1.19), specifically, that of order $1/k$ coming from the third term of $W_R^K(A)$ and the first term of $e^{iI'}$, *i.e.*,

$$V_1 = \frac{1}{N} \int DAD\phi DcD\bar{c} e^{i(I_0 + I_g)} \iint\limits_{s_1<s_2} ds_1 ds_2 A_{i_1}(K(s_1)) A_{i_2}(K(s_2)) \dot{K}^{i_1}(s_1) \dot{K}^{i_2}(s_2), \tag{1.20}$$

where $N = \int DAD\phi DcD\bar{c} e^{i(I_0 + I_g)}$. The previous equation gives rise to the self-linking invariant or the Vassiliev invariant of first order. Also important are those of order $1/k^2$ coming from the fourth term of $W_R^K(A)$ and the part without ghosts of the second term of $e^{iI'}$, *i.e.*,

$$V_{21} = \frac{1}{N} \left(\frac{i}{4\pi}\right) \int DAD\phi DcD\bar{c} e^{i(I_0 + I_g)} \int_{\mathbb{R}^3} \mathrm{Tr}\left(\frac{1}{3} \varepsilon^{ijk} A_i [A_j, A_k]\right) \iiint\limits_{s_1<s_2<s_3} ds_1 ds_2 ds_3$$
$$\times \Big\{ A_{i_1}(K(s_1)) A_{i_2}(K(s_2)) A_{i_3}(K(s_3)) \dot{K}^{i_1}(s_1) \dot{K}^{i_2}(s_2) \dot{K}^{i_3}(s_3) \Big\}, \tag{1.21}$$

and from the fifth term of $W_R^K(A)$ and the first term of $e^{iI'}$, *i.e.*,

$$V_{22} = \frac{1}{N} \int DAD\phi DcD\bar{c} e^{i(I_0 + I_g)} \iiiint\limits_{s_1<s_2<s_3<s_4} ds_1 ds_2 ds_3 ds_4 \Big\{ A_{i_1}(K(s_1)) A_{i_2}(K(s_2))$$
$$\times A_{i_3}(K(s_3)) A_{i_4}(K(s_4)) \dot{K}^{i_1}(s_1) \dot{K}^{i_2}(s_2) \dot{K}^{i_3}(s_3) \dot{K}^{i_4}(s_4) \Big\}, \tag{1.22}$$

that constitute the Vassiliev invariant of second order. There are other three terms of order $1/k^2$ but according to Ref. [13] they do not contribute to the invariant.

The three integrals above have ϕ and ghosts dependence only in the $e^{i(I_0+I_g)}$ factor (see 1.8). Thus the factor obtained by performing integration of ϕ, c and $\bar{c}$ fields will cancel that from the normalization factor N yielding

$$V_1 = \frac{1}{N_A}\int DAe^{iI_A}\iint_{s_1<s_2} ds_1 ds_2 A_{i_1}(K(s_1))A_{i_2}(K(s_2))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2), \tag{1.23}$$

$$\begin{aligned}V_{21}=&\frac{1}{N_A}\int DAe^{iI_A}\int_{\mathbb{R}^3}\mathrm{Tr}\left(\frac{i}{12\pi}\varepsilon^{ijk}A_i[A_j,A_k]\right)\iiint_{s_1<s_2<s_3} ds_1 ds_2 ds_3\Big\{A_{i_1}(K(s_1))\\ &\times A_{i_2}(K(s_2))A_{i_3}(K(s_3))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\Big\},\end{aligned} \tag{1.24}$$

$$\begin{aligned}V_{22}=&\frac{1}{N_A}\int DAe^{iI_A}\iiiint_{s_1<s_2<s_3<s_4} ds_1 ds_2 ds_3 ds_4\Big\{A_{i_1}(K(s_1))A_{i_2}(K(s_2))A_{i_3}(K(s_3))\\ &\times A_{i_4}(K(s_4))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\dot{K}^{i_4}(s_4)\Big\},\end{aligned} \tag{1.25}$$

where

$$I_A = \frac{k}{4\pi}\int_{\mathbb{R}^3}\mathrm{Tr}\left(\varepsilon^{ijk}A_i\partial_j A_k\right), \tag{1.26}$$

$$N_A = \int DAe^{iI_A}. \tag{1.27}$$

Following a similar procedure to get the first and second order expressions it can be shown that the third order expressions (those associated with $1/k^3$) can be written as

$$\begin{aligned}V_{31} = &\frac{1}{N_A}\int DAe^{I_A}\left[\int_{\mathbb{R}^3}\mathrm{Tr}\left(\frac{i}{12\pi}\varepsilon^{ijk}A_i[A_j,A_k]\right)\right]^2\iiiint_{s_1<s_2<s_3<s_4} ds_1 ds_2 ds_3 ds_4\\ &\times\Big\{A_{i_1}(K(s_1))A_{i_2}(K(s_2))A_{i_3}(K(s_3))A_{i_4}(K(s_4))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\dot{K}^{i_4}(s_4)\Big\},\end{aligned} \tag{1.28}$$

$$\begin{aligned}V_{32} = &\frac{1}{N_A}\int DAe^{iI_A}\int_{\mathbb{R}^3}\mathrm{Tr}\left(\frac{i}{12\pi}\varepsilon^{ijk}A_i[A_j,A_k]\right)\int\iiiint_{s_1<s_2<s_3<s_4<s_5} ds_1 ds_2 ds_3 ds_4 ds_5\\ &\times\Big\{A_{i_1}(K(s_1))A_{i_2}(K(s_2))A_{i_3}(K(s_3))A_{i_4}(K(s_4))A_{i_5}(K(s_5))\\ &\times\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\dot{K}^{i_4}(s_4)\dot{K}^{i_5}(s_5)\Big\},\end{aligned} \tag{1.29}$$

$$V_{33} = \frac{1}{N_A}\int DAe^{iI_A}\int\iiiiint_{s_1<s_2<s_3<s_4<s_5<s_6} ds_1 ds_2 ds_3 ds_4 ds_5 ds_6$$

$$\times\Big\{A_{i_1}(K(s_1))A_{i_2}(K(s_2))A_{i_3}(K(s_3))A_{i_4}(K(s_4))A_{i_5}(K(s_5))A_{i_6}(K(s_6))$$
$$\times\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\dot{K}^{i_4}(s_4)\dot{K}^{i_5}(s_5)\dot{K}^{i_6}(s_6)\Big\}, \tag{1.30}$$

$$V_{34} = \frac{1}{N_A}\int DAe^{iI_A}\int\!\!\!\!\!\int\!\!\!\!\!\int\!\!\!\!\!\int\!\!\!\!\!\int\!\!\!\!\!\int_{s_1<s_2<s_3<s_4<s_5<s_6} ds_1ds_2ds_3ds_4ds_5ds_6$$
$$\times\Big\{A_{i_1}(K(s_1))A_{i_2}(K(s_2))A_{i_3}(K(s_3))A_{i_4}(K(s_4))A_{i_5}(K(s_5))A_{i_6}(K(s_6))$$
$$\times\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\dot{K}^{i_4}(s_4)\dot{K}^{i_5}(s_5)\dot{K}^{i_6}(s_6)\Big\}; \tag{1.31}$$

the Vassiliev invariant of third order is just constituted by these four expressions. Similar identifications can be done to obtain Vassiliev invariants for higher orders. In Ref. [17] the perturbative analysis is used to give integral expressions for Vassiliev invariants for all prime knots up to six crossings up to order six. In Ref. [18] the same ideas are applied to all two-component links up to six crossings up to order four. In both works it is used a semi-simple gauge group G because a simple one is not enough to capture all invariants.

1.1.1 Feynman diagrams for knots

The vacuum expectation value (1.12) can be written as the following perturbative series expansion [17]

$$\left\langle W_R^K(A)\right\rangle = d(R)\sum_{i=0}^{\infty}\sum_{j=1}^{d_i}\alpha_{ij}(K)r_{ij}(G,R)x^i, \tag{1.32}$$

where $x = 2\pi i/k$, $d(R)$ is the dimension of the representation R, $d_0 = \alpha_{01} = r_{01} = 1$ and $d_1 = 0$. There are many features of (1.32) to discuss:

- The factor $\alpha_{ij}(K)$ is called the geometrical factor and it depends only on the knot K while being independent of the group and the representation chosen. The factor $r_{ij}(G,R)$ is called the group factor and it depends only on the group and representation but it is independent of the geometry of the knot.
- The index i is the order of the perturbation while the index j accounts for the contributions of the group factors at that order. Actually, there are d_i independent group factors at order i and this quantity is called the dimension of the space of invariants at that order.
- Vacuum diagrams are not included.
- Diagrams with collapsible propagators are not considered in this expression because they all contribute to the framing and this is not an intrinsic property of the knot. For

example, there is no linear term in (1.32) (condition $d_1 = 0$) because at this order the only contribution is a diagram with one collapsible propagator.

- Diagrams that include loops in the two-point or three-point subdiagrams are also excluded because they only contribute to the shift $k \longrightarrow k + h^\vee$, a quantum correction in the non-perturbative analysis.

- It is important to describe how the independent factors arise. First, all Feynman diagrams at a given order must be written. Second, ignore the diagrams that contain the structure described in the latter two points above. Third, write the group factors corresponding to the remaining diagrams (this is done by finding the Casimirs of the gauge group). Fourth, use commutator relations and Jacobi identity to relate them and to find the independent true group factors.

The perturbative expansion (1.32) can be normalized by dividing by the vacuum expectation value of the unknot K_0 at the same representation to give

$$\frac{\langle W_R^K(A)\rangle}{\langle W_R^{K_0}(A)\rangle} = \sum_{i=0}^{\infty}\sum_{j=1}^{d_i} \widetilde{\alpha}_{ij}(K) r_{ij}(G,R) x^i, \tag{1.33}$$

where $d(R)$ does not longer appear. As $r_{ij}(G,R)$ does not depend on the knot but only on the group and its representation it can be calculated by means of group theory. The factors $\widetilde{\alpha}_{ij}(K)$ are the Vassiliev invariants; once all the Feynman diagrams at certain order are given the integral expression of these factors can be built.

From Ref. [17], expression (1.33) can be written as

$$\begin{aligned}\frac{\langle W_R^K(A)\rangle}{\langle W_R^{K_0}(A)\rangle} = 1 + &\ \widetilde{\alpha}_{21} r_{21} x^2 + \widetilde{\alpha}_{31} r_{31} x^3 \\ &+ \big[\widetilde{\alpha}_{41}(r_{21})^2 + \widetilde{\alpha}_{42} r_{42} + \widetilde{\alpha}_{43} r_{43}\big] x^4 \\ &+ \big[\widetilde{\alpha}_{51} r_{21} r_{31} + \widetilde{\alpha}_{52} r_{52} + \widetilde{\alpha}_{53} r_{53} + \widetilde{\alpha}_{54} r_{54}\big] x^5 \\ &+ \big[\widetilde{\alpha}_{61}(r_{21})^3 + \widetilde{\alpha}_{62}(r_{31})^2 + \widetilde{\alpha}_{63} r_{21} r_{42} + \widetilde{\alpha}_{64} r_{21} r_{43} + \widetilde{\alpha}_{65} r_{65} + \widetilde{\alpha}_{66} r_{66} \\ &+ \widetilde{\alpha}_{67} r_{67} + \widetilde{\alpha}_{68} r_{68} + \widetilde{\alpha}_{69} r_{69}\big] x^6 + O(x^7).\end{aligned} \tag{1.34}$$

Our work will be focused in orders up to three of (1.34), that is, an analysis of geometrical factors $\widetilde{\alpha}_{21}(K)$ and $\widetilde{\alpha}_{31}(K)$ will be done. As stated at the beginning of this section diagrams with collapsible propagators are not considered in (1.32) and that is why there is not a linear term there. However, in our work this associated linear term $\widetilde{\alpha}_{11}(K)$ will be also analysed.

The integral expressions for many geometrical factors of (1.34) are given in [17], for instance for $\widetilde{\alpha}_{21}(K)$ one has

$$\begin{aligned}\widetilde{\alpha}_{21}(K) &= \frac{\alpha_{21}(K)}{\langle W_R^{K_0}(A)\rangle} \\ &= \frac{1}{\langle W_R^{K_0}(A)\rangle}\frac{1}{4\pi^2} \oint_K dx_\mu \int^x dy_\nu \int^y dz_\rho \int^z dw_\tau \Big[\varepsilon^{\mu\sigma_1\rho}\varepsilon^{\nu\sigma_2\tau}\frac{(x-z)_{\sigma_1}}{|x-z|^3}\frac{(y-w)_{\sigma_2}}{|y-w|^3}\Big]\end{aligned}$$

$$-\frac{1}{\langle W_R^{K_0}(A)\rangle}\frac{1}{16\pi^3}\oint_K dx_\mu\int^x dy_\nu\int^y dz_\rho\int_{\mathbb{R}^3}d^3w\Big[\varepsilon^{\mu\rho_1\sigma_1}\varepsilon^{\nu\rho_2\sigma_2}\varepsilon^{\rho\rho_3\sigma_3}\varepsilon_{\sigma_1\sigma_2\sigma_3}\times\frac{(x-w)_{\rho_1}}{|x-w|^3}\frac{(y-w)_{\rho_2}}{|y-w|^3}\frac{(z-w)_{\rho_3}}{|z-w|^3}\Big], \tag{1.35}$$

while for $\widetilde{\alpha}_{31}(K)$ the expression is

$$\begin{aligned}\widetilde{\alpha}_{31}(K) =\ & \frac{1}{\langle W_R^{K_0}(A)\rangle}\frac{1}{64\pi^5}\oint_K dx_\mu\int^x dy_\nu\int^y dt_\rho\int^t dz_\tau\int_{\mathbb{R}^3}d^3w_1\int_{\mathbb{R}^3}d^3w_2\Big[\varepsilon_{\alpha\beta\gamma}\varepsilon_{\eta\xi\zeta}\\ &\times\varepsilon^{\mu\sigma_1\alpha}\varepsilon^{\nu\sigma_2\beta}\varepsilon^{\gamma\sigma_3\zeta}\varepsilon^{\rho\sigma_4\eta}\varepsilon^{\tau\sigma_5\xi}\frac{(x-w_1)_{\sigma_1}}{|x-w_1|^3}\frac{(y-w_1)_{\sigma_2}}{|y-w_1|^3}\frac{(w_1-w_2)_{\sigma_3}}{|w_1-w_2|^3}\frac{(t-w_2)_{\sigma_4}}{|t-w_2|^3}\frac{(z-w_2)_{\sigma_5}}{|z-w_2|^3}\Big]\\ &+\frac{1}{\langle W_R^{K_0}(A)\rangle}\frac{5}{32\pi^4}\oint_K dx_\mu\int^x dy_\nu\int^y dt_\rho\int^t dz_\tau\int^z dv_\eta\int_{\mathbb{R}^3}d^3w\\ &\times\Big[\varepsilon^{\nu\sigma\eta}\varepsilon_{\alpha\beta\gamma}\varepsilon^{\mu\sigma_1\alpha}\varepsilon^{\rho\sigma_2\beta}\varepsilon^{\tau\sigma_3\gamma}\frac{(y-v)_\sigma}{|y-v|^3}\frac{(x-w)_{\sigma_1}}{|x-w|^3}\frac{(t-w)_{\sigma_2}}{|t-w|^3}\frac{(z-w)_{\sigma_3}}{|z-w|^3}\Big]\\ &+\frac{1}{\langle W_R^{K_0}(A)\rangle}\frac{3}{8\pi^3}\oint_K dx_\mu\int^x dy_\nu\int^y dt_\rho\int^t dz_\tau\int^z dv_\eta\int^v dw_\zeta\\ &\times\Big[\varepsilon^{\mu\sigma_1\tau}\varepsilon^{\nu\sigma_2\zeta}\varepsilon^{\rho\sigma_3\eta}\frac{(x-z)_{\sigma_1}}{|x-z|^3}\frac{(y-w)_{\sigma_2}}{|y-w|^3}\frac{(t-v)_{\sigma_3}}{|t-v|^3}\Big]\\ &+\frac{1}{\langle W_R^{K_0}(A)\rangle}\frac{1}{4\pi^3}\oint_K dx_\mu\int^x dy_\nu\int^y dt_\rho\int^t dz_\tau\int^z dv_\eta\int^v dw_\zeta\\ &\times\Big[\varepsilon^{\mu\sigma_1\tau}\varepsilon^{\nu\sigma_2\eta}\varepsilon^{\rho\sigma_3\zeta}\frac{(x-z)_{\sigma_1}}{|x-z|^3}\frac{(y-v)_{\sigma_2}}{|y-v|^3}\frac{(t-w)_{\sigma_3}}{|t-w|^3}\Big],\end{aligned} \tag{1.36}$$

where the information of the connection is given through the propagator (this object will play a relevant role in the subsequent sections)

$$\left\langle A_i^a(x)A_j^b(y)\right\rangle = \varepsilon^{ijk}\delta_{ab}\left(\frac{i}{4\pi}\right)\frac{(x-y)_k}{|x-y|^3}. \tag{1.37}$$

Note that, after using propagator (1.67), expressions (1.24)-(1.25) are (up to multiplicative constants) precisely those for second order Vassiliev invariant $\widetilde{\alpha}_{21}(K)$ in (1.35) while expressions (1.28)-(1.31) correspond to third order Vassiliev invariant $\widetilde{\alpha}_{31}(K)$ in (1.36). Analogously, expression (1.23) is associated with first order Vassiliev invariant $\widetilde{\alpha}_{11}(K)$.

1.1.2 Feynman diagrams for links

Let $L = K_1 \cup K_2$ be a link with components K_1 and K_2 and representations R_1 and R_2 of the gauge group, respectively. The important vacuum expectation value will be, from (1.13),

$$\langle W_{R_1}^{K_1}(A) W_{R_2}^{K_2}(A) \rangle = \int DAD\phi DcD\bar{c}\; e^{i(I_0 + I_g)} e^{iI'} W_{R_1}^{K_1}(A) W_{R_2}^{K_2}(A). \tag{1.38}$$

The generalization of Eq. (1.32) from knots to links is not trivial because the group factors in the latter case have a more complicated structure. In [18] via a factorization theorem for Wilson lines it is found that such a generalization is given by

$$\langle W_{R_1}^{K_1}(A) W_{R_2}^{K_2}(A) \rangle = \langle W_{R_1}^{K_1}(A) \rangle \langle W_{R_2}^{K_2}(A) \rangle \langle \mathcal{Z}_{R_1,R_2}^{K_1,K_2}(A) \rangle, \tag{1.39}$$

where $\langle W_{R_1}^{K_1}(A) \rangle$ and $\langle W_{R_2}^{K_2}(A) \rangle$ are written as in Eq. (1.32) and

$$\langle \mathcal{Z}_{R_1,R_2}^{K_1,K_2}(A) \rangle = \sum_{i=0}^{\infty} \sum_{j=1}^{\delta_i} {\gamma_i}^j (K_1, K_2) s_{ij}(G, R_1, R_2) x^i \tag{1.40}$$

is the pure link contribution. The objects ${\gamma_i}^j(K_1, K_2)$ are called the Vassiliev link invariants and they depend only on the knots K_1 and K_2, the objects $s_{ij}(G, R_1, R_2)$ are the new group factors that depend on the gauge group and its representations R_1 and R_2. Here again the index i is the order of the perturbative expansion while j stands for the contributions of the group factors at that order, $x = 2\pi i/k$ and δ_i is the number of independent group factors at order i or the dimension of the space of invariants at that order. Similar considerations to those given in Eq. (1.32), concerning the type of diagrams appearing in the expansion as well as the independence of the group factors, also apply in this case. Expression (1.40) at order four can be written as

$$\langle \mathcal{Z}_{R_1,R_2}^{K_1,K_2}(A) \rangle = 1 + \left[\frac{({\gamma_1}^1)^2}{2!} s_{21} \right] x^2 + \left[\frac{({\gamma_1}^1)^3}{3!} s_{31} + {\gamma_3}^2 s_{32} \right] x^3 + \left[\frac{({\gamma_1}^1)^4}{4!} s_{41} + \frac{{\gamma_1}^1 {\gamma_3}^2}{2} s_{42} + {\gamma_4}^3 s_{43} \right] x^4 + O(x^5). \tag{1.41}$$

At this order the primitive Vassiliev invariants are then ${\gamma_1}^1$, ${\gamma_3}^2$ and ${\gamma_4}^3$. In Ref. [18] the explicit integral expressions for these three γ's are given. For example for ${\gamma_1}^1$ one has

$${\gamma_1}^1 = \frac{1}{2} \oint dx \oint dy\; p(x, y), \tag{1.42}$$

where

$$p(x, y) = \Delta_{\mu\nu}(x - y) = \frac{1}{\pi} \varepsilon_{\mu\rho\nu} \frac{(x - y)^\rho}{|x - y|^3}. \tag{1.43}$$

Expression (1.42) is twice the linking number of the link while ${\gamma_3}^2$ and ${\gamma_4}^3$ are new invariants found in Ref. [18] that are not clear to be related with known numerical link invariants. Up to order three in (1.41) the invariants are ${\gamma_1}^1$ and ${\gamma_3}^2$. The Feynman diagram corresponding to ${\gamma_1}^1$ is that of figure 1.1 (note again that there is no linear term in (1.40) but this is exactly the diagram for that order) while figure 1.2 stands for the diagrams corresponding to ${\gamma_3}^2$.

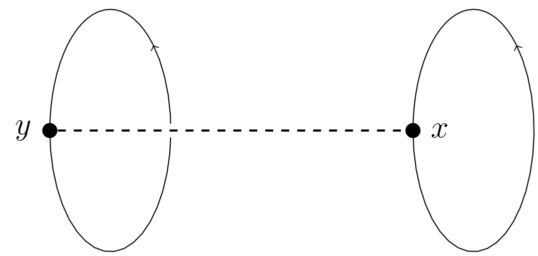


Figure 1.1: Feynman diagram for ${\gamma_1}^1$.

1.1.3 About numerical calculations

The invariants $\widetilde{\alpha}_{ij}(K)$ can be calculated by performing the integrals defining them. Alternatively, expression (1.34) can be used because its left hand side can be found in the literature as they are the quantum group invariants obtained from the non-perturbative analysis.

As an example, the values of $\widetilde{\alpha}_{21}(K)$ and $\widetilde{\alpha}_{31}(K)$ are explicitly calculated for the right-handed trefoil 3_1 with gauge group $G = SU(2)$. The normalized HOMFLY-PT polynomial is used to rewrite the left hand side of (1.34) as

$$\begin{aligned}
\lambda\left(1+q^2-\lambda q^2\right) &= q\left(1+q^2-q^3\right) \\
&= q+q^3-q^4 \\
&= \exp(x)+\exp(3x)-\exp(4x) \\
&= \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)+\left(1+3x+\frac{9x^2}{2}+\frac{27x^3}{6}\right) \\
&\quad -\left(1+4x+\frac{16x^2}{2}+\frac{64x^3}{6}\right)+O(x^4) \\
&= 1-3x^2-6x^3+O(x^4),
\end{aligned} \tag{1.44}$$

where $\lambda = q^{N-1} = q$ because $N = 2$. In the right hand side of (1.34) the values [17]

$$r_{21} = \sum_{k=1}^{n} C_3^{(k)} = -\frac{1}{4}\left(N^2-1\right) = -\frac{3}{4}, \tag{1.45}$$

$$r_{31} = \sum_{k=1}^{n} \left(C_3^{(k)}\right)^2 \left(C_2^{(k)}\right)^{-1} = \frac{\left(-\frac{1}{4}\left(N^2-1\right)\right)^2}{-\frac{1}{2N}\left(N^2-1\right)} = -\frac{2N}{16}\left(N^2-1\right) = -\frac{3}{4}, \tag{1.46}$$

for the group factors (the Casimirs are given in the fundamental representation) are used, *i.e.*, the right hand side looks like

$$1+\widetilde{\alpha}_{21}\left(-\frac{3}{4}\right)x^2+\widetilde{\alpha}_{31}\left(-\frac{3}{4}\right)x^3+O(x^4). \tag{1.47}$$

By equating both sides at order three it is found that

$$1-3x^2-6x^3 = 1+\widetilde{\alpha}_{21}\left(-\frac{3}{4}\right)x^2+\widetilde{\alpha}_{31}\left(-\frac{3}{4}\right)x^3, \tag{1.48}$$

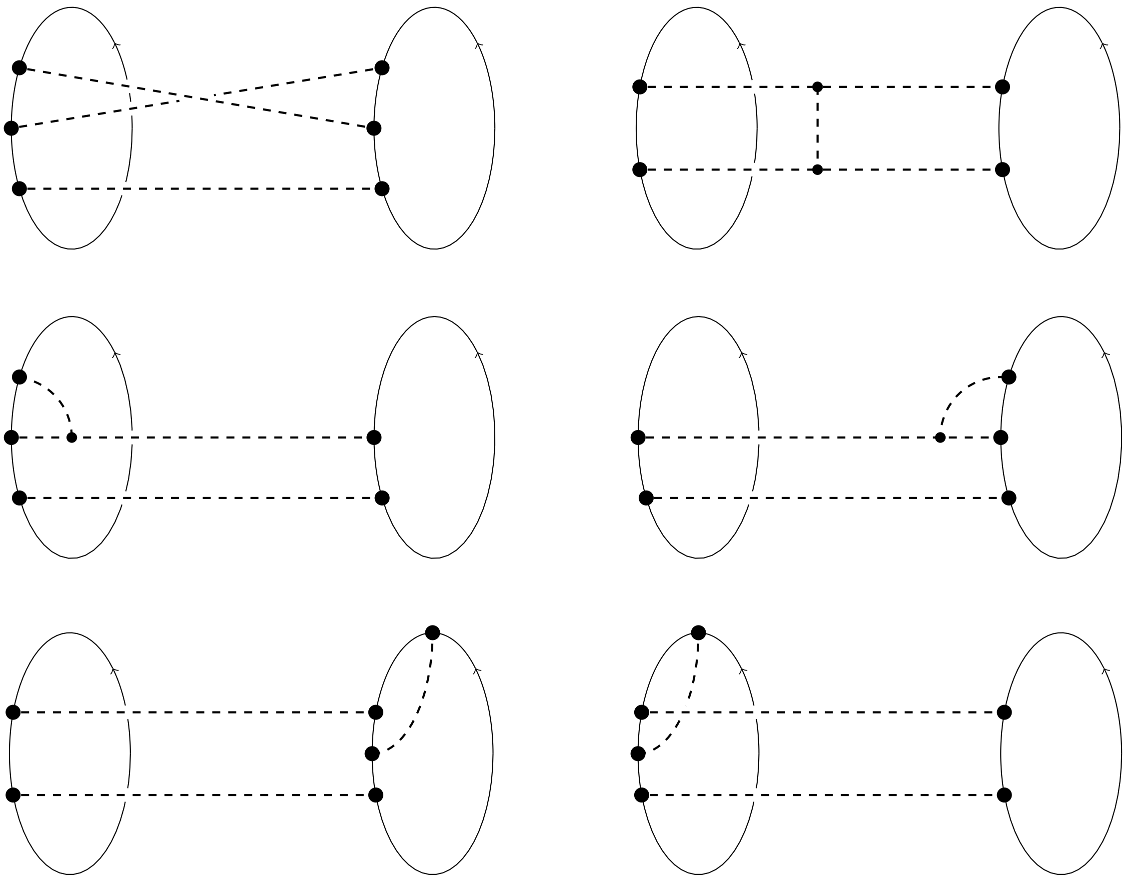


Figure 1.2: Feynman diagrams for ${\gamma_3}^2$.

or simply

$$\widetilde{\alpha}_{21} = 4, \tag{1.49}$$
$$\widetilde{\alpha}_{31} = 8, \tag{1.50}$$

in accordance with [17].

1.2 Bott-Taubes integration and volume-preserving vector fields

In this section we briefly overview general aspects of Bott-Taubes integrals and divergence-free vector fields. This is applied to construct average asymptotic Vassiliev invariants. In particular, along this section, we follow notations and conventions used in Refs. [72] and [41].

Some efforts to describe Vassiliev invariants (of finite type) in a geometrical framework were made by Kontsevich [36] and by Bott and Taubes [38], where the identification of the correct spaces in which Feynman integrals could be rewriting was one of the key achievements of such a description.

Since the 19$^{\text{th}}$ century Gauss work on electromagnetic theory showed that there is an integral formula for the linking number of two curves γ_0 and γ_1 in $\mathbb{R}^3$, which also represents a homotopy invariant of both curves under some assumptions of transversal intersection. It is given by

$$\frac{1}{4\pi}\int_{\mathbf{S}^1\times\mathbf{S}^1}\Phi^*\omega, \tag{1.51}$$

where $\Phi^*\omega$ is the pullback of $\omega\in\Omega^2\left(\mathbf{S}^2\right)$, which is the volume 2-form given in Ref. [72]. Moreover, given a pair of curves $\gamma_0,\gamma_1:\mathbf{S}^1\to\mathbb{R}^3$, the map $\Phi:\mathbf{S}^1\times\mathbf{S}^1\to\mathbf{S}^2$ is given by $\Phi=\phi\circ(\gamma_0\times\gamma_1)$, where $\phi:\mathbb{R}^3\times\mathbb{R}^3\to\mathbf{S}^2$ is the map defined via $\phi(x_1,x_2)=\dfrac{x_1-x_2}{|x_1-x_2|^3}$ with $x_1,x_2\in\mathbb{R}^3$. This map is known as the Gauss map and in components it is written as

$$\omega_{\mu\nu}=\frac{\varepsilon_{\mu\nu\sigma}}{4\pi}\frac{x^\sigma}{|x|^3}. \tag{1.52}$$

This procedure gives a new way to formulate the question of building up a homotopy invariant for a single knot in the same way as the linking number was developed. The answer to that question requires the introduction of *configuration spaces*, $C(n,M)$, defined by

$$C(n,M)=\left\{(x_1,\dots,x_n)\in\prod_{i=1}^{n}M\mid x_i\neq x_j\Leftrightarrow i\neq j\right\} \tag{1.53}$$

as the natural framework where such an integral has to be defined (because of the explicit form of ϕ). To ensure that an integral defined on this kind of spaces converges, some new features such as compactification of the configuration space $C(n,M)$ are required. Here we will use the well known *Fulton-MacPherson compactification* for $C(n,M)$ with some refinements worked out in [73].

In Ref. [41] there is a description of how to obtain a finite type knot invariant via a linear combination of integrals on some bundles (pullback bundles, see Appendix A.1) of *compactified configuration spaces* $C[n,M]$. The building blocks for writing these linear combinations depend on a trivalent diagram D and in the mathematical diagram of figure 1.3. In this diagram $P(D)$ denotes the set of dashed lines in each Feynman diagram, ϕ is a product of the Gauss maps each one associated to a line in $P(D)$ and Φ is an extension of the function ϕ to the corresponding compactified space [74].

In figure 1.3, $\mathcal{K}=\{\beta:\mathbf{S}^1\to\mathbf{S}^3\mid K \text{ is a smooth embedding}\}$ is the space of all smooth knots in $\mathbf{S}^3$ and K is one of those knots. The map ev is given by

$$ev((s_1,\dots,s_p),K)=(K(s_1),\dots,K(s_p))\in C(p,\mathbf{S}^3), \tag{1.54}$$

so that ev_K can be expressed as $ev_K:=ev(\cdot,K)$. Also π_p and pr are projections defined via

$$\pi_p(x_1,\dots,x_p,x_{p+1},\dots,x_{p+q})=(x_1,\dots,x_p),\qquad pr(v,K)=K, \tag{1.55}$$

respectively. As stated in Ref. [41] the maps α_m^n are the inclusion of configuration spaces into their compactifications while m and n denote, respectively, the number of points in $C(m,\mathbf{S}^n)$ and the dimension of the underlaying sphere. Finally, $\overline{ev}_K$ and $\overline{\pi}_p$ are extensions of the maps ev_K and π_p to the corresponding compactifications of their domains and codomains.

The building blocks are actually functions $I_D:\mathcal{K}\to\mathbb{R}$ on the space of knots $\mathcal{K}$ defined via

$$I_D(K)=(pr\circ pr_1)_*\left(\Phi\circ pr_2\right)^*\overline{\omega},\quad K\in\mathcal{K}, \tag{1.56}$$

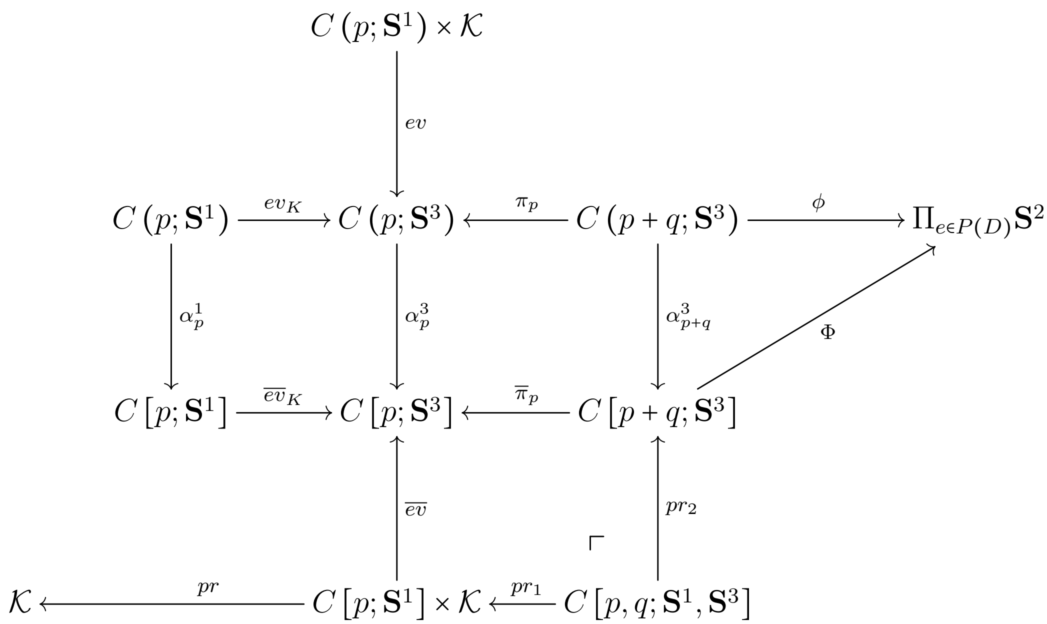


Figure 1.3: Mathematical diagram for a trivalent diagram D with p points on the knot and q points out of it.

where the mapping $pr : C[p;\mathbf{S}^1]\times\mathcal{K}\to\mathcal{K}$ is the projection in the second entry and

$$\overline{\omega} := \bigwedge_{e\in P(D)} \omega \in \Omega^{2P(D)}\left(\prod_{e\in P(D)} \mathbf{S}^2\right) \tag{1.57}$$

is the product of the unit volume form ω given as in Eq. (1.52), $(pr\circ pr_1)_*$ stands for the pushforward (or integration over the fibre, see Appendix A.2 for the definition) of the form $(\Phi\circ pr_2)^*\,\overline{\omega}$, and the fibres of $(pr\circ pr_1)$ are compact smooth manifolds with corners [39].

The next proposition, proved in Ref.[41], asserts that for each diagram D the value of these blocks in a specific knot K can also be calculated by integration of the function

$$\begin{aligned} f_{D,K}(\overline{s}) &= \left(\left(\alpha_p^3\circ ev_K\right)^*(\overline{\pi}_p)_*\,\Phi^*\overline{\omega}\right)_{\overline{s}}(\partial_{\overline{s}}) \\ &= \left(\left(\alpha_p^3\right)^*(\overline{\pi}_p)_*\,\Phi^*\overline{\omega}\right)_{K(\overline{s})}(\dot{K}(s_1),\dots,\dot{K}(s_p)), \end{aligned} \tag{1.58}$$

on the original configuration space $C\left(p,\mathbf{S}^1\right)$ of only the points that belong to K. In the above expression $\overline{s}\in C\left(p,\mathbf{S}^1\right)$ and $\partial_{\overline{s}}$ is a p-tuple where each element is the canonical vector field ∂_{s_i} on $\mathbf{S}^1$, also $\left(\dot{K}(s_1),\dots,\dot{K}(s_p)\right)$ is given by the pushforward of $\partial_{\overline{s}}$ by ev_K. This way $\dot{K}$ defines a vector field along the curve K in $\mathbb{R}^3$ and also, for each point $\overline{s}\in C(p,\mathbf{S}^1)$, it determines a frame $\left(\dot{K}(s_1),\dots,\dot{K}(s_p)\right)$ in $C(p,\mathbf{S}^3)$.

Proposition 1 ***(Proposition 3.7 in Ref. [41])*** *With $f_{D,K}$ as defined in (1.58), we have the following identity for $I_D(K)$:*

$$I_D(K) = \int_{C(p,\mathbf{S}^1)} f_{D,K}(\overline{s})d\overline{s}. \tag{1.59}$$

These basic blocks can also be used to define invariants associated with a volume-preserving vector field X on a compact domain $\mathcal{S}$ of $\mathbb{R}^3$ and tangent to its boundary by making an appropriate generalization of Eq. (1.58).

Explicitly, let X be a volume-preserving vector field on a domain $\mathcal{S}$ in $\left(\mathbb{R}^3, \mu\right)$ with flow $\theta : \mathbb{R} \times \mathcal{S} \to \mathcal{S}$ and let $x \in \mathcal{S}$, then $\theta^x := \theta(\cdot, x)$ defines a curve on $\mathcal{S}$. Moreover, μ is a Borel probability measure invariant under the flow. Now for every $T \in \mathbb{R}$, by taking $\sigma(x, \theta^x(T))$ to be the set of uniformly bounded curves between x and $\theta^x(T)$ and $\gamma \in \sigma(x, \theta^x(T))$, then $\gamma_T^x : \mathbf{S}^1 \to \theta([0,T], x) \cup \gamma$ is a piecewise smooth closed curve on $\mathcal{S}$ that can be defined in the interval $[0, T+1]$, where $[T, T+1]$ is a parametrization of the points in γ. This construction is illustrated in figure 1.4 and corresponds to the asymptotic cycles of Schwartzman [20, 23].

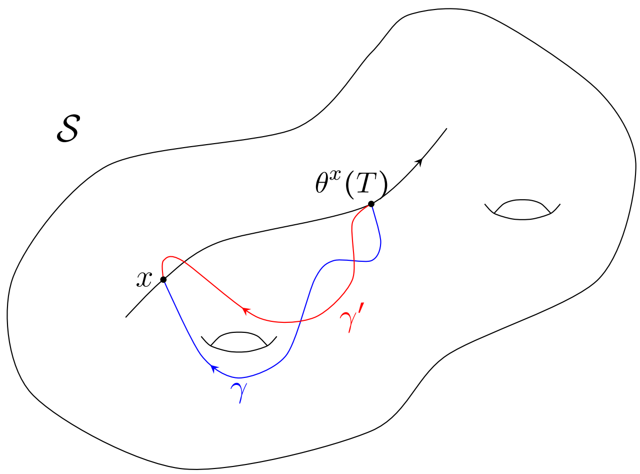


Figure 1.4: Construction of asymptotic cycles for γ and γ' defined via the flow θ in the domain $\mathcal{S}$.

In a similar way, $f_{D,K}$ given in Eq. (1.58) is generalized to $f_{D,X}$ who belongs to $\Omega^0\left(C(p, \mathcal{S})\right)$ and it is defined by

$$f_{D,X}(x_1, \ldots, x_p) := \left(\left(\alpha_p^3\right)^* (\overline{\pi}_p)_* \Phi^* \overline{\omega}\right)_{(x_1, \ldots, x_p)} (X_{x_1}, \ldots, X_{x_p}). \tag{1.60}$$

In order to obtain the average asymptotic invariants of a vector field X, asymptotic values of knot invariants along the flow of X (of order p) are defined as

$$\mathcal{F}^p(X) = \lim_{T \to \infty} \int_{x \in \mathcal{S}} \frac{1}{T^p} \mathcal{F}(\gamma_T^x), \tag{1.61}$$

where $\mathcal{F}$ is a real-valued function, typically a knot invariant, on the space of knots $\mathcal{K}$, and $\mathcal{F}(\gamma_T^x)$ is the restriction of such a function $\mathcal{F}$ to the space of curves $\{\gamma_T^x\}_{x \in \mathcal{S}}$ seen as real-valued functions in the domain $\mathcal{S}$ on $\mathbb{R}^3$.

This construction can be performed in the particular case of the function I_D described in Eq. (1.56) for which its time average at a point $x \in \mathcal{S}$ is given by

$$\overline{\lambda}_D(x) = \lim_{T \to \infty} \frac{1}{T^p} I_D\left(\gamma_T^x\right)$$

$$= \lim_{T\to\infty} \frac{1}{T^p} \int_0^{T+1} \cdots \int_0^{T+1} f_{D,X}(\gamma_T^x(t_1),\ldots,\gamma_T^x(t_p))dt_1 \wedge \cdots \wedge dt_p. \tag{1.62}$$

Here $k = k(D)$ is the amount of vertices over the circle in the diagram D and γ_T^x are the closed up orbits of X. The last identity is satisfiled when Eq. (1.59) is applied to $I_D(\gamma_T^x)$.

Then the following pair of lemmas stated in Ref. [41] proof some facts about this time average $\overline{\lambda}_D(x)$.

Lemma 1 ***(Key Lemma in Ref. [41]).*** *Let μ be the underlying measure on the domain $\mathcal{S} \subset \mathbb{R}^3$, invariant under the flow of X. Consider the time average of $f_{D,X}$ over $\mathcal{F}^p(X)$ defined as*

$$\lambda_D(x) = \lim_{T\to\infty} \frac{1}{T^p} \int_0^{T} \cdots \int_0^{T} f_{D,X}(\theta^x(t_1),\ldots,\theta^x(t_p))dt_1 \wedge \cdots \wedge dt_p,\ x \in \mathcal{S}, \tag{1.63}$$

where in comparison to Eq. (1.62), we skipped the integrals over paths γ. Then this limit exists almost everywhere on $\mathcal{S}$ and λ_D is $L^1(\mathcal{S},\mu)$.

Lemma 2 ***(Lemma 4.5 in Ref. [41]).*** *We have*

$$\overline{\lambda}_D(x) = \lambda_D(x)\ ,\ \textit{almost everywhere.} \tag{1.64}$$

The previous function as stated in the Key Lemma of Ref. [41] belongs to $L^1(\mathcal{S},\mu)$, and when integrated via the invariant measure, it gives a new kind of flow-invariant quantity that can be rewriting as

$$\int_{\mathcal{S}} \lambda_D \mu = \int_{\mathcal{S}^p} f_{D,X} \overline{\mu}_\Delta, \tag{1.65}$$

where, by taking θ^p to be the p-fold product of the flow θ, we have that

$$\overline{\mu}_\Delta = \lim_{T\to\infty} \frac{1}{T^p} \int_0^{T} \cdots \int_0^{T} \left((\theta^p)_* \, \mu_\Delta\right) dt_1 \wedge \cdots \wedge dt_p \tag{1.66}$$

is a well defined limit measure.

Finally, theorems A and B from Ref. [41] assert that this generalization also works as the basic building blocks for average asymptotic invariants of the vector field X, *i.e.*, quantities that are invariant under the action of their own flow θ that are going to be calculated through asymptotic values of some functions defined on the space of knots $\mathcal{K}$.

Theorem 1.2.0.1 ***(Theorem A,(i) & (ii) in Ref. [41])*** *Let X be a volume-preserving nonvanishing vector field on a compact domain $\mathcal{S} \subset \mathbb{R}^3$, tangent to the boundary. We then have:*

(i) For any diagram D in the set of trivalent diagrams of degree n, the asymptotic value $\mathcal{I}_D^k(X)$, $k = k(D)$ of I_D along the flow of X exist.

(ii) For any invariant V_W of type n, the asymptotic value of invariant $\mathcal{V}_W$ of order $2n$ exists and equals the asymptotic value $\mathcal{V}_W^{2n}(X)$ of V_W^{2n} along the flow X.

1.3 Correspondence between Feynman diagrams and Bott-Taubes integrals

Before working out with knots in the configuration space formalism it would be useful to rewrite the expressions for Vassiliev invariants up to order 3 (see section 1.1) by using figures 1.5b, 1.6 and 1.9, and the propagator

$$\langle A_{i_1}(K(s_1))A_{i_2}(K(s_2))\rangle = \frac{1}{N_A}\int DAe^{iI_A}A_{i_1}(K(s_1))A_{i_2}(K(s_2)). \tag{1.67}$$

Specifically, expression (1.23) rewrites as

$$\begin{aligned} V_1 =&\frac{1}{N_A}\int DAe^{iI_A}\iint\limits_{s_1<s_2} ds_1 ds_2 A_{i_1}(K(s_1))A_{i_2}(K(s_2))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\\ =&\iint\limits_{s_1<s_2} ds_1ds_2\left[\frac{1}{N_A}\int DAe^{iI_A}A_{i_1}(K(s_1))A_{i_2}(K(s_2))\right]\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\\ =&\iint\limits_{s_1<s_2} ds_1ds_2\langle A_{i_1}(K(s_1))A_{i_2}(K(s_2))\rangle\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2). \end{aligned}\tag{1.68}$$

Hence Eq. (1.24) looks like

$$\begin{aligned} V_{21} =&\frac{1}{N_A}\int DAe^{iI_A}\int_{\mathbb{R}^3}\mathrm{Tr}\left(\frac{i}{12\pi}\varepsilon^{ijk}A_i[A_j,A_k]\right)\iiint\limits_{s_1<s_2<s_3} ds_1ds_2ds_3\Big\{A_{i_1}(K(s_1))\\ &\times A_{i_2}(K(s_2))A_{i_3}(K(s_3))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\Big\}\\ =&\iiint\limits_{s_1<s_2<s_3} ds_1ds_2ds_3\int\limits_{\mathbb{R}^3}\varepsilon^{ijk}d^3x_4\Big\{\langle A_{i_1}(K(s_1))A_i(x_4)\rangle\langle A_{i_2}(K(s_2))A_j(x_4)\rangle\\ &\times\langle A_{i_3}(K(s_3))A_k(x_4)\rangle\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\Big\}. \end{aligned}\tag{1.69}$$

Expression (1.25) can be seen to be

$$\begin{aligned} V_{22} =&\frac{1}{N_A}\int DAe^{iI_A}\iiiint\limits_{s_1<s_2<s_3<s_4} ds_1ds_2ds_3ds_4\Big\{A_{i_1}(K(s_1))A_{i_2}(K(s_2))A_{i_3}(K(s_3))\\ &\times A_{i_4}(K(s_4))\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\dot{K}^{i_4}(s_4)\Big\}\\ =&\iiiint\limits_{s_1<s_2<s_3<s_4} ds_1ds_2ds_3ds_4\Big\{\langle A_{i_1}(K(s_1))A_{i_3}(K(s_3))\rangle\langle A_{i_2}(K(s_2))A_{i_4}(K(s_4))\rangle\\ &\times\dot{K}^{i_1}(s_1)\dot{K}^{i_2}(s_2)\dot{K}^{i_3}(s_3)\dot{K}^{i_4}(s_4)\Big\}. \end{aligned}\tag{1.70}$$

Analogously, expressions (1.28) - (1.31) rewrite also as

$$V_{31} = \iiiint\limits_{s_1<s_2<s_3<s_4} ds_1 ds_2 ds_3 ds_4 \int\limits_{\mathbb{R}^3} \varepsilon^{ijk} d^3x_5 \int\limits_{\mathbb{R}^3} \varepsilon^{lmn} d^3x_6 \Big\{ \big\langle A_m(x_5) A_n(x_6) \big\rangle \\ \times \big\langle A_{i_1}(K(s_1)) A_i(x_6) \big\rangle \big\langle A_{i_2}(K(s_2)) A_j(x_5) \big\rangle \big\langle A_{i_3}(K(s_3)) A_k(x_5) \big\rangle \\ \times \big\langle A_{i_4}(K(s_4)) A_l(x_6) \big\rangle \dot{K}^{i_1}(s_1) \dot{K}^{i_2}(s_2) \dot{K}^{i_3}(s_3) \dot{K}^{i_4}(s_4) \Big\}, \tag{1.71}$$

$$V_{32} = \int \iiiint\limits_{s_1<s_2<s_3<s_4<s_5} ds_1 ds_2 ds_3 ds_4 ds_5 \int\limits_{\mathbb{R}^3} \varepsilon^{ijk} d^3x_6 \Big\{ \big\langle A_{i_2}(K(s_2)) A_{i_5}(K(s_5)) \big\rangle \\ \times \big\langle A_{i_1}(K(s_1)) A_i(x_6) \big\rangle \big\langle A_{i_3}(K(s_3)) A_j(x_6) \big\rangle \big\langle A_{i_4}(K(s_4)) A_k(x_6) \big\rangle \\ \times \dot{K}^{i_1}(s_1) \dot{K}^{i_2}(s_2) \dot{K}^{i_3}(s_3) \dot{K}^{i_4}(s_4) \dot{K}^{i_5}(s_5) \Big\}, \tag{1.72}$$

$$V_{33} = \int \int\!\!\iiiint\limits_{s_1<s_2<s_3<s_4<s_5<s_6} ds_1 ds_2 ds_3 ds_4 ds_5 ds_6 \Big\{ \big\langle A_{i_1}(K(s_1)) A_{i_4}(K(s_4)) \big\rangle \\ \times \big\langle A_{i_2}(K(s_2)) A_{i_6}(K(s_6)) \big\rangle \big\langle A_{i_3}(K(s_3)) A_{i_5}(K(s_5)) \big\rangle \\ \times \dot{K}^{i_1}(s_1) \dot{K}^{i_2}(s_2) \dot{K}^{i_3}(s_3) \dot{K}^{i_4}(s_4) \dot{K}^{i_5}(s_5) \dot{K}^{i_6}(s_6) \Big\}, \tag{1.73}$$

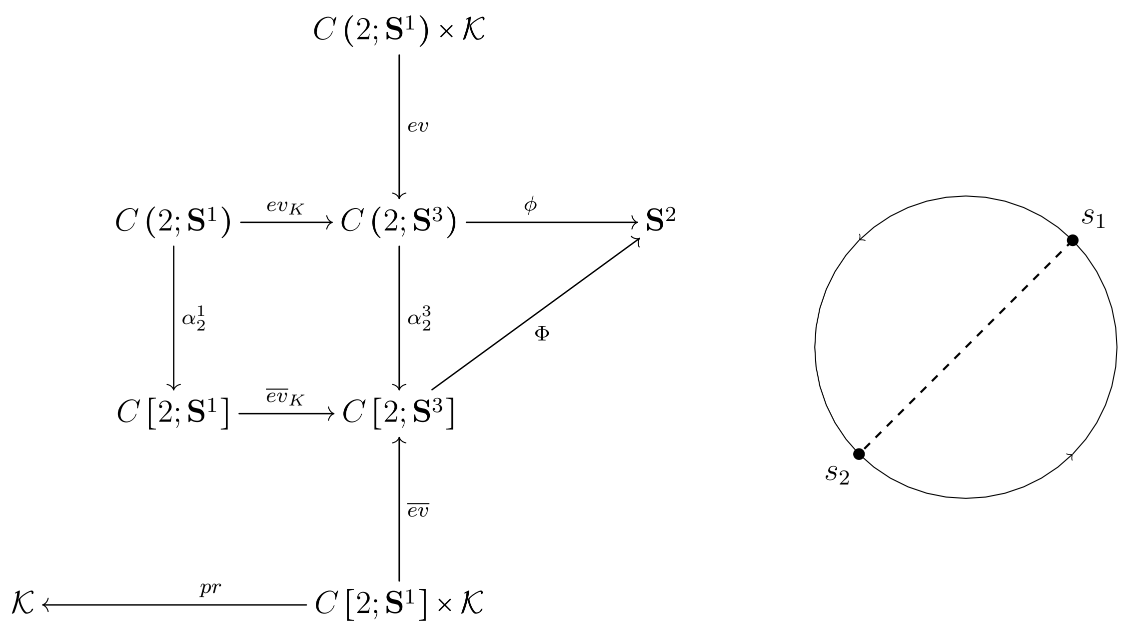
$$V_{34} = \int \int\!\!\iiiint\limits_{s_1<s_2<s_3<s_4<s_5<s_6} ds_1 ds_2 ds_3 ds_4 ds_5 ds_6 \Big\{ \big\langle A_{i_1}(K(s_1)) A_{i_4}(K(s_4)) \big\rangle \\ \times \big\langle A_{i_2}(K(s_2)) A_{i_5}(K(s_5)) \big\rangle \big\langle A_{i_3}(K(s_3)) A_{i_6}(K(s_6)) \big\rangle \\ \times \dot{K}^{i_1}(s_1) \dot{K}^{i_2}(s_2) \dot{K}^{i_3}(s_3) \dot{K}^{i_4}(s_4) \dot{K}^{i_5}(s_5) \dot{K}^{i_6}(s_6) \Big\}. \tag{1.74}$$

Expressions (1.68), (1.69), (1.70) and (1.71) - (1.74) will be the subject of the following subsections.

1.3.1 First order Vassiliev knot invariant: self-linking of a knot

The Feynman diagram D_1 corresponding to the term (1.68) is that of figure (1.5b) while the mathematical one is, according to the theory of section 1.2, that of figure (1.5a), where the column corresponding to internal points has been supressed (see figure 1.3). The integral corresponding to this diagram in the configuration space formalism is thus[3]

[3] From now on $\overline{I}_D$ denotes the Bott-Taubes integral for diagram D in the compactified configuration space while I_D stands for the integral without boundaries.



(a) Mathematical diagram for D_1.

(b) Feynman diagram D_1.

Figure 1.5: First order Vassiliev knot invariant: self-linking of a knot.

$$
\begin{aligned}
\overline{I}_{D_1} &= \int_{C[2,\mathbf{S}^1]} \left(\Phi\circ\overline{ev}_K\right)^*\omega = \int_{C(2,\mathbf{S}^1)} \left(\phi\circ ev_K\right)^*\omega + \int_{\partial C[2,\mathbf{S}^1]} \left(\Phi\circ\overline{ev}_K\right)^*\omega \\
&= \int_{C(2,\mathbf{S}^1)} \left(\frac{K(x_2)-K(x_1)}{|K(x_2)-K(x_1)|}\right)^*\omega + \int_{\partial C[2,\mathbf{S}^1]} \left(\pm\frac{\dot K(x_1)}{|\dot K(x_1)|}\right)^*\omega,
\end{aligned}
\tag{1.75}
$$

where (see Ref. [39])

$$
\Phi\circ\overline{ev}_K = \pm\frac{\dot K(x_1)}{|\dot K(x_1)|}
\tag{1.76}
$$

is the expression for $\Phi\circ\overline{ev}_K$ at the boundary of $C[2,\mathbf{S}^1]$. The sign in Eq. (1.76) depends on whether the collapse of points x_1 and x_2 is in one direction or the other.

To explicitly make contact of (1.75) with (1.68) coming from the Chern-Simons theory it is necessary to calculate the pullback of ω under ϕ, $\phi^*\omega$. This result (see Appendix A.3 for the derivation) is given by

$$
\phi^*_{a,b}\omega = \frac{\varepsilon_{\mu\nu\sigma}}{4\pi}\frac{(x_b-x_a)^\mu}{|x_b-x_a|^3}\left(\frac{1}{2}dx_a^\nu\wedge dx_a^\sigma - dx_a^\nu\wedge dx_b^\sigma + \frac{1}{2}dx_b^\nu\wedge dx_b^\sigma\right).
\tag{1.77}
$$

According to Ref. [40] the integral over the configuration space is non-vanishing only if there appears exactly one dx_a and one dx_b, *i.e.*, it is enough to consider

$$
\phi^*_{a,b}\omega = -\frac{\varepsilon_{\mu\nu\sigma}}{4\pi}\frac{(x_b-x_a)^\mu}{|x_b-x_a|^3}dx_a^\nu\wedge dx_b^\sigma.
\tag{1.78}
$$

Actually the important expression is that for $(\phi \circ ev_K)^* \omega = ev_K^* \phi^* \omega$, to be precise

$$ev_K^* \phi_{ab}^* \omega = -\frac{\varepsilon_{\mu\nu\sigma}}{4\pi} \frac{(K(x_b) - K(x_a))^\mu}{|K(x_b) - K(x_a)|^3} \frac{dK^\nu(x_a)}{dx_a} \frac{dK^\sigma(x_b)}{dx_b} dx_a \wedge dx_b. \tag{1.79}$$

By using this expression in Eq. (1.75) it is obtained

$$\begin{aligned} \overline{I}_{D_1} &= \int_{C(2,\mathbf{S}^1)} \left[-\frac{\varepsilon_{\mu\nu\sigma}}{4\pi} \frac{(K(x_2) - K(x_1))^\mu}{|K(x_2) - K(x_1)|^3} \frac{dK^\nu(x_1)}{dx_1} \frac{dK^\sigma(x_2)}{dx_2} \right] dx_1 \wedge dx_2 \\ &+ \int_{\partial C[2,\mathbf{S}^1]} \left(\pm \frac{\dot{K}(x_1)}{|\dot{K}(x_1)|} \right)^* \omega \\ &= \int_{C(2,\mathbf{S}^1)} \Delta_{\nu\sigma}(K(x_1) - K(x_2)) \dot{K}^\nu(x_1) \dot{K}^\sigma(x_2) dx_1 \wedge dx_2 \\ &+ \int_{\partial C[2,\mathbf{S}^1]} \left(\pm \frac{\dot{K}(x_1)}{|\dot{K}(x_1)|} \right)^* \omega, \end{aligned} \tag{1.80}$$

where the last equality used standard notation for the derivatives and the following expression for the propagator [40]

$$\Delta_{\mu\nu}(\overline{x}) = \frac{\varepsilon_{\mu\nu\sigma}}{4\pi} \frac{x^\sigma}{|\overline{x}|^3}. \tag{1.81}$$

The boundary term is exactly cancelled with a framing term in order to obtain an invariant of knots with framing, *i.e.*, the real topological invariant is

$$I_{D_1} = \int_{C(2,\mathbf{S}^1)} \Delta_{\nu\sigma}(K(x_1) - K(x_2)) \dot{K}^\nu(x_1) \dot{K}^\sigma(x_2) dx_1 \wedge dx_2. \tag{1.82}$$

The matching between expressions (1.68) coming from Chern-Simons theory and (1.82) coming from the configuration space construction, *i.e.*, $I_{D_1} = V_1$, establishes a deep correspondence between formalisms.

1.3.2 Second order Vassiliev knot invariant

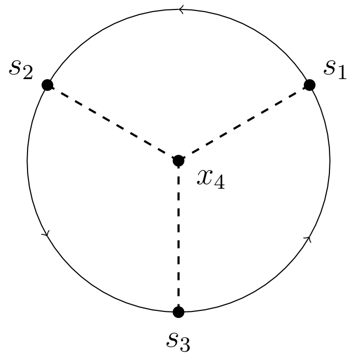
The second order Vassiliev invariant comes from the contributions of Feynman diagrams D_{21} and D_{22} (figures 1.6a and 1.6b, respectively) corresponding to terms (1.69) and (1.70), in that order.

The first step is to analyse diagram D_{21}. In this case the map ϕ in figure 1.7 is given by the restriction of

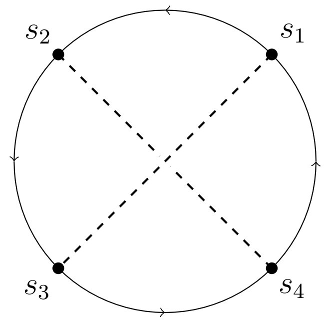
$$\phi_{1,4} \times \phi_{2,4} \times \phi_{3,4} : \prod_{i=1}^{4} \mathbf{S}^3 \longrightarrow \prod_{i=1}^{3} \mathbf{S}^2 \tag{1.83}$$

to $C(3+1, \mathbf{S}^3)$ where each of these $\phi_{a,b}$ corresponds to a Gauss map. By taking the pullback of $\overline{\omega} := \omega \times \omega \times \omega \in \Omega^6 \left(\mathbf{S}^2 \times \mathbf{S}^2 \times \mathbf{S}^2 \right)$ under the previous map it is obtained

$$\phi^* (\overline{\omega}) = (\phi_{1,4} \times \phi_{2,4} \times \phi_{3,4})^* (\overline{\omega})$$



(a) Feynman diagram D_{21}.



(b) Feynman diagram D_{22}.

Figure 1.6: Feynman diagrams for the second order Vassiliev invariant.

$$=\phi_{1,4}^{*}\omega \wedge \phi_{2,4}^{*}\omega \wedge \phi_{3,4}^{*}\omega, \tag{1.84}$$

where each of these pullbacks is given in the same way as in Eq. (1.78).

Let $\overline{s} \in C(3,\mathbf{S}^1)$ and $\overline{x} = \alpha_3^3 \circ ev_K(\overline{s})$, then $\overline{\pi}_3^{-1}(\{\overline{x}\})$ is the homotopy fibre of $\overline{\pi}_3$. By integrating $\Phi^*\omega$ along this homotopy fibre (or equivalently by taking the pushforward under $\overline{\pi}_3$) and by performing the pullback by $\alpha_3^3 \circ ev_K$ then (see also Eq. (1.58))

$$f_{D_{21},K}(\overline{s}) = \left(\left(\alpha_3^3 \circ ev_K\right)^* (\overline{\pi}_3)_* \, \Phi^*\overline{\omega}\right)_{\overline{s}} (\partial_{\overline{s}}) \tag{1.85}$$

is a 3-form in $C(3,\mathbf{S}^1)$. Thus

$$\begin{aligned}\overline{I}_{D_{21}} = \int\limits_{C(3,\mathbf{S}^1)} f_{D_{21},K}(\overline{s}) d\overline{s} &= \int\limits_{C(3,\mathbf{S}^1)} d\overline{s} \int\limits_{\overline{\pi}_3^{-1}(\{\overline{x}\})} \Phi^*\overline{\omega}\left(\left[\left(\alpha_3^3\right)_* \dot{K}(\overline{s})\right]_\ell, \dots\right) \\ &= \int\limits_{C(3,\mathbf{S}^1)} d\overline{s} \int\limits_{\mathbf{S}^3} \phi^*\overline{\omega}\left(\left[\dot{K}(\overline{s})\right]_\ell, \dots\right) + B_{21},\end{aligned} \tag{1.86}$$

where $\left[\dot{K}(\overline{s})\right]_\ell$ is the lift of the tangent vectors of the knot at each point $\overline{s} = (s_1, s_2, s_3)$ and B_{12} stands for all the boundary terms. The non-boundary contribution of (1.86) is given by the pullback of $\overline{\omega}$ under ϕ but now the contribution of the knot is given through the lifts $\left[\dot{K}(\overline{s})\right]_\ell$, that is

$$\phi^*\overline{\omega}\left(\left[\dot{K}(\overline{s})\right]_\ell, \dots\right) = \phi_{1,4}^*\omega\left(\left[\dot{K}(s_1)\right]_\ell, \dots\right) \wedge \phi_{2,4}^*\omega\left(\left[\dot{K}(s_2)\right]_\ell, \dots\right) \wedge \phi_{3,4}^*\omega\left(\left[\dot{K}(s_3)\right]_\ell, \dots\right), \tag{1.87}$$

where each of these pullbacks has the form

$$\begin{aligned}\phi_{i,4}^*\omega\left(\left[\dot{K}(s_i)\right]_\ell, \dots\right) &= -\frac{\varepsilon_{\mu\nu\sigma}}{4\pi} \frac{(x_4 - x_i)^\mu}{|x_4 - x_i|^3} dx_i^\nu \wedge dx_4^\sigma \left(\left[\dot{K}(s_i)\right]_\ell, \dots\right) \\ &= -\frac{\varepsilon_{\mu\nu\sigma}}{4\pi} \frac{(x_4 - K(s_i))^\mu}{|x_4 - K(s_i)|^3} \dot{K}^\nu(s_i)\, dx_4^\sigma\end{aligned}$$

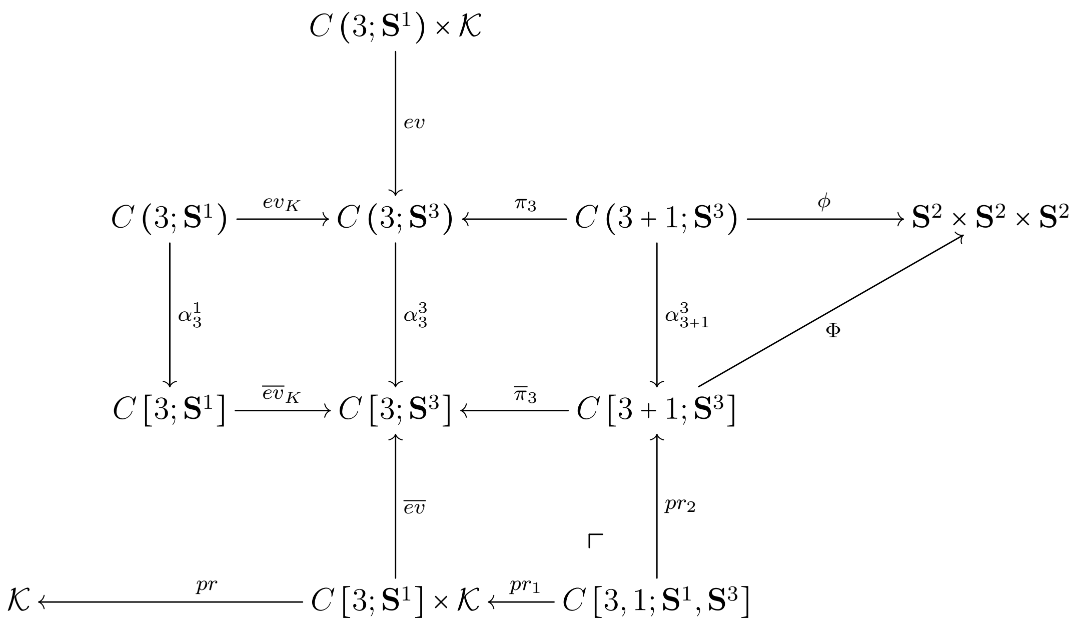


Figure 1.7: Mathematical diagram for D_{21}.

$$= \Delta_{\nu\sigma}(K(s_i) - x_4)\dot{K}^{\nu}(s_i)\, dx_4^{\sigma}, \tag{1.88}$$

with $i = 1, 2, 3$. Substitution of this expression into the non-boundary part of Eq. (1.86) yields

$$\begin{aligned}
&\int\limits_{C(3,\mathbf{S}^1)} d\overline{s} \int\limits_{\mathbf{S}^3} \phi^*\overline{\omega}\left(\left[\dot{K}(\overline{s})\right]_{\ell}, \ldots\right) \\
&= \int\limits_{C(3,\mathbf{S}^1)} d\overline{s} \int\limits_{\mathbf{S}^3} \Big[\Delta_{\nu_1\sigma_1}(K(s_1) - x_4)\Delta_{\nu_2\sigma_2}(K(s_2) - x_4)\Delta_{\nu_3\sigma_3}(K(s_3) - x_4) \\
&\qquad\qquad \times\dot{K}^{\sigma_1}(s_1)\dot{K}^{\sigma_2}(s_2)\dot{K}^{\sigma_3}(s_3)\Big] dx_4^{\nu_1} \wedge dx_4^{\nu_2} \wedge dx_4^{\nu_3} \\
&= \int\limits_{C(3,\mathbf{S}^1)} \dot{K}^{\sigma_1}(s_1)\dot{K}^{\sigma_2}(s_2)\dot{K}^{\sigma_3}(s_3) ds_1 \wedge ds_2 \wedge ds_3 \int\limits_{\mathbf{S}^3} \varepsilon^{\nu_1\nu_2\nu_3} d^3x_4 \Delta_{\nu_1\sigma_1}(K(s_1) - x_4) \\
&\qquad\qquad \times\Delta_{\nu_2\sigma_2}(K(s_2) - x_4)\Delta_{\nu_3\sigma_3}(K(s_3) - x_4),
\end{aligned} \tag{1.89}$$

and then $I_{D_{21}}$ is written as

$$\begin{aligned}
I_{D_{21}} = &\int\limits_{C(3,\mathbf{S}^1)} \dot{K}^{\sigma_1}(s_1)\dot{K}^{\sigma_2}(s_2)\dot{K}^{\sigma_3}(s_3) ds_1 \wedge ds_2 \wedge ds_3 \int\limits_{\mathbf{S}^3} \varepsilon^{\nu_1\nu_2\nu_3} d^3x_4 \Delta_{\nu_1\sigma_1}(K(s_1) - x_4) \\
&\qquad\qquad \times\Delta_{\nu_2\sigma_2}(K(s_2) - x_4)\Delta_{\nu_3\sigma_3}(K(s_3) - x_4).
\end{aligned} \tag{1.90}$$

In analogy to the self-linking case, this expression can be regarded to match with Eq. (1.69).

Now we proceed to discuss diagram D_{22} by using figure 1.8. In this case the ϕ map is given by $\phi = \phi_{1,3} \times \phi_{2,4}$, where all the points are on the knot, $\overline{\pi}_4 = id$ as in the self-linking case and

$\overline{\omega} = \omega \wedge \omega \in \Omega^4 \left(\mathbf{S}^2 \times \mathbf{S}^2\right)$. This time the configuration space integral is given by

$$\overline{I}_{D_{22}} = \int_{C(4,\mathbf{S}^1)} d\overline{s} \left(\Phi \circ \alpha_4^3 \circ ev_K\right)^* \overline{\omega} + B_{22}$$
$$= \int_{C(4,\mathbf{S}^1)} d\overline{s} \left(\phi \circ ev_K\right)^* \overline{\omega} + B_{22}, \tag{1.91}$$

where B_{22} stands for the boundary terms.

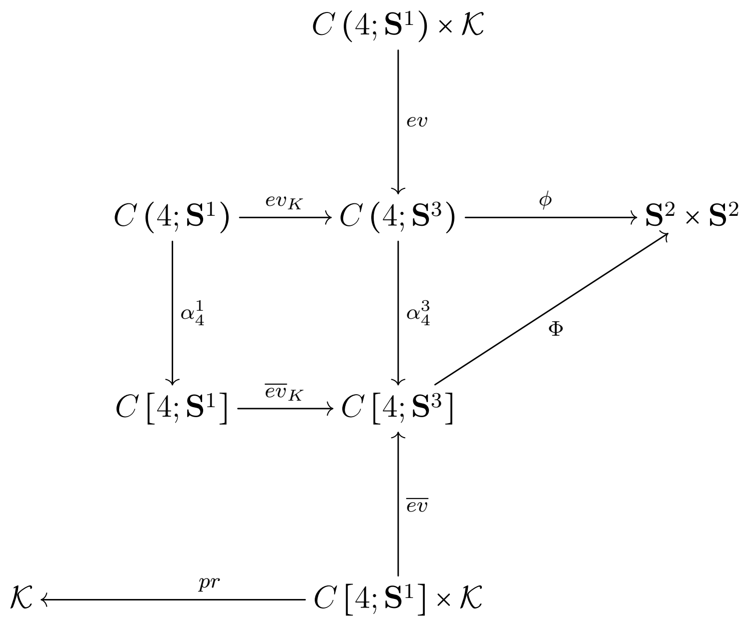


Figure 1.8: Mathematical diagram for D_{22}.

By using Eq. (1.79) the pullback under $\phi \circ ev_K$ of $\overline{\omega}$ is written as

$$(\phi \circ ev_k)^* \overline{\omega} = \left(ev_K^* \phi_{1,3}^* \omega\right) \wedge \left(ev_K^* \phi_{2,4}^* \omega\right)$$

$$= \left[\frac{\varepsilon_{\mu_1\nu_1\sigma_1}}{4\pi} \frac{(K(s_1) - K(s_3))^{\mu_1}}{|K(s_1) - K(s_3)|^3} \frac{\varepsilon_{\mu_2\nu_2\sigma_2}}{4\pi} \frac{(K(s_2) - K(s_4))^{\mu_2}}{|K(s_2) - K(s_4)|^3}\right]$$
$$\times \dot{K}^{\nu_1}(s_1) \dot{K}^{\sigma_1}(s_3) \dot{K}^{\nu_2}(s_2) \dot{K}^{\sigma_2}(s_4)\, ds_1 \wedge ds_3 \wedge ds_2 \wedge ds_4$$

$$= \left[\Delta_{\nu_1\sigma_1}(K(s_1) - K(s_3)) \Delta_{\nu_2\sigma_2}(K(s_2) - K(s_4))\right]$$
$$\times (-1) \dot{K}^{\nu_1}(s_1) \dot{K}^{\nu_2}(s_2) \dot{K}^{\sigma_1}(s_3) \dot{K}^{\sigma_2}(s_4)\, ds_1 \wedge ds_2 \wedge ds_3 \wedge ds_4, \tag{1.92}$$

and consequently $I_{D_{22}}$ is written as

$$I_{D_{22}} = -\int_{C(4,\mathbf{S}^1)} \Big[\Delta_{\nu_1\sigma_1}(K(s_1) - K(s_3)) \Delta_{\nu_2\sigma_2}(K(s_2) - K(s_4)) \dot{K}^{\nu_1}(s_1) \dot{K}^{\nu_2}(s_2)$$

$$\times \dot{K}^{\sigma_1}(s_3)\,\dot{K}^{\sigma_2}(s_4)\Big] ds_1 \wedge ds_2 \wedge ds_3 \wedge ds_4, \tag{1.93}$$

which can be seen to match with Eq. (1.70). Boundary contributions B_{21} and B_{22} cancel each other and so the second order Vassiliev topological invariant is the sum $I_{D_{21}} + I_{D_{22}}$ from expressions (1.90) and (1.93).

1.3.3 Third order Vassiliev knot invariant

The third order Vassiliev invariant has an effective contribution given by diagrams D_{31}, D_{32}, D_{33} and D_{34} (figures 1.9a, 1.9b, 1.9c and 1.9d, respectively) corresponding to terms (1.71), (1.72), (1.73) and (1.74), in that order.

In a completely analogous way to the analysis of the first and second order Vassiliev invariants, mathematical diagrams of figures 1.10 - 1.13 are to be used to build the corresponding configuration space expressions. The unitary volume form $\omega \in \Omega^2\left(S^2\right)$ and the corresponding products for each diagram will be used.

The first step is to analyse diagram D_{31}. In this case the ϕ map in figure 1.10 is given by the restriction of

$$\phi_{1,6} \times \phi_{2,5} \times \phi_{3,5} \times \phi_{4,6} \times \phi_{5,6} : \prod_{i=1}^{6} \mathbf{S}^3 \longrightarrow \prod_{i=1}^{5} \mathbf{S}^2 \tag{1.94}$$

to $C(4+2, \mathbf{S}^3)$. By taking the pullback of $\overline{\omega} := \omega^5 \in \Omega^{10}\left(\left(\mathbf{S}^2\right)^5\right)$ under the previous map it is obtained

$$\begin{aligned}\phi^*\left(\overline{\omega}\right) &= \left(\phi_{1,6} \times \phi_{2,5} \times \phi_{3,5} \times \phi_{4,6} \times \phi_{5,6}\right)^*\left(\overline{\omega}\right)\\ &= \phi_{1,6}^*\omega \wedge \phi_{2,5}^*\omega \wedge \phi_{3,5}^*\omega \wedge \phi_{4,6}^*\omega \wedge \phi_{5,6}^*\omega,\end{aligned} \tag{1.95}$$

where again each of these pullbacks are given by Eq. (1.78).

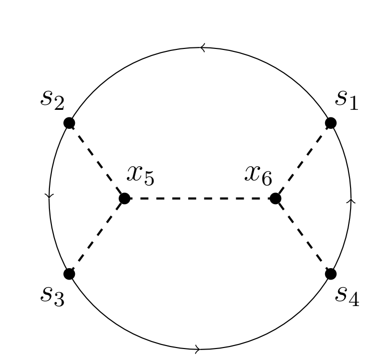
By applying Eq. (1.58) to a point $\overline{s} \in C(4, \mathbf{S}^1)$ and according with figure 1.10 we have

$$f_{D_{31},K}(\overline{s}) = \left(\left(\alpha_4^3 \circ ev_K\right)^* \left(\overline{\pi}_4\right)_* \Phi^*\overline{\omega}\right)_{\overline{s}} \left(\partial_{\overline{s}}\right), \tag{1.96}$$

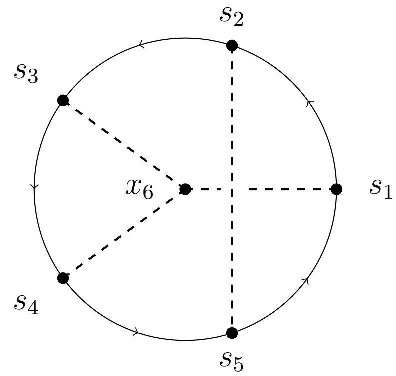
which is a 4-form in $C(4, \mathbf{S}^1)$. The fiber of $\overline{\pi}_4$ is a space of the same homotopy type than $\mathbb{R}^3 \times \mathbb{R}^3$ and so

$$\begin{aligned}\overline{I}_{D_{31}} &= \int_{C(4,\mathbf{S}^1)} f_{D_{31},K}(\overline{s}) d\overline{s} = \int_{C(4,\mathbf{S}^1)} d\overline{s} \int_{\overline{\pi}_4^{-1}(\{\overline{x}\})} \Phi^*\overline{\omega}\left(\left[(\alpha_4^3)^*\dot{K}(\overline{s})\right]_\ell, \ldots\right)\\ &= \int_{C(4,\mathbf{S}^1)} d\overline{s} \int_{\mathbb{R}^3\times\mathbb{R}^3} \phi^*\overline{\omega}\left(\left[\dot{K}(\overline{s})\right]_\ell, \ldots\right) + B_{31},\end{aligned} \tag{1.97}$$

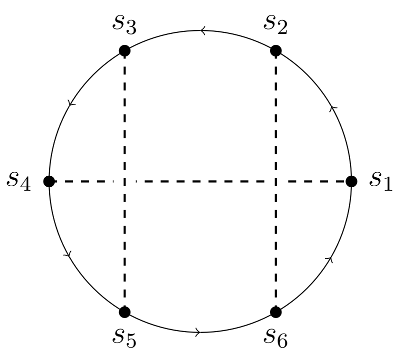
where B_{31} represents the boundary terms. For this diagram the non-boundary contribution of (1.97) is given by the pullback of $\overline{\omega}$ under ϕ evaluated at lifts $\left[\dot{K}(\overline{s})\right]_\ell$ of the tangent vectors of the knot at each point in $\overline{s} = (s_1, \ldots, s_4)$, that is



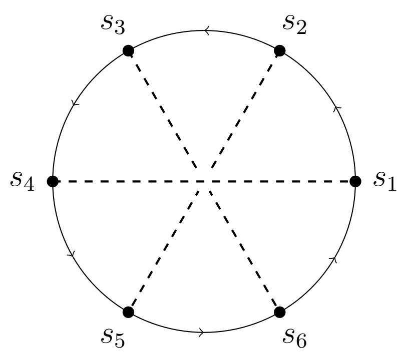
(a) Feynman diagram D_{31}.



(b) Feynman diagram D_{32}.



(c) Feynman diagram D_{33}.



(d) Feynman diagram D_{34}.

Figure 1.9: Feynman diagrams for the third order Vassiliev invariant.

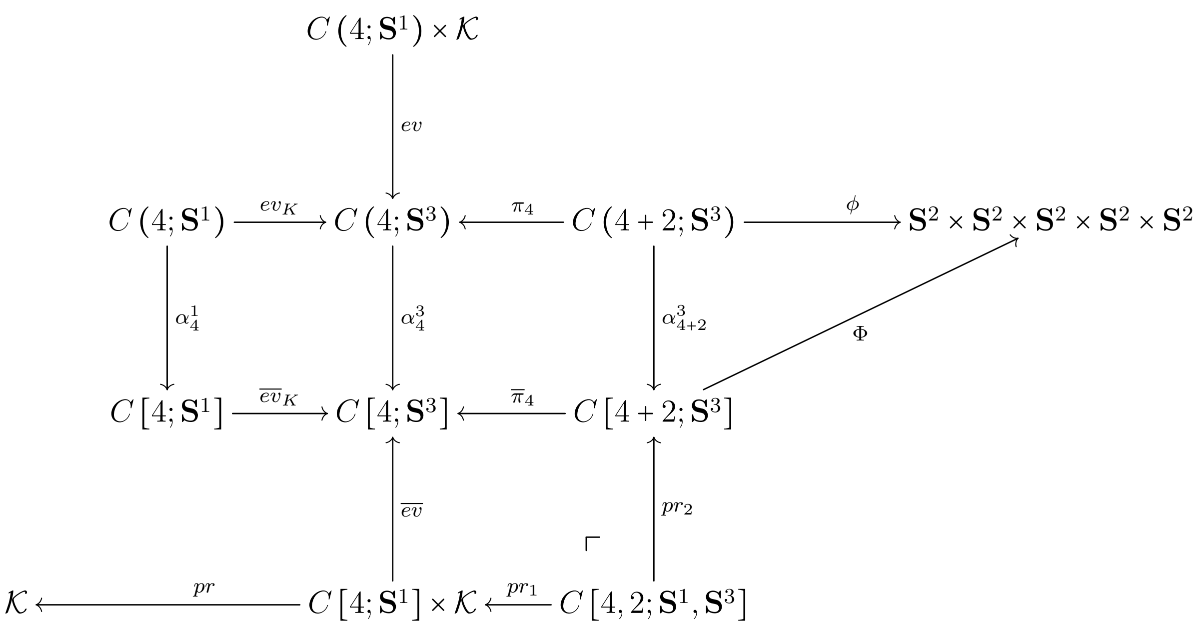


Figure 1.10: Mathematical diagram for D_{31}.

$$\phi^*\overline{\omega}\left(\left[\dot{K}(\overline{s})\right]_\ell,\dots\right) = \phi^*_{1,6}\omega\left(\left[\dot{K}(s_1)\right]_\ell,\dots\right)\wedge\phi^*_{2,5}\omega\left(\left[\dot{K}(s_2)\right]_\ell,\dots\right)\wedge\phi^*_{3,5}\omega\left(\left[\dot{K}(s_3)\right]_\ell,\dots\right)$$
$$\wedge\phi^*_{4,6}\omega\left(\left[\dot{K}(s_4)\right]_\ell,\dots\right)\wedge\phi^*_{5,6}\omega, \tag{1.98}$$

where these pullbacks are rewritten as

$$\begin{aligned}\phi^*_{i,j}\omega\left(\left[\dot{K}(s_i)\right]_\ell,\dots\right) &= -\frac{\varepsilon_{\mu\nu\sigma}}{4\pi}\frac{(x_j-x_i)^\mu}{|x_j-x_i|^3}dx_i^\nu\wedge dx_j^\sigma\left(\left[\dot{K}(s_i)\right]_\ell,\dots\right)\\ &= -\frac{\varepsilon_{\mu\nu\sigma}}{4\pi}\frac{(x_j-K(s_i))^\mu}{|x_j-K(s_i)|^3}\dot{K}^\nu(s_i)\,dx_j^\sigma\\ &= \Delta_{\nu\sigma}(K(s_i)-x_j)\dot{K}^\nu(s_i)\,dx_j^\sigma,\end{aligned} \tag{1.99}$$

with $i=1,2,3,4$, $j=5,6$ and $\phi^*_{5,6}\omega$ as in Eq. (1.78). Substitution of this expression into the non-boundary part of (1.97) directly yields to

$$\begin{aligned}&\int_{C(4,\mathbf{S}^1)} d\overline{s}\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi^*\overline{\omega}\left(\left[\dot{K}(\overline{s})\right]_\ell,\dots\right)\\ &=\int_{C(4,\mathbf{S}^1)} d\overline{s}\int_{\mathbb{R}^3\times\mathbb{R}^3}\Big[\Delta_{\nu_1\sigma_1}(K(s_1)-x_6)\Delta_{\nu_2\sigma_2}(K(s_2)-x_5)\Delta_{\nu_3\sigma_3}(K(s_3)-x_5)\\ &\quad\times\Delta_{\nu_4\sigma_4}(K(s_4)-x_6)\Delta_{\nu_5\sigma_5}(x_5-x_6)\dot{K}^{\nu_1}(s_1)\dot{K}^{\nu_2}(s_2)\dot{K}^{\nu_3}(s_3)\dot{K}^{\nu_4}(s_4)\Big]\\ &\quad\times dx_6^{\sigma_1}\wedge dx_5^{\sigma_2}\wedge dx_5^{\sigma_3}\wedge dx_6^{\sigma_4}\wedge dx_5^{\nu_5}\wedge dx_6^{\sigma_5},\end{aligned} \tag{1.100}$$

and then $I_{D_{31}}$ is written as

$$\begin{aligned}I_{D_{31}} = &\int_{C(4,\mathbf{S}^1)}\dot{K}^{\nu_1}(s_1)\dot{K}^{\nu_2}(s_2)\dot{K}^{\nu_3}(s_3)\dot{K}^{\nu_4}(s_4)ds_1\wedge ds_2\wedge ds_3\wedge ds_4\int_{\mathbb{R}^3}\varepsilon^{\sigma_2\sigma_3\nu_5}d^3x_5\\ &\times\int_{\mathbb{R}^3}\varepsilon^{\sigma_1\sigma_4\sigma_5}d^3x_6\Delta_{\nu_1\sigma_1}(K(s_1)-x_6)\Delta_{\nu_2\sigma_2}(K(s_2)-x_5)\\ &\times\Delta_{\nu_3\sigma_3}(K(s_3)-x_5)\Delta_{\nu_4\sigma_4}(K(s_4)-x_6)\Delta_{\nu_5\sigma_5}(x_5-x_6).\end{aligned} \tag{1.101}$$

It can be regarded that this expression matches with (1.71) from Chern-Simons theory.

Now we analyse diagram D_{32} by using the mathematical construction of figure 1.11. Here the ϕ map is given by the restriction of

$$\phi_{1,6}\times\phi_{3,6}\times\phi_{4,6}\times\phi_{2,5}:\prod_{i=1}^{6}\mathbf{S}^3\longrightarrow\prod_{i=1}^{4}\mathbf{S}^2 \tag{1.102}$$

to $C(5+1,\mathbf{S}^3)$. This time the fibre of $\overline{\pi}_5$ is again a space of the same homotopy type than $\mathbb{R}^3$ and so

$$\overline{I}_{D_{32}} = \int_{C(5,\mathbf{S}^1)} f_{D_{32},K}(\overline{s})d\overline{s} = \int_{C(5,\mathbf{S}^1)} d\overline{s}\int_{\overline{\pi}_5^{-1}(\{\overline{x}\})}\Phi^*\overline{\omega}\left(\left[(\alpha_5^3)^*\dot{K}(\overline{s})\right]_\ell,\dots\right)$$

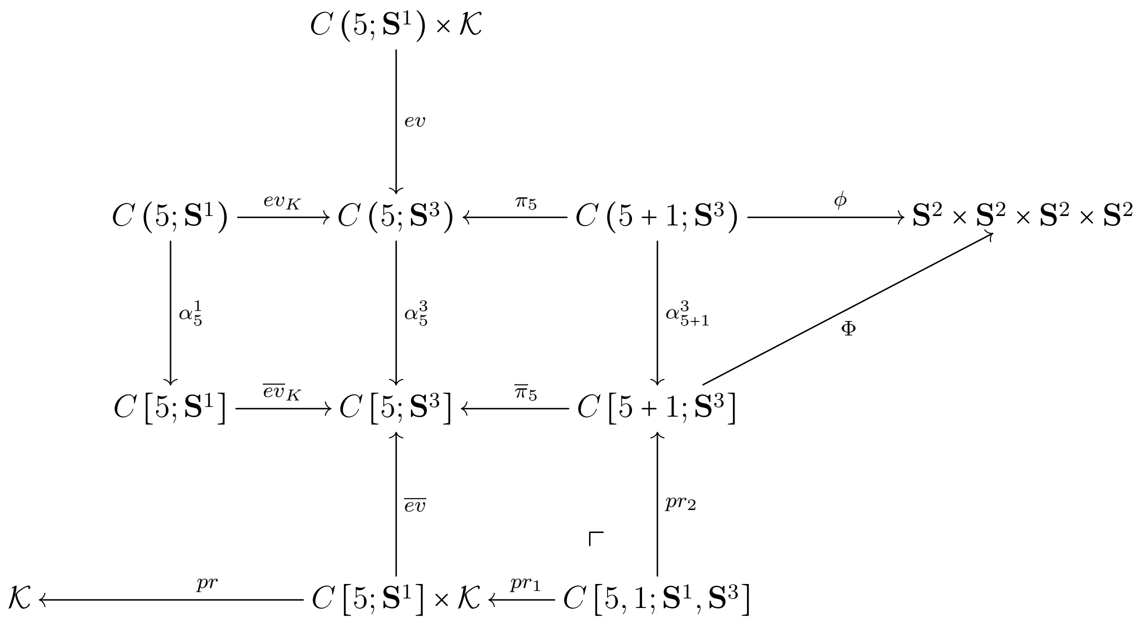


Figure 1.11: Mathematical diagram for D_{32}.

$$= \int\limits_{C(5,\mathbf{S}^1)} d\overline{s} \int\limits_{\mathbb{R}^3} \phi^*\overline{\omega}\left(\left[\dot{K}(\overline{s})\right]_\ell, \ldots\right) + B_{32}, \tag{1.103}$$

where B_{32} stands for all the boundary contributions. In this case the non-boundary part of (1.103) is expressed as

$$\phi^*\overline{\omega}\left(\left[\dot{K}(\overline{s})\right]_\ell, \ldots\right) = \phi_{1,6}^*\omega\left(\left[\dot{K}(s_1)\right]_\ell, \ldots\right) \wedge \phi_{2,5}^*\omega\left(\dot{K}(s_2), \dot{K}(s_5)\right) \wedge \phi_{3,6}^*\omega\left(\left[\dot{K}(s_3)\right]_\ell, \ldots\right) \wedge \phi_{4,6}^*\omega\left(\left[\dot{K}(s_4)\right]_\ell, \ldots\right), \tag{1.104}$$

where each of the pullbaks are given by Eq. (1.99), *i.e.*, by

$$\phi_{i,j}^*\omega\left(\left[\dot{K}(s_i)\right]_\ell, \ldots\right) = \Delta_{\nu\sigma}(K(s_i) - x_j)\dot{K}^\nu(s_i)\, dx_j^\sigma, \tag{1.105}$$

with $i = 1, 3, 4$, $j = 6$ and where $\phi_{2,5}^*\omega\left(\dot{K}(s_2), \dot{K}(s_5)\right)$ is rewritten as

$$\phi_{2,5}^*\omega\left(\dot{K}(s_2), \dot{K}(s_5)\right) = \frac{\varepsilon_{\mu\nu\sigma}}{4\pi} \frac{(K(s_2) - K(s_5))^\mu}{|K(s_2) - K(s_5)|^3} \dot{K}^\nu(s_2)\, \dot{K}^\sigma(s_5).$$

Analogously to the case of diagram D_{31}, the non-boundary part of (1.103) is given by

$$\int\limits_{C(5,\mathbf{S}^1)} d\overline{s} \int\limits_{\mathbb{R}^3} \phi^*\overline{\omega}\left(\left[\dot{K}(\overline{s})\right]_\ell, \ldots\right)$$
$$= \int\limits_{C(5,\mathbf{S}^1)} d\overline{s} \int\limits_{\mathbb{R}^3} \Big[\Delta_{\nu_1\sigma_1}(K(s_1) - x_6)\Delta_{\nu_2\sigma_2}(K(s_2) - K(s_5))\Delta_{\nu_3\sigma_3}(K(s_3) - x_6)\Delta_{\nu_4\sigma_4}(K(s_4) - x_6)$$

$$\times \dot{K}^{\nu_1}(s_1)\dot{K}^{\nu_2}(s_2)\dot{K}^{\sigma_2}(s_5)\dot{K}^{\nu_3}(s_3)\dot{K}^{\nu_4}(s_4)\Big] dx_6^{\sigma_1}\wedge dx_6^{\sigma_3}\wedge dx_6^{\sigma_4}, \tag{1.106}$$

and then $I_{D_{32}}$ is written as

$$I_{D_{32}} = \int_{C(5,\mathbf{S}^1)} \dot{K}^{\nu_1}(s_1)\dot{K}^{\nu_2}(s_2)\dot{K}^{\nu_3}(s_3)\dot{K}^{\nu_4}(s_4)\dot{K}^{\sigma_2}(s_5) ds_1\wedge ds_2\wedge ds_3\wedge ds_4\wedge ds_5 \int_{\mathbb{R}^3} \varepsilon^{\sigma_1\sigma_3\sigma_4} d^3x_6$$
$$\times \Delta_{\nu_1\sigma_1}(K(s_1)-x_6)\Delta_{\nu_2\sigma_2}(K(s_2)-K(s_5))\Delta_{\nu_3\sigma_3}(K(s_3)-x_6)\Delta_{\nu_4\sigma_4}(K(s_4)-x_6). \tag{1.107}$$

Again this expression matches with Eq. (1.72) which comes from Chern-Simons theory.

Now we continue our analysis with diagram D_{33}. This time the map ϕ in figure 1.12 is given by the restriction of

$$\phi_{1,4}\times\phi_{2,6}\times\phi_{3,5} : \prod_{i=1}^{6}\mathbf{S}^3 \longrightarrow \prod_{i=1}^{3}\mathbf{S}^2 \tag{1.108}$$

to $C(6,\mathbf{S}^3)$. Here all the points belong to the knot, therefore $\overline{\omega} = \omega\times\omega\times\omega \in \Omega^6\left((\mathbf{S}^2)^3\right)$ and $\overline{\pi}_6 = id$. This time the configuration space integral is given by

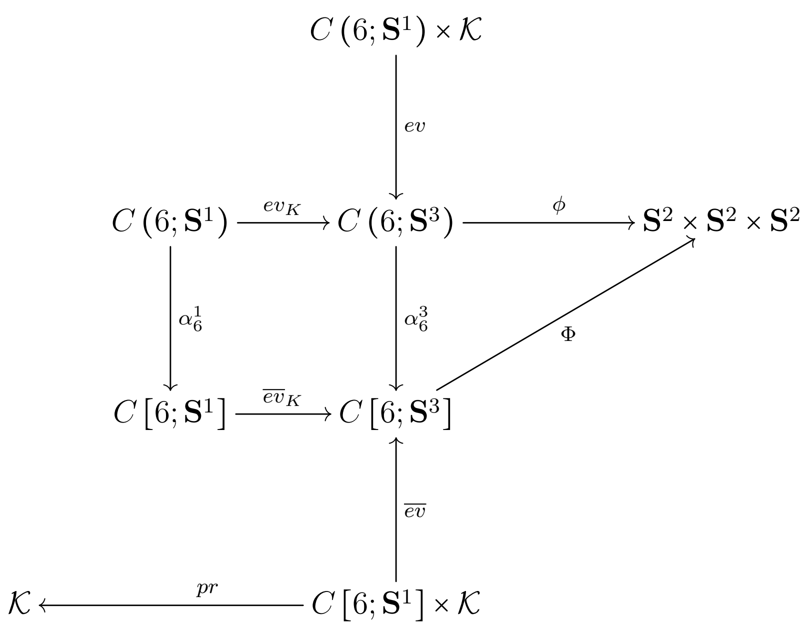


Figure 1.12: Mathematical diagram for D_{33}.

$$\begin{aligned}\overline{I}_{D_{33}} &= \int_{C(6,\mathbf{S}^1)} d\overline{s}\left(\Phi\circ\alpha_6^3\circ ev_K\right)^*\overline{\omega} + B_{33} \\ &= \int_{C(6,\mathbf{S}^1)} d\overline{s}\left(\phi\circ ev_K\right)^*\overline{\omega} + B_{33},\end{aligned} \tag{1.109}$$

where B_{33} stands for all boundary terms. By using Eq. (1.79), the pullback of $\overline{\omega}$ under $\phi \circ ev_K$ for this case is written as

$$
\begin{aligned}
&(\phi \circ ev_k)^* \, \overline{\omega} \\
&= (ev_K)^* \, \phi^*_{1,4}\omega \times (ev_K)^* \, \phi^*_{2,6}\omega \times (ev_K)^* \, \phi^*_{3,5}\omega \\
&= \left[\frac{\varepsilon_{\mu_1\nu_1\sigma_1}}{4\pi} \frac{(K(s_1) - K(s_4))^{\mu_1}}{|K(s_1) - K(s_4)|^3} \frac{\varepsilon_{\mu_2\nu_2\sigma_2}}{4\pi} \frac{(K(s_2) - K(s_6))^{\mu_2}}{|K(s_2) - K(s_6)|^3} \frac{\varepsilon_{\mu_3\nu_3\sigma_3}}{4\pi} \frac{(K(s_3) - K(s_5))^{\mu_3}}{|K(s_3) - K(s_5)|^3} \right] \\
&\qquad \times \dot{K}^{\nu_1}(s_1) \, \dot{K}^{\sigma_1}(s_4) \, \dot{K}^{\nu_2}(s_2) \, \dot{K}^{\sigma_2}(s_6) \, \dot{K}^{\nu_3}(s_3) \, \dot{K}^{\sigma_3}(s_5) \\
&\qquad \times ds_1 \wedge ds_4 \wedge ds_2 \wedge ds_6 \wedge ds_3 \wedge ds_5 \\
&= \Big[\Delta_{\nu_1\sigma_1}(K(s_1) - K(s_4)) \Delta_{\nu_2\sigma_2}(K(s_2) - K(s_6)) \Delta_{\nu_3\sigma_3}(K(s_3) - K(s_5)) \Big] \\
&\qquad \times \dot{K}^{\nu_1}(s_1) \, \dot{K}^{\sigma_1}(s_4) \, \dot{K}^{\nu_2}(s_2) \, \dot{K}^{\sigma_2}(s_6) \, \dot{K}^{\nu_3}(s_3) \, \dot{K}^{\sigma_3}(s_5) \\
&\qquad \times ds_1 \wedge ds_2 \wedge ds_3 \wedge ds_4 \wedge ds_5 \wedge ds_6,
\end{aligned} \tag{1.110}
$$

and then $I_{D_{33}}$ is given by

$$
\begin{aligned}
I_{D_{33}} = \int_{C(6,\mathbf{S}^1)} \Big[&\Delta_{\nu_1\sigma_1}(K(s_1) - K(s_4)) \, \Delta_{\nu_2\sigma_2}(K(s_2) - K(s_6)) \, \Delta_{\nu_3\sigma_3}(K(s_3) - K(s_5)) \\
&\times \dot{K}^{\nu_1}(s_1) \dot{K}^{\nu_2}(s_2) \dot{K}^{\nu_3}(s_3) \dot{K}^{\sigma_1}(s_4) \dot{K}^{\sigma_2}(s_6) \dot{K}^{\sigma_3}(s_5) \Big] \\
&\times ds_1 \wedge ds_2 \wedge ds_3 \wedge ds_4 \wedge ds_5 \wedge ds_6,
\end{aligned} \tag{1.111}
$$

which can be identified with Eq. (1.73) from Chern-Simons theory.

The final step is to analyse diagram D_{34}. The map ϕ in figure 1.13 is given now by the restriction of the following map to $C(6, \mathbf{S}^3)$,

$$
\phi_{1,4} \times \phi_{2,5} \times \phi_{3,6} : \prod_{i=1}^{6} \mathbf{S}^3 \longrightarrow \prod_{i=1}^{3} \mathbf{S}^2. \tag{1.112}
$$

Again all the points are defined on the knot so there is no integration on internal points. In this case $\overline{\omega} = \omega \times \omega \times \omega \in \Omega^6\left((\mathbf{S}^2)^3\right)$ and the configuration space integral reads

$$
\begin{aligned}
\overline{I}_{D_{34}} &= \int_{C(6,\mathbf{S}^1)} d\overline{s} \left(\Phi \circ \alpha^3_6 \circ ev_K\right)^* \overline{\omega} + B_{34} \\
&= \int_{C(6,\mathbf{S}^1)} d\overline{s} \, (\phi \circ ev_K)^* \, \overline{\omega} + B_{34},
\end{aligned} \tag{1.113}
$$

where B_{34} again stands for the boundary terms. The same pullback from diagram D_{33} applies to this case and $\phi \circ ev_K$ is rewritten as

$$
(\phi \circ ev_k)^* \, \overline{\omega}
$$

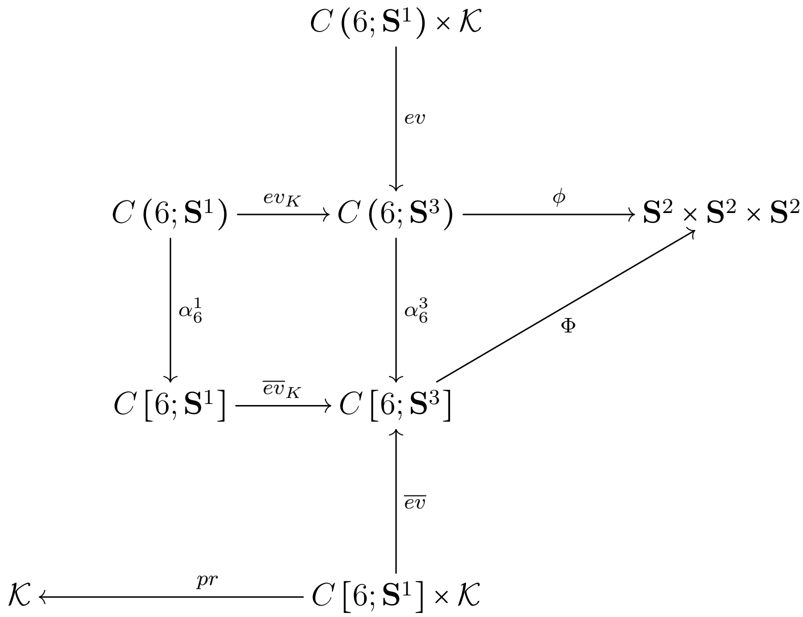


Figure 1.13: Mathematical diagram for D_{34}.

$$=\left(ev_K^*\phi_{1,4}^*\omega\right)\wedge\left(ev_K^*\phi_{2,5}^*\omega\right)\wedge\left(ev_K^*\phi_{3,6}^*\omega\right)$$

$$=\left[\frac{\varepsilon_{\mu_1\nu_1\sigma_1}}{4\pi}\frac{(K(s_1)-K(s_4))^{\mu_1}}{|K(s_1)-K(s_4)|^3}\frac{\varepsilon_{\mu_2\nu_2\sigma_2}}{4\pi}\frac{(K(s_2)-K(s_5))^{\mu_2}}{|K(s_2)-K(s_5)|^3}\frac{\varepsilon_{\mu_3\nu_3\sigma_3}}{4\pi}\frac{(K(s_3)-K(s_6))^{\mu_3}}{|K(s_3)-K(s_6)|^3}\right]$$
$$\times\dot{K}^{\nu_1}(s_1)\,\dot{K}^{\sigma_1}(s_4)\,\dot{K}^{\nu_2}(s_2)\,\dot{K}^{\sigma_2}(s_5)\,\dot{K}^{\nu_3}(s_3)\,\dot{K}^{\sigma_3}(s_6)$$
$$\times ds_1\wedge ds_4\wedge ds_2\wedge ds_5\wedge ds_3\wedge ds_6$$

$$=\left[\Delta_{\nu_1\sigma_1}(K(s_1)-K(s_4))\Delta_{\nu_2\sigma_2}(K(s_2)-K(s_5))\Delta_{\nu_3\sigma_3}(K(s_3)-K(s_6))\right]$$
$$\times(-1)\dot{K}^{\nu_1}(s_1)\cdot\dot{K}^{\sigma_1}(s_4)\,\dot{K}^{\nu_2}(s_2)\,\dot{K}^{\sigma_2}(s_5)\,\dot{K}^{\nu_3}(s_3)\,\dot{K}^{\sigma_3}(s_6)$$
$$\times ds_1\wedge ds_2\wedge ds_3\wedge ds_4\wedge ds_5\wedge ds_6, \quad (1.114)$$

and then $I_{D_{34}}$ can be written as

$$I_{D_{34}}=-\int_{C(6,\mathbf{S}^1)}\Big[\Delta_{\nu_1\sigma_1}(K(s_1)-K(s_4))\,\Delta_{\nu_2\sigma_2}(K(s_2)-K(s_5))\,\Delta_{\nu_3\sigma_3}(K(s_3)-K(s_6))$$
$$\times\dot{K}^{\nu_1}(s_1)\dot{K}^{\nu_2}(s_2)\dot{K}^{\nu_3}(s_3)\dot{K}^{\sigma_1}(s_4)\dot{K}^{\sigma_2}(s_5)\dot{K}^{\sigma_3}(s_6)\Big]$$
$$\times ds_1\wedge ds_2\wedge ds_3\wedge ds_4\wedge ds_5\wedge ds_6, \quad (1.115)$$

which again matches with the corresponding integral in Eq. (1.74) from Chern-Simons theory. Boundary contributions B_{31}, B_{32}, B_{33} and B_{34} cancel each other and so the third order

Vassiliev topological invariant is the sum $I_{D_{31}} + I_{D_{32}} + I_{D_{33}} + I_{D_{34}}$ from equations (1.101), (1.107), (1.111) and (1.115).

1.3.4 Boundary cancellation

There are important aspects to consider about the boundary terms appearing in Eqs. (1.75), (1.86) and (1.91). In the self-linking case the boundary term in (1.75) is cancelled via the introduction of a framing term of the knot [72]. The case of the second order Vassiliev invariant needs some more considerations.

In general for a given knot diagram there will be many boundary terms depending on the number and the rates of point collapses [74]. If two points collapse the face is called *principal*, if three or more points (but not all) collapse the face is called *hidden*. If all points collapse the face is called *anomalous* and if one or more collapsing points are considered to be at infinity the face is called *face at infinity*.

For the case of figure 1.6b the integrals corresponding to hidden and anomalous faces as well as to faces at infinity vanish [74], while those corresponding to principal faces do not necessarily vanish. The procedure to obtain a topological invariant is then to find another Feynman diagram such that the contribution of its non-vanishing faces are exactly the same and then subtract them. The choice is just diagram 1.6a. The integrals corresponding to hidden and anomalous faces, faces at infinity and principal faces coming from the collapse of two points on the knot vanish [74]. The remaining non-vanishing boundary contributions are the ones coming from the collapse of the internal point x_4 and a point on the knot, and they can be seen to exactly coincide with those from the first diagram. Thus, subtraction of diagrams 1.6a and 1.6b makes all boundary contributions cancel and the result is a topological invariant, this is precisely the second order Vassiliev invariant. It can be shown that for the third order Vassiliev invariant an analogous boundary cancellation occurs between the four diagrams in figure 1.9, actually this is also true for higher orders [74]. Consequently, the real topological invariant is the sum of contributions of all diagrams in that figure.

1.4 Average asymptotic Vassiliev invariants from Chern-Simons perturbation theory

In this section we incorporate a divergence-free vector field X in the domain $\mathcal{S}$ of $\mathbb{R}^3$ where Chern-Simons theory is defined. The key idea is to replace $\dot{K}$ by the vector field X, *i.e.*, one has the identification

$$\dot{K} \longleftrightarrow X \tag{1.116}$$

in the sense of section 1.2. As in the case without flow information the appropriate boundary cancellation will be assumed.

1.4.1 First order flow invariant

As indicated in section 1.2, θ^p is the p-fold product of the flow generated by the vector field X defined by $\theta^p((x_1,t_1),\dots,(x_p,t_p)) = (\theta(x_1,t_1),\dots,\theta(x_p,t_p)) \in C(p,\mathcal{S})$, where $x_1,\dots,x_p \in \mathcal{S}$

and $(t_1,\dots,t_p) \in C(p,\mathbf{S}^1)$, also $\theta^x(t) = \theta(x,t)$ with $x \in \mathcal{S}$ and $t \in \mathbf{S}^1$. From now on $\overline{x}$ stands for a p-tuple $(x,\dots,x)$ in $\mathcal{S}^p$.

The first example will be the self-linking expression (1.82) coming from figure 1.5. For this case, Eq. (1.60) reads

$$f_{D_1,X}(\theta^x(t_1),\theta^x(t_2)) = \left(\left(\alpha_2^3\right)^* \Phi^*\omega\right)_{(\theta^x(t_1),\theta^x(t_2))} (X_{\theta^x(t_1)}, X_{\theta^x(t_2)}). \tag{1.117}$$

By using a theorem from Ref. [41], the average asymptotic invariant has the form

$$\int_{\mathcal{S}} \lambda_{D_1}\mu = \int_{\mathcal{S}\times\mathcal{S}} \lim_{T\to\infty} \frac{1}{T^2} \left\{ \int_0^T \int_0^T f_{D_1,X}(\theta^{x_1}(t_1),\theta^{x_2}(t_2)) dt_1 \wedge dt_2 \right\} \mu_\Delta, \tag{1.118}$$

where the part within braces in the integrand can be rewritten as

$$\begin{aligned}\int_0^T \int_0^T f_{D_1,X}(\theta^{x_1}(t_1),\theta^{x_2}(t_2)) dt_1 \wedge dt_2 &= \int_{C(2,S^1)} \left(\phi\circ(\theta^{x_1}\times\theta^{x_2})\right)^* \omega \\ &= \int_{C(2,S^1)} \frac{\varepsilon_{\mu\nu\sigma}}{4\pi} \frac{(\theta^{x_1}(t_1)-\theta^{x_2}(t_2))^\mu}{|\theta^{x_1}(t_1)-\theta^{x_2}(t_2)|^3} (\dot{\theta}^{x_1})^\nu (\dot{\theta}^{x_2})^\sigma dt_1\wedge dt_2 \\ &= \mathrm{lk}\left(\theta^{x_1},\theta^{x_2}\right),\end{aligned} \tag{1.119}$$

and then by substituting this expression into Eq. (1.118) one gets

$$\mathcal{H}(X) = \int_{\mathcal{S}} \lambda_{D_1}\mu = \int_{\mathcal{S}\times\mathcal{S}} \lim_{T\to\infty} \frac{1}{T^2} \mathrm{lk}\left(\theta^{x_1},\theta^{x_2}\right) \mu_\Delta, \tag{1.120}$$

which is the called average asymptotic linking invariant (or average Hopf invariant) for the vector field X.

1.4.2 Second order flow invariant

The flow contributions of the diagrams in figure 1.6 once the vector field X was introduced are again an application of Eq. (1.64).

In the case of diagram 1.6b we have to use figure 1.8 for which again $\phi = \phi_{1,3}\times\phi_{2,4}$. Then Eq. (1.60) is written as

$$f_{D_{22},X}(\theta^x(t_1),\dots,\theta^x(t_4)) = \left(\left(\alpha_4^3\right)^* \Phi^*\overline{\omega}\right)_{(\theta^x(t_1),\dots,\theta^x(t_4))} (X_{\theta^x(t_1)},\dots,X_{\theta^x(t_4)}), \tag{1.121}$$

and then the integral (1.65) takes the form

$$\int_{\mathcal{S}} \lambda_{D_{22}}\mu = \int_{\mathcal{S}^4} \lim_{T\to\infty} \frac{1}{T^4} \left\{ \int_0^T \cdots \int_0^T f_{D_{22},X}(\theta^{x_1}(t_1),\dots,\theta^{x_4}(t_4)) dt_1\wedge\cdots\wedge dt_4 \right\} \mu_\Delta. \tag{1.122}$$

Once again, by first making an analysis of the term between braces in the previous equation one gets

$$\int_0^T \cdots \int_0^T f_{D_{22},X}(\theta^{x_1}(t_1),\ldots,\theta^{x_4}(t_4))dt_1\wedge\cdots\wedge dt_4 = \int_{C(4,\mathbf{S}^1)} \left(\phi\circ\theta^4\right)^*\overline{\omega} + \widetilde{B}_{22}, \tag{1.123}$$

where $\widetilde{B}_{22}$ stands for the boundary terms.

The integrand in the first term of the right hand side of the last expression takes the form

$$\begin{aligned}\left(\phi\circ\theta^4\right)^*\overline{\omega} =&(\theta^4)^*(\phi_{1,3}^*\omega\wedge\phi_{2,4}^*\omega)\\ =&\left[\frac{\varepsilon_{\mu_1\nu_1\sigma_1}}{4\pi}\frac{(\theta^{x_1}(t_1)-\theta^{x_3}(t_2))^{\mu_1}}{|\theta^{x_1}(t_1)-\theta^{x_3}(t_2)|^3}\right]\left[\frac{\varepsilon_{\mu_2\nu_2\sigma_2}}{4\pi}\frac{(\theta^{x_2}(t_3)-\theta^{x_4}(t_4))^{\mu_2}}{|\theta^{x_2}(t_3)-\theta^{x_4}(t_4)|^3}\right]\\ &\times(\dot\theta^{x_1})^{\nu_1}(\dot\theta^{x_3})^{\sigma_1}(\dot\theta^{x_2})^{\nu_2}(\dot\theta^{x_4})^{\sigma_2}dt_1\wedge dt_2\wedge dt_3\wedge dt_4\\ =&\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-\theta^{x_3}(t_2))\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_3)-\theta^{x_4}(t_4))\\ &\times X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\sigma_1}_{\theta^{x_3}(t_2)}X^{\nu_2}_{\theta^{x_2}(t_3)}X^{\sigma_2}_{\theta^{x_4}(t_4)}dt_1\wedge dt_2\wedge dt_3\wedge dt_4.\end{aligned} \tag{1.124}$$

Then, the average asymptotic integral of Eq. (1.124) has the form

$$\begin{aligned}\int_{\mathcal{S}}\lambda_{D_{22}}\mu =\int_{\mathcal{S}^4}\lim_{T\to\infty}\frac{1}{T^4}\Bigg\{\int_{C(4,\mathbf{S}^1)}&\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-\theta^{x_3}(t_2))\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_3)-\theta^{x_4}(t_4))\\ &\times X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\sigma_1}_{\theta^{x_3}(t_2)}X^{\nu_2}_{\theta^{x_2}(t_3)}X^{\sigma_2}_{\theta^{x_4}(t_4)}dt_1\wedge dt_2\wedge dt_3\wedge dt_4+\widetilde{B}_{22}\Bigg\}\mu_\Delta.\end{aligned} \tag{1.125}$$

The diagram 1.6a needs different considerations because now there is a point outside the knot and it is necessary to take this information into account. These considerations imply the use of figure 1.7 where, analogously to the previous analysis, ϕ and $\overline{\omega}$ are given by $\phi=\phi_{1,4}\times\phi_{2,4}\times\phi_{3,4}$ and $\overline{\omega}=\omega\wedge\omega\wedge\omega$ then one gets

$$\begin{aligned}&f_{D_{21},X}(\theta^3((x,t_1),\ldots,(x,t_3)))\\ &=\left((\alpha_3^3)^*(\overline{\pi}_3)_*\Phi^*\overline{\omega}\right)_{(\theta^3((x,t_1),\ldots,(x,t_3)))}\left(X_{\theta^x(t_1)},X_{\theta^x(t_2)},X_{\theta^x(t_3)}\right)\\ &=(\alpha_3^3)^*\left(\int_{(\overline{\pi}_3)^{-1}(\theta^3((x,t_1),\ldots,(x,t_3)))}\Phi^*\overline{\omega}([X_{\theta^x(t_1)}]_\ell,[X_{\theta^x(t_2)}]_\ell,[X_{\theta^x(t_3)}]_\ell,\ldots)\right)\\ &=\int_{(\overline{\pi}_3)^{-1}(\alpha_3^3(\theta^3((x,t_1),\ldots,(x,t_3))))}\Phi^*\overline{\omega}([(\alpha_3^3)_*\left(X_{\theta^x(t_1)}\right)]_\ell,[(\alpha_3^3)_*\left(X_{\theta^x(t_2)}\right)]_\ell,[(\alpha_3^3)_*\left(X_{\theta^x(t_3)}\right)]_\ell,\ldots)\\ &=\int_{\mathbb{R}^3}\Phi^*\overline{\omega}([(\alpha_3^3)_*\left(X_{\theta^x(t_1)}\right)]_\ell,[(\alpha_3^3)_*\left(X_{\theta^x(t_2)}\right)]_\ell,[(\alpha_3^3)_*\left(X_{\theta^x(t_3)}\right)]_\ell,\ldots),\end{aligned} \tag{1.126}$$

where $\left[(\alpha_3^3)_*\left(X_{\theta^x(t_i)}\right)\right]_\ell$ denotes any lift of the pushforward by α_3^3 of the tangent vectors $X_{\theta^x(t_i)}$ to tangent vectors in $C[3+1,\mathbb{R}^3]$.

Then the average asymptotic term for this diagram can be written as

$$\int_{\mathcal{S}}\lambda_{D_{21}}\mu = \int_{\mathcal{S}^3}\lim_{T\to\infty}\frac{1}{T^3}\left\{\int_0^T\cdots\int_0^T f_{D_{21},X}(\theta^{x_1}(t_1),\theta^{x_2}(t_2),\theta^{x_3}(t_3))dt_1\wedge dt_2\wedge dt_3\right\}\mu_\Delta. \quad (1.127)$$

By using Eq. (1.126) in the last expression it is possible to rewrite the term between braces as

$$\int_0^T\cdots\int_0^T\int_{\mathbb{R}^3}\Phi^*\overline{\omega}(\left[(\alpha_3^3)_*\left(X_{\theta^{x_1}(t_1)}\right)\right]_\ell,\left[(\alpha_3^3)_*\left(X_{\theta^{x_2}(t_2)}\right)\right]_\ell,\left[(\alpha_3^3)_*\left(X_{\theta^{x_3}(t_3)}\right)\right]_\ell,\ldots)dt_1\wedge dt_2\wedge dt_3$$
$$=\int_{C(3,\mathbf{S}^1)}\int_{\mathbb{R}^3}(\phi\circ(\theta^3\times id^3_{\mathbb{R}}))^*\overline{\omega}+\widetilde{B}_{21}, \quad (1.128)$$

where $\widetilde{B}_{21}$ stands for the boundary terms.

By using again Eq. (1.78) the integrand in the first term of the right hand side of the last expression takes the form

$$\begin{aligned}
\left(\phi\circ\left(\theta^3\times id_{\mathbb{R}^3}\right)\right)^*\overline{\omega} =& \left(\theta^3\times id_{\mathbb{R}^3}\right)^*\left(\phi_{1,4}^*\omega\wedge\phi_{2,4}^*\omega\wedge\phi_{3,4}^*\omega\right)\\
=&\left[\frac{\varepsilon_{\mu_1\nu_1\sigma_1}}{4\pi}\frac{(\theta^{x_1}(t_1)-x_4)^{\mu_1}}{|\theta^{x_1}(t_1)-x_4|^3}\left(\dot{\theta}^{x_1}\right)^{\nu_1}dt_1\wedge dx_4^{\sigma_1}\right]\\
&\wedge\left[\frac{\varepsilon_{\mu_2\nu_2\sigma_2}}{4\pi}\frac{(\theta^{x_2}(t_2)-x_4)^{\mu_2}}{|\theta^{x_2}(t_2)-x_4|^3}\left(\dot{\theta}^{x_2}\right)^{\nu_2}dt_2\wedge dx_4^{\sigma_2}\right]\\
&\wedge\left[\frac{\varepsilon_{\mu_3\nu_3\sigma_3}}{4\pi}\frac{(\theta^{x_3}(t_3)-x_4)^{\mu_3}}{|\theta^{x_3}(t_3)-x_4|^3}\left(\dot{\theta}^{x_3}\right)^{\nu_3}dt_3\wedge dx_4^{\sigma_3}\right]\\
=&\quad\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-x_4)\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_2)-x_4)\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_3)-x_4)\\
&\times X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\nu_2}_{\theta^{x_2}(t_2)}X^{\nu_3}_{\theta^{x_3}(t_3)}dt_1\wedge dx_4^{\sigma_1}\wedge dt_2\wedge dx_4^{\sigma_2}\wedge dt_3\wedge dx_4^{\sigma_3}\\
=&\quad\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-x_4)\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_2)-x_4)\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_3)-x_4)\\
&\times X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\nu_2}_{\theta^{x_2}(t_2)}X^{\nu_3}_{\theta^{x_3}(t_3)}dt_1\wedge dt_2\wedge dt_3\wedge(-1)^3\varepsilon^{\sigma_1\sigma_2\sigma_3}d^3x_4. \quad (1.129)
\end{aligned}$$

Therefore the average asymptotic integral finally reads

$$\begin{aligned}
\int_{\mathcal{S}}\lambda_{D_{21}}\mu = \int_{\mathcal{S}^3}\lim_{T\to\infty}\frac{1}{T^3}\Bigg\{\int_{C(3,\mathbf{S}^1)}\int_{\mathbb{R}^3}&\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-x_4)\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_2)-x_4)\\
&\times\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_3)-x_4)X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\nu_2}_{\theta^{x_2}(t_2)}X^{\nu_3}_{\theta^{x_3}(t_3)}\\
&\times(-1)^3\varepsilon^{\sigma_1\sigma_2\sigma_3}dt_1\wedge dt_2\wedge dt_3\wedge d^3x_4+\widetilde{B}_{21}\Bigg\}\mu_\Delta. \quad (1.130)
\end{aligned}$$

The average asymptotic second order Vassiliev invariant is then the sum of expressions (1.125) and (1.130) with the corresponding boundary cancellation, as stated at the begining of this section.

1.4.3 Third order flow invariant

In the case of diagrams 1.9a and 1.9b there are inner points and a pushforward process has to be performed for each one.

For diagram 1.9a, the contribution of Eq. (1.60) is given by

$$
\begin{aligned}
& f_{D_{31},X}(\theta^4((x,t_1),\dots,(x,t_4))) \\
&= \left((\alpha_4^3)^*(\overline{\pi}_4)_*\Phi^*\overline{\omega}\right)_{(\theta^4((x,t_1),\dots,(x,t_4)))}\left(X_{\theta^x(t_1)},X_{\theta^x(t_2)},X_{\theta^x(t_3)},X_{\theta^x(t_4)}\right) \\
&= (\alpha_4^3)^*\left(\int\limits_{(\overline{\pi}_4)^{-1}(\theta^4((x,t_1),\dots,(x,t_4)))} \Phi^*\overline{\omega}([X_{\theta^x(t_1)}]_\ell,[X_{\theta^x(t_2)}]_\ell,[X_{\theta^x(t_3)}]_\ell,[X_{\theta^x(t_4)}]_\ell,\dots)\right) \\
&= \int\limits_{(\overline{\pi}_4)^{-1}\left(\alpha_4^3(\theta^4((x,t_1),\dots,(x,t_4)))\right)} \Phi^*\overline{\omega}\left(\left[(\alpha_4^3)_*\left(X_{\theta^x(t_1)}\right)\right]_\ell,\dots,\left[(\alpha_4^3)_*\left(X_{\theta^x(t_4)}\right)\right]_\ell,\dots\right) \\
&= \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \Phi^*\overline{\omega}\left(\left[(\alpha_4^3)_*\left(X_{\theta^x(t_1)}\right)\right]_\ell,\dots,\left[(\alpha_4^3)_*\left(X_{\theta^x(t_4)}\right)\right]_\ell,\dots\right),
\end{aligned}
\tag{1.131}
$$

and then, in the average asymptotic integral this takes the form

$$
\begin{aligned}
\int_{\mathcal{S}}\lambda_{D_{31}}\mu &= \int_{\mathcal{S}^4}\lim_{T\to\infty}\frac{1}{T^4}\left\{\int_0^T\cdots\int_0^T f_{D_{31},X}(\theta^{x_1}(t_1),\dots,\theta^{x_4}(t_4))dt_1\dots\wedge dt_4\right\}\mu_\Delta \\
&= \int_{\mathcal{S}^4}\lim_{T\to\infty}\frac{1}{T^4}\left\{\int_0^T\cdots\int_0^T\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\Phi^*\overline{\omega}\left(\left[(\alpha_4^3)_*\left(X_{\theta^{x_1}(t_1)}\right)\right]_\ell,\dots,\right.\right. \\
&\qquad \left.\left.\times\left[(\alpha_4^3)_*\left(X_{\theta^{x_4}(t_4)}\right)\right]_\ell,\dots\right)dt_1\wedge\dots\wedge dt_4\right\}\mu_\Delta,
\end{aligned}
\tag{1.132}
$$

where once again the integrand form can be separated into its inner and its boundary parts, the former one is given by

$$
\begin{aligned}
& \phi^*\overline{\omega}\left(\left[\left(\alpha_4^3\right)_*\left(X_{\theta^{x_1}(t_1)}\right)\right]_\ell,\dots,\left[\left(\alpha_4^3\right)_*\left(X_{\theta^{x_4}(t_4)}\right)\right]_\ell,\dots\right) \\
&= (\phi_{1,6}^*\omega\wedge\phi_{2,5}^*\omega\wedge\phi_{3,5}^*\omega\wedge\phi_{4,6}^*\omega\wedge\phi_{5,6}^*\omega)\left(\left[\left(\alpha_4^3\right)_*\left(X_{\theta^{x_1}(t_1)}\right)\right]_\ell,\dots,\left[\left(\alpha_4^3\right)_*\left(X_{\theta^{x_4}(t_4)}\right)\right]_\ell,\dots\right) \\
&= \left[\frac{\varepsilon_{\mu_1\nu_1\sigma_1}}{4\pi}\frac{(\theta^{x_1}(t_1)-x_6)^{\mu_1}}{|\theta^{x_1}(t_1)-x_6|^3}\left(\dot{\theta}^{x_1}\right)^{\nu_1}dx_6^{\sigma_1}\right]\wedge\left[\frac{\varepsilon_{\mu_2\nu_2\sigma_2}}{4\pi}\frac{(\theta^{x_2}(t_2)-x_5)^{\mu_2}}{|\theta^{x_2}(t_2)-x_5|^3}\left(\dot{\theta}^{x_2}\right)^{\nu_2}dx_5^{\sigma_2}\right] \\
&\quad\wedge\left[\frac{\varepsilon_{\mu_3\nu_3\sigma_3}}{4\pi}\frac{(\theta^{x_3}(t_3)-x_5)^{\mu_3}}{|\theta^{x_3}(t_3)-x_5|^3}\left(\dot{\theta}^{x_3}\right)^{\nu_3}dx_5^{\sigma_3}\right]\wedge\left[\frac{\varepsilon_{\mu_4\nu_4\sigma_4}}{4\pi}\frac{(\theta^{x_4}(t_4)-x_6)^{\mu_4}}{|\theta^{x_4}(t_4)-x_6|^3}\left(\dot{\theta}^{x_4}\right)^{\nu_4}dx_6^{\sigma_4}\right] \\
&\quad\wedge\left[\frac{\varepsilon_{\mu_5\nu_5\sigma_5}}{4\pi}\frac{(x_5-x_6)^{\mu_5}}{|x_5-x_6|^3}dx_5^{\nu_5}\wedge dx_6^{\sigma_5}\right] \\
&= \Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-x_6)\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_2)-x_5)\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_3)-x_5)\Delta_{\nu_4\sigma_4}(\theta^{x_4}(t_4)-x_6) \\
&\quad\times\Delta_{\nu_5\sigma_5}(x_5-x_6)X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\nu_2}_{\theta^{x_2}(t_2)}X^{\nu_3}_{\theta^{x_3}(t_3)}X^{\nu_4}_{\theta^{x_4}(t_4)} \\
&\quad\times dx_6^{\sigma_1}\wedge dx_5^{\sigma_2}\wedge dx_5^{\sigma_3}\wedge dx_6^{\sigma_4}\wedge dx_5^{\nu_5}\wedge dx_6^{\sigma_5}
\end{aligned}
$$

$$
\begin{aligned}
=&\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-x_6)\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_2)-x_5)\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_3)-x_5)\Delta_{\nu_4\sigma_4}(\theta^{x_4}(t_4)-x_6)\\
&\times \Delta_{\nu_5,\sigma_5}(x_5-x_6)X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\nu_2}_{\theta^{x_2}(t_2)}X^{\nu_3}_{\theta^{x_3}(t_3)}X^{\nu_4}_{\theta^{x_4}(t_4)}\\
&\times \varepsilon^{\sigma_2\sigma_3\nu_5}d^3x_5\wedge\varepsilon^{\sigma_1\sigma_4\sigma_5}d^3x_6. \qquad (1.133)
\end{aligned}
$$

From Eq. (1.65), the average asymptotic integral reads

$$
\begin{aligned}
\int_{\mathcal{S}}\lambda_{D_{31}}\mu=\int_{\mathcal{S}^4}\lim_{T\to\infty}\frac{1}{T^4}\Bigg\{&\int_{C(4,\mathbf{S}^1)}dt_1\wedge\ldots\wedge dt_4\int_{\mathbb{R}^3\times\mathbb{R}^3}\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-x_6)\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_2)-x_5)\\
&\times\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_3)-x_5)\Delta_{\nu_4\sigma_4}(\theta^{x_4}(t_4)-x_6)\Delta_{\nu_5\sigma_5}(x_5-x_6)\\
&\times X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\nu_2}_{\theta^{x_2}(t_2)}X^{\nu_3}_{\theta^{x_3}(t_3)}X^{\nu_4}_{\theta^{x_4}(t_4)}\varepsilon^{\sigma_2\sigma_3\nu_5}d^3x_5\wedge\varepsilon^{\sigma_1\sigma_4\sigma_5}d^3x_6\\
&+\widetilde{B}_{31}\Bigg\}\mu_\Delta. \qquad (1.134)
\end{aligned}
$$

Now we proceed to study diagram 1.9b in a similar way as in the previous case. We have

$$
\begin{aligned}
&f_{D_{32},X}(\theta^5((x,t_1),\ldots,(x,t_5)))\\
=&\left((\alpha_5^3)^*(\overline{\pi}_5)_*\Phi^*\overline{\omega}\right)_{(\theta^5((x,t_1),\ldots,(x,t_5)))}\left(X_{\theta^x(t_1)},X_{\theta^x(t_2)},X_{\theta^x(t_3)},X_{\theta^x(t_4)},X_{\theta^x(t_5)}\right)\\
=&(\alpha_5^3)^*\left(\int_{(\overline{\pi}_5)^{-1}(\theta^5((x,t_1),\ldots,(x,t_5)))}\Phi^*\overline{\omega}(\left[X_{\theta^x(t_1)}\right]_\ell,\left[X_{\theta^x(t_2)}\right]_\ell,\left[X_{\theta^x(t_3)}\right]_\ell,\left[X_{\theta^x(t_4)}\right]_\ell,\left[X_{\theta^x(t_5)}\right]_\ell,\ldots)\right)\\
=&\int_{(\overline{\pi}_5)^{-1}(\alpha_5^3(\theta^5(\overline{x},(t_1,\ldots,t_5))))}\Phi^*\overline{\omega}\left(\left[(\alpha_5^3)_*\left(X_{\theta^x(t_1)}\right)\right]_\ell,\ldots,\left[(\alpha_5^3)_*\left(X_{\theta^x(t_5)}\right)\right]_\ell,\ldots\right)\\
=&\int_{\mathbb{R}^3}\Phi^*\overline{\omega}\left(\left[(\alpha_5^3)_*\left(X_{\theta^x(t_1)}\right)\right]_\ell,\ldots,\left[(\alpha_5^3)_*\left(X_{\theta^x(t_5)}\right)\right]_\ell,\ldots\right), \qquad (1.135)
\end{aligned}
$$

and then the average asymptotic integral reads

$$
\begin{aligned}
\int_{\mathcal{S}}\lambda_{D_{32}}\mu&=\int_{\mathcal{S}^5}\lim_{T\to\infty}\frac{1}{T^5}\left\{\int_0^T\cdots\int_0^T f_{D_{32},X}(\theta^{x_1}(t_1),\ldots,\theta^{x_5}(t_5))dt_1\cdots\wedge dt_5\right\}\mu_\Delta\\
&=\int_{\mathcal{S}^4}\lim_{T\to\infty}\frac{1}{T^4}\Bigg\{\int_0^T\cdots\int_0^T\int_{\mathbb{R}^3}\Phi^*\overline{\omega}\left(\left[(\alpha_5^3)_*\left(X_{\theta^{x_1}(t_1)}\right)\right]_\ell,\ldots,\left[(\alpha_5^3)_*\left(X_{\theta^{x_5}(t_5)}\right)\right]_\ell,\ldots\right)\\
&\qquad\times dt_1\wedge\cdots\wedge dt_5\Bigg\}\mu_\Delta. \qquad (1.136)
\end{aligned}
$$

The inner part in the integrand can be evaluated and it yields

$$
\begin{aligned}
&\phi^*\overline{\omega}\left(\left[\left(\alpha_5^3\right)_*\left(X_{\theta^{x_1}(t_1)}\right)\right]_\ell,\ldots\left[(\alpha_5^3)_*\left(X_{\theta^{x_5}(t_5)}\right)\right]_\ell,\ldots\right)\\
=&(\phi_{1,6}^*\omega\wedge\phi_{2,5}^*\omega\wedge\phi_{3,6}^*\omega\wedge\phi_{4,6}^*\omega)\left(\left[\left(\alpha_5^3\right)_*\left(X_{\theta^{x_1}(t_1)}\right),\ldots,\left[(\alpha_5^3)_*\left(X_{\theta^{x_5}(t_5)}\right)\right]_\ell,\ldots\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
=&\left[\frac{\varepsilon_{\mu_1\nu_1\sigma_1}}{4\pi}\frac{(\theta^{x_1}(t_1)-x_6)^{\mu_1}}{|\theta^{x_1}(t_1)-x_6|^3}\left(\dot\theta^{x_1}\right)^{\nu_1}dx_6^{\sigma_1}\right]\wedge\left[\frac{\varepsilon_{\mu_2\nu_2\sigma_2}}{4\pi}\frac{(\theta^{x_2}(t_2)-\theta^{x_5}(t_3))^{\mu_2}}{|\theta^{x_2}(t_2)-\theta^{x_5}(t_3)|^3}\left(\dot\theta^{x_2}\right)^{\nu_2}\left(\dot\theta^{x_5}\right)^{\sigma_2}\right]\\
&\wedge\left[\frac{\varepsilon_{\mu_3\nu_3\sigma_3}}{4\pi}\frac{(\theta^{x_3}(t_4)-x_6)^{\mu_3}}{|\theta^{x_3}(t_4)-x_6|^3}\left(\dot\theta^{x_3}\right)^{\nu_3}dx_6^{\sigma_3}\right]\wedge\left[\frac{\varepsilon_{\mu_4\nu_4\sigma_4}}{4\pi}\frac{(\theta^{x_4}(t_5)-x_6)^{\mu_4}}{|\theta^{x_4}(t_5)-x_6|^3}\left(\dot\theta^{x_6}\right)^{\nu_4}dx_6^{\sigma_4}\right]\\
=&\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-x_6)\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_2)-\theta^{x_5}(t_3))\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_4)-x_6)\Delta_{\nu_4\sigma_4}(\theta^{x_4}(t_5)-x_6)\\
&\times X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\nu_2}_{\theta^{x_2}(t_2)}X^{\sigma_2}_{\theta^{x_5}(t_3)}X^{\nu_3}_{\theta^{x_3}(t_4)}X^{\nu_4}_{\theta^{x_4}(t_5)}dx_6^{\sigma_1}\wedge dx_6^{\sigma_3}\wedge dx_6^{\sigma_4}\\
=&\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-x_6)\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_2)-\theta^{x_5}(t_3))\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_4)-x_6)\Delta_{\nu_4\sigma_4}(\theta^{x_4}(t_5)-x_6)\\
&\times X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\nu_2}_{\theta^{x_2}(t_2)}X^{\sigma_2}_{\theta^{x_5}(t_5)}X^{\nu_3}_{\theta^{x_3}(t_3)}X^{\nu_4}_{\theta^{x_4}(t_4)}\varepsilon^{\sigma_1\sigma_3\sigma_4}d^3x_6.
\end{aligned}\tag{1.137}
$$

From Eq. (1.65) again, the average asymptotic integral reads

$$
\begin{aligned}
\int_{\mathcal S}\lambda_{D_{32}}\mu=\int_{\mathcal S^5}\lim_{T\to\infty}\frac{1}{T^5}\Bigg\{&\int_{C(5,\mathbf S^1)}dt_1\wedge\ldots\wedge dt_5\int_{\mathbb R^3}\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-x_6)\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_2)-\theta^{x_5}(t_3))\\
&\times\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_4)-x_6)\Delta_{\nu_4\sigma_4}(\theta^{x_4}(t_5)-x_6)X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\nu_2}_{\theta^{x_2}(t_2)}X^{\sigma_2}_{\theta^{x_5}(t_3)}\\
&\times X^{\nu_3}_{\theta^{x_3}(t_4)}X^{\nu_4}_{\theta^{x_4}(t_5)}\varepsilon^{\sigma_1\sigma_3\sigma_4}d^3x_6+\widetilde B_{32}\Bigg\}\mu_\Delta,
\end{aligned}\tag{1.138}
$$

where $\widetilde B_{32}$ takes into account all the boundary terms.

For diagrams 1.9c and 1.9d that just have points on the knot the expressions for Eq. (1.65) (use the notation $(X_{\theta^{x_i}(t_i)})_{i=1}^n=(X_{\theta^{x_1}(t_1)},\ldots,X_{\theta^{x_n}(t_n)})$ and consider $\overline\pi_p=\mathrm{Id}$) are the same, namely,

$$
\begin{aligned}
f_{D_{33},X}(\theta^6((x_1,t_1),\ldots,(x_6,t_6)))&=\left((\alpha_6^3)^*(\overline\pi_6)_*\Phi^*\overline\omega\right)_{(\theta^6((x_1,t_1),\ldots,(x_6,t_6)))}\left((X_{\theta^{x_i}(t_i)})_{i=1}^6\right)\\
&=(\overline\pi_6)_*\,\Phi^*\overline\omega_{\alpha_6^3(\theta^6((x_1,t_1),\ldots,(x_6,t_6)))}\left(((\alpha_6^3)_*X_{\theta^{x_i}(t_i)})_{i=1}^6\right)\\
&=\phi^*\overline\omega_{\theta^6((x_1,t_1),\ldots,(x_6,t_6))}\left((X_{\theta^{x_i}(t_i)})_{i=1}^6\right)+\widetilde B_{33}\\
&=(\phi_{1,4}^*\omega\wedge\phi_{2,6}^*\omega\wedge\phi_{3,5}^*\omega)\left((X_{\theta^{x_i}(t_i)})_{i=1}^6\right)+\widetilde B_{33},
\end{aligned}\tag{1.139}
$$

and

$$
\begin{aligned}
f_{D_{34},X}(\theta^6((x_1,t_1),\ldots,(x_6,t_6)))&=\left((\alpha_6^3)^*(\overline\pi_6)_*\Phi^*\overline\omega\right)_{(\theta^6((x_1,t_1),\ldots,(x_6,t_6)))}\left((X_{\theta^{x_i}(t_i)})_{i=1}^6\right)\\
&=(\overline\pi_6)_*\,\Phi^*\overline\omega_{\alpha_6^3(\theta^6((x_1,t_1),\ldots,(x_6,t_6)))}\left(((\alpha_6^3)_*X_{\theta^{x_i}(t_i)})_{i=1}^6\right)\\
&=\phi^*\overline\omega_{\theta^6((x_1,t_1),\ldots,(x_6,t_6))}\left((X_{\theta^{x_i}(t_i)})_{i=1}^6\right)+\widetilde B_{34}\\
&=(\phi_{1,4}^*\omega\wedge\phi_{2,5}^*\omega\wedge\phi_{3,6}^*\omega)\left((X_{\theta^{x_i}(t_i)})_{i=1}^6\right)+\widetilde B_{34},
\end{aligned}\tag{1.140}
$$

where $\widetilde B_{33}$ and $\widetilde B_{34}$ are the corresponding contributions of the boundary terms.

After evaluating $f_{D_{33}}$ and $f_{D_{34}}$ in the 6-fold integral in Eq. (1.63), $\phi^*\overline\omega_{\theta^6(\bar t)}\left((X_{\theta^{x_i}(t_i)})_{i=1}^6\right)$ is respectively given by

$$
\begin{aligned}
&(\phi_{1,4}^*\omega\wedge\phi_{2,6}^*\omega\wedge\phi_{3,5}^*\omega)\left(X_{\theta^{x_1}(t_1)},\ldots,X_{\theta^{x_6}(t_6)}\right)\\
=&\left[\frac{\varepsilon_{\mu_1\nu_1\sigma_1}}{4\pi}\frac{(\theta^{x_1}(t_1)-\theta^{x_4}(t_2))^{\mu_1}}{|\theta^{x_1}(t_1)-\theta^{x_4}(t_2)|^3}\left(\dot\theta^{x_1}\right)^{\nu_1}\left(\dot\theta^{x_4}\right)^{\sigma_1}\right]\left[\frac{\varepsilon_{\mu_2\nu_2\sigma_2}}{4\pi}\frac{(\theta^{x_2}(t_3)-\theta^{x_6}(t_4))^{\mu_2}}{|\theta^{x_2}(t_3)-\theta^{x_6}(t_4)|^3}\left(\dot\theta^{x_2}\right)^{\nu_2}\left(\dot\theta^{x_6}\right)^{\sigma_2}\right]
\end{aligned}
$$

$$\times\left[\frac{\varepsilon_{\mu_3\nu_3\sigma_3}}{4\pi}\frac{(\theta^{x_3}(t_5)-\theta^{x_5}(t_6))^{\mu_3}}{|\theta^{x_3}(t_5)-\theta^{x_5}(t_6)|^3}\left(\dot\theta^{x_3}\right)^{\nu_3}\left(\dot\theta^{x_5}\right)^{\sigma_3}\right]$$
$$=\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-\theta^{x_4}(t_2))\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_3)-\theta^{x_6}(t_4))\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_5)-\theta^{x_5}(t_6))$$
$$\times X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\sigma_1}_{\theta^{x_4}(t_2)}X^{\nu_2}_{\theta^{x_2}(t_3)}X^{\sigma_2}_{\theta^{x_6}(t_4)}X^{\nu_3}_{\theta^{x_3}(t_5)}X^{\sigma_3}_{\theta^{x_5}(t_6)}, \tag{1.141}$$

and

$$\left(\phi^*_{1,4}\omega\wedge\phi^*_{2,5}\omega\wedge\phi^*_{3,6}\omega\right)\left(X_{\theta^{x_1}(t_1)},\dots,X_{\theta^{x_6}(t_6)}\right)$$
$$=\left[\frac{\varepsilon_{\mu_1\nu_1\sigma_1}}{4\pi}\frac{(\theta^{x_1}(t_1)-\theta^{x_4}(t_2))^{\mu_1}}{|\theta^{x_1}(t_1)-\theta^{x_4}(t_2)|^3}\left(\dot\theta^{x_1}\right)^{\nu_1}\left(\dot\theta^{x_4}\right)^{\sigma_1}\right]\left[\frac{\varepsilon_{\mu_2\nu_2\sigma_2}}{4\pi}\frac{(\theta^{x_2}(t_3)-\theta^{x_5}(t_4))^{\mu_2}}{|\theta^{x_2}(t_3)-\theta^{x_5}(t_4)|^3}\left(\dot\theta^{x_2}\right)^{\nu_2}\left(\dot\theta^{x_5}\right)^{\sigma_2}\right]$$
$$\times\left[\frac{\varepsilon_{\mu_3\nu_3\sigma_3}}{4\pi}\frac{(\theta^{x_3}(t_5)-\theta^{x_6}(t_6))^{\mu_3}}{|\theta^{x_3}(t_5)-\theta^{x_6}(t_6)|^3}\left(\dot\theta^{x_3}\right)^{\nu_3}\left(\dot\theta^{x_6}\right)^{\sigma_3}\right]$$
$$=\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-\theta^{x_4}(t_2))\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_3)-\theta^{x_5}(t_4))\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_5)-\theta^{x_6}(t_6))$$
$$\times X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\sigma_1}_{\theta^{x_4}(t_2)}X^{\nu_2}_{\theta^{x_2}(t_3)}X^{\sigma_2}_{\theta^{x_5}(t_4)}X^{\nu_3}_{\theta^{x_3}(t_5)}X^{\sigma_3}_{\theta^{x_6}(t_6)}. \tag{1.142}$$

Thus, the average asymptotic integrals are then given as

$$\int_{\mathcal{S}}\lambda_{D_{33}}\mu=\int_{\mathcal{S}^6}\lim_{T\to\infty}\frac{1}{T^6}\Bigg\{\int_{C(6,\mathbf{S}^1)}dt_1\wedge\dots\wedge dt_6\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-\theta^{x_4}(t_2))\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_3)-\theta^{x_6}(t_4))$$
$$\times\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_5)-\theta^{x_5}(t_6))X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\sigma_1}_{\theta^{x_4}(t_2)}X^{\nu_2}_{\theta^{x_2}(t_3)}X^{\sigma_2}_{\theta^{x_6}(t_4)}X^{\nu_3}_{\theta^{x_3}(t_5)}X^{\sigma_3}_{\theta^{x_5}(t_6)}+\widetilde{B}_{33}\Bigg\}\mu_\Delta, \tag{1.143}$$

and

$$\int_{\mathcal{S}}\lambda_{D_{34}}\mu=\int_{\mathcal{S}^6}\lim_{T\to\infty}\frac{1}{T^6}\Bigg\{\int_{C(6,\mathbf{S}^1)}dt_1\wedge\dots\wedge dt_5\Delta_{\nu_1\sigma_1}(\theta^{x_1}(t_1)-\theta^{x_4}(t_2))\Delta_{\nu_2\sigma_2}(\theta^{x_2}(t_3)-\theta^{x_5}(t_4))$$
$$\times\Delta_{\nu_3\sigma_3}(\theta^{x_3}(t_5)-\theta^{x_6}(t_6))X^{\nu_1}_{\theta^{x_1}(t_1)}X^{\sigma_1}_{\theta^{x_4}(t_2)}X^{\nu_2}_{\theta^{x_2}(t_3)}X^{\sigma_2}_{\theta^{x_5}(t_4)}X^{\nu_3}_{\theta^{x_3}(t_5)}X^{\sigma_3}_{\theta^{x_6}(t_6)}+\widetilde{B}_{34}\Bigg\}\mu_\Delta, \tag{1.144}$$

where as stated at the begining of the section boundary cancellations lead to the average asymptotic third order Vassiliev invariant as the sum of (1.134), (1.138), (1.143) and (1.144).

CHAPTER 2

Star-triangle type relations from $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ dualities

This chapter concerns the research article "Star-triangle type relations from $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ dualities" (see reference [43]) done in collaboration with my thesis supervisor and accepted in Journal of High Energy Physics on November 2020. Inspired by the gauge/YBE correspondence the authors derived star-triangle type relations for certain $2d$ $\mathcal{N} = (0,2)$ supersymmetric quiver gauge theory dualities. The interest of the authors in this topic comes from the fact that the correspondence has been done for different dimensions, amounts of sypersymmetry, gauge groups and manifolds, but as far as we know there is no explicit integrable model associated with $2d$ $\mathcal{N} = (0,2)$ theories despite a triality has been found for them (there is a speculation that triality corresponds to tetrahedron equation in a similar way Seiberg duality is related with Yang-Baxter equation). As a first step in this direction we concentrate in studying what kind of star-triangle relations (or their variants) can be associated to some of the dualities obeyed for these supersymmetric quiver gauge theories. To be precise, we analyse two cases. The first case is $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ Intriligator-Pouliot duality coming from dimensional reduction of $4d$ $\mathcal{N} = 1$ $USp(2N)$ Intriligator-Pouliot duality. The description is performed explicitly for $N = 1,2,3,4,5$ and for $N = 3k+2$, which generalizes the situation in $N = 2,5$. A triangle identity is obtained for $N = 1$ while for $N = 2,5$ it is found that the realization of duality implies slight variations of a star-triangle relation type (STR type). The values $N = 3,4$ are shown to be associated to a similar version of the asymmetric STR. The second case is a duality for $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ theories with matter in the antisymmetric tensor representation that comes from dimensional reduction of $4d$ $\mathcal{N} = 1$ $USp(2N)$ Csáki-Skiba-Schmaltz duality, and the conclusion is that this duality is associated to a triangle type identity for any value of N. For both cases the precise determination of Boltzmann weights, interaction and normalization factors was done. The obtained expressions were also compared with those already present in the literature.

2.1 Overview of gauge/YBE correspondence

In the present section we provide a brief overview of the gauge/YBE correspondence. Our aim will not intend to be exhaustive but only to introduce the notation and conventions that will be useful in the subsequent sections.

As stated in [50], an integrable model is considered to be a solution of the Yang-Baxter equation with spectral parameters that satisfies the rapidity difference property in their R-matrices

$$R_{23}(z_2 - z_3)R_{13}(z_1 - z_3)R_{12}(z_1 - z_2) = R_{12}(z_1 - z_2)R_{13}(z_1 - z_3)R_{23}(z_2 - z_3), \tag{2.1}$$

where z_1, z_2 and z_3 are the spectral parameters and

$$R_{ij} \in \mathrm{End}(V_i \otimes V_j) \tag{2.2}$$

for all $i,j \in \{1,2,3\}$ with $i \neq j$. Note that in equation (2.1) operators (2.2) are actually promoted to operators in $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ by an adequate insertion of an identity.

One of the best known integrable models is the $2d$ Ising model in statistical mechanics which is part of the Ising-type integrable models that can be obtained from the YBE depending on the values taken by the spin variables, which can be discrete, continuous or a combination of both of them. There are two relations from statistical mechanics, known as star-star relation and star-triangle relation (SSR and STR from now on, respectively), such that a solution of one of them is immediately a solution of the YBE. In constructing integrable models it is preferable to solve one of those relations instead because of the highly constrained nature of the YBE.

The so called gauge/YBE correspondence is given between supersymmetric quiver gauge theories and integrable models in statistical mechanics. It is then necessary to roughly describe such theories and their relation with statistical mechanics.

2.1.1 Construction of the correspondence

The review of this subsection is carried out mainly following Ref. [50]. For a quiver gauge theory in dimension d with gauge group G, let V and E be the sets of all vertices and edges in its associated quiver diagram, respectively. Each vertex v contains gauge fields[1] $A_v^\mu(x)$ with values in the associated Lie algebra of the gauge group G_v while each edge e from v' to v'' contains a matter field ϕ_e transforming in the bifundamental representation $(\square,\overline{\square})$ of $G_{v'} \times G_{v''}$. The partition function for the quiver gauge theory with gauge group G is given by the partition function which is the product of partition functions for all possible vertices and edges

$$\widetilde{\mathcal{Z}} = \int \prod_{v\in V} DA_v \prod_{e\in E} D\phi_e \; e^{-\widetilde{\mathcal{L}}\left(\{A_v\}_{v\in V},\{\phi_e\}_{e\in E}\right)}, \tag{2.3}$$

where

$$\widetilde{\mathcal{L}}\left(\{A_v\}_{v\in V},\{\phi_e\}_{e\in E}\right) = \sum_{v\in V} \widetilde{\mathcal{L}}^v\left(A_v\right) + \sum_{e\in E} \widetilde{\mathcal{L}}^e\left(\{A_v\}_{v\in e},\phi_e\right), \tag{2.4}$$

with $\widetilde{\mathcal{L}}^v\left(A_v\right)$ the kinetic term for A_v and $\widetilde{\mathcal{L}}^e\left(\{A_v\}_{v\in e},\phi_e\right)$ the interaction term between ϕ_e and the gauge fields $\{A_v\}_{v\in e}$. To obtain a supersymmetric quiver gauge theory one must supersymmetrize the theory described in (2.3). After this procedure is applied to $4d$ $\mathcal{N}=1$ Yang-Mills theory, each vertex v has now associated a $\mathcal{N}=1$ vector multiplet[2] $\mathcal{V}_v = (A_v,\lambda_v,F_v)$ where λ_v and F_v are a gaugino and an auxiliary field, respectively, while each edge e from v' to v'' has now a $\mathcal{N}=1$ chiral multiplet[3] $\Phi_e = (\phi_e,\psi_e,H_e)$ where ψ_e and H_e are a fermion and an

[1] Here, $\mu = 1,\ldots,d$ and x is a point in the d-dimensional space. From now on it will be written as A_v.
[2] Each field is in the adjoint representation of G_v.
[3] The multiplet is in a non-trivial representation of $G_{v'} \times G_{v''}$.

auxiliary field, respectively. The supersymmetric theory has partition function

$$\mathcal{Z} = \int \prod_{v\in V} DA_v D\lambda_v DF_v \prod_{e\in E} D\phi_e D\psi_e DH_e \; e^{-\mathcal{L}\left(\{\mathcal{V}_v\}_{v\in V},\, \{\Phi_e\}_{e\in E}\right)}. \tag{2.5}$$

According to [50], after regularization by integration in a compact manifold M this partition function can be reduced to

$$\mathcal{Z}[M] = \sum\!\!\!\!\!\!\int \prod_{v\in V} d\sigma_v \; e^{-\mathcal{L}\left(\{\sigma_v\}_{v\in V}\right)}, \tag{2.6}$$

where

$$\mathcal{L}\left(\{\sigma_v\}_{v\in V}\right) = \sum_{v\in V} \mathcal{L}^v\left(\sigma_v\right) + \sum_{e\in E} \mathcal{L}^e\left(\{\sigma_v\}_{v\in e}\right), \tag{2.7}$$

with $\{\sigma_v\}_{v\in V}$ a set of finite-dimensional variables associated with holonomies of A_v along non-trivial homology cycles of M, the choice of this manifold can make the variables either continuous or both continuous and discrete. Equations (2.6) and (2.7) nicely match with statistical mechanics because in a typical statistical lattice the vertices contain spin variables s_v so that the partition function is given by

$$\mathcal{Z} = \sum_{\{s_v\}_{v\in V}} e^{-\xi\left(\{s_v\}_{v\in V}\right)}, \tag{2.8}$$

where

$$\xi\left(\{s_v\}_{v\in V}\right) = \sum_{v\in V} \xi^v\left(s_v\right) + \sum_{e\in E} \xi^e\left(\{s_v\}_{v\in e}\right), \tag{2.9}$$

with $\xi^v\left(s_v\right)$ the self-interaction term at vertex v and $\xi^e\left(\{s_v\}_{v\in e}\right)$ the nearest-neighbour interaction of the spins. Comparison of Eqs. (2.6) and (2.7) with (2.8) and (2.9), respectively, provides a deep connection between supersymmetric quiver gauge theories and statistical mechanical theories, and this is an important point of the gauge/YBE correspondence.

Until now the description has made manifest the correspondence between quiver diagram and statistical lattice, supersymmetric quiver gauge partition function and statistical partition function, vector multiplet in the adjoint representation and self-interaction term, chiral multiplet in the bifundamental representation and nearest-neighbour interaction, and holonomies of gauge fields and spin variables; but the relation between these sets of theories is actually deeper. Dualities in supersymmetric gauge theories play a very important role [50]. In particular, we will see in the following subsection that Seiberg-like duality is related to the star-star relation.

2.1.2 From Seiberg-like duality to star-star relation

Original Seiberg duality [75] is a strong/weak (or S) duality between two $4d$ $\mathcal{N} = 1$ gauge theories in the infrared regime, one with gauge group $SU(N_c)$ and the other one with gauge group $SU(N_f - N_c)$, where N_c and N_f are the number of colors and flavours. For example, for an original theory with $SU(2)$ gauge group and $SU(6)$ flavour group (this means 6 flavours or chiral multiplets transforming in the fundamental representation of both the gauge and the flavour groups, and the vector multiplets transforming in the adjoint representation of the gauge group), the dual theory is that with gauge group $SU(4)$, 15 chiral multiplets in the totally antisymmetric tensor representation of the flavour group and without gauge degrees of freedom. There are several generalizations of this duality depending on the dimension of the theory and the amount of supersymmetry. In the rest of this section we will consider Seiberg-like duality in two dimensions.

As stated at the beginning of this section, a solution of the SSR is also a solution of the YBE. This means that the correspondence between Seiberg-like duality and SSR can be used to build and to study integrable models from the point of view of supersymmetric quiver gauge theories. One way for constructing integrable models is to find the correspondence between supersymmetric indices (*a.k.a.* flavoured elliptic genera) of Seiberg-like dual theories and then directly compare them with SSR or STR expressions in order to find the associated Boltzmann weights.

In Ref. [52] an integrable model is derived from Seiberg-like duality of $2d$ $\mathcal{N}=(2,2)$ supersymmetric quiver gauge theories on $\mathbb{T}^2$. This model is shown to be a dimensional reduction of $4d$ $\mathcal{N}=1$ supersymmetric quiver gauge theories on $\mathbb{T}^2\times\mathbb{S}^2$ [76]. Gauge theories $2d$ $\mathcal{N}=(2,2)$ are described in [77, 78] as dimensional reduction of $4d$ $\mathcal{N}=1$ theories. The spectrum of $(2,2)$ theories in two dimensions consists of two different multiplets, namely, the chiral multiplet with fermions ψ_+ and ψ_- of opposite chirality and a complex scalar ϕ, and the vector multiplet V containing Majorana fermions λ_+ and λ_-, a complex scalar σ and gauge bosons $\{v_\alpha\}_{\alpha=0,1}$. The analysis of the index (flavoured elliptic genus in the NS-NS sector) of $2d$ $\mathcal{N}=(2,2)$ supersymmetric gauge theories is carried out in [79, 80]. Moreover, the contribution due to chiral and vector multiplets is given in terms of Jacobi theta functions, $\theta(y;q)$. The index duality of these $2d$ $\mathcal{N}=(2,2)$ theories is given as follows [52]

$$\frac{1}{2}\left(\frac{(q,q)_\infty^2}{\theta(y;q)}\right)\int\frac{dz}{2\pi iz}\left[\frac{\prod_{i=1}^6\Delta(a_iz^{\pm1};q,y)}{\Delta(z^{\pm2};q,y)}\right]=\prod_{1\le i<j\le6}\Delta(a_ia_j;q,y), \tag{2.10}$$

where

$$\Delta(a;q,y)=\frac{\theta(ay;q)}{\theta(a;q)}, \tag{2.11}$$

here, the left hand side consists of a theory with gauge group $SU(2)$ and flavour group $SU(6)$ while the right hand one is a theory with only 15 chiral multiplets and no gauge symmetry (this is actually the reason why there is no integration in the right hand side of (2.10), it appears only when gauge symmetry is present). Note that the field content is essentially the same as in the $4d$ $\mathcal{N}=1$ Seiberg duality [4]. This $2d$ $\mathcal{N}=(2,2)$ duality corresponds in the statistical mechanical side to STR for continuous spin variables [52] (see Appendix B.1 for the explicit derivation)

$$\int d\sigma S(\sigma)W_{\eta-\gamma}(\sigma,\sigma_i)W_{\eta-\beta}(\sigma,\sigma_j)W_{\eta-\alpha}(\sigma,\sigma_k)=R(\alpha,\beta,\gamma)W_\alpha(\sigma_i,\sigma_j)W_\beta(\sigma_i,\sigma_k)W_\gamma(\sigma_j,\sigma_k), \tag{2.12}$$

[4]The superpotential is important to determine the R-charge of the theory. However the determination of the Boltzmann weights does not require it and one just focuses on the field content.

where $S(\sigma)$ and $R(\alpha,\beta,\gamma)$ stand for the interaction and normalization factors, respectively, and they are given by

$$S(\sigma)=\frac{1}{4\pi}\left(\frac{(q,q)_\infty^2}{\theta(y;q)}\right)\frac{\theta\left(e^{\pm 2i\sigma;q}\right)}{\theta\left(e^{-2\eta\pm 2i\sigma;q}\right)},$$

$$R(\alpha,\beta,\gamma)=\frac{\theta\left(e^{-2\alpha-2\gamma};q\right)}{\theta\left(e^{-2\beta};q\right)}\frac{\theta\left(e^{-2\beta-2\gamma};q\right)}{\theta\left(e^{-2\alpha};q\right)}\frac{\theta\left(e^{-2\alpha-2\beta};q\right)}{\theta\left(e^{-2\gamma};q\right)},\tag{2.13}$$

while the associated Boltzmann weights are defined by

$$\begin{aligned}
W_\alpha(\sigma_j,\sigma_k)&=\frac{\theta\left(e^{-\alpha-\eta\mp i(\sigma_j\pm\sigma_k)};q\right)}{\theta\left(e^{\alpha-\eta\pm i(\sigma_j\pm\sigma_k)};q\right)}, &\quad W_{\eta-\alpha}(\sigma_i,\sigma)&=\frac{\theta\left(e^{-(\eta-\alpha)-\eta\mp i(\sigma_i\pm\sigma)};q\right)}{\theta\left(e^{(\eta-\alpha)-\eta\pm i(\sigma_i\pm\sigma)};q\right)},\\
W_\beta(\sigma_i,\sigma_k)&=\frac{\theta\left(e^{-\beta-\eta\mp i(\sigma_i\pm\sigma_k)};q\right)}{\theta\left(e^{\beta-\eta\pm i(\sigma_i\pm\sigma_k)};q\right)}, &\quad W_{\eta-\beta}(\sigma_j,\sigma)&=\frac{\theta\left(e^{-(\eta-\beta)-\eta\mp i(\sigma_j\pm\sigma)};q\right)}{\theta\left(e^{(\eta-\beta)-\eta\pm i(\sigma_j\pm\sigma)};q\right)},\\
W_\gamma(\sigma_i,\sigma_j)&=\frac{\theta\left(e^{-\gamma-\eta\mp i(\sigma_i\pm\sigma_j)};q\right)}{\theta\left(e^{\gamma-\eta\pm i(\sigma_i\pm\sigma_j)};q\right)}, &\quad W_{\eta-\gamma}(\sigma_k,\sigma)&=\frac{\theta\left(e^{-(\eta-\gamma)-\eta\mp i(\sigma_k\pm\sigma)};q\right)}{\theta\left(e^{(\eta-\gamma)-\eta\pm i(\sigma_k\pm\sigma)};q\right)},
\end{aligned}\tag{2.14}$$

where the identifications given in (2.13) and (2.14) required the balancing condition[5]

$$\prod_{i=1}^{6}a_i=\frac{q}{y}.\tag{2.15}$$

The next section contains a derivation of the star-triangle type expression discussed in [55] obtained from $4d$ $\mathcal{N}=1$ $USp(2N)$ Csáki-Skiba-Schmaltz duality for supersymmetric quiver gauge theories with matter in the antisymmetric tensor representation first studied in [62], whose index duality is given in terms of standard elliptic gamma functions.

2.2 Star-triangle type relation for $4d$ $\mathcal{N}=1$ $USp(2N)$ Csáki -Skiba-Schmaltz duality

In [55] the star-triangle type relation associated with $4d$ $\mathcal{N}=1$ $USp(2N)$ duality for theories with matter in the antisymmetric tensor representation is determined. It is convenient to realize the duality for these theories through the following expression

$$\begin{aligned}
&\frac{(p,p)_\infty^n(q,q)_\infty^n}{(4\pi)^n n!}\int_{\mathbb{T}^n}\prod_{j=1}^{n}\left[\frac{dz_j}{iz_j}\right]\prod_{1\leq j<k\leq n}\left[\frac{\Gamma\left(tz_j^{\pm1}z_k^{\pm1};p,q\right)}{\Gamma\left(z_j^{\pm1}z_k^{\pm1};p,q\right)}\right]\prod_{j=1}^{n}\left[\frac{\prod_{m=1}^{6}\Gamma\left(t_mz_j^{\pm1};p,q\right)}{\Gamma\left(z_j^{\pm2};p,q\right)}\right]\\
&=\prod_{j=1}^{n}\left[\frac{\Gamma\left(t^j;p,q\right)}{\Gamma\left(t;p,q\right)}\right]\prod_{j=1}^{n}\left[\prod_{1\leq m<s\leq 6}\Gamma\left(t^{j-1}t_mt_s;p,q\right)\right],
\end{aligned}\tag{2.16}$$

[5] The balancing condition is a constraint relating the fugacities associated with the global symmetry group of the theory and the parameters of the special functions present in the index of the theory. Physically, the fugacities can be defined in terms of spectral parameters and spin variables, so the balancing condition is the bridge between variables of the supersymmetric quiver gauge theory and the associated statistical model.

where Γ is the standard elliptic gamma function. This 4d $USp(2N)$ duality corresponds in the statistical mechanics side to the following star-triangle type relation stated in [55]

$$\int_{[0,2\pi]^n} [d\mathbf{u}] S(\mathbf{u};t,p,q) W_{\eta-\alpha}(x,\mathbf{u}) W_{\alpha+\gamma}(y,\mathbf{u}) W_{\eta-\gamma}(w,\mathbf{u})$$
$$= R(\alpha,\gamma,\eta;t,p,q) W^t_{\alpha}(y,w) W^t_{\eta-\alpha-\gamma}(x,w) W^t_{\gamma}(x,y), \tag{2.17}$$

where $S(\mathbf{u};t,p,q)$ and $R(\alpha,\gamma,\eta;t,p,q)$ are the interaction and normalization factors, respectively, while $W_{\eta-\alpha}(x,\mathbf{u})$ and $W^t_{\alpha}(y,w)$ are the two types of associated Boltzmann weights. The Boltzmann weights for this model are explicitly calculated[6] by comparing the supersymmetric duality (2.16) and the STR type expression (2.17). To this end, consider the following definitions

$$\begin{aligned} &t_1 = \sqrt{pq}\; e^{\eta-\alpha+ix}, && t_3 = \sqrt{pq}\; e^{\alpha+\gamma+iy}, && t_5 = \sqrt{pq}\; e^{\eta-\gamma+iw},\\ &t_2 = \sqrt{pq}\; e^{\eta-\alpha-ix}, && t_4 = \sqrt{pq}\; e^{\alpha+\gamma-iy}, && t_6 = \sqrt{pq}\; e^{\eta-\gamma-iw},\\ &z_j = e^{iu_j}, && pq = t^{2n-2}\prod_{m=1}^{6} t_m, && pq = t^{-n+1}e^{-2\eta}, \end{aligned} \tag{2.18}$$

where equations involving pq are the balancing condition and the definition of the crossing parameter η, in that order. Let's work explicitly both sides of equation (2.16). First, rewrite this equation by using (2.18) as

$$\int_{[0,2\pi]^n} \left[\frac{(p,p)^n_\infty (q,q)^n_\infty}{(4\pi)^n n!} \prod_{j=1}^{n} \left[\frac{\Gamma(t;p,q)}{\Gamma(t^j;p,q)}\right] \prod_{j=1}^{n} du_j \right] \prod_{1\le j<k\le n} \left[\frac{\Gamma\left(te^{\pm iu_j}e^{\pm iu_k};p,q\right)}{\Gamma\left(e^{\pm iu_j}e^{\pm iu_k};p,q\right)}\right]$$
$$\times \prod_{j=1}^{n} \frac{1}{\Gamma\left(e^{\pm 2iu_j};p,q\right)} \prod_{j=1}^{n}\prod_{m=1}^{6} \Gamma\left(t_m e^{\pm iu_j};p,q\right)$$
$$= \prod_{j=1}^{n} \prod_{1\le m<s\le 6} \Gamma\left(t^{j-1}t_m t_s;p,q\right). \tag{2.19}$$

By defining the measure $[d\mathbf{u}]$ and the interaction term $S(\mathbf{u};t,p,q)$ as

$$\begin{aligned} [d\mathbf{u}] &= \frac{(p,p)^n_\infty (q,q)^n_\infty}{(4\pi)^n n!} \prod_{j=1}^{n} \left[\frac{\Gamma(t;p,q)}{\Gamma(t^j;p,q)}\right] \prod_{j=1}^{n} du_j,\\ S(\mathbf{u};t,p,q) &= \prod_{1\le j<k\le n} \left[\frac{\Gamma\left(te^{\pm iu_j}e^{\pm iu_k};p,q\right)}{\Gamma\left(e^{\pm iu_j}e^{\pm iu_k};p,q\right)}\right] \prod_{j=1}^{n} \frac{1}{\Gamma\left(e^{\pm 2iu_j};p,q\right)}, \end{aligned} \tag{2.20}$$

where $\mathbf{u}=(u_1,\dots,u_n)$, it is possible to express (2.19) as

$$\int_{[0,2\pi]^n} [d\mathbf{u}] S(\mathbf{u};t,p,q) \prod_{j=1}^{n}\prod_{m=1}^{6} \Gamma\left(t_m e^{\pm iu_j};p,q\right) = \prod_{j=1}^{n} \prod_{1\le m<s\le 6} \Gamma\left(t^{j-1}t_m t_s;p,q\right). \tag{2.21}$$

[6]Calculations here contain a slightly different definition of the measure and of the interaction and normalization factors from those in reference [55].

For the left hand of (2.21) we note that

$$\begin{aligned}\prod_{m=1}^{6}\Gamma\left(t_m e^{\pm iu_j};p,q\right) =&\ \Gamma\left(\sqrt{pq}\ e^{\eta-\alpha+ix}e^{\pm iu_j};p,q\right)\Gamma\left(\sqrt{pq}\ e^{\eta-\alpha-ix}e^{\pm iu_j};p,q\right)\\ &\times\Gamma\left(\sqrt{pq}\ e^{\alpha+\gamma+iy}e^{\pm iu_j};p,q\right)\Gamma\left(\sqrt{pq}\ e^{\alpha+\gamma-iy}e^{\pm iu_j};p,q\right)\\ &\times\Gamma\left(\sqrt{pq}\ e^{\eta-\gamma+iw}e^{\pm iu_j};p,q\right)\Gamma\left(\sqrt{pq}\ e^{\eta-\gamma-iw}e^{\pm iu_j};p,q\right)\\ =&\ \Gamma\left(\sqrt{pq}\ e^{\eta-\alpha}e^{\pm ix}e^{\pm iu_j};p,q\right)\Gamma\left(\sqrt{pq}\ e^{\alpha+\gamma}e^{\pm iy}e^{\pm iu_j};p,q\right)\\ &\times\Gamma\left(\sqrt{pq}\ e^{\eta-\gamma}e^{\pm iw}e^{\pm iu_j};p,q\right). \end{aligned}\tag{2.22}$$

Now, for the right hand side of (2.21) one has

$$\begin{aligned}\prod_{1\le m<s\le 6}\Gamma\left(t^{j-1}t_m t_s;p,q\right) =&\ \Gamma\left(t^{j-1}t_1t_2;p,q\right)\Gamma\left(t^{j-1}t_1t_3;p,q\right)\Gamma\left(t^{j-1}t_1t_4;p,q\right)\Gamma\left(t^{j-1}t_1t_5;p,q\right)\\ &\times\Gamma\left(t^{j-1}t_1t_6;p,q\right)\Gamma\left(t^{j-1}t_2t_3;p,q\right)\Gamma\left(t^{j-1}t_2t_4;p,q\right)\Gamma\left(t^{j-1}t_2t_5;p,q\right)\\ &\times\Gamma\left(t^{j-1}t_2t_6;p,q\right)\Gamma\left(t^{j-1}t_3t_4;p,q\right)\Gamma\left(t^{j-1}t_3t_5;p,q\right)\Gamma\left(t^{j-1}t_3t_6;p,q\right)\\ &\times\Gamma\left(t^{j-1}t_4t_5;p,q\right)\Gamma\left(t^{j-1}t_4t_6;p,q\right)\Gamma\left(t^{j-1}t_5t_6;p,q\right)\\ =&\ \Gamma\left(t^{j-1}(pq)\ e^{2(\eta-\alpha)};p,q\right)\Gamma\left(t^{j-1}(pq)\ e^{\eta+\gamma+ix+iy};p,q\right)\\ &\times\Gamma\left(t^{j-1}(pq)\ e^{\eta+\gamma+ix-iy};p,q\right)\Gamma\left(t^{j-1}(pq)\ e^{2\eta-\alpha-\gamma+ix+iw};p,q\right)\\ &\times\Gamma\left(t^{j-1}(pq)\ e^{2\eta-\alpha-\gamma+ix-iw};p,q\right)\Gamma\left(t^{j-1}(pq)\ e^{\eta+\gamma-ix+iy};p,q\right)\\ &\times\Gamma\left(t^{j-1}(pq)\ e^{\eta+\gamma-ix-iy};p,q\right)\Gamma\left(t^{j-1}(pq)\ e^{2\eta-\alpha-\gamma-ix+iw};p,q\right)\\ &\times\Gamma\left(t^{j-1}(pq)\ e^{2\eta-\alpha-\gamma-ix-iw};p,q\right)\Gamma\left(t^{j-1}(pq)\ e^{2(\alpha+\gamma)};p,q\right)\\ &\times\Gamma\left(t^{j-1}(pq)\ e^{\eta+\alpha+iy+iw};p,q\right)\Gamma\left(t^{j-1}(pq)\ e^{\eta+\alpha+iy-iw};p,q\right)\\ &\times\Gamma\left(t^{j-1}(pq)\ e^{\eta+\alpha-iy+iw};p,q\right)\Gamma\left(t^{j-1}(pq)\ e^{\eta+\alpha-iy-iw};p,q\right)\\ &\times\Gamma\left(t^{j-1}(pq)\ e^{2(\eta-\gamma)};p,q\right)\\ =&\ \Bigg[\Gamma\left(t^{j-1}(pq)\ e^{2(\eta-\alpha)};p,q\right)\Gamma\left(t^{j-1}(pq)\ e^{2(\alpha+\gamma)};p,q\right)\end{aligned}$$

$$
\begin{aligned}
&\times\Gamma\left(t^{j-1}(pq)\ e^{2(\eta-\gamma)};p,q\right)\Big] \\
&\times\Gamma\left(t^{j-1}(pq)\ e^{\eta+\gamma}e^{\pm ix}e^{\pm iy};p,q\right) \\
&\times\Gamma\left(t^{j-1}(pq)\ e^{2\eta-\alpha-\gamma}e^{\pm ix}e^{\pm iw};p,q\right) \\
&\times\Gamma\left(t^{j-1}(pq)\ e^{\eta+\alpha}e^{\pm iy}e^{\pm iw};p,q\right) \\
\\
= \ & R(\alpha,\gamma,\eta;t,p,q) \\
&\times\Gamma\left(\sqrt{pq}\left(t^{j-\frac{n+1}{2}}\right)e^{\alpha}e^{\pm iy}e^{\pm iw};p,q\right) \\
&\times\Gamma\left(\sqrt{pq}\left(t^{j-\frac{n+1}{2}}\right)e^{\gamma}e^{\pm ix}e^{\pm iy};p,q\right) \\
&\times\Gamma\left(\sqrt{pq}\left(t^{j-\frac{n+1}{2}}\right)e^{\eta-\alpha-\gamma}e^{\pm ix}e^{\pm iw};p,q\right),
\end{aligned}
\tag{2.23}
$$

where the last equality introduced the normalization factor $R(\alpha,\gamma,\eta;t,p,q)$ as

$$
R(\alpha,\gamma,\eta;t,p,q) = \Gamma\left(t^{j-1}(pq)\ e^{2(\eta-\alpha)};p,q\right)\Gamma\left(t^{j-1}(pq)\ e^{2(\alpha+\gamma)};p,q\right)\Gamma\left(t^{j-1}(pq)\ e^{2(\eta-\gamma)};p,q\right)
\tag{2.24}
$$

and it was used the equality

$$
(pq)\left(t^{j-1}\right)e^{\eta+\alpha} = \sqrt{pq}\left(t^{j-\frac{n+1}{2}}\right)e^{\alpha},
$$

which can be obtained from the last expression in (2.18) as follows

$$
\begin{aligned}
&\left(t^{-n+1}\right)e^{-2\eta} = pq \\
\Leftrightarrow\quad &(pq)\left(t^{n-1}t^{2}\right)e^{2\eta} = t^{2} \\
\Leftrightarrow\quad &\sqrt{pq}\left(t^{\frac{n+1}{2}}\right)e^{\eta} = t \\
\Leftrightarrow\quad &\sqrt{pq}\left(t^{-1}\right)e^{\eta} = t^{-\frac{n+1}{2}} \\
\Leftrightarrow\quad &(pq)\left(t^{j-1}\right)e^{\eta} = \sqrt{pq}\left(t^{j-\frac{n+1}{2}}\right) \\
\Leftrightarrow\quad &(pq)\left(t^{j-1}\right)e^{\eta+\alpha} = \sqrt{pq}\left(t^{j-\frac{n+1}{2}}\right)e^{\alpha}.
\end{aligned}
\tag{2.25}
$$

Thus, by keeping (2.22) and (2.23) in mind, and taking the definition of the Boltzmann weights as follows

$$
\begin{aligned}
W_{\eta-\alpha}(x,\mathbf{u}) &= \prod_{j=1}^{n}\Gamma\left(\sqrt{pq}\ e^{\eta-\alpha}e^{\pm ix}e^{\pm iu_j};p,q\right), \\
W_{\alpha+\gamma}(y,\mathbf{u}) &= \prod_{j=1}^{n}\Gamma\left(\sqrt{pq}\ e^{\alpha+\gamma}e^{\pm iy}e^{\pm iu_j};p,q\right), \\
W_{\eta-\gamma}(w,\mathbf{u}) &= \prod_{j=1}^{n}\Gamma\left(\sqrt{pq}\ e^{\eta-\gamma}e^{\pm iw}e^{\pm iu_j};p,q\right), \\
W^{t}_{\alpha}(y,w) &= \prod_{j=1}^{n}\Gamma\left(\sqrt{pq}\left(t^{j-\frac{n+1}{2}}\right)e^{\alpha}e^{\pm iy}e^{\pm iw};p,q\right),
\end{aligned}
$$

$$W^t_{\eta-\alpha-\gamma}(x,w)=\prod_{j=1}^{n}\Gamma\left(\sqrt{pq}\left(t^{j-\frac{n+1}{2}}\right)e^{\eta-\alpha-\gamma}e^{\pm ix}e^{\pm iw};p,q\right),$$
$$W^t_{\gamma}(x,y)=\prod_{j=1}^{n}\Gamma\left(\sqrt{pq}\left(t^{j-\frac{n+1}{2}}\right)e^{\gamma}e^{\pm ix}e^{\pm iy};p,q\right), \tag{2.26}$$

it is possible to rewrite (2.21) exactly as (2.17), as desired. Note that the right hand side Boltzmann weights contain an extra parameter t that is not present in the left hand side ones. This feature will be shared with our result in section 2.3.2.

2.3 STR type expressions for $2d\ \mathcal{N}=(0,2)$ $USp(2N)$ dualities

Gauge theories $2d\ \mathcal{N}=(0,2)$ are nicely described in [77, 78]. The spectrum of these theories consist of three different multiplets, namely, the chiral multiplet Φ with one chiral fermion Ψ_+ and one complex scalar ϕ, the vector multiplet V with one fermion χ_- and gauge bosons $\{v_\alpha\}_{\alpha=0,1}$, and the Fermi multiplet Λ with one chiral spinor λ_-, and an auxiliary field G.

The Fermi multiplet is required to satisfy the constraint $\overline{\mathcal{D}}_+\Lambda=\sqrt{2}E(\Phi)$, where E is a holomorphic function of the chiral superfields Φ_i. This theory also has a set of holomorphic functions J^a of the chiral superfields Φ_i, one for each Fermi superfield, that satisfies the relation

$$\sum_a E_a(\Phi_i)J^a(\Phi_i)=0. \tag{2.27}$$

An interacting term in the Lagrangian can be constructed with the Fermi multiplets and the J^a functions

$$L_J=-\frac{1}{\sqrt{2}}\int d^2x d\theta^+\sum_a\left(\Lambda_a J^a\Big|_{\overline{\theta}^+=0}\right)-\text{h.c.}$$

This term can be thought of as the $\mathcal{N}=(0,2)$ analog of the superpotential. The suitable choice of holomorphic functions E and J^a defines completely the scalar potential and the Yukawa couplings of the theory. Although those holomorphic functions are important in the definition of $(0,2)$ theories, in the identification of Boltzmann weights just the field content of the theories is relevant so throughout this section we will concentrate on it. The contributions to the index (elliptic flavored genus in the NS-NS sector) of $2d\ \mathcal{N}=(0,2)$ theories coming from chiral, vector and Fermi multiplets are calculated in [79, 81] and they are given in terms of the Jacobi theta functions.

2.3.1 $2d\ \mathcal{N}=(0,2)$ $USp(2N)$ Intriligator-Pouliot duality

In this subsection we analyse $2d\ \mathcal{N}=(0,2)$ $USp(2N)$ Intriligator-Pouliot duality and we build star-triangle type relations for different values of N. As stated in [59], this duality comes from dimensional reduction on $\mathbb{S}^2$ of $4d\ \mathcal{N}=1$ $USp(2N)$ confining Intriligator-Pouliot duality (this one, first studied in [62], is the $USp(2N)$ version of Seiberg duality). The $2d$ $\mathcal{N}=(0,2)$ $USp(2N)$ Intriligator-Pouliot duality is realized between a $USp(2N)$ gauge theory with $2N+2$ chiral multiplets in the fundamental representation, and a Laudau-Ginzburg model with $(N+1)(2N+1)$ chiral multiplets and a Fermi multiplet. The elliptic flavoured genera expression for the duality of these $2d$ $\mathcal{N}=(0,2)$ $USp(2N)$ supersymmetric quiver gauge theories is, from [61][7],

$$\int \frac{d\hat{z}_N}{\prod_{i=1}^{N}\prod_{a=1}^{2N+2}\theta\left(su_a z_i^{\pm 1};q\right)} = \frac{\theta\left(qs^{-2(N+1)};q\right)}{\prod_{1\leq a<b\leq 2N+2}\theta\left(s^2 u_a u_b;q\right)}, \tag{2.28}$$

where

$$d\hat{z}_N = \frac{(q;q)_\infty^{2N}}{N!(4\pi)^N}\prod_{i=1}^{N}\left[\frac{dz_i}{iz_i}\theta\left(z_i^{\pm 2};q\right)\right]\prod_{1\leq i<j\leq N}\theta\left(z_i^{\pm 1}z_j^{\pm 1};q\right) \tag{2.29}$$

is the measure associated with $USp(2N)$. Here, $\{u_a\}_{a=1,\dots,2N+2}$ and $\{s\}$ are the sets of fugacities associated with the global symmetry group $SU(2N+2)_u\times U(1)_s$ of these theories. Thus, to match with star-triangle type relations the three spectral parameters α, β and γ have to be distributed into $N+1$ pairs of fugacities, each pair having an associated spin variable x_i, $i=1,\dots,N+1$, as in the subsequent expressions (2.37), (2.47), (2.56), (2.65), (2.75) and (2.85). There will be then $4N(N+1)$ theta functions having spin variables x_i in the left hand side of (2.28) while there will be $2N(N+1)$ in the right hand one. It will also be useful to define the following expressions

$$z_i = e^{i\Omega_i}, \tag{2.30}$$

$$[d\mathbf{\Omega}] = \prod_{i=1}^{N} d\Omega_i, \tag{2.31}$$

$$S\left(\mathbf{\Omega};q\right) = \prod_{i=1}^{N}\theta\left(e^{\pm 2i\Omega_i};q\right)\prod_{1\leq i<j\leq N}\theta\left(e^{\pm i\Omega_i}e^{\pm i\Omega_j};q\right), \tag{2.32}$$

where (2.32) will stand for the interaction factor for any value of N, so that (2.28) can be written as

$$\int \frac{[d\mathbf{\Omega}]\,S\left(\mathbf{\Omega};q\right)}{\prod_{i=1}^{N}\prod_{a=1}^{2N+2}\theta\left(su_a e^{\pm i\Omega_i};q\right)} = \frac{N!(4\pi)^N}{(q;q)_\infty^{2N}}\left[\frac{\theta\left(qs^{-2(N+1)};q\right)}{\prod_{1\leq a<b\leq 2N+2}\theta\left(s^2 u_a u_b;q\right)}\right]. \tag{2.33}$$

Define also the crossing parameter η as

$$\eta = \alpha+\beta+\gamma \tag{2.34}$$

and the following notation that will be used throughout the whole work

$$\theta\left(ae^{\pm b};q\right) = \theta\left(ae^{b};q\right)\theta\left(ae^{-b};q\right). \tag{2.35}$$

In the following subsections we obtain STR type expressions for $2d$ $\mathcal{N}=(0,2)$ $USp(2N)$ Intriligator-Pouliot duality (2.28). In section 2.3.1.1 we obtain an expression analogous to the so called triangle identity identified in [65] in the context of Yang–Baxter/$3D$-consistency correspondence. In sections 2.3.1.3 and 2.3.1.4 we observe a similarity with the asymmetric form of the star-triangle relation [64, 65, 66]. Finally, in sections 2.3.1.2, 2.3.1.5 and 2.3.1.6 an attempt to build an STR type expression is carried out.

[7] As stated in this reference, equality (2.28) can be tested perturbatively in variable q.

2.3.1.1 Case $N=1$

The analysis of $N=1$ case is interesting because duality (2.28) reduces, by using (2.29), to

$$\frac{(q;q)_\infty^2}{4\pi}\int\left[\frac{dz}{iz}\theta\left(z^{\pm 2};q\right)\right]\prod_{a=1}^{4}\left[\theta\left(su_a z^{\pm 1};q\right)\right]^{-1}=\theta\left(qs^{-4};q\right)\prod_{1\leq a<b\leq 4}\left[\theta\left(s^2u_au_b;q\right)\right]^{-1}, \quad (2.36)$$

which is precisely $2d$ $\mathcal{N}=(0,2)$ $SU(2)$ duality considered in [70, 59] between a $SU(2)$ gauge theory with 4 chiral multiplets in the fundamental representation and a Landau-Ginzburg model with 6 chiral multiplets and a Fermi multiplet.

By using (2.35) and defining the following relations between fugacities, spectral parameters and spin variables as

$$u_1=s^{-1}e^{-\alpha+ix_1}, \qquad u_3=s^{-1}e^{-\beta+ix_2},$$
$$u_2=s^{-1}e^{-\alpha-ix_1}, \qquad u_4=s^{-1}e^{-\beta-ix_2}, \quad (2.37)$$

where x_1 and x_2 are the spin variables defined previously in the paragraph before Eq. (2.30) and the balancing condition as

$$\prod_{a=1}^{4}u_a=\frac{1}{q}, \quad (2.38)$$

it is possible to write some factors in (2.36) as

$$\begin{aligned}\prod_{a=1}^{4}\theta\left(su_az^{\pm1};q\right)&=\theta\left(e^{-\alpha+ix_1}e^{\pm i\Omega};q\right)\theta\left(e^{-\alpha-ix_1}e^{\pm i\Omega};q\right)\theta\left(e^{-\beta+ix_2}e^{\pm i\Omega};q\right)\theta\left(e^{-\beta-ix_2}e^{\pm i\Omega};q\right)\\&=\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega};q\right)\theta\left(e^{-\beta}e^{\pm ix_2}e^{\pm i\Omega};q\right)\end{aligned} \quad (2.39)$$

and

$$\begin{aligned}\prod_{1\leq a<b\leq 4}\theta\left(s^2u_au_b;q\right)=&\;\theta\left(e^{-2\alpha};q\right)\theta\left(e^{-2\beta};q\right)\theta\left(e^{-(\alpha+\beta)+ix_1+ix_2};q\right)\\&\times\theta\left(e^{-(\alpha+\beta)+ix_1-ix_2};q\right)\theta\left(e^{-(\alpha+\beta)-ix_1+ix_2};q\right)\theta\left(e^{-(\alpha+\beta)-ix_1-ix_2};q\right)\\=&\;\theta\left(e^{-2\alpha};q\right)\theta\left(e^{-2\beta};q\right)\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_2};q\right).\end{aligned} \quad (2.40)$$

Also, note that the balancing condition (2.38) implies the following relation

$$qs^{-4}=e^{2(\alpha+\beta)}. \quad (2.41)$$

Now, by using Eqs. (2.30), (2.39), (2.40) and (2.41), the index duality (2.36) can be rewritten as

$$\begin{aligned}&\frac{(q;q)_\infty^2}{4\pi}\int[d\mathbf{\Omega}]\left[\theta\left(e^{\pm 2i\Omega};q\right)\right]\left[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega};q\right)\theta\left(e^{-\beta}e^{\pm ix_2}e^{\pm i\Omega};q\right)\right]^{-1}\\&=\theta\left(e^{2(\alpha+\beta)};q\right)\left[\theta\left(e^{-2\alpha};q\right)\theta\left(e^{-2\beta};q\right)\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_2};q\right)\right]^{-1}.\end{aligned} \quad (2.42)$$

The definition of the interaction and normalization factors, $S(\mathbf{\Omega};q)$ and $R(\alpha,\beta)$, respectively, as

$$\begin{aligned} S(\mathbf{\Omega};q) &= \left(e^{\pm 2i\Omega};q\right), \\ R(\alpha,\beta) &= \frac{1!(4\pi)^1}{(q;q)_\infty^{2(1)}}\theta\left(e^{2(\alpha+\beta)};q\right)\left[\theta\left(e^{-2\alpha};q\right)\theta\left(e^{-2\beta};q\right)\right]^{-1}, \end{aligned} \tag{2.43}$$

and the Boltzmann weights as

$$\begin{aligned} W_\alpha(x_1,\mathbf{\Omega}) &= \left[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega};q\right)\right]^{-1}, \\ W_\beta(x_2,\mathbf{\Omega}) &= \left[\theta\left(e^{-\beta}e^{\pm ix_2}e^{\pm i\Omega};q\right)\right]^{-1}, \\ W_{\alpha+\beta}(x_1,x_2) &= \left[\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_2};q\right)\right]^{-1}, \end{aligned} \tag{2.44}$$

allows us to write the index duality (2.42) as

$$\int [d\mathbf{\Omega}]\, S(\mathbf{\Omega};q)\, W_\alpha(x_1,\mathbf{\Omega})\, W_\beta(x_2,\mathbf{\Omega}) = R(\alpha,\beta) W_{\alpha+\beta}(x_1,x_2). \tag{2.45}$$

Expression (2.45) is very interesting because its form is analogous to that of the triangle identity considered in the context of Yang–Baxter/3D-consistency correspondence [65].

2.3.1.2 Case $N = 2$

Now let's consider the case $N = 2$ for which (2.28) can be written, by using (2.29), as

$$\begin{aligned} \int \left[\frac{(q;q)_\infty^{2(2)}}{2!(4\pi)^2}\left[\prod_{i=1}^{2}\frac{dz_i}{iz_i}\right]\prod_{i=1}^{2}\theta\left(z_i^{\pm 2};q\right)\prod_{1\leq i<j\leq 2}\theta\left(z_i^{\pm 1}z_j^{\pm 1};q\right)\right]\prod_{i=1}^{2}\prod_{a=1}^{6}\left[\theta\left(su_a z_i^{\pm 1};q\right)\right]^{-1} \\ = \theta\left(qs^{-6};q\right)\prod_{1\leq a<b\leq 6}\left[\theta\left(s^2u_au_b;q\right)\right]^{-1}. \end{aligned} \tag{2.46}$$

In order to work explicitly both sides of expression (2.46) define the following relations

$$\begin{aligned} u_1 &= s^{-1}e^{-\alpha+ix_1}, & u_3 &= s^{-1}e^{-\beta+ix_2}, & u_5 &= s^{-1}e^{-\gamma+ix_3}, \\ u_2 &= s^{-1}e^{-\alpha-ix_1}, & u_4 &= s^{-1}e^{-\beta-ix_2}, & u_6 &= s^{-1}e^{-\gamma-ix_3}, \end{aligned} \tag{2.47}$$

as well as the balancing condition

$$\prod_{a=1}^{6} u_a = \frac{1}{q}. \tag{2.48}$$

By using (2.30) and (2.47), the left and right hand sides of (2.46) can be rewritten as

$$\begin{aligned} \prod_{i=1}^{2}\prod_{a=1}^{6}\theta\left(su_a z_i^{\pm 1};q\right) = \prod_{i=1}^{2}\Big[&\theta\left(su_1e^{\pm i\Omega_i};q\right)\theta\left(su_2e^{\pm i\Omega_i};q\right)\theta\left(su_3e^{\pm i\Omega_i};q\right) \\ &\times\theta\left(su_4e^{\pm i\Omega_i};q\right)\theta\left(su_5e^{\pm i\Omega_i};q\right)\theta\left(su_6e^{\pm i\Omega_i};q\right)\Big] \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{2} \Bigg[\theta\left(e^{-\alpha} e^{ix_1} e^{\pm i\Omega_i}; q\right) \theta\left(e^{-\alpha} e^{-ix_1} e^{\pm i\Omega_i}; q\right) \\
&\quad \times \theta\left(e^{-\beta} e^{ix_2} e^{\pm i\Omega_i}; q\right) \theta\left(e^{-\beta} e^{-ix_2} e^{\pm i\Omega_i}; q\right) \\
&\quad \times \theta\left(e^{-\gamma} e^{ix_3} e^{\pm i\Omega_i}; q\right) \theta\left(e^{-\gamma} e^{-ix_3} e^{\pm i\Omega_i}; q\right) \Bigg] \\
&= \prod_{i=1}^{2} \Bigg[\theta\left(e^{-\alpha} e^{\pm ix_1} e^{\pm i\Omega_i}; q\right) \theta\left(e^{-\beta} e^{\pm ix_2} e^{\pm i\Omega_i}; q\right) \theta\left(e^{-\gamma} e^{\pm ix_3} e^{\pm i\Omega_i}; q\right) \Bigg],
\end{aligned}
\tag{2.49}
$$

and

$$
\begin{aligned}
\prod_{1 \le a < b \le 6} \theta\left(s^2 u_a u_b; q\right) = {} & \theta\left(s^2 u_1 u_2; q\right) \theta\left(s^2 u_1 u_3; q\right) \theta\left(s^2 u_1 u_4; q\right) \theta\left(s^2 u_1 u_5; q\right) \theta\left(s^2 u_1 u_6; q\right) \\
& \times \theta\left(s^2 u_2 u_3; q\right) \theta\left(s^2 u_2 u_4; q\right) \theta\left(s^2 u_2 u_5; q\right) \theta\left(s^2 u_2 u_6; q\right) \theta\left(s^2 u_3 u_4; q\right) \\
& \times \theta\left(s^2 u_3 u_5; q\right) \theta\left(s^2 u_3 u_6; q\right) \theta\left(s^2 u_4 u_5; q\right) \theta\left(s^2 u_4 u_6; q\right) \theta\left(s^2 u_5 u_6; q\right) \\
= {} & \left[\theta\left(e^{-2\alpha}; q\right) \theta\left(e^{-2\beta}; q\right) \theta\left(e^{-2\gamma}; q\right) \right] \\
& \times \Big[\theta\left(e^{-(\alpha+\beta)+ix_1+ix_2}; q\right) \theta\left(e^{-(\alpha+\beta)+ix_1-ix_2}; q\right) \theta\left(e^{-(\alpha+\beta)-ix_1+ix_2}; q\right) \\
& \times \theta\left(e^{-(\alpha+\beta)-ix_1-ix_2}; q\right) \Big] \\
& \times \Big[\theta\left(e^{-(\alpha+\gamma)+ix_1+ix_3}; q\right) \theta\left(e^{-(\alpha+\gamma)+ix_1-ix_3}; q\right) \theta\left(e^{-(\alpha+\gamma)-ix_1+ix_3}; q\right) \\
& \times \theta\left(e^{-(\alpha+\gamma)-ix_1-ix_3}; q\right) \Big] \\
& \times \Big[\theta\left(e^{-(\beta+\gamma)+ix_2+ix_3}; q\right) \theta\left(e^{-(\beta+\gamma)+ix_2-ix_3}; q\right) \theta\left(e^{-(\beta+\gamma)-ix_2+ix_3}; q\right) \\
& \times \theta\left(e^{-(\beta+\gamma)-ix_2-ix_3}; q\right) \Big] \\
= {} & \left[\theta\left(e^{-2\alpha}; q\right) \theta\left(e^{-2\beta}; q\right) \theta\left(e^{-2\gamma}; q\right) \right] \\
& \times \Big[\theta\left(e^{-(\alpha+\beta)} e^{\pm ix_1} e^{\pm ix_2}; q\right) \theta\left(e^{-(\alpha+\gamma)} e^{\pm ix_1} e^{\pm ix_3}; q\right) \theta\left(e^{-(\beta+\gamma)} e^{\pm ix_2} e^{\pm ix_3}; q\right) \Big],
\end{aligned}
\tag{2.50}
$$

respectively. It is also important to remark that balancing condition (2.48) implies the relation

$$
q s^{-6} = e^{2(\alpha+\beta+\gamma)}. \tag{2.51}
$$

Then, by using Eqs. (2.49), (2.50) and (2.51), the index duality (2.46) can be rewritten as

$$\begin{aligned}
&\int \left[\prod_{i=1}^{2} d\Omega_i\right]\left[\prod_{i=1}^{2}\theta\left(e^{\pm 2i\Omega_i};q\right)\prod_{1\le i<j\le 2}\theta\left(e^{\pm i\Omega_i}e^{\pm i\Omega_j};q\right)\right]\\
&\times\prod_{i=1}^{2}\left[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\gamma}e^{\pm ix_3}e^{\pm i\Omega_i};q\right)\right]^{-1}\\
&=\frac{2!(4\pi)^2}{(q;q)_\infty^4}\left[\theta\left(e^{-2(\alpha+\beta+\gamma)};q\right)\right]\left[\theta\left(e^{-2\alpha};q\right)\theta\left(e^{-2\beta};q\right)\theta\left(e^{-2\gamma};q\right)\right]^{-1}\\
&\times\left[\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_2};q\right)\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_1}e^{\pm ix_3};q\right)\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_2}e^{\pm ix_3};q\right)\right]^{-1}. \qquad (2.52)
\end{aligned}$$

Thus, by taking expression (2.31) as well as the definition of interaction and normalization factors, $S\left(\mathbf{\Omega};q\right)$ and $R(\alpha,\beta,\gamma)$, respectively, as

$$\begin{aligned}
S\left(\mathbf{\Omega};q\right) &= \prod_{i=1}^{2}\theta\left(e^{\pm 2i\Omega_i};q\right)\prod_{1\le i<j\le 2}\theta\left(e^{\pm i\Omega_i}e^{\pm i\Omega_j};q\right),\\
R(\alpha,\beta,\gamma) &= \frac{2!(4\pi)^2}{(q;q)_\infty^4}\left[\theta\left(e^{2(\alpha+\beta+\gamma)};q\right)\right]\left[\theta\left(e^{-2\alpha};q\right)\theta\left(e^{-2\beta};q\right)\theta\left(e^{-2\gamma};q\right)\right]^{-1}, \qquad (2.53)
\end{aligned}$$

the crossing parameter as (2.34) and the Boltzmann weights as

$$\begin{aligned}
W_\alpha^{\pm}\left(x_1,\mathbf{\Omega}\right) &= \prod_{i=1}^{2}\left[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\right]^{-1},\\
W_\beta^{\pm}\left(x_2,\mathbf{\Omega}\right) &= \prod_{i=1}^{2}\left[\theta\left(e^{-\beta}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\right]^{-1},\\
W_\gamma^{\pm}\left(x_3,\mathbf{\Omega}\right) &= \prod_{i=1}^{2}\left[\theta\left(e^{-\gamma}e^{\pm ix_3}e^{\pm i\Omega_i};q\right)\right]^{-1},
\end{aligned}$$

$$\begin{aligned}
W_{\eta-\gamma}(x_1,x_2,_) &= \left[\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_2};q\right)\right]^{-1},\\
W_{\eta-\beta}(x_1,_,x_3) &= \left[\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_1}e^{\pm ix_3};q\right)\right]^{-1},\\
W_{\eta-\alpha}(_,x_2,x_3) &= \left[\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_2}e^{\pm ix_3};q\right)\right]^{-1}, \qquad (2.54)
\end{aligned}$$

it is possible to rewrite (2.52) exactly as the following STR type expression

$$\begin{aligned}
&\int [d\mathbf{\Omega}]\, S\left(\mathbf{\Omega};q\right)W_\alpha^{\pm}\left(x_1,\mathbf{\Omega}\right)W_\beta^{\pm}\left(x_2,\mathbf{\Omega}\right)W_\gamma^{\pm}\left(x_3,\mathbf{\Omega}\right)\\
&\qquad = R(\alpha,\beta,\gamma)W_{\eta-\alpha}(x_2,x_3)W_{\eta-\beta}(x_1,x_3)W_{\eta-\gamma}(x_1,x_2), \qquad (2.55)
\end{aligned}$$

which resembles an STR expression because of the distribution of the spin variables as well as the spectral parameters in both sides of the relation despite the different definition of left and right hand side Boltzmann weights.

2.3.1.3 Case $N = 3$

The analysis is done in a similar way to those of the previous subsections. For this case it is convenient to define the following expressions

$$\begin{aligned} &u_1 = s^{-1}e^{-\alpha+ix_1}, \quad &&u_3 = s^{-1}e^{-\alpha+ix_2}, \quad &&u_5 = s^{-1}e^{-\beta+ix_3}, \quad &&u_7 = s^{-1}e^{-\gamma+ix_4}, \\ &u_2 = s^{-1}e^{-\alpha-ix_1}, \quad &&u_4 = s^{-1}e^{-\alpha-ix_2}, \quad &&u_6 = s^{-1}e^{-\beta-ix_3}, \quad &&u_8 = s^{-1}e^{-\gamma-ix_4}, \end{aligned} \tag{2.56}$$

and the balancing condition

$$\prod_{a=1}^{8} u_a = \frac{1}{q}, \tag{2.57}$$

which in turn implies

$$qs^{-8} = e^{2(2\alpha+\beta+\gamma)}. \tag{2.58}$$

By using the same notation that we followed in the previous subsections it is possible to write

$$\begin{aligned} \prod_{i=1}^{N}\prod_{a=1}^{2N+2}\theta\left(su_a z_i^{\pm1};q\right) = &\prod_{i=1}^{3}\Big[\theta\left(e^{-\alpha+ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha-ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha+ix_2}e^{\pm i\Omega_i};q\right) \\ &\times\theta\left(e^{-\alpha-ix_2}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta+ix_3}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta-ix_3}e^{\pm i\Omega_i};q\right) \\ &\times\theta\left(e^{-\gamma+ix_4}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\gamma-ix_4}e^{\pm i\Omega_i};q\right)\Big] \\ = &\prod_{i=1}^{3}\Big[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha}e^{\pm ix_2}e^{\pm i\Omega_i};q\right) \\ &\times\theta\left(e^{-\beta}e^{\pm ix_3}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\gamma}e^{\pm ix_4}e^{\pm i\Omega_i};q\right)\Big] \end{aligned} \tag{2.59}$$

and

$$\begin{aligned} \prod_{1\le a<b\le 2N+2}\theta\left(s^2u_au_b;q\right)\times = &\left[\theta\left(e^{-2\alpha};q\right)\right]^2\theta\left(e^{-2\beta};q\right)\theta\left(e^{-2\gamma};q\right) \\ &\times\theta\left(e^{-2\alpha\pm ix_1\pm ix_2};q\right)\theta\left(e^{-(\alpha+\beta)\pm ix_1\pm ix_3};q\right)\theta\left(e^{-(\alpha+\gamma)\pm ix_1\pm ix_4};q\right) \\ &\times\theta\left(e^{-(\alpha+\beta)\pm ix_2\pm ix_3};q\right)\theta\left(e^{-(\alpha+\gamma)\pm ix_2\pm ix_4};q\right) \\ &\times\theta\left(e^{-(\beta+\gamma)\pm ix_3\pm ix_4};q\right). \end{aligned} \tag{2.60}$$

We use again (2.31), the interaction factor (2.32), and the normalization factor $R(\alpha,\beta,\gamma)$ given by

$$R(\alpha,\beta,\gamma) = \frac{3!(4\pi)^3}{(q;q)_\infty^{2(3)}}\theta\left(e^{2(2\alpha+\beta+\gamma)};q\right)\left[\left[\theta\left(e^{-2\alpha};q\right)\right]^2\theta\left(e^{-2\beta};q\right)\theta\left(e^{-2\gamma};q\right)\right]^{-1} \tag{2.61}$$

as well as the crossing parameter (2.34) and expressions (2.59) and (2.60) to write (2.33) as

$$\int[d\mathbf{\Omega}]\,S\left(\mathbf{\Omega};q\right)\prod_{i=1}^{3}\Bigg[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)$$

$$\times\theta\left(e^{-\beta}e^{\pm ix_3}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\gamma}e^{\pm ix_4}e^{\pm i\Omega_i};q\right)\Bigg]^{-1}$$

$$=R(\alpha,\beta,\gamma)\Bigg[\theta\left(e^{-2\alpha\pm ix_1\pm ix_2};q\right)\theta\left(e^{-(\eta-\gamma)\pm ix_1\pm ix_3};q\right)\theta\left(e^{-(\eta-\beta)\pm ix_1\pm ix_4};q\right)$$

$$\times\theta\left(e^{-(\eta-\gamma)\pm ix_2\pm ix_3};q\right)\theta\left(e^{-(\eta-\beta)\pm ix_2\pm ix_4};q\right)\theta\left(e^{-(\eta-\alpha)\pm ix_3\pm ix_4};q\right)\Bigg]^{-1}. \quad (2.62)$$

The last equation is quite suggestive and definition of the Boltzmann weights as

$$W_\alpha\left(x_1,x_2,\mathbf{\Omega}\right)=\frac{\prod_{i=1}^{3}\left[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\right]^{-1}}{\left[\theta\left(e^{-2\alpha\pm ix_1\pm ix_2};q\right)\right]^{-1}},$$

$$V_\beta\left(x_3,\mathbf{\Omega}\right)=\prod_{i=1}^{3}\left[\theta\left(e^{-\beta}e^{\pm ix_3}e^{\pm i\Omega_i};q\right)\right]^{-1},$$

$$V_\gamma\left(x_4,\mathbf{\Omega}\right)=\prod_{i=1}^{3}\left[\theta\left(e^{-\gamma}e^{\pm ix_4}e^{\pm i\Omega_i};q\right)\right]^{-1},$$

$$\overline{W}_{\eta-\alpha}\left(\_,\_,x_3,x_4\right)=\left[\theta\left(e^{-(\eta-\alpha)\pm ix_3\pm ix_4};q\right)\right]^{-1},$$

$$\overline{V}_{\eta-\beta}\left(x_1,x_2,\_,x_4\right)=\left[\theta\left(e^{-(\eta-\beta)\pm ix_1\pm ix_4};q\right)\theta\left(e^{-(\eta-\beta)\pm ix_2\pm ix_4};q\right)\right]^{-1},$$

$$\overline{V}_{\eta-\gamma}\left(x_1,x_2,x_3,\_\right)=\left[\theta\left(e^{-(\eta-\gamma)\pm ix_1\pm ix_3};q\right)\theta\left(e^{-(\eta-\gamma)\pm ix_2\pm ix_3};q\right)\right]^{-1}, \quad (2.63)$$

leads us to write down the index duality (2.62) as

$$\int[d\mathbf{\Omega}]\,S\left(\mathbf{\Omega};q\right)W_\alpha\left(x_1,x_2,\mathbf{\Omega}\right)V_\beta\left(x_3,\mathbf{\Omega}\right)V_\gamma\left(x_4,\mathbf{\Omega}\right)$$
$$=R(\alpha,\beta,\gamma)\overline{W}_{\eta-\alpha}\left(\_,\_,x_3,x_4\right)\overline{V}_{\eta-\beta}\left(x_1,x_2,\_,x_4\right)\overline{V}_{\eta-\gamma}\left(x_1,x_2,x_3,\_\right). \quad (2.64)$$

Note that each side of (2.64) have the same definition for two Boltzmann weights while the third one is different. This feature resembles the graphical representation of the asymmetric form of the star-triangle relation [65]; unfortunately here there are more spin variables and the position of V and $\overline{V}$ is not the same as in [64, 65, 66] for the asymmetric star-triangle relation in the context of Yang–Baxter/$3D$-consistency correspondence.

2.3.1.4 Case $N=4$

The analysis is quite similar to that of the previous case. Define the expressions

$$u_1=s^{-1}e^{-\alpha+ix_1},\quad u_3=s^{-1}e^{-\alpha+ix_2},\quad u_5=s^{-1}e^{-\beta+ix_3},\quad u_7=s^{-1}e^{-\beta+ix_4},\quad u_9=s^{-1}e^{-\gamma+ix_5},$$

$$u_2 = s^{-1}e^{-\alpha-ix_1}, \quad u_4 = s^{-1}e^{-\alpha-ix_2}, \quad u_6 = s^{-1}e^{-\beta-ix_3}, \quad u_8 = s^{-1}e^{-\beta-ix_4}, \quad u_{10} = s^{-1}e^{-\gamma-ix_5}, \tag{2.65}$$

and the balancing condition

$$\prod_{a=1}^{10} u_a = \frac{1}{q}, \tag{2.66}$$

which now implies

$$qs^{-10} = e^{2(2\alpha+2\beta+\gamma)}. \tag{2.67}$$

This time, factors in the left and right hand sides of (2.33) can be worked out as

$$\begin{aligned}\prod_{i=1}^{N}\prod_{a=1}^{2N+2}\theta\left(su_a z_i^{\pm1};q\right) = \prod_{i=1}^{4}\Big[&\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\\ &\times\theta\left(e^{-\beta}e^{\pm ix_3}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta}e^{\pm ix_4}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\gamma}e^{\pm ix_5}e^{\pm i\Omega_i};q\right)\Big]\end{aligned} \tag{2.68}$$

and

$$\begin{aligned}\prod_{1\le a<b\le 2N+2}\theta\left(s^2u_au_b;q\right)\times = \;&\left[\theta\left(e^{-2\alpha};q\right)\right]^2\left[\theta\left(e^{-2\beta};q\right)\right]^2\theta\left(e^{-2\gamma};q\right)\\ &\times\theta\left(e^{-2\alpha\pm ix_1\pm ix_2};q\right)\theta\left(e^{-(\alpha+\beta)\pm ix_1\pm ix_3};q\right)\theta\left(e^{-(\alpha+\beta)\pm ix_1\pm ix_4};q\right)\\ &\times\theta\left(e^{-(\alpha+\gamma)\pm ix_1\pm ix_5};q\right)\theta\left(e^{-(\alpha+\beta)\pm ix_2\pm ix_3};q\right)\\ &\times\theta\left(e^{-(\alpha+\beta)\pm ix_2\pm ix_4};q\right)\theta\left(e^{-(\alpha+\gamma)\pm ix_2\pm ix_5};q\right)\\ &\times\theta\left(e^{-2\beta\pm ix_3\pm ix_4};q\right)\theta\left(e^{-(\beta+\gamma)\pm ix_3\pm ix_5};q\right)\theta\left(e^{-(\beta+\gamma)\pm ix_4\pm ix_5};q\right),\end{aligned} \tag{2.69}$$

respectively. Again, definitions (2.31) and (2.32), crossing parameter (2.34), and normalization factor $R(\alpha,\beta,\gamma)$ given by

$$R(\alpha,\beta,\gamma) = \frac{4!(4\pi)^4}{(q;q)_\infty^{2(4)}}\theta\left(e^{2(2\alpha+2\beta+\gamma)};q\right)\left[\left[\theta\left(e^{-2\alpha};q\right)\right]^2\left[\theta\left(e^{-2\beta};q\right)\right]^2\theta\left(e^{-2\gamma};q\right)\right]^{-1}, \tag{2.70}$$

are considered. Moreover, expressions (2.68) and (2.69) lead to write (2.33) as

$$\begin{aligned}\int[d\mathbf{\Omega}]\,S\left(\mathbf{\Omega};q\right)\prod_{i=1}^{4}\Big[&\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\\ &\times\theta\left(e^{-\beta}e^{\pm ix_3}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta}e^{\pm ix_4}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\gamma}e^{\pm ix_5}e^{\pm i\Omega_i};q\right)\Big]^{-1}\\ = R(\alpha,\beta,\gamma)\Big[&\theta\left(e^{-2\alpha\pm ix_1\pm ix_2};q\right)\theta\left(e^{-2\beta\pm ix_3\pm ix_4};q\right)\\ &\times\theta\left(e^{-(\eta-\gamma)\pm ix_1\pm ix_3};q\right)\theta\left(e^{-(\eta-\gamma)\pm ix_1\pm ix_4};q\right)\theta\left(e^{-(\eta-\gamma)\pm ix_2\pm ix_3};q\right)\\ &\times\theta\left(e^{-(\eta-\gamma)\pm ix_2\pm ix_4};q\right)\theta\left(e^{-(\eta-\beta)\pm ix_1\pm ix_5};q\right)\theta\left(e^{-(\eta-\beta)\pm ix_2\pm ix_5};q\right)\end{aligned}$$

$$\times\theta\left(e^{-(\eta-\alpha)\pm ix_3\pm ix_5};q\right)\theta\left(e^{-(\eta-\alpha)\pm ix_4\pm ix_5};q\right)\Bigg]^{-1}. \tag{2.71}$$

Finally, definition of the Boltzmann weights as

$$V_\alpha\left(x_1,x_2,\mathbf{\Omega}\right) = \frac{\prod_{i=1}^{4}\left[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\right]^{-1}}{\left[\theta\left(e^{-2\alpha\pm ix_1\pm ix_2};q\right)\right]^{-1}},$$

$$V_\beta\left(x_3,x_4,\mathbf{\Omega}\right) = \frac{\prod_{i=1}^{4}\left[\theta\left(e^{-\beta}e^{\pm ix_3}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta}e^{\pm ix_4}e^{\pm i\Omega_i};q\right)\right]^{-1}}{\left[\theta\left(e^{-2\beta\pm ix_3\pm ix_4};q\right)\right]^{-1}},$$

$$W_\gamma\left(x_5,\mathbf{\Omega}\right) = \prod_{i=1}^{4}\left[\theta\left(e^{-\gamma}e^{\pm ix_5}e^{\pm i\Omega_i};q\right)\right]^{-1},$$

$$\overline{V}_{\eta-\alpha}\left(\_,\_,x_3,x_4,x_5\right) = \left[\theta\left(e^{-(\eta-\alpha)\pm ix_3\pm ix_5};q\right)\theta\left(e^{-(\eta-\alpha)\pm ix_4\pm ix_5};q\right)\right]^{-1},$$

$$\overline{V}_{\eta-\beta}\left(x_1,x_2,\_,\_,x_5\right) = \left[\theta\left(e^{-(\eta-\beta)\pm ix_1\pm ix_5};q\right)\theta\left(e^{-(\eta-\beta)\pm ix_2\pm ix_5};q\right)\right]^{-1},$$

$$\begin{aligned}\overline{W}_{\eta-\gamma}\left(x_1,x_2,x_3,x_4,\_\right) = &\Big[\theta\left(e^{-(\eta-\gamma)\pm ix_1\pm ix_3};q\right)\theta\left(e^{-(\eta-\gamma)\pm ix_1\pm ix_4};q\right)\\ &\times\theta\left(e^{-(\eta-\gamma)\pm ix_2\pm ix_3};q\right)\theta\left(e^{-(\eta-\gamma)\pm ix_2\pm ix_4};q\right)\Big]^{-1},\end{aligned} \tag{2.72}$$

allows us to write the index duality (2.71) as

$$\begin{aligned}&\int[d\mathbf{\Omega}]S\left(\mathbf{\Omega};q\right)V_\alpha\left(x_1,x_2,\mathbf{\Omega}\right)V_\beta\left(x_3,x_4,\mathbf{\Omega}\right)W_\gamma\left(x_5,\mathbf{\Omega}\right)\\ &=R(\alpha,\beta,\gamma)\overline{V}_{\eta-\alpha}\left(\_,\_,x_3,x_4,x_5\right)\overline{V}_{\eta-\beta}\left(x_1,x_2,\_,\_,x_5\right)\overline{W}_{\eta-\gamma}\left(x_1,x_2,x_3,x_4,\_\right),\end{aligned} \tag{2.73}$$

whose analysis is analogous to that of $N=3$ case but with more spin variables.

2.3.1.5 Case $N = 5$

In this case, expression (2.28) can be rewritten as

$$\begin{aligned}\int\frac{(q;q)_\infty^{2(5)}}{(5)!(4\pi)^5}\left[\prod_{i=1}^{5}\frac{dz_i}{iz_i}\right]\left[\prod_{i=1}^{5}\theta\left(z_i^{\pm2};q\right)\prod_{1\le i<j\le5}\theta\left(z_i^{\pm1}z_j^{\pm1};q\right)\right]\prod_{i=1}^{5}\prod_{a=1}^{12}\left[\theta\left(su_az_i^{\pm1};q\right)\right]^{-1}\\ =\theta\left(qs^{-12};q\right)\prod_{1\le a<b\le12}\left[\theta\left(s^2u_au_b;q\right)\right]^{-1}.\end{aligned} \tag{2.74}$$

A generalization of relations (2.47) is given by

$$u_1 = s^{-1}e^{-\alpha+ix_1}, \qquad u_3 = s^{-1}e^{-\alpha+ix_2},$$

$$
\begin{aligned}
u_2 &= s^{-1}e^{-\alpha-ix_1}, & u_4 &= s^{-1}e^{-\alpha-ix_2},\\
u_5 &= s^{-1}e^{-\beta+ix_3}, & u_7 &= s^{-1}e^{-\beta+ix_4},\\
u_6 &= s^{-1}e^{-\beta-ix_3}, & u_8 &= s^{-1}e^{-\beta-ix_4},\\
u_9 &= s^{-1}e^{-\gamma+ix_5}, & u_{11} &= s^{-1}e^{-\gamma+ix_6},\\
u_{10} &= s^{-1}e^{-\gamma-ix_5}, & u_{12} &= s^{-1}e^{-\gamma-ix_6},
\end{aligned}
\tag{2.75}
$$

while the new balancing condition reads

$$
\prod_{a=1}^{12} u_a = \frac{1}{q}, \tag{2.76}
$$

which in turn implies

$$
qs^{-12} = e^{4(\alpha+\beta+\gamma)}. \tag{2.77}
$$

Now, by using Eqs. (2.30) and (2.75) for the left and right hand sides of (2.74), one has

$$
\begin{aligned}
\prod_{i=1}^{5}\prod_{a=1}^{12}\theta\left(su_a z_i^{\pm1};q\right) = \prod_{i=1}^{5}\Big[&\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\\
&\times\theta\left(e^{-\beta}e^{\pm ix_3}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta}e^{\pm ix_4}e^{\pm i\Omega_i};q\right)\\
&\times\theta\left(e^{-\gamma}e^{\pm ix_5}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\gamma}e^{\pm ix_6}e^{\pm i\Omega_i};q\right)\Big]
\end{aligned}
\tag{2.78}
$$

and

$$
\begin{aligned}
\prod_{1\le a<b\le 12}\theta\left(s^2u_au_b;q\right) = &\Big[\theta\left(e^{-2\alpha};q\right)\theta\left(e^{-2\alpha};q\right)\theta\left(e^{-2\beta};q\right)\theta\left(e^{-2\beta};q\right)\theta\left(e^{-2\gamma};q\right)\theta\left(e^{-2\gamma};q\right)\Big]\\
&\times\Big[\theta\left(e^{-(\alpha+\alpha)}e^{\pm ix_1}e^{\pm ix_2};q\right)\\
&\quad\times\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_3};q\right)\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_4};q\right)\\
&\quad\times\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_2}e^{\pm ix_3};q\right)\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_2}e^{\pm ix_4};q\right)\Big]\\
&\times\Big[\theta\left(e^{-(\beta+\beta)}e^{\pm ix_3}e^{\pm ix_4};q\right)\\
&\quad\times\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_3}e^{\pm ix_5};q\right)\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_3}e^{\pm ix_6};q\right)\\
&\quad\times\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_4}e^{\pm ix_5};q\right)\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_4}e^{\pm ix_6};q\right)\Big]\\
&\times\Big[\theta\left(e^{-(\gamma+\gamma)}e^{\pm ix_5}e^{\pm ix_6};q\right)\\
&\quad\times\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_1}e^{\pm ix_5};q\right)\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_1}e^{\pm ix_6};q\right)\\
&\quad\times\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_2}e^{\pm ix_5};q\right)\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_2}e^{\pm ix_6};q\right)\Big],
\end{aligned}
\tag{2.79}
$$

respectively. Thus, by keeping expressions (2.31), (2.77), (2.78) and (2.79) in mind, definition of the interaction factor $S(\mathbf{\Omega};q)$ and the normalization factor $R(\alpha,\beta,\gamma)$ as

$$S(\mathbf{\Omega};q) = \prod_{i=1}^{5}\theta\left(e^{\pm 2i\Omega_i};q\right)\prod_{1\le i<j\le 5}\theta\left(e^{\pm i\Omega_i}e^{\pm i\Omega_j};q\right),$$
$$R(\alpha,\beta,\gamma) = \frac{5!(4\pi)^5}{(q;q)_\infty^{2(5)}}\left[\theta\left(e^{4(\alpha+\beta+\gamma)};q\right)\right]\left[\theta\left(e^{-2\alpha};q\right)\theta\left(e^{-2\beta};q\right)\theta\left(e^{-2\gamma};q\right)\right]^{-2}, \tag{2.80}$$

the crossing parameter as (2.34) and the Boltzmann weights as

$$W_\alpha^\pm(x_1,x_2,\mathbf{\Omega}) = \frac{\prod_{i=1}^{5}\left[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\right]^{-1}}{\left[\theta\left(e^{-2\alpha}e^{\pm ix_1}e^{\pm ix_2};q\right)\right]^{-1}},$$

$$W_\beta^\pm(x_3,x_4,\mathbf{\Omega}) = \frac{\prod_{i=1}^{5}\left[\theta\left(e^{-\beta}e^{\pm ix_3}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta}e^{\pm ix_4}e^{\pm i\Omega_i};q\right)\right]^{-1}}{\left[\theta\left(e^{-2\beta}e^{\pm ix_3}e^{\pm ix_4};q\right)\right]^{-1}},$$

$$W_\gamma^\pm(x_5,x_6,\mathbf{\Omega}) = \frac{\prod_{i=1}^{5}\left[\theta\left(e^{-\gamma}e^{\pm ix_5}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\gamma}e^{\pm ix_6}e^{\pm i\Omega_i};q\right)\right]^{-1}}{\left[\theta\left(e^{-2\gamma}e^{\pm ix_5}e^{\pm ix_6};q\right)\right]^{-1}},$$

$$\begin{aligned}W_{\eta-\gamma}(x_1,x_2,x_3,x_4,\_,\_) = \Big[&\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_3};q\right)\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_4};q\right)\\ &\times\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_2}e^{\pm ix_3};q\right)\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_2}e^{\pm ix_4};q\right)\Big]^{-1},\\ W_{\eta-\alpha}(\_,\_,x_3,x_4,x_5,x_6) = \Big[&\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_3}e^{\pm ix_5};q\right)\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_3}e^{\pm ix_6};q\right)\\ &\times\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_4}e^{\pm ix_5};q\right)\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_4}e^{\pm ix_6};q\right)\Big]^{-1},\\ W_{\eta-\beta}(x_1,x_2,\_,\_,x_5,x_6) = \Big[&\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_1}e^{\pm ix_5};q\right)\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_1}e^{\pm ix_6};q\right)\\ &\times\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_2}e^{\pm ix_5};q\right)\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_2}e^{\pm ix_6};q\right)\Big]^{-1},\end{aligned} \tag{2.81}$$

makes it possible to rewrite (2.74) exactly as the STR type expression

$$\int[d\mathbf{\Omega}]\,S(\mathbf{\Omega};q)\,W_\alpha^\pm(x_1,x_2,\mathbf{\Omega})\,W_\beta^\pm(x_3,x_4,\mathbf{\Omega})\,W_\gamma^\pm(x_5,x_6,\mathbf{\Omega})$$

$$= R(\alpha,\beta,\gamma)W_{\eta-\alpha}(\_,\_,x_3,x_4,x_5,x_6)W_{\eta-\beta}(x_1,x_2,\_,\_,x_5,x_6)W_{\eta-\gamma}(x_1,x_2,x_3,x_4,\_,\_), \tag{2.82}$$

which can be thought of as a generalization of $N=2$ case for more spin variables.

2.3.1.6 General case $N=3k+2$

Calculations made for the cases $N=2$ and $N=5$ can be naturally extended to all values of N such that $N=3k+2$ for any $k\in\mathbb{Z}^+\cup\{0\}$[8]. In this case, (2.28) can be rewritten as

$$\int R(k)\left[\prod_{i=1}^{3k+2}\frac{dz_i}{iz_i}\right]\left[\prod_{i=1}^{3k+2}\theta\left(z_i^{\pm2};q\right)\prod_{1\le i<j\le 3k+2}\theta\left(z_i^{\pm1}z_j^{\pm1};q\right)\right]\prod_{i=1}^{3k+2}\prod_{a=1}^{6(k+1)}\left[\theta\left(su_az_i^{\pm1};q\right)\right]^{-1}$$
$$=\theta\left(qs^{-6(k+1)};q\right)\prod_{1\le a<b\le 6(k+1)}\left[\theta\left(s^2u_au_b;q\right)\right]^{-1}, \tag{2.83}$$

where

$$R(k)=\frac{(q;q)_\infty^{2(3k+2)}}{(3k+2)!(4\pi)^{(3k+2)}}. \tag{2.84}$$

Generalization of relations (2.47) is

$$\begin{aligned}
u_1 &= s^{-1}e^{-\alpha+ix_1}, & \cdots && u_{2(k+1)-1} &= s^{-1}e^{-\alpha+ix_{(k+1)}},\\
u_2 &= s^{-1}e^{-\alpha-ix_1}, & \cdots && u_{2(k+1)} &= s^{-1}e^{-\alpha-ix_{(k+1)}},\\
u_{2(k+1)+1} &= s^{-1}e^{-\beta+ix_{(k+1)+1}}, & \cdots && u_{4(k+1)-1} &= s^{-1}e^{-\beta+ix_{2(k+1)}},\\
u_{2(k+1)+2} &= s^{-1}e^{-\beta-ix_{(k+1)+1}}, & \cdots && u_{4(k+1)} &= s^{-1}e^{-\beta-ix_{2(k+1)}},\\
u_{4(k+1)+1} &= s^{-1}e^{-\gamma+ix_{2(k+1)+1}}, & \cdots && u_{6(k+1)-1} &= s^{-1}e^{-\gamma+ix_{3(k+1)}},\\
u_{4(k+1)+2} &= s^{-1}e^{-\gamma-ix_{2(k+1)+1}}, & \cdots && u_{6(k+1)} &= s^{-1}e^{-\gamma-ix_{3(k+1)}},
\end{aligned} \tag{2.85}$$

while the new balancing condition reads

$$\prod_{a=1}^{6(k+1)}u_a=\frac{1}{q}. \tag{2.86}$$

Then, by keeping crossing parameter (2.34) and expressions (2.31), (2.85) and (2.86) in mind, generalization of interaction and normalization factors, $S(\mathbf{\Omega};q)$ and $R(\alpha,\beta,\gamma)$, respectively, to

$$S(\mathbf{\Omega};q)=\prod_{i=1}^{3k+2}\theta\left(e^{\pm2i\Omega_i};q\right)\prod_{1\le i<j\le 3k+2}\theta\left(e^{\pm i\Omega_i}e^{\pm i\Omega_j};q\right),$$

[8] The cases $k=0$ and $k=1$ correspond precisely to the cases $N=2$ and $N=5$, respectively.

$$R(\alpha,\beta,\gamma)=\frac{(3k+2)!(4\pi)^{3k+2}}{(q;q)_\infty^{2(3k+2)}}\left[\theta\left(e^{2(k+1)(\alpha+\beta+\gamma)};q\right)\right]\left[\theta\left(e^{-2\alpha};q\right)\theta\left(e^{-2\beta};q\right)\theta\left(e^{-2\gamma};q\right)\right]^{-(k+1)},\tag{2.87}$$

and Boltzmann weights to

$$W_\alpha^\pm\left(x_1,\ldots,x_{k+1},\mathbf{\Omega}\right)=\frac{\prod_{i=1}^{3k+2}\prod_{j=1}^{k+1}\left[\theta\left(e^{-\alpha}e^{\pm ix_j}e^{\pm i\Omega_i};q\right)\right]^{-1}}{\prod_{\substack{m<n\\ m,n=1,\ldots,(k+1)}}\left[\theta\left(e^{-2\alpha}e^{\pm ix_m}e^{\pm ix_n};q\right)\right]^{-1}},$$

$$W_\beta^\pm\left(x_{(k+1)+1},\ldots,x_{2(k+1)},\mathbf{\Omega}\right)=\frac{\prod_{i=1}^{3k+2}\prod_{j=(k+1)+1}^{2(k+1)}\left[\theta\left(e^{-\beta}e^{\pm ix_j}e^{\pm i\Omega_i};q\right)\right]^{-1}}{\prod_{\substack{m<n\\ m,n=(k+1)+1,\ldots,2(k+1)}}\left[\theta\left(e^{-2\beta}e^{\pm ix_m}e^{\pm ix_n};q\right)\right]^{-1}},$$

$$W_\gamma^\pm\left(x_{2(k+1)+1},\ldots,x_{3(k+1)},\mathbf{\Omega}\right)=\frac{\prod_{i=1}^{3k+2}\prod_{j=2(k+1)+1}^{3(k+1)}\left[\theta\left(e^{-\gamma}e^{\pm ix_j}e^{\pm i\Omega_i};q\right)\right]^{-1}}{\prod_{\substack{m<n\\ m,n=2(k+1)+1,\ldots,3(k+1)}}\left[\theta\left(e^{-2\gamma}e^{\pm ix_m}e^{\pm ix_n};q\right)\right]^{-1}},$$

$$W_{\eta-\gamma}\left(\{x_l\}_{l=1,\ldots,3(k+1)}\backslash\{x_{2(k+1)+1,\ldots,3(k+1)}\}\right)=\prod_{\substack{m=1,\ldots,(k+1)\\ n=(k+1)+1,\ldots,2(k+1)}}\left[\theta\left(e^{-(\alpha+\beta)}e^{\pm ix_m}e^{\pm ix_n};q\right)\right]^{-1},$$

$$W_{\eta-\alpha}\left(\{x_l\}_{l=1,\ldots,3(k+1)}\backslash\{x_1,\ldots,x_{(k+1)}\}\right)=\prod_{\substack{m=(k+1)+1,\ldots,2(k+1)\\ n=2(k+1)+1,\ldots,3(k+1)}}\left[\theta\left(e^{-(\beta+\gamma)}e^{\pm ix_m}e^{\pm ix_n};q\right)\right]^{-1},$$

$$W_{\eta-\beta}\left(\{x_l\}_{l=1,\ldots,3(k+1)}\backslash\{x_{(k+1)+1,\ldots,2(k+1)}\}\right)=\prod_{\substack{m=1,\ldots,(k+1)\\ n=2(k+1)+1,\ldots,3(k+1)}}\left[\theta\left(e^{-(\alpha+\gamma)}e^{\pm ix_m}e^{\pm ix_n};q\right)\right]^{-1},\tag{2.88}$$

leads us to write down the index duality (2.83) as the STR type expression

$$\int[d\mathbf{\Omega}]\,S\left(\mathbf{\Omega};q\right)W_\alpha^\pm\left(x_1,\ldots,x_{k+1},\mathbf{\Omega}\right)W_\beta^\pm\left(x_{(k+1)+1},\ldots,x_{2(k+1)},\mathbf{\Omega}\right)$$

$$\begin{aligned} &\times W^{\pm}_{\gamma}\left(x_{2(k+1)+1},\dots,x_{3(k+1)},\mathbf{\Omega}\right) \\ = R(\alpha,\beta,\gamma) W_{\eta-\alpha}&\left(\{x_l\}_{l=1,\dots,3(k+1)}\backslash\{x_1,\dots,x_{(k+1)}\}\right) W_{\eta-\beta}\left(\{x_l\}_{l=1,\dots,3(k+1)}\backslash\{x_{(k+1)+1,\dots,2(k+1)}\}\right) \\ &\times W_{\eta-\gamma}\left(\{x_l\}_{l=1,\dots,3(k+1)}\backslash\{x_{2(k+1)+1,\dots,3(k+1)}\}\right), \end{aligned} \tag{2.89}$$

which is a generalization of the results obtained in subsections 2.3.1.2 and 2.3.1.5.

2.3.2 $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ Csáki-Skiba-Schmaltz duality

In this subsection we study a $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ Csáki-Skiba-Schmaltz duality for theories with matter in the antisymmetric tensor representation. This duality is obtained in [61][9] from dimensional reduction of $4d$ $\mathcal{N} = 1$ $USp(2N)$ Csáki-Skiba-Schmaltz duality for theories with matter in the antisymmetric tensor representation first studied in [62]. The $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ Csáki-Skiba-Schmaltz duality is given between a $USp(2N)$ gauge theory with 4 chiral multiplets in the fundamental representation, N Fermi multiplets and one antisymmetric chiral, and a Laudau-Ginzburg model with $6N$ chiral multiplets and N Fermi multiplets. The elliptic flavoured genera expression for the duality of these $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ supersymmetric quiver gauge theories is, from [61],

$$\prod_{i=1}^{N}\theta\left(qx^{-i};q\right)\int\left[\frac{d\hat{z}_N}{[\theta(x;q)]^N\prod_{1\le i<j\le N}\theta\left(xz_i^{\pm1}z_j^{\pm1};q\right)}\right]\left[\frac{1}{\prod_{i=1}^{N}\prod_{a=1}^{4}\theta\left(sx^{\frac{1-N}{3}}u_a z_i^{\pm1};q\right)}\right] = \prod_{i=1}^{N}\frac{\theta\left(qs^{-4}x^{i-\frac{2N+1}{3}};q\right)}{\prod_{1\le a<b\le 4}\theta\left(s^2x^{i-\frac{2N+1}{3}}u_a u_b;q\right)}, \tag{2.90}$$

where, again,

$$d\hat{z}_N = \frac{(q;q)_\infty^{2N}}{N!(4\pi)^N}\prod_{i=1}^{N}\left[\frac{dz_i}{iz_i}\theta\left(z_i^{\pm2};q\right)\right]\prod_{1\le i<j\le N}\theta\left(z_i^{\pm1}z_j^{\pm1};q\right) \tag{2.91}$$

is the measure associated with $USp(2N)$. Here, $\{u_a\}_{a=1,\dots,4}$, $\{s\}$ and $\{x\}$ are the sets of fugacities associated with the global symmetry group $SU(4)_u \times U(1)_s \times U(1)_x$ of the theories. Note that for $N = 1$ the duality is, as well as in the Intriligator-Pouliot case of section 2.3.1.1, reduced to the $2d$ $\mathcal{N} = (0,2)$ $SU(2)$ duality considered in [59, 70].

Index duality (2.90) can be analysed for general N by defining, in analogy with (2.37), the following relations between fugacities, spectral parameters and spin variables

$$\begin{aligned} u_1 &= s^{-1}x^{\frac{N-1}{3}}e^{-\alpha+ix_1}, & u_3 &= s^{-1}x^{\frac{N-1}{3}}e^{-\beta+ix_2}, \\ u_2 &= s^{-1}x^{\frac{N-1}{3}}e^{-\alpha-ix_1}, & u_4 &= s^{-1}x^{\frac{N-1}{3}}e^{-\beta-ix_2}, \end{aligned} \tag{2.92}$$

and the balancing condition

$$\prod_{a=1}^{4}u_a = \frac{1}{q}. \tag{2.93}$$

[9] This reference actually found two different index dualities, we refer here to the one given by $N_b = 4$ and $N_f = 0$.

First of all, use (2.30) to define

$$S'\left(\mathbf{\Omega};x,q\right) = \prod_{i=1}^{N}\theta\left(qx^{-i};q\right)\left[\frac{\prod_{i=1}^{N}\theta\left(e^{\pm 2i\Omega_i};q\right)\prod_{1\le i<j\le N}\theta\left(e^{\pm i\Omega_i}e^{\pm i\Omega_j};q\right)}{\left[\theta(x;q)\right]^N\prod_{1\le i<j\le N}\theta\left(xe^{\pm i\Omega_i}e^{\pm i\Omega_j};q\right)}\right] \tag{2.94}$$

and, in turn, use it to rewrite (2.90) as

$$\frac{(q;q)_\infty^{2N}}{N!(4\pi)^N}\int\left[\prod_{i=1}^{N}\frac{dz_i}{iz_i}\right]\left[S'\left(\mathbf{\Omega};x,q\right)\right]\left[\frac{1}{\prod_{i=1}^{N}\prod_{a=1}^{4}\theta\left(sx^{\frac{1-N}{3}}u_a z_i^{\pm1};q\right)}\right]$$
$$= \prod_{i=1}^{N}\frac{\theta\left(qs^{-4}x^{i-\frac{2N+1}{3}};q\right)}{\prod_{1\le a<b\le 4}\theta\left(s^2x^{i-\frac{2N+1}{3}}u_a u_b;q\right)}. \tag{2.95}$$

By using expressions (2.92) it is possible to rewrite some factors in (2.95) as

$$\prod_{i=1}^{N}\prod_{a=1}^{4}\theta\left(sx^{\frac{1-N}{3}}u_a z_i^{\pm1};q\right) = \prod_{i=1}^{N}\Big[\theta\left(e^{-\alpha+ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\alpha-ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta+ix_2}e^{\pm i\Omega_i};q\right)$$
$$\times\theta\left(e^{-\beta-ix_2}e^{\pm i\Omega_i};q\right)\Big]$$
$$= \prod_{i=1}^{N}\left[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\right] \tag{2.96}$$

and

$$\prod_{i=1}^{N}\prod_{1\le a<b\le 4}\theta\left(s^2x^{i-\frac{2N+1}{3}}u_a u_b;q\right) = \prod_{i=1}^{N}\Big[\theta\left(x^{i-1}e^{-2\alpha};q\right)\theta\left(x^{i-1}e^{-2\beta};q\right)$$
$$\times\theta\left(x^{i-1}e^{-(\alpha+\beta)+ix_1+ix_2};q\right)\theta\left(x^{i-1}e^{-(\alpha+\beta)+ix_1-ix_2};q\right)$$
$$\times\theta\left(x^{i-1}e^{-(\alpha+\beta)-ix_1+ix_2};q\right)\theta\left(x^{i-1}e^{-(\alpha+\beta)-ix_1-ix_2};q\right)\Big]$$
$$= \prod_{i=1}^{N}\left[\theta\left(x^{i-1}e^{-2\alpha};q\right)\theta\left(x^{i-1}e^{-2\beta};q\right)\theta\left(x^{i-1}e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_2};q\right)\right]. \tag{2.97}$$

Also, we note that the balancing condition (2.93) implies the relation

$$qs^{-4}x^{\frac{4(N-1)}{3}} = e^{2(\alpha+\beta)}, \tag{2.98}$$

from where we have

$$\theta\left(qs^{-4}x^{i-\frac{2N+1}{3}};q\right) = \theta\left(x^{i-(2N-1)}e^{2(\alpha+\beta)};q\right). \tag{2.99}$$

By using expressions (2.96), (2.97) and (2.99), the index duality (2.95) rewrites as

$$\frac{(q;q)_\infty^{2N}}{N!(4\pi)^N}\int\left[\prod_{i=1}^N d\Omega_i\right]\left[S'\left(\mathbf{\Omega};x,q\right)\right]\prod_{i=1}^N\left[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\theta\left(e^{-\beta}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\right]^{-1}$$
$$=\theta\left(x^{i-(2N-1)}e^{2(\alpha+\beta)};q\right)\prod_{i=1}^N\left[\theta\left(x^{i-1}e^{-2\alpha};q\right)\theta\left(x^{i-1}e^{-2\beta};q\right)\theta\left(x^{i-1}e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_2};q\right)\right]^{-1}. \tag{2.100}$$

Thus, by keeping (2.31) in mind, identification of the interaction factor $S\left(\mathbf{\Omega};x,q\right)$ and the normalization factor $R(\alpha,\beta;x)$ as

$$S\left(\mathbf{\Omega};x,q\right) = S'\left(\mathbf{\Omega};x,q\right),$$
$$R(\alpha,\beta;x) = \frac{N!(4\pi)^N}{(q;q)_\infty^{2N}}\left[\theta\left(x^{i-(2N-1)}e^{2(\alpha+\beta)};q\right)\right]\prod_{i=1}^N\left[\theta\left(x^{i-1}e^{-2\alpha};q\right)\theta\left(x^{i-1}e^{-2\beta};q\right)\right]^{-1}, \tag{2.101}$$

and the Boltzmann weights as

$$W_\alpha\left(x_1,\mathbf{\Omega}\right) = \prod_{i=1}^N\left[\theta\left(e^{-\alpha}e^{\pm ix_1}e^{\pm i\Omega_i};q\right)\right]^{-1},$$
$$W_\beta\left(x_2,\mathbf{\Omega}\right) = \prod_{i=1}^N\left[\theta\left(e^{-\beta}e^{\pm ix_2}e^{\pm i\Omega_i};q\right)\right]^{-1},$$
$$W^x_{\alpha+\beta}\left(x_1,x_2\right) = \prod_{i=1}^N\left[\theta\left(x^{i-1}e^{-(\alpha+\beta)}e^{\pm ix_1}e^{\pm ix_2};q\right)\right]^{-1}, \tag{2.102}$$

leads us to put the index duality (2.100) in the form

$$\int\left[d\mathbf{\Omega}\right]S\left(\mathbf{\Omega};x,q\right)W_\alpha\left(x_1,\mathbf{\Omega}\right)W_\beta\left(x_2,\mathbf{\Omega}\right) = R(\alpha,\beta;x)W^x_{\alpha+\beta}\left(x_1,x_2\right). \tag{2.103}$$

Again, as in the Intriligator-Pouliot case discussed in section 2.3.1.1, expression (2.103) has an analogous form to that of the triangle identity considered in [65] but with a slight distinction between left and right hand side Boltzmann weights given by an extra parameter x in the latter one, this situation is quite similar to that in expression (2.26) for Boltzmann weights found in [55].

Conclusions and future research lines

This thesis concerns research work done in mathematical aspects of quantum field theories, on the one hand the construction of higher-order average asymptotic Vassiliev invariants in Chern-Simons gauge theory and on the other hand the derivation of star-triangle type relations from $2d$ $\mathcal{N} = (0, 2)$ $USp(2N)$ supersymmetric quiver gauge theory dualities.

In chapter 1 we pursued the calculation of average asymptotic Vassiliev invariants for flows associated to higher order terms of the perturbative Chern-Simons theory by using Bott-Taubes integration in the configuration space. The traditional way of obtaining asymptotic invariants is to give a partial foliation on the underlying manifold with leaves of certain dimension [20, 21, 22]. One can endow the manifold with a collection of flow boxes and orient the set of flow boxes along the foliation. On the submanifolds transversal to the leaves one gives a transverse Borel measure of the foliation preserved by the flow. These data gives rise to a *geometric current* and it is used as an object dual to differential forms defined on the leaves of the foliation. That determines a homology cycle dependent on the flow or the associated vector field. Consequently, in order to define topological invariants for flows we integrate differential forms on the transverse measure. Jones-Witten theory is precisely an example where this construction can be applied [19], here, observables are defined as integrals of the pullback of the connection one-forms on M over the one-dimensional asymptotic cycle. In the original version of perturbative Chern-Simons theory it was very difficult to write Vassiliev invariants as integrals of certain differential forms. However the formulation of perturbative Chern-Simons theory using Bott-Taubes integrals on configuration spaces [38] gives rise to a natural way of determining the cohomology of such spaces and in consequence the Vassiliev invariants can be easily rewritten as integrals of certain differential forms on these spaces. In this chapter we also studied in a systematic way the correspondence between Feynman diagrams in perturbative Chern-Simons theory and the associated Bott-Taubes integrals. For the Feynman diagrams of order one in $1/k$, we regained the self-linking number (1.82). For second order in the expansion of $1/k$ there are two relevant contributions to the Vassiliev invariant which come from Eqs. (1.90) and (1.93). For order three, there are four diagrams which contribute to the Vassiliev invariant whose asosiated Bott-Taubes integrals are given in Eqs. (1.101), (1.107), (1.111) and (1.115). The analysis of the first order contribution for Feynman diagrams having all the marked points lying on the knot and the discussion of one second order diagram with three points lying on the knot and one outside from it was worked out in Ref. [40]. In the present chapter we regained the Vassiliev invariant at second order in perturbation theory constructed from the relevant diagrams D_{21} and D_{22}. This was obtained after a proper discussion of the behavior of the boundary terms. Moreover, we go further to third order and regain the corresponding Vassiliev invariant; an analysis of the boundary terms of the Bott-Taubes integrals is also discussed. The problem arising in the computation of the Jones-Witten invariants for flows [19] involving the distinction between the Abelian and non-Abelian cases does not appear here. Even if we are discussing the non-Abelian case, the Vassiliev invariants are obtained as a perturbative series and then the exponential in the Wilson loop operators are expanded leaving all the terms as Lie algebra valued objects.

We have used the previous results and the advantage of writing Vassiliev invariants as

Bott-Taubes integrals in order to introduce flows on the underlying manifold. Thus we were able to incorporate easily the non-singular and non-divergence smooth vector field X on M ($\mathbb{R}^3$ or $\mathbf{S}^3$) to obtain invariants of triplets $(M, \mathcal{F}, \mu)$. This approach was followed in Ref. [41] to compute some average asymptotic Vassiliev invariants, namely the average asymptotic self-linking number was obtained. This invariant was obtained at higher-order with all marked points lying on the knot. For the first order in $1/k$ the average asymptotic Vassiliev invariant corresponds precisely with the average asymptotic self-linking number or helicity (1.120) obtained in [41]. Furthermore, at second order there are two contributions to the average asymptotic Vassiliev invariant, which is given by the sum of Eqs. (1.125) and (1.130). The boundary terms cancel by the same reason that in the case without flows. Moreover, the average asymptotic third order Vassiliev invariant is given by the superposition of four average integrals in Eqs. (1.134), (1.138), (1.143) and (1.144). From the previous results it is clear that average asymptotic Vassiliev invariants obtained from higher-order diagrams in Chern-Simons theory will be constructed following a similar procedure. An algorithm for the construction of any order diagram is not given here and it is a subject of future work. Also, it would be interesting to generalize these explicit constructions to the case of the two component links described in reference [18]. It has to be noticed that the match between amplitudes coming directly from perturbative Chern-Simons theory and those arising from Bott-Taubes integrals in configuration spaces is given in this work up to signature. The reason is that Chern-Simons theory was expressed in Lorentzian signature while Bott-Taubes integration is shown to be compatible with the Euclidean signature.

Among the different future directions for the work described in chapter 1 my interest is mainly focused in categorification. It is well known that knot and link invariants can be categorified to obtain knot homology invariants, particularly, from the Jones polynomial [2] it can be built the so called Khovanov homology [82]. The physical approach to that homology in terms of gauge theory and brane theory was studied, for instance, in [83, 84, 85, 86]. My future plan is to find an average asymptotic version of the categorified Jones polynomial (it is not clear if the formalism of Bott-Taubes integrals will play an important role also in this case) and to study a potential relation with Vassiliev invariants via categorified Vassiliev skein relations [87, 88]. If this asymptotic version is found, then, a generalization related to HOMFLY-PT homology would also be worthy. It would also be interesting to extend the work done in this chapter towards two possible directions, the first one is to construct the average asymptotic Vassiliev invariants for knots at order four and beyond, while the second one is to include the description of section 1.1.2 to obtain higher-order average asymptotic Vassiliev invariants for links; both possibilities will face up some mathematical subtleties, in the first one it would be nice to find a mathematical framework that takes into account the ghost fields (it is not clear if Bott-Taubes integrals in configuration spaces are enough to capture this information) even though those associated Feynman diagrams are supossed to be cancelled as in order two, while the second one requires the inclusion of configuration spaces of products of manifolds and many of their properties.

In chapter 2, a brief overview of the gauge/YBE correspondence is provided. The work performed in this subject is brand new, the dictionary of this correspondence is incomplete and it is yet under construction. For example, references [89, 90] contain a large list of $4d$ $\mathcal{N} = 1$ dualities and their corresponding supersymmetric index equalities, this is done for many gauge groups, but it is not clear if there are star-triangle type relations associated to all of them. In particular, star-triangle type relations associated to $2d$ $\mathcal{N} = (0, 2)$ supersymmetric quiver gauge theory dualities had not been found before and so they are not included yet into this context. It was the purpose of this chapter to provide them for the $2d$ $\mathcal{N} = (0, 2)$ $USp(2N)$ dualities presented in Ref. [61], that is, Intriligator-Pouliot and Csáki-Skiba-Schmaltz dualities in two dimensions.

The derivation coming from Intriligator-Pouliot duality for $\mathcal{N} = (0, 2)$ supersymmetric quiver gauge theories was carried out explicitly for different values of N. In particular, for $N = 2, 5$ we found that the realization of the duality conditions (2.46) and (2.74) implied the corresponding STR type expressions (2.55) and (2.82), respectively. These cases were generalized to the value $N = 3k + 2$, where duality condition (2.83) implied STR type expression (2.89). It should be noticed that the generalization is only straightforward in this case because there are $6k + 6$ fugacities associated with the symmetry group $SU(2N + 2)_u$ and so the spectral parameters α, β and γ can be equally distributed. That is, there are $2k+2$ fugacities (or, alternatively, $k+1$ spin variables) for each parameter as it can be seen in (2.85). All these STR type expressions have two different definitions for Boltzmann weights, one for the left hand side and other for the right hand side ones. The cases with $N = 3, 4$ (expressions (2.64) and (2.73), respectively) are somewhat similar to the asymmetric form of the star-triangle relation already reported in Refs. [64, 65, 66] although they are not exactly the same. The value $N = 1$, our expression (2.45), is more interesting because it highly resembles the triangle identity reported previously [65] in the literature of Yang–Baxter/$3D$-consistency correspondence.

The derivation of expression (2.103) from Csáki-Skiba-Schmaltz duality for $\mathcal{N} = (0, 2)$ supersymmetric quiver gauge theories with an antisymmetric tensor is valid for all values of N, this expression also resembles the triangle identity found in [65] but the right hand side Boltzmann weight have an extra parameter similar in spirit to definitions (2.26) coming from [55]. The existence of an extra parameter is a feature preserved under dimensional reduction but it is quite interesting to notice that even when in four dimensions the associated expression is the star-triangle type relation (2.17), for two dimensions one obtains the triangle identity (2.103), this is due to the fact that the amount of fugacities is clearly different in each dimension.

As part of the work, the Boltzmann weights as well as the interaction and normalization factors were completely determined for all cases. It is worthy to remark that all examples we found here have not exactly the form of a SSR or a STR expression, which we certainly know give rise to integrable models. Thus, although we have shown that the diverse dualities of certain $2d$ $\mathcal{N} = (0, 2)$ models considered here have an associated STR type expression they probably do not represent integrable models. Actually, in the context of Yang–Baxter/$3D$-consistency correspondence the relation of triangle identity with integrability is still unclear [65]. We hope our expressions could give insights in the study of integrability in an alternative direction to that of Ref. [69] were a triality between $2d$ $\mathcal{N} = (0, 2)$ models was found and it was conjectured that a tetrahedron equation of certain integrable systems would be associated with those models. Precise determination of integrability properties of our STR type expressions is an interesting topic for future work.

There are many striking future directions for the work presented in chapter 2, but I am specially interested in the determination of precise integrability properties of the obtained star-triangle type relations and their potential connection with $(0, 2)$ trialities [69] and with some works in Yang-Baxter/$3D$-consistency correspondence (see for example [65]). The first possible connection I glimpse with trialities is related to the space of $2d$ $\mathcal{N} = (0,2)$ theories preserving supersymmetry which can be schematized as a triangle (see reference [69]), the theories in the vertices and in the edges of that triangle could be simple enough to have a star-triangle type relation similar to the ones found in the work of this chapter. The second possible connection is due to the speculation that trialities are related with a tetrahedron equation in the same sense Seiberg-like dualities are related with Yang-Baxter equation, then it would be interesting to study if the star-triangle type relations found in this chapter are or not a special case of a tetrahedron equation. Another research lines I want to work on are the relation of those expressions, no matter whether they have an integrable model associated or not, with topological knot and link invariants [90, 91], and the possible description of gauge/YBE correspondence in terms of brane box configurations discussed in Refs. [92, 93].

It is worth mentioning that my work will not be limited to the personal interests described above. I truly believe this thesis is just an opportunity to promote and to encourage collaborative work with other students and researchers either in one of the projects described here or in another ones that we can elaborate together. Finally, it can be added that I also have research interest in collaborative work with other areas of science through subjects like DNA modeling via knot theory, topological data analysis and machine learning.

Appendices

APPENDICES A

Pullbacks and integration

A.1 Pullback bundle

A pullback (or fibre product) of a pair $(f : X \to Y, g : Z \to Y)$ is a subspace of the product $X \times Z$ defined by

$$X \times_Y Z := \{(x, z) \in X \times Z \mid f(x) = g(z)\}. \tag{A.1}$$

Then by considering the projections $\pi_1 : X \times Z \to X$ and $\pi_2 : X \times Z \to Z$ from $X \times Z$ into their first and second coordinates, respectively, the restriction of these maps to the fibre product

$$pr_1 = \pi_1|_{X\times_Y Z} : X \times_Y Z \to X, \qquad pr_2 = \pi_2|_{X\times_Y Z} : X \times_Y Z \to Z, \tag{A.2}$$

makes diagram in figure A.1 commutative.

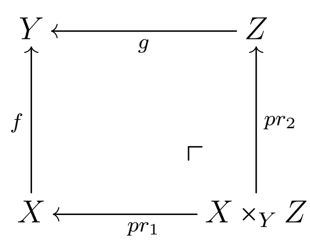


Figure A.1: Pullback of a pair (f, g).

In the case that $f : X \to Y$ is a kind of bundle and $g : Z \to Y$ is a morphism between the spaces, then the fibre product is usually denoted by g^*X and $pr_2 : g^*X \to Z$ is called *the pullback bundle of the bundle f over Z* [26]. Since this is a commutative diagram, the icon inside it is a usual notation to identify the fibre product $X \times_Y Z$ and the corner of this icon indicates the direction of all the arrows in the diagram.

A.2 Integration along the fibres

Let $\pi : E \to B$ be a smooth fibre bundle with homotopy compact fibre $F_b := \pi^{-1}(\{b\}) \simeq F$ with $\dim(F) = n$. Let $\omega \in \Omega^k(E)$. There is a map $\pi_* : \Omega^k(E) \to \Omega^{k-n}(B)$ called *integration along the fibre of π* given by

$$(\pi_*\omega)_b \left(V_b^1, V_b^2, \ldots, V_b^{k-n}\right) = \int_{F_b} i^*\omega_\pi, \tag{A.3}$$

where ω_π is an n-form in the total space E whose pullback through the inclusion map $i : F_b \hookrightarrow E$ is now an n-form in the fibre F_b which is given, for a point $p \in \pi^{-1}(\{b\})$, by

$$(i^*(\omega_\pi))_p(W_1, \dots, W_n) := \omega\left(W_1, \dots, W_n, \left[V_b^1\right]_\ell, \dots, \left[V_b^{k-n}\right]_\ell\right), \tag{A.4}$$

with $\left[V_b^i\right]_\ell \in T_pE$ any lift of the tangent vector $V_b^i \in T_bB$ and $\{W_1, \dots, W_n\}$ a set of vectors tangent to F_b at the point p.

To ensure that this definition is independent on the choice of the specific lifts consider two different lifts $\left[V\right]_\ell$ and $\left[V'\right]_\ell$ of V_b^i over the point $p \in F_b$. Since both of them are lifts then

$$d\pi_p\left(\left[V\right]_\ell - \left[V'\right]_\ell\right) = V_b^i - V_b^i = 0, \tag{A.5}$$

thus $\left[V\right]_\ell - \left[V'\right]_\ell \in \mathrm{Ker}\,(d\pi_p) = T_a\pi^{-1}(\{b\})$. Now the set $\left\{W_1, \dots, W_n, \left[V\right]_\ell - \left[V'\right]_\ell\right\}$ with $n+1$ different tangent vectors on $\pi^{-1}(\{b\})$ (whose dimension is n) has to be linearly dependent, and since ω is an alternating tensor then

$$\omega\left(W_1, \dots, W_{n-k}, [V_b^1]_\ell, \dots, [V]_\ell - [V']_\ell, \dots, [V_b^{k-n}]_\ell\right) = 0. \tag{A.6}$$

The previous equation asserts that $\pi_*\omega$ is independent of the choice of the lifts of the tangent vectors [26].

A.3 Gauss map pullback

Explicit calculation will be done here for points s_1 and x_4 in diagram of figure 1.6a. Note that a generalization for any pair of points in any diagram is straightforward.

The volume form in $\mathbf{S}^2$ can be taken as [40]

$$\omega = \frac{\varepsilon_{\mu\nu\sigma}}{8\pi}\frac{x^\mu dx^\nu \wedge dx^\sigma}{|\overline{x}|^3}. \tag{A.7}$$

It will be useful to write ω explicitly in the coordinate system $\{x, y, z\}$ in $\mathbb{R}^3$ as

$$\begin{aligned} \omega =& \frac{1}{4\pi}\left[\frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}\right] \\ =& \omega_{12}((x, y, z))dx \wedge dy + \omega_{23}((x, y, z))dy \wedge dz + \omega_{13}((x, y, z))dx \wedge dz, \end{aligned} \tag{A.8}$$

where the coefficient functions are given by

$$\begin{aligned} &\omega_{12} : \mathbf{S}^2 \longrightarrow \mathbb{R} \\ &\omega_{12}((x, y, z)) = \frac{1}{4\pi}\left[\frac{z}{(x^2 + y^2 + z^2)^{3/2}}\right], \end{aligned} \tag{A.9a}$$

$$\begin{aligned} &\omega_{23} : \mathbf{S}^2 \longrightarrow \mathbb{R} \\ &\omega_{23}((x, y, z)) = \frac{1}{4\pi}\left[\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right], \end{aligned} \tag{A.9b}$$

$$\omega_{13} : \mathbf{S}^2 \longrightarrow \mathbb{R}$$
$$\omega_{13}((x,y,z)) = \frac{1}{4\pi}\left[\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right]. \tag{A.9c}$$

Remember that the Gauss map

$$\phi : C(3+1,\mathbf{S}^3) \longrightarrow \mathbf{S}^2 \times \mathbf{S}^2 \times \mathbf{S}^2 \tag{A.10}$$

factors for this diagram as

$$\phi = \phi_{1,4} \times \phi_{2,4} \times \phi_{3,4}, \tag{A.11}$$

where the indices refer to the coordinate system $\{x_1, x_2, x_3, x_4\}$ on $C(3+1,\mathbf{S}^3)$ seen as a subset of $\mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^3$. Each of these coordinates has three indices (for example x_1 represents the coordinates $\{x_1^1, x_1^2, x_1^3\}$ in the first $\mathbf{S}^3$ factor) thus the coordinate system on $C(3+1,\mathbf{S}^3)$ is really taken to be $\{x_1^1, x_1^2, x_1^3, x_2^1, x_2^2, x_2^3, x_3^1, x_3^2, x_3^3, x_4^1, x_4^2, x_4^3\}$.

The factors of ϕ are explicitly given by

$$\phi_{1,4} : C(3+1,\mathbf{S}^3) \longrightarrow \mathbf{S}^2$$
$$\phi_{1,4}((x_1,x_2,x_3,x_4)) = \frac{x_4 - x_1}{|x_4 - x_1|} = \frac{(x_4^1 - x_1^1, x_4^2 - x_1^2, x_4^3 - x_1^3)}{[(x_4^1 - x_1^1)^2 + (x_4^2 - x_1^2)^2 + (x_4^3 - x_1^3)^2]^{1/2}}, \tag{A.12a}$$

$$\phi_{2,4} : C(3+1,\mathbf{S}^3) \longrightarrow \mathbf{S}^2$$
$$\phi_{2,4}((x_1,x_2,x_3,x_4)) = \frac{x_4 - x_2}{|x_4 - x_2|} = \frac{(x_4^1 - x_2^1, x_4^2 - x_2^2, x_4^3 - x_2^3)}{[(x_4^1 - x_2^1)^2 + (x_4^2 - x_2^2)^2 + (x_4^3 - x_2^3)^2]^{1/2}}, \tag{A.12b}$$

$$\phi_{3,4} : C(3+1,\mathbf{S}^3) \longrightarrow \mathbf{S}^2$$
$$\phi_{3,4}((x_1,x_2,x_3,x_4)) = \frac{x_4 - x_3}{|x_4 - x_3|} = \frac{(x_4^1 - x_3^1, x_4^2 - x_3^2, x_4^3 - x_3^3)}{[(x_4^1 - x_3^1)^2 + (x_4^2 - x_3^2)^2 + (x_4^3 - x_3^3)^2]^{1/2}}. \tag{A.12c}$$

In what follows the function $\phi_{1,4}$ is studied in detail. First write

$$\phi_{1,4}((x_1,x_2,x_3,x_4)) = (\phi_x((x_1,x_4)), \phi_y((x_1,x_4)), \phi_z((x_1,x_4))), \tag{A.13}$$

where (see (A.12a))

$$\phi_x((x_1,x_4)) = \frac{x_4^1 - x_1^1}{[(x_4^1 - x_1^1)^2 + (x_4^2 - x_1^2)^2 + (x_4^3 - x_1^3)^2]^{1/2}}, \tag{A.14a}$$

$$\phi_y((x_1,x_4)) = \frac{x_4^2 - x_1^2}{[(x_4^1 - x_1^1)^2 + (x_4^2 - x_1^2)^2 + (x_4^3 - x_1^3)^2]^{1/2}}, \tag{A.14b}$$

$$\phi_z((x_1,x_4)) = \frac{x_4^3 - x_1^3}{[(x_4^1 - x_1^1)^2 + (x_4^2 - x_1^2)^2 + (x_4^3 - x_1^3)^2]^{1/2}}. \tag{A.14c}$$

We assume that (A.8) is the volume form in the first $\mathbf{S}^2$ factor of the codomain in (A.10); then its pullback to $C(3+1,\mathbf{S}^3)$ under $\phi_{1,4}$ is given by

$$\phi_{1,4}^*\omega = \omega_{12}(\phi_{1,4}((x_1,x_2,x_3,x_4)))d\phi_x \wedge d\phi_y + \omega_{23}(\phi_{1,4}((x_1,x_2,x_3,x_4)))d\phi_y \wedge d\phi_z$$

$$+\omega_{13}(\phi_{1,4}((x_1,x_2,x_3,x_4)))d\phi_x \wedge d\phi_z. \qquad \text{(A.15)}$$

By defining

$$\Theta \equiv (x_4^1 - x_1^1)^2 + (x_4^2 - x_1^2)^2 + (x_4^3 - x_1^3)^2, \qquad \text{(A.16)}$$

the above equation reads

$$\begin{aligned}
\phi_{1,4}^*\omega = \ & \omega_{12}\left(\frac{x_4^1 - x_1^1}{\Theta^{1/2}}, \frac{x_4^2 - x_1^2}{\Theta^{1/2}}, \frac{x_4^3 - x_1^3}{\Theta^{1/2}}\right) d\phi_x \wedge d\phi_y \\
& +\omega_{23}\left(\frac{x_4^1 - x_1^1}{\Theta^{1/2}}, \frac{x_4^2 - x_1^2}{\Theta^{1/2}}, \frac{x_4^3 - x_1^3}{\Theta^{1/2}}\right) d\phi_y \wedge d\phi_z \\
& +\omega_{13}\left(\frac{x_4^1 - x_1^1}{\Theta^{1/2}}, \frac{x_4^2 - x_1^2}{\Theta^{1/2}}, \frac{x_4^3 - x_1^3}{\Theta^{1/2}}\right) d\phi_x \wedge d\phi_z \\
= \ & \frac{1}{4\pi}\left[\frac{\frac{x_4^3 - x_1^3}{\Theta^{1/2}}}{\left[\left(\frac{x_4^1 - x_1^1}{\Theta^{1/2}}\right)^2 + \left(\frac{x_4^2 - x_1^2}{\Theta^{1/2}}\right)^2 + \left(\frac{x_4^3 - x_1^3}{\Theta^{1/2}}\right)^2\right]^{3/2}}\right] d\phi_x \wedge d\phi_y \\
& +\frac{1}{4\pi}\left[\frac{\frac{x_4^1 - x_1^1}{\Theta^{1/2}}}{\left[\left(\frac{x_4^1 - x_1^1}{\Theta^{1/2}}\right)^2 + \left(\frac{x_4^2 - x_1^2}{\Theta^{1/2}}\right)^2 + \left(\frac{x_4^3 - x_1^3}{\Theta^{1/2}}\right)^2\right]^{3/2}}\right] d\phi_y \wedge d\phi_z \\
& +\frac{1}{4\pi}\left[\frac{-\left(\frac{x_4^2 - x_1^2}{\Theta^{1/2}}\right)}{\left[\left(\frac{x_4^1 - x_1^1}{\Theta^{1/2}}\right)^2 + \left(\frac{x_4^2 - x_1^2}{\Theta^{1/2}}\right)^2 + \left(\frac{x_4^3 - x_1^3}{\Theta^{1/2}}\right)^2\right]^{3/2}}\right] d\phi_x \wedge d\phi_z \\
= \ & \frac{1}{4\pi}\left[\frac{x_4^3 - x_1^3}{\Theta^{1/2}}\right] d\phi_x \wedge d\phi_y + \frac{1}{4\pi}\left[\frac{x_4^1 - x_1^1}{\Theta^{1/2}}\right] d\phi_y \wedge d\phi_z - \frac{1}{4\pi}\left[\frac{x_4^2 - x_1^2}{\Theta^{1/2}}\right] d\phi_x \wedge d\phi_z. \qquad \text{(A.17)}
\end{aligned}$$

Thus, the pullback of ω to $C(3+1,S^3)$ under ϕ_{14} is given by

$$\phi_{14}^*\omega = \frac{1}{4\pi}\left[\frac{x_4^3-x_1^3}{\Theta^{1/2}}\right]d\phi_x\wedge d\phi_y + \frac{1}{4\pi}\left[\frac{x_4^1-x_1^1}{\Theta^{1/2}}\right]d\phi_y\wedge d\phi_z - \frac{1}{4\pi}\left[\frac{x_4^2-x_1^2}{\Theta^{1/2}}\right]d\phi_x\wedge d\phi_z. \quad \text{(A.18)}$$

The next step to write $\phi_{1,4}^*\omega$ explicitly is to analyse the forms $d\phi_x\wedge d\phi_y$, $d\phi_y\wedge d\phi_z$ and $d\phi_x\wedge d\phi_z$ with ϕ_x, ϕ_y and ϕ_z defined from (A.14a) to (A.14c). Due to the fact that these functions do not depend on coordinates with subindices 2 and 3 the following simplifications apply

$$\begin{aligned} d\phi_x =& \frac{\partial\phi_x}{\partial x_1^1}dx_1^1 + \frac{\partial\phi_x}{\partial x_1^2}dx_1^2 + \frac{\partial\phi_x}{\partial x_1^3}dx_1^3 + \frac{\partial\phi_x}{\partial x_2^1}dx_2^1 + \frac{\partial\phi_x}{\partial x_2^2}dx_2^2 + \frac{\partial\phi_x}{\partial x_2^3}dx_2^3 \\ &+\frac{\partial\phi_x}{\partial x_3^1}dx_3^1 + \frac{\partial\phi_x}{\partial x_3^2}dx_3^2 + \frac{\partial\phi_x}{\partial x_3^3}dx_3^3 + \frac{\partial\phi_x}{\partial x_4^1}dx_4^1 + \frac{\partial\phi_x}{\partial x_4^2}dx_4^2 + \frac{\partial\phi_x}{\partial x_4^3}dx_4^3 \\ =& \frac{\partial\phi_x}{\partial x_1^1}dx_1^1 + \frac{\partial\phi_x}{\partial x_1^2}dx_1^2 + \frac{\partial\phi_x}{\partial x_1^3}dx_1^3 + \frac{\partial\phi_x}{\partial x_4^1}dx_4^1 + \frac{\partial\phi_x}{\partial x_4^2}dx_4^2 + \frac{\partial\phi_x}{\partial x_4^3}dx_4^3, \end{aligned} \quad \text{(A.19a)}$$

$$d\phi_y = \frac{\partial\phi_y}{\partial x_1^1}dx_1^1 + \frac{\partial\phi_y}{\partial x_1^2}dx_1^2 + \frac{\partial\phi_y}{\partial x_1^3}dx_1^3 + \frac{\partial\phi_y}{\partial x_4^1}dx_4^1 + \frac{\partial\phi_y}{\partial x_4^2}dx_4^2 + \frac{\partial\phi_y}{\partial x_4^3}dx_4^3, \quad \text{(A.19b)}$$

$$d\phi_z = \frac{\partial\phi_z}{\partial x_1^1}dx_1^1 + \frac{\partial\phi_z}{\partial x_1^2}dx_1^2 + \frac{\partial\phi_z}{\partial x_1^3}dx_1^3 + \frac{\partial\phi_z}{\partial x_4^1}dx_4^1 + \frac{\partial\phi_z}{\partial x_4^2}dx_4^2 + \frac{\partial\phi_z}{\partial x_4^3}dx_4^3. \quad \text{(A.19c)}$$

It is clear from the above equations that the forms $d\phi_x\wedge d\phi_y$, $d\phi_y\wedge d\phi_z$ and $d\phi_x\wedge d\phi_z$ have many mixed terms. In what follows the interest will be concentrated in coordinates x_4^1 and x_4^2, *i.e.*, just the part $dx_4^1\wedge dx_4^2$ of $\phi_{1,4}^*\omega$ will be analysed. The notation for this part will be $\left[\phi_{1,4}^*\omega\right]_{4,4}^{1,2}$. Thus from (A.18) we have

$$\begin{aligned} \left[\phi_{1,4}^*\omega\right]_{4,4}^{1,2} =& \frac{1}{4\pi}\left[\frac{x_4^3-x_1^3}{\Theta^{1/2}}\right]\left[d\phi_x\wedge d\phi_y\right]_{4,4}^{1,2} + \frac{1}{4\pi}\left[\frac{x_4^1-x_1^1}{\Theta^{1/2}}\right]\left[d\phi_y\wedge d\phi_z\right]_{4,4}^{1,2} \\ &-\frac{1}{4\pi}\left[\frac{x_4^2-x_1^2}{\Theta^{1/2}}\right]\left[d\phi_x\wedge d\phi_z\right]_{4,4}^{1,2}, \end{aligned} \quad \text{(A.20)}$$

where

$$\begin{aligned} \left[d\phi_x\wedge d\phi_y\right]_{4,4}^{1,2} =& \frac{\partial\phi_x}{\partial x_4^1}\frac{\partial\phi_y}{\partial x_4^2}dx_4^1\wedge dx_4^2 - \frac{\partial\phi_x}{\partial x_4^2}\frac{\partial\phi_y}{\partial x_4^1}dx_4^1\wedge dx_4^2 \\ =& \left[\frac{\Theta^{\frac{1}{2}}-(x_4^1-x_1^1)\Theta^{-\frac{1}{2}}(x_4^1-x_1^1)}{\Theta}\right]\left[\frac{\Theta^{\frac{1}{2}}-(x_4^2-x_1^2)\Theta^{-\frac{1}{2}}(x_4^2-x_1^2)}{\Theta}\right]dx_4^1\wedge dx_4^2 \\ &-\left[\frac{0-(x_4^1-x_1^1)\Theta^{-\frac{1}{2}}(x_4^2-x_1^2)}{\Theta}\right]\left[\frac{0-(x_4^2-x_1^2)\Theta^{-\frac{1}{2}}(x_4^1-x_1^1)}{\Theta}\right]dx_4^1\wedge dx_4^2 \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\Theta^{-\frac{1}{2}}}{\Theta}\right)^2 \left[\left(\Theta - (x_4^1 - x_1^1)(x_4^1 - x_1^1)\right)\left(\Theta - (x_4^2 - x_1^2)(x_4^2 - x_1^2)\right)\right] dx_4^1 \wedge dx_4^2 \\
&\quad - \left(\frac{\Theta^{-\frac{1}{2}}}{\Theta}\right)^2 \left[(x_4^1 - x_1^1)^2 (x_4^2 - x_1^2)^2\right] dx_4^1 \wedge dx_4^2 \\
&= \frac{1}{\Theta^3}\left[\Theta^2 - \Theta(x_4^1 - x_1^1)^2 - \Theta(x_4^2 - x_1^2)^2\right] dx_4^1 \wedge dx_4^2, \qquad \text{(A.21a)}
\end{aligned}
$$

$$
\begin{aligned}
\left[d\phi_y \wedge d\phi_z\right]_{4,4}^{1,2} &= \frac{\partial\phi_y}{\partial x_4^1}\frac{\partial\phi_z}{\partial x_4^2} dx_4^1 \wedge dx_4^2 - \frac{\partial\phi_y}{\partial x_4^2}\frac{\partial\phi_z}{\partial x_4^1} dx_4^1 \wedge dx_4^2 \\
&= \left[\frac{0 - (x_4^2 - x_1^2)\Theta^{-\frac{1}{2}}(x_4^1 - x_1^1)}{\Theta}\right]\left[\frac{0 - (x_4^3 - x_1^3)\Theta^{-\frac{1}{2}}(x_4^2 - x_1^2)}{\Theta}\right] dx_4^1 \wedge dx_4^2 \\
&\quad - \left[\frac{\Theta^{\frac{1}{2}} - (x_4^2 - x_1^2)\Theta^{-\frac{1}{2}}(x_4^2 - x_1^2)}{\Theta}\right]\left[\frac{0 - (x_4^3 - x_1^3)\Theta^{-\frac{1}{2}}(x_4^1 - x_1^1)}{\Theta}\right] dx_4^1 \wedge dx_4^2 \\
&= \left(\frac{\Theta^{-\frac{1}{2}}}{\Theta}\right)^2 \left[(x_4^1 - x_1^1)(x_4^2 - x_1^2)(x_4^2 - x_1^2)(x_4^3 - x_1^3)\right] dx_4^1 \wedge dx_4^2 \\
&\quad - \left(\frac{\Theta^{-\frac{1}{2}}}{\Theta}\right)^2 \left[\left(\Theta - (x_4^2 - x_1^2)(x_4^2 - x_1^2)\right)\left(-(x_4^1 - x_1^1)(x_4^3 - x_1^3)\right)\right] dx_4^1 \wedge dx_4^2 \\
&= \frac{1}{\Theta^3}\left[\Theta(x_4^1 - x_1^1)(x_4^3 - x_1^3)\right] dx_4^1 \wedge dx_4^2, \qquad \text{(A.21b)}
\end{aligned}
$$

$$
\begin{aligned}
\left[d\phi_x \wedge d\phi_z\right]_{4,4}^{1,2} &= \frac{\partial\phi_x}{\partial x_4^1}\frac{\partial\phi_z}{\partial x_4^2} dx_4^1 \wedge dx_4^2 - \frac{\partial\phi_x}{\partial x_4^2}\frac{\partial\phi_z}{\partial x_4^1} dx_4^1 \wedge dx_4^2 \\
&= \left[\frac{\Theta^{\frac{1}{2}} - (x_4^1 - x_1^1)\Theta^{-\frac{1}{2}}(x_4^1 - x_1^1)}{\Theta}\right]\left[\frac{0 - (x_4^3 - x_1^3)\Theta^{-\frac{1}{2}}(x_4^2 - x_1^2)}{\Theta}\right] dx_4^1 \wedge dx_4^2 \\
&\quad - \left[\frac{0 - (x_4^1 - x_1^1)\Theta^{-\frac{1}{2}}(x_4^2 - x_1^2)}{\Theta}\right]\left[\frac{0 - (x_4^3 - x_1^3)\Theta^{-\frac{1}{2}}(x_4^1 - x_1^1)}{\Theta}\right] dx_4^1 \wedge dx_4^2 \\
&= \left(\frac{\Theta^{-\frac{1}{2}}}{\Theta}\right)^2 \left[\left(\Theta - (x_4^1 - x_1^1)(x_4^1 - x_1^1)\right)\left(-(x_4^2 - x_1^2)(x_4^3 - x_1^3)\right)\right] dx_4^1 \wedge dx_4^2 \\
&\quad - \left(\frac{\Theta^{-\frac{1}{2}}}{\Theta}\right)^2 \left[(x_4^1 - x_1^1)^2 (x_4^2 - x_1^2)(x_4^3 - x_1^3)\right] dx_4^1 \wedge dx_4^2
\end{aligned}
$$

$$= \frac{1}{\Theta^3}\left[-\Theta(x_4^2 - x_1^2)(x_4^3 - x_1^3)\right] dx_4^1 \wedge dx_4^2. \tag{A.21c}$$

By substituting these expressions in (A.20) it is straightforward to find

$$\begin{aligned}
[\phi_{14}^*\omega]_{4,4}^{1,2} = &\ \frac{1}{4\pi}\left[\frac{x_4^3 - x_1^3}{\Theta^{1/2}}\right]\frac{1}{\Theta^3}\left[\Theta^2 - \Theta(x_4^1 - x_1^1)^2 - \Theta(x_4^2 - x_1^2)^2\right] dx_4^1 \wedge dx_4^2 \\
&+ \frac{1}{4\pi}\left[\frac{x_4^1 - x_1^1}{\Theta^{1/2}}\right]\frac{1}{\Theta^3}\left[\Theta(x_4^1 - x_1^1)(x_4^3 - x_1^3)\right] dx_4^1 \wedge dx_4^2 \\
&- \frac{1}{4\pi}\left[\frac{x_4^2 - x_1^2}{\Theta^{1/2}}\right]\frac{1}{\Theta^3}\left[-\Theta(x_4^2 - x_1^2)(x_4^3 - x_1^3)\right] dx_4^1 \wedge dx_4^2 \\
= &\ \frac{1}{4\pi}\left[\frac{1}{\Theta^{5/2}}\right]\Big[\Theta(x_4^3 - x_1^3) - (x_4^1 - x_1^1)^2(x_4^3 - x_1^3) - (x_4^2 - x_1^2)^2(x_4^3 - x_1^3) \\
&\qquad + (x_4^1 - x_1^1)^2(x_4^3 - x_1^3) + (x_4^2 - x_1^2)^2(x_4^3 - x_1^3)\Big] dx_4^1 \wedge dx_4^2 \\
= &\ \frac{1}{4\pi}\left[\frac{x_4^3 - x_1^3}{\Theta^{3/2}}\right] dx_4^1 \wedge dx_4^2 \\
= &\ \frac{1}{4\pi}\left[\frac{x_4^3 - x_1^3}{\left[(x_4^1 - x_1^1)^2 + (x_4^2 - x_1^2)^2 + (x_4^3 - x_1^3)^2\right]^{3/2}}\right] dx_4^1 \wedge dx_4^2 \\
= &\ \frac{1}{4\pi}\frac{x_4^3 - x_1^3}{|x_4 - x_1|^3} dx_4^1 \wedge dx_4^2 \\
= &\ \frac{1}{4\pi}\frac{x_4^3 - x_1^3}{|x_4 - x_1|^3}\left(\frac{1}{2}\left(dx_4^1 \wedge dx_4^2 - dx_4^2 \wedge dx_4^1\right)\right) \\
= &\ \frac{1}{4\pi}\frac{x_4^3 - x_1^3}{|x_4 - x_1|^3}\left(\frac{1}{2}\epsilon_{312} dx_4^1 \wedge dx_4^2 + \frac{1}{2}\epsilon_{321} dx_4^2 \wedge dx_4^1\right) \\
= &\ \frac{\epsilon_{3\nu\sigma}}{4\pi}\frac{x_4^3 - x_1^3}{|x_4 - x_1|^3}\left(\frac{1}{2} dx_4^\nu \wedge dx_4^\sigma\right),
\end{aligned} \tag{A.22}$$

i.e.,

$$\left[\phi_{1,4}^*\omega\right]_{4,4}^{1,2} = \frac{\varepsilon_{3\nu\sigma}}{4\pi}\frac{x_4^3 - x_1^3}{|x_4 - x_1|^3}\left(\frac{1}{2} dx_4^\nu \wedge dx_4^\sigma\right), \tag{A.23}$$

with $\nu, \sigma = 1, 2, 3$, which is to be compared with (1.77).

APPENDICES B

A star-triangle relation in two dimensions

B.1 Integrable model for $2d$ $\mathcal{N} = (2,2)$ $SU(2)$ duality

Reference [52] builds the integrable model associated with $2d$ $\mathcal{N} = (2,2)$ $SU(2)$ duality. The Boltzmann weights for this model are explicitly calculated by direct identification of Seiberg duality (2.10) and star-triangle relation (2.12). These two expressions are rewritten for convenience, namely,

$$\frac{1}{2}\left(\frac{(q,q)_\infty^2}{\theta(y;q)}\right)\int\frac{dz}{2\pi iz}\left[\frac{\prod_{i=1}^{6}\Delta(a_i z^{\pm1};q,y)}{\Delta(z^{\pm2};q,y)}\right] = \prod_{1\le i<j\le6}\Delta(a_ia_j;q,y) \tag{B.1}$$

and

$$\int d\sigma S(\sigma)W_{\eta-\gamma}(\sigma,\sigma_i)W_{\eta-\beta}(\sigma,\sigma_j)W_{\eta-\alpha}(\sigma,\sigma_k) = R(\alpha,\beta,\gamma)W_\alpha(\sigma_i,\sigma_j)W_\beta(\sigma_i,\sigma_k)W_\gamma(\sigma_j,\sigma_k). \tag{B.2}$$

Let's work explicitly both sides of equation (B.1). Define the following relations between different parameters of the gauge/YBE correspondence

$$\begin{aligned} a_1 &= e^{-\alpha+i\sigma_i}, & a_3 &= e^{-\beta+i\sigma_j}, & a_5 &= e^{-\gamma+i\sigma_k}, & z &= e^{i\sigma},\\ a_2 &= e^{-\alpha-i\sigma_i}, & a_4 &= e^{-\beta-i\sigma_j}, & a_6 &= e^{-\gamma-i\sigma_k}, & \frac{q}{y} &= e^{-2\eta}. \end{aligned} \tag{B.3}$$

Note that the last equality involving q/y is actually the balancing condition

$$\frac{q}{y} = \prod_{i=1}^{6} a_i = e^{-2\eta}. \tag{B.4}$$

It will also be useful to consider the following expresions

$$\begin{aligned} \Delta(a;q,y) &= \frac{\theta(ay;q)}{\theta(a;q)},\\ \theta(az^{\pm1};q) &= \theta(az;q)\theta(az^{-1};q),\\ \theta(zq;q) &= \theta(z^{-1};q). \end{aligned} \tag{B.5}$$

For the right hand side of (B.1) one has, by using (B.3) and (B.5),

$$A = \prod_{1\le i<j\le6}\Delta(a_ia_j;q,y) = \prod_{1\le i<j\le6}\frac{\theta(a_ia_jy;q)}{\theta(a_ia_j;q)}, \tag{B.6}$$

where

$$\theta(a_i a_j y;q)=\theta(a_i a_j e^{2\eta}q;q)=\theta\left(\left(a_i a_j e^{2\eta}\right)^{-1};q\right)=\theta\left(a_i^{-1}a_j^{-1}e^{-2\eta};q\right). \tag{B.7}$$

To work out (B.6) explicitly calculate

$$\begin{aligned}
&a_1a_2=e^{-2\alpha},\\
&a_1a_3=e^{-(\alpha+\beta)+i(\sigma_i+\sigma_j)}, && a_2a_3=e^{-(\alpha+\beta)-i(\sigma_i-\sigma_j)},\\
&a_1a_4=e^{-(\alpha+\beta)+i(\sigma_i-\sigma_j)}, && a_2a_4=e^{-(\alpha+\beta)-i(\sigma_i+\sigma_j)}, && a_3a_4=e^{-2\beta},\\
&a_1a_5=e^{-(\alpha+\gamma)+i(\sigma_i+\sigma_k)}, && a_2a_5=e^{-(\alpha+\gamma)-i(\sigma_i-\sigma_k)}, && a_3a_5=e^{-(\beta+\gamma)+i(\sigma_j+\sigma_k)},\\
&a_1a_6=e^{-(\alpha+\gamma)+i(\sigma_i-\sigma_k)}, && a_2a_6=e^{-(\alpha+\gamma)-i(\sigma_i+\sigma_k)}, && a_3a_6=e^{-(\beta+\gamma)+i(\sigma_j-\sigma_k)},\\
\\
&a_4a_5=e^{-(\beta+\gamma)-i(\sigma_j-\sigma_k)},\\
&a_4a_6=e^{-(\beta+\gamma)-i(\sigma_j+\sigma_k)}, && a_5a_6=e^{-2\gamma},
\end{aligned} \tag{B.8}$$

and use them along with (B.7) to obtain

$$\begin{aligned}
&\theta(a_1a_2y;q),=\theta\left(e^{-2\beta-2\gamma};q\right),\\
&\theta(a_1a_3y;q),=\theta\left(e^{-\gamma-\eta-i(\sigma_i+\sigma_j)};q\right), && \theta(a_2a_3y;q)=\theta\left(e^{-\gamma-\eta+i(\sigma_i-\sigma_j)};q\right),\\
&\theta(a_1a_4y;q),=\theta\left(e^{-\gamma-\eta-i(\sigma_i-\sigma_j)};q\right), && \theta(a_2a_4y;q)=\theta\left(e^{-\gamma-\eta+i(\sigma_i+\sigma_j)};q\right),\\
&\theta(a_1a_5y;q),=\theta\left(e^{-\beta-\eta-i(\sigma_i+\sigma_k)};q\right), && \theta(a_2a_5y;q)=\theta\left(e^{-\beta-\eta+i(\sigma_i-\sigma_k)};q\right),\\
&\theta(a_1a_6y;q),=\theta\left(e^{-\beta-\eta-i(\sigma_i-\sigma_k)};q\right), && \theta(a_2a_6y;q)=\theta\left(e^{-\beta-\eta+i(\sigma_i+\sigma_k)};q\right),\\
\\
&\theta(a_3a_4y;q),=\theta\left(e^{-2\alpha-2\gamma};q\right),\\
&\theta(a_3a_5y;q),=\theta\left(e^{-\alpha-\eta-i(\sigma_j+\sigma_k)};q\right), && \theta(a_4a_5y;q)=\theta\left(e^{-\alpha-\eta+i(\sigma_j-\sigma_k)};q\right),\\
&\theta(a_3a_6y;q),=\theta\left(e^{-\alpha-\eta-i(\sigma_j-\sigma_k)};q\right), && \theta(a_4a_6y;q)=\theta\left(e^{-\alpha-\eta+i(\sigma_j+\sigma_k)};q\right),\\
\\
&\theta(a_5a_6y;q),=\theta\left(e^{-2\alpha-2\beta};q\right),
\end{aligned} \tag{B.9}$$

and

$$\begin{aligned}
&\theta(a_1a_2;q),=\theta\left(e^{-2\alpha};q\right),\\
&\theta(a_1a_3;q),=\theta\left(e^{\gamma-\eta+i(\sigma_i+\sigma_j)};q\right), && \theta(a_2a_3;q)=\theta\left(e^{\gamma-\eta-i(\sigma_i-\sigma_j)};q\right),\\
&\theta(a_1a_4;q),=\theta\left(e^{\gamma-\eta+i(\sigma_i-\sigma_j)};q\right), && \theta(a_2a_4;q)=\theta\left(e^{\gamma-\eta-i(\sigma_i+\sigma_j)};q\right),\\
&\theta(a_1a_5;q),=\theta\left(e^{\beta-\eta+i(\sigma_i+\sigma_k)};q\right), && \theta(a_2a_5;q)=\theta\left(e^{\beta-\eta-i(\sigma_i-\sigma_k)};q\right),\\
&\theta(a_1a_6;q),=\theta\left(e^{\beta-\eta+i(\sigma_i-\sigma_k)};q\right), && \theta(a_2a_6;q)=\theta\left(e^{\beta-\eta-i(\sigma_i+\sigma_k)};q\right),\\
\\
&\theta(a_3a_4;q),=\theta\left(e^{-2\beta};q\right),\\
&\theta(a_3a_5;q),=\theta\left(e^{\alpha-\eta+i(\sigma_j+\sigma_k)};q\right), && \theta(a_4a_5;q)=\theta\left(e^{\alpha-\eta-i(\sigma_j-\sigma_k)};q\right),
\end{aligned}$$

$$\theta(a_3a_6;q), = \theta\left(e^{\alpha-\eta+i(\sigma_j-\sigma_k)};q\right), \qquad \theta(a_4a_6;q) = \theta\left(e^{\alpha-\eta-i(\sigma_j+\sigma_k)};q\right),$$

$$\theta(a_5a_6;q) = \theta\left(e^{-2\gamma};q\right). \tag{B.10}$$

Combination of these last two sets of expressions leads to write (B.6) as

$$A = \frac{\theta\left(e^{-2\alpha-2\gamma};q\right)}{\theta\left(e^{-2\beta};q\right)}\frac{\theta\left(e^{-2\beta-2\gamma};q\right)}{\theta\left(e^{-2\alpha};q\right)}\frac{\theta\left(e^{-2\alpha-2\beta};q\right)}{\theta\left(e^{-2\gamma};q\right)}\times$$

$$\times\frac{\theta\left(e^{-\alpha-\eta-i(\sigma_j+\sigma_k)};q\right)}{\theta\left(e^{\alpha-\eta+i(\sigma_j+\sigma_k)};q\right)}\frac{\theta\left(e^{-\alpha-\eta-i(\sigma_j-\sigma_k)};q\right)}{\theta\left(e^{\alpha-\eta+i(\sigma_j-\sigma_k)};q\right)}\frac{\theta\left(e^{-\alpha-\eta+i(\sigma_j+\sigma_k)};q\right)}{\theta\left(e^{\alpha-\eta-i(\sigma_j+\sigma_k)};q\right)}\frac{\theta\left(e^{-\alpha-\eta+i(\sigma_j-\sigma_k)};q\right)}{\theta\left(e^{\alpha-\eta-i(\sigma_j-\sigma_k)};q\right)}\times$$

$$\times\frac{\theta\left(e^{-\beta-\eta-i(\sigma_i+\sigma_k)};q\right)}{\theta\left(e^{\beta-\eta+i(\sigma_i+\sigma_k)};q\right)}\frac{\theta\left(e^{-\beta-\eta-i(\sigma_i-\sigma_k)};q\right)}{\theta\left(e^{\beta-\eta+i(\sigma_i-\sigma_k)};q\right)}\frac{\theta\left(e^{-\beta-\eta+i(\sigma_i+\sigma_k)};q\right)}{\theta\left(e^{\beta-\eta-i(\sigma_i+\sigma_k)};q\right)}\frac{\theta\left(e^{-\beta-\eta+i(\sigma_i-\sigma_k)};q\right)}{\theta\left(e^{\beta-\eta-i(\sigma_i-\sigma_k)};q\right)}\times$$

$$\times\frac{\theta\left(e^{-\gamma-\eta-i(\sigma_i+\sigma_j)};q\right)}{\theta\left(e^{\gamma-\eta+i(\sigma_i+\sigma_j)};q\right)}\frac{\theta\left(e^{-\gamma-\eta-i(\sigma_i-\sigma_j)};q\right)}{\theta\left(e^{\gamma-\eta+i(\sigma_i-\sigma_j)};q\right)}\frac{\theta\left(e^{-\gamma-\eta+i(\sigma_i+\sigma_j)};q\right)}{\theta\left(e^{\gamma-\eta-i(\sigma_i+\sigma_j)};q\right)}\frac{\theta\left(e^{-\gamma-\eta+i(\sigma_i-\sigma_j)};q\right)}{\theta\left(e^{\gamma-\eta-i(\sigma_i-\sigma_j)};q\right)}\times$$

$$= \frac{\theta\left(e^{-2\alpha-2\gamma};q\right)}{\theta\left(e^{-2\beta};q\right)}\frac{\theta\left(e^{-2\beta-2\gamma};q\right)}{\theta\left(e^{-2\alpha};q\right)}\frac{\theta\left(e^{-2\alpha-2\beta};q\right)}{\theta\left(e^{-2\gamma};q\right)}\times$$

$$\times\frac{\theta\left(e^{-\alpha-\eta\mp i(\sigma_j\pm\sigma_k)};q\right)}{\theta\left(e^{\alpha-\eta\pm i(\sigma_j\pm\sigma_k)};q\right)}\frac{\theta\left(e^{-\beta-\eta\mp i(\sigma_i\pm\sigma_k)};q\right)}{\theta\left(e^{\beta-\eta\pm i(\sigma_i\pm\sigma_k)};q\right)}\frac{\theta\left(e^{-\gamma-\eta\mp i(\sigma_i\pm\sigma_j)};q\right)}{\theta\left(e^{\gamma-\eta\pm i(\sigma_i\pm\sigma_j)};q\right)}. \tag{B.11}$$

For the left hand side of (B.1) one has, by using (B.5),

$$B = \frac{1}{2}\left(\frac{(q,q)_\infty^2}{\theta(y;q)}\right)\int\frac{dz}{2\pi iz}\left[\frac{\prod_{i=1}^6\Delta(a_iz^{\pm1};q,y)}{\Delta(z^{\pm2};q,y)}\right] \tag{B.12}$$

$$= \frac{1}{2}\left(\frac{(q,q)_\infty^2}{\theta(y;q)}\right)\int\frac{dz}{2\pi iz}\left[\frac{\theta(z^{\pm2};q)}{\theta(z^{\pm2}y;q)}\prod_{i=1}^6\frac{\theta(a_iz^{\pm1}y;q)}{\theta(a_iz^{\pm1};q)}\right] \tag{B.13}$$

$$= \frac{1}{2}\left(\frac{(q,q)_\infty^2}{\theta(y;q)}\right)\int\frac{dz}{2\pi iz}\left[\frac{\theta(z^{\pm2};q)}{\theta(z^{\pm2}y;q)}\prod_{i=1}^6\frac{\theta(a_izy;q)\;\theta(a_iz^{-1}y;q)}{\theta(a_iz;q)\;\theta(a_iz^{-1};q)}\right], \tag{B.14}$$

where

$$\theta(z^{\pm2}y;q) = \theta\left(z^{\pm2}qe^{2\eta};q\right) = \theta\left(\left(z^{\pm2}e^{2\eta}\right)^{-1};q\right) = \theta\left(z^{\mp2}e^{-2\eta};q\right),$$

$$\theta(a_izy;q) = \theta\left(a_izqe^{2\eta};q\right) = \theta\left(\left(a_ize^{2\eta}\right)^{-1};q\right) = \theta\left(a_i^{-1}z^{-1}e^{-2\eta};q\right),$$

$$\theta(a_iz^{-1}y;q) = \theta\left(a_iz^{-1}qe^{2\eta};q\right) = \theta\left(\left(a_iz^{-1}e^{2\eta}\right)^{-1};q\right) = \theta\left(a_i^{-1}ze^{-2\eta};q\right). \tag{B.15}$$

Now use (B.3) and (B.15) to obtain

$$\theta(a_1zy;q) = \theta\left(e^{-(\eta-\alpha)-\eta-i(\sigma_i+\sigma)};q\right), \qquad \theta(a_1z;q) = \theta\left(e^{(\eta-\alpha)-\eta+i(\sigma_i+\sigma)};q\right),$$

$$\begin{aligned}
\theta(a_1 z^{-1} y;q) &= \theta\left(e^{-(\eta-\alpha)-\eta-i(\sigma_i-\sigma)};q\right), & \theta(a_1 z^{-1};q) &= \theta\left(e^{(\eta-\alpha)-\eta+i(\sigma_i-\sigma)};q\right),\\
\theta(a_2 zy;q), &= \theta\left(e^{-(\eta-\alpha)-\eta+i(\sigma_i-\sigma)};q\right), & \theta(a_2 z;q) &= \theta\left(e^{(\eta-\alpha)-\eta-i(\sigma_i-\sigma)};q\right),\\
\theta(a_2 z^{-1} y;q) &= \theta\left(e^{-(\eta-\alpha)-\eta+i(\sigma_i+\sigma)};q\right), & \theta(a_2 z^{-1};q) &= \theta\left(e^{(\eta-\alpha)-\eta-i(\sigma_i+\sigma)};q\right),\\
\theta(a_3 zy;q) &= \theta\left(e^{-(\eta-\beta)-\eta-i(\sigma_j+\sigma)};q\right), & \theta(a_3 z;q) &= \theta\left(e^{(\eta-\beta)-\eta+i(\sigma_j+\sigma)};q\right),\\
\theta(a_3 z^{-1} y;q) &= \theta\left(e^{-(\eta-\beta)-\eta-i(\sigma_j-\sigma)};q\right), & \theta(a_3 z^{-1};q) &= \theta\left(e^{(\eta-\beta)-\eta+i(\sigma_j-\sigma)};q\right),\\
\theta(a_4 zy;q) &= \theta\left(e^{-(\eta-\beta)-\eta+i(\sigma_j-\sigma)};q\right), & \theta(a_4 z;q) &= \theta\left(e^{(\eta-\beta)-\eta-i(\sigma_j-\sigma)};q\right),\\
\theta(a_4 z^{-1} y;q) &= \theta\left(e^{-(\eta-\beta)-\eta+i(\sigma_j+\sigma)};q\right), & \theta(a_4 z^{-1};q) &= \theta\left(e^{(\eta-\beta)-\eta-i(\sigma_j+\sigma)};q\right),\\
\theta(a_5 zy;q) &= \theta\left(e^{-(\eta-\gamma)-\eta-i(\sigma_k+\sigma)};q\right), & \theta(a_5 z;q) &= \theta\left(e^{(\eta-\gamma)-\eta+i(\sigma_k+\sigma)};q\right),\\
\theta(a_5 z^{-1} y;q) &= \theta\left(e^{-(\eta-\gamma)-\eta-i(\sigma_k-\sigma)};q\right), & \theta(a_5 z^{-1};q) &= \theta\left(e^{(\eta-\gamma)-\eta+i(\sigma_k-\sigma)};q\right),\\
\theta(a_6 zy;q) &= \theta\left(e^{-(\eta-\gamma)-\eta+i(\sigma_k-\sigma)};q\right), & \theta(a_6 z;q) &= \theta\left(e^{(\eta-\gamma)-\eta-i(\sigma_k-\sigma)};q\right),\\
\theta(a_6 z^{-1} y;q) &= \theta\left(e^{-(\eta-\gamma)-\eta+i(\sigma_k+\sigma)};q\right), & \theta(a_6 z^{-1};q) &= \theta\left(e^{(\eta-\gamma)-\eta-i(\sigma_k+\sigma)};q\right).
\end{aligned} \tag{B.16}$$

By using these expressions, (B.14) can be written as

$$\begin{aligned}
B = \frac{1}{2}\left(\frac{(q,q)^2_\infty}{\theta(y;q)}\right)\int \frac{dz}{2\pi i z}\Bigg[&\frac{\theta\left(e^{\pm 2i\sigma;q}\right)}{\theta\left(e^{-2\eta\pm 2i\sigma;q}\right)}\times\\
&\times\frac{\theta\left(e^{-(\eta-\alpha)-\eta-i(\sigma_i+\sigma)};q\right)\theta\left(e^{-(\eta-\alpha)-\eta-i(\sigma_i-\sigma)};q\right)}{\theta\left(e^{(\eta-\alpha)-\eta+i(\sigma_i+\sigma)};q\right)\theta\left(e^{(\eta-\alpha)-\eta+i(\sigma_i-\sigma)};q\right)}\times\\
&\times\frac{\theta\left(e^{-(\eta-\alpha)-\eta+i(\sigma_i-\sigma)};q\right)\theta\left(e^{-(\eta-\alpha)-\eta+i(\sigma_i+\sigma)};q\right)}{\theta\left(e^{(\eta-\alpha)-\eta-i(\sigma_i-\sigma)};q\right)\theta\left(e^{(\eta-\alpha)-\eta-i(\sigma_i+\sigma)};q\right)}\times\\
&\times\frac{\theta\left(e^{-(\eta-\beta)-\eta-i(\sigma_j+\sigma)};q\right)\theta\left(e^{-(\eta-\beta)-\eta-i(\sigma_j-\sigma)};q\right)}{\theta\left(e^{(\eta-\beta)-\eta+i(\sigma_j+\sigma)};q\right)\theta\left(e^{(\eta-\beta)-\eta+i(\sigma_j-\sigma)};q\right)}\times\\
&\times\frac{\theta\left(e^{-(\eta-\beta)-\eta+i(\sigma_j-\sigma)};q\right)\theta\left(e^{-(\eta-\beta)-\eta+i(\sigma_j+\sigma)};q\right)}{\theta\left(e^{(\eta-\beta)-\eta-i(\sigma_j-\sigma)};q\right)\theta\left(e^{(\eta-\beta)-\eta-i(\sigma_j+\sigma)};q\right)}\times\\
&\times\frac{\theta\left(e^{-(\eta-\gamma)-\eta-i(\sigma_k+\sigma)};q\right)\theta\left(e^{-(\eta-\gamma)-\eta-i(\sigma_k-\sigma)};q\right)}{\theta\left(e^{(\eta-\gamma)-\eta+i(\sigma_k+\sigma)};q\right)\theta\left(e^{(\eta-\gamma)-\eta+i(\sigma_k-\sigma)};q\right)}\times\\
&\times\frac{\theta\left(e^{-(\eta-\gamma)-\eta+i(\sigma_k-\sigma)};q\right)\theta\left(e^{-(\eta-\gamma)-\eta+i(\sigma_k+\sigma)};q\right)}{\theta\left(e^{(\eta-\gamma)-\eta-i(\sigma_k-\sigma)};q\right)\theta\left(e^{(\eta-\gamma)-\eta-i(\sigma_k+\sigma)};q\right)}\Bigg]
\end{aligned}$$

$$= \frac{1}{2}\left(\frac{(q,q)^2_\infty}{\theta(y;q)}\right)\int \frac{dz}{2\pi i z}\Bigg[\frac{\theta\left(e^{\pm 2i\sigma;q}\right)}{\theta\left(e^{-2\eta\pm 2i\sigma;q}\right)}\frac{\theta\left(e^{-(\eta-\alpha)-\eta\mp i(\sigma_i\pm\sigma)};q\right)}{\theta\left(e^{(\eta-\alpha)-\eta\pm i(\sigma_i\pm\sigma)};q\right)}\times$$

$$\times \frac{\theta\left(e^{-(\eta-\beta)-\eta\mp i(\sigma_j\pm\sigma)};q\right)}{\theta\left(e^{(\eta-\beta)-\eta\pm i(\sigma_j\pm\sigma)};q\right)}\frac{\theta\left(e^{-(\eta-\gamma)-\eta\mp i(\sigma_k\pm\sigma)};q\right)}{\theta\left(e^{(\eta-\gamma)-\eta\pm i(\sigma_k\pm\sigma)};q\right)}\Bigg]. \tag{B.17}$$

Identification of (B.11) and (B.17) with the left and right hand sides of (B.2), respectively, yields to the following definition of Boltzmann weights (use also $z = e^{i\sigma}$ from (B.3))

$$\begin{aligned}
W_\alpha(\sigma_j,\sigma_k) &= \frac{\theta\left(e^{-\alpha-\eta\mp i(\sigma_j\pm\sigma_k)};q\right)}{\theta\left(e^{\alpha-\eta\pm i(\sigma_j\pm\sigma_k)};q\right)}, & W_{\eta-\alpha}(\sigma_i,\sigma) &= \frac{\theta\left(e^{-(\eta-\alpha)-\eta\mp i(\sigma_i\pm\sigma)};q\right)}{\theta\left(e^{(\eta-\alpha)-\eta\pm i(\sigma_i\pm\sigma)};q\right)},\\
W_\beta(\sigma_i,\sigma_k) &= \frac{\theta\left(e^{-\beta-\eta\mp i(\sigma_i\pm\sigma_k)};q\right)}{\theta\left(e^{\beta-\eta\pm i(\sigma_i\pm\sigma_k)};q\right)}, & W_{\eta-\beta}(\sigma_j,\sigma) &= \frac{\theta\left(e^{-(\eta-\beta)-\eta\mp i(\sigma_j\pm\sigma)};q\right)}{\theta\left(e^{(\eta-\beta)-\eta\pm i(\sigma_j\pm\sigma)};q\right)},\\
W_\gamma(\sigma_i,\sigma_j) &= \frac{\theta\left(e^{-\gamma-\eta\mp i(\sigma_i\pm\sigma_j)};q\right)}{\theta\left(e^{\gamma-\eta\pm i(\sigma_i\pm\sigma_j)};q\right)}, & W_{\eta-\gamma}(\sigma_k,\sigma) &= \frac{\theta\left(e^{-(\eta-\gamma)-\eta\mp i(\sigma_k\pm\sigma)};q\right)}{\theta\left(e^{(\eta-\gamma)-\eta\pm i(\sigma_k\pm\sigma)};q\right)},
\end{aligned} \tag{B.18}$$

and the interaction, $S(\sigma)$, and normalization, $R(\alpha,\beta,\gamma)$, factors

$$S(\sigma) = \frac{1}{4\pi}\left(\frac{(q,q)^2_\infty}{\theta(y;q)}\right)\frac{\theta\left(e^{\pm 2i\sigma;q}\right)}{\theta\left(e^{-2\eta\pm 2i\sigma;q}\right)},$$

$$R(\alpha,\beta,\gamma) = \frac{\theta\left(e^{-2\alpha-2\gamma};q\right)}{\theta\left(e^{-2\beta};q\right)}\frac{\theta\left(e^{-2\beta-2\gamma};q\right)}{\theta\left(e^{-2\alpha};q\right)}\frac{\theta\left(e^{-2\alpha-2\beta};q\right)}{\theta\left(e^{-2\gamma};q\right)}, \tag{B.19}$$

this is exactly the integrable model described in [52].

References

[1] E. Witten, "Supersymmetry and Morse theory", J. Differential Geom. **17**, No. 4, 661-692 (1982).

[2] E. Witten, "Quantum Field Theory and the Jones Polynomial", Commun. Math. Phys. **121**, 351-399 (1989).

[3] J. de-la-Cruz-Moreno, H. García-Compeán and E. López, "Vassiliev Invariants for Flows Via Chern-Simons Perturbation Theory", arXiv:2004.13893 [hep-th].

[4] A. Schwarz, "The Partition Function of Degenerate Quadratic Functional and Ray-Singer Invariants", Lett. Math. Phys. **2**, 247-252 (1978); "The Partition Function of a Degenerate Functional", Commun. Math. Phys. **67**, 1-16 (1979).

[5] D. B. Ray and I. M. Singer, "R-torsion and the Laplacian on Riemannian Manifolds", Adv. Math. **7**, 145 (1971).

[6] E. Guadagnini, M. Martellini and M. Mintchev, "Wilson Lines in Chern-Simons Theory and Link Invariants", Nucl. Phys. B **330**, 575 (1990), doi:10.1016/0550-3213(90)90124-V.

[7] D. Altschuler and L. Freidel, "Vassiliev Knot Invariants and Chern-Simons Perturbation Theory to All Orders", Commun. Math. Phys. **187**, 261–287 (1997).

[8] E. Guadagnini, *Link Invariants of the Chern-Simons Field Theory*, Expositions in Mathematics, De Gruyer (1993).

[9] Sen Hu, *Lecture Notes on Chern-Simons-Witten Theory*, World Scientific (2001).

[10] M. Polyak, "Feynman Diagrams for Pedestrians and Mathematicians", Proc. Symp. Pure Math. **73**, 15 (2005), [math/0406251 [math.GT]].

[11] J. Sawon, "Perturbative Expansion of Chern-Simons Theory", Geom. Topol. Monogr. **8**, 145-166 (2006).

[12] D. Bar-Natan, "Perturbative Aspects of the Chern-Simons Topological Quantum Field Theory", Ph. D. thesis, Department of Mathematics, Princeton University (1991).

[13] D. Bar-Natan, "Perturbative Chern-Simons Theory", J. Knot Theo. Ram. **04**, no. 04, 503-547 (1995).

[14] S. Axelrod and I. M. Singer, "Chern-Simons Perturbation Theory", hep-th/9110056.

[15] S. Axelrod and I. M. Singer, "Chern-Simons Perturbation Theory. II", J. Diff. Geom. **39**, no. 1, 173 (1994), [hep-th/9304087].

[16] S. Chmutov, S. Duzhin and J. Mostovoy, *Introduction to Vassiliev Knot Invariants*, Cambridge University Press (2012).

[17] M. Alvarez and J. M. F. Labastida, "Numerical Knot Invariants of Finite Type from Chern-Simons Perturbation Theory", Nucl. Phys. B **433**, 555-596 (1995).

[18] M. Alvarez, J. M. F. Labastida and E. Pérez, "Vassiliev Invariants for Links from Chern-Simons Perturbation Theory", Nucl. Phys. B **488**, no. 3, 677-718 (1997).

[19] A. Verjovsky and R. F. Vila-Freyer, "The Jones-Witten Invariant for Flows on a 3-Dimensional Manifold", Commun. Math. Phys. **163**, 73-88 (1994).

[20] S. Schwartzman, "Asymptotic Cycles", Ann. Math. **66**(2), 270-284 (1957).

[21] D. Ruelle and D. Sullivan, "Currents, Flows and Diffeomorphisms", Topology **14**, 319-327 (1975).

[22] D. Sullivan, "Cycles for the Dynamical Study of Foliated Manifolds and Complex Manifolds", Invent. Math. **36**, 225-255 (1976).

[23] S. Schwartzman, "Higher Dimensional Asymptotic Cycles", Canad. J. Math. **55**(3), 636-648 (2003).

[24] V. Muñoz and R. Pérez-Marco, "Intersection Theory for Ergodic Solenoids", in *Frontiers in Complex Dynamics: in Celebration of John Milnor's 80th Birthday*, A. Bonifant, M. Lyubich and S. Sutherland (Eds.), Princeton Mathematical Series, Princeton University Press, pp. 623-644 (2014).

[25] V. I. Arnold, "The Asymptotic Hopf Invariant and its Applications", Proc. Summer School in Differential Equations at Dilizhan, 1973 (1974), Erevan (in Russian); English translation: Sel. Math. Sov. **5**, 327-345 (1986).

[26] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer Verlag (1982).

[27] H. K. Moffat, "The Degree of Knottedness of Tangled Vortex Lines", J. Fluid. Mech. **35**, 1213-1216 (1969).

[28] L. Woltjer, "A Theorem on Force-free Magnetic Fields", Proc. Natn. Acad. Sci. **44**, 489-491 (1958).

[29] T. Vogel, "On the Asymptotic Linking Number", arXiv:math/0011159[math.DS].

[30] A. Enciso, D. Peralta-Salas and F. Torres de Lizaur, "Helicity is the Only Integral Invariant of Volume-preserving Transformations", Proc. Natl. Acad. Sci. USA **113**:8, 2035-2040 (2016).

[31] V. I. Arnold and B. Khesin, *Topological Methods in Hydrodynamics*, Springer Verlag (1998).

[32] B. Khesin, "Topological Fluid Dynamics", Notices of the AMS **52**:1, 9-19 (2005).

[33] P. Dehornoy, "Asymptotic Invariants of 3-Dimensional Fields", Winter Braids Lecture Notes **2**, 1-19 (2015).

[34] H. García-Compeán and R. Santos-Silva, "Link Invariants for Flows in Higher Dimensions", J. Math. Phys. **51**:6, 063506-1 - 063506-17 (2010).

[35] H. García-Compeán, R. Santos-Silva and A. Verjovsky, "Invariants of Four-manifolds with Flows Via Cohomological Field Theory", in *Frontiers in Complex Dynamics: in Celebration of John Milnor's 80th Birthday*, A. Bonifant, M. Lyubich and S. Sutherland (Eds.), Princeton Mathematical Series, Princeton University Press, pp. 645-676 (2014).

[36] M. Kontsevich, "Feynman Diagrams and Low-dimensional Topology", in *Progress in Mathematics*, Vol. II, pp. 120 (1994).

[37] A. S. Cattaneo, P. Cotta-Ramusino and R. Longoni, "Configuration Spaces and Vassiliev Classes in any Dimension", Algebr. Geom. Topol. **2**, 949 (2002), doi:10.2140/agt.2002.2.949, [math/9910139 [math.GT]].

[38] R. Bott and C. Taubes, "On the Self-linking of Knots", J. Math. Phys. **35**:10, 5247-5287 (1994).

[39] I. Volić, "A Survey of Bott-Taubes Integration", J. Knot Theo. Ram. **16**:01, 1-42 (2007).

[40] D. Thurston, "Integral Expressions for the Vassiliev Knot Invariants", arXiv:math/9901110v2 [math.QA] (1999).

[41] R. Komendarczyk and I. Volić, "On Volume-preserving Vector Fields and Finite-type Invariants of Knots", Ergod. Th. and Dynam. Sys. **36**, 832–859 (2016).

[42] S. Baader and J. Marché, "Asymptotic Vassiliev Invariants for Vector Fields", Bull. Soc. Math. France **140**:4, 569–582 (2012).

[43] J. de-la-Cruz-Moreno and H. García-Compeán, "Star-triangle type relations from $2d$ $\mathcal{N} = (0,2)$ $USp(2N)$ dualities", JHEP **01**, 023 (2021), doi:10.1007/JHEP01(2021)023 [arXiv:2008.02419 [hep-th]].

[44] R. J. Baxter, "Exactly solved models in statistical mechanics", Academic Press (1982).

[45] M. Jimbo and T. Miwa, "Nonlinear integrable systems - classical theory and quantum theory. Proceedings, RIMS Symposium, Kyoto, Japan, May 13-16, 1981", World Scientific (1983).

[46] M. Jimbo, "Yang-Baxter equation in integrable systems", Adv. Ser. Math. Phys. **10**, 1-715 (1989).

[47] C. N. Yang and M. L. Ge, "Braid group, knot theory and statistical mechanics", Adv. Ser. Math. Phys. **9**, 1-329 (1989).

[48] C. N. Yang and M. L. Ge, "Braid group, knot theory and statistical mechanics. Vol. 2", Adv. Ser. Math. Phys. **17**, 1-467 (1994).

[49] C. Gomez, G. Sierra and M. Ruiz-Altaba, "Quantum groups in two-dimensional physics", doi:10.1017/CBO9780511628825.

[50] M. Yamazaki, "Integrability as duality: the gauge/YBE correspondence", Phys. Rept. **859**, 1-20 (2020), doi:10.1016/j.physrep.2020.01.006 [arXiv:1808.04374 [hep-th]].

[51] M. Yamazaki and W. Yan, "Integrability from 2d $\mathcal{N} = (2,2)$ dualities", J. Phys. A **48**, 394001 (2015), doi:10.1088/1751-8113/48/39/394001 [arXiv:1504.05540 [hep-th]].

[52] S. Jafarzade and Z. Nazari, "A new integrable Ising-type model from 2d $\mathcal{N}$=(2,2) dualities", [arXiv:1709.00070 [hep-th]].

[53] E. Eren, I. Gahramanov, S. Jafarzade and G. Mogol, "Gamma function solutions to the star-triangle equation", [arXiv:1912.12271 [math-ph]].

[54] I. Gahramanov and A. P. Kels, "The star-triangle relation, lens partition function, and hypergeometric sum/integrals", JHEP **02**, 040 (2017), doi:10.1007/JHEP02(2017)040 [arXiv:1610.09229 [math-ph]].

[55] V. P. Spiridonov, "Elliptic beta integrals and solvable models of statistical mechanics", Contemp. Math. **563**, 181-211 (2012), [arXiv:1011.3798 [hep-th]].

[56] M. Yamazaki, "New integrable models from the gauge/YBE correspondence", J. Statist. Phys. **154**, 895 (2014), doi:10.1007/s10955-013-0884-8 [arXiv:1307.1128 [hep-th]].

[57] I. Gahramanov and S. Jafarzade, "Integrable lattice spin models from supersymmetric dualities", Phys. Part. Nucl. Lett. **15**, no.6, 650-667 (2018), doi:10.1134/S1547477118060079 [arXiv:1712.09651 [math-ph]].

[58] J. Yagi, "Quiver gauge theories and integrable lattice models", JHEP **10**, 065 (2015), doi:10.1007/JHEP10(2015)065 [arXiv:1504.04055 [hep-th]].

[59] A. Gadde, S. S. Razamat and B. Willett, "On the reduction of 4d $\mathcal{N} = 1$ theories on $\mathbb{S}^2$", JHEP **11**, 163 (2015), doi:10.1007/JHEP11(2015)163 [arXiv:1506.08795 [hep-th]].

[60] K. A. Intriligator and P. Pouliot, "Exact superpotentials, quantum vacua and duality in supersymmetric $SP(N_c)$ gauge theories", Phys. Lett. B **353**, 471-476 (1995), doi:10.1016/0370-2693(95)00618-U [arXiv:hep-th/9505006 [hep-th]].

[61] M. Sacchi, "New 2d $\mathcal{N} = (0,2)$ dualities from four dimensions", [arXiv:2004.13672 [hep-th]].

[62] C. Csaki, W. Skiba and M. Schmaltz, "Exact results and duality for $Sp(2N)$ SUSY gauge theories with an antisymmetric tensor", Nucl. Phys. B **487**, 128-140 (1997), doi:10.1016/S0550-3213(96)00709-2 [arXiv:hep-th/9607210 [hep-th]].

[63] V. V. Bazhanov, A. P. Kels and S. M. Sergeev, "Quasi-classical expansion of the star-triangle relation and integrable systems on quad-graphs", J. Phys. A **49**, no.46, 464001 (2016), doi:10.1088/1751-8113/49/46/464001 [arXiv:1602.07076 [math-ph]].

[64] A. P. Kels, "Two-component Yang-Baxter maps associated to integrable quad equations", [arXiv:1910.03562 [math-ph]].

[65] A. P. Kels, "Integrable quad equations derived from the quantum Yang-Baxter equation", Lett. Math. Phys. **110**, 1477–1557 (2020), doi:10.1007/s11005-020-01255-3 [arXiv:1803.03219 [math-ph]].

[66] A. P. Kels, "Interaction-around-a-face and consistency-around-a-face-centered-cube", [arXiv:2003.08883 [math-ph]].

[67] V. P. Spiridonov and G. S. Vartanov, "Superconformal indices for $N = 1$ theories with multiple duals", Nucl. Phys. B **824**, 192-216 (2010), doi:10.1016/j.nuclphysb.2009.08.022 [arXiv:0811.1909 [hep-th]].

[68] V. P. Spiridonov and G. S. Vartanov, "Vanishing superconformal indices and the chiral symmetry breaking", JHEP **06**, 062 (2014), doi:10.1007/JHEP06(2014)062 [arXiv:1402.2312 [hep-th]].

[69] A. Gadde, S. Gukov and P. Putrov, "(0,2) trialities," JHEP **03**, 076 (2014), doi:10.1007/JHEP03(2014)076 [arXiv:1310.0818 [hep-th]].

[70] M. Dedushenko and S. Gukov, "IR duality in 2D $N = (0, 2)$ gauge theory with noncompact dynamics", Phys. Rev. D **99**, no.6, 066005 (2019), doi:10.1103/PhysRevD.99.066005 [arXiv:1712.07659 [hep-th]].

[71] Y. G. Stroganov, "Tetrahedron equation and spin integrable models on a cubic lattice," Theor. Math. Phys. **110**, 141-167 (1997), doi:10.1007/BF02630441.

[72] D. Moskovich, "Framing and the Self-linking Integral", Far East J. Math. Sci. **14**:2, 165-183 (2004).

[73] D. P. Sinha, "Manifold-theoretic Compactifications of Configuration Spaces", Sel. Math. New Ser. **10**, 391-428 (2004).

[74] I. Volić, "Configuration Space Integrals and the Topology of Knot and Link Spaces", Morfismos **17**:2, 1-56 (2013).

[75] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories", Nucl. Phys. B **435**, 129-146 (1995), doi:10.1016/0550-3213(94)00023-8 [arXiv:hep-th/9411149 [hep-th]].

[76] M. Honda and Y. Yoshida, "Supersymmetric index on $T^2 \times S^2$ and elliptic genus", [arXiv:1504.04355 [hep-th]].

[77] E. Witten, "Phases of $N = 2$ theories in two-dimensions", AMS/IP Stud. Adv. Math. **1**, 143-211 (1996), doi:10.1016/0550-3213(93)90033-L [arXiv:hep-th/9301042 [hep-th]].

[78] I. V. Melnikov, "An introduction to two-dimensional quantum field theory with (0,2) supersymmetry", Lect. Notes Phys. **951**, pp.-1-482 (2019), doi:10.1007/978-3-030-05085-6.

[79] F. Benini, R. Eager, K. Hori and Y. Tachikawa, "Elliptic genera of 2d $\mathcal{N} = 2$ gauge theories", Commun. Math. Phys. **333**, 1241–1286 (2015), doi:10.1007/s00220-014-2210-y, [arXiv:1308.4896 [hep-th]].

[80] A. Gadde and S. Gukov, "2d index and surface operators", JHEP **03**, 080 (2014), doi:10.1007/JHEP03(2014)080 [arXiv:1305.0266 [hep-th]].

[81] A. Gadde, S. Gukov and P. Putrov, "Walls, lines, and spectral dualities in 3d gauge theories", JHEP **05**, 047 (2014), doi:10.1007/JHEP05(2014)047 [arXiv:1302.0015 [hep-th]].

[82] M. Khovanov, "A Categorification of the Jones Polynomial", Duke Math. J. **101**, 359 (2000), doi:10.1215/S0012-7094-00-10131-7, [math/9908171 [math.QA]].

[83] E. Witten, "Fivebranes and Knots", arXiv:1101.3216 [hep-th].

[84] E. Witten, "Khovanov Homology and Gauge Theory", arXiv:1108.3103 [math.GT].

[85] E. Witten, "Two Lectures on the Jones Polynomial and Khovanov Homology", arXiv:1401.6996 [math.GT].

[86] S. Gukov and I. Saberi, "Lectures on Knot Homology and Quantum Curves", doi:10.1090/conm/613/12235, arXiv:1211.6075 [hep-th].

[87] N. Ito and M. Kameyama, "On a Poincaré Polynomial from Khovanov Homology and Vassiliev Invariants", arXiv:1905.05664 [math.GT].

[88] N. Ito and J. Yoshida, "Crossing Change on Khovanov Homology and a Categorified Vassiliev Skein Relation", arXiv:1911.09308 [math.GT].

[89] V. P. Spiridonov and G. S. Vartanov, "Elliptic hypergeometry of supersymmetric dualities", Commun. Math. Phys. **304**, 797-874 (2011), doi:10.1007/s00220-011-1218-9 [arXiv:0910.5944 [hep-th]].

[90] V. P. Spiridonov and G. S. Vartanov, "Elliptic hypergeometry of supersymmetric dualities II. Orthogonal groups, knots, and vortices", Commun. Math. Phys. **325**, 421-486 (2014), doi:10.1007/s00220-013-1861-4 [arXiv:1107.5788 [hep-th]].

[91] F. Y. Wu, "Knot theory and statistical mechanics", Rev. Mod. Phys. **64**, 1099-1131 (1992), doi:10.1103/RevModPhys.64.1099.

[92] H. Garcia-Compean and A. M. Uranga, "Brane box realization of chiral gauge theories in two-dimensions", Nucl. Phys. B **539**, 329-366 (1999), doi:10.1016/S0550-3213(98)00725-1 [arXiv:hep-th/9806177 [hep-th]].

[93] S. Franco, D. Ghim, S. Lee and R. K. Seong, "Elliptic genera of 2d (0,2) gauge theories from brane brick models", JHEP **06**, 068 (2017), doi:10.1007/JHEP06(2017)068 [arXiv:1702.02948 [hep-th]].