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T-Dualidad para modelos sigma lineales normados dos dimensionales con torsión

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"T-duality for gauged linear sigma models in two dimensions with torsion"

Thesis submitted by

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Master of Science

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Abstract

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Applying the T-duality algorithm to a gauged lineal sigma model in two dimensions with supersymmetry (2,2) and torsion given by the semi-chiral fields representation and we obtained the dual model. Using these results we describe the geometry and torsion associated.

Resumen
Aplicamos el algoritmo de dualidad-T a un modelo sigma lineal normado en dos dimensiones con supersimetría (2,2) y con torsión por medio de la representación de los campos semi-quirales se obtuvo el modelo dual. Usando los resultados se encontró la geometría y la torsión asociada.

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Contents

Chapter 1

Introduction

Before the 1990s, string theorists believed that there were five distinct superstring theories, but research has shown that these five were not so different, that these are related by dualities. The relation between different string theories is shown in the figure (1.1), where the symmetry map of superstring theories that relates type IIA with IIB and heterotic string theory HE with HO is, in general, an Abelian or non-Abelian T-duality. Another map is given by S-duality, which relates SUGRA $n = 11$ with HE and type IIA and Ω-duality that relates type I and IIB. This means that duality relations connect different limits [1].

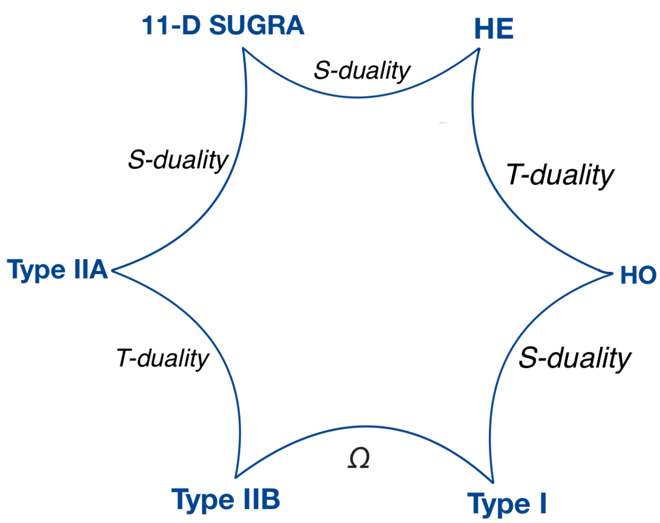

Figure 1.1: Sketch of duality in string theory.

In the paper [1] Y. Lozano et al. (1994), showed that Buscher's Abelian T-duality transformation rules can be recovered by performing a canonical transformation, first suggested by Giveon et al. (1989), see [2]. Furthermore, in paper [3], Giveon and Roček, (1994), proposed the general method for sigma models in non-Abelian T-duality.

In 2000 Hori and Vafa [4] studied the relation between mirror symmetry and T-duality, where for the first time they found an Abelian T-dual model of a gauged linear sigma model (GLSM). However, Cabo et al. (2017), found the generalization for a non-Abelian T-duality in GLSM [5].

The motivation of the present work was given by Roček et al. [6], who studied a broad class of two dimensional gauged linear sigma models (GLSMs) off-shell with $\mathcal{N} = (2, 2)$ supersymmetry that flow to nonlinear sigma models (NLSMs) on non-compact geometries with torsion. These models require to use chiral, twisted chiral, and semichiral multiplets to construct a new $\mathcal{N} = (2, 2)$ vector multiplet [7].

This thesis work is focused on the study of GLSM's with torsion, their Abelian T-duality and the background geometry. It would be interesting to study them because one can find a new point of view, for this GLSM with torsion that might give us more information about the mirror symmetry for these models.

The organization of this thesis is as follows: In the second chapter, we review supersymmetry $\mathcal{N} = (2, 2)$ and introduce the semi-vector multiplet given in [7]. Moreover, we define this GLSM with symmetry group $U(1)$ with the semi-vector multiplet and explain the cases given by this vector multiplet. Also, we discuss the method given by Giveon et al. (1994) [3] for making the Abelian T-duality. In the third chapter, we review the target space geometry and rewrite the Lagrangians in bosonic and fermionic terms for GLSM chiral and twisted chiral multiplets. In the fourth chapter, we find the dual Lagrangias for both cases using the general fixing of the symmetry. In the fifth chapter, we compute the scalar potential for two possible cases and analyze their geometry. In the sixth chapter, we finish obtaining the background geometry for each model, where we find the metric and torsion. Finally, in the last chapter, we give the conclusions and remarks.

Chapter 2

Supersymmetry

On this chapter we review the $\mathcal{N} = (2, 2)$ supersymmetry in 2D and their representations. Therefore, we provide a systematic way to obtain the GLSM Lagrangians and a method to do Abelian T-duality.

2.1 $\mathcal{N} = (2, 2)$ supersymmetry in 2D

Let us consider a field theory in $d = 2$ dimensions with space-time coordinates: x^0 as time coordinate and x^1 as space coordinate [8]. We take the flat Minkowski metric

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1}$$

We introduce the four fermionic coordinates $\theta^{\pm}$ and $\bar{\theta}^{\pm}$, where the $\pm$ index denotes chirality under Lorentz transformation [8], [9]. This coordinates anti-commute as we review in the appendix A. The $\mathcal{N} = (2, 2)$ superspace is the space with the coordinates: $x^0, x^1, \theta^{\pm}, \bar{\theta}^{\pm}$ [8]. Furthermore, we define in this superspace a set of differential operators given in [8], [10] as

$$\mathbb{D}_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - i\bar{\theta}^{\pm}\partial_{\pm}, \tag{2.2}$$

$$\bar{\mathbb{D}}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}} + i\theta^{\pm}\partial_{\pm}, \tag{2.3}$$

Here $\partial_{\pm}$ are differentiations by $x^{\pm} := x^0 \pm x^1$

$$\partial_{\pm} = \frac{\partial}{\partial x^{\pm}} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right), \tag{2.4}$$

which obey the following relation

$$\{\mathbb{D}_{\pm}, \bar{\mathbb{D}}_{\pm}\} = 2i\partial_{\pm\pm}. \tag{2.5}$$

The most general, linear, SUSY-invariant, constraints one can impose are chiral superfield Φ and anti-chiral $\bar{\Phi}$ that are defined in [8] as

$$\bar{\mathbb{D}}_{\pm}\Phi = 0, \tag{2.6}$$
$$\mathbb{D}_{\pm}\bar{\Phi} = 0. \tag{2.7}$$

For twisted chiral χ and anti-twisted chiral $\bar{\chi}$, which are defined in [8], we have

$$\bar{\mathbb{D}}_{+}\chi = \mathbb{D}_{-}\chi = 0, \tag{2.8}$$
$$\mathbb{D}_{+}\bar{\chi} = \bar{\mathbb{D}}_{-}\bar{\chi} = 0. \tag{2.9}$$

These are supersymmetric representations that only occur in $d = 2$ dimensions. As well as their dual complex linear superfields [8], [11] we have

$$\bar{\mathbb{D}}_{+}\bar{\mathbb{D}}_{-}\Sigma = \mathbb{D}_{+}\mathbb{D}_{-}\bar{\Sigma} = 0, \tag{2.10}$$

while for twisted linear superfields it is know

$$\bar{\mathbb{D}}_{+}\mathbb{D}_{-}\tilde{\Sigma} = \mathbb{D}_{+}\bar{\mathbb{D}}_{-}\bar{\tilde{\Sigma}} = 0. \tag{2.11}$$

This kind of fields have been extensively studied. Moreover, there is new kind of fields, the left, and right semi-chiral superfields, defined in [7], [11] as

$$\bar{\mathbb{D}}_{+}\mathbb{X}_{L} = \mathbb{D}_{+}\bar{\mathbb{X}}_{L} = 0, \quad \bar{\mathbb{D}}_{-}\mathbb{X}_{R} = \mathbb{D}_{-}\bar{\mathbb{X}}_{R} = 0. \tag{2.12}$$

These are the semi-chiral multiplets that contain 3 scalars, 4 Weyl fermions, 1 chiral vector, which are physical and auxiliary fields [7].

2.2 Vector multiplets

Motivated by the fact that σ-models always admit a local description in $\mathcal{N} = (2,2)$ superspace in terms of complex chiral superfields ϕ_i, twisted chiral superfields χ_j and semichiral superfields $(\mathbb{X}_L^A, \mathbb{X}_R^A)$ [7], the most general description of a Lagrange density with $\mathcal{N} = (2,2)$ supersymmetry [12] can be written as:

$$K = K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R). \tag{2.13}$$

The isometries acting on a Kähler manifold, correspond to changes in the chiral superfields Φ or twisted chiral superfields χ. We assume $U(1)$ isometry away from a fixed point so that the choice of coordinates give us the Killing vectors:

$$k_{\phi} = i(\partial_{\phi} - \partial_{\bar{\phi}}), \quad k_{\chi} = i(\partial_{\chi} - \partial_{\bar{\chi}}). \tag{2.14}$$

We obtain new isometries from the mixing of chiral and twisted chiral superfields, that were discover in [11], [13],

$$k_{\phi\chi} = k_\phi - k_\chi, \tag{2.15}$$
$$k_{LR} = i(\partial_L - \partial_{\bar{R}} - \partial_R + \partial_{\bar{R}}), \tag{2.16}$$

if the vector field has a component along k_ϕ, k_χ or $k_{\phi\chi}$, we can (locally) redefine $\mathbb{X}$ to eliminate any component along the new vector field k_{LR}.

The isometries corresponding to the invariant Lagrange density (2.13) can be promoted to local gauge symmetries. In general, the isometries act on the coordinates under local transformations with some constant parameter λ as [7]

$$\delta z = [\lambda k, z], \tag{2.17}$$

where z is $\phi, \chi, \mathbb{X}_R$, etc. Now, the parameter λ fulfills a local parameter Λ, that follows the constraints for all the fields in the form

$$\begin{aligned}
\delta_g \Phi = i\Lambda &\Rightarrow \bar{\mathbb{D}}_\pm \Lambda = 0,\\
\delta_g \bar{\Phi} = -i\bar{\Lambda} &\Rightarrow \mathbb{D}_\pm \bar{\Lambda} = 0,\\
\delta_g \chi = i\tilde{\Lambda} &\Rightarrow \bar{\mathbb{D}}_+ \tilde{\Lambda} = \mathbb{D}_- \tilde{\Lambda} = 0,\\
\delta_g \bar{\chi} = -i\bar{\tilde{\Lambda}} &\Rightarrow \mathbb{D}_+ \bar{\tilde{\Lambda}} = \bar{\mathbb{D}}_- \bar{\tilde{\Lambda}} = 0,\\
\delta_g \mathbb{X}_L = i\Lambda_L &\Rightarrow \bar{\mathbb{D}}_+ \Lambda_L = 0,\\
\delta_g \bar{\mathbb{X}}_L = -i\bar{\Lambda}_L &\Rightarrow \mathbb{D}_+ \bar{\Lambda}_L = 0,\\
\delta_g \mathbb{X}_R = i\Lambda_R &\Rightarrow \bar{\mathbb{D}}_- \Lambda_R = 0,\\
\delta_g \bar{\mathbb{X}}_R = -i\bar{\Lambda}_R &\Rightarrow \mathbb{D}_- \bar{\Lambda}_R = 0.
\end{aligned} \tag{2.18}$$

Under the local transformation given by the isometries, we introduce the vector multiplets, for each killing vector. For k_ϕ, k_χ and $k_{\phi\chi}$, respectively:

$$\delta_g V^\Phi = i(\bar{\Lambda} - \Lambda), \tag{2.19}$$
$$\delta_g V^\chi = i(\bar{\tilde{\Lambda}} - \tilde{\Lambda}), \tag{2.20}$$
$$\delta_g V^{'} = \bar{\Lambda} + \Lambda + \bar{\tilde{\Lambda}} + \tilde{\Lambda}, \tag{2.21}$$
$$\tag{2.22}$$

for k_L, k_R and k_{LR}, we have

$$\delta_g \mathbb{V}^L = i(\bar{\Lambda}_L - \Lambda_L), \tag{2.23}$$
$$\delta_g \mathbb{V}^R = i(\bar{\Lambda}_R - \Lambda_R), \tag{2.24}$$
$$\delta_g \mathbb{V}' = \bar{\Lambda}_R + \Lambda_R + \bar{\Lambda}_L + \Lambda_L. \tag{2.25}$$

In the case for semichiral fields it is more convenient to introduce the complex combinations [11] given by the gauge transformations (2.23), (2.24) and (2.25):

$$\mathbb{V} = \frac{1}{2}\left(\mathbb{V}' + i\left(\mathbb{V}^L + \mathbb{V}^R\right)\right), \tag{2.26}$$

$$\tilde{\mathbb{V}} = \frac{1}{2}\left(\mathbb{V}' + i\left(\mathbb{V}^L - \mathbb{V}^R\right)\right), \tag{2.27}$$

with gauge transformations

$$\delta\mathbb{V} = \Lambda_L - \Lambda_R, \delta\mathbb{V} = \Lambda_L - \bar{\Lambda}_R. \tag{2.28}$$

2.3 Gauging

Since the isometry acts in the semi-chiral fields, we want to promote λ to a pair of left and right semi-chiral gauge parameters [7], [11]. We have seen it in the previous section, introducing the real vector multiplet Λ_L and Λ_R, where the gauge transformation for these fields are in [11] and are given by

$$\mathbb{X}_L \to e^{iQ_L\Lambda_L}\mathbb{X}_L, \quad \mathbb{X}_R \to e^{iQ_R\Lambda_R}\mathbb{X}_R. \tag{2.29}$$

The gauge transformation of the vector multiplets are given from (2.23) to (2.25), where $\mathbb{V}_{L,R}$ are real, but $\mathbb{V}$ and $\tilde{\mathbb{V}}$ are complex by definition in the equations (2.26) and (2.27).

2.4 Field strenghts

Likewise in the chiral and twisted chiral case, we introduce the field strength that comes from the multiplets. The following complex field-strengths are full gauge invariant [7], [11]

$$\mathbb{F} = \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\mathbb{V}, \quad \bar{\mathbb{F}} = -\mathbb{D}_+\mathbb{D}_-\bar{\mathbb{V}}, \tag{2.30}$$

$$\tilde{\mathbb{F}} = \bar{\mathbb{D}}_+\mathbb{D}_-\tilde{\mathbb{V}}, \quad \bar{\tilde{\mathbb{F}}} = -\mathbb{D}_+\bar{\mathbb{D}}_-\bar{\tilde{\mathbb{V}}}. \tag{2.31}$$

where $\mathbb{F}$ and $\tilde{\mathbb{F}}$ are chiral fields and twisted chiral fields, respectively. Thus, we can define the fields in $2d$ with $\mathcal{N} = (2,2)$ flat space with Lorentzian signature. The algebra of $\mathcal{N} = (2,2)$ spinor derivatives are on (2.5). We called $\mathbb{V}$ and $\tilde{\mathbb{V}}$ semichiral vector multiplet (SVM), if we constrain these we obtain the constrain semichiral vector multiplet (CSVM). The CSVM is obtained by constraining one of these field strengths (but not both) to vanish [6]:

$$\mathbb{F} = 0, \quad \tilde{\mathbb{F}} = 0. \tag{2.32}$$

Given the two field strengths from equation (2.31), we can write down the kinetic action for the SVM, this is in [6] and is given by

$$\mathcal{L}_{SVM} = -\frac{1}{2e^2}\int d^4\theta \left(\bar{\tilde{\mathbb{F}}}\tilde{\mathbb{F}} - \bar{\mathbb{F}}\mathbb{F}\right). \tag{2.33}$$

Now, the kinetic action for SVM is constrained by adding chiral Φ and semichiral χ Lagrange multipliers [6], as follows

$$\begin{aligned}\mathcal{L}_{CSVM} =& -\frac{1}{2e^2}\int d^4\theta \left(\bar{\tilde{\mathbb{F}}}\tilde{\mathbb{F}} - \bar{\mathbb{F}}\mathbb{F}\right) + i\left(\int d^2\theta \Phi\mathbb{F} + c.c.\right) \\ &+ i\left(\int d^2\theta\chi\tilde{\mathbb{F}} + c.c.\right).\end{aligned} \tag{2.34}$$

The constraint implies (locally) that $\mathbb{V}$ or $\tilde{\mathbb{V}}$ is pure gauge. It is also possible to add Fayet-Iliopoulos (FI) terms, with the form

$$\mathcal{L}_{FI} = it\left(\int d^2\tilde{\theta}\tilde{\mathbb{F}} + c.c.\right) + is\left(\int d^2\theta\mathbb{F} + c.c.\right). \tag{2.35}$$

Here, the D-term can be written as $t = \frac{1}{2}(\xi - i\frac{\theta}{2\pi})$ and $s = \frac{1}{2}(\tilde{\xi} - i\frac{\tilde{\theta}}{2\pi})$, where $\xi, \tilde{\xi}$ are real FI parameters and $\theta, \tilde{\theta}$ are the topological theta angles.

2.5 GLSM with vector multiplets

In order to give a kinetic term, we consider an equal number of left and right semi-chiral fields, with couplings between them [6]. It takes the form

$$\mathcal{L}_{Matter} = \int d^4\theta \left[a\bar{\mathbb{X}}_L\mathbb{X}_L + b\bar{\mathbb{X}}_R\mathbb{X}_R + \left(c\bar{\mathbb{X}}_L\mathbb{X}_R + c.c.\right) + \left(d\mathbb{X}_L\mathbb{X}_R + c.c.\right)\right], \tag{2.36}$$

where a, b are real and c, d are complex parameters. The parameters c, d are not both zero [6]. As well as $a, b \neq 0$ can be scaled to ± 1, also, the parameters can't be absorbed by redefining the fields. The requirement of a positive definite kinetic term imposes minor conditions on the signs of a, b and the range of the remaining parameters.

Semi-chiral models typically have fewer (rigid) flavor symmetries than models for chiral fields [6]. For instance, $c, d = 0$, has no U(1) isometries. In the special case $c = 0$ there is a U(1) isometry acting by

$$\mathbb{X}_L \to e^{iq\lambda}\mathbb{X}_L, \quad \mathbb{X}_R \to e^{-iq\lambda}\mathbb{X}_R, \tag{2.37}$$

where λ is a real parameter. In the special case $d = 0$ there is a U(1) isometry acting by

$$\mathbb{X}_L \to e^{iq\lambda}\mathbb{X}_L, \quad \mathbb{X}_R \to e^{iq\lambda}\mathbb{X}_R, \tag{2.38}$$

More generally, if $(\mathbb{X}_L, \mathbb{X}_R)$ are valued in the Lie algebra of a group G, one can write a linear sigma model with an isometry if they are either in a representation $(\mathfrak{R}, \bar{\mathfrak{R}})$ or $(\mathfrak{R}, \mathfrak{R})$ of G.

Then we have two cases of interest in models with isometries, where $d = 0$ or $c = 0$. In fact, these two cases are equivalent due to semichiral-semichiral duality [12]. Finally, the requirement of a positive definite kinetic term imposes $|a|= |b|= 1$ and $d = \beta > 1|\beta \in \mathbb{R}$ in (2.36) give us the follow cases.

When the isometry acts on (2.37), the gauge-invariant matter Lagrangian is given by

$$\mathcal{L}_{Matter} = \int d^4\theta \left[\bar{\mathbb{X}}_L e^{QV_L}\mathbb{X}_L + \bar{\mathbb{X}}_R e^{-QV_R}\mathbb{X}_R + \beta\left(\mathbb{X}_L e^{-iQ\mathbb{V}}\mathbb{X}_R + c.c.\right)\right], \tag{2.39}$$

and when the isometry acts as (2.38), the gauge-invariant matter Lagrangian goes as

$$\mathcal{L}_{Matter} = \int d^4\theta \left[\bar{\mathbb{X}}_L e^{QV_L}\mathbb{X}_L + \bar{\mathbb{X}}_R e^{QV_R}\mathbb{X}_R + \alpha\left(\mathbb{X}_L e^{iQ\bar{\tilde{\mathbb{V}}}}\mathbb{X}_R + c.c.\right)\right]. \tag{2.40}$$

Then, the general Lagrangian with semi-chiral fields and CVSM is given by

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{CSVM} + \mathcal{L}_{FI} + \mathcal{L}_{Matter} \\
&= -\frac{1}{2e^2}\int d^4\theta\left(\bar{\tilde{\mathbb{F}}}\tilde{\mathbb{F}} - \bar{\mathbb{F}}\mathbb{F}\right) \\
&+ i\left(\int d^2\theta\Phi\mathbb{F} + c.c.\right) + i\left(\int d^2\tilde{\theta}\chi\tilde{\mathbb{F}} + c.c.\right) \\
&+ it\left(\int d^2\tilde{\theta}\tilde{\mathbb{F}} + c.c.\right) + is\left(\int d^2\theta\mathbb{F} + c.c.\right) \\
&+ \int d^4\theta\left[\bar{\mathbb{X}}_L\mathbb{X}_L + \bar{\mathbb{X}}_R\mathbb{X}_R + \alpha\left(\mathbb{X}_L\mathbb{X}_R + c.c.\right)\right].
\end{aligned} \tag{2.41}$$

Here we consider a pair of semi-chiral fields with opposite charges, coupled to the SVM and constrained by $\mathbb{F} = 0$ or $\tilde{\mathbb{F}} = 0$, but not both [6]. This leaves us two cases to analyze their dualities.

2.6 T-duality in GLSM's

In general, duality means an exact quantum equivalence between two theories $\mathcal{T}$ and $\mathcal{T}'$ such that both represent only one theory, albeit in different guises [3], [14]. The symmetry known as T-duality can be described employing a gauge theory, obtained by gauging the global symmetry T-duality group to a local one and adding Lagrange multipliers [3]. The global group can be Abelian or non-Abelian.

Given a symmetry group G, if a theory is invariant under G, this theory is adequate to carry out a T-duality [1]. In particular, the Lagrangian in the equation (2.41) is invariant under the action of $U(1)$ global group.

The general method (algorithm) for T-duality is described in the work by Giveon and Roček [3] for some global symmetry group G, this transformation proceeds in few steps:

1. One gauges the isometry group, thus introducing a gauge field $V = V_a T_a$, where T_a are the generators, and adding a Lagrange multiplier field Ψ. Integrating over the field Ψ constrains the gauged field V to be pure gauge with vanishing field-strength obtaining back the original model.

2. If we integrate out the gauge field V this leads to the dual model, as we can see in the figure (2.1).

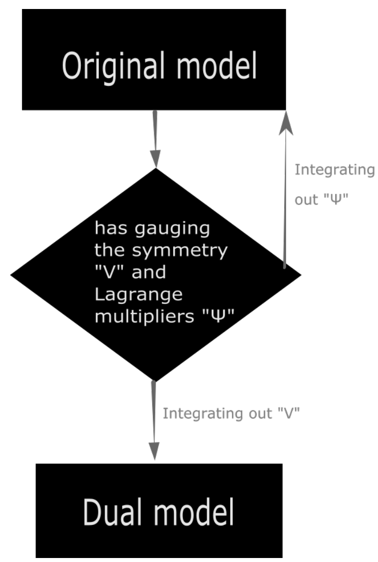

Figure 2.1: Scheme of the T-Duality process.

With these steps, we can work out the mapping between the operators and variables from the original to the dual model. Unfortunately, there are some problems with non-Abelian T-duality when it is compared with Abelian T-duality. In the case of Abelian T-duality, as we see in figure (2.1), we can recover the original model from the dual model [3]. However, in the non-Abelian duality, it is not always possible to obtain back the original theory, as the dual theory may loose the original global symmetry [3].

2.7 Example for Abelian T-duality for GLSM

We begin with the example of a simple GLSM of (anti)chiral fields with $U(1)$ global symmetry and employ the duality algorithm described in the last section in order to obtain the dual model.

We implement T-duality for a $(2,2)$ GLSM with gauge group $U(1)$ given by [4], that has a vector superfield field V_0, two chiral superfields Φ_1 and Φ_2 with equal charges Q_1 and Q_2 under the $U(1)$ gauge group. The global symmetry can be gauged by adding the vector superfield V, and the Lagrange multipliers Ψ and $\bar{\Psi}$ to the GLSM Lagrangian. This yield us

$$\begin{aligned}\mathcal{L}_{U(1)} &= \int d^4\theta \left(\bar{\Phi}_1 e^{2Q_0V_0+2QV}\Phi_1 + \bar{\Phi}_2 e^{2Q_{0,2}V_0}\Phi_2\right) \\ &+ \int d^4\theta \left(-\frac{1}{2e^2}\bar{\Sigma}_0\Sigma_0 + \Psi\Sigma + \bar{\Psi}\bar{\Sigma}\right) + \frac{1}{2}\left(-t\int d^2\tilde{\theta}\Sigma_0 + c.c.\right).\end{aligned} \tag{2.42}$$

Here Σ is the gauge field strength associated to the gauged field V, $\Sigma = D_+D_-V$. The next step is to proceed to the integration of the gauge field to obtain the dual model, we search the equation of motion by doing $\frac{\delta\mathcal{L}}{\delta V} = 0$. Employing the definition for the twisted field strength in the equation (2.8) and integrating by parts the terms with the Lagrange multipliers, we have

$$\bar{\Phi}_1 e^{2Q_0V_0+2QV}\Phi_1 = \frac{\Lambda+\bar{\Lambda}}{2Q}, \tag{2.43}$$

where Λ is given by

$$\Lambda = \frac{1}{2}\bar{D}_+D_-\Psi. \tag{2.44}$$

Now, from equation (2.43) will obtain V and we eliminate it from the Lagrangian (2.42) to get

$$\begin{aligned}\mathcal{L}_{U(1),dual} &= \int d^4\theta \left(-\frac{\Lambda+\bar{\Lambda}}{2Q}\ln\left(\frac{\Lambda+\bar{\Lambda}}{2Q}\right) - \frac{\Lambda+\bar{\Lambda}}{2Q}\ln\left(\bar{\Phi}_1 e^{2Q_0V_0}\Phi_1\right)\right) \\ &+ \int d^4\theta \left(\bar{\Phi}_2 e^{2Q_{0,2}V_0}\Phi_2 - \frac{1}{2e^2}\bar{\Sigma}_0\Sigma_0\right) + \frac{1}{2}\left(-t\int d^2\tilde{\theta}\Sigma_0 + c.c.\right).\end{aligned} \tag{2.45}$$

Next, we can work out the second term in (2.45), integrating by parts and using the definition of the gauge twisted chiral strength, as follows,

$$\begin{aligned} I &= \int d^4 \frac{\Lambda+\bar{\Lambda}}{2Q}\ln\left(\bar{\Phi}_1 e^{2Q_0V_0}\Phi_1\right) \\ &= \int d\theta^+ d\theta^- d\bar{\theta}^- d\bar{\theta}^+ \frac{\Lambda+\bar{\Lambda}}{2Q}\left(2Q_0V_0 + \ln\left(\bar{\Phi}_1\Phi_1\right)\right) \\ &= \int d\theta^+ d\theta^- d\bar{\theta}^- d\bar{\theta}^+ \frac{Q_0}{Q}(\Lambda+\bar{\Lambda})V_0 \\ &= \frac{Q_0}{Q}\left(\int d\theta^+ d\bar{\theta}^- \Sigma\Lambda + \int d\theta^- d\bar{\theta}^+ \bar{\Sigma}\bar{\Lambda}\right).\end{aligned} \tag{2.46}$$

Then we obtain the dual Lagrangian

$$
\begin{aligned}
\mathcal{L}_{U(1),dual} = & \int d^4\theta \left(-\frac{\Lambda + \bar{\Lambda}}{2Q} \ln \left(\frac{\Lambda + \bar{\Lambda}}{2Q} \right) + \bar{\Phi}_2 e^{2Q_{0,2}V_0} \Phi_2 - \frac{1}{2e^2} \bar{\Sigma}_0 \Sigma_0 \right) \\
& - \frac{1}{2} \left(\int d\tilde{\theta} \Sigma_0 (\frac{Q_0}{Q} \Lambda - t) + \int d\bar{\tilde{\theta}} \bar{\Sigma}_0 (\frac{Q_0}{Q} \bar{\Lambda} - \bar{t}) \right) .
\end{aligned}
\tag{2.47}
$$

The can be expanded in terms of auxiliary, fermionic and scalar components of the superfields $\Phi_2, \Lambda, \Sigma_0$. From this Lagrangian, we can obtain the equations of motion as well as the scalar potential. The supersymmetric vacua of this potential will lead to the dual target space.

In the same way that we find the dual Lagrangian for an extra $U(1)$ global symmetry, we can make it for multiple $U(1)$ global symmetries with N chiral superfields and particular charges.

Chapter 3

Target space geometry

Here, we review the tools to obtain the target space geometry from a sigma model depending on the commutator's kernel of the complex forms $J_\pm$. Moreover, we discuss the method to leave chiral and twisted-chiral Lagrangians in terms of the bosonic components to use it later.

3.1 Geometry for sigma models

The geometry of the target space in each sigma model is different, it depends on the supersymmetry $\mathcal{N}$ and its dimension d [15]. This situation can be summarized by the following table

SUSY	(1,1)	(2,2)	(2,2)
$E = g + B$	g, B	g	g, B
Geometry	Riemannian	Kähler	Bihermitean

Table 3.1: The geometries of sigma models [15].

where we denote the left- and right-moving sectors by $\mathcal{N} = (p, q)$. We focus in the case $\mathcal{N} = (2, 2)$ in $d = 2$, where the background space is flat and may contain an antisymmetric b-field given by the semi-chiral representation [6].

In the paper [15] Lindstrom et al. reduced from $\mathcal{N} = (2, 2)$ to $\mathcal{N} = (1, 1)$, this leaves a non-linear sigma model, such that it is easy to identify the metric and b-field of the target space. They shown that the metric and b-field are non-linear functions of second derivatives of the Kähler potential K, so they wrote the complex structures $J_\pm$ and the 2-form Ω in canonical coordinates (q, P, z, z'), with the definition of the Kähler potential as $K = K(q, P, z, z')$. Finally, they obtained $J_\pm$ by doing a change of coordinates, from the canonical coordinates to the basis

$$
\begin{pmatrix} X_L^A \\ X_R^{A'} \\ \phi^{\mathcal{A}} \\ \chi^{\mathcal{A}'} \end{pmatrix}, \tag{3.1}
$$

where we use the same collective notation like [15], where $\mathcal{A} \equiv \{\alpha, \bar{\alpha}\}$, $\mathcal{A}' \equiv \{\alpha', \bar{\alpha}'\}$, $A \equiv \{a, \bar{a}\}$ and $A' \equiv \{a', \bar{a}'\}$. These fields carry indices

$$
\begin{aligned}
&\Phi^\alpha, \bar{\Phi}^{\bar{\alpha}}, \alpha = 1, ..., d_c \quad , \quad \chi^{\alpha'}, \bar{\chi}^{\bar{\alpha}'}, \alpha' = 1, ..., d_t, \\
&\mathbb{X}_L^a, \bar{\mathbb{X}}_L^{\bar{a}}, a = 1, ..., d_s \quad , \quad \mathbb{X}_R^{a'}, \bar{\mathbb{X}}_R^{\bar{a}'}, a' = 1, ..., d_s.
\end{aligned} \tag{3.2}
$$

such that their dimension is given by the number of the superfields, we have d_s, d_c and d_t for left and right semi-chiral, twisted chiral and chiral, respectively. In a basis where the coordinates are arranged as (3.1), we introduce the notation suppressing the index structure for the matrices,

$$
K_{AB}^{-1} = (K_{BA})^{-1}, \tag{3.3}
$$

$$
\begin{aligned}
C =& JK - KJ = \begin{pmatrix} 0 & 2iK \\ -2iK & 0 \end{pmatrix}, \\
A =& JK + KJ = \begin{pmatrix} 2iK & 0 \\ 0 & -2iK \end{pmatrix}.
\end{aligned} \tag{3.4}
$$

where, e.g., K_{AB} is the matrix of second derivatives along L-, R-, t- and c-directions, which are left, right semichiral, twisted chiral and chiral, respectively. The complex structures and the 2-form are completely determined given the generalized Kähler potential K by

$$
J_+ = \begin{pmatrix} J_s & 0 & 0 & 0 \\ K_{RL}^{-1} C_{LL} & K_{RL}^{-1} J_s K_{LR} & K_{RL}^{-1} C_{Lc} & K_{RL}^{-1} C_{Lt} \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \\ , & & & \end{pmatrix} \tag{3.5}
$$

and

$$
J_- = \begin{pmatrix} K_{LR}^{-1} J_s K_{RL} & K_{LR}^{-1} C_{RR} & K_{LR}^{-1} C_{Rc} & K_{LR}^{-1} A_{Lt} \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix}. \tag{3.6}
$$

These matrices have been taken from the paper [15], and the matrices J_s, J_c and J_t are given by

$$
J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{3.7}
$$

3.2 General Kähler Geometry

We have the special case for bihermitian geometry where the metric g and torsion b [15] are given by

$$g = \Omega[J_+, J_-], \quad b = \{J_+, J_-\} \tag{3.8}$$

We start analyzing the geometry of the original model. The following conditions are obtained from the equations (3.5), (3.6) and (3.8)

- J_+ and J_- are almost complex structures, i.e; $J_\pm^2 = -1$.
- They are integrable, i.e; Nijenhuis tensor vanishes.
- The metric is hermitean with respect to both complex structures: $J_\pm^T g J_\pm = g$.
- J's are covariantly constant with respect to a torsionful connection: $\nabla_\pm J^\pm = 0$

Given the above, our original model in (2.41) represents a bihermitean target space geometry with a b-field, and result from requiring invariance of the action (2.39) and (2.40) under the transformations as well as the closure of the algebra (2.17). The closure is only achieved on-shell, however. Only under the special condition that the two complex structures commute does the algebra close off-shell. In that case, there is a manifestly $\mathcal{N} = (2, 2)$ action for the model given in terms of chiral and twisted chiral $\mathcal{N} = (2, 2)$ superfields.

The general form to obtain $J_\pm$ are given in the equation (3.5) and (3.6), but we just gave the form to obtain the metric and the b-field in the case of the bihermitian geometry that is a particular case when $\ker[J_+, J_-] = \emptyset$ [6], [15].

In the general case when we have the condition $\ker[J_+, J_-] \neq \emptyset$, we apply a similar method [15]. First, we obtain the forms $J_\pm$ with the equations (3.5) and (3.6). The second step is to find the metric g, but the equation given in (3.8) is just for the bihermitian metric. For general Kähler geometry the metric [15] is given by

$$g = \begin{pmatrix} g_{AB} & g_{AB'} & g_{A\mathcal{B}} & g_{A\mathcal{B}'} \\ g_{A'B} & g_{A'B'} & g_{A'\mathcal{B}} & g_{A'\mathcal{B}'} \\ g_{\mathcal{A}B} & g_{\mathcal{A}B'} & g_{\mathcal{A}\mathcal{B}} & g_{\mathcal{A}\mathcal{B}'} \\ g_{\mathcal{A}'B} & g_{\mathcal{A}'B'} & g_{\mathcal{A}'\mathcal{B}} & g_{\mathcal{A}'\mathcal{B}'} \end{pmatrix}, \tag{3.9}$$

where the definition of the Poisson structure Ω determines all the components, except along the kernel [15]

$$\begin{pmatrix} g_{\mathcal{A}\mathcal{B}} & g_{\mathcal{A}\mathcal{B}'} \\ g_{\mathcal{A}'\mathcal{B}} & g_{\mathcal{A}'\mathcal{B}'} \end{pmatrix}. \tag{3.10}$$

These components can be obtained by solving the partial differential equation (PDE) [15], in the canonical coordinates (q, P, z, z')

$$J^\lambda_{+\mu} J^\sigma_{+\nu} J^\gamma_{+\rho} (d\omega_+)_{\lambda\sigma\gamma} = -J^\lambda_{-\mu} J^\sigma_{-\nu} J^\gamma_{-\rho} (d\omega_-)_{\lambda\sigma\gamma} \tag{3.11}$$

However, some components on (3.10) can be obtained using the symmetry of g and anti-symmetry of b on $E = \frac{1}{2}(g + b)$.

3.3 Target space geometry from Kähler potential

In the previous section, we talked about how the geometry can be understood in these sigma models, where we have a special form of the metric and the b-field but we did not mention how to obtain these. Starting from a given function of the form of (2.13) we obtain the Lagrangian

$$\mathcal{L} = \int d^4\theta K(\bar{\mathbb{X}}_i, \mathbb{X}_i; \Phi, \bar{\Phi}; \chi, \bar{\chi}), \tag{3.12}$$

where the function K is known as the generalized Kähler potential and besides obeying mild conditions for the metric g to be positive definite [15], it is otherwise arbitrary. The function $E = \frac{1}{2}(g + b)$ is completely determined by these functions, as

$$\begin{aligned}
E_{LL} =& C_{LL}K^{-1}_{LR}J_sK_{RL}\\
E_{LR} =& J_sK_{LR}J_s + C_{LL}K^{-1}_{LR}J_sC_{RR}\\
E_{Lc} =& K_{Lc} + J_sK_{Lc}J_c + C_{LL}K^{-1}_{LR}C_{Rc}\\
E_{Lt} =& -K_{Lt} - J_sK_{Lt}J_t + C_{LL}K^{-1}_{LR}A_{Rt}\\
E_{RL} =& -K_{RL}J_sK^{-1}_{LR}J_sK_{RL}\\
E_{RR} =& -K_{RL}J_sK^{-1}_{LR}C_{RR}\\
E_{Rc} =& K_{Rc} - K_{RL}J_sK^{-1}_{LR}C_{Rc}\\
E_{Rt} =& -K_{Rt} - K_{RL}J_sK^{-1}_{LR}A_{Rt}\\
E_{cL} =& C_{cL}K^{-1}_{LR}J_sK_{RL}\\
E_{cR} =& J_cK_{cR}J_s + C_{cL}K^{-1}_{LR}C_{RR}\\
E_{cc} =& K_{cc} + J_cK_{cc}J_c + C_{cL}K^{-1}_{LR}C_{Rc}\\
E_{ct} =& -K_{ct} - J_cK_{ct}J_t + C_{cL}K^{-1}_{LR}A_{Rt}\\
E_{tL} =& C_{tL}K^{-1}_{LR}J_sK_{RL}\\
E_{tR} =& J_tK_{tR}J_s + C_{tL}K^{-1}_{LR}C_{RR}\\
E_{tc} =& K_{tc} + J_tK_{tc}J_c + C_{tL}K^{1}_{LR}C_{Rc}\\
E_{tt} =& -K_{tt} - J_tK_{tt}J_t + C_{tL}K^{-1}_{LR}A_{Rt}
\end{aligned} \tag{3.13}$$

where we take K, C and A as in the equations (3.3) and (3.4). From this, the metric g and anti-symmetric b-field may be in terms of non-linear second derivatives of K.

3.4 Kähler for chiral fields

The last set of equations describes a general Lagrange density (2.41), but the particular case where K is a function of chiral and anti-chiral superfields is on the book of supersymmetry and supergravity [9]. We find that the Lagrangian can be written in terms of the Kähler potential and its derivatives. It lets us write the Lagrangian in terms of their geometry.

The most general Lagrangian that can be built from chrial superfiels Φ^i, for $i = 1, 2, \dots, n$ [9] is

$$\mathcal{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, K\big(\Phi^i, \Phi^{+j}\big) + \bigg[\int \mathrm{d}^2\theta\, P(\Phi^i) + \text{h.c.}\bigg], \tag{3.14}$$

where P is a holomorphic function (i.e. only is function of Φ^i). K is a function that depends on n holomorphic and anti-holomorphic variables Φ^i and $\bar{\Phi}^i$,respectively, with power series expansions in terms of chrial superfields Φ^i:

$$K(\Phi, \Phi^+) = \sum c_{i_1 \cdots i_N, j_1 \cdots j_N} \Phi^{i_1} \cdots \Phi^{i_N} \Phi^{+j_1} \cdots \Phi^{+j_M}, \tag{3.15}$$

$$P(\Phi) = \sum g_{i_1 \cdots i_N} \Phi^{i_1} \cdots \Phi^{i_N}. \tag{3.16}$$

To find the component Lagrangian, we must expand K and P in terms of the θ variables. For the superpotential P, we have the expansion

$$P(\Phi) = P(A) + \sqrt{2}\theta\chi^i \frac{\partial P(A)}{\partial A^i} + \theta\theta \bigg\{ F^i \frac{\partial P(A)}{\partial A^i} - \frac{1}{2}\chi^i\chi^j \frac{\partial^2 P(A)}{\partial A^i \partial A^j} \bigg\}. \tag{3.17}$$

The θ expansion of $K(\Phi, \Phi^+)$ can be obtained from the monomial:

$$K_{NM} = \Phi^{i_1} \cdots \Phi^{i_N} \Phi^{+j_1} \cdots \Phi^{+j_M}. \tag{3.18}$$

This expression simplifies the notation of a Kähler manifold [9], where:

$$g_{ij*} = \frac{\partial}{\partial A^i} \frac{\partial}{\partial A^{*j}} K|, \tag{3.19}$$

$$g_{ij*,k} = \frac{\partial}{\partial A^k} g_{ij*} = g_{mj*} \Gamma^m_{ik}, \tag{3.20}$$

$$g_{ij*,k*} = \frac{\partial}{\partial A^{*k}} g_{ij*} = g_{im*} \Gamma^{m*}_{j*k*}. \tag{3.21}$$

Moreover, the Lagrangian in terms of the metric, superpotential and auxiliary fields is

$$\begin{aligned}
\mathcal{L} =& g_{ij^*} F^i F^{j^*} + \frac{1}{4} g_{ij^*,kl^*} \chi^i \chi^k \bar{\chi}^{j^*} \bar{\chi}^{l^*} \\
& - F^i \left(\frac{1}{2} g_{im^*} \Gamma^{m^*}_{j^*k^*} \bar{\chi}^{j^*} \bar{\chi}^{k^*} - \frac{\partial P}{\partial A^i} \right) \\
& - F^{i^*} \left(\frac{1}{2} g_{mi^*} \Gamma^m_{jk} \chi^j \chi^k - \frac{\partial \bar{P}}{\partial \bar{A}^{i^*}} \right) \\
& - g_{ij^*} \partial_m A^i \partial^m A^{j^*} - i g_{ij^*} \bar{\chi}^{j^*} \bar{\sigma}^m D_m \chi^i \\
& - \frac{1}{2} \frac{\partial^2 P}{\partial A^i \partial A^j} \chi^i \chi^j - \frac{1}{2} \frac{\partial^2 P}{\partial \bar{A}^{i^*} \partial \bar{A}^{j^*}} \bar{\chi}^{i^*} \bar{\chi}^{j^*},
\end{aligned} \tag{3.22}$$

where the auxiliary fields are given by

$$F^i = g^{ij^*}\left(\frac{1}{2}g_{kj^*}\Gamma^k_{ml}\chi^m\chi^l - \frac{\partial\bar{P}}{\partial\bar{A}^{j^*}}\right). \tag{3.23}$$

3.5 Kähler for twisted-chiral fields

In the paper of Cabo et al [16], they used the same methods given in [9] to find the Lagrangian in terms of their geometry. The general Lagrangian that we can build with twisted-chiral superfields Ψ^μ, for $i = 1, 2, \ldots, n$ [9] is

$$\mathcal{L} = \int \mathrm{d}^2\widetilde{\theta}\,\mathrm{d}^2\bar{\widetilde{\theta}}\,K\big(\Psi^\mu, \bar{\Psi}^\mu\big) + \left(\int \mathrm{d}^2\widetilde{\theta}\,W(\Psi^\mu) + \text{c.c.}\right), \tag{3.24}$$

where W is a holomorphic function (i.e. it is only a function of Ψ^μ). K is a function that depends on n holomorphic and anti-holomorphic variables Ψ^μ and $\bar{\Psi}^\mu$, respectively, with power series expansions in terms of twisted-chiral superfields Ψ^i:

$$K\big(\Psi^\mu, \bar{\Psi}^\mu\big) = \sum_{i,j}\sum_{\mu_1,\cdots,\mu_i,\nu_1,\cdots,\nu_j} k_{\mu_1\cdots\mu_i\nu_1\cdots\nu_j}\Psi^{\mu_1}\cdots\Psi^{\mu_i}\Psi^{\nu_1}\cdots\Psi^{\nu_j}, \tag{3.25}$$

$$W(\Psi^\mu) = \sum_i\sum_{\mu_1,\cdots,\mu_i} p_{\mu_1\cdots\mu_i}\Psi^{\mu_1}\cdots\Psi^{\mu_i}. \tag{3.26}$$

The expansion in θ let us rewrite the Lagrangian in terms of the following variables

$$g_{\mu\bar{\nu}} = \frac{\partial}{\partial\psi^\mu\bar{\psi}^\nu}k, \qquad k = \sum_{ij} k_{ij} \tag{3.27}$$

$$\frac{\partial}{\partial\psi^\rho}g_{\mu\bar{\nu}} = g_{\sigma\bar{\nu}}\Gamma^\sigma_{\mu\rho}. \tag{3.28}$$

Finally the Lagrangian can be written as

$$\begin{aligned}\mathcal{L} = {} & G^\mu\bar{G}^{\bar{\nu}}g_{\mu\bar{\nu}} - \frac{1}{2}G^\mu\Big(\bar{\widetilde{\chi}}^{\bar{\nu}}\bar{\widetilde{\chi}}^{\bar{\rho}}g_{\mu\bar{\sigma}}\Gamma^{\bar{\sigma}}_{\bar{\nu}\bar{\rho}} - 2\partial_\mu W\Big) - \frac{1}{2}\bar{G}^{\bar{\mu}}\big(\widetilde{\chi}^\nu\widetilde{\chi}^\rho g_{\mu\bar{\sigma}}\Gamma^\sigma_{\nu\rho} - 2\partial_{\bar{\mu}}W\big) \\ & + \frac{1}{4}\widetilde{\chi}^\mu\widetilde{\chi}^\nu\bar{\widetilde{\chi}}^{\bar{\rho}}\bar{\widetilde{\chi}}^{\bar{\sigma}}\partial_\nu\partial_{bar\sigma}g_{\mu\bar{\rho}} - \big(\partial_m\psi^\mu\partial^m\bar{\psi}^{\bar{\nu}}\big)g_{\mu\bar{\nu}} - i\bar{\widetilde{\chi}}^{\bar{\mu}}\bar{\sigma}^m g_{\nu\bar{\mu}}D_m\widetilde{\chi}^\nu \\ & - \frac{1}{2}(\partial_\mu\partial_\nu W)\widetilde{\chi}^{\alpha\mu}\widetilde{\chi}^\nu_\alpha - \frac{1}{2}\big(\partial_{\bar{\mu}}\partial_{\bar{\nu}}\bar{W}\big)\bar{\widetilde{\chi}}^{\bar{\mu}}_{\dot{\alpha}}\bar{\widetilde{\chi}}^{\dot{\alpha}\bar{\nu}},\end{aligned} \tag{3.29}$$

where the auxiliary fields can be integrated using the Euler-Lagrange equations, thus giving us

$$\bar{G}^{\bar{\nu}} = \frac{1}{2}g^{\mu\bar{\nu}}\Big(\bar{\widetilde{\chi}}^{\bar{\theta}}\bar{\widetilde{\chi}}^{\bar{\rho}}g_{\mu\bar{\sigma}}\Gamma^{\bar{\sigma}}_{\bar{\theta}\bar{\rho}} - 2\partial_\mu W\Big), \tag{3.30}$$

$$G^{\nu} = \frac{1}{2} g^{\nu\bar{\mu}} \big(\widetilde{\chi}^{\theta} \widetilde{\chi}^{\rho} g_{\bar{\mu}\sigma} \Gamma^{\sigma}_{\theta\rho} - 2\partial_{\bar{\mu}} \bar{W} \big). \tag{3.31}$$

This leads to the Lagrangian in terms of fermionic and bosonic components, where it is easy to compute the scalar potential, just taking the bosonic terms in (3.22) and (3.29).

Chapter 4

Abelian T-Duality for GLSM in $\mathcal{N} = (2,2)$ with Torsion

In this chapter, we compute the dual Lagrangian for both cases $\mathbb{F} = 0$ and $\tilde{\mathbb{F}} = 0$, using the method given in chapter 2 and doing a gauge fixing to the original fields to obtain the dual Lagrangian.

4.1 Cases for the field strength

We found the matter Lagrangian with the semi-chiral representation when isometry acts on these superfields in the equations (2.39) and (2.40), then we have two cases to analyze : $\mathbb{F} = 0$ and $\tilde{\mathbb{F}} = 0$.

The method that we follow to do the Abelian T-duality is gauging the field $\mathbb{X}$ and add a viewer field that preserves the $U(1)$ symmetry. We use the definition for the vector multiplets (2.26) and (2.27) to rewrite V_L and V_R as

$$V_R = i(\mathbb{V} - \tilde{\mathbb{V}}) = i(\bar{\tilde{\mathbb{V}}} - \bar{\mathbb{V}}) \tag{4.1}$$

$$V_L = i(\bar{\tilde{\mathbb{V}}} - \mathbb{V}) = i(\bar{\mathbb{V}} - \tilde{\mathbb{V}}). \tag{4.2}$$

4.1.1 Case $\mathbb{F} = 0$

The first case is for $\mathbb{F} = 0$, we choose the gauge $\mathbb{V} = 0$, this gives us the condition

$$\mathbb{V}_R = \mathbb{V}_L \equiv V, \tag{4.3}$$

where we observe that V is the usual vector multiplet. Then, the Lagrangian becomes invariant under gauge group $U(1)$ with the same charge,

$$\mathcal{L}_{\mathbb{F}=0} = \int d^4\theta \left[-\frac{1}{2e^2}\bar{\Sigma}\Sigma + \bar{\mathbb{X}}_L e^{QV_0}\mathbb{X}_L + \bar{\mathbb{X}}_R e^{-QV_0}\mathbb{X}_R \right. \\ \left. + \beta(\mathbb{X}_L\mathbb{X}_R + c.c.) \right] + it\int d^4\tilde{\theta}\Sigma_0 + c.c. \tag{4.4}$$

We obtain the equation of motion for Φ in (2.41), then it sets $\mathbb{F}=0$, using (4.3) let us $\tilde{\mathbb{F}} = \Sigma$. Eventually, we promote the global symmetry to a local one by introducing a vector superfield V_1 for the fields $\mathbb{X}_{L,R}$ to implement T-duality, but we fix the gauge to remove the phase transformation of $\mathbb{X}_{2,LR}$, where we use the gauging given in (2.29). This is invariant under the residual, chiral, gauge symmetry, that in this case is written as

$$\delta V = i\left(\bar{\Lambda} - \Lambda\right), \quad \mathbb{X}_L \to e^{iQ\Lambda}\mathbb{X}_L, \quad \mathbb{X}_R \to e^{-iQ\Lambda}\mathbb{X}_R. \tag{4.5}$$

Then, the Lagrangian becomes

$$\mathcal{L}_{\mathbb{F}=0} = \int d^4\theta \left[\bar{\mathbb{X}}_{1,L} e^{Q_0V_0+QV_1}\mathbb{X}_{1,L} + \bar{\mathbb{X}}_{1,R} e^{-(Q_0V_0+QV_1)}\mathbb{X}_{1,R} + \beta(\mathbb{X}_{1,L}\mathbb{X}_{1,R} + c.c.)\right] \\ + \int d^4\theta \left[\bar{\mathbb{X}}_{2,L} e^{Q_0V_0}\mathbb{X}_{2,L} + \bar{\mathbb{X}}_{2,R} e^{-Q_0V_0}\mathbb{X}_{2,R} + \beta_2(\mathbb{X}_{2.L}\mathbb{X}_{2,R} + c.c.)\right] \\ + \int d^4\theta\Psi\Sigma_1 + \int d^4\theta\bar{\Psi}\bar{\Sigma}_1 + \int d^4\theta - \frac{1}{2e^2}\bar{\Sigma}_0\Sigma_0 + it\int d^4\tilde{\theta}\Sigma_0 + c.c. \tag{4.6}$$

where Ψ is an unconstrained superfield. Σ_1 is the fields strength associated to V_1. Integratiing Ψ, one gets a pure vector superfield V_1 and returns to the original Lagrangian. As we discussed before the field strength corresponds to $\tilde{\mathbb{F}} = \Sigma$, this is a function of derivatives of V_1 as

$$\Sigma_1 = \frac{1}{2}\bar{\mathbb{D}}_+\mathbb{D}_-V_1 \quad \bar{\Sigma}_1 = \frac{1}{2}\bar{\mathbb{D}}_-\mathbb{D}_+V_1. \tag{4.7}$$

Then, the Lagrange multiplier is

$$\begin{aligned} \mathcal{L}_{LM1} &= \int d^4\theta\Psi\Sigma_1 + \int d^4\theta\bar{\Psi}\bar{\Sigma}_1 \\ &= \frac{1}{2}\int d^4\theta\Psi\bar{\mathbb{D}}_+\mathbb{D}_-V_1 + \frac{1}{2}\int d^4\theta\bar{\Psi}\bar{\mathbb{D}}_-\mathbb{D}_+V_1 \\ &= \frac{1}{2}\int d^4\theta V_1\bar{\mathbb{D}}_+\mathbb{D}_-\Psi + \frac{1}{2}\int d^4\theta V_1\bar{\mathbb{D}}_-\mathbb{D}_+\bar{\Psi}. \end{aligned} \tag{4.8}$$

and the variation of the whole Lagrangian (4.6) will be

$$Q\left[\bar{\mathbb{X}}_{1,L} e^{Q_0V_0+QV_1}\mathbb{X}_{1,L} - \bar{\mathbb{X}}_{1,R} e^{-(Q_0V_0+QV_1)}\mathbb{X}_{1,R}\right] + \frac{1}{2}\left(\Upsilon + \bar{\Upsilon}\right) = 0, \tag{4.9}$$

where $\Upsilon = \frac{1}{2}\bar{\mathbb{D}}_+\mathbb{D}_-\Psi$ and $\bar{\Upsilon} = \frac{1}{2}\bar{\mathbb{D}}_-\mathbb{D}_+\bar{\Psi}$. If we multiply by e^{QV_1}, we obtain a second-degree equation and hence, two real solutions, they are

$$V_1 = \frac{\ln\left(\sqrt{g^2+16LR}-g\right)}{Q} - \frac{\ln(2L)}{Q}, \tag{4.10}$$

$$V_1 = \frac{\ln\left(\sqrt{g^2+16LR}+g\right)}{Q} - \frac{\ln(2L)}{Q}, \tag{4.11}$$

where $g = \frac{1}{2Q}\left(\Upsilon+\bar{\Upsilon}\right)$, $L = \bar{\mathbb{X}}_{1,L}e^{Q_0V_0}\mathbb{X}_{1,L}$ and $R = \bar{\mathbb{X}}_{1,R}e^{-Q_0V_0}\mathbb{X}_{1,R}$. Now, we will find the dual Lagrangian. We compute the kinetic Lagrangian in terms of the dual fields, to get

$$\begin{aligned}
L_k =& \int d^4\theta \left[\bar{\mathbb{X}}_{1,L}e^{Q_0V_0+QV_1}\mathbb{X}_{1,L} + \bar{\mathbb{X}}_{1,R}e^{-(Q_0V_0+QV_1)}\mathbb{X}_{1,R} + \beta(\mathbb{X}_{1,L}\mathbb{X}_{1,R}+c.c.)\right]\\
=& \int d^4\theta \left[\frac{1}{2Q}\left(\Upsilon+\bar{\Upsilon}\right) + 2\bar{\mathbb{X}}_{1,L}e^{Q_0V_0+QV_1}\mathbb{X}_{1,L} + \beta(\mathbb{X}_{1,L}\mathbb{X}_{1,R}+c.c.)\right]\\
=& \int d^4\theta \left[2\bar{\mathbb{X}}_{1,L}e^{Q_0V_0+QV_1}\mathbb{X}_{1,L}\right] = 2\int d^4\theta L e^{QV_1}\\
=&2\int d^4\theta L \exp\left[\ln\left(\sqrt{g^2+16LR}\pm g\right) - \ln(2L)\right]\\
=& \int d^4\theta \sqrt{g^2+16LR}
\end{aligned} \tag{4.12}$$

For the Lagrange multipliers, by using the definition of V_1 in (4.10) and (4.11), we obtain the follow

$$\begin{aligned}
L_{LM} =& \frac{1}{2}\int d^4\theta V_1\bar{\mathbb{D}}_+\mathbb{D}_-\Psi + \frac{1}{2}\int d^4\theta V_1\bar{\mathbb{D}}_-\mathbb{D}_+\bar{\Psi}\\
=& \int d^4\theta V_1\Upsilon + \int d^4\theta V_1\bar{\Upsilon}\\
=& \int d^4\theta V_1\left(\Upsilon+\bar{\Upsilon}\right)\\
=& \int d^4\theta \left(\frac{\ln\left(\sqrt{g^2+4LR}\pm g\right)}{Q} - \frac{\ln(2L)}{Q}\right)\left(\Upsilon+\bar{\Upsilon}\right)\\
=& \frac{1}{Q}\int d^4\theta \left(\ln\left(\sqrt{g^2+4LR}\pm g\right) - \ln(2L)\right)\left(\Upsilon+\bar{\Upsilon}\right)\\
=& \frac{1}{Q}\int d^4\theta \ln\left(\sqrt{g^2+4LR}\pm g\right)\left(\Upsilon+\bar{\Upsilon}\right) - C_1\\
=&2\int d^4\theta \ln\left(\sqrt{g^2+4LR}\pm g\right) g - C_1,
\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}
C_1 &= -\frac{1}{Q}\int d^4\theta \left(\ln(2L)\right)\left(\Upsilon+\bar{\Upsilon}\right) \\
&= -\frac{1}{Q}\int d^4\theta \left(\ln\left(2\bar{\mathbb{X}}_L e^{Q_0 V_0}\mathbb{X}_L\right)\right)\left(\Upsilon+\bar{\Upsilon}\right) \\
&= -\frac{1}{Q}\int d^4\theta \left(\ln\left(2\bar{\mathbb{X}}_L \mathbb{X}_L\right)+\ln\left(e^{Q_0 V_0}\right)\right)\left(\Upsilon+\bar{\Upsilon}\right) \\
&= -\frac{1}{Q}\int d^4\theta \left(\ln\left(2\bar{\mathbb{X}}_L \mathbb{X}_L\right)\right)\left(\Upsilon+\bar{\Upsilon}\right) - \frac{Q_0}{2Q}\int d^4\theta V_0\left(\Upsilon+\bar{\Upsilon}\right).
\end{aligned} \tag{4.14}$$

The first term in (4.14) can be obtained integrating by parts and using the relations (2.12), to obtain

$$\begin{aligned}
-\frac{1}{Q}\int d^4\theta \left(\ln\left(2\bar{\mathbb{X}}_L \mathbb{X}_L\right)\right)\left(\Upsilon+\bar{\Upsilon}\right) &= -\frac{1}{Q}\int d\theta^- d\bar{\theta}^- \left(\Upsilon+\bar{\Upsilon}\right)\bar{\mathbb{D}}_+\mathbb{D}_+\left(\ln\left(2\bar{\mathbb{X}}_L \mathbb{X}_L\right)\right) \\
&= -\frac{1}{Q}\int d\theta^- d\bar{\theta}^- \left(\Upsilon+\bar{\Upsilon}\right)\frac{\bar{\mathbb{D}}_+\mathbb{D}_+\left(\bar{\mathbb{X}}_L \mathbb{X}_L\right)}{\bar{\mathbb{X}}_L \mathbb{X}_L} \\
&= 0.
\end{aligned} \tag{4.15}$$

Similary the second term in (4.14) can be integrated by parts again, to get

$$\begin{aligned}
-\frac{Q_0}{Q}\int d^4\theta V_0\left(\Upsilon+\bar{\Upsilon}\right) &= -\frac{Q_0}{Q}\int d^4\theta V_0\left(\frac{1}{2}\bar{\mathbb{D}}_+\mathbb{D}_-\Psi+\frac{1}{2}\bar{\mathbb{D}}_-\mathbb{D}_+\bar{\Psi}\right) \\
&= -\frac{Q_0}{2Q}\int d^4\theta \left(\Psi\bar{\mathbb{D}}_+\mathbb{D}_- V_0+\bar{\Psi}\bar{\mathbb{D}}_-\mathbb{D}_+ V_0\right) \\
&= \frac{Q_0}{2Q}\left[\int d\theta^+ d\bar{\theta}^- \Upsilon\bar{\mathbb{D}}_+\mathbb{D}_- V_0+\int d\bar{\theta}^+ d\theta^- \bar{\Upsilon}\bar{\mathbb{D}}_-\mathbb{D}_+ V_0\right] \\
&= \frac{Q_0}{Q}\left[\int d\theta^+ d\bar{\theta}^- \Upsilon\Sigma_0+\int d\bar{\theta}^+ d\theta^- \bar{\Upsilon}\bar{\Sigma}_0\right].
\end{aligned} \tag{4.16}$$

Then, the first term in (4.12) is zero, because we have only chiral fields, and thus, the Lagrangian (4.6) becomes

$$\begin{aligned}
\mathcal{L}_{\mathbb{F}=0,dual} &= \int d^4\theta \left[\sqrt{g^2+4LR}+2g\ln\left(\sqrt{g^2+4LR}\pm g\right)\right]-\int d^4\theta\frac{1}{2e^2}\bar{\Sigma}_0\Sigma_0 \\
&+\int d^4\theta \left[\bar{\mathbb{X}}_{2,L}e^{Q_0 V_0}\mathbb{X}_{2,L}+\bar{\mathbb{X}}_{2,R}e^{-Q_0 V_0}\mathbb{X}_{2,R}+\beta(\mathbb{X}_{2.L}\mathbb{X}_{2,R}+c.c.)\right] \\
&+\int d\theta^+ d\bar{\theta}^- \left(\frac{Q_0}{Q}\Upsilon+it\right)\Sigma_0+\int d\bar{\theta}^+ d\theta^- \left(\frac{Q_0}{Q}\bar{\Upsilon}-i\bar{t}\right)\bar{\Sigma}_0.
\end{aligned} \tag{4.17}$$

4.1.2 Case $\tilde{\mathbb{F}} = 0$

In the second case, we take the Lagrangian (2.41) with the term of matter (2.40), coupling semi-chiral fields to the SVM constrained by the equation of motion of χ on (2.41) sets $\tilde{\mathbb{F}} = 0$ and $\tilde{\mathbb{V}} = 0$. The corresponding vector multiplet V is invariant under the residual, twisted chiral, gauge symmetry

$$\delta_g \tilde{V} = i\left(\bar{\tilde{\Lambda}} - \tilde{\Lambda}\right), \quad \mathbb{X}_L \to e^{iQ\tilde{\Lambda}}\mathbb{X}_L, \quad \mathbb{X}_R \to e^{-iQ\bar{\tilde{\Lambda}}}\mathbb{X}_R. \tag{4.18}$$

Then, the gauge theory with matter effectively reduces to a gauge theory of semi-chiral fields coupled to the twisted vector multiplet, where $\tilde{V} = \tilde{V}^\dagger$. Thus, the gauge invariant field strength is given by

$$\Theta = \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde{V}, \quad \bar{\Theta} = \mathbb{D}_-\mathbb{D}_+\tilde{V}. \tag{4.19}$$

Now, as in the last case, we can add a viewer field and a Lagrange multiplier for our field strength, that looks like

$$\begin{aligned}
\mathcal{L}_{\tilde{\mathbb{F}}=0} = &\int d^4\theta \left[\bar{\mathbb{X}}_{1,L}e^{Q_0\tilde{V}_0+Q\tilde{V}_1}\mathbb{X}_{1,L} + \bar{\mathbb{X}}_{1,R}e^{Q_0\tilde{V}_0+Q\tilde{V}_1}\mathbb{X}_{1,R} + \alpha\left(\mathbb{X}_{1,L}e^{Q_0\tilde{V}_0+Q\tilde{V}_1}\mathbb{X}_{1,R} + c.c.\right)\right] \\
&+ \int d^4\theta \left[\bar{\mathbb{X}}_{L,2}e^{Q_0\tilde{V}_0}\mathbb{X}_{L,2} + \bar{\mathbb{X}}_{R,2}e^{Q_0\tilde{V}_0}\mathbb{X}_{R,2} + \alpha_2\left(\mathbb{X}_{L,2}e^{Q\tilde{V}_0}\mathbb{X}_{R,2} + c.c.\right)\right] \\
&+ \frac{1}{2e^2}\int d^4\theta\left(\bar{\Theta}_0\Theta_0\right) + i\left(\int d^4\theta\tilde{\Psi}\Theta_1 + c.c.\right) + is\left(\int d^2\theta\Theta_0 + c.c.\right).
\end{aligned} \tag{4.20}$$

Here, $\tilde{\Psi}$ is an unconstrained superfield and Θ_1 is the field strength associated to V_1. By integrating out $\tilde{\Psi}$ one gets a pure vector superfield V_1 and returns to the original Lagrangian. Furthermore, integrating by parts the Lagrange multipliers using the definition on (4.19), gives

$$\begin{aligned}
\mathcal{L}_{LM,2} &= \int d^4\theta\tilde{\Psi}\Theta_1 + \int d^4\theta\bar{\tilde{\Psi}}\bar{\Theta}_1 \\
&= \int d^4\theta\tilde{\Psi}\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde{V}_1 + \int d^4\theta\bar{\tilde{\Psi}}\mathbb{D}_-\mathbb{D}_+\tilde{V}_1 \\
&= \int d^4\theta\tilde{V}_1\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde{\Psi} + \int d^4\theta\tilde{V}_1\mathbb{D}_-\mathbb{D}_+\bar{\tilde{\Psi}}.
\end{aligned} \tag{4.21}$$

Then, the equation of motion is given as

$$Qe^{Q\tilde{V}_1}\left[\bar{\mathbb{X}}_{1,L}e^{Q_0\tilde{V}_0}\mathbb{X}_{1,L} + \bar{\mathbb{X}}_{1,R}e^{Q_0\tilde{V}_0}\mathbb{X}_{1,R} + \alpha\left(\mathbb{X}_{1,L}e^{Q_0\tilde{V}_0}\mathbb{X}_{1,R} + c.c.\right)\right] + \frac{1}{2}\left(\tilde{\Upsilon} + \bar{\tilde{\Upsilon}}\right) = 0, \tag{4.22}$$

where $\tilde{\Upsilon} = \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde{\Psi}$ and $\bar{\tilde{\Upsilon}} = \mathbb{D}_-\mathbb{D}_+\bar{\tilde{\Psi}}$. Therefore, the solution to V_1 becomes

$$
\begin{aligned}
\tilde{V}_1 =& -\frac{1}{Q}\ln\left[\bar{\mathbb{X}}_{1,L}e^{Q_0\tilde{V}_0}\mathbb{X}_{1,L}+\bar{\mathbb{X}}_{1,R}e^{Q_0\tilde{V}_0}\mathbb{X}_{1,R}+\alpha\left(\mathbb{X}_{1,L}e^{Q_0\tilde{V}_0}\mathbb{X}_{1,R}+c.c.\right)\right] \\
&+\frac{1}{Q}\ln\left(\frac{\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}}{2Q}\right).
\end{aligned} \tag{4.23}
$$

As in the first case by using this solution for the gauge field V_1, in the first term of the Lagrangian (4.20), the kinetic term reduce to

$$
\begin{aligned}
L_{k,2} =& \int d^4\theta\left[\bar{\mathbb{X}}_{1,L}e^{Q_0\tilde{V}_0+Q\tilde{V}_1)}\mathbb{X}_{1,L}+\bar{\mathbb{X}}_{1,R}e^{Q_0\tilde{V}_0+Q\tilde{V}_1}\mathbb{X}_{1,R}+\alpha\left(\mathbb{X}_{1,L}e^{Q_0\tilde{V}_0+Q\tilde{V}_1}\mathbb{X}_{1,R}+c.c.\right)\right] \\
=& -\frac{1}{2Q}\int d^4\theta\left(\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}\right).
\end{aligned} \tag{4.24}
$$

Next, for Lagrange multipliers we get

$$
\begin{aligned}
\mathcal{L}_{LM,2} =& \int d^4\theta V_1\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde{\Psi}+\int d^4\theta V_1\mathbb{D}_-\mathbb{D}_+\bar{\tilde{\Psi}} \\
=& \int d^4\theta V_1\left(\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}\right) \\
=& \int d^4\theta\left(-\frac{Q_0\tilde{V}_0+K}{Q}+\frac{1}{Q}\ln\left(\frac{\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}}{2Q}\right)\right)\left(\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}\right) \\
=& -\int d^4\theta\frac{Q_0\tilde{V}_0+K}{Q}\left(\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}\right)+\frac{1}{Q}\int d^4\theta\ln\left(\frac{\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}}{2Q}\right)\left(\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}\right),
\end{aligned} \tag{4.25}
$$

where

$$
K=\ln\left(\bar{\mathbb{X}}_{1,L}\mathbb{X}_{1,L}+\bar{\mathbb{X}}_{1,R}\mathbb{X}_{1,R}+\alpha\left(\mathbb{X}_{1,L}\mathbb{X}_{1,R}+c.c.\right)\right). \tag{4.26}
$$

In (4.25) we can integrate the first term by parts in the same manner as in (4.16), such that it becomes

$$
\begin{aligned}
-\int d^4\theta\frac{Q_0\tilde{V}_0}{Q}\left(\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}\right) =& -\frac{Q_0}{Q}\int d^4\theta\tilde{V}_0\left(\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}\right) \\
=& -\left(\frac{Q_0}{2Q}\int d^4\theta\tilde{V}_0\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde{\Psi}+\int d^4\theta\tilde{V}_0\mathbb{D}_-\mathbb{D}_+\bar{\tilde{\Psi}}\right) \\
=& \left(\frac{Q_0}{2Q}\int d\theta^+d\bar{\theta}^-\tilde{\Upsilon}\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde{V}_0+\int d\bar{\theta}^+d\theta^-\bar{\tilde{\Upsilon}}\mathbb{D}_-\mathbb{D}_+\tilde{V}_0\right) \\
=& \left(\frac{Q_0}{Q}\int d\theta^+d\bar{\theta}^-\tilde{\Upsilon}\Theta_0+\int d\bar{\theta}^+d\theta^-\bar{\tilde{\Upsilon}}\bar{\Theta}_0\right).
\end{aligned} \tag{4.27}
$$

Henceforth (4.25), becomes

$$\begin{aligned}\mathcal{L}_{LM,2} =& -\int d^4\theta \frac{Q_0\tilde{V}_0 + K}{Q}\left(\tilde{\Upsilon} + \bar{\tilde{\Upsilon}}\right) + \frac{1}{Q}\int d^4\theta \ln\left(\frac{\tilde{\Upsilon} + \bar{\tilde{\Upsilon}}}{2Q}\right)\left(\tilde{\Upsilon} + \bar{\tilde{\Upsilon}}\right) \\ =& \frac{1}{Q}\int d^4\theta \ln\left(\frac{\tilde{\Upsilon} + \bar{\tilde{\Upsilon}}}{2Q}\right)\left(\tilde{\Upsilon} + \bar{\tilde{\Upsilon}}\right) + \frac{Q_0}{Q}\left(\int d\theta^+ d\bar{\theta}^- \tilde{\Upsilon}\Theta_0 + \int d\bar{\theta}^+ d\theta^- \bar{\tilde{\Upsilon}}\bar{\Theta}_0\right) \\ & - \frac{1}{Q}\int d^4\theta K\left(\tilde{\Upsilon} + \bar{\tilde{\Upsilon}}\right).\end{aligned} \tag{4.28}$$

Finally the dual Lagrangian can be written as

$$\begin{aligned}\mathcal{L}_{\tilde{\mathbb{F}}=0} =& \frac{1}{Q}\int d^4\theta \ln\left(\frac{\tilde{\Upsilon} + \bar{\tilde{\Upsilon}}}{2Q}\right)\left(\tilde{\Upsilon} + \bar{\tilde{\Upsilon}}\right) + \frac{1}{2e^2}\int d^4\theta \left(\bar{\Theta}_0\Theta_0\right) - \frac{1}{Q}\int d^4\theta K\left(\tilde{\Upsilon} + \bar{\tilde{\Upsilon}}\right) \\ & + \int d^4\theta \left[\bar{\mathbb{X}}_{L,2} e^{Q_0\tilde{V}_0}\mathbb{X}_{L,2} + \bar{\mathbb{X}}_{R,2} e^{Q_0\tilde{V}_0}\mathbb{X}_{R,2} + \alpha\left(\mathbb{X}_{L,2} e^{Q\tilde{V}_0}\mathbb{X}_{R,2} + c.c.\right)\right] \\ & + \int d^2\theta \left(\frac{Q_0}{16Q}\tilde{\Upsilon} + is\right)\Theta_0 + \int d^2\theta \left(\frac{Q_0}{16Q}\bar{\tilde{\Upsilon}} - i\bar{s}\right)\bar{\Theta}_0.\end{aligned} \tag{4.29}$$

4.1.3 Fixing gauge symmetry

Furthermore, with this result it is possible to fix the gauge as in the paper [3], letting both Lagrangians (4.17) and (4.29) in terms of the dual fields. To do so, we start from the terms in (4.17) and (4.29), which do not depend on the original fields, so they can be eliminated by this method. We begin with the $\mathbb{F} = 0$ case, where we can find one term which depends of the original fields in (4.17), we called it an extra term

$$\sqrt{g^2 + 4LR}, \tag{4.30}$$

where $g = \frac{1}{2Q}\left(\Upsilon + \bar{\Upsilon}\right)$, $L = \bar{\mathbb{X}}_L e^{Q_0 V_0}\mathbb{X}_L$, $R = \bar{\mathbb{X}}_R e^{-Q_0 V_0}\mathbb{X}_R$. Here, we omitted the index 1 in the semi-chiral fields. We know from the transformation of these fields, giving by (4.5), that the LR term under this symmetry is given by

$$\begin{aligned}LR &= \left(\bar{\mathbb{X}}_L\mathbb{X}_L\right)\left(\bar{\mathbb{X}}_R\mathbb{X}_R\right) \\ &\to \left(e^{iQ\Lambda}\mathbb{X}_L e^{-iQ\bar{\Lambda}}\bar{\mathbb{X}}_L\right)\left(e^{-iQ\Lambda}\mathbb{X}_R e^{iQ\bar{\Lambda}}\bar{\mathbb{X}}_R\right) \\ &= e^{iQ\Lambda - iQ\bar{\Lambda} - iQ\Lambda + iQ\bar{\Lambda}}\left(\mathbb{X}_L\bar{\mathbb{X}}_L\right)\left(\mathbb{X}_R\bar{\mathbb{X}}_R\right) \\ &= \left(\mathbb{X}_L\bar{\mathbb{X}}_L\right)\left(\mathbb{X}_R\bar{\mathbb{X}}_R\right).\end{aligned} \tag{4.31}$$

This term is invariant under $U(1)$ transformation as we expected. Nevertheless, these are strange kinds of fields that do not let us gauge away semi-chiral fields [6], [11].

One cannot use them to go to a unitary gauge, so, we need think in partial gauge fixing [6]. Nonetheless, we can keep all the fields of the CSVM; such that we can choose a unitary gauge, e.g., $\mathbb{X}_R = \mathbb{X}_L = 1$ [6]. This let us the extra term equal to (4.30).

$$\left(2 + \frac{1}{Q}\right)\left(\sqrt{g^2 + 4LR} \pm g\right), \tag{4.32}$$

In the second case, the extra term is given by

$$T = \ln\left(\bar{\mathbb{X}}_L \mathbb{X}_L + \bar{\mathbb{X}}_R \mathbb{X}_R + \alpha\left(\mathbb{X}_L \mathbb{X}_R + \bar{\mathbb{X}}_L \bar{\mathbb{X}}_R\right)\right), \tag{4.33}$$

Thus, by using the rules of transformation in (4.18), we obtain

$$\begin{aligned}
s &= \bar{\mathbb{X}}_L \mathbb{X}_L + \bar{\mathbb{X}}_R \mathbb{X}_R + \alpha\left(\mathbb{X}_L \mathbb{X}_R + \bar{\mathbb{X}}_L \bar{\mathbb{X}}_R\right) \\
&\to e^{-iQ\bar{\tilde{\Lambda}}} \bar{\mathbb{X}}_L e^{iQ\tilde{\Lambda}} \mathbb{X}_L + e^{iQ\tilde{\Lambda}} \bar{\mathbb{X}}_R e^{-iQ\bar{\tilde{\Lambda}}} \mathbb{X}_R \\
&+ \alpha\left(e^{iQ\tilde{\Lambda}} \mathbb{X}_L e^{-iQ\bar{\tilde{\Lambda}}} \mathbb{X}_R + e^{iQ\bar{\tilde{\Lambda}}} \bar{\mathbb{X}}_L e^{iQ\tilde{\Lambda}} \bar{\mathbb{X}}_R\right) \\
&= e^{iQ\tilde{\Lambda} - iQ\bar{\tilde{\Lambda}}} \bar{\mathbb{X}}_L \mathbb{X}_L + e^{iQ\tilde{\Lambda} - iQ\bar{\tilde{\Lambda}}} \bar{\mathbb{X}}_R \mathbb{X}_R \\
&+ \alpha\left(e^{iQ\tilde{\Lambda} - iQ\bar{\tilde{\Lambda}}} \mathbb{X}_L \mathbb{X}_R + e^{iQ\bar{\tilde{\Lambda}} + iQ\tilde{\Lambda}} \bar{\mathbb{X}}_L \bar{\mathbb{X}}_R\right) \\
&= e^{iQ(\tilde{\Lambda} - \bar{\tilde{\Lambda}})} \bar{\mathbb{X}}_L \mathbb{X}_L + e^{iQ(\tilde{\Lambda} - \bar{\tilde{\Lambda}})} \bar{\mathbb{X}}_R \mathbb{X}_R \\
&+ \alpha\left(e^{iQ(\tilde{\Lambda} - \bar{\tilde{\Lambda}})} \mathbb{X}_L \mathbb{X}_R + e^{iQ(-\bar{\tilde{\Lambda}} + \tilde{\Lambda})} \bar{\mathbb{X}}_L \bar{\mathbb{X}}_R\right) \\
&= e^{iQ(\tilde{\Lambda} - \bar{\tilde{\Lambda}})}\left(\bar{\mathbb{X}}_L \mathbb{X}_L + \bar{\mathbb{X}}_R \mathbb{X}_R\right. \\
&\left. + \alpha\left(\mathbb{X}_L \mathbb{X}_R + \bar{\mathbb{X}}_L \bar{\mathbb{X}}_R\right)\right),
\end{aligned} \tag{4.34}$$

so, we can choose an unitary global fixing $s = 1$, then substituting in the equation (4.33) give us $T = 0$.

4.2 T-dual model and gauge fixing

We have seen in the last section that the extra term of the Lagrangian (4.17) can not be fixed at all, we can keep all the fields of the CSVM and choose a unitary gauged or apply a CSVM and not fix the fields to a unitary gauged [3], in this case we choose apply a CSVM and not fix the fields to a unitary gauged. For the second Lagrangian (4.29) can be fix to 1 all the term in the logarithm, which leave us $T = 0$ in (4.33). Furthermore, this allows us to simplify both Lagrangians and leave us the dual Lagrangian in terms of dual fields and the field that preserves the symmetry. For the first case, we have that the dual Lagrangian is

$$
\begin{aligned}
\mathcal{L}_{\mathbb{F}=0,dual} =& \int d^4\theta \left[\sqrt{g^2+4LR} + 2g\ln\left(\sqrt{g^2+4LR}\pm g\right)\right] - \int d^4\theta \frac{1}{2e^2}\bar{\Sigma}_0\Sigma_0 \\
&+ \int d^4\theta \left[\bar{\mathbb{X}}_{2,L}e^{Q_0V_0}\mathbb{X}_{2,L} + \bar{\mathbb{X}}_{2,R}e^{-Q_0V_0}\mathbb{X}_{2,R} + \beta(\mathbb{X}_{2.L}\mathbb{X}_{2,R} + c.c.)\right] \\
&+ \int d\theta^+ d\bar{\theta}^- \left(\frac{Q_0}{Q}\Upsilon + it\right)\Sigma_0 + \int d\bar{\theta}^+ d\theta^- \left(\frac{Q_0}{Q}\bar{\Upsilon} - i\bar{t}\right)\bar{\Sigma}_0.
\end{aligned} \tag{4.35}
$$

For the second case we obtain the Lagrangian in (4.29) without the K term that is eliminated when we fix the gauge, letting us to

$$
\begin{aligned}
\mathcal{L}_{\tilde{\mathbb{F}}=0,dual} =& 2\int d^4\theta \ln\left(\frac{\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}}{2Q}\right)\left(\frac{\tilde{\Upsilon}+\bar{\tilde{\Upsilon}}}{2Q}\right) + \frac{1}{2e^2}\int d^4\theta \left(\bar{\Theta}_0\Theta_0\right) \\
&+ \int d^4\theta \left[\bar{\mathbb{X}}_{L,2}e^{Q_0\tilde{V}_0}\mathbb{X}_{L,2} + \bar{\mathbb{X}}_{R,2}e^{Q_0\tilde{V}_0}\mathbb{X}_{R,2} + \alpha_2\left(\mathbb{X}_{L,2}e^{Q\tilde{V}_0}\mathbb{X}_{R,2} + c.c.\right)\right] \\
&+ \int d^2\theta \left(\frac{Q_0}{16Q}\tilde{\Upsilon} + is\right)\Theta_0 + \int d^2\theta \left(\frac{Q_0}{16Q}\bar{\tilde{\Upsilon}} - i\bar{s}\right)\bar{\Theta}_0,
\end{aligned} \tag{4.36}
$$

where $\tilde{\Upsilon}, \bar{\tilde{\Upsilon}}$ are twisted chiral and anti-twisted chiral, respectively.

Let us compare now with the dual model for chiral and twisted chiral fields without torsion, we can observe that this dual Lagrangians seem so similar to the dual chiral and twisted chiral Lagrangians given in [16] with extra terms. That makes sense because we must have some extra term given by the semi-chiral fields.

Chapter 5

Scalar potential of the GLSM's with torsion

In this chapter we start by obtaining the T-dual Lagrangian for a GLSM with semi-chiral superfields. We apply the method given in [9], [16], writing the Lagrangian in terms of the Kähler potential and its derivatives. Furthermore, we separate the Lagrangian in the bosonic and the fermionic components, so that the bosonic part contains the classical scalar potential U.

5.1 Case $\mathbb{F} = 0$

We separate the problem; the semi-chiral Kähler potential that preserves the symmetry $U(1)$ and the part of the extra term, but we omitted the part given by the combination of the semi-chiral with the dual fields, we let it for future work. In the appendix B and C, we do a SUSY reduction from $\mathcal{N} = (2, 2)$ to $\mathcal{N} = 0$. This method gives us the potential of one part of Kähler term and for the superpotential in the Lagrangian (4.35) as

$$\begin{aligned}
&\int d^4\theta \left[\bar{\mathbb{X}}_{2,L} e^{Q_0 V_0} \mathbb{X}_{2,L} + \bar{\mathbb{X}}_{2,R} e^{-Q_0 V_0} \mathbb{X}_{2,R} + \beta(\mathbb{X}_{2.L}\mathbb{X}_{2,R} + c.c.)\right] \\
&+ \int d\theta^+ d\bar{\theta}^- \left(\frac{Q_0}{Q}\Upsilon + it\right)\Sigma_0 + \int d\bar{\theta}^+ d\theta^- \left(\frac{Q_0}{Q}\bar{\Upsilon} - i\bar{t}\right)\bar{\Sigma}_0 \\
&- \int d^4\theta \frac{1}{2e^2}\bar{\Sigma}_0\Sigma_0
\end{aligned} \tag{5.1}$$

.

The above Lagrangian give us the potential same potential as the given in (C.4), but we take the special case when $\mathbb{F} = M = 0$, this let us

$$U = 2e^2(\mu_2 - r_2)^2 + \frac{\beta^2}{2(\beta^2 - 1)}|\tilde{\sigma}|^2|X_2|^2 \tag{5.2}$$

where $\mu_2 \equiv -\left(\bar{X}_L X_L - \bar{X}_R X_R\right)$, $r_2 = -2\,\mathrm{Re}\{t\}$ and $|X|^2 = X^\mu X^\nu g_{\mu\nu}$. At this point we can use the method in the appendix B to make the SUSY reduction, we redefine the Lagrangian using the equation (B.8) as

$$\begin{aligned}
&\int d^4\theta \left[\sqrt{g^2+4LR} + 2g\ln\left(\sqrt{g^2+4LR} \pm g\right)\right] \\
&= \int \mathcal{D}_+\mathcal{D}_- \left[\mathcal{Q}_+\mathcal{Q}_- \left(\sqrt{g^2+4LR} + 2g\ln\left(\sqrt{g^2+4LR} \pm g\right)\right)\right]| \\
&= h(X_1, \bar{X}_1, g|, \bar{g}|) = h
\end{aligned} \tag{5.3}$$

where $g|$ and X_1 denote the dual field and the original fields with all the fermionic coordinates set to zero. This analysis allow us rewrite the potential as

$$U = 2e^2(\mu_2 - r_2)^2 + \frac{\beta^2}{2(\beta^2-1)}|\tilde{\sigma}_0|^2|X|^2 + h \tag{5.4}$$

The geometry for this scalar potential, can be obtained by using the method presented on [17], [18], with this aim analyze each branch and take the D-term as $t = \frac{1}{2}(\xi - i\frac{\theta}{2\pi})$. We have:

- For $\tilde{\sigma}_0 \neq 0$ and $\mu_2 \neq 0$ the SUSY is broken and there is no moduli space.
- This potential has SUSY vacua at $\tilde{\sigma}_0 = 0$ and $\mu_2 \neq 0$, this implies that there are three subsets of vacua:
  - First, where $X_1 = 0$ and $2e^2(\mu_2 - r_2)^2 + h(g, X_1 = 0) = 0$.
  - second, while $\mu_2 = 0$ and $2e^2 r_2^2 + h(g, X_1) = 0$.
  - third, while $g = 0$ and $2e^2(\mu_2 - r_2)^2 + h(g = 0, X_1) = 0$.
- On the Higgs branch, we take $\sigma_0 = 0$ and $\mu_2 = 0$, then we have

$$\mathcal{M}_{\mathbb{F}=0} = \{2e^2 r_2^2 + h(g, X_1) = 0\}/U(1). \tag{5.5}$$

5.2 Case $\tilde{\mathbb{F}} = 0$

Starting with the metric that has been obtained using the equation (3.27), we just take the Kähler potential given by the twisted-chiral and twisted anti-chiral parts of the Lagrangian, in this case (4.36)

$$g_{\alpha\beta,\tilde{\mathbb{F}}=0} = \begin{pmatrix} \frac{1}{2Q(\tilde{y}+\tilde{\bar{y}})} & 0 \\ 0 & \frac{1}{2e^2} \end{pmatrix}, \tag{5.6}$$

which inverse is

$$g^{\alpha\beta}_{\tilde{\mathbb{F}}=0} = \begin{pmatrix} 2Q(\tilde{y}+\bar{\tilde{y}}) & 0 \\ 0 & 2e^2 \end{pmatrix}, \tag{5.7}$$

where we take the bosonic terms of the chiral $\tilde{\Upsilon}$ and anti-chiral $\bar{\tilde{\Upsilon}}$ fields to be $\tilde{y}, \bar{\tilde{y}}$, respectively.

The non zero Christoffel symbols are computed from equations (3.30) and (3.31), they are

$$\Gamma^1_{1,1} = \bar{\Gamma}^1_{1,1} = -\frac{1}{2(\tilde{y}+\bar{\tilde{y}})}. \tag{5.8}$$

As well as in the first case, we obtain the potential using the Lagrangian (3.29), where we just take the bosonic terms, then, we obtain

$$U = \frac{80Q_0}{Q^2}\left(\frac{10}{e^2}-1\right)|i\frac{Q}{Q_0}s+\tilde{y}|^2 + 6Q\left(\frac{Q_0\tilde{\sigma_0}}{2Q}\right)^2(\tilde{y}+\bar{\tilde{y}}). \tag{5.9}$$

In this case, as above, the semi-chiral fields contribute to the scalar potential as

$$\begin{aligned} U =& \frac{80Q_0}{Q^2}\left(\frac{10}{e^2}-1\right)|i\frac{Q}{Q_0}s+\tilde{y}|^2 + 6Q\left(\frac{Q_0\tilde{\sigma_0}}{2Q}\right)^2(\tilde{y}+\bar{\tilde{y}}) \\ &- 2e^2\left(\mu_1^2+\mu_2^2-\frac{1}{2}\mu_3^2\right), \end{aligned} \tag{5.10}$$

where the μ_i are defined in (B.22). Then, we can analyze the geometry vacua, the analysis is the same as the previous. We take the D-term as $s = \frac{1}{2}(\tilde{\xi} - i\frac{\tilde{\theta}}{2\pi})$ and $\tilde{x} = \mu_2^2 - \frac{1}{2}\mu_3^2$, the cases are:

- For $\tilde{\sigma_0} \neq 0$ and $\mu_1 \neq 0$ the SUSY is broken and there is no moduli space.
- This potential has SUSY vacua at $\tilde{\sigma_0} = 0$ and $\mu_1 \neq 0$, this implies that there are four subsets of vacua:
  - where $y = 0$ and $\frac{80Q_0}{Q^2}\left(\frac{10}{e^2}-1\right)|i\frac{Q}{Q_0}s|^2 - 2e^2(\mu_1^2 + x) = 0$,
  - where $\mu_1 = 0$ and $|i\frac{Q}{Q_0}s+\tilde{y}|^2 - 2e^2\tilde{x} = 0$
  - where $\mu_2 = 0$ and $|i\frac{Q}{Q_0}s+\tilde{y}|^2 - 2e^2\left(\mu_1^2 - \frac{1}{2}\mu_3^2\right) = 0$.
  - while $\mu_3 = 0$ and $|i\frac{Q}{Q_0}s+\tilde{y}|^2 - 2e^2\left(\mu_1^2+\mu_2^2\right) = 0$.
- On the Higgs branch we take $\sigma_0 = 0$ and $\mu_1 = 0$, then we have

$$\mathcal{M}_{\tilde{\mathbb{F}}=0} = \{|i\frac{Q}{Q_0}s+\tilde{y}|^2 - 2e^2x = 0\}/U(1). \tag{5.11}$$

We have obtained the locus given the scalar potential for each case. This is an important result because it represents the SUSY vacuum geometry in the dual theory. With this analysis we cannot see the torsion, with the study presented in next chapter we will see its origin.

Chapter 6

Geometry of the GLSM's with torsion

In this chapter, we analyze the target space geometry of the original model. Besides, using the same method given in [6], [15], we compute the geometry of the dual target space given the classical scalar potential in both Lagrangians (4.35) and (4.36).

6.1 Original model

The structure of the moduli space of the GLSM's, which involves SVM and CSVM, is very different, so we analyze them separately.

We start as in the paper [6], by reducing partially the $\mathcal{N} = (2,2)$ superspace to $\mathcal{N} = (1,1)$ superspace. The Lagrangian is given by $\mathcal{L} = \mathcal{L}_{SVM} + \mathcal{L}_{FI} + \mathcal{L}_{matter}$, where each term of it are (2.34), (2.35) and (2.39), respectively. The superspace reduction allows us to identify the metric g and b-field easily. Here, we denote as $\mathcal{D}_\pm$ the gauge covariant derivatives on the language $\mathcal{N} = (1,1)$ and the basic multiplets are described by the unconstrained bosonic and fermionic superfields, and the vector multiplet with superfield strength f. From the reduction, we obtain the Lagrangian:

$$\mathcal{L} = \mathcal{L}_{SVM} + \int \mathcal{D}_+\mathcal{D}_- \left[\frac{1}{2}(g_{\mu\nu} + b_{\mu\nu})\mathcal{D}_+X^\mu\mathcal{D}_-X^\nu + 2i\sigma_I(\mu_I - r_I)\right], \tag{6.1}$$

where the indexes are $I = 1, 2, 3$, $\mathcal{L}_{SVM}$ is given in (B.13), $X^\mu = (X_L, \bar{X}_L, X_R, \bar{X}_R)$, we define r_I as $r_1 = 2\,\mathrm{Re}(s)$, $r_2 = -2\,\mathrm{Re}(t)$, $r_3 = \mathrm{Im}(s + t)$. The functions μ_I are given in terms of the bosonic part of the semi-chiral superfields as

$$\mu_1 \equiv \bar{X}_L X_L + \bar{X}_R X_R + \alpha_2\left(X_L X_R + \bar{X}_L \bar{X}_R\right), \tag{6.2}$$

$$\mu_2 \equiv -\left(\bar{X}_L X_L - \bar{X}_R X_R\right), \tag{6.3}$$

$$\mu_2 \equiv -\frac{i\alpha_2}{2}\left(X_L X_R - \bar{X}_L \bar{X}_R\right). \tag{6.4}$$

and g, b are the flat-space metric and b-field, setting $(a = b = 1, d = \alpha_2)$. In this basis one has

$$J_+ = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & \frac{2i}{\beta} & i & 0 \\ -\frac{2i}{\beta} & 0 & 0 & -i \end{pmatrix}, \quad J_- = \begin{pmatrix} i & 0 & 0 & \frac{2i}{\beta} \\ 0 & -i & -\frac{2i}{\beta} & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{6.5}$$

and

$$g = 4\begin{pmatrix} 0 & 1 & \frac{1}{\beta} & 0 \\ 1 & 0 & 0 & \frac{1}{\beta} \\ \frac{1}{\beta} & 0 & 0 & 1 \\ 0 & \frac{1}{\beta} & 1 & 0 \end{pmatrix}, \quad b = 2\left(\frac{2}{\beta} - \beta\right)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{6.6}$$

Finally, reducing to $\mathcal{N} = 0$ components and integrating out the auxiliary fields as in appendix B, we obtain the following scalar potential

$$U = 2e^2\left[(\mu_1 - r_1)^2 + (\mu_2 - r_2)^2 + (\mu_3 - r_3)^2\right] + \beta^2\left(|\sigma|^2 + \frac{1}{\beta^2 - 1}|\tilde{\sigma}|^2\right)\frac{1}{2}|X|^2, \tag{6.7}$$

where $|X|^2 \equiv g_{\mu\nu}X^\mu X^\nu$, with g as in (6.6) and from the appendix C we see that $\sigma = \mathbb{F}|$, $\tilde{\sigma} = \tilde{\mathbb{F}}|$ are the complex scalars fields of the SVM.

There are two branches: the Coulomb branch and the Higgs branch:

1. The Coulomb branch is parametrized by the VEVs of σ and $\tilde{\sigma}$ and by $X_L = X_R = 0$. This branch exists only for $r_1 = r_2 = r_3 = 0$.

2. The Higgs branch is given by $\sigma = \tilde{\sigma} = 0$, and the space of solutions to

$$\mu_I = r_I, \tag{6.8}$$

modulo U(1) gauge transformations.

In this simple model with a single pair of semichiral fields, the Higgs branch is a point. We can obtain something more interesting when we consider GLSMs with opposite charges coupled to the CSVM with a twisted mass and FI parameters. This Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{SVM} + \mathcal{L}_{\Phi;M} + \mathcal{L}_{FI} + \mathcal{L}_{matter}, \tag{6.9}$$

where each term is given by (2.33), (B.14), (2.35) and (2.39), respectively. The only FI parameter is $t = \frac{1}{2}(\xi - i\frac{\theta}{2\pi})$, the FI parameter s given in (2.35) can be set to $s = 0$ by a shift in Φ. In terms the $\mathcal{N} = (1,1)$ superspace, the CSVM $\mathbb{F} = M$ corresponds to setting

$$\begin{aligned} \sigma_1 &= 2\,\mathrm{Re}(M), \\ \sigma_3 &= f - 4\,\mathrm{Im}(M), \end{aligned} \tag{6.10}$$

while $\tilde{\mathbb{F}} = \tilde{M}$ corresponds to setting

$$\begin{aligned} \sigma_2 &= 2\,\mathrm{Re}(\tilde{M}), \\ \sigma_3 &= -f + 4\,\mathrm{Im}(\tilde{M}). \end{aligned} \tag{6.11}$$

Now, using the Lagrangian in (6.1), taking $\mathcal{L}_{gauge}$ as the action for the standard vector and substituting on (6.1) the equations (6.10) and (6.11), these gives

$$\begin{aligned} \mathcal{L} =& \mathcal{L}_{gauge} + \int \mathcal{D}_+\mathcal{D}_- \left(\frac{1}{2}(g_{\mu\nu} + b_{\mu\nu})\mathcal{D}_+X^\mu\mathcal{D}_-X^\nu + 2i\sigma_2(\mu_2 - \xi) \right. \\ & \left. + 2i\left(f\mu_3 + 2\mu_1\,\mathrm{Re}(M) - 4\mu_3\,\mathrm{Im}(M)\right)\right). \end{aligned} \tag{6.12}$$

Finally, reducing to $\mathcal{N} = 0$ using the set of equations (C.1) in (6.12) leads to (C.2), where integrating out the auxiliary fields D_I in (C.2) gives the same potential as (6.7).

For the case where $\mathbb{F} = M$ one must set $D_1 = D_3 = 0$ and $\sigma = M$ in (C.2) and integrate out the auxiliary field D_2 to obtain

$$U = 2e^2(\mu_2 + \xi)^2 + \beta^2\left(|M|^2 + \frac{1}{\beta^2 - 1}|\tilde{\sigma}|^2\right)\frac{1}{2}|X|^2. \tag{6.13}$$

As the SVM case we analize the different cases for SUSY vacua, which depends on the values of the parameters ξ and M:

1. For $M \neq 0$ and $\xi \neq 0$ supersymmetry is broken and there is no moduli space.
2. For $M \neq 0$ and $\xi = 0$ there is only the Coulomb branch paramaterized by $\tilde{\sigma}$, and $X_L = X_R = 0$.
3. For $M = 0$ there can be two branches.
   (a) The Higgs branch, given by $\tilde{\sigma} = 0$ and the space of solutions to
   $$|X_L|^2 - |X_R|^2 = \xi, \tag{6.14}$$
   modulo gauge transformations.
   (b) For $\xi = 0$ there is also a Coulomb branch, parametrized by the VEV of $\tilde{\sigma}$ and $X_L = X_R = 0$.

The case where $\tilde{\mathbb{F}} = \tilde{M}$ is analogous. Then, we take a shift in χ and set $t = 0$, thus, we take $s = \frac{1}{2}(\tilde{\xi} - i\frac{\tilde{\theta}}{2\pi})$, then one must set $D_2 = D_3 = 0$ and $\tilde{\sigma} = \tilde{M}$ in (C.2) and integrate out the auxiliary field D_1 to obtain

$$\tilde{U} = 2e^2(\mu_1 - \tilde{\xi})^2 + \beta^2\left(|\sigma|^2 + \frac{1}{\beta^2 - 1}|\tilde{M}|^2\right)\frac{1}{2}|X|^2. \tag{6.15}$$

This depends on the values of the parameters $\tilde{\xi}$ and $\tilde{M}$.

1. For $\tilde{M} \neq 0$ and $\tilde{\xi} \neq 0$ supersymmetry is broken and there is no moduli space.

2. For $\tilde{M} \neq 0$ and $\tilde{\xi} = 0$ there is only the Coulomb branch paramaterized by σ, and $X_L = X_R = 0$.

3. For $\tilde{M} = 0$ there can be two branches.

   (a) The Higgs branch, given by $\sigma = 0$ and the space of solutions to

   $$|X_L|^2 + |X_R|^2 + \beta\left(X_L X_R + \bar{X}_L \bar{X}_R\right) = \tilde{\xi}, \tag{6.16}$$

   modulo gauge transformations.

   (b) For $\tilde{\xi} = 0$ there is also a Coulomb branch, parametrized by the VEV of σ and $X_L = X_R = 0$.

We focus on the Higgs branch that is a non-compact generalized Kähler manifold whose geometric structure depends on data such as the number of multiplets and their charges.

6.2 Model with two pairs semi-chiral fields

Consider the Lagrangian (4.6), which have two pairs of semi-chiral fields $(\bar{\mathbb{X}}_R, \bar{\mathbb{X}}_L, \bar{\mathbb{X}}_{2,R}, \bar{\mathbb{X}}_{2,L})$ with charges $(1, -1, 1, -1)$, respectively. As discussed, the low-energy dynamics on the Higgs branch is given by a NLSM as in the Appendix B and C, we can do a SUSY reduction and adding these fields in a summation as

$$\begin{aligned}\mathcal{L}_k &= \int d^4\theta \left[\bar{\mathbb{X}}_{1,L} e^{Q_0 V_0 + Q V_1} \mathbb{X}_{1,L} + \bar{\mathbb{X}}_{1,R} e^{-(Q_0 V_0 + Q V_1)} \mathbb{X}_{1,R} + \beta_1 (\mathbb{X}_{1,L}\mathbb{X}_{1,R} + c.c.)\right] \\ &+ \int d^4\theta \left[\bar{\mathbb{X}}_{2,L} e^{Q_0 V_0} \mathbb{X}_{2,L} + \bar{\mathbb{X}}_{2,R} e^{-Q_0 V_0} \mathbb{X}_{2,R} + \beta_2 (\mathbb{X}_{2.L}\mathbb{X}_{2,R} + c.c.)\right] \\ &= \int d^4\theta \sum_{i=1}^{2} \left[\bar{\mathbb{X}}_{i,L} e^{Q_0 V_0 + Q V_1} \mathbb{X}_{i,L} + \bar{\mathbb{X}}_{i,R} e^{-(Q_0 V_0 + Q V_1)} \mathbb{X}_{i,R} + \beta_i (\mathbb{X}_{i,L}\mathbb{X}_{i,R} + c.c.)\right],\end{aligned} \tag{6.17}$$

we get the same Higgs branch as in (6.14), but with the sum of these fields

$$\mathcal{M} = \{\sum_{i=1}^{2} |X_{i,L}|^2 - |X_{i,R}|^2 = \xi\}/U(1). \tag{6.18}$$

Topologically, this space coincides with the conifold, which admits a Calabi-Yau metric with $SU(2) \times SU(2) \times U(1)$ symmetry and can be realized as a GLSM for chiral fields with charges $(1, -1, 1, -1)$.

6.3 Case $\mathbb{F} = 0$

Let us in this section to analyze the dual model geometry for the gauge given by $\mathbb{F} = 0$.

In previous section we followed the method given in the paper [6], were they started from the objects $(J_\pm, g, b)$, which they used to completely describe the geometry of the target space, but using the scalar potential instead of $(J_\pm, g, b)$. They analyzed the geometry using the parametrized potential, giving the different branches. We apply the same method in this chapter, but the full expressions for $(J_\pm, g, b)$ are lengthy and not particularly enlightening. Then we just show the value of the H-field

$$
\begin{aligned}
H = & \Bigg[\frac{32\,\mathrm{Re}\{\phi\}}{Q^2 \left(32\,\mathrm{Re}\{X_L\} + \frac{(2\,\mathrm{Re}\{\phi\})^2}{Q^2}\right)^{3/2}} \\
& - \frac{6(\mathrm{Re}\{\phi\})\,(64|X_L|^2 Q^2 + 8(2\,\mathrm{Re}\{\Phi\})^2)}{Q^4 \left(16|X_L|^2 + \frac{(2\,\mathrm{Re}\{\phi\})^2}{Q^2}\right)^{5/2}} \Bigg] \big(d\phi \wedge dX_L \wedge dX_R \\
& + d\bar{\phi} \wedge dX_L \wedge dX_R\big) + c.c. + \mathcal{O}\Big(\frac{1}{\beta_1}, \frac{1}{\beta_2}\Big).
\end{aligned}
\tag{6.19}
$$

In this model we start from a conifold to this geometry. This model realizes a family of generalized Kähler structures, but we leave a careful study of this case for future work.

6.4 Case $\tilde{\mathbb{F}} = 0$

Let us analyze in this section the dual model geometry for the gauge given by $\tilde{\mathbb{F}} = 0$.

In this case we followed the method given in the paper [6], were they started from the objects $(J_\pm, g, b)$, where the forms $J_\pm$ are given by the equations (3.5) and (3.6), we apply the Kähler potential in (4.36) to give us

$$
J_+ = \begin{pmatrix}
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2ie^{QV}}{\alpha_2} & i & 0 & 0 & 0 & 0 & 0 \\
-\frac{2ie^{QV}}{\alpha_2} & 0 & 0 & -i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -i
\end{pmatrix}
\tag{6.20}
$$

and

$$J_- = \begin{pmatrix} i & 0 & 0 & \frac{2ie^{-QV}}{\alpha_2} & 0 & 0 & 0 & 0 \\ 0 & -i & -\frac{2ie^{-QV}}{\alpha_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \end{pmatrix}. \tag{6.21}$$

As we mentioned in chapter 3, the metric in the generalized Kähler case cannot be obtained using the equations in (3.8). The part of the metric that we can determinate is given by

$$g = \left(\begin{array}{cccc|cccc} 0 & 4e^{QV_0} & \frac{4}{\beta_2} & 0 & & & & \\ 4e^{QV_0} & 0 & 0 & \frac{4}{\beta_2} & & \multicolumn{2}{c}{\Large 0} & \\ \frac{4}{\beta_2} & 0 & 0 & 4e^{-QV_0} & & & & \\ 0 & \frac{4}{\beta_2} & 4e^{-QV_0} & 0 & & & & \\ \hline & & & & g_{1,1} & g_{1,2} & g_{1,3} & g_{1,4} \\ & \multicolumn{2}{c}{\Large 0} & & g_{2,1} & g_{2,2} & g_{2,3} & g_{2,4} \\ & & & & g_{3,1} & g_{3,2} & g_{3,3} & g_{3,4} \\ & & & & g_{4,1} & g_{4,2} & g_{4,3} & g_{4,4} \end{array}\right), \tag{6.22}$$

while the part of the co-kernel we cannot determinate is denoted by $g_{a,b}$ with $a, b = 1, 2, 3, 4$. Furthermore, we can obtain the torsion b. The set of equations (3.13), where $E = \frac{1}{2}(g + b)$ is given by

$$E = \begin{pmatrix} E_{LL} & E_{LR} & E_{Lc} & E_{Lt} \\ E_{RL} & E_{RR} & E_{Rc} & E_{Rt} \\ E_{cL} & E_{cR} & E_{cc} & E_{ct} \\ E_{tL} & E_{tR} & E_{tc} & E_{tt}. \end{pmatrix}. \tag{6.23}$$

Using the set of equations (3.13) and the equation above we can obtain the following matrix

$$E = \left(\begin{array}{cccc|cccc} 0 & 2e^{QV_0} & \frac{4}{\beta_2} - \beta_2 & 0 & & & & \\ 2e^{QV_0} & 0 & 0 & \frac{4}{\beta_2} - \beta_2 & & \multicolumn{2}{c}{\Large 0} & \\ \beta_2 & 0 & 0 & 2e^{-QV_0} & & & & \\ 0 & \beta_2 & 2e^{-QV_0} & 0 & & & & \\ \hline & & & & 0 & s_1 & 0 & 0 \\ & \multicolumn{2}{c}{\Large 0} & & s_1 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & \frac{1}{e^2} \\ & & & & 0 & 0 & \frac{1}{e^2} & 0 \end{array}\right), \tag{6.24}$$

where

$$s_2 = -\frac{1}{Q^2\tilde{g}|}. \tag{6.25}$$

Finally the b-field is easy to obtain from $E = \frac{1}{2}(g+b)$ as

$$b = \left(\begin{array}{cccc|c} 0 & 0 & \frac{4}{\beta_2} - 2\beta_2 & 0 & \\ 0 & 0 & 0 & \frac{4}{\beta_2} - 2\beta_2 & \\ 2\beta_2 - \frac{4}{\beta_2} & 0 & 0 & 0 & \mathbf{0} \\ 0 & 2\beta_2 - \frac{4}{\beta_2} & 0 & 0 & \\ \hline & & \mathbf{0} & & S_0 \end{array}\right), \tag{6.26}$$

where

$$S_0 = \begin{pmatrix} -g_{1,1} & \frac{1}{e^2} - g_{1,2} & -g_{1,3} & -g_{1,4} \\ \frac{1}{e^2} - g_{2,1} & -g_{2,2} & -g_{2,3} & -g_{2,4} \\ -g_{3,1} & -g_{3,2} & -g_{3,3} & 2s_2 - g_{3,4} \\ -g_{4,1} & -g_{4,2} & 2s_2 - g_{4,3} & -g_{4,4} \end{pmatrix}. \tag{6.27}$$

We know that the metric g is symmetric and the b-field is anti-symmetric, so, we use this fact to find the value on the co-kernel region $G = g_{a,b}$. From the fact that the metric is symmetric we have $g_{a,b} = g_{b,a}$ in the equations (6.27) and (6.22), where $a, b = 1, 2, 3, 4$. Then we can obtain more information about S_0 in (6.27) and $G = g_{a,b}$ using that the b-field is anti-symmetric, then $g_{a,b} = -g_{b,a}$ in (6.27) where for $a \neq b$ the components $g_{a,b}$ are zero except for

$$\begin{aligned} \frac{1}{e^2} - g_{1,2} &= -\frac{1}{e^2} + g_{2,1}, \\ 2s_2 - g_{4,3} &= -2s_2 + g_{3,4}. \end{aligned}$$

This gives us $g_{3,4} = 2s_2$ and $g_{1,2} = \frac{1}{e^2}$, then the matrices will be

$$g = \left(\begin{array}{cccc|cccc} 0 & 4e^{QV_0} & \frac{4}{\alpha_2} & 0 & & & & \\ 4e^{QV_0} & 0 & 0 & \frac{4}{\alpha_2} & & \mathbf{0} & & \\ \frac{4}{\alpha_2} & 0 & 0 & 4e^{-QV_0} & & & & \\ 0 & \frac{4}{\alpha_2} & 4e^{-QV_0} & 0 & & & & \\ \hline & & & & g_{1,1} & \frac{1}{e^2} & 0 & 0 \\ & \mathbf{0} & & & \frac{1}{e^2} & g_{2,2} & 0 & 0 \\ & & & & 0 & 0 & g_{3,3} & s_2 \\ & & & & 0 & 0 & s_2 & g_{4,4} \end{array}\right), \tag{6.28}$$

$$
b = \left(\begin{array}{cccc|cccc}
0 & 0 & \frac{4}{\alpha_2} - 2\alpha_2 & 0 & & & & \\
0 & 0 & 0 & \frac{4}{\alpha_2} - 2\alpha_2 & & \multicolumn{2}{c}{\text{\Huge 0}} & \\
2\alpha_2 - \frac{4}{\alpha_2} & 0 & 0 & 0 & & & & \\
0 & 2\alpha_2 - \frac{4}{\alpha_2} & 0 & 0 & & & & \\
\hline
 & & & & -g_{1,1} & 0 & 0 & 0 \\
 & \multicolumn{2}{c}{\text{\Huge 0}} & & 0 & -g_{2,2} & 0 & \\
 & & & & 0 & 0 & -g_{3,3} & 0 \\
 & & & & 0 & 0 & 0 & -g_{4,4}
\end{array}\right), \tag{6.29}
$$

and as 2-form the b-field is

$$
b = s\left(\frac{4}{\alpha_2} - 2\alpha_2\right)(dX_L \wedge dX_R + dX_L \wedge dX_R)\,. \tag{6.30}
$$

As we have seen in chapter 5 when we reduced to $\mathcal{N} = 0$, it is easy observed that we have a scalar potential for twisted chiral and chiral superfields, for each case. The term that gave us more information is the term of the CSVM in the appendix C.1, which disappears when we made the T-duality, but due the spectator semi-shiral superfields $(\mathbb{X}_{2,R}, \mathbb{X}_{2,L})$ the torsion is preserved.

Chapter 7

Conclusions and Remarks

We have described T-duality for a GLSM for a general Lagrangian with CVSM [6]. We divide the analysis by the cases $\mathbb{F} = 0$ and $\tilde{\mathbb{F}} = 0$. Besides we use the method given in [3] to promote the global symmetry to local, by gauging the fields, and with the subsequent addition of the Lagrangian multipliers. Integrating the Lagrange multiplier fields one goes to the original model. Integrating out the vector multiplet we have found the T-dual GLSM's with torsion. We have also found the geometry of these models with the analysis of each branch and, finally, we have found the metric g and the b-field.

The T-dual Lagrangian for the gauge fixing $\tilde{\mathbb{F}} = 0$ is (4.36) which is easy to obtain by applying the method for T-duality and fixing the partially gauge to a unitary gauge. But for the case of the gauge fixing $\mathbb{F} = 0$ it cannot be possible to (4.35), due to that we choose the gauge $\mathbb{F} = 0$. In addition to using the method for T-duality, we need to make a global gauge fixing so that we may keep all the fields of the CSVM.

In chapter 5 we have obtained the geometry for each branch of vacua in the dual theory. In the Higgs branch if we take $\xi > 0$, the geometry has the structure of $\mathcal{M}_{\mathbb{F}=0} = \{2e^2 r_2^2 + h(g, X_1) = 0\}/U(1)$ and $\mathcal{M}_{\tilde{\mathbb{F}}=0} = \{|i\frac{Q}{Q_0}s + \tilde{y}|^2 - 2e^2(\mu_2^2 - \frac{1}{2}\mu_3^2) = 0\}/U(1)$, respectively, for both cases. Moreover, in chapter 6 we have obtained the g metric and the b-field for the original model [6], which for the SVM is a point on the Higgs branch, but for the CSVM gives us a Kähler geometry $\mathcal{M} = \{\mu_2 + \xi = 0\}/U(1)$. We have also described the model with two expectator superfields, which leads to the conifold with torsion. We can compare (6.6) with the kernel of the semi-chiral part given the b-fields in the case $\tilde{\mathbb{F}} = 0$ in (6.29). Furthermore, in the case $\tilde{\mathbb{F}} = 0$, which is equal in the kernel for semi-chiral fields, we see how the semi-chiral fields gives us the anti-symmetric part of E explicitly. In the case $\mathbb{F} = 0$, we observe that the explicitly form of the metric and b-field are lengthy, but we emphasize that the H-field is non-zero.

In the case $\beta \to \infty$ the endpoint of the flow is a NLSM on a Calabi-Yau; this type of model is an string theory. In general, the study of the topologically version of NLSMs with H-flux is a good approach to a realistic theory [19], [20].

For future work we leave to obtain the solution for the general case of the metric given the PDE in (3.11) [15], and applying it to the general case with d_s, d_c and d_t fields [6], [15]. Another problem left opened is the T-dual model in the general case [6], without gauge fixing, and also to explore the mirror symmetry using the existence of the b-field [21].

Appendix A

Notation

The convention that we use for this work is the usual notation given in [9] and [8]. For the integrals in Grassman variables over the superspace we use

$$\int d\theta = \int d\bar{\theta} = \int d\theta\bar{\theta} = \int d\bar{\theta}\theta = 0, \tag{A.1}$$

$$\int d\theta^{\alpha}\theta_{\beta} = \delta^{\alpha}_{\beta}, \int d\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}, \tag{A.2}$$

$$\int d^2\theta\theta^2 = \int d^2\bar{\theta}\bar{\theta}^2 \tag{A.3}$$

$$\int d^4\theta^2\bar{\theta}^2 = 1, \tag{A.4}$$

where the differential form in terms of the grassman variables are

$$d^2\theta = -d\theta^{\alpha}d\theta^{\beta}\epsilon_{\alpha\beta}, \tag{A.5}$$

$$d^2\bar{\theta} = -d\bar{\theta}_{\dot{\alpha}}d\bar{\theta}_{\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}, \tag{A.6}$$

$$d^4\theta = d^2\theta d^2\bar{\theta}, \tag{A.7}$$

and the anticommutation relations are

$$\{\theta_{\alpha}, \bar{\theta}_{\dot{\beta}}\} = \{\theta_{\alpha}, \theta_{\beta}\} = \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0. \tag{A.8}$$

Furthermore, we take the functions $A = A(\theta^{\pm}, \bar{\theta}^{\pm}, x^{\mu})$, which depend of the coordinates of superspace, evaluated all θ's in zero. We use the following notation

$$A| = A(\theta^{\pm} = 0, \bar{\theta}^{\pm} = 0, x^{\mu}), \tag{A.9}$$

but in special cases where we just evaluated some θ coordinates, we write it explicitly. For example $A|_{\theta^{\pm}=0} = A(\theta^{\pm} = 0, \bar{\theta}^{\pm}, x^{\mu})$.

Appendix B

SUSY Reduction

B.1 Reduction from $\mathcal{N} = (2,2)$ to $\mathcal{N} = (1,1)$

To determine the scalar potential given the Lagrangian (2.36) for CVSM, we start by doing the reduction from SUSY $(2,2)$ to $(1,1)$ for the SVM. It is more convenient to work in the covariant approach [6], rather than writing the derivatives with a phase [4]. We take as gauge-covariant derivatives: $\nabla_\pm, \bar{\nabla}_\pm$. We then write the gauge-covariant derivatives, in terms of $\mathcal{N} = (1,1)$ by

$$\nabla_\pm = \frac{1}{2}\left(\mathcal{D}_\pm - i\mathcal{Q}_\pm\right), \quad \bar{\nabla}_\pm = \frac{1}{2}\left(\mathcal{D}_\pm + i\mathcal{Q}_\pm\right), \tag{B.1}$$

where $\mathcal{D}_\pm$ and $\mathcal{Q}_\pm$ are gauge-covariant fermionic derivatives and the gauge-covariant generators, respectively. These fermionic derivatives satisfy the algebra of $(1,1)$

$$\{\mathcal{D}_\pm, \mathcal{D}_\pm\} = i\mathcal{D}_{\pm\pm}, \quad \{\mathcal{D}_+, \mathcal{D}_-\} = f, \tag{B.2}$$

where $\mathcal{D}_{\pm\pm}$ is the gauge-covariant space derivative and f is the $(1,1)$ field strength.

The gauge-invariant field strength related to the vector multiplet V is defined by $\Sigma = i\{\bar{\nabla}_+, \nabla_-\}$, which is a twisted chiral field. The reduction to $(1,1)$ is given by

$$\Sigma| = \frac{1}{2}(\sigma + if), \tag{B.3}$$

where σ is a real bosonic superfield. The non-manifest SUSY acts by

$$\{\mathcal{Q}_+, \mathcal{D}_-\} = -\sigma, \quad \{\mathcal{Q}_-, \mathcal{D}_+\} = \sigma, \tag{B.4}$$

and obeys the algebra

$$\{\mathcal{Q}_+, \mathcal{Q}_-\} = f. \tag{B.5}$$

For the twisted vector multiplet $\tilde{V}$ we take as field strength $\Theta = i\{\bar{\nabla}_+, \bar{\nabla}_-\}$, which is chiral. The reduction to $(1,1)$ is given in terms of a bosonic superfield σ' and f, given by

$$\theta| = \frac{1}{2}(\sigma' + if), \tag{B.6}$$

which obeys the algebra

$$\{\mathcal{Q}_+, \mathcal{D}_-\} = \sigma', \quad \{\mathcal{Q}_-, \mathcal{D}_+\} = -\sigma', \{\mathcal{Q}_+, \mathcal{Q}_-\} = -f. \tag{B.7}$$

Furthermore, the form to reduce a Lagrangian from $\mathcal{N} = (2,2)$ superspace to $\mathcal{N} = (1,1)$ is

$$\int d^4\theta K = \int \mathcal{D}_+\mathcal{D}_- \left(\mathcal{Q}_+\mathcal{Q}_- K\right)|_{\bar{\theta}^\pm - \theta^\pm = 0} \tag{B.8}$$

and F-terms reduce by

$$\int d^2\mathcal{U}(\Phi) = \int \mathcal{D}_+\mathcal{D}_-\mathcal{U}(\phi). \tag{B.9}$$

B.2 Lagrangian for SVM

In chapter 2, we have explained that there are two cases of interest, therefore there are two gauge-invariant field strengths in the SVM: $\mathbb{F} = i\{\bar{\nabla}_+, \bar{\nabla}_-\}$ and $\bar{\mathbb{F}} = i\{\bar{\nabla}_+, \nabla_-\}$. But, in terms of the $\mathcal{N} = (1,1)$ fields, the SVM consists of three real bosonic superfields $\{\sigma_i | i = 1,2,3\}$ and the vector multiplet f, given by

$$\begin{aligned} \sigma_1 &= \left(\mathbb{F} + \bar{\mathbb{F}}\right)|, \quad \sigma_1 = \left(\tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)|, \\ \sigma_3 &= i\left(\mathbb{F} - \bar{\mathbb{F}} - \tilde{\mathbb{F}} + \bar{\tilde{\mathbb{F}}}\right)|, \\ f &= -i\left(\mathbb{F} - \bar{\mathbb{F}} + \tilde{\mathbb{F}} - \bar{\tilde{\mathbb{F}}}\right)|. \end{aligned} \tag{B.10}$$

Solving for $\mathbb{F}|$ and $\tilde{\mathbb{F}}|$, we obtain

$$\mathbb{F}| = \frac{1}{2}\left(\sigma_1 + \frac{i}{2}(f - \sigma_3)\right), \quad \tilde{\mathbb{F}}| = \frac{1}{2}\left(\sigma_2 + \frac{i}{2}(f + \sigma_3)\right), \tag{B.11}$$

which obey the following algebra

$$\begin{aligned} \{\mathcal{Q}_+, \mathcal{D}_-\} &= -(\sigma_1 + \sigma_2), \\ \{\mathcal{D}_+, \mathcal{Q}_-\} &= -(\sigma_1 - \sigma_2), \\ \{\mathcal{Q}_+, \mathcal{Q}_-\} &= \sigma_3. \end{aligned} \tag{B.12}$$

The reduction on SUSY of the kinetic action SVM in (2.33), in the Abelian case, gives

$$
\begin{aligned}
\mathcal{L}_{SVM,reduced} =& \frac{1}{2e^2}\int \mathcal{D}_+\mathcal{D}_-\left(\frac{1}{2}\mathcal{D}_+ f\mathcal{D}_- f + \mathcal{D}_+\sigma_1\mathcal{D}_-\sigma_1\right.\\
&\left.+\mathcal{D}_+\sigma_2\mathcal{D}_-\sigma_2 + \frac{1}{2}\mathcal{D}_+\sigma_3\mathcal{D}_-\sigma_3\right).
\end{aligned} \tag{B.13}
$$

B.3 Twisted masses

It is possible to introduce twisted masses by gauging the flavor symmetry and setting the associated field strength to a constant background. In semi-chiral models, we have fewer flavor symmetries, but due to the enlarged gauge symmetry of the SVM one may introduce a new kind of mass parameter, which does not require additional flavor symmetries. This constrains the field strengths in the SVM not to vanish but instead to be constant.

For the first case $\mathbb{F}$, this can be imposed by

$$
i\int d^2\theta\Phi\left(\mathbb{F}-M\right)+c.c. \tag{B.14}
$$

Thus, integrating out Φ we set $\mathbb{F}=M$, with $\bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\mathbb{V}=M$. Similarly, for the chiral field strength $\tilde{\mathbb{F}}$, one could constrain this to a constant $\tilde{\mathbb{F}}=\tilde{M}$ by the term

$$
i\int d^2\theta\chi\left(\tilde{\mathbb{F}}-\tilde{M}\right)+c.c. \tag{B.15}
$$

B.4 Lagrangian CSVM

Given the reduction in terms of language $(1,1)$ we could rewrite the twisted mass constraint (B.14) as

$$
\mathcal{L}_\phi = \frac{i}{2}\int \mathcal{D}_+\mathcal{D}_-\left[\phi\left(\sigma_1 - 2\,\mathrm{Re}(M) + i(f-\sigma_3-4\,\mathrm{Im}(M))\right)\right]+c.c., \tag{B.16}
$$

where it is easy to follow from (B.11), that

- Constraining $\mathbb{F}=M$ corresponds to setting $\sigma_1 = 2\,\mathrm{Re}(M)$ and $\sigma_3 = f - 4\,\mathrm{Im}(M)$.
- Constraining $\tilde{\mathbb{F}}=\tilde{M}$ corresponds to setting $\sigma_2 = 2\,\mathrm{Re}(\tilde{M})$ and $\sigma_3 = -f + 4\,\mathrm{Im}(\tilde{M})$.

B.5 Lagrangian Semi-chiral fields

Using the previous definition of semi-chiral fields, we have defined these by

$$\bar{\nabla}_+\mathbb{X}_L = \bar{\nabla}_-\mathbb{X}_R = 0. \tag{B.17}$$

From the gauge-covariant derivatives in terms of fermionic derivatives it is easy to see the action of $\mathcal{Q}_\pm$ on $X_{L,R}$, which is simply defined as independent fermionic multiplets $\Psi_\pm$. Then, in $(1,1)$ language the semi-chiral multiplets consist of:

$$X_L = \mathbb{X}_L|, \quad \Psi_- = \mathcal{Q}_-\mathbb{X}_L, \quad \bar{X}_L = \mathbb{X}_L|, \quad \bar{\Psi}_- = \mathcal{Q}_-\bar{\mathbb{X}}_L|, \tag{B.18}$$

$$X_R = \mathbb{X}_R|, \quad \Psi_+ = \mathcal{Q}_+\mathbb{X}_R, \quad \bar{X}_R = \mathbb{X}_R|, \quad \bar{\Psi}_+ = \mathcal{Q}_+\bar{\mathbb{X}}_R|. \tag{B.19}$$

Then, the action in (2.39) can be reduced as

$$\mathcal{L}_{matter} = \int \mathcal{D}_+\mathcal{D}_- \left[\mathcal{Q}_+\mathcal{Q}_-(\bar{\mathbb{X}}_L\mathbb{X}_L + \bar{\mathbb{X}}_R\mathbb{X}_R + \beta(\mathbb{X}_L\mathbb{X}_R + c.c.))\right]|. \tag{B.20}$$

Using the equations in (B.18) and (B.19) in the above Lagrangian, we finally integrate out the auxiliary superfields $\Psi_\pm$. This gives

$$\mathcal{L}_{matter} = \int \mathcal{D}_+\mathcal{D}_- \left(\frac{1}{2}(g_{\mu\nu} + b_{\mu\nu})\mathcal{D}_+X^\mu\mathcal{D}_-X^\nu + 2i\sigma_I\mu_I\right). \tag{B.21}$$

where g and b are the flat-space metric and b-field of the ungauged case, respectively , and

$$\begin{aligned}
\mu_1 &\equiv \bar{X}_LX_L + \bar{X}_RX_R + \beta(X_LX_R + \bar{X}_L\bar{X}_R)\\
\mu_2 &\equiv -\left(\bar{X}_LX_L - \bar{X}_RX_R\right)\\
\mu_3 &\equiv -\frac{i\beta}{2}\left(X_LX_R + \bar{X}_L\bar{X}_R\right)
\end{aligned} \tag{B.22}$$

Appendix C

Scalar potential CSVM

C.1 Reduction from $\mathcal{N} = (1,1)$ to $\mathcal{N} = 0$

To compute the scalar potential in a GLSM one may reduce to $\mathcal{N} = 0$ components. The reduction of a bosonic $(1,1)$ to $\mathcal{N} = 0$ components is given by

$$\begin{aligned} X &= X|, \quad \psi_\pm = \mathcal{D}_\pm X|, \quad g = \mathcal{D}_+\mathcal{D}_- X|, \\ \bar{X} &= \bar{X}|, \quad \bar{\psi}_\pm = \mathcal{D}_\pm \bar{X}|, \quad \bar{g} = \mathcal{D}_+\mathcal{D}_- \bar{X}|. \end{aligned} \tag{C.1}$$

where we set the remaining $\mathcal{N} = (1,1)$ Grassman variables to zero. It is easy to reduce the Lagrangian (B.21) to $\mathcal{N} = 0$ components and integrate out the remaining auxiliary fields. The lowest scalars are given by:

- $\mathbb{F}$ and $\tilde{\mathbb{F}}$ by σ and $\tilde{\sigma}$, respectively.
- the auxiliary scalars by $D_I = \mathcal{D}_+\mathcal{D}_-\sigma_I|$.
- and the auxiliary scalars in the semi-chiral multiplet by $g_{L,R} = \mathcal{D}_+\mathcal{D}_- X_{L,R}$.

Integrating out $g_{L,R}$ we obtain the Lagrangian

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2e^2}\left(D_1^2 + D_2^2 + \frac{1}{2}D_3^2\right) - \beta^2\left(|\sigma|^2 + \frac{1}{\beta^2-1}|\tilde{\sigma}|^2\right)\frac{1}{2}|X|^2 \\ &+ 2i\left(\bar{X}_L D_1 X_L - \bar{X}_R D_1 X_R - \beta X_L D_1 X_R + \beta \bar{X}_L D_1 \bar{X}_R\right) \\ &- 2i\left(\bar{X}_L D_2 X_L + \bar{X}_R D_2 X_R\right) \\ &- \beta\left(X_L D_3 X_R + \bar{X}_L D_3 \bar{X}_R\right) + \ldots \end{aligned} \tag{C.2}$$

where $|X|^2 > 0$ and the ellipses represent kinetic and fermionic terms that do not contribute to the scalar potential.

C.2 Scalar potential CSVM by cases

The scalar potential associated to the SVM can be obtained by integrating out D_I's, this is given by

$$\begin{aligned} U =& 2e^2(\mu_1 - r_1)^2 + 2e^2(\mu_2 - r_2)^2 + 2e^2(\mu_3 - r_3)^2 \\ &+ \beta \left(|\sigma|^2 + \frac{1}{\beta^2 - 1} |\tilde{\sigma}| \right), \end{aligned} \tag{C.3}$$

where μ_I are defined in (B.22).

In the case $\tilde{\mathbb{F}} = 0$, for the constrained SVM with $\mathbb{F} = M$ one must set $D_1 = D_3 = 0$ and $\sigma = M$ in (C.2). To obtain the scalar potential one must integrate out D_2. In the other case, for the constrained SVM with $\tilde{\mathbb{F}} = \tilde{M}$ is obtained by setting $D_2 = D_3 = 0$ and $\tilde{\sigma} = \tilde{M}$ in (C.2), leading to the scalar potential

$$U = 2e^2(\mu_2 - r_2)^2 + \beta^2 \left(|M|^2 + \frac{1}{\beta^2 - 1} |\tilde{\sigma}|^2 \right) \frac{1}{2} |X|^2 \tag{C.4}$$

and

$$\tilde{U} = 2e^2(\mu_1 - r_1)^2 + \beta^2 \left(|\sigma|^2 + \frac{1}{\beta^2 - 1} |\tilde{M}|^2 \right) \frac{1}{2} |X|^2, \tag{C.5}$$

respectively.

Finally, the case of equal charges and a CSVM with $\mathbb{F} = M$:

$$U = 2e^2(\mu_2' - r_2)^2 + \alpha^2 \left(\frac{1}{\alpha^2 - 1} |M|^2 + |\tilde{\sigma}|^2 \right) \frac{1}{2} |X|^2 \tag{C.6}$$

where $|X|^2$ is contracted with the corresponding flat-space metric and $\mu_2' = \bar{X}_L X_L + \bar{X}_R X_R + \alpha(\bar{X}_L X_R + c.c.)$.

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