Centro de Investigación y de Estudios Avanzados
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# Regulación Robusta por Modos Deslizantes de Sistemas no Lineales de Fase no Mínima

Tesis que presenta:

**Marcos Israel Galicia Cueva**

para obtener el grado de:

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Directores de Tesis

**Dr. Alexander Georgievich Loukianov**
**Dr. Jorge Rivera Dominguez**

CINVESTAV del IPN Unidad Guadalajara, Guadalajara, Jalisco, Diciembre de 2013.

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**Ingeniería Eléctrica**


Por:
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CINVESTAV del IPN -- Unidad Guadalajara 2007-2009



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# Robust Sliding Mode Regulation of Nonminimum Phase Nonlinear Systems

A thesis presented by·
**M. Sc. Marcos Israel Galicia Cueva**

to obtain the degree of:
**Doctor in Science**

in the subject of:
**Electrical Engineering**

Thesis Advisors:
**Dr. Alexander Georgievich Loukianov**
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**CINVESTAV del IPN Unidad Guadalajara, Guadalajara, Jalisco, December 2013**

# Robust Sliding Mode Regulation of Nonminimum Phase Nonlinear Systems

**Doctor of Science Thesis**
**In Electrical Engineering**

By:
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**CINVESTAV del IPN Unidad Guadalajara, December, 2013.**

# Contents

# RESUMEN

En este trabajo se aborda el Problema de Regulación utilizando Control por Bloques (CB) y Modos Deslizantes (MD) para sistemas no lineales de fase no mínima con perturbación. Comparado con soluciones de otros trabajos, en las soluciones aquí presentadas se presenta una metodología más simple para el diseño de reguladores que puede aplicarse a sistemas sin restricción en el grado relativo. Además, se logra la robustez ante perturbaciones que no cumplen la condición de acoplamiento afectando al sistema.

Para la solución del problema establecido se consideran varios casos:

- Considerando la representación del sistema se divide en: a) **Estructurado** cuando en el diseño se considera alguna estructura del sistema como la forma Controlable por Bloques; b) **sin estructura** no se considera ninguna estructura especial para el diseño del regulador, en este último caso el sistema puede estar en forma general o se puede transformar a la forma Regular.

- Considerando el tipo de referencias para la salida se divide en: a) **Caso no causal** cuando las referencias son generadas por un exosistema; b) **Causal** cuando las referencias son dadas como funciones del tiempo, en este caso no se tiene ningún exosistema.

- Considerando que las perturbaciones que afectan al sistema se divide en: a) El sistema es afectado por perturbaciones que **cumplen** la condición de acoplamiento (CA); b) el sistema es afectado por dos tipos de perturbaciones, las que cumplen tanto como las que **no cumplen** la condición de acoplamiento.

Para el caso no causal de sistemas sin estructura con perturbaciones que cumplen la CA, se define el problema de Regulación por MD Integrales y se proponen dos soluciones robustas. Para sistemas estructurados se presentan dos soluciones: una para el caso de sistemas con perturbaciones que cumplen la CA. La otra basada en *Lyapunov Redesign* para cuando hay perturbaciones que no cumplen tanto como las que cumplen la CA.

Para el caso no causal y estructurado, se define también una extensión al Problema de Regulación en tiempo continuo para sistemas en tiempo discreto y se presenta una solución para sistemas con perturbaciones que cumplen la CA.

Para la solución al problema de Regulación en el caso causal, se tiene que obtener referencias estables para la dinámica interna inestable del sistema. Para esto se utiliza un estimador adaptable el cuál sirve como un supuesto exosistema. Utilizando los estados del estimador dos métodos son empleados para obtener las referencias estables para la dinámica interna. El primero esta basado en una solución a una ecuación diferencial lineal inestable. El segundo utiliza *System Center Method*. El estimador adaptable también es utilizado para estimar perturbaciones afectando a la dinámica interna.

Con las referencias estables dadas, se proponen tres soluciones al Problema de Regulación. Dos son para sistemas con perturbaciones que cumplen la CA y la otra es para sistemas con perturbaciones de los dos tipos antes mencionados.

También se presentan dos ejemplos de aplicación de los reguladores propuestos: Control del par eléctrico de un motor de inducción, control de posición para el Pendubot.

# ABSTRACT

This work addresses the Sliding Mode Output Regulation (SMOR) problem for nonlinear nonminimum phase systems (NPS). The proposed solution is based on the Block Control (BC) and Sliding Mode (SM) techniques. In contrast with other works, we present an improved method for the regulator design and the respectively solutions do not have constraint with respect to the relative degree of the system. Moreover, robustness properties are achieved for system with matched and unmatched perturbations.

To establish the solutions to SMOR problem, we consider three cases:

- With respect to the representation of the system, we divide the systems in two: a) **Structured** when the BC is applied to the system; b) **unstructured** when we do not use the BC linearization. In thi case the system can be expressed in general or Regular form.

- With respect to the output references we divide in: a) **Noncausal case** when the references are generated for an exosystem; b) **Causal** if the reference signal is an arbitrary function of the time and there is no any exosystem.

- With respect to the perturbations affecting the system, we divide in: a) Systems with matched perturbations; b) systems with both matched and unmatched perturbations

In the noncausal case, for unstructured systems with matched perturbation, we introduce Integral SMOR Problem for systems with matched perturbations. Solution conditions are derived for NPS in structured form and two solutions are presented.

Analogously to the SMOR problem in continuous time we introduce Discrete-Time SM Output Regulation Problem for discrete time NPS with matched perturbations for the non causal case. Solution conditions are derived for NPS in structured form. The proposed controller for Discrete-Time NPS presented in unstructured form for the noncausal case.

For structured systems, we present two solutions: The first one using BC technique is for NPS with matched perturbations. And the second one based on Lyapunov Redesign is for systems with unmatched perturbations.

To propose a solution in the causal case, we have to find stable references for the unstable internal dynamics of the system. For that, we use an adaptive estimator which serves as a kind of exosystem. Based on the states of the estimator, two methods are used to obtain stable references for the internal dynamics. The first one is based on the solution of a linear unstable differential equation. The second one is based on the System Center Method. The adaptive estimator is used to achieve the estimation for unmatched perturbations affecting the internal dynamics.

Once is given the stable references, we propose three solutions to SMOR problem. Two are for systems with matched perturbations and the other one is for systems with both, matched and unmatched perturbations.

As illustrative examples, we present the a Sliding Mode Sensorless Torque Regulator for Induction Motor and position control for Pendubot.

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# Chapter 1

# Introduction

Roughly speaking, the classical Output Regulation problem [Isidori and Byrnes, 1990], consists in designing a continuous state or error feedback controller such that the output of a system tracks a reference signal in the presence of a known disturbance signal. The reference and perturbations considered in the problem are generated by an exosystem. To improve classical Output Regulation, two main research directions were proposed:

- to expand the class of perturbations affecting the dynamic system;
- to facilitate the design of the regulator.

To increase the class of perturbations considered in Output Regulation, several robust nonlinear controllers were proposed to substitute the linear state feedback presented by Isidori. One of the most used is Sliding Mode (SM) control that consists of the design of discontinuous state feedback [Utkin, 1992b]. The SM control is recognized as an efficient tool to deal with a complex nonlinear system in presence of an uncertainty, since its main advantages are:

- the possibility of decoupling the original system into two subsystems of lower dimension due to finite time convergence to a sliding manifold, and
- low sensitivity with respect to perturbations.

Thus, analogously to classical OR the Sliding Mode Output Regulation (SMOR) was stated in [Loukianov et al., 1999b]. The SMOR problem is defined as the problem of designing

- first, a sliding manifold which contents the steady state (center) manifold , and on which the equilibrium point of the closed loop system is asymptotically stable, and the output tracking error goes asymptotically to zero;
- secondly, a discontinuous controller which drives the state of the closed-loop system to the designed sliding manifold

Moreover, the SM regulator provides

- robustness property to the system with respect to unknown matched perturbations, and
- semi-global stabilization while the classical regulator ensures only a local stability.

The last fact is due to linearization of the full system used in the classical OR while the SM regulator design needs to linearise only the reduced order sliding mode equation [Utkin, 1992a].

To distinguish between different OR problems, we say that a system is in an *unstructured* form when the system model does not present the internal dynamics in the explicit form. On the other hand, a *structured* system is when the internal dynamics are expressed in the explicit form. It can be noted that the last *structured* form presentation allows to apply directly a feedback linearization technique, however that is only for minimum phase systems.

Also, to expand the problems to deal, we referred to as *noncausal* case problem if the considered output reference signal is generated by an exosystem as in classical OR. And the problem is referred to as *causal* case if the reference signal is an arbitrary function of the time, that is, there is no any exosystem.

For dynamical nonlinear systems presented in the unstructured General or Regular form, a SMOR problem solution for the noncausal case, in absence of plant model uncertainty, was studied in [Loukianov et al., 1999b], and [Memon and Khalil, 2010], and considering a plant model uncertainty in [Castillo-Toledo and Castro-Linares, 1995].

For the structured but minimum phase nonlinear systems (noncausal case) a SMOR problem solution using the Input-Output (I-O) linearization, was presented in [Elmali and Olgac, 1992]. The class of structured nonminimum phase systems, again in the noncausal case, was studied in [Bonivento et al., 2001], however, that is only for nonlinear systems with relative degree one. A state feedback SM regulator have been designed in [Gopalswamy and Hedrick, 1993] for structured nonminimum phase systems but that is for the case of constant output reference (noncausal case).

In the causal case, (structured) nonminimum phase systems have been studied in [Shtessel et al., 2012] using the I-O linearization and SM techniques, taking an approximation of the characteristic polynomial of an exosystem. Another approach based on an inverse model in absence of perturbations, have been presented in [Zou and Devasia, 2004].

There are several results in the literature which deal with the Output Regulation problem for nonlinear system with matched perturbations for example: [Elmali and Olgac, 1992] for a minimum phase system, [Loukianov et al., 1999b], for the unstructured nonlinear system, [Memon and Khalil, 2008] with a Lyapunov redesign approach, [Gopalswamy and Hedrick, 1993] for nonminimum phase, [Bonivento et al., 2001] for structured nonlinear nonminimum phase systems with unitary relative degree, and, recently, in [Shtessel et al., 2012] for the structured causal case. On the other hand, it is well known that the SM is a robust control technique, however, that is only with respect to the matched perturbation [Drazenovich, 1969], [Utkin, 1992a]. To overcome this drawback, the robust nested Block Control technique (e.g. [Huerta-Avila et al., 2007]) have been proposed for structured minimum phase systems to design a sliding manifold on which unmatched perturbation effect, is rejected. To deal with unmatched perturbations, also a control scheme based on Block Control and quasi-continuous HOSM techniques was proposed in [Estrada and Fridman, 2008], however, in that work only full relative degree systems are considered. In [Castillo-Toledo and Castro-Linares, 1995], it is proposed a solution where the matching condition is weakened and replaced by a new condition that describes intrinsic structural invariance properties of the nominal and uncertain system.

In this work, we consider that the output tracking problem for perturbed nonminimum phase systems has two main challenges. The first one is the output tracking in presence of the both matched and unmatched perturbations. The second one is the internal dynamics stabilization. While for the classical and SM Output Regulation, there are several solutions, the perturbed case was not completely studied. Moreover, we propose to use Block Control FL approach. Comparing with the Input Output technique [Isidori, 1995] which is often used in the regulation problem, the Block Control linearization approach [Loukianov, 1998] is more attractive since that allows directly place the poles of the system with the controller gains introduced in the design. Additionally, in contrast with the Input Output, the Block Control can be used for nonlinear MIMO systems with different relative degree with respect to output vector components. Nevertheless, in the case of a nonminimum phase system, FL cannot be applied directly due to the unstable internal dynamics which are unobservable from the output.

On the other hand, the standard Sliding Mode discontinuous control can produce the chattering phenomenon, that is generally characterized by small oscillations with finite frequency and amplitude at the output of the system that can result harmful because it leads to low control accuracy and high wear of mechanical parts. In order to overcome the chattering phenomenon, the High-Order SM concept was introduced by [Levant, 1993]. Recently, High-Order SM (HOSM) controllers are most often preferred due to they keep the main advantages of the standard SM control, moreover their control signal is a continuous function instead of the standard discontinuous SM case, that feature allows to HOSM reducing the chattering effect on the output.

## 1.1 Thesis Objectives

The objectives stated in this work are the following.

**General Objective:**

To propose a robust solution for the Output Regulation problem considering both matched and unmatched perturbations for nonlinear nonminimum phase systems with arbitrary relative degree vector for the causal and noncausal cases.

**Specific Objectives:**

- To design a robust regulator for nonminimum phase systems (NPS) presented in the unstructured form with matched perturbations for the noncausal case.
- To design a robust regulator for NPS presented in the structured form with both matched and unmatched perturbations for the noncausal case.
- To design a robust regulator for NPS presented in the structured form with matched perturbations for the causal case.
- To design a robust regulator for NPS presented in the structured form with both matched and unmatched perturbations for the causal case.
- To design a robust regulator for Discrete-Time NPS presented in the unstructured form for the noncausal case.

## 1.2 Contributions of this Work

The following contributions were made by this thesis in the fields of: robustness against perturbations in both the causal and the noncausal cases, SM Regulator design, constraints with respect to relative degree of the systems.

### Robustness

We present solutions for the SMOR problem for nonminimum phase systems with both perturbations, matched and unmatched. On Discrete Time dynamical systems, we present a solution for the SMOR problem for nonminimum phase systems with matched perturbation.

### SM Regulator design

We define an iterative form to design a suitable sliding mode surface using the Block Control (BC) linearization. With the BC technique we increase the class of nonlinear systems which can be dealt. The Integral Sliding Mode Output Regulation Problem has been introduced. In this approach compared to the classical solutions the steady state control needs not to be calculated. Moreover, the Discrete Time SMOR was defined and solved for discrete time nonlinear NMP systems.

### Constraints on relative degree

Introducing the BC linearization in the regulator design, we relax the constraints of relative degree unitary of the system, imposed in [Bonivento et al., 2001] and equal relative degree with respect to output vector components imposed in other works which use the Input Output linearization.

### Causal and noncausal cases

Proposed solutions were presented for both cases: when the reference signal is generated by a known exosystem, and also when there is no any exosystem. Considering a class of arbitrary references.

## 1.3 Thesis Organization

The rest of the document is organized as follows.

In Chapter 2 the main tools used during this work are introduced.

In Chapter 3 we address Sliding Mode Output Regulation problem for the noncausal case. We first present the SMOR problem. Secondly, we focus on NPS presented in the unstructured form. In section 3.3, we propose a robust controller based on Integral SM control (ISMC) for NPS in the general form, then conditions for the solution are presented and the robust controller is designed. In Section 3.3, to simplify the design of ISMC, we present a solution for SMOR problem for NPS presented in the unstructured Regular form. In Section 3.4, we focus on NPS presented in the structured form. A robust SM regulator is designed for NPS with matched perturbations in Section 3.4. Finally, in section 3.5 we present a solution based on Lyapunov Redesign concepts for SMOR problem for NPS with both matched and unmatched perturbations.

In Chapter 4 we address Sliding Mode Output Regulation problem for the causal case. We first present the problem statement. In Section 4.1 we design a suitable sliding manifold and introduce conditions for the solution of the stated problem. The SMOR problem is solvable if there is a bounded solution for unstable internal dynamics. For that, we present to approaches to find a bounded solution: the first one is using a linear differential equation [Jeong and Utkin, 1999]. The second one is based on the System Center technique [Shtessel and Shkolnikov, 1999]. For both of the above mentioned approaches we use an adaptive estimator [Obregon-Pulido, 2003] to estimate the reference and its derivatives. In Section 4.2, we present a solution for NPS with matched perturbations. In Section 4.3 a solution for NPS with both matched and unmatched perturbations is proposed.

In Chapter 5 we present the Discrete-time Sliding Mode Output Regulation (DTSMOR) problem for NPS, for the noncausal case. Firstly, in Section 5.2, analogously to continuous version (SMOR), we first present the DTSMOR problem for systems presented in the structured form. Secondly, in Section 5.2 a sliding manifold is designed and the solvability conditions of the DTSMOR are given. In section 5.2, the discrete time SM regulator is presented.

In Chapter 6 we show three examples of the proposed solutions presented in this work. Firstly in Section 6 a second order SM sensorless torque regulator for Induction Motor is presented. Then we continue in Section 6.2 with academic examples for the Robust SMRP for systems with both matched and unmatched perturbations. In Section 6.3, a Discrete-time Sliding Mode Regulator for Pendubot is presented.

Finally, in Chapter 7 the conclusions and the future work are presented.

# Chapter 2

# Preliminaries

## 2.1 Classical State Feedback Output Regulation

The regulator problem, in the classical setup, consists in designing a continuous state or error feedback controller such that the output of a system tracks a reference signal possibly in the presence of a disturbance signal. As first established in Isidori and Byrnes [Isidori and Byrnes, 1990], the main condition for the solution of this problem via state-feedback or output-feedback control is the solvability of the so called regulator equations. If this equations are solvable, under some standard assumptions, there exists a state-feedback or output-feedback control law such that the closed-loop system is internally stable, and the tracking error will asymptotically approach to zero for all sufficiently small initial conditions of the plant and sufficiently small reference inputs and/or disturbances. This section presents the classical Output Regulation Problem as well as its solution. The linear output regulation problem is a special case, and was completely solved based on the existence of a solution for a set of algebraic matrix equations with the collective efforts of several researchers, including Davison, Francis, and Wohnam, among others.

In order to formulate the Output Regulation Problem formally, consider a system of the form

$$\begin{aligned} \dot{x} &= f(x, w, u) \\ \dot{w} &= s(w) \\ e &= h(x, w) \end{aligned} \tag{2.1}$$

with the state $x(t)$ defined in a neighborhood $U$ near the origin in $\mathbb{R}^n$, the input space $\mathbb{R}^m$ and the state $w(t)$ defined in a neighborhood $W$ near the origin $\mathbb{R}^q$. Two scenarios can be considered, depending on the available information as follows. Consider that the plant states $x$ and the exosystem states $w$ are measured; that is, the controller has all the information available. The nonlinear state feedback output regulation is stated as follows.
Given a nonlinear system of the form (2.1), determine, if possible, a control law $u = \alpha(x, w)$ such that:

$S_{FI}$ The equilibrium point $x = 0$ of

$$\dot{x} = f(x, 0, \alpha(x, 0)) \tag{2.2}$$

is asymptotically stable on the first approximation.

$R_{FI}$ There exists a neighborhood $W \in U \times \Omega$ near $(0, 0)$ such that, for every initial condition $(x(0), w(0)) \in \Omega$ the solution of

$$\begin{aligned} \dot{x} &= f(x, w, \alpha(x, w)) \\ \dot{w} &= s(w) \end{aligned} \tag{2.3}$$

satisfies

$$\lim_{t \to \infty} e(t) = 0 \tag{2.4}$$

The properties of the lineal approximation for the controlled plant play an important role in the solution of the output regulation problem; hence, it is convenient to introduce a notation where the parameters of this approximation appear explicit. Notice that the closed loop system (2.3) can be formulated as:

$$\begin{aligned} \dot{x} &= (A + BK)x + (P + BL)w + \varphi(x, w) \\ \dot{w} &= Sw + \psi(x, w) \end{aligned}$$

where $\varphi(x, w)$ and $\psi(x, w)$ vanish in the origin along with its first order derivatives and $A, B, P, K, L, S$ are matrices defined by

$$\begin{aligned} A &= \left[\frac{\partial f}{\partial x}\right]_{0,0,0} & B &= \left[\frac{\partial f}{\partial u}\right]_{0,0,0} \\ P &= \left[\frac{\partial f}{\partial w}\right]_{0,0,0} & K &= \left[\frac{\partial \alpha}{\partial x}\right]_{0,0,0} \\ L &= \left[\frac{\partial \alpha}{\partial w}\right]_{0,0,0} & S &= \left[\frac{\partial s}{\partial w}\right]_{0,0,0} \end{aligned} \tag{2.5}$$

for every $w \subset \Omega_0$.

The necessary and sufficient conditions for the solution of the state feedback output regulator are established in the following theorem.

**Theorem 2.1.** *The state feedback output regulation problem has a solution if and only if the pair $(A, B)$ is stabilizable and there exists mappings such that $\pi(w)$ and $u = c(w)$, with $\pi(0) = 0$ and $c(0) = 0$, both defined on a neighborhood $\Omega_0 \subset \Omega$, from the origin such that:*

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), w, \alpha(\pi(w), w)) \\ 0 &= h(\pi(w), w) \end{aligned} \tag{2.6}$$

for every $w \subset \Omega_0$.
Proof: See [Isidori, 1995].

Once $\pi(w)$ and $c(w)$ are known from equation (2.6), the classical control law which solves the output regulation problem is:

$$\alpha(x, w) = c(w) + K(x - \pi(w)) \tag{2.7}$$

where $K$ is a matrix such that $(A + BK)$ is Hurwitz. The block diagram for the classical control law is presented in Figure 2.1.

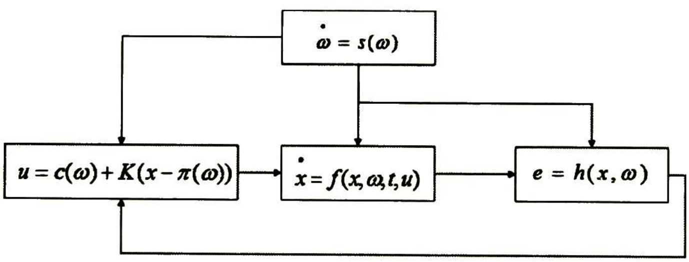


Figure 2.1: Classical nonlinear output regulation problem

## 2.2 Block Control Linearization

Assume that the nonlinear system (2.1) can be represented (maybe after a transformation) in the following Nonlinear Block Controllable (NBC) form:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + B_1(x_1)x_2 + D_1(x_1)w(t) \\
\dot{x}_i &= f_i(\bar{x}_i) + B_i(\bar{x}_i)x_{i+1} + D_i(\bar{x}_i)w(t), \;\; i = 2, ..., r-1. \qquad (2.8)\\
\dot{x}_r &= f_r(\bar{x}_r, x_{r+1}) + B_r(\bar{x}_r, x_{r+1})u + D_r(\bar{x}_{r+1})w(t) \qquad (2.9)\\
y &= h(x) = x_1
\end{aligned}$$

where the vector $x$ is decomposed as $x = (x_1, .., x_r)^T$, $\bar{x}_j = (x_1, ..., x_j)^T$, $j = 2, ..., r$, and $x_i$ is a $n_i \times 1$ vector. The vector $w$ is generated for an exosystem (2.1) rewritten in the following equation

$$\dot{w} = \xi(w) \qquad (2.10)$$

In this case, to design a sliding manifold on which the tracking error $e = y - q(w)$ tends asymptotically to zero, according to the block control linearization we introduce the following recursive nonlinear transformation [Luk'yanov and Utkin, 1981]:

$$z_1 = x_1 := \Phi_1(x_1) \qquad (2.11)$$

$$z_2 = f_1(x_1) + B_1(x_1)x_2 + D_1(x_1, w) + k_1 x_1 := \Phi_2(x_1, x_2, w),$$

$$z_3 = \tilde{B}_3(\bar{x}_2)x_3 + \begin{bmatrix} \bar{f}_2(\bar{x}_2) + d_2(\bar{x}_2, w) + k_2\Phi_2(\bar{x}_2, w) \\ 0 \end{bmatrix} := \Phi_3(\bar{x}_3, w)$$

$$\text{with } d_2(\cdot) = \sum_{l=1}^{2} \frac{\partial \Phi_2}{\partial x_l} D_l w + \frac{\partial \Phi_2}{\partial w}\xi(w)$$

$$z_{i+1} = \tilde{B}_{i+1}(\bar{x}_i)x_{i+1} + \begin{bmatrix} \bar{f}_i(\bar{x}_i) + d_i(\bar{x}_i, w) + k_i\Phi_i(\bar{x}_i, w) \\ 0 \end{bmatrix} \qquad (2.12)$$

$$:= \Phi_{i+1}(\bar{x}_{i+1}, w) \;\; i = 3, 4, ..., r-1,$$

where $z_i$ is a vector of new variables of dimension $n_1 \times 1$, $k_i > 0$, $\bar{f}_i(\bar{x}_i) = \sum\limits_{j=1}^{i-1} \left[\frac{\partial \Phi_i}{\partial x_j} f_j + B_j x_{j+1}\right] + \frac{\partial \Phi_i}{\partial x_i} f_i$, $\bar{B}_i = \tilde{B}_{i-1}B_i$, $\tilde{B}_{i+1} = \begin{bmatrix} \bar{B}_i \\ E_{i,1} \end{bmatrix}$, $E_{i,1} = \begin{bmatrix} 0 & I_{n_{i+1}-n_i} \end{bmatrix}$, $E_{i,1} \in R^{(n_{i+1}-n_i)\times n_{i+1}}$, $I_{n_{i+1}-n_i}$ is the indentity matrix $d_i = \sum\limits_{j=1}^{i} \left[\frac{\partial \Phi_i}{\partial x_j} D_j w\right] + \frac{\partial \Phi_i}{\partial w}\xi(w)$. The system (2.8)-(2.9) using the recursive transformation (2.11)-(2.12) can be represented in the following form:

$$\begin{aligned} \dot{z}_1 &= -k_1 z_1 + z_2 \\ \dot{z}_i &= -k_i z_i + E_{i,1} z_{i+1}, \; i = 2, ..., r-1 \\ \dot{z}_r &= \bar{f}_r(z) + \bar{B}_r(z) u + d_r(z, w) \end{aligned} \tag{2.13}$$

where $z = (z_1, ..., z_r)^T$, $\bar{f}_r(z)$ is a bounded function, *rango* $\bar{B}_r = m$ and $\bar{B}_r = \tilde{B}_{r-1} B_r$. Finally, the transformed system (2.13)-(2.14) will be used to design an advisable manifold to solve the Sliding Mode Output Regulation Problem for nonlinear nonminimum phase systems.

## 2.3 Super-Twisting Controller

The main disadvantage of the standard Sliding Mode is the chattering phenomenon, that is characterized generally by small oscillations with finite frequency and amplitude at the output of the system that can result harmful to the system because it leads to low control accuracy and high wear of mechanical parts. The chattering can be developed due to neglected fast dynamics and to digital implementation issues.

In order to overcome the chattering phenomenon, the high-order sliding mode concept was introduced by [Levant, 1993]. Let us consider a smooth dynamic system with an output function $S$ of class $C^{r-1}$ closed by some static or dynamic discontinuous feedback as in [Levant, 2001]. Then, the calculated time derivatives $S$, $\dot{S}$, $\ldots$, $S^{r-1}$, are continuous functions of the system state, where the set $S = \dot{S} = \ldots = S^{r-1} = 0$ is non-empty and consists locally of Filippov trajectories. The motion on the set above mentioned is said to exist in $r$-sliding mode or $r_{th}$ order sliding mode. The $r_{th}$ derivative $S^r$ is considered to be discontinuous or non-existent. Therefore the high-order sliding mode removes the relative-degree restriction and can practically eliminate the chattering problem.

There are several algorithms to realize HOSM. In particular, the $2_{nd}$ order sliding mode controllers are used to zero outputs with relative degree two or to avoid chattering while zeroing outputs with relative degree one. Among $2_{nd}$ order algorithms one can find the sub-optimal controller, the terminal sliding mode controllers, the twisting controller and the super-twisting controller. In particular, the twisting algorithm forces the sliding variable $S$ of relative degree two in to the 2-sliding set, requiring knowledge of $\dot{S}$. The super-twisting algorithm does not require $\dot{S}$, but the sliding variable has relative degree one. Therefore, the super-twisting algorithm is nowadays preferable over the classical siding mode, since it eliminates the chattering phenomenon.

The actual disadvantage of HOSM is that the stability proofs are based on geometrical methods, since the Lyapunov function proposal results in a difficult task, [Levant, 2005]. The work presented in [Moreno and Osorio, 2008] proposes quadratic like Lyapunov functions for the super-twisting controller, making possible to obtain an explicit relation for the controller design parameters. In the following lines this analysis will be revisited.

Let us consider the following SISO nonlinear scalar system

$$\dot{\sigma} = f(t,\sigma) + u \tag{2.14}$$

where $f(t,\sigma)$ is an unknown bounded perturbation term and globally bounded by $|f(t,\sigma)| \leq \delta|\sigma|^{1/2}$ for some constant $\delta > 0$. The super-twisting sliding mode controller for perturbation and chattering elimination is given by

$$\begin{aligned} u &= -k_1\sqrt{|\sigma|}sign(\sigma) + v \\ \dot{v} &= -k_2 sign(\sigma). \end{aligned} \tag{2.15}$$

System (2.14) closed by control (2.15) results in

$$\begin{aligned} \dot{\sigma} &= -k_1\sqrt{|\sigma|}sign(\sigma) + v + f(t,\sigma) \\ \dot{v} &= -k_2 sign(\sigma). \end{aligned} \tag{2.16}$$

Proposing the following candidate Lyapunov function:

$$\begin{aligned} V &= 2k_2|\sigma| + \frac{1}{2}v^2 + \frac{1}{2}(k_1|\sigma|^{1/2}sign(\sigma) - v)^2 \\ &= \xi^T P \xi \end{aligned}$$

where $\xi^T = \begin{pmatrix} |\sigma|^{1/2}sign(\sigma) & v \end{pmatrix}$ and

$$P = \frac{1}{2}\begin{pmatrix} 4k_2 + k_1^2 & -k_1 \\ -k_1 & 2 \end{pmatrix}$$

Its time derivative along the solution of (2.16) results as follows:

$$\dot{V} = -\frac{1}{|\sigma^{1/2}|}\xi^T Q \xi + \frac{f(t,\sigma)}{|\sigma^{1/2}|}q_1^T \xi$$

where

$$Q = \frac{k_1}{2}\begin{pmatrix} 2k_2 + k_1^2 & -k_1 \\ -k_1 & 1 \end{pmatrix},$$

$$q_1^T = \begin{pmatrix} 2k_2 + \frac{1}{2}k_1^2 & -\frac{1}{2}k_1 \end{pmatrix}.$$

It is considered the next assumption for the perturbation terms,

**A. 1.** ***The perturbation term $f(t,\sigma)$ in (2.16) is bounded by***

$$|f(t,\sigma)| \le \delta_i \, |\sigma_i|^{1/2} \quad \delta_i > 0 \tag{2.17}$$

If assumption A.1 is satisfied, the expression for the derivative of the Lyapunov function is reduced to

$$\dot{V} = -\frac{k_1}{2|\sigma^{1/2}|}\xi^T \tilde{Q} \xi$$

where

$$\tilde{Q} = \begin{pmatrix} 2k_2 + k_1^2 - (\frac{4k_2}{k_1} + k_1)\delta & -k_1 + 2\delta \\ -k_1 + 2\delta & 1 \end{pmatrix}.$$

In this case, if the controller gains satisfy the following relations

$$k_1 > 2\delta, \quad k_2 > k_1 \frac{5\delta k_1 + 4\delta^2}{2(k_1 - 2\delta)},$$

then, $\tilde{Q} > 0$, implying that the derivative of the Lyapunov function is negative definite.

## 2.4 Robust High Order Sliding Mode Differentiator

In order to estimate the derivatives for the arbitrary sinusoidal reference and/or perturbation signal considered in Section 4.3 of this work, we propose to use a robust differentiator. The fundamentals of that are shown in this section.

Let $f(t) = f_0(t) + \eta(t)$ be a signal consisting of a bounded noise $\eta(t)$ with unknown magnitude $\varepsilon$, and of an unknown base signal $f_0(t)$, whose $(k+1)$ derivative satisfies $L \ge |f^{(k+1)}(t)|$ for a known constant $L$. The problem of estimating in real-time the derivatives $\dot{f}_0(t), \ddot{f}_0(t), \ldots, f_0^{(k)}(t)$ was shown to be solved by the recursive algorithm [Levant, 2003]

$$\begin{aligned}
\dot{z}_0 &= v_0, & v_0 &= -\lambda_k L^{1/(k+1)}|z_0 - f(t)|^{k/(k+1)} sign(z_0 - f(t)) + z_1 \\
\dot{z}_1 &= v_1, & v_1 &= -\lambda_{k-1} L^{1/(k)}|z_1 - v_0|^{(k-1)/(k)} sign(z_1 - v_0) + z_2 \\
& & &\vdots \\
\dot{z}_{k-1} &= v_{k-1}, & v_{k-1} &= -\lambda_1 L^{1/2}|z_{k-1} - v_{k-2}|^{1/2} sign(z_{k-1} - v_{k-2}) + z_k \\
\dot{z}_k &= v_k, & v_k &= -\lambda_0 L sign(z_k - v_{k-1}).
\end{aligned} \tag{2.18}$$

The parameters being properly chosen, the following equalities are true in the absence of input noises after a finite time of a transient process

$$z_i = f_0^{(i)}(t) \quad i = 0, \ldots, k.$$

The $\lambda_i$ parameters are calculated recursively, i.e. once $\lambda_1, \cdots, \lambda_k - 1$ are chosen for the $(k-1)$-th order differentiator, the only parameter that needs to be tuned for the $k$-th differentiator is $\lambda_k$. In particular, the parameter $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$ are enough for the construction of differentiators up to the 5-th order.

The algorithm (2.18) can also be stated in a non-recursive form by substituting $v_i$ in $v_{i+1}$, $i = 0, \ldots, k-1$, which yields to

$$\begin{aligned} \dot{z}_0 &= z_1 + l_0\rho|z_0 - f(t)|^{k/(k+1)} sign(z_0 - f(t)) \\ \dot{z}_{j-1} &= z_j + l_{j-1}\rho^j|z_0 - f(t)|^{(k-j)/(k+1)} sign(z_0 - f(t)) \text{ for } j = 1, \ldots, k-1. \\ \dot{z}_k &= l_k\rho^{(k+1)} sign(z_0 - f(t)). \end{aligned} \tag{2.19}$$

where $\rho = L^{1/(k+1)}$ and the $l_i$ gains can be calculated in the basis of the $\lambda_i$'s. A selection of the $l_i$ gains can be such that the matrix

$$\begin{bmatrix} -l_0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -l_{k-1} & 0 & \cdots & 1 \\ -l_k & 0 & \cdots & 0 \end{bmatrix} \tag{2.20}$$

is Hurwitz. That is, given the set $\{\alpha_0, \ldots, \alpha_k\}$, $\alpha_i \in \mathbb{C}$, of symmetric poles (with respect to the real axis) lying in the open left half-plane of the complex plane, the gains $\{l_0, \ldots, l_k\}$ are such that

$$s^k + l_0 s^{k-1} + \cdots + l_{k-1}s + l_k = (s - \alpha_0)\cdots(s - \alpha_k).$$

Thus, with the gains $l_j$ being properly chosen and in the absence of noise and discrete sampling, (2.18) gives an exact estimate in finite-time of the first $k$ derivatives of $f(t)$, which implies that the estimation error dynamics

$$\begin{aligned} \dot{e}_0 &= e_1 - l_0\rho|e_0|^{k/(k+1)} sign(e_0) \\ \dot{e}_{j-1} &= e_j - l_{j-1}\rho^j|e_0|^{(k-j)/(k+1)} sign(e_0) \text{ for } j = 1, \ldots, k-1. \\ \dot{e}_k &= f^{(k+1)}(t) - l_k\rho^{k+1} sign(e_0) \end{aligned} \tag{2.21}$$

where $e_i(t) = f^{(i)}(t) - z_i(t)$, $i = 0, \ldots, k$, goes to the origin $e(t) = [e_0 \cdots e_k]^T = 0$ in finite-time [Levant, 2003]. On the other hand, when either noise or sampling is present, (2.18) gives the best possible asymptotical accuracy [Levant, 2003]. That is, on one hand, if the noise magnitude is $\varepsilon$, then for some constant $\mu_i$ the estimation error satisfies

$$e_i = |z_i - f_0^{(i)}| \leq \mu_i \varepsilon^{(n-i+1)/(n+1)}, \quad i = 0, \ldots, n.$$

On the other hand, given the constant sampling interval $\tau$ in the absence of noise, then for some constant $v_i$ the estimation error satisfies

$$e_i = |z_i - f_0^{(i)}| \leq v_i \tau^{(n-i+1)}, \quad i = 0, \ldots, n.$$

The constants $\mu_i$ and $v_i$ depend exclusively on the parameters of the differentiator. This means that the best accuracy is provided by this differentiator structure. However, to reduce $\mu_i$ and $v_i$, some manual tuning is still required, and trade-offs have to be made between fast convergence and noise filtering.

Notice, that (2.18) is a continuous–time algorithm. However, the same accuracy can be obtained, as pointed out in [Levant, 2011], by using Euler's discretization on (2.18) or (2.19) and sampling with zero–order–hold on $f(t)$. Let $t_1, t_2, \ldots, t_{i-1}, t_i$ be the sampling times with $t_i - t_{i-1} = \tau_i < \tau$ (with $\tau_i$'s possible different, i.e. under variable sampling rate). Then, Euler's discretization gives

$$\dot{z}_j \approx \frac{z_j(t_i) - z_j(t_{i-1})}{\tau_i} = v_j(z(t_{i-1}), f(t_{i-1}))$$

or equivalently

$$z_j(t_i) = z_j(t_{i-1}) + v_j(z(t_{i-1}), f(t_{i-1}))\tau_i. \tag{2.22}$$

When the sampling periods are constant or slowly changing, (2.22) can be replaced by

$$z_j(t_i) = z_j(t_{i-1}) + v_j(z(t_{i-1}), f(t_i))\tau_i.$$

## 2.5 Adaptive Estimator

In this section we present the work developed by doctor Obregon in his doctoral thesis [Obregon-Pulido, 2003]. As an alternative to robust differentiator to estimate the derivatives of the arbitrary sinusoidal reference signal, we use an adaptive estimator which can estimate the reference signal and his derivatives. Then the adaptive estimator serves as a kind of exosystem. Here we assume that a sinusoidal signal could be generated for an unknown exosystem. Then an adaptive estimator is used to estimate the states and the frequency parameter of the unknown exosystem.

Consider the signal

$$y(t) = Asin(\alpha t + \phi)$$

that can be produced for an exosystem

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad (2.23)$$
$$y(t) = c_1 w_1 + c_2 w_2$$

with unknown parameter (frecuency) $\alpha$, then we can use the adaptive estimator to estimate that parameter and the states of the exosystem.

Define the variables

$$z_1 = w_1 \quad z_2 = \frac{w_2}{\lambda} \quad c_1 = \frac{k_1}{\lambda} \quad c_2 = \frac{k_2}{\lambda}$$

where $\lambda$, $k_1$, $k_2$ are constants used to scale the state $w$ of the exosystem (2.23). Noting that under this configuration, in order to estimate exactly the state variables $w_1$ $w_2$, the constants $c_1$, $c_2$ are assumed to be known:

**A. 2.** *The constant $c_1$ and $c_2$ are known.*

Using the new variables the exosystem is rewritten as:

$$\dot{z}_1 = \lambda z_2$$
$$\dot{z}_2 = -\frac{\alpha^2}{\lambda} z_1$$
$$y(t) = \frac{k_1}{\lambda} w_1 + k_2 w_2$$

The following proposition shows the adaptive estimator and establish its stability conditions.

**Proposition 2.2.** *[Obregon-Pulido, 2003] The estimator*

$$\dot{\xi}_1 = \lambda \xi_2 + \frac{\lambda}{k_2}(y - \hat{y}) \qquad (2.24)$$
$$\dot{\xi}_2 = -\frac{\sigma \xi_1 \xi_3}{\lambda} + \zeta(y - \hat{y}) \qquad (2.25)$$
$$\dot{\xi}_3 = -\gamma \xi_1 (y - \hat{y})$$
$$\hat{y} = \frac{k_1}{\lambda}\xi_1 + k_2 \xi_2$$
$$e = y - \hat{y}$$

*with $\sigma$, $\lambda$, $\zeta$, $\lambda$, $k_1$, $k_2 > 0$, is such that $lim_{t\to\infty} e(t) = 0$, $\xi_1 \to w_1$, $\xi_2 \to \frac{w2}{\lambda}$, $\xi_3 \to \frac{\alpha^2}{\sigma}$ for any initial conditions of (2.23).*

*Proof.* The proof is given in [Obregon-Pulido, 2003]. □

In this work the adaptive estimator is used to estimate sinusoidal signals instead the state variables $w_1$ and $w_2$. Thus, unlike to the work developed in [Obregon-Pulido et al., 2010], we do not have to know the values of the constants $c_1$ and $c_2$. Those are considered design parameters.

# Chapter 3

# Sliding Mode Output Regulation

## 3.1 Introduction

In this chapter we present different solutions to the Sliding Mode output regulation problem for nonlinear (NL) nonminimum phase (NP) systems.

Four designed regulators are proposed for the *noncausal case* where the reference tracking profile is produced by an exosystem. In this case a bounded steady state for the internal dynamics is computed using the Francis-Isidori-Byrnes equation.

## 3.2 Problem Statement

Consider the perturbed nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + B(x)u + D(x)w(t) + g(x,t) \\ y &= h(x) \end{aligned} \tag{3.1}$$

where $x \in X \subset R^n$ is the state vector, $u \in U \subset R^m$ is the control vector, $y \in V \subset R^p$ is the output vector. The vector field $f(x)$ and the columns of $B(x)$ and $D(x)$ are smooth and bounded mappings and $f(0) = 0$, $h(0) = 0$, $RankB(x) = m$ for all $x \in X$. The vector $g(x,t)$ is the unmodeled disturbance vector of unknown perturbations.

The output tracking error is defined as

$$e = y - q(w) \tag{3.2}$$

where $w \in W \subset R^q$ is a vector generated by the exosystem:

$$\dot{w} = \xi(w). \tag{3.3}$$

The problem to deal is controlling the output $y$ of system (3.1), to achieve asymptotic tracking of prescribed trajectories $q(w)$, that is, the $lim_{t\to\infty}e(t)$. Moreover, to achieve asymptotic rejection of the undesired disturbances $w(t)$ generated by the exosystem (3.3) and finite time rejection of arbitrary disturbance $g(x,t)$ via Sliding Mode control.

In this chapter we consider the following assumptions:

**A. 3.** *The state vectors $x$ and $w$ are available for measurement.*

**A. 4.** *The Jacobian matrix $S = \left[\frac{\partial \xi}{\partial w}\right]_{(0)}$ at the equilibrium point $w = 0$ has all eigenvalues on the imaginary axis.*

**A. 5.** *The unknown perturbation $g(x,t)$ satisfies the matching condition (Drazenovic, 1969) There exist a vector $\gamma(x,t) \in R^m$ such that the following relation holds:*

$$g(x,t) = B(x)\gamma(x,t), \quad \gamma \in R^m. \tag{3.4}$$

Assumption A.3 is introduced because we focus our attention on the solvability of the state feedback SM problem with the knowledge of $x$ and $w$ while the error feedback problem can be solved by addtional design of a compensator or an observer. Assumption A.4 is from classical output regulation theory. The last assumption A.5 it is common in a robust SM control system design.

If the system (3.1) is a minimum phase then we can apply feedback linearization technique to achieve reference output tracking. In the case when the system is a nonminimum phase then the feedback linearization technique cannot be applied directly due to the unstable internal dynamics. To work with nonminimum phase system we consider that the plant model can be expressed in two forms: **structured** and **unstructured**.

**Definition 3.1.** *The system (3.1) where the internal dynamics are not expressed in explicit form, is refered to as an **unstructured system**.*

**Definition 3.2.** *A system where the internal dynamics are expressed in explicit form, is refered to as an **structured system**.*

To illustrate definitions 1 and 2, we present the following example. Consider a nonlinear system expressed as:

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 + x_3 \\ \dot{x}_2 &= x_1 + x_2 + x_3 \\ \dot{x}_3 &= x_1 + x_2^2 + x_3 + b_3 u \\ y &= x_1 \end{aligned} \tag{3.5}$$

In this case the system (3.5) is in ***unstructured*** general form. Also we can express the same system (3.5) in the form:

$$\begin{aligned} \dot{x}_{12} &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\ \dot{x}_3 &= x_1 + x_2^2 + x_3 + b_3 u \end{aligned}$$

which is ***unstructured*** Regular form.

On the other hand, the system (3.5) can be presented as a ***structured*** one, that representation consists in two subsystems:

$$\begin{cases} \dot{x}_1 = x_1^2 + x_2 + x_3 \\ \dot{x}_3 = x_1 + x_2^2 + x_3 + b_3 u \end{cases} \tag{3.6}$$

$$\{\dot{x}_2 = x_1 + x_2 + x_3 \tag{3.7}$$

The first subsystem (3.6) has block controllable form, the second subsystem (3.7) presents the internal dynamics in explicit form.

To solve the output tracking problem for nonminimum phase systems, there are two ways:

1. The first way is to deal directly with the system as *unstructured* one which does not have the structure of nonminimum phase system explicitly. For example: in the general form (3.1) or in Regular Form.

2. The second way is to deal with the system presented in the *structured* form. For example the nonlinear Block Controllable Form.

Other classification used in this work is related to the reference signal for the output. That classification is made to expand on the subject of control for nonminimum phase nonlinear systems. Then we use the following definitions:

**Definition 3.3.** *We referred to as* **noncausal** *case for the problem if the considered output reference for the output is generated by an exosystem.*

**Definition 3.4.** *The problem is referred to as* **causal** *if the reference signal is an arbitrary function of the time and there is no any exosystem.*

In this work, for the structured system we deal both cases *causal* and *noncausal* while for unstructured systems, we only address the noncausal case.

In the following subsections we first show the classical Output Regulation, then the SMOR problem is briefly showed.

### A. Classical Output Regulation

For the nonlinear system (3.1) in absence of perturbation $g(x,t)$, the Output Regulation Problem presented in [Isidori and Byrnes, 1990] (See Section 2.1), the control action was proposed as a state feedback in the form $u = \alpha(x,w)$ and the solvability of the Output Regulation Problem was stated in terms of the existence of a pair of mappings $\pi(w)$ and $c(w)$ with $\pi(0) = 0$ and $c(0) = 0$, which solves the partial differential equation (FIB equation)

$$\frac{\partial \pi(w)}{\partial w}\xi(w) = f(\pi(w)) + Bc(w) + D(x)w \tag{3.8}$$

where $\pi(w)$ is the steady state, and $c(w)$ is the steady state input.

To compare between the Classical and Sliding Mode Output Regulation problems, we present an example for a system presented in the unstructured form. Consider the following nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 + x_3 \\ \dot{x}_2 &= x_1 + x_2 + x_3 \\ \dot{x}_3 &= x_1 + x_2^2 + x_3 + b_3 u \\ y &= x_1 \end{aligned} \tag{3.9}$$

where $x_i$, $i = 1, 2, 3$, are the state variables, $y$ is the output and $u$ is the input of the system. The output tracking error is defined as:

$$\varepsilon_1 = y - y_{ref}(w) \tag{3.10}$$

where $y_{ref} = q^T w$ is the reference signal, the state vector $w$ is generated by the exosystem:

$$\dot{w} = \xi(w), \quad w \in R^2 \tag{3.11}$$

The control objective is to design a control law such that the output tracking error (3.10) goes asymptotically to zero. We consider that the assumptions A.3 and A.4 hold. To make less extensive this example we consider the system (3.9) is not perturbed, then we have $D(x)w(t) = 0$ and $g(x,t) = 0$ in the general unstructured presentation of the system (3.1).

For this example, the system (3.9) can be represented as unstructured in general form as

$$\dot{x} = f(x) + Bu \tag{3.12}$$

with $B = \begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix}$, $f(x) = \begin{pmatrix} x_1^2 + x_2 + x_3 \\ x_2^2 + x_3 \\ x_1 + x_2^2 + x_3 \end{pmatrix}$.

In order to apply the classical regulator design for the system (3.9) we define the steady state error in the form

$$\varepsilon = x - \pi(w)$$

$\varepsilon = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix}^T$, $\varepsilon_i = x_i - \pi_i(w)$, $i = 1,2,3$, $\pi(w) = \begin{bmatrix} \pi_1(w) & \pi_2(w) & \pi_3(w) \end{bmatrix}^T$

Considering (3.12) the error dynamics are:

$$\dot{\varepsilon} = f(\varepsilon, w) + B(\varepsilon, w)u - \frac{\partial \pi(w)}{\partial w}\xi(w) \tag{3.13}$$

where $f(\varepsilon, w) = f(\varepsilon + \pi(w)) = f(x)_{x=\varepsilon+\pi(w)}$, $B(\varepsilon, w) = B(\varepsilon + \pi(w)) = B(x)_{x=\varepsilon+\pi(w)}$.

We can also express the system (3.13) in the form:

$$\dot{\varepsilon}_1 = \varepsilon_1^2 + \varepsilon_2 + \varepsilon_3 + 2\pi_1\varepsilon_1 + \pi_1^2(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_1(w)}{\partial w}\xi(w) \tag{3.14}$$

$$\dot{\varepsilon}_2 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \pi_1(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_2(w)}{\partial w}\xi(w) \tag{3.15}$$

$$\dot{\varepsilon}_3 = \varepsilon_1 + \varepsilon_2^2 + \varepsilon_3 + 2\pi_2\varepsilon_2 + \pi_1(w) + \pi_3(w) + \pi_2^2(w) + b_3 u - \frac{\partial \pi_3(w)}{\partial w}\xi(w) \tag{3.16}$$

Note, the last part of equations (3.14)-(3.16) corresponds to the FIB equation. Considering that $\pi(w)$ is a solution of (3.8), then the error dynamics (3.13) are reduced to:

$$\begin{aligned} \dot{\varepsilon}_1 &= \varepsilon_1^2 + \varepsilon_2 + \varepsilon_3 + 2\pi_1\varepsilon_1 \\ \dot{\varepsilon}_2 &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ \dot{\varepsilon}_3 &= \varepsilon_1 + \varepsilon_2^2 + \varepsilon_3 + 2\pi_2\varepsilon_2 + b_3 u \end{aligned} \tag{3.17}$$

Denoting $A := \left[\frac{\partial f(\varepsilon,w)}{\partial \varepsilon}\right]_{(0)}$, $B := B(0)$, the linearization of the system (3.17) at origin (0,0,0) is expressed as

$$\dot{\varepsilon} = A\varepsilon + Bu + \phi(\varepsilon, w) \tag{3.18}$$

where the vector $\phi(\varepsilon, w)$ vanishes at the origin with its first order derivatives. Then, the classical state feedback control input to achieve output regulation $u = \alpha(x, w)$ is defined in the form

$$u = K\varepsilon + c(w)$$

where $K$ is a matrix such that $(A + BK)$ is Hurwitz in the linear approximation (3.18), and $c(w)$ is a solution to (3.8).

### Sliding Mode Output Regulation for systems in unstructured general form

Using the Sliding Mode Regulation approach, for the system presented in unstructured general form (3.12) we can propose a sliding surface as

$$s = C^T \varepsilon$$

with $C = \begin{bmatrix} c_1 & c_2 & 1 \end{bmatrix}^T$ The dynamics of $s$ along the trajectories of the error variables (3.13) are

$$\begin{aligned} \dot{s} &= c_1 \left( \varepsilon_1^2 + \varepsilon_2 + \varepsilon_3 + 2\pi_1\varepsilon_1 + \pi_1^2(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_1(w)}{\partial w}\xi(w) \right) \\ &\quad + c_2 \left( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \pi_1(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_2(w)}{\partial w}\xi(w) \right) \\ &\quad + \varepsilon_1 + \varepsilon_2^2 + \varepsilon_3 + \pi_1(w) + \pi_3(w) + 2\pi_2\varepsilon_2 + \pi_2^2(w) + b_3 u - \frac{\partial \pi_3(w)}{\partial w}\xi(w). \end{aligned}$$

To determine the sliding mode dynamics of the error system (3.13) under the action of some discontinuous sliding mode control, we use the equivalent control technique [Utkin et al., 1999]. The equivalent control $u_{eq}$ is obtained by solving $\dot{s} = 0$ for $u$, i.e.

$$\begin{aligned} u_{eq} &= -b_3^{-1} c_1 \left( \varepsilon_1^2 + \varepsilon_2 + \varepsilon_3 + 2\pi_1\varepsilon_1 + \pi_1^2(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_1(w)}{\partial w}\xi(w) \right) \\ &\quad - b_3^{-1} c_2 \left( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \pi_1(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_2(w)}{\partial w}\xi(w) \right) \\ &\quad - b_3^{-1} \left( \varepsilon_1 + \varepsilon_2^2 + \varepsilon_3 + \pi_1(w) + \pi_3(w) + 2\pi_2\varepsilon_2 + \pi_2^2(w) - \frac{\partial \pi_3(w)}{\partial w}\xi(w) \right) \end{aligned}$$

Substituting $u_{eq}$ in the error dynamics (3.15) we obtain the sliding mode equation

$$\dot{\varepsilon} = P\left(f(\varepsilon + \pi(w)) - \frac{\partial \pi(w)}{\partial w}\xi(w)\right) \tag{3.19}$$

where $P = \left(I_3 - B\left(C^T B\right)^{-1} C^T\right)$. Or expressed using scalar equations as:

$$\begin{aligned}
\dot{\varepsilon}_1 &= \varepsilon_1^2 - c_1\varepsilon_1 + k_2\varepsilon_2 + 2\pi_1\varepsilon_1 + \pi_1^2(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_1(w)}{\partial w}\xi(w) \\
\dot{\varepsilon}_2 &= k_1\varepsilon_1 + k_2\varepsilon_2 + \pi_1(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_2(w)}{\partial w}\xi(w) \\
\dot{\varepsilon}_3 &= -b_3^{-1}c_1\left(\varepsilon_1^2 - c_1\varepsilon_1 + k_2\varepsilon_2 + 2\pi_1\varepsilon_1 + \pi_1^2(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_1(w)}{\partial w}\xi(w)\right) \\
&\quad -b_3^{-1}c_2\left(k_1\varepsilon_1 + k_2\varepsilon_2 + \pi_1(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_2(w)}{\partial w}\xi(w)\right)
\end{aligned} \tag{3.20}$$

where $k_1 = (1 - c_1)$ and $k_2 = (1 - c_2)$.

Note that on the sliding manifold $s = 0$, the sliding mode equation (3.20) does not depend on $\frac{\partial \pi_3(w)}{\partial w}\xi(w)$ and $c(w)$. It will be shown later that we do not need to solve the FIB equation (3.8) for $c(w)$. In this case, the sliding mode equation just have two partial differential equations, those are:

$$\begin{aligned}
\frac{\partial \pi_1(w)}{\partial w}\xi(w) &= \pi_1(w)^2 + \pi_2(w) + \pi_3(w) \\
\frac{\partial \pi_2(w)}{\partial w}\xi(w) &= \pi_1(w) + \pi_2(w) + \pi_3(w)
\end{aligned} \tag{3.21}$$

where $\pi_3(w)$ is determined as a function of $\pi_1(w)$ and $\pi_2(w)$. For this case when the system is in the unstructured form, the Sliding Mode Output Regulation problem reduces the order of the partial differential equation (3.8). The variable $\pi_3(w)$ can be computed from an algebraic equation as a function of $\pi_1(w)$ and $\pi_2(w)$, that is implied in the equation (3.19). This fact can be seen better with the system presented in unstructured Regular form.

### Sliding Mode Output Regulation for systems in unstructured Regular form

Consider the system (3.12) presented as unstructured one in Regular Form, i.e.

$$\begin{aligned}
\dot{x}_{12} &= A_{11}x_{12} + A_{12}x_3 + f_{12}(x) \\
\dot{x}_3 &= A_{31}x_{12} + A_{32}x_3 + B_3 u + f_3(x).
\end{aligned} \tag{3.22}$$

where $x_{12} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$ $A_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A_{12} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $f_{12}(x) = \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$, $A_{31} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $A_{32} = 1$, $f_3(x) = x_2^2$ and $B_3 = b_3$.

To analyze the output error behavior we define the error vector $\varepsilon_{12} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \end{bmatrix}$. We can represent the error dynamics of the system (3.22) as

$$\begin{aligned} \dot{\varepsilon}_{12} &= A_{11}\varepsilon_{12} + A_{12}\varepsilon_3 + A_{11}\pi_{12} + A_{12}\pi_3 + f_{12}(\varepsilon + \pi) - \frac{\partial \pi_{12}(w)}{\partial w}\xi(w) \\ \dot{\varepsilon}_3 &= A_{31}\varepsilon_{12} + A_{32}\varepsilon_3 + B_3 u + A_{31}\pi_{12} + A_{32}\pi_3 + f_3(\varepsilon + \pi) - \frac{\partial \pi_3(w)}{\partial w}\xi(w) \end{aligned} \tag{3.23}$$

with $\frac{\partial \pi_{12}(w)}{\partial w} = \begin{bmatrix} \frac{\partial \pi_1(w)}{\partial w} & \frac{\partial \pi_2(w)}{\partial w} \end{bmatrix}^T$

Now, we define a sliding surface in the form

$$s_2 = \varepsilon_3 + C_1^T \varepsilon_{12} \tag{3.24}$$

where $C_1 = \begin{bmatrix} c_{11} & c_{21} \end{bmatrix}^T$ The dynamics of the sliding surface $s_2$ on the trajectories of (3.23) are

$$\begin{aligned} \dot{s}_2 &= A_{31}\varepsilon_{12} + A_{32}\varepsilon_3 + b_3 u + f_3(\varepsilon, \pi(w)) + A_{31}\pi_{12} + A_{32}\pi_3 \\ &\quad + C_1^T\left(A_{11}\varepsilon_{12} + A_{12}\varepsilon_3 + f_{12}(\varepsilon, \pi(w))\right) + C_1^T\left(A_{11}\pi_{12} + A_{12}\pi_3\right) \end{aligned}$$

and the equivalent control $u_{eq}$ obtained as a solution of $\dot{s}_2 = 0$ is

$$\begin{aligned} u_{eq} &= -b_3^{-1}\left(A_{31}\varepsilon_{12} + A_{32}\varepsilon_3 + f_3(\varepsilon, \pi(w))\right) - b_3^{-1}\left(+A_{31}\pi_{12} + A_{32}\pi_3\right) \\ &\quad - b_3^{-1}C_1^T\left(A_{11}\varepsilon_{12} + A_{12}\varepsilon_3 + f_{12}(\varepsilon, \pi(w))\right) - b_3^{-1}C_1^T\left(A_{11}\pi_{12} + A_{12}\pi_3\right). \end{aligned}$$

Substituting $u_{eq}$ in (3.23) then, the sliding mode equation is

$$\dot{\varepsilon}_{12} = \left(A_{11} - A_{12}C_1^T\right)\varepsilon_{12} + f_{12}(\varepsilon + \pi(w)) + A_{11}\pi_{12} + A_{12}\pi_3 - \frac{\partial \pi_{12}(w)}{\partial w}\xi(w) \tag{3.25}$$

$$\dot{\varepsilon}_3 = C_1^T\left[\left(A_{11} - A_{12}C_1^T\right)\varepsilon_{12} + f_{12}(\varepsilon, \pi(w)) + A_{11}\pi_{12} + A_{12}\pi_3 - \frac{\partial \pi_{12}(w)}{\partial w}\xi(w)\right] \tag{3.26}$$

we can see that the second equation (3.26) is linear combination of the first one (3.25). For the Regular Form of the reduced system the sliding mode equation is

$$\begin{aligned} \dot{\varepsilon}_1 &= \varepsilon_1^2 - c_{11}\varepsilon_1 + (1 - c_{12})\varepsilon_2 + 2\pi_1\varepsilon_1 + \pi_1(w)^2 + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_1(w)}{\partial w}\xi(w) \\ \dot{\varepsilon}_2 &= (1 - c_{11})\varepsilon_1 + (1 - c_{12})\varepsilon_2 + \pi_1(w) + \pi_2(w) + \pi_3(w) - \frac{\partial \pi_2(w)}{\partial w}\xi(w) \end{aligned} \tag{3.27}$$

or expressed in vectorial form

$$\dot{\varepsilon}_{12} = \left(A_{11} - A_{12}C_1^T\right)\varepsilon_{12} + f_{12}(\varepsilon + \pi(w)) + A_{11}\pi_{12} + A_{12}\pi_3 - \frac{\partial \pi_{12}(w)}{\partial w}\xi(w)$$

At this point, we can see explicitly that the sliding mode equation is reduced and has a reduced vector of partial differential equations $\frac{\partial \pi_{12}(w)}{\partial w}$. Considering that $\pi_1(w)$ and $\pi_2(w)$ are solution of the partial differential equation (3.21), in other words, if

$$f_{12}(\pi(w)) + A_{11}\pi_{12} + A_{12}\pi_3 = \frac{\partial \pi_{12}(w)}{\partial w}\xi(w)$$

then the system (3.25)-(3.26) is reduced to

$$\dot{\varepsilon}_{12} = \left(A_{11} - A_{12}C_1^T\right)\varepsilon_{12} + \varphi(\varepsilon + \pi(w)) \tag{3.28}$$

where $\varphi(\varepsilon + \pi(w))$ is a function which vanishes at the origin with its first order derivatives. Now, $c_{11}$ and $c_{12}$ are chosen to place the poles in a linearized approximation of (3.15), thus, $\lim_{t\to\infty}\varepsilon_{12} = C_1^T\varepsilon_3 = 0$.

It was shown that the sliding mode approach reduces the order of the partial differential FIB equation (3.8) to be analyzed. Also, we show that the Regular form the sliding mode equation is in explicit form.

## 3.3 Integral SM Regulation for Nonlinear NP Systems in Unstructured Form

In this section we present a solution for Sliding Mode Output Regulation (SMOR) problem for perturbed nonlinear systems presented in unstructured general and Regular forms, for the noncausal case. This approach is an extension of the work [Loukianov et al., 1999b].

To deal with the matched perturbations we first use the Integral Sliding Mode technique combined with second order SM Super-Twisting algorithm. On the proposed integral sliding manifold the perturbation term $g(x,t)$ is rejected. Moreover, the use of super-twisting algorithm allows to ensure chattering free sliding mode motion. Secondly, a sliding manifold which contents the steady state manifold will be designed. On that manifold the output tracking error tends to zero.

Consider the nonlinear system

$$\dot{x} = f(x) + B(x)u + D(x)w + g(x,t) \tag{3.29}$$

$$y = h(x) \tag{3.30}$$

where $x \in X \subset R^n$ is the state vector, $u \in U \subset R^m$ is the control vector, $y \in V \subset R^p$ is the output vector. The vector field $f(x)$ and the columns of $B(x)$ and $D(x)$ are smooth and bounded mappings and $f(0) = 0$, $h(0) = 0$, $Rank B(x) = m$ for all $x \in X$. The vector $g(x,t)$ is the unmodeled disturbance vector of unknown matched perturbations (A.5), $w \in W \subset R^q$ is a vector generated by the exosystem

$$\dot{w} = \xi(w), \quad \xi(0) = 0. \tag{3.31}$$

The output tracking error is defined as the difference between the output of the system $y$, and a reference $y_{ref} = q(w)$ in the form

$$e = h(x) - y_{ref} \tag{3.32}$$

Consider that assumptions A.3, A.4 and A.5 hold. Denoting $A := \left[\frac{\partial f}{\partial x}\right]_{(0)}$, $C := \left[\frac{\partial h}{\partial x}\right]_{(0)}$, $B := B(0)$, the following assumption is introduced:

**A. 6.** *The pair $\{A, B\}$ is stabilizable.*

Assumption A.6 is clearly needed to stabilize locally sliding mode dynamics.

Following the Output Regulation theory [Isidori, 1995] and integral SM control technique [Utkin et al., 1999], we consequently introduce a local center manifold [Isidori, 1995]

$$\varepsilon(x,w) = 0, \; \varepsilon = x - \pi(w) \;\text{ with }\; \pi(0) = 0 \tag{3.33}$$

and define the control as a combination of two parts [Utkin et al., 1999]

$$u = u_0 + u_1. \tag{3.34}$$

Under the matched condition A.5 (3.4) for the perturbation $g(x,t)$, substituting $u$ (3.34) into (3.30) results in

$$\dot{x} = f(x) + B(x)u_0 + D(x)w + B(x)\left(u_1 + \gamma(x,t)\right). \tag{3.35}$$

Then the change of variables $\varepsilon = x - \pi(w)$ (3.33) transforms (3.35) and (3.32) in

$$\dot{\varepsilon} = f(\varepsilon,w) + B(\varepsilon,w)u_0 + B(\varepsilon,w)\left[u_1 + \gamma(\varepsilon,w,t)\right] + D(\varepsilon,w)w - \frac{\partial w}{\partial w}\xi(w) \tag{3.36}$$

$$e = h(\varepsilon + \pi(w)) - q(w) \tag{3.37}$$

where $f(\varepsilon,w) = f(\varepsilon + \pi(w)) = f(x)_{x=\varepsilon+\pi(w)}$, $B(\varepsilon,w) = B(\varepsilon + \pi(w)) = B(x)_{x=\varepsilon+\pi(w)}$, $D(\varepsilon,w) = D(\varepsilon + \pi(w)) = D(x)_{x=\varepsilon+\pi(w)}$, $\gamma(\varepsilon,w,t) = \gamma(\varepsilon + \pi(w),t) = \gamma(x,t)_{x=\varepsilon+\pi(w)}$ and $h(\varepsilon,w) = h(\varepsilon + \pi(w)) = h(x)_{x=\varepsilon+\pi(w)}$.

The *Integral Sliding Mode Regulation Problem* (ISMR-problem) is defined as the problem of finding smooth sliding functions $\sigma(\varepsilon)$, $\sigma \in \Re^m$and $s(\varepsilon)$, $s \in \Re^m$ such that the following conditions hold:

- (*ISMS*) (Integral Sliding Mode Stability). The state of the system (3.35) with a discontinuous state feedback with SM super-twisting control $u_1(x, w)$ in the presence of unknown matched perturbation $g(x,t) = B(x)\gamma(x,t)$, converges in finite time to the sliding manifold

$$\sigma(\varepsilon) = 0, \quad \sigma = (\sigma_1, ..., \sigma_m)^T \tag{3.38}$$

In this case the unknown perturbation $g(x,t)$ is rejected. That results in the following integral SM equation which describes a motion on the manifold (3.38):

$$\dot{x} = f(x) + B(x)u_0 + D(x)w \tag{3.39}$$

Using the error variable $\varepsilon$ the system (3.39) is

$$\dot{\varepsilon} = f(\varepsilon, w) + B(\varepsilon, w)u_0 + D(\varepsilon, w)w - \frac{\partial \pi(w)}{\partial w}\xi(w) \tag{3.40}$$

- (*SMS*) (Sliding Mode Stability). The state of the system (3.39) with super-twisting controller $u_0(\varepsilon)$ converges in finite time to the sliding manifold

$$s(\varepsilon) = 0, \quad s = (s_1, ..., s_m)^T \tag{3.41}$$

which contains the steady-state (central) manifold (3.33), and the dynamics of the closed-loop system tend asymptotically along the sliding manifold (3.41) to the steady-state behavior.

- (*S*). The equilibrium $\varepsilon = 0$ of the sliding mode dynamics on $\sigma(\varepsilon, z) = 0$ (3.41) governed by

$$\dot{x} = f(x) + B(x)u_{0eq}(\varepsilon, w) + D(x)w \tag{3.42}$$

or

$$\dot{\varepsilon} \;=\; f(\varepsilon + \pi(w)) + B(\varepsilon + \pi(w))u_{0eq}(\varepsilon, w) + D(\varepsilon + \pi(w))w - \frac{\partial \pi(w)}{\partial w}\xi(w)$$

are asymptotically stable, where $u_{0eq}$ is the equivalent control calculated as a solution for $\dot{s} = 0$ [Utkin et al., 1999];

- $(R)$. There exists a neighborhood $V_0 \subset X \times W$ of $(0,0)$ such that, for each initial condition $(x_0, w_0) \in V_0$, the output tracking error (3.32) despite the presence of unknown but bounded matched perturbation $g(x,t)$ goes asymptotically to zero, i.e. $lim_{t\to\infty} e(t) = 0$.

In the following subsection, a solution to Integral Sliding Mode Output Regulation problem for nonlinear system considering the *noncausal* case will be presented.

### ISMR Problem Solution

In this section, firstly a control law which ensures the requirements $SMS$ and $S$ will be designed, and then the ISMR-problem solvability conditions under which the requirement $R$ is satisfied, that will be derived for a nonlinear system described by (3.29) (3.31). In the sequel, a SM regulator will be developed for nonlinear systems.

### Integral SM Controller Design (requirement $ISMS$)

To reject the bounded perturbation $g(x,t)$ and to ensure the convergence of the state vector to SM manifold $\sigma(\varepsilon) = 0$ (which will be shaped later), the Integral SM technique based on the super-twisting algorithm is used. According to this philosophy, we suppose that the sliding function $s(\varepsilon)$ is designed (see subsection 3.3), then the auxiliar sliding function $\sigma(\varepsilon)$ is formulated as

$$\begin{aligned}\sigma(\varepsilon) &= s(\varepsilon) - \int_0^t G(\varepsilon(\tau))\left(f(\varepsilon(\tau), w(\tau))\right) d\tau - \int_0^t G(\varepsilon(\tau))\left(B(\varepsilon(\tau), w(\tau)) u_0(\tau)\right) d\tau \\ &- \int_0^t G(\varepsilon(\tau))\left(D(\varepsilon(\tau), w(\tau)) w(\tau)\right) d\tau + \int_0^t G(\varepsilon(\tau))\left(\frac{\partial \pi(w)}{\partial w}\xi(w(\tau))\right) d\tau \qquad (3.43)\end{aligned}$$

where initial conditions for the integrators are set to $-s(\varepsilon(0))$ in order to the sliding mode to occur from the initial time instant.

Using (3.43) and (3.36) the straightford calculation yields

$$\dot{\sigma} = G(\varepsilon) B(\varepsilon, w)\left[u_1 + \gamma(\varepsilon, w, t)\right]$$

where $G(\varepsilon) = \dfrac{\partial s}{\partial \varepsilon}$, $G(0) = \Sigma$ and $\text{rank}[G(\varepsilon)] = m$, results in

$$\dot{\sigma} = B_1(\varepsilon, w)\left[u_1 + \gamma(\varepsilon, w, t)\right]$$

or

$$\dot{\sigma} = v + \gamma_1(\varepsilon, w, t). \tag{3.44}$$

where $v = B_1(\varepsilon, w)u_1$, $v = (v_1, ..., v_m)$, the matrix $B_1(\varepsilon, w) = G(\varepsilon)B(\varepsilon, w)$ is assumed to be nonsingular, and $\gamma_1(\varepsilon, w, t) = B_1(\varepsilon, w)\gamma(\varepsilon, w, t)$. Following [Moreno and Osorio, 2008] we assume

**A. 7.** *The perturbation*

$$\|\gamma_1(\varepsilon, w, t)\| \leq \delta_1 \|\varepsilon\|^{\frac{1}{2}}, \delta_1 > 0, \forall (x, w) \in X \times W$$

*and* $t \geq 0$.

Now, to enforce chattering-free SM motion on the auxiliary manifold $\sigma = 0$ (3.43) a super-twisting control algorithm is chosen, given by [Fridman and Levant, 2002]

$$\begin{aligned} v_i &= -k_{1i}\sqrt{|\sigma_i|}sign(\sigma_i) + v_{0i}, \\ \dot{v}_{0i} &= -k_{2i}sign(\sigma_i), \ i = 1, ..., m. \end{aligned} \tag{3.45}$$

System (3.44) closed-loop by (3.45) results in

$$\begin{aligned} \dot{\sigma}_i &= -k_{1i}\sqrt{|\sigma_i|}sign(\sigma_i) + v_{0i} + \gamma_{1i}(\varepsilon, w, t), \\ \dot{v}_{0i} &= -k_{2i}sign(\sigma_i), \ i = 1, ..., m \end{aligned}$$

where $v_0 = (v_{01}, \ldots, v_{0m})^T$. $\gamma_1 = (\gamma_{11}, \ldots, \gamma_{1m})^T$, $k_1 = (k_{11,\ldots,k_{1m}})$ and $k_2 = (k_{21}, \ldots, k_{2m})$.

**Proposition 3.5.** *[Moreno and Osorio, 2008]: Using assumption A.7 and under the following conditions:*

$$k_{1i} > 2\delta_1, \quad k_{2i} > k_1 \frac{5\delta_1 + 4\delta_1^2}{2(k_{1i} - 2\delta_1)}, \quad i = 1, \ldots, m$$

*the state of the closed-loop system converges to the sliding manifold $\sigma = 0$ (3.43) in finite time, ensuring the requirement ($ISMS_{ef}$).*

On this manifold formally setting

$$\dot{\sigma} = B_1(\varepsilon, w)\left[u_{1eq} + \gamma(\varepsilon, w, t)\right] = 0$$

one calculates

$$u_{1eq} = -\gamma(\varepsilon, w, t) \tag{3.46}$$

where $u_{1eq}$ is refered to as the equivalent control [Utkin et al., 1999]. Substituting (3.46) in (3.36), the full order integral SM dynamics on $\sigma = 0$ governed by (3.39) or (3.40) are invariant with respect to the perturbation $\gamma(\varepsilon, w, t)$.

### SM Regulator Design (requirement $SMS$)

To enforce now sliding mode on $s(\varepsilon) = 0$ (3.41) the projection motion on $s$ is first derived using (3.40) of the form

$$\dot{s} \; = \; G(\varepsilon)\left(f(\varepsilon, w) + D(\varepsilon, w)w - \frac{\partial \pi(w)}{\partial w}\xi(w)\right) + G(\varepsilon)B(\varepsilon, w)u_0 \qquad (3.47)$$

where $\text{rank}[G(\cdot)B(\cdot)] = m \;\; \forall x \in X \subset \Re^n$. We choose the nominal control part, $u_0$ as

$$u_0 \; = \; u_{0eq}(\varepsilon, w) - [G(\varepsilon)B(\varepsilon, w)]^{-1}\left(k_3\sqrt{\|s\|}sign(s) + k_4 s\right)$$

where the equivalent control $u_{0,eq}(\cdot)$ is

$$u_{0,eq}(\varepsilon, w) \; = \; -\left[G(\varepsilon)B(\varepsilon, w)\right]^{-1} G(\varepsilon)\left[f(\varepsilon, w) + D(\varepsilon, w)w - \frac{\partial \pi(w)}{\partial w}\xi(w)\right] \qquad (3.48)$$

this $u_{0,eq}(\cdot)$ is calculated as a solution of $\dot{s} = 0$, then substituting $u_0$ in (3.47) yields the closed-loop system

$$\dot{s} = -k_3 \|s\|^{\frac{1}{2}} sign(s) - k_4 s.$$

It is easy to see that if $k_3 > 0$ and $k_4 > 0$ then a sliding mode motion occurs on the nominal manifold $s(\varepsilon) = 0$ in finite time, then the requirement ($SMS$) is fulfilled.

### SM Dynamics

Sliding motion on $s(\varepsilon) = 0$ is described by (3.42) or using the error variable is

$$\dot{\varepsilon} \; = \; f(\varepsilon, w) + B(\varepsilon, w)u_{0eq}(\varepsilon, w) + D(\varepsilon, w)w - \frac{\partial \pi(w)}{\partial w}\xi(w) \qquad (3.49)$$

Substituting (3.48) into (3.49) the sliding mode dynamics on $s(\varepsilon) = 0$ result in the following form:

$$\dot{\varepsilon} = P(\varepsilon, w)\left[f(\varepsilon, w) + D(\varepsilon, w)w - \frac{\partial \pi(w)}{\partial w}\xi(w)\right] \qquad (3.50)$$

where the nonlinear projector operator $P(\varepsilon, w) = P(\varepsilon + \pi(w)) = P(x)_{x=\varepsilon+\pi(w)}$ is defined as

$$P(\varepsilon, w) = I_n - B(\varepsilon, w)\left[G(\varepsilon)B(\varepsilon, w)\right]^{-1} G(\varepsilon). \qquad (3.51)$$

**Lemma 3.6.** *Consider the operator (3.51). The condition*

$$P(\pi(w))\left[f(\pi(w))+D(\pi(w))w-\frac{\partial\pi(w)}{\partial w}\xi(w)\right]=0 \qquad (3.52)$$

*holds true if and only if there are mappings $\pi(w)$ and $\lambda(w)$, such that*

$$f(\pi(w))+D(\pi(w))w-\frac{\partial\pi(w)}{\partial w}\xi(w)=B(\pi(w))\lambda(w). \qquad (3.53)$$

*Proof.* The operator $P(\cdot)$ is a projector operator along the subspace of $range[B(\cdot)]$ for each point $w$ over the subspace of $ker[B(\cdot)]$ i. e.

$$\begin{aligned} P(\pi(w))B(\pi(w)) &= \left(I_n - B(\pi(w))\left[G(0)B(\pi(w))\right]^{-1}G(0)\right)B(\pi(w)) = 0 \qquad (3.54)\\ P(\pi(w))\varepsilon &= \varepsilon, \qquad \forall\varepsilon\in\aleph \qquad ,\aleph=\{\varepsilon\in\Re^n|s(\varepsilon)=0\}. \end{aligned}$$

Thus, if condition (3.53) holds, then from (3.54) it follows that

$$P(\pi(w))\left(f(\pi(w))+D(\pi(w))w-\frac{\partial\pi(w)}{\partial w}\xi(w)\right)=P(\pi(w))B(\pi(w))\lambda(w)=0.$$

Therefore condition (3.52) is satisfied. Conversely, if condition (3.52) is satisfied, then

$$\left(f(\pi(w))+D(\pi(w))w-\frac{\partial\pi(w)}{\partial w}\xi(w)\right)$$

must be in the range of $B(\pi(w))$, i. e. must to satisfy the matching condition [Drazenovich, 1969]

$$f(\pi(w))+D(\pi(w))w-\frac{\partial\pi(w)}{\partial w}\xi(w)=B(\pi(w))\lambda(w) \qquad (3.55)$$

for some vector $\lambda(w)$. □

## Conditions for solution of the ISMR Problem

Define the sliding function $s(\varepsilon)$ as

$$s(\varepsilon)=\Sigma\varepsilon=\Sigma(x-\pi(w))$$

where $\Sigma$ is a constant matrix of proper dimension.

On the other hand, using the a linear approximation, the system (3.36)-(3.37) that can be represented as

$$\begin{aligned} \dot{\zeta} &= \bar{A}\zeta + \bar{B}u + \Phi(\zeta, u) \\ e &= \bar{C}\zeta + \phi_e(\zeta) \end{aligned} \tag{3.56}$$

where $\zeta = (\varepsilon, w)^T$ $\bar{A} = \begin{pmatrix} A & A\Pi - \Pi S + D_0 \\ 0 & S \end{pmatrix}$ $\bar{B} = \begin{pmatrix} B_0 \\ 0 \end{pmatrix}$ $\bar{C} = \begin{pmatrix} C & C\Pi - Q \end{pmatrix}$, $\Phi(\zeta, u) = \begin{pmatrix} \phi(\varepsilon, w, u) \\ \phi_w(w) \end{pmatrix}$, and the functions $\phi(\varepsilon, w, u)$, $\phi_w(w)$, $\phi_e(\varepsilon, w)$ and its first derivatives vanish at the origin.

$A = \left[\dfrac{\partial f(\zeta)}{\partial \zeta}\right]_{(0)}$ $B_0 = B(0)$, $D_0 = D(0)$, $S = \left[\dfrac{\partial \phi(w)}{\partial w}\right]_{(0)}$ $C = \left[\dfrac{\partial h}{\partial \zeta}\right]_{(0)}$ and $Q = \left[\dfrac{\partial q}{\partial w}\right]_{(0)}$

The sliding mode dynamics (3.50) can be thus represented as

$$\dot{\varepsilon} = P_0 A\varepsilon + P_0(A\Pi - \Pi S + D_0)w + \phi_s(\varepsilon, w)$$

where $\phi_s(\varepsilon, w)$ and its first derivative vanish at the origin, $P_0 = \left[\dfrac{\partial P(\zeta)}{\partial \zeta}\right]_{\zeta=0} = I_n - B_0(\Sigma B_0)^{-1}\Sigma$, is the linear approximation of the nonlinear operator (3.51).

**Proposition 3.7.** *[Loukianov et al., 1999b] Consider assumptions A.3 and A.4 hold. If there exist $C^k$ $(k \geq 2)$ mapping $x = \pi(w)$ with $\pi(0) = 0$, defined in a neighborhood $\tilde{W}$ of 0 satisfying the following conditions:*

$$f(\pi(w)) + B(\pi(w))\lambda(w) + D(\pi(w))w = \frac{\partial \pi(w)}{\partial w}\xi(w) \tag{3.57}$$

$$h(\pi(w)) - q(w) = 0 \tag{3.58}$$

*then, the nonlinear ISMR problem is solvable.*

*Proof.* The closed-loop system motion on this manifold can be described by

$$\dot{\varepsilon} = P_0 A\varepsilon + P_0(A\Pi - \Pi S + D_0)w + \phi_s(\varepsilon, w),$$

$$\Sigma\varepsilon = \Sigma(x - \pi(w)) = 0 \tag{3.59}$$

$$\dot{w} = Sw + \phi_w(w) \tag{3.60}$$

$$e = h(\varepsilon + \pi(w)) - q(w) \tag{3.61}$$

where $\zeta = (\varepsilon, w)^T$ and $P_0 = \left[\frac{\partial P(\zeta)}{\partial \zeta}\right]_{\zeta=0} = I_n - B_0(\Sigma B_0)^{-1}\Sigma$ is the linear approximation of the nonlinear operator (3.51); $\phi_s(\varepsilon, w)$ and $\phi_w(w)$, and its first derivatives vanish at the origin.

The matrix $\Sigma$ can be chosen (by assumption A.4) such that the $(n-m)$ eigenvalues of $P_0A$ are in $C^-$ while the others $m$ eigenvalues are equal to zero, [Utkin and Young, 1978]. We can easily see that for sufficiently small initial state $(\varepsilon(0),\ w(0))$, the condition $(S_{ef})$ is satisfied.

Now, if the partial differential equation (3.57) holds, then by Lemma 1 it follows

$$P_0(A\Pi - \Pi S + D)w + \phi_s(w) \quad = \quad P(\pi(w))\left[f(\pi(w)) + D(\pi(w))w - \frac{\partial \pi(w)}{\partial w}\xi(w)\right] = 0$$

Therefore, under assumption A.6, the system (3.59) - (3.61) has a center manifold [Carr, 1981] contained in the sliding manifold

$$\sigma(\varepsilon) = 0,\ \varepsilon = 0 \tag{3.62}$$

or in the original variables the graph of mappings

$$\sigma(x - \pi(w)) = 0,\ x = \pi(w) \tag{3.63}$$

which is locally invariant and attractive under the flow of (3.50). The restriction of this flow to manifold (3.62) or (3.63) is a diffeomorphic copy of the flow of the exosystem (3.60). Thus, $lim_{t\to\infty}\varepsilon(t) = 0$, and if condition (3.58) holds, then by continuity of $h(\varepsilon + \pi(w))$ (3.61), $e(t) \to 0$ as $t \to \infty$, i.e. condition $(R_{ef})$ is satisfied. □

**Remark 3.8.** *It is worth noting that in the case of the classical regulator, the steady state input $u_{ss} = \lambda(w)$ is used in the construction of the controller,*

$$u = K\left(x - \pi(w)\right) + \lambda(w)$$

*where $(A + BK)$ is a Hurwitz matrix, while in our present approach we need to find the mapping $\pi(w)$ which achieves the matching condition (3.55). The proposed SM controller does not use directly the signal $u_{ss} = \lambda(w)$, but it is generated by the the controller (3.34)* *when the flow of the system is on the sliding manifold. Indeed, the sliding center manifold (3.63) is rendered locally invariant by the effect of a suitable equivalent control which in the steady state is equal to, indeed using (3.55) we have*

$$\begin{aligned} u_{0,eq}(0,w) &= -\left[G(0)B(\pi(w))\right]^{-1} G(0)\left[f(\pi(w)) + D(\pi(w))w - \frac{\partial \pi(w)}{\partial w}\xi(w)\right] \\ &= -\left[G(0)B(\pi(w))\right]^{-1} G(0)B(\pi(w))\lambda(w) \\ &= \lambda(w) \end{aligned}$$

*and this manifold is annihilated by the error map $e = h(x) - q(w)$ in a similar way taken place in the classical regulator formulation.*

## Integral SM Regulation for Nonlinear NP Systems in Unstructured Regular form

Consider the nonlinear system (3.29) under the following assumptions:

**A. 8.** *The matrix $B(x)$ has a block $B_2(x_{11}, x_2)$ such that $rank[B_2(x_{11}, x_2)] = m \; \forall x \in X \subset \Re^n$ where $B(\cdot) = (B_1(\cdot), B_2(\cdot))^T$ $x = (x_{11}, x_2)^T$, $x_{11} \in X_1 \subset \Re^{n-m}$, $x_2 \in X_2 \subset \Re^m$*

**A. 9.** *The Pfaffian system*

$$dx_{11} - B_1(\cdot)B_2^{-1}(\cdot)dx_2 = 0$$

*is completely integrable [Luk'yanov and Utkin, 1981] that is, there is a smooth solution to*

$$x_{11} = \bar{g}(x_2, c), \;\; \bar{g} = (\bar{g}_1, ..., \bar{g}_{n-m})^T$$

*which can be presented by the Implicit Function Theorem into*

$$\hat{g}(x_{11}, x_2) = c, \;\; \hat{g} = (\hat{g}_1, ..., \hat{g}_{n-m})^T$$

*where $c = (c_1, .., c_{n-m})^T$ is a vector of integration constants.*

Under assumptions A.5, A.8 and A.9, the local diffeomorphism

$$x' := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \hat{g}(x_{11}, x_2) \\ x_2 \end{bmatrix}$$

reduces the original system into Regular form [Luk'yanov and Utkin, 1981]:

$$\begin{aligned} \dot{x}_1 &= f_1(x') + D_1(x')w & (3.64)\\ \dot{x}_2 &= f_2(x') + B_2(x')u + D_2(x')w + d_2(x',t) & \\ \dot{w} &= \xi(w) & (3.65)\\ e &= h(x') - q(w) & (3.66) \end{aligned}$$

where $\text{rank}[B_2(x')] = m \ \forall x' \in X \subset \Re^n$ Note that the Regular form presents the external matched perturbation $d_2(x',t)$ in explicit form, affecting just to $x_2$. That is very suitable for the control design.

Let us now introduce the steady state for $x_1$ and $x_2$ as $\pi_1(w)$ and $\pi_2(w)$, respectively. Then, defining the steady state error

$$\varepsilon = x' - \pi(w) = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \pi_1(w) \\ \pi_2(w) \end{bmatrix} \tag{3.67}$$

the dynamic equation for (3.67) with tracking error $e$ can be obtained from (3.64) - (3.66) as

$$\begin{aligned} \dot{\varepsilon}_1 &= f_1(\varepsilon + \pi(w)) + D_1(\varepsilon + \pi(w))w - \frac{\partial \pi_1(w)}{\partial w}\xi(w) & (3.68)\\ \dot{\varepsilon}_2 &= B_2(\varepsilon + \pi(w))u + d_2'(\varepsilon, w, t) & (3.69)\\ e &= h(\varepsilon + \pi(w)) - q(w) & (3.70) \end{aligned}$$

where $d_2'(\varepsilon, w, t) = f_2(\varepsilon + \pi(w)) + D_2(\varepsilon + \pi(w))w + g(\varepsilon + \pi(w), t) - \frac{\partial \pi_2(w)}{\partial w}\xi(w)$. Where $g_2(\cdot)$ is the transformed $g(t)$. The proposed sliding manifold is based in the integral sliding mode philosophy

$$\begin{aligned} \sigma &= s_2(\varepsilon) + \zeta_2, \quad s_2(\varepsilon) = \varepsilon_2 - s_1(\varepsilon_1),\\ s_1(0) &= 0, \ \left[\frac{\partial s_1}{\partial \varepsilon_1}\right]_{(0)} = G_1(\varepsilon_1)_{(0)} = \Sigma_1, \end{aligned}$$

where $\sigma = (\sigma_1, \ldots, \sigma_m)^T$ Taking the derivative of the sliding function along the trajectories of system (3.69) and replacing $u = u_0 + u_1$ results in the following expression:

$$\begin{aligned} \dot{\sigma} &= B_2(\varepsilon + \pi(w))u_0 + B_2(\varepsilon + \pi(w))u_1 + d_2'(\varepsilon, w, t) - G_1(\varepsilon_1)\left(f_1(\varepsilon + \pi(w)) + D_1(\varepsilon + \pi(w))w\right)\\ &+ G_1(\varepsilon_1)\left(\frac{\partial \pi_1(w)}{\partial w}\xi(w)\right) + \dot{\zeta}_2 \end{aligned}$$

choosing $\dot{\zeta}_2$ as

$$\dot{\zeta}_2 = G_1(\varepsilon_1)\left(f_1(\varepsilon+\pi(w)) + D_1(\varepsilon+\pi(w))w\right) - G_1(\varepsilon_1)\left(\frac{\partial\pi_1(w)}{\partial w}\xi(w)\right) - B_2(\varepsilon+\pi(w))u_0,$$

with $\zeta_2(0) = -s_2(\varepsilon(0))$, then the derivative of the sliding function reduces to

$$\dot{\sigma} = \nu + d_2'(\varepsilon, w, t)$$

with $\nu = (\nu_1, \ldots, \nu_m)^T = B_2(\varepsilon+\pi(w))u_1$. Applying the super-twisting algorithm:

$$\begin{aligned} \nu_i &= -k_{1i}\sqrt{|\sigma_i|}sign(\sigma_i) + \nu_{1i} \\ \dot{\nu}_{1i} &= -k_{2i}sign(\sigma_i), \end{aligned} \tag{3.71}$$

with $\nu_1 = (\nu_{11}, \ldots, \nu_{1m})^T$ $k_1 = (k_{11}, \ldots, k_{1m})^T$ and $k_2 = (k_{21}, \ldots, k_{2m})^T$. the closed-loop sliding mode function results in

$$\begin{aligned} \dot{\sigma}_i &= -k_{1i}\sqrt{|\sigma_i|}sign(\sigma_i) + \nu_{1i} + \varrho_{1i}(\varepsilon, w, t) \\ \dot{\nu}_{1i} &= -k_{2i}sign(\sigma_i), \quad i = 1, \ldots, m, \end{aligned}$$

where $\varrho_1 = (\varrho_{11}, \ldots, \varrho_{1m})^T = d_2'(z, w, t)$. As in the general nonlinear case, it is possible to show that there exist $k_{1i} > 0$ and $k_{2i} > 0$ such that, an sliding mode occurs on the sliding manifold $\sigma = 0$ in finite time. In this case, the dynamic of the nominal sliding function is of the following form:

$$\dot{s}_2(\varepsilon) = B_2(\varepsilon+\pi(w))u_0 + \varrho_0(\varepsilon, w)$$

with

$$\varrho_0(\varepsilon, w) = -G_1(\varepsilon_1)\left(f_1(\varepsilon+\pi(w)) + D_1(\varepsilon+\pi(w))w\right) + G_1(\varepsilon_1)\left(\frac{\partial\pi_1(w)}{\partial w}\xi(w)\right)$$

Selecting the nominal control part, $u_0$ as

$$u_0 = u_{0,eq} - k_3\left[B_2(\cdot)\right]^{-1}\left(\sqrt{\|s_2\|}sign(s_2) + k_4 v_0\right) \tag{3.72}$$

where $u_{0,eq}$ is the equivalent control calculated as a solution of $\dot{s_2} = 0$ as

$$u_{0,eq} = -\left[B_2(\cdot)\right]^{-1}\varrho_0(\varepsilon, w).$$

Again, it is possible to show that there exist $k_3 > 0$ and $k_4 > 0$ such that after a finite time an sliding mode occurs on the nominal manifold $s(\varepsilon) = 0$ then the requirement $(SMS_ef)$ is fulfilled, and the $(n-m)th$ order sliding mode equation is given by

$$\begin{aligned} \dot{\varepsilon}_1 &= f_1(\varepsilon_1 + \pi_1(w), s_1(\varepsilon_1) + \pi_2(w)) \\ &+ D_1(\varepsilon_1 + \pi_1(w), s_1(\varepsilon_1) + \pi_2(w))w - \frac{\partial\pi_1(w)}{\partial w}\xi(w). \end{aligned} \tag{3.73}$$

To analyze the stability of the sliding dynamics (3.73), the systems (3.68) - (3.70) and (3.65) are represented in the form:

$$\begin{aligned} \begin{pmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} u + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} w + \begin{pmatrix} \phi_1(\varepsilon, w) \\ \phi_2(\varepsilon, w, u) \end{pmatrix} \\ \dot{w} &= Sw + \phi_w(w) \\ e &= C_1\varepsilon_1 + C_2\varepsilon_2 + (C_1\Pi_1 + C_2\Pi_2 - Q)w \\ &+ \phi_e(\varepsilon, w). \end{aligned}$$

Then, the sliding mode equation (3.73) can be rewritten as

$$\dot{\varepsilon}_1 = (A_{11} - A_{12}\Sigma_1)\varepsilon_1 + R_1 w + \phi_{1s}(\varepsilon_1, w)$$

where $R_1 = A_{11}\Pi_1 + A_{12}\Pi_2 - \Pi_1 S + D_1$ and $R_2 = A_{21}\Pi_1 + A_{22}\Pi_2 - \Pi_2 S + D_2$, with $A_{ij} = \left[\frac{\partial f_i}{\partial x_i}\right]_{(0,0)}$ $B_2 = g_2(0)$, $C_i = \left[\frac{\partial h}{\partial x_j}\right]_{(0,0)}$, $D_i = d_i(0,0)$, $\Pi_i = \left[\frac{\partial \pi_i}{\partial w}\right]_{(0)}$; the functions $\phi_1(\cdot)$, $\phi_2(\cdot)$, $\phi_w(\cdot)$, $\phi_e(\cdot)$ and $\phi_{1s}(\cdot)$ vanish at the origin with its first derivatives; and the constant matrices $S$ and $Q$ are already defined in assumption (4) and equation (3.56) respectively. Now the solvability conditions of the ISMR problem for the nonlinear system in Regular form will be presented.

**Proposition 3.9.** *Under assumptions A.6, A.4, if there exists $C^k$ $(k \geq 2)$ mappings $x_1 = \pi_1(w)$ and $x_2 = \pi_2(w)$, with $\pi_1(0) = 0$ and $\pi_2(0) = 0$, defined in neighborhood $W$ of $(0,0)$ which satisfy the following conditions:*

$$f_1(\pi_1(w), \pi_2(w)) + D_1(\pi_1(w), \pi_2(w))w = \frac{\partial \pi_1(w)}{\partial w}\xi(w) \qquad (3.74)$$

$$h(\pi_1(w), \pi_2(w)) - q(w) = 0 \qquad (3.75)$$

*then, the ISMR problem for nonlinear systems in Regular form is solvable.*

*Proof.* After sliding mode occurs, we have $\varepsilon_2 = s_1(\varepsilon_1)$, and the motion of the closed-loop system will be governed by

$$\begin{aligned} \dot{\varepsilon}_1 &= (A_{11} - A_{12}\Sigma_1)\varepsilon_1 + R_1 w + \phi_{1s}(\varepsilon_1, w) \\ \dot{w} &= Sw + \phi_w(w) \\ e &= h(\varepsilon_1 + \pi_1(w), \sigma_1(z_1) + \pi_2(w)) - q(w) \end{aligned}$$

were $\phi_{1s}(\varepsilon_1, w)$ vanishes at the origin with it first derivative. The matrix $(A_{11} + A_{12}\Sigma_1)$ is Hurwitz by a proper choice of $\Sigma_1$ and if condition (3.74) holds, then $R_1 w + \phi_{1s}(z_1, w) = f_1(\pi_1(w), \pi_2(w)) + D_1(\pi_1(w), \pi_2(w))w - \frac{\partial \pi_1(w)}{\partial w}\xi(w) = 0$. Hence, under the property of center manifolds, we have $\varepsilon_1(t) \to 0 \Rightarrow x_1(t) \to \pi_1(w(t))$, and $\varepsilon_2(t) \to 0 \Rightarrow x_2(t) \to \pi_2(w(t))$ with $t \to \infty$. Thus, the requirement $(S)$ is fulfilled. So, by continuity, if condition (3.75) holds, then the output tracking error (3.66) converges to zero and condition $(R)$ holds too. □

In the following sections, only structured nonminimum phase systems are considered.

## 3.4 SM Regulator for Nonlinear NP Systems in Structured Form with MP

This section presents an approach to solve the SM output regulation problem for a class of nonlinear nonminimum phase systems which are presented in the structured form. The structured form can be expressed in the block controllable form with internal dynamics in explicit manner. Based on decomposition block control technique and Sliding Mode control, we propose a sliding manifold that contains a steady state manifold on that the residual dynamics become asymptotically stable. To enforce the SM motion on the designed sliding manifold a super-twisting SM algorithm is used. Only the *noncausal* case for the reference is considered here. Moreover, we consider there are unknown matched perturbations (MP) $g(x,t)$ affecting the system.

This approach can be considered as an extension of the work [Bonivento et al., 2001], which solves the SMOR problem only for systems with relative degree one. The effectiveness of the proposed methodology is verified via the design of a torque tracking controller for an induction motor presented as illustrative example in Chapter 6.

Consider the perturbed nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + B(x)u + D(x)w(t) + g(x,t) \\ y &= h(x) \end{aligned} \tag{3.76}$$

which does not have a full relative degree vector and is a nonminimum phase system. The output tracking error is defined as:

$$e = y - y_{ref} \tag{3.77}$$

where $y_{ref} = q(w)$, $w$ contains known perturbations and reference signal, $w$ and is produced by the exosystem

$$\dot{w} = \xi(w). \tag{3.78}$$

Then we assume that the system (3.76) under matching condition A.5 can be represented in the following Nonlinear Block Controllable (NBC) form:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + B_1(x_1)x_2 + D_1(x_1)w(t) \\ \dot{x}_i &= f_i(\bar{x}_i) + B_i(\bar{x}_i)x_{i+1} + D_i(\bar{x}_i)w(t), \ \ i = 2, ..., r-1. \\ \dot{x}_r &= f_r(x) + B_r(x)(u+\gamma) + D_r(x)w(t) \\ \dot{x}_{r+1} &= f_{r+1}(\bar{x}_r, x_{r+1}) \\ y &= h(x) = x_1 \end{aligned} \tag{3.79}$$

which contains unmatched known perturbations $D(x)w$ and matched unknown perturbations $B(x)\gamma$ and internal dynamics $x_{r+1}$. The vector $x$ is decomposed as $x = (x_1, .., x_r, x_{r+1})^T$ $\bar{x}_j = (x_1, ..., x_j)^T$, $j = 2, ..., r$, where $r$ is the relative degree and $x_i$ is a $n_i \times 1$. The elements of $f_i(\bar{x}_i)$, $B_i(\bar{x}_i)x_{i+1}$ and $D_i(\bar{x}_i)$ are continuously differentiable functions of $(i-1)th$ order, $i = 1, ..., r$, with respect to all arguments in interval $t \in [0, \infty)$, and all the derivatives are bounded; the matrix $B_i(\cdot)$ in each block has full rank, that is

$$rank B_i = n_i \quad \forall x \in X \subset \Re^n \tag{3.80}$$

The indices $(n_1, n_2, ..., n_r)$ define the structure of the subsystem (3.79) and satisfy the following relation:

$$n_1 \le n_2 \le ... \le n_r = m, \quad \sum_{i=1}^{r+1} n_i = n. \tag{3.81}$$

The relation (3.81) means $n_i = n_{i+1}$ or $n_i < n_{i+1}$, therefore, we consider the plant with the structure $n_1 = n_2 < ... < n_r = m$, that includes both cases.

**Remark 3.10.** *For a nonminimum phase system, the internal dynamics $x_{r+1}$ are unstable.*

In the following section the problem to deal is presented.

## Robust Sliding Mode Regulation Problem

The *Robust Sliding Mode Regulation Problem* is defined as the problem of finding a sliding manifold $\sigma(x, w)$, $\sigma \in R^m$,

$$\sigma(x, w) = 0 \tag{3.82}$$

$\sigma = (\sigma_1, \dots, \sigma_m)^T$ and design a Sliding Mode controller $u = (u_1, \dots, u_m)^T$. where $\sigma(x, w)$ and $u$ are chosen to induce local asymptotic convergence of the state vector to the manifold (3.82), such that the following conditions hold:

- ($SMS$) (Sliding Mode Stability). The state of the closed-loop system (3.76)-(3.78), with the controller $u$, converges to the manifold (3.82) in a finite time,
- ($S$) The equilibrium $(x, w) = (0, 0)$, when $\sigma = 0$ of the sliding mode dynamics

$$\dot{x} = f(x) + Bu_{eq} + Dw + g(x, t)$$

is asymptotically stable in spite of the perturbation $g(x, t)$, where $u_{eq}$ is the equivalent control defined as a solution of $\dot{\sigma} = 0$,
- ($R$) There exists a neighborhood $V \subset X \times W$ of $(0, 0)$ such that, for each initial condition $(x_0, w_0) \in V$. the output tracking error (3.77) goes asymptotically to zero, i.e. $lim_{t \to \infty} e(t) = 0$.

## Block Control Linearization for Nonlinear Nonminimum Phase Systems

In order to design a sliding manifold $\sigma = 0$ on which the tracking error $e = y - y_{ref}$ (3.77) tends asymptotically to zero, we introduce the following recursive nonlinear transformation [Luk'yanov and Utkin, 1981]:

$$z_1 = e = x_1 - q(w) := \Phi_1(x_1, w) \tag{3.83}$$

$$\begin{aligned} z_2 &= f_1(x_1) + B_1(x_1)x_2 + d_1(x_1, w) + K_1(x_1 - q(w)) \\ &:= \Phi_2(x_1, x_2, w), \quad d_1 = D_1(x_1)w - (\partial q/\partial w)\xi(w) \end{aligned} \tag{3.84}$$

$$\begin{aligned} z_3 &= \tilde{B}_3(\bar{x}_2)x_3 + \begin{bmatrix} \bar{f}_2(\bar{x}_2) + d_2(\bar{x}_2, w) + K_2\Phi_2(\bar{x}_2, w) \\ 0 \end{bmatrix} \\ &:= \Phi_3(\bar{x}_3, w), \text{ con } d_2(\cdot) = \sum_{l=1}^{2} \frac{\partial \Phi_2}{\partial x_l} D_l w + \frac{\partial \Phi_2}{\partial w}\xi(w) \end{aligned} \tag{3.85}$$

$$\begin{aligned} z_{i+1} &= \tilde{B}_{i+1}(\bar{x}_i)x_{i+1} + \begin{bmatrix} \bar{f}_i(\bar{x}_i) + d_i(\bar{x}_i, w) + K_i\Phi_i(\bar{x}_i, w) \\ 0 \end{bmatrix} \\ &:= \Phi_{i+1}(\bar{x}_{i+1}, w) \end{aligned} \tag{3.86}$$

with $i = 3, 4, ..., r-1$, where $z_i$ is a vector of new variables of dimension $n_1 \times 1$, $K_i > 0$, $\bar{f}_i(\bar{x}_i) = \sum_{j=1}^{i-1}\left[\frac{\partial \Phi_i}{\partial x_j} f_j + B_j x_{j+1}\right] + \frac{\partial \Phi_i}{\partial x_i} f_i$, $\bar{B}_i = \tilde{B}_{i-1} B_i$, $\tilde{B}_{i+1} = \begin{bmatrix} \bar{B}_i \\ E_{i,1} \end{bmatrix}$, $E_{i,1} = \begin{bmatrix} 0 & I_{n_{i+1}-n_i} \end{bmatrix}$, $E_{i,1} \in R^{(n_{i+1}-n_i) \times n_{i+1}}$, $I_{n_{i+1}-n_i}$ is the indentity matrix, $d_i = \sum_{j=1}^{i}\left[\frac{\partial \Phi_i}{\partial x_j} D_j w\right] + \frac{\partial \Phi_i}{\partial w}\xi(w)$.

The system (3.79) using the recursive transformation (3.83)-(3.86) can be represented in the following form:

$$\begin{aligned} \dot{z}_1 &= -K_1 z_1 + z_2 \\ \dot{z}_i &= -K_i z_i + E_{i,1} z_{i+1}, \quad i = 2, ..., r-1 \\ \dot{z}_r &= \bar{f}_r(z) + \bar{B}_r(z)(u + \gamma) + d_r(z, w) \end{aligned} \tag{3.87}$$

$$\dot{x}_{r+1} = \bar{f}_{r+1}(z, x_{r+1}, w) \tag{3.88}$$

where $z = (z_1, ..., z_r)^T$, $\bar{f}_r(z) = \left[\sum_{j=1}^{r-1}\left[\frac{\partial \Phi_r}{\partial x_j} f_j + B_j x_{j+1}\right] + \frac{\partial \Phi_r}{\partial x_r} f_r\right]_{x=\Phi^{-1}}$ is a bounded function, $rank\ \bar{B}_r = m$ and $\bar{B}_r = \tilde{B}_{r-1} B_r$. Finally, the transformed system (3.87)-(3.88) will be used to design an advisable manifold to solve the Sliding Mode Output Regulation Problem.

At this point we have a part of the system (3.87)-(3.88) linearized by the feedback linearization. Noting that a solution of the zero dynamics

$$\dot{x}_{r+1} = \bar{f}_{r+1}(0, x_{r+1}, w)$$

is not stable.

## Sliding Manifold Design

Denoting $\eta := x_{r+1}$ and rewriting $\bar{f}_\eta(\bar{z}_{r-1}, z_r, \eta, w) := \bar{f}_{r+1}(z, \eta, w)$ where $\bar{z}_{r-1} = (z_1, ..., z_{r-1})^T$ we obtain

$$\dot{\eta} = \bar{f}_\eta(\bar{z}_{r-1}, z_r, \eta, w). \tag{3.89}$$

Considering that there exist a steady state for the internal dynamic $\eta$, in other words, we assume:

**A. 10.** *There exist a smooth mapping $\eta_{ss} = \pi_\eta(w)$ with $\pi_\eta(0) = 0$ defined in a neighborhood $W^o \in W$ of the origin which is solution to the following equation*

$$\frac{\partial \pi_\eta}{\partial w}\xi(w) = \bar{f}_\eta\left(0, \alpha_\eta(\pi_\eta, w)\right). \tag{3.90}$$

Defining $z_\eta = \eta - \pi_\eta(w)$. A linear approximation at origin of the dynamics of $z_\eta$ is

$$\dot{z_\eta} = A_\eta z_\eta + B_\eta z_r + A_{r-1}\bar{z}_{r-1} + A_\eta \pi_\eta(w) + D_\eta w - \frac{\partial \pi_\eta}{\partial w}\xi(w) + \psi_\eta(z,\eta,w) \tag{3.91}$$

where $\bar{z}_{r-1} = (z_1, ..., z_{r-1})^T$ $A_\eta = \frac{\partial}{\partial \eta}\bar{f}_\eta(0)$, $A_{r-1} = \frac{\partial}{\partial \bar{z}_{r-1}}\bar{f}_\eta(0)$, $B_\eta = \frac{\partial}{\partial z_r}\bar{f}_\eta(0)$, $D_\eta = \frac{\partial}{\partial w}\bar{f}_\eta(0)$. Then, the following assumptions are introduced,

**A. 11.** *Assume that function $\psi_\eta(\bar{z},\eta,w)$ vanish at origin with its first order derivatives and is bounded by $\psi_\eta(\bar{z},\eta,w) < \beta\|z_\eta\|$.*

**A. 12.** *The pair $(A_\eta, B_\eta)$ is controllable.*

### SM Controller Design

Considering $z_r$ as virtual control input to stabilize the residual dynamics $\eta$, the vector $z_r$ is chosen of the form

$$z_r = \alpha_\eta(\eta, w), \quad \alpha_\eta(\eta,w) = K_\eta z_\eta. \tag{3.92}$$

Then we define a sliding variable $\sigma$ in the form

$$\sigma = z_r - \alpha_\eta(\eta,w), \quad \sigma = [\sigma_1, ..., \sigma_{n-r}]^T \tag{3.93}$$

with dynamics

$$\dot{\sigma} = f_\sigma(z,\eta) + \bar{B}_r(z,\eta)u \tag{3.94}$$

where $f_\sigma(z,\eta) = \bar{f}_r(z,x_{r+1}) + \bar{B}_r(z,x_{r+1})\gamma + d_r(z,x_{r+1},w) - K_\eta \dot{z}_\eta(\eta,w)$,
$f_\sigma(z,\eta) = \begin{pmatrix} f_{\sigma 1}(z,\eta) & f_{\sigma 2}(z,\eta) & \cdots & f_{\sigma_{n-r}}(z,\eta) \end{pmatrix}$.
Now, considering that

**A. 13.** *The perturbation term $f_\sigma(z,\eta) = \bar{f}_r + d_r - \dot{\alpha}_\eta$, $f_\sigma = (f_{\sigma 1}, ...., f_{\sigma n_r})^T$ where the functions $f_{\sigma i}(z,\eta)$ in (3.94) are bounded by*

$$|f_{\sigma i}(z,\eta)| \leq \delta_i \, |\sigma_i|^{1/2} \quad \delta_i > 0, i = 1, ..., n_r. \tag{3.95}$$

To achieve sliding mode motion on the manifold $\sigma = 0$ we use the super-twisting SM control [Fridman and Levant, 2002]:

$$\begin{aligned} u &= \bar{B}_r^{-1}(z,\eta)v, \quad v = [v_1, ..., v_{n-r}]^T \\ v_i &= -k_{i1}\,|\sigma_i|^{1/2}\, sign(\sigma_i) + v_{i1} \\ \dot{v}_{i1} &= -k_{i2}\, sign(\sigma_i), \quad i = 1, ..., n-r. \end{aligned} \tag{3.96}$$

## Stability of the SM equation

**Theorem 3.11.** *Consider system (3.87) with control (3.96). If the assumptions A.10, A.13 and the following conditions given in (Moreno et al., 2008) hold:*

$$k_{i1} > 2\delta_i \qquad k_{i2} > k_{i1}\frac{5\delta_i k_{i1} + 4\delta_1^2}{2\left(k_{i1} - 2\delta_1\right)} \tag{3.97}$$

*then the overall system state converges to a sliding manifold where the output tracking error* $z_1$ *asymptotically tends to zero.*

*Proof.* The closed loop system (3.94)-(3.96) is:

$$\begin{aligned} \dot{\sigma}_i &= f_{\sigma i}(z, x_{r+1}) - k_{i1}\left|\sigma_i\right|^{1/2} sign(\sigma_i) + v_{i1}, \\ \dot{v}_{i1} &= -k_{i2} sign(\sigma_i), \quad i = 1, ..., n_r. \end{aligned} \tag{3.98}$$

Under condition (3.97) the overall system state converges to the manifold $\sigma = 0$ and $z_r = -K_\eta z_\eta$ in a finite time. On the manifold $\sigma = 0$ and under assumption A.10 the SM dynamics are governed by the reduced order system

$$\begin{aligned} \dot{z}_1 &= -K_1 z_1 + z_2 \\ \dot{z}_i &= -K_i z_i + E_{i,1} z_{i+1}, \quad i = 2, ..., r-2 \\ \dot{z}_{r-1} &= -K_{r-1} z_{r-1} - K_\eta z_\eta \end{aligned} \tag{3.99}$$

$$\dot{z}_\eta = (A_\eta + B_\eta K_\eta) z_\eta + A_{r-1}\bar{z}_{r-1} + \psi_1(z, w) \tag{3.100}$$

or rewritten in the form:

$$\dot{\bar{z}} = \bar{A}\bar{z} + \psi_{\bar{z}} \tag{3.101}$$

where $\bar{z} = (\bar{z}_{r-1}, z_\eta)^T$, and

$$\bar{A} = \begin{pmatrix} -K_1 & I_{n_1} & 0 & \dots & & 0 \\ 0 & -K_2 & I_{n_2} & 0 & \dots & \vdots \\ \vdots & 0 & & & 0 & \\ & \vdots & & & I_{n_{R-2}} & 0 \\ 0 & 0 & \dots & 0 & -K_{r-1} & -K_\eta \\ \bar{A}_1 & \bar{A}_2 & \dots & \bar{A}_{r-2} & \bar{A}_{r-1} & (A_\eta - B_\eta K_\eta) \end{pmatrix}$$

Note that we can choose the gains $K_i$, $i = 1, 2, ..., r-1, \eta$, such that $\bar{A}$ is Hurwitz. The perturbation term $\psi_{\bar{z}}$ only contains the term $\psi_1$, then considering the constraint A.11 results that

$$\|\psi_{\bar{z}}\| < \beta_1 \|z\| \tag{3.102}$$

for all $t \geq 0$ and all $\bar{z} \in \Re^{n-1}$ with $\beta_1 > 0$ constant.

Let $Q = Q^T > 0$ and solving the Lyapunov equation $P\bar{A} + \bar{A}^T P = -Q$ for P. Then, consider a candidate Lyapunov function in the form:

$$V(\bar{z}) = \bar{z}^T P \bar{z} \tag{3.103}$$

taking the derivatives of $V(\bar{z})$ along the trajectories of (3.101) results

$$\dot{V}(z) = -z^T Q z + 2z^T P \tag{3.104}$$

$$\leq -\lambda_{min}(Q)\|\bar{z}\|_2^2 + 2\lambda_{max}(P)\beta_1\|\bar{z}\|_2 \tag{3.105}$$

Now, as the perturbation term is vanishing bounded, and the matrix $\bar{A}$ is Hurwitz because can be modified through the gains $K_i$, $i = 1, 2, ..., r-1, \eta$, then there exist a matrix $Q$ such that

$$\beta_1 < \lambda_{min}(Q)/(2\lambda_{max}(P)) \tag{3.106}$$

[Khalil, 1996], where $\lambda(N)_{min/max}$ denotes the *minimum/maximum* eigenvalue of $N$. Under the last stated conditions and (3.106), the origin $z = 0$ is semiglobally stable. □

As an illustrative example a Second Order Sliding Mode Sensorless Torque Regulator for Induction Motor is presented in section 6.

## 3.5 SM Regulator for Nonlinear NP Systems in Structured Form with UP

In Section 3.4 it was presented a solution for SM Output Regulation problem for systems with matched perturbations. In this section we extend that result and present a robust sliding mode controller for nonlinear nonminimum phase systems subject to both unknown matched and unmatched perturbations (UP). Based on Regulation and Lyapunov Redesign theories we design a suitable sliding manifold. On this manifold the perturbed dynamics are stable namely they remain bounded whereas the output tracking error is bounded in spite of the presence of external perturbations. To enforce the SM motion on the designed manifold a super-twisting SM controller is used. The effectiveness of the proposed methodology is verified via a simple example in the section 6.

Consider the perturbed nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + B(x)u + D(x)w(t) + g(x,t) \\ y &= h(x) \end{aligned} \tag{3.107}$$

which does not have a full relative degree vector and is a nonminimum phase system. The output tracking error is defined as:

$$e = y - q(w) \tag{3.108}$$

where $w$ contains known perturbations and/or reference signal and is produced by the exosystem

$$\dot{w} = \xi(w). \tag{3.109}$$

In this section it is considered a class of nonlinear systems (3.107) which can be presented in the following perturbed nonlinear block controllable form (NBC-form)

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + B_1(x_1)x_2 + D_1(x_1)w + g_1(x_1,t) \\ \dot{x}_i &= f_i(\bar{x}_i) + B_i(\bar{x}_i)x_{i+1} + D_i(\bar{x}_i)(w) + g_i(\bar{x}_i,t) \qquad (3.110) \\ \dot{x}_r &= f_r(x) + B_r(x)u + D_r(x)(w) + g_r(x,t) \\ \dot{x}_{r+1} &= f_{r+1}(x) + D_{r+1}(x)w + g_{r+1}(x,t) \qquad (3.111) \\ y &= h(x) = x_1, \qquad i = 2,...,r-1. \end{aligned}$$

where the vector $x$ is decomposed as $x = (x_1,..,x_r,x_{r+1})^T$, $\bar{x}_j = (x_1,...,x_j)^T$, $j = 2,...,r$, and $x_i$ is a $n_i \times 1$. The terms $g_j(\bar{x}_j,t)$, $j = 1,...,r+1$, are bounded unknown perturbations. The elements of $f_i(\bar{x}_i)$, $B_i(\bar{x}_i)x_{i+1}$ and $D_i(\bar{x}_i)$ are continuously differentiable functions of $(i-1)th$ order, $i = 1, ..., r$, with respect to all arguments in interval $t \in [0, \infty)$, and all the derivatives are bounded; the matrix $B_i(\cdot)$ in each block has full rank, that is

$$rankB_i = n_i \quad \forall x \in X \subset \Re^n \tag{3.112}$$

The indices $(n_1, n_2, ..., n_r)$ define the structure of the subsystem (3.110) and satisfy the following relation:

$$n_1 \leq n_2 \leq ... \leq n_r = m, \quad \sum_{i=1}^{r+1} n_i = n. \tag{3.113}$$

The relation (3.113) means $n_i = n_{i+1}$ or $n_i < n_{i+1}$, therefore, we consider the plant with the structure $n_1 = n_2 < ... < n_r = m$, that includes both cases.

The control objective is to ensure ultimately bounded output tracking error $e$ (3.108) in spite of both, unknown matched and unmatched perturbations.

The used procedure to obtain a Sliding Mode Regulator to achieve the control objective is:

1. First, the nominal part of system (3.107) presented in NBC-form (3.110) is linearized applying the block control technique [Loukianov, 1998] combined with Lyapunov redesign approach [Khalil, 1996]; as result a standard sliding function $s(x, w)$ is obtained.

2. Secondly, using the designed sliding variable $s$ as a virtual control input in the residual dynamics (3.111), a stabilizing control law for this subsystem is designed applying a Lyapunov redesign approach;

3. Finally, a sliding manifold $\sigma(x, w) = 0$ is formulated, on that the output tracking error (3.108) tends to a neighborhood $V \subset X \times W$ of zero in spite of the presence of unmatched perturbations. Then, the SM super-twisting control algorithm is implemented to ensure the designed manifold be attractive.

## Block Linearizing Transformation with Lyapunov Redesign

In this section we present the linearization of the system. To simplify the notation we omit the arguments of some functions when no confusion arises.

We define the first variable $z_1$ as

$$z_1 = e = x_1 - q(w) := \Phi_1(x_1, w) \tag{3.114}$$

the dynamics of $z_1$ are then given by

$$\dot{z}_1 = \bar{f}_1(x_1, w) + B_1(x_1)x_2 + g_1(x_1, t)$$

where $\bar{f}_1(x_1, w) = f_1(x_1) + D_1(x)w - \frac{\partial q(w)}{\partial w}\xi(w)$ and $x_2$ is considered as a virtual control for $z_1$. Then, we can impose the desired dynamics $(K_1 z_1 - \rho_1 sigm(s_1))$ for this block considering the desired value $x_{2,d}$ for $x_2$ in the form

$$\begin{aligned} x_{2,d} &= B_1^+(x_1)\left(-\bar{f}_1(x_1, w) + K_1 z_1 - \rho_1 sigm(s_1)\right) \\ s_1 &= P_1 z_1 \end{aligned} \tag{3.115}$$

where $\rho_i = diag[\rho_{i1}, ..., \rho_{in_i}]$, $\rho_{i_1} > 0$, $sigm(s_i) = [sigm(s_{i1}), ..., sigm(s_{in_i})]^T$ for $i = 1, 2, ..., r-1$, $B_1^+ = (B_1^T B_1)^{-1} B_1^T$ is the right pseudo-inverse matrix of $B_1$, $B_1(z_1, w) = B_1(x_1)\Big|_{x_1 = \Phi_1^{-1}}$, the matrix $K_1$ is a Hurwitz , $P_1$ is a positive defined solution of the Lyapunov equation

$$P_1 K_1 + K_1^T P_1 = -Q_1,\ Q_1 > 0 \tag{3.116}$$

and $sigm(s_{ij}) := sigm(\epsilon, s_{ij})$ for some real number $\epsilon$, $i = 1, 2, ..., r-1$ and $j = 1, 2, ..., n_i$. Where the continuously differentiable sigmoid function $sigm(\epsilon, s_{ij})$ can approximate to the sign function $sign(s_{ij})$, in particular the sigmoid function used in this work is $sigm(\epsilon, s_{ij}) = tanh(\epsilon s_{ij})$, $\epsilon$ defines the sigmoid function slope near to zero.

Now, defining

$$z_2 = x_2 - x_{2,d}(x_1, w) := \Phi_2(\bar{x}_2, w) \tag{3.117}$$

the first block of (3.110) is then represented in the new variables $z_1$ and $z_2$ as

$$\dot{z}_1 = K_1 z_1 - \rho_1 sigm(s_1) + B_1(z_1, w)z_2 + g_1(z_1, w, t)$$

On the second step, using (3.110), (3.115) and (3.117), the dynamics for $z_2$ are derived of the form

$$\dot{z}_2 = \bar{f}_2(\bar{x}_2, w) + B_2(\bar{x}_2)x_3 + \bar{g}_2(\bar{z}_2, w, t) \tag{3.118}$$

where $\bar{z}_2 = (z_1, z_2)^T$ $\bar{f}_2(\bar{x}_2, w) = f_2(\bar{x}_2) + D_2(\bar{x}_2)w$,

$$\bar{g}_2(\bar{z}_2, w, t) = [g_2(\bar{x}_2, t) - \dot{x}_{2,d}(x_1, w)]_{x_1 = \Phi_1^{-1},\ x_2 = \Phi_2^{-1}}$$

and again, we regard $x_3$ as a virtual control for $z_2$. Thus, choosing the desired dynamics for $z_2$ (similar to the first step) as $(K_2 z_2 - \rho_2 sigm(s_2))$, the desired value $x_{3,d}$ for $x_3$ is asigned similar to (3.115) as:

$$\begin{aligned} x_{3,d} &= B_2^+(\bar{x}_2)\left(-\bar{f}_2(\bar{x}_2, w) + K_2 z_2 - \rho_2 sigm(s_2)\right) \\ s_2 &= P_2 z_2 \end{aligned}$$

where $B_2(\bar{z}_2, w) = B_2(\bar{x}_2)\Big|_{x_1=\Phi_1^{-1},\ x_2=\Phi_2^{-1}}$, $K_2$ is a Hurwitz matrix, $\rho_2 > 0$ and the matrix $P_2$ is a positive defined solution of the Lyapunov equation

$$P_2K_2 + K_2^TP_2 = -Q_2,\ Q_2 > 0 \tag{3.119}$$

Defining

$$z_3 = x_3 - x_{3,d}(\bar{x}_2, w) := \Phi_3(\bar{x}_3, w) \tag{3.120}$$

the second transformed block is obtained in the new variables $z_1$, $z_2$ and $z_3$ of the form

$$\dot{z}_2 = K_2z_2 - \rho_2 sigm(s_2) + B_2(\bar{z}_2, w)z_3 + \bar{g}_2(\bar{z}_2, w, t)$$

This procedure is iterated considering the $i^{th}$ new variable

$$z_i = x_i - x_{i,d} := \Phi_i(\bar{x}_i, w),\ \ i = 3, \dots, r-1, \tag{3.121}$$

and the $i^{th}$ dynamics of the form

$$\dot{z}_i = \bar{f}_i(\cdot) + B_i(\cdot)x_{i+1} + \bar{g}_i(\bar{z}_i, w, t) \tag{3.122}$$

with $\bar{g}_i(\bar{z}_i, w, t) = [g_i(\bar{x}_i, t) - \dot{x}_{i,d}(x_{i-1}, w)]_{x_1=\Phi_1^{-1}, \dots, x_i=\Phi_i^{-1}}$, $\bar{f}_i(\cdot) = f_i(\bar{x}_i) + D_i(\bar{x}_i)w$.

Then, the desired value $x_{i+1,d}$ for $x_{i+1}$ is chosen as

$$\begin{aligned} x_{i+1,d} &= B_i^+(\cdot)\left(-\bar{f}_i(\cdot) + K_iz_i - \rho_i sigm(s_i)\right) \\ s_i &= P_iz_i \end{aligned}$$

where $K_i$ is a Hurwitz matrix, $\rho_i > 0$ and the matrix $P_i$ is a positive defined solution of the Lyapunov equation

$$P_iK_i + K_i^TP_i = -Q_i,\ Q_i > 0 \tag{3.123}$$

Defining

$$z_{i+1} = x_{i+1} - x_{i+1,d} \tag{3.124}$$

we obtain the $i^{th}$ dynamics of $z_i$ in the form

$$\dot{z}_i = K_iz_i - \rho_i sigm(s_i) + B_2(\bar{z}_i, w)z_{i+1} + \bar{g}_i(\bar{z}_i, w, t)$$

and using the transformation (3.114), (3.117) and (3.121) the system (3.110)-(3.111) in the new variables $z = (z_1, ..., z_r)^T$, can be represented of the form

$$\begin{aligned}
\dot{z}_1 &= K_1 z_1 - \rho_1 sigm(s_1) + B_1(z_1, w) z_2 + \bar{g}_1(z_1, w, t) \\
\dot{z}_i &= K_i z_i - \rho_i sigm(s_i) + B_i(\bar{z}_i, w) z_{i+1} + \bar{g}_i(\bar{z}_i, w, t) \qquad (3.125) \\
i &= 2, ..., r-1, \\
\dot{z}_r &= \bar{f}_r(z, x_{r+1}, w) + B_r(z, x_{r+1}, w) u + \bar{g}_r(z, x_{r+1}, w, t) \\
\dot{x}_{r+1} &= f_{r+1}(z, x_{r+1}, w) + g_{r+1}(z, x_{r+1}, t) \qquad (3.126)
\end{aligned}$$

where $\bar{z}_j = (z_1, ..., z_j)^T, j = 2, ..., r-1$,

The natural choice of a sliding variable is $s = z_r$. However, a sliding motion on the manifold $s = 0$ is not stable since the original system (3.110)-(3.111) is a nonminimum phase system.

## Sliding Manifold Design

To stabilize the internal dynamics, the subsystem (3.126) is first represented as

$$\eta := x_{r+1}, \quad f_\eta(\bar{z}_{r-1}, z_r, \eta, w) := f_{r+1}(z, x_{r+1}, w), \quad g_\eta(z, \eta, t) := g_{r+1}(z, x_{r+1}, t)$$

we have $\dot{\eta} = f_\eta(\bar{z}_{r-1}, z_r, \eta, w) + g_\eta(z, \eta, t)$, now we obtain a linear approximation of the last system $\dot{\eta}$ at origin as

$$\dot{\eta} = A_\eta \eta + B_\eta z_r + D_\eta w + \psi_1(\bar{z}_r, \eta, w) + g_\eta(z, \eta, t) \qquad (3.127)$$

where $A_\eta = \frac{\partial}{\partial \eta} f_\eta(0)$, $B_\eta = \frac{\partial}{\partial z_r} f_\eta(0)$, $D_\eta = \frac{\partial}{\partial w} f_\eta(0)$, $\psi_1(0,0,0) = 0$ and $g_\eta(z, \eta, t) = g_{r+1}(z, x_{r+1}, t)$.

Considering $z_r$ as virtual control in (3.127), the state feedback control is chosen as

$$z_r = \alpha_\eta(\eta, w)$$

where $\alpha_\eta(\eta, w)$ is smooth mapping defined on $X^{n_{r+1}} \times W$ and $\alpha_\eta(0,0) = 0$. The the following assumptions are needed:

**A. 14.** *The pair $(A_\eta, B_\eta)$ is controllable.*

**A. 15.** *There is a vector $\gamma_\eta(\cdot)$ such that [Drazenovich, 1969]*

$$g_\eta(z,\eta,t) = B_\eta \gamma_\eta(z,\eta,t)$$

and $\|\gamma_\eta(z,\eta,t)\| \leq c_1$, $c_1 > 0$.

**A. 16.** *There exist a smooth mapping $\eta_{ss} = \pi_\eta(w)$ with $\pi_\eta(0) = 0$ defined in a neighborhood $W_0 \subset W$ of the origin that satisfied the following equation [Isidori and Byrnes, 1990]:*

$$\frac{\partial \pi_\eta}{\partial w}\xi(w) = f_\eta(0, \alpha_\eta(\pi_\eta(w), w), \pi_\eta(w), w)) \tag{3.128}$$

In this way, we propose:

$$\begin{aligned} z_r &= \alpha_\eta(\eta, w) && (3.129)\\ \alpha_\eta(\eta, w) &= C_\eta\left(\eta - \pi_\eta(w)\right) - \rho_\eta sigm(s_\eta) \\ s_\eta &= P_\eta z_\eta && (3.130)\end{aligned}$$

where $z_\eta := \eta - \pi_\eta(w)$, $C_\eta$ is such that $K_\eta$=$(A_\eta + B_\eta C_\eta)$ is Hurwitz $\rho_\eta > 0$ and the matrix $P_\eta$ is a positive defined solution of the Lyapunov equation

$$P_\eta K_\eta + K_\eta^T P_\eta = -Q_\eta,\ Q_\eta > 0 \tag{3.131}$$

In order to accomplish the desired value (3.129) for $z_r$ we propose a sliding variable in the form:

$$\sigma = z_r - \alpha_\eta(\eta, w), \quad \sigma = [\sigma_1, ..., \sigma_{n_r}]^T \tag{3.132}$$

taking the derivative of (3.132) considering dynamics (3.125), thus we have

$$\dot{\sigma} = \bar{f}_r(\cdot) + B_r(\cdot)u + \bar{g}_r(\cdot) - \dot{\alpha}_\eta(\cdot). \tag{3.133}$$

To induce chattering-reduced sliding mode on $\sigma = 0$ we use the super-twisting algorithm [Levant, 2001]

$$\begin{aligned} u &= B_r^+(z,\eta)v, \quad v = [v_1, ..., v_{n_r}]^T \\ v_i &= -k_{i1}\left|\sigma_i\right|^{1/2} sign(\sigma_i) + v_{i1} && (3.134)\\ \dot{v}_{i1} &= -k_{i2}\, sign(\sigma_i), \quad i = 1, ..., n_r, \end{aligned}$$

the closed loop system (3.133)-(3.134) becomes

$$\begin{aligned} \dot{\sigma}_i &= f_{\sigma i}(z, x_{r+1}) - k_{i1}\left|\sigma_i\right|^{1/2} sign(\sigma_i) + v_{i1} \\ \dot{v}_{i1} &= -k_{i2}\, sign(\sigma_i), \quad i = 1, ..., n_r, && (3.135)\end{aligned}$$

where $f_\sigma = \bar{f}_r + \bar{g}_r - \dot{\alpha}_\eta$ , $f_\sigma = (f_{\sigma 1}, ...., f_{\sigma n_r})^T$

We assume that

$$|f_{\sigma i}(z, x_{r+1})| \leq \delta_i |\sigma_i|^{\frac{1}{2}} \quad \delta_i > 0 \tag{3.136}$$

then, if equation (3.136) and the following conditions [Moreno and Osorio, 2008] hold

$$k_{i1} > 2\delta_i \qquad k_{i2} > k_{i1}\frac{5\delta_i k_{i1} + 4\delta_1^2}{2(k_{i1} - 2\delta_1)} \tag{3.137}$$

then the state of the system (3.133) converges to the manifold $\sigma = 0$ in a finite time.

## SM Dynamics

On the manifold $\sigma = 0$ the SM dynamics are governed by equations

$$\begin{aligned} \dot{z}_1 &= K_1 z_1 - \rho_1 sigm(s_1) + B_1(z_1, w) z_2 + \bar{g}_1(\bar{z}_i, w, t) \\ \dot{z}_i &= K_i z_i - \rho_i sigm(s_i) + B_i(\bar{z}_i, w) z_{i+1} \bar{g}_i(\bar{z}_i, w, t) \\ & \quad i = 1, ..., r-1 \\ \dot{z}_\eta &= K_\eta z_\eta - B_\eta \rho_\eta sigm(s_\eta) + g_\eta(\bar{z}_i, w, t) \end{aligned} \tag{3.138}$$

which is the sliding mode equation considering the perturbation terms and the steady state error $z_\eta$. In order to make an analysis, we consider that each term $\varphi_i = -\rho_i sigm(s_i)$, $i = 1, 2, ..., r-1, \eta$, has two stages: a discontinuous one and a linear one. This is represented as

$$\varphi_i = \begin{cases} -\rho_i \left(\frac{s_i}{\|s_i\|_2}\right) & if \ \ \rho_i \|s_i\|_2 \geq \mu_i \\ -\rho_i^2 (s_i/\mu_i) & if \ \ \rho_i \|s_i\|_2 < \mu_i \end{cases} \tag{3.139}$$

for some $\mu_i > 0$.

Now we propose a candidate Lyapunov function $V = \sum V_i$ where $V_i = z_i^T P_i z_i$, $i = 1, 2..., r-1, \eta$. In the first case (3.139), considering that $\rho_i \|s_i\|_2 \geq \mu_i$, the derivative of each $V_i$ is in the form:

$$\begin{aligned} &\dot{V}_i = -z_i^T Q_i z_i - 2\rho_i z_i^T P_i \tfrac{P_i z_i}{\|P_i z_i\|} + 2 z_i^T P_i g_i + 2 z_i^T P_i B_i z_{i+1} \\ &\leq -z_i^T Q_i z_i - 2\|\rho_i\| \|z_i^T P_i\| \tfrac{P_i z_i}{\|P_i z_i\|} + 2\|z_i^T P_i\| \|g_i\| + 2\|z_i^T P_i B_i z_{i+1}\| \end{aligned}$$

we consider the following assumption

**A. 17.** *The perturbation terms are bounded* $\|g_i(z, t)\| \leq \lambda_i(z, t)$, $i = 1, 2..., r-1, \eta$.

Considering the inequality $\|z_i^T P_i B_i z_{i+1}\|_2 \leq \beta_{i,1}\|z_i\|_2^2 + \beta_{i+1,2}\|z_{i+1}\|_2^2$ with $\beta_{i,1}, \beta_{i+1,2} > 0$, then we have

$$\begin{aligned} \dot{V}_i \;\leq\; & -\left[\lambda_{min}(Q_i) - 2\beta_{i,1}\right]\|z_i\|^2 - 2\|z_i^T P_i\|\left[\|\rho_i\| - \lambda_i\right] + 2\beta_{i+1,2}\|z_{i+1}\|^2 \\ & i = 1, 2 ..., r-1. \end{aligned} \tag{3.140}$$

Note that the third term $2\beta_{i+1,2}\|z_{i+1}\|^2$ can be associated with the terms of function $\dot{V}_{i+1}$, $i = 1, 2, ..., r-2$, then we have

$$\begin{aligned} \dot{V} \;\leq\; & -\sum_{1}^{r-1}\left[\lambda_{min}(Q_i) - 2\left(\beta_{i,1} + \beta_{i,2}\right)\right]\|z_i\|^2 - 2\sum_{1}^{r-1}\|z_i^T P_i\|\left(\|\rho_i\| - \lambda_i\right) \\ & + 2\beta_{r,2}\|z_\eta\|^2 + \dot{V}_\eta \end{aligned} \tag{3.141}$$

with $\beta_{1,2} = 0$, and the derivative of $V_\eta$ is

$$\begin{aligned} \dot{V}_\eta \;=\; & -z_\eta^T Q_\eta z_\eta - 2\rho_\eta z_\eta^T P_\eta \frac{P_\eta z_\eta}{\|P_\eta z_\eta\|} + 2z_\eta^T P_\eta\left[g_\eta(z,\eta,t) + \psi_1(\bar{z}_r,\eta,w)\right] \\ \leq\; & -\lambda_{min}(Q_\eta)\|z_\eta\|^2 - 2\|z_\eta^T P_\eta\|\left[\|\rho_\eta\| - \tilde{\lambda}_\eta\right] \end{aligned} \tag{3.142}$$

with $\tilde{\lambda}_\eta = \lambda_\eta + \lambda_\psi$, $\|\psi_1\| \leq \lambda_\psi$, then (3.141) is

$$\begin{aligned} \dot{V} \;\leq\; & -\sum_{1}^{r-1}\left[\lambda_{min}(Q_i) - 2\left(\beta_{i,1} + \beta_{i,2}\right)\right]\|z_i\|^2 - 2\sum_{1}^{r-1}\|z_i^T P_i\|\left[\rho_i - \lambda_i\right] - \|z_\eta^T P_\eta\|\left[3\|\rho_\eta\| - 2\tilde{\lambda}_\eta\right] \\ - \; & \left[\lambda_{min}(Q_\eta) - 2\beta_{r,2}\|C_\eta\|\right]\|z_\eta\|^2 \end{aligned} \tag{3.143}$$

then, the derivative of $\dot{V}$ is negative if $\lambda_{min}(Q_i) > (\beta_{i,1} + \beta_{i,2})$, $\rho_i > \lambda_i$ for $i = 1, 2, ..., r-1$, $\lambda_{min}(Q_\eta) > 2\beta_{r,2}\|C_\eta\|$ and $3\rho_\eta > 2\tilde{\lambda}_\eta$.

For the second case (3.139) when $\rho_i\|s_i\| < \mu_i$, we have

$$\begin{aligned} \dot{V} \;\leq\; & -\sum_{1}^{r-1}\left[\lambda_{min}(Q_i) - 2\left(\beta_{i,1} + \beta_{i,2}\right)\right]\|z_i\|^2 + 2\sum_{1}^{r-1}\|z_i^T P_i\|\left[(-\rho_i^2/\mu_i)\|z_i^T P_i\| + \lambda_i\right] \\ & + 2\beta_{r,2} z_r + \dot{V}_\eta \end{aligned}$$

and $\dot{V}_\eta$ is

$$\begin{aligned} \dot{V}_\eta \;=\; & -z_\eta^T Q_\eta z_\eta + 2z_\eta^T P_\eta(-\rho_\eta^2/\mu_\eta)\|z_\eta^T P_\eta\| + 2z_\eta^T P_\eta\left[g_\eta(z,\eta,t) + \psi_1(\bar{z}_r,\eta,w)\right] \\ \leq\; & -\lambda_{min}(Q_\eta)\|z_\eta\|^2 - 2\|z_\eta^T P_\eta\|\left[\left(\rho_\eta^2/\mu_\eta\right)\|z_\eta^T P_\eta\| - \tilde{\lambda}_\eta\right] \end{aligned}$$

substituting $\dot{V}_\eta$ in $\dot{V}$ we have

$$\begin{aligned}\dot{V} \leq & -\sum_1^{r-1}\left[\lambda_{min}(Q_i) - 2\left(\beta_{i,1}+\beta_{i,2}\right)\right]\|z_i\|^2 + 2\sum_1^{r-1}\|z_iP_i\|\left[(-\rho_i^2/\mu_i)\|z_iP_i\| + \lambda_i\right] \\ & -\left[\lambda_{min}(Q_\eta) - 2\beta_{r,2}\|C_\eta\|\right]\|z_\eta\|^2 - 3\|z_\eta^TP_\eta\|\left[\left(\rho_\eta^2/\mu_\eta\right)\|z_\eta^TP_\eta\| - \tilde{\lambda}_\eta\right] \end{aligned} \quad (3.144)$$

then, on a region where $\left(\|z_i^TP_i\|\right) > \left(\lambda_i\mu_i/\rho_i^2\right)$, $i = 1, 2, ..., r-1, \eta$, the derivative $\dot{V}$ is defined negative if $\lambda_{min}(Q_i) > (\beta_{i,1} + \beta_{i,2})$, for $i = 1, 2, ..., r-1$, and $\lambda_{min}(Q_\eta) > 2\beta_{r,2}\|C_\eta\|$. Then the system (3.138) is ultimately bounded in presence of matched and unmatched perturbations [Khalil, 2002].

To show the performance of this regulator some examples are shown in Section 6 for both cases matched and unmatched perturbation. In order to show the performance of this regulator, we use non vanishing functions of the time in the simulations too, these perturbations were not considered in the theoretical analysis.

# Chapter 4

# SM Output Regulation Causal Case

In this section we present the SM Output Regulation for structured Nonminimum Phase Systems, the main difference between those SM regulators presented in Chapter 3 for *noncausal case* and these here presented for the *causal case* is that here we consider more general reference signals and perturbations affecting the system, and we do not consider to have a known exosystem, we deal with reference signals for the output and perturbations with arbitrary sinusoidal form. In this case a solution of the Francis-Isidori-Byrnes (FIB) equation is not computed instead that we use two approaches and the adaptive estimator (see Section 2.5) to obtain a bounded solution for the unstable internal dynamics of nonminimum phase systems and the derivatives of the reference signal. Moreover, we deal with both unmatched and matched unknown perturbations affecting the nonlinear nonminimum phase system.

## 4.1 Problem Statement

Consider the perturbed nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + B(x)u + g(x,t) \\ y &= h(x) \end{aligned} \tag{4.1}$$

where $x \in X \subset R^n$ is the state vector, $u \in U \subset R^m$ is the control vector, $y \in V \subset R^p$ is the output vector. The vector field $f(x)$ and the columns of $B(x)$ are smooth and bounded mappings. The vector $g(x,t)$ presents unknown disturbances.

Now, we consider systems (4.1) which can be expressed in the perturbed block controllable form:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1) + B_1(x_1)x_2 \\
\dot{x}_i &= f_i(\bar{x}_i) + B_i(\bar{x}_i)x_{i+1}, \ i = 2, ..., r-1. \\
\dot{x}_r &= f_r(x) + B_r(x)u + g_r(x,t) \qquad (4.2) \\
\dot{x}_{r+1} &= f_{r+1}(\bar{x}_r, x_{r+1}) + g_{r+1}(t) \qquad (4.3) \\
y &= h(x) = x_1
\end{aligned}
$$

where the vector $x$ is decomposed as $x = (x_1, .., x_r, x_{r+1})^T$, $\bar{x}_j = (x_1, ..., x_j)^T$, $j = 2, ..., r$, and $x_i$ is a $n_i \times 1$ vector, $g_r(x,t) = B\lambda_s$ is a matched external perturbation. For the external unknown perturbation $g_{r+1}(t)$ we consider the following assumption:

**A. 18.** *The unmatched perturbation $g_{r+1}(t)$ has sinusoidal shape, i.e.*

$$g_{r+1}(t) = A_g sin(\alpha_g t + \phi_g).$$

The output tracking error is defined as

$$z_1 = y - y_{ref}(t) \qquad (4.4)$$

where $y_{ref}(t) = Asin(\alpha t + \phi)$ is an arbitrary sinusoidal reference signal with $A, \alpha, \phi \in R$.

We define the **SM Output Regulation Problem** for *causal* case as the problem of finding a smooth sliding surface $\sigma$, $\sigma \in \Re^m$ and a controller $u$ to render attractive the manifold $\sigma = 0$ such that

- (SMS) The state of the closed-loop system (3.76)-(3.78), with the controller $u$, converges to the manifold (3.82) in a finite time,
- ($S$) The equilibrium $x = 0$, when $\sigma = 0$ of the sliding mode dynamics

  $$\dot{x} = f(x) + Bu_{eq} + g(x,t)$$

  is stable in spite of the perturbation $g(x,t)$, where $u_{eq}$ is the equivalent control defined as a solution of $\dot{\sigma} = 0$,
- ($R$) There exists a neighborhood $V \subset X$ of the origin such that, for each initial condition $(x_0) \in V$, the output tracking error $z_1 = y - y_{ref}(t)$ goes asymptotically to zero, i.e. $lim_{t \to \infty} z_1(t) = 0$.

### Block Control Linearization for Nonlinear Nonminimum Phase Systems

To design a sliding manifold $\sigma = 0$ we introduce the following recursive nonlinear transformation [Luk'yanov and Utkin, 1981]:

$$z_1 = e = x_1 - y_{ref} := \Phi_1(x_1, t) \tag{4.5}$$

$$z_2 = f_1(x_1) + B_1(x_1)x_2 - d_1(x_1, t) + K_1(x_1 - y_{ref}) \tag{4.6}$$

$$:= \Phi_2(x_1, x_2, t), \quad d_1 = \dot{y}_{ref}$$

$$z_3 = \bar{B}_3(\bar{x}_2)x_3 + \begin{bmatrix} \bar{f}_2(\bar{x}_2) - d_2(\bar{x}_2) + K_2\Phi_2(\bar{x}_2, t) \\ 0 \end{bmatrix}$$

$$:= \Phi_3(\bar{x}_3, t), \text{ con } d_2(\cdot) = \frac{d^2 y_{ref}}{dt^2} \tag{4.7}$$

$$z_{i+1} = \bar{B}_{i+1}(\bar{x}_i)x_{i+1} + \begin{bmatrix} \bar{f}_i(\bar{x}_i) + d_i(\bar{x}_i) + K_i\Phi_i(\bar{x}_i) \\ 0 \end{bmatrix} \tag{4.8}$$

$$:= \Phi_{i+1}(\bar{x}_{i+1})$$

with $i = 3, 4, ..., r-1$, where $z_i$ is a vector of new variables of dimension $n_1 \times 1$, $K_i > 0$, $\bar{f}_i(\bar{x}_i) = \sum_{j=1}^{i-1} \left[\frac{\partial \Phi_i}{\partial x_j} f_j + B_j x_{j+1}\right] + \frac{\partial \Phi_i}{\partial x_i} f_i$, $\bar{B}_i = \bar{B}_{i-1}B_i$, $\bar{B}_{i+1} = \begin{bmatrix} \bar{B}_i \\ E_{i,1} \end{bmatrix}$, $E_{i,1} = \begin{bmatrix} 0 & I_{n_{i+1}-n_i} \end{bmatrix}$, $E_{i,1} \in R^{(n_{i+1}-n_i) \times n_{i+1}}$, $I_{n_{i+1}-n_i}$ is the indentity matrix, $d_i = \frac{d^i y_{ref}}{dt^i}$.

According to the block control nonlinear transformation the system (4.2)-(4.3) can be rewritten as:

$$\begin{aligned} \dot{z}_1 &= -K_1 z_1 + z_2 \\ \dot{z}_i &= -K_i z_i + E_{i,1} z_{i+1}, \quad i = 2, ..., r-1 \\ \dot{z}_r &= \bar{f}_r(z) + \bar{B}_r(z)u + g_r(z, t) \end{aligned} \tag{4.9}$$

$$\dot{x}_{r+1} = f_{r+1}(z, x_{r+1}) + g_{r+1}(t) \tag{4.10}$$

**A. 19.** *The unknown perturbation $g_r(x, t)$ satisfies the matching condition (Drazenovic, 1969) There exist a vector $\gamma(x, t) \in R^m$ such that the following relation holds:*

$$g_r(x, t) = B_r(x)\gamma(x, t), \quad \gamma \in R^m. \tag{4.11}$$

### Sliding Manifold Design

To stabilize the internal dynamics, we define

$$\eta := x_{r+1}, \quad f_\eta(\bar{z}_{r-1}, z_r, \eta, t) := f_{r+1}(z, x_{r+1}, t), \quad g_\eta(t) := g_{r+1}(t)$$

now the subsystem (4.10) is represented as $\dot{\eta} = f_\eta(\bar{z}_{r-1}, z_r, \eta, t) + g_\eta(t)$, then we obtain its linear approximation at origin as

$$\dot{\eta} = A_\eta \eta + B_\eta z_r + A_{r-1}\bar{z}_{r-1} + B_\eta x_r^* + \psi_\eta(\bar{z}, \eta, t) + g_\eta(t) \tag{4.12}$$

where $x_r^*$ is a known reference signal obtained as a desired value for $x_r$ by the block control linearization $\bar{z}_{r-1} = (z_1, ..., z_{r-1})^T$. $A_\eta = \frac{\partial}{\partial \eta}\bar{f}_\eta(0)$, $A_{r-1} = \frac{\partial}{\partial \bar{z}_{r-1}}\bar{f}_\eta(0)$, $B_\eta = \frac{\partial}{\partial z_r}\bar{f}_\eta(0)$. The following assumption is introduced

**A. 20.** *The function $\psi_\eta(\bar{z}, \eta, t)$ vanishes at origin with its first order derivatives and is bounded by $\psi_\eta(\bar{z}, \eta, t) < \beta\|\bar{z}\|$.*

Defining

$$z_\eta = \eta - x_{r+1}^*(t) \tag{4.13}$$

where $x_{r+1}^*$ is a reference to be computed later. The dynamics of $z_\eta$ along the trajectories of (4.10)-(4.12) are in the form

$$\dot{z}_\eta = A_\eta z_\eta + B_\eta z_r + A_{r-1}\bar{z}_{r-1} + A_\eta x_{r+1}^* + B_\eta x_r^* - \dot{x}_{r+1}^* + \psi_\eta(\bar{z}, \eta, t) + g_\eta(t) \tag{4.14}$$

In this case, we consider to $\dot{z}_\eta$ as a subsystem, where $z_r$ is considered as input. Since the system is a nonminimum phase $A_\eta$ is a matrix with real part positive eigenvalues. On the other hand, the vectors $x_r^*$ and $x_{r+1}^*$ are considered references for $x_r$ and $x_{r+1}$ respectively, the reference $x_{r+1}^*$ will be obtained later in the chapter.

Consider the following assumption

**A. 21.** *The pair $(A_\eta, B_\eta)$ is controllable.*

Following the SM technique a sliding variable $\sigma$ is proposed in the form

$$\sigma = z_r - K_\eta z_\eta \tag{4.15}$$

where $K_\eta$ is a matrix to design of dimension $n_{r+1} \times n_r$ and $\sigma = [\sigma_1 \ \sigma_2 \ ... \ \sigma_{n_r}]^T$ The dynamics of $\sigma$ along the trajectories of (4.9)-(4.10) are

$$\dot{\sigma} = f_\sigma(z, \eta) + \bar{B}_r u \tag{4.16}$$

where $f_\sigma(z,\eta) = \bar{f}_r(z) + \bar{B}_r(z)\lambda_s + d_r(z,t) - K_\eta \dot{z}_\eta(\eta, w)$,
$f_\sigma(z,\eta) = \begin{pmatrix} f_{\sigma 1}(z,\eta) & f_{\sigma 2}(z,\eta) & \cdots & f_{\sigma_{n\_r}}(z,\eta) \end{pmatrix}^T$ complies

**A. 22.** *The functions $f_{\sigma i}(z,\eta)$ in (4.16) are bounded by*

$$|f_{\sigma i}(z,\eta)| \leq \delta_i \, |\sigma_i|^{1/2} \qquad \delta_i > 0, i = 1, ..., n_r. \tag{4.17}$$

To induce chattering-free sliding mode on (4.15) we use the super-twisting algorithm [Fridman and Levant, 2002]

$$\begin{aligned} u &= \bar{B}_r^{-1}(z)v, \quad v = [v_1, ..., v_{n_r}]^T \\ v_i &= -k_{i1} \, |\sigma_i|^{1/2} \, sign(\sigma_i) + v_{i1} \\ \dot{v}_{i1} &= -k_{i2} \, sign(\sigma_i), \quad i = 1, ..., n_r. \end{aligned} \tag{4.18}$$

Assuming the conditions given in [Moreno and Osorio, 2008] hold:

$$k_{i1} > 2\delta_i \qquad k_{i2} > k_{i1} \frac{5\delta_i k_{i1} + 4\delta_1^2}{2\,(k_{i1} - 2\delta_1)} \tag{4.19}$$

then the overall system state converges to a sliding manifold $\sigma = 0$ and $z_r = -K_\eta z_\eta$.

## SM Dynamics

On the manifold $\sigma = 0$, the sliding mode equation is governed by

$$\begin{aligned} \dot{z}_1 &= -K_1 z_1 + z_2 \\ \dot{z}_i &= -K_i z_i + E_{i,1} z_{i+1}, \quad i = 2, ..., r-2 \\ \dot{z}_{r-1} &= -K_{r-1} z_{r-1} + K_\eta z_\eta \end{aligned} \tag{4.20}$$

$$\dot{z}_\eta = (A_\eta + B_\eta K_\eta) z_\eta + A_{r-1} \bar{z}_{r-1} + A_\eta x^*_{r+1} + B_\eta x^*_r - \dot{x}^*_{r+1} + \psi_\eta(\bar{z}, t) + g_\eta(t) \tag{4.21}$$

In order to show the stability of the sliding mode equation (4.20)-(4.21), we first have to find the reference $x^*_{r+1}(t)$ and analyse the resulting system. In the following subsections, we consider two cases presented in the sliding mode equation, specifically in the subsystem (4.21), then we introduce the proposed solutions to both cases using different approaches. First let us introduce the two considered cases:

Case 1: The subsystem (4.21) is not perturbed, namely $g_\eta(t) = 0$;

Case 2: The subsystem (4.21) has unmatched unknown perturbation, which is a sinusoidal shaped signal, that is $g_\eta(t) = A_g sin(\alpha_g t + \phi_g)$.

## 4.2 Case 1: Nonlinear NP Systems with Matched Perturbation

Consider $g_\eta(t) = 0$. Assigning an arbitrary $x^*_{r+1}$, such that

$$A_{r+1}x^*_{r+1} + B_{r+1}x^*_r - \dot{x}^*_{r+1} = 0, \tag{4.22}$$

under condition A.21 we can choose properly gains $K_\eta$, $K_i$, $i = 1, 2, ...r-1$ such that a solution of the sliding mode equation (4.20)-(4.21) is asymptotically stable, consequently the output tracking error $z_1$ is too (See theorem 3.11).

However, due to the considered system is a nonminimum phase, the matrix $A_{r+1}$ has eigenvalues with positive real part, then an arbitrary reference $x^*_{r+1}$ in (4.22) will increase and the control signal will be unbounded too.

### Solvability Conditions for SMOR in the Causal Case

To achieve the stability of the closed loop system (4.9)-(4.18), we also need to give a bounded reference $x^*_{r+1}$ to the unstable internal dynamics $x_{r+1}$. The mentioned bounded reference has to be a stable solution for the unstable differential equation

$$\dot{x}^*_{r+1} = A_{r+1}x^*_{r+1} + B_{r+1}x^*_r. \tag{4.23}$$

Considering the sliding mode equation (4.20)-(4.21), the SM Output Regulation problem in the causal case is transformed in the problem of obtain a bounded solution $x^*_{r+1}$ for the unstable differential equation (4.23). In the following subsections, we present two approaches to obtain a bounded reference $x^*_{r+1}$.

### Bounded Solution for Unstable Linear Differential Equations

In this subsection the basis of a first approach developed in [Jeong and Utkin, 1999] to obtain a bounded reference $x_{r+1}$ is presented. Let us consider a linear system

$$\dot{\eta}(t) = Q\eta(t) + Bu(t) \tag{4.24}$$

where matrix Q has no eigenvalues on imaginary axis. When Q is stable, the solution satisfying the initial condition is in the following form

$$\eta(t) = e^{Qt}\eta(0) + \int_0^t e^{Q(t-\tau)}Bu(\tau)d\tau \tag{4.25}$$

which is bounded if the input $u$ is bounded. If all eigenvalues of Q have positive real part, the bounded solution satisfying the following boundary condition can also be obtained

$$\eta(T) = 0, \quad t < T < \infty \tag{4.26}$$

**Theorem 4.1.** *[Jeong and Utkin, 1999] The bounded solution $\eta^o$ of system (4.24) which contents unstable modes, satisfying boundary condition (4.26) with $T$ = constant is*

$$\eta^o(t) = -\int_t^T e^{Q(t-\tau)} Bu(\tau) d\tau \tag{4.27}$$

*Proof.* Differentiating (4.27) shows that it is indeed the solution of the unstable system (4.24).

$$\begin{aligned} \dot{\eta}(t)^o &= -Qe^{Qt} \int_t^T e^{-Q\tau} Bu(\tau) d\tau - e^{Qt} \left[0 - e^{-Qt} Bu(t)\right] \\ &= Q\eta^o(t) + Bu(t) \end{aligned} \tag{4.28}$$

The boundedness can be seen from the facts that, Q is unstable and $t - \tau < 0$. □

The bounded solution of the linear system (4.24), either stable or unstable, can also be represented with the derivatives of the input signal.

**Theorem 4.2.** ***Bounded Solution in Derivative Form.*** *Suppose the matrix $Q$ of the linear system (4.24) has no imaginary axis eigenvalues, then a bounded solution $\eta^o(t)$ of the system with boundary condition $\eta^o(\infty) = 0$ can be given in the following derivative form*

$$\eta^o = -\sum_{n=0}^{\infty} Q^{-(n+1)} Bu^{(n)}(t) \tag{4.29}$$

*Proof.* From (4.27) with $T = \infty$, integrating by parts

$$\begin{aligned} \eta^o(t) &= -\int_t^\infty e^{Q(t-\tau)} Bu(\tau) d\tau \\ &= Q^{-1} e^{Q(t-\tau)} Bu(\tau) \mid_t^\infty - \int_t^\infty Q^{-1} e^{Q(t-\tau)} B\dot{u}(\tau) d\tau \\ &= -Q^{-1} Bu(t) - Q^{-2} B\dot{u}(t) - \ldots \\ &= -\sum_{n=0}^{\infty} Q^{-(n+1)} Bu^{(n)}(t) \end{aligned}$$

□

It is important to note that for the nonminimum phase case, the nominal trajectory $\eta^o$ is noncausal (the integral (4.27) is defined from $t$ to $\infty$) and can be obtained through a convolution integral (4.27). However, in order to avoid the noncausal condition, we can use the bounded solution in derivative form (4.29) and use a robust differentiator, or the adaptive estimator (see Section 2.5) to obtain the derivatives of the input $u$.

Let come back to the output regulation problem for systems without unmatched perturbation. When $g_{r+1}(t) = 0$ then we can use the bounded solution (4.29) directly to obtain a bounded reference for $x^*_{r+1}$ (4.23). Replacing $Q$ by $A_{r+1}$, $B$ by $B_{r+1}$ and the input $u$ by $x^*_r$ in solution (4.29) the bounded output reference which satisfies the unstable equation (4.23) is:

$$x^*_{r+1} = -\sum_{n=0}^{\infty} A_{r+1}^{-(n+1)} B_{r+1} x_r^{*(n)}(t) \tag{4.30}$$

In order to estimate the derivatives of the reference signal $x^*_r$ we use an adaptive estimator (see subsection 2.5). For that, as the output reference signal $y_{ref}$ is a sinusoidal shaped one, we assume that $x^*_r$ has the same shape $x^*_r = A_{mr} sin(\alpha_r t + \phi_r)$.

## SM Regulator and the Adaptive Estimator

In this section we estimate $x^*_r$ with the adaptive estimator (2.24), for that we propose an estimator in the form:

$$\begin{aligned} \dot{\xi}_1 &= \lambda\xi_2 + \frac{\lambda}{l_2}(x^*_r - \hat{y}) \\ \dot{\xi}_2 &= -\frac{\sigma\xi_1\xi_3}{\lambda} + \zeta(x^*_r - \hat{y}) \\ \dot{\xi}_3 &= -\gamma\xi_1(x^*_r - \hat{y}) \end{aligned} \tag{4.31}$$

(4.32)

where the output $\hat{y}$

$$\begin{aligned} \hat{y}(t) &= \frac{l_1}{\lambda}\xi_1 + l_2\xi_2 \\ \tilde{e} &= x^*_r - \hat{y} \end{aligned} \tag{4.33}$$

is the estimated value for $x^*_r$, and $\tilde{e}$ is the estimation error (4.33). Choosing $\sigma$, $\lambda$, $\zeta$, $\gamma$, $l_1$, $l_2 >$ 0, we can modify the convergence of the estimator and the error $\tilde{e}$ asymptotically goes to zero, namely $lim_{t\to\infty}\tilde{e}(t) = 0$. Moreover, the derivatives of the estimation $\hat{y}^{(n)}(t)$ tends to the derivatives of the $x_r^{*(n)}$, $n = 1, 2, ..., \infty$. Equation (4.30) is an infinite series but that could be approximated taking a finite number of the series elements.

Then we can obtain the bounded reference signal for $x_{r+1}^*$ in the form:

$$x_{r+1}^* = -A_{r+1}^{-1} B_{r+1} x_r(t) - A_{r+1}^{-2} B_{r+1} \dot{y}(t) - A_{r+1}^{-3} B_{r+1} \ddot{y}(t) - ... \tag{4.34}$$

with this, if we substitute (4.34) in (4.21) we obtain

$$\dot{z}_\eta = (A_\eta + B_\eta K_\eta) z_\eta + A_{r-1} \bar{z}_{r-1} + \psi_\eta(\bar{z}, t) \tag{4.35}$$

then, the sliding mode equation can be presented as

$$\dot{\bar{z}} = \bar{A} \bar{z} + \psi_{\bar{z}} \tag{4.36}$$

where $\bar{z} = (\bar{z}_{r-1}, z_\eta)^T$, and $\bar{A}$ is

$$\bar{A} = \begin{pmatrix} -K_1 & I_{n_1} & 0 & \cdots & & 0 \\ 0 & -K_2 & I_{n_2} & 0 & \cdots & \vdots \\ \vdots & 0 & & \ddots & -K_{r-1} & -K_\eta \\ \bar{A}_1 & \bar{A}_2 & \cdots & \bar{A}_{r-2} & \bar{A}_{r-1} & (A_\eta - B_\eta K_\eta) \end{pmatrix}$$

The perturbation term $\psi_{\bar{z}}$ only contains the term $\psi_1$, then considering the constraint A.11 results that $\|\psi_{\bar{z}}\| < \beta_1 \|z\|$ for all $t \geq 0$ and all $\bar{z} \in \Re^{n-1}$ with $\beta_1 > 0$ constant.

We can choose the gains $K_i$ such that $\bar{A}$ is Hurwitz, then using the proof of Theorem 3.11 presented in Section 3.4 the reduced order system is asymptotically stable (implies condition $(S)$). Consequently, the output tracking error $z_1$ tends asymptotically to zero (condition $(R)$).

### SM Regulator with System Center and Adaptive Estimator

In this section, we present another approach to solve the SM Output Regulation problem in *causal* case. Based on the work [Gopalswamy and Hedrick, 1993] and using the extensions given by [Shtessel and Shkolnikov, 1999], [Shtessel et al., 2012] we present an alternative form using the adaptive estimator. The main idea is based on System Center approach [Shtessel, 1994]. As we mentioned before, the output regulation problem is transformed to obtain a bounded reference $x_{r+1}^*$ for $x_{r+1}$. Here we present the System Center equation [Shtessel et al., 2012] to obtain that reference.

### System Center Design

Consider a linear system in the form

$$\dot{\eta} = Q\eta + \theta \tag{4.37}$$

where $\theta$ is considered as an external input generated for an exosystem defined in the form

$$\dot{w} = Sw \tag{4.38}$$

$$\theta = Dw \tag{4.39}$$

where $w \in R^k$, $\theta \in R^p$, $k > p$, and $S \in R^{k \times k}$

The Extended Method of Stable System Center [Shtessel et al., 2012] computes an estimated $\hat{\eta}_b$ of a bounded solution $\eta_b$ for the unstable equation (4.37) using the solution $\hat{\eta}_b$ obtained of:

$$\hat{\eta}_b^{(k)} + c_{k-1}\hat{\eta}_b^{(k-1)} + ... + c_1\hat{\eta}_b^{(1)} + c_0\hat{\eta}_b = -\left(P_{k-1}\theta^{(k-1)} + ... + P_1\dot{\theta} + P_0\theta\right) \tag{4.40}$$

where $k$ is the order of the exosystem, and the output $\theta$ as defined in equation (4.39). The numbers $c_0, c_1, ..., c_{k-1}$ are choosen to provide desired eigenvalue placement of convergence $\hat{\eta}_b \to \eta_b$, and the matrices $P_{k-1}, ...P_1, P_0 \in R^{p \times p}$ are given by:

$$\begin{aligned}
P_{k-1} &= (I + c_{k-1}Q^{-1} + ... + c_0Q^{-k})(I + p_{k-1}Q^{-1} + ... + p_0Q^{-k}) - I \\
P_{k-2} &= c_{k-2}Q^{-1} + ... + c_0Q^{-(k-1)} - (P_{k-1} + I)(p_{k-2}Q^{-1} + ... + p_0Q^{-(k-1)}) \\
&\vdots \\
P_1 &= c_1Q^{-1} + c_0Q^{-2} - (P_{k-1} + I)(p_1Q^{-1} + p_0Q^{-2}) \\
P_0 &= c_0Q^{-1} - (P_{k-1} + I)p_0Q^{-1}
\end{aligned} \tag{4.41}$$

where the constants $p_0,\ p_1, ..., p_{k-1}$ are the coefficients in the characteristic polynomial $p(\lambda) = \lambda^k + p_{k-1}\lambda^{k-1} + ... + p_1\lambda + p_0$ of the exosystem (4.38).

Noting that in this approach, we need to know the coefficients $p_0,\ p_1, ..., p_{k-1}$. It is necessary then to estimate the characteristic polynomial and the state of an exosystem. Unlike to the solution presented in [Shtessel et al., 2012], we use the adaptive estimator to obtain the necessary characteristic polynomial for the estimation of $\eta_b$. We assume the exosystem (4.38) has the form

$$\begin{aligned}
\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
\theta &= Dw
\end{aligned} \tag{4.42}$$

with characteristic polynomial $p_k(\lambda) = \lambda^2 + \alpha^2$ The presented adaptive estimator (4.31) can use the known output $\theta$ and estimate the frequency $\alpha$ (see section 2.5). Then we use the estimator (4.43)

$$\begin{aligned} \dot{\xi}_1 &= \lambda\xi_2 + \frac{\lambda}{l_2}(\theta - \hat{y}) \\ \dot{\xi}_2 &= -\frac{\mu\xi_1\xi_3}{\lambda} + \zeta(\theta - \hat{y}) \\ \dot{\xi}_3 &= -\gamma\xi_1(\theta - \hat{y}) \end{aligned} \tag{4.43}$$

with $\mu, \lambda,\ \zeta,\ \gamma,\ l_1,\ l_2 > 0$ as design parameters, the output of the estimator $\hat{y}$ is

$$\hat{y} = \frac{l_1}{\lambda}\xi_1 + l_2\xi_2$$

is the estimation for $\theta$, and $\xi_3 \to \mu\alpha^2$. Then we use the estimation $\hat{\alpha}^2 = \xi_3/\mu$ to substitute $p_0$ in the computation of $P_{k-1}, ...P_1, P_0$ (4.41). Then we have a bounded estimated solution $\hat{\eta}_b$ for $x^*_{r+1}$ substituting this in (4.21), that equation results:

$$\dot{z}_\eta = (A_\eta + B_\eta K_\eta)z_\eta + A_{r-1}\bar{z}_{r-1} + \psi_\eta(\bar{z}, t).$$

Then, we obtain the sliding mode equation (4.36), which is asymptotically stable.

A particular case for the SM Output Regulation is when the reference $y_{ref}$ is constant. For that case in [Gopalswamy and Hedrick, 1993] was shown that a bounded reference $x^*_{r+1}$ for the internal dynamics can be obtained as a solution of the unstable differential equation (4.23) with a change in the sign, i.e.

$$\dot{x}^*_{r+1} = -A_{r+1}x^*_{r+1} - B_{r+1}x^*_r. \tag{4.44}$$

this is an asymptotically stable system because of the negative sign in (4.44). This is seen as we run the unstable parts backward in the time, and we should converge to a solution on a stable manifold.

## 4.3 Case 2: Nonlinear NP Systems with Matched and Unmatched Perturbation

Now, we consider the case when $g_{r+1}(t) \neq 0$, in this case we consider that the perturbation can be seen as an additional input affecting the unstable linear equation (4.23) in the form:

$$\dot{x}^*_{r+1} = A_{r+1}x^*_{r+1} + B_{r+1}x^*_r + g_{r+1}(t) \tag{4.45}$$

for that problem, we also can use the bounded solution (4.29) considering an additional input $g_{r+1}(t)$ affecting to the subsystem (4.3).

### Solution to the SMORP for Nonlinear NP Systems with Unmatched Perturbation

To overcome the new case of the perturbed subsystem (4.21), we propose to use an observer for the perturbation term $g_{r+1}(t)$ and add its estimation to the solution (4.29).

Now considering that the state $x(t)$ is known (A.3) we use an observer based on robust differentiator [Levant, 2003], to estimate the unknown unmatched perturbation term $g_{r+1}(t)$. Based on the known nominal dynamic system for $x_{r+1}$ (4.3) the proposed observer is:

$$\begin{aligned} \dot{\hat{x}}_{r+1} &= f_{r+1}(\bar{x}_r, x_{r+1}) + l_1|\tilde{x}_{r+1}|sign(\tilde{x}_{r+1}) + g_e \\ \dot{g}_e &= l_2 sign(\tilde{x}_{r+1}) \end{aligned} \tag{4.46}$$

where $\tilde{x}_{r+1} = x_{r+1} - \hat{x}_{r+1}$ is the estimation error of the internal state, and $g_e$ is the estimation for $g_{r+1}$. Considering conditions for the gains $l_1$, $l_2$ presented in section 2.3, the error $\tilde{x}_{r+1} = x_{r+1} - \hat{x}_{r+1}$ is zero in a finite time, and the estimation $g_e(t)$ is equal to $g_{r+1}(t)$.

To rejects the perturbation term $g_{r+1}(t)$ and obtain a bounded reference $x^*_{r+1}$ in the perturbed subsystem (4.3), we add the estimation $g_e(t)$ in the solution (4.29).

Assuming that estimation $g_e$ and its derivatives $\dot{g}_e, \ddot{g}_e$, $g_e^{(n)}$, $n = 3, 4, ..., \infty$, are available, we can obtain a bounded reference $x^*_{r+1}$ which satisfies the unstable equation (4.45) as:

$$x^*_{r+1} = -\sum_{n=0}^{\infty} \left( A_{r+1}^{-(n+l)} B_{r+1} x_r^{*(n)}(t) + A_{r+1}^{-(n+l)} g_e^{(n)} \right) \tag{4.47}$$

As the output reference $y_{ref}$ and perturbation $g_{r+1}$ signals are a sinusoidal shaped ones (A.18), namely $x^*_r = A_{mr} sin(\alpha_r t + \phi_r)$ and $g_{r+1}(t) = A_g sin(\alpha_g t + \phi_g)$, we can estimate their derivatives $x_r^{*(n)}$ and $g_e^{(n)}$ $n = 1, 2, ..., \infty$, using two adaptive estimators. The estimator for $x^*_r$ is (4.31), and the estimator for $g_{r+1}$ is:

$$\begin{aligned} \dot{\xi}_{1g} &= \lambda_1 \xi_{2g} + \frac{\lambda_1}{k_{2g}}(g_e - \hat{y}_1) \\ \dot{\xi}_{2g} &= -\frac{\mu_1 \xi_{1g} \xi_{3g}}{\lambda_1} + \zeta_1(g_e - \hat{y}_1) \\ \dot{\xi}_{3g} &= -\gamma_1 \xi_{1g}(g_e - \hat{y}_1) \end{aligned} \tag{4.48}$$

where the output $\hat{y}_1$

$$\begin{aligned} \hat{y}_1 &= \frac{l_{1g}}{\lambda_1}\xi_{1g} + l_{2g}\xi_{2g} \\ \tilde{e}_g &= g_e - \hat{y}_1 \end{aligned} \tag{4.49}$$

is the estimation for $g_{r+1}$ based on the estimated value $g_e(t)$. Choosing $\mu_1$, $\gamma_1$, $\zeta_1$, $\lambda_1$, $l_{1g}$, $l_{2g} > 0$, we achieve $lim_{t\to\infty}\tilde{e}_g(t) = 0$ [Obregon-Pulido et al., 2010], and the derivatives of the estimation $\hat{y}_1^{(n)}$ tends to the derivatives of the function $g_e(t)$, $n = 1, 2, ..., \infty$.

Then we have a bounded reference signal for $x_{r+1}^*$ which rejects the unmatched perturbation term $g_{r+1}(t)$ in the form:

$$x_{r+1}^* = -A_{r+1}^{-1}B_{r+1}x_r(t) - A_{r+1}^{-1}g_e(t) - A_{r+1}^{-2}B_{r+1}\dot{\hat{y}}(t) - A_{r+1}^{-2}\dot{\hat{y}}_1(t) \quad (4.50)$$

$$- \; A_{r+1}^{-3}B_{r+1}\ddot{\hat{y}}(t) - A_{r+1}^{-3}\ddot{\hat{y}}_1(t)... \quad (4.51)$$

## SM Dynamics

Substituting (4.51) in (4.21) the sliding mode equation results

$$\dot{z}_\eta = (A_\eta + B_\eta K_\eta)z_\eta + A_{r-1}\bar{z}_{r-1} + \psi_\eta(\bar{z}, t).$$

Then, we obtain the sliding mode equation (4.36), which is asymptotically stable (see Theorem 3.11).

# Chapter 5

# Discrete-Time SM Regulator for Nonminimum Phase Systems

This chapter presents an approach to solve the output regulation problem for a class of nonlinear discrete-time nonminimum phase perturbed systems. Based on feedback linearization Block Control technique and discrete-time sliding mode (SM) control, we propose a sliding manifold on which the zero dynamics are stabilized. To enforce the robust SM motion on the designed manifold, a discrete-time super-twisting SM algorithm is implemented. The effectiveness of the proposed methodology is verified in section 6 via the design of a position tracking controller for an under-actuated robotic system, the Pendubot.

## 5.1 Discrete-Time Classical Output Regulation Problem

Consider a nonlinear discrete-time SISO system

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, w_k) \\ y_k &= h(x_k) \end{aligned} \tag{5.1}$$

where $k \in Z$ denotes the discrete time instants, with $Z$ the set of the nonnegative integers. The state vector $x_k$ is defined on a neighborhood $X$ of the origin of $R^n$, $u_k \in R$ is the input vector, and $y_k \in R$ is the vector of the output variables to be controlled. Here $f(\cdot,\cdot)$, $h(\cdot)$ are smooth vector fields of class $C^{\infty}_{[t,\infty)}$, with $f(0,0) = 0$, $h(0) = 0$. It is worth mentioning that system (5.1) can also be considered as a sampled data system, in that case, $x_k \simeq x(t_k)$, and under the assumption of zero-order hold, $u_k = u(t_k)$, where $t_k = k\delta$, with $\delta$ as the sampling period.

The tracking error is defined as the difference between $y_k$ and a reference signal $y_r(w_k)$ to be tracked i.e.

$$e_k = y_k - y^r(w_k). \tag{5.2}$$

The reference signal $y^r(w_k)$ is assumed to be bounded, with bounded increments, and generated by a given external system described by

$$\begin{aligned} w_{k+1} &= s(w_k), \quad w_k \in W \subset R^s \\ y_k^r &= y^r(w_k), \end{aligned} \tag{5.3}$$

with $y^r(w_k)$ as a known variable.

The control problem is to design a controller using full information, which enables to bring the tracking error (5.2) to zero. Isidori et al., have proposed in [Isidori and Byrnes, 1990] a solution to this problem in the continuous-time setting. We now consider this solution to extend a version for the discrete-time setting. The solution to the above mentioned problem can be provided by a state feedback $u_k = \alpha(x_k, w_k)$, where the pair of mappings $x_k = \pi_k(w_k)$ and $u_k = c_k(w_k)$ solve the difference equations

$$\begin{aligned} \pi_{k+1}(w_{k+1}) &= f(\pi_k, c_k) \\ 0 &= h(\pi_k) - y_k^r. \end{aligned} \tag{5.4}$$

with $\pi_k(0) = 0$ and $c_k(0) = 0$.

In this section, we consider the system (5.1) which contents both the known $w_k$ and unknown $d_k$ disturbance terms, namely

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, w_k, d_k) \\ y_k &= h(x_k) \end{aligned} \tag{5.5}$$

where $d_k = d(x_k, k)$ represents internal and external disturbances.

The classical regulator $u_k = c_k(w_k) + k^T(x_k - \pi_k(w_k))$ with $(A + bk^T)$ Shur matrix, $A = \frac{\partial f}{\partial x}|_{x=0}$, $b = \frac{\partial f}{\partial u}|_{x=0}$ , can achieve only local stability of the system (5.5) around the tracking trajectory in absence of the unknown perturbation $d_k$.

To overcome these problems and increase the stability region also as to achieve robustness of the closed-loop system, we propose to apply SM control technique [Utkin et al., 1999] combined with the block control (BC) feedback linearization one [Loukianov, 2002].

## 5.2 Discrete-Time Sliding Mode Regulation Problem

Using the before established concepts for the continuous time setting, we propose an extension of the sliding mode regulator problem (SMRP). In this case (discrete time), the SMRP is defined as the problem of finding a sliding function $\sigma_k(z_k, w_k)$ (with $z_k = x_k - \pi_k(w_k)$) such that the following conditions hold:

**SMS)** The state of the closed-loop system (5.1), with the static discrete-time sliding mode controller $u_k(z_k, w_k)$, makes the state error $z_k$ converges to the manifold $\sigma_k = 0$ in a finite time.

**S)** The equilibrium $x_k = 0$ of the sliding mode dynamics under $w_k = 0$ and $d_k = 0$ :

$$x_{k+1} = f(x_k, u_{eq,k}, 0, 0)_{\sigma_k=0}$$

is stable. The term $u_{eq,k}$ is the equivalent control obtained from $\sigma_{k+1} = 0$.

**R)** There exists a neighborhood $V_1 \subset X \times W$ of $(0,0)$ such that, for each initial condition $(x_0, w_0) \in V_1$ the output tracking error (5.2) goes asymptotically to zero, i. e.

$$\lim_{k\to\infty} e_k = 0.$$

As it is commonly in the SM control design, we introduce the following assumption.

**A. 23.** *The unknown but bounded disturbance $d_k$ satisfies the matching condition [Drazenovich, 1969].*

### Nonlinear Nonminimum Phase Discrete-time System

In this work, we consider a class of nonlinear SISO affine control systems (5.5), which can be presented (possibly after an appropriate diffeomorphic transformation) in the regular form [Luk'yanov and Utkin, 1981]:

$$x'_{1,k+1} = f'_1(x'_{1,k}, x'_{2,k}) \tag{5.6}$$

$$x'_{2,k+1} = f'_2(x'_{1,k}, x'_{2,k}) + b'_2(x'_{1,k}, x'_{2,k})u_k + d'_k \tag{5.7}$$

where the part of subsystem (5.6) with subsystem (5.7) has NBC-form with respect to the output $y_k = x_{1,k}$, while the rest part of the subsystem (5.6) describes the residual dynamics (subsystem (5.9):

$$\begin{aligned} x_{1,k+1} &= f_1(x_{1,k}) + b_1(x_{1,k})x_{2,k} && (5.8)\\ x_{i,k+1} &= f_i(x_{1,k},\dots,x_{i,k}) + b_i(x_{1,k},\dots,x_{i,k})x_{i+1,k} &&\\ x_{r,k+1} &= f_r(\bar{x}_k, x_{r+1,k}) + b_r(\bar{x}_k, x_{r+1,k})u_k + d_k, && (5.9)\\ i &= 2,\cdots,r-1 &&\\ x_{r+1,k+1} &= f_{r+1}(\bar{x}_k, x_{r+1,k}) && (5.10) \end{aligned}$$

$$y_k = x_{1,k} \quad (5.11)$$

where $\bar{x}_k = (x_{1,k},\dots,x_{r,k})^T \in X \subset R^n$, $dim(x_{r+1,k}) = n - r$ where $r$ is the relative degree, moreover,

$$f_1'(x_{1,k+1}', x_{2,k+1}') = \begin{pmatrix} f_1(x_{1,k}) + b_1(x_{1,k})x_{2,k} \\ \cdots \\ f_{r-1}(x_{1,k},\dots,x_{r-1,k}) + b_{r-1}(x_{1,k},\dots,x_{r-1,k})x_{r,k} \\ f_{r+1}(\bar{x}_k, x_{r+1,k}) \end{pmatrix}$$

$d_k = d_k'$, $f_r(\cdot) = f_2'(\cdot)$, $b_r(\cdot) = b_2'(\cdot)$,
and we assume

**A. 24.** $b_i(\cdot) \neq 0$, $i = 1,\dots,r$, $\forall x' \in X \subset R^n$

Dynamics of many control plants, for example, electro-mechanical under- actuated systems, can be presented (possible after a nonlinear transformation) in the form (5.8)- (5.11).

A solution of the zero dynamics

$$x_{r+1,k+1} = f_{r+1}(0, x_{r+1,k})$$

is not required to be asymptotically stable, i. e., the system (5.8)- (5.11) can be a nonminimum phase system. In this case, the direct implementation of the combined SM and BC method cannot to stabilize the closed-loop system. Therefore, a special sliding manifold on which the residual dynamics are stable, should be designed.

## Sliding Mode Manifold Design

Consider the subsystem (5.8). The sliding manifold design procedure consists of a step–by–step construction of a new system with states $z_{i,k} = x_{i,k} - x^d_{i,k}$, $i = 1, \cdots, r$, where $x^d_{i,k}$ is the desired value for $x_{i,k}$, which will be defined by such a construction.

We start by defining as new variable the tracking error (5.2)

$$z_{1,k} = e_k = x_{1,k} - x^d_{i,k}$$

with $x^d_{1,k} = y^r(w_k)$ the reference value for $x_{1,k}$, having dynamics

$$z_{1,k+1} = f_1(x_{1,k}) + B_1(x_{1,k})x_{2,k} - x^d_{i,k+1}. \tag{5.12}$$

In the system (5.12), $x_{2,k}$ is viewed as a virtual control input used to impose the following desired dynamics

$$z_{1,k+1} = k_1 z_{1,k}. \tag{5.13}$$

with $|k_1| < 1$ to ensure the asymptotic stability of (5.13). Therefore, on the basis of assumption A.24, one determines the solution in $x_{2,k}$ for the equation

$$x^d_{2,k} = b_1^{-1}(x_{1,k})(k_1 z_{1,k} - f_1(x_{1,k}) + x^d_{i,k+1})$$

which represents the reference behavior for $x_{2,k}$. Proceeding in the same way, one introduces $z_{2,k} = x_{2,k} - x^d_{2,k}$, having dynamics

$$z_{2,k+1} = f_2(x_{1,k}, x_{2,k}) + b_2(x_{1,k}, x_{2,k})x_{3,k} - x^d_{2,k+1}.$$

One imposes the desired dynamics

$$z_{2,k+1} = k_2 z_{2,k} \tag{5.14}$$

where $|k_2| < 1$. By assumption A.24, the solution in $x_{3,k}$ given by

$$x^d_{3,k} = b_2^{-1}(x_{1,k}, x_{2,k})(k_2 z_{2,k} - f_2(x_{1,k}, x_{2,k}) + x^d_{2,k+1})$$

which is the reference value for $x_{3,k}$. Iterating these steps, one finally introduces the variable $z_{r,k} = x_{r,k} - x^d_{r,k}$,

$$x^d_{r,k} = b_{r-1}^{-1}(\cdot)(k_{r-1} z_{r-1,k} - f_{r-1}(\cdot) + x^d_{r-1,k+1}),$$

with dynamics

$$\begin{aligned} z_{r,k+1} &= f_r(\bar{x}_k, x_{r+1}) + b_r(\bar{x}_k, x_{r+1})u_k \\ &+ d_k - x^d_{r,k+1}. \end{aligned}$$

It is worth mentioning that the new variables $z_{i,k}$, $i = 0, 1, \cdots, r+1$ are determined by the following nonlinear transformation:

$$\begin{aligned} z_{1,k} &= x_{1,k} - x^d_{1,k} = \varphi_1(x_{1,k}, w_k) \\ z_{2,k} &= x_{2,k} - x^d_{2,k} \\ &= \varphi_2(x_{1,k}, x_{2,k}, w_k) \\ z_{3,k} &= x_{3,k} - x^d_{3,k} \\ &= \varphi_3(x_{1,k}, x_{2,k}, x_{3,k}, w_k) \\ &\vdots \\ z_{r,k} &= x_{r,k} - x^d_{r,k} \\ &= \varphi_q(x_{1,k}, x_{2,k}, \cdots, x_{r,k}, w_k) \\ z_{r+1,k} &= x_{r+1,k} - x^d_{r+1,k} \\ &= \varphi_{r+1}(x_{1,k}, x_{2,k}, \cdots, x_{r+1,k}). \end{aligned} \tag{5.15}$$

Now, for the the residual dynamics (5.9) we need the following assumption:

**A. 25.** *There exist a smooth mapping $x_{r+1,k} = x^d_{r+1,k}(w_k)$ with $x^d_{r+1,k}(0) = 0$ defined in a neighborhood $W^o \subset W$ of the origin, such that*

$$x^d_{r+1,k+1}(w_{k+1}) = f_{r+1}(\bar{x}^d_k, x^d_{r+1,k}(w_k)) \tag{5.16}$$

with $\bar{x}^d_k = \begin{pmatrix} x^T_{d,1,k} & x^T_{d,2,k} & \cdots & x^T_{d,r,k} \end{pmatrix}^T$ is part of the solution to the partial equation (5.4)

It is easy to check that, by means of this transformation $z_k = \varphi(x_k, w_k) = \begin{pmatrix} \varphi^T_1 & \varphi^T_1 & \cdots & \varphi^T_{r+1} \end{pmatrix}'$ the system (5.8)- (5.9) is diffeomorphic to

$$\begin{aligned} z_{1,k+1} &= k_1 z_{1,k} + b_1(z_{1,k}) z_{2,k} \\ z_{2,k+1} &= k_2 z_{2,k} + b_2(z_{1,k}, z_{2,k}) z_{3,k} \\ &\vdots \\ z_{r-1,k+1} &= k_{r-1} z_{r-1,k} + b_{r-1}(z_{1,k}, ..., z_{r-1,k}) z_{r,k} \\ z_{r,k+1} &= f_r(z_k) + b_r(z_k) u_k - x^d_{r,k+1} + d_k \\ z_{r+1,k+1} &= \bar{f}_{r+1}(z_{1,k}, \cdots, z_{r+1,k}). \end{aligned} \tag{5.17}$$

Having the system of the form (5.17), a natural choice of a sliding variable $s_k$ is $s_k = z_{r,k}$. In this case, however, the zero dynamics of $x_{r+1,k}$ or $z_{r+1,k}$ on the manifold $s_k = z_{r,k} = 0$

is considered to be unstable. Therefore, in order to stabilize such dynamic, we propose to formulate the sliding manifold $\sigma_k = 0$ of the following form:

$$\sigma_k = 0, \quad \sigma_k = s_k - \sigma_{0,k}(z_{r+1,k}, w_k), \tag{5.18}$$

where $\sigma_{0,k}(z_{r+1,k}, w_k)$ is a smooth function to be selected with $\sigma_{0,k}(0,0) = 0$.

## Sliding Mode Controller Design

Taking one step ahead of (5.18) results in

$$\begin{aligned} \sigma_{k+1} &= f_r(z_k) + b_r(z_k)u_k \\ &+ d_k - x^d_{r,k+1} - \sigma_{0,k+1} \end{aligned}$$

To induce a sliding mode on $\sigma_k = 0$ we discretize the super-twisting algorithm [Levant, 2001]. If the continuous time version of this algorithm is just approximated by means of Euler method, then, the control action will be as a modulated discontinuous signal. To avoid this problem, we approximate the dynamics of the integral part of the super-twisting algorithm (integral of the sign function) by Gao's [Gao et al., 1995] extension of the reaching law [Hung et al., 1993] for discrete-time systems, resulting in

$$\begin{aligned} u_k &= -k_{11}\sqrt{|\sigma_k|}sign(\sigma_k) + \zeta_k \\ \zeta_{k+1} &= \zeta_k - \delta(k_{12}\sqrt{|\sigma_k|}sign(\sigma_k) + q_1\zeta_k) \end{aligned} \tag{5.19}$$

with $k_{11} > 0$, $k_{12} > 0$, $q_1 > 0$ and $1 + \delta q_1 > 0$, where $\delta$ is the sampling period. It is possible to show that there are $k_{11} > 0$ and $k_{12} > 0$ such that the state vector of the closed-loop system converges to the sliding manifold $\sigma_k = 0$ (5.18) in finite time. On this manifold we have $s_k = \sigma_{0,k}$, and the SM dynamics are governed by the reduced order system

$$\begin{aligned} z_{1,k+1} &= k_1 z_{1,k} + b_1(z_{1,k})z_{2,k} \\ z_{2,k+1} &= k_2 z_{2,k} + b_2(z_{1,k}, z_{2,k})z_{3,k} \\ &\vdots \\ z_{r-2,k+1} &= k_{r-2}z_{2,k} + b_{r-2}(z_{1,k}, .., z_{r-2,k})z_{r-1,k} \\ z_{r-1,k+1} &= k_{r-1}z_{r-1,k} \qquad (5.20) \\ z_{r+1,k+1} &= A_{r+1}z_{r+1,k} + B_{r+1}s_k \\ &+ \psi(z_{1,k}, \ldots, z_{r-1,k}, z_{r+1,k}, w_k) \qquad (5.21) \\ e_k &= z_{1,k} \qquad (5.22) \end{aligned}$$

where

$$A_{r+1} = \frac{\partial \bar{f}}{\partial z_{r+1,k}}\Big|_{z_{r+1,k}=0}, \quad B_{r+1} = \frac{\partial \bar{f}}{\partial z_{r,k}}\Big|_{z_{r,k}=0}$$

with $\psi(\cdot)$ as a function that vanishes at the origin. Now, we assume

**A. 26.** *The functions $b_1(z_{1,k})$ and $b_i(z_{1,k},..,z_{i,k})$, $i = 2,...,r-2$ are bounded.*

**A. 27.** *The pair $\{A_{r+1}, B_{r+1}\}$ is controllable.*

Therefore, under the assumption A.27, by proper selecting the sliding manifold $\sigma_{0,k} = k_{r+1}z_{r+1,k}$ one can locally stabilize the residual dynamics (5.21) by asigning the matrix $(A_{r+1} + B_{r+1}k_{r+1})$ be Schur. In this case, there exists a locally stable central manifold $x_{r+1,k} = x^r_{r+1,k}(w_k)$ satisfying condition (5.16). Moreover, with the proper selection of $k_1, \ldots, k_{r-1}$, under the assumption A.26 we have

$$\lim_{k\to\infty} \bar{z}_{r-1k} = 0$$

where $\bar{z}_{r-1,k} = \begin{pmatrix} z^T_{1,k} & z^T_{2,k} & \cdots & z^T_{r-1,k} \end{pmatrix}^T$, satisfying condition (S), and as a consequence condition (R) is also satisfied.

# Chapter 6

# Illustrative Examples

## 6.1 Second Order Sliding Mode Sensorless Torque Regulator for Induction Motor

The developed methodology in section 3.4 is illustrated here via the design a torque tracking controller for an induction motor (IM). A dynamic model of an IM defined in the stationary reference frame $(\alpha, \beta)$ is described by

$$
\begin{aligned}
\frac{dw_r}{dt} &= \frac{3}{2}\frac{L_m n_p}{J_r L_r}(\psi_a i_b - \psi_b i_a) - \frac{T_l}{J_r} \\
\frac{d\psi_a}{dt} &= -\alpha\psi_a - n_p w_r \psi_b + \alpha L_m i_a \\
\frac{d\psi_b}{dt} &= -\alpha\psi_b + n_p w_r \psi_a + \alpha L_m i_b \\
\frac{di_a}{dt} &= -\gamma i_a + \alpha\beta\psi_a + n_p \beta w_r \psi_b + u_a/\sigma_m \\
\frac{di_b}{dt} &= -\gamma i_b + \alpha\beta\psi_b - n_p \beta w_r \psi_a + u_b/\sigma_m
\end{aligned} \tag{6.1}
$$

where $\psi_a$ and $\psi_b$ are the rotor magnetic flux linkage components, $i_a$ and $i_b$ are the stator current components and $w_r$ is the mechanical rotor speed; $J_r$ is the rotor moment of inertia, $T_l$ is the load torque, $n_p$ is the number of pole pairs. Given full state measurements, the control aim is to achieve the torque $T_e = \frac{3L_m n_p}{2L_r}(\psi_a i_b - \psi_b i_a)$ tracks a reference and to keep the rotor flux magnitude $\psi = \psi_a^2 + \psi_b^2$ constant.

We define error variables as:

$$z_1 = \psi - \psi_{ref} \qquad y \qquad z_2 = T_e - T_{ref}$$

applying the block control technique and introducing desired dynamics the error system can be transformed in the form

$$\begin{aligned} \dot{z}_1 &= -(k_1 + 2\alpha)z_1 + z_3 & (6.2) \\ \dot{z}_2 &= f_2 + b_{11}u_a + b_{12}u_b \\ \dot{z}_3 &= f_3 + b_{21}u_a + b_{22}u_b \\ \dot{\eta}_1 &= (1/J_r)(z_2 + T_{ref} - T_l) & (6.3) \end{aligned}$$

with $z_3 = 2\alpha\psi_{ref} + 2\alpha L_m(\psi_a x_2 + \psi_b x_3) + k_1 z_1$, $\eta_1 = w_r$. As an special case due to the stable dynamics of the speed $\eta_1$ in order to stabilize the dynamics of the speed and avoid the solving of the corresponding FIB equations we introduce the steady state speed error as

$$\varepsilon_1 = \eta_1 - \hat{w}_{ss} \tag{6.4}$$

where $\hat{w}_{ss}$ is an estimate of the rotor speed steady state value. In this way, using (6.3) the dynamics of the error variable (6.4) are obtained of the form

$$\dot{\varepsilon}_1 = (1/J_r)(z_2 + T_{ref} - T_l) - \dot{\hat{w}}_{ss} \tag{6.5}$$

Now we define the sliding variables $\sigma_1$ and $\sigma_2$ as

$$\begin{aligned} \sigma_1 &= z_2 + c_4\varepsilon_1 \qquad c_4 > 0 \\ \sigma_2 &= z_3 \end{aligned}$$

using (6.1), (6.2), (6.5), $v = Bu$ and $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ the projection motion on the subspaces $\sigma_1$ and $\sigma_2$ can be written as

$$\begin{aligned} \dot{\sigma}_1 &= \bar{f}_1 + v_1 & (6.6) \\ \dot{\sigma}_2 &= f_3 + v_2 & (6.7) \end{aligned}$$

to achieve chattering free SM motion on the manifold $\sigma_1 = \sigma_2 = 0$ we use the super-twisting algorithm:

$$\begin{aligned} v_1 &= -k_{11}\left|\sigma_1\right|^{1/2} sign(\sigma_1) + v_{11} & (6.8) \\ \dot{v}_{11} &= -k_{12} sign(\sigma_1) \\ v_2 &= -k_{21}\left|\sigma_2\right|^{1/2} sign(\sigma_2) + v_{21} & (6.9) \\ \dot{v}_{21} &= -k_{22} sign(\sigma_2) \end{aligned}$$

under conditions similar to (3.95) and (3.97) the closed-loop system state converges to the manifold $\sigma_1 = \sigma_2 = 0$ in finite time.

The SM motion on this manifold is governed by

$$\dot{z}_1 = -(k_1 + 2\alpha) z_1 \quad (6.10)$$

$$z_2 = -c_4\varepsilon_1 \quad (6.11)$$

then the flux error $z_1$ tends asymptotically to zero, while the torque tracking error $z_2$ depends on the speed steady state error $\varepsilon_1$.

The dynamics of the rotor speed error (6.5) can be represented on the manifold $z_2 = -c_4\varepsilon_1$ as

$$\dot{\varepsilon}_1 = (z_2 + T_{ref} - T_l)/J_r - \dot{\hat{w}}_{ss} = (-c_4\varepsilon_1 + T_{ref} - T_l)/J_r - \dot{\hat{w}}_{ss} \quad (6.12)$$

thus, we choose

$$\dot{\hat{w}}_{ss} = (T_{ref} - T_l)/J_r + c_5 sign(\varepsilon_1) \quad (6.13)$$

then substituting (6.13) in (6.12) yields

$$\dot{\varepsilon}_1 = -(c_4/J_r)\varepsilon_1 - c_5 sign(\varepsilon_1) \quad (6.14)$$

if $c_4 > 0$ and $c_5 > 0$ then the speed error $\varepsilon_1$ converges to zero in finite time.

### Simulations

A three-phase, four pole machine was simulated. The motor parameters used are: $R_s = 14\Omega$, $L_s = 0.4$, $L_m = 0.377H$, $R_r = 10.1\Omega$, $L_r = 0.4128H$, $J_r = 0.01Kgm^2$, $n_p = 2$. The control parameters used are: $k_{11} = 2100$, $k_{12} = 7220$, $k_{21} = 400$, $k_{22} = 203$, $c_4 = 0.3$, $c_5 = 4$. In this simulation results we add unknown perturbations in the last two equations of (6.1), thus we have:

$$\frac{di_a}{dt} = -\gamma i_a + \alpha\beta\psi_a + n_p\beta w_r\psi_b + u_a/\sigma_m + 0.1sin(t)$$

$$\frac{di_b}{dt} = -\gamma i_b + \alpha\beta\psi_b - n_p\beta w_r\psi_a + u_b/\sigma_m + 0.1cos(t)$$

The Figure 6.1 shows the reference tracking result of torque using the proposed controller. The torque reference is proposed as a sinusoidal signal i.e. $T_{ref} = 1 + 0.2sin(\pi t)$ and the load torque is $T_l = 1$ Nm. Figure 6.2 shows the rotor flux magnitude response. The rotor fluxes magnitude reference is $\Psi_{ref} = 0.15Wb^2$. The rotor speed $w_r$ is shown in 6.3. As expected, $w_r$ asymptotically reaches the designed variable $\hat{w}_{ss}$ and it remains bounded.

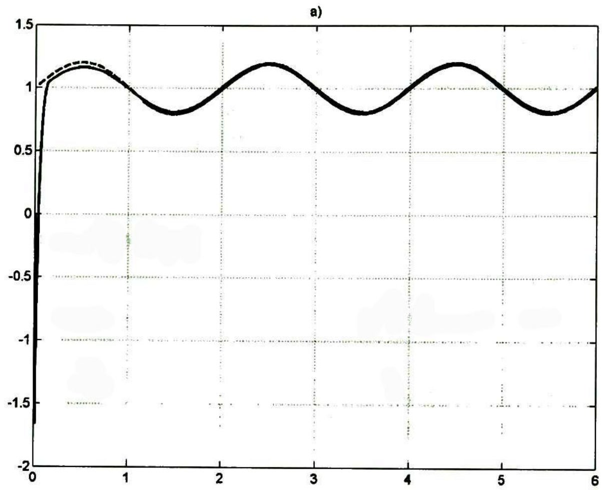


Figure 6.1: *a*) Electrical torque $T_e$ (solid) and reference $T_{ref}$ (dotted) [Nm vs s]

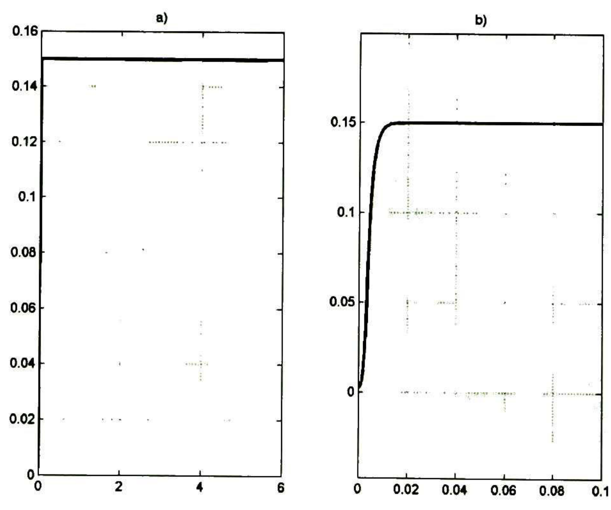


Figure 6.2: *a*) Rotor fluxes magnitude $\Psi$ response [ $Wb^2$ vs s]; b) zoom of a)

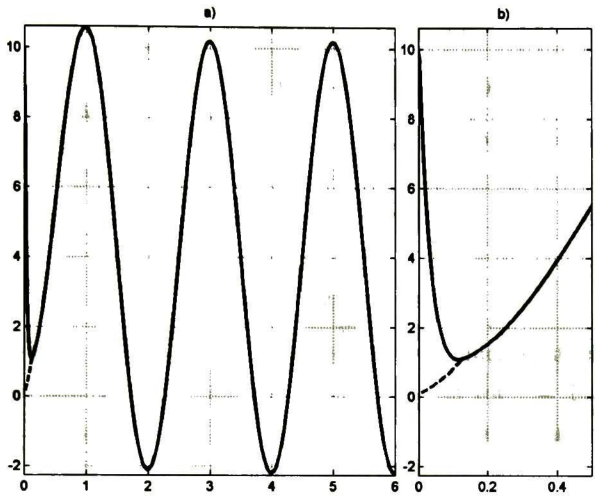


Figure 6.3: *a*) Motor speed $w_r$ (solid) and the integral variable $\hat{w}_{ss}$ (dotted) [rad/s vs s]; b) zoom of a)

## 6.2 Robust SM Regulator for Perturbed Nonminimum Phase System

In this section we present an example applying the proposed methodology shown in 3.5, the unknown perturbations are considered of two types: vanishing and non-vanishing ones. Consider a third order system modeled by the equations

$$\begin{aligned}
\dot{x}_1 &= x_2 + 2sin(x_1) + g_1(x,t) \\
\dot{x}_2 &= u + g_2(x_2,t) \\
\dot{x}_3 &= x_1 + x_2 + x_3 + g_3(x,t) \\
y &= x_1
\end{aligned} \tag{6.15}$$

and the exosystem

$$\begin{aligned}
\dot{w}_1 &= w_2 \\
\dot{w}_2 &= -w_1
\end{aligned} \tag{6.16}$$

first, we define the error variable $z_1 = x_1 - y_{ref}$ where $y_{ref} = w_1$, the dynamic of $z_1$ is given by

$$\dot{z}_1 = x_2 + 2sin(x_1) + g_1(x,t) - \dot{y}_{ref}$$

then, we take $x_2$ as a virtual control for $z_1$. Now, we impose the desired dynamics for this block and the desired value for $x_2$ is $x_{2,d} = -k_1 z_1 - w_2 + 2sin(x_1) - \rho_1 sigm(p_1 z_1)$ where $k_1 > 0$, then following the iterative transformation (3.114)-(3.124) we define $z_2 = x_2 - x_{2,d}$, the dynamics for $z_2$ are

$$\dot{z}_2 = u + g_2(x,t) - \dot{x}_{2,d}$$

now we define the sliding variable $\sigma = z_2 - \alpha_3$, where $\alpha_3 = c(x_3 - \pi_3(w)) - \rho_3 sigm(p_3 z_3)$ with $c > 0$ and $\pi_3(w) = -w_2$ is the steady state for $x_3$, that was calculated solving the Francis-Isidori-Byrnes equation (3.90). The super-twisting controller [Levant, 2001] is in the form

$$\begin{aligned} u &= -k_{11} |\sigma|^{1/2} sign(\sigma) + v_{11} \\ \dot{v}_{11} &= -k_{12} sign(\sigma) \end{aligned} \qquad (6.17)$$

under conditions (3.97), when the sliding mode occurs on the manifold $\sigma = 0$ then the dynamics of the system are described by

$$\begin{aligned} \dot{z}_1 &= -k_1 z_1 + z_2 - \rho_1 sigm(p_1 z_1) + g_1(z,w,t) \\ z_2 &= \alpha_3(x_3, w) \\ \dot{z}_3 &= (1-c)z_3 - \rho_3 sigm(p_3 z_3) + g_3(z,w,t) \end{aligned} \qquad (6.18)$$

where $z = [z_1, z_2]^T$ The selected control parameters are $c = 65$, $k_1 = 20$, $k_{11} = 160$, $k_{12} = 420$. For the first block in (6.18) we propose the Lyapunov candidate function $V_1 = z_1^2 p_1$ and we obtain $p_1$ as the solution of the Lyapunov equation $2k_1 p_1 = 1$ which is scalar in this case. It is easy to see that the result is $p_1 = 0.0078$. In the same way, for $z_3$ we propose the Lyapunov candidate function $V_3 = z_3^2 p_3$ and $\rho_3 = 3$ and $p_3 = 0.025$. The simulations results are shown in Figures 6.4-5, the first three ones show the results in the case when the perturbations are non-vanishing, then $g_1(x,w,t) = 2sin(t)$ and $g_3(x,w,t) = 2cos(t) + 3sin(t)$. For the vanishing case, the results are in the last two figures, and for that we define $g_1(x,w,t) = 1.5(x_3 + w_2)$, $g_3(x,w,t) = 2.3(x_1 - w_1)$. The output tracking performance is shown in Figure 6.4 and Figure 6.5 presents the output tracking error $z_1$. Figure 6.6 shows the tracking error $e_3 = x_3 - \pi_3(w)$ and the variable $z_2$.

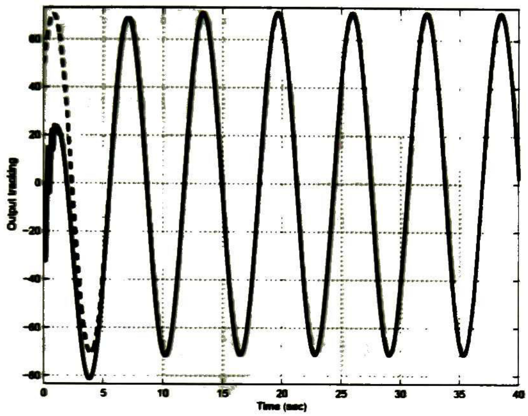


Figure 6.4: Output $x_1$ (solid) and reference signal $w_1$ (dotted), under non-vanishing perturbation.

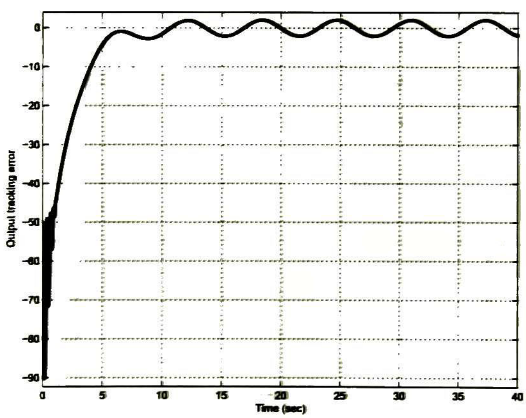


Figure 6.5: Error $z_1$, under non-vanishing perturbation.

In Figure 6.7 we present the result for the output tracking error when the system is under vanishing perturbation. For this case we change the gains $c = 6$ and $k_1 = 2$. Finally, in Figure 6.8 we present the steady state error for $x_3$ and the variable $z_2$.

## 6.3 Discrete-Time Sliding Mode Regulator for Pendubot

In this section, we apply the proposed control scheme presented in the section 5 to a discrete version of the Pendubot [Spong and Vidyasagar, 1989] which is set as a nonlinear affine discrete-time system. This model was obtained in [Rivera et al., 2010] by means of the Symplectic Euler method outlined in [Stern and Desbrun, 2006]:

$$\begin{aligned}
x_{1,k+1} &= x_{1,k} + \delta x_{3,k} + \delta^2(b_{3,k}p_{1,k} + b_{3,k}u_k) \\
x_{2,k+1} &= x_{2,k} + \delta x_{4,k} + \delta^2(b_{4,k}p_{2,k} + b_{4,k}u_k) \\
x_{3,k+1} &= x_{3,k} + \delta(b_{3,k}p_{1,k} + b_{3,k}u_k) \\
x_{4,k+1} &= x_{4,k} + \delta(b_{4,k}p_{2,k} + b_{4,k}u_k) \\
y_k &= x_{2,k}.
\end{aligned} \tag{6.19}$$

where $x_{i,k} = x_i(k\delta)$, $i = 1,2,3,4$, $b_{3,k} = b_3(x_{2,k})$, $b_{4,k} = b_4(x_{2,k})$, $p_{1,k} = p_1(x_k)$, $p_{2,k} = p_2(x_k)$, $x_k = x(k\delta)$. Please refer to [Rivera et al., 2010] for the detailed description of the parameters. This model will be used for the control law design

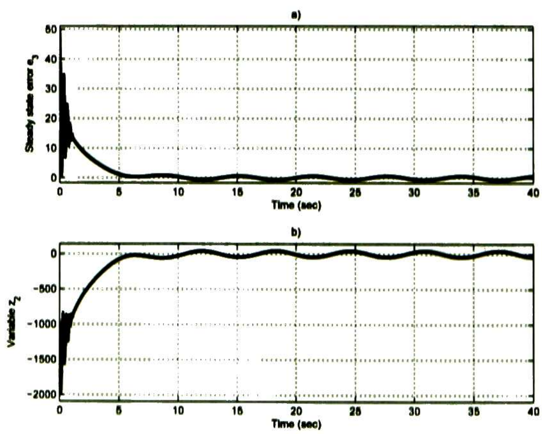


Figure 6.6: a) Error $e_3$; b) variable $z_2$. Under non-vanishing perturbation.

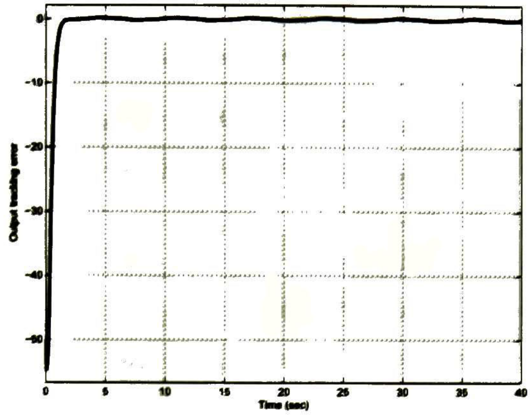

Figure 6.7: Error $z_1$, under vanishing perturbation.

### Discrete-time SM Regulator for Pendubot

To represent the system (6.19) to the regular form, we define a nonlinear transformation $\bar{x}_k = (\bar{x}_{1,k}, \bar{x}_{2,k}, \bar{x}_{3,k}, \bar{x}_{4,k})^T = \psi(x_k)$ of the following form:

$$\begin{aligned}\bar{x}_{1,k} &= x_{1,k} - b_{3,k-1}b_{4,k-1}^{-1}x_{2,k}\\ \bar{x}_{2,k} &= x_{2,k} - \delta x_{4,k}\\ \bar{x}_{3,k} &= x_{3,k} - b_{3,k-1}b_{4,k-1}^{-1}x_{4,k}\\ \bar{x}_{4,k} &= x_{4,k}.\end{aligned} \tag{6.20}$$

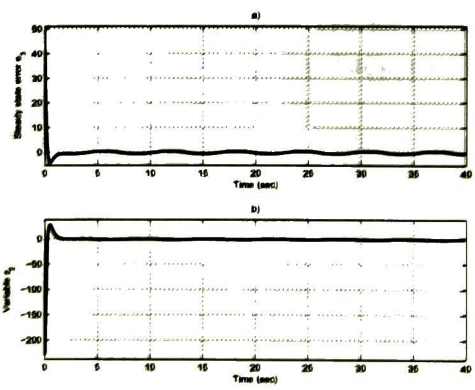

Figure 6.8: Error $e_3$ and variable $z_2$, under vanishing perturbation.

The discrete-time model of the Pendubot (6.19) is now represented in the new variables $\bar{x}_k$ by taking one step ahead in (6.20) in the regular form

$$\begin{aligned}
\bar{x}_{1,k+1} &= \bar{x}_{1,k} + (\rho_{k-1} - \rho_k)(\bar{x}_{2,k} + 2\delta\bar{x}_{4,k}) + \delta\bar{x}_{3,k} \\
&+ \delta^2\bar{b}_{3,k}(\bar{p}_{1,k} - \bar{p}_{2,k}) \\
\bar{x}_{2,k+1} &= \bar{x}_{2,k} + \delta\bar{x}_{4,k} \\
\bar{x}_{3,k+1} &= \bar{x}_{3,k} + (\rho_{k-1} - \rho_k)\bar{x}_{4,k} + \delta\bar{b}_{3,k}(\bar{p}_{1,k} - \bar{p}_{2,k}) \\
\bar{x}_{4,k+1} &= \bar{x}_{4,k} + \delta\bar{b}_{4,k}(\bar{p}_{2,k} + u_k) \\
y_k &= \bar{x}_{2,k}
\end{aligned}$$

where $\rho_k = \bar{b}_{3,k}\bar{b}_{4,k}^{-1}$, $\bar{b}_{3,k} = b_3(\bar{x}_{2,k})$, $\bar{b}_{4,k} = b_4(\bar{x}_{2,k})$, $\bar{p}_{1,k} = p_1(\bar{x}_k)$, $\bar{p}_{2,k} = p_2(\bar{x}_k)$.

Now, the steady-state for system (6.19), $x_k^r = (x_{1,k}^r, x_{2,k}^r, x_{3,k}^r, x_{4,k}^r)^T$ will be determined. For that, we consider the following exosystem that will generate a sinusoidal shape output reference signal:

$$\begin{aligned}
w_{1,k+1} &= \cos(\alpha\delta)w_{1,k} + \sin(\alpha\delta)w_{2,k} \\
w_{2,k+1} &= -\sin(\alpha\delta)w_{1,k} + \cos(\alpha\delta)w_{2,k},
\end{aligned} \tag{6.21}$$

where $\alpha$ is the frequency of the generated signals and if the initial conditions are chosen as $w_{1,0} = w_{2,0}$, then, the amplitude is $\sqrt{2}w_{1,0}$.

The steady state for the output is assigned as $x_{2,k}^r = w_{2,k}$. Making use of a natural steady-state constraint given in [J. et al., 2008], that states that, the sum of the two angles, $q_1$ and $q_2$ equals $\pi/2$, one can easily determine the steady-state for $x_{1,k}$ as $x_{1,k}^r = \pi/2 - x_{2,k}^r$. Finally, the steady-state values for $x_{3,k}$ and $x_{4,k}$ can be determined by using the first two equations in (6.19), in the form of difference equations, i. e., $x_{3,k+1}^r = (x_{1,k+1}^r - x_{1,k}^r)/\delta$ and $x_{4,k+1}^r = (x_{2,k+1}^r - x_{2,k}^r)/\delta$.

Transforming $x_k^r$ through the diffeomorphism (6.20) results in the steady state vector $\bar{x}_k^r = (\bar{x}_{1,k}^r, \bar{x}_{2,k}^r, \bar{x}_{3,k}^r, \bar{x}_{4,k}^r)^T$:

$$\begin{aligned}
\bar{x}_{1,k}^r &= x_{1,k}^r - b_{3,k-1}^r b_{4,k-1}^{-1r} x_{2,k}^r \\
\bar{x}_{2,k}^r &= x_{2,k}^r - \delta x_{4,k}^r \\
\bar{x}_{3,k}^r &= x_{3,k}^r - b_{3,k-1}^r b_{4,k-1}^{-1r} x_{4,k}^r \\
\bar{x}_{4,k}^r &= x_{4,k}^r.
\end{aligned} \tag{6.22}$$

where $b_{3,k}^r = b_3(x_{2,k}^r)$, $b_{4,k}^r = b_4(x_{2,k}^r)$.

Now, we introduce the error variable vector $z_k = \bar{x}_k - \bar{x}_k^r$, and taking one step ahead on $z_k$, yields in the following error system:

$$z_{k+1} = \phi(z_k, \bar{x}_k^r) \tag{6.23}$$

where $\phi(\cdot) = (\phi_1(\cdot), \phi_2(\cdot), \phi_3(\cdot), \phi_4(\cdot))^T$ with

$$\begin{aligned}
\phi_{1,k}(\cdot) &= z_{1,k} + \bar{x}_{1,k}^r + (\rho_{k-1} - \rho_k)(z_{2,k} + \bar{x}_{2,k}^r) \\
&+ 2\delta(\rho_{k-1} - \rho_k)(z_{4,k} + \bar{x}_{4,k}^r) \\
&+ \delta(z_{3,k} + \bar{x}_{3,r,k}) + \bar{c}_k, \\
\phi_{2,k}(\cdot) &= z_{2,k} + \bar{x}_{2,k}^r + \delta(z_{4,k} + \bar{x}_{4,r,k}) - \bar{x}_{2,k+1}^r \\
\phi_{3,k}(\cdot) &= z_{3,k} + \bar{x}_{3,k}^r + (\rho_{k-1} - \rho_k)(z_{4,k} + \bar{x}_{4,k}^r) \\
&+ \delta\bar{b}_{3,k}(\bar{p}_{1,k} - \bar{p}_{2,k}) - \bar{x}_{3,k+1}^r, \\
\phi_{4,k}(\cdot) &= z_{4,k} + \bar{x}_{4,k}^r + \delta\bar{b}_{4,k}\bar{p}_{2,k} - \bar{x}_{4,k+1}^r + \delta\bar{b}_{4,k}u_k.
\end{aligned}$$

and $\bar{c}_k = \delta^2\bar{b}_{3,k}(\bar{p}_{1,k} - \bar{p}_{2,k}) - \bar{x}_{1,k+1}^r$. We can see that the part of error system (6.23) has the BC form, therefore, to design a sliding manifold we first apply the BC technique. Define a new variable $\varepsilon_{2,k} = z_{2,k}$, and taking one step ahead we have

$$\varepsilon_{2,k+1} = \varepsilon_{2,k} + \bar{x}_{2,k}^r + \delta\bar{x}_{4,k}^r + \delta z_{4,k} - \bar{x}_{2,k+1}^r$$

Considering $z_{4,k}$ as virtual control, we formulate its desired value $z_{4,k}^d$ as

$$z_{4,k}^d = -(1/\delta)\left(\bar{x}_{2,r,k} + \delta\bar{x}_{4,k}^r + k_2\varepsilon_{2k} - \bar{x}_{2,k+1}^r\right)$$

to induce the desired dynamics $k_2\varepsilon_{2k}$ with $|k_2| < 1$ for $\varepsilon_{2k}$. Defining now a new error variable $\varepsilon_{4,k} = z_{4,k} - z_{4,k}^d$, and taking a step ahead yields

$$\begin{aligned}
\varepsilon_{4,k+1} &= z_{4,k} + \bar{x}_{4,k}^r + \delta\bar{b}_{4,k}\bar{p}_{2,k} - \bar{x}_{4,k+1}^r + \delta\bar{b}_{4,k}u_k \\
&\quad +(1/\delta)\left(\bar{x}_{2,k+1}^r + \delta\bar{x}_{4,k+1}^r\right) \\
&\quad +(1/\delta)\left(k_2\varepsilon_{2,k+1} - \bar{x}_{2,k+2}^r\right)
\end{aligned}$$

For the residual dynamics, we also define $\varepsilon_{1,k} = z_{1,k}$ and $\varepsilon_{3,k} = z_{3,k}$. Then, the system (6.23) is represented in the new variables as

$$\varepsilon_{1,k+1} = \psi_1(\cdot) \tag{6.24}$$

$$\varepsilon_{2,k+1} = \psi_2(\cdot)$$

$$\varepsilon_{3,k+1} = \psi_3(\cdot)$$

$$\varepsilon_{4,k+1} = \psi_4(\cdot) + \delta\bar{b}_{4,k}u_k \tag{6.25}$$

$$\begin{aligned}
\psi_1(\cdot) &= \varepsilon_{1,k} + \bar{x}^r_{1,k} + (\rho_{k-1} - \rho_k)(\varepsilon_{2,k} + \bar{x}^r_{2,k}) \\
&\quad +2\delta(\rho_{k-1} - \rho_k)(\varepsilon_{4,k} - (1/\delta) \\
&\quad \left(\bar{x}^r_{2,k} + k_2\varepsilon_{2k} - \bar{x}^r_{2,k+1}\right)) \\
&\quad +\delta(\varepsilon_{3,k} + \bar{x}^r_{3,k}) + \delta^2\bar{b}_{3,k}(\bar{p}_{1,k} - \bar{p}_{2,k}) \\
&\quad -\bar{x}^r_{1,k+1}, \\
\psi_2(\cdot) &= \varepsilon_{2,k} + \bar{x}^r_{2,k} + \delta\bar{x}_{4,r,k} + \delta(\varepsilon_{4,k} + z^d_{4,k}) \\
&\quad -\bar{x}^d_{2,k+1}, \\
\psi_3(\cdot) &= \varepsilon_{3,k} + \bar{x}^r_{3,k} + (\rho_{k-1} - \rho_k)(\varepsilon_{4,k} + z^d_{4,k}) \\
&\quad +(\rho_{k-1} - \rho_k)(\bar{x}^r_{4,k}) \\
&\quad +\delta\bar{b}_{3,k}(\bar{p}_{1,k} - \bar{p}_{2,k}) - \bar{x}^r_{3,k+1}, \\
\psi_4(\cdot) &= \varepsilon_{4,k} - (1/\delta)\left(\bar{x}^r_{2,k} + \delta\bar{x}^r_{4,k} + k_2\varepsilon_{2k}\right) \\
&\quad -(1/\delta)\left(-\bar{x}^r_{2,k+1}\right) \\
&\quad +\bar{x}^r_{4,k} + \delta\bar{b}_{4,k}\bar{p}_{2,k} \\
&\quad +(1/\delta)(\bar{x}^r_{2,k+1} - \bar{x}^r_{2,k+2}) \\
&\quad +(k_2/\delta)(\varepsilon_{2,k} + \bar{x}^r_{2,k} + \delta\bar{x}^r_{4,k} - \bar{x}^r_{2,k+1}) \\
&\quad +k_2(\varepsilon_{4,k} - (1/\delta)\left(\bar{x}^r_{2,k} + \delta\bar{x}^r_{4,k}\right)) \\
&\quad +k_2(\varepsilon_{4,k} - (1/\delta)\left(k_2\varepsilon_{2k} - \bar{x}^r_{2,k+1}\right))
\end{aligned}$$

Now, we regard the system (6.25) in the form $\varepsilon_{k+1} = \left(\varepsilon^1_{k+1}, \varepsilon^2_{k+1}\right)^T$ with

$$\begin{aligned}
\varepsilon^1_{k+1} &= \left(\varepsilon_{1,k+1}, \varepsilon_{2,k+1}, \varepsilon_{3,k+1}\right)^T \\
\varepsilon^2_{k+1} &= \left(\varepsilon_{4,k+1}\right).
\end{aligned}$$

Also we have

$$\varepsilon_{k+1} = \psi(\varepsilon_k, \bar{x}^r_k) + \gamma(\varepsilon_k, \bar{x}^r_{2,k})u_k$$

where where $\psi(\cdot) = (\psi^1(\cdot), \psi^2(\cdot))^T$ with $\psi^1(\cdot) = (\psi_1(\cdot), \psi_2(\cdot), \psi_3(\cdot))^T$ $\psi^2(\cdot) = \psi_4(\cdot)$ and $\gamma(\cdot) = (\gamma^1(\cdot), \gamma^2(\cdot))^T$ . $\gamma^1(\cdot) = (0, 0, 0)^T$ $\gamma^2(\cdot) = \delta\bar{b}_{4,k}$, $\bar{\rho}_k = (\rho_{k-1} - \rho_k)$, and $\bar{p}_{3,k} = \bar{p}_{1,k} - \bar{p}_{2,k}$. Now, one defines the sliding manifold as follows:

$$\sigma_k = 0, \ \sigma_k = \varepsilon_{4,k} + k_1\varepsilon_{1,k} + k_3\varepsilon_{3,k} \tag{6.26}$$

with parametrs $k_1$ and $k_3$. To force the states of the system (6.25) to the sliding manifold (6.26) one make use of the proposed discrete-time super-twisting controller

$$\begin{aligned} u_k &= -\rho_1\sqrt{|\sigma_k|}sign(\sigma_k) + \zeta_k \\ \zeta_{k+1} &= \zeta_k - \delta(\rho_2\sqrt{|\sigma_k|}sign(\sigma_k) + q\zeta_k) \end{aligned}$$

When the sliding mode occurs on $\sigma_k = 0$, one can calculate $\varepsilon_{4,k}$ from (6.26) as

$$\varepsilon_{4,k} = -k_1\varepsilon_{1,k} - k_3\varepsilon_{3,k}. \tag{6.27}$$

Then, by replacing (6.27) in the three first equations of (6.25) yields to the follwing SM equation:

$$\begin{aligned} \varepsilon^1_{k+1} &= \psi_k \\ \psi_k &= \psi^1(\varepsilon^1_k, \varepsilon^2_k, \bar{x}^r_k) \mid_{\sigma_k=0} . \end{aligned} \tag{6.28}$$

The parameters $k_1$, $k_2$ and $k_3$ should stabilize the sliding mode dynamics (6.28). For a proper choice of such constant parameters one can linearize the SM equation (6.28) as

$$\varepsilon^1_{k+1} = A_{sm}(\kappa)\varepsilon^1_k$$

where $A_{sm}(\kappa) = \partial\psi_k/\partial\varepsilon^1_k \mid_{z^1_k=0}$, with $\kappa = (k_1, k_2, k_3)$. To choose the design parameters, a polynomial with desired poles is proposed as $p_d(z) = (z-\lambda_1)(z-\lambda_2)(z-\lambda_3)$. The coefficients of the characteristic equation that results from the matrix $A_{sm}$ are equalized with the ones related with $p_d(z)$, i. e., $det(zI - A_{sm}) = p_d(z)$. So, in such manner one can find explicit relations for $\kappa$. In this case $lim_{k\to\infty}\varepsilon_k = 0$, accomplishing with the control objective.

### Simulations

In order to show the effectiveness of the control methodology here proposed, simulations have been carried out. The nominal values of the parameters of the Pendubot are defined as follows: $m_1 = 0.8293$, $m_2 = 0.3402$, $l_1 = 0.2032$, $l_{c1} = 0.1551$, $l_{c2} = 0.1635125$, $g = 9.81$, $I_1 = 0.00595035$, $I_2 = 0.00043001254$, $\mu_1 = 0.00545$, $\mu_2 = 0.00047$. The constant parameters used in the control law are $\lambda_1 = 0.9941 + 0.0030j$, $\lambda_2 = 0.9941 - 0.0030j$ and $\lambda_3 = 0.9978$. The vector $\kappa$ depends on the different values assigned to $\delta$ and therefore it is only shown for the particular value of $\delta = 0.001$, resulting in $k_1 = 3710.0$, $k_2 = 0.037281$, $k_3 = 103.17$. The controller gains are selected as $\rho_1 = 3$, $\rho_2 = 1$ and $q = 2$. The parameters used in the exosystem (6.21) are $\alpha = 0.3$, $w_{1,0} = w_{2,0} = 0.09$ and the reference signal is given by $w_{2,k}$. It is worth mentioning that the Pendubot has been simulated as a continuous time system, in order to consider a more realistic condition. In Figure 6.9 is shown the output $x_{2,k}$ performance. We can observe that the angle $x_{2,k}$ tends asymptotically to the reference.

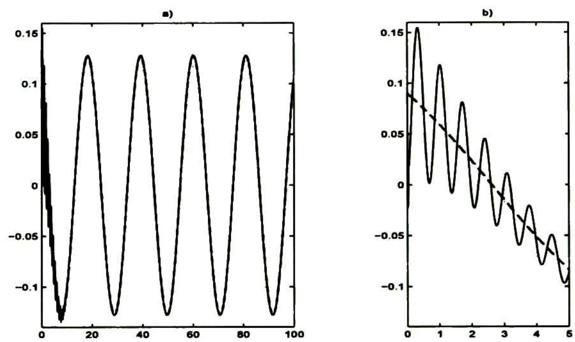


Figure 6.9: a) Angle $x_{2,k}$ (solid) and reference signal $w_{2,k}$ (dotted) [rad vs s]; b) zoom of a)

In Figure 6.10 is shown the residual dynamics $x_1$ and $x_3$. We can see that those variables become stable.

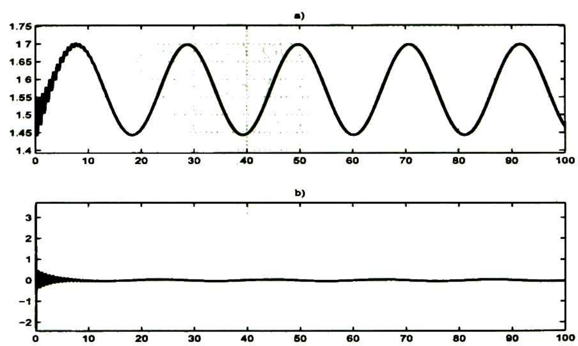


Figure 6.10: a) Angle $x_{1,k}$ performance [rad vs s]; b)speed $x_{3,k}$ [(rad/s) vs s]

# Chapter 7

# Conclusions and Future Work

## 7.1 Conclusions

Solutions for Sliding Mode Output Regulation problem for nonlinear nonminimum phase (NP) systems with both matched and unmatched perturbations have been presented. The causal and noncausal cases for the reference signals are considered in the presented solutions. The presented solutions can be applied to the systems with arbitrary relative degree vector.

The Integral Sliding Mode Output Regulation Problem for NP perturbed nonlinear systems presented in the unstructured General and Regular forms has been introduced for the noncausal case, and the solvability conditions are derived.

Two robust regulators designed based on the block control linearization for a class of NP nonlinear systems presented in structured form have been proposed for the noncausal case. In the first regulator design the matched perturbation is rejected while in the second one both matched and unmatched perturbations are rejected.

Three robust regulators for NP perturbed nonlinear systems presented in structured form have been proposed for the causal case. On one of these three regulators designs both matched and unmatched perturbations are rejected, in the other two designs the matched perturbation is rejected. Two approaches to obtain a bounded solution for unstable internal dynamics have been presented.

Discrete-Time SM Output Regulation Problem for discrete time NP nonlinear systems presented in structured form with matched perturbations has been introduced for the noncausal case and the solvability conditions are derived.

## 7.2 Future Work

As future work, the following topics are considered:

- The real time application of the proposed regulators to induction motor and pendubot.
- To extend the Robust SM Output Regulation in the noncausal case for the problem when only the output is available.
- To extend the Robust SM Output Regulation in the causal case for the problem when only the output is available.
- To design a adaptive observer which generates the full steady state when only the output is available for the noncausal case and the system has unknown perturbations.
- To design a adaptive observer which generates the full steady state when only the output is available for the noncausal case and the system has unknown perturbations.
- To design a adaptive observer which generates the full steady state when only the output is available for the noncausal case and the system has unknown perturbations.
- To design a adaptive observer which generates the full steady state when only the output is available for the causal case and the system has unknown perturbations.

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# Appendix A

# Publications

A list of the works published during the development of this thesis is presented in this section.

## Journal

- *J. D. Sánchez-Torres, A. G. Loukianov, M. I. Galicia y Jorge Rivera.* **Robust Nested Sliding Modes Integral Control for Anti-lock Brake System.** International Journal Vehicle Design. International Journal Vehicle Design. Vol. 62, Nos. 2/3/4, 2013

## International Conferences

- *Marcos I. Galicia, Alexander G. Loukianov and Jorge Rivera,* **Robust regulator for nonminimum phase systems by Lyapunov Redesign**, World Automation Congress (WAC), 2012

- *Marcos I. Galicia, Alexander G. Loukianov and Jorge Rivera and Vadim I. Utkin,* **Discrete-time sliding mode regulator for nonminimum phase systems**, IEEE 51st Annual Conference on Decision and Control (CDC), 2012

- *Marcos I. Galicia, Alexander G. Loukianov, B. Castillo-Toledo y Jorge Rivera.* **Second order SM Regulator for Nonlinear Non Minimum Phase Systems.** 18th World Congress of the International Federation of Automatic Control, IFAC 2011.

- *Marcos I. Galicia, Alexander G. Loukianov and Jorge Rivera.* **Second Order Sliding Mode Sensorless Torque Regulator for Induction Motor**. 49th IEEE Conference on Decision and Control, CDC 2010.

- *B. Castillo-Toledo, S. Di Gennaro, M. I. Galicia, A. G. Loukianov, and J. Rivera.* **Indirect Discrete-Time Sliding Mode Torque Control of Induction Motors**. 19th International Conference on Electrical Machines, ICEM 2010.

- *Marcos I. Galicia, B. Castillo-Toledo, Alexander G. Loukianov, S. Di Gennaro y Jorge Rivera.* **Discrete Time Sliding Mode Torque Control of Induction Motor** World Automation Congress, WAC 2010.

- *Marcos I. Galicia, Juan Diego Sánchez, Alexander G. Loukianov y Jorge Rivera.* **Sliding Mode Control for Antilock Brake System**. 7th International Conference on Electrical Engineering, Computing Science and Automatic Control, CCE 2010.

- *Alexander G. Loukianov, Victor Utkin, Marcos Galicia y Jorge Rivera.* **Automotive High Order SM Integral Nested Control**. Congreso Nacional 2010 de la Asociación de México de Control Automático, AMCA 2010.

- *Juan Diego Sánchez, Alexander G. Loukianov, Marcos I. Galicia, Javier Ruiz y Jorge Rivera.* **ABS and Active Suspension Control via High Order Sliding Modes and Linear Geometric Methods for Disturbance Rejection**. 8th International Conference on Electrical Engineering, Computing Science and Automatic Control, CCE 2011.

- *Juan Diego Sánchez, Alexander G. Loukianov, Marcos I. Galicia, Jorge Rivera* **A Sliding Mode Regulator for Antilock Brake System**. 18th World Congress of the International Federation of Automatic Control, IFAC 2011.

- *Sanchez-Torres, J.D.; Ferreira de Loza, A.; Galicia, M.I.; Loukianov, A.G.,* **ABS design and active suspension control based on HOSM**. American Control Conference (ACC), 2013

- *Iván Vázquez, Marcos I. Galicia, Juan Diego Sánchez, Alexander G. Loukianov, Pavel A. Kruchinin.* **Integral Nested Sliding Mode Control for Antilock Brake System**. The International Symposium on Advances in Automotive Control AAC 2010.

# CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL I.P.N.
## UNIDAD GUADALAJARA

El Jurado designado por la Unidad Guadalajara del Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional aprobó la tesis

**Regulación Robusta por Modos Deslizantes de Sistemas no Lineales de Fase no Mínima–Robust Sliding Mode Regulation of Nonminimum Phase Nonlinear Systems**

del (la) C.

**Marcos Israel GALICIA CUEVA**

el día 05 de Diciembre de 2013.

| | |
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| Dr. Jorge Rivera Domínguez<br>Profesor Investigador C<br>Universidad de Guadalajara | Dra. Alma Yolanda Alanis García<br>Profesor<br>CUCEI UDG |