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“Singularidades de Curvatura y Perturbaciones Gravitacionales en Agujeros de Gusano”

Tesis que presenta

Juan Carlos Del Águila Rodríguez

para obtener el Grado de

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en la Especialidad de

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Director de tesis: Dr. Tonatiuh Matos Chassin

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"Curvature Singularities and Gravitational Perturbations in Wormholes"

by

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"No esperes por el juicio final. Se lleva a cabo cada día."

Albert Camus

Singularidades de Curvatura y Perturbaciones Gravitacionales en Agujeros de Gusano

Resumen

En esta tesis dos temas principales de investigación dentro de Relatividad General son abordados: singularidades de curvatura y perturbaciones gravitacionales en agujeros de gusano.

Primeramente, estudiaremos el comportamiento local de geodésicas en la vecindad de una singularidad de curvatura contenida en un espacio-tiempo estacionario y axialmente simétrico. Además de estas propiedades, se requerirá que las métricas en las que nos enfocaremos admitan primeras integrales de orden cuadrático en los vectores tangentes de sus geodésicas. En particular, buscaremos las condiciones en la geometría del espacio-tiempo para las cuales se puedan encontrar geodésicas nulas y de tipo tiempo que se acerquen arbitrariamente a la singularidad. Dichas condiciones estarán determinadas por las ecuaciones de movimiento de partículas en caída libre. También se analiza la existencia de geodésicas que llegan directamente a la singularidad y que por consiguiente se vuelven incompletas. Los resultados son enunciados en forma de criterios que dependen del tensor métrico inverso junto con cantidades conservadas como energía y momento angular. Como ejemplo, los criterios encontrados son aplicados a la clase de espacio-tiempos de Plebański-Carter. Adicionalmente se propone un elemento de línea que describe a un agujero de gusano cuyas singularidades de curvatura son, de acuerdo a nuestros resultados, inaccesibles a geodésicas causales. Para finalizar se estudian dos agujeros de gusano rotantes y de campo escalar que poseen una singularidad de anillo. Los resultados precedentes se usarán como guía para determinar si estos espacio-tiempos pueden ser considerados regulares.

Después de ello, se tratará el problema de la estabilidad lineal ante perturbaciones gravitacionales en agujeros de gusano estacionarios y esféricamente simétricos. Para esto se hace uso del formalismo de Newman-Penrose, el cual es especialmente útil para expresar tanto radiación gravitacional en Relatividad General, así como el aspecto geométrico de esta teoría. Mediante este método se obtiene una "ecuación maestra" que describe el comportamiento de perturbaciones impares en la norma de Regge-Wheeler. Esta ecuación es posteriormente aplicada a una clase específica de agujeros de gusano de Morris-Thorne y también a la métrica de un agujero de gusano asintóticamente plano con campo escalar. Este último ejemplo es conocido por ser inestable ante perturbaciones radiales, éstas no se tratarán específicamente en este trabajo. El análisis de las ecuaciones que estos espacio-tiempos generan revela que no existen modos inestables de vibración provocados por el tipo de perturbaciones aquí estudiadas.

Curvature Singularities and Gravitational Perturbations in Wormholes

Abstract

In this thesis two principal research topics within General Relativity are explored: curvature singularities and gravitational perturbations in wormholes.

First, we study the local behavior of geodesics in the neighborhood of a curvature singularity contained in a stationary and axially symmetric space-time. Apart from these properties, the metrics we shall focus on will also be required to admit a first integral of quadratic order in the tangent vectors of their geodesics. In particular, we search for the conditions on the geometry of the space-time for which null and time-like geodesics can arbitrarily approach the singularity. These conditions are determined by the equations of motion of a freely-falling particle. We also analyze the existence of geodesics that can directly reach the singularity and then become incomplete. The results are stated as criteria that depend on the inverse metric tensor along with conserved quantities such as energy and angular momentum. As an example, the derived criteria are applied to the Plebański-Carter class of space-times. We additionally propose a line element that describes a wormhole whose curvature singularities are, according to our results, inaccessible to causal geodesics. To finalize we study two rotating scalar field wormholes that possess a ring singularity. The preceding results will be used as a guide to determine if this space-times can be considered as regular.

After that, the problem of linear stability of gravitational perturbations in stationary and spherically symmetric wormholes is treated. For this purpose, we employ the Newman-Penrose formalism which is well-suited for expressing gravitational radiation in General Relativity, as well as the geometrical aspect of this theory. With this method we obtain a “master equation” that describes the behavior of gravitational perturbations that are of odd-parity in the Regge-Wheeler gauge. This equation is later applied to a specific class of Morris-Thorne wormholes and also to the metric of an asymptotically flat scalar field wormhole. The last example is known to be unstable under radial perturbations, these are not specifically treated in this work. The analysis of the equations that these space-times yield reveals that there are no unstable vibrational modes generated by the type of perturbations here studied.

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General Notation and Conventions

$g_{\mu\nu}$	Metric tensor
$R^{\alpha}{}_{\beta\mu\nu}$	Riemann tensor
$R_{\mu\nu}$	Ricci tensor
R	Ricci scalar
$G_{\mu\nu}$	Einstein tensor
$T_{\mu\nu}$	Stress-energy tensor
$C_{\alpha\beta\mu\nu}$	Weyl tensor
$\Psi_i \;\; (i = 0, 1, \ldots, 4)$	Weyl scalars
∇	Covariant derivative
$\Box = \nabla^{\mu}\nabla_{\mu}$	D'Alambertian operator
$\pounds$	Lie derivative
$\otimes$	Outer product
$\wedge$	Wedge product
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\varnothing$	The empty set
C^{∞}	The set of infinitely continuously differentiable (smooth) functions
$z_m^{\ \mu} = \{l^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu}\}$	Null tetrad
Speed of light	$c = 2.997\,924\,58 \times 10^{8}\,\mathrm{m/s}$ ($c = 1$ in geometrized units)
Gravitational constant	$G = 6.674\,301\,5 \times 10^{-11}\,\mathrm{m^3 kg/s^2}$ ($G = 1$ in geometrized units)
Acceleration due to Earth's gravity	$g_{\oplus} = 9.806\,65\,\mathrm{m/s^2}$

Throughout the whole thesis the following conventions are used:

- The Einstein summation convention for repeated indices. E.g., $k^{\mu}k_{\mu} = \sum_{\mu} k^{\mu}k_{\mu}$, and similarly for any pair of repeated indices.
- Greek indices are tensor indices that take values $\mu, \nu = 0, \ldots, 3$. In Chapter 3, lower-case Latin indices take values $i, j = 0, 3$, while upper-case Latin indices take values $A, B = 1, 2$. In Section 2.2 and Chapter 4, lower-case indices are tetrad indices that take values $m, n = 0, \ldots, 3$.
- The $(-,+,+,+)$ signature is utilized in all chapters, with the exceptions of Section 2.2 and Chapter 4, in which the $(+,-,-,-)$ signature is adopted instead.
- Round brackets denote symmetrization of the indices enclosed, while squared brackets denote anti-symmetrization. For instance,

$$S_{(\mu\nu)} = \frac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu}\right), \quad S_{[\mu\nu]} = \frac{1}{2}\left(S_{\mu\nu} - S_{\nu\mu}\right).$$

Chapter 1

Introduction

During approximately the last half century, the theoretical development of General Relativity, the classical field theory of gravitation, has experienced a momentous growth that has led to its consolidation as one of the most relevant fields in Modern Physics. Despite it being by now more than a century old and very well-established within the scientific knowledge, the theory still contains open questions that are nowadays subject of active research. Among them, the fields of space-time singularities and wormholes can be mentioned. These are the two main topics to which this thesis is devoted to.

Singularities are perhaps the least understood aspect of General Relativity. They are believed to describe fascinating matters such as the origin of the Universe and the ultimate fate of gravitational collapse. Even so, singularities are often seen as inherently ill objects in the theory and their appearance in space-times is considered unpleasant, specially if they can be perceived by its distant observers. In fact, in order to deal with this problem, Roger Penrose has proposed the "cosmic censorship" conjecture which, roughly speaking, forbids the appearance of such observable singularities. In this work we will study if it is possible that the curves followed by free-falling physical observers (geodesics) can avoid contact with the singularities of a given space-time. The opposite behavior, this is, geodesics reaching or traveling near the singularity will also be of importance. Under what conditions on the geometry of the space-time either of both situations happen will be the main focus. Our final intention will be to contribute mathematical arguments on the possibility that a space-time possessing singularities could be ultimately considered as regular and well-behaved.

On a separate matter, but not completely unrelated, wormholes have gained the interest of physicists due to their peculiar characteristics. In particular, their ability to communicate distant regions of the same universe (or two different universes) through a kind of space-time "shortcut" is rather attractive. Their existence, however, encounters significant troubles due to the fact that the most known and simple wormhole solutions need to be supported by exotic matter (matter that violates the energy conditions), and seem to be highly unstable when traversed by any test particle. In this last regard, the mathematical analysis needed to describe the gravitational perturbations of even the simplest wormholes, i.e., spherically symmetric and static, can be unwieldy. In this thesis, a framework will be introduced that can ease the calculations needed and that may enable to study perturbations in more complicated wormholes such as rotating and axially symmetric. This in turn, can help in the objective of finding, at least theoretically, stable wormhole solutions. Both of the topics treated in this thesis come together in some recently found wormhole space-times that contain curvature singularities.

This work is structured in the following way. In this chapter a brief introduction to General Relativity is given along with a historical development that emphasizes black hole solutions and wormholes. Chapter 2 contains the background mathematical tools used during this thesis, here the basic notions related to singularities and gravitational perturbations are also introduced. Readers familiar with General Relativity may skip the first two chapters, although, sections 2.2, 2.3, and 2.4 are a recommended read since they provide the adequate context for the rest of the thesis. Original research is presented in chapters 3 and 4 where the main results of the thesis regarding curvature singularities and gravitational perturbations, respectively, are discussed. Finally, chapter 5 closes this work by drawing conclusions and reflecting on perspectives, as well as proposing possible future directions to pursue.

1.1 General Relativity: A New Conception of Time and Space

In 1915, the introduction by Albert Einstein of the theory of Relativity to the scientific discourse changed drastically the previously established notion of time and space. In it, both of these concepts are combined in a continuum that receives the name of "space-time" which cannot be separated into two independent entities. This signified abandoning the idea that the universe exists in a three-dimensional Euclidean space governed by an absolute time, replacing it with a curved or deformed four-dimensional space-time (3 spatial dimensions and 1 temporal dimension). Einstein proposed the matter content of the universe as the responsible of this deformation. The renowned theoretical physicist John Wheeler described Einstein's theory in the following way: "Space-time tells matter how to move; matter tells space-time how to curve."

To express the relationship between the space-time geometry and matter content, Einstein postulated the famous field equation

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.1}$$

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein tensor and the stress-energy tensor, respectively. The first one contains information about the curvature of space-time and the second one describes the matter and energy of the physical system. Additionally, $\kappa = 8\pi G/c^4$ is a proportionality constant, G being the universal gravitational constant and c the speed of light in the vacuum.

The Einstein tensor $G_{\mu\nu}$ depends on the metric tensor of the space-time (see Chapter 2 for the mathematical details), which defines the geometry of the space and assigns to it some sense of distance or "measure". It is commonly expressed through the line element as $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. In the context of General Relativity, the necessity of a concept of metric comes from the fact that space is deformed, the metric is a mathematical object that provides a measure of this deformation. As a contrast, the case of three-dimension Euclidean space has a trivial metric $g_{ij} = diag(1,1,1)$, this means that it is a flat space or, equivalently, a space without curvature.

Equation (1.1) revolutionized the world of Physics as understood in the 20th century. It made possible to predict how space-time would curve in the presence of certain type of matter, or the converse, the type of matter that could produce a desired deformation of the geometry. Unfortunately, (1.1) is actually a set of 10 non-linear, second order partial differential equations, hence finding exact solutions to it has not been an easy task.

The success of General Relativity as a physical theory is undeniable and constitutes, together with Quantum Physics, the two pillars of Modern Physics. Its applications reside majorly in the fields of Astrophysics and Cosmology, where it has predicted the existence of black holes, neutron stars, and gravitational waves to name a few. Throughout the years, this theory has undergone a great number of tests, surpassing everyone of them and finally showing that the physical formulation of Einstein is correct. Among these tests, the gravitational redshift, the bending of light paths due to the mass of the Sun, and the perihelion precession of Mercury can be mentioned. There is however, one phenomenon that cannot be explained entirely through classical Relativity: the accelerated expansion of the universe[1]. It has remained, until know, as one of the greatest mysteries for Cosmology and current Physics.

1.1.1 The Schwarzschild Geometry

The first solution to (1.1) was found by the German physicist Karl Schwarzschild in 1916. It is a static and spherically symmetric solution in the vacuum, i.e., $T_{\mu\nu} = 0$ and hence, $G_{\mu\nu}$ vanishes too. In this solution, the space-time metric $g_{\mu\nu}$ is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - 2m/r} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1.2}$$

here the parameter m is the mass of the spherical object that curves the space-time, for example, a star. This geometry has been of great physical importance since it can mathematically describe a black hole. The principal features of metric (1.2) are:

- Coordinate singularity at $r = 2m$. The hyper-surface $r = 2m$ defines the distinctive property of a black hole: the event horizon. The curvature in this frontier of space-time is such that, once an object (even light itself) enters it, said object cannot escape back to the outer region. More properly, an event horizon is the boundary of the region that is causally disconnected from the rest of the space-time.

- Essential singularity at $r = 0$. This singularity can be observed when crossing the event horizon and is reached unavoidably by all objects inside the black hole. In this region curvature is infinite and, consequently, any observer is crushed by its enormous tidal gravitational forces.

- It is asymptotically flat. When $r \to \infty$, the flat space-time metric $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$ is recovered.

Is worth mentioning that the singularity at $r = 0$ is, in some way, protected by the event horizon. This is due to the fact that even if a physical observer crosses the horizon and runs into the singularity, it would not be able to return to the outer region of the black hole with this "information". The same happens for anything coming out from the singularity: it would be trapped within the black hole. Years later (1969), this phenomenon would be named by Roger Penrose as "cosmic censorship". In the next chapter, singularities and cosmic censorship will be discussed in more detail.

1.1.2 The Kerr Metric

In 1963 the mathematician Roy Kerr found another solution to the Einstein field equations (1.1) in the vacuum. Unlike the Schwarzschild solution, the Kerr metric is axially symmetric and non-static, i.e., it describes a rotating body. It preserves,

[1]Even the incorporation of a cosmological constant into the field equation (1.1) does not fully resolve this issue, this matter is still nowadays subject of continuous research.

though, the property of being stationary (it does not evolve with time, a more precise definition of this concept is given in subsection 2.1.7). Possibly the most simple way of expressing the Kerr metric is using Boyer-Lindquist coordinates, which are essentially oblate spheroidal coordinates. The line element is

$$\begin{aligned} ds^2 = &- \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4aMr\sin^2\theta}{\rho^2} dt d\varphi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\ &+ \left(r^2 + a^2 + \frac{2a^2 Mr \sin^2\theta}{\rho^2}\right) \sin^2\theta d\varphi^2, \end{aligned} \tag{1.3}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Just like in the Schwarzschild metric, M is a mass parameter, while the additional parameter $a = J/M$ is related to the angular momentum J of the rotating body. Metric (1.3) can then be used to represent a spinning black hole, its principal characteristics are:

- When $|M| > |a|$, it possesses two horizons at the hyper-surfaces $r = r_\pm = M \pm \sqrt{M^2 - a^2}$.
- The outer horizon r_+ is an event horizon.
- The inner horizon r_- is a Cauchy horizon. Inside this region, future events are no longer uniquely determined by past events outside the black hole.
- It has an essential ring singularity at $r = 0$ and $\theta = \pi/2$.
- The angular momentum J of the black hole gives rise to a phenomenon known as "frame dragging" in which observers close enough to the black hole are forced to rotate with it[2].
- When $a = 0$ the Schwarzschild metric is recovered.

Besides its event horizons, the Kerr black hole also features another hyper-surface named as ergosphere at $r = M + \sqrt{M^2 - a^2\cos^2\theta}$. The ergosphere is localized outside the event horizon and denotes the region in which any physical observer (including a photon) rotates in the same direction as the black hole, regardless of having an arbitrarily large angular momentum in the opposite direction.

Just like in the Schwarzschild case, the space-time singularity is protected by an event horizon and hence, causally disconnected from the outer region. The cosmic censorship conjecture then holds. Nevertheless, if $|a| > |M|$, the event horizon disappears and the singularity is, in theory, left visible to far away observers. This is known as a "naked singularity".

1.2 The Rise of Wormholes

While Einstein's theory predicted the deformation of space-time due to matter, in its origins it was not considered that curvature could enable communication between two different universes (or even distant regions of the same universe). Twenty years after proposing the theory, Einstein himself along with Nathan Rosen, suggested the

[2]In more precise terms, frame dragging occurs when the inner product of the two Killing vectors $\xi = \partial/\partial t$ and $\zeta = \partial/\partial\varphi$ is non-vanishing, i.e., $g(\xi, \zeta) \neq 0$ (see next chapter, particularly subsections 2.1.6 and 2.1.7, for a proper introduction of these concepts).

theoretical existence of a similar concept. In fact, the name "wormhole" was presumably coined by Wheeler in 1955, who considered multiply connected universes with a non-trivial topology. In any case, it will be seen that wormholes can arise as exact solutions to the Einstein field equations.

1.2.1 Einstein-Rosen Bridges

The first ones to come up with a space-time composed of two joint identical congruent parts, or "sheets" connected by a "bridge", were Einstein and Rosen in 1935. This concept would later receive the name of Einstein-Rosen bridges for obvious reasons.

This was done as an attempt to describe either charged or neutral particles in the context of a field theory such as General Relativity. Another purpose in mind was to unify Electromagnetism with the theory of Einstein. Since they were dealing with field theories, Einstein and Rosen made use only of the electromagnetic four-potential A^{μ} and the space-time metric $g_{\mu\nu}$. However, in their prescription, the metric should be free of any kind of singularity.

To obtain non-singular metric components, the Schwarzschild metric (1.2) was taken, but with the following change of variable $u^2 = r - 2m$. This yielded the first metric that could be said to explicitly describe what would eventually be called a wormhole,

$$ds^2 = \frac{-u^2 dt^2}{u^2 + 2m} + 4(u^2 + 2m)du^2 + (u^2 + 2m)^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.4}$$

The change of variable performed implies that $u = \pm\sqrt{r - 2m}$. This admits the interpretation that the space-time can be described by two sheets, one with $u > 0$ and another one with $u < 0$. The region $u = 0$ is the connection between both parts and corresponds to the so-called bridge.

By this reasoning, Einstein and Rosen concluded that an elementary neutral particle is characterized by one of these bridges in space-time [1]. Finally, the attempt of Einstein and Rosen to unify General Relativity with Electromagnetism was not successful, but from this failure the idea of bridges connecting different parts of the space-time was born.

Despite the change of variable, the line element (1.2) as well as (1.4), describe both the geometry of a black hole. This means that a Schwarzschild black hole can be seen too as a space-time containing an Einstein-Rosen bridge. Fuller and Wheeler would later show that, in order for a physical object to cross this bridge, it would need a speed greater than that of light, violating therefore, the causality principle [2]. In some intuitive level this was expected since the bridge $u = 0$ corresponds to the event horizon $r = 2m$. A black hole is not a traversable wormhole.

1.2.2 The Analytic Extension of Kerr Space-Time

The Kerr black hole also admits a wormhole structure just as the Schwarzschild case. Metric (1.3) can be analytically extended to cover an infinite number of asymptotically flat universes connected through bridges or "throats" [3]. The analytic continuation is done to add extensions beyond the Cauchy horizon $r = r_-$ and $r = 0$. This is achieved by fixing $\theta = 0$ and φ as constant, a null coordinate transformation $u = t + r$ and $u + w = F(r)$ is also required. Here,

$$F(r) = 2r + A\ln|r - r_+| - B\ln|r - r_-|, \tag{1.5}$$

with $A = 2m(1/\sqrt{1-a^2/m^2}+1)$ and $B = 2m(1/\sqrt{1-a^2/m^2}-1)$. With these modifications the Kerr metric (1.3) reads

$$ds^2 = \left(1 - \frac{2mr}{r^2+a^2}\right) dudw. \tag{1.6}$$

With the aid of a Penrose diagram (see figure 1.1), the global structure of the geometry (1.6) and the way it connects distinct universes, either through the Cauchy horizon or the $r = 0$ hyper-surface ($\theta \neq \pi/2$), can be observed. It is also possible to distinguish which regions of space-time are accessible following causal curves. In this type of diagrams, light follows straight lines with unitary slopes forming thus a light cone. Causal observers move always within the light cone. The diagram found in figure 1.1 can be infinitely extended to include an arbitrary number of universes with the same structure. Non-trivial topologies of the diagram itself can be done by suitably joining the event horizons of different universes. This however leads to closed time-like curves (time traveling) which violate causality.

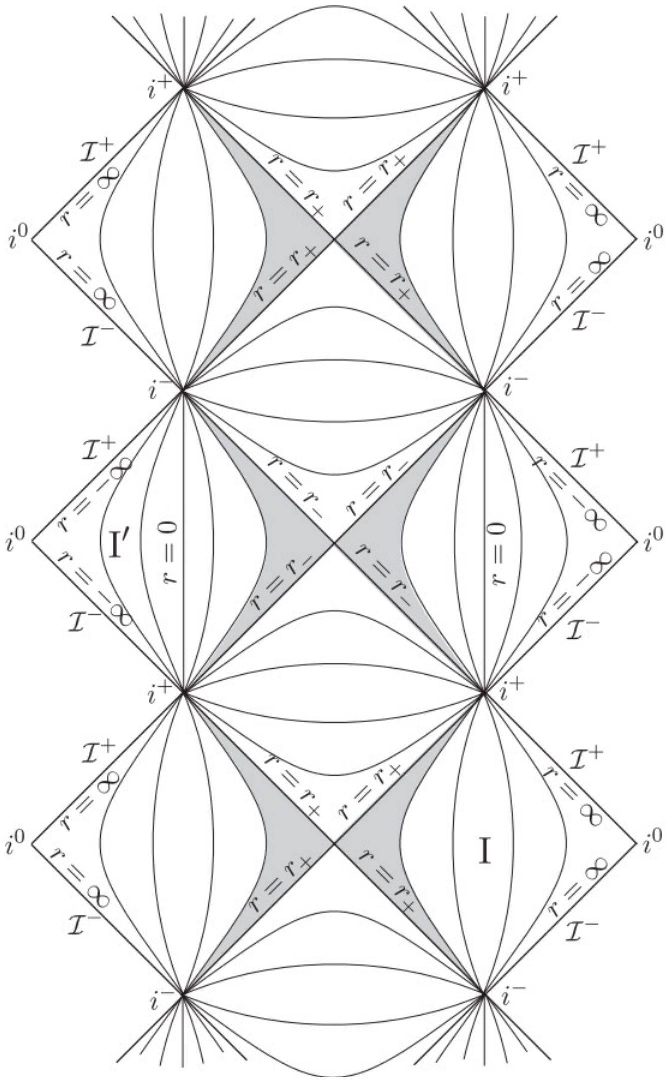

FIGURE 1.1: The Penrose Diagram of a rotating black hole [4]. Regions I and I′ are examples of asymptotically flat universes, the event horizons are labeled as $r = r_+$ and the Cauchy horizons as $r = r_-$. Universes I and I′ are connected through the throat at $r = 0$. When $\theta = \pi/2$, the $r = 0$ line turns into a curvature singularity.

It is important to notice that, unlike a Schwarzschild black hole, it is not necessary to travel faster than the speed of light to enter the other universes that exist in this theoretical construction. This is due to the analytical extension performed, combined with the interesting geometrical structure that the Cauchy horizon provides in this space-time.

1.2.3 Wheeler and the Concept of Wormhole

In 1955, Wheeler and Misner tried to build a classical model of electric charges using the sourceless Maxwell equations in the context of gravity described by General Relativity. The novelty of their model was the inclusion of topologically non-trivial spaces [5]. Wheeler and Misner desired to describe Classical Physics (gravity, electromagnetism, unquantized mass and unquantized charge) in terms of a curved space-time in the vacuum, that is to say, in purely geometrical terms.

Under this treatment electric charges were interpreted as electromagnetic fields that satisfy the Maxwell equations in the vacuum, but that were immersed in a space with a multiply connected topology. In this topology, the electromagnetic field lines would flow through a tunnel that doubly connects the space, as shown in figure 1.2. The field lines then enter from one side of the tunnel and exit on the other side with an intensity of equal magnitude but opposite direction, manifesting therefore the concept of electric charge.

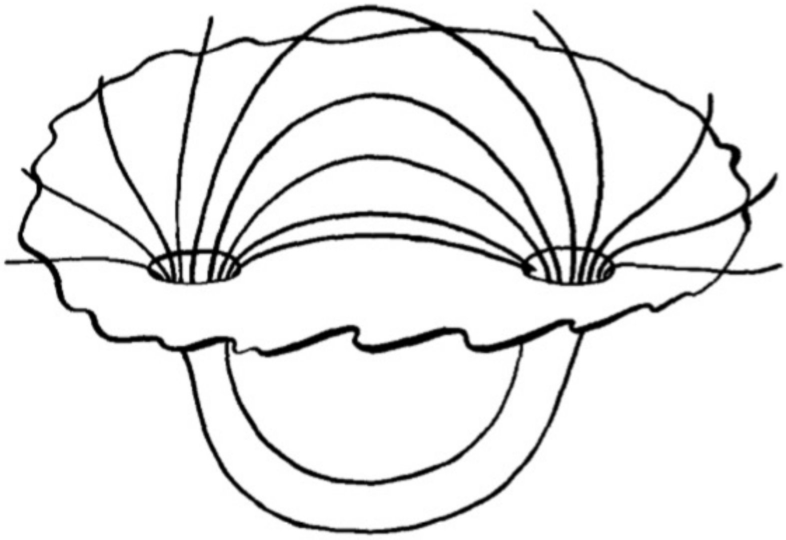

FIGURE 1.2: An illustration of a "wormhole" or tunnel through which electromagnetic flux travels to a distant region of the same space. Topologically this object is known as a handle.

Accordingly, Wheeler named this concept as "wormhole", instead of using the mathematical term "handle" used normally in the field of Topology. It is worth mentioning that the original idea of Wheeler was for these wormholes to communicate regions of the same space. Meanwhile, the wormholes presented in the past subsections join two different universes. Nowadays, the name wormhole is utilized to refer to either of these cases.

1.2.4 The Ellis Model

The idea of Einstein and Rosen of seeing particles as bridges in space-time was taken by H.G. Ellis, who in 1973 coupled a scalar field ϕ to the Einstein field equations (1.1) and found a static, spherically symmetric and horizon-free space-time with a topological hole in its center. Ellis called this particular type of solution a "drainhole" [6]. A justification for this name can be argued due to metric (1.7) admitting a vector field that could be interpreted as the velocity of an "ether" or substance that drains through the hole.

The line element of the solution has a spherically symmetric form and it is given by

$$ds^2 = -\left[1 - f^2(\rho)\right] dt^2 + \frac{d\rho^2}{1 - f^2(\rho)} + r^2(\rho)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1.7}$$

where the functions f and r are determined by the mentioned field equations. Since there is a coupled scalar field to the solution, the field equations (1.1) now adopt the form $R_{\mu\nu} = -2\partial_\mu\phi\partial_\nu\phi$, with $R_{\mu\nu}$ being the components of the Ricci tensor (introduced in the next chapter) and $\phi = \phi(\rho)$. The negative sign in this equation is opposed to the usual one and implies that the scalar field has a negative kinetic term. Additionally, the scalar field must also satisfy the wave equation $\Box\phi = 0$. Note that if $r(\rho) = \rho$ and $f^2(\rho) = 2m/\rho$, the Schwarzschild metric (1.2) is obtained, and thus $\phi = 0$.

When solving the presented set of field equations several different cases may rise, here only the most interesting one will be shown. This case is characterized by the functions

$$\begin{aligned} f^2(\rho) =& 1 - e^{-2m\phi/n}, \quad r^2(\rho) = (\rho^2 + a^2)e^{2m\phi/n}, \\ \phi(\rho) =& \frac{n}{a}\left[\frac{\pi}{2} - \arctan\left(\frac{\rho}{a}\right)\right], \end{aligned} \tag{1.8}$$

where $n^2 > m^2$ are two integration constants and $a^2 = n^2 - m^2$. The parameter a can be analyzed and shown to be related to the size of the throat of the wormhole.

The intention of coupling the Einstein field equations to a scalar field was to remove the essential singularity that appears in the Schwarzschild geometry. Thereby allowing the space-time to be geodesically complete (see next chapter for a definition of this concept).

Metric (1.7) describes a space-time that covers both universes, each of them asymptotically behave as the Schwarzschild metric with corresponding mass parameters $m_\pm$. On one side of the drainhole ($\rho > 0$), and sufficiently far away from the topological hole, the asymptotic parameter mass m_+ is positive. On the other side ($\rho < 0$) m_- is negative. Therefore, one side of the drainhole attracts matters while the opposite side expels it.

Finally, the ether flowing through the hole was the illustrative manner in which Ellis explained gravity in this space-time. The ether completely filled the space and was never at rest, its continuous flow manifested as gravity. Consequently, particles with mass could be represented as sources or sinks of this kind of ether flow.

The simplest model of a wormhole may be found by setting $m = 0$ in the functions (1.8). This indicates that there is no ether flow and both sides of the wormhole era exactly the same. Then the line element (1.7) becomes

$$ds^2 = -dt^2 + d\rho^2 + (\rho^2 + a^2)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1.9}$$

This metric is sometimes known as the Ellis-Bronnikov wormhole with reflection symmetry. The symmetric part refers to the fact that the line element is invariant to the reflection $\rho \to -\rho$. Bronnikov independently proposed similar scalar fields models around the same time [7]. It is not uncommon, thus, to find the name Ellis-Bronnikov drainholes for these general space-times.

1.3 Traversable Wormholes

In section 1.2 the first and more important wormhole models were introduced. Nonetheless, despite its respective authors deeply analyzed the physical characteristics of each metric, none of them had considered these objects as a way to travel great distances (of the order of light years) across the universe. The first ones to consider this possibility were M.S. Morris and K.S. Thorne in 1988, who were interested in the properties that a wormhole must possess in order for a human being to safely travel through it, this is, to be traversable [8].

1.3.1 Properties

Among the principal features that a traversable wormhole should have are:

1. A spherically symmetric, stationary, and static metric is recommended.
2. The Einstein field equations (1.1) must be satisfied in every region of space-time.
3. It must contain a throat that connects two asymptotically flat space-times.
4. It should be free of event horizons, otherwise a physical object could not return to the universe it came from.
5. The tidal gravitational forces experienced by a traveler must be equal or less to those of the Earth.
6. The time required to travel through the wormhole should be finite, this includes that which is measured by the traveler, as well as that measured by static observers in any of the two universes.
7. The matter or field that generates the wormhole must be physically reasonable.
8. It must be stable with respect to small perturbations.
9. Its hypothetical construction should not require more mass than that of the universe, nor more time than the age of the universe.

Property 1 is only requested for simplicity in the calculations that need to be done, it does not mean that a traversable wormhole without spherical symmetry cannot exist. Initiating from the statements of properties 1-4, which are related to the geometry of the space-time, Morris and Thorne analyzed the physical implications that these restrictions would impose. Through this analysis, it could be determined if a space-time of these characteristics may actually exist.

1.3.2 General Metric

Morris and Thorne utilized the following metric as a general form of a spherically symmetric, stationary, and static wormhole:

$$ds^2 = -e^{2\Phi}dt^2 + \frac{dr^2}{1-b/r} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1.10}$$

where $\Phi = \Phi(r)$ and $b = b(r)$ are functions that specify the geometry of the space-time. The spatial shape of the wormhole is related to $b(r)$, and is therefore known as the shape function, while $\Phi(r)$ specifies the gravitational redshift, and hence is called the redshift function. The properties mentioned earlier establish some constraints on the functions $\Phi(r)$ and $b(r)$. For example, the no event horizon condition implies that the redshift function is finite in all points of space-time, if metric (1.10) is demanded to be asymptotically flat, then one needs that

$$\lim_{r\to\infty} \phi(r) = 0 \quad \text{and} \quad \lim_{r\to\infty} \frac{b(r)}{r} = 0. \tag{1.11}$$

To have a better understanding of the wormhole's geometry, it is helpful to embed the general metric into three-dimensional Euclidean space expressed in cylindrical coordinates, i.e., $ds^2 = dr^2 + dz^2 + r^2d\varphi^2$. For this purpose, and without loss of generality due to the fact that the metric is spherically symmetric and stationary, the coordinate t is taken as constant and $\theta = \pi/2$ in (1.10). This yields $ds^2 = dr^2/(1-b/r) + r^2d\varphi^2$. Assuming the existence of a profile $z(r)$ in the Euclidean space that describes the embedded surface of the wormhole, and by comparison of the two treated metrics, it is seen that

$$\frac{dz}{dr} = \frac{\pm 1}{\sqrt{r/b(r) - 1}}. \tag{1.12}$$

Every wormhole by definition must have a minimum radius $r = b_0 > 0$ that localizes its throat. The $z(r)$ profile can be used to indicate such a radius since, at the throat, the slope of its tangent would be totally vertical, this is, (1.12) diverges. This implies that $b(b_0) = b_0$. This condition can be equivalently expressed as $dr/dz = 0$ at $r = b_0$. Furthermore, as r_0 is a minimum radius, then $d^2r/dz^2 > 0$ at the throat. Thus,

$$\frac{1 - b'(b_0)}{2b_0} > 0, \tag{1.13}$$

where $b' = db/dr$. The domain of the radial coordinate is then $r \in [b_0, \infty)$. It should be clarified that this coordinate decreases from positive infinity to b_0 as the throat is approached from one of the two universes it connects, and then increases back to infinity when emerging in the other universe. In the following it will be shown that inequality (1.13) causes several physical complications.

Before ending this subsection it should be noticed that the Ellis-Bronnikov wormhole with reflection symmetry is included in the general metric (1.10). Indeed, this wormhole constitutes a very simple example of a traversable one, for this case, $\Phi(r) = 0$ and $b(r) = b_0^2/r$. The relation between the radial coordinate r and the ρ coordinate (which is the proper radial distance) introduced earlier is $r^2 = \rho^2 + b_0^2$.

1.3.3 Violation of the Energy Conditions

Using the Einstein field equations, the type of matter that generates a wormhole with metric (1.10) can be found. It is convenient, though, to switch to the reference frame of an observer at rest (an orthonormal frame). This is achieved through the following change of basis,

$$\begin{aligned} \mathbf{e}_{\hat{t}} &= e^{-\phi}\mathbf{e}_t, & \mathbf{e}_{\hat{r}} &= \sqrt{1-\frac{b}{r}}\mathbf{e}_r, \\ \mathbf{e}_{\hat{\theta}} &= \frac{1}{r}\mathbf{e}_\theta, & \mathbf{e}_{\hat{\varphi}} &= \frac{1}{r\sin\theta}\mathbf{e}_\varphi. \end{aligned} \tag{1.14}$$

In this basis the components of the metric are simply $g_{\hat{\mu}\hat{\nu}} = diag[-1,1,1,1] = \eta_{\hat{\mu}\hat{\nu}}$. For the non-vanishing components of the Einstein tensor $G_{\hat{\mu}\hat{\nu}}$, it can be found that

$$\begin{aligned} G_{\hat{t}\hat{t}} &= \frac{b'}{r^2}, \qquad G_{\hat{r}\hat{r}} = -\frac{b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}, \qquad (1.15) \\ G_{\hat{\theta}\hat{\theta}} &= G_{\hat{\varphi}\hat{\varphi}} = \left[1-\frac{b}{r}\right]\left[\Phi'' - \frac{b'r-b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r-b}{2r^2(r-b)}\right]. \end{aligned}$$

In geometrized units ($G = 1$, $c = 1$), the components of the stress-energy tensor in this basis $T_{\hat{\mu}\hat{\nu}}$ can be identified with physical quantities such as

$$T_{\hat{t}\hat{t}} = \rho(r), \quad T_{\hat{r}\hat{r}} = -\tau(r), \quad \text{and} \quad T_{\hat{\theta}\hat{\theta}} = T_{\hat{\varphi}\hat{\varphi}} = p(r), \tag{1.16}$$

where $\rho(r)$ is the total mass-energy density, $\tau(r)$ is the tension per unit area in the radial direction, and $p(r)$ is the pressure in the lateral directions (orthogonal to the radial direction). All of these quantities are expressed as measured by a static observer.

The components of both tensors are related by the Einstein equations (1.1), yielding expressions for ρ, τ and p in terms of the metric functions $\Phi(r)$ and $b(r)$,

$$\rho = \frac{b'}{8\pi r^2}, \quad \tau = \frac{1}{8\pi r^2}\left[\frac{b}{r} - 2(r-b)\Phi'\right], \quad p = \frac{r}{2}\left[(\rho-\tau)\Phi' - \tau'\right] - \tau. \tag{1.17}$$

Evaluating the energy density ρ and the radial tension τ at the throat of the wormhole where $b(b_0) = b_0$, it can be obtained that

$$\rho_0 = \frac{b'(b_0)}{8\pi b_0^2} \quad \text{and} \quad \tau_0 = \frac{1}{8\pi b_0^2}. \tag{1.18}$$

From these two previous equations the next expression can be written:

$$\rho_0 = b'(b_0)\tau_0,$$

but from inequality (1.13) one has that $b'(b_0) < 1$. Hence, $\rho_0 < \tau_0$. Any type of matter that generates this wormhole has to satisfy this condition, unfortunately, this is a characteristic of matter that is referred to as "exotic". This exotic material is physically problematic since the condition $\rho_0 < \tau_0$ implies that an observer moving with sufficiently large velocity could measure a negative energy density at the throat of the wormhole.

The restriction on the positive definiteness of the total mass-energy density as seen by an arbitrary observer is established by several so-called energy conditions, two of them are the weak energy condition (WEC) and the null energy condition (NEC). Mathematically they can be expressed as

$$T_{\mu\nu}u^{\mu}u^{\nu} \geq 0,$$

where u^{μ} is a time-like vector for the WEC, or a null vector for the NEC. These energy conditions are said to hold when the previous inequality holds for all time-like vectors u^{μ} in the case of the WEC, and for all null vectors u^{μ} for the case of the NEC. Considering a radial null vector $\mathbf{u} = \mathbf{e}_{\hat{t}} \pm \mathbf{e}_{\hat{r}}$, then the NEC imposes that $T_{\mu\nu}u^{\mu}u^{\nu} = T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}} = \rho - \tau > 0$. This inequality is the exact opposite as the one fulfilled by the exotic matter of a wormhole. Thereby, the existence of a traversable wormhole with properties 1-3 directly violates the NEC.

Years later, M. Visser and D. Hochberg showed that the energy conditions must be violated in any regular traversable wormhole, not necessarily spherically symmetric or stationary, whose throat consists of a two-dimensional surface of minimal area. Furthermore, this violation is due only to the geometrical structure of the throat [9, 10]. Even more generally, "topological censor" theorems that do not allow observers to probe the topology of an asymptotically flat space-time satisfying the null energy condition, would forbid the existence of traversable wormholes as well [11].

1.3.4 Duration of the Trip

Property 6 of subsection 1.3.1 specifies that, in a traversable wormhole, a traveler must take a finite time to cross from one universe to another. In the following, this restriction is analyzed. First, since using the coordinate r near the throat ($r = b_0$) can lead to ill-behavior in the metric components and related quantities, a proper radial distance l needs to be defined as

$$\frac{dl}{dr} = \pm \frac{1}{\sqrt{1 - b(r)/r}}. \tag{1.19}$$

The plus sign in the previous equation is used to describe one of the universes (upper universe), and the minus sign to describe the other universe (lower universe). If proper radial distance is to be well-defined everywhere, then the inequality

$$1 - \frac{b(r)}{r} \geq 0,$$

must hold for all regions of the space-time. Also, when using the proper distance l instead of the radial coordinate r, the metric can be written as $ds^2 = -e^{2\Phi}dt^2 + dl^2 + r^2(l)d\Omega^2$.

Now let $v(r)$ be the radial velocity of a traveler measured by a static observer, l the proper distance traveled, r the coordinate radius traveled, t the coordinate time, and τ the proper time measured by the traveler. Then[3],

[3]From this point forward in this subsection, and also during the next one, geometrized units will be partially abandoned so that the numeric calculations carried out yield physically meaningful quantities. Hence, c and G are no longer set to unity.

$$v = e^{-\phi}\frac{dl}{dt} \quad \text{and} \quad \frac{dl}{d\tau} = v\gamma, \tag{1.20}$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$ is the Lorentz factor. Assuming that the trip begins in the lower universe ($l < 0$) at $l = -l_1$ and ends in the upper universe ($l > 0$) at $l = l_2$, the time $\Delta\tau$ a traveler takes to arrive from $-l_1$ to l_2 measured by the same traveler, and the time Δt measured by a static observer will be

$$\Delta\tau = \int_{-l_1}^{l_2} \frac{dl}{v\gamma} \quad \text{and} \quad \Delta t = \int_{-l_1}^{l_2} \frac{dl}{ve^{\Phi}}. \tag{1.21}$$

These times are demanded to be finite and reasonable when compared to the average life of a human being, for example, $\Delta\tau \leq 1$ *year* and $\Delta t \leq 1$ *year*

1.3.5 Acceleration due to Tidal Gravitational Forces

When crossing the wormhole, a traveler will feel gravitational accelerations that emerge because of the non-vanishing space-time curvature. According to property 5 of subsection 1.3.1, these should not exceed the acceleration due to Earth's gravity, therefore guaranteeing the physical integrity of the traveler. If too large, the tidal gravitational forces would provoke an acceleration in the traveler's body such that it would lethally stretch the unfortunate adventurer.

The next change of basis describes the reference frame of a traveler with respect to an observer at rest

$$\begin{aligned} \mathbf{e}_{\hat{0}'} &= \gamma\left(\mathbf{e}_{\hat{t}} \pm \frac{v}{c}\mathbf{e}_{\hat{r}}\right), & \mathbf{e}_{\hat{1}'} &= \gamma\left(\pm\mathbf{e}_{\hat{r}} + \frac{v}{c}\mathbf{e}_{\hat{t}}\right), \\ \mathbf{e}_{\hat{2}'} &= \mathbf{e}_{\hat{\theta}}, & \mathbf{e}_{\hat{3}'} &= \mathbf{e}_{\hat{\varphi}}. \end{aligned} \tag{1.22}$$

The change of basis (1.22) is basically a Lorentz transformation between the frames of an observer at rest and the traveler, where the 4-velocity $\mathbf{u}$ of the traveler is given by $\mathbf{u} = \mathbf{e}_{\hat{0}'}$.

Since the acceleration $\mathbf{a}$ of the traveler is only radial in its reference frame, then $\mathbf{a} = a\mathbf{e}_{\hat{1}'}$, with a being the magnitude of the acceleration. Meanwhile, 4-acceleration is defined as $a^{\mu} = c^2 u^{\nu}\nabla_{\nu}u^{\mu}$, where ∇_{ν} is the covariant derivative (cf. subsection 2.1.2). The temporal component a_t in the coordinate frame is related to the magnitude a by $a_t = -\gamma v e^{\Phi} a/c$. Thus,

$$a = e^{-\Phi}\frac{d}{dl}(\gamma e^{\Phi})c^2. \tag{1.23}$$

Bounding now the acceleration in (1.23) so that it does not exceed that of the Earth due to gravity $g_{\oplus} = 9.8\ m/s^2$, the next restriction is obtained

$$\left| e^{-\Phi}\frac{d}{dl}(\gamma e^{\Phi}) \right| \leq \frac{g_{\oplus}}{c^2} \approx \frac{1}{0.97\ light\ years}. \tag{1.24}$$

On the other hand, the acceleration Δa^{μ} due to tidal gravitational forces is given by

$$\Delta a^{\hat{\mu}'} = -c^2 R^{\hat{\mu}'}{}_{\hat{\nu}'\hat{\alpha}'\hat{\beta}'} u^{\hat{\nu}'} \xi^{\hat{\alpha}'} u^{\hat{\beta}'}, \tag{1.25}$$

where $R^{\hat{\mu}'}{}_{\hat{\nu}'\hat{\alpha}'\hat{\beta}'}$ are the Riemann tensor components and $\xi^{\hat{\nu}'}$ are the components of a separation vector that joins both ends of the traveler's body (the magnitude of this vector is the height of the traveler). Both quantities are measured in the reference frame of the traveler, hence, $\xi^{\hat{0}'} = 0$. The expression shown in (1.25) comes from the geodesic deviation equation, which will be later treated along with the Riemann tensor (section 2.1).

Each spatial component of $\Delta a^{\hat{\mu}'}$ in (1.25) can be further reduced to

$$\Delta a^{\hat{1}'} = -c^2 R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}\xi^{\hat{1}'}, \quad \Delta a^{\hat{2}'} = -c^2 R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}\xi^{\hat{2}'}, \quad \Delta a^{\hat{3}'} = -c^2 R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'}\xi^{\hat{3}'}.$$

The past components of the Riemann tensor are

$$\begin{aligned} R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'} &= -\left[1 - \frac{b}{r}\right]\left[-\Phi'' + \frac{b'r - b}{2r(r-b)}\Phi' - (\Phi')^2\right], \\ R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'} &= R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'} = \frac{\gamma^2}{2r^2}\left[\left(\frac{v}{c}\right)^2\left(b' - \frac{b}{r}\right) + 2(r-b)\Phi'\right]. \end{aligned} \tag{1.26}$$

The following is now demanded for the spatial components of the acceleration: $\Delta a^{\hat{i}'} \leq g_{\oplus}$ with $i = 1, 2, 3$. This implies that,

$$|R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}| \leq \frac{g_{\oplus}}{c^2(2\ m)} \approx \frac{1}{2 \times 10^{16}\ m^2}, \quad |R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}| \leq \frac{g_{\oplus}}{c^2(2\ m)} \approx \frac{1}{2 \times 10^{16}\ m^2}, \tag{1.27}$$

where a height of $2\ m$ has been assumed for the traveler. The first condition in (1.27) is related to the radial tidal force and constraints the redshift function Φ, this condition is easily satisfied if, for example, $\Phi' = 0$ everywhere. The second conditions is associated to the lateral tidal force and restricts the velocity in which a traveler crosses the wormhole.

1.3.6 The Challenging Existence of Wormholes

In the past subsections the most important aspects of a traversable wormhole were reviewed as analyzed by Morris and Thorne. The requisites on the geometry of space-time regarding the safe travel of any human through the wormhole, e.g., a finite trip and weak enough induced accelerations, were shown to be possible to achieve by demanding some restrictions on the metric components and the velocity of the traveler. Unfortunately, by far, the most problematic issue is that of the exotic matter needed to support the throat of the wormhole. This theoretical fact constitutes the principal obstacle to the existence of a wormhole.

Negative energy densities are not allowed by Classical Physics. However, quantum mechanically (and even in semi-classical approaches) there are known violations of the energy conditions which can be of potential interest to wormhole applications. Some examples of these violations are (see [12] and references therein for a more detailed review on the subject):

- **The Casimir Effect.** Two conducting and parallel plates separated by a small distance (of the order of nanometers) can induce fluctuations on the zero-point energy of the quantum vacuum. This induction is known as the Casimir effect and it has been experimentally tested. A stress-energy tensor that describes the vacuum expectation value of the energy associated with the Casimir effect may be calculated. One of its features is a negative energy density that implies the violation of the WEC. The NEC can also be shown to be violated.

- **Squeezed Vacuum States.** These are special quantum states of the electromagnetic field. Experimentally it involves the manipulation, through non-linear optics techniques, of the quantum fluctuations of a laser beam. With the help of these techniques, energy can be "extracted" from a vacuum state and accumulated into another region. As a result, one ends up with a state whose energy level is lower than that of the vacuum, i.e., it has a negative expectation value for its energy density. A squeezed vacuum state can be seen as a traveling electromagnetic wave that oscillates between negative and positive energy densities. It violates the null and weak energy conditions, however, its averaged density is still positive.

- **Hawking Evaporation.** A black hole may radiate away in what is known as evaporation process. This is a process in which the surface area of its event horizon shrinks, implying then a breakdown of the area increase theorem. Thus, at a quantum level, one of the assumptions of the theorem must not hold: the NEC. It is possible to find, in a neighborhood of the event horizon, the expectation value of the quantum stress-energy tensor and verify that it violates said condition.

- General particle creation is also a situation in which the energy conditions are often not fulfilled.

While these examples indicate that, contrary to standard and more traditional ideas in Physics, there exist cases in which the energy conditions are not necessarily met, there is still a long way for any of these phenomena to be exploited as exotic sources for a wormhole. The Casimir effect is thought be the most likely candidate for such an objective, nevertheless, a physical realistic implementation of a pair of plates maintaining a wormhole opened is yet problematic due to the small size of the induced effect. This raises the question of how much exotic material is needed to support the throat of a wormhole. In this regard, traversable wormholes which need only arbitrarily small quantities of exotic matter to be generated have been presented as a possibility [13].

Alternatively, rather than recurring to quantum effects for the necessary energy violations, another exotic source of cosmological nature has been gaining recent attention. The discovery that the Universe is expanding at an accelerated rate has deeply impacted Cosmology and related areas of Physics. The entity responsible for such an expansion is known as dark energy and is yet to be fully understood [14]. There are numerous physical candidates that aim to successfully explain this mysterious phenomenon, one of them are the so-called phantom scalar fields [15, 16], i.e., the scalar fields with a negative sign considered in subsection 1.2.4. Due to the minus sign, this type of scalar fields imply a violation of the energy conditions and they are believed to lead to gravitational repulsion (instead of the usual attraction), which in turn would explain the accelerated expansion of the Universe.

Whether wormholes can be naturally formed as a consequence of the dark energy component of the Universe, or they must be created in laboratories by taking advantage of some of the discussed quantum effects discussed above is still an open question to this very day. Another wormhole formation mechanism can be that of a black hole accreting phantom scalar fields. This has been shown to be considered as an evaporation process involving exotic matter [17].

Finally, there is one property of traversable wormholes that has not been discussed here, namely, its stability under small perturbations. The treatment of this extensive subject will be left for the next chapter where it will be seen that it adds obstacles, besides the exotic matter problem, to the pursued existence of wormholes.

1.3.7 Some Modern Wormhole Solutions

Despite the discouraging discovery due to Morris and Thorne that wormholes in General Relativity require exotic matter to exist, there is great interest in pushing forward the theoretical understanding of these fascinating non-trivial topological structures. As a result, there is a vast quantity of research exploring the further possibilities that gravitational theories may allow. In the following, only a few of these works are outlined in a brief manner (this is not by any means, a comprehensive list).

In [18], Bronnikov and Fabris abandoned the spherical symmetry assumption and found static and axially symmetric wormholes in D-dimensional gravity with a dilatonic field. A particular feature of these solutions is the appearance of a ring singularity bounding the throat of the wormhole. Hence, they are often referred to as ring wormholes.

Just like black holes, wormholes can also admit a rotating metric that describes its geometry. A generalization to a non-static wormhole of the Ellis-Bronnikov model (1.7, 1.8) was carried over in [19]. Unfortunately, the obtained space-time is not asymptotically flat, though it can be smoothly matched to the outside region of the Kerr metric. Rotating scalar field wormholes were also considered in [20], but with the difference of its metric having a cylindrical symmetry.

An alternate geometrical formalism can be applied to wormholes to deal with space-times of more complexity than those with spherical symmetry. This formalism is the thin shell formalism and, as the name suggests, is a suitable technique in the analysis of gravitational fields that consist of a thin layer of matter [12].

Interesting results regarding wormholes in modified theories of gravity (generalizations of Einstein's theory) combined with the mentioned thin shell formalism are also worth commenting. Thin shell wormholes that do not need of exotic matter to support them (even no matter at all) were found in the context of Gauss-Bonnet gravity (this particular theory yields a non-trivial generalization of General Relativity for more than three spatial dimensions) [21]. Without recurring to the thin shell formalism, some traversable Gauss-Bonnet wormholes that share the absence of exotic matter were later found too in [22].

The presence of an electromagnetic field coupled to a wormholes is also possible. In fact, a class of rotating wormholes with a magnetic field were presented in [23]. These axially symmetric space-times are solutions to the Einstein-Maxwell equations with scalar and dilatonic fields. A ring singularity is also found in these metrics just as in the ring wormholes.

As it can be seen, wormholes constitute a very interesting aspect of General Relativity which is still subject to an active development. Only time will tell, along with further advancements in science and technology, if wormholes will stay as theoretical speculations or if someday they may become part of astrophysical observations, or otherwise obtained from experiments in a laboratory.

Chapter 2

Preliminary Concepts and Results

2.1 The Mathematical Background of General Relativity

General Relativity is a theory of great mathematical beauty, it utilizes fundamental concepts of geometry to express physical properties in a uniquely elegant manner. It is a combination of physical intuition and mathematical sophistication whose ultimate goal is none other than the objective description of two basic components of our reality, namely, time and space.

This section will be entirely devoted to giving a concise description of the basic mathematical tools that are used in the theory of General Relativity. The approach adopted here will not be that of an in-depth review on the topic, but of a survey on the essential elements of the technical aspect of relativistic physics. Almost any Relativity book contains at least an introductory mathematical background, standard references are the classical textbooks [24–26]. Other more formal mathematical approaches can be found in [27, 28]. At the end, some more specialized concepts that are relevant to this thesis will also be described.

2.1.1 The Definition of a Space-Time

The whole mathematical framework in which General Relativity is expressed is Pseudo-Riemannian Geometry. This field of mathematics (specifically of Differential Geometry) is concerned with the study of the mathematical objects known as manifolds when endowed with a metric.

Definition 2.1. M is a differentiable manifold of dimension m if

1. M is provided with a family of pairs $\{(U_i, \varphi_i)\}$.
2. $\{U_i\}$ is a family of subsets such that $\cup_i U_i = M$, i.e., $\{U_i\}$ covers M.
3. φ_i is a one-to-one, onto map $\varphi_i : U_i \to U'_i$, such that $U'_i \subset \mathbb{R}^m$, i.e., U'_i is an open subset of $\mathbb{R}^m$.
4. Given U_i and U_j such that $U_i \cap U_j \neq \varnothing$, the composite map $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable.

Very loosely speaking, a differentiable manifold can be seen as a set of "patches" that locally resemble $\mathbb{R}^m$ and that can be joined together smoothly. The pair (U_i, φ_i) is called a chart of the manifold, and the family of charts $\{(U_i, \varphi_i)\}$ is known as an atlas. It is also common to refer to the maps φ_i as coordinate systems (they are also expressed more informally as $\{x^\mu\}$ with $\mu = 1, 2, \ldots, m$).

A variety of mathematical entities can be defined over a manifold, some of these include general tensors. Particular cases of tensors are vectors and their duals (also referred to as one-forms). They are however only defined locally at a point $p \in M$. An illustrative example is that of vectors. On a manifold, as opposed to Euclidean space, they cannot be obtained by joining two points $p_1, p_2 \in M$, they can instead be seen as the tangents to a curve[1] in M, which then in turn are associated to directional derivatives (infinitesimal displacements about p). See any of the references mentioned in the introduction of this chapter for a more detailed and precise exposition of this non-trivial matter. Consider a point $p \in M$ and the set of all tangent vectors to curves that pass through p. These vectors form a linear vector space T_pM, which is called the tangent space of M at p, and it is of the same dimension as M. A vector $V \in T_pM$ can be written as

$$V = V^\mu e_\mu,$$

where $\{e_\mu\}$ is the coordinate basis and V^μ the components of the vector. For a coordinate system $\{x^\mu\}$, the coordinate basis is $e_\mu = \partial/\partial x^\mu$.

Similarly, dual vectors form a dual vector space to T_pM, which is denoted by T_p^*M and is called the cotangent space of M at p. A one-form $w \in T_p^*M$ is expressed as

$$w = w_\mu e^\mu,$$

where $\{e^\mu\} = \{dx^\mu\}$ is the coordinate basis of T_p^*M and w_μ the components of the one-form. With these two objects, an inner product $\langle\ ,\ \rangle : T_p^*M \times T_pM \to \mathbb{R}$ can be defined as

$$\langle w, V\rangle = w_\mu V^\mu,$$

where the condition $\langle dx^\nu, \partial/\partial x^\mu\rangle = \delta^\nu_\mu$ for a dual basis is used (δ^ν_μ denotes the Kronecker delta).

Having discussed the basic notions of vectors and one-forms on a manifold, tensors can now be introduced in the same context.

Definition 2.2. A tensor T of rank (r,s) is a multilinear map $T : \otimes^r T_p^*M \otimes^s T_pM \to \mathbb{R}$.

In the previous definition the symbol $\otimes^n$ denotes the outer product of n spaces, this is for instance, $\otimes^r T_p^*M = T_p^*M \otimes \ldots \otimes T_p^*M$ (r products of T_p^*M). A vector can be seen as a type $(1,0)$ tensor, while a dual vector is a $(0,1)$ tensor. In a coordinate basis and with explicit components, a tensor is expressed as

$$T = T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}\frac{\partial}{\partial x^{\mu_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\mu_r}} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_s}.$$

Another basic property of a tensor is its transformation law, this is, the way in which the components of a tensor are affected under a change of basis. Passing from a coordinate system $\{x^\mu\}$ to a primed one $\{x'^\mu\}$, a tensor transforms as

$$T'^{\mu'_1\ldots\mu'_r}{}_{\nu'_1\ldots\nu'_s} = T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}\frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x'^{\mu'_r}}{\partial x^{\mu_r}}\frac{\partial x^{\nu_1}}{\partial x'^{\nu'_1}}\cdots\frac{\partial x^{\nu_s}}{\partial x'^{\nu'_s}}. \tag{2.1}$$

An indispensable tensor within Riemannian Geometry is the metric tensor. Conceptually, a metric contains all the necessary information to measure distances on a space. In a related way, it can be employed to define the inner product between two vectors V_1 and V_2.

[1]An open curve in a manifold M is a map $c : (a,b) \to M$, where $(a,b) \in \mathbb{R}$ is an open interval.

Definition 2.3. Let M be a differentiable manifold. A Riemannian metric g on M is a symmetric positive-definite bi-linear form, this is, a rank $(0, 2)$ tensor that for arbitrary vectors $V_1, V_2 \in T_pM$, satisfies the following axioms at each point $p \in M$:

1. $g(V_1, V_2) = g(V_2, V_1)$,
2. $g(V_1, V_1) \geq 0$ (the equality holds if and only if $V_1 = 0$)

A metric induces in a natural way an isomorphism between T_pM and T_p^*M. This is very simply seen in the following equations

$$w_\mu = g_{\mu\nu} V^\nu, \quad V^\mu = g^{\mu\nu} w_\nu,$$

where $V \in T_pM$, $w \in T_p^*M$ and $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, i.e., $g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu^\nu$. In practical terms, these operations are respectively known as "lowering" and "raising" indices with the metric.

In General Relativity, though, the metrics of interest are those which are pseudo-Riemannian. They are defined similarly but the positive-definite property is abandoned. Thus, condition 2 must be replaced by

2. If $g(V_1, V_2) = 0$ for any $V_1 \in T_pM$, then $V_2 = 0$.

For this kind of metrics condition 1 remains unchanged. Pseudo-Riemannian metrics allow therefore for the output of $g(V_1, V_2)$ to be negative. This fact has direct physical implications and is so important that it assigns a type to a vector $V \in T_pM$ according to:

- If $g(V, V) > 0$ then V is said to be space-like,
- If $g(V, V) = 0$ then V is said to be light-like or null,
- If $g(V, V) < 0$ then V is said to be time-like.

The metric can be diagonalized through the use of an orthonormal basis and scaled such that its diagonal entries are equal to ± 1. The set of signs of these entries is called the signature of the metric. A Riemannian metric has $(+, \ldots, +)$ signature, while a pseudo-Riemannian metric has commonly a $(-, +, \ldots, +)$ signature (a $(+, -, \ldots, -)$ signature may be used equivalently, but changing the inequality signs in the above vector classification).

After presenting the elementary concepts of Riemannian Geometry, the definition of a space-time can finally be given.

Definition 2.4. A space-time is a differentiable four-dimensional pseudo-Riemannian manifold, i.e., a pair $(M, g_{\mu\nu})$.

It is very common to refer to the line element of a space-time, this mathematical quantity gives the small change of the squared distance when a small displacement on the coordinates is done. On a dual basis $\{dx^\mu\}$ it is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

2.1.2 The Covariant Derivative

The next necessary task is to introduce a derivative operator on the manifold such that it maps tensors to tensors (not necessarily of the same rank). It can be verified that a simple partial derivation $\partial_\mu \equiv \partial/\partial x^\mu$ applied to the components of a tensor does not fulfill the desired property because the obtained object does not follow the transformation law (2.1).

As in the last subsection, the case of vectors will be first treated and then generalized to tensors. For this purpose, an affine connection ∇ is defined. In the following, the set of all the vector fields on M will be denoted by $\mathcal{X}(M)$, and the set of all smooth functions on M by $\mathcal{F}(M)$.

Definition 2.5. An affine connection ∇ is a map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ satisfying the following conditions for $f \in \mathcal{F}(M)$ and $X, Y, Z \in \mathcal{X}(M)$:

- $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$
- $\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z$
- $\nabla_{(fX)} Y = f \nabla_X Y$
- $\nabla_X(fY) = X[f] + f\nabla_X Y$

For a coordinate basis $\{e_\mu\}$ in T_pM, the connection coefficients $\Gamma^\alpha_{\mu\nu}$ can be defined as

$$\nabla_{e_\mu} e_\nu = \nabla_\mu e_\nu = e_\alpha \Gamma^\alpha_{\mu\nu},$$

these coefficients describe how the vector basis $\{e_\mu\}$ changes from point to point on the manifold. For two vectors $V = V^\mu e_\mu$ and $W = W^\mu e_\mu$, it is then easy to see that

$$\nabla_V W = V^\mu (\partial_\mu W^\alpha + W^\nu \Gamma^\alpha_{\mu\nu}) e_\alpha. \tag{2.2}$$

This is the covariant derivative of a vector. In Physics, however, the form of equation (2.2) is slightly modified to express only the μ component of $\nabla_{e_\mu} W = \nabla_\mu W$, leading to the widely used notation within General Relativity,

$$\nabla_\mu W^\alpha = \partial_\mu W^\alpha + W^\nu \Gamma^\alpha_{\mu\nu}.$$

Additional properties of ∇ are:

- The Liebnitz rule for the product of two tensor fields T_1 and T_2 must hold, $\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2)$,
- When applied to a function $f \in \mathcal{F}(M)$, it must reduce to the directional derivative
$$\nabla_X f = X[f] = X^\mu \frac{\partial f}{\partial x^\mu}.$$

By taking into account these last two properties and the covariant derivative of a vector, an expression for the covariant derivative of a one-form w can be found by considering $\nabla_\mu(w_\nu W^\nu)$. This yields,

$$\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma^\alpha_{\mu\nu} w_\alpha.$$

One can also verify that the past quantities obtained by application of the covariant derivative do transform as tensors. Hence, one can generalize ∇ as an operator that maps tensor of type (r, s) to tensor of type $(r, s+1)$. It is possible to write an equation for the components of the covariant derivative of a tensor T,

$$\begin{aligned}\nabla_\alpha T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} =& \partial_\alpha T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} + \Gamma^{\mu_1}_{\alpha\beta} T^{\beta\mu_2\ldots\mu_r}{}_{\nu_1\ldots\nu_s} + \ldots + \Gamma^{\mu_r}_{\alpha\beta} T^{\mu_1\ldots\mu_{r-1}\beta}{}_{\nu_1\ldots\nu_s} \\ & - \Gamma^{\beta}_{\alpha\nu_1} T^{\mu_1\ldots\mu_r}{}_{\beta\nu_2\ldots\nu_s} - \ldots - \Gamma^{\beta}_{\alpha\nu_s} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{s-1}\beta}. \end{aligned} \tag{2.3}$$

There are many different ways to choose the connection coefficients $\Gamma^\alpha_{\mu\nu}$ on M, i.e., they are not uniquely specified for a given manifold. Nevertheless, no connection coefficients are preferred above others and there is no natural condition to fix them, except for one, and it is called the Levi-Civita connection. This type of connection imposes two restrictions:

1. $\nabla_\alpha g_{\mu\nu} = 0$ (compatibility with the metric)
2. $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$ (torsion-free condition)

The connection coefficients that satisfy both of these restrictions always exist provided M is endowed with a (pseudo)-Riemannian metric. They also are unique and, as expected, are given in terms of the metric,

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}). \tag{2.4}$$

When the connection coefficients are written as in equation (2.4), they are known too as the Christoffel symbols. These connection coefficients are those which are used in the classical theory of General Relativity. However, there are other gravitational theories which consider modifications to conditions 1 and 2, allowing thus torsion or the so-called metric affine formulations.

There is one final idea related to the covariant derivative with geometrical and physical importance, that of parallel transport. Consider a curve c in M whose tangent vector has components W^μ, a vector V is said to be parallel transported along the curve c if,

$$W^\nu \nabla_\nu V^\mu = 0. \tag{2.5}$$

This notion can be used to introduce the concept of curvature on a manifold. Consider a closed loop and a tangent vector defined in some starting point of the loop. It is possible to "carry" the vector along the loop making sure that when transporting it, said vector remains parallel to its version of the previous point by satisfying (2.5) (it also must be kept tangent to the manifold at each point). Eventually, the starting point will be reached due to the curve being a closed loop. In a flat space it would be natural to expect that the final vector has the same direction as the starting vector. On the contrary, if the directions are different, the curvature of the manifold can be considered as the responsible of that discrepancy.

2.1.3 Curvature

In the previous subsection the connection coefficients $\Gamma^\alpha_{\mu\nu}$ were presented along with their expressions for a compatible metric and torsion-free manifold (Levi-Civita connection). Since $\Gamma^\alpha_{\mu\nu}$ is not a tensor, and even though it has the geometric meaning of an affine connection, these coefficients are hardly quantities that can be used to intrinsically describe curvature. This can be seen from the fact that in flat space the connection vanishes if Cartesian coordinates are chosen, but are non-zero if spherical coordinates are used instead. Thus, adequate tensors need to be found that can give a precise measurement of curvature.

A curvature tensor can be introduced by examining the commutator of two covariant derivatives applied to a vector field V^α,

$$[\nabla_\mu, \nabla_\nu]V^\alpha = R^\alpha{}_{\beta\mu\nu}V^\beta. \tag{2.6}$$

The meaning of this equation is that the tensor $R^\alpha{}_{\beta\mu\nu}$ measures the failure of the vector V^μ to return to its initial state when parallel-transported around a closed loop. This is due to the interpretation that can be given to the commutator of two covariant derivatives. It can be thought of as first computing the change of the vector in one direction, then in another one, and finally comparing the result of doing so in the opposite order. It is clear that the property of the manifold responsible for this failure on a closed loop must be curvature. Therefore, $R^\alpha{}_{\beta\mu\nu}$ is a curvature tensor and it is known as the Riemann tensor, while the set of equations yielded by (2.6) are the so-called Ricci identities. In terms of the Christoffel symbols in a coordinate basis $\{x^\mu\}$, the Riemann tensor is explicitly given by

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\delta\mu}\Gamma^\delta_{\beta\nu} - \Gamma^\alpha_{\delta\nu}\Gamma^\delta_{\beta\mu}. \tag{2.7}$$

The Riemann tensor has the following set of properties:

1. $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$,
2. $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$,
3. The Bianchi identities hold, i.e., $\nabla_\delta R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\delta\mu} + \nabla_\mu R_{\alpha\beta\nu\delta} = 0$.

This tensor can be decomposed into a part with trace and another one that is trace-free. Due to the anti-symmetries of the tensor, the only non-vanishing trace that can be obtained from it is

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}, \tag{2.8}$$

where $R_{\mu\nu} = R_{\nu\mu}$ is called the Ricci tensor. The corresponding trace of said tensor $R = R^\mu{}_\mu$ is also of great importance and is known as the Ricci scalar. The trace-free part is the Weyl tensor $C_{\alpha\beta\mu\nu}$ and it can be expressed for manifolds of dimension $n \geq 3$ as

$$C_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} + \frac{2}{n-2}\left(g_{\beta[\mu}R_{\nu]\alpha} - g_{\alpha[\mu}R_{\nu]\beta}\right) + \frac{2}{(n-1)(n-2)}Rg_{\alpha[\mu}g_{\nu]\beta}. \tag{2.9}$$

Properties 1 to 3 of the Riemann tensor hold for the Weyl tensor as well. This tensor is important in its own right since, among other features, it can be shown that $C^\alpha{}_{\beta\mu\nu}$ is invariant under conformal transformations of the metric, this is, if $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ then $C^\alpha{}_{\beta\mu\nu} \to C^\alpha{}_{\beta\mu\nu}$. It is also very helpful in the description of gravitational perturbations (see subsections 2.2.2, 2.2.3 and section 2.4).

There is one final tensor with deep physical relevance that can be constructed, namely, the Einstein tensor $G_{\mu\nu}$. It is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \tag{2.10}$$

and it is the tensor that appears in the "geometrical" side of the field equations (1.1). Manipulating the Bianchi identities it can be verified that

$$\nabla_\mu G^{\mu\nu} = 0. \tag{2.11}$$

Then, with the aid of the Einstein field equations, expression (2.11) can be interpreted as a manifestation of the energy conservation law.

2.1.4 The Lie Derivative and Gauge Freedom

Comparing vectors at two different points of a manifold is not a straightforward task. Naively one may think of subtracting the components of the vectors as a means of performing such comparison. However, as the vectors are only defined locally at each point, they do not belong to the same tangent space and hence, this operation is not well defined. There is no absolute notion of parallelism on a curved manifold without affine connection. The introduction of an operator called the Lie derivative allows for a manner to compare vectors, and any arbitrary tensor too, along a congruence of curves.

To give an expression for the Lie derivative denoted by $\pounds$, the help of a one-parameter group of diffeomorphisms ϕ_t generated by a vector field v^μ is needed. In this sense, the integral curves of v^μ are the orbits of ϕ_t. Then, for a tensor T of rank (r,s), the Lie derivative can be defined as

$$\pounds_v T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} = \lim_{t\to 0} \frac{(\phi^*_{-t}T)^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} - T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}}{t}. \tag{2.12}$$

It can be realized that this definition of a Lie derivative effectively captures the sought purpose. Consider the case of a vector $T^\mu|_p \in T_pM$ in equation (2.12), where $p \in M$. The vector $T^\mu|_{\phi_t(p)} \in T_{\phi_t(p)}M$ is "dragged" back to the tangent space T_pM by the map ϕ^*_{-t}, thus, $(\phi^*_{-t}T)^\mu|_{\phi_t(p)}, T^\mu|_p \in T_pM$. After that, the difference between both vectors in the same tangent space can be taken for small t.

If a coordinate system $\{x^\mu\}$ is chosen so that $x^1 = t$, the action of the map ϕ_t corresponds to a small displacement on said coordinate, this is, $x^1 \to x^1 + t$. With this, and conveniently starting from the lowest rank tensor (first a scalar, then a vector and a one-form, and so on), a general equation for the Lie derivative can be found by induction from its definition (2.12). Its explicit components are somewhat cumbersome and are given by

$$\begin{aligned}\pounds_v T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} =& v^\alpha \nabla_\alpha T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} - \sum_{i=1}^{r} T^{\mu_1\ldots\alpha\ldots\mu_r}{}_{\nu_1\ldots\nu_s} \nabla_\alpha v^{\mu_i} \\ &+ \sum_{j=1}^{s} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\alpha\ldots\nu_s} \nabla_{\nu_j} v^\alpha. \end{aligned} \tag{2.13}$$

The specific form of the Lie derivative applied to a scalar f, a vector u^μ, and a one-form w_μ is presented in the following equations,

$$\begin{aligned}\pounds_v f &= v(f) = v^\mu \nabla_\mu f, \quad \text{(the directional derivative)} \\ \pounds_v u^\mu &= v^\nu \nabla_\nu u^\mu - u^\nu \nabla_\nu v^\mu, \\ \pounds_v w_\mu &= v^\nu \nabla_\nu w_\mu + w_\nu \nabla_\mu v^\nu.\end{aligned}$$

One can verify that the Lie derivative is linear and maps tensors of rank (r, s) to tensors of that same rank. It also satisfies Liebnitz rule of derivation for outer products of tensors.

One of the most important applications of the Lie derivative in General Relativity is related to the gauge freedom that exists within the theory due to diffeomorphisms $\phi : M \to M$. This means that two space-times, $(M, g_{\mu\nu})$ and $(M, (\phi^* g)_{\mu\nu})$, are physically equivalent, i.e., they possess the exact same physical properties. A one-parameter family ϕ_t of diffeomorhpisms can be introduced such that they represent small coordinate displacements $x^\mu \to x^\mu + v^\mu$, where v^μ is the vector field which generates ϕ_t. Consider linear expansions of both metrics around the parameter $t = 0$ (do not confuse it with coordinate time) so that $g_{\mu\nu} = g_{\mu\nu}|_{t=0} + t\gamma_{\mu\nu}$ and $(\phi_t^* g)_{\mu\nu} = (\phi_t^* g)_{\mu\nu}|_{t=0} + t\gamma'_{\mu\nu}$, where

$$\gamma_{\mu\nu} = \left.\frac{dg_{\mu\nu}}{dt}\right|_{t=0}, \quad \gamma'_{\mu\nu} = \left.\frac{d(\phi_t^* g)_{\mu\nu}}{dt}\right|_{t=0}.$$

From the definition of the Lie derivative (2.12) and since $(\phi_t^* g)_{\mu\nu}\big|_{t=0} = g_{\mu\nu}\big|_{t=0}$, both of the previous terms are related by $\pounds_v g_{\mu\nu} = \gamma_{\mu\nu} - \gamma'_{\mu\nu}$. Consequently, by applying expression (2.13) to the metric, the gauge transformation becomes

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu} - \nabla_\mu v_\nu - \nabla_\nu v_\mu. \tag{2.14}$$

Thus, the gauge freedom in General Relativity consisting on the liberty given by diffeomorhpisms to choose a physical space-time $(M, g_{\mu\nu})$, manifests itself through the Lie derivative of the metric.

2.1.5 Geodesics

Geodesics are probably the most important curves on Riemannian Geometry. The basic concept behind them is really simple: they are the "straightest" curves that can be drawn from one point to another on a manifold [24]. They also represent extremal curves in the sense that a geodesic minimizes the length between any two given points. In mathematical terms, a geodesic is a curve whose tangent T^μ is parallel-transported along itself, i.e.,

$$T^\mu \nabla_\mu T^\nu = 0. \tag{2.15}$$

In a coordinate basis $\{x^\mu\}$, a geodesic $x^\mu(\lambda)$ with tangent $T^\mu = dx^\mu/d\lambda \equiv \dot{x}^\mu$ can also be described by rewriting equation (2.15) as

$$\frac{dT^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} T^\alpha T^\beta = 0. \tag{2.16}$$

A parameter λ such that the previous equations hold is called the affine parameter of the geodesic. Both equations are invariant to reparametrizations of the type $\lambda' = a\lambda + b$, where a and b are constants. Hence, an affine parameter is uniquely determined up to linear transformations of itself. From the theory of ordinary differential equations, given a set of initial data $x^\mu(0)$ and $T^\mu(0)$, the geodesic equation yields an unique, local solution for $\lambda = 0$.

Another important concept regarding geodesics is the geodesic deviation equation. Given a one-parameter family of geodesics with tangent T^μ, this equation describes how the curves of said family draw near or away from each other as its affine parameter varies. For this purpose a deviation vector ξ^μ is introduced that connects infinitesimally nearby geodesics and measures the separation along them. As it can be expected, curvature plays a determinant role in this description since the path of a curve on the manifold is subject to this property. The geodesic deviation equation is

$$T^{\alpha}\nabla_{\alpha}(T^{\beta}\nabla_{\beta}\xi^{\nu}) = R^{\nu}{}_{\alpha\beta\mu}T^{\alpha}T^{\beta}\xi^{\mu}. \tag{2.17}$$

The solution ξ^{μ} to equation (2.17) is sometimes called a Jacobi field. The quantities appearing in the left-hand side of equation (2.17) can be given a direct interpretation. The relative velocity between nearby geodesics is given by

$$v^{\mu} = T^{\alpha}\nabla_{\alpha}\xi^{\mu},$$

while their relative acceleration by

$$a^{\mu} = T^{\alpha}\nabla_{\alpha}v^{\mu} = T^{\alpha}\nabla_{\alpha}(T^{\beta}\nabla_{\beta}\xi^{\mu}).$$

It is then easy to realize the impact of curvature on geodesics. For instance, on flat space the Riemann tensor vanishes ($R^{\nu}_{\alpha\beta\mu} = 0$) and hence, geodesics of a family with the same tangent vector T^{μ} do not accelerate away or toward each other, this is, they remain strictly parallel. On the contrary, curvature ($R^{\nu}_{\alpha\beta\mu} \neq 0$) causes that geodesics of the same family fail to remain parallel with respect to each other.

To finalize this subsection, the notion of conjugate points in geodesics is introduced since they are utilized in the formulation of singularity theorems. Conjugate points are useful because they can appear when geodesics are locally no longer curves of minimum length between two points [25].

Definition 2.6. Let γ be a geodesic on a manifold M with tangent T^{μ}. A pair of points $p, q \in \gamma$ are said to be conjugate if there exists a Jacobi field ξ^{μ} which is not identically zero but vanishes at both p and q.

A simple example of this are the poles of a sphere, they are conjugate points of the geodesics that join them.

2.1.6 Isometries and Killing Vectors

In the geometrical description of space-time and its geodesics, it is often useful to take advantage of a set of special vectors called Kiling vectors. These vectors are related to the isometries of the space-time.

Definition 2.7. Let $(M, g_{\mu\nu})$ be a (pseudo)-Riemannian manifold. A diffeomorphism $f : M \to M$ is an isometry if $(f^{*}g)_{\mu\nu} = g_{\mu\nu}$, where f^{*} denotes the pullback of f.

Definition 2.8. If $\phi_t : M \to M$ is a one-parameter group of isometries, the vector field ζ^{μ} which generates ϕ_t is called a Killing vector field.

From these definitions and that of the Lie derivative, it can be seen that a Killing vector must satisfy the equation,

$$\pounds_{\zeta}g_{\mu\nu} = 0. \tag{2.18}$$

Killing vectors can then be interpreted as representations of the direction of a symmetry in the manifold. Condition (2.18) can be further rewritten by using the expression of the Lie derivative applied on a second-rank tensor, obtaining hence

$$\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = 0, \tag{2.19}$$

which is known as the Killing equation.

The importance of Killing vectors and its relation to geodesics is due to the next statement. Given a Killing vector ζ^μ, a geodesic γ with affine parameter λ whose tangent is T^μ, and defining $K = T^\mu \zeta_\mu$, one can easily verify that

$$\dot{K} = \frac{dK}{d\lambda} = T^\mu \nabla_\mu K = 0.$$

Thus, the quantity K is constant along the geodesic γ, i.e., it is a conserved quantity (or constant of motion). In a variety of physically significant situations, the existence of these conserved quantities is essential to the integration of geodesic curves on the space-time.

Finally, there exists a natural generalization of equation (2.19) to tensors of higher order such that

$$\nabla_{(\mu} K_{\nu_1 \ldots \nu_m)} = 0, \tag{2.20}$$

where $K_{\nu_1 \ldots \nu_m}$ is called a Killing tensor field of order (or rank) m. Though this type of tensors do not possess an intuitive interpretation associated to isometries of the space-time as their rank 1 counterparts, they do keep the nice property that the scalar constructed from contracting the tensor with the tangent of geodesics $K = K_{\nu_1 \ldots \nu_m} T^{\nu_1} \ldots T^{\nu_m}$ is a constant of motion (also known as a first integral of degree m).

For the case of type D space-times (see subsection 2.1.8), Walker and Penrose demonstrated the existence of at least a quadratic (second-rank) conformal Killing tensor field [29], i.e., a tensor $K_{\mu\nu}$ such that

$$\nabla_{(\alpha} K_{\mu\nu)} = k_\alpha g_{\mu\nu},$$

where k_α is a one-form. If $k_\alpha = 0$, then $K_{\mu\nu}$ is a proper Killing tensor. The quantity $K = K_{\mu\nu} T^\mu T^\nu$ is called a quadratic first integral.

2.1.7 Stationary and Axially Symmetric Space-Times

Often, when trying to find solutions to the Einstein field equations, assumptions about the geometry of the space-time are done in order to simplify the problem. The most common of them include spherical or axial symmetry, the last one being of more complexity than the first one. In addition, if the object or model of study is not dynamic, the stationary assumption is also imposed. This type of space-times are physically meaningful since they can describe rotating objects such as non-static black holes. In the following, the precise statements of these properties will be described in more detail.

Definition 2.9. A space-time $(M, g_{\mu\nu})$ is said to be stationary if there exists a one-parameter group of isometries σ_t whose orbits are time-like curves.

Definition 2.10. A space-time $(M, g_{\mu\nu})$ is said to be axisymmetric, or axially symmetric, if there exists a one-parameter group of isometries χ_φ whose orbits are closed space-like curves.

These definitions and the discussion in the last subsection imply that there are two Killing vectors, ξ^μ and ζ^μ, related to the stationary condition and axial symmetry, respectively. If the time translations commute with the rotations (each of them associated to the previous properties), this is, if

$$[\xi, \zeta] = 0$$

then a space-time is referred to as being both stationary and axially symmetric. Moreover, the system of coordinates $\{x^\mu\}$ can be chosen as $x^0 = t$ and $x^3 = \varphi$, so that $\xi = \partial/\partial t$ and $\zeta = \partial/\partial\varphi$. The metric can then be expressed as independent of these coordinates, hence,

$$g_{\mu\nu} = g_{\mu\nu}(x^1, x^2).$$

The addition of a further and independent condition called circularity is a relevant property when describing the geometry of this type of space-times.

Definition 2.11. An axially symmetric and stationary space-time $(M, g_{\mu\nu})$ is said to be circular if, for its associated Killing vectors ξ^μ and ζ^μ, the following holds

$$\xi \wedge \zeta \wedge d\xi = \xi \wedge \zeta \wedge d\zeta = 0. \tag{2.21}$$

Equation (2.21) can be alternatively expressed in components as $\xi_{[\mu}\zeta_\nu \nabla_\alpha \xi_{\beta]} = \xi_{[\mu}\zeta_\nu \nabla_\alpha \zeta_{\beta]} = 0$. When under the action of a rotation there exist fixed points in which ζ^μ vanishes (a symmetry axis), then (2.21) can be shown to be equivalent to the Ricci circularity condition [30], this is,

$$\xi \wedge \zeta \wedge R(\xi) = \xi \wedge \zeta \wedge R(\zeta) = 0,$$

where the Ricci form $R(u)$ is defined as $R(u) = R_{\mu\nu}u^\nu$. Similarly, Ricci circularity can be written with explicit components as $\xi^\mu R_{\mu[\nu}\xi_\alpha \zeta_{\beta]} = \zeta^\mu R_{\mu[\nu}\xi_\alpha \zeta_{\beta]} = 0$. The circularity condition holds for a wide class of energy-momentum tensors of great physical interest. For instance, vacuum space-times, Einstein-Maxwell fields, perfect fluid solutions with circular flow, and real scalar fields solutions. This helpful property can be used to additionally reduce the form of the space-time metric through the following theorem.

Theorem 2.1. Stationary axisymmetric space-times admit 2-spaces integrable and orthogonal to ξ^μ and ζ^μ if, and only if, Ricci circularity is satisfied.

Applying this theorem, the remaining coordinates x^1 and x^2 of the space-time can be chosen as the coordinates on the 2-spaces. Due to the fact that the 2-spaces are orthogonal to the Killing vectors $\xi = \partial/\partial t$ and $\zeta = \partial/\partial\varphi$, then the metric can be decomposed as

$$ds^2 = g_{AB}(x^1, x^2)dx^A dx^B + g_{ij}(x^1, x^2)dx^i dx^j, \tag{2.22}$$

with $A, B = 1, 2$ and $i, j = 0, 3$. One final general reduction to (2.22) can yet be accomplished. This is done by conveniently picking the coordinates $x^1 = \rho = \sqrt{\det[g_{ij}]}$ and $x^2 = z$ such that $\nabla_\mu \rho$ and $\nabla_\mu z$ are orthogonal too. Thus, in the coordinate system $\{t, \rho, z, \varphi\}$ the line element can be cast as

$$ds^2 = -V(dt^2 - \Omega d\varphi)^2 + \frac{\rho^2}{V}d\varphi^2 + K^2(d\rho^2 + \Lambda dz^2), \tag{2.23}$$

where V, Ω, K and Λ are four unknown metric components. This form of the metric is sometimes called the Papetrou line element.

2.1.8 The Algebraic Classification of Space-Times

In the study of the geometrical properties of a space-time, it is of interest and utility to be able to identify it according to its general features within a certain "catalog", or more formally, a classification. One of the most popular classification of space-times was introduced by Petrov in 1954 based on the algebraic characteristics of the Weyl tensor.

The classification is done according to the eigenvalues and their multiplicities of the Weyl tensor. Its eigenvalue equation is

$$\frac{1}{2}C_{\alpha\beta\mu\nu}X^{\mu\nu} = \lambda X_{\alpha\beta}, \tag{2.24}$$

where λ is an eigenvalue and $X^{\mu\nu}$ is an eigenbivector (an antisymmetric tensor of second order). Using a unit time-like vector u^{μ}, the eigenvalue problem (2.24) can be reduced to

$$Q_{\mu\nu}X^{\nu} = \lambda X_{\mu}. \tag{2.25}$$

In equation (2.25) the following set of definitions for the Weyl tensor has been done,

$$C^{*}_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + i\widetilde{C}_{\alpha\beta\mu\nu}, \quad \widetilde{C}_{\alpha\beta\mu\nu} = \frac{1}{2}\varepsilon_{\alpha\beta\rho\sigma}C^{\rho\sigma}{}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}C_{\alpha\beta}{}^{\rho\sigma}, \quad Q_{\mu\nu} = -C^{*}_{\mu\alpha\nu\beta}u^{\alpha}u^{\beta},$$

where $\varepsilon_{\alpha\beta\rho\sigma}$ is the Levi-Civita 4-form. Similar definitions hold for the bivector $X^{\mu\nu}$,

$$X^{*}_{\mu\nu} = X_{\mu\nu} + i\widetilde{X}_{\mu\nu}, \quad \widetilde{X}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}X^{\alpha\beta}, \quad X_{\mu} = X^{*}_{\mu\nu}u^{\nu}.$$

While the eigenvalue problem (2.25) can be studied in the 4-dimensional Lorentz space, it is far more convenient to treat it in a three-dimensional complex space with Euclidean metric. This is possible because the group $SO(3,\mathbb{C})$ of orthogonal transformation is isomorphic to the group of proper orthochronous Lorentz transformations (transformations that preserve orientation and time direction) $O^{\uparrow}_{+} = \{\Lambda \in O(1,3) \mid \det[\Lambda] = 1, \Lambda_{00} > 0\}$.

Depending on the algebraic and geometric multiplicity of the eigenvalues of the 3×3 matrix $\mathbf{Q}$, a classification of space-times can be achieved.

- **Type I**. Algebraically general space-time with three different eigenvalues ($\lambda_1 \neq \lambda_2 \neq \lambda_3$),
$$(\mathbf{Q} - \lambda_1\mathbf{I})(\mathbf{Q} - \lambda_2\mathbf{I})(\mathbf{Q} - \lambda_3\mathbf{I}) = 0.$$
- **Type II**. Two repeated eigenvalues with geometric multiplicity of 1,
$$\left(\mathbf{Q} + \frac{1}{2}\lambda\mathbf{I}\right)^2 (\mathbf{Q} - \lambda\mathbf{I}) = 0.$$
- **Type D**. Two repeated eigenvalues with geometric multiplicity of 2.
$$\left(\mathbf{Q} + \frac{1}{2}\lambda\mathbf{I}\right) (\mathbf{Q} - \lambda\mathbf{I}) = 0.$$
- **Type III**. Three repeated eigenvalues equal to zero with geometric multiplicity of 1, $\mathbf{Q}^3 = 0$ with $\mathbf{Q}^2 \neq 0$.

- **Type N**. Three repeated eigenvalues equal to zero with geometric multiplicity of 2, $\mathbf{Q}^2 = 0$ with $\mathbf{Q} \neq 0$.
- **Type O**. $\mathbf{Q} = 0$ (Conformally flat space-times).

Here, the 3×3 identity matrix is denoted by $\mathbf{I}$. Every Petrov type, except type I, is said to be algebraically special.

This classification can also be equivalently expressed in terms of the so-called principal null directions of a space-time, this is, null vectors k^μ such that

$$k_{[\rho} C_{\alpha]\beta\mu[\nu} k_{\sigma]} k^\beta k^\mu = 0. \tag{2.26}$$

In a four-dimensional space-time there exists in general four vectors of this kind. These vectors are completely determined up to a scaling $k^\mu \to c k^\mu$ and, in some particular cases, their directions can indeed coincide. In fact, the existence of repeated directions is the base of the classification.

- **Type I**. There exist 4 different principal null directions and equation (2.26) holds for each of them.
- **Type II**. Two principal null directions coincide, while the other two are different from each other. The repeated directions satisfy

$$C_{\alpha\beta\mu[\nu} k_{\sigma]} k^\beta k^\mu = 0. \tag{2.27}$$

- **Type D**. There are two principal null directions, both of which are twice repeated. The two vectors satisfy equation (2.27).
- **Type III**. Three principal null directions coincide, while the other one is different. The repeated directions satisfy

$$C_{\alpha\beta\mu[\nu} k_{\sigma]} k^\mu = 0.$$

- **Type N**. The four principal null directions coincide and they satisfy

$$C_{\alpha\beta\mu\nu} k^\mu = 0.$$

- **Type O**. There are no special directions on the space-time, i.e., $C_{\alpha\beta\mu\nu} = 0$.

Various examples of physical interest can be found for each of the previously described types of space-times. There exist perfect fluid solutions of the Robinson-Trautman field that are of type II, some black holes can be mentioned as relevant examples of type D space-times, metrics that describe longitudinal and transverse radiation are of type III and type N, respectively. Finally, Minkowski space-time is of type O, as well as the Friedmann-Robertson-Walker metrics used extensively in Cosmology.

Lastly, regarding algebraically special vacuum space-times, a theorem is known that relates them to the shear of a geodesic null congruence (this can be thought of as the collective distortion of the geodesics when propagating). This important result is known as the Goldberg-Sachs theorem.

Theorem 2.2. A vacuum metric is algebraically special if, and only if, it contains a shear-free geodesic null congruence.

2.2 The Newman-Penrose Formalism

In 1962 Newman and Penrose proposed an alternative approach to the geometrical description of General Relativity [31]. In this novel treatment they developed a formalism by introducing a null vector tetrad in a given space-time, along with the use of Ricci rotation coefficients. This formalism also admits an equivalent form in terms of a spinor dyad. Here, the tetrad version of the formalism will be mostly utilized. Throughout the years, this approach has been proved to be of great aid when treating the problem of gravitational radiation within General Relativity, as well as the geometrical aspect of this theory.

In the Newman-Penrose formalism a null tetrad $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ is introduced into every point of a four-dimensional pseudo-Riemannian manifold of signature $(+1, -1, -1, -1)$ and metric $g_{\mu\nu}$. The vectors l^μ and n^μ are real, while m^μ and $\bar{m}^\mu$ are complex. A bar over any given quantity will be used to denote its complex conjugate. The vectors of the tetrad must also satisfy the orthogonal property $l^\mu n_\mu = -m^\mu \bar{m}_\mu = 1$, with the rest of the vector combinations being zero. The space-time metric can then be expressed as

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu. \tag{2.28}$$

This relation can be rewritten in a more compact way as $g_{\mu\nu} = z_{m\mu} z_{n\nu} \gamma^{mn}$, if one conveniently defines[2]

$$\begin{aligned} z_{m\mu} &= (l_\mu, n_\mu, m_\mu, \bar{m}_\mu), \\ z_m^{\ \mu} &= (l^\mu, n^\mu, m^\mu, \bar{m}^\mu), \\ \gamma_{mn} = \gamma^{mn} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \end{aligned} \tag{2.29}$$

where $\gamma^{mp}\gamma_{pn} = \delta^m_{\ n}$. Using (2.29) we can also write the orthogonality properties simply as $z_m^{\ \mu} z_{n\mu} = \gamma_{mn}$. The metric γ will be used to raise or lower tetrad indices. Newman and Penrose define 12 complex spin coefficients that depend on linear combinations of the quantities $\mathcal{Z}_{mnp} = z_m^{\ \mu} z_n^{\ \nu} \nabla_\mu z_{p\nu}$, which are anti-symmetrical in their last two indices. These quantities are explicitly

$$\begin{aligned} \kappa &= \mathcal{Z}_{020} = l^\mu m^\nu \nabla_\mu l_\nu, & \pi &= -\mathcal{Z}_{031} = -l^\mu \bar{m}^\nu \nabla_\mu n_\nu, \\ \rho &= \mathcal{Z}_{320} = \bar{m}^\mu m^\nu \nabla_\mu l_\nu, & \lambda &= -\mathcal{Z}_{331} = -\bar{m}^\mu \bar{m}^\nu \nabla_\mu n_\nu, \\ \sigma &= \mathcal{Z}_{220} = m^\mu m^\nu \nabla_\mu l_\nu, & \mu &= -\mathcal{Z}_{231} = -m^\mu \bar{m}^\nu \nabla_\mu n_\nu, \\ \tau &= \mathcal{Z}_{120} = n^\mu m^\nu \nabla_\mu l_\nu, & \nu &= -\mathcal{Z}_{131} = -n^\mu \bar{m}^\nu \nabla_\mu n_\nu, \\ \varepsilon &= \frac{1}{2}(\mathcal{Z}_{010} - \mathcal{Z}_{032}) = \frac{1}{2}(l^\mu n^\nu \nabla_\mu l_\nu - l^\mu \bar{m}^\nu \nabla_\mu m_\nu), \\ \gamma &= \frac{1}{2}(\mathcal{Z}_{110} - \mathcal{Z}_{132}) = \frac{1}{2}(n^\mu n^\nu \nabla_\mu l_\nu - n^\mu \bar{m}^\nu \nabla_\mu m_\nu), \\ \alpha &= \frac{1}{2}(\mathcal{Z}_{310} - \mathcal{Z}_{332}) = \frac{1}{2}(\bar{m}^\mu n^\nu \nabla_\mu l_\nu - \bar{m}^\mu \bar{m}^\nu \nabla_\mu m_\nu), \end{aligned}$$

[2] Greek indices $(\mu, \nu = 0, 1, 2, 3)$ will be used to denote tensor indices and lower-case Latin indices $(a, b, m, n = 0, 1, 2, 3)$ to denote tetrad indices.

$$\beta = \frac{1}{2}(\mathcal{Z}_{210} - \mathcal{Z}_{232}) = \frac{1}{2}(m^{\mu}n^{\nu}\nabla_{\mu}l_{\nu} - m^{\mu}\bar{m}^{\nu}\nabla_{\mu}m_{\nu}). \tag{2.30}$$

Additionally, four differential operators are introduced

$$D = l^{\mu}\nabla_{\mu}, \quad \Delta = n^{\mu}\nabla_{\mu}, \quad \delta = m^{\mu}\nabla_{\mu}, \quad \delta^* = \bar{m}^{\mu}\nabla_{\mu}, \tag{2.31}$$

or more compactly $D_m = z_m^{\ \mu}\nabla_{\mu}$ with $D_m = (D, \Delta, \delta, \delta^*)$. These differential operators obey the following commutation relations,

$$\begin{aligned}
[\Delta, D] &= (\gamma + \gamma^*)D + (\varepsilon + \varepsilon^*)\Delta - (\tau + \pi^*)\delta^* - (\tau^* + \pi)\delta, \\
[\delta, D] &= (\alpha^* + \beta - \pi^*)D + \kappa\Delta - \sigma\delta^* - (\rho^* + \varepsilon - \varepsilon^*)\delta, \\
[\delta, \Delta] &= -\nu^* D + (\tau - \alpha^* - \beta)\Delta + \lambda^*\delta^* + (\mu - \gamma + \gamma^*)\delta, \\
[\delta^*, \delta] &= (\mu^* - \mu)D + (\rho^* - \rho)\Delta - (\alpha^* - \beta)\delta^* - (\beta^* - \alpha)\delta.
\end{aligned} \tag{2.32}$$

Using the 12 spin coefficients, along with the operators (2.31), Newman and Penrose obtained a set of numerous equations that are the equivalent of the Bianchi identities and the components of the Ricci and Weyl tensors in tetrad form, this is now known as the Newman-Penrose formalism. Since the Einstein field equations make use of the curvature tensors yielded by a given space-time metric, one can discuss any problem in General Relativity (at least its geometrical aspects) within this formalism.

To describe the mentioned tensors, which are characteristic of Riemannian Geometry, 10 curvature related quantities Φ_{AB} $(A, B = 0, 1, 2)$ and $\Lambda = R/24$ are defined. These are merely the projection of the tetrad vectors into the Ricci tensor, i.e., $R_{\mu\nu}z_m^{\ \mu}z_n^{\ \nu}$, and a rescaling of the Ricci scalar R, respectively. Each of these curvature quantities are given by

$$\begin{aligned}
\Phi_{00} &= -\frac{1}{2}R_{\mu\nu}l^{\mu}l^{\nu} = \Phi_{00}^*, & \Phi_{01} &= -\frac{1}{2}R_{\mu\nu}l^{\mu}m^{\nu} = \Phi_{10}^*, \\
\Phi_{11} &= -\frac{1}{4}R_{\mu\nu}\left(l^{\mu}n^{\nu} + m^{\mu}\bar{m}^{\nu}\right), & \Phi_{10} &= -\frac{1}{2}R_{\mu\nu}l^{\mu}\bar{m}^{\nu} = \Phi_{01}^*, \\
\Phi_{02} &= -\frac{1}{2}R_{\mu\nu}m^{\mu}m^{\nu} = \Phi_{20}^*, & \Phi_{12} &= -\frac{1}{2}R_{\mu\nu}n^{\mu}m^{\nu} = \Phi_{21}^*, \\
\Phi_{20} &= -\frac{1}{2}R_{\mu\nu}\bar{m}^{\mu}\bar{m}^{\nu} = \Phi_{02}^*, & \Phi_{21} &= -\frac{1}{2}R_{\mu\nu}n^{\mu}\bar{m}^{\nu} = \Phi_{12}^*, \\
\Phi_{22} &= -\frac{1}{2}R_{\mu\nu}n^{\mu}n^{\nu} = \Phi_{22}^*.
\end{aligned} \tag{2.33}$$

Another set of very important quantities of the formalism are the Weyl Scalars Ψ_N $(N = 0, 1, 2, 3, 4)$, which are obtained similarly by projecting the tetrad vectors, but now into the Weyl tensor. They are defined as

$$\begin{aligned}
\Psi_0 &= -C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}l^{\gamma}m^{\delta}, & \Psi_1 &= -C_{\alpha\beta\gamma\delta}l^{\alpha}n^{\beta}l^{\gamma}m^{\delta}, \\
\Psi_2 &= -C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}\bar{m}^{\gamma}n^{\delta}, & \Psi_3 &= -C_{\alpha\beta\gamma\delta}n^{\alpha}l^{\beta}n^{\gamma}\bar{m}^{\delta}, \\
\Psi_4 &= -C_{\alpha\beta\gamma\delta}n^{\alpha}\bar{m}^{\beta}n^{\gamma}\bar{m}^{\delta}.
\end{aligned} \tag{2.34}$$

The Weyl scalars are generally complex and contain all of the information of the 10 independent components of they Weyl tensor. As it will be later seen, the importance of these quantities lies in the fact that they are deeply related to the description of gravitational radiation in a space-time. The definition of the Weyl scalars changes by a sign if the $(-1,+1,+1,+1)$ signature is used instead.

Having introduced all of the necessary quantities and operators of the formalism, the previously mentioned Ricci identities (2.6) can be written

$$D\rho-\delta^*\kappa=\rho^2+\sigma\sigma^*+(\varepsilon+\varepsilon^*)\rho-\kappa^*\tau-\kappa(3\alpha+\beta^*-\pi)+\Phi_{00}, \tag{2.35a}$$
$$D\sigma-\delta\kappa=(\rho+\rho^*)\sigma+(3\varepsilon-\varepsilon^*)\sigma-(\tau-\pi^*+\alpha^*+3\beta)\kappa+\Psi_0, \tag{2.35b}$$
$$D\tau-\Delta\kappa=(\tau+\pi^*)\rho+(\tau^*+\pi)\sigma+(\varepsilon-\varepsilon^*)\tau-(3\gamma+\gamma^*)\kappa+\Psi_1+\Phi_{01}, \tag{2.35c}$$
$$D\alpha-\delta^*\varepsilon=(\rho+\varepsilon^*-2\varepsilon)\alpha+\beta\sigma^*-\beta^*\varepsilon-\kappa\lambda-\kappa^*\gamma+(\varepsilon+\rho)\pi+\Phi_{10}, \tag{2.35d}$$
$$D\beta-\delta\varepsilon=(\alpha+\pi)\sigma+(\rho^*-\varepsilon^*)\beta-(\mu+\gamma)\kappa-(\alpha^*-\pi^*)\varepsilon+\Psi_1, \tag{2.35e}$$
$$\begin{aligned}D\gamma-\Delta\varepsilon=&(\tau+\pi^*)\alpha+(\tau^*+\pi)\beta-(\varepsilon+\varepsilon^*)\gamma-(\gamma+\gamma^*)\varepsilon+\tau\pi-\nu\kappa\\&+\Psi_2-\Lambda+\Phi_{11},\end{aligned} \tag{2.35f}$$
$$D\lambda-\delta^*\pi=\rho\lambda+\sigma^*\mu+\pi^2+(\alpha-\beta^*)\pi-\nu\kappa^*-(3\varepsilon-\varepsilon^*)\lambda+\Phi_{20}, \tag{2.35g}$$
$$D\mu-\delta\pi=\rho^*\mu+\sigma\lambda+\pi\pi^*-(\varepsilon+\varepsilon^*)\mu-\pi(\alpha^*-\beta)-\nu\kappa+\Psi_2+2\Lambda, \tag{2.35h}$$
$$D\nu-\Delta\pi=(\pi+\tau^*)\mu+(\pi^*+\tau)\lambda+(\gamma-\gamma^*)\pi-(3\varepsilon+\varepsilon^*)\nu+\Psi_3+\Phi_{21}, \tag{2.35i}$$
$$\Delta\lambda-\delta^*\nu=-(\mu+\mu^*)\lambda-(3\gamma-\gamma^*)\lambda+(3\alpha+\beta^*+\pi-\tau^*)\nu-\Psi_4, \tag{2.35j}$$
$$\delta\rho-\delta^*\sigma=\rho(\alpha^*+\beta)-\sigma(3\alpha-\beta^*)+(\rho-\rho^*)\tau+(\mu-\mu^*)\kappa-\Psi_1+\Phi_{01}, \tag{2.35k}$$
$$\begin{aligned}\delta\alpha-\delta^*\beta=&\mu\rho-\lambda\sigma+\alpha\alpha^*+\beta\beta^*-2\alpha\beta+\gamma(\rho-\rho^*)+\varepsilon(\mu-\mu^*)\\&-\Psi_2+\Lambda+\Phi_{11},\end{aligned} \tag{2.35l}$$
$$\delta\lambda-\delta^*\mu=(\rho-\rho^*)\nu+(\mu-\mu^*)\pi+\mu(\alpha+\beta^*)+\lambda(\alpha^*-3\beta)-\Psi_3+\Phi_{21}, \tag{2.35m}$$
$$\delta\nu-\Delta\mu=\mu^2+\lambda\lambda^*+(\gamma+\gamma^*)\mu-\nu^*\pi+(\tau-3\beta-\alpha^*)\nu+\Phi_{22}, \tag{2.35n}$$
$$\delta\gamma-\Delta\beta=(\tau-\alpha^*-\beta)\gamma+\mu\tau-\sigma\nu-\varepsilon\nu^*-\beta(\gamma-\gamma^*-\mu)+\alpha\lambda^*+\Phi_{12}, \tag{2.35o}$$
$$\delta\tau-\Delta\sigma=\mu\sigma+\lambda^*\rho+(\tau+\beta-\alpha^*)\tau-(3\gamma-\gamma^*)\sigma-\kappa\nu^*+\Phi_{02}, \tag{2.35p}$$
$$\Delta\rho-\delta^*\tau=-\rho\mu^*-\sigma\lambda+(\beta^*-\alpha-\tau^*)\tau+(\gamma+\gamma^*)\rho+\nu\kappa-\Psi_2-2\Lambda, \tag{2.35q}$$
$$\Delta\alpha-\delta^*\gamma=(\rho+\varepsilon)\nu-(\tau+\beta)\lambda+(\gamma^*-\mu^*)\alpha+(\beta^*-\tau^*)\gamma-\Psi_3. \tag{2.35r}$$

For completeness sake, the form that the Bianchi identities adopt in the formalism is presented in what follows [32]. However, they are not used in the remainder of this thesis. These relations have been helpful though, for instance, in the perturbation theory of rotating black holes where the equation governing gravitational perturbations in the Kerr metric is derived from this version of the Bianchi identities (cf. subsection 2.4.4). Hence, they constitute too an important aspect of the Newman-Penrose formalism.

$$\begin{aligned}
\delta^*\Psi_0-D\Psi_1+D\Phi_{01}-\delta\Phi_{00}=&\ (4\alpha-\pi)\Psi_0-2(2\rho+\varepsilon)\Psi_1+3\kappa\Psi_2+2(\varepsilon+\rho^*)\Phi_{01}\\
&+(\pi^*-2\alpha^*-2\beta)\Phi_{00}+2\sigma\Phi_{10}-2\kappa\Phi_{11}-\kappa^*\Phi_{02},\\
\Delta\Psi_0-\delta\Psi_1+D\Phi_{02}-\delta\Phi_{01}=&\ (4\gamma-\mu)\Psi_0-2(2\tau+\beta)\Psi_1+3\sigma\Psi_2+2(\pi^*-\beta)\Phi_{01}\\
&+(2\varepsilon-2\varepsilon^*+\rho^*)\Phi_{02}+2\sigma\Phi_{11}-2\kappa\Phi_{12}-\lambda^*\Phi_{00},\\
\delta^*\Psi_3-D\Psi_4+\delta^*\Phi_{21}-\Delta\Phi_{20}=&\ (4\varepsilon-\rho)\Psi_4-2(2\pi+\alpha)\Psi_3+3\lambda\Psi_2+2(\tau^*-\alpha)\Phi_{21}\\
&+(\mu^*+2\gamma-2\gamma^*)\Phi_{20}+2\lambda\Phi_{11}-2\nu\Phi_{10}-\sigma^*\Phi_{22},\\
D\Psi_2-\delta^*\Psi_1+\Delta\Phi_{00}-\delta^*\Phi_{01}+2D\Lambda=&\ -\lambda\Psi_0+2(\pi-\alpha)\Psi_1+3\rho\Psi_2-2\kappa\Psi_3
\end{aligned}$$

$$
\begin{aligned}
&-2\tau\Phi_{10}+2\rho\Phi_{11}+\sigma^*\Phi_{02}-2(\tau^*+\alpha)\Phi_{01}+(2\gamma+2\gamma^*-\mu^*)\Phi_{00},\\
\Delta\Psi_2-\delta\Psi_3+D\Phi_{22}-\delta\Phi_{21}+2\Delta\Lambda &= \sigma\Psi_4+2(\beta-\tau)\Psi_3-3\mu\Psi_2+2\nu\Psi_1+2\pi\Phi_{12}\\
&\quad -2\mu\Phi_{11}-\lambda^*\Phi_{20}+2(\pi^*+\beta)\Phi_{21}+(\rho^*-2\varepsilon-2\varepsilon^*)\Phi_{22},\\
D\Psi_3-\delta^*\Psi_2-D\Phi_{21}+\delta\Phi_{20}-2\delta^*\Lambda &= -\kappa\Psi_4+2(\rho-\varepsilon)\Psi_3+3\pi\Psi_2-2\lambda\Psi_1-2\pi\Phi_{11}\\
&\quad +2\mu\Phi_{10}+\kappa^*\Phi_{22}-2(\rho^*-\varepsilon)\Phi_{21}+(2\alpha^*-2\beta-\pi^*)\Phi_{20},\\
\Delta\Psi_1-\delta\Psi_2-\Delta\Phi_{01}+\delta^*\Phi_{02}-2\delta\Lambda &= \nu\Psi_0+2(\gamma-\mu)\Psi_1-3\tau\Psi_2+2\sigma\Psi_3+2\tau\Phi_{11}\\
&\quad -2\rho\Phi_{12}-\nu^*\Phi_{00}+2(\mu^*-\gamma)\Phi_{01}+(\tau^*-2\beta^*+2\alpha)\Phi_{02},\\
D\Phi_{11}-\delta\Phi_{10}-\delta^*\Phi_{01}+\Delta\Phi_{00}+3D\Lambda &= (2\gamma-\mu+2\gamma^*-\mu^*)\Phi_{00}+(\pi-2\alpha-2\tau^*)\Phi_{01}\\
&+(\pi^*-2\alpha^*-2\tau)\Phi_{10}+2(\rho+\rho^*)\Phi_{11}+\sigma^*\Phi_{02}+\sigma\Phi_{20}-\kappa^*\Phi_{12}-\kappa\Phi_{21},\\
D\Phi_{12}-\delta\Phi_{11}-\delta^*\Phi_{02}+\Delta\Phi_{01}+3\delta\Lambda &= (2\beta^*+\pi-2\alpha-\tau^*)\Phi_{02}+(\rho^*+2\rho-2\varepsilon^*)\Phi_{12}\\
&+(2\gamma-2\mu^*-\mu)\Phi_{01}+2(\pi^*-\tau)\Phi_{11}+\nu^*\Phi_{00}-\lambda^*\Phi_{10}+\sigma\Phi_{21}-\kappa\Phi_{22},\\
D\Phi_{22}-\delta\Phi_{21}-\delta^*\Phi_{12}+\Delta\Phi_{11}+3\Delta\Lambda &= (\rho+\rho^*-2\varepsilon-2\varepsilon^*)\Phi_{22}+(2\beta^*+2\pi-\tau^*)\Phi_{12}\\
&+(2\beta+2\pi^*-\tau)\Phi_{21}-2(\mu+\mu^*)\Phi_{11}+\nu\Phi_{01}+\nu^*\Phi_{10}-\lambda^*\Phi_{20}-\lambda\Phi_{02}.
\end{aligned}
\tag{2.36}
$$

Generally it turns out to be convenient to use this formalism when dealing with algebraically special space-times. For example, the proof of the Goldberg-Sachs theorem becomes relatively easy by utilizing a suitable tetrad, and its corresponding spin coefficients, combined with the Ricci and Bianchi identities. Another important result that the Newman-Penrose formalism yields is that of the “peeling theorem”, which will be discussed in a later subsection.

2.2.1 Tetrad Transformations

There exists certain freedom in picking a null vector tetrad for a given space-time. This freedom corresponds to the group of Lorentz transformations that leave the orthogonality properties of the formalism invariant. These transformations are the so-called null rotations of the tetrad and suitable rescalings of the vectors that constitute it [32]. In fact, each type represents an Abelian subgroup of the Lorentz group, namely,

- Null rotations with n^μ fixed:

$$l'^\mu = l^\mu + a_1\bar{m}^\mu + a_1^* m^\mu + \|a_1\|^2 n^\mu, \quad m'^\mu = m^\mu + a_1 n^\mu, \quad n'^\mu = n^\mu. \tag{2.37}$$

- Null rotations with l^μ fixed:

$$n'^\mu = n^\mu + a_2\bar{m}^\mu + a_2^* m^\mu + \|a_2\|^2 l^\mu, \quad m'^\mu = m^\mu + a_2 l^\mu, \quad l'^\mu = l^\mu. \tag{2.38}$$

- Rescaling of the tetrad (Lorentz boosts):

$$l'^\mu = A l^\mu, \quad n'^\mu = \frac{1}{A} n^\mu, \quad m'^\mu = e^{i\theta} m^\mu. \tag{2.39}$$

The quantities $a_1, a_2 \in \mathbb{C}$ and $A, \theta \in \mathbb{R}$ are the parameters of the rotations. From here, it can be seen that there exist 6 real parameters that represent the liberty of choosing a null tetrad.

As one would expect, when performing any of these transformations the Newman-Penrose quantities will be modified in general. It is then possible to write transformation laws for the spin coefficients and curvature quantities Φ_{AB}. However, the most physically interesting quantities to study are the Weyl scalars since they give information regarding gravitational radiation. Under each of these type of transformations the Weyl scalars can be shown to change as

- Null rotations with n^μ fixed:

$$\begin{aligned}\Psi_0' =& \Psi_0 + 4a_1\Psi_1 + 6a_1^2\Psi_2 + 4a_1^3\Psi_3 + a_1^4\Psi_4,\\ \Psi_1' =& \Psi_1 + 3a_1\Psi_2 + 3a_1^2\Psi_3 + a_1^3\Psi_4, \quad \Psi_2' = \Psi_2 + 2a_1\Psi_3 + a_1^2\Psi_4,\\ \Psi_3' =& \Psi_3 + a_1\Psi_4, \quad \Psi_4' = \Psi_4.\end{aligned} \tag{2.40}$$

- Null rotations with l^μ fixed:

$$\begin{aligned}\Psi_4' =& \Psi_4 + 4a_2^*\Psi_3 + 6a_2^{*2}\Psi_2 + 4a_2^{*3}\Psi_1 + a_2^{*4}\Psi_0,\\ \Psi_3' =& \Psi_3 + 3a_2^*\Psi_2 + 3a_2^{*2}\Psi_3 + a_2^{*3}\Psi_0, \quad \Psi_2' = \Psi_2 + 2a_2^*\Psi_1 + a_2^{*2}\Psi_0,\\ \Psi_1' =& \Psi_1 + a_2^*\Psi_0, \quad \Psi_0' = \Psi_0.\end{aligned} \tag{2.41}$$

- Rescaling of the tetrad (Lorentz boosts):

$$\begin{aligned}\Psi_0' =& A^2 e^{2i\theta}\Psi_0, \quad \Psi_1' = Ae^{i\theta}\Psi_1, \quad \Psi_2' = \Psi_2,\\ \Psi_3' =& \frac{e^{-i\theta}}{A}\Psi_3, \quad \Psi_4' = \frac{e^{-2i\theta}}{A^2}\Psi_4.\end{aligned} \tag{2.42}$$

The importance of applying these rotations comes from the convenience of being able to find an adapted tetrad to a certain metric. For instance, in algebraically special space-times of type D, there always exists an adapted tetrad such that the only non-vanishing Weyl scalar is Ψ_2. This leads to simpler calculations in the Newman-Penrose formalism. In this case the vectors l^μ and n^μ are aligned to the twice degenerated principal null directions of the metric. Even in an algebraically general space-time it is always possible to find an adapted tetrad in which $\Psi_0 = \Psi_4 \neq 0$, $\Psi_2 \neq 0$, and $\Psi_1 = \Psi_3 = 0$. Null rotations are tools to finding such kind of tetrads starting with an initial non-adapted one. This process can nevertheless run into some difficulties since, as it can be seen from equations (2.40) and (2.41), the parameters of the rotations can have a quartic dependence in a general case.

2.2.2 The Peeling Theorem

In General Relativity it is of special interest to describe the properties of asymptotically flat space-times. The reason behind this is that one can expect that a compact physical object consisting of a strong gravitational source would considerably curve the space-time around it. Sufficiently far enough, though, gravity would diminish almost completely and consequently, the curvature too, i.e., the space-time would become nearly flat. A number of mathematical implications arise from this asymptotic condition. Such implications are naturally related to the metric and curvature tensors of the metric. One of them, for example, is that there exists coordinate systems such that $g_{\mu\nu} \to \eta_{\mu\nu}$ at large distances from the source. That is to say, the metric reduces to that of flat space-time in a suitable coordinate system, e.g., Cartesian or spherical coordinates. Unfortunately this is a rather unsatisfactory (even though simple) way to define asymptotical flatness due to it being coordinate dependent. Roughly speaking, a more appropriate definition can be made by means of a non-physical space-time $(\widetilde{M}, \widetilde{g}_{\mu\nu})$ for which $g_{\mu\nu} = \Omega^2 \widetilde{g}_{\mu\nu}$ and by identifying a boundary of the unphysical space-time constituted of the points at infinity of the physical space-time $(M, g_{\mu\nu})$. The physical space-time is thus mapped into the unphysical one through a conformal isometry. An additional requirement for asymptotical flatness is that every null geodesic must intersect said boundary at two points, one of them belonging to past null infinity $\mathcal{J}^-$ and the other to future null infinity $\mathcal{J}^+$. This definition is manifestly coordinate free [25].

On a related matter, it is also important to characterize the asymptotic behavior of the curvature tensors. In this context, the Weyl tensor can be considered as the more relevant of them since it is conformally invariant, i.e., $C^{\alpha}{}_{\beta\mu\nu} = \widetilde{C}^{\alpha}{}_{\beta\mu\nu}$. The peeling theorem, which is given here without proof, establishes precisely this behavior in asymptotically flat space-times using the Weyl scalars [31].

Theorem 2.3. Let γ_0 be a null geodesic in an asymptotically flat space-time $(M, g_{\mu\nu})$ going from a point $p \in M$ to a point q on future null infinity $\mathcal{J}^+ \subset M$, with affine parameter λ_0. Then, as $\lambda_0 \to \infty$, the Weyl scalars have the following asymptotic behavior

$$\Psi_n = \mathcal{O}(\lambda_0^{n-5}), \quad (n = 0, 1, \ldots, 4).$$

In order for this theorem to be valid, the use of a tetrad in which one of its vectors is aligned with the tangent of γ_0 is required. Theorem 2.3 allows one to divide the space-time into five regions depending on which Weyl scalars become negligible there. In this classification there is a near zone (the closest region to the gravitational source) where all of the Weyl scalars have to be considered. Then, as one moves further away, Ψ_0 becomes small, followed by Ψ_1 and eventually by Ψ_2. These regions can be understood as three respective transition zones. Finally, in the so-called radiation zone, Ψ_3 becomes negligible too and only Ψ_4 dominates there. So, as the null geodesic γ_0 travels across these regions in its way to future null infinity, each of the Weyl scalars become small in turn. This is called the peeling property of the Weyl scalars, justifying therefore the name of the theorem. It is notable that there exists a clear analogy between this description and the case of electromagnetic radiation in which, similarly, three zones can be defined (near, transition and radiation zone).

Furthermore, as the peeling theorem describes the fall-off rates of the Weyl scalars, a type of the algebraic classification of space-times can be associated to each of the above regions. The near zone is of type I, i.e., algebraically general. The first transition zone is of type III, the second of type II or type D, and the third of type III. The radiation zone is essentially null, this is, of type N. The tangent vector k^μ of the null geodesic γ_0 is aligned to the degenerate null directions of the space-time in each case (in type I it is aligned to one of the four non-degenerate directions). See figure 2.1.

2.2.3 Physical Interpretation of the Weyl Scalars

As mentioned earlier in this chapter, of all the Newman-Penrose quantities characteristic of the formalism, maybe the most important of them are the Weyl scalars.

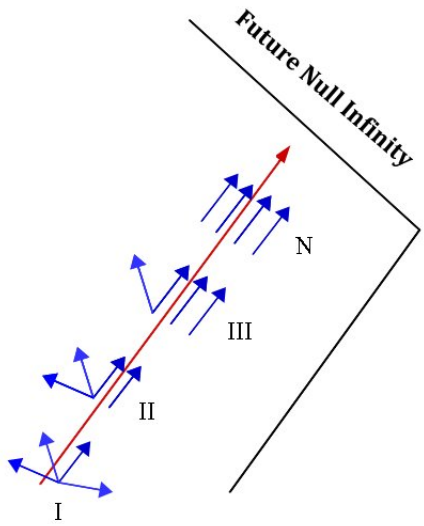

FIGURE 2.1: The peeling property of the Weyl scalars in an asymptotically flat space-time [33].

This owes in large part to the physical interpretation they admit. This interpretation can be revealed through the following analysis [34].

Consider an orthonormal frame $\{e_0^\mu, e_1^\mu, e_2^\mu, e_3^\mu\}$ of a given metric and the tangent $u^\mu = e_0^\mu$ of a time-like geodesic observer. With the aid of the Riemann and Weyl tensors decomposition, the geodesic deviation equation of the integral curves of the vector u^μ, with deviation vector δx^μ, can be studied separately for each case of vanishing Weyl scalars (assuming the use of an adapted null tetrad). This in turn corresponds to a certain algebraic type of space-time. The time-like observer will measure the effect of the gravitational field by the distortion it induces on its neighboring geodesics, that is, by the geometrical properties of the deviation vector δx^μ. One may think of this distortion as caused by a gravitational wave propagating in the k^μ direction (a principal null direction of the space-time), orthogonal to the plane in which e_1^μ and e_2^μ lie. For each of the types of space-times, the relevant term in the geodesic deviation equation (2.17) is,

- For type N ($\Psi_4 \neq 0, \Psi_3 = \Psi_2 = \Psi_1 = \Psi_0 = 0$):

$$\delta \ddot{x}^\mu = \frac{1}{2}(\Psi_4 + \Psi_4^*)(e_1^\mu e_{1\nu} - e_2^\mu e_{2\nu})\delta x^\nu.$$

  Since the terms appearing in this equation are related to vectors orthogonal to the direction of propagation, this gravitational field is considered as a pure transverse gravitational wave.

- For type III ($\Psi_3 \neq 0, \Psi_4 = \Psi_2 = \Psi_1 = \Psi_0 = 0$):

$$\delta \ddot{x}^\mu = \frac{1}{2}(\Psi_3 + \Psi_3^*)(e_3^\mu e_{1\nu} - e_1^\mu e_{3\nu})\delta x^\nu.$$

  In this type it can be seen that there appear terms consisting of e_3^μ vectors, these are longitudinal wave components.

- For type D ($\Psi_2 \neq 0, \Psi_4 = \Psi_3 = \Psi_1 = \Psi_0 = 0$):

$$\delta \ddot{x}^{\mu} = \Psi_2 \left[e_3^{\mu} e_{3\nu} - \frac{1}{2} \left(e_1^{\mu} e_{1\nu} - e_2^{\mu} e_{2\nu} \right) \right] \delta x^{\nu}.$$

This particular deviation vector causes a sphere of particles around the time-like observer to deform into an ellipsoid with the direction e_3^{μ} as principal axis. This component is named as Coulomb-like since particles falling towards a central body with an inverse square law are expected to behave this way.

- For type II ($\Psi_4, \Psi_2, \neq 0$, $\Psi_3 = \Psi_1 = \Psi_0 = 0$) the previous results for type N and type D can be used. In this case there will be a combination of terms corresponding to an outgoing transverse wave and a Coulomb component.
- For type I ($\Psi_4, \Psi_2, \Psi_0 \neq 0$, $\Psi_3 = \Psi_1 = 0$) the gravitational field is a mixture of outgoing and ingoing transverse waves, as well as a Coulomb term.

Therefore, it can be concluded that the Ψ_4 and Ψ_3 scalars describe, respectively, transverse and longitudinal waves in the direction of the null vector $k^{\mu} = l^{\mu}$, where l^{μ} is a vector of the tetrad. On the other hand, Ψ_0 and Ψ_1 also represent respectively transverse and longitudinal waves, but in the direction of the null tetrad vector n^{μ}. The Ψ_2 scalar is a Coulomb component of the wave.

Another important application of the discussed scalars, which is consequence of the foregoing analysis, resides in the gravitational perturbations of a background space-time. In section 2.4 it will be seen that in perturbation theory, the Weyl scalars Ψ_0 and Ψ_4 serve also to describe the energy flux of gravitational radiation for ingoing and outgoing waves, respectively, as measured by a distant observer. It is clear then that the Weyl scalars represent quantities of great physical interest.

2.3 The Problem of Space-Time Singularities

We now turn to the particular topic of space-time singularities. Space-time singularities are, to this day, one of the aspects of General Relativity which still hold several unanswered questions. Difficulties arise even from the supposedly simple task of exactly defining what a singularity is and whether a space-time is singular or not. In fact, historically, the concepts of event horizons and essential singularities were frequently mixed up and confused. Fortunately, the notion of a singularity within General Relativity is nowadays somewhat clearer. Nevertheless, this problematic part of the theory is yet to be fully understood.

Many widely known exact solutions to the Einstein equations contain a singularity, for instance, those who describe black holes. In fact, it is thought that once gravitational collapse takes place, the formation of a space-time singularity is unavoidable [35]. In the case of black holes solutions in the vacuum, its singularities are commonly associated to the divergence of curvature scalars in the space-time metric. This has led to the notion of the necessity of a quantum theory of gravity to adequately describe regions where said scalars approach to the Planck scale. Nevertheless, these particular singularities are physically accepted since they are hidden behind event horizons and hence, causally disconnected from the outer region of the black hole. This is illustrated in figure 2.2.

On the contrary, space-times with unbounded curvature scalars, and that are not equipped with an event horizon (the so-called naked singularities), are often dismissed as non-physical and thus, considered as pathological. As examples of this

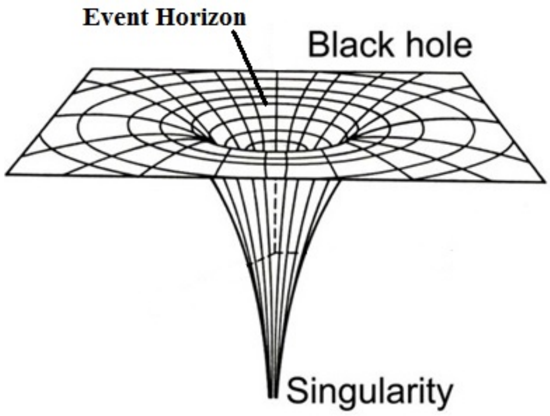

FIGURE 2.2: A schematic representation of a singularity in the Schwarzschild black hole. The region of infinite curvature is covered by the event horizon.

type of space-times, the extremal Kerr metric ($|a| > |m|$) and the axially symmetric ring wormholes can be mentioned [18].

In this section the main issues regarding space-time singularities will be discussed. Related helpful concepts centered around their description are also presented.

2.3.1 Identifying a Singularity

Loosely speaking, one may think of a singularity as a "place" where something "goes wrong", for instance, a set of coordinate values for which physical quantities or fields diverge or are otherwise ill-defined. This is barely a proper definition and is certainly very ambiguous. Apart from that, in the context of General Relativity it is an incorrect idea too.

The theory of Relativity attempts to describe not only physical phenomena (specially that of the gravitational type), but also space-time itself. In this sense, General Relativity has a very particular nature since it intrinsically ties gravity with geometry. When encountering a singularity in this theory, hence, the implication is that even space-time is not well-defined along with any additional physical quantities. A space-time singularity should thus not be seen as a "place" [25].

To identify a singularity then it might be worth to examine properties that arise from the metric of the space-time and search for any ill behavior. This must be done carefully since wrong conclusions can be reached from an incorrect interpretation. Among some of the problems that can be found are [36]:

- The entries of the metric tensor by themselves are by no means a correct way of finding singularities. A basic example is g_{rr} in the Schwarzschild metric: it is ill-defined at the event horizon, even though that hypersurface is not actually singular.
- The components of any given curvature tensor are bad indicators of essential singularities too. Their expressions necessarily depend on a coordinate basis and therefore, a poor or unfortunate choice of basis can artificially create irregular quantities.

- Curvature scalars, such as $R^{\mu}{}_{\mu}$, $R^{\mu\nu}R_{\mu\nu}$, etc., are more reliable since they are basis independent. Unbounded scalars of this type are often signs of singularities. These will be referred to as curvature singularities. However, if these scalars vanish, singularities can still be present. Some plane gravitational wave solutions can indeed present this feature. It can also be the case that curvature diverges as infinity is approached even if it is not singular. Curvature scalars are thus not free of any issues.

- Singularities can also appear from topological defects of the space-time manifold without them manifesting in curvature quantities. For instance: conical singularities and cosmic strings.

To make matters worse, in a strict sense, singularities do not even belong to the space-time, which is by definition constituted only of regular points. As it can be seen, the problem of generally defining a singularity is quite troublesome due to the variety of situations that can be characterized as non-regular. In this spirit, a classification due to Ellis and Schmidt describes all the possible ill behaviors that the geometry of a space-time may possess [37]. While the exploration of irregularities in the metric that can occur within General Relativity is a helpful approach, the lack of a general way to define, or at least to identify, a space-time singularity remained as unsatisfactory.

The main aim of this work is focused on the previously described curvature singularities, specifically those found in black holes and wormholes. For this case, curvature scalars can be seen as the relevant quantities to examine. Even so, the following characterization of singular space-times is also of great importance to them, as for any kind of singularity.

2.3.2 Incomplete Curves on the Space-Time

Despite all of the difficulties explained for properly defining a singularity, over the years, an alternative approach has been proposed to establish the non-regularity of a space-time. This approach is that of examining the curves of the space-time manifold. Physical observers move following curves exclusively within the manifold, hence a good tool to detect singularities is by the ill effects that they may cause on these observers, particularly on geodesics.

A very important property of geodesics plays a key role in this characterization: its affine completeness. This property refers to the ability to extend such curves to, either past or future, arbitrary values of its affine parameter. This can be defined in more formal terms [38].

Definition 2.12. A geodesic on the space-time manifold M with affine parameter λ is complete if it is defined for all values $\lambda \in \mathbb{R}$ of its affine parameter. The manifold M is said to be geodesically complete if all geodesics on M are complete.

This definition must be restricted to inextendible geodesics, i.e., curves without past or future endpoints[3]. Otherwise, the arbitrary removal of a point in a curve would lead to incompleteness.

Definition 2.13. A space-time is geodesically incomplete if it contains an incomplete geodesic.

[3] A point $p \in M$ is said to be a future endpoint of a curve γ if for every neighborhood N of p there exists a finite affine parameter λ_0 such that $\gamma(\lambda) \in N$ for all $\lambda > \lambda_0$ [25]. Similarly for a past endpoint.

The definitions of geodesic completeness and incompleteness allow a natural particularization to a specific type of geodesics: space-like, null or time-like. For instance, a manifold M is time-like geodesically complete if all time-like geodesics on M are complete, etc., and similarly for geodesic incompleteness. An extension to general curves on the manifold (not necessarily geodesics) can also be easily done.

To obtain a better understanding of the notion of incompleteness in the context of curves, it might serve well to think of a graphical representation as in figure 2.3. Because incomplete curves are not defined for all values of their affine parameter, an observer following such a curve disappears off the manifold in a finite amount of the affine parameter λ. The time reversal situation describes an observer suddenly appearing into the manifold after a certain value of λ is reached. On the opposite situation, a complete curve is always defined and hence, no matter the value of λ, the observer is surely localized within the space-time manifold (it does not suddenly appear or disappear).

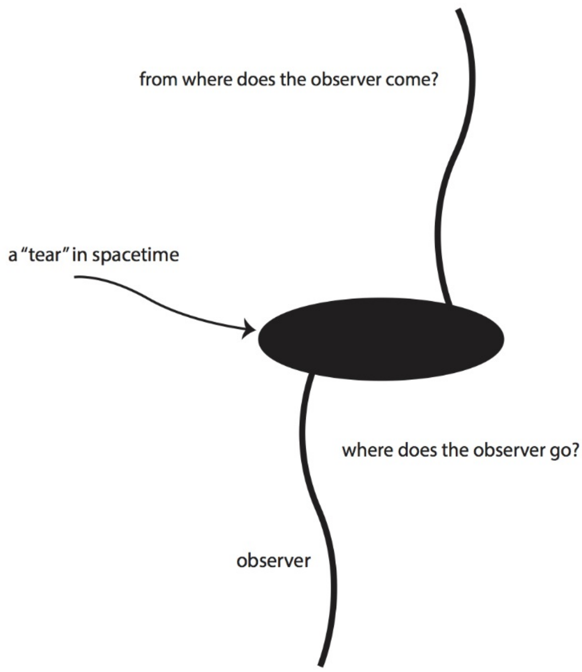

FIGURE 2.3: An incomplete curve on the space-time [39].

There is of course something inherently wrong with geodesically incomplete space-times and a reasonable relation between this global property of the manifold and singularities can be made. Since singular points do not belong to the space-time, they would leave out "holes" on it (similar to the one shown in figure 2.3) where otherwise regular points should be. The presence of these holes can then be inferred if geodesics, as well as other curves, become incomplete when reaching them. This means that the curves end or begin their trajectory upon intersecting said holes. Taking advantage of the previous definitions, then, there is general agreement on considering that a geodesically incomplete space-time is, by all means, singular [25, 40].

Though this might seem as a satisfactory definition of a singularity, it is unfortunately not without flaws. Some additional considerations must be taken, or at least, one must be aware of its limitations. The first one has to do with extensions of a given space-time [41].

Definition 2.14. An extension of a space-time $(M, g_{\mu\nu})$ is an isometric embedding $\zeta : M \to \widetilde{M}$, where $(\widetilde{M}, \widetilde{g}_{\mu\nu})$ denotes a space-time and ζ is onto a proper subset of $\widetilde{M}$. A space-time is extendible if it has an extension.

Arbitrary points can be removed from a regular space-time inducing, in a somewhat artificial manner, incompleteness in its curves. However, the original space-time should not be deemed in any way as singular. The reason for this is that the apparently singular space-time can be extended to a regular one. Black hole space-times are clear examples of why extensions are helpful tools in General Relativity. While their event horizon may seem as a singular hypersurface that provokes geodesic incompleteness, a suitable extension (see subsection 1.2.2) of the space-time shows that geodesics can be further continued after entering the horizon. Nevertheless, no extension is able to remove the true curvature singularity inside the event horizon. Therefore, the characterization of singularities from geodesic incompleteness is consequently limited to non-extendible space-times.

Another problem is the possibility that in geodesically complete space-times, a congruence of time-like curves of limited acceleration can still become incomplete. In this case, it can be argued that the space-time must be considered as singular despite its geodesic completeness.

Finally, it turns out that geodesic incompleteness cannot always be associated to the presence of the described "holes" on the space-time manifold. Misner gives an example of this consisting of a compact, yet geodesically incomplete, space-time [42]. Further description of this metric will be given in the next subsection. Due to compactness, it is expected that the manifold is free of holes that correspond to the removal of a singularity. Thus, incompleteness in this case is not related at all to the existence of holes, yet it is still a sign of pathological behavior of freely falling particles or photons.

Regardless of the outlined limitations, geodesic incompleteness provides a way of determining if a given space-time can be considered as singular. It should be remarked, though, that the completeness of geodesics does not guarantee regularity, i.e., geodesic completeness is not equivalent to the notion of a space-time being non-singular. Geroch has constructed a complex example of this, see [40] for details.

2.3.3 Curvature Singularities and Geodesic Incompleteness

In the previous subsection geodesic incompleteness was established as a sufficient condition for a space-time to be singular. Since this work focuses on curvature singularities of the metric, the particular relation between these two concepts will be discussed now.

One may be tempted to think that geodesic incompleteness implies the existence of curvature singularities. This is not generally true, incompleteness does lead to singularities, however and as mentioned in subsection 2.3.1, there are various types of pathological behaviors of the space-time not necessarily related to unbounded curvature. The following specific space-time illustrates this point.

a. **Geodesic incompleteness does not imply unbounded curvature.**

   A simple example of this claim is the above mentioned metric due to Misner. Its line element is the following [42]

$$ds^2 = \cos x (dy^2 - dx^2) + 2 \sin x dx dy. \tag{2.43}$$

This two-dimensional metric of signature $(-1,+1)$ is analytic on the manifold M, which is defined to be a torus $S^1 \times S^1$. Here, x and y are angular coordinates ranging over the values $0 \leq x, y \leq 2\pi$. Its curvature scalars can be easily computed and shown to be completely regular. The Ricci scalar R and other quadratic scalars, such as the Kretschamnn scalar $K = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, are given by

$$R^2 = 2R^{\mu\nu}R_{\mu\nu} = K = \cos^2 x. \tag{2.44}$$

Curvature in this metric is clearly bounded. However, incomplete geodesics can still be found despite its seemingly regularity. Taking advantage on the fact that the metric components do not depend on the coordinate y, the geodesics of the space-time can be given in terms of the first integrals

$$\begin{aligned} \dot{y}\cos x + \dot{x}\sin x &= p, \\ \dot{x}^2 + \kappa \cos x &= p^2. \end{aligned} \tag{2.45}$$

In equations (2.45), the constant of motion p was introduced and $\kappa = -1, 0, 1$ for time-like, null and space-like geodesics, respectively. Consider the geodesics moving with constant $x = \pi/2$, the above system of equations reduces identically to zero with $p = 0$. For this case the geodesic equation can be reduced to

$$\ddot{y} + \frac{1}{2}\dot{y}^2 = 0.$$

Its solution is $y = \ln \lambda^2$ with affine parameter λ. Since y is not well-defined for $\lambda = 0$, this geodesic is incomplete. In fact, the curve cannot be extended from positive to negative values of the affine parameter. This is just the simplest example of an incomplete geodesic but, due to the $\cos x$ factor that multiplies the coordinate velocity $\dot{y}$ in (2.45), troublesome behavior is caused in the function $y(\lambda)$. A further example is that given by null geodesics ($\kappa = 0$), integration of equations (2.45) gives

$$\begin{aligned} x &= p\lambda + x_0, \\ y &= \ln\left[\sqrt{2}\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)\right] - \ln\left[\sqrt{2}\cos\left(\frac{x}{2} + \frac{\pi}{4}\right)\right] + \ln\left[\cos x\right] + y_0, \end{aligned}$$

where x_0 and y_0 are integration constants. Once again, there are values for which the y part of the solutions is not defined, e.g., $x = \pi/2$. Therefore, this space-time is geodesically incomplete but with regular curvature scalars, thereby proving that geodesic incompleteness does not imply unbounded curvature.

Another interesting example of this is the two-dimensional model of the Taub-NUT vacuum space-time by Misner. Indeed, Misner himself described this metric as a "counterexample to almost anything" [43].

b. **Does unbounded curvature imply geodesic incompleteness?**

Conversely, the question of whether curvature singularities imply geodesic incompleteness seems to defy the intuition generated by the idea that, once a geodesic runs into a hole left by the singularity, the curve cannot be continued further. Examples of presumably geodesic completeness despite the presence of curvature singularities have been reported in recent years. A metric with this feature is provided by the wormhole geometries found in [44]. The line element is a solution to high-energy quadratic extensions of General Relativity coupled to Maxwell electrodynamics. Further study of said metric has shown that even the curves of observers with bounded acceleration are complete [45], therefore extending the argumentation of the regularity of those space-times. This example seem to indicate that, in regards to geodesic incompleteness and the divergence of curvature scalars, one does not imply the other. An analysis of these two features, as well as unbounded energy density, is further discussed in [46] for the case of space-times in a quadratic $f(R)$ gravity theory.

One additional example of curvature divergences along complete geodesics is given by some spherically symmetric shear-free perfect fluid metrics [47]. In this case, curves take an infinite amount of affine parameter to run into the singular regions of space-time.

Both properties are hence apparently independent of each other: neither geodesic incompleteness implies the presence of curvature singularities, nor unbounded curvature implies geodesic incompleteness [48].

2.3.4 The Cosmic Censorship Conjecture

While the curvature singularities of the wormholes described in the past subsection seem to be "harmless", at least for causal curves on the manifold, this may not be the case for the whole generality of "naked singularities". Thus provoking ill effects on the space-time and its physical observers. To deal with this unfortunate possibility, in 1969 Roger Penrose had already conjectured the existence of a "cosmic censor" that forbids that such singularities can be perceived by observer on the space-time [49]. This is now called the "cosmic censorship" conjecture and it can be formulated in a variety of ways depending on how heavy are the restrictions it imposes:

- **Weak Cosmic Censorship.** No singularity is ever visible to asymptotically distant observers. This means that the ultimate result of gravitational collapse must be a black hole so that the singularity it produces, always lies behind an event horizon. The singularity is hence, causally disconnected from the rest of the space-time, as well as any observer that can actually "see" it.

- **Strong Cosmic Censorship.** Excluding a possible initial singularity (the hypothetical origin of the universe), no singularity is ever visible to any observer [50]. This leads to the assertion that a physically reasonable space-time is globally hyperbolic, i.e., the whole evolution of the space-time can be predicted entirely from an appropriate set of initial data.

Both versions of the conjecture can be put in more formal mathematical terms, rather than the less precise, but easier to understand, previous statements. This allows for a suitable demonstration of their validity, or otherwise to disprove them. Some assumptions made in these formulations regard conditions on the matter sources of the space-time, these include the fulfillment of some energy condition and that its field equations admit a well-posed initial value problem with a suitable behavior at infinity. These impositions are part of deeming a space-time as "physically reasonable". The complete mathematical statements of the conjecture, though, will not be presented here since they involve several technical concepts that are not discussed within this work. Nevertheless, further details can be found in the cited references, as well as in [25].

It is worth highlighting too that, maybe against basic intuition, strong cosmic censorship does not imply its weak counterpart. For instance, a gravitational collapse that results in a wave of singularity that propagates to null infinity without affecting global hyperbolicity violates weak cosmic censorship, but not the strong version of it.

There is still not a rigorous mathematical argument in favor or against cosmic censorship and it remains to be seen if the conjecture is a fundamental component of the theory, this is, if nature itself really does work such that it conceals possible space-time singularities from physical observers. The censorship mechanism in black holes is very clear, however, the issue of whether this conjecture is generally true or false is yet an open question in General Relativity. Penrose, himself, acknowledges that quite possibly more advanced mathematical tools are needed to finally give an answer to this question. According to Penrose, the key might be in the development of twistor theory and its geometric applications to the manifolds of space-times [51].

2.3.5 The Singularity Theorems

One of the most remarkable and elegant accomplishments of General Relativity are the singularity theorems. The first of them is due to Penrose and, among other conclusions, it shows that singularities are indeed an inherent part of the theory, not simply consequences of the high degree of symmetry assumed for the solutions of the Einstein field equations [35]. They possess extreme relevance with respect to the problem of gravitational collapse as well as important cosmological implications.

Penrose formulated his theorem in 1965 when trying to prove that deviations from spherical symmetry in gravitational collapse would not avoid the formation of a singularity. Here, this first theorem is given without proof.

Theorem 2.4. A space-time $(M, g_{\mu\nu})$ cannot be null geodesically complete if

1. $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ for all null vectors k^{μ},
2. there is a non-compact Cauchy surface Σ in M,
3. there is a close trapped surface $\mathcal{T}$ in M.

In theorem 2.4 some new terms are introduced which will now be explained. A Cauchy surface Σ is a space-like hypersurface in M whose initial data predicts the future (and past) of the entire manifold. The existence of a Cauchy surface in M was later shown to be equivalent to M being globally hyperbolic [52]. Meanwhile, a closed trapped surface $\mathcal{T}$ is a compact, two-dimensional, space-like submanifold of M in which future directed null geodesics orthogonal to $\mathcal{T}$ (ingoing and outgoing null rays) are all converging [25]. In other words, light is dragged back into $\mathcal{T}$ and confined there. Since even light cannot escape from that surface, ordinary matter is said to be trapped within $\mathcal{T}$ too. All spheres of constant $r < 2m$ in the Schwarzschild black hole are examples of closed trapped surfaces.

Further work by Hawking managed to find a similar theorem but including time-like geodesic incompleteness along with slightly less strict conditions [53]. For instance, global hyperbolicity is replaced with the so-called strong causality condition on the space-time. In contrast to the first theorem of Penrose, this result by Hawking was aimed at singularities in a cosmological context. It can be used to prove that, given suitable energy conditions on the matter of the universe, along with a causality principle and its expansion, the universe began a finite time ago starting from a singular state.

The first singularity theorem by Penrose motivated a series of developments in the field in which the relaxation (or alternatives) of conditions 1 to 3 were considered in the context of other physical scenarios. As a result, one of the stronger singularity theorems was obtained by Hawking and himself.

Theorem 2.5. The following three conditions cannot hold simultaneously in a space-time $(M, g_{\mu\nu})$:

1. every inextendible causal geodesic contains a pair of conjugate points,
2. the chronology condition holds on M,
3. there is a trapped set ς.

Where again, several new concepts are involved in the theorem. The chronology condition means that there are no closed future-directed time-like curves through any point $p \in M$. Additionally, a trapped set ς is an achronal set of points of M (i.e., a set constituted of points that cannot be joined by causal curves) such that $E^{+}(\varsigma)$ or $E^{-}(\varsigma)$ is compact. The set of points $E^{\pm}(p)$ for $p \in M$ is defined as

$$E^{\pm}(p) \equiv \{q \in M \mid \text{there is a future (past)-directed null curve from } p \text{ to } q\}.$$

In this definition the future-directed part applies for the plus sign and the past-directed for the minus sign. The natural extension of this definition for a set of points $\xi \subset M$ is

$$E^{\pm}(\xi) = \bigcup_{p \in \xi} E^{\pm}(p).$$

In this subsection two of the most important theorems are shown since they serve as good examples of what they prove and the assumptions they require. Indeed, most of the singularity theorems follow a typical structure such as the following [54]: A space-time will contain at least an incomplete causal geodesics if it satisfies

1. an energy condition,
2. a causality condition,
3. a boundary or initial condition.

It can be seen then that, unfortunately, the singularity theorems go as far as asserting the existence of incomplete causal geodesics. As explained in subsection 2.3.2, this is sufficient to consider the space-time as singular provided it is inextendible. Nevertheless, the theorems do not give any sort of information about extensions, and neither regarding the nature, characteristics, or location of the possible singularity.

The derivation of the singularity theorems contains a great intrinsic value for physics and is the result of a deep comprehension and advanced development of the theory. While they definitely represent a step forward in attempting to fully comprehend space-time singularities, much yet needs to be done. For the time being, singularities remain as elusive and mysterious entities in General Relativity.

2.4 Gravitational Perturbations in General Relativity

In General Relativity a considerable amount of solutions to the Einstein field equations (1.1) describe isolated and idealized systems, providing them as a result with a metric $g_{\mu\nu}$. While these models are relatively simple and of enough physical generality, in some cases they might not suffice to adequately describe a realistic situation. For example, the Schwarzschild metric can be used to analyze the geometry of a non-rotating black hole, just as long as there are no other objects possessing a significant gravitational field orbiting it, or otherwise disrupting its geometry. To account for these possibilities it would be required to consider dynamical and more complex models. However, these models might prove to be too difficult to study, or even to properly interpret. The objective of perturbation theory in General Relativity is to give a more complete description of a given background space-time when it is disrupted by some "change" of small magnitude that deviates it from its original state.

A perturbation of a background metric $g_{\mu\nu}$ is often proposed as an addition of a perturbation term $h_{\mu\nu}$ to it, i.e., $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$. The perturbation is then assumed to be small compared to the background metric so, in some suitable sense $|g_{\mu\nu}| \gg |h_{\mu\nu}|$. With this perturbed metric one can next proceed to compute the corresponding curvature tensors and express the Einstein field equations. In this process only the linear order terms of the perturbation are considered to be significant. Superior orders are therefore neglected, which is justified by the assumption of small perturbations. Even though only first order terms of the perturbation are kept, the latter is not an easy task to perform due to the non-linearity and tensorial nature of the theory.

From an analysis of this kind of perturbations, results of great physical relevance can be obtained such as the linear stability of the considered background space-time and the appearance of gravitational waves. In the following, some of the most important works regarding linearized gravitational perturbations are outlined and briefly discussed. A particular emphasis is done at the end towards wormhole space-times.

2.4.1 Linearized Gravity and Gravitational Waves

A natural starting point for gravitational perturbations is to study first a flat background space-time with a small perturbation term in the metric, this is,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.46}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. Though this is the simplest example that one can imagine, it yields one of the most impressive predictions of General Relativity, that of gravitational waves, it also helps to develop a sense of intuition on the relevant physical aspects involved in the problem of gravitational perturbations.

Beginning with metric (2.46), the components of the perturbed Riemann tensor can be computed to first order of $h_{\mu\nu}$, obtaining [24]

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_{\beta\gamma}h_{\alpha\delta} + \partial_{\alpha\delta}h_{\beta\gamma} - \partial_{\beta\delta}h_{\alpha\gamma} - \partial_{\alpha\gamma}h_{\beta\delta}\right) + \mathcal{O}(h^2). \tag{2.47}$$

In equation (2.47) the explicit dependence of higher order terms of $h_{\mu\nu}$ has been indicated. However, hereafter this dependence will be omitted for compactness. One should bear in mind that the equations of the rest of this chapter will be valid only to first order of the perturbation.

From (2.47), the Einstein tensor can be easily calculated as

$$G_{\mu\nu} = -\frac{1}{2}\left(\partial^{\alpha}\partial_{\alpha}\widetilde{h}_{\mu\nu} + \eta_{\mu\nu}\partial^{\alpha}\partial^{\beta}\widetilde{h}_{\alpha\beta} - \partial^{\alpha}\partial_{\nu}\widetilde{h}_{\mu\alpha} - \partial^{\alpha}\partial_{\mu}\widetilde{h}_{\nu\alpha}\right), \tag{2.48}$$

with the following definitions

$$\widetilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h,$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. Choosing a convenient gauge (cf. subsection 2.1.4) in which

$$\partial^{\mu}\widetilde{h}_{\mu\nu} = 0,$$

and using the Einstein field equations it can be found that

$$\Box\widetilde{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \tag{2.49}$$

where $\Box = \partial^{\alpha}\partial_{\alpha}$ is the D'Alambertian operator. This gauge is sometimes known as the Lorentz, de Donder, or harmonic gauge and it can be shown to always exist for this case. Equation (2.49) is commonly referred to as the equation of linearized gravity.

The Newtonian limit of gravity may be obtained from the equation of linearized gravity assuming a non-vanishing stationary source $T^{00} = \rho$ and $T^{\mu\nu} \approx 0$ for the rest of the components of the stress-energy tensor. However, here the solutions of linearized gravity in the vacuum, i.e., with $T_{\mu\nu} = 0$, will be of special interest. This would correspond to the weak gravitational field of a source that a distant observer measures in an otherwise flat space-time. This field is not necessarily stationary due to the general coordinate dependence of equation (2.49). The solutions have the form

$$\widetilde{h}_{\mu\nu} = A_{\mu\nu}e^{ik_{\alpha}x^{\alpha}}, \tag{2.50}$$

with k^{α} being a null vector in Minkowski space and $A_{\mu\nu}$ a symmetric tensor of constant amplitudes. These solutions represent plane waves moving at the speed of light and traveling in the direction of the null vector k^{α}. The components of the $A_{\mu\nu}$ tensor are not completely independent, the Lorentz gauge imposes that $A_{\mu\nu}k^{\nu} = 0$. There is also further liberty within this gauge so that

$$A^{\mu}{}_{\mu} = 0, \quad A_{\mu\nu}u^{\nu} = 0,$$

where u^{μ} is a unit time-like vector. These last two equations are the conditions of the more specialized radiation gauge, it is sometimes called transverse-traceless gauge too.

Summing up all of the above conditions, an explicit expression for the perturbation metric $h_{\mu\nu}$ can be found. Note that $A^{\mu}{}_{\mu} = 0$ implies that $\widetilde{h} = -h = 0$ and hence, $\widetilde{h}_{\mu\nu} = h_{\mu\nu}$. Choosing the time-like vector $u = \partial/\partial t$ as the time basis vector and aligning the spatial coordinates so that the wave travels in the z-direction with a frequency ω, one can obtain that $h_{\mu\nu} = A_{\mu\nu}e^{ik_{\alpha}x^{\alpha}}$ with

$$
\begin{aligned}
A_{\mu\nu} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
k^{\mu} &= \omega(1, 0, 0, \pm 1).
\end{aligned} \tag{2.51}
$$

The positive sign in the k^{μ} vector describes outgoing waves, whereas the negative sign ingoing waves. There are only two independent components in the amplitude of the wave, they cannot be reduced by means of any gauge transformation and therefore, can be associated to a significant physical interpretation. Namely, these components are related to the polarization of the gravitational wave.

To arrive at this conclusion it is helpful to consider, as done somewhat similarly in subsection 2.2.3, the geodesic deviation equation of two rest particles (time-like observers whose tangent to their world line is $u = \partial/\partial t$). Using the linearized Riemann tensor (2.47) and the derived explicit form of the perturbation metric, equation (2.17) can be reduced to

$$
\ddot{\xi}^{\mu} = \frac{\partial^2 \xi^{\mu}}{\partial^2 t} = \frac{1}{2}\eta^{\mu\nu}\frac{\partial^2 h_{\nu\alpha}}{\partial^2 t}\xi^{\alpha}, \tag{2.52}
$$

where ξ^{μ} is the deviation vector between neighboring geodesics and assumed to lie on a plane orthogonal to the z-direction. The deviation vector measures the relative distance between the two rest particles. Thus, splitting equation (2.52) into components and inserting the expression for $h_{\mu\nu}$, one has that

$$
\begin{aligned}
\frac{\partial^2 \xi^1}{\partial^2 t} &= -\frac{\omega^2}{2}\left(A_{11}\xi^1 + A_{12}\xi^2\right) e^{ik_{\alpha}x^{\alpha}}, \\
\frac{\partial^2 \xi^2}{\partial^2 t} &= -\frac{\omega^2}{2}\left(A_{12}\xi^1 - A_{11}\xi^2\right) e^{ik_{\alpha}x^{\alpha}}.
\end{aligned} \tag{2.53}
$$

By making separately $A_{11} = 0$ and $A_{12} = 0$ in the previous system of equations, the effect of a gravitational wave passing through a ring of particles located at rest in the x-y plane can be analyzed. Suppose, for instance, that $A_{12} = 0$ and $A_{11} \neq 0$, then two particles initially separated only in the x-direction ($\xi^2|_{t=0} = 0$) will undergo a displacement in that precise direction. The same goes for two particles with an initial separation in the y-direction ($\xi^1|_{t=0} = 0$), they will be deformed only in their direction of initial separation. This is known as "plus" polarization and is illustrated in figure 2.4. On the contrary, if $A_{11} = 0$ and $A_{12} \neq 0$, the past two cases of a pair of particles will be deviated in both directions. This type of polarization is also shown in figure 2.4 and is named as "cross" polarization. It is important to notice too that the movement of the particles will oscillate about their initial position.

With the physical meaning of the amplitudes $A_{11} = A_{+}$ and $A_{12} = A_{\times}$ established, an illustrative application of the Weyl scalars can now be treated. The main goal here will be that of seeking a connection between the perturbations of the spacetime and those scalars with the particular intention of extracting physical information from them. For this purpose it will be necessary to introduce a null tetrad on Minkowski space,

$$
l^{\mu} = (1, 0, 0, 1), \quad n^{\mu} = \frac{1}{2}(1, 0, 0, -1), \quad m^{\mu} = \frac{1}{\sqrt{2}}(0, 1, i, 0). \tag{2.54}
$$

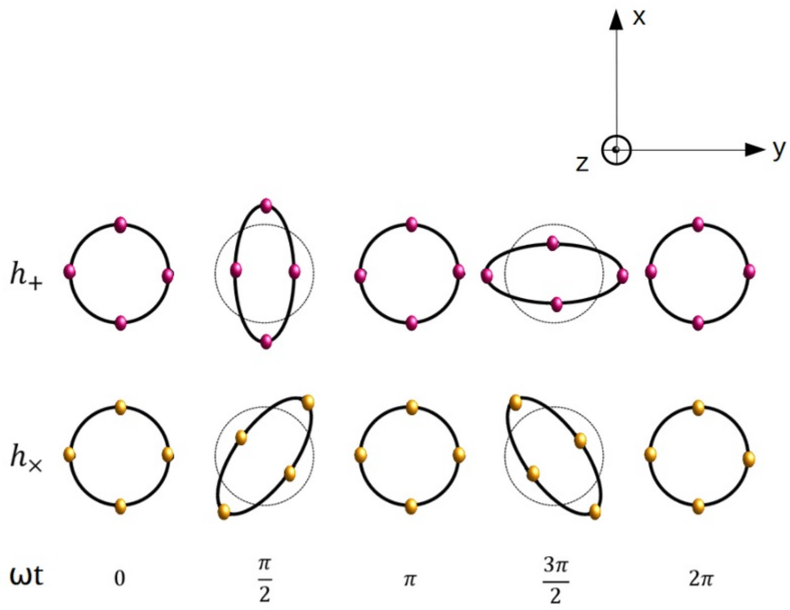

FIGURE 2.4: The distortion caused by a gravitational wave traveling in the z-direction in an array of particles located in the x-y plane. Plus $(+)$ polarization is denoted by h_+ and cross $(\times)$ polarization by $h_\times$ [55].

The tetrad (2.54) is expressed in the $\{t, x, y, z\}$ basis. Since the background space-time is flat, all of its non-perturbed corresponding Weyl scalars vanish. However, as a consequence of the addition of the small term $h_{\mu\nu}$ in the metric, some perturbation terms will now appear in their expressions. The Weyl tensor is equal to the Riemann tensor in the vacuum. Using tetrad (2.54) to evaluate Ψ_0 and Ψ_4 in (2.34), it can then be obtained that [56]

$$\begin{aligned}\Psi_0 =&\frac{1}{2}\left(R_{0101}+2R_{0131}+R_{3131}-R_{0202}-2R_{0232}-R_{3232}\right)\\&+i(R_{0102}+R_{3132}+R_{3102}+R_{0132}),\\ \Psi_4 =&\frac{1}{4}\left(R_{0101}-2R_{0131}+R_{3131}-R_{0202}+2R_{0232}-R_{3232}\right)\\&+\frac{i}{2}\left(-R_{0102}-R_{3132}+R_{3102}+R_{0132}\right).\end{aligned} \tag{2.55}$$

Substituting the expression of the perturbed Riemann tensor (2.47) in (2.55) yields the following remarkable result

$$\begin{aligned}\Psi_0 &= \begin{cases} 0 & \text{for outgoing gravitational waves,} \\ 2\omega^2(A_+ + iA_\times)e^{ik_\alpha x^\alpha} & \text{for ingoing gravitational waves.}\end{cases}\\ \Psi_4 &= \begin{cases} \frac{1}{2}\omega^2\left(A_+ - iA_\times\right)e^{ik_\alpha x^\alpha} & \text{for outgoing gravitational waves,} \\ 0 & \text{for ingoing gravitational waves.}\end{cases}\end{aligned} \tag{2.56}$$

This shows that, as mentioned at the end of subsection 2.2.3, the scalar Ψ_0 describes ingoing waves and the Ψ_4 describes outgoing waves. Furthermore, note that the real part of both quantities is related to the plus polarization, while the imaginary part to the cross polarization (here only the real part of the exponential is taken into account as done typically in the physical analysis of wave solutions).

Relations (2.56) can be used to express the energy flux carried by gravitational waves. Due to the fact that the energy of the wave cannot be exactly localized in a certain region within the wavelength, there are some issues regarding its precise definition. A common solution to this problem is to take an average $\langle \ldots \rangle$ over wavelengths and use it to define an effective stress-energy tensor $T^{(GW)}_{\mu\nu}$ of gravitational waves. This tensor is also known as the Isaacson tensor and, in the transverse-traceless (TT) gauge, is simply given by [57]

$$T^{(GW)}_{\mu\nu} = \frac{1}{32\pi}\left\langle \sum_{j,k} \partial_\mu h^{TT}_{jk} \partial_\nu h^{TT}_{jk} \right\rangle, \tag{2.57}$$

here the lowercase Latin indices take the values $j,k = 1,2$, i.e., the sum in equation (2.57) is over the transverse-traceless part of the wave. The energy flux can be determined from (2.57) by considering the appropriate components of the tensor. For plane waves propagating in the z-direction, the flux of energy per unit area is related to the component

$$T^z_{\ t} = \frac{1}{32\pi}\omega^2(|A_+|^2 + |A_\times|^2),$$

or in terms of the previous Weyl scalars

$$T^z_{\ t} = \begin{cases} \frac{1}{8\pi\omega^2}||\Psi_4||^2 & \text{for outgoing gravitational waves,} \\ \frac{1}{128\pi\omega^2}||\Psi_0||^2 & \text{for ingoing gravitational waves.} \end{cases}$$

This idea can be extended to the total energy flux per unit solid angle of spherical waves, in which case,

$$\frac{d^2E}{dtd\Omega} = \lim_{r\to\infty} r^2 T^r_{\ t} = \frac{\omega^2}{16\pi}\lim_{r\to\infty} r^2(|h_{\theta\theta}|^2 + |h_{\theta\varphi}|^2),$$

where $\{t, r, \theta, \varphi\}$ are spherical coordinates. Likewise, the energy flux can be expressed in terms of the discussed scalars,

$$\frac{d^2E}{dtd\Omega} = \begin{cases} \frac{1}{4\pi\omega^2}\lim_{r\to\infty} r^2||\Psi_4||^2 & \text{for outgoing gravitational waves,} \\ \frac{1}{64\pi\omega^2}\lim_{r\to\infty} r^2||\Psi_0||^2 & \text{for ingoing gravitational waves.} \end{cases} \tag{2.58}$$

The present analysis has been oriented towards highlighting the physical importance of the Weyl scalars (at least two of them) in the theory of gravitational waves, or even more generally, in the perturbation theory of General Relativity. Therefore they should be seen as key quantities, or at the minimum as very useful tools, when attempting to study linearly perturbed space-times.

2.4.2 Gravitational Wave Detectors

Although the main focus of this thesis is not the experimental aspect of the various predictions of General Relativity, it might be worth to briefly comment on the most important concepts related to the detection of gravitational waves. It is clear that an objective of the sort is an extremely relevant issue when thinking about experimental tests of General Relativity. Albert Einstein himself believed that such prediction was untestable, impossible to detect on Earth, because of the extreme sensitivity required for the measurement equipment. Almost a century later, the novel detections of gravitational wave disproved Einstein in this regard, but at the same time, reconfirmed his revolutionary theory.

In subsection 2.4.1 the most significant physical effects of gravitational waves on matter were discussed, in particular, the small deviation induced on test particles by a passing wave and the energy carried by their radiation. A manner to detect a wave of this type would be then by seeking for these effects and effectively measuring them. Unfortunately this implies severe practical and realistic complications, as it will be explained in the following. Throughout the years, the advanced development of technology has allowed science to overcome these obstacles, leading to the functional wave detectors that exist nowadays and that have been successfully monitoring the arrival of gravitational radiation to Earth to the present day.

Maybe the most straightforward way to attempt to detect a gravitational wave is by measuring the slight deformation of some material caused by its incidence. This is not at all simple. Such detectors are called resonant mass detectors, or simply bar detectors, and their aim is to respond to the gravitational wave by vibrating (stretching and compacting) with a certain resonant amplitude. A pioneer detector of this kind was introduced by Webler during the 1960s [58], and consisted of a cylindrical aluminum bar in vacuum, isolated from external disturbances such as small mechanical vibrations. The length of the bar was of $153\,cm$ with a diameter of $66\,cm$ and it weighted $1.4 \times 10^6\,g$. Roughly speaking, a gravitational wave of amplitude of order 10^{-20} would make the bar resonate with an amplitude of the order of the diameter of an atomic nucleus. This is of course nearly impossible to measure directly, hence, piezoelectric strain transducers were implemented so that the deformation of the bar could be observed as electrical signals. Weber was able to detect in 1969 simultaneous events with two bars, one in Washington while the other one in Chicago [59], however, it is not completely certain that those events corresponded to gravitational waves.

While bar detectors were the first ones to be proposed, nowadays, there are other types of modern detectors that surpass the sensitivity of their resonant counterparts. Nevertheless, the former can serve to exemplify how hard it is to measure the passing of a gravitational wave by looking into the deformation of solid materials. Perhaps the most important detectors of the mentioned other types are those which use laser interferometry as their working principle.

An illustrative diagram of the general setup of a laser interferometer can be found in figure 2.5 and is described in the following. A laser beam goes through a beam splitter, each divided light path then reaches a suspended mass (a long arm) that has a mirror attached to one of its ends. The two arms are of the same length and perpendicular to each other. Both laser beams are reflected back off their corresponding mirror, interfering hence with each other at some point of their path. Initially the phase of the two laser beams is correlated, if the proper length of both arms is the same then the interference will be constructive. On the contrary, if the proper length changes by virtue of a gravitational wave that compresses or stretches the arms, the laser beams will arrive at the photodetector out of phase, presenting then a different interference pattern.

Such a sensible device has to be properly isolated in order to avoid measuring otherwise regular phenomena not at all related to gravitational radiation, i.e., noise.

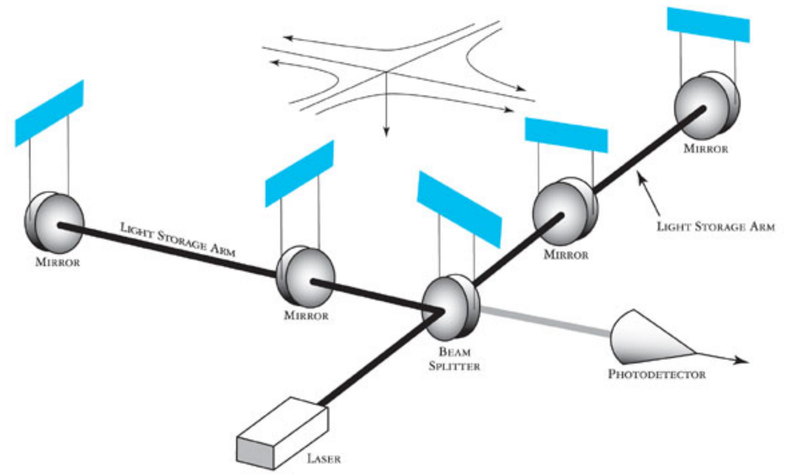

FIGURE 2.5: A laser interferometer array used for the detection of gravitational waves. (Image: LIGO)

Thus, to protect the device from external mechanical perturbations, all of its components are placed inside the vacuum and suspended as pendulums. Additionally, hydraulic systems that protect the interferometer from seismic vibrations are implemented. Another source of noise are thermal vibrations. Though being in the vacuum helps to minimize this problem, special materials (ultra-high-Q) are used for the mirrors and suspension wires so that their resonant frequencies are above the expected spectrum of a gravitational wave.

There are several gravitational wave interferometers across the world. Two of them in the United States, which are referred to as LIGO [60] (the largest of them with arms of $4\,km$ in length), VIRGO in Italy ($3\,km$), GEO in Germany and KAGRA in Japan ($3\,km$).

FIGURE 2.6: Aerial view of the VIRGO installations in Italy. (Image: The VIRGO Collaboration)

On September 14, 2015, a remarkable event took place in both of the LIGO interferometers: the first detection of a gravitational wave GW150914. The signal was registered by the two observatories, separated by a distance of 3.000 km, with a delay of 7 ms. This ground-breaking discovery was later announced on February 11, 2016 by the LIGO and VIRGO collaboration [61]. According to the properties of the observed wave, it was generated by a binary black hole merger located 1.3 billion light years away from Earth. This direct observation signified yet another proof of the theory of General Relativity and the beginning of a new age in Astronomy and Astrophysics. Since then, more detections have been constantly reported by LIGO and VIRGO, for example, at least 11 black hole merger events and a binary neutron star [62]. Not only do these measurements represent an enormous international scientific effort, but they are also one of the greatest accomplishments of modern physics.

2.4.3 Perturbations of the Schwarzschild Black Hole

After studying the way in which a small perturbation term in the metric of flat space-time creates gravitational waves, one of the next levels in increasing order of difficulty can be to consider a static and spherically symmetric space-time in the vacuum. Birkhoff's theorem establishes that such a solution is unique, and is that of Schwarzschild [63]. In this case, unlike Minkowski space-time, now the metric components will depend at least on a spatial coordinate. This will evidently complicate the calculations, but it is worth it. By analyzing the problem of gravitational perturbations in this metric, interesting results are found, specially about the linear stability of a black hole.

The first ones who treated this problem were Regge and Wheeler in their pioneering work [64]. To find a solution, they took advantage of the symmetry of the space-time and proposed the most general perturbation whose angular part consisted of products of scalar, vector, and tensor spherical harmonics $Y_{\ell,m}(\theta,\varphi)$ on the 2-sphere. The perturbation was further simplified through gauge transformations and its φ-dependence was eliminated without significant loss of information by setting $m=0$. This is possible due to the spherical symmetry of the background space-time. Additionally, Regge and Wheeler divided the perturbation into two cases: perturbations of odd and even parity.

In an appropriate basis of spherical coordinates $\{t,r,\theta,\varphi\}$, and assuming a harmonic dependence on time, the explicit form of both perturbations is

$$h_{\mu\nu}^{(odd)} = A_{\mu\nu}e^{i\omega t}\sin\theta\frac{d}{d\theta}P_\ell(\cos\theta), \quad h_{\mu\nu}^{(even)} = B_{\mu\nu}e^{i\omega t}P_\ell(\cos\theta), \tag{2.59}$$

with

$$\begin{aligned} A_{\mu\nu} &= \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{bmatrix}, \\ B_{\mu\nu} &= \begin{bmatrix} H_0(1-2m/r) & H_1 & 0 & 0 \\ H_1 & H_2/(1-2m/r) & 0 & 0 \\ 0 & 0 & r^2K & 0 \\ 0 & 0 & 0 & r^2K\sin^2\theta \end{bmatrix}. \end{aligned} \tag{2.60}$$

In (2.60) the parameter m denotes the mass of the black hole. The functions H_0, H_1, H_2 and K also depend only on the radial coordinate r. This gauge is sometimes called the Regge-Wheeler gauge. The parity of the perturbations refers to their symmetry under reflections about the origin, $(-1)^{\ell+1}$ for odd-parity and $(-1)^{\ell}$ for even-parity. Both cases were later associated to other types of perturbations, odd-parity represents axial perturbations, and even-parity polar perturbations.

The carried out procedure is the same as in linear gravity: compute the relevant curvature tensors with the benefit of having now explicit equations for $h_{\mu\nu}$. Nevertheless, the calculations become considerably more complicated because of the own curvature of the background metric. On the positive side, it is still a highly symmetrical vacuum space-time. The perturbed Ricci tensor $\delta R_{\mu\nu}$ can then be used to find suitable perturbed field equations, in this case, $\delta R_{\mu\nu} = 0$. Consequently, the problem may be as hard as finding 10 field equations to solve. In what follows and for the sake of brevity, the totality of these equations shall not be presented, only the most relevant ones. The two types of perturbations will be treated separately.

For odd-parity perturbations only 3 non-trivial field equations appear for the two perturbation functions h_0 and h_1. It is possible to solve the resulting system of equations, finding thus a wave-like expression that describes the behavior of the perturbation

$$\frac{d^2Q}{dr_*^2} - \left[V_{odd}(r) - \omega^2\right] Q = 0, \tag{2.61}$$

where $Q = (1-2m/r)h_1/r$ and $V_{odd}(r) = (1-2m/r)[\ell(\ell+1) - 6m/r]/r^2$ can be considered as a potential for reasons explained below. Also, r_* is the so-called tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{r}{r-2m}.$$

The significant quantity here is now the function Q, while h_0 is expressed by one of the non-trivial field equations in terms of Q. This relation, though, is not of vital importance.

For the even-parity case, 7 non-trivial field equations are obtained. After several cumbersome manipulations, Zerilli managed to find a perturbation equation that can also be cast into a simple form as (2.61), but with a more intricate potential [65]

$$V_{even}(r) = \left(1 - \frac{2m}{r}\right) \frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 m r^2 + 18\lambda m^2 r + 18m^3}{r^3(\lambda r + 3m)^2}, \tag{2.62}$$

with $\lambda = (\ell-1)(\ell+2)/2$. For the even-parity perturbations, the relevant function Q is given by a combination of H_1 and K.

The problem of gravitational perturbations in the Schwarzschild geometry has been therefore reduced to ordinary differential equations of second-order. Even more specifically, to Schrödinger-like equations of the type $\mathcal{H}Q = \omega^2 Q$, where the differential operator,

$$\mathcal{H} = -\frac{d^2}{dr_*^2} + V(r),$$

is called a Schrödinger operator with potential $V(r)$.

The equations found by Regge, Wheeler and Zerilli play a center role when attempting to describe the gravitational radiation of a perturbed static black hole. The stability of such an astrophysical object can also be deduced from them. To conclude this, suitable boundary conditions must be imposed on the radial function Q. Roughly speaking, Q needs to be regular at infinity ($r \to \infty$) and at the event horizon

($r \to 2m$), or equivalently for the tortoise coordinate, $r_* \to \infty$ and $r_* \to -\infty$, respectively. From (2.61) and the form of the potentials, it can be seen that asymptotically the radial solutions behave as

$$Q \sim e^{\pm i\omega r} \quad \text{as } r \to \infty, \qquad Q \sim e^{\pm i\omega r_*} \quad \text{as } r \to 2m.$$

For real values of ω, the solutions at infinity are incoming and outgoing gravitational waves. When approaching the black hole, the waves will encounter the potential $V(r)$ and depending on their frequency ω, will continue its way on to the event horizon, or will be reflected back to infinity. On the other hand, purely imaginary values of ω can lead to instabilities.

Unlike the plane waves described in subsection 2.4.1, these spherical waves admit different multipolar modes determined by the parameter ℓ. The modes corresponding to different values of ℓ need to be examined separately. Vibrational modes with $\ell \geq 2$ are true dynamical radiative modes, however, the dynamical modes generated by $\ell = 0, 1$ can be eliminated by gauge transformations [66, 67], i.e., they have no physical meaning. Despite this, the static version ($\omega = 0$) of these two previous modes does have some physical significance. They represent small angular momentum deviations of the metric (odd-parity mode with $\ell = 1$), small changes of the black hole mass (even-parity mode with $\ell = 0$) and small displacements of the origin (even-parity mode with $\ell = 1$) [68, 69]. Hence, the lowest multipole of gravitational radiation is the quadrupole ($\ell = 2$).

Since dynamical modes are the ones that potentially affect the stability of the metric, one can focus on the values $\ell \geq 2$ for the potentials $V_{odd}(r)$ and $V_{even}(r)$. It follows that due to them being strictly positive for all values $0 < 2m < r < \infty$, there are no physically regular solutions of the perturbation function Q that represent unstable modes, this is, modes with purely imaginary ω. Thus, the Schwarzschild black hole has linear mode stability against gravitational perturbations [68].

Additional results may be derived from the Regge-Wheeler and Zerilli equations. For instance, the numerical study of the quasi-normal modes of a static black hole, this is, discrete complex frequencies ω with boundary conditions consisting of purely ingoing waves at the horizon and purely outgoing waves at infinity. The determination of the damping time and oscillation rate of these modes can help establish the physical parameters of a black hole by detecting the gravitational waves it produces when perturbed. Unfortunately, because the odd and even potentials do no admit a known analytical solution, the calculation of the quasi-normal modes has been done numerically [70], and also with a semi-analytical approach based on the WKB method [71].

2.4.4 The Master Equation for the Kerr Metric

The perturbation analysis of the Schwarzschild black hole was still manageable starting from a small extra term $h_{\mu\nu}$ in the metric, which was followed by writing the modified Ricci tensor components. This was possible in large part because of the high degree of symmetry of the space-time. While this approach is easy to understand, it can become extremely tedious for more complex geometries. Such is the case of a rotating black hole due to the frame-dragging effect of the metric (see subsection 1.1.2) and its axial symmetry.

With a less straightforward method, Teukolsky was able to find a so-called "master equation" that describes gravitational, electromagnetic, and neutrino field perturbations of a spinning black hole [72]. To accomplish this, Teukolsky exploited the full potential of the Newman-Penrose formalism [31] and the underlying geometric properties of the Kerr metric, in particular, the fact that it is of type D in the algebraic classification of space-times (cf. subsection 2.1.8). The procedure is briefly outlined here.

If the metric of a space-time is perturbed by a small $h_{\mu\nu}$, then all of the Newman-Penrose quantities will also undergo a first-order change in their form, e.g.,

$$\rho = \rho^A + \rho^B, \quad \Psi_2 = \Psi_2^A + \Psi_2^B.$$

Here, an A superscript is used to denote a background term, and a B superscript a perturbation term. One can therefore extract information from the perturbation by finding an equation that involves one, and only one, suitable quantity with a B superscript, with the rest of the expressions appearing in the equation being background quantities. Strong candidates for this objective are the perturbed Weyl scalars, namely Ψ_0^B and Ψ_4^B since, as explained in subsection 2.4.1 they describe outgoing and ingoing gravitational radiation. Another non-trivial reason for using these quantities is the fact that their respective perturbed terms are invariant to gauge transformations as well as tetrad rotations.

Consider an adapted tetrad such that the only non-vanishing background Weyl scalar is Ψ_2^A. This tetrad always exists because the space-time is of type D and it is achieved by choosing the l^μ and n^μ vectors along the repeated principal null directions of the Weyl tensor. From the background Bianchi identities in the vacuum (2.36), it can be seen that the following spin coefficients vanish: $\kappa^A = \sigma^A = \nu^A = \lambda^A = 0$.

To ease the notation, the A superscript will now be dropped from the unperturbed quantities. With the linearized Bianchi identities in the vacuum, and after further manipulation, the desired equations for gravitational perturbations can be found

$$\begin{aligned}
&[(D - 3\varepsilon + \varepsilon^* - 4\rho - \rho^*)(\Delta - 4\gamma + \mu) \\
&\quad - (\delta + \pi^* - \alpha^* - 3\beta - 4\tau)(\delta^* + \pi - 4\alpha) - 3\Psi_2]\Psi_0^B = 0, \\
&[(\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)(D + 4\varepsilon - \rho) \\
&\quad - (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi)(\delta - \tau + 4\beta) - 3\Psi_2]\Psi_4^B = 0.
\end{aligned} \tag{2.63}$$

In equations (2.63), as in the perturbations of the Schwarzschild metric, the assumption $R_{\mu\nu}^B = 0$ has been made. A non-empty perturbed Ricci tensor can also be considered, in which case, the equations should be equal to some matter content instead of zero.

To obtain an explicit expression for equations (2.63), consider the adapted tetrad given in the basis of Boyer-Lindquist coordinates $\{t, r, \theta, \varphi\}$ by[4]

$$\begin{aligned}
l^\mu &= \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad n^\mu = \frac{1}{2\Sigma}\left(r^2 + a^2, -\Delta, 0, a\right), \\
m^\mu &= \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right).
\end{aligned} \tag{2.64}$$

[4]In (2.64) and in the following, the symbol Δ will no longer refer to the operator of the formalism, instead it will denote the usual definition $\Delta = r^2 - 2Mr + a^2$, where M will be used to represent the mass of the black hole.

This tetrad yields the following non-vanishing background spin coefficients

$$\rho = -\frac{1}{r - ia\cos\theta}, \quad \beta = -\frac{\rho^* \cot\theta}{2\sqrt{2}}, \quad \pi = \frac{ia\rho^2 \sin\theta}{\sqrt{2}}, \quad \tau = -\frac{ia\rho\rho^* \sin\theta}{\sqrt{2}},$$
$$\mu = \frac{\rho^2\rho^*\Delta}{2}, \quad \gamma = \mu + \frac{\rho\rho^*(r-M)}{2}, \quad \alpha = \pi - \beta^*. \tag{2.65}$$

Inserting the spin coefficients in equations (2.63) and recalling the definition of the differential operators of the formalism, the master equation for the Kerr black hole is found,

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\Psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\Psi}{\partial t\partial\varphi} + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\frac{\partial^2\Psi}{\partial\varphi^2}$$
$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\Psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\Psi}{\partial\varphi}$$
$$-2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\Psi}{\partial t} + s(s\cot^2\theta - 1)\Psi = 0. \tag{2.66}$$

It is noteworthy that the two equations in (2.63) can be expressed as one. In fact, the introduced parameter s called the spin weight, admits the physically meaningful values $s = 0, \pm1/2, \pm1, \pm2$. Namely, with $s = 0$ the master equation reduces to that of a scalar field in the Kerr geometry, with $s = \pm1/2$ it describes a neutrino field, with $s = \pm1$ an electromagnetic perturbation, and with $s = \pm2$ a gravitational perturbation. Only the equations related to the gravitational perturbations were shown here. The interpretation of the test field Ψ then changes according to the spin weight parameter s. For instance, if $s = \pm1$ then Ψ contains a Maxwell scalar defined by $\Phi_0 = F_{\mu\nu}l^\mu m^\nu$ and $\Phi_2 = F_{\mu\nu}\bar{m}^\mu n^\nu$, where $F_{\mu\nu}$ is the electromagnetic field tensor. Some other examples are presented in table 2.1.

TABLE 2.1: The test fields Ψ for some values of the spin weight s.

s	Ψ
0	Φ (A scalar field)
1	ϕ_0^B
-1	ϕ_2^B/ρ^2
2	Ψ_0^B
-2	Ψ_4^B/ρ^4

Remarkably, equation (2.66) is separable when assuming an ansatz of the type $\Psi = e^{-i\omega t}R(r)\Theta(\theta)e^{im\varphi}$, thereby finding two ordinary differential equations

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \left[\frac{\mathcal{C}^2 - 2is(r-M)\mathcal{C}}{\Delta} + 4is\omega r - \mathcal{A}\right]R = 0, \tag{2.67}$$
$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$
$$+\left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + A\right)\Theta = 0.$$

Here, $\mathcal{C} = (r^2 + a^2)\omega - am$ and $\mathcal{A} = A + a^2\omega^2 - 2am\omega$, with A being the separation constant. When the angular equation of (2.67) is complemented with proper boundary conditions at $\theta = 0$ and $\theta = \pi$, its general solutions are the so-called spin-weighted spheroidal harmonics. The separation constant is then an eigenvalue $A = A^m_{s\ell}(a\omega)$ labeled by ℓ, whose smallest value is given by $\ell_{min} = \max(|s|, |m|)$. This is consistent with the fact that for dynamic gravitational perturbations $s = \pm 2$ and hence, the lowest mode of vibration is $\ell = 2$.

As for the radial equation $R(r)$, just as in the Schwarzschild black hole, it is subject as well to boundary conditions of regularity at infinity and at the horizon. As $r \to \infty$, its asymptotic solutions are $R \sim e^{i\omega r}/r^{2s+1}$ for outgoing waves and $R \sim e^{-i\omega r}/r$ for ingoing waves. Particularly for the case of gravitational perturbations ($s = \pm 2$), it is seen that

$$\Psi_4^B \sim \frac{e^{i\omega r}}{r}, \quad \Psi_0^B \sim \frac{e^{i\omega r}}{r^5} \text{ for outgoing waves,}$$
$$\Psi_0^B \sim \frac{e^{-i\omega r}}{r}, \quad \Psi_4^B \sim \frac{e^{-i\omega r}}{r^5} \text{ for ingoing waves.}$$

One can then use the appropriate expressions in (2.58) to find the energy flux per unit angle of the gravitational waves generated by the perturbation.

On the other hand, at the outer horizon $r_+ = M + \sqrt{M^2 - a^2}$, the solution behaves asymptotically as

$$R \sim \Delta^{-s} e^{-ikr_*},$$

with $k = \omega - ma/2Mr_+$. In this case, the tortoise coordinate r_* is defined by the relation

$$\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}.$$

This asymptotic behavior can be shown to be the correct one since it indicates that waves can only travel into the black hole, however, they cannot come out of it.

A requirement to finally answer the question of the stability of a Kerr black hole is to make sure that there are no frequencies ω in the upper half complex plane that correspond to physically well-behaved solutions at infinity and at the horizon. Said frequencies would evolve without bound in time. Unluckily, the radial equation in (2.67) has no analytic solution. Using numerical techniques, Teukolsky and Press examined the first vibrational modes ($\ell = 2$ through $\ell = 4$) of rotating black holes whose value of angular momentum is within the physical range of $0 \leq a < M$. They found no signs of instabilities present in these modes [73]. The stability of all physically regular modes was later shown in [74], based on equations (2.67), and without the need of numerical methods.

From a practical point of view, this general result was of great relevance to the existence and possible observation of a realistic black hole. From a theoretical standpoint, the most interesting aspect of this approach could be that of the derivation of the master equation.

2.4.5 The Stability of Charged Black Hole Solutions

So far the discussion regarding gravitational perturbations has focused on solutions of the Einstein field equations in the vacuum. These are of course fundamental pieces of the theory. Nevertheless, there are also several physically interesting solutions with a non-zero stress-energy tensor. One of them being the Reissner-Nordström solution, i.e., an electrically charged static black hole.

The core of the perturbation scheme remains the same: add a perturbation term to the background metric and obtain adequate perturbed field equations. There are, though, additional complications. Specifically, Maxwell equations must be perturbed too and still hold up to linear order when introducing the perturbation. The relevant field equations to be satisfied are now

$$R_{\mu\nu} = F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F^2, \quad \nabla_{\nu}F^{\nu\mu} = -J^{\mu}, \quad \nabla_{\nu}G^{\nu\mu} = 0, \tag{2.68}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor, $G_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}/2$ its dual, $F^2 = F_{\mu\nu}F^{\mu\nu}$ is the electromagnetic scalar, and J^{μ} the electromagnetic 4-current, which is zero when no sources are present. In the linearization of equations (2.68) not only must the metric be perturbed $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, but also the electromagnetic tensor $F_{\mu\nu} \to F_{\mu\nu} + \delta F_{\mu\nu}$. The background electric field is given by

$$F = -2(q/r^2)dt \wedge dr,$$

with electric charge q.

With these considerations, Zerilli was able to linearize and solve the set of Einstein-Maxwell equations [75]. Once again, the gravitational perturbations can be divided into odd and even parities. As a result, a decoupled second-order system of two ordinary differential equations is obtained:

- For odd-parity

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right) Z_i^{(odd)} = V_i^{(odd)} Z_i^{(odd)} \quad (i = 1, 2), \tag{2.69}$$

  where the potentials $V_i^{(odd)}$ are given by

$$\begin{aligned} V_1^{(odd)} &= \frac{1}{r^2}\left(1 - \frac{2m}{r}\right)\left[2\lambda + 2 - \frac{q_-}{r}\left(1 + \frac{q_+}{2\lambda r}\right)\right], \\ V_2^{(odd)} &= \frac{1}{r^2}\left(1 - \frac{2m}{r}\right)\left[2\lambda + 2 - \frac{q_+}{r}\left(1 + \frac{q_-}{2\lambda r}\right)\right]. \end{aligned}$$

  The following definitions have been done $q_{\pm} = 3m \pm \sqrt{9m^2 + 8q^2\lambda}$, and $\lambda = (\ell - 1)(\ell + 2)/2$.

- For even-parity

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right) Z_i^{(even)} = V_i^{(even)} Z_i^{(even)} \quad (i = 1, 2), \tag{2.70}$$

  where the potentials $V_i^{(even)}$ are given by

$$\begin{aligned} V_1^{(even)} &= \frac{1}{r^3}\left(1 - \frac{2m}{r}\right)\left[U + \frac{1}{2}\left(q_+ - q_-\right) W\right], \\ V_2^{(even)} &= \frac{1}{r^3}\left(1 - \frac{2m}{r}\right)\left[U - \frac{1}{2}\left(q_+ - q_-\right) W\right]. \end{aligned}$$

The following definitions have been done $U = (2\lambda r + 3m)W + (v - \lambda r - m) - 2\lambda r(r - 2m)/v$, with $W = (r - 2m)(2\lambda r + 3m)/v^2 + (\lambda r + m)/v$, and $v = \lambda r + 3m - 2q^2/r$.

The functions Z_i for both parities contain combinations of the perturbations of the metric and of the electric field. When $q = 0$ (a Schwarzschild black hole) and $Z_1 = 0$, the potentials V_2 of the remaining equations can be verified to reduce to those of Regge-Wheeler and Zerilli for each corresponding parity.

Thus, the problem of gravitational perturbations in the Reissner-Nordström black hole has been reduced to wave-like equations with Schrödinger operators and its corresponding potentials. Around the same time, Moncrief alternatively derived the same equations through the use of a Hamiltonian method [76, 77]. By the same arguments that have been given in the past subsections, the mode stability of this non-vacuum metric follows too.

2.4.6 Wormholes and Issues with their Stability

It is clear that the problem of stability for black holes has been thoroughly studied. The fact that they arise as vacuum solutions of the Einstein field equations greatly aids in the analysis of their gravitational perturbations. Over the years, the developments discussed during the last subsections have contributed to the physical relevance of this outstanding prediction of General Relativity.

On the other hand, wormholes are still only theoretical entities. They are commonly, but not uniquely, proposed as stationary and spherically symmetric spacetimes supported by a phantom scalar field, i.e., a scalar field whose kinetic energy has a reversed sign (sometimes referred too as ghost scalar field). Maybe the most simple model of such a wormhole is that of Ellis and Bronnikov with reflection symmetry [6, 7] (see subsection 1.2.4). This model is obtained by making $f(\rho) = 0$ and $r(\rho) = \sqrt{\rho^2 + b^2}$ in (1.7), hence its line element is

$$ds^2 = -dt^2 + d\rho^2 + (\rho^2 + b^2)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.71}$$

where the parameter b is related to the size of the throat of the wormhole. When $\rho = 0$, the function $r(\rho)$ acquires its minimal value, this locates the throat (a surface of minimal area). In this case the scalar field is given by $\phi = \arctan(\rho/b)$.

In recent years, many works have been written about the question of the stability of these type of wormholes and a handful of them report that they are generally unstable. Thus adding another problematic issue to their set of particular properties.

One of the first ones to consider this problem were Shinkai and Hayward who showed numerically that the Ellis-Bronnikov wormhole would collapse to a black hole or form an inflationary universe, depending on the type of matter that causes the perturbation [78]. Analytically, it is also firmly established that the radial monopole mode of phantom wormholes unavoidably leads to instabilities [79–81].

Particularly in [81], the simplest case of the Ellis wormhole (2.71) was studied exhaustively. In fact, this space-time serves as a very illustrative and easy enough example. Consider purely radial (and dynamic) perturbations to the metric of the form

$$h_{\mu\nu} = H_{\mu\nu}e^{i\omega t},$$

with the components of $H_{\mu\nu}$ expressed in the $\{t, \rho, \theta, \varphi\}$ coordinate basis as

$$H_{\mu\nu} = diag[h_0(\rho), h_1(\rho), h_2(\rho), h_2(\rho)\sin^2\theta]. \tag{2.72}$$

For a wormhole with a phantom scalar field as a source, the field equations that need to be linearized and later solved are

$$R_{\mu\nu} = -\nabla_\mu \phi \nabla_\nu \phi, \quad \nabla^\mu \nabla_\mu \phi = 0.$$

An adequate gauge in which the scalar field need not be perturbed, i.e., $\delta\phi = 0$, can be chosen. Performing the necessary calculations, one can reduce the perturbed field equations to a single master equation containing the typical Schrödinger operator, this is,

$$\frac{d^2 Q}{d\rho^2} - \left[V(\rho) - \omega^2 \right] Q = 0.$$

Here $Q = h_0/r(\rho) - h_2/2r^3(\rho)$ and $V(\rho) = -3b^2/r^4(\rho)$. Unlike the case of a perturbed non-rotating black hole, this potential is strictly negative for all $\rho \in \mathbb{R}$. This means that if $V(\rho)$ is sufficiently negative, the Schrödinger operator $\mathcal{H} = -d^2/d\rho^2 + V(\rho)$ acting on a $L^2(d\rho, \mathbb{R})$ space would have at least a negative eigenvalue $\omega_-^2 < 0$. Unfortunately, such is the case for the analyzed wormhole as shown in [79] by using a variational method. This corresponds to an unstable mode since then, ω_- is purely imaginary. Also, the eigenfunction Q_- is square-integrable because $Q_- \in L^2(d\rho, \mathbb{R})$, and therefore, the perturbation is physically regular.

Apart from this somewhat brief example, similar results were later extended to include a phantom scalar field with a self-interacting potential [82, 83]. In both of these references specific metrics of scalar field wormholes consisting either of asymptotically flat, or (A)dS ends (or a mixture of them), were also considered. An attempt has been made to stabilize these kind of wormholes by coupling an electromagnetic field to their configuration. Unfortunately, the analysis still led to modes growing exponentially in time [84].

On the contrary, a stable wormhole was reported in [85]. Curiously enough, its metric is that of Ellis given in equation (2.71), however, instead of being supported by the usual phantom scalar field, its gravitational source is constituted by two elements: a radial electric field (a magnetic field is also possible) and a perfect fluid with negative density. Thus, the perturbed space-time must obey field equations similar to the ones presented in subsection 2.4.5, rather than those of a scalar field. Other stable models are obtained through the use of thin shells of matter [86].

Alternatively, it has been proposed as a conjecture that rotation in the metric of wormholes may induce stability against gravitational perturbations. Several wormholes models have been generalized to include said rotation, see [18, 19] for just a few examples. Despite this, and mainly due to the complexity on the geometry of the space-time, the treatment of the perturbation equations can become rather cumbersome.

Chapter 3

Geodesics Near a Curvature Singularity in Stationary and Axisymmetric Space-Times

In this chapter the main results of the thesis concerning curvature singularities are presented. As mentioned in subsection 2.3.3, there is not a clear relation between unbounded curvature and geodesic incompleteness. Following this idea, we first try to establish which geodesics of a space-time containing diverging scalars can be found within every neighborhood of the curvature singularity. This property is of interest because the absence of any such curves would imply there could exist space-times whose causal geodesics are not able to reach the singularity. We also consider the possibility that unavoidably said singularity is met by a given curve, this could be interpreted as an indication of incompleteness. Here, two theorems that serve mainly as criteria for the occurrence of these two particular behaviors are developed. We will consider four-dimensional, stationary and axially symmetric space-times, with some additional requirements that will be explicitly mentioned in the next section. Before stating both theorems with their respective proofs, some concepts and results that shall be helpful throughout the analysis will be discussed. The formulated criteria will be applied then to a physically relevant class of space-times, that of Plebański-Carter, obtaining thereby some well-known results. Afterward, based on the first derived theorem, we construct a metric in which causal geodesics are unable to touch its curvature singularity. Proving thus, with the aid of some additional arguments, its causal geodesic completeness despite the presence of unbounded curvature. The content of these sections is majorly based on reference [87]. We study lastly two rotating scalar field wormholes that serve as examples of how our results can be used for space-times that are outside the scope of the developed criteria. For a complete analysis, though, we will be forced to recur in this final section to different resources other than the two theorems.

3.1 Definitions and Auxiliary Results

We begin by establishing various concepts that will allow us to deal with singularities throughout this chapter. As mentioned previously we shall focus on the so-called curvature singularities. Let $x^\mu = \{x^0, x^1, x^2, x^3\}$ be any coordinate system on a (pseudo)-Riemannian manifold M. We will say a space-time $(M, g_{\mu\nu})$ contains a curvature singularity if any of its curvature scalars R_X diverge at some coordinate values x_0^μ. The scalars R_X may be constructed from index contractions or from polynomial expressions of the Riemann or Ricci tensors. In the case of a n-dimensional manifold, the curvature scalars can be considered as a map $R_X : \mathbb{R}^n \to \mathbb{R}$. With this in mind, we make the following definitions for a four-dimensional space-time.

Definition 3.1. Let $R_X : \mathbb{R}^4 \to \mathbb{R}$ denote a curvature scalar of a given metric $g_{\mu\nu}$ that contains a curvature singularity labeled as σ. The singular curvature set is defined as

$$\sigma_X = \left\{ (x_0^0, x_0^1, x_0^2, x_0^3) \in \mathbb{R}^4 \mid \lim_{x^\mu \to x_0^\mu} R_X(x^\mu) \text{ does not exist or is infinite} \right\}.$$

In some cases it might be helpful to consider a Cartesian (or "Cartesian-like") coordinate system $u^\mu = \{u^0 = t, u^1, u^2, u^3\}$, if existent for a given space-time, to express the points of σ_X. This allows to obtain a more precise idea of the "shape", in some suitable meaning, of the singular region in said space-time. Here, the coordinate system $\{u^\mu\}$ is referred to as Cartesian-like in the sense that there exist appropriate limits on the parameters of the space-time for which $\{u^1, u^2, u^3\}$ become regular Cartesian coordinates in Euclidean three-space. For such a space-time we make the next definition.

Definition 3.2. The singular curvature set will be said to be spatially compact, bounded, or open, if its subset $(u^1, u^2, u^3) \in \mathbb{R}^3$ is compact, bounded, or open, respectively.

The singular curvature set of an asymptotically flat space-time is either empty or spatially bounded. In this work we will treat only singularities whose σ_X is spatially compact.

Since by definition, a space-time is constituted only of regular points, a singularity does not properly belong to it. This implies that a neighborhood of the singular points cannot be defined in the usual topological sense. However, using an auxiliary manifold $\tilde{M}$, the neighborhood of a curvature singularity may be ultimately defined.

Definition 3.3. Let $(M, g_{\mu\nu})$ be a space-time that contains a curvature singularity σ. Also, let $\zeta : M \to \tilde{M}$ be a non-isometric embedding, being $\tilde{M}$ a manifold containing all the points of the set σ_X and so, $M \subset \tilde{M}$, i.e., M is a proper subset of $\tilde{M}$. Then, the neighborhood N of the singularity is $N = \tilde{N} \cap M$, where $\tilde{N}$ is a neighborhood of σ_X in $\tilde{M}$.

Note that ζ must be a non-isometric embedding so that σ_X is not singular in $\tilde{M}$. With the neighborhood of the singularity properly defined, we can distinguish between certain types of singularities depending on the nature of space-time events that take place in N.

Definition 3.4. Let S be an hyper-surface such that at least a pair of points $p, q \in S$ can be joined by a causal curve in M. A curvature singularity σ will be called time-like if there exists a neighborhood N of σ in which every hyper-surface $S \subset N$ is time-like.

From this definition, one can see that a particle lying inside a neighborhood of a time-like singularity will not necessarily meet the singularity in the future of its world line.

These concepts shall be later applied to axially symmetric line elements. In this thesis we will be interested in four-dimensional space-times $(M, g_{\mu\nu})$ that possess the following set of properties:

1. Stationary, axially symmetric and satisfying the circularity condition[1].

[1]See subsection 2.1.7.

2. Its geodesics admit a non-trivial[2] quadratic first integral.
3. Contains a time-like curvature singularity σ whose singular curvature set σ_X is non-empty.
4. There exists an unphysical space-time $(\tilde{M}, \tilde{g}_{\mu\nu})$ such that $M \cup \sigma_X \subseteq \tilde{M}$ and $g^{\mu\nu} = \tilde{g}^{\mu\nu}/\tau$, where $\tilde{g}^{\mu\nu}$ and τ are analytic in a neighborhood $\tilde{N} \subset \tilde{M}$ of σ_X.

Since the circularity condition of property 1 holds for a wide class of energy-momentum tensors, the results presented here are not restricted to specific solutions of the Einstein field equations. It is worth to point out too that property 2, at least for null geodesics, is fulfilled for any algebraically special space-time of type D [29]. Finally, property 4 may seem arbitrary and even unjustified, however there are space-times of physical interest in which it is satisfied (see section 3.4). It is also mathematically necessary so that meaningful and well-defined results are yielded when evaluating quantities at the points of σ_X.

We now develop some auxiliary results regarding the implications of properties 1 to 4, which will be later used in the proof of the main theorem.

Lemma 1. If the geodesics of a four-dimensional, stationary, axially symmetric, and circular space-time $(M, g_{\mu\nu})$ admit a non-trivial quadratic first integral, then there exists a coordinate system $\{x^\mu\}$ in which

$$g^{\mu\nu} = \frac{L^{\mu\nu}(x^1) + \Theta^{\mu\nu}(x^2)}{f(x^1) + h(x^2)},$$

with $L = L^{ij}\partial_i \otimes \partial_j + L^{11}\partial_1 \otimes \partial_1$ and $\Theta = \Theta^{ij}\partial_i \otimes \partial_j + \Theta^{22}\partial_2 \otimes \partial_2$ $(i, j = 0, 3)$.

Proof. It follows from stationarity and axial symmetry that $g_{\mu\nu}$ is characterized by two commuting Killing vector fields $X_0 = \partial/\partial t$ and $X_3 = \partial/\partial\varphi$, where we have introduced coordinates $x^0 = t$ and $x^3 = \varphi$, which in flat space-time can be given the physical interpretation of coordinate time and azimuthal angle, respectively. Furthermore, to each corresponding Killing vector field there is an associated momentum $p_i = \partial\mathcal{L}/\partial\dot{x}^i$ that remains constant along geodesic curves[3]. Here, $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu/2$ is the Lagrangian of a freely falling particle in the space-time and $\dot{x}^\mu = dx^\mu/d\lambda$ are its coordinate velocities with respect to an affine parameter λ. The Killing vectors X_i represent isometries of the space-time in the directions x^i. Hence, having previously stated the physical meaning of these coordinates, we can relate the momenta $p_0 = -\mathcal{E}$ and $p_3 = L_z$ to the energy of the test particle and its projection of angular momentum on the z-axis.

Also, from the fulfillment of the circularity condition it follows that the 2-planes orthogonal to the Killing vectors X_i are integrable (cf. subsection 2.1.7). Thus, there exist adapted coordinates y^1 and y^2 such that the metric tensor g can be divided into two subspaces $g = \gamma \oplus G$, where $\gamma = \gamma_{ij}dx^i \otimes dx^j$ and $G = G_{AB}dy^A \otimes dy^B$. Furthermore, $g_{\mu\nu}$ depends only on the y^A coordinates.

If the space-time $(M, g_{\mu\nu})$ admits a non-trivial quadratic first integral, then there exists a quadratic (or second-rank) Killing tensor $K^{\mu\nu}$ [29]. This tensor will yield a fourth constant of motion when contracted twice with the momenta p_μ, that is,

[2] By non-trivial we mean a quadratic first integral other than the one yielded by the metric itself, since $g^{\mu\nu}p_\mu p_\nu$ is constant.

[3] Throughout the rest of this chapter Greek indices will run from 0 to 3 (as used conventionally), while lower-case Latin indices will only take the values of 0 and 3 $(i = 0, 3)$, and upper-case Latin indices will take the values of 1 and 2 $(A = 1, 2)$.

$K = K^{\mu\nu} p_\mu p_\nu$. The other three conserved quantities are the pair of momenta p_i, and the Hamiltonian of a freely falling test particle $2\mathcal{H} = g^{\mu\nu} p_\mu p_\nu = \kappa$, where $\kappa = 0$ for null geodesics and $\kappa = -1$ for time-like geodesics.

Using the Hamilton-Jacobi equation it can be proven that the fourth conserved quantity comes from the separability of the Hamiltonian into two terms, each depending on the coordinates x^A of some special coordinate system $\{t, x^1, x^2, \varphi\}$, and expressed as

$$2\mathcal{H} = \kappa = [F_1(x^1) + F_2(x^2)]/[f(x^1) + h(x^2)], \tag{3.1}$$

with $p_1 = p_1(x^1)$ and $p_2 = p_2(x^2)$. Since $g^{\mu\nu} p_\mu p_\nu = 2\mathcal{H}$, equation (3.1) constraints the form of the inverse metric tensor in the following way

$$g^{\mu\nu} = \frac{L^{\mu\nu}(x^1) + \Theta^{\mu\nu}(x^2)}{f(x^1) + h(x^2)}, \tag{3.2}$$

where $L^{\mu\nu}$ and $\Theta^{\mu\nu}$ are symmetrical tensors with the restriction $L^{2\mu} = L^{\mu 2} = \Theta^{1\nu} = \Theta^{\nu 1} = 0$ and $f(x^1)$, $h(x^2)$ are one parameter functions. Notice that if the mentioned restriction on the symmetrical tensors would not be imposed, separability could not be achieved. Also note that the coordinates x^A need not be the same as the previously introduced adapted coordinates y^A, however, it can be seen that the $\{x^\mu\}$ system can consist of adapted coordinates too. Suppose (3.1) is not separable in the y^A coordinates, then a change of basis from y^A to x^A using $y^A = y^A(x^1, x^2)$, would only affect the subspace of the metric orthogonal to both X_i and hence, x^A are still adapted coordinates. So, not any system of adapted coordinates will make equation (3.1) separable, but those who do can also be adapted to the metric. Equation (3.1) is separable too if a coordinate change of the form $x'^1 = x'^1(x^1)$ and $x'^2 = x'^2(x^2)$ is performed. Additionally, taking into account that x^A are adapted coordinates of the metric, we have the further restriction on the symmetrical tensors that the only non-vanishing components of $L^{1\mu} = L^{\mu 1}$ and $\Theta^{2\mu} = \Theta^{\mu 2}$ are L^{11} and Θ^{22}, respectively. Adding up these restrictions, the symmetrical tensors can finally be written as

$$L = L^{ij}\partial_i \otimes \partial_j + L^{11}\partial_1 \otimes \partial_1 \quad \text{and} \quad \Theta = \Theta^{ij}\partial_i \otimes \partial_j + \Theta^{22}\partial_2 \otimes \partial_2.$$

□

Lemma 2. A space-time $(M, g_{\mu\nu})$ with properties 1-2 admits a second-rank Killing tensor $K^{\mu\nu}$ given in the coordinate system $\{x^\mu\}$ by

$$K^{\mu\nu} = f(x^1) g^{\mu\nu} - L^{\mu\nu}(x^1) = \Theta^{\mu\nu}(x^2) - h(x^2) g^{\mu\nu}.$$

Proof. Inserting the form of $g^{\mu\nu}$ established by lemma 1 in the Hamiltonian $2\mathcal{H} = \kappa = g^{\mu\nu} p_\mu p_\nu$ we get,

$$\left[f(x^1) + h(x^2)\right]\kappa = \left[L^{\mu\nu}(x^1) + \Theta^{\mu\nu}(x^2)\right] p_\mu p_\nu. \tag{3.3}$$

Applying the Hamilton-Jacobi theory, we choose Hamilton's principal function as $S(u^\mu, p_\mu, \lambda) = p_i u^i + W_1(x^1) + W_2(x^2) - \lambda\kappa/2$, with $W_A(x^A)$ being auxiliary functions and λ the affine parameter. Now, the Hamilton-Jacobi equation $\mathcal{H} + \partial S/\partial\lambda = 0$ yields the Hamiltonian of the geodesics. From Hamilton-Jacobi theory we also have that $p_\mu = \partial S/\partial u^\mu$, so $p_A = dW_A(x^A)/dx$. With this conditions the variables x^A in (3.3) can be easily separated, implying the existence of a new constant K that reads

$$K = f(x^1)\kappa - L^{ij}(x^1)p_i p_j - L^{11}(x^1)p_1^2 = \Theta^{ij}(x^2)p_i p_j + \Theta^{22}(x^2)p_2^2 - h(x^2)\kappa. \quad (3.4)$$

Note that the last two sides of equation (3.4) depend each on a single different coordinate. Assuming there exists a second rank tensor such that $K = K^{\mu\nu}p_\mu p_\nu$ then, by comparison with (3.4), it has to be that

$$K^{\mu\nu} = f(x^1)g^{\mu\nu} - L^{\mu\nu}(x^1) = \Theta^{\mu\nu}(x^2) - h(x^2)g^{\mu\nu}. \quad (3.5)$$

Alternatively, this tensor can also be expressed as

$$K^{\mu\nu} = \frac{f(x^1)\Theta^{\mu\nu}(x^2) - h(x^2)L^{\mu\nu}(x^1)}{f(x^1) + h(x^2)}. \quad (3.6)$$

A straightforward (but extremely tedious) computation reveals that $\nabla_{(\sigma}K_{\mu\nu)} = 0$, i.e. $K_{\mu\nu}$ is a second rank Killing tensor. The conserved quantity K is associated with this hidden symmetry of the metric.

□

Similar results regarding separability and the general form of the Killing tensor were independently found in [88] using a theorem by Benenti and Frankaviglia (see references therein).

Lemma 3. In a space-time $(M, g_{\mu\nu})$ with properties 1 to 4, the factor τ can be defined to be strictly positive in a sufficiently small neighborhood N of σ, consequently $\tilde{g}^{AA}|_N > 0$ for both $A = 1, 2$.

Proof. In a stationary, axially symmetric, and circular space-time with a time-like singularity σ, the adapted coordinate vectors $\partial/\partial x^A$ are everywhere space-like in a sufficiently small neighborhood N of σ. This follows when one considers hypersurfaces of constant x^A infinitesimally close to the singularity, which is stated to be time-like. Using the specified form of the metric of property 4, we have that $g_{AA}|_N = \tau/\tilde{g}^{AA}|_N > 0$. We can identify the quantities appearing in the inverse metric (3.2) with the τ factor and unphysical metric as $\tau = f(x^1) + h(x^2)$, and $\tilde{g}^{\mu\nu} = L^{\mu\nu}(x^1) + \Theta^{\mu\nu}(x^2)$. Thus, $\nabla\tau|_N$ is space-like.

Now the lemma can be proven by contradiction. Consider a pair of points x_1^A in N such that $\tau(x_1^1, x_1^2) > 0$. Then, $L^{11}(x_1^1), \Theta^{22}(x_1^2) \geq 0$ since the vectors $\partial/\partial x^A$ must be space-like in N. Assume now there exist a different pair of points in N, say x_1^1 and x_2^2, for which $\tau(x_1^1, x_2^2) < 0$. We now have that $g_{11} = \tau(x_1^1, x_2^2)/L^{11}(x_1^1) < 0$ which clearly contradicts the hypothesis of σ being a time-like singularity. The same can be done for the g_{22} component by considering other pair of points, x_1^2 and x_2^1, for which $\tau(x_2^1, x_1^2) < 0$, thereby discarding also a change of sign of τ when keeping the point x_1^2 constant. As a result, we have that τ can be expressed as strictly positive or strictly negative in N, this implies that $\tilde{g}^{AA}|_N > 0$ or $\tilde{g}^{AA}|_N < 0$, respectively. We choose the positive option.

□

Since τ has been shown to be positive in an appropriate neighborhood of the singularity, it can be considered as a conformal factor in such a region.

Regarding the third property of the space-time, the form of the inverse metric (3.2) that resulted from lemma 1 can be utilized to compute the curvature scalars of the manifold. Their general expression is

$$R_X = F_X(L^{\mu\nu}, \Theta^{\mu\nu}, f, h)/\Gamma^n \tau^m, \tag{3.7}$$

where F_X is a rather complicated, but regular, function that depends on the curvature invariant. The expressions for F_X are not included here mainly due to them being extremely large, but also because they are uninteresting and contain no useful information. Additionally,

$$\Gamma = det(\widetilde{\gamma}^{-1}) = (L^{00} + \Theta^{00})(L^{33} + \Theta^{33}) - (L^{03} + \Theta^{03})^2,$$

and $n, m \in \mathbb{Z}^+$. For the Ricci scalar, for example, $n = 2$ and $m = 3$. Examining (3.7) it can be observed that, if there exists a pair of points x_0^A for which $\tau(x_0^A) = 0$, then $q(x_0^\mu) \in \sigma_X$ and a curvature singularity can emerge in a common case. This hypothetical pair of points are later going to be of great relevance to the problem of affine completeness. Note that the existence of such a pair of points is not in contradiction with lemma 3 because, in a strict sense, the singularity does not belong to the neighborhood $N \subset M$ constituted only of regular points.

Remark. Other curvature singularities may arise apart from that of the previously mentioned pair x_0^A. For instance, if F_X does not contain any powers of Γ such that it cancels the determinant in the denominator of (3.7), then a region in which Γ vanishes will yield another curvature singularity. This case will in general define a singular hyper-surface, and hence, $\sigma_X = \Sigma^2 \times \mathbb{R}$, where Σ^2 is a two-manifold. See the metric of section 3.5 for an explicit example of this.

Finally, we can particularize the previously defined singular curvature set σ_X to a stationary and axially symmetric space-time that admits Cartesian-(like) coordinates $\{u^\mu\}$, e.g., one that is asymptotically flat. Since $g_{\mu\nu} = g_{\mu\nu}(x^A)$, then

$$\sigma_X = \left\{ \left(t, v\cos\varphi, v\sin\varphi, u^3\right) \in \mathbb{R}^4 \mid 1/R_X(u^\mu) = 0 \right\}, \tag{3.8}$$

with the quantities v and u^3 depending only on the coordinates x^A. These coordinates need not be adapted to the metric. A point $q(x_0^\mu) \in \sigma_X$ can be expressed as $q(x_0^\mu) = (t, v_0\cos\varphi, v_0\sin\varphi, u_0^3)$, where $v_0 = v(x_0^A)$ and $u_0^3 = u^3(x_0^A)$. It is readily seen that if the pair of points x_0^A is unique for a given space-time we have that

$$\sigma_X = \left\{ (t, v_0\cos\varphi, v_0\sin\varphi, u_0^3) \mid -\infty < t < \infty, 0 \leq \varphi < 2\pi \right\},$$

i.e., $\sigma_X = S^1 \times \mathbb{R}$ and spatially compact provided that $v_0 \neq 0$. This will be the case for the class of metrics presented in section 3.4.

3.2 Geodesics in Every Neighborhood of the Curvature Singularity

We are now ready to present the first theorem enlisting once again the properties of the space-times of interest.

Theorem 3.1. Let $(M, g_{\mu\nu})$ be a four-dimensional space-time with $(-,+,+,+)$ signature and the following set of properties:

1. Stationary, axially symmetric and satisfying the circularity condition.

2. Its geodesics admit a non-trivial quadratic first integral.

3. Contains a time-like curvature singularity σ whose singular curvature set σ_X is non-empty.

4. There exists an unphysical space-time $(\tilde{M}, \tilde{g}_{\mu\nu})$ such that $M \cup \sigma_X \subseteq \tilde{M}$ and $g^{\mu\nu} = \tilde{g}^{\mu\nu}/\tau$ with $\tilde{g}^{\mu\nu}, \tau \in C^\infty$ in a neighborhood $\tilde{N} \subset \tilde{M}$ of σ_X.

Choose a coordinate system $\{x^\mu\}$ in which $g^{\mu\nu}$ admits the separable structure (3.2), and such that $X_i = \partial/\partial x^i$ $(i = 0, 3)$ are Killing vectors with p_i denoting their associated momenta. Define $\psi(p_i) = \tau(\kappa - g^{ij} p_i p_j)$, where $\kappa = 0, -1$ for null and time-like geodesics, respectively. Then, at least a curve (or a segment of it) of the family $\eta(p_i)$ of causal geodesics defined by a given pair $p_i \in \mathbb{R}$ can be found in every neighborhood $N \subset M$ of σ, if and only if, starting from $n = 0$, for any point $q \in \sigma_X$ and any $A = 1, 2$, there exists a first non-vanishing derivative $\partial^n_A \psi(p_0, p_3)\big|_q$ such that either:

a. n is odd or,

b. the derivative is positive with n even.

Proof. From lemma 1 and the subsequent equation (3.4) we can express the separated equations of motion in terms of the velocities $\dot{x}^1$ and $\dot{x}^2$,

$$\Big[(f+h)\dot{x}^1\Big]^2 = L^{11}(f\kappa - L^{ij} p_i p_j - K) := \Xi^1(x^1), \tag{3.9}$$

$$\big[(f+h)\dot{x}^2\big]^2 = \Theta^{22}(K + h\kappa - \Theta^{ij} p_i p_j) := \Xi^2(x^2), \tag{3.10}$$

here we have used $\dot{x}^A = g^{AA} p_A$. Note that trajectories defined by these equations of motion will only be possible for coordinate values such that $\Xi^A(x^A) \geq 0$, where the $\Xi^A(x^A)$ functions are continuous due to property 4 of the space-time. Thus, in order for geodesics to exist within every neighborhood N of the curvature singularity σ, the condition $\Xi^A(x^A_0) \geq 0$ has to hold for some point $q(x^\mu_0) \in \sigma_X$. If necessary, suitable impositions on the derivatives $\partial_A \Xi^B\big|_q$ must also be considered. This is done so that these functions do not become negative everywhere in a sufficiently small neighborhood N_1 of σ. It will be helpful to now introduce the following notation: $f(x^1) = j_1$, $h(x^2) = j_2$ and $\tau = j_1 + j_2$. Please be aware that the superscript or subscript in the quantities Ξ^A and j_A is a tag for values $A = 1, 2$. However, it is not a tensorial index.

In the singularity σ, the equations of motion yield for some point $q(x^\mu_0) \in \sigma_X$:

$$\Xi^1(x^1_0) = -L^{11}(x^1_0)(\alpha + K), \tag{3.11}$$

$$\Xi^2(x^2_0) = -\Theta^{22}(x^2_0)(\beta - K), \tag{3.12}$$

with $\alpha = L^{ij}(x^1_0) p_i p_j - f(x^1_0)\kappa$ and $\beta = \Theta^{ij}(x^2_0) p_i p_j - h(x^2_0)\kappa$, which are quantities that are only in terms of parameters of the space-time (e.g. mass, angular momentum, etc.) and the constants of motion p_i. Hence, real and non-trivial solutions to equations (3.9) and (3.10) will exist in a small neighborhood of the singular point q only if there is a non-zero set of conserved quantities $p_0, p_3, K \in \mathbb{R}$ for which $\Xi^A(x^A_0) \geq 0$ for both $A = 1, 2$. Such solutions represent geodesics that may be able to arbitrarily approach the singularity.

By lemma 3 we have that[4] $L^{11}(x_0^1), \Theta^{22}(x_0^2) > 0$. Using these last conditions, one can easily realize that $\Xi^A(x_0^A) \geq 0$ if, and only if, $\alpha + \beta \leq 0$ for some real values of p_0, p_3. This last inequality can be rewritten as

$$\tau(g^{ij} p_i p_j - \kappa)\Big|_q \leq 0. \tag{3.13}$$

It is important to remark that $\alpha + \beta \leq 0$ is considered also as a sufficient condition in the above implication because one can always choose $\beta \leq K \leq -\alpha$ so that $\Xi^A(x_0^A) \geq 0$ for both values of A. The singularity could appear, then, in the trajectory of a geodesic (though not guaranteed yet). On the contrary, if $\alpha + \beta > 0$, there will not exist $K \in \mathbb{R}$ such that $\Xi^A(x_0^A) \geq 0$ simultaneously and thus, geodesics near the point q of the singularity cannot be found.

Case $K = -\alpha = \beta$. This special situation needs to be considered since we have three independent conserved quantities that determine the motion of the test particle. Out of the four existing constants of motion, κ is fixed depending on the nature of the geodesics (time-like or light-like), we are therefore left with three degrees of freedom. This means we can impose restrictions on p_i such that $-\alpha = \beta$ and then K can be chosen to be equal to those expressions, leaving us with one undetermined conserved quantity. Despite this, and once the explicit restrictions are known, one should verify they correspond to physically realistic scenarios.

The equations of motion for this case become

$$\left(\dot{x}^A\right)^2 = \frac{\hat{\Xi}^A(x^A)}{\tau^2(x^1, x^2)}, \tag{3.14}$$

here we have defined $\hat{\Xi}^1(x^1) := L^{11}(f\kappa - L^{ij} p_i p_j + \alpha)$ and $\hat{\Xi}^2(x^2) := \Theta^{22}(h\kappa - \Theta^{ij} p_i p_j - \alpha)$. It is clear that in this case, both $\hat{\Xi}^A(x_0^A) = 0$. If at least a geodesic exists in every neighborhood of σ, this implies that the $\hat{\Xi}^A$ functions are positive at least in some region within the close proximity of the singular point q. An expansion in power series of the $\hat{\Xi}^A$ functions around the discussed point of the singularity helps to describe their behavior in a sufficiently small neighborhood N of said point:

$$\hat{\Xi}^A = \sum_n c_n^A (x^A - x_0^A)^n.$$

To guarantee that the functions $\hat{\Xi}^A$ do not become both negative everywhere in N, the coefficient c_n^A of the leading term of each series must satisfy one of the following conditions for any $A = 1, 2$:

a. either n is odd or,

b. $c_n^A > 0$ if n is even.

In general, the coefficients c_n^A of each series will be given by

$$c_n^1 = \frac{1}{n!} L^{11} \partial_1^n \left(j_1\kappa - L^{ij} p_i p_j\right)\Big|_q, \quad c_n^2 = \frac{1}{n!} \Theta^{22} \partial_2^n \left(j_2\kappa - \Theta^{ij} p_i p_j\right)\Big|_q, \tag{3.15}$$

as long as the preceding terms vanish, that is, $c_m^A = 0$ for all $m < n$ with $m, n \in \mathbb{Z}^+$ (implying that c_n^A is the coefficient of the leading term). The expressions given

[4] If τ were to be chosen as negative definite, the proof could carry on but with $L^{11}(x_0^1), \Theta^{22}(x_0^2) < 0$. With this slight difference the theorem would still be valid, but with the final criterion for the first non-vanishing derivative $\partial_A^n \psi(p_0, p_3)\big|_q$ changed to negative in the case of n even.

by (3.15) constitute the relevant derivatives that were mentioned to need certain constraints at the beginning of this proof. The condition of interest for n even, $c_n^A > 0$, may be rewritten in the general form

$$\partial_A^n \left[\tau(\kappa - g^{ij} p_i p_j) \right] \Big|_q > 0. \tag{3.16}$$

Note that inequality (3.13) is also included here by making $n = 0$. Evidently, when the equality sign holds in (3.13), the important quantities to examine are the derivatives appearing in a. and b.

With this, we end the discussion of the considerations that need to be taken into account for the case $K = -\alpha = \beta$.

Thus far, we have proven that the fulfillment of either a. or b. is a necessary condition so that geodesics can be found in every neighborhood N of a singular point q. Now we show that it is also a sufficient condition.

There are several combinations on the signs of c_n^A for which a. or b. can be satisfied. There is one that should be set apart from the others as it will differ slightly from the next argumentation, the referred combination occurs when $c_n^1 > 0$ and $c_m^2 < 0$ with n and m even, or vice versa.

Assuming that either a. or b. hold through some any other combination different than the above, a regular point $q'(x_0'^\mu)$ can always be found in any neighborhood N of the singularity such that $\Xi^A(x_0'^A) > 0$ and $\tau = j_1(x_0'^1) + j_2(x_0'^2) \neq 0$. In fact, these two properties also apply everywhere in some additional neighborhood $N' \subset N$ of point q'. The following initial value problem may then be defined in N',

$$\dot{x}^A = \pm \frac{\sqrt{\Xi^A(x^A)}}{j_1(x^1) + j_2(x^2)} = \pm F^A(x^1, x^2), \quad x^A(\lambda_0) = x_0'^A, \quad \text{for } A = 1, 2, \tag{3.17}$$

where the $\pm$ signs apply for separate solutions. For a fixed value of $A = 1, 2$ (there is no sum over repeated indices in the next equation), the partial derivatives

$$\partial_A F^A(x^1, x^2) = \pm \frac{1}{2\sqrt{\Xi^A(x^A)}\tau^2} \left[\tau \partial_A \Xi^A(x^A) - 2\Xi^A(x^A) \partial_A j_A(x^A) \right] \tag{3.18}$$

are continuous in N'. This implies that the functions $F^A(x^1, x^2)$ are locally Lipschitz continuous with respect to the corresponding x^A variable. Therefore, utilizing the well-known Cauchy-Lipschitz theorem for a system of coupled first-order ordinary differential equations, the initial value problem (3.17) has a unique solution defined in the interval $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ for some real $\epsilon > 0$ and inside the neighborhood N'. When complemented with the remaining equations of motion for $\dot{x}^i$ and respective initial conditions, namely

$$\dot{x}^i = \frac{\tilde{g}^{ij} p_j}{j_1(x^1) + j_2(x^2)} = F^i(x^1, x^2), \quad x^i(\lambda_0) = x_0'^i, \quad \text{for } i = 0, 3, \tag{3.19}$$

a geodesic in $N' \subset N$ is obtained. A solution to the initial value problem (3.19) is guaranteed to exist by the same arguments as those used for (3.17).

It is also possible to obtain a geodesic within any N for the specific combination of leading coefficients $c_n^1 > 0$ and $c_m^2 < 0$ with $n, m \neq 0$ and even. The procedure is similar to the above discussion, the only difference is that now $x_0'^2 = x_0^2$ is required for the point $q'(x_0'^\mu)$. The reason for this is that $\Xi^2(x^2) \leq 0$ in a sufficiently small neighborhood of q, where the equality sign holds when $x^2 = x_0^2$ due to c_0^2 being zero. Hence, only geodesics with constant $x^2 = x_0^2$ and $\dot{x}^2 = \ddot{x}^2 = 0$ are possible. This implies that $F^2(x^1, x_0^2) = 0$ along such curves. These geodesic may be found, yet again, by solving suitable initial value problems (3.17) and (3.19) in some neighborhood N' of q' in which $\tau \neq 0$, $\Xi^1(x^1) > 0$, and fixed $\Xi^2(x_0^2) = 0$. Of course, the same applies for the opposite case $c_n^1 < 0$ and $c_m^2 > 0$ by interchanging the indices in this previous example. These are the sought results regarding the existence of geodesics.

Finally, we may regard the set of all causal geodesics in the space-time $(M, g_{\mu\nu})$ as a six-parameter family of curves[5]. For convenience, these parameters are chosen to be the conserved quantities $p_i, K \in \mathbb{R}$, $\kappa = 0, -1$, and a pair of initial conditions $x^A(0) \in \mathbb{R}$. Fixing the values of momenta p_i hence, defines a 4-parameter subfamily $\eta(p_i)$ of causal geodesics. So, making $\psi(p_0, p_3) = \tau(\kappa - g^{ij} p_i p_j)$ and summing up the above analysis, we find that there exists a curve of the subfamily $\eta(p_i)$ in every neighborhood N of the singularity if, and only if, starting from $n = 0$, for any point $q \in \sigma_X$ and any $A = 1, 2$, there exists a first non-vanishing derivative $\partial_A^n \psi(p_0, p_3)|_q$ such that either:

a. n is odd or,

b. the derivative is positive with n even.

This concludes the proof. □

Note that in this theorem we were not able to guarantee that any given geodesic $\xi(\lambda)$ arbitrarily close to the singularity will indeed meet it, i.e., that there exists a finite affine parameter λ_0 such that $\xi(\lambda_0) \in \sigma_X$. This situation, though, cannot be discarded either and the incompleteness of these curves is plausible, specially considering their closeness to σ. Nevertheless, these results are still useful and the next statement does follow immediately from the previous theorem.

Corollary. In a space-time $(M, g_{\mu\nu})$ with conditions 1-4, there will not exist geodesics of the subfamily $\eta(p_i)$ inside a sufficiently small neighborhood of the singularity σ if, for every point $q \in \sigma_X$, conditions a. and b. do not hold for a given pair $p_i \in \mathbb{R}$, i.e., if the first non-vanishing derivative $\partial_A^n \psi(p_0, p_3)|_q$ is negative with n even for both $A = 1, 2$.

This corollary can lead to interesting implications. Consider a space-time with all the properties mentioned above such that for all non-trivial sets of conserved quantities[6], conditions a. and b. do not hold. Thus, there would not exist any causal geodesics that are able to reach the singular point q.

It might be worth to make some additional comments about the past statement. The fact that causal geodesics are not able to run into the singularity does not mean that one cannot find a solution to the geodesic equation, belonging to a subfamily $\eta(p_i)$, that is arbitrarily close to q. This is always possible in a pseudo-Riemannian manifold for any type of curves (c.f. subsection 2.1.5). However, if the previous corollary holds for all non-trivial sets of conserved quantities then, no matter how initially close these solutions are, there will always exist a sufficiently small neighborhood N of the singular point q which they would not be able to access. One could further find causal geodesics, but of another subfamily $\eta(p_i')$, at a point $p \in N$. For

[5]Since there are two Killing vectors X_i, the number of parameters is reduced from the standard 8 (the set of initial conditions $x^\mu(0)$ and $\dot{x}^\mu(0)$) to 6 due to the isometries of the space-time in the directions x^i.

[6]By non-trivial we mean a set of conserved quantities that do not simultaneously vanish.

them there will exist another corresponding sufficiently small neighborhood N' of q that does not contain any segments of the geodesics of $\eta(p_i')$. This process can continue indefinitely without ever truly finding a causal geodesic that reaches the singular point. This can provide arguments for the existence of cases where curvature singularities do not necessarily imply causal geodesic incompleteness. An example illustrating this possibility for only the time-like case is presented in subsection 3.4.1, while in section 3.5 another one is constructed for all causal geodesics.

3.2.1 Coordinate Transformations

We should emphasize the fact that the results found so far demand the use of a convenient coordinate system $\{x^\mu\}$, one which makes the inverse metric separable, in order for them to be meaningful. Otherwise mixed derivatives (e.g., $\partial^2_{12}\psi$) would need to be considered, complicating thus the analysis. Such a coordinate system always exists for a space-time with properties 1 and 2 according to lemma 1. However, it is not uniquely determined since simple coordinate transformations of the type:

$$x^1 = x^1(x'^1), \quad x^2 = x^2(x'^2), \tag{3.20}$$

trivially preserve separability. In this case, the first non-vanishing derivatives of order n in the primed system $\{x'^\mu\}$ at point q follow the basic transformation law,

$$\partial^n_{A'}\psi|_q = (\partial_{A'}x^A)^n\Big|_q \partial^n_A\psi|_q\,. \tag{3.21}$$

The previous change of coordinates is assumed to be invertible at q, therefore, $\partial_{1'}x^1\big|_q, \partial_{2'}x^2\big|_q \neq 0$. This implies that $\partial^n_{A'}\psi\big|_q = 0$ if, and only if, $\partial^n_A\psi\big|_q = 0$ for $A = 1,2$. Furthermore the sign of even order derivatives, which is the important property established by the criterion for this type of order, does not change between coordinate systems. The signs of odd order derivatives can indeed be modified, nevertheless this is unimportant regarding theorem 3.1 and the subsequent corollary because, for this order, the relevant fact is whether they vanish or not.

One can consider the more complex change of coordinates (assuming it exists and that it does not breakdown the separability of the inverse metric, this is not at all trival):

$$x^1 = x^1(x'^1, x'^2), \quad x^2 = x^2(x'^1, x'^2). \tag{3.22}$$

The transformation law (3.21) is still valid, and so is the inverse law,

$$\partial^n_A\psi|_q = (\partial_A x'^{A'})^n\Big|_q \partial^n_{A'}\psi|_q\,.$$

We can alternatively write both transformations in matrix notation as $\Psi'_n = \mathbb{T}_n\Psi_n$ and $\Psi_n = \mathbb{T}'_n\Psi'_n$, where

$$\Psi_n = \begin{bmatrix}\partial^n_1\psi\\ \partial^n_2\psi\end{bmatrix}\Bigg|_q, \quad \Psi'_n = \begin{bmatrix}\partial^n_{1'}\psi\\ \partial^n_{2'}\psi\end{bmatrix}\Bigg|_q, \quad \mathbb{T}_n = \begin{bmatrix}(\partial_{1'}x^1)^n & (\partial_{1'}x^2)^n\\ (\partial_{2'}x^1)^n & (\partial_{2'}x^2)^n\end{bmatrix}\Bigg|_q,$$
$$\mathbb{T}'_n = \begin{bmatrix}(\partial_1 x'^{1'})^n & (\partial_1 x'^{2'})^n\\ (\partial_2 x'^{1'})^n & (\partial_2 x'^{2'})^n\end{bmatrix}\Bigg|_q.$$

Clearly $\mathbb{T}' = \mathbb{T}^{-1}$ has to hold for consistency. However, for n even there is an evident problem with these relations. This follows because the entries of $\mathbb{T}^{-1}$ cannot be positive definite if $\mathbb{T}$ is not purely diagonal (or anti-diagonal) and with positive entries as well. Note that a diagonal $\mathbb{T}$ matrix corresponds to the already analyzed transformation (3.20). The only other way then to avoid a contradiction is if $\mathbb{T}_n$ and $\mathbb{T}'_n$ are singular matrices for n even, i.e., $\det \mathbb{T}_n = \det \mathbb{T}'_n = 0$, and thus implying that the system of equations is linearly dependent. This of course imposes a constraint on the coordinate transformation. Bear in mind, though, that $\mathbb{T}_1$ and $\mathbb{T}'_1$ are the Jacobian matrices of the change of coordinates and, as such, their determinants cannot vanish in order for the transformation to be invertible at least near q. With this assumption, and despite the particularity of the matrices with n even, similar conclusions can be drawn as those obtained for the change of coordinates (3.20). Specifically, the signs of the components of Ψ_n suffer no modification when transformed to Ψ'_n for n even, and $\Psi_n = 0$ if and only if $\Psi'_n = 0$.

After this discussion we can realize that the conclusions yielded by the application of theorem 3.1 do not depend on the coordinate system used to describe the g_{AB} components of the metric, as long as the requested separable properties are fulfilled.

3.3 Geodesics Terminating at the Singularity

Up until now, we have been careful enough to avoid referring to geodesics running into singular points, which properly are not part of the space-time manifold. This unfortunate fate is a common source of affine incompleteness and thus, of possible ill-behavior of general curves in the space-time. Instead, we have analyzed the existence of the mentioned type of curves in neighborhoods of the singular points. In this section we shall discuss some specific cases in which geodesics do meet the curvature singularity and the possible effects this has on their completeness.

Theorem 3.2. Let $(M, g_{\mu\nu})$ be a space-time with properties 1 to 4 and $\eta'(p'_i)$ a family of causal geodesics in a small neighborhood of the singularity σ. Consider an expansion in power series of $\tau(x^A)$ around the singular point $q(x^\mu_0)$, and let δ_A be the order of the leading terms that go as $(x^A - x^A_0)^{\delta_A}$ in the series for each A. Then, there will exist a curve $\xi(\lambda)$ of $\eta'(p'_i)$ defined on an interval $[\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ such that $\xi(\lambda_0) = q$ for some finite value affine parameter λ_0, if for some momenta $p'_i \in \mathbb{R}$ the following holds for fixed values of A and B ($A \neq B$):

- i $\partial_B \psi(p'_0, p'_3)|_q = 0$ and the lowest non-vanishing derivative $\partial^n_A \psi(p'_0, p'_3)|_q$ is positive with $n = 2\delta_A$,
- ii $\partial^n_A \tilde{g}^{ij}|_q \, p_j = 0$ for $0 \leq n < \delta_A$ and both $i = 0, 3$.

Proof. We start by distinguishing between two specific possibilities for a singular point $q(x^\mu_0) \in \sigma_X$, namely $\tau(x^A_0) \neq 0$ and $\tau(x^A_0) = 0$. Later, both of them will be generalized. When approaching the singularity, the analysis of the first case is fairly simple. Consider the equations of motion (3.9) and (3.10), as well as the two remaining ones $\dot{x}^i = \tilde{g}^{ij} p_j / \tau$. It can be easily seen that

$$\lim_{x^A \to x^A_0} (\dot{x}^A)^2 = \mathcal{C}^A, \quad \lim_{x^A \to x^A_0} \dot{x}^i = \mathcal{D}^i \quad \text{if } \tau(x^A_0) \neq 0, \tag{3.23}$$

where $\mathcal{C}^A = \Xi^A(x^A_0)/\tau^2(x^1_0, x^2_0)$ and $\mathcal{D}^i = \tilde{g}^{ij}(x^A_0) p_j / \tau(x^1_0, x^2_0)$ are well-defined constants. Hence, the coordinate velocities of any geodesic approaching the singularity through point q remain finite. This already is a good indication that we may find geodesics $\xi(\lambda)$ such that $\xi(\lambda_0) = q(x_0^\mu)$ for some finite affine parameter λ_0. For the case $\tau(x_0^A) = 0$, the analysis increases in complexity but similar conclusions can be drawn.

Focus now on a power series expansion of the function

$$\tau = \sum_{A,n} d_n^A (x^A - x_0^A)^n,$$

here we have written explicitly the sum on the x^A coordinates to avoid any sort of confusion. Since τ is strictly positive in a neighborhood N of the singularity, it follows that the leading terms d_n^A of the series are positive with n even. Let δ_A and ε_A be the order of the leading term of the j_A functions and $\Xi^A(x^A)$ functions, respectively, and define for compactness $\mathcal{A}_A = c^A_{\varepsilon_A}$, $\mathcal{B}_A = d^A_{\delta_A}$.

Consider the case of geodesics with constant $x^B = x_0^B$ and hence, $\dot{x}^B = \ddot{x}^B = 0$ for some fixed value of $B = 0, 1$. This is possible provided that $\Xi^B(x_0^B)$ and $\partial_B \psi|_q$ vanish. When inserting the corresponding series in $F^A(x^1, x^2)$ (the right-hand side of equation (3.17)) for some fixed $A \neq B$, the following limit can be evaluated

$$\lim_{x^A \to x_0^A} \frac{\left[\sum_{n=\varepsilon_A}^{\infty} c_n^A (x^A - x_0^A)^n\right]^s}{\sum_{m=\delta_A}^{\infty} d_m^A (x^A - x_0^A)^m} = \begin{cases} \infty & \text{for } s\varepsilon_A < \delta_A, \\ w & \text{for } s\varepsilon_A = \delta_A, \\ 0 & \text{for } s\varepsilon_A > \delta_A, \end{cases} \tag{3.24}$$

with $w = (\mathcal{A}_A)^s / \mathcal{B}_A$ and $s \in \mathbb{R}^+$.

Examining equation (3.24) along with (3.14), and since the limits shown there are bounded for $s\varepsilon_A \geq \delta_A$, a hint towards possible geodesics passing through σ can be found, just as in the $\tau(x_0^A) \neq 0$ case. In fact, note that this situation is included in the previous limit ($\delta_A = 0$).

We now show that if $\varepsilon_A = 2\delta_A$, then $F^A(x^1, x_0^2)$ is locally Lipschitz continuous at $x^A = x_0^A$. It suffices to verify that the derivative $\partial_A F^A$ given by (3.18) is continuous in some interval centered around x_0^A. One way to realize this is by inserting the necessary power series in (3.18). Thus finding that the order of the leading term of $\partial_A F^A$, seen as a series that includes negative powers if required so, is $\varepsilon_A/2 - \delta_A = 0$. To arrive at this expression one needs only to examine the lowest powers of the series in the numerator and the denominator of (3.18). Additionally, the term of order $\delta_A + \varepsilon_A - 1$ that appears in the whole numerator vanishes since its coefficient in the series is given by $(\varepsilon_A - 2\delta_A)\mathcal{A}_A \mathcal{B}_A = 0$. Furthermore, if $\varepsilon_A = 2\delta_A$, then $F^A(x^A, x_0^B)$ can be written as

$$F^A(x^A, x_0^B) = \frac{\sqrt{\mathcal{A}_A + c^A_{\varepsilon_A+1}(x^A - x_0^A) + \dots}}{\mathcal{B}_A + d^A_{\delta_A+1}(x^A - x_0^A) + \dots},$$

which is analytic at $x^A = x_0^A$. Therefore, it admits its own series expansion at $x^A = x_0^A$ starting with a zero order term, the same applies to its derivative $\partial_A F^A$. This implies that both functions are smooth, at least in a neighborhood of x_0^A, and consequently also locally Lipschitz continuous.

With the previously used Cauchy-Lipschitz theorem, a unique real solution can be found for the initial value problem

$$\dot{x}^A = \pm F^A(x^A, x_0^B), \quad x^A(\lambda_0) = x_0^A, \tag{3.25}$$

as long as $\mathcal{A}_A > 0$. This is consistent with the requirement that there exists a geodesic in every neighborhood of the singular point q. At this point, we should not forget about the other pair of equations of motion $\dot{x}^i = \tilde{g}^{ij} p_j / \tau$, which need to be solved as well in order to obtain a geodesic on the space-time manifold. For this purpose we perform yet another expansion in power series around the points x_0^A. In this last instance for $\tilde{g}^{ij} p_j = \sum_{A,n} b_n^{Ai} (x^A - x_0^A)^n$, where

$$b_n^{Ai} = \frac{1}{n!} \partial_A^n \tilde{g}^{ij} \Big|_q p_j. \tag{3.26}$$

We may apply limit (3.24) once again with the expansion for $\tilde{g}^{ij} p_j$ in the numerator and $s = 1$. By similar arguments as those used for the $\dot{x}^A$ equations, it can be seen that if the coefficients b_n^{Ai} vanish for both $i = 0, 3$ and $0 \le n < \delta_A$ with some fixed value of A, then the following initial value problem will be well-posed when complemented by (3.25):

$$\dot{x}^i = F^i(x^A, x_0^B), \quad x^i(\lambda_0) = x_0^i, \quad \text{for } i = 0, 3.$$

Here,

$$F^i(x^A, x_0^B) = \frac{b_{\delta_A}^{Ai} + b_{\delta_A+1}^{Ai} (x^A - x_0^A) + \dots}{\mathcal{B}_A + d_{\delta_A+1}^A (x^A - x_0^A) + \dots}.$$

We have thus found the desired curves which are defined on the interval $[\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ for some real $\epsilon > 0$. Hence, there will exist geodesics $\xi(\lambda)$ of a familiy $\eta'(p_i')$ such that $\xi(\lambda_0) = q \in \sigma_X$, if for some momenta $p_i' \in \mathbb{R}$ and fixed values $A \neq B$ the following holds:

i $\partial_B \psi(p_0', p_3')|_q = 0$ and the lowest non-vanishing derivative $\partial_A^n \psi(p_0', p_3')|_q$ is positive with $n = 2\delta_A$,

ii $\partial_A^n \tilde{g}^{ij}|_q p_j = 0$ for $0 \le n < \delta_A$ and both $i = 0, 3$.

This concludes the proof.

□

Note that in this theorem the $\tau \neq 0$ case is included when $\delta_A = 0$. Also, the second condition of the theorem can be expressed in an alternative way. The vanishing of $\partial_A^n \psi(p_0, p_3)|_q$ for $0 \le n < 2\delta_A$ yields the following relation for the momenta p_0 and p_3,

$$\left[\partial_A^n (\tilde{g}^{ii} p_i + \tilde{g}^{ij} p_j) \Big|_q \right]^2 = \left[-\det(\partial_A^n \tilde{\gamma}^{-1}) p_j^2 + \kappa (\partial_A^n \tilde{g}^{ii})(\partial_A^n \tau) \right] \Big|_q, \tag{3.27}$$

with $\det(\partial_A^n \tilde{\gamma}^{-1}) = (\partial_A^n \tilde{g}^{00})(\partial_A^n \tilde{g}^{33}) - (\partial_A^n \tilde{g}^{03})^2$. Warning: in equation (3.27) we have temporarily abandoned the summation convention for repeated indices, the intended use of this expression is for fixed values $i, j = 0, 3$ and $i \neq j$. This liberty is taken only in this equation and in the following one. This particular form of the equation is used due to it being easily substituted in (3.26), obtaining thus

$$b_n^{Ai} = \pm \frac{1}{n!} \sqrt{-\det(\partial_A^n \tilde{\gamma}^{-1}) p_j^2 + \kappa (\partial_A^n \tilde{g}^{ii})(\partial_A^n \tau)} \Big|_q. \tag{3.28}$$

The advantage of using (3.28) over (3.26) lies in the reduction of one free parameter in the equation (one of the constants of motion), despite this, the latter is way more compact.

The result of theorem 3.2 states a sufficient condition for the existence of causal geodesics that reach a singular curvature point q in a finite affine parameter λ_0. Such curves are yielded by solutions of ordinary differential equations and are even defined for small future and past values of λ_0. Unfortunately, this may not be enough to not consider them as incomplete geodesics in that reduced interval of λ. Since singularities are not part of the space-time manifold, when $\lambda = \lambda_0$ these curves terminate (or begin) at σ and cannot be continued further. It is true, however, that there exists a future-directed geodesic ξ_1 ending at the singular point q when $\lambda = \lambda_0$, as well as its past-directed counterpart ξ_2 starting at q in that same value of affine parameter. Were it not for the missing point in M corresponding to the singularity, ξ_1 and ξ_2 could be joined into a single curve. This suggests that incompleteness could be avoided recurring to a construction in which singular points are added as boundary points in the manifold (see figure 3.1 in the next section for a very rudimentary example of this).

A particularly interesting construction as the one mentioned above has been carried out in [89] and receives the name of "abstract boundary" (or a-boundary). Very briefly, and sparing the deep technical details, the a-boundary of a manifold M contains all the types of idealized points of M, this includes points at infinity, suitable regular points and singularities. This boundary is built up of equivalence classes of boundary points in all possible open embeddings of M. The matter of whether this procedure, or a similar one, would lead to the recovery of completeness for the case of the ξ_1 and ξ_2 geodesics just outlined, is left here only as a mere conjecture.

Finally, the transformation law under change of coordinates of the type (3.22) for the quantities specified in condition i of theorem 3.2 was already studied in the past section. It is evident that the expressions of condition ii follow an identical transformation. Thus, said quantities will vanish if and only if they also do in another coordinate system in which the inverse metric admits the form (3.2).

3.4 The Plebański-Carter Class of Space-times

In this section we apply our results to the so-called Plebański-Carter class of space-times. This name owes to the fact that this family of metrics was found independently by Carter [90] and Plebański [91]. The class consists of solutions to the Einstein-Maxwell field equations with a generally non-zero cosmological constant. Some physically relevant space-times, such as black holes, are contained within this family of six parameters, namely:

- Mass m
- Angular momentum per unit mass a
- Electric charge q_e
- Magnetic charge q_m
- The Taub-NUT parameter l
- Cosmological constant Λ

This family of metrics is also known as the Kerr-Newman-NUT-(A)dS solutions. There exists an even more general class of space-times, that of Plebański-Demiański [92], which includes an additional seventh parameter: acceleration $\mathcal{A}$. However for the case of non-vanishing $\mathcal{A}$, only a conformal Killing tensor may be found, allowing the integrability of the equations of motion for null geodesics exclusively. This falls beyond the scope of the derived criteria. If additionally $l = \Lambda = 0$, then the resulting four parameter space-times are asymptotically flat. In Boyern-Lindquist coordinates, the line element of the Plebański-Carter metrics is given by [4]

$$\begin{aligned} ds^2 = &- \frac{1}{\Sigma}\left[(\Delta_r - a^2\Delta_\theta \sin^2\theta)dt^2 + 2(\Delta_r\chi - a[\Sigma + a\chi]\Delta_\theta \sin^2\theta)dtd\varphi \right. \\ &+ \left.\left([\Sigma + a\chi]^2\Delta_\theta \sin^2\theta - \Delta_r\chi^2\right) d\varphi^2\right] + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2, \end{aligned} \tag{3.29}$$

where $\Sigma = r^2 + (l + a\cos\theta)^2$ and $\chi = a\sin^2\theta - 2l(\cos\theta - 1)$. Additionally,

$$\begin{aligned} \Delta_r &= -\frac{\Lambda}{3}r^4 - \left[\frac{\Lambda}{3}\left(6l^2 + a^2\right) - 1\right] r^2 - 2mr + (1 - l^2\Lambda)(a^2 - l^2) + Q^2, \\ \Delta_\theta &= \frac{\Lambda}{3}a\cos\theta\left(a\cos\theta + 4l\right) + 1, \end{aligned}$$

with $Q^2 = q_e^2 + q_m^2$. By setting the appropriate parameters to zero in (3.29) we can obtain some thoroughly studied metrics such as Schwarzschild, Reissner-Nordström, Kerr, etc. The roots of Δ_r and Δ_θ define Killing horizons. When $\Lambda = 0$, these horizons are located at $r_{1,2} = m \pm \sqrt{m^2 - (a^2 - l^2 + Q^2)}$ and the outer one is an event horizon. If $\Lambda \neq 0$, then Δ_r becomes a quartic polynomial that can yield cosmological horizons too. On the other hand, any set of space-time parameters that allow real roots of Δ_θ will not be treated during the following. The reason for this is that they would generate horizons consisting of cones of constant θ, such a situation is not considered of physical interest.

These space-times contain a ring singularity when $\Sigma = 0$, i.e. $r = 0$ and $\cos\theta = -l/a$. Their singular curvature set can be expressed using the Cartesian-like Kerr-Schild coordinates[7] for which $v = \sqrt{r^2 + a^2}\sin\theta$ and $u^3 = r\cos\theta$ in equation (3.8). Since the singularity is defined by a single pair of points, $r = 0$ and $\cos\theta = -l/a$, then σ_X can be written as

$$\sigma_X = \left\{(t, v_0\cos\varphi, v_0\sin\varphi, 0) \mid -\infty < t < \infty, 0 \leq \varphi < 2\pi\right\},$$

where $v_0 = \sqrt{a^2 - l^2}$. If $\Lambda \neq 0$ and $a^2 - l^2 + Q^2 > m^2$, there are is no event horizon and the ring singularity is, in principle, left visible to any asymptotically distant observer. Also, note that the ring singularity will only exist if $|a| > |l|$. This inequality is consistent with the previous no event horizon condition.

Notice that $\Delta_r|_{r=0} = Q^2 + (1 - l^2\Lambda)(a^2 - l^2)$ and $\Delta_\theta|_{\cos\theta = -l/a} = 1 - l^2\Lambda$. Thus, the curvature singularity will be time-like only if $1 - l^2\Lambda > 0$. This follows due to the fact that both coordinate vector fields ∂/∂_r and ∂/∂_θ are space-like everywhere in a sufficiently small neighborhood of the singularity only if said inequality holds. For this reason we will consider here only metrics that satisfy $1 - l^2\Lambda > 0$ and, since Δ_θ is required to not possess real roots, this implies that $\Delta_\theta > 0$. In this matter,

[7]This set of coordinates is particularly helpful in realizing that the singularity of this type of black holes is indeed a ring [93].

the singularity of the Schwarzschild black hole ($a = l = Q = \Lambda = 0$) needs to be excluded from our analysis since it is known to be space-like. Hence, once a causal observer crosses the event horizon, the singularity will unavoidably appear in the future of its world-line.

It is simple to compute the inverse metric $g^{\mu\nu}$ of this class of space-times and express it in the separated form

$$g^{\mu\nu} = \frac{\mathcal{R}^{\mu\nu}(r) + \Theta^{\mu\nu}(\theta)}{f(r) + h(\theta)}. \tag{3.30}$$

Explicitly we have,

$$\begin{aligned}
\mathcal{R}^{\mu\nu} &= \begin{bmatrix} -(\Sigma + a\chi)^2/\Delta_r & 0 & 0 & -a(\Sigma + a\chi)/\Delta_r \\ 0 & \Delta_r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a(\Sigma + a\chi)/\Delta_r & 0 & 0 & -a^2/\Delta_r \end{bmatrix}, \\
\Theta^{\mu\nu} &= \begin{bmatrix} \chi^2/\Delta_\theta \sin^2\theta & 0 & 0 & \chi/\Delta_\theta \sin^2\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_\theta & 0 \\ \chi/\Delta_\theta \sin^2\theta & 0 & 0 & 1/\Delta_\theta \sin^2\theta \end{bmatrix},
\end{aligned} \tag{3.31}$$

with $f(r) = r^2$ and $h(\theta) = (l + a\cos\theta)^2$. Observe that the sum $\Sigma + a\chi$ depends only on the coordinate r.

One of the most important features of this class of solutions is that they are algebraically special in the Petrov classification of space-times. In particular, they are of type D, meaning they possess two principal null directions which are repeated twice (see subsection 2.1.8). This implies the existence of a second rank Killing tensor and thus, the integrability of the equations of motion in the space-time [29]. The Killing tensor has the structure of equation (3.5).

Since, when imposing $1 - l^2\Lambda > 0$, the Plebański-Carter class of space-times satisfies the properties stated in both of the theorems of this chapter, we can check if there are any causal geodesics arbitrarily close to the curvature singularity. If this is the case, we can further establish by using theorem 3.2 if any of these curves can be a solution that runs directly into σ. Evaluating $\psi = \Sigma(\kappa - g^{ij}p_ip_j)$ in the ring singularity and defining for convenience a constant $C = (1 - l^2\Lambda)(a^2 - l^2) > 0$, we obtain

$$\psi|_q = -\frac{Q^2P^2}{C(C + Q^2)}, \tag{3.32}$$

where we have introduced a constant of motion $P = (a + l)^2\mathcal{E} - aL_z$, which is in terms of the conserved quantities $p_0 = -\mathcal{E}$ and $p_3 = L_z$. In the notation of section 3.2, we have that

$$\alpha = -\frac{P^2}{C + Q^2}, \quad \beta = \frac{P^2}{C},$$

if $Q = 0$ then $-\alpha = \beta$. Therefore, it shall be useful to consider at this point two different cases of this class of space-times, as they will possess some unique interesting properties.

3.4.1 Case $Q \neq 0$

For a non-vanishing electromagnetic charge, clearly $\psi|_q$ is strictly negative and so, there will exist a sufficiently small neighborhood of σ in which there are no geodesics of arbitrary values of energy and angular momentum. Nonetheless, consider the particular case $P = 0$ which establishes an exact relation between angular momentum, energy and the space-time parameters, leading to $\psi|_q = 0$. We therefore have to analyze the first non-vanishing derivatives of ψ. It can be seen that $(\partial_r\psi)|_q = (\partial_\theta\psi)|_q = 0$ when the condition $P = 0$ holds. For the second derivatives we have

$$\left(\partial_r^2\psi\right)\big|_q = \frac{1}{a^2 - l^2}\left(\partial_\theta^2\psi\right)\big|_q = 2\kappa. \tag{3.33}$$

From (3.33) it can be concluded that time-like geodesics ($\kappa = -1$) do not reach the singularity in the space-times of the Plebański-Carter class. This is in full agreement with one of the results from [94], where the singular region of a Reissner-Nordström black hole is shown to be physically inaccessible to time-like curves of limited acceleration[8]. However, for null geodesics the second derivatives reduce to zero once again. The same goes for the third derivatives $\left(\partial_r^3\psi\right)\big|_q = \left(\partial_\theta^3\psi\right)\big|_q = 0$. It is until the fourth derivation that a non-vanishing constant can be found,

$$(C + Q^2)\left(\partial_r^4\psi\right)\Big|_q = -\frac{1 - l^2\Lambda}{a^2 - l^2}\left(\partial_\theta^4\psi\right)\Big|_q = 4!\mathcal{E}^2. \tag{3.34}$$

Hence, a light-like geodesic defined by $P = 0$ can always be found within every neighborhood of the singularity in the Plebański-Carter class of space-times with $Q \neq 0$. These previous calculations are presented in table 3.1, where the special case $K = -\alpha = \beta$ is considered when $n = 0$.

TABLE 3.1: The first criterion for arbitrarily close approach to the singularity in Plebański-Carter space-times with $Q \neq 0$.

		$\partial^n_A\psi(p_0, p_3)\vert_q$	Conditions	Geodesics Approaching σ
$n = 0$		$-\frac{Q^2P^2}{C(C+Q^2)}$	-	Only if $P = K = 0$
$n = 1$	$A = 1$	0	$P = K = 0$	-
	$A = 2$	0		
$n = 2$	$A = 1$	2κ	$P = K = 0$	Non-existent if $\kappa = -1$
	$A = 2$	$2(a^2 - l^2)\kappa$		
$n = 3$	$A = 1$	0	$P = K = \kappa = 0$	-
	$A = 2$	0		
$n = 4$	$A = 1$	$\frac{4!\mathcal{E}^2}{C+Q^2}$	$P = K = \kappa = 0$	Existent
	$A = 2$	$-\frac{4!(a^2-l^2)\mathcal{E}^2}{1-l^2\Lambda}$		

The equations of motion for the previous null geodesics of interest are simply

$$(\Sigma\dot{r})^2 = \mathcal{E}^2 r^4, \quad (\Sigma\dot{\theta})^2 = -\mathcal{E}^2(l + a\cos\theta)^4/a^2\sin^2\theta. \tag{3.35}$$

[8] In fact, it is easy to realize this for time-like geodesics from the expression $\dot{r}^2 = \mathcal{E}^2 + V(r)$, where $V(r) = 2m/r - q_e^2/r^2 - 1$, which becomes an infinitely repulsive potential when $r \to 0$.

It can be easily seen that motion is only possible within the plane $\cos\theta = -l/a$. A null observer constrained to this plane will closely approach the ring singularity coming from infinity. For this to be the case, specific conditions on the constants of motion have to be met, namely $P = \kappa = K = 0$, which define a family $\eta(\mathcal{E}, L_z, K = 0)$ of null geodesics. Since the curves of this family are constrained to the discussed plane, η is a one-parametric family of null geodesics, being the initial condition $r(\lambda_0)$ the only degree of freedom. These curves are the principal null rays, the rest of the causal geodesics will never reach the singularity. It is worth pointing out that this property regarding the principal null rays of the Kerr metric was already mentioned in [95].

3.4.2 Case $Q = 0$

Focusing now on the metrics (3.29) without electromagnetic charge, it can be seen in (3.32) that $\psi|_q$ is zero without imposing restrictions on the conserved quantities $\mathcal{E}$ and L_z. In this case the first non-vanishing derivatives are

$$\frac{C^2}{2m}\left(\partial_r\psi\right)|_q = -\frac{3C(1-l^2\Lambda)\sqrt{a^2-l^2}}{2l\left[3+\Lambda(a^2-4l^2)\right]}\left(\partial_\theta\psi\right)|_q = P^2. \tag{3.36}$$

These are derivatives of odd order and thus, there are causal geodesics in every neighborhood of the ring singularity. Assuming a positive mass, as is the case for physical black holes, these geodesics are the ones coming (going) from (to) $r > 0$. Hence, they remain within the innermost horizon of the space-time ($0 < r < r_1$) with no chance of escaping to the domain of outer communications. For the region of negative values of r, which is obtained by performing a maximal analytic extension of the space-time manifold for metrics with $a \neq 0$ or $l \neq 0$ (similar to that of [95]), we have the opposite situation and causal geodesics will actually be repelled from the singularity.

One may still try to obtain causal geodesics that could meet σ coming from either positive or negative values of the radial coordinate by setting the conjugate momenta to $P = 0$. This would result in the vanishing of the first derivatives in (3.36), leading then to the same results shown in equations (3.33) and (3.34), but with $Q = 0$. The computations for this case are concisely shown in table 3.2. Note that in comparison with table 3.1, the special case $K = -\alpha = \beta$ now corresponds to $P/C = K$.

3.4.3 Principal Null Rays Encountering the Singularity

We have already shown that there indeed exists a family of null geodesics that get arbitrarily close to the singularity in the Plebański-Carter class of space-times. In what follows we wonder if those curves can indeed meet it. For this purpose we apply theorem 3.2.

Consider the family $\eta(\mathcal{E}, L_z, K = 0)$ of null geodesics with $P = 0$. In the above calculations we saw that for this family, the first non-vanishing derivatives of $\partial^n_A\psi|_q$ were those of order $n = 4$ for both coordinates r and θ. Furthermore, $\partial^4_r\psi|_q$ is positive and $\partial^4_\theta\psi|_q$ is negative. Since $\delta_r = \delta_\theta = 2$ because $\Sigma = r^2 + (l + a\cos\theta)^2$, condition i of theorem 3.2 is satisfied when choosing $x^A = r$ for a fixed value of A. To verify condition ii, all that is left to be done is to compute the $\partial^n_A\tilde{g}^{ij}|_q\, p_j$ derivatives. An evaluation of these expressions yields

TABLE 3.2: The first criterion for arbitrarily close approach to the singularity in Plebański-Carter space-times with $Q = 0$.

		$\partial^n_A \psi(p_0, p_3)\|_q$	Conditions	Geodesics Approaching σ
$n = 0$		0	-	Only if $P/C = K$
$n = 1$	$A = 1$	$\dfrac{2mP^2}{C^2}$	$P/C = K$	Existent in $r \geq 0$, Non-existent in $r < 0$, unless $P = K = 0$
	$A = 2$	$\dfrac{-2l\left[3 + \Lambda(a^2 - 4l^2)\right] P^2}{3C(1 - l^2\Lambda)\sqrt{a^2 - l^2}}$		
$n = 2$	$A = 1$	2κ	$P = K = 0$	Non-existent if $\kappa = -1$
	$A = 2$	$2(a^2 - l^2)\kappa$		
$n = 3$	$A = 1$	0	$P = K = \kappa = 0$	-
	$A = 2$	0		
$n = 4$	$A = 1$	$4!\mathcal{E}^2/C$	$P = K = \kappa = 0$	Existent
	$A = 2$	$-\frac{4!(a^2-l^2)\mathcal{E}^2}{1-l^2\Lambda}$		

$$
\begin{aligned}
\frac{1}{(a+l)^2}\, \tilde{g}^{0j}\Big|_q p_j &= \frac{1}{a}\, \tilde{g}^{3j}\Big|_q p_j = -\frac{Q^2 P}{C(C+Q^2)}, \\
\frac{1}{(a+l)^2}\, \partial_r \tilde{g}^{0j}\Big|_q p_j &= \frac{1}{a}\, \partial_r \tilde{g}^{3j}\Big|_q p_j = \frac{2mP}{(C+Q^2)^2}, \\
\frac{1}{(a+l)^2}\, \partial_\theta \tilde{g}^{0j}\Big|_q p_j &= \frac{1}{a}\, \partial_\theta \tilde{g}^{3j}\Big|_q p_j = -\frac{2l\left[3 + \Lambda(a^2 - 4l^2)\right] P}{3C(1 - l^2\Lambda)\sqrt{a^2 - l^2}}.
\end{aligned} \tag{3.37}
$$

Condition ii is clearly satisfied too by the family $\eta(\mathcal{E}, L_z, K = 0)$ of null geodesics. Hence, these curves encounter the singularity in some finite value of their affine parameter. Indeed, from the equations of motion (3.35) and the fact that these causal geodesics are constrained to the plane $\cos\theta = -l/a$, we have for the radial coordinate velocity $\dot{r} = \pm\mathcal{E}$, which describes ingoing and outgoing radial null rays. This equation is easily integrated and shows that a geodesic coming from either positive or negative values of r touches the curvature singularity σ. It is remarkable that the tangents of these geodesics are aligned with the principal null directions $k^\mu_\pm$ of the space-time, namely

$$
k^\mu_\pm = \left[\frac{r^2 + (a+l)^2}{\Delta_r}, \pm 1, 0, \frac{a}{\Delta_r}\right], \tag{3.38}
$$

expressed in the $\{t, r, \theta, \varphi\}$ basis of Boyer-Lindquist coordinates.

It is worth now summarizing the results here obtained for each case. In the space-times with non-vanishing electromagnetic charge of the Plebański-Carter class, time-like geodesics will in general avoid the ring singularity, while the principal null rays shall indeed meet it. On the other hand, for metrics with $Q = 0$, we found that there are causal geodesics with arbitrary values of momenta coming from $r > 0$ that are able to infinitesimally approach the singular region. These are curves that potentially may reach the singularity and become incomplete in the process. Unfortunately, the results stated by the derived criteria are not able to go as far as to confirm this. It is reasonable, though, to expect that curvature provokes ill-effects on these geodesics. By contrast, the encountering of the principal null rays with the singular ring was found in both cases of electromagnetic charge value and $a \neq 0$. These geodesics are in fact mathematically defined for future values of its affine parameter after striking the singularity. Nevertheless, because singularities are to be excluded from the space-time manifold, said curves will reach the ring singularity in a finite amount of its affine parameter and then disappear off the manifold. The reverse situation is possible as well, this is, an observer could suddenly appear in the singularity and then follow its way into the manifold.

To finalize, based on the argument that the principal null rays are defined for an interval of affine parameter before and after meeting the singularity, we propose a very basic construction that could avoid their incompleteness. The idea, which is not uncommon, is to add boundary points to the manifold that represent the singularity. The following construction is specific for this example and by no means it should be considered as a procedure that can be adopted in a general way. We consider two separate patches, one with $r > 0$ and another one with $r < 0$, describing a space-time slice of constant $\cos\theta = -l/a$. Of course, at $r = 0$, a curvature singularity is found in both patches. For each and every future-directed null geodesic ξ_+ in the positive r patch with tangent $k^\mu_\pm$ and that terminates in the singularity when $\lambda = \lambda_0$, there will exist in the negative r patch a past-directed null geodesic ξ_- with the same tangent vector starting at an equal value of affine parameter. Singular boundaries are added to both patches at $r = 0$, which are then to be identified. By choosing coinciding initial conditions at $r = 0$ for the two curves ξ_+ and ξ_-, the joint boundaries allow for both geodesics to become a single curve which can be continued further after running into the singular boundary. The same union can be done for future-directed curves in $r < 0$ with suitable past-directed curves in $r > 0$. Figure 3.1 illustrates this process.

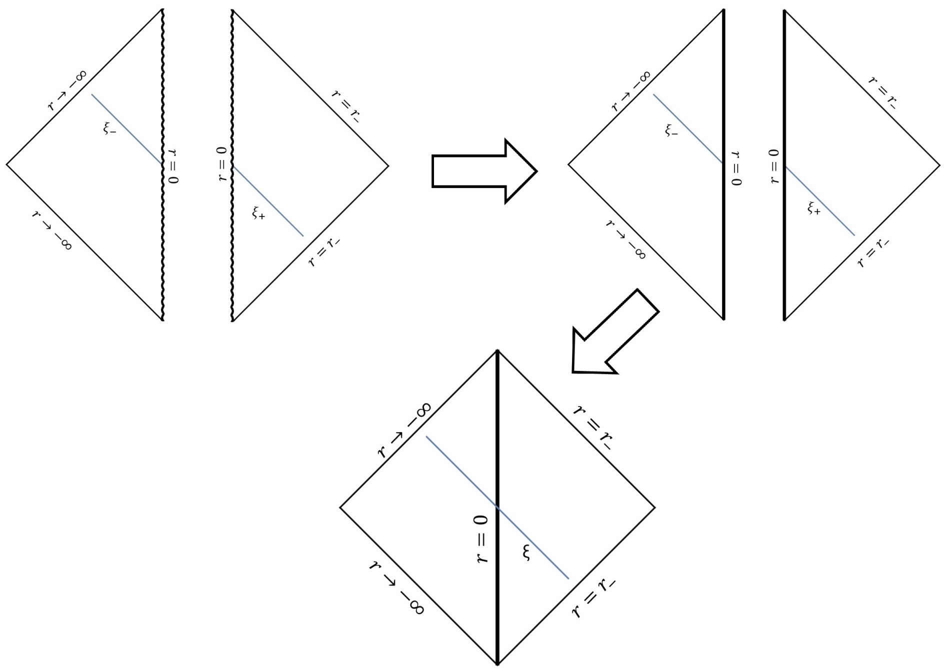

FIGURE 3.1: A very basic construction of a singular boundary that would allow initially incomplete geodesics to be continued after meeting the singularity in the Plebański-Carter space-times studied here ($\cos\theta = -l/a$).

Whether the previously described singular boundary may be allowed under any physical grounds, as well as the non-trivial geometrical and topological implications that it carries, is left for future analysis. If such a construction could be shown to be physically valid and mathematically well-posed, then causal geodesic completeness would be confirmed for a particular metric of the class of space-times studied in this section, namely the Kerr-Newman-(A)dS metric. This is due to the fact that its maximal analytic extension is known to include negative values of r, and also because it does not possess any other kind of singularity within its space-time manifold. Both of these properties are also shared with the Kerr-(A)dS black hole, but since $Q = 0$ for this case, there could be other incomplete causal geodesics (besides the principal null rays) as mentioned before. Unfortunately, in space-times with $l \neq 0$, e.g. Kerr-NUT, a conical singularity in the symmetry axis is formed which can provoke geodesic incompleteness. This falls beyond the scope of our results. In this cases, our analysis would at most tell that geodesics do not become incomplete inside a sufficiently small neighborhood of the ring singularity. For the sake of clarity, it should be remarked that these affirmations strictly depend on the process shown in figure 3.1, or something similar, being possible.

3.4.4 The Deviation of the Principal Null Rays

In this subsection we will be interested in the effects that unbounded curvature has over the two null geodesic congruences whose tangents are the principal null directions (3.38) of the space-time. We have seen by now that these curves are capable of encountering the singularity. Therefore, when approaching this problematic region, it is natural to expect that the deviation vector ξ^μ between infinitesimally close geodesics of any of these congruences must exhibit some kind of ill-behavior. In what follows, we will examine the geodesic deviation equation (2.17) for null geodesics with tangent vectors $k^\mu_\pm$, this is,

$$k^\alpha_\pm \nabla_\alpha (k^\beta_\pm \nabla_\beta \xi^\nu) = R^\nu{}_{\alpha\beta\mu} k^\alpha_\pm k^\beta_\pm \xi^\mu. \tag{3.39}$$

The analysis is most easily done when using some of the tools provided by the Newman-Penrose formalism of section 2.2. It is then necessary to introduce a complex null tetrad $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, where[9]

$$l^\mu = k^\mu_+, \quad n^\mu = \frac{\Delta_r}{2\Sigma} k^\mu_-, \quad m^\mu = \frac{1}{\sqrt{2\Delta_\theta}\,[r + i(l + a\cos\theta)]} \left[\frac{i\chi}{\sin\theta}, 0, \Delta_\theta, \frac{i}{\sin\theta} \right]. \tag{3.40}$$

Note that the l^μ and n^μ vectors are chosen to be aligned with the principal null directions. This tetrad yields the following set of non-vanishing spin coefficients (similar to those of (2.65)):

$$\begin{aligned} \rho =& \frac{1}{r - i(l + a\cos\theta)}, \quad \beta = -\frac{\rho^* P_\theta}{2\sqrt{2\Delta_\theta}\sin\theta}, \quad \pi = -ia\sqrt{\frac{\Delta_\theta}{2}}\rho^2 \sin\theta, \\ \tau =& ia\sqrt{\frac{\Delta_\theta}{2}}\rho\rho^* \sin\theta, \quad \mu = \frac{\rho^2\rho^*\Delta_r}{2}, \quad \gamma = \mu - \frac{\rho\rho^*}{4}\frac{d\Delta_r}{dr}, \quad \alpha = \pi - \beta^*, \end{aligned}$$

[9] In this chapter a signature $(-,+,+,+)$ is being used, contrary to section 2.2 where the signature with the signs inverted is utilized as convention. This modifies the sign of the orthogonal properties between the vectors of the tetrad as follows: $l^\mu n_\mu = -m^\mu \bar{m}_\mu = -1$. It also changes the sign in the definitions of the spin coefficients (2.30), the curvature quantities (2.33), and the Weyl scalars (2.34).

where $P_\theta = 2a\Lambda(a\cos\theta + 3l)\cos^2\theta/3 + (1 - a^2\Lambda/3)\cos\theta - 2al\Lambda/3$. Consider first the congruence of outgoing null geodesics whose tangent is k_+^μ, with the help of the derivative operator $D = l^\mu\nabla_\mu$ of the formalism, equation (3.39) can be compactly rewritten as

$$D^2\xi^\nu = R^\nu{}_{\alpha\beta\mu} l^\alpha l^\beta \xi^\mu. \tag{3.41}$$

The deviation vector ξ^μ can now be expanded in the basis of the null tetrad. However, some considerations need to be taken first. Orthogonality between l^μ and the deviation vector will be demanded since the non-orthogonal part of ξ^μ is not physically interesting, hence $\xi^\mu l_\mu = 0$. Also, two deviation vectors that differ only by a term in the direction of l^μ represent a displacement to the same geodesic [25], this term is then unimportant too. We may therefore expand ξ^μ as

$$\xi^\mu = Am^\mu + B\bar{m}^\mu, \tag{3.42}$$

with $B = A^*$ so that the deviation vector is real. For such a vector, it can be shown that the right-hand side of equation (3.41) admits the following expansion in the tetrad basis,

$$R^\nu{}_{\alpha\beta\mu} l^\alpha l^\beta \xi^\mu = [A(\Psi_1 + \Phi_{01}) + B(\Psi_1^* + \Phi_{10})]\, l^\nu - (A\Phi_{00} + B\Psi_0^*)m^\nu - (A\Psi_0 + B\Phi_{00})\bar{m}^\nu.$$

To find this past relation, the curvature quantities of the formalism are needed, along with the aid of the Weyl tensor expression (2.9). However, because (3.40) is an adapted null tetrad of a type D space-time, then $\Psi_0 = \Psi_1 = 0$. Furthermore, using the Ricci identities (2.35) it can be seen that Φ_{00} and $\Phi_{01} = \Phi_{10}^*$ also vanish. Thus,

$$D^2\xi^\mu = 0,$$

which means that there is no relative acceleration between neighboring geodesics. This is already a good indication that unbounded curvature does not negatively affect this congruence of curves. Notice how this was only possible due to $l^\mu = k_+^\mu$ being a twice repeated principal null directions.

The geodesic deviation equation may be solved now explicitly. To do this, the left-hand side of (3.41) needs to be expanded as well in the tetrad basis. This is accomplished by using the following specific expressions (and their complex conjugates) taken from the sometimes called propagation equations [96],

$$Dl^\mu = (\varepsilon + \varepsilon^*)l^\mu - \kappa^* m^\mu - \kappa\bar{m}^\mu, \quad Dm^\mu = (\varepsilon - \varepsilon^*)m^\mu + \pi^* l^\mu - \kappa n^\mu. \tag{3.43}$$

Applying then equations (3.43) to the deviation vector ξ^μ we obtain,

$$\begin{aligned} D\xi^\mu &= l^\mu(A\pi^* + B\pi) + m^\mu DA + \bar{m}^\mu DB, \\ D^2\xi^\mu &= l^\mu\left[AD\pi^* + BD\pi + 2\pi^* DA + 2\pi DB\right] + m^\mu D^2A + \bar{m}^\mu D^2B = 0. \end{aligned} \tag{3.44}$$

Note that since A and B cannot depend on the coordinates t and φ, the D operator acting on any of these coefficients reduces simply to $DA = \partial A/\partial r$ (similarly for B). The tetrad basis is linearly independent, hence the last two terms in (3.44) imply trivially that $A = A_1 r + A_0$ and $B = B_1 r + B_0$, where $A_{0,1}$ and $B_{0,1}$ are integration constants.

At this point, an additional constraint must be imposed on the A and B coefficients. The congruence of curves discussed here spans the submanifold M' obtained by fixing $\cos\theta = -l/a$ in the manifold M. At each point p of M', the tangent l^μ lies in the tangent subspace T_pM', and so must the deviation vector ξ^μ. To guarantee this, the θ component of ξ^μ needs to vanish, which is achieved by setting $A = -B$. Combining this condition with the previous restriction $A = B^*$, we have that A must be purely imaginary.

Returning to the deviation equation in (3.44), the quantities in squared brackets can be reduced using the explicit expression of the spin coefficients evaluated at $\cos\theta = -l/a$, along with the results previously obtained for A and B. This leads to a final restriction on the integration constants, namely, $A_0 = -B_0 = 0$. Therefore,

$$\xi^\mu = A_1 r(m^\mu - \bar{m}^\mu).$$

Explicitly, the deviation vector may be expressed in the $\{t, r, \theta, \varphi\}$ basis as

$$\xi^\mu = \xi_0 \left[(a+l)^2, 0, 0, a\right], \tag{3.45}$$

where we have defined $\xi_0 = iA_1\sqrt{2/C}$, which is a real quantity because A_1 is purely imaginary and $C = (1 - l^2\Lambda)(a^2 - l^2) > 0$. The squared norms of the deviation vector ξ^μ and relative velocity $v^\mu = D\xi^\mu$ are easily computed from (3.42) and (3.44), obtaining thus,

$$\xi^\mu \xi_\mu = C\xi_0^2 r^2, \quad v^\mu v_\mu = C\xi_0^2.$$

The vectors ξ^μ and v^μ are space-like, except for the deviation vector at $r = 0$, where it becomes null. These past quantities are also consistent with the fact that the relative acceleration $\mathcal{A}^\mu = D^2\xi^\mu$ vanishes.

An analogous procedure can be performed for the ingoing principal null rays, i.e., for the curves with tangent k^μ_-. A simple way to carry out this analysis is by making a slight change in the vectors of tetrad (3.40). Choose now

$$l^\mu = k^\mu_-, \quad n^\mu = \frac{\Delta_r}{2\Sigma} k^\mu_+,$$

and leave m^μ unmodified. Expression (3.44) for the geodesic deviation equation is still valid and the Newman-Penrose quantities appearing there (D and π) change only by a sign. This difference has no relevant effect on the end result and hence, the vector ξ^μ shown in (3.45) is a deviation vector for this congruence of curves too.

As it can be seen, ξ^μ shows surprisingly explicit regular behavior despite the fact of being the deviation vector of geodesics encountering the ring singularity. Even when closely approaching it, infinite curvature seems to have no deep effect in the deviation of these geodesics. This suggests, combined with the hypothetical construction of a singular boundary shown in figure 3.1, that it could be possible that the curvature singularity does not induce effects related to incompleteness on the two congruences of principal null geodesics. This, of course, provided such a procedure is physically meaningful.

3.5 A Geodesically Complete Wormhole Space-time

In the previous section we analyzed a physically relevant class of space-times and found that its singularity lies in the path of some null geodesics with specific constraints on the constants of motion. It might be natural now to wonder if a space-time whose singularity is inaccessible for both, null and time-like geodesics, does actually exist. In this spirit, we propose an example of such a case in what follows.

The line element is roughly based on the axially symmetrical wormhole space-times (specifically the so-called ring wormholes) found in [97, 98], but with the element g^{tt} of the inverse metric tensor modified so that the property $\psi|_q < 0$ holds for any conserved quantity p_i. Evidently, the construction of this metric is guided by geometrical arguments rather than physical significance. Hence, the gravitational source that could produce such a space-time geometry could not bear any physical relevance. We use spheroidal oblate coordinates x, y to express the corresponding line element,

$$\begin{aligned} ds^2 = &-\frac{\Delta^2}{\Delta_s}dt^2 + L^2\frac{\Delta}{\Delta_1}dx^2 + \frac{\Delta}{1-y^2}dy^2 - 2aLx(1-y^2)\frac{\Delta}{\Delta_s}dtd\varphi \\ &+ (1-y^2)(\Delta\Delta_1 - a^2)\frac{\Delta}{\Delta_s}d\varphi^2, \end{aligned} \tag{3.46}$$

where we have defined $\Delta = L^2(x^2+y^2)$, $\Delta_1 = L^2(x^2+1)$ and $\Delta_s = \Delta^2 - a^2y^2$. This set of coordinates is related to those of Boyer-Lindquist through $Lx = r - r_1$ and $y = \cos\theta$. Also, L is defined as $L^2 = r_0^2 - r_1^2$ with r_0 and r_1 being constant length parameters, and a a parameter with units of angular momentum. The inverse metric $g^{\mu\nu}$ has the more compact expression $g^{\mu\nu} = [\mathcal{X}^{\mu\nu}(x) + \mathcal{Y}^{\mu\nu}(y)]/[f(x)+h(y)]$. These functions are given by $f(x) = L^2x^2$ and $h(y) = L^2y^2$, while the tensors $\mathcal{X}^{\mu\nu}$ and $\mathcal{Y}^{\mu\nu}$ by

$$\begin{aligned} \mathcal{X}^{\mu\nu} &= \begin{bmatrix} -L^2x^2 + a^2/\Delta_1 & 0 & 0 & -aLx/\Delta_1 \\ 0 & \Delta_1/L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -aLx/\Delta_1 & 0 & 0 & -L^2/\Delta_1 \end{bmatrix}, \\ \mathcal{Y}^{\mu\nu} &= diag\left[-L^2y^2, 0, 1-y^2, \frac{1}{1-y^2}\right]. \end{aligned} \tag{3.47}$$

We now present the embedding profiles of the metric in three-dimensional Euclidean space (figure 3.2). These profiles show that the line element (3.46) has indeed a wormhole geometry whose throat is a disc of radius L located at $x = 0$. The throat connects two different universes (or possibly distant regions of the same universe), one with $x > 0$ and another with $x < 0$.

The singular regions of this metric can be revealed by computing the Ricci scalar, which is

$$R = \frac{-3a^2L^2(3a^2y^2 + \Delta^2)\left[y^2(1-y^2) + x^2(1+3y^2)\right]}{2\Delta^2\Delta_s^2}. \tag{3.48}$$

From (3.48) we notice that $y = 1$ is nothing but a coordinate singularity due to the choice of our spheroidal coordinates. The root $\Delta = 0$ corresponds to a ring singularity σ ($x = y = 0$) similar to that of the Plebański-Demiański class of space-times. While $\Delta_s = 0$ yields an additional singularity σ' with very interesting properties that shall be discussed later in this section. For the time being, we focus our attention on the ring singularity.

We proceed by calculating the quantity $\psi = \Delta(\kappa - g^{ij}p_ip_j)$ in $x = y = 0$, thus finding (in the notation of section 3.2, $\alpha = (a\mathcal{E}/L)^2 + \mathcal{L}^2$ and $\beta = -\mathcal{L}^2$):

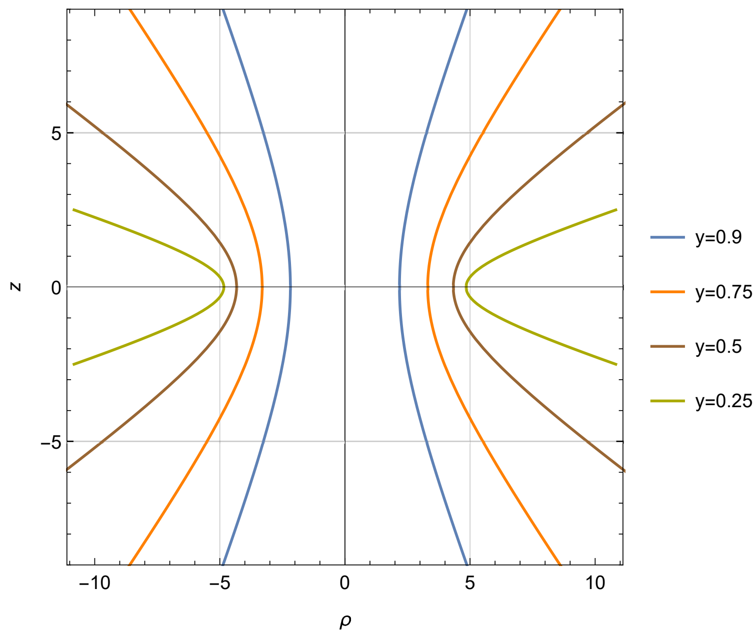

FIGURE 3.2: Embedding diagram of the wormhole in three-dimensional Euclidean space for different constant values of y with $a = 0.1$ and $L = 5$. Here, z and ρ are the usual cylindrical coordinates.

$$\psi|_q = -(a\mathcal{E}/L)^2. \tag{3.49}$$

For non-zero values of energy, any observer traveling in geodesic motion will be repelled from the ring singularity. Following the same procedure as in the Plebański-Demiański case, $\psi|_q$ can be set to zero, which implies $\mathcal{E} = 0$. Then, the derivatives $(\partial^n_A \psi)|_q$ for $A = 1, 2$ will determine if the ring singularity is accessible. We obtain for the lowest order non-vanishing derivative the following expression

$$(\partial^2_x \psi)|_q = \left(\partial^2_y \psi\right)\Big|_q = 2(L^2\kappa - \mathcal{L}^2). \tag{3.50}$$

This, yet again, is a negative quantity for time-like geodesics of arbitrary angular momentum $p_3 = \mathcal{L}$. For null geodesics with zero angular momentum, (3.50) vanishes. However, with said restriction, we have set all of the constants of motion to zero and hence have reduced the motion of the particle to a trivial case. That is, the particle remains at a constant set of coordinates, including time itself. Motion such as this can be considered as unphysical behavior. Thus, we can conclude that no causal geodesic can reach the ring singularity σ in this space-time. Table 3.3 summarizes these calculations, the special case $K = -\alpha = \beta$ becomes $\mathcal{E} = K - \mathcal{L}^2 = 0$.

At this point we should not forget about the remaining singularity σ' which occurs when $\Delta_s = 0$. This equation can be rearranged to a more familiar form $x^2 + (y \pm a/2L^2)^2 = a^2/4L^4$, i.e., σ' is described by two circles in the $x - y$ plane. We can write the singular curvature set for this singularity as $\sigma'_X = \sigma'_+ \cup \sigma'_-$, where

$$\begin{aligned}\sigma'_\pm = \{\, (t, v\cos\varphi, v\sin\varphi, u^3) \mid -\infty < t < \infty, 0 \leq \varphi < 2\pi, \\ x^2 + (y \pm a/2L^2)^2 = a^2/4L^4 \Big\}\,,\end{aligned} \tag{3.51}$$

TABLE 3.3: The first criterion for arbitrarily close approach to the ring singularity in wormhole (3.46).

		$\partial_A^n \psi(p_0, p_3)\rvert_q$	Conditions	Geodesics Approaching σ
$n=0$		$-(a\mathcal{E}/L)^2$	-	Only if $\mathcal{E}=K-\mathcal{L}^2=0$
$n=1$	$A=1$	0	$\mathcal{E}=K-\mathcal{L}^2=0$	-
	$A=2$	0		
$n=2$	$A=1$	$2(L^2\kappa-\mathcal{L}^2)$	$\mathcal{E}=K-\mathcal{L}^2=0$	Non-existent unless, $\mathcal{L}=\kappa=0$ (Trivial motion)
	$A=2$	$2(L^2\kappa-\mathcal{L}^2)$		

with $v = L\sqrt{(x^2+1)(1-y^2)}$ and $u^3 = Lxy$. Interestingly enough, this singular curvature set contains that of the ring singularity such that, $\sigma_X = \sigma'_+ \cap \sigma'_-$. By changing back to Boyer-Lindquist coordinates, and then to the Cartesian-like coordinates $\{u_1, u_2, u_3\}$, we can correctly visualize the shape of the singularity. Before doing so though, and since $y \in [-1, 1]$, one can realize by examining (3.51) that $a/L^2 = 1$ is a limiting case for the topology of $\sigma'_\pm$. For values $a/L^2 > 1$, we will obtain two closed line segments in the $x-y$ plane, rather than the previously described pair of circles that occur only when $a/L^2 \leq 1$. As a result, depending on the parameter a/L^2 the singular curvature set σ'_X can have different geometrical properties. Furthermore, taking into account the azimuthal symmetry of the metric, we have that

$$\sigma'_\pm \cong \begin{cases} S^1 \times S^1 \times \mathbb{R} & \text{if } a/L^2 \leq 1, \\ S^2 \times \mathbb{R} & \text{if } a/L^2 > 1. \end{cases} \tag{3.52}$$

In the first case of (3.52), the "spatial part" of the singular curvature set σ'_X is homeomorphic to two tori (one for each universe) which intersect at the ring singularity and at some other point of the throat. For the case $a/L^2 > 1$, this singularity consists of two deformed two-spheres that completely surround the throat of the wormhole, making it impossible for any test particle to cross it. This is shown in figure 3.3.

Unfortunately, some complications arise from the structure of this singularity. Its topology is no longer $\Sigma^1 \times \mathbb{R}$, instead it is $\Sigma^2 \times \mathbb{R}$, i.e., a two-manifold $\times$ "time". From the previous argumentation, we already know that causal geodesics cannot meet the ring singularity, but what about the rest of the points of σ'_X? Note that the size of the singularity depends on the unit-less space-time parameter a/L^2. So, for $a/L^2 \geq 1$ there will always exist points $q'(x_s, y_s) \in \sigma'_X$ far from the ring singularity where $\psi\rvert_{q'} \geq 0$. Therefore, by virtue of theorem 3.1, geodesic curves exist within every neighborhood of the singularity σ'.

Nevertheless, restricting the parameter to $a/L^2 \ll 1$ which corresponds to a slowly rotating wormhole, the region of the singularity σ' is shrunk to a small neighborhood of σ (see right panel of figure 3.3). That singularity has already been proven to be inaccessible to observers in geodesic motion. Additionally, it can be seen that the singular regions σ and σ' are the only possible source of affine incompleteness

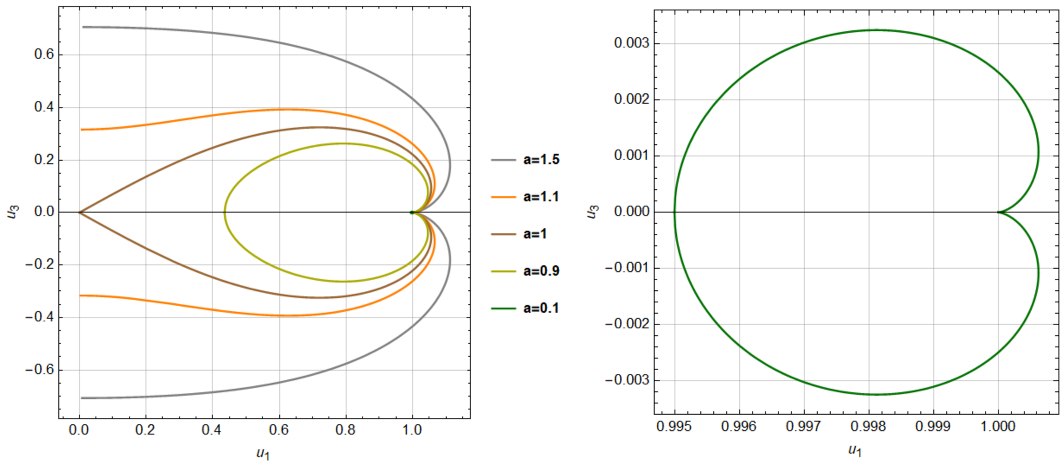

FIGURE 3.3: Cross section of the singularity σ' in the plane u_1-u_3 for different values of a/L^2, for simplicity we have made $L = 1$. Here, u_3 is the symmetry axis. In the right panel we show a particular case where $a = 0.1$, and hence $a/L^2 \ll 1$.

since, substituting the tensors (3.47) in (3.10), the equations of motion show no ill-behavior for other points of the space-time[10]. Thus, and because the coordinate system $\{t, x, y, \varphi\}$ covers completely both universes, metric (3.46) describes a geodesically complete space-time, both for null and time-like curves, only for parameter values $a/L^2 \ll 1$.

Despite the absence of an event horizon in this metric, the curvature singularities of the space-time cannot be observed by test particles in free-fall through the wormhole. As a passing note we point out that the Killing vector $X_0 = \partial/\partial t$ becomes space-like inside the compact hyper-surface defined by the singularity σ', while outside of it, is time-like as expected in an asymptotically flat space-time. In fact, this is due to the metric becoming Riemannian, i.e. of signature $(+, +, +, +)$, inside the region bounded by σ'. This can be seen from the determinant of the wormhole metric

$$\det(g) = -\frac{L^2\Delta^4}{\Delta_s},$$

which is manifestly discontinuous at the singularity σ'. In the outer region the determinant is negative, while in the inner region is positive. As a result, both of these regions of the space-time can be seen as disjointed from each other.

3.6 Two Rotating Scalar Field Wormholes

So far, we have examined cases of space-times with singular curvature and whose exact geodesics admit first integrals. This last condition was crucial for the analysis performed in sections 3.2 and 3.3. While there exist interesting metrics that fulfill this requirement (any Type D space-time, for example), it can be regarded as too much of a restrictive property. In this section we will deal with two wormholes that lack the mentioned first integrals, serving thus as examples of how the previous results can still give information for more general space-times.

[10]The divergence of the tensor $\mathcal{Y}^{\mu\nu}$ for $y = 1$ is a consequence of the spheroidal coordinates here used and can be eliminated through a suitable change of coordinate system, e.g. the Cartesian-like coordinates $\{u_1, u_2, u_3\}$ mentioned previously in this section.

Both wormholes belong to the class of Einstein-Maxwell scalar fields described in [23]. The general Lagrangian of these solutions is

$$\mathscr{L} = R - 2\varepsilon\nabla_{\mu}\Phi\nabla^{\mu}\Phi - e^{-2\alpha\Phi}F_{\mu\nu}F^{\mu\nu}, \tag{3.53}$$

where R is the Ricci scalar, $F_{\mu\nu}$ is the electromagnetic field tensor, and Φ the scalar field of a zero spin (composed) particle. Here, $\varepsilon = +1$ for a dilatonic field and $\varepsilon = -1$ for a phantom (or ghost) field. The Einstein-Maxwell-Dilaton field equations from Lagrangian (3.53) are

$$R_{\mu\nu} = 2\varepsilon\nabla_{\mu}\Phi\nabla_{\nu}\Phi + 2e^{-2\alpha\Phi}\left(F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\delta\gamma}F^{\delta\gamma}\right),$$

with coupling constant α. The individual features of the two solutions are the following:

1. **The Kerr-like phantom wormhole.** This is a solution consisting purely of a scalar field with a negative sign in its kinetic energy, therefore $F_{\mu\nu} = 0$ and $\varepsilon = -1$. Its line element in oblate spheroidal coordinates is

$$ds^2 = -f(dt + \Omega d\varphi)^2 + \frac{1}{f}\left[\Delta\left(\frac{L^2dx^2}{\Delta_1} + \frac{dy^2}{1-y^2}\right) + \Delta_1(1-y^2)d\varphi^2\right], \tag{3.54}$$

with $\Delta = L^2(x^2+y^2)$, $\Delta_1 = L^2(x^2+1)$, and

$$\Omega = \frac{ax(1-y^2)}{L(x^2+y^2)}, \qquad \lambda = \frac{(a^2+k_1^2)y}{2k_1L^2(x^2+y^2)}, \qquad f = \frac{(a^2+k_1^2)e^{\lambda}}{a^2+k_1^2e^{2\lambda}}.$$

The scalar field is given by $\Phi = \lambda/\sqrt{2}$. The parameters L, a, and k_1 are constants whose physical significance will be given later. The general form of the scalar curvature invariants is

$$R_X = \sum_{n=1}^{n<2\alpha} \frac{e^{n\lambda}F_n(x,y)}{(a^2+k_1^2e^{2\lambda})^{\alpha}(x^2+y^2)^{\beta}}, \tag{3.55}$$

where α and β are positive integers, and $F_n(x,y)$ are polynomials of degree less than the degree of $(x^2+y^2)^{\beta}$. For instance, for the Ricci scalar $\alpha = 1$, $\beta = 4$, and $F_1(x,y) = -(a^2+k_1^2)^3\left[y^2(1-y^2) + x^2(1+3y^2)\right]/8k_1^2L^6$. It is evident that curvature is ill-defined at the singular point $x = y = 0$. Nevertheless when approaching it from certain directions, for example from $y = 0$ such that $f = 1$, it will diverge. Whereas there exist other directions, e.g. a curve with $x = 0$, in which curvature is observed to be bounded (in fact, with a limit tending to zero in the singularity).

2. **The electromagnetic dipole wormhole.** The line element is again presented in oblate spheroidal coordinates as

$$ds^2 = -(dt + \Omega d\varphi)^2 + e^{K}\Delta\left(\frac{L^2dx^2}{\Delta_1} + \frac{dy^2}{1-y^2}\right) + \Delta_1(1-y^2)d\varphi^2, \tag{3.56}$$

where

$$K = \frac{k}{L^4} \frac{\left[1-y^2\right]\left[8x^2y^2(x^2+1)-(1-y^2)(x^2+y^2)^2\right]}{(x^2+y^2)^4}, \tag{3.57}$$

and the constant k is defined by

$$k = \frac{a^2}{8}\left(1-\frac{4\varepsilon}{\alpha^2}\right).$$

Interesting special cases for the coupling constant are $\alpha^2 = 1$, which represents a low-energy string theory, and $\alpha^2 = 3$, in which the Lagrangian (3.53) reduces to that of a 5D Kaluza-Klein theory. In Table 3.4 we show the values of the constant k in the previous cases for the dilatonic and ghost fields. It also contains the value of α^2 for which $k = 0$ (only the dilatonic field is possible).

TABLE 3.4: Real values of k for some cases of α^2 for both, dilatonic and ghost scalar fields.

	k	
α^2	Dilatonic Field ($\varepsilon = 1$)	Ghost Field ($\varepsilon = -1$)
1	$-3a^2/8$	$5a^2/8$
3	$-a^2/24$	$7a^2/24$
4	0	-

All of the distinguishing physical quantities characteristic of the class are non-trivial in this case. Solutions for the dilatonic field and the phantom field are both possible. For the scalar field Φ and the electromagnetic vector potential A_μ we have that,

$$\Phi = \frac{ay}{\alpha L^2(x^2+y^2)}, \quad A = -\frac{e^{\alpha\Phi}}{2}\left[(1-e^{-\alpha\Phi})dt + \Omega d\varphi\right].$$

The definitions of the quantities appearing in the line element and in the vector four-potential remain unchanged from the previous space-time. The electromagnetic field $F_{\mu\nu}$ is given by

$$F = \frac{aLe^{\alpha\Phi}}{\Delta^2}\left[2Lxydt\wedge dx + \frac{L^2}{\Delta}(1-y^2)\left(L^2[y^4-x^4]-2ax^2y\right)dx\wedge d\varphi \right.$$
$$\left. + L(y^2-x^2)dt\wedge dy + \frac{x}{\Delta}\left(aL^2[x^2-y^2][1-y^2]-2y\Delta\Delta_1\right)dy\wedge d\varphi\right].$$

It is, however, more illustrative to consider the asymptotically dominant components of this field tensor in an orthonormal frame $\{\hat{t},\hat{x},\hat{y},\hat{\varphi}\}$, namely,

$$F_{\hat{\mu}\hat{\nu}} = \frac{a}{L^3x^3}\begin{bmatrix} 0 & y & -\sqrt{1-y^2}/2 & 0 \\ -y & 0 & 0 & -\sqrt{1-y^2}/2 \\ \sqrt{1-y^2}/2 & 0 & 0 & -y \\ 0 & \sqrt{1-y^2}/2 & y & 0 \end{bmatrix} + \mathcal{O}\left(\frac{1}{x^4}\right).$$

Since asymptotically $Lx \sim r$ and $y = \cos\theta$, being r and θ regular spherical coordinates in flat space-time, then the form of this electromagnetic field can be immediately identified with those of an electric and magnetic dipole. Their electric and magnetic dipole moments, p and μ respectively, are $p = \mu = a/2$. Hence, in the following this metric will be referred to as the electromagnetic dipole wormhole, or electromagnetic wormhole for short.

The general form of the scalar curvature invariants is similar to the Kerr-like phantom wormhole:

$$R_X = \frac{e^{-\delta K} F(x,y)}{(x^2+y^2)^\beta}, \tag{3.58}$$

here again, δ and β are positive integers, and $F(x,y)$ a polynomial of degree less than the degree of $(x^2+y^2)^\beta$. Particularly, $\delta = 1$ for invariants of linear order in the curvature tensors, e.g., the Ricci scalar $R = R^\mu{}_\mu$, and $\delta = 2$ for quadratic invariants, e.g., $R^{\mu\nu}R_{\mu\nu}$. For the Ricci scalar, additionally, $\beta = 4$ and $F(x,y) = (a^2 - 8k)\left[y^2(1-y^2) + x^2(1+3y^2)\right]/2L^6$. Curvature is then also not well-defined at the point $x = y = 0$ and its limit depends too on the direction of approach. From (3.58) it can be seen that an observer will encounter an infinite or vanishing curvature depending on the sign that K takes on its path (details of this can be found in figures 3.11 and 3.12).

These two wormholes share some principal characteristics:

- The parameter $L > 0$ has units of length and is related to the size of the throat of the wormholes, while a and k_1 have units of angular momentum.
- Their mass m and angular momentum J, which are found by using Komar integrals evaluated on two-spheres of arbitrarily large radius, are $m = 0$ and $J = a$.
- The throat of the wormholes is located at $x = 0$.
- They possess a curvature ring singularity at $x = y = 0$ of radius L, which bounds the throat.
- They are asymptotically flat.

Further individual details of the Kerr-like phantom wormhole and the electromagnetic one can be found in [98, 99] and [97], respectively.

By computing the inverse metric tensor, it can be readily seen that the separable expression (3.2) does not hold for these space-times. More specifically, and in terms of the underlying geometric structure, an irreducible quadratic Killing tensor cannot be found for these cases. The question of whether higher order tensors of this kind may exist or not in these wormholes is left opened here. Nevertheless, by taking a physically meaningful limit on the metric parameters, we can obtain the desired separable form as an approximate description. This particular limit is that of a slowly rotating wormhole.

3.6.1 The Slowly Rotating Limit

In this subsection the slowly rotating limit will be applied to the wormholes (3.54) and (3.56). Mathematically, it is expressed as a condition on the metric parameters,

$a/L^2 \ll 1$. By keeping terms up to first order of a/L^2 we then find for the slowly rotating inverse metric tensor $g^{\mu\nu}_{SR}$ that

$$g^{\mu\nu}_{SR} = \frac{\mathcal{X}^{\mu\nu}(x) + \mathcal{Y}^{\mu\nu}(y)}{h_1(x) + h_2(y)}.$$

For each of the two wormholes of interest the explicit above expressions are,

1. **For the Kerr-like phantom wormhole:**

$$h_1(x) = L^2 x^2, \quad h_2(y) = L^2 y^2 + k_1 y/2,$$

$$\begin{aligned} \mathcal{X}^{\mu\nu} &= \begin{bmatrix} -L^2x^2 & 0 & 0 & -axL/\Delta_1 \\ 0 & \Delta_1/L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -axL/\Delta_1 & 0 & 0 & -L^2/\Delta_1 \end{bmatrix}, \\ \mathcal{Y}^{\mu\nu} &= diag\left[-(L^2y + k_1)y, 0, 1 - y^2, \frac{1}{1-y^2}\right]. \end{aligned} \tag{3.59}$$

To obtain this result we have also assumed a weak scalar field limit $k_1^2/L^2 \ll 1$.

2. **For the electromagnetic dipole wormhole:** $h_1(x) = L^2x^2, \quad h_2(y) = L^2y^2,$

$$\begin{aligned} \mathcal{X}^{\mu\nu} &= \begin{bmatrix} -L^2x^2 & 0 & 0 & -aLx/\Delta_1 \\ 0 & \Delta_1/L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -aLx/\Delta_1 & 0 & 0 & -L^2/\Delta_1 \end{bmatrix}, \\ \mathcal{Y}^{\mu\nu} &= diag\left[-L^2y^2, 0, 1 - y^2, \frac{1}{1-y^2}\right]. \end{aligned}$$

Notice that for both examples these quantities are almost identical. In fact, the only difference lies in the presence of an additional parameter k_1 in the Kerr-like phantom wormhole. Thus, for convenience, we will use equations (3.59) as a way of describing both slowly rotating wormholes. To obtain such version of the electromagnetic wormhole we simply set $k_1 = 0$ in (3.59). It should be remarked though, that this is a purely mathematical identification that only makes sense in the slowly rotating limit, it has no physical meaning. In other words, one does not obtain by any means the same physical properties of metric (3.56) by making $k_1 = 0$ in metric (3.54).

While the slowly rotating limit can be helpful, it must be kept in mind that it has a very important inherent restriction: it is not valid throughout the whole space-time. The limit is based on the approximations of the f and e^K functions as

$$\begin{aligned} f &\approx 1 + \frac{a^2 - k_1^2}{a^2 + k_1^2}\lambda + \mathcal{O}(\lambda^2) \approx 1 - \frac{k_1 y}{2L^2(x^2+y^2)} + \mathcal{O}\left(\frac{a^2}{L^4}, \frac{k_1^2}{L^4}\right), \\ e^K &\approx 1 + \mathcal{O}\left(\frac{a^2}{L^4}\right). \end{aligned}$$

The first order term in these approximations, as well as the higher order terms, depend on the coordinates x and y. Because of this, it is important to discuss in which regions of the space-time this approximation describes adequately the two original functions. The analysis is done separately for each case.

1. **The Kerr-like phantom wormhole.** To consider valid the series expansion for f the first step would consist on checking for convergence, for this purpose Cauchy's ratio test can be proven useful. However, given the functional form of f and as we are interested in a first order approximation, a more helpful criterion can be considered to be $|a_2/a_1| < 1$, where a_n is the n-th order term of the approximation. This is basically Cauchy's criterion with $n = 1$ and will determine whether the first order term suffices to approximate f. For this case at least, this condition is more restrictive than the general ratio test $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$, so convergence is guaranteed.

   The first and second order terms obtained by expanding f are

   $$a_1 = \frac{a^2 - k_1^2}{a^2 + k_1^2}\lambda, \quad a_2 = \frac{a^4 - 6a^2k_1^2 + k_1^4}{2(a^2 + k_1^2)^2}\lambda^2.$$

   So the above criterion becomes

   $$\left| \frac{2cy}{x^2 + y^2} \right| < 1, \tag{3.60}$$

   with $c = (a^4 - 6a^2k_1^2 + k_1^4)/8L^2(a^2 - k_1^2)k_1$ for $a \neq k_1$.

   Inequality (3.60) could be rearranged to the more familiar form $x^2 + (y \pm c)^2 > c^2$ for $x, y \neq 0$. For the ring singularity, and very close to it, the numerator in (3.60) dominates over the denominator and hence the inequality is not satisfied. Thus, by means of the defined criterion, the domain of validity of the slowly rotating limit is described in the x-y plane by the region outside of two circles with their centers located at $(0, \pm c)$ and radius c. See figure 3.4. Notice that as $k_1 \to a$, $c \to \infty$ since the first order term vanishes. The particular case $a = k_1$ would need a different analysis.

2. **The electromagnetic dipole wormhole.** To determine the region of validity of the slowly rotating approximation we introduce the criterion $|K| < 0.1$. This follows from the fact that $e^K \approx 1 + K$ to second order in a/L^2. The objective of this criterion is to indicate for which values of x and y the zeroth order term is the leading term and thus, the zeroth order approximation suffices. See figure 3.4.

Naturally, in both cases, as we get closer to the ring singularity the approximations begin to disagree with respect to the original functions. Nevertheless, it can be argued that the valid domain of the approximation is close enough to the singularity to correctly describe the nearby region.

From (3.9), (3.10) and (3.59), the separated equations of motion for the two slowly rotating wormholes can be written as,

$$\left[\Delta + \frac{k_1}{2}y\right]^2 \dot{x}^2 = X(x), \quad \left[\Delta + \frac{k_1}{2}y\right]^2 \dot{y}^2 = Y(y), \tag{3.61}$$

where

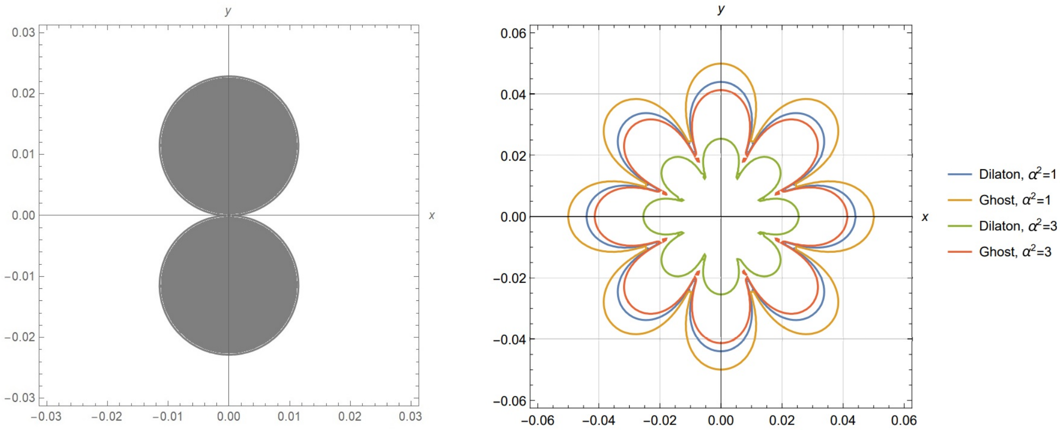

FIGURE 3.4: Validity of the slowly rotating limit in the x-y plane. In the left panel the case of the Kerr-like phantom wormhole is shown with $L = 5$, $a = 0.11$ and $k_1 = 0.1$, hence $c = 0.01141$. The right panel corresponds to different coupling constants α of the electromagnetic wormhole with values $a = 0.1$ and $L = 10$. Inside the area of the closed curves the slowly rotating approximation is no longer valid.

$$X(x) = \Delta_1 \left[(\kappa + \mathcal{E}^2) x^2 + \frac{\mathcal{K}}{L^2} \right] - \frac{2a\mathcal{E}\mathcal{L}x}{L} + \mathcal{L}^2, \tag{3.62}$$

$$Y(y) = [1 - y^2] \left[L^2(\kappa + \mathcal{E}^2) y^2 - \mathcal{K} + \frac{k_1}{2} \left(\frac{\kappa}{2} + \mathcal{E}^2 \right) y \right] - \mathcal{L}^2, \tag{3.63}$$

with $p_0 = -\mathcal{E}$ and $p_3 = \mathcal{L}$ (same notation as past section). Note that $X(0) = -Y(0) = \mathcal{K} + \mathcal{L}^2$. This could be a hint towards the possibility that, for arbitrary conserved quantities $\mathcal{K} \neq -\mathcal{L}^2$, geodesics would not be able to touch the ring singularity at $x = y = 0$. We should be careful, though, since at exactly this region the slowly rotating approximation is not valid. Indeed, this inconvenience prevents us from constructing the ψ quantity as defined by theorem 3.1 and then drawing conclusions solely from it, just as in the previous sections of this chapter. To obtain more information on the behavior of the polynomials (3.62) and (3.63), an analysis of their roots is needed.

In a fourth degree polynomial the nature of its roots is determined by the discriminants Δ_x, P_1 and P_2, which of course depend on the coefficients of the polynomial [100]. In the case of $X(x)$, these discriminants are given by:

$$\begin{aligned} \Delta_x =& 16A(16A^2B^3 - 8AB^2D^2 + 36ABDC^2 - 27AC^4 + BD^4 - C^2D^3), \\ P_1 =& 8AD, \qquad P_2 = 16A^2(4AB - D^2), \end{aligned} \tag{3.64}$$

with $A = L^2(\mathcal{E}^2 + \kappa)$, $B = \mathcal{K} + \mathcal{L}^2$, $C = a\mathcal{L}\mathcal{E}/L$ and $D = A + \mathcal{K}$.

Consider geodesics that cross the throat of the wormhole and freely travel through both universes. This implies that $X(x) > 0 \quad \forall x \in \mathbb{R}$, which is accomplished by demanding that $B = \mathcal{K} + \mathcal{L}^2 > 0$ and that (3.62) has four complex conjugate roots. To obtain such roots, the conditions on the discriminants (3.64) are: $\Delta_x > 0$ and either $P_1 > 0$ or $P_2 > 0$.

Thus, if $X(x) > 0$ for all $x \in \mathbb{R}$, then additionally to $B > 0$, the following conditions should be necessarily met:

1. $A > 0$ (Note that this is trivially satisfied for null geodesics).

2. Either $D > 0$ or $4AB > D^2$.

Now, for the polynomial $Y(y)$ its discriminants are somewhat more complicated than for $X(x)$:

$$P_1 = -8AD - 3E^2, \quad P_2 = 16A^2(4AB - D^2) - E^2(3E^2 + 16A\mathcal{K}), \tag{3.65}$$

with $E = (\kappa/2 + \mathcal{E}^2)k_1$. It is important to mention that here we omitted a discriminant Δ_y, mainly due to it being a large expression but also because independently of its sign, (3.63) can have real roots, which is what we desire for this polynomial. The reason for this requirement is that, in order for geodesics to cross the wormhole, there should be some values $y_0 \in [-1, 1]$ for which $Y(y_0) > 0$.

The conditions on the discriminants (3.65) for four real roots in $Y(y) = 0$ are: $P_1 < 0$ and $P_2 < 0$. So, if $X(x) > 0$ for all $x \in \mathbb{R}$ then $A > 0$, and it is sufficient to impose that: $D > 0$, $D^2 > 4AB$ and $\mathcal{K} > 0$ in order to fulfill said conditions on (3.65). See figure 3.5 for explicit examples.

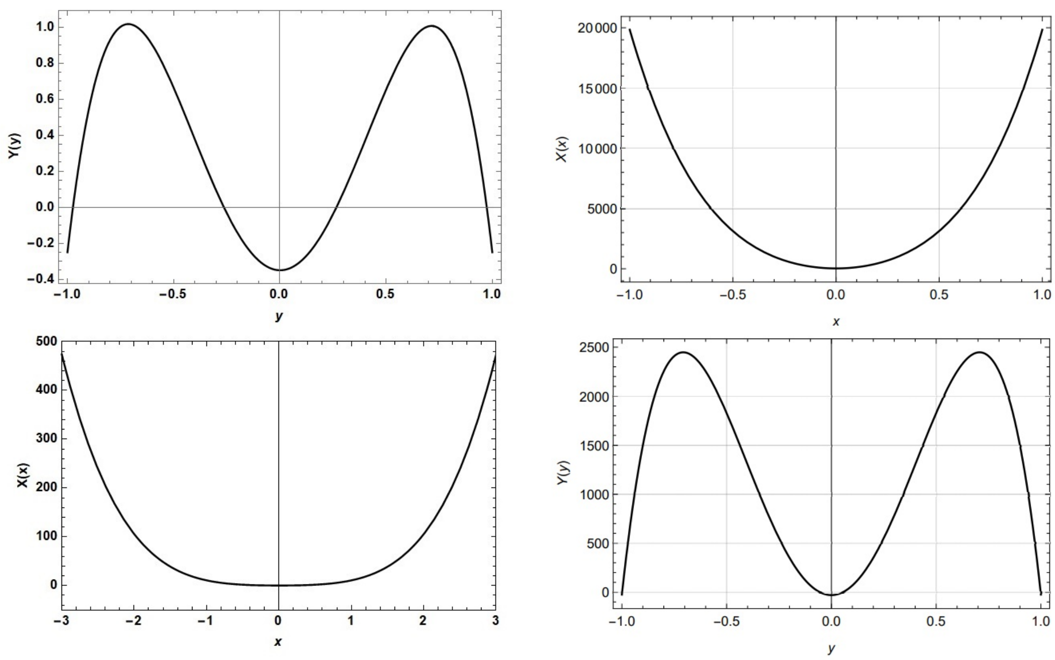

FIGURE 3.5: The polynomials $X(x)$ and $Y(y)$ for time-like geodesics. In the left panel (top and bottom) the functions of the Kerr-like phantom wormhole are presented with $\mathcal{L} = 0.5$, $\mathcal{E} = 1.1$, $k_1 = 0.1$, $a = 0.11$, $L = 5$ and $\mathcal{K} = 0.1$. The roots of $Y(y)$ are at: $y_1 = -0.272331$, $y_2 = 0.258018$, $y_3 = -0.973654$ and $y_4 = 0.974443$. The right panel (top and bottom) shows the case of the electromagnetic wormhole with $\mathcal{E} = 10$, $\mathcal{L} = 5$, $a = 0.1$, $L = 10$ and $\mathcal{K} = 5$. The roots of $Y(y)$ are at: $y_{1,2} = \pm 0.055$ and $y_{3,4} = \pm 0.998$.

At this point we may discuss the validity of our results. Even though the slowly rotating approximation breaks inside and very near the ring singularity, its valid domain is close enough from it to observe its repulsive effects, namely the negative behavior of the polynomial $Y(y)$. This can be seen explicitly in figure 3.6.

We can therefore conclude that, in the slowly rotating limit, geodesics that cross the wormhole and communicate both universes, are not able to touch the singularity

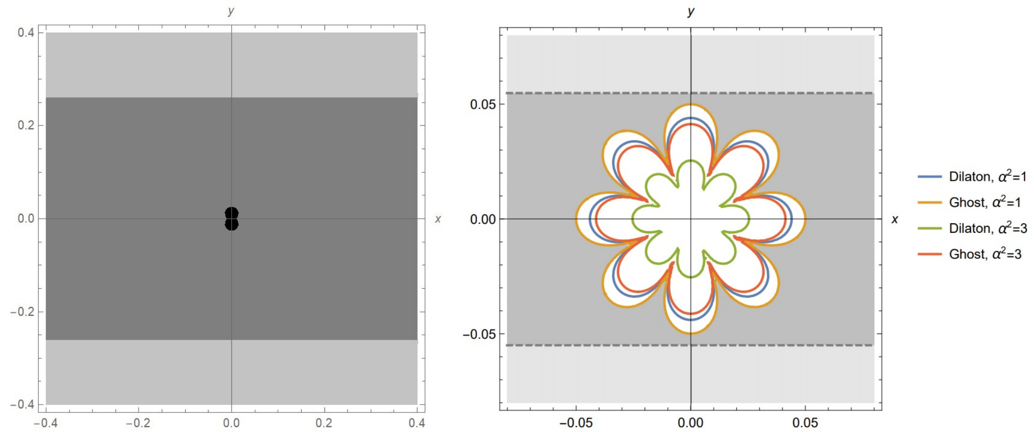

FIGURE 3.6: Main regions of interest in the x-y plane close to the ring singularity for the Kerr-like phantom wormhole (left panel) and the electromagnetic wormhole (right panel). In the light gray regions $Y(y) > 0$ and geodesics are allowed to move, while in the darker gray regions $Y(y) < 0$ and repulsive effects emerge. For both wormholes the values of the space-time parameters and constants of motion are the same as those in figure 3.5.

of the space-time. For this to be the case, the conditions on the conserved quantities studied above should be satisfied. Of course, the behavior of the curves (3.62) and (3.63) shown in figure 3.5 is not unique and will change depending on the four constants of motion through the discriminants (3.64) and (3.65). However, given the form of the coefficients of these polynomials is not possible to find curves where: $X(x) > 0$ for all $x \in \mathbb{R}$ and $Y(y) > 0$ for all $y \in [-1, 1]$.

Note that geodesics with the opposite condition assumed during the past analysis, i.e., $\mathcal{K} + \mathcal{L}^2 < 0$, experience the same repulsive behavior when approaching the singularity. The difference lies in the fact that these geodesics are unable to cross the throat, as they remain in their universe of origin.

3.6.2 Beyond the Validity of the Slowly Rotating Limit

The region of validity of the slowly rotating limit allowed us to determine the behavior of a set of geodesics near the ring singularity, specifically those for which $\mathcal{K} + \mathcal{L}^2 > 0$ or $\mathcal{K} + \mathcal{L}^2 < 0$. Unfortunately, the special case $\mathcal{K} + \mathcal{L}^2 = 0$ cannot be properly described by a similar procedure. This case actually corresponds to that considered in section 3.2 ($\mathcal{K} = -\alpha = \beta$). The reason behind this inconvenience is that, with said condition and according to the equations of motion (3.61), the negative potential barrier disappears and consequently, curves can continue further into the singularity. Eventually the slowly rotating limit will breakdown, and the separable equations (3.61) will no longer give reliable information about the motion of test particles.

The Hamiltonian $2\mathcal{H} = \kappa$, from which separability in the slowly rotating limit was deduced earlier, keeps holding without the need of approximations. It can be rearranged in the following convenient form,

$$\frac{e^K \Delta}{f}\left(\frac{L^2 \dot{x}^2}{\Delta_1} + \frac{\dot{y}^2}{1-y^2}\right) = \kappa + \frac{\mathcal{E}^2}{f} - \frac{f(\Omega\mathcal{E} + \mathcal{L})^2}{\Delta_1(1-y^2)}. \tag{3.66}$$

In addition to the previously defined metric parameters for each individual wormhole, set $K = 0$ to obtain the Kerr-like phantom wormhole in (3.66), and set $f = 1$ to obtain the electromagnetic one. This equation, though, is not enough to determine the path geodesics will follow in a general case. The only resource left then is to consider the geodesic equations for each wormhole, which are of course valid everywhere but extremely difficult (if not impossible) to study in full detail analytically. However, some special and interesting cases can be considered with the help of (3.66).

- **Motion in the equatorial plane outside the throat.** In order for geodesics to be constrained to the plane $y = 0$, we also need that $\dot{y}$ and $\ddot{y}$ both vanish. The first condition can only be achieved in $y = 0$ if $\mathcal{K} + \mathcal{L}^2 = 0$. Then, when making $y = \dot{y} = 0$, the $\ddot{y}$ component of the geodesic equation yields

$$\ddot{y} + \frac{a^2 - k_1^2}{4k_1 L^4 x^4}\left(2\mathcal{E}^2 + \kappa\right) = 0 \quad \text{and} \quad \ddot{y} = 0,$$

respectively, for the Kerr-like phantom and the electromagnetic wormholes. Thus, equatorial geodesics are not possible for the Kerr-like case unless $a = k_1$, unlike the electromagnetic dipole wormhole where they do exist and need not require any special condition on the constants of motion. Keep in mind that the $2\mathcal{E}^2 + \kappa$ factor cannot be made to vanish, at least for geodesics that reach asymptotic infinity, because $\mathcal{E} \geq 1$ for time-like observers and $\mathcal{E} \neq 0$ for light.

Instead of writing the $\ddot{x}$ component of the geodesic equation, we can use (3.66) with $y = \dot{y} = 0$ to express $\dot{x}$ as

$$e^{K_x}\frac{L^4 x^2 \dot{x}^2}{\Delta_1} = \kappa + \mathcal{E}^2 - \frac{1}{\Delta_1}\left(\frac{a\mathcal{E}}{Lx} + \mathcal{L}\right)^2, \tag{3.67}$$

with $K_x = K|_{y=0}$. Note that the left hand side of the previous equation is positive. However, as $x \to 0$, the term in round brackets of the right hand side dominates over the others. This is a negative term, thus implying a contradiction since clearly, a positive quantity cannot be equal to a negative one. By this argument, it can be established that there are no solutions of (3.67) that reach the ring singularity.

- **Motion within the throat.** Geodesics that are constrained to the throat of the wormhole satisfy $x = \dot{x} = \ddot{x} = 0$. Again, if $\mathcal{K} + \mathcal{L}^2 = 0$, then $\dot{x} = 0$ in the throat. From the $\ddot{x}$ component of the geodesic equation with vanishing x and $\dot{x}$ we have that,

$$\ddot{x} - f_y^2 \frac{a\mathcal{E}\mathcal{L}}{L^5 y^4} = 0 \quad \text{and} \quad \ddot{x} - e^{-K_y}\frac{a\mathcal{E}\mathcal{L}}{L^5 y^4} = 0,$$

respectively, for the Kerr-like phantom and the electromagnetic wormholes. Here, $f_y = f|_{x=0}$ and $K_y = K|_{x=0}$. It can be seen that this type of motion is then only possible if any of the conserved quantities $\mathcal{E}$ or $\mathcal{L}$ is zero.

Applying the same analysis as before, we utilize (3.66) now with $x = \dot{x} = 0$, obtaining thus,

$$\dot{y}^2 = \frac{e^{-K_y}}{L^4 y^2}\left[L^2(1 - y^2)(\mathcal{E}^2 + \kappa f_y) - f_y^2 \mathcal{L}^2\right]. \tag{3.68}$$

In what follows we will consider that $\mathcal{E} \neq 0$ and $\mathcal{L} = 0$, due to the fact that this corresponds to a familiar and physically realistic case, in contrast to the other possibility of a test particle with vanishing energy. With the help of equation (3.68) it can now be determined if geodesics that lie in the throat can get arbitrarily close to the singularity for each wormhole.

In the Kerr-like phantom wormhole ($K_y = 0$), $\dot{y}^2 \to \infty$ as $y \to 0$. Hence, these geodesics can infinitesimally approach the singularity, possibly becoming incomplete. In fact, equation (3.68) can be easily integrated for the particular case of null geodesics with vanishing angular momentum in this wormhole. The solution simply reads

$$y = \pm\sqrt{1 - \lambda'^2}, \tag{3.69}$$

where the affine parameter λ' was rescaled though a linear transformation of the original parameter λ for convenience. When $\lambda' = 1$ the singularity is clearly reached by these curves, which are only defined[11] for $\lambda' \in [-1, 1]$.

A similar behavior in every neighborhood of the singularity is replicated in the electromagnetic dipole wormhole ($f_y = 1$) with $k \geq 0$. On the other hand, those with $k < 0$ behave in the opposite way, as $y \to 0$ we have that $\dot{y}^2 \to 0$. In fact, $\ddot{y} \to 0$ as $y \to 0$ too. This can be seen from the $\ddot{y}$ component of the geodesic equation of these particular curves:

$$\ddot{y} + \frac{\left[L^4 y^4 + 2k(1 - y^2)^2\right] \dot{y}^2}{L^4 (1 - y^2) y^5} = 0.$$

The combination of these kinematic conditions indicates that geodesics constrained to the throat and that start their path at some value $y = y_0$ will slow down when approaching the singularity. As they go closer, their coordinate velocity $\dot{y}$ will become smaller, almost completely decreasing to zero. As a result, they will never reach the ring singularity in a finite amount of their affine parameter, in other words, an infinite affine parameter is needed so that they can meet the singularity.

This past conclusion can also be obtained from the following approximate solution of equation (3.68) for $y \ll 1$,

$$\lambda' = \pm \int y e^{-k/2L^4 y^4} dy, \tag{3.70}$$

here the original affine parameter λ was properly rescaled again for simplicity. It is readily seen that if $k \geq 0$, the integrand vanishes as $y \to 0$, leading to a finite affine parameter at which the singularity is met. On the contrary, with $k < 0$ the integrand becomes infinite at $y = 0$ and consequently, so does

[11] An important remark should be done here. If one considers geodesics in Minkowski space-time with the same conditions as above ($\mathcal{L} = \kappa = 0$), an identical expression for $y(\lambda')$ is obtained. A similar situation occurs when considering the same type of solutions of (3.67) in flat space-time ($a = K_x = 0$), i.e., $x = \pm\sqrt{\lambda'^2 - 1}$. Both of these results could suggest incompleteness. Flat space-time, though, is known for being geodesically complete. Hence, this particular expressions are merely an indication that a space-time extension must be performed in order to continue the path of these geodesics when they arrive to the problematic points. Put another way, this is just a consequence of using an inconvenient choice of coordinates for describing flat space-time. Whether for the case of the Kerr-like wormhole an extension could be possible or not, shall be discussed very briefly later in this subsection.

λ' when trying to reach the singular region. It should be remarked that, of the interesting cases for the coupling constant α shown in table 3.4, those that have a negative k correspond to a dilatonic field, whereas those with positive k represent a ghost field.

The two previous simple types of motion have already given valuable information. The most important is that equatorial geodesics, if any, are not in contact with the ring singularity. On the other hand, in some of the wormholes (Kerr-like phantom and electromagnetic with $k \geq 0$), those that lie within the throat are able to encounter it in a finite affine parameter. Indeed, equations (3.69) and (3.70) indicate that this is the case for geodesics with zero angular momentum. This rules out said wormholes as space-times possessing a curvature singularity without geodesics touching it. In fact, only the electromagnetic dipole wormhole metric with negative k stands now as a possible candidate of such a space-time.

There are of course more general geodesics other than those constrained to the equatorial plane or the throat. However, due to the complicated expressions of the geodesic equation for these wormholes, it is not possible to analytically solve these curves. Nevertheless, we can extend the study of equation (3.66) to less restrictive curves and try to obtain some insight on their behavior. This equation can be rearranged as,

$$e^K \Delta^3 \left[L^2(1-y^2)\dot{x}^2 + \Delta_1 \dot{y}^2\right] = \mathcal{P}(x,y), \tag{3.71}$$

with

$$\mathcal{P}(x,y) = (\mathcal{E}^2 + \kappa f)\Delta^2 \Delta_1 (1-y^2) - f^2 \left[aL\mathcal{E}x(1-y^2) + \mathcal{L}\Delta\right]^2.$$

Notice yet again that the left hand side of (3.71) is strictly positive. Therefore, any values $x \in \mathbb{R}$ and $y \in [-1,1]$ such that $\mathcal{P}(x,y) < 0$ will constitute a forbidden region for geodesics in the space-time. Because $\mathcal{P}(x,y)$ is everywhere well-defined and continuous, except at $x = y = 0$, this function has to vanish for non-trivial x_s and y_s in order to change from positive to negative values. Thus, we search for the hyper-surfaces $\mathcal{S}(\mathcal{E},\kappa,\mathcal{L})$ defined by $\mathcal{P}(x,y) = 0$ for coordinate values $\{x,y\} \in \mathbb{R} \times [-1,1] \setminus \{0,0\}$ and for given constants of motion $\mathcal{E}$, κ and $\mathcal{L}$. The roots of $\mathcal{P}(x,y)$ do not admit an explicit expression due to it being non-linear, especially because of the f function. We can consider first the approximation of the simple case $f = 1$ (the electromagnetic wormhole, for example) with the additional assumption that $|x|, |y| \ll 1$, which reduces $\mathcal{P}(x,y)$ to

$$\mathcal{P}(x,y) \approx \Delta^2 L^2(\mathcal{E}^2 + \kappa) - (aL\mathcal{E}x + \mathcal{L}\Delta)^2.$$

The roots of this equation can then be written in a quadratic form as

$$(x - C_{1,2})^2 + y^2 = C_{1,2}^2$$

with $C_{1,2} = \pm a\mathcal{E}/2L(L\sqrt{\mathcal{E}^2+\kappa} \mp \mathcal{L})$. Thus, for $L^2(\mathcal{E}^2+\kappa) \neq \mathcal{L}^2$, the hyper-surfaces $\mathcal{S}(\mathcal{E},\kappa,\mathcal{L})$ are defined by two circles in the x-y plane of radius $C_{1,2}$, which intersect at $x = y = 0$, and whose centers lie in the x axis. It should be mentioned that strictly speaking, the point $x = y = 0$ does not belong to $\mathcal{S}$ as it was previously discarded from the analysis. It might now be helpful to focus on angular momentum $\mathcal{L}$ such that $L^2(\mathcal{E}^2+\kappa) > \mathcal{L}^2$. The reason for this is that the only incoming curves from infinity with the opposite inequality, combined with $\mathcal{K} + \mathcal{L}^2 = 0$, are those that travel in the $y = 0$ plane. Motion such as this has already been studied. This follows from $Y(y)$ in (3.61) being negative everywhere, except at $y = 0$, with said conditions.

If $L^2(\mathcal{E}^2+\kappa) > \mathcal{L}^2$, both quantities $C_{1,2}$ are always well-defined and $C_1 > 0$, $C_2 < 0$, or vice versa. It can be readily verified that for any point x_{in} and y_{in} inside any of the circles (e.g., $x_{in} = C_1$, $y_{in} = 0$, and $x_{in} = C_2$, $y_{in} = 0$), we have that $\mathcal{P}(x_{in}, y_{in}) < 0$. Meanwhile, for points x_{out} and y_{out} outside both circles, $\mathcal{P}(x_{out}, y_{out}) > 0$. Geodesics are repelled then from the inside region of $\mathcal{S}(\mathcal{E}, \kappa, \mathcal{L})$. The existence of such a region can be mathematically related to the rotation of the wormhole exclusively since the radii of the circles will vanish, hence disappearing $\mathcal{S}$, if and only if $a = 0$.

Figure 3.7 illustrates some examples of the hyper-surfaces $\mathcal{S}(\mathcal{E}, \kappa, \mathcal{L})$ in the x-y plane using the exact expression for $\mathcal{P}(x, y)$. It can be seen that the approximation we made yields a reliable result in comparison to the complete function. Also, for the Kerr-like phantom wormhole, $\mathcal{S}$ appears as two deformed circles. In both wormholes and for a given set of constants of motion, $\mathcal{S}(\mathcal{E}, \kappa, \mathcal{L})$ is unique, at least near the curvature singularity.

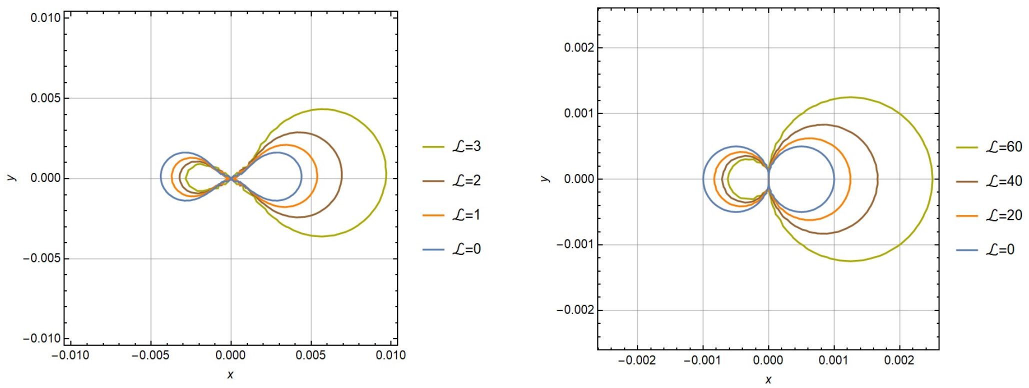

FIGURE 3.7: The hyper-surfaces $\mathcal{P}(x, y) = 0$ in the x-y plane for different values of $\mathcal{L}$ ($L^2(\mathcal{E}^2+\kappa) > \mathcal{L}^2$). The left panel corresponds to the Kerr-like phantom wormhole with $L = 5$, $a = 0.11$, $k_1 = 0.1$, $\mathcal{E} = 1.1$ and $\kappa = 0$. In the right panel the contours of the electromagnetic wormhole with $a = 0.1$, $L = 10$, $\mathcal{E} = 10$ and $\kappa = 0$ are shown. Inside the area of any of the circles, $\mathcal{P}(x, y)$ is negative.

For the sake of completeness, the cases $L^2(\mathcal{E}^2+\kappa) = \mathcal{L}^2$ and $L^2(\mathcal{E}^2+\kappa) < \mathcal{L}^2$ will be roughly described. If $L^2(\mathcal{E}^2+\kappa) = \mathcal{L}^2$, then only one circle is well-defined and the other approximate solution is $x = 0$. Unlike the first case, here, there are more roots of $\mathcal{P}(x, y)$ outside the approximation $|x|, |y| \ll 1$ that generate other hyper-surfaces. On the other hand, if $L^2(\mathcal{E}^2+\kappa) < \mathcal{L}^2$, then $C_{1,2} > 0$ or $C_{1,2} < 0$, and as a consequence, the two circles overlap. Only in the area inside the bigger circle and outside the smaller one, the function $\mathcal{P}(x, y)$ is positive. Hence, geodesics with these characteristics are bounded to this region, without possibility of escaping to infinity. The corresponding plots are shown in figures 3.8 and 3.9.

From figures 3.7 to 3.9 we observe that in all cases there are always forbidden regions near the curvature singularity, unfortunately these regions do not completely surround it. In this sense, the singularity is partially protected by a potential barrier that prevents geodesics from reaching it when coming from almost any direction, except for those that approach closely from inside the throat (close to the line defined by $x = 0$ in the x-y plane). This is consistent with the two simple examples of motion examined earlier during this subsection.

Since according to the previous results, there exists the possibility that geodesics can be found arbitrarily close to the ring singularity by traveling near the disc bounded

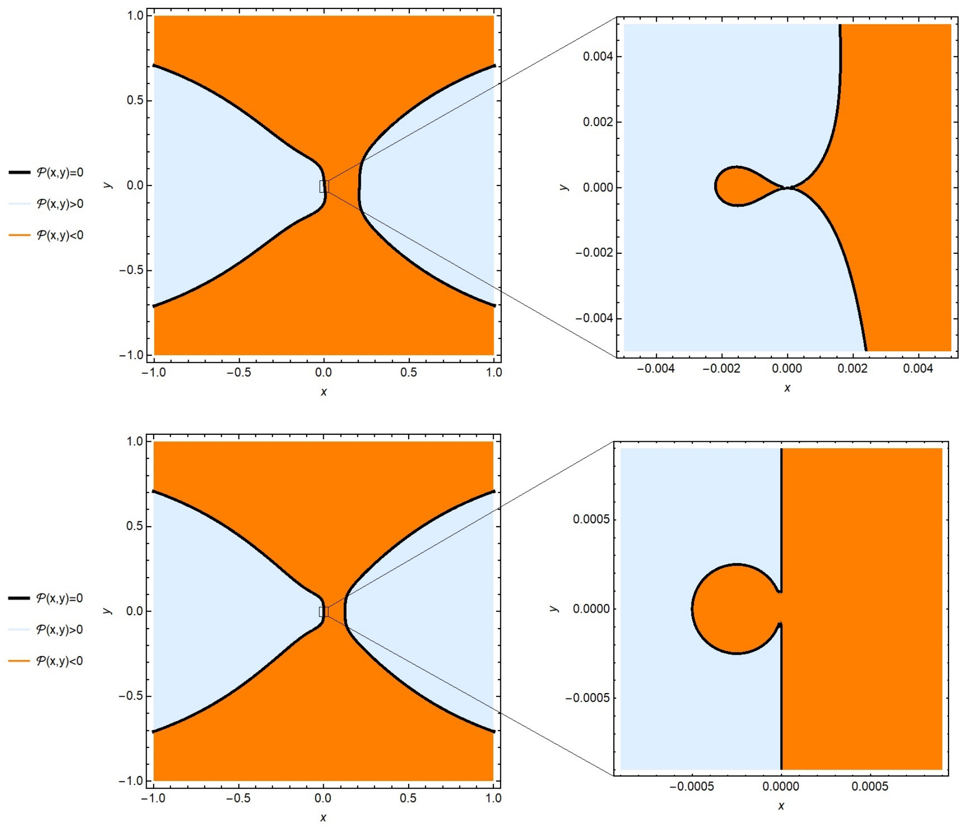

FIGURE 3.8: The hyper-surfaces $\mathcal{P}(x,y) = 0$ for the case $L^2(\mathcal{E}^2 + \kappa) = \mathcal{L}^2$. Same numerical values as in figure 3.7 are used. The Kerr-like phantom wormhole is shown above ($\mathcal{L} = 5.5$), and the electromagnetic space-time below ($\mathcal{L} = 100$).

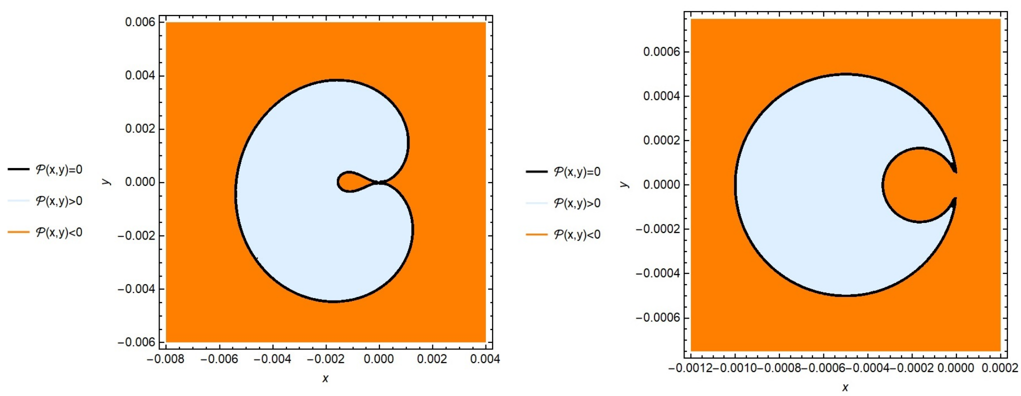

FIGURE 3.9: A hyper-surface $\mathcal{P}(x,y) = 0$ for the case $L^2(\mathcal{E}^2 + \kappa) < \mathcal{L}^2$. Same numerical values as in figure 3.7 are used. The Kerr-like phantom wormhole is shown in the left panel with $\mathcal{L} = 10$, while the electromagnetic space-time in the right panel with $\mathcal{L} = 200$.

by it, we need to determine if there are general geodesics (others than those of constant $x = 0$) that follow this path. The only resource left is to study them numerically with the inherent and unfortunate restrictions of these methods.

The procedure we shall follow is now described. We are interested in geodesics for which $\mathcal{K} + \mathcal{L}^2 = 0$ in the slowly rotating limit. Therefore, initial conditions for the coordinates x and y are chosen in a region where this limit is valid. In turn, for given values of the constants of motion $\mathcal{E}$, $\mathcal{L}$ and $\kappa = 0, -1$, as well as the corresponding space-time parameters, the initial position x_0 and y_0 will fix the initial velocities $\dot{x}_0$ and $\dot{y}_0$ according to equations (3.61). An additional constraint will be imposed on the angular momentum $\mathcal{L}$ so that $L^2(\mathcal{E}^2 + \kappa) > \mathcal{L}^2$, which as previously explained, is a necessary condition for unbounded curves outside the $y = 0$ plane. Finally, as is evident, our principal objective is to examine geodesics that advance towards the singularity as their affine parameter increases. With this set of initial data and conditions the numerical calculation of geodesics is performed. Several results are shown in the following figures.

For the Kerr-like phantom wormhole (figure 3.10), curves with $\mathcal{L} \neq 0$ approach initially the singularity but are eventually repelled, turning back to infinity. This repulsive effect is more significant as angular momentum increases. The fact that effectively, the curves with $\mathcal{L} = 0$ do encounter the singularity, is consistent with the previous statement. Nevertheless, by looking closely into their path of approach, it can be seen that these geodesics first get near the vertical line $x = 0$, and from there, they move close within said line into the problematic point $x = y = 0$. This is in whole agreement with the results regarding the hyper-surfaces $\mathcal{S}(\mathcal{E}, \kappa, \mathcal{L})$ just discussed. One curve that does not touch the singularity is that which starts at the point $x = -0.1$ and $y = 0$ with vanishing $\dot{y}$ (recall that for this space-time motion constrained to $y = 0$ is not possible). The rest of them seem to be able to reach the singular ring if they have zero angular momentum. To make matter worse, reversing their affine parameter, these curves can be seen to come from past infinity, making the singularity visible to asymptotic observers. This would be more than enough to deem a space-time as ill-behaved.

Despite there being singularity encountering for geodesics with vanishing $\mathcal{L}$, there can still be a way to consider this fact somewhat acceptable. According to the general form of the scalar invariants (3.55), these curves observe a vanishing curvature when reaching this region. They do however seem to become incomplete, which is troublesome. A clear example of this is provided by solution (3.69). This leaves open the possibility of this singularity having a quasi-regular structure along these geodesics[12]. The question now is if in this wormhole this apparent incompleteness is due to a bad choice of coordinates (as explained in footnote 11 for Minkowski space-time) or due to problematic curvature. This is a rather complex problem that will not be treated here. As mere speculation we put forward the idea that, since curvature does not diverge in this particular direction, a space-time extension can be done in order to continue the path of the aforementioned geodesics. In fact, if we are dealing with incomplete curves in a quasi-regular curvature singularity, a local extension of those curves has been shown to exist [101]. Thereby, after striking the singularity they can perhaps remain within the throat or follow a path into a non-forbidden region outside of $\mathcal{S}(\mathcal{E}, \kappa, 0)$. This assertion of course needs further study if one wishes to consider it as true. If such a extension exists and is physically reasonable, then this space-time may not be considered as geodesically incomplete (at

[12] In a singularity with a quasi-regular structure, the components of the Riemann tensor in parallel-propagated frames and its scalar invariants converge to finite values along an incomplete curve [48].

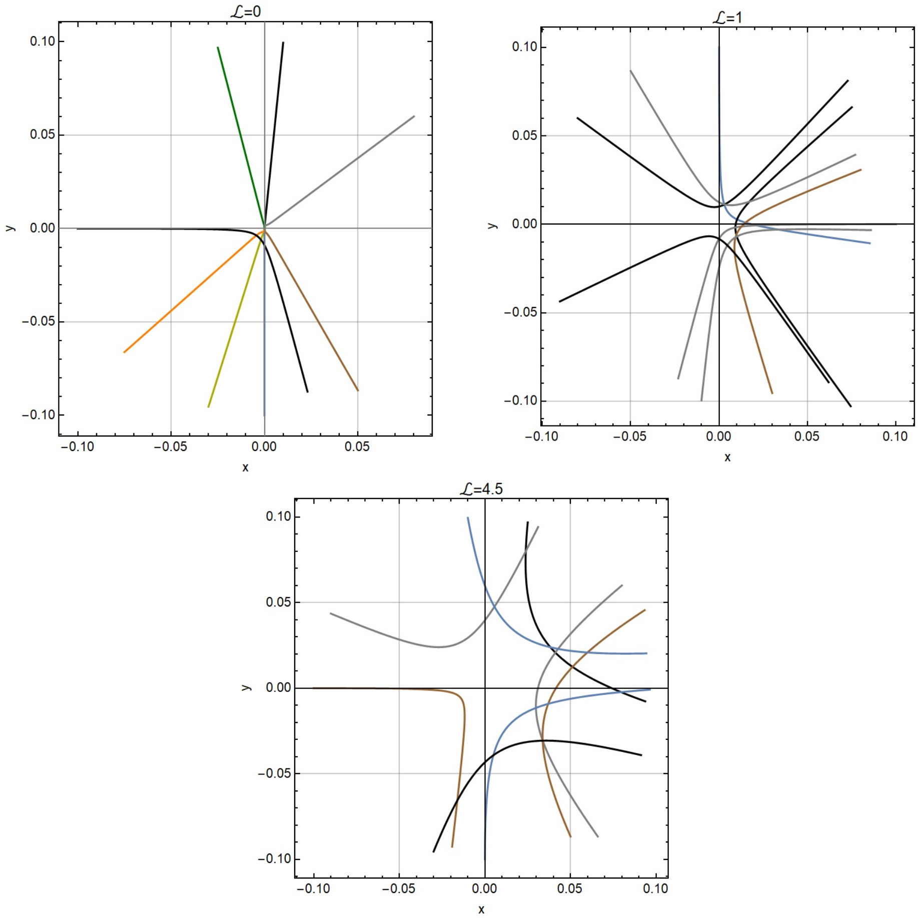

FIGURE 3.10: Null geodesics in the Kerr-like phantom wormhole for three values of angular momentum $\mathcal{L}$. The numerical values of the space-time parameters and constants of motions are the same as those of previous figures. Different colors are used to distinguish between intersecting curves.

least for causal geodesics).

The geodesics of the electromagnetic dipole wormhole are now examined. As discussed previously in the analysis of the two simple types of motion, we can expect different behaviors in this wormhole depending on the sign of the constant k of equation (3.57). We thus start with the case of a positive k (figure 3.11). An interesting property that seems to dominate the path taken by geodesics is whether K is positive or negative in the e^{-K} factor appearing, for example, in equations (3.58) and (3.68). Curves whose initial position is in the $K > 0$ area tend to stay within it, and curves whose initial position is in the $K < 0$ area tend to cross their starting region into the first area. Furthermore there are four points in the x-y plane, one for each $K > 0$ zone, at which all curves converge to. The only found exceptions to this behavior were already described, i.e., motion constrained to $y = 0$ or $x = 0$ (not shown in figure 3.11). It must be mentioned that although it appears that geodesics reach one of these convergence points and then stop there, it is not quite exactly what happens. Instead, their velocities and accelerations become increasingly small as they advance, most likely as a consequence of the e^{-K} factor severely decreasing as well. This is not at all unfamiliar since we found earlier that some geodesics in the throat of the wormhole with $k < 0$ exhibit this type of behavior. Other than geodesics constrained to the throat, which can exist arbitrarily close to the singularity, no other curves were found that could touch it in a finite amount of affine parameter. Due to the fact that geodesics in the throat become incomplete, we should definitely discard this kind of wormhole as being regular. One must keep in mind, though, that the problematic geodesics are bounded by the ring, they do not escape to infinity. In this sense, one can think of this space-time as not so badly behaved for distant observers.

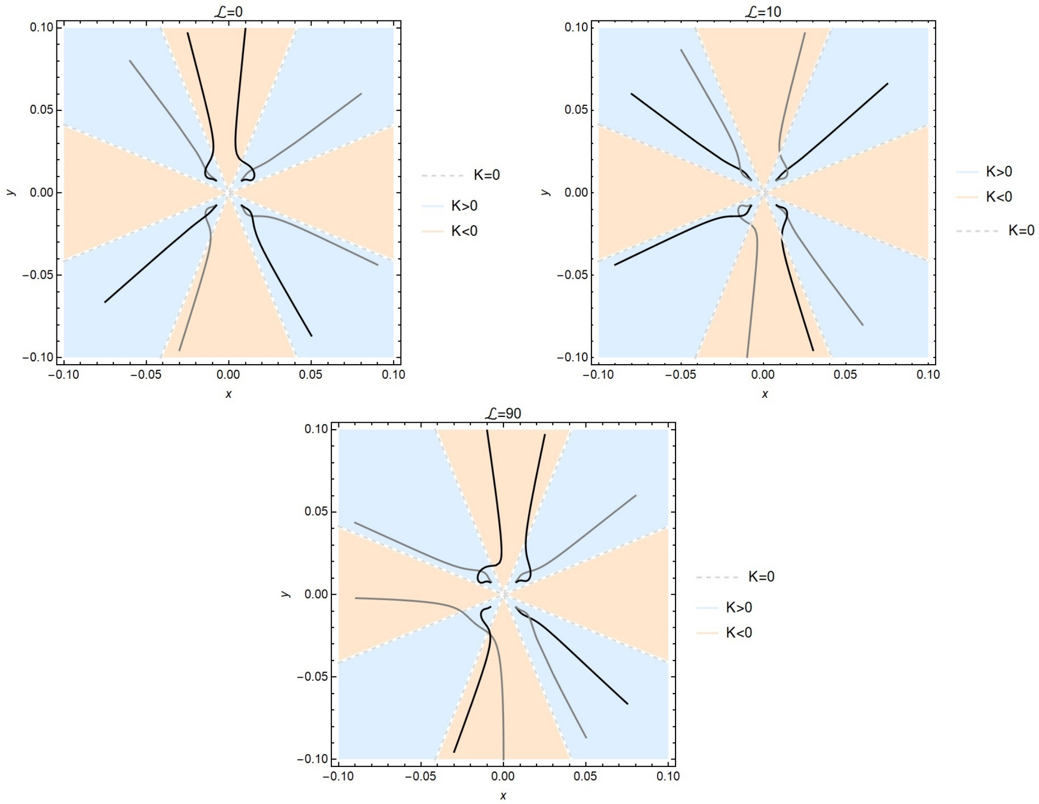

FIGURE 3.11: Null geodesics in the electromagnetic wormhole with $k = 7a^2/24$ (a ghost field) for three values of angular momentum $\mathcal{L}$. The numerical values of the space-time parameters and constants of motions are the same as those of previous figures. Different colors are used to distinguish between intersecting curves.

There are some similarities between the electromagnetic dipole wormhole with $k > 0$ and that with $k < 0$ (figure 3.12). The behavior of the curves is naturally also heavily influenced by the e^{-K} factor, but as the sign of the constant k has been changed, so have the areas of positive and negative K. We also observe here convergence points that were rotated so that they are now located in the $K > 0$ region too. From these points forward, and in direction to the singularity, curves seem to greatly slow down as well. The difference in this wormholes is that the inversion of the K regions with respect to the $k > 0$ case allows for the x and y axis to be found within a positive K area. This leads to the past result that geodesics constrained to $x = 0$ take an infinite amount of affine parameter to meet the ring singularity. Of the wormholes analyzed during this section, this is the only class of space-time whose causal geodesics were not found to encounter the curvature singularity in a strict sense. Furthermore, none of this type of curves were found to be incomplete either. Hence, the electromagnetic wormhole with a negative constant k remains as a candidate of a space-time with complete curves despite the presence of curvature singularities.

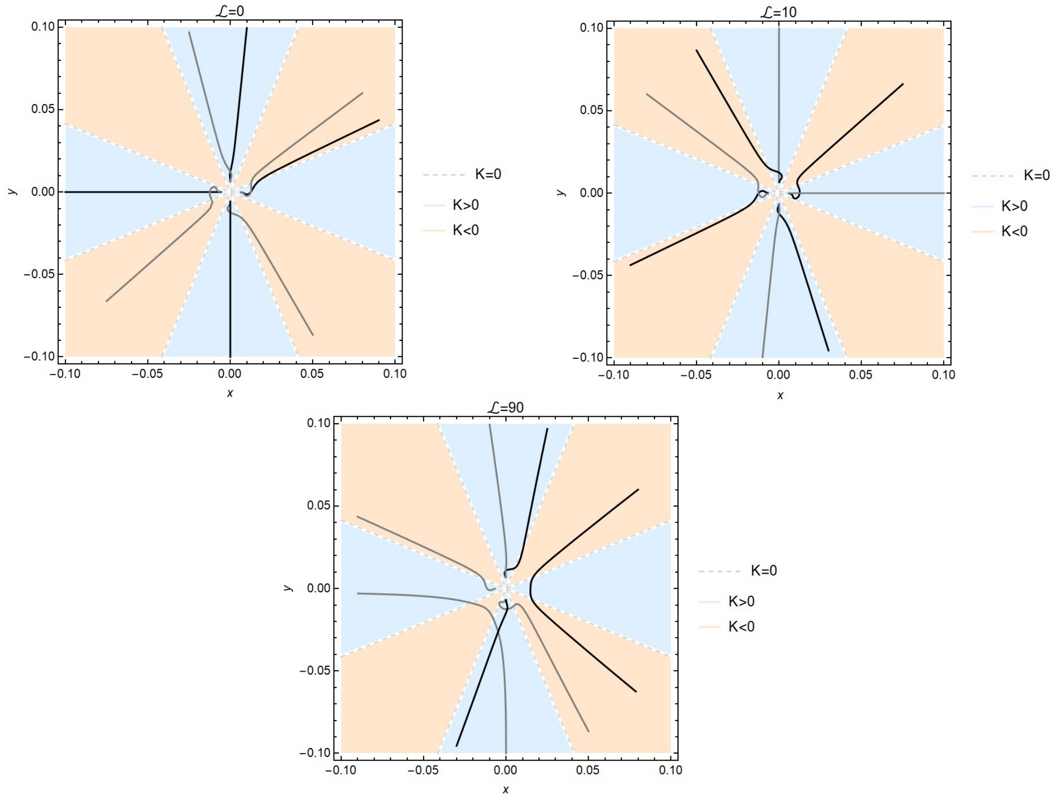

FIGURE 3.12: Null geodesics in the electromagnetic wormhole with $k = -a^2/24$ (a dilatonic field) for three values of angular momentum $\mathcal{L}$. The numerical values of the space-time parameters and constants of motions are the same as those of previous figures. Different colors are used to distinguish between intersecting curves.

This specific manner of achieving completeness in the analyzed geodesics leads to peculiar conclusions that resemble properties of the so-called "bag of gold" singularities introduced by Wheeler in [102]. Namely, the counter-intuitive notion of having an infinite volume bounded by a finite superficial area. To see this, consider a surface S_ϵ of t and $x = \epsilon$ constant. It is evident that S_ϵ has a finite superficial area. Now take a space-like geodesic ζ with t, φ and $x = 0$ constant ($\mathcal{E} = \mathcal{L} = 0$ and $\kappa = 1$) in (3.68). The approximate solution in (3.70) is equally valid for this case, hence, the geodesic length of this curve becomes infinite when approaching the singularity. The geodesic ζ extends across S_ϵ and thus, it would be reasonable to compute the volume enclosed by the surface S_ϵ using the geodesic length of said curve. However, by doing this, one ends up with an infinite volume bounded by a finite superficial area. This unusual property is a direct consequence of having geodesics directed toward the singularity, but never completely reaching it.

Some final remarks are now outlined. Regarding the hyper-surfaces $\mathcal{S}(\mathcal{E}, \kappa, \mathcal{L})$, as expected, none of the numerically obtained geodesics were able to enter their inside regions. They are not shown in the three past figures because the utilized scale in the plots was too large for a proper visualization. On a separate matter, these figures only show null geodesics, the reason for this is that there was no significant qualitative difference between said geodesics and their time-like counterparts.

Lastly, the results described here are numerous, in order to concisely present them, table 3.5 summarizes their most important aspects.

TABLE 3.5: The properties of geodesics that closely approach the ring singularity σ of the two scalar field wormholes.

	Geodesics that encounter σ		
Wormhole	Conditions	Curvature	Regularity
Kerr-like phantom	$\mathcal{L} = 0$ (finite λ required, incomplete if non-extendible)	Vanishing	Singular if non-extendible (singularity visible to distant observers)
Electromagnetic with $k > 0$	Constrained to $x = 0$ (finite λ required, incomplete)	Unbounded	Singular (singularity visible to observers in the throat)
Electromagnetic with $k < 0$	Infinite λ required to reach σ	Vanishing	Complete causal geodesics

Chapter 4

Gravitational Perturbations in the Newman-Penrose Formalism: Applications to Wormholes

The intention in this chapter is to develop a basic framework for treating linear gravitational perturbations using the Newman-Penrose formalism. The use of this formulation will benefit us with some previously used tools and results that have been found over the years for the problem of gravitational radiation in General Relativity. Additionally, in this first approach, we will specialize to the odd-parity perturbations in the Regge-Wheeler gauge (cf. subsection 2.4.3) of spherically symmetric space-times. Though not treated yet in this chapter, we believe that through this scheme, a description of the perturbations in geometrically more complicated space-times could be facilitated. First, we will study the general problem of gravitational perturbations within the tetrad formalism. This treatment will be then particularized to stationary and spherically symmetric space-times, for which a master equation that describes the behavior of the perturbations shall be found. The meaning of physical regularity that must be imposed on the mentioned perturbations will be later discussed. Finally, examples of the application of our master equation will be given, first on the Morris-Thorne wormholes, and then on a specific phantom scalar field wormhole. Some laborious calculations that are related to the contents of this chapter can be consulted in two separate appendixes at the end of the thesis. This whole chapter, as well as both appendixes, are entirely based on reference [103].

4.1 Gravitational Perturbations in the Tetrad Formalism

In this section we develop the basic notions of the sought general framework for perturbation theory using the Newman-Penrose formalism. The scheme we follow is the typical one for linear gravitational perturbations described in section 2.4, that is, we add a perturbation term $h_{\mu\nu}$ to a certain background metric $g_{\mu\nu}$, and then compute the components of the Ricci tensor keeping terms up to first order of the perturbation. The perturbation term is assumed to be small compared to its background counterpart.

In this formalism, the perturbation term of the metric will be represented by a perturbation in the null tetrad, for example, $l_\mu = \widetilde{l}_\mu + \hat{l}_\mu$, where we will establish the convention that a tilde denotes any given background quantity and the hat denotes the perturbation term of said quantity. To proceed, we expand the perturbation terms of the tetrad in the basis of the background tetrad, hence,

$$
\begin{aligned}
z_m^{\ \mu} &= \tilde{z}_m^{\ \mu} + \hat{z}_m^{\ \mu} = \tilde{z}_m^{\ \mu} + \hat{\Sigma}_m^{\ n} \tilde{z}_n^{\ \mu}, \\
z_{m\mu} &= \tilde{z}_{m\mu} + \hat{z}_{m\mu} = \tilde{z}_{m\mu} + \hat{\Omega}_m^{\ n} \tilde{z}_{n\mu}.
\end{aligned} \tag{4.1}
$$

To maintain the vectors l^μ and n^μ real, the $\hat{\Sigma}_m^{\ n}$ matrix has to satisfy $\hat{\Sigma}_m^{\ n} \in \mathbb{R}$ and $\hat{\Sigma}_m^{\ 2} = \hat{\Sigma}_m^{*3}$ for $m, n = 0, 1$. Additionally, we require that $\hat{\Sigma}_2^{\ 3} = \hat{\Sigma}_3^{*2}$, $\hat{\Sigma}_2^{\ 2} = \hat{\Sigma}_3^{*3}$, and that $\hat{\Sigma}_2^{\ m} = \hat{\Sigma}_3^{*m}$ for $m = 0, 1$ in order for m^μ and $\bar{m}^\mu$ to remain as complex conjugates of each other. To simplify notation we drop the hat off the perturbation terms $\hat{\Sigma}_m^{\ n}, \hat{\Omega}_m^{\ n}$ and keep in mind through the rest of this section that now $\Sigma_m^{\ n}$ and $\Omega_m^{\ n}$ carry exclusively perturbed quantities. Then, the metric can be written as[1]

$$
g_{\mu\nu} = \gamma^{mn} z_{m\mu} z_{n\nu} = \tilde{g}_{\mu\nu} + \Omega^{mn} (\tilde{z}_{m\mu} \tilde{z}_{n\nu} + \tilde{z}_{m\nu} \tilde{z}_{n\mu}) + \mathcal{O}(\Omega^2), \tag{4.2}
$$

where $\Omega^{mn} = \gamma^{mp} \Omega_p^{\ n}$ and $\gamma^{mn} = \gamma_{mn}$ is used as given by (2.29). Our first task will be to find a relation between $\Sigma_m^{\ n}$ and $\Omega_m^{\ n}$ such that the orthogonal properties of the tetrad formalism hold to first order of Σ, Ω. Of course, these properties are assumed to be satisfied for the background tetrad. It is not difficult to prove that the relation we are looking for is $\Omega_m^{\ n} = -\gamma_{mp} \Sigma_q^{\ p} \gamma^{qn}$, or alternatively, $\Omega^{mn} = -\Sigma^{nm}$. Using this result one can next verify that $g^{\mu\rho} g_{\rho\nu} = \delta^\mu_{\ \nu}$, and so, the fundamental equations of the formalism are consistent.

With the tetrad given by equation (4.1) the quantities $\mathcal{Z}_{abc}$ related to the spin coefficients may be computed. However, note that the connection Γ associated to the operator ∇ appearing in these quantities is compatible with the metric g, not with the background metric $\tilde{g}$. Naturally, the components of the connection Γ can be expressed as $\Gamma^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\mu\nu} + \hat{\Gamma}^\rho_{\mu\nu}$. We obtain, thus,

$$
\mathcal{Z}_{abc} = \tilde{\mathcal{Z}}_{abc} - \hat{\Gamma}_{cab} + \tilde{D}_a \Omega_{cb} + \tilde{\mathcal{Z}}_{abp} \Omega_c^{\ p} + \tilde{\mathcal{Z}}_{apc} \Sigma_b^{\ p} + \tilde{\mathcal{Z}}_{pbc} \Sigma_a^{\ p}, \tag{4.3}
$$

where we have defined $\hat{\Gamma}_{cab} = \tilde{z}_{c\alpha} \hat{\Gamma}^\alpha_{\mu\nu} \tilde{z}_a^{\ \mu} \tilde{z}_b^{\ \nu}$. The components of the perturbed connection may be found by using the compatibility condition $\nabla_\alpha g_{\mu\nu} = 0$ and the torsion free symmetry $\hat{\Gamma}_{abc} = \hat{\Gamma}_{acb}$. A straightforward, but somewhat long, calculation yields

$$
\hat{\Gamma}_{abc} = \tilde{D}_{(b} \Omega_{c)a} + \tilde{D}_{[b} \Omega_{a]c} + \tilde{D}_{[c} \Omega_{a]b} + \tilde{\mathcal{Z}}_{(bc)p} \Xi_a^{\ p} + \tilde{\mathcal{Z}}_{[ba]p} \Xi_c^{\ p} + \tilde{\mathcal{Z}}_{[ca]p} \Xi_b^{\ p}, \tag{4.4}
$$

with $\Xi_m^{\ n} = \Omega_m^{\ n} - \Sigma_m^{\ n}$, and also $\Pi_m^{\ n} = \Omega_m^{\ n} + \Sigma_m^{\ n}$. Substituting (4.4) in (4.3), and after some algebraic simplifications, we get $\mathcal{Z}_{abc}$ in terms only of background quantities and metric perturbations,

$$
\begin{aligned}
\mathcal{Z}_{abc} =& \tilde{\mathcal{Z}}_{abc} + \tilde{D}_{[b} \Sigma_{a]c} - \tilde{D}_{[c} \Sigma_{a]b} + \tilde{D}_{[b} \Sigma_{c]a} + \Xi_{[b}^{\ m} \tilde{\mathcal{Z}}_{c]am} + \tilde{\mathcal{Z}}_{am[c} \Pi_{b]}^{\ m} \\
& + \Xi_a^{\ m} \tilde{\mathcal{Z}}_{[cb]m} + \tilde{\mathcal{Z}}_{mbc} \Sigma_a^{\ m}.
\end{aligned} \tag{4.5}
$$

This equation is manifestly anti-symmetric in its last two indices as the quantity $\mathcal{Z}_{abc}$ should be. Though lengthy, equation (4.5) describes how the spin coefficients, which are necessary for the Newman-Penrose formalism, change to first order for any given perturbation $\Sigma_m^{\ n}$.

[1] In equation (4.2) we have explicitly indicated there are second order terms of $\Omega_m^{\ n}$. From this point forward we will omit the second order dependency in every equation for compactness and, unless otherwise noted, every equal sign should be understood as such only to first order of Σ or Ω.

4.1.1 Perturbed Tetrad Rotations

Consider a transformation of the perturbation terms $\Omega^{mn} \rightarrow \Omega^{mn} + \Omega'^{mn}$. From (4.2) it can be seen that

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \Omega'^{mn}(\widetilde{z}_{m\mu}\widetilde{z}_{n\nu} + \widetilde{z}_{m\nu}\widetilde{z}_{n\mu}).$$

Since the expression in parenthesis is symmetric in its tetrad indices, the metric will then be invariant under these type of transformations if we demand that $\Omega'^{mn} = -\Omega'^{nm}$. Not only will the metric be invariant, but naturally, also any other scalar or tensor derived from it, so long as the tensor does not possess tetrad indices. Therefore, there exists liberty in choosing the perturbation tetrad $\hat{z}_m^{\ \mu} = \Sigma_m^{\ n}\widetilde{z}_n^{\ \mu}$ since the Ω^{mn} that corresponds to a certain perturbed metric is not unique (recall that $\Omega_m^{\ n} = -\gamma_{mp}\Sigma_q^{\ p}\gamma^{qn}$). This of course is related to the group of Lorentz transformations that leave invariant the orthogonality properties of the formalism (see subsection 2.2.1). However, for this case, the parameters of the Lorentz group should be taken as infinitesimal.

Under the transformation $\Omega^{mn} \rightarrow \Omega^{mn} + \Omega'^{mn}$, the previously defined $\Xi_m^{\ n}$ is invariant, while

$$\Pi_m^{\ n} \rightarrow \Pi_m^{\ n} + 2\Omega'^{n}_{m},$$

with $\Omega'^{n}_{m} = \gamma_{mp}\Omega'^{pn}$. Note that $\Omega'^{mn} = -\Omega'^{nm}$ implies that $\Omega'^{mn} = \Sigma'^{mn}$. Using these relations, we have that the quantities $\mathcal{Z}_{abc}$ transform as

$$\mathcal{Z}_{abc} \rightarrow \mathcal{Z}_{abc} + \widetilde{D}_a\Omega'_{cb} + 2\widetilde{\mathcal{Z}}_{am[c}\Omega'^{m}_{b]} + \widetilde{\mathcal{Z}}_{mbc}\Omega'^{m}_{a}.$$

Perhaps the most important benefit that the perturbed tetrad rotations provide lies in the differential operators D_m. They evidently change as $D_m \rightarrow D_m + \Sigma'^{n}_{m}\widetilde{D}_n$, but because there is some freedom in choosing the perturbation tetrad vectors, we may then conveniently pick them so that, for instance, $D_m = \widetilde{D}_m + \chi\widetilde{D}_n$ for some fixed $m \neq n$, and a scalar field χ. In the following section we take advantage of this particular property, simplifying thus our calculations.

It is important to notice that, when performing any rotation through $\Omega^{mn} \rightarrow \Omega^{mn} + \Omega'^{mn}$, one has to be careful that the rotated vectors l^μ and n^μ end up being real, and that m^μ and $\bar{m}^\mu$ remain as complex conjugates. This restricts the possible valid rotations that can be done. Taking into account these constraints, one can be convinced that there is a total of six degrees of freedom, which is consistent with the fact that the group of Lorentz transformations is a six parameter group.

4.1.2 Gauge Transformations

There is one additional freedom involved in the election of a perturbed tetrad, that of gauge transformations corresponding to diffeomorphisms in the space-time. For the sake of completeness, and though actually none of these transformations are used in the remainder of this work, we give a brief description of them in the context of our framework. From equation (2.14) in subsection 2.1.4, the perturbed metric $h_{\mu\nu}$ transforms as

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu,$$

here ξ^μ is the vector generating the diffeomorphism (in this case, a small coordinate translation). To express this liberty within the tetrad formalism, said vector may be expanded in the background tetrad basis as $\xi^\mu = \mathcal{X}^m\widetilde{z}_m^{\ \mu}$, where the four entries of

$\mathcal{X}^m$ contain all of the gauge information. Contracting the tensor indices of the above transformation with vectors of the background tetrad we have that

$$\Omega_{(mn)} \to \Omega_{(mn)} - \widetilde{D}_{(m}\mathcal{X}_{n)} - \mathcal{X}^p \widetilde{\mathcal{Z}}_{(mn)p},$$

since $\widetilde{z}_m^{\ \mu}\widetilde{z}_n^{\ \nu} h_{\mu\nu} = 2\Omega_{(mn)}$. Hence, a gauge transformation in the tetrad formalism can be seen as the addition of a symmetric term to the Ω_{mn} matrix, i.e.,

$$\Omega_{mn} \to \Omega_{mn} + \Omega'_{mn},$$

with $\Omega'_{mn} = -\widetilde{D}_{(m}\mathcal{X}_{n)} - \mathcal{X}^p \widetilde{\mathcal{Z}}_{(mn)p}$. Note that there is no loss of generality involved in considering only symmetric Ω'_{mn} matrices because, as previously shown in the past subsection, arbitrary antisymmetric terms actually correspond to null rotations. Due to the mentioned symmetry property, we have that $\Sigma'_{mn} = -\Omega'_{mn}$ for gauge transformations and therefore, $\Pi_m^{\ n}$ can be shown to be invariant. On the other hand,

$$\Xi_m^{\ n} \to \Xi_m^{\ n} + 2\Omega'^{n}_{m}.$$

The quantities related to spin coefficients then change as,

$$\mathcal{Z}_{abc} \to \mathcal{Z}_{abc} + 2\widetilde{D}_{[c}\Omega'_{b]a} + 2\Omega'^{m}_{[b}\widetilde{\mathcal{Z}}_{c]am} + 2\Omega'^{m}_{a}\widetilde{\mathcal{Z}}_{[cb]m} - \widetilde{\mathcal{Z}}_{mbc}\Omega'^{m}_{a}.$$

While these relations are not fully exploited in this chapter, they can be of utility for future works that try to further explore the role of gauge and its treatment in the tetrad formalism.

It is important to keep in mind the role of gauge during the analysis of linearized perturbations. Otherwise one could obtain perturbed solutions that can be eliminated through an adequate gauge transformation. On this matter, in section 4.3 we associate the solution of our final perturbation equation to suitable gauge invariant quantities, in this case certain Weyl scalars as measured by background null geodesics. In the following section we will also fix the gauge to that of Regge-Wheeler.

4.2 Gravitational Perturbations in Spherically Symmetric Space-Times

For the remainder of this chapter we focus on four-dimensional stationary and spherically symmetric space-times $(M, g_{\mu\nu})$ whose line element, without loss of generality, can be written in the form

$$ds^2 = g_0(r)dt^2 - g_1(r)dr^2 - g_2(r)d\Omega^2, \tag{4.6}$$

where we have introduced a radial coordinate r and the metric elements $g_{0,1,2}(r)$, which are arbitrary functions of said coordinate. Also, $d\Omega^2$ is the standard metric on the two-sphere. Note that the number of free functions in line element (4.6) can be reduced from three to two by choosing coordinates $\{t, R, \theta, \varphi\}$ on M such that $R^2 = g_2(r)$ is an areal coordinate, and then redefining accordingly $g_1(R)$. This is the standard procedure when dealing with these highly symmetrical space-times. However, we choose to keep the form of (4.6) since it is well-adapted to some wormhole metrics, and hence, greatly simplifies their analysis (see for instance section 4.5).

An orthonormal frame for the metric of interest is simply given by

$$X_0 = \frac{1}{\sqrt{g_0}}\partial_t, \quad X_1 = \frac{1}{\sqrt{g_1}}\partial_r, \quad X_2 = \frac{1}{\sqrt{g_2}}\partial_\theta, \quad X_3 = \frac{1}{\sqrt{g_2}\sin\theta}\partial_\varphi. \tag{4.7}$$

From frame (4.7), a null tetrad can be constructed by taking appropriate linear combinations of the X vectors. In this work we will take advantage of the symmetries of the space-time, namely the fact that ∂_t and ∂_φ are Killing vectors, and choose $\widetilde{l}^\mu$ and $\widetilde{n}^\mu$ so that they lie in the subspace spanned by said Killing vectors. This can also be extended to axially symmetric space-times. Hence, the vectors of the null tetrad will be[2]

$$\widetilde{l}^\mu = \frac{1}{\sqrt{2}}(X_0^\mu + X_3^\mu), \quad \widetilde{n}^\mu = \frac{1}{\sqrt{2}}(X_0^\mu - X_3^\mu), \quad \widetilde{m}^\mu = \frac{1}{\sqrt{2}}(X_1^\mu + iX_2^\mu). \tag{4.8}$$

A direct evaluation of the spin coefficients of the Newman-Penrose formalism with metric (4.6) and tetrad (4.8) yields that the only non-vanishing coefficients are $\widetilde{\kappa}$, $\widetilde{\nu}$, $\widetilde{\tau}$, $\widetilde{\pi}$, $\widetilde{\alpha}$ and $\widetilde{\beta}$. Additionally, the following properties hold

$$\widetilde{\kappa} + \widetilde{\nu}^* = \widetilde{\tau} + \widetilde{\pi}^* = \widetilde{\alpha} + \widetilde{\beta} = 0, \quad \widetilde{\kappa} + \widetilde{\nu} = -\widetilde{\tau} - \widetilde{\pi}, \quad \widetilde{\alpha} = \frac{1}{4}\left(\widetilde{\nu} + \widetilde{\nu}^* + \widetilde{\tau} + \widetilde{\tau}^*\right), \tag{4.9}$$

with $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{R}$. Notice that as a consequence of our choice of vectors $\widetilde{l}^\mu$ and $\widetilde{n}^\mu$ we will have that $\widetilde{D}\widetilde{\phi} = \widetilde{\Delta}\widetilde{\phi} = 0$ for any background scalar quantity $\widetilde{\phi}$, including these spin coefficients.

We now add a perturbation term $h_{\mu\nu}$ to the background metric introduced in this section. Following the work of Regge and Wheeler described in subsection 2.4.3, we consider an odd-parity perturbation in the Regge-Wheeler gauge, this is,

$$h_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & h_0 \\ 0 & 0 & 0 & h_1 \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{bmatrix}, \tag{4.10}$$

with $h_{\mu\nu}$ expressed in the coordinate basis $\{t, r, \theta, \varphi\}$ and $h_{0,1} = h_{0,1}(t, r, \theta)$. Using the one-forms of the background tetrad $\left\{\widetilde{l}_\mu, \widetilde{n}_\mu, \widetilde{m}_\mu, \widetilde{\bar{m}}_\mu\right\}$ as a basis, we can write

$$h_{\mu\nu} = f_0(\widetilde{n}_\mu\widetilde{n}_\nu - \widetilde{l}_\mu\widetilde{l}_\nu) + 2f_1[\widetilde{l}_{(\mu}\widetilde{m}_{\nu)} + \widetilde{l}_{(\mu}\widetilde{\bar{m}}_{\nu)} - \widetilde{n}_{(\mu}\widetilde{m}_{\nu)} - \widetilde{n}_{(\mu}\widetilde{\bar{m}}_{\nu)}],$$

where $f_0 = h_0/\sqrt{g_0 g_2}\sin\theta$ and $f_1 = h_1/2\sqrt{g_1 g_2}\sin\theta$. It can be verified that an acceptable tetrad for the perturbed metric $g_{\mu\nu} = \widetilde{g}_{\mu\nu} + h_{\mu\nu}$ is given by

$$\begin{aligned} l_\mu &= \widetilde{l}_\mu + \frac{1}{2}f_0\widetilde{n}_\mu - f_1(\widetilde{m}_\mu + \widetilde{\bar{m}}_\mu), \quad n_\mu = \widetilde{n}_\mu - \frac{1}{2}f_0\widetilde{l}_\mu + f_1(\widetilde{m}_\mu + \widetilde{\bar{m}}_\mu), \\ m_\mu &= \widetilde{m}_\mu, \end{aligned} \tag{4.11}$$

from which the elements of $\Omega_m^{\ n}$ can be easily read off as

[2] This choice of tetrad differs from the usual, for instance in [104], in which $\widetilde{l}^\mu$ and $\widetilde{n}^\mu$ are combination of the X_0 and X_1 vectors. Our choice will come with certain advantages that will later be seen.

$$\Omega_m^{\ n} = \begin{bmatrix} 0 & f_0/2 & -f_1 & -f_1 \\ -f_0/2 & 0 & f_1 & f_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It will more helpful, though, to represent the perturbation in terms of $\Sigma_m^{\ n} = -\gamma_{mp}\Omega_q^{\ p}\gamma^{qn}$, obtaining thus

$$\Sigma_m^{\ n} = \begin{bmatrix} 0 & -f_0/2 & 0 & 0 \\ f_0/2 & 0 & 0 & 0 \\ f_1 & -f_1 & 0 & 0 \\ f_1 & -f_1 & 0 & 0 \end{bmatrix}. \tag{4.12}$$

As it was already stated in the previous section, any given metric perturbation does not uniquely define $\Omega_m^{\ n}$, and consequently, $\Sigma_m^{\ n}$. We will be interested in a perturbed tetrad such that

$$D\widetilde{\phi} = (\chi_1\widetilde{D} + \chi_2\widetilde{\Delta})\widetilde{\phi} = 0, \quad \Delta\widetilde{\phi} = (\xi_1\widetilde{D} + \xi_2\widetilde{\Delta})\widetilde{\phi} = 0, \tag{4.13}$$

where again, $\widetilde{\phi}$ is any background scalar quantity and $\chi_{1,2}$, $\xi_{1,2}$ are elements of $\Sigma_m^{\ n}$. It turns out that precisely the matrix given by (4.12) describes the perturbation tetrad with this desired property. Nonetheless, it is important to mention that one can always find, through an adequate tetrad rotation, a perturbation tetrad such that (4.13) holds in a spherically symmetric (even in an axially symmetric, for that matter) background space-time. This is possible too due to our previous election of background vectors $\widetilde{l}^\mu$ and $\widetilde{n}^\mu$, namely, the fact that they lie in the subspace spanned by Killing vectors. Another advantage that this $\Sigma_m^{\ n}$ possess is that $\delta\widetilde{\phi} = \widetilde{\delta}\widetilde{\phi}$. However, this will not always be the case for an arbitrary metric perturbation, even performing a perturbed tetrad rotation. Additionally, it can be seen that $\widetilde{D}f_{0,1} = \widetilde{\Delta}f_{0,1}$ because of the φ independence of those functions.

With an explicit expression for the perturbation matrix $\Sigma_m^{\ n}$, we can now proceed to compute the perturbed spin coefficients using equation (4.5). Our objective then will be to write the components of the Ricci tensor $R_{\mu\nu}$ in terms of these spin coefficients using the equations of the Newman-Penrose formalism. It can already be foreseen that we will obtain second-order partial differential equations for the perturbation functions f_0 and f_1 due to the fact that the formalism provides first-order partial differential equations for the spin coefficients, and these in turn, have first-order derivatives of said functions. Since the calculation of the NP quantities is pretty much straightforward and the results are numerous, they will be shown separately in Appendix A, and we should cite them in the following as needed.

We now make use of the curvature related quantities, Φ_{AB} $(A, B = 0, 1, 2)$ and $\Lambda = R/24$, of the Newman-Penrose formalism. See equations (2.33) for their definition. By either a direct calculation of these expressions, or by the vanishing of the background spin coefficients, the following holds for the background metric

$$\widetilde{\Phi}_{01} = \widetilde{\Phi}_{10} = \widetilde{\Phi}_{12} = \widetilde{\Phi}_{21} = 0, \quad \widetilde{\Phi}_{00} = \widetilde{\Phi}_{22}. \tag{4.14}$$

This also follows from the fact that the background Ricci tensor admits the form $\widetilde{R}_{\mu\nu} = diag\left[\widetilde{R}_{00}, \widetilde{R}_{11}, \widetilde{R}_{22}, \widetilde{R}_{33}\right]$ in the $\{t, r, \theta, \varphi\}$ basis. Using (2.35a), (2.35n) and (4.9), it can be seen that the last equality in (4.14) implies that

$$\widetilde{\delta}\widetilde{\nu}+\widetilde{\delta}^{*}\widetilde{\kappa}=2\widetilde{\alpha}(\widetilde{\nu}+\widetilde{\kappa}). \tag{4.15}$$

Note that, with the exception of Λ, the Φ_{AB} quantities depend manifestly on the tetrad choice. Even upon fixing the background tetrad, Φ_{AB} will vary with perturbed rotations such as the ones described in the previous section. Since the Ricci tensor itself is invariant to these type of transformations, we look for expressions of its components in the coordinate basis and constructed from the quantities Φ_{AB} and Λ. More precisely, we will look for the components of $R_{\mu\nu}$ in the orthonormal frame (4.7).

In terms of the background tetrad basis, the orthonormal basis can be written as $X_\alpha^{\ \mu}=\widetilde{\Gamma}_\alpha^{\ m}\widetilde{z}_n^{\ \mu}$. From (4.8) it can be easily seen that,

$$\widetilde{\Gamma}_\alpha^{\ m}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & -i & i\\ 1 & -1 & 0 & 0\end{bmatrix}.$$

Similarly, in terms of the perturbed tetrad we have that $X_\alpha^{\ \mu}=\Gamma_\alpha^{\ m}z_m^{\ \mu}$. Using the fact that $z_m^{\ \mu}=(\delta_m^{\ n}+\Sigma_m^{\ n})\widetilde{z}_n^{\ \mu}$, we obtain $\Gamma_\alpha^{\ n}=\widetilde{\Gamma}_\alpha^{\ m}(\delta_m^{\ n}-\Sigma_m^{\ n})$ to first order in Σ, or explicitly,

$$\Gamma_\alpha^{\ n}=\frac{1}{\sqrt{2}}\begin{bmatrix}1-f_0/2 & 1+f_0/2 & 0 & 0\\ -2f_1 & 2f_1 & 1 & 1\\ 0 & 0 & -i & i\\ 1+f_0/2 & -1+f_0/2 & 0 & 0\end{bmatrix}. \tag{4.16}$$

We can then write,

$$\begin{aligned}
X_0^{\mu}=&\frac{1}{\sqrt{2}}\left[\left(1-\frac{f_0}{2}\right)l^{\mu}+\left(1+\frac{f_0}{2}\right)n^{\mu}\right], \quad X_1^{\mu}=\frac{1}{\sqrt{2}}\left[m^{\mu}+\bar{m}^{\mu}+2f_1(n^{\mu}-l^{\mu})\right],\\
X_2^{\mu}=&\frac{i}{\sqrt{2}}\left[\bar{m}^{\mu}-m^{\mu}\right], \quad X_3^{\mu}=\frac{1}{\sqrt{2}}\left[\left(1+\frac{f_0}{2}\right)l^{\mu}-\left(1-\frac{f_0}{2}\right)n^{\mu}\right].
\end{aligned} \tag{4.17}$$

The X vectors of equation (4.17) can be shown to be invariant (up to first order in Σ) under transformations $\Sigma_m^{\ n}\to\Sigma_m^{\ n}+\Sigma_m'^{n}$, by noting that $\Gamma_\alpha^{\ n}\to\Gamma_\alpha^{\ n}-\widetilde{\Gamma}_\alpha^{\ m}\Sigma_m'^{n}$, and $z_m^{\ \mu}\to z_m^{\ \mu}+\Sigma_m'^{n}\widetilde{z}_n^{\ \mu}$. Thus, we may find the desired invariant equations for the Ricci components by contracting these vectors with the Ricci tensor field. Unfortunately, this has to be done for the 10 independent components of said tensor, yielding the following relations

$$\begin{aligned}
\hat{\mathcal{R}}_{00}=&-\hat{\Phi}_{00}-\hat{\Phi}_{22}+2(3\hat{\Lambda}-\hat{\Phi}_{11}), & \hat{\mathcal{R}}_{01}=&-\hat{\Phi}_{01}-\hat{\Phi}_{10}-\hat{\Phi}_{12}-\hat{\Phi}_{21},\\
\hat{\mathcal{R}}_{02}=&i(\hat{\Phi}_{01}-\hat{\Phi}_{10}+\hat{\Phi}_{12}-\hat{\Phi}_{21}), & \hat{\mathcal{R}}_{03}=&-\hat{\Phi}_{00}+\hat{\Phi}_{22}+2(3\widetilde{\Lambda}-\widetilde{\Phi}_{11})f_0,\\
\hat{\mathcal{R}}_{11}=&-\hat{\Phi}_{02}-\hat{\Phi}_{20}-2(3\hat{\Lambda}+\hat{\Phi}_{11}), & \hat{\mathcal{R}}_{12}=&i(\hat{\Phi}_{02}-\hat{\Phi}_{20}),\\
\hat{\mathcal{R}}_{13}=&-\hat{\Phi}_{01}+\hat{\Phi}_{12}-\hat{\Phi}_{10}+\hat{\Phi}_{21} & \hat{\mathcal{R}}_{22}=&\hat{\Phi}_{02}+\hat{\Phi}_{20}-2(3\hat{\Lambda}+\hat{\Phi}_{11}),\\
&+4(3\widetilde{\Lambda}-\widetilde{\Phi}_{11}+\widetilde{\Phi}_{00})f_1, & &\\
\hat{\mathcal{R}}_{23}=&i(\hat{\Phi}_{01}-\hat{\Phi}_{10}+\hat{\Phi}_{21}-\hat{\Phi}_{12}), & \hat{\mathcal{R}}_{33}=&-\hat{\Phi}_{00}-\hat{\Phi}_{22}-2(3\hat{\Lambda}-\hat{\Phi}_{11}),
\end{aligned} \tag{4.18}$$

where we have defined $\hat{\mathcal{R}}_{\alpha\beta} = \hat{R}_{\mu\nu} X_\alpha^{\ \mu} X_\beta^{\ \nu}$. In equations (4.18) we have written only the perturbation terms (denoted by a hat), that is, the terms of first order in $f_{0,1}$. Naturally, the background terms that should appear on both sides of the equations, which are of order zero, cancel each other out. Hereafter, we drop the tilde off the background quantities and so, any quantity or operator without a hat should be understood to be of the background space-time, except for the perturbation functions f_0 and f_1 (same convention as Appendix A).

Taking the results (A.6) from Appendix A, one can realize that the only non-vanishing components of $\hat{\mathcal{R}}_{\alpha\beta}$ are

$$\begin{aligned} \hat{\mathcal{R}}_{03} &= \left[(\delta_- + 2\kappa_+)\delta_- - (\delta_+ + \kappa_- + 3\pi_-)\delta_+ + 4(\kappa_-^2 - \kappa_+^2) + 2(3\Lambda - \Phi_{11})\right] f_0 \\ &\quad +2(\delta_+ - 6\alpha) D f_1, \\ \hat{\mathcal{R}}_{13} &= 2\left[D^2 + (\delta_- + 4\kappa_+)(\delta_- - 2\kappa_+) + 2(3\Lambda - \Phi_{11} + \Phi_{00})\right] f_1 \\ &\quad -(\delta_+ - 3\kappa_- + \pi_-) D f_0, \\ \hat{\mathcal{R}}_{23} &= i(\delta_- - 2\kappa_+) D f_0 - 2i(\delta_+ + \kappa_- + 3\pi_-)(\delta_- - 2\kappa_+) f_1. \end{aligned} \tag{4.19}$$

With the help of the commutator $[\delta_- - 2\kappa_+, \delta_+] = (\kappa_- + \pi_-)(\delta_- + 2\kappa_+)$, the $\hat{\mathcal{R}}_{23}$ component of the past equations can be rewritten as

$$\hat{\mathcal{R}}_{23} = i(\delta_- - 2\kappa_+)\left[D f_0 - 2(\delta_+ + 2\pi_-) f_1\right].$$

So far we have focused on describing how the space-time geometry is modified when adding a small term to the background metric. In fact, equations (4.19) describe precisely this change up to linear order. To obtain a set of suitable perturbation equations, however, one must take into account both sides of the field equations that a particular gravitational source yields. Namely, equations (4.19) will give information of one side of the field equations, whereas the other side will be determined by the physical variables of the source.

As an initial approach, consider background field equations that consist of a simple structure such as

$$R_{\mu\nu} = S_{\mu\nu} + S g_{\mu\nu}, \tag{4.20}$$

where $S_{\mu\nu}$ is a symmetrical tensor and S a scalar, both containing the physical properties of the source. Spherical symmetry, combined with expression (4.20) for the Ricci tensor, imposes that the tensor $S_{\mu\nu}$ must be written as

$$S_{\mu\nu} = diag[S_{00}(r), S_{11}(r), S_{22}(r), S_{22}(r)\sin^2\theta] \tag{4.21}$$

in the $\{t, r, \theta, \varphi\}$ basis. The gravitational source may in principle be also perturbed, i.e., $S_{\mu\nu} \to S_{\mu\nu} + \hat{S}_{\mu\nu}$ and $S \to S + \hat{S}$. The most general linearized perturbed field equations will then be

$$\hat{R}_{\mu\nu} = \hat{S}_{\mu\nu} + S h_{\mu\nu} + \hat{S} g_{\mu\nu}. \tag{4.22}$$

The first step for solving this system of equations is to recall from (4.19) that there are only three non-vanishing components of the perturbed Ricci tensor. Taking into account the diagonal form of $g_{\mu\nu}$ and that of $h_{\mu\nu}$ in equation (4.10), it is easy to realize that only the $\hat{S}_{03}$, $\hat{S}_{13}$ and $\hat{S}_{23}$ components are important for the problem of odd-parity perturbations. The rest of the $\hat{S}_{\mu\nu}$ quantities are irrelevant as they have no influence over the perturbations f_0 and f_1 of the metric.

In this thesis we will treat the simple case in which $\hat{S}_{03} = \hat{S}_{13} = \hat{S}_{23} = 0$. As a consequence, the metric functions f_0 and f_1 decouple from $\hat{S}$ and the $\hat{S}_{\mu\nu}$ components in the perturbed field equations. There exists matter content of physical interest whose perturbations fulfill these conditions, for instance, scalar fields (even self-interacting) in spherically symmetric and stationary space-times and perfect fluids. This is readily seen from the expressions of $S_{\mu\nu}$ in their respective field equations,

$$S_{\mu\nu} = \pm\nabla_\mu\phi\nabla_\nu\phi, \quad \text{and} \quad S_{\mu\nu} = (\rho + p)u_\mu u_\nu,$$

where $u_\mu = (l_\mu + n_\mu)/\sqrt{2}$ in the tetrad basis. The addition of a small term in the physical parameters of the space-times, this is, $\phi(r) \to \phi(r) + \hat{\phi}(t, r, \theta)$, $\rho \to \rho + \hat{\rho}$ and $p \to p + \hat{p}$, yields

$$\hat{S}_{\mu\nu} = \pm 2\nabla_{(\mu}\phi\nabla_{\nu)}\hat{\phi}, \quad \text{and} \quad \hat{S}_{\mu\nu} = (\hat{\rho} + \hat{p})u_\mu u_\nu.$$

Note that due to the type of perturbation considered during the present work, there is no dependency on the coordinate φ and $\hat{u}_\mu = 0$ from equation (4.11). Therefore, $\hat{S}_{03} = \hat{S}_{13} = \hat{S}_{23} = 0$. In fact, further analysis of the linearized field equations reveals that $\hat{\phi} = 0$ and $\hat{\rho} = \hat{p} = 0$ for odd-parity perturbations.

We should remark here that perturbations in space-times that solve the Einstein-Maxwell equations are, in the most common case, excluded from the treatment that is described in the following. The reason for this is that the set of Maxwell's equations fall beyond the scope of the present discussion. Hence, they are not guaranteed to hold up to liner order when perturbing the metric and the gravitational source with the constraint $\hat{S}_{03} = \hat{S}_{13} = \hat{S}_{23} = 0$. The same applies to electromagnetic perturbations.

Thus, considering the above conditions and contracting the necessary X vectors with the perturbed Ricci tensor (4.22), the non-vanishing components of the field equations become

$$\begin{aligned}
0 &= \left[(\delta_- + 2\kappa_+)\delta_- - (\delta_+ + \kappa_- + 3\pi_-)\delta_+ + 4(\kappa_-^2 - \kappa_+^2) + 2(3\Lambda - \Phi_{11}) - S\right] f_0 \\
&\quad + 2(\delta_+ - 6\alpha)Df_1, \\
0 &= 2\left[D^2 + (\delta_- + 4\kappa_+)(\delta_- - 2\kappa_+) + 2\Lambda_s - S\right] f_1 - (\delta_+ - 3\kappa_- + \pi_-)Df_0, \\
0 &= i(\delta_- - 2\kappa_+)\left[Df_0 - 2(\delta_+ + 2\pi_-)f_1\right],
\end{aligned} \tag{4.23}$$

with $\Lambda_s = 3\Lambda - \Phi_{11} + \Phi_{00}$. We may now attempt to solve the system of equations (4.23). Consider the last equation, we have already factored it in a way that can be easily solved. Notice that the expression in parentheses cannot vanish since δ_- is a differential operator and κ_+ is a scalar quantity. Thus, the left side of this equation will vanish if the expression in square brackets also does, or by the application of the operator in parentheses to the quantity in square brackets. We will examine the first possibility, that is[3],

$$Df_0 = 2(\delta_+ + 2\pi_-)f_1. \tag{4.24}$$

[3]The most general solution is actually $Df_0 - 2(\delta_+ + 2\pi_-)f_1 = C(t, r)\sin\theta$. This severely restricts the angular part of f_0 and f_1. Additionally, when considering system (4.23) as a whole and after numerous cumbersome algebraic steps, the first equation of the system can be reduced to the form $d(r)C(t, r)\sin\theta = 0$ (see Appendix B), where $d(r)$ is a generally non-zero function depending on the background metric components. This implies that $C(t, r) = 0$. For these reasons, the $C(t, r)$ quantity will not be of interest.

By inserting (4.24) in the $\hat{\mathcal{R}}_{13}$ component of (4.23), an equation for the perturbation function f_1 can finally be found,

$$\left[D^2 + (\delta_- + 4\kappa_+)(\delta_- - 2\kappa_+) - (\delta_+ - 3\kappa_- + \pi_-)(\delta_+ + 2\pi_-) + 2\Lambda_s - S\right] f_1 = 0. \tag{4.25}$$

We are left, though, with the first equation in (4.23) yet to be solved with the inconvenient that the perturbation functions f_0 and f_1 have already been used to satisfy the other two equations in the system. The $\hat{\mathcal{R}}_{03}$ component can be shown to vanish only if

$$(\delta_+ - 4\alpha)(S - 2\Lambda_s) = 0. \tag{4.26}$$

This expression then implies that $S_{22}(r) = c$ in equation (4.21), being c an integration constant (see Appendix B for details). It turns out that examples that fulfill this background condition, besides vacuum space-times, were already mentioned. These are the solutions of the Einstein-scalar field equations with a self-interacting potential $R_{\mu\nu} = \pm\nabla_\mu\phi\nabla_\nu\phi + \mathcal{V}(\phi)g_{\mu\nu}$, as well as perfect fluid solutions. For both of these cases, $c = 0$. Many wormhole space-times arise as solutions to the first type of field equations, hence, our results can be applied to them.

For reasons explained in the next section, we shall opt to replace the perturbation function f_1 with $Q = 2\sqrt{g_0}\sin\theta f_1$. To do so, the following helpful relations can be verified to be true by examining the spin coefficients and operators of (A.2),

$$\delta_-\left(\frac{1}{\sin\theta}\right) = -\frac{2\kappa_+}{\sin\theta}, \quad \delta_+\left(\frac{1}{\sqrt{g_0}}\right) = \frac{\kappa_- - \pi_-}{\sqrt{g_0}}.$$

Substituting $f_1 = Q/2\sqrt{g_0}\sin\theta$ in (4.25), and employing these two equalities, we at last arrive to our master equation for odd-parity perturbations,

$$\left[D^2 + (\delta_- + 2\kappa_+)(\delta_- - 4\kappa_+) - (\delta_+ - 2\kappa_-)(\delta_+ + \pi_- + \kappa_-) + 2\Lambda_s - S\right] Q = 0. \tag{4.27}$$

The notation introduced throughout this paper allows us to easily identify the terms appearing in the master equation. The D operator is associated to the time dependence of the perturbation, the second term is associated with the angular part due to it containing the δ_- operators, and the third term is related to the radial part because of the δ_+ operators. In (4.27) there also appears a background matter term which is purely radial. It is natural then to propose a separable ansatz of the form $Q = T(t)R(r)\Theta(\theta)$. With such a proposed solution, the angular part of the master equation will yield the following differential equation when inserting the explicit expressions for the spin coefficients and operators,

$$\frac{d^2\Theta}{d\theta^2} - \frac{1}{\tan\theta}\frac{d\Theta}{d\theta} = -\ell(\ell+1)\Theta. \tag{4.28}$$

Equation (4.28) has for solution $\Theta_\ell = \sin\theta dP_\ell(\cos\theta)/d\theta$, where $P_\ell(\cos\theta)$ are the well-known Legendre polynomials. This result was of course, expected, owing to the spherical symmetry of the line element (4.6) and to the decomposition in tensor spherical harmonics of the perturbation that Regge and Wheeler previously used. In fact, this part of the solution is, obviously, the same that appears in their paper.

The radial part of the master equation is obtained by computing all of the terms appearing in (4.27), this is,

$$\frac{1}{g_0}\frac{\partial^2 Q}{\partial t^2}-\frac{1}{g_2}\left(\frac{\partial^2 Q}{\partial\theta^2}-\frac{1}{\tan\theta}\frac{\partial Q}{\partial\theta}+2Q\right)-\frac{1}{\sqrt{g_1}}\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{g_1}}\frac{\partial Q}{\partial r}\right)$$
$$-\frac{g_0'}{2g_0g_1}\frac{\partial Q}{\partial r}+\sqrt{\frac{g_2}{g_0g_1}}\frac{d}{dr}\left[\sqrt{\frac{g_0}{g_1}}\frac{d}{dr}\left(\frac{1}{\sqrt{g_2}}\right)\right]Q+2(2\Lambda_s-S)Q=0.$$

The above second-order partial differential equation can be further simplified by considering the previously introduced ansatz for Q, whose angular part has already been solved, and with the additional assumption of an harmonic dependence on time, i.e., $Q=e^{i\omega t}R(r)\sin\theta dP_l(\cos\theta)/d\theta$. Furthermore, introducing the tortoise coordinate r_* defined by

$$\frac{d}{dr_*}=\sqrt{\frac{g_0}{g_1}}\frac{d}{dr},$$

and substituting equations (B.5, 4.28), the master equation can finally be rewritten in a very compact form as,

$$\frac{d^2R}{dr_*^2}-\left[V(r)-\omega^2\right]R=0, \tag{4.29}$$

with

$$V(r)=\frac{g_0}{g_2}[\ell(\ell+1)-2(c+1)]+\sqrt{g_2}\frac{d^2}{dr_*^2}\left(\frac{1}{\sqrt{g_2}}\right). \tag{4.30}$$

We have reduced the master equation to an eigenvalue problem for ω^2 and the operator $\mathcal{H}=-d^2/dr_*^2+V(r)$, which is linear and self-adjoint. An operator of this type is sometimes called a Schrödinger operator with effective potential $V(r)$. The potential found here can now be compared to some previous results. For instance, it can be verified that this potential reduces to that of Regge-Wheeler when inserting the corresponding metric components of the Schwarzschild metric (compare to equation (2.61)). It also coincides with the potential of axial and uncharged perturbations for electrically neutral background space-times found in [82]. The potential derived here can be seen as a slight generalization of the uncharged result to arbitrary spherically symmetric line elements[4] that admit field equations of the type (4.20), with the exception of Einstein-Maxwell solutions. Finally, it is contained in the general odd-parity potential for perturbations in spherically symmetric space-times coupled to matter fields given in [105]. During the next sections of this work we will analyze the potential $V(r)$ of some wormhole examples, along with their properties.

However, before ending this section it might be worth discussing the perturbed stress-energy tensor $T_{\mu\nu}+\hat{T}_{\mu\nu}$ and its conservation law. Our analysis was focused on the case in which the metric perturbations decoupled from those of the gravitational source in the field equations, thus we can make $\hat{S}_{\mu\nu}=\hat{S}=0$ for simplicity. From (4.20), the background and perturbation terms of the stress-energy tensor are respectively given in geometrized units ($G=c=1$) by

$$8\pi T_{\mu\nu}=S_{\mu\nu}-\left(\frac{1}{2}s+S\right)g_{\mu\nu}\quad\text{and}\quad 8\pi\hat{T}_{\mu\nu}=-\left(\frac{1}{2}s+S\right)h_{\mu\nu},$$

[4] In [82] the space-times considered are supported by a phantom scalar field (hence, $c=0$ in equation (4.30)), with the assumption that $g_{tt}=-1/g_{rr}$ for the metric tensor.

where $s = S_{\mu\nu}g^{\mu\nu}$. Note that $\hat{s} = S_{\mu\nu}h^{\mu\nu} = 0$. It can be verified then that $\nabla_\mu(T^{\mu\nu} + \hat{T}^{\mu\nu}) = 0$ up to linear order of the perturbation, where the covariant differentiation ∇ is compatible with the perturbed metric. This was expected, of course, as a result of solving in a consistent way the system (4.23) of field equations. For the case of the Einstein-scalar field equations with a self-interacting potential $\mathcal{V}(\phi)$, we have that $S_{\mu\nu} = \pm\nabla_\mu\phi\nabla_\nu\phi$ and $S = \mathcal{V}(\phi)$. It can be shown that the Klein-Gordon equation is implied by the conservation law of an arbitrary scalar field ϕ and hence,

$$\pm\nabla^\mu\nabla_\mu\phi = \frac{d\mathcal{V}(\phi)}{d\phi},$$

up to first order of the perturbation too.

4.3 Perturbed Weyl Scalars and Their Meaning

We turn our attention now to the perturbed Weyl scalars of the formalism. As explained in subsection 2.2.3, they possess an important physical significance in perturbation theory. Therefore, during this section the linear change of said scalars will be analyzed with the objective of associating a physical meaning to the perturbation function Q. This will lead us to discuss first the physical regularity of the perturbation, and then some of the physical properties that can be described through the perturbed Weyl scalars.

4.3.1 Physical Regularity of the Perturbation

In order for the gravitational perturbation to be of any physical relevance, it has to display an "acceptable" behavior throughout space-time, or at least asymptotically. One might naturally impose the condition that the metric perturbation functions of $h_{\mu\nu}$ do not grow without bound as $r \to \infty$ and deem that as physical regularity. Nevertheless, due to the gauge freedom that exists in General Relativity, this condition is not quite precise (see subsections 2.1.4 and 2.2.2). Fortunately, the Newman-Penrose formalism can also be used to describe more accurately what this acceptable behavior is expected to be by means of the previously presented “peeling theorem”.

Consider the following vectors tangent to ingoing and outgoing radial null geodesics of the background metric,

$$k^\mu_\pm = X^\mu_0 \pm X^\mu_1 = \frac{1}{\sqrt{2}}\left(\tilde{l}^\mu + \tilde{n}^\mu \pm \tilde{m}^\mu \pm \tilde{\bar{m}}^\mu\right).$$

The next null rotations of our initial tetrad yield a new one such that the unperturbed part of l''^μ and n''^μ is aligned to the k^μ_+ and k^μ_- vectors, respectively,

$$\begin{aligned} l'^\mu = l^\mu + a_1\bar{m}^\mu + a_1^* m^\mu + \|a_1\|^2 n^\mu, \quad m'^\mu = m^\mu + a_1 n^\mu, \quad n'^\mu = n^\mu, \\ n''^\mu = n'^\mu + a_2\bar{m}'^\mu + a_2^* m'^\mu + \|a_2\|^2 l'^\mu, \quad l''^\mu = l'^\mu, \end{aligned} \tag{4.31}$$

with $a_1 = 1$ and $a_2 = -1/2$. In equations (4.31) and (4.32), and only in those equations, we temporarily restore the convention of section 4.1 in which any given quantity ξ of the space-time is written as the sum of a background term and a perturbation term, i.e., $\xi = \tilde{\xi} + \hat{\xi}$. Under rotations (4.31), the transformation laws of the Weyl scalars we will need are (cf. subsection 2.2.1)

$$\Psi_0' = \Psi_0 + 4a_1\Psi_1 + 6a_1^2\Psi_2 + 4a_1^3\Psi_3 + a_1^4\Psi_4, \quad \Psi_1' = \Psi_1 + 3a_1\Psi_2 + 3a_1^2\Psi_3 + a_1^3\Psi_4,$$
$$\Psi_2' = \Psi_2 + 2a_1\Psi_3 + a_1^2\Psi_4, \quad \Psi_2'' = \Psi_2' + 2a_2^*\Psi_1' + a_2^{*2}\Psi_0'. \tag{4.32}$$

If $a_1 = 1$ and $a_2 = -1/2$, then $\Psi_2'' = \Psi_0/4 - \Psi_2/2 + \Psi_4/4$. After substituting the expressions found in (A.7), the perturbed part of this Weyl scalar reduces to

$$\hat{\Psi}_2'' = \frac{1}{2}(\delta_- + 2\kappa_+)\left[(\delta_+ - 2\kappa_-)f_0 - 2Df_1\right]. \tag{4.33}$$

The physical significance of $\hat{\Psi}_2''$ can be revealed by applying the operator D to (4.33), and then reducing it accordingly with some of the relations of the formalism derived in this chapter along with the master equation, thus obtaining

$$D\hat{\Psi}_2'' = \frac{1}{2\sqrt{g_0}\sin\theta}\delta_-\left[(\delta_- + 2\kappa_+)(\delta_- - 4\kappa_+) + 2\Lambda_s - S)\right]Q. \tag{4.34}$$

We have already solved the angular part of the master equation whose terms appear again in (4.34). By making use of said solution and some properties of the Legendre equation, the past expression can be rewritten as

$$\frac{\partial\hat{\Psi}_2''}{\partial t} = -\frac{i\ell(\ell+1)}{g_2^{3/2}}\left[(\ell-1)(\ell+2) - c\right]T(t)R(r)P_\ell(\cos\theta), \tag{4.35}$$

where we have made use of restriction (B.5) too. In the case of space-times that solve the Einstein-scalar field equations (and vacuum space-times too) we have that $c = 0$, and the meaning of

$$\frac{\partial\hat{\Psi}_2''}{\partial t} = -\frac{i(\ell+2)!}{g_2^{3/2}(\ell-2)!}T(t)R(r)P_\ell(\cos\theta) \tag{4.36}$$

becomes clearer, as well as the reason behind the use of the perturbation function Q. The peeling theorem establishes that the Weyl scalar Ψ_2 asymptotically decays at null infinity as $1/\lambda'^3$, where λ' is the affine parameter of a null geodesic that reaches said infinity (c.f. subsection 2.2.2). Considering the background radial null geodesics to which the unperturbed part of l''^μ and n''^μ are tangent to, we have that asymptotically $\lambda' \sim r$, due to r being an appropriate radial coordinate. Asymptotically too, the metric component appearing in (4.36) goes as $g_2(r) \sim r^2$. The perturbation function $Q = T(t)R(r)\Theta_\ell(\theta)$, hence, manifestly describes the peeling property that the Ψ_2 scalar should display at null infinity (here we are assuming that the perturbation is dynamic, this implies that $T(t)$ cannot be constant). From this analysis we can state that a regular behavior of Q is one that does not alter the $1/r^3$ decay of the Weyl scalar $\hat{\Psi}_2''$ when $r \to \infty$. Also in this case, and from the reduced form of $\partial_t\hat{\psi}_2''$, it can be seen that the $\ell = 0$ and $\ell = 1$ solutions will not yield radiative multipoles due to the vanishing of this Weyl scalar, i.e., as mentioned in subsection 2.4.3, the lowest multipole of gravitational radiation is the quadrupole ($\ell = 2$). The relation shown in equation (4.36) was previously found in the case of perturbations of the Schwarzschild black hole in [67]. There, it was also shown that $\hat{\Psi}_2''$ is invariant under infinitesimal null tetrad rotations and under gauge transformations as well, making this quantity measurable by any observer. Such properties are also valid for the $\hat{\Psi}_2''$ of the gravitational perturbations discussed here.

4.3.2 Energy Flux of Gravitational Radiation

With the conditions of physical regularity established, some other interesting Weyl scalars are examined in the following. Specifically, we will focus on Ψ_0 and Ψ_4 since they describe outgoing and ingoing gravitational waves (see subsection 2.4.1). In this analysis we maintain the twice rotated frame for the background tetrad of the previous subsection. The missing transformed Weyl scalars we require are,

$$\begin{aligned} \Psi_3' =& \Psi_3 + a_1\Psi_4, \quad \Psi_4' = \Psi_4, \\ \Psi_0'' =& \Psi_0', \quad \Psi_4'' = \Psi_4' + 4a_2^*\Psi_3' + 6a_2^{*2}\Psi_2' + 4a_2^{*3}\Psi_1' + a_2^{*4}\Psi_0', \end{aligned} \tag{4.37}$$

with $a_1 = 1$ and $a_2 = -1/2$. For our purposes it shall be convenient to express the perturbed part of the desired scalars as

$$D\hat{\Psi}_0'' = 4D(\hat{\Psi}_2'' + \hat{\Psi}_1 - \hat{\Psi}_1^* + 2\hat{\Psi}_2), \quad D\hat{\Psi}_4'' = \frac{1}{4}D\left(\hat{\Psi}_2'' - \hat{\Psi}_1 + \hat{\Psi}_1^* + 2\hat{\Psi}_2\right). \tag{4.38}$$

The explicit expressions for the $\hat{\Psi}_1$ scalar can be found in (A.7). After several manipulations, which include the use of the master equation and the commutators (A.5), the terms appearing in (4.38) can be rewritten as

$$\begin{aligned} D(\hat{\Psi}_1 - \hat{\Psi}_1^*) =& 2(\delta_+ + 3\pi_- + \kappa_-)(\delta_- - 2\kappa_+)Df_1, \\ D\hat{\Psi}_2 =& \left[D^2 + 2\kappa_-(\delta_+ + 3\pi_- + \kappa_-) - 2\kappa_+(\delta_- + 4\kappa_+)\right](\delta_- - 2\kappa_+)f_1 \\ & - 2\kappa_+(2\Lambda_s - S)f_1. \end{aligned}$$

Notice that the angular operator $\delta_- - 2\kappa_+$ is being applied to the perturbation function f_1 in both expressions. Consider then only the relevant angular part $f_\theta = dP_\ell(\cos\theta)/d\theta$ of this function. From (A.2) it follows that,

$$(\delta_- - 2\kappa_+)f_\theta = \frac{i\sin\theta}{\sqrt{2g_2}}\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{dP_\ell(\cos\theta)}{d\theta}\right], \tag{4.39}$$

which clearly vanishes for the $\ell = 0, 1$ modes and is also regular for all $\theta \in [0, \pi]$. We once again have that if $c = g_2(S - 2\Lambda_s) = 0$, then the expressions for the Weyl scalars in (4.38), reduce to zero for the monopolar and dipolar modes. This is in agreement with the result of the past subsection where it was mentioned that these modes are not dynamical.

We will now be interested in the asymptotic behavior of the $D\hat{\Psi}_0''$ and $D\hat{\Psi}_4''$ scalars keeping $c = 0$. It can be seen, using the explicit forms of the spin coefficients and operators, that for $r \to \infty$ the relevant term for both scalars is

$$D^2(\delta_- - 2\kappa_+)f_1 = -\frac{i\omega^2}{4g_0^{3/2}\sqrt{2g_2}}e^{i\omega t}R(r)\sin\theta\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{dP_\ell(\cos\theta)}{d\theta}\right].$$

Assuming asymptotical flatness ($g_0 \to 1$ and $g_2 \to r^2$ as $r \to \infty$), then these Weyl scalars decay appropriately as,

$$\hat{\Psi}_{0,4}'' \sim e^{i\omega t}\frac{R(r)}{r}\sin\theta\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{dP_\ell(\cos\theta)}{d\theta}\right]. \tag{4.40}$$

When imposing the required conditions of physical regularity to $R(r)$, equation (4.40) allows to find meaningful expressions of energy flux for each multipolar mode $\ell \geq 2$ through the use of (2.58). This energy flux is related to ingoing and outgoing gravitational radiation, namely,

$$\frac{d^2E}{dtd\Omega} = \begin{cases} \frac{1}{4\pi\omega^2} \lim_{r\to\infty} r^2 \|\hat{\Psi}_4''\|^2 & \text{for outgoing gravitational waves,} \\ \frac{1}{64\pi\omega^2} \lim_{r\to\infty} r^2 \|\hat{\Psi}_0''\|^2 & \text{for ingoing gravitational waves.} \end{cases}$$

We have therefore succeeded at associating the expected physical significance of energy flux to the modified Weyl scalars due to odd-parity perturbations. As in the previous subsection for $\hat{\Psi}_2''$, these Weyl scalars are also invariant to gauge transformations and infinitesimal null tetrad rotations. See Appendix A of [72] for a very simple proof of this which applies to our case as well. Finally the remaining scalars, $\hat{\Psi}_1''$ and $\hat{\Psi}_3''$, need not be considered since they are quantities of no physical interest due to them not being invariant under infinitesimal null rotations. This concludes, thus, the analysis of the perturbed Weyl scalars.

4.4 The Morris-Thorne Wormholes

Next we will apply the master equation found in this work to the Morris-Thorne wormhole space-times introduced in section 1.3. Because throughout this chapter the $(+1, -1, -1, -1)$ signature has been employed, the general line element in geometrized units ($G = c = 1$) has to be written as

$$ds^2 = e^{2\Phi(r)}dt^2 - \frac{dr^2}{1 - b(r)/r} - r^2 d\Omega^2. \tag{4.41}$$

This represents just an overall change of sign in the metric components with signature $(-1, +1, +1, +1)$ of equation (1.10). The rest of the geometrical properties and conditions described in section 1.3 need not be modified, they will also be frequently used during the subsequent calculations.

Following Morris and Thorne, we consider matter whose stress-energy tensor in an orthonormal frame is

$$T_{\hat{\mu}\hat{\nu}} = diag\left[\rho, -\tau, p, p\right],$$

whose components are given a physical interpretation in which ρ is the energy density, τ is the tension per unit area in the radial direction, and p is the pressure in the lateral directions[5]. All of these quantities are expressed as measured by a static observer and depend on the coordinate r. Morris and Thorne demonstrated that in order for the space-time to have the geometric properties of a wormhole, the null energy condition must be violated at least near its throat, this is, $\tau(b_0) > \rho(b_0)$ (see subsection 1.3.3). The implication is that there exist observers that measure a negative energy density, this could include a static observer too.

In the canonical $\{t, r, \theta, \varphi\}$ frame, the past stress-energy tensor can be written as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - (\tau + p)v_\mu v_\nu - p g_{\mu\nu}, \tag{4.42}$$

[5]For the sake of clarity we outline that, hereafter, the symbols ρ and τ are no longer used for the spin coefficients of the Newman-Penrose formalism, but instead to represent now the mentioned physical quantities. The same will happen from equation (4.43) forward, where π will denote the usual geometrical constant instead of the spin coefficient.

where u^μ is the 4-velocity of the matter in a co-moving frame and v^μ a unit space-like vector orthogonal to u^μ and pointing in the $x^1 = r$ direction. If $\tau = -p$, the stress-energy tensor of a perfect fluid is recovered. From (4.42), the Ricci tensor is found to be

$$\frac{1}{8\pi} R_{\mu\nu} = (\rho + p) u_\mu u_\nu - (\tau + p) v_\mu v_\nu - \frac{1}{2}(\rho + \tau) g_{\mu\nu}. \tag{4.43}$$

Comparing this expression to (4.20) and (4.21), we have that $S_{\mu\nu}/8\pi = (\rho + p)u_\mu u_\nu - (\tau + p)v_\mu v_\nu$, and hence $c = 0$. Another important matter to take into account is the fact that, as seen from equation (4.11), the one-forms $u_\mu = (l_\mu + n_\mu)/\sqrt{2}$ and $v_\mu = (m_\mu + \bar{m}_\mu)/\sqrt{2}$ are not modified when perturbing the wormholes with (4.10). This is similar to the discussed case in section 4.2 of a perfect fluid and owes to the use of the Regge-Wheeler gauge. Thus, the master equation is valid for this matter model of the Morris-Thorne wormholes, so long as the space-time is not electrically charged (see discussion after equation (4.22)), and when the Ricci tensor can be cast in the form (4.43).

The tortoise coordinate is then defined by

$$\frac{d}{dr_*} = \pm e^{\Phi} \sqrt{1 - \frac{b}{r}} \frac{d}{dr}.$$

In the coordinate transformation performed, one can always choose an adequate integration constant so that the throat of the wormhole is located at $r_* = 0$. Moreover, the r_* coordinate takes the positive sign on one side of the throat, and the negative sign on the other side. When $r \to \infty$, one has that $r_* \to \pm r$. The coordinate r_* therefore takes values on the whole real line, i.e., $r_* \in (-\infty, \infty)$.

Inserting the metric components of line element (4.6) in the general expression (4.30) of the potential $V(r)$ we obtain,

$$V(r) = \frac{e^{2\Phi}}{r^2} \left[\ell(\ell+1) - \frac{5b}{2r} - r\Phi' \left(1 - \frac{b}{r}\right) + \frac{b'}{2} \right].$$

Using the relations (1.17) for the energy density and the tension in terms of the shape and redshift functions, the potential can be rearranged as

$$V(r) = \frac{e^{2\Phi}}{r^2} \left[\ell(\ell+1) - \frac{3b}{r} + 4\pi r^2(\rho + \tau) \right]. \tag{4.44}$$

With the domain of the tortoise coordinate established and an explicit expression for the potential $V(r)$, the stability analysis consists now in studying the eigenvalue equation $\mathcal{H}R = \omega^2 R$ with $\mathcal{H} = -d^2/dr_*^2 + V(r)$. Specifically, if there exist eigenvalues which represent perturbations that grow without bound as $t \to \infty$, but are physically regular otherwise. Since the operator $\mathcal{H}$ is self-adjoint, the eigenvalues ω^2 must be real. Hence, considering the time dependent part of the proposed ansatz, any instability will appear as a purely imaginary ω, this is, as a negative eigenvalue.

A qualitative discussion of the eigenvalue spectrum of the operator $\mathcal{H}$ follows in a fairly simple manner based on the properties of the potential $V(r)$. If the potential is strictly positive there cannot exist negative eigenvalues (energy bound states) whose eigenfunctions are physically regular and thus, all of the vibrational modes of this class of wormholes are linearly stable, at least under odd-parity perturbations. In this case the eigenvalue spectrum is continuous. On the other hand, if $V(r) < 0$ at some region of the space-time, it is possible that regular solutions with negative eigenvalues arise, leading to the instability of at least one of the vibrational modes of the wormhole.

By examining the individual terms that appear in the potential (4.44) one can realize that a sufficiently negative energy density, which is possible due to the violation of the energy conditions, can make $V(r) < 0$ for some coordinate values r. Thus, stability can be seen to strongly depend on the physical parameters ρ and τ. In what follows we will focus on a particular class of Morris-Thorne metrics, those for which $\rho + \tau = 0$, as they will be proven to describe wormholes with no unstable modes of odd-parity gravitational perturbations.

The condition $\rho + \tau = 0$ means physically that the energy density matches the radial pressure of the matter (the negative of the tension). This condition determines a constraint on the redshift and shape functions which can be expressed through equations (1.17), namely,

$$\frac{rb'(r) + b(r)}{2r} + [b(r) - r]\,\Phi'(r) = 0. \tag{4.45}$$

We will now show that the class of Morris-Thorne metrics defined by (4.45), satisfies the conditions that a wormhole must possess. The most compelling way to accomplish this is to rearrange the defining constraint of the class so that the shape function, without its first derivative, is in terms only of the redshift function. This will allow us to pick a suitable $\Phi(r)$, specifically an everywhere finite function, and find the corresponding expression for $b(r)$. Using the basic theory of first-order differential equations one can show that the desired relation between these functions is

$$b(r) = r - \frac{2e^{-2\Phi(r)}}{r}\left[F(r) + c_1\right],$$

where c_1 is an integration constant and

$$F(r) = \int re^{2\Phi(r)}dr.$$

The integration constant can be chosen so that the condition $b(b_0) = b_0$ on the minimum radius $r = b_0$ is fulfilled. Obtaining thus,

$$b(r) = r - \frac{2e^{-2\Phi(r)}}{r}\int_{b_0}^{r} r'e^{2\Phi(r')}dr'. \tag{4.46}$$

From (4.46) and the fact that the integrand there is strictly positive in the domain of integration, it can be seen that the condition $1 - b(r)/r \geq 0$ is satisfied. This also implies that the vector $\partial/\partial r$ remains everywhere space-like. Furthermore, by examining the limit $r \to \infty$ for which $\Phi(r) \to 0$, one can realize that $b(r)/r \to 0$. Therefore, the wormhole fulfills the asymptotically flatness condition too. We have obtained thus a relation for the shape function in which, given an appropriate redshift function, the metrics of interest possess indeed the geometry of a wormhole.

The potential of the Schrödinger operator for this restricted class of Morris-Thorne wormholes is now simply

$$V(r) = \frac{e^{2\Phi}}{r^2}\left[\ell(\ell+1) - \frac{3b}{r}\right].$$

Recall that, for a vanishing value of the constant c (as is the case), we concluded from the analysis of the Weyl scalar Ψ_2 in the previous section that the lowest radiative multipole is the quadrupole. Then, it is readily seen that $V(r) > 0$ for all $r \in [b_0, \infty)$, due to the $1 - b(r)/r \geq 0$ condition and to the fact that ℓ takes positive integer values starting from $\ell = 2$. It can be shown that the asymptotic solution of the eigenfunctions is

$$R \sim r_* [h_\ell^{(1)}(r_* \omega_\pm) + h_\ell^{(2)}(r_* \omega_\pm)] \quad \text{as } r_* \to \pm\infty,$$

where $h_\ell^{(1)}$ and $h_\ell^{(2)}$ denote the spherical Hankel functions of the first kind and of the second kind, respectively, and of order $\ell \geq 2$. Given this behavior and since the potential is strictly positive, the eigenvalue spectrum of the operator $\mathcal{H}$ is positive and continuous. Also, by equation (4.36) and the peeling theorem, the eigenfunctions R with positive eigenvalues ($\omega^2 \geq 0$) will describe physically regular perturbations. Thus, there are no linearly unstable vibrational modes generated by perturbations of odd-parity in this class of wormholes.

To finalize this section we provide some examples of this class of Morris-Thorne wormholes in table 4.1. They are easily obtained utilizing equation (4.46) for the shape function. This process requires only of a well-behaved and bounded redshift function as input and so, can be used to yield as many space-times as functions that exist of this type. Note that the $\Phi(r) = 0$ case reduces to the well-known Ellis-Bronnikov wormhole[6], which additionally is a solution of the Einstein-scalar field equations with a negative sign. Unfortunately, since all of these wormholes belong to the family of Morris-Thorne metrics, they violate the energy conditions at least near their throats. From the metric functions found in table 4.1, one can compute the relevant physical parameters of the wormholes such as energy density $\rho = -\tau$ and lateral pressure p using relations (1.17).

TABLE 4.1: Metric components of a few examples from the class of Morris-Thorne wormholes studied in section 4.4.

$e^{2\Phi(r)}$	$1 - b(r)/r$
1	$1 - b_0^2/r^2$
$1 + e^{-(r/b_0)^2}$	$e^{-2\Phi(r)}[1 - b_0^2(e^{2\Phi(r)} - e^{-1})/r^2]$
$1 + b_0^2/(x^2 + b_0^2)$	$e^{-2\Phi(r)}\left[1 + b_0^2(\ln\left[1/2 + r^2/2b_0^2\right] - 1)/r^2\right]$
$1/2 + \arctan(r/b_0 - 1)/\pi$	$e^{-2\Phi(r)}\left[e^{2\Phi(r)} - b_0/\pi r - b_0^2\left(\pi/2 - 1 + \ln[1 + (1 - r/b_0)^2]\right)/\pi r^2\right]$

Interestingly enough, and although the $\ell = 0$ modes do not yield gravitational radiation as a result of the perturbation, the potential $V(r)$ we deduce here reduces to that studied in [79] for the Ellis-Bronnikov metric with reflection symmetry when inserting the $\ell = 0$ value. In those works the instability of that wormhole follows due to their corresponding potential being negative. This indicates that the angular dependance of the solution proposed here is crucial to deduce stability, at least for the odd-parity case. Of course, the reason why we obtain a different result lies in the type of perturbation we have analyzed during this thesis.

[6] To obtain its more familiar line element (2.71), the transformation from the radial coordinate r to $r_* = \pm\sqrt{r^2 - b_0^2}$ is needed. In this case, the r_* coordinate is the proper radial distance.

4.5 Odd-Parity Perturbations in a Phantom Scalar Field Wormhole

In section 4.2 it was mentioned that the master equation derived there is valid for solutions of the Einstein-scalar field equations. In fact, one of the examples of Morris-Thorne wormholes shown in table 4.1 is indeed a solution of this type, namely the Ellis-Bronnikov metric. In what follows we will examine the perturbation equation of one last example of a wormhole supported by a phantom scalar field, the Ellis-Bronnikov "drainhole" model (1.7) described in subsection 1.2.4. We will, however, rewrite the line element in Boyer-Lindquist coordinates with the pertinent signature change and the redefinition of some constants. Thus,

$$ds^2 = f dt^2 - \frac{1}{f}\left[dr^2 + (r^2 - 2rr_1 + r_0^2)d\Omega^2\right],$$

with $f = e^{-\phi_0(\lambda - \pi/2)}$ and $\lambda = \arctan\left[(r - r_1)/\sqrt{r_0^2 - r_1^2}\right]$. In this coordinate system we have for the Boyer-Lindquist radius that $-\infty < r < \infty$, covering thereby both universes. The quantities r_0 and r_1 are constant parameters whose units are that of length, and for which $r_0^2 > r_1^2$. The scalar field is given by

$$\phi = \sqrt{2 + \phi_0^2/2}(\lambda - \pi/2),$$

being ϕ_0 a constant without units. In this wormhole the throat joins two asymptotically flat sides, nevertheless, these sides are not symmetrical. This can be seen when taking the asymptotic limits of the f function,

$$\lim_{r\to\infty} f = 1, \quad \lim_{r\to-\infty} f = e^{\phi_0 \pi}.$$

By rescaling the t and r coordinates to $t_- = e^{\phi_0\pi/2}t$ and $r_- = e^{-\phi_0\pi/2}r$, it can be realized that indeed the other side of the throat is asymptotically flat as well. The wormhole becomes symmetric only if $\phi_0 = 0$, in which case, the line element reduces to that of the simple Ellis-Bronnikov wormhole. It will be convenient to replace the coordinate r with $x = (r - r_1)/L$, where $L^2 = r_0^2 - r_1^2$. Therefore,

$$ds^2 = f dt^2 - \frac{L^2}{f}\left[dx^2 + (x^2 + 1)d\Omega^2\right], \tag{4.47}$$

and $\lambda = \arctan x$. In these coordinates the throat of the wormhole is located at $x = 0$, while the upper and lower universes are described by $x > 0$ and $x < 0$, respectively.

To obtain the equation that governs the odd-parity gravitational perturbations of this space-time we proceed with the same scheme as in the previous section. The tortoise coordinate is given by

$$\frac{d}{dx_*} = \frac{f}{L}\frac{d}{dx}.$$

Since f is regular for all $x \in \mathbb{R}$ and because of the asymptotic form of said function at both infinities, the new coordinate ranges over the values $-\infty < x_* < \infty$. A suitable integration constant can also be picked so that the throat is described by $x_* = 0$. Assuming a similar ansatz as the one used throughout this paper, $Q = e^{i\omega t}X(x)\sin\theta dP_\ell(\cos\theta)/d\theta$, and substituting the metric functions of the phantom wormhole in (4.30), we have that

$$\frac{d^2X}{dx_*^2} - \left[V(x) - \omega^2\right] X = 0, \tag{4.48}$$

where now

$$V(x) = \frac{f^2}{L^2(x^2+1)} \left[\ell(\ell+1) + \frac{3}{x^2+1}\left(\phi_0 x + \frac{\phi_0^2}{4} - 1\right)\right].$$

In this case equation (4.48) defines an eigenvalue problem for the operator $\mathcal{H} = -d^2/dx_*^2 + V(x)$. This potential coincides with that found in [106] for the case of axial perturbations. Its properties are the same as those of the previous examples in section 4.4. Additionally, it can be easily verified that the second term that appears inside brackets in the expression of $V(x)$ has a global minimum $u_{min} = -3$ at the coordinate value $x = -\phi_0/2$. Hence, appealing to the fact that the $\ell = 2$ vibrational modes are the lowest possible, the potential $V(x)$ is strictly positive for all $x \in \mathbb{R}$. By the same arguments as those mentioned for the former class of Morris-Thorne wormholes, we can conclude that this scalar field wormhole is stable when its metric is perturbed by a small term of odd-parity.

As mentioned before, and just like in the class of Morris-Thorne metrics previously discussed, the symmetric Ellis-Bronnikov space-time is again a particular case of this phantom scalar field wormhole when the parameter ϕ_0 vanishes[7]. The wormhole presented here also violates the energy conditions as a result of it being a solution of the Einstein-scalar field equations with a negative sign.

[7]For this particular case the relation between the proper radial length r_* and the coordinates $x = x_*$ is $Lx = r_*$, with $L = b_0$.

Chapter 5

Conclusions

In this final chapter the conclusions of the thesis will be presented. During this work two particular topics within General Relativity were explored: curvature singularities and its relation with geodesic incompleteness, and gravitational perturbations in wormholes. The principal results obtained regarding both subjects are described in the following section. Some concluding remarks, which include perspectives and possible future work, are given to finalize the thesis along with a reflection of the overall significance of these results.

5.1 Main Results

In chapter 3 we formulated a series of criteria regarding causal geodesics and curvature singularities in stationary and axially symmetric space-times with a quadratic first integral. The criteria were stated in two theorems. The first one establishes the sufficient and necessary conditions for which time-like and null geodesics in such space-times can be found withing every neighborhood of the singularity. The second one determines the conditions for a special situation in which causal geodesics indeed make contact with the singular region in a finite amount of their affine parameter. Both criteria are in terms of the inverse metric tensor and constants of motion along geodesics.

Afterward, the theorems were applied to the general class of Plebański-Carter metrics some of which physically describe black holes, and geometrically belong to the type D algebraic classification of space-times. It was found that in the electromagnetically charged space-times of that class, the singularity is only reached by null geodesics with a specific relation of energy and angular momentum. These curves correspond to the principal null rays of the metric. A further analysis of the geodesic deviation equation for this special congruence of null curves showed that curvature itself does not cause singular behavior on their deviation. This is despite the fact that scalar invariants are unbounded in these space-times. Such a feature is shared with the Plebański-Demiański space-times without electromagnetic charge too. A very basic (and hypothetical) construction based on the addition of a boundary in the singular region was carried over as a proposal to avoid incompleteness in the principal null rays when encountering the singularity. On the other hand, time-like geodesics are repelled from the singularity in the charged metrics, while in the uncharged ones, they can arbitrarily approach it along with other null geodesics different than the principal null rays. Said time-like and null curves constitute principal candidates of incomplete geodesics.

Additionally, and based on the derived theorems, we presented an example of a causal geodesically complete space-time that has a wormhole geometry with unbounded curvature. Finally, two rotating scalar field wormholes with a ring singularity were examined: the Kerr-like phantom and the electromagnetic dipole wormholes. Since these metrics do not admit a quadratic first integral, the first criterion offered only a guide instead of direct results. The analysis was complemented with the search of hyper-surfaces whose inside region is forbidden to geodesics, and also with numerical solutions of this kind of curves. Such numerical tests were necessary because the hyper-surfaces found in these wormholes did not completely cover the singularity. The results revealed that the causal geodesics of the electromagnetic wormholes with a dilatonic field studied here require an infinite amount of their affine parameter to reach the singular ring, leaving opened the possibility of them being space-times with complete causal curves. The rest of the wormholes have incomplete geodesics due to the singularity. In the Kerr-like wormhole these curves come from asymptotic infinity, and in the electromagnetic wormholes with a ghost field they are constrained to the throat.

In chapter 4 we utilized the Newman-Penrose formalism to obtain a so-called master equation that describes the linear behavior of gravitational perturbations in stationary and spherically symmetric space-times. The perturbations were assumed to be of odd-parity in the Regge-Wheeler gauge. This framework allowed us to write the derived master equation in a compact manner through the use of the spin coefficients and operators that characterize the formalism. One of the advantages of using this method against the standard procedure (i.e., computing the perturbed curvature tensors using a coordinate basis) is that the formalism directly provides the equations needed to geometrically describe curvature in the most general setting. In this sense, the fact that Weyl scalars can be used as gauge invariant quantities and are easily accessible within the formalism represents another benefit. Also, a convenient tetrad which exploits the symmetries or algebraic type of the space-time can greatly simplify the computations required when compared to other methods.

On the negative side, our master equation is not applicable to the whole generality of space-times with spherical symmetry, this is due to a constraint on certain components of the Ricci tensor that has to be obeyed. Despite this, we showed that it is well-suited for analyzing some interesting examples of metrics that describe wormholes, for instance, the solutions of the Einstein-scalar field equations. Other space-times that were found to be within the range of validity of our master equation belong to the family of Morris-Thorne wormholes.

In particular, we focused on wormholes whose gravitational source has the distinguishing property that its energy density is equal to its radial pressure. After applying the aforementioned master equation to them, we found that there are no unstable modes of vibration due to odd-parity perturbations. The explicit metric components of some of this type of space-times were presented too. Finally, we gave one last example of a static scalar field wormhole that, according to the properties of its corresponding master equation, is not unstable against the perturbations here studied.

5.2 Future Work and Perspectives

There are unfortunately some limitations in our results. In what concerns curvature singularities and geodesics, the applicability of the criteria here developed is quite restricted as our analysis relies heavily on the separability of the equations of motion. Therefore, a generalization to any axially symmetric line element seems unlikely through this approach. While an attempt was made to apply these results to less restrictive space-times (the two scalar field wormholes of section 3.6), these criteria alone were not enough to determine the singularity encountering or avoidance of all causal geodesics. They served, though, as a guide to study only some of those curves and conclude that they indeed avoid the singularity. The study of the whole totality of causal geodesics in such space-times is impossible with this particular method. On the positive side, these theorems can always be used for stationary space-times with spherical symmetry.

Another important issue lies in the probability that curves of bounded acceleration can also be in contact with curvature singularities and may consequently be incomplete. Here we have focused on geodesics only since this offers a huge simplification on the general analysis and equations. A strictly complete study of a seemingly singular space-time should include all of these considerations. The possibility of the existence of singularities that are unreachable by every physical observer, even an accelerated one, could pose the question of whether there may be other "censorship" mechanisms besides the well-known event horizons. Candidates of such a space-time, provided accelerated observers avoid the singularity as causal geodesics were proven to do, are the wormhole example constructed in section 3.5 and the electromagnetic dipole wormhole with $k < 0$ of section 3.6. On this matter, these space-times provide good examples of cases in which unbounded curvature scalars do not necessarily imply causal geodesic incompleteness in a strict sense. If other non-geodesic curves are shown to reach the singularity, then we are left with particular and interesting instances of singular space-times that are also geodesically complete.

Furthermore, the study of the global and causal properties (global hyperbolicity for instance) in this kind of hypothetical space-times, and in the context of the singularity theorems, can lead to highly interesting results. Aiding thus, in the understanding of the mysterious and elusive objects that are space-time singularities.

In regards to our study of gravitational perturbations in wormholes, it should be borne in mind that, while our results indicate stability for some of the Morris-Thorne metrics, it is only with respect to perturbations of odd-parity. Future developments of this particular work include the treatment of their even-parity counterparts within the Newman-Penrose formalism as well. However, the complexity of the calculations involved for this purpose increases compared to the odd case. For phantom wormholes one would expect to find instabilities in the even parity sector, at least according to recent results mentioned during this work. Nevertheless, the possibility remains that a specific combination of energy density, radial, and lateral pressures in a wormhole can help to stabilize it.

Another interesting aspect to determine is the possibility to generalize the perturbation scheme presented here in the context of the tetrad formalism to axially symmetric space-times. This in turn may imply a generalization of the Regge-Wheeler gauge to this kind of metrics. Yet again, the whole process may require of lengthy calculations that hopefully are still manageable from an analytical approach. If this is successfully achieved, one might be able to mathematically confirm or disprove the conjecture that a rotating wormhole can be stable. For this purpose, the consideration of a slowly rotating wormhole can be helpful.

Finally, two interesting cases of rotating scalar field wormholes were discussed in this thesis. They represent good examples of part of the motivations of this work as they contain curvature singularities and their instability has not been confirmed yet. In order for them to be realistic space-times, their singularities should not induce ill-effects on physical observers traveling through their throats, and then the perturbations caused by gravitational interactions should not collapse them as well. The results of this thesis can be seen as steps forward to prove whether this pair of wormholes, and others with similar characteristics too, are of genuine astrophysical interest or not.

Appendix A

Perturbed NP Quantities

In this appendix we show all the relevant quantities of the Newman-Penrose formalism that appear in chapter 4. They are calculated for metric (4.6) with background tetrad (4.8) and perturbation matrix (4.12). To simplify notation, the use of the tilde for background quantities will be dropped and the hat will be kept for the perturbation terms. Thus, any quantity or operator without a hat should be understood to be of the background space-time, except for the perturbations functions f_0 and f_1.

From (2.30) and equation (4.5) of our text, the perturbation term of the spin coefficients is given by

$$
\begin{aligned}
\hat{\kappa} &= \hat{\mathcal{Z}}_{020} = Df_1 - \frac{1}{2}\delta f_0, & \hat{\pi} &= -\hat{\mathcal{Z}}_{031} = 0, \\
\hat{\nu} &= -\hat{\mathcal{Z}}_{131} = \Delta f_1 - \frac{1}{2}\delta^* f_0, & \hat{\tau} &= \hat{\mathcal{Z}}_{120} = 0, \\
\hat{\rho} &= \hat{\mathcal{Z}}_{320} = (-\delta_- + \kappa_+ - \pi_+)f_1, & \hat{\lambda} &= -\hat{\mathcal{Z}}_{331} = 0, \\
\hat{\mu} &= -\hat{\mathcal{Z}}_{231} = (\delta_- - \kappa_+ + \pi_+)f_1, & \hat{\sigma} &= \hat{\mathcal{Z}}_{220} = 0, \\
\hat{\alpha} &= \frac{1}{2}(\hat{\mathcal{Z}}_{310} - \hat{\mathcal{Z}}_{332}) = \frac{1}{2}(\nu f_0 - Df_1), \\
\hat{\beta} &= \frac{1}{2}(\hat{\mathcal{Z}}_{210} - \hat{\mathcal{Z}}_{232}) = -\frac{1}{2}(\kappa f_0 - Df_1), \\
\hat{\varepsilon} &= \frac{1}{2}(\hat{\mathcal{Z}}_{010} - \hat{\mathcal{Z}}_{032}) = \frac{1}{2}\left[(-\delta_- + \kappa_+ - \pi_+)f_1 - \Delta f_0\right], \\
\hat{\gamma} &= \frac{1}{2}(\hat{\mathcal{Z}}_{110} - \hat{\mathcal{Z}}_{132}) = \frac{1}{2}\left[(\delta_- - \kappa_+ + \pi_+)f_1 - Df_0\right],
\end{aligned}
\tag{A.1}
$$

with the definitions $\delta_\pm = (\delta \pm \delta^*)/2$, $\kappa_\pm = (\kappa \pm \nu)/2$, and $\pi_\pm = (\pi \pm \tau)/2$. In terms of the metric components, these newly defined coefficients and operators take the explicit form,

$$
\begin{aligned}
&\kappa_+ = -\pi_+ = \frac{i\cot\theta}{2\sqrt{2g_2}}, \quad \pi_- + \kappa_- = -2\alpha = \frac{g_2'}{2g_2\sqrt{2g_1}}, \quad \pi_- - \kappa_- = \frac{g_0'}{2g_0\sqrt{2g_1}}, \\
&\delta_+ = \frac{1}{\sqrt{2g_1}}\frac{\partial}{\partial r}, \quad \delta_- = \frac{i}{\sqrt{2g_2}}\frac{\partial}{\partial\theta}, \quad D = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{g_0}}\frac{\partial}{\partial t} + \frac{1}{\sqrt{g_2}\sin\theta}\frac{\partial}{\partial\varphi}\right),
\end{aligned}
\tag{A.2}
$$

where a prime in this set of equations denotes derivation with respect to the radial coordinate r. Note that $\delta_+^* = \delta_+$, $\delta_-^* = -\delta_-$, and that $\kappa_-, \pi_- \in \mathbb{R}$ while κ_+, π_+ are purely imaginary. With this notation, identity (4.15) can be expressed as

$$
\delta_+\kappa_+ = 2\alpha\kappa_+. \tag{A.3}
$$

The linearized Ricci identities can be computed utilizing the Newman-Penrose equations (2.35). Thereby, we obtain (the labels next to each of the following equations refer to the corresponding Ricci identity that was linearized)

$$\begin{aligned}
D\hat{\rho} - \delta^*\hat{\kappa} =& -\kappa^*\hat{\tau} - \hat{\kappa}^*\tau - \kappa(3\hat{\alpha} + \hat{\beta}^* - \hat{\pi}) - \hat{\kappa}(3\alpha + \beta^* - \pi) + \hat{\Phi}_{00}, && \text{(2.35a)}\\
D\hat{\sigma} - \delta\hat{\kappa} =& -(\tau - \pi^* + \alpha^* + 3\beta)\hat{\kappa} - (\hat{\tau} - \hat{\pi}^* + \hat{\alpha}^* + 3\hat{\beta})\kappa + \hat{\psi}_0, && \text{(2.35b)}\\
D\hat{\alpha} - \delta^*\hat{\varepsilon} =& (\hat{\rho} + \hat{\varepsilon}^* - 2\hat{\varepsilon})\alpha + \beta\hat{\sigma}^* - \beta^*\hat{\varepsilon} - \kappa\hat{\lambda} - \kappa^*\hat{\gamma} + (\hat{\varepsilon} + \hat{\rho})\pi + \hat{\Phi}_{10}, && \text{(2.35d)}\\
D\hat{\gamma} - \Delta\hat{\varepsilon} =& (\hat{\tau} + \hat{\pi}^*)\alpha + (\hat{\tau}^* + \hat{\pi})\beta + \tau\hat{\pi} + \hat{\tau}\pi - \nu\hat{\kappa} - \hat{\nu}\kappa\\
&+ \hat{\Psi}_2 - \hat{\Lambda} + \hat{\Phi}_{11}, && \text{(2.35f)}\\
D\hat{\lambda} - \delta^*\hat{\pi} =& 2\pi\hat{\pi} + (\alpha - \beta^*)\hat{\pi} + (\hat{\alpha} - \hat{\beta}^*)\pi - \nu\hat{\kappa}^* - \hat{\nu}\kappa^* + \hat{\Phi}_{20}, && \text{(2.35g)}\\
D\hat{\mu} - \delta\hat{\pi} =& \pi\hat{\pi}^* + \hat{\pi}\pi^* - \pi(\hat{\alpha}^* - \hat{\beta}) - \hat{\pi}(\alpha^* - \beta) - \nu\hat{\kappa} - \hat{\nu}\kappa\\
&+ \hat{\Psi}_2 + 2\hat{\Lambda}, && \text{(2.35h)}\\
\Delta\hat{\lambda} - \delta^*\hat{\nu} =& (3\alpha + \beta^* + \pi - \tau^*)\hat{\nu} + (3\hat{\alpha} + \hat{\beta}^* + \hat{\pi} - \hat{\tau}^*)\nu + \hat{\psi}_4, && \text{(2.35j)}\\
\delta\hat{\alpha} - \delta^*\hat{\beta} =& \alpha\hat{\alpha}^* + \hat{\alpha}\alpha^* + \beta\hat{\beta}^* + \hat{\beta}\beta^* - 2\alpha\hat{\beta} - 2\hat{\alpha}\beta - \hat{\Psi}_2 + \hat{\Lambda} + \hat{\Phi}_{11}, && \text{(2.35l)}\\
\delta\hat{\nu} - \Delta\hat{\mu} =& -\nu^*\hat{\pi} - \hat{\nu}^*\pi + (\tau - 3\beta - \alpha^*)\hat{\nu} + (\hat{\tau} - 3\hat{\beta} - \hat{\alpha}^*)\nu + \hat{\Phi}_{22}, && \text{(2.35n)}\\
\delta\hat{\gamma} - \Delta\hat{\beta} =& (\tau - \beta - \alpha^*)\hat{\gamma} + \hat{\mu}\tau - \hat{\sigma}\nu - \hat{\varepsilon}\nu^* - \beta(\hat{\gamma} - \hat{\gamma}^* - \hat{\mu}) + \alpha\hat{\lambda}^*\\
&+ \hat{\Phi}_{12}, && \text{(2.35o)}\\
\delta\hat{\tau} - \Delta\hat{\sigma} =& (\tau + \beta - \alpha^*)\hat{\tau} + (\hat{\tau} + \hat{\beta} - \hat{\alpha}^*)\tau - \kappa\hat{\nu}^* - \hat{\kappa}\nu^* + \hat{\Phi}_{02}, && \text{(2.35p)}
\end{aligned}$$

where we have taken advantage of the property $\hat{D}_m\phi = 0$ that our particular choice of tetrad gives us for arbitrary background scalars ϕ. We have also omitted the background terms that should appear on both sides of these equations since they cancel each other out.

The commutators (2.32) of the background differential operators of the formalism are,

$$\begin{aligned}
[\Delta, D] =& 0, \quad [\delta, D] = -\pi^* D + \kappa\Delta, \quad [\delta, \Delta] = -\nu^* D + \tau\Delta,\\
[\delta^*, \delta] =& 2\alpha(\delta - \delta^*). && \text{(A.4)}
\end{aligned}$$

These expressions can be utilized to derive the commutation relations for our previously introduced operators $\delta_\pm$,

$$[\delta_\pm, D] = \mp\pi_\mp D + \kappa_\mp\Delta, \quad [\delta_\pm, \Delta] = \kappa_\mp D \mp \pi_\mp\Delta, \quad [\delta_-, \delta_+] = -2\alpha\delta_-. \qquad \text{(A.5)}$$

When applying these commutators to φ-independent scalar quantities ϕ (as will always be the case in chapter 4) there is a further simplification

$$[\delta_-, D]\phi = [\delta_-, \Delta]\phi = 0,$$

since $D\phi = \Delta\phi$ and $\kappa_+ + \pi_+ = 0$.

After some considerable algebraic steps, reduced equations for the linearized Ricci identities can be obtained by inserting the perturbed spin coefficients (A.1) into the mentioned identities presented above, along with the further aid of the commutators in (A.5) and the spin coefficient properties (4.9, 4.14). Doing so yields

$$
\begin{aligned}
\hat{\Phi}_{00} = -\hat{\Phi}_{22} =& \frac{1}{2}\left[(\delta_+ + \kappa_- + 3\pi_-)\delta_+ - (\delta_- + 2\kappa_+)\delta_- + 4(\kappa_+^2 - \kappa_-^2)\right] f_0 \\
& - (\delta_+ - 6\alpha) D f_1, \\
\hat{\Phi}_{12} = \hat{\Phi}_{21}^* = -\hat{\Phi}_{01} = -\hat{\Phi}_{10}^* =& \frac{1}{2}\left[D^2 + (\delta_+ + \delta_- + \kappa_- + 3\pi_- + 4\kappa_+)(\delta_- - 2\kappa_+)\right] f_1 \\
& - \frac{1}{4}(\delta_+ + \delta_- + \pi_- - 3\kappa_- - 2\kappa_+) D f_0, \\
\hat{\Phi}_{11} = \hat{\Phi}_{20} = \hat{\Phi}_{02}^* = \hat{\Lambda} =& 0.
\end{aligned}
\tag{A.6}
$$

For the Weyl scalars we obtain,

$$
\begin{aligned}
\hat{\Psi}_0 = -\hat{\Psi}_4^* =& \frac{1}{2}(\delta_+ + \delta_- - \kappa_- + \pi_- + 2\kappa_+)\left[(\delta_+ + \delta_-) f_0 - 2 D f_1\right] \\
& - 2(\kappa_+ + \kappa_-)\left[D f_1 + (\kappa_+ + \kappa_-) f_0\right], \\
\hat{\Psi}_1 = -\hat{\Psi}_3^* =& \frac{1}{4}(\delta_+ + \delta_- + \pi_- - 3\kappa_- - 2\kappa_+) D f_0 \\
& - \frac{1}{2}\left[D^2 - (\delta_+ + \delta_- + 3\kappa_- + 4\kappa_+ + \pi_-)(\delta_- - 2\kappa_+)\right] f_1, \\
\hat{\Psi}_2 =& \delta_- D f_1 + (\kappa_- \delta_- - \kappa_+ \delta_+) f_0.
\end{aligned}
\tag{A.7}
$$

Appendix B

The Constraint on the $S_{\mu\nu}$ Tensor

A more detailed proof of the consistency condition (4.26) of the linearized Einstein field equations that were treated in section 4.2 is presented in this appendix.

When applying the operator D to it, and using the commutators (A.4), the component $\hat{\mathcal{R}}_{03}$ of system (4.23) reduces to

$$\begin{aligned} 0 =& 2(\delta_+ - 4\alpha + 2\pi_-)D^2 f_1 \\ &+ \left[(\delta_- + 2\kappa_+)\delta_- - (\delta_+ + 4\pi_-)(\delta_+ + \pi_- - \kappa_-) + 4(\kappa_-^2 - \kappa_+^2)\right. \\ &\left. +2(3\Lambda - \Phi_{11}) - S\right] D f_0. \end{aligned} \tag{B.1}$$

Expression (4.24) for the perturbation function f_0 can now be substituted in (B.1). The resulting terms can be rearranged as

$$\begin{aligned} 0 =& - D f_0 \left[(\delta_+ + \kappa_- + 3\pi_-)\kappa_- + 2\kappa_+^2 - 3\Lambda + \Phi_{11} + S/2\right] \\ &+ (\delta_+ - 4\alpha + 2\pi_-)\left([D^2 + (\delta_- + 4\kappa_+)(\delta_- - 2\kappa_+)\right. \\ &\left. - (\delta_+ - 3\kappa_- + \pi_-)(\delta_+ + 2\pi_-)]f_1 + 2f_1[\delta_- + 4\kappa_+]\kappa_+\right). \end{aligned}$$

Careful attention must be paid on the order in which the operators are being applied. The previous equation can be simplified by using (4.25) and defining the quantities $\mathcal{A} = 2(\delta_- + 4\kappa_+)\kappa_+ - 2(3\Lambda - \Phi_{11} + \Phi_{00}) + S$, as well as $\mathcal{B} = 2(\delta_+ + \kappa_- + 3\pi_-)\kappa_- + 4\kappa_+^2 - 2(3\Lambda + \Phi_{11}) + S$. Despite the appearance of differential operators in these quantities, $\mathcal{A}$ and $\mathcal{B}$ should not be understood as such. They are merely scalar quantities, the operators $\delta_\pm$ in them are meant to be applied only to the spin coefficients $\kappa_\pm$. Hence, we can write

$$\left[(\delta_+ - 4\alpha + 2\pi_-)\mathcal{A} - \mathcal{B}(\delta_+ + 2\pi_-)\right] f_1 = 0. \tag{B.2}$$

Expanding the first term of (B.2) leads to

$$f_1\left[(\delta_+ - 4\alpha + 2\pi_-)\mathcal{A} - 2\pi_-\mathcal{B}\right] + 2(\mathcal{A} - \mathcal{B})\delta_+ f_1 = 0. \tag{B.3}$$

Using the background Ricci identities of the Newman-Penrose formalism (2.35a) and (2.35b), it can be proven that

$$\mathcal{A} = \mathcal{B} = 4\kappa_+^2 - (\psi_0 + \psi_0^*)/2 - \Phi_{00} - 2(3\Lambda - \Phi_{11}) + S.$$

Another helpful identity, consequence of (A.3) and (A.5), is $(\delta_+ - 4\alpha)(\delta_- + 4\kappa_+)\kappa_+ = 0$. Equation (B.3) thereby simplifies to[1]

[1] When considering the more general relation $Df_0 - 2(\delta_+ + 2\pi_-)f_1 = C(t,r)\sin\theta$ mentioned in footnote 3 of section 4.2, an additional term $[\mathcal{D}_r + 1/2g_2(r)]\,C(t,r)$ appears in the left-hand side of

$$f_1(\delta_+ - 4\alpha)(S - 2\Lambda_s) = 0, \tag{B.4}$$

with $\Lambda_s = 3\Lambda - \Phi_{11} + \Phi_{00}$. By considering the explicit form of the spin coefficients and operators shown in (A.2), and since f_1 cannot vanish, this past condition can be rewritten as

$$\frac{1}{g_2\sqrt{2g_1}}\frac{d}{dr}\left[g_2(S - 2\Lambda_s)\right] = 0.$$

If this equation is true everywhere in space-time the implication is that

$$S - 2\Lambda_s = c/g_2, \tag{B.5}$$

where c is an integration constant. Furthermore, using equations (2.33), we have that $R_{\mu\nu}l^\mu n^\nu = 2(3\Lambda - \Phi_{11})$ and $R_{\mu\nu}l^\mu l^\nu = -2\Phi_{00}$. From the particular tetrad considered here and the expressions (4.20) and (4.21) for $R_{\mu\nu}$ and $S_{\mu\nu}$, it can be seen that

$$S - 2\Lambda_s = S_{22}/g_2. \tag{B.6}$$

Comparing (B.5) with (B.6), the result that was anticipated in section 4.2 of the main text is obtained, i.e., $S_{22} = c$.

(B.4), where $\mathcal{D}_r = 2(3\Lambda - \Phi_{11}) - S - 2(\delta_+ + 3\pi_- + \kappa_-)\kappa_-$. Since the space-times analyzed during chapter 4 all fulfill $(\delta_+ - 4\alpha)(S - 2\Lambda_s) = 0$, then equation (B.4) becomes $[\mathcal{D}_r + 1/2g_2(r)]\, C(t,r) = 0$.

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