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Reconstrucción y Automorfismos de las Gráficas de Fichas

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Reconstruction and Automorphisms of Token Graphs

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Abstract

This thesis is devoted to the study of token graphs. For a simple graph G of order n and an integer k with $1 \leq k \leq n-1$, the *k-token graph $F_k(G)$ of G* is the graph whose vertices are all the k-subsets of vertices of G in which two such k-subsets are adjacent whenever their symmetric difference is a pair of adjacent vertices of G. An example of token graphs are the Johnson graphs. For n and k with $n > k \geq 1$, the Johnson graph $J(n,k)$ is the graph whose vertices are the k-subsets of the set $\{1, 2, \ldots, n\}$, where two of these vertices are adjacent if they intersect in $k-1$ elements. Thus, the Johnson graph $J(n,k)$ is the k-token graph of the complete graph K_n. As far as we know, token graphs have been defined, independently, four times since 1988; they have several applications, for example in Physics and Coding theory. In this thesis we study the following problems: reconstruction of token graphs, automorphism group of token graphs, and connectivity of token graphs of trees. Besides of the study of these three problems, this thesis aims to provide new strategies and tools for the study of other parameters of token graphs in the future.

The reconstruction problem of token graphs can be stated as follows: Given a graph F isomorphic to the k-token graph of some graph G, determine if G is unique (up to isomorphism), and if so, to construct a graph isomorphic to G. Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia and Wood (GC 2012) conjectured the following. "Given two graphs G and H, if $F_k(G)$ is isomorphic to $F_k(H)$ for some k, then G and H are isomorphic". In this thesis we prove this conjecture for an infinite family of graphs: the (C_4, D_4)-free graphs, where C_4 denotes the cycle graph of four vertices and the diamond graph D_4 is a 4-cycle with one chord. More specifically, we show that if F is a graph isomorphic to the k-token graph of G, for some (C_4, D_4)-free graph G, then we can construct a graph J isomorphic to G. Moreover, if G is connected, such construction can be done in polynomial time.

Regarding the problem of determining the automorphism group of token graphs, there are only a few results in the literature. It is known that the automorphism group $\mathrm{Aut}(G)$ of G is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(F_k(G))$ of $F_k(G)$ when $k \neq \frac{|G|}{2}$, and that the direct product $\mathbb{Z}_2 \times \mathrm{Aut}(G)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(F_k(G))$ of $F_k(G)$ when $k = \frac{|G|}{2}$. For the complete graph K_n, it is known that $\mathrm{Aut}(K_n)$ is isomorphic to $\mathrm{Aut}(F_k(K_n))$ when $k \neq \frac{n}{2}$, and $\mathbb{Z}_2 \times \mathrm{Aut}(K_n)$ is isomorphic to $\mathrm{Aut}(F_k(K_n))$; see, e.g., the work of Jones (EJC 2005). Recently, Ibarra and Rivera (Arxiv 2019) showed that if G is a cycle, star, fan or wheel graph, then $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{Aut}(F_2(G))$, and for the path graph P_n of order n, they showed that $\mathrm{Aut}(P_n)$ is isomorphic to $\mathrm{Aut}(F_k(P_n))$, for any $k \neq n/2$. In this thesis we show a more general result: If G is a connected (C_4, D_4)-free graph, then $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{Aut}(F_k(G))$ when $k \neq \frac{|G|}{2}$, and $\mathbb{Z}_2 \times \mathrm{Aut}(G)$ is isomorphic to $\mathrm{Aut}(F_k(G))$ when $k = \frac{|G|}{2}$. Moreover, we also show that the automorphism group of token graphs is strongly related to the problem of

reconstruction of token graphs. This relationship is also developed in this thesis. We also study the automorphism group of token graphs of two families of graphs: complete bipartite graphs and Cartesian product of graphs. With these two families we exhibit an infinite number of graphs G for which $\mathrm{Aut}(G)$ is isomorphic to a proper subgroup of $\mathrm{Aut}(F_k(G))$ when $k \neq \frac{|G|}{2}$, and $\mathbb{Z}_2 \times \mathrm{Aut}(G)$ is isomorphic to a proper subgroup of $\mathrm{Aut}(F_k(G))$ when $k = \frac{|G|}{2}$. Before this work, only a finite number of examples satisfying this property were known.

Finally, we study the connectivity of token graphs of trees. Let us first mention some results on the connectivity of token graphs. Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia and Wood (GC 2012) showed that a graph G is connected if and only if $F_k(G)$ is connected, and moreover, they showed that the connectivity of $F_k(G)$ is at least the connectivity of G, and conjectured that if G is t-connected, for $t \geq k$, then $F_k(G)$ is $k(t-k+1)$-connected. They also exhibited an infinite family of graphs attaining this lower bound. Leaños and Trujillo-Negrete (GC 2018) proved this conjecture. Later, Leaños and Ndjatchi (GC 2021) proved an analogous result for the edge-connectivity of token graphs. Notice that connectivity and edge-connectivity of trees do not hold these properties, so the best known result for the connectivity and edge-connectivity of token graphs of trees was that they are 1-connected and 1-edge-connected. In this thesis we show that the connectivity of token graphs of trees is best possible, that is, the connectivity of the k-token graph of a tree T is equal to the minimum degree of the k-token graph of T. We believe this result can be generalized to other connected graphs.

Resumen

Esta tesis está dedicada al estudio de las gráficas de fichas. Dada una gráfica simple G de orden n y un entero k con $1 \leq k \leq n-1$, la *gráfica de k-fichas $F_k(G)$ de G* es la gráfica cuyos vértices son todos los k-conjuntos de vértices de G y donde dos k-conjuntos son adyacentes si su diferencia simétrica es un par de vértices adyacentes en G. Un ejemplo de las gráficas de fichas son las gráficas de Johnson. Dados n y k con $n > k \geq 1$, la gráfica de Johnson $J(n,k)$ es la gráfica cuyos vértices son los k-conjuntos del conjunto $\{1, 2, \ldots, n\}$, donde dos k-conjuntos son adyacentes si intersectan en $k-1$ elementos. Por tanto, la gráfica de Johnson $J(n,k)$ es la gráfica de k-fichas de la gráfica completa K_n. Hasta donde sabemos, las gráficas de fichas se han definido cuatro veces, de manera independiente, desde 1988; tienen distintas aplicaciones, por ejemplo en Física y Teoría de Códigos. En esta tesis estudiamos los siguientes problemas: reconstrucción de gráficas de fichas, grupo de automorfismos de gráficas de fichas y conexidad de las gráficas de fichas de los árboles. Además del estudio de estos tres problemas, el objetivo de esta tesis es proporcionar nuevas estrategias y herramientas para el estudio de otros parámetros de las gráficas de fichas en un futuro.

El problema de reconstrucción de gráficas de fichas puede enunciarse de la siguiente manera: Dada una gráfica F isomorfa a la gráfica de k-fichas de alguna gráfica G, determinar si la gráfica G es única (salvo isomorfismos), y en dicho caso, construir una gráfica isomorfa a G. Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia y Wood (GC 2012) conjeturaron lo siguiente. "Dadas dos gráficas G y H, si $F_k(G)$ es isomorfa a $F_k(H)$ para algún k, entonces G y H son isomorfas". En esta tesis demostramos esta conjetura para una familia infinita de gráficas: las gráficas libres de 4-ciclos y diamantes, donde la gráfica diamante es un 4-ciclo con una cuerda. Mas específicamente, demostramos que si F es una gráfica isomorfa a la gráfica de k-fichas de G, para alguna gráfica G libre de 4-ciclos y diamantes, entonces podemos construir una gráfica J isomorfa a G. Más aún, si G es conexa, dicha construcción puede hacerse en tiempo polinomial.

Con respecto al problema de determinar el grupo de automorfismos de las gráficas de fichas, hay pocos resultados en la literatura. Se sabe que el grupo de automorfismos $\mathrm{Aut}(G)$ de G es isomorfo a un subgrupo del grupo de automorphismos $\mathrm{Aut}(F_k(G))$ de $F_k(G)$ cuando $k \neq \frac{|G|}{2}$, y que el producto directo $\mathbb{Z}_2 \times \mathrm{Aut}(G)$ es isomorfo a un subgrupo de $\mathrm{Aut}(F_k(G))$ cuando $k = \frac{|G|}{2}$. Para la gráfica completa K_n, se sabe que $\mathrm{Aut}(K_n)$ es isomorfo a $\mathrm{Aut}(F_k(K_n))$ cuando $k \neq \frac{n}{2}$, y que $\mathbb{Z}_2 \times \mathrm{Aut}(K_n)$ es isomorfo a $\mathrm{Aut}(F_k(K_n))$ cuando $k = \frac{n}{2}$; vea, por ejemplo, el trabajo de Jones (EJC 2005). Recientemente, Ibarra y Rivera (Arxiv 2019) demostraron que si G es un ciclo, una estrella, una gráfica abanico o una gráfica rueda, entonces $\mathrm{Aut}(G)$ es isomorfo a $\mathrm{Aut}(F_2(G))$; y para el camino P_n de orden n mostraron que $\mathrm{Aut}(P_n)$ es isomorfo a $\mathrm{Aut}(F_k(P_n))$, para cualquier $k \neq n/2$. En esta tesis mostramos un resultado más general: Si G es una gráfica conexa y libre de 4-ciclos y diamantes, entonces $\mathrm{Aut}(G)$ es isomorfo a $\mathrm{Aut}(F_k(G))$ cuando $k \neq \frac{|G|}{2}$, y

$\mathbb{Z}_2 \times \mathrm{Aut}(G)$ es isomorfo a $\mathrm{Aut}(F_k(G))$ cuando $k = \frac{|G|}{2}$. Más aún, demostramos que el grupo de automorfismos de las gráficas de fichas está fuertemente relacionado al problema de reconstrucción de gráficas de fichas. En esta tesis desarrollamos dicha relación. También estudiamos el grupo de automorfismos de las gráficas de fichas de dos familias de gráficas: las gráficas bipartitas completas y el producto Cartesiano de gráficas. Con estas dos familias exhibimos un número infinito de gráficas G para las cuales $\mathrm{Aut}(G)$ es un subgrupo propio de $\mathrm{Aut}(F_k(G))$ cuando $k \neq \frac{|G|}{2}$, y $\mathbb{Z}_2 \times \mathrm{Aut}(G)$ es un subgrupo propio de $\mathrm{Aut}(F_k(G))$ cuando $k = \frac{|G|}{2}$. Previo a este trabajo, solo se conocían un número finito de ejemplos con esta propiedad.

Finalmente, estudiamos la conexidad de las gráficas de fichas de los árboles. Mencionamos primero algunos resultados sobre la conexidad de gráficas de fichas. Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia y Wood (GC 2012) demostraron que una gráfica G es conexa si y solo si $F_k(G)$ es conexa, y más aún, que la conexidad de $F_k(G)$ es al menos la conexidad de G. Ellos conjeturaron que si G es una gráfica t-conexa con $t \geq k$, entonces $F_k(G)$ es $k(t-k+1)$-conexa; además, exhibieron una familia infinita de gráficas para las cuales esta cota inferior es justa. Leaños y Trujillo-Negrete (GC 2018) demostraron esta conjetura. Más adelante, Leaños y Ndjatchi (GC 2021) demostraron un resultado análogo para la arista-conexidad de las gráficas de fichas. Note que la conexidad y arista-conexidad de los árboles no satisfacen estas propiedades, por lo que la mejor cota conocida para las gráficas de fichas de los árboles es que son 1-conexas y 1-arista-conexas. En esta tesis demostramos que la conexidad de las gráficas de fichas de los árboles es lo mejor posible, es decir, que la conexidad de la gráfica de k-fichas de un árbol T es igual al grado mínimo de la gráfica de k-fichas de T. Creemos que este resultado puede generalizarse para otras gráficas conexas.

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Contents

Chapter 1

Introduction

The interest in graph theory has been increasing along the time, and one reason is that there have been discovered several applications of graph theory to distinct areas, such as physics, chemistry, biology, computer science, electrical engineering and operational research. The main reason for this is that any system involving a binary relation can be modeled by a graph. Graph theory is intimately related to other branches of mathematics, such as group theory, probability, matrix theory and combinatorics.

The origin of graph theory can be traced to the works of Euler in the decade of 1730, with the famous *Königsberg Bridge Problem*. Euler is considered to be the most prolific mathematician in history, and is also known as the father of graph theory as well as topology. There was a puzzle involving the bridges of the city of Königsberg, Germany. Königsberg was divided by a river into four distinct regions connected by seven bridges. The puzzle consists in the following: beginning at any of the four regions, walk across each bridge exactly once and return to the initial point. Before Euler, no citizen of Königsberg could provide a successful route, neither proving that it was impossible. Rather than treating this specific situation, Euler generalized this problem to any number of regions and any number of bridges. In 1736, Euler presented a paper containing the solution to the Königsberg Bridge Problem, as well as the solution to the generalized problem. In his honor, a graph is said to be *Eulerian* if it contains a closed walk containing each edge exactly once.

This thesis is devoted to the study of some parameters of token graphs. Consider a simple finite graph G of order $n \geq 2$ and k an integer such that $1 \leq k \leq n-1$. The *k-token graph* $F_k(G)$ of G is the graph whose vertices are the k-subsets of $V(G)$; two of which are adjacent if their symmetric difference is a pair of adjacent vertices in G. In Figure 1.1 is depicted an example. An example of token graphs are the Johnson graphs. For $n > k \geq 1$, the Johnson graph $J(n,k)$ is the graph whose vertices are the k-subsets of the set $[n] := \{1, 2, \ldots, n\}$, where two of these vertices are adjacent if they intersect in $k-1$

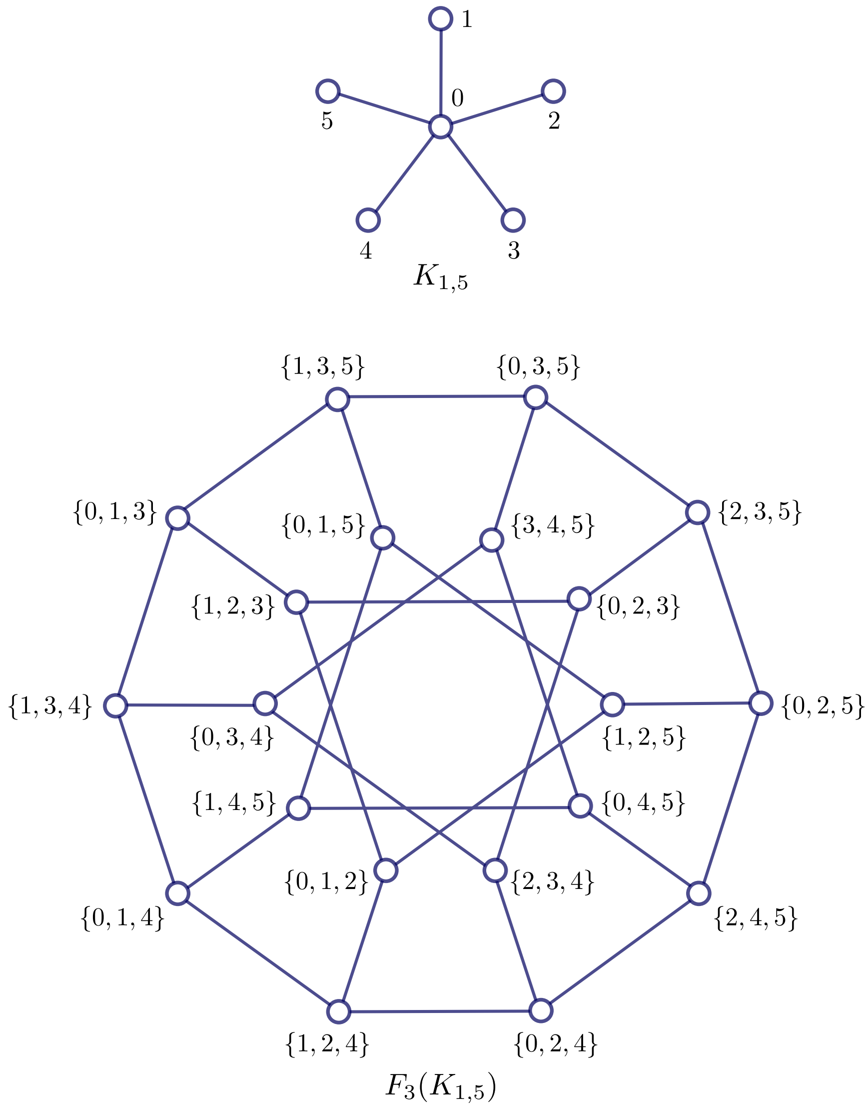


Figure 1.1: Graph $K_{1,5}$ and its 3-token graph $F_3(K_{1,5})$.

elements. Thus, we have $J(n,k) = F_k(K_n)$. In this thesis we study the following three problems: the reconstruction of token graphs, the automorphism group of token graphs as well as the connectivity of token graphs of trees.

The name of "token graph" is motivated by the following interpretation. Take k indistinguishable tokens and place them on the vertices of G (at most one token per vertex); define a new graph whose vertices are all the possible token configurations, and make two configurations adjacent if one can be reached from the other by taking a token and sliding it along an edge to an unoccupied vertex. The resulting graph is isomorphic to $F_k(G)$. In Figure 1.2 is depicted the 3-token graph of $K_{1,5}$ as this model of tokens sliding along the edges of the graph. Indeed, this interpretation of token graphs has been very useful to show many of the results presented in this thesis.

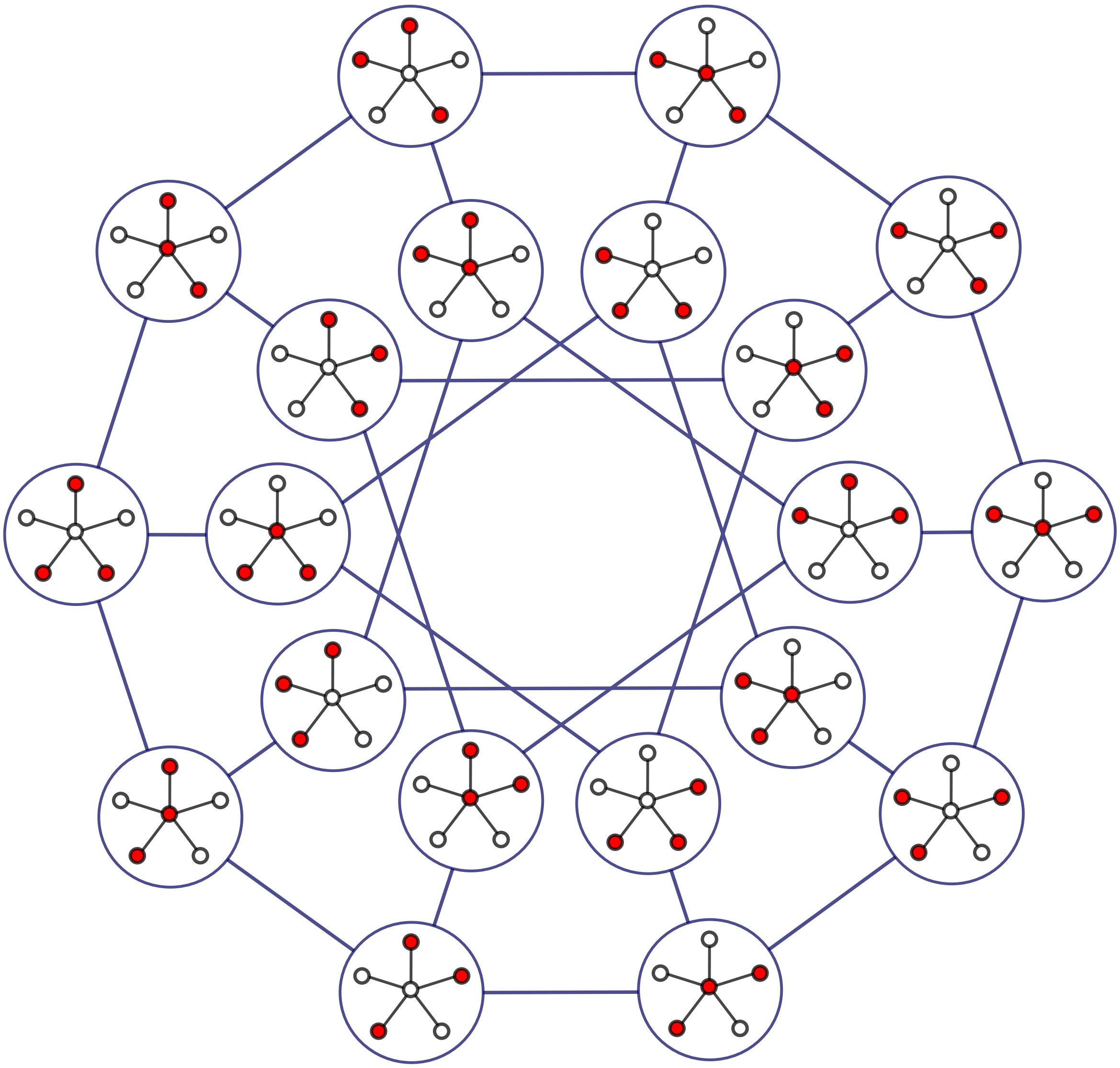


Figure 1.2: The 3-token graph of $K_{1,5}$ represented as a model of tokens sliding along the edges of the graph.

1.1 Background

To our knowledge, token graphs have been defined, independently, four times, each of them with different approaches and different aims. Next, we give a short summary of these lines of research.

1. In 1988, in his PhD thesis, Johns [32] defined the k-token graph of G under the name of *k-subgraph graph* of G, where the vertices of the k-subgraph graph are the subgraphs of k vertices of G, and two subgraphs F and H are adjacent if their distance is one; that is, if there exist adjacent vertices u and v in G such that $u \in F$, $v \in H$ and $V(F \setminus \{u\}) = V(H \setminus \{v\})$.

2. In 1991, Alavi, Behzad, Erdős and Lick [3] defined the 2-token graph under the name of *double vertex graph*. In 1992, Zhu, Liu, Lick and Alavi [54] extended the definition to $k \geq 2$ and called it the *k-tuple vertex graph*. They defined the k-tuple vertex graph of G as the graph whose vertices are the k-subsets of vertices of G, two of which are adjacent if their symmetric difference is a pair of adjacent vertices of G. Following this line of research, several authors studied combinatorial parameters of token graphs, such as Eulerianess, Hamiltonicity, chromatic number, connectivity, regularity and planarity, see e.g., [3, 4, 5, 31, 54].

3. In 2002, Rudolph [45] defined the k-token graph, calling it the k-level matrix, with the following physical interpretation. Consider a cluster of n interacting qubits (2-level atoms). Each qubit can be in the *ground state* $|0\rangle$ or in the *excited state* $|1\rangle$. At any given moment, exactly k qubits are in the excited state. Represent this system of qubits with a graph G in which the qubits are the vertices of G, being two qubits adjacent if they interact. The k-token graph of G represents the possible evolution of this cluster of qubits. His aim was to study the Isomorphism problem of graphs by translating physical quantities of a cluster of qubits to graph invariants of token graphs, in particular he payed special attention to the spectra of token graphs. In his work, Rudolph gave two cospectral graphs such that their 2-token graphs are not cospectral, showing with this that the original graphs are not isomorphic. He was wondering if the spectra of token graphs could be sufficient to distinguish isomorphic graphs. As expected, this was answered in the negative. First, in 2007, Audenaert, Godsil, Royle and Rudolph [8] showed the existence of pairs of non-isomorphic cospectral graphs such that their 2-token graphs are cospectral. Later, in 2009, Barghi and Ponomarenko [12], and independently, in 2010, Alzaga, Iglesias and Pignol [7], showed that for each k, there exist infinitely many pairs of non-isomorphic graphs such that their k-token graphs are cospectral. In [8], the authors renamed the k-level matrix as the *symmetric k-th power*. Following this line of research, several authors have continued with the study of the possible applications of token graphs to Physics, see e.g., [23, 24, 40].

4. In 2012, Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia and Wood [21] defined token graphs as the model presented above of k indistinguishable tokens moving on a graph along its edges. They studied several combinatorial parameters of token graphs, such as connectivity, diameter, chromatic and clique numbers, and Hamiltonian paths. This line of research has been continued by different authors, see, e.g., [15, 17, 34, 35, 39]. Other parameters of token graphs that have been studied are: Hamiltonicity, Eulerianess, connectivity, edge-connectivity, regularity, planarity, independence number, matching number, well-coveredness and automorphism group.

Let us now mention some basic properties of token graphs. Let G be a graph of order n and size m, and let k be an integer with $1 \leq k \leq n-1$.

a) The vertices of $F_k(G)$ are all the k-subsets of vertices of G, so the order of $F_k(G)$ is $\binom{n}{k}$.

b) Note that for each edge ab of G, there exist $\binom{n-2}{k-1}$ edges AB in $F_k(G)$ such that $A \triangle B = \{a, b\}$ (here, the elements in $A \cap B$ can be taken from the elements in $V(G) \setminus \{a, b\}$, where $|A \cap B| = k-1$), and conversely, each edge of $F_k(G)$ is of this type. Now, since G has m edges, it follows that the size of $F_k(G)$ is $m\binom{n-2}{k-1}$.

c) Let $\varphi : F_k(G) \to F_{n-k}(G)$ be the function that maps each k-subset A to its complement $V(G) \setminus A$. This map φ is bijective, and for any two distinct vertices $A, B \in F_k(G)$ we have $A \triangle B = (V(G) \setminus A) \triangle (V(G) \setminus B)$, implying that φ is an isomorphism, and so, $F_k(G) \simeq F_{n-k}(G)$. This property allows us to assume that $k \leq n/2$ if necessary.

d) Note that $G \simeq F_1(G)$, with the trivial isomorphism that maps each vertex $x \in V(G)$ to the 1-subset $\{x\} \in V(F_1(G))$. Combining this fact with the previous fact we have $G \simeq F_1(G) \simeq F_{n-1}(G)$. To avoid trivial cases, by this property we may assume that $2 \leq k \leq n-2$.

For completeness of this background, next we list some relevant known results of token graphs, assuming $2 \leq k \leq n-2$. We also include in this list the results obtained in this thesis.

Diameter: In 2012, Fabila-Monroy et. al. [21] showed that if G is connected with diameter δ, then $F_k(G)$ is connected with diameter at least $k(\delta - k + 1)$ and at most $k\delta$; these bounds are tight.

Connectivity: In 2012, Fabila-Monroy et. al. [21] showed that G is connected if and only if $F_k(G)$ is connected. Moreover, in [21], the authors showed that the connectivity of $F_k(G)$ is at least the connectivity of the graph G. Besides, they conjectured that if G is t-connected and $t \geq k$, then the connectivity of $F_k(G)$ is at least $k(t-k+1)$, they also showed an infinite family attaining this lower bound. This conjecture was proven in 2018 by Leaños and Trujillo-Negrete [35].

In this thesis we study the connectivity of token graphs of trees. We show that the connectivity of token graphs of trees is best possible, that is, we show that the connectivity of $F_k(G)$ is equal to its minimum degree, when G is a tree. We believe this result also holds for other families of graphs, and our conjecture is that if G is a connected graph with girth at least five, then $F_k(G)$ has connectivity equal to its minimum degree.

Edge-connectivity: In 2019, Leaños and Ndjatchi [34] showed that if G is ℓ-edge-connected and $\ell \geq k$, then the edge-connectivity of $F_k(G)$ is at least $k(\ell - k + 1)$; they also provided an infinite family of graphs attaining this lower bound.

Our result for the connectivity of token graphs of trees implies trivially that the edge-connectivity of $F_k(G)$ is also best possible, when G is a tree.

Chromatic number: Let us denote the chromatic number of a graph H by $\chi(H)$.

For the Johnson graph $J(n,k)$ [18] (recall that $J(n,k) \simeq F_k(K_n)$) it is known that $\chi(J(n,k)) \leq n$, and there are some cases with $\chi(J(n,k)) < n$, for example, if n is even then $\chi(J(n,2)) = n-1$; for more results on $\chi(J(n,k))$ we refer to [18].

In 2012, Fabila-Monroy et. al. [21] showed the following results:

(1) $\chi(F_k(G)) \leq \chi(G)$;

(2) $\chi(F_k(G)) = 2$ if and only if $\chi(G) = 2$;

(3) $\chi(F_k(G)) \geq \frac{n-k+2}{n}\chi(G) - 1$; and,

(4) $\chi(F_k(G)) \geq (\frac{1}{2} + \frac{2}{n})\chi(G) - 1$.

The following open question was posed in [21]: "Does there exist a constant $c > 0$ such that $\chi(F_k(G)) > \chi(G) - c$ for every graph G and integer $k \geq 1$?" To our knowledge, this question remains open.

Clique number: Let us denote the clique number of a graph H by $\omega(H)$.

This parameter has been completely determined in [21]. The authors showed that for a subset X of $F_k(G)$, X is a clique if and only if there is a clique K of G and a set S of $G - K$ and either:

(a) $X = \{S \cup \{v\} : v \in K\}$ and $|S| = k-1$, or,

(b) $X = \{(S \cup K) \setminus \{v\} : v \in K\}$ and $|S| + |K| = k+1$.

For the clique number, the authors showed that

$$\omega(F_k(G)) = \min\{\omega(G), \max\{n-k+1, k+1\}\}.$$

Hamiltonicity: It is known that the Johnson graph $J(n,k)$ (which is isomorphic to $F_k(K_n)$) is Hamiltonian, see, e.g., [6].

It is well known that the Hamiltonicity of G does not imply the Hamiltonicity of $F_k(G)$. For example, for the complete bipartite graph $K_{m,m}$, Fabila-Monroy et al. [21] showed that if k is even, then $F_k(K_{m,m})$ is non-Hamiltonian. An easier example is the case of the cycle graph C_n; it is known that if $n = 4$ or $n \geq 6$, then $F_2(C_n)$ is not Hamiltonian, see, e.g., [5]. On the other hand, there exist non-Hamiltonian graphs for which its k-token graph is Hamiltonian, for example, $F_2(K_{1,3})$. Recently, Adame, Rivera and Trujillo-Negrete [2] showed for the generalized fan graph $F_{m,n}$ that $F_k(F_{m,n})$ is Hamiltonian if $n \geq k$ and $1 \leq m \leq 2n$; this family provides an infinite number of non-Hamiltonian graphs with Hamiltonian k-token graphs.

For the existence of Hamiltonian paths, Fabila-Monroy et al. [21] showed, for the path graph P_n, that $F_k(P_n)$ has a Hamiltonian path if and only if n is even and k is odd. As the authors showed, a consequence of this result is that if G has a Hamiltonian path of order n even and k is odd, then $F_k(G)$ has a Hamiltonian path.

The Hamiltonicity of token graphs of some graphs have a direct relationship with Gray codes for combinations. A detailed explanation of this fact can be found in Section 1.2.3.

Eulerianess: This parameter has been completely determined. In [5], the authors showed that $F_2(G)$ is Eulerian if and only if G is connected and all the vertices of G have the same parity. Later, in 2016, Mirajkar and Priyanka [39] showed that $F_k(G)$ is Eulerian if and only if one of the following holds:

(a) every vertex in G is of even degree, or,

(b) every vertex in G is of odd degree and k is even.

Planarity: In 1991, Alavi et. al. [4] showed for G with $|G| > 10$, that $F_2(G)$ is planar if and only if G is isomorphic to the path graph P_n. The graphs of order at most ten for which $F_2(G)$ is planar are also presented in [4].

In 2017, independently, Carballosa, Fabila-Monroy, Leaños and Rivera [15] showed for G, with $|G| > 10$, that $F_k(G)$ is planar if and only if $k = 2$ or $k = n - 2$ and G is isomorphic to the path graph P_n. The graphs of order at most ten for which $F_k(G)$ is planar are shown explicitly in [15].

Regularity: This parameter was solved by Carballosa, Fabila-Monroy, Leaños and Rivera [15] in 2017, where they showed that $F_k(G)$ is regular if and only if one of the following cases holds:

(a) G is isomorphic to the complete graph K_n;

(b) G is isomorphic to the the graph of n isolated vertices E_n;

(c) G is isomorphic to the complete bipartite graph $K_{1,n-1}$ and $k = n/2$;

(d) G is isomorphic to the complement of $K_{1,n-1}$ and $k = n/2$.

Automorphism group: The automorphism group of a graph H is denoted by $\mathrm{Aut}(H)$.

For the Johnson graph $J(n,k)$ (which is isomorphic to $F_k(K_n)$) it is known that $\mathrm{Aut}(J(n,k)) \simeq S_n \simeq \mathrm{Aut}(K_n)$ if $k \neq n/2$, and $\mathrm{Aut}(J(n,k)) \simeq \mathbb{Z}_2 \times S_n \simeq \mathbb{Z}_2 \times \mathrm{Aut}(K_n)$ if $k = n/2$, see e.g., [14, 38].

Ibarra and Rivera [29] showed that $\mathrm{Aut}(G)$ is a subgroup of $\mathrm{Aut}(F_k(G))$ if $k \neq n/2$, and $\mathbb{Z}_2 \times \mathrm{Aut}(G)$ is a subgroup of $\mathrm{Aut}(F_k(G))$ if $k = n/2$. They also showed that if G is a cycle, a star graph, a fan graph or a wheel graph, then $\mathrm{Aut}(F_2(G)) \simeq \mathrm{Aut}(G)$; and for the path graph P_n, the authors showed that $\mathrm{Aut}(F_k(P_n)) \simeq \mathrm{Aut}(P_n)$, for any k with $k \neq n/2$.

In this thesis we show that if G is a connected (C_4, D_4)-free graph then $\mathrm{Aut}(F_k(G)) \simeq \mathrm{Aut}(G)$ if $k \neq n/2$, and $\mathrm{Aut}(F_k(G)) \simeq \mathbb{Z}_2 \times \mathrm{Aut}(G)$ if $k = n/2$, where C_4 is a 4-cycle and D_4 is the diamond graph (a 4-cycle with one chord). We also show that the automorphism group of $F_k(G)$ is strongly related to the reconstruction problem of token graphs. Moreover, we study the automorphisms of the k-token graphs of two families of graphs: complete bipartite graphs and Cartesian product of graphs. Specifically, we compute the number of automorphisms of:

(1) $F_k(K_{m,n})$, for any $k \in \{2, \ldots, m+n-2\}$, and,

(2) $F_2(G)$, where G is the Cartesian product of at least two non-trivial graphs.

These families provide graphs for which $|\operatorname{Aut}(F_k(G))| > |\operatorname{Aut}(G)|$ and others for which $|\operatorname{Aut}(F_k(G))| = |\operatorname{Aut}(G)|$.

Spectra: It seems that the study of spectra of token graphs began in 2002, when Rudolph [45] defined token graphs as a model of qubits interacting via an (excitation)-exchange Hamiltonian. As we mentioned before, his aim was to study the Isomorphism Problem of graphs through the spectra of token graphs. He exhibited two cospectral graphs G and H with non cospectral 2-token graphs, so, as he stated, the spectra of token graphs is a more powerful invariant than the spectra of the original graphs. In 2007, Audenaert, Godsil, Royle and Rudolph showed that the spectra of the 2-token graphs of strongly regular graphs with the same parameters are equal. Later, in 2009 Barghi and Ponomarenko [12], and independently, in 2010 Alzaga, Iglesias and Pignol [7], showed that for any value of k, there are infinitely many pairs of graphs with cospectral k-token graphs.

Laplacian spectra: Let us denote the Laplacian spectra of a graph H by $LS(H)$.

In [16], Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete and Zaragoza Martínez showed that the Laplacian spectra of G is contained in the Laplacian spectra of its k-token graph $F_k(G)$, that is, $LS(G) \subseteq LS(F_k(G))$, for any $k \in \{1, 2, \ldots, |G|-1\}$. Moreover, the authors showed that for any graph G and any integers t and k with $1 \leq t \leq k \leq |G|/2$, $LS(F_t(G)) \subseteq LS(F_k(G))$. Also, for the complementation of graphs, the authors showed that there is a matching between $LS(F_k(G))$ and $LS(F_k(\overline{G}))$, such that when we sum over this matching, we obtain $LS(F_k(K_n))$, where $\overline{G}$ denotes the complement of G. Besides, for the algebraic connectivity of graphs, the authors showed that the algebraic connectivity of $F_k(G)$ is at most the algebraic connectivity of G, for any admissible value of k. They also showed that this upper bound is tight for complete graphs, complete bipartite graphs and path graphs, and conjectured that the equality is satisfied for any graph G and any k.

Independence number: Let us denote the independence number of a graph H by $\alpha(H)$.

In [17], de Alba, Carballosa, Leaños and Rivera showed the following:

(1) for the complete bipartite graph $K_{m,n}$ they showed that

$$\alpha(F_2(K_{m,n})) = \max\left\{mn, \binom{m+n}{2} - mn\right\};$$

(2) if G is a bipartite graph with a perfect matching and k is odd, then

$$\alpha(F_k(G)) = \frac{\binom{m+n}{k}}{2};$$

(3) for the star graph $K_{1,n}$ and $k \leq (n+1)/2$, they showed that

$$\alpha(F_k(K_{1,n})) = \binom{n}{k};$$

(4) for the cycle graph P_n they showed that

$$\alpha(F_2(C_n)) = \left\lfloor \frac{p\lfloor \frac{p}{2} \rfloor}{2} \right\rfloor.$$

In [1], Abdelmalek, Meulen, Meulen and Van Tuyl showed the following bounds:

$$\binom{\alpha(G)}{k} \leq \alpha(F_k(G)) \leq \frac{1}{k}\binom{n}{k-1}\alpha(G).$$

Matching number: Let us denote the matching number of a graph H by $\nu(H)$.

In [17], de Alba, Carballosa, Leaños and Rivera showed that if G is a graph with $\nu(G) = \lfloor n/2 \rfloor$, then

(1) $\nu(F_k(G)) = \binom{n}{k}/2$, if n is even and k id odd; and,

(2) $\nu(F_k(G)) \geq (\binom{n}{k} - \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor})/2$, otherwise,

moreover, the bound (2) is tight when G is a perfect matching or an almost perfect matching.

Reconstruction: In [21] the authors conjectured that if $F_k(G) \simeq F_k(H)$ for some graphs G and H and some positive integer k, then $G \simeq H$. This conjecture was posed as a question for 2-token graphs in [31]. In [31], the authors showed that if G is regular and does not contain a 4-cycle as a subgraph, or if G is a cubic graph, then G is reconstructible from its 2-token graph. In [3] it is claimed that if G is a tree, then G is reconstructible from its 2-token graph.

In this thesis we show that if G is a (C_4, D_4)-free graph, then G is reconstructible from its k-token graph, for any k with $2 \leq k \leq n-2$, where C_4 denotes the cycle graph of four vertices and D_4 is the diamond graph (a 4-cycle with one chord). Moreover, if G is connected, this reconstruction can be done in polynomial time. As we mentioned before, we also show that the automorphism group of token graphs is highly related to the reconstruction of token graphs.

Packing number: Let us denote the packing number of a graph H by $\rho(H)$.

In 2018 Gómez Soto, Leaños, Ríos-Castro and Rivera [27] showed for the path graph P_{n+1} of $n+1$ vertices, with $n \geq 6$, that

$$\rho(F_2(P_{n+1})) = \begin{cases} \frac{1}{10}(n^2+n+20) & \text{if } n \equiv 0 \pmod 5 \text{ or } n \equiv 4 \pmod 5, \\ \frac{1}{10}(n^2+n+18) & \text{if } n \equiv 1 \pmod 5 \text{ or } n \equiv 3 \pmod 5, \\ \frac{1}{10}(n^2+n+14) & \text{if } n \equiv 2 \pmod 5. \end{cases}$$

The packing number of $F_2(P_{n+1})$ has a direct relationship with certain error correcting codes. This relationship is explained in Section 1.2.2.

Before go further, let us present some applications of token graphs to Physics and Coding Theory. Several researchers are currently exploring more applications of token graphs to Physics.

1.2 Applications

In the last three decades, several connections of token graphs with other research areas have been discovered, such as Physics and Coding Theory. In this section we give an explanation of three direct applications of token graphs:

1) token graphs modeling a system of qubits interacting via an (excitation)-exchange Hamiltonian;
2) the packing number of the k-token graph of the path graph P_n corresponds to the largest code of length n and constant weight k that can correct a single adjacent transposition; and
3) Hamiltonian paths of token graphs corresponds to Gray codes for combinations.

Besides, since Johnson graphs are a particular case of token graphs, any application of Johnson graphs can be considered as an application of token graphs.

1.2.1 Token graphs modeling a system in Quantum Mechanics

As we mentioned before, Rudolph defined token graphs (under the name of k-level matrices) as a model of qubits interacting via an (excitation)-exchange Hamiltonian. This section is devoted to explaining this interpretation.

Consider a system of n interacting qubits (two-level atoms), each of ground state $|0\rangle$ and excited state $|1\rangle$. Assume that these qubits are interacting via an (excitation)-exchange Hamiltonian. The system of qubits is represented by a graph G, in which the vertices of G are the n qubits and two vertices are adjacent if their corresponding qubits interact. The *generic interaction Hamiltonian* of this system is of the form:

$$H_{int}(G) = g_{i,j} \sum_{i \sim j} (S_i^+ S_j^- + S_i^- S_j^+),$$

where $S_i^+ = |1\rangle\langle 0|$ and $S_i^- = |0\rangle\langle 1|$ are the raising and lowering operators, respectively, $i \sim j$ means that vertices i and j are adjacent in G, and $g_{i,j}$ is a coupling constant, which we take equal to 1 if i and j are interacting qubits, and equal to 0 otherwise.

There are 2^n possible states of the system, and we can label each state as the subset of qubits which are in the excited state, so the 2^n possible states can be seen as the subsets of $V(G)$. The nature of the Hamiltonian H_{int} is such that it conserves excitation, that is, if one qubit goes "up", the other must come "down". Thus, the matrix element of $H_{int}(G)$ between two states with different number of excited qubits is always zero; indeed, from the definition of H_{int} it can be deduced that the matrix element of $H_{int}(G)$ between two states can be non-zero only if the two states differ in one, and only one, pair of qubits, say a and b; in such a case, it is equal to 1 if a and b are interacting qubits, and 0 otherwise. From this fact we observe that $H_{int}(G)$ (seen as a matrix) satisfies the following:

(1) its diagonal elements are equal to 0,

(2) it is symmetric, and

(3) it is block diagonal.

These three observations together imply that $H_{int}(G)$ is the adjacency matrix of a disconnected graph H. The connected components of this graph are all the possible token graphs of G, that is,

$$H \simeq F_0(G) \cup F_1(G) \cup \cdots \cup F_n(G),$$

where $F_k(G)$ denotes the k-token graph of G for $1 \leq k \leq n-1$, and $F_0(G)$ is the graph consisting of one isolated vertex, and the same holds for $F_n(G)$. The Hamiltonian $H_{int}(G)$ (seen as a matrix) is then, the direct sum of the adjacency matrix of each possible token graph of G.

Roughly speaking, the Hamiltonian H_{int} describes the energy transferring among the qubits. In [8], the authors provided the following examples for the initial system of qubits: two-level atoms in a molecule, interacting via a dipole-dipole interaction; spins on a lattice interacting via an "XY" spin-exchange interaction; or, hard-core bosons hopping around some lattice structure (Bose-Hubbard model). For more details on this application of token graphs to the Hamiltonian H_{int} we refer the reader to [8, 40, 45]. Besides, other researchers are currently exploring more applications of token graphs to Physics, see, e.g. [23, 24].

1.2.2 Packing sets of token graphs and error correcting codes

In this section we explain a relationship between token graphs and error correcting codes.

Let n be a positive integer and let $k \in \{1, \ldots, n-1\}$. A *binary code of length n and weight k* is a set S of n-vectors in $\{0,1\}^n$ with exactly k 1's. The elements of S are called *codewords*. Let e denote an error; for example, e can be the deletion of some bits, or the transposition of some bits. For a codeword u, let $B_e(u)$ be the set of binary vectors in $\{0,1\}^n$ that can be obtained from u as a consequence of the error e.

We are interested in the error e that consists of a single adjacent transposition of bits. For example, for $n = 5$ and $k = 2$, consider the binary vector $x = (0, 0, 1, 1, 0)$, the vectors $(0, 1, 0, 1, 0)$ and $(0, 0, 1, 0, 1)$ can be obtained from x by the error of a single adjacent transposition of bits. A subset $C \subseteq \{0, 1\}^n$ is said that *can correct a single adjacent transposition* if $B_e(x) \cap B_e(y) = \emptyset$, for any distinct $x, y \in C$. One can consider the problem of determining the size of largest code of length n and constant weight k than can correct a single adjacent transposition. The case $k = 2$ of this problem corresponds to the sequence A085680(n) in OEIS [49]. This case has been studied since 2003, and the exact values of this sequence was known from 2 to 50 until 2018, when Gómez Soto, Leaños, Ríos-Castro and Rivera [27] solved the problem using 2-token graphs. Next, we explain the relationship between 2-token graphs and largest codes of length n and constant weight 2 than can correct a single adjacent transposition.

Sloane[1] formulated the problem of determining the values of A085680(n) as the following problem in graph theory: construct a graph Γ_n whose vertices are all the binary vectors of length n and constant weight 2, and let two vertices be adjacent if one can be obtained from the other by transposing a pair of adjacent coordinates. The value A085680(n) is the maximal size of a subset S of $V(\Gamma_n)$ such that any two vertices in S are at distance at least 3 in Γ_n. This formulation can be naturally generalized to codes of length n and constant weight k, for $2 < k < n$. Let $\Gamma_{n,k}$ be the graph obtained given this generalization. Let us now explain the fact that the graph $\Gamma_{n,k}$ is isomorphic to the k-token graph of P_n. Notice that the binary vectors of length n with constant weight k and the k-subsets of the set $[n] := \{1, 2, \ldots, n\}$ are in a one to one correspondence (since any of such k-subsets X can be transformed into a binary vector x by letting the i-th coordinate of x be 1 if $i \in X$, and 0 otherwise; and vice versa, a binary vector x can be transformed into a k-subset X). Now, consider two binary vectors x and y in $\Gamma_{n,k}$ and their corresponding k-subsets X and Y, respectively. The vectors x and y are adjacent in $\Gamma_{n,k}$ if and only if the symmetric difference $X \triangle Y$ is a pair $\{i, i+1\}$, for $1 \leq i \leq n-1$, and this last holds if and only if the vertices X and Y are adjacent in $F_k(P_n)$. This gives that $\Gamma_{n,k} \simeq F_k(P_n)$.

For a graph G, a set $S \subseteq V(G)$ is a *packing set* of G if every pair of distinct vertices $u, v \in S$ are at distance at least three. The *packing number* $\rho(G)$ of G is the maximum cardinality of a packing set of G. It is straightforward to see that the size of a largest code of length n and constant weight k that can correct a single adjacent transposition corresponds to the packing number of $F_k(P_n) \simeq \Gamma_{n,k}$. The problem of determining the size of largest code of length n and constant weight k than can correct a single adjacent transposition corresponds to sequence A085684 in OEIS [49], and so far, it remains open for any $2 < k < n-2$. A variant of this problem is to determine the size of largest code of length n and constant weight k than can correct a single adjacent transposition where the end-around transposition is allowed (initial and final bits can be swapped). This variant corresponds to sequence A085685 in OEIS [49], and it also corresponds to the packing number of the k-token graph of cycle graph C_n. To our knowledge, this problem is open even for $k = 2$.

[1] https://oeis.org/A085680

1.2.3 Hamiltonicity of token graphs and Gray codes for combinations

Consider the problem of generating all the subsets of an n-set; this can be reduced to the problem of generating all possible binary strings of length n (since each k-subset can be transformed into a n-binary string by placing an 1 in the j-th entry if j belongs to the subset, and 0 otherwise). The most straightforward way of generating all these n-binary strings is counting in binary; however, many elements may change from one string to the next. Thus, it is desirable that only a few elements change between successive strings. The case when successive strings differ by a single bit, is commonly known as *Gray codes*. Similarly, the problem of generating all the k-subsets of an n-set is reduced to the problem of generating all the n-binary strings of constant weight k (with exactly k 1's).

The term "Gray" derives from Frank Gray, a research physicist at the Bell Telephone Laboratories, who used these codes in a patent he obtained for *pulse code communication*. Gray codes are known to have applications in different areas, such as cryptography, circuit testing, statistics and exhaustive combinatorial searches. For a more detailed information on Gray codes, we refer the reader to [13, 46, 48]. Next, we present a formal definition of Gray codes.

Let S be a set of n combinatorial objects and C a relation on S, C is called the *closeness relation*. A *Combinatorial Gray Code* (or simply *Gray code*) for S with respect to C is a sequence $s_1, s_2, \ldots, s_n$ of the elements of S such that $(s_i, s_{i+1}) \in C$, for $i = 1, 2, \ldots, n-1$. If, additionally, $(s_n, s_1) \in C$ then the Gray code is said to be *cyclic*. In other words, a Gray code for S with respect to C is a sequence of the elements of S in which successive elements are close (with respect to C). There is a digraph $G(S, C)$, the *closeness graph*, associated to S with respect to C, where the vertex set and edge set of $G(S, C)$ are S and C, respectively. If the closeness relation is symmetric, $G(S, C)$ is an undirected graph. A Gray code (resp. cyclic Gray code) for S with respect to C is a Hamiltonian path (resp. a Hamiltonian cycle) in $G(S, C)$.

We are interested in Gray codes for combinations. A k-combination of the set $[n] = \{1, 2, \ldots, n\}$ is a k-subset of $[n]$, which in turn, can be thought as a binary string of length n and constant weight k (it has k 1's and $n-k$ 0's). Consider the set $S = S(n, k)$ of all the k-combinations of $[n]$. Next, we mention three closeness relations that can be applied to S; for other closeness relations we refer the reader to [46].

1) The *transposition* condition: two k-subsets are close if they differ in exactly two elements. Example: $\{1, 2, 5\}$ and $\{2, 4, 5\}$ are close, while $\{1, 2, 5\}$ and $\{1, 3, 4\}$ are not.

2) The *adjacent transposition* condition: two k-subsets are close if they differ in exactly two consecutive elements i and $i+1$. Example: $\{1, 2, 5\}$ and $\{1, 3, 5\}$ are close, while $\{1, 2, 5\}$ and $\{1, 4, 5\}$ are not.

3) The *one or two apart transposition* condition: two k-subsets are close if they differ in exactly two elements i and j, with $|i-j| \leq 2$. Example: $\{1,2,5\}$ and $\{1,4,5\}$ are close, while $\{1,2,5\}$ and $\{2,4,5\}$ are not.

The relationship between the closeness graph associated to S with respect to one of these closeness conditions and some token graphs is the following:

- for the transposition condition, the closeness graph associated is isomorphic to the k-token graph of the complete graph K_n;
- for the adjacent transposition condition, the closeness graph associated is isomorphic to the k-token graph of the path graph P_n; and,
- for the one or two apart transposition condition, the closeness graph associated is isomorphic to the k-token graph of P_n^2, where P_n^2 denotes the square of the path graph P_n.

This relationship was first noted by Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia and Wood [21] in 2012, and can be generalized to other closeness relations. The following is a direct consequence of this relationship. Suppose that for a closeness relation C, the closeness graph associated to S with respect to C is isomorphic to the k-token graph of some graph G. Then, a Gray code and a cyclic Gray code for S with respect to C correspond to a Hamiltonian path and a Hamiltonian cycle of the k-token graph of G.

1.3 Contributions and outline

This thesis is devoted to the study of token graphs. Specifically, we study the following three problems:

1. reconstruction of token graphs;
2. automorphism group of token graphs; and
3. connectivity of token graphs of trees.

Along the study of these three problems, we develop several techniques and tools that can be used to study other parameters of token graphs in the future. Let us present the main contributions of this thesis.

Chapter 2 is devoted to the study of reconstruction of token graphs. Constructing graphs from an initial graph is a common practice in Graph Theory. When constructing graphs, we have that if two new constructed graphs are non-isomorphic, then the initial ones are non-isomorphic. However, it may be the case that for two initial non-isomorphic graphs, the constructed ones are isomorphic. This general approach arises the following question:

Is the constructed graph completely determined (up to isomorphism) by its initial graph?

This question corresponds to the *reconstruction problem* associated to the given construction. In this thesis we are interested in the problem of reconstructing a graph from its token graph. This problem is stated as follows:

Given a token graph F, find a graph G and an integer k, such that $F_k(G)$ is isomorphic to F.

This can be posed as an existential question: is G unique up to isomorphism? Or it can be an algorithmic problem: What is the complexity of finding such a graph G? Another variant would be to have k as part of the input. In this thesis we consider the problem of reconstructing a graph G from its token graph, when G is a (C_4, D_4)-free graph.

In 2012, Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia and Wood [21] made the following conjecture.

Conjecture 2.1. *Let G and H be two graphs such that, for some k, their k-token graphs are isomorphic. Then G and H are isomorphic.*

Conjecture 2.1 is equivalent to the reconstruction problem of token graphs. In 2007, Conjecture 2.1 was posed as a question for 2-token graphs by Jacob, Goddard and Laskar [31]. A reformulation of Conjecture 2.1 is that $F_k(G)$ determines G completely (up to isomorphism). If this is the case for some graph G, we say that G is *reconstructible from its k-token graph*. We believe this to be a hard problem, even in the case of only two tokens. There are very few results in this direction. We mention some of them. In [31], the authors showed that if G is regular and does not contain a 4-cycle as a subgraph, then G is reconstructible from its 2-token graph. They also showed that cubic graphs are reconstructible from their 2-token graphs. In [3] it is claimed (without proof) that trees are reconstructible from their 2-token graphs.

In order to state our main contributions for the reconstruction problem of token graphs, we need the following definitions. The *diamond* graph D_4 is a 4-cycle with one chord; C_4 is the cycle of four vertices. The class of (C_4, D_4)-*free graphs* is the class of graphs without a C_4 nor a D_4 as an induced subgraph. This is the class of graphs studied in Chapter 2. One of the main results of this thesis is that Conjecture 2.1 is true for the class of connected (C_4, D_4)-free graphs.

Theorem 2.1. *Let G be a connected (C_4, D_4)-free graph. Given only a graph isomorphic to $F_k(G)$, we can compute in polynomial time a graph isomorphic to G.*

Let F be a graph. Let φ be an isomorphism from F to $F_k(G)$. We call the pair (G, φ) a *k-token reconstruction* of F. We say that a graph G is *k-token reconstructible from its k-token graph* if for every (G', φ'), k-token reconstruction of $F_k(G)$, we have that $G \simeq G'$. Thus, Conjecture 2.1 states that all graphs are k-token reconstructible from their token graphs. In this thesis we prove the following result, which is stronger than Theorem 2.1.

Theorem 2.2. *Let G be a connected (C_4, D_4)-free graph. Given only a graph F, isomorphic to $F_k(G)$, we can compute in polynomial time a k-token reconstruction of F.*

In this thesis we introduce the notion of a graph F being uniquely k-token reconstructible as the k-token graph of G. Informally, a graph F is *uniquely reconstructible as the k-token graph of G* if its k-token reconstruction, as the k-token graph of G, is unique up to automorphisms of G. We show the following result.

Theorem 2.3. *Let G be a connected (C_4, D_4)-free graph. Then $F_k(G)$ is uniquely reconstructible as the k-token graph of G.*

Let $\mathrm{Aut}(G)$ denote the automorphism group of G. The property of being uniquely reconstructible as a k-token graph is strongly related to the automorphism group of token graphs. This relationship is depicted in the following result.

Theorem 2.4. *Let G be a graph on at least 3 vertices. Then $F_k(G)$ is uniquely k-token reconstructible as the k-token graph of G if and only if*

$$\mathrm{Aut}(F_k(G)) \simeq \begin{cases} \mathrm{Aut}(G) \times \mathbb{Z}_2 & \text{for } k = n/2 \text{ and } n \geq 4, \\ \mathrm{Aut}(G) & \text{otherwise.} \end{cases} \tag{1}$$

The reconstruction problem of token graphs can also be considered for distinct number of tokens, say k and k'. Trujillo-Negrete [51] in her Master's thesis gave an example of two non-isomorphic graphs G and H, and a pair of distinct integers k and k', such that $F_k(G)$ and $F_{k'}(H)$ are isomorphic (and non-trivial). This example shows that Conjecture 2.1 is not true for distinct number of tokens. For completeness, we provide this example in Section 2.7. On the positive side, Conjecture 2.1 is also true for the class of disconnected (C_4, D_4)-free graphs.

Theorem 2.5. *Let G and H be two (C_4, D_4)-free graphs. If $F_k(G)$ and $F_k(H)$ are isomorphic for some k, then G and H are isomorphic.*

All these results concerning the reconstruction of token graphs are presented in Chapter 2.

When constructing graphs, the following approach is often followed:

For a given graph invariant η, what can be said about $\eta(F_k(G))$ in terms of G and $\eta(G)$?

Following this approach, we have studied two graph invariants: the automorphism group and connectivity.

We now turn to the second problem we study in this thesis: the automorphism group of token graphs. The automorphism group of a graph characterizes its symmetries. Sometimes, it is easy to find some automorphisms of a graph, but it may be quite difficult to determine all its automorphisms. Determining the automorphism group of a graph is closely related to the Graph Isomorphism Problem. The Graph Isomorphism Problem is the algorithmic problem of determining whether two given graphs are isomorphic. The current best published algorithm for this problem was given by Babai and Luks [11]. This algorithm runs in $\exp\left(O(\sqrt{n \log n})\right)$ time for graphs on n vertices. In 2015, Babai [9, 10] announced a $\exp\left((\log n)^{O(1)}\right)$ time algorithm for the Graph Isomorphism Problem. Helfgott discovered an error in the proof. In 2017, Babai announced a correction[2], which Helfgott verified[3].

Consider an automorphism ψ of G. Define a function $\iota : F_k(G) \to F_k(G)$ as follows. For every vertex $A \in F_k(G)$, let

$$\iota(\psi)(A) := \{\psi(v) : v \in A\}.$$

We call $\iota(\psi)$ the automorphism induced by ψ. It is straightforward to show that $\iota(\psi)$ is an automorphism of $F_k(G)$, and for $\phi \in \mathrm{Aut}(G)$ we have

$$\iota(\phi \circ \psi) = \iota(\phi) \circ \iota(\psi).$$

Ibarra and Rivera recently showed that ι is an injective group homomorphism from $\mathrm{Aut}(G)$ to $\mathrm{Aut}(F_k(G))$, implying that

$$\mathrm{Aut}(G) \leq \mathrm{Aut}(F_k(G)). \tag{1}$$

Let $\mathfrak{c} : F_k(G) \to F_{n-k}(G)$ be the map that sends every vertex $A \in F_k(G)$ to its complement

$$\mathfrak{c}(A) := V(G) \setminus A.$$

This map is, indeed, an isomorphism from $F_k(G)$ to $F_{n-k}(G)$, and so, if $k = n/2$ then this map is an automorphism of $F_k(G)$, which we call the *complement automorphism of* $F_k(G)$. If $|G| \geq 3$ and $k = n/2$, then $\mathfrak{c} \notin \imath(\mathrm{Aut}(G))$. Thus, when $k = n/2$ we have that

$$\mathrm{Aut}(G) \times \mathbb{Z}_2 \leq \mathrm{Aut}(F_k(G)). \tag{2}$$

Inclusions (1) and (2) may be proper. For example, $\mathrm{Aut}(K_{2,3}) = \mathbb{Z}_2 \times S_3 < \mathbb{Z}_2 \times S_4 = \mathrm{Aut}(F_2(K_{2,3}))$ and $\mathrm{Aut}(C_4) \times \mathbb{Z}_2 = D_4 \times \mathbb{Z}_2 < S_4 \times \mathbb{Z}_2 = \mathrm{Aut}(F_2(C_4))$.

[2] http://people.cs.uchicago.edu/~laci/update.html
[3] https://valuevar.wordpress.com/2017/01/04/graph-isomorphism-in-subexponential-time/

As we show in Theorem 2.3, $F_k(G)$ is uniquely reconstructible as the k-token graph of G if and only if

$$\mathrm{Aut}(F_k(G)) \simeq \begin{cases} \mathrm{Aut}(G) \times \mathbb{Z}_2 & \text{if } k = n/2 \text{ and } n \geq 4, \\ \mathrm{Aut}(G) & \text{otherwise.} \end{cases}$$

In other words, $F_k(G)$ is uniquely k-reconstructible as the k-token graph of G if $\mathrm{Aut}(F_k(G))$ consists only of induced automorphisms when $k \neq n/2$, or if consists only of induced automorphisms, the complement automorphism, and the composition of these two types, when $k = n/2$.

The existence of graphs G and values k for which $F_k(G)$ is not uniquely reconstructible as the k-token graph of G motivated us to study the automorphism group of token graphs, and as we mentioned before, by Theorem 2.4, we know that if G is a connected (C_4, D_4)-free graph, then for any admissible k, $F_k(G)$ is uniquely reconstructible as the k-token graph of G. To our knowledge, before this work, the only families of graphs for which $\mathrm{Aut}(F_k(G))$ has been studied are:

- $F_k(K_n)$ (which is isomorphic to the Johnson graph $J(n,k)$), for each admissible k, see, e.g., [25, 33, 42];
- $F_k(P_n)$, for $2 \leq k < n/2$, see [29];
- $F_2(G)$, where G is a cycle, a star, a fan or a wheel graph, see [29].

It is remarkable that for all these families, $F_k(G)$ is uniquely reconstructible as the k-token graph of G, and so far, only some examples of graphs G and values k are known for which $F_k(G)$ is not uniquely reconstructible as the k-token graph of G. A natural problem then, is the following.

To characterize the graphs G and values k for which $F_k(G)$ is not uniquely reconstructible as the k-token graph of G.

Motivated by this problem, in Chapter 3 we study the automorphism group of the k-token graphs of two families of graphs: complete bipartite graphs and Cartesian product of graphs. In these two families, in some cases $F_k(G)$ is uniquely reconstructible as the k-token graph of G, and in other cases it is not. Surprisingly, we will see that, sometimes, this depends only on G, and in others, for the same graph G, this depends only on k.

For the family of complete bipartite graphs we show the following result.

Theorem 3.1. *Let $1 \leq m \leq n$ and $1 \leq k \leq m+n-1$. Then $F_k(K_{m,n})$ is uniquely reconstructible as the k-token graph of $K_{m,n}$ if and only if $m \neq 2$. Moreover,*

$$|\mathrm{Aut}(F_k(K_{2,n}))| = \begin{cases} 2^{\binom{n}{k-1}-1} |\mathrm{Aut}(K_{2,n})| & \text{if } k \neq \frac{n+2}{2}, \text{ and} \\ 2^{\binom{n}{k-1}} |\mathrm{Aut}(K_{2,n})| & \text{if } k = \frac{n+2}{2}. \end{cases}$$

For the Cartesian product of graphs we show the following.

Theorem 3.2. *Let G be a connected graph with prime factor decomposition $G = G_1 \square \ldots \square G_r$, where $r > 1$, $n = |G|$ and $2 \leq k \leq n/2$. Then*

$$|\operatorname{Aut}(F_k(G))| \geq \begin{cases} 2^{r-1}\,|\operatorname{Aut}(G)| & \text{if } k = 2, \\ 2\,|\operatorname{Aut}(G)| & \text{if } k = \frac{n}{2}, \\ |\operatorname{Aut}(G)| & \text{if } 2 < k < \frac{n}{2}. \end{cases}$$

Moreover, this lower bound is tight.

In Chapter 4 we turn our attention to the connectivity of token graphs. In this thesis we focus on the connectivity of token graphs of trees. Let $\kappa(G)$, $\lambda(G)$ and $\delta(G)$ be denote the connectivity, edge-connectivity and minimum degree of G, respectively. These three parameters are related as follows:

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

In [21] the authors showed that G is connected if and only if $F_k(G)$ is connected, for any k with $2 \leq k \leq |G| - 2$. Moreover, they showed that the connectivity of $F_k(G)$ is at least the connectivity of G. Also, in [21] the authors conjectured that if G is t-connected and $t \geq k$, then $F_k(G)$ is $k(t-k+1)$-connected, being this lower bound tight. This conjecture was proved by Leaños and Trujillo-Negrete [35]. Recently, a similar lower bound was proven for the edge-connectivity of $F_k(G)$ by Leaños and Ndjatchi [34]. Our main contribution in this direction is the following.

Theorem 4.1. *If G is a tree of order n and $1 \leq k \leq n-1$ then*

$$\kappa(F_k(G)) = \lambda(F_k(G)) = \delta(F_k(G)).$$

Some remarks on the connectivity of token graphs are also presented in Chapter 4.

1.4 Preliminaries and basic results

In this section, we define basic concepts and notations used throughout this thesis. Also, we prove some basic results which are used in both Chapter 2 and Chapter 3.

Let $G = (V, E)$ be a graph. We denote with $|G|$ and $||G||$ the number of vertices and edges of G, respectively. Let U, W be two subsets of vertices of G or two subgraphs of G. We use: $G \setminus W$ to denote the subgraph of G that results by removing W from G; $U \setminus W$ to denote set subtraction; and $U \triangle W$ to denote symmetric difference. We denote with $E(U, W)$ the set of edges of G with one endpoint in U and the other endpoint in W. If uw is an edge in $E(U, W)$ we always assume that $u \in U$ and $w \in W$. We refer to the edges in $E(U, W)$ as $U - W$ edges. For a vertex u of G, the *neighbourhood* of u is the set $N(u) := \{v \in V(G) : uv \in E(G)\}$. The degree $\deg(u)$ of u in G is the number $|N(u)|$. The number $\delta(G) := \min\{\deg(u) : u \in V(G)\}$ is the *minimum degree* of G. Let u and v be distinct vertices of G. The distance between u and v in G is denoted by $d_G(u, v)$ (we sometimes write $d(u, v)$ when G is understood from the context); we usually write uv or $u \sim v$ when u and v are adjacent. A *$u - v$ path of* G is a path starting at u and ending in v.

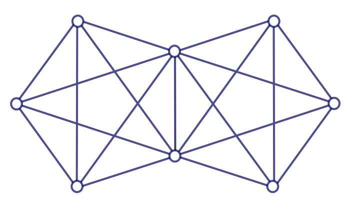
A graph G is *connected* if any two of its vertices are linked by a path in G. More generally, a graph G is *t-vertex-connected* (or simply, t-connected) if $|G| > t$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$. The greatest integer t such that G is t-connected is the *connectivity* $\kappa(G)$ of G. Similarly, a graph G is *ℓ-edge-connected* if $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E(G)$ with $|F| < \ell$. The greatest integer ℓ such that G is ℓ-edge-connected is the *edge-connectivity* $\lambda(G)$ of G. One of the well known results on the connectivity of graphs is probably Menger's Theorem:

Theorem. *A graph G is t-connected if and only if for any two vertices a and b of G, there are t internally disjoint $a - b$ paths in G.*

There is an analogous formulation of Menger's Theorem for the edge-connectivity of graphs. It is well known that if G is connected then

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

In Figure 1.3 is depicted a graph G with vertex-connectivity $\kappa(G) = 2$, edge-connectivity $\lambda(G) = 4$ and minimum degree $\delta(G) = 4$.



G

Figure 1.3: A graph G with $\kappa(G) = 2$, $\lambda(G) = 4$ and $\delta(G) = 4$.

Let us now define a product of graphs. Let $G_1, \ldots, G_n$ be graphs. The *cartesian product* of $G_1, \ldots, G_n$ is the graph $G_1 \,\square \cdots \square\, G_n$ with vertex set $V(G_1) \times \cdots \times V(G_n)$; where $(x_1, \ldots, x_n)$ is adjacent to $(y_1, \ldots, y_n)$ if and only if there exists and index $1 \leq i \leq n$ such that x_i is adjacent to y_i and $x_j = y_j$ for all $j \neq i$. Let $v := (x_1, \ldots, x_n)$ be a vertex of $G_1 \,\square \cdots \square\, G_n$, we denote the i-th coordinate of $v = (x_1, \ldots, x_n)$ with $v(i) := x_i$. A graph is *composite* if it is isomorphic to the Cartesian product of two or more non-trivial graphs. Otherwise, we say it is a *prime* graph. For a composite graph G with $G \simeq G_1 \,\square \cdots \square\, G_r$, where each G_i is a prime non-trivial graph, $G_1 \,\square \cdots \square\, G_r$ is called the *prime factor decomposition* of G. The *d-dimensional hypercube* Q_d (or simply d-cube) is the Cartesian product of d copies of K_2, that is,

$$Q_d := \underbrace{K_2 \square K_2 \square \ldots \square K_2}_{d \text{ times}}.$$

In Figure 1.4 are depicted the d-cubes for $d \in \{1, 2, 3, 4\}$. Cartesian products of graphs, and specifically, the d-cubes, are used in Chapters 2 and 3.

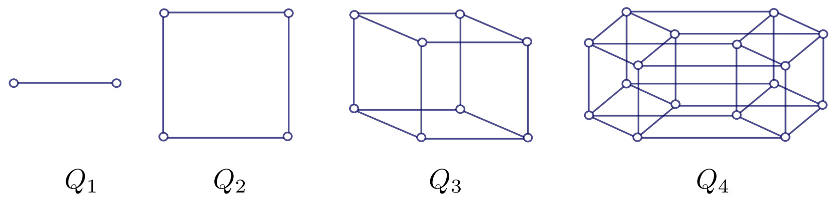


Figure 1.4: Some cube graphs.

The *line graph* $L(G)$ of G is the graph whose vertex set is the edge set of G. Two vertices of $L(G)$ are adjacent if, as edges of G, they have an end in common. In Figure 1.5 is depicted an example. We are interested in reconstructions of graphs from their line graphs. More precisely, such reconstruction exists and can be done in polynomial time. Whitney [53] showed that, except for the cases of a triangle and $K_{1,3}$, if G and G' are two graphs such that $L(G) \simeq L(G')$, then $G \simeq G'$. For $|G| > 3$, Roussopoulos [44] and Lehot [36] gave an $O(|G| + ||G||)$ time algorithm that, given a graph isomorphic to $L(G)$, constructs a graph isomorphic to G.

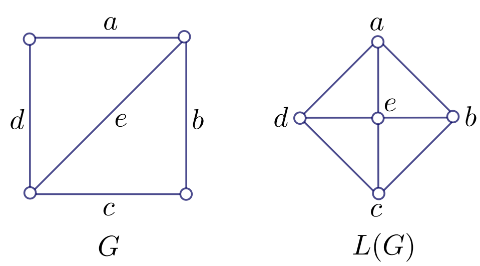


Figure 1.5: A graph G and its line graph $L(G)$.

Given two graphs G and H, we say that G is an H-free graph if G does not contain a copy of H as an induced subgraph. In general, for a finite number of graphs $H_1, H_2, \ldots, H_t$, a graph G is a $(H_1, H_2, \ldots, H_t)$-free graph if G does not contain a copy of H_i as an induced subgraph, for any $i \in \{1, 2, \ldots, t\}$. In Chapter 2 we focus on the class of (C_4, D_4)-free graphs, where C_4 is the cycle graph of four vertices and D_4 is the diamond graph (a 4-cycle with one chord). Note that the class of (C_4, D_4)-free graphs can be seen as the class of graphs G in which any 4-cycle of G induces a complete graph. In Figure 1.6 are depicted all the connected (C_4, D_4)-free graphs on five vertices.

Two graphs G and H are *isomorphic* if there exists a bijection φ between the vertices of G and the vertices of H that satisfies the following. A vertex x is adjacent to a vertex y in

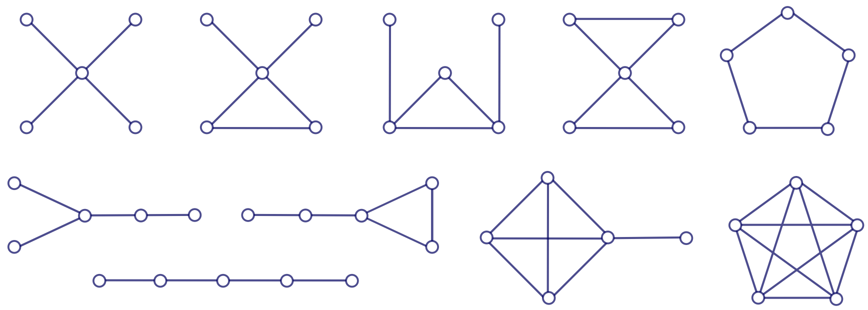


Figure 1.6: All the connected (C_4, D_4)-free graphs on five vertices.

G if and only if $\varphi(x)$ is adjacent to $\varphi(y)$ in H. We say that φ is an *isomorphism* between G and H. We write $G \simeq H$ to denote that G and H are isomorphic (as an example, graphs G and H depicted in Figure 1.7 are isomorphic). We denote with $\mathrm{Iso}(G, H)$ the set of isomorphisms from G to H. An isomorphism of G with itself is called an *automorphism*. The set of automorphisms of G form a group under function composition; we denote this group by $\mathrm{Aut}(G)$.

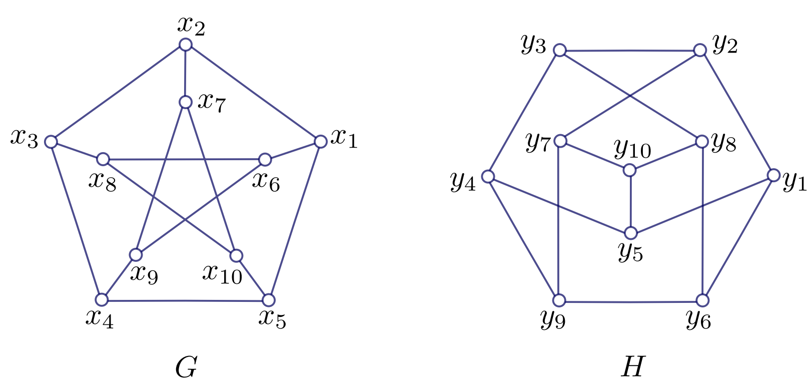


Figure 1.7: Graphs G and H are isomorphic, where the isomorphism is $\varphi(x_i) = y_i$, for $1 \leq i \leq 10$.

Let Δ and Γ be two groups. If Δ is a subgroup of Γ we write $\Delta \leq \Gamma$; if, in addition, Δ is a proper subgroup of Γ we write $\Delta < \Gamma$, otherwise we write $\Delta = \Gamma$.

For brevity, if m is a positive integer, then we use $[m]$ to denote $\{1, \ldots, m\}$. We follow the convention that $[m] = \emptyset$ if $m = 0$.

1.4.1 Automorphisms of token graphs

We now mention some results on the automorphisms of token graphs. We start by considering the general case of the isomorphisms of token graphs.

Fix a graph G, and consider a graph H isomorphic to G. We define a function ι :

$\mathrm{Iso}(H,G) \to \mathrm{Iso}(F_k(H), F_k(G))$ as follows. Let $\psi \in \mathrm{Iso}(H,G)$. Let $\iota(\psi)$ be the function that maps every $A \in V(F_k(H))$ to

$$\iota(\psi)(A) := \{\psi(v) : v \in A\}.$$

Given two vertices A and B of $F_k(H)$, note that

$$\iota(\psi)(A) \triangle \iota(\psi)(B) = \{\psi(v) : v \in A \triangle B\}.$$

Then,

$$\begin{aligned} AB \in E(F_k(H)) &\iff A\triangle B = \{a,b\} \text{ with } ab \in E(H) \\ &\iff \iota(\psi)(A) \triangle \iota(\psi)(B) = \{\psi(a), \psi(b)\} \text{ with } \psi(a)\psi(b) \in E(G) \\ &\iff \iota(\psi)(A)\, \iota(\psi)(B) \in E(F_k(G)) \end{aligned}$$

thus, $\iota(\psi) \in \mathrm{Iso}(F_k(H), F_k(G))$. When $G = H$, $\iota(\psi)$ is an automorphism of $F_k(G)$, which we call the *automorphism induced by* ψ. Let us show that ι is injective. Let $\phi \in \mathrm{Iso}(H,G)$ with $\psi \neq \phi$. Let $v \in V(H)$ such that $\phi(v) \neq \psi(v)$ and let $u \in V(G)$ be such that $\phi(u) = \psi(v)$. Thus, $u = \phi^{-1}\psi(v)$ and $u \neq v$. Let $A \in V(F_k(H))$ such that $v \in A$ and $u \notin A$. We have that $\psi(v) \notin \iota(\phi)(A)$ and $\psi(v) \in \iota(\psi)(A)$. Therefore, $\iota(\phi)(A) \neq \iota(\psi)(A)$.

Let J be a graph isomorphic to G, and let ϕ now be an isomorphism from G to J. Note that for every vertex A of $F_k(H)$ we have

$$\begin{aligned} \iota(\phi \circ \psi)(A) &= \{(\phi \circ \psi)(v) : v \in A\} \\ &= \phi(\{\psi(v) : v \in A\}) \\ &= (\iota(\phi) \circ \iota(\psi))\,(A). \end{aligned}$$

Thus,

$$\iota(\phi \circ \psi) = \iota(\phi) \circ \iota(\psi).$$

Ibarra and Rivera [29] recently showed that, when $G = H$, $\imath$ is an injective group homomorphism from $\mathrm{Aut}(G)$ to $\mathrm{Aut}(F_k(G))$. Thus,

$$\mathrm{Aut}(G) \leq \mathrm{Aut}(F_k(G)). \tag{1}$$

Let $\mathfrak{c}$ be the map that sends every set A of k vertices of G to its complement $V(G) \setminus A$. Note that $A\triangle B = \mathfrak{c}(A) \triangle \mathfrak{c}(B)$, which implies that

$$A \text{ and } B \text{ are adjacent in } F_k(G) \iff \mathfrak{c}(A) \text{ and } \mathfrak{c}(B) \text{ are adjacent in } F_{n-k}(G),$$

and so, $\mathfrak{c}$ is an isomorphism from $F_k(G)$ to $F_{n-k}(G)$. If $k = n/2$ then this map is an automorphism of $F_k(G)$, which we call the *complement automorphism of* $F_k(G)$.

Proposition 1.1. *If $n \geq 3$ and $k = n/2$ then $\mathfrak{c} \notin \imath(\mathrm{Aut}(G))$.*

Proof. Suppose for a contradiction that there exists $\phi \in \mathrm{Aut}(G)$ such that $\imath(\phi) = \mathfrak{c}$. Note that ϕ is not the identity; thus, there exists a vertex v_1 of G such that $v_1 \neq \phi(v_1)$. Let $A := \{v_1, \phi(v_1), v_2, \ldots, v_{k-1}\}$ be a vertex in $F_k(G)$. Then $V(G)\backslash A = \{\phi(v_1), \phi(\phi(v_1)), \ldots, \phi(v_{k-1})\}$. This implies that $\phi(v_1) \in A$ and $\phi(v_1) \notin A$—a contradiction. □

Note that for every $\psi \in \mathrm{Aut}(G)$ we have that $\mathfrak{c} \circ \iota(\psi) = \iota(\psi) \circ \mathfrak{c}$. Since $\mathfrak{c}^2$ is the identity, the group generated by $\mathrm{Aut}(G)$ and $\mathfrak{c}$ is isomorphic to $\mathrm{Aut}(G) \times \mathbb{Z}_2$. Thus, when $k = n/2$ we have that

$$\mathrm{Aut}(G) \times \mathbb{Z}_2 \leq \mathrm{Aut}(F_k(G)). \tag{2}$$

The inclusions (1) and (2) may be proper. Using the SageMath [50] and GAP [26] softwares it can be shown that

$$\mathrm{Aut}(K_{2,3}) = \mathbb{Z}_2 \times S_3 < \mathbb{Z}_2 \times S_4 = \mathrm{Aut}(F_2(K_{2,3}))$$

and

$$\mathrm{Aut}(C_4) \times \mathbb{Z}_2 = D_4 \times \mathbb{Z}_2 < S_4 \times \mathbb{Z}_2 = \mathrm{Aut}(F_2(C_4)).$$

We say that $F_k(G)$ is *uniquely reconstructible as the k-token graph of G* if any two k-token reconstructions (G, φ) and (G, ψ) of $F_k(G)$ are equivalent, that is, there exists an automorphism $\mathfrak{s}(\varphi, \psi)$ of G such that

$$\psi = \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi \quad \text{or} \quad \psi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi.$$

As we will show in Theorem 2.4, the property of $F_k(G)$ being uniquely reconstructible as the k-token graph of G is equivalent to the following property of $\mathrm{Aut}(F_k(G))$:

$$\mathrm{Aut}(F_k(G)) = \begin{cases} \mathrm{Aut}(G) & \text{when } k \neq n/2, \\ \mathrm{Aut}(G) \times \mathbb{Z}_2 & \text{when } k = n/2. \end{cases}$$

So far, the families of graphs for which $\mathrm{Aut}(F_k(G))$ has been studied are:

- $F_k(K_n)$ (which is isomorphic to the Johnson graph $J(n, k)$), for each admissible k, see, e.g., [33, 25, 42];
- $F_k(P_n)$, for $2 \leq k < n/2$, see [29];
- $F_2(G)$, where G is a cycle, a star, a fan or a wheel graph, see [29].

For all these families, $F_k(G)$ is uniquely reconstructible as the k-token graph of G. As we mentioned before, in this thesis we study the automorphism group of token graphs of the following families of graphs: connected (C_4, D_4)-free graphs, complete bipartite graphs and Cartesian product of graphs.

Chapter 2

Reconstruction of token graphs

A common practice in Graph Theory is the construction of graphs from an initial graph. When constructing graphs, we have that if two new constructed graphs are non-isomorphic, then the initial ones are non-isomorphic. However, it may be the case that for two initial non-isomorphic graphs, the constructed ones are isomorphic. This general approach arises the following question:

Is the constructed graph completely determined (up to isomorphism) by its initial graph?

This question corresponds to a *reconstruction problem* associated to the given construction. In this chapter we are interested in the problem of reconstructing a graph from its token graph. This problem is stated as follows:

Given a token graph F, find a graph G and an integer k, such that $F_k(G)$ is isomorphic to F.

This can be posed as an existential question: is G unique up to isomorphism? Or it can be an algorithmic problem: What is the complexity of finding such a graph G? Another variant would be to have k as part of the input. In this chapter we consider the problem of reconstructing a graph G from its token graph, when G is a (C_4, D_4)-free graph.

The problem of reconstructing a graph from its token graph seems to be related to the Graph Isomorphism Problem. The Graph Isomorphism Problem is the algorithmic problem of determining whether two given graphs are isomorphic. The current best published algorithm for this problem was given by Babai and Luks [11]. This algorithm runs in $\exp\big(O(\sqrt{n \log n})\big)$ time for graphs on n vertices. In 2015, Babai [9, 10] announced an $\exp\big((\log n)^{O(1)}\big)$ time algorithm for the Graph Isomorphism Problem. Helfgott discovered an error in the proof. In 2017, Babai announced a correction[1], which Helfgott verified[2].

There are many graph invariants, computable in polynomial time, that in many instances distinguish pairs of non isomorphic graphs. One of these is the spectra of a graph (the eigenvalues of its adjacency matrix). Two graphs are *cospectral* if they have the same spectra. As expected, there are pairs of non-isomorphic cospectral graphs. In [45], Rudolph noted that the spectra of 2-token graphs may help in distinguishing two graphs. He gave an example of a pair of non-isomorphic cospectral graphs whose 2-token graphs are not cospectral. In [8], the authors showed that the 2-token graphs of any two strongly regular graphs with the same parameters are cospectral. Thus, yielding a plethora of examples of pairs of non-isomorphic graphs whose 2-token graphs are cospectral. In the same paper it is noted that, if for some k it would be the case that two graphs are isomorphic if and only if their k-token graphs are cospectral, then this would provide a polynomial time algorithm for the Graph Isomorphism Problem. This was shown not to be the case independently by Barghi and Ponomarenko [12], and Alzaga, Iglesias and Pignol [7]. Recently, Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete and Zaragoza-Martínez [16], considered the Laplacian spectra of token graphs. They showed that the Laplacian spectra of a graph is closely related to the Laplacian spectra of its token graphs. There is no known example of a pair of non-isomorphic graphs whose token graphs have the same Laplacian spectra.

Underlying the question of whether token graphs may help in distinguishing pairs of non-isomorphic graphs, is the question of how much information from G is carried out to the k-token graphs of G. In [21] the authors made the following conjecture.

Conjecture 2.1. *Let G and H be two graphs such that, for some k, their k-token graphs are isomorphic. Then G and H are isomorphic.*

Conjecture 2.1 was posed as a question for 2-token graphs by Jacob, Goddard and Laskar [31]. An equivalent reformulation of the conjecture is that $F_k(G)$ determines G completely (up to isomorphism). If this is the case for some graph G, we say that G can be *reconstructed from its k-token graph*. We believe this to be a hard problem, even in the case of only two tokens. There are very few results in this direction. We mention some of them. In [31], it is shown that if G is regular and does not contain a 4-cycle as a subgraph then G is reconstructible from its 2-token graph. They also show that cubic graphs can be reconstructed from their 2-token graphs. In [3] it is claimed (without proof) that trees can be reconstructed from their 2-token graphs. Trujillo-Negrete [51] in her Master's thesis gave an example of two non-isomorphic graphs G and H, and a pair of distinct integers k and k', such that $F_k(G)$ and $F_{k'}(H)$ are isomorphic (and non-trivial). For completeness we provide this example in Section 2.7.

This chapter is based on a joint work in progress with Ruy Fabila-Monroy [19]. In order to be precise, we expose the main results of this chapter in the following section.

[1] http://people.cs.uchicago.edu/~laci/update.html
[2] https://valuevar.wordpress.com/2017/01/04/graph-isomorphism-in-subexponential-time/

2.1 Main results

One of the main results of this chapter is that Conjecture 2.1 is true for the class of (C_4, D_4)-free graphs with an extra property when the initial graph is connected.

Theorem 2.1. *Let G be a connected (C_4, D_4)-free graph. Given only a graph isomorphic to $F_k(G)$, we can compute in polynomial time a graph isomorphic to G.*

Let F be a graph. Let φ be an isomorphism from F to $F_k(G)$. We call the pair (G, φ) a *k-token reconstruction* of F. We say that a graph G is *k-token reconstructible* from its k-token graph if for every (G', φ'), k-token reconstruction of $F_k(G)$, we have that $G \simeq G'$. Thus, Conjecture 2.1 states that all graphs are k-token reconstructible from their token graphs. We prove the following result; which is stronger than Theorem 2.1 for the case of connected graphs.

Theorem 2.2. *Let G be a connected (C_4, D_4)-free graph. Given only a graph F, isomorphic to $F_k(G)$, we can compute in polynomial time a k-token reconstruction of F.*

In Section 2.2, we introduce the notion of a graph F being uniquely k-token reconstructible as the k-token graph of G. Informally, a graph F is uniquely reconstructible as the k-token graph of G if its k-token reconstruction as the k-token graph of G, is unique up to automorphisms of G. We show the following.

Theorem 2.3. *Let G be a connected (C_4, D_4)-free graph. Then $F_k(G)$ is uniquely reconstructible as the k-token graph of G.*

Besides, we show that the property of $F_k(G)$ being uniquely reconstructible as the k-token graph of G is highly related to the automorphism group of $F_k(G)$. This special relationship is depicted in the following result.

Theorem 2.4. *Let G be a graph on at least 3 vertices. Then $F_k(G)$ is uniquely k-token reconstructible as the k-token graph of G if and only if*

$$\mathrm{Aut}(F_k(G)) \simeq \begin{cases} \mathrm{Aut}(G) \times \mathbb{Z}_2 & \text{for } k = n/2 \text{ and } n \geq 4, \\ \mathrm{Aut}(G) & \text{otherwise.} \end{cases}$$

For the class of disconnected (C_4, D_4)-free graphs, we prove the following result.

Theorem 2.5. *Let G and H be two (C_4, D_4)-free graphs. If $F_k(G)$ and $F_k(H)$ are isomorphic for some k, then G and H are isomorphic.*

2.1.1 Roadmap

In Section 2.2 we introduce the notion of unique k-token reconstructibility. Informally a graph F is uniquely reconstructible as the k-token graph of G if has a unique, up to isomorphisms of G, reconstruction as the k-token graph of G. In Section 2.2 we also present three conditions equivalent to being uniquely reconstructible as the k-token graph of some graph. Besides, in Section 2.2 we introduce the notion of k-token reconstruction families of F. These k-token reconstruction families are related to the k-token reconstructions of F. Conjecture 2.1 can be rephrased using k-token reconstruction families. Moreover, the property of F being uniquely reconstructible as the k-token graph of a graph G is equivalent to a condition using k-token reconstruction families.

In Section 2.3, we consider the token graph of stars. Token graphs of stars play an instrumental role in our reconstruction algorithm. We show that token graphs of stars are uniquely reconstructible as the k-token graph of $K_{1,n}$. We also show that if F is isomorphic to $F_k(K_{1,n})$, then a reconstruction of F as the k-token graph of $K_{1,n}$ can be found in polynomial time.

In Section 2.4 we study the induced 4-cycles and ladders of $F_k(G)$. A *ladder* is a graph isomorphic to the Cartesian product $P_n \square K_2$. We show how induced 4-cycles of $F_k(G)$ are generated; moreover, if G is a (C_4, D_4)-free graph then any induced 4-cycle of $F_k(G)$ is generated by moving two tokens on two disjoint edges of G, while the remaining tokens kept fixed at some other vertices of G. This gives an equivalence relation on the edges of $F_k(G)$ and can be extended to certain subgraphs of $F_k(G)$ isomorphic to the Cartesian product of some graphs. As a corollary we obtain that if G is a connected (C_4, D_4)-free graph, then $F_k(G)$ is a prime graph.

In Section 2.5 we present a polynomial time algorithm that given a graph F, isomorphic to $F_k(G)$, constructs a graph J isomorphic to G. In Section 2.6 we provide a k-token reconstruction of F (involving the graph J constructed in Section 2.5). Besides, we prove that F is uniquely reconstructible as the k-token graph of G.

Finally, in Section 2.7 we study the problem of reconstructing G when it is a disconnected (C_4, D_4)-free graph. Although we show that if $F_k(G) \simeq F_k(H)$ then $G \simeq H$, we are unable to reconstruct G in polynomial time. Another significant difference to the connected case, is that if G is disconnected, $F_k(G)$ is not uniquely reconstructible as the k-token graph of G.

2.2 Uniquely k-token Reconstructible Graphs

In this section we use some basic results showed in Section 1.4. Here we recall those results without proofs.

Let H be a graph isomorphic to G and let $\psi \in \mathrm{Iso}(H, G)$. Let $\iota : \mathrm{Iso}(H, G) \to \mathrm{Iso}(F_k(H), F_k(G))$ defined as

$$\iota(\psi)(A) := \{\psi(v) : v \in A\} \qquad \text{for each } A \in F_k(H).$$

We have that $\iota(\psi)$ is an isomorphism from $F_k(H)$ to $F_k(G)$, being ι injective. When $G = H$ then ι maps automorphisms of G to automorphisms of $F_k(G)$, and moreover, we have that

$$\mathrm{Aut}(G) \leq \mathrm{Aut}(F_k(G)). \tag{1}$$

In this case, $\iota(\psi)$ is called the automorphism induced by ψ.

Let $\mathfrak{c}$ be the map that sends every vertex $A \in F_k(G)$ to its complement

$$\mathfrak{c}(A) := V(G) \setminus A.$$

We have that $\mathfrak{c}$ is an isomorphism from $F_k(G)$ to $F_{n-k}(G)$, and if $k = n/2$ then this map is an automorphism of $F_k(G)$, which we call the *complement automorphism of* $F_k(G)$. As we saw in Section 1.4, if $n \geq 3$ and $k = n/2$ then $\mathfrak{c} \notin \iota(\mathrm{Aut}(G))$. Besides, the group generated by $\iota(\mathrm{Aut}(G))$ and $\mathfrak{c}$ is isomorphic to $\mathrm{Aut}(G) \times \mathbb{Z}_2$. Thus, when $k = n/2$ we have

$$\mathrm{Aut}(G) \times \mathbb{Z}_2 \leq \mathrm{Aut}(F_k(G)). \tag{2}$$

We define a notion of equivalence between k-token reconstructions. Let (G, ψ) and (G, φ) be two k-token reconstructions of a graph F. We say that (G, φ) and (G, ψ) are *equivalent* k-token reconstructions of F if there exists an automorphism $\mathfrak{s}(\varphi, \psi)$ of G such that

$$\psi = \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi \quad \text{or} \quad \psi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi.$$

We say that F is *uniquely reconstructible as the k-token graph of G* if any two k-token reconstructions of F as the k-token graph of G are equivalent. Next, we provide an example of a graph G such that $F_k(G)$ is not uniquely reconstructible as the k-token graph of G.

Example 2.1. *Consider the graph G, that is a 4-cycle with vertex set $\{a, b, c, d\}$, and consider its 2-token graph $F_2(G)$. Then, (G, ϕ), (G, φ) and (G, ψ) are three 2-token reconstructions of $F_2(G)$, where $\phi(F)$, $\varphi(F)$ and $\psi(F)$ are depicted in Figure 2.1. Note that (G, ϕ) and (G, φ) are equivalent: for $\mathfrak{s}(\varphi, \phi) = \mathrm{id}$ (the identity map) we have $\phi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi, \phi)) \circ \varphi$; while (G, ψ) is neither equivalent to (G, ϕ) nor to (G, φ). Thus, $F_2(G)$ is not uniquely reconstructible as the k-token graph of G.*

Suppose that F is uniquely reconstructible as the k-token graph of G. For a fixed $\varphi \in \mathrm{Iso}(F, F_k(G))$ let

$$I_\varphi := \{\psi \in \mathrm{Iso}(F, F_k(G)) : \psi = \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi\}$$

and

$$C_\varphi := \{\psi \in \mathrm{Iso}(F, F_k(G)) : \psi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi\}.$$

By Proposition 1.1 we have that I_φ and C_φ are disjoint. The proof of the following result is straightforward.

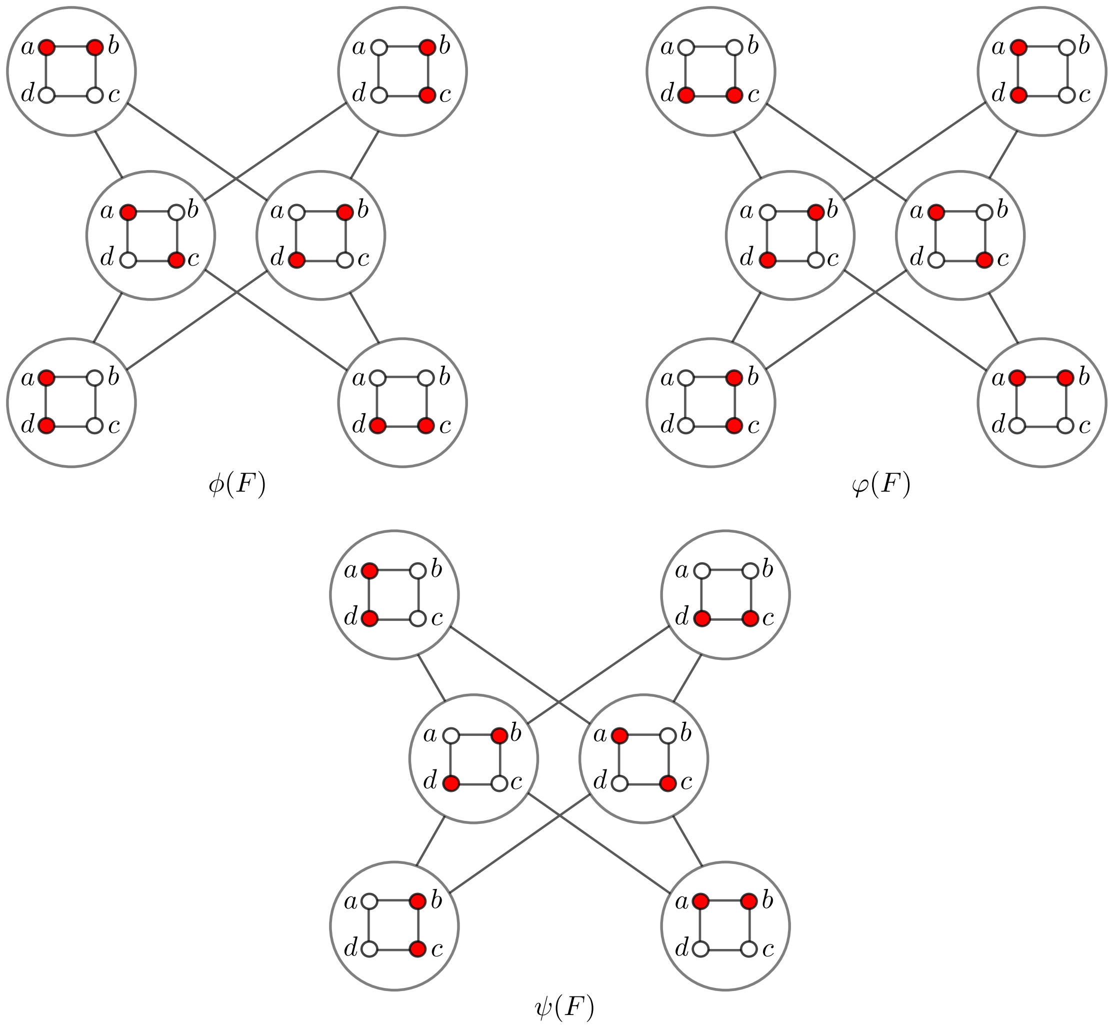


Figure 2.1: For $G = C_4$ and its 2-token graph $F_2(G)$, (G, ϕ), (G, φ) and (G, ψ) are k-token reconstructions of F. (G, ϕ) is equivalent to (G, φ), while (G, ψ) is neither equivalent to (G, ϕ) nor to (G, φ).

Lemma 2.6. *Suppose that F is uniquely reconstructible as the k-token graph of G and let $\varphi \in \mathrm{Iso}(F, F_k(G))$. Then*

$$\mathfrak{s}(\varphi, \cdot) : I_\varphi \to \mathrm{Aut}(G)$$

and

$$\mathfrak{s}(\varphi, \cdot) : C_\varphi \to \mathrm{Aut}(G)$$

are injective.

For a given vertex $u \in G$ let

$$\kappa_G(u, k) := \{A \in F_k(G) : u \in A\}$$

and

$$\overline{\kappa_G}(u, k) := \{A \in F_k(G) : u \notin A\}.$$

Seeing the k-token graph of G as the model of k indistinguishable tokens moving along the edges of G, the set $\kappa_G(u,k)$ corresponds to all the k-token configurations with a token fixed at vertex u and the remaining $k-1$ tokens placed at vertices in $G-\{u\}$; and the set $\overline{\kappa_G}(u,k)$ corresponds to all the k-token configurations with the k tokens placed at vertices of $G-\{u\}$ and no token placed at vertex u. With this in mind, it is easy to see that the subgraph induced by $\kappa_G(u,k)$ is isomorphic to $F_{k-1}\big(G-\{u\}\big)$, while the subgraph induced by $\overline{\kappa_G}(u,k)$ is isomorphic to $F_k\big(G-\{u\}\big)$.

In the following theorem we give three equivalent conditions for $F_k(G)$ to be uniquely reconstructible as the k-token graph of G.

Theorem 2.7. *Let G and H be isomorphic graphs on at least 3 vertices. The following assertions are equivalent:*

1) $F_k(G)$ is uniquely reconstructible as the k-token graph of G.

2)

$$\mathrm{Aut}(F_k(G)) \simeq \begin{cases} \mathrm{Aut}(G)\times \mathbb{Z}_2 & \textit{for } k = n/2, \\ \mathrm{Aut}(G) & \textit{otherwise.} \end{cases}$$

3) For every $\psi \in \mathrm{Iso}(F_k(H), F_k(G))$ there exist a unique $\iota^{-1}(\psi) \in \mathrm{Iso}(H,G)$ such that

$$\psi = \iota(\iota^{-1}(\psi)) \textit{ or } \psi = \mathfrak{c}\circ\iota(\iota^{-1}(\psi)).$$

4) There exists a function f that assigns to every $\psi \in \mathrm{Iso}(F_k(H), F_k(G))$ a function $f(\psi) : V(H) \to V(G)$ such that the following holds. For every vertex $u \in H$ either

$$\psi(\kappa_H(u,k)) = \kappa_G(f(\psi)(u),k) \textit{ or } \psi(\kappa_H(u,k)) = \overline{\kappa_G}(f(\psi)(u),k).$$

Proof.

1) $\Rightarrow$ 2): Suppose that $k \neq n/2$. By Lemma 2.6 we know that $|\mathrm{Aut}(F_k(G))| \le |\mathrm{Aut}(G)|$. Since $\mathrm{Aut}(G) \le \mathrm{Aut}(F_k(G))$, it follows that $\mathrm{Aut}(G) = \mathrm{Aut}(F_k(G))$. Suppose that $k = n/2$. By Lemma 2.6 we have that $|\mathrm{Aut}(F_k(G))| \le 2|\mathrm{Aut}(G)|$. Since $\mathrm{Aut}(G)\times\mathbb{Z}_2 \le \mathrm{Aut}(F_k(G))$, we have that $\mathrm{Aut}(G)\times\mathbb{Z}_2 = \mathrm{Aut}(F_k(G))$.

2) $\Rightarrow$ 3): Note that $|\mathrm{Iso}(H,G)| = |\mathrm{Aut}(G)|$ and $|\mathrm{Iso}(F_k(H),F_k(G))| = |\mathrm{Aut}(F_k(G))|$. Suppose that $k \neq n/2$. We have that $|\mathrm{Iso}(H,G)| = |\mathrm{Iso}(F_k(H),F_k(G)|$. Since ι is an injection from $\mathrm{Iso}(H,G)$ to $\mathrm{Iso}(F_k(H),F_k(G))$ it is also a bijection and we have 3). Suppose that $k = n/2$. Let

$$X := \{\iota(\phi) : \phi \in \mathrm{Iso}(H,G)\} \text{ and } Y := \{\mathfrak{c}\circ\iota(\phi) : \phi \in \mathrm{Iso}(H,G)\}.$$

Note that $|X| = |\mathrm{Iso}(H,G)| = |Y|$. By Proposition 1.1 we have that $X\cap Y = \emptyset$. Therefore, $\mathrm{Iso}(F_k(H),F_k(G)) = X\cup Y$ which implies 3).

3) $\Rightarrow$ 4): Let $f(\psi) = \iota^{-1}(\psi)$ and let u be a vertex of H. If $\psi = \iota(f(\psi))$, then

$$\psi(\kappa_H(u,k)) = \iota(f(\psi))(\kappa_H(u,k)) = \kappa_G(f(\psi)(u),k).$$

If $\psi = \mathfrak{c} \circ \iota(f)$, then

$$\psi(\kappa_H(u,k)) = \mathfrak{c} \circ \iota(f)(\kappa_H(u,k)) = \mathfrak{c} \circ \kappa_G(f(\psi)(u),k) = \overline{\kappa_G}(f(\psi)(u),k).$$

4) $\Rightarrow$ 1): Note that for any $v \in H$ we have

$$|\kappa_H(v,k)| = \binom{n-1}{k-1} \text{ and } |\overline{\kappa_H}(v,k)| = \binom{n-1}{k}.$$

Therefore, if for some vertex v of H we have that $\psi(\kappa_H(v,k)) = \overline{\kappa_G}(f(\psi)(v),k)$, we would have that $\binom{n-1}{k-1} = \binom{n-1}{k}$. This would imply that n is even and $k = n/2$. Suppose that for some pair of vertices $u, v \in H$ we have that $\psi(\kappa_H(u,k)) = \kappa_G(f(\psi)(u),k)$ and $\psi(\kappa_H(v,k)) = \overline{\kappa_G}(f(\psi)(v),k)$. The set $\kappa_H(u,k) \cap \kappa_H(v,k)$ can be understood as the k-token configurations with two tokens fixed at vertices u and v and the remaining vertices placed at vertices of $G - \{u,v\}$, so

$$|\kappa_H(u,k) \cap \kappa_H(v,k)| = \binom{n-2}{k-2};$$

on the other hand, the set $\kappa_G(f(\psi)(u),k) \cap \overline{\kappa_G}(f(\psi)(v),k)$ can be understood as all the k-token configurations with one token fixed at vertex $f(\psi)(v)$ and no token at vertex $f(\psi)(u)$, so

$$|\kappa_G(f(\psi)(u),k) \cap \overline{\kappa_G}(f(\psi)(v),k)| = \binom{n-2}{k-1};$$

and then,

$$|\varphi\big(\kappa_H(u,k) \cap \kappa_H(v,k)\big)| = |\varphi(\kappa_H(u,k)) \cap \varphi(\kappa_H(v,k))| = |\kappa_G(f(\psi)(u),k) \cap \overline{\kappa_G}(f(\psi)(v),k)|.$$

Thus, $\binom{n-2}{k-2} = \binom{n-2}{k-1}$, and n is odd—a contradiction. Therefore, for all vertices $u \in H$ either $\psi(\kappa_H(u,k)) = \kappa_G(f(\psi)(u),k)$ or $\psi(\kappa_H(u,k)) = \overline{\kappa_G}(f(\psi)(u),k)$.

Let u, v be two vertices of H. We have that

$$\binom{n-2}{k-2} = |\kappa_H(u,k) \cap \kappa_H(v,k)| = |\psi(\kappa_H(u,k) \cap \kappa_H(v,k))| = |\psi(\kappa_H(u,k)) \cap \psi(\kappa_H(v,k))|.$$

Therefore,

$$|\kappa_G(f(\psi)(u),k) \cap \kappa_G(f(\psi)(v),k)| = \binom{n-2}{k-2}$$

or

$$|\overline{\kappa_G}(f(\psi)(u),k) \cap \overline{\kappa_G}(f(\psi)(v),k)| = \binom{n-2}{k-2}.$$

If would be the case that $f(\psi)(u) = f(\psi)(v)$, we would have $\binom{n-1}{k-1} = \binom{n-2}{k-2}$ and then $n = k$, a contradiction. Thus, $f(\psi)(u) \neq f(\psi)(v)$ and $f(\psi)$ is injective; thus, it is also bijective.

Note that u is not adjacent to v if and only if

$$E\left(\kappa_H(u,k) \setminus \kappa_H(v,k), \kappa_H(u,k) \setminus \kappa_H(v,k)\right) = \emptyset.$$

Similarly, u is not adjacent to v if and only if

$$E\left(\overline{\kappa_H}(u,k) \setminus \overline{\kappa_H}(v,k), \overline{\kappa_H}(u,k) \setminus \overline{\kappa_H}(v,k)\right) = \emptyset.$$

Let $X := \kappa_H(u,k)$ and $Y := \kappa_H(v,k)$. Since

$$|E\left(X \setminus Y, Y \setminus X\right)| = |\psi\left(E\left(X \setminus Y, Y \setminus X\right)\right)| = |E\left(\psi(X) \setminus \psi(Y), \psi(Y) \setminus \psi(X)\right)|,$$

we have that u is adjacent to v if and only if $f(\psi)(u)$ is adjacent to $f(\psi)(v)$. Thus, $f(\psi) \in \mathrm{Iso}(H,G)$.

Moreover, if $A \in F_k(H)$ then

$$\psi(A) = \psi\left(\bigcap_{v\in A} \kappa_H(v,k)\right) = \bigcap_{v\in A} \psi(\kappa_H(v,k)) = \bigcap_{v\in A} \kappa_G(f(\psi)(v),k) = \iota(f(\psi))(A)$$

or

$$\psi(A) = \psi\left(\bigcap_{v\in A} \kappa_H(v,k)\right) = \bigcap_{v\in A} \psi(\kappa_H(v,k)) = \bigcap_{v\in A} \overline{\kappa_G}(f(\psi)(v),k) = \mathfrak{c} \circ \iota(f(\psi))(A).$$

Fix an isomorphism ψ from $F_k(H)$ to $F_k(G)$ and let (G,φ) and (G,ϕ) be two k-token reconstructions of $F_k(G)$. Let $\mathfrak{s}(\varphi,\phi) := f(\phi\psi) \circ f(\varphi\psi)^{-1}$. Note that if $\psi(\kappa_H(u,k)) = \kappa_G(f(\psi)(u),k)$ then

$$\begin{aligned}
\iota(\mathfrak{s}(\varphi,\phi)) \circ \varphi &= \iota(f(\phi\psi) \circ f(\varphi\psi)^{-1}) \circ \varphi \\
&= (\phi\psi) \circ (\psi^{-1}\varphi^{-1}) \circ \varphi \\
&= \phi
\end{aligned}$$

and if $\psi(\kappa_H(u,k)) = \overline{\kappa_G}(f(\psi)(u),k)$ we have

$$\begin{aligned}
\mathfrak{c} \circ \iota(\mathfrak{s}(\varphi,\phi)) \circ \varphi &= \mathfrak{c} \circ \iota(f(\phi\psi) \circ f(\varphi\psi)^{-1}) \circ \varphi \\
&= \mathfrak{c} \circ (\mathfrak{c} \circ \phi) \\
&= \phi
\end{aligned}$$

Thus, $\phi = \iota(\mathfrak{s}(\varphi,\phi)) \circ \varphi$ or $\phi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi,\phi)) \circ \varphi$ and we have 1). □

Example 2.2. *Let us show that $F_2(K_{2,3})$ is not uniquely reconstructible as the 2-token graph of $K_{2,3}$. To see this, we make use of the equivalence 1) $\iff$ 4) of Theorem 2.7.*

Let us denote the graph $K_{2,3}$ by G and let $\{X,Y\}$ be the bipartition of G with $X := \{1,2\}$ and $Y := \{3,4,5\}$. Let $\psi : F_2(G) \to F_2(G)$ be the function such that

- $\psi(\{1,3\}) = \{2,3\}$,
- $\psi(\{2,3\}) = \{1,3\}$, *and*
- $\psi(A) = A$ *for any* $A \in V(F_2(G))$ *other than* $\{1,3\}$ *and* $\{2,3\}$.

We have

$$\kappa_G(1,k) = \big\{\{1,2\},\{1,3\},\{1,4\},\{1,5\}\big\}, \quad \kappa_G(2,k) = \big\{\{1,2\},\{2,3\},\{2,4\},\{2,5\}\big\},$$
$$\kappa_G(3,k) = \big\{\{1,3\},\{2,3\},\{3,4\},\{3,5\}\big\}, \quad \kappa_G(4,k) = \big\{\{1,4\},\{2,4\},\{3,4\},\{4,5\}\big\},$$
$$\kappa_G(5,k) = \big\{\{1,5\},\{2,5\},\{3,5\},\{4,5\}\big\}.$$

Then,

$$\psi(\kappa_G(1,k)) = \big\{\{1,2\},\{2,3\},\{1,4\},\{1,5\}\big\}$$

and so $\psi(\kappa_G(1,k)) \neq \kappa_G(i,k)$ *for any* $i \in \{1,2,3,4,5\}$. *This implies that there is no function* f *such that* $\psi(\kappa_G(1,k)) = \kappa_G(f(\psi)(1),k)$, *and thus, by Theorem 2.7,* $F_2(K_{2,3})$ *is not uniquely reconstructible as the* 2*-token graph of* $K_{2,3}$.

We remark that, as we will show in Chapter 3, the fact that $F_2(K_{2,3})$ is not uniquely reconstructible as the 2-token graph of $K_{2,3}$ is just a simple case of a more general result, that $F_k(K_{2,n})$ is not uniquely reconstructible as the k-token graph of $K_{2,n}$, for any $n \geq 2$ and $2 \leq k \leq n$.

2.2.1 k-reconstruction Families

We now present some consequences of Theorem 2.7.

Suppose that F is a graph on $\binom{n}{k}$ vertices. Inspired by property 4) of Theorem 2.7, we define the concept of a k-reconstruction family of F. Let $\mathcal{R}$ be a family of subsets of vertices of F. For every vertex $A \in F$, let

$$S_{\mathcal{R}}(A) := \{X \in \mathcal{R} : A \in X\}.$$

We say that a family $\mathcal{R}$ of subsets of vertices of F is a *k-reconstruction family* of F if it satisfies the following properties.

1) $|X| = \binom{n-1}{k-1}$ for all $X \in \mathcal{R}$;
2) $|S_{\mathcal{R}}(A)| = k$ for all $A \in V(F)$; and
3) for every edge $AB \in F$ we have that $|S_{\mathcal{R}}(A) \cap S_{\mathcal{R}}(B)| = k - 1$.

Note that 1) and 2) imply that $|\mathcal{R}| = n$. Let (G, φ) be a k-reconstruction of F. Note that

$$\mathcal{R}_\varphi := \{\varphi^{-1}(\kappa_G(u,k)) : u \in V(G)\}$$

is a k-reconstruction family of F. Conversely, from a k-reconstruction family we can obtain a k-token reconstruction of F as follows. Let $G_\mathcal{R}$ be the graph whose vertex set is $\mathcal{R}$; and such that X is adjacent to Y in $G_\mathcal{R}$ if and only if there exists an edge AB of F such that

$$S_\mathcal{R}(A) \triangle S_\mathcal{R}(B) = \{X, Y\}.$$

Proposition 2.8. *If $\mathcal{R}$ is a k-reconstruction family of F, then $(G_\mathcal{R}, S_\mathcal{R})$ is a k-token reconstruction of F.*

Proof. By 2), for every $A \in V(F)$ we have that $S_\mathcal{R}(A)$ is a k-subset of vertices of $G_\mathcal{R}$ and so $S_\mathcal{R}(A) \in V(F_k(G_\mathcal{R}))$. Note that $S_\mathcal{R}$ maps vertices of F to vertices of $F_k(G_\mathcal{R})$, and moreover, this map is injective, which implies that $|F| \leq |F_k(G_\mathcal{R})|$, and since $|F_k(G_\mathcal{R})| = \binom{n}{k}$, it follows that $S_\mathcal{R}$ is a bijection from $V(F)$ to $V(F_k(G_\mathcal{R}))$.

Let us now see that $S_\mathcal{R}$ is an isomorphism from F to $F_k(G_\mathcal{R})$. Let $A, B \in F$. We have

$$\begin{aligned} AB \in E(F) &\iff S_\mathcal{R}(A) \triangle S_\mathcal{R}(B) = \{X, Y\} \text{ for some } X, Y \in \mathcal{R} \\ &\qquad\qquad \text{with } A \in X,\ B \in Y \text{ and } XY \in E(G_\mathcal{R}) \\ &\iff S_\mathcal{R}(A) S_\mathcal{R}(B) \in E(F_k(G_\mathcal{R})). \end{aligned}$$

This completes the proof. □

Consider a k-reconstruction family $\mathcal{R}$ of F; let

$$\overline{\mathcal{R}} := \{V(F) \setminus X : X \in \mathcal{R}\}.$$

As is expected, $\overline{\mathcal{R}}$ is a $(n-k)$-reconstruction family of F, and if $k = n/2$, then $\overline{\mathcal{R}}$ is a k-reconstruction family of F.

Proposition 2.9. *Suppose that $\mathcal{R}$ is a k-reconstruction family of F and that $k = n/2$. Then $\overline{\mathcal{R}}$ is a k-reconstruction family of F.*

Proof. For every $X \in \mathcal{R}$ we have that $|V(F) \setminus X| = \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k-1}$; thus, $\overline{\mathcal{R}}$ satisfies 1). For every $A \in V(F)$ we have that $|S_{\overline{\mathcal{R}}}(A)| = |\mathcal{R} \setminus S_\mathcal{R}(A)| = n - k = k$; thus, $\overline{\mathcal{R}}$ satisfies 2). For every edge $AB \in F$ we have

$$\begin{aligned} |S_{\overline{\mathcal{R}}}(A) \cap S_{\overline{\mathcal{R}}}(B)| &= |\big(\mathcal{R} \setminus S_\mathcal{R}(A)\big) \cap \big(\mathcal{R} \setminus S_\mathcal{R}(B)\big)| \\ &= |\mathcal{R} \setminus \big(S_\mathcal{R}(A) \cup S_\mathcal{R}(B)\big)| \\ &= n - (k+1) \\ &= k - 1 \end{aligned}$$

and so, $\overline{\mathcal{R}}$ satisfies 3). □

The property of two k-reconstructions being equivalent can be deduced using with k-reconstruction families.

Proposition 2.10. *Let (G,φ) and (G,ψ) be two k-token reconstructions of F. Then, (G,φ) and (G,ψ) are equivalent k-token reconstructions of G if and only if $\mathcal{R}_\varphi = \mathcal{R}_\psi$ or $\mathcal{R}_\varphi = \overline{\mathcal{R}}_\psi$.*

Proof. Suppose that (G,φ) and (G,ψ) are equivalent k-token reconstructions of G. Then there exists an automorphism $\mathfrak{s}(\varphi,\psi)$ of G such that

$$\psi = \iota(\mathfrak{s}(\varphi,\psi)) \circ \varphi \quad \text{or} \quad \psi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi,\psi)) \circ \varphi.$$

In the first case we have that

$$\begin{aligned} R_\varphi &= \{\varphi^{-1}(\kappa_G(u,k) : u \in V(G)\} \\ &= \{\psi^{-1} \circ \iota(\mathfrak{s}(\varphi,\psi))(\kappa_G(u,k)) : u \in V(G)\} \\ &= \{\psi^{-1}(\kappa_G(\mathfrak{s}(\varphi,\psi)(u),k)) : u \in V(G)\} \\ &= \{\psi^{-1}(\kappa_G(u,k) : u \in V(G)\} \\ &= \mathcal{R}_\psi. \end{aligned}$$

In the second case we have that

$$\begin{aligned} R_\varphi &= \{\varphi^{-1}(\kappa_G(u,k) : u \in V(G)\} \\ &= \{\psi^{-1} \circ \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi,\psi))(\kappa_G(u,k)) : u \in V(G)\} \\ &= \{\mathfrak{c} \circ \psi^{-1}(\kappa_G(\mathfrak{s}(\varphi,\psi)(u),k)) : u \in V(G)\} \\ &= \{V(F) \setminus \psi^{-1}(\kappa_G(u,k) : u \in V(G)\} \\ &= \overline{\mathcal{R}}_\psi. \end{aligned}$$

Suppose that $\mathcal{R}_\varphi = \mathcal{R}_\psi$ or $\mathcal{R}_\varphi = \overline{\mathcal{R}}_\psi$. We define an automorphism f of G as follows. Let $u \in V(G)$ and let $f(u)$ be the vertex of G such that

$$\varphi^{-1}(\kappa_G(u,k)) = \psi^{-1}(\kappa_G(f(u),k)) \text{ or } \varphi^{-1}(\kappa_G(u,k)) = V(F) \setminus \psi^{-1}(\kappa_G(f(u),k)).$$

Condition 3) in the definition of k-reconstruction family implies that f is an automorphism of G. We have that

$$\psi = \iota(f) \circ \varphi \quad \text{or} \quad \psi = \mathfrak{c} \circ \iota(f) \circ \varphi.$$

Thus, (G,φ) and (G,ψ) are equivalent k-reconstructions of F. □

We next present an equivalent condition to $F_k(G)$ being uniquely reconstructible as the k-token graph of G, but now using reconstruction families.

Proposition 2.11. *Let $\mathcal{R}$ be a k-reconstruction family of F. F is uniquely reconstructible as the k-token graph of $G_\mathcal{R}$ if and only if for every automorphism φ of F we have that*

$$\mathcal{R} = \{\varphi(X) : X \in \mathcal{R}\} \textit{ or } \overline{\mathcal{R}} = \{\varphi(X) : X \in \mathcal{R}\}.$$

Proof. Let $X \in \mathcal{R}$. Note that

$$\begin{aligned}\kappa_{G_{\mathcal{R}}}(X,k) &= \{W \subset \mathcal{R} : |W| = k \text{ and } X \in W\}\\ &= \{S_{\mathcal{R}}(A) : A \in X\}\\ &= S_{\mathcal{R}}(X).\end{aligned}$$

If $k = n/2$, we also have that

$$\begin{aligned}\overline{\kappa_{G_{\mathcal{R}}}}(X,k) &= \{W \subset \mathcal{R} : |W| = k \text{ and } X \notin W\}\\ &= \{S_{\mathcal{R}}(A) : A \notin X\}\\ &= S_{\mathcal{R}}(V(F) \setminus X).\end{aligned}$$

Let

$$\varphi' := S_{\mathcal{R}} \circ \varphi \circ S_{\mathcal{R}}^{-1}.$$

Note that φ' is an automorphism of $F_k(G_{\mathcal{R}})$.

Suppose that F is uniquely reconstructible as the k-token graph of $G_{\mathcal{R}}$. Thus, $F_k(G_{\mathcal{R}})$ is uniquely reconstructible as the k-token graph of $G_{\mathcal{R}}$. Let $X \in \mathcal{R}$. By 4) of Theorem 2.7 we have that $\varphi'(\kappa_{G_{\mathcal{R}}}(X,k)) = \kappa_{G_{\mathcal{R}}}(Y,k)$ or $\varphi'(\kappa_{G_{\mathcal{R}}}(X,k)) = \overline{\kappa_{G_{\mathcal{R}}}}(Y,k)$, for some $Y \in \mathcal{R}$. In the first case we have

$$\begin{aligned}\varphi'(\kappa_{G_{\mathcal{R}}}(X,k)) &= (S_{\mathcal{R}} \circ \varphi \circ S_{\mathcal{R}}^{-1})(\kappa_{G_{\mathcal{R}}}(X,k))\\ &= (S_{\mathcal{R}} \circ \varphi \circ S_{\mathcal{R}}^{-1})(S_{\mathcal{R}}(X))\\ &= (S_{\mathcal{R}} \circ \varphi)(X)\\ &= S_{\mathcal{R}}(\varphi(X),k)\\ &= \kappa_{G_{\mathcal{R}}}(\varphi(X),k)\end{aligned}$$

and so

$$\varphi(X) = Y.$$

By similar arguments, in the second case we have

$$\varphi(X) = V(F) \setminus Y.$$

As in the proof of 4) $\Rightarrow$ 1) in Theorem 2.7, we have either the first case happens for all $X \in \mathcal{R}$ or the second case happens for all $X \in \mathcal{R}$. Thus,

$$\mathcal{R} = \{\varphi(X) : X \in \mathcal{R}\} \text{ or } \overline{\mathcal{R}} = \{\varphi(X) : X \in \mathcal{R}\}. \tag{3}$$

Suppose now that (3) holds. For all $X \in \mathcal{R}$ we have that

$$\varphi'(\kappa_{G_{\mathcal{R}}}(X,k)) = \kappa_{G_{\mathcal{R}}}(Y,k) \text{ or } \varphi'(\kappa_{G_{\mathcal{R}}}(X,k)) = \overline{\kappa_{G_{\mathcal{R}}}}(Y,k), \text{ for some } Y \in \mathcal{R}$$

and so, by Theorem 2.7 it follows that $F \simeq F_k(G_{\mathcal{R}})$ is uniquely reconstructible as the k-token graph of $G_{\mathcal{R}}$. □

The following result allows us to determine when F is reconstructible by means of reconstruction families.

Proposition 2.12. *Let (G, φ) and (H, ϕ) be two k-token reconstructions of F. Then $G \simeq H$ if and only if there exists an automorphism ψ of F such that*

$$\mathcal{R}_\phi = \{\psi(X) : X \in \mathcal{R}_\varphi\}.$$

Proof. Suppose that $G \simeq H$. Let f be an isomorphism from G to H. Let

$$\psi := \phi^{-1} \circ \iota(f) \circ \varphi.$$

Let $X \in \mathcal{R}_\varphi$. Let $x \in V(G)$ be such that $\varphi(X) = \kappa_G(x, k)$. We have that $\iota(f) \circ \varphi(X) = \kappa_H(f(x), k)$. Let $Y \in \mathcal{R}_\phi$ be such that $Y = \phi^{-1}(\kappa_H(f(x), k)$. Thus, $Y = \psi(X)$, and $\mathcal{R}_\phi = \{\psi(X) : X \in \mathcal{R}_\varphi\}$.

Suppose that there exists $\psi \, \mathrm{Aut}(F)$ such that $\mathcal{R}_\phi = \{\psi(X) : X \in \mathcal{R}_\varphi\}$. We define an isomorphism, f, from G to H. Let $x \in V(G)$. Let $Y = \psi(\varphi^{-1}(\kappa_G(x, k))$. Let $f(x)$ be the vertex of H such that $\phi^{-1}(\kappa_H(f(x), k)) = Y$. Condition 3) in the definition of k-reconstructible family implies that f is an isomorphism. □

As we have seen, k-token reconstructions are related to k-token reconstruction families, and so, we can use Proposition 2.12 to rephrase Conjecture 2.1:

Conjecture 2.2. *Let G be a graph. For every two k-token reconstruction families $\mathcal{R}$ and $\mathcal{R}'$ of $F_k(G)$, there exists an automorphism ψ of $F_k(G)$ such that*

$$\mathcal{R}' = \{\psi(X) : X \in \mathcal{R}\}.$$

As we saw in Example 2.1, $F_2(C_4)$ is not uniquely reconstructible as the 2-token graph of C_4. Next we show the same result, but now using k-token reconstruction families.

Example 2.3. *Let $C_4 =: (a, b, c, d)$ and let*

$$X_1 := \big\{\{b,c\},\{a,c\},\{a,d\}\big\},\ X_2 := \big\{\{a,b\},\{b,c\},\{b,d\}\big\},$$
$$X_3 := \big\{\{a,b\},\{a,c\},\{c,d\}\big\},\ X_4 := \big\{\{a,d\},\{b,d\},\{c,d\}\big\}.$$

Let $\mathcal{R} := \{X_1, X_2, X_3, X_4\}$. It is straightforward to show that $\mathcal{R}$ is a 2-token reconstruction family of $F_2(C_4)$. For $i \in \{1, 2, 3, 4\}$ let $\overline{X}_i := V(F_2(C_4)) \setminus X_i$, so

$$\overline{X}_1 = \big\{\{a,b\},\{b,d\},\{c,d\}\big\},\ \overline{X}_2 = \big\{\{a,c\},\{a,d\},\{c,d\}\big\},$$
$$\overline{X}_3 = \big\{\{a,d\},\{b,c\},\{b,d\}\big\},\ \overline{X}_4 = \big\{\{a,b\},\{a,c\},\{b,c\}\big\}.$$

Let $\overline{\mathcal{R}} := \{\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4\}$. By Proposition 2.9 we know that $\overline{\mathcal{R}}$ is also a k-reconstruction family of $F_2(C_4)$.

To see that $F_2(C_4)$ is not uniquely reconstructible as the 2-token graph of C_4, we are going to exhibit an automorphism φ of $F_2(C_4)$ such that $\mathcal{R} \neq \{\varphi(X) : X \in \mathcal{R}\}$ and $\overline{\mathcal{R}} \neq \{\varphi(X) : X \in \mathcal{R}\}$, and according to Proposition 2.11, this will imply that $F_2(C_4)$ is not uniquely reconstructible as the k-token graph of C_4. Consider the automorphism φ of $F_2(C_4)$ such that

- $\varphi(\{a,b\}) = \{b,c\}$,
- $\varphi(\{b,c\}) = \{a,b\}$, *and,*
- $\varphi(A) = A$ *for any* $A \in V(F_2(C_4))$ *other than* $\{a,b\}$ *and* $\{b,c\}$.

Then we have

$$\varphi(X_1) = \big\{\{a,b\},\{a,c\},\{a,d\}\big\},\ \varphi(X_2) = \big\{\{a,b\},\{b,c\},\{b,d\}\big\},$$
$$\varphi(X_3) = \big\{\{a,c\},\{b,c\},\{c,d\}\big\},\ \varphi(X_4) = \big\{\{a,d\},\{b,d\},\{c,d\}\big\},$$

and so, it is clear that $\mathcal{R} \neq \{\varphi(X) : X \in \mathcal{R}\}$ and $\overline{\mathcal{R}} \neq \{\varphi(X) : X \in \mathcal{R}\}$, as we wanted.

2.2.2 Boolean Combinations

Let $\mathcal{F}$ be a family of subsets of a set S. A *Boolean combination* on $\mathcal{F}$ is defined recursively as follows.

(1) For every $X \in \mathcal{F}$, X is Boolean combination on $\mathcal{F}$;

(2) If Γ_1 and Γ_2 are Boolean combinations on $\mathcal{F}$, then so are $S \setminus \Gamma_1$, $\Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2$.

In Section 2.6 we use the following proposition to show that F is uniquely reconstructible; it follows from Theorem 2.7 by induction on Γ.

Proposition 2.13. *Let G and H be isomorphic graphs on at least 3 vertices. Suppose that $F_k(G)$ is uniquely reconstructible as the the k-token graph of G. Let Γ be a Boolean combination on $V(F_k(H))$, and let $\psi \in \operatorname{Iso}(F_k(H), F_k(G))$. Let Γ_ψ be the Boolean combination on $V(F_k(G))$ that is obtained by replacing every term A in Γ with $\psi(A)$; and let $\overline{\Gamma}_\psi$ be the Boolean combination on $V(F_k(G))$ that is obtained by replacing each term A in Γ with $V(G) \setminus \psi(A)$. Then*

$$\psi(\Gamma) = \Gamma_\psi \text{ or } \psi(\Gamma) = \overline{\Gamma}_\psi.$$

We now take a brief detour and consider the token graphs of stars; these graphs play a crucial part in our reconstruction algorithm.

2.3 Token graphs of stars

We call the complete bipartite graph $K_{1,n}$ a *star*. Throughout this section let $n > 2$ and $k \leq (n+1)/2$. Let $\{x_0, x_1, \ldots, x_n\}$ be the vertices of $K_{1,n}$ so that x_0 is the vertex of degree greater than one. Note that $F_k(K_{1,n})$ is a bipartite graph; one set in the partition corresponds to the token configurations without a token at x_0 and the other set corresponds to the token configurations with a token at x_0. Let V_0 and V_1 be these sets, respectively. Every vertex in V_0 has degree equal to k and every vertex in V_1 has degree equal to $n-k+1$.

In the following lemmas we show that it is possible to determine whether a graph is isomorphic (or not) to the k-token graph of a star. If this is the case, then we can also compute an isomorphism.

Lemma 2.14. *Let F be a graph isomorphic to a token graph of a star. Then there exist unique positive integers n and $k \leq (n+1)/2$, such that $F \simeq F_k(K_{1,n})$; these integers can be found in polynomial time.*

Proof. We may assume that $n > 2$ as otherwise $K_{1,n}$ is an edge and we are done. Let W_0 and W_1 be the two sets in the bipartition of $V(F)$. Note that every vertex in W_0 has the same degree d_0, and every vertex in W_1 had the same degree d_1. Without loss of generality assume that $d_0 \leq d_1$. If $d_0 < d_1$, an isomorphism from F to $F_k(K_{1,n})$ must map W_0 to V_0 and W_1 to V_1. If $d_0 = d_1$, an isomorphism from F to $F_k(K_{1,n})$ can map W_0 to V_0 or to V_1. In both cases $d_0 = k$ and $d_1 = n - k + 1$. Therefore, k and n are uniquely determined, and computable in polynomial time. $\square$

Lemma 2.15. *Let F be a bipartite graph, and let W_0 and W_1 be its partition sets. Suppose that every vertex in W_0 has degree equal to k and that every vertex in W_1 has degree equal to $n - k + 1$.*

- *Let v^* be a vertex in W_0;*
- *let $w_1, \ldots, w_k$ be the neighbours of v^*;*
- *let $v_{k+1}, v_{k+2}, \ldots, v_n$ be the neighbours of w_1 distinct from v^*.*

Let f be any injective function that maps $\{v^*\} \cup \{w_1, \ldots, w_k\} \cup \{v_{k+1}, v_{k+2}, \ldots, v_n\}$ to the vertices of $F_k(K_{1,n})$ such that

- *$f(N(v^*)) = f(\{w_1, \ldots, w_k\}) = N(f(v^*))$ and*
- *$f(N(w_1) \setminus \{v^*\}) = f(\{v_{k+1}, v_{k+2}, \ldots, v_n\}) = N(f(w_1)) \setminus \{f(v^*)\}$.*

If F and $F_k(K_{1,n})$ are isomorphic, then in polynomial time we can extend f to an isomorphism from F to $F_k(K_{1,n})$. Moreover, if F and $F_k(K_{1,n})$ are not isomorphic then we can determine in polynomial time that such an extension does not exist.

Proof. We may assume that $n > 2$ as otherwise we are done. We provide an algorithm that attempts to extend f to an isomorphism from F to $F_k(K_{1,n})$. The algorithm succeeds if and only if F and $F_k(K_{1,n})$ are isomorphic. Our algorithm proceeds by labelling the vertices of F. Let v be a vertex of F. If v is in W_0 then v will be labelled with a string of integers $s_1 s_2 \cdots s_k$; this means that isomorphism maps v to the token configuration $\{x_{s_1}, \ldots, x_{s_k}\}$. If v is in W_1 then v will be labelled with a string of integers $s_1 \cdot s_2 \cdots s_{k-1}$; this means that our isomorphism maps v to the token configuration $\{x_0, x_{s_1}, \ldots, x_{s_{k-1}}\}$. We denote with $\ell(v)$ the label assigned to a vertex v. Let s be one of these labellings. For a given integer j, we denote with $s \ominus j$ the label that results from s by removing the appearance of j. Similarly, we denote with $s \oplus j$ the label that results from adding j to s.

If necessary we relabel the vertices of $K_{1,n}$ so that $\ell(v^*) = 1 \cdot 2 \cdots k$. Note that the neighbours of v^* receive a label of the form $\ell(v^*) \ominus j$ for some $1 \leq j \leq k$. We relabel the neighbours of v so that $\ell(w_j) := \ell(v^*) \ominus j$. Note that the neighbours of w_1 distinct from v^* receive a label of the form $\ell(v^*) \ominus 1 \oplus j$ for some $k+1 \leq j \leq n$. We relabel the neighbours of w_1 distinct from v^* so that $\ell(v_j) = \ell(v^*) \ominus 1 \oplus j$. This first labelling can be made if F and $F_k(K_{1,n})$ are indeed isomorphic. In what follows, we show that this first labelling determines the labels of the remaining vertices of F.

We now label the neighbours of each w_j with $j \neq 1$. Note that the neighbourhoods of distinct w_j only intersect at v^*. Let $j \neq 1$. Let u be an unlabelled neighbour of w_j. Note that u should receive a label of the form $\ell(v^*) \ominus j \oplus t$ for some $k+1 \leq t \leq n$. Further note that u should receive the label $\ell(v^*) \ominus j \oplus t$ if and only if there is a path of length two from u to v_t. This corresponds to the following token moves: starting from the token configuration assigned to v_t move the token at x_j to x_0; then move this token from x_0 to x_1 to arrive to the token configuration assigned to u. We label each such u by checking the paths of length 2 from u to $v_{k+1}, v_{k+2}, \ldots, v_n$. In the process we check whether there are conflicting labellings for u, in which case F and $F_k(K_{1,n})$ are not isomorphic.

So far we have labelled all the vertices in W_1 at distance one from v^* and all vertices in W_0 at distance two from v^*. Let $d \geq 3$ be an odd integer. Suppose we have labelled all the vertices in W_1 at distance at most $d-2$ from v^* and all the vertices in W_0 at distance at most $d-1$ from v^*. We now label the vertices in W_1 at distance d from v^* and the vertices in W_0 at distance $d+1$ from v^*.

Let u be a vertex in W_1 at distance d from v^*. Let y_1 and y_2 be two neighbours of u at distance $d-1$ from v^*. Note that there exists two integers t_1 and t_2 such that $\ell(y_2) = \ell(y_1) \ominus t_1 \oplus t_2$; thus, u should be labelled with $s := \ell(y_1) \ominus t_1 = \ell(y_2) \ominus t_2$. We label each such u by checking all its pairs of neighbours at distance $d-1$ from v^*. In the process we check whether there are conflicting labellings for u, in which case F and $F_k(K_{1,n})$ are not isomorphic.

Let now u be a vertex in W_0 at distance $d+1$ from v^*. Let y_1 and y_2 be two neighbours of u at distance d from v^*. Note that there exists two integers t_1 and t_2 such that $\ell(y_2) =$

$\ell(y_1) \ominus t_1 \oplus t_2$; thus, u should be labelled with $s := \ell(y_1) \oplus t_2 = \ell(y_2) \oplus t_1$. We label each such u by checking all its pairs of neighbours at distance d from v^*. In the process we check whether there are conflicting labellings for u, in which case F and $F_k(K_{1,n})$ are not isomorphic. If the algorithm succeeds in labelling the vertices of F, then F and $F_k(K_{1,n})$ are isomorphic. □

Lemma 2.16. *We can determine in polynomial time whether F and $F_k(K_{1,n})$ are isomorphic. Moreover, if $F \simeq F_k(K_{1,n})$ we have the following.*

- *we can find an isomorphism between F and $F_k(K_{1,n})$ in polynomial time;*
- *F is uniquely reconstructible as the k-token graph of $K_{1,n}$.*

Proof. We use Lemma 2.14 to compute the only possible values for k and n (with $k \leq (n+1)/2$). Assume that F is bipartite as otherwise, F and $F_k(K_{1,n})$ are not isomorphic. Let W_0 and W_1 the bipartition of $V(F)$. We may assume that every vertex of W_0 has degree equal to k and that every vertex in W_1 has degree equal to $n-k+1$, or that every vertex of W_1 has degree equal to k and that every vertex in W_0 has degree equal to $n-k+1$. Otherwise, F and $F_k(K_{1,n})$ are not isomorphic. Assume without loss of generality that every vertex of W_0 has degree equal to k and that every vertex in W_1 has degree equal to $n-k+1$. Pick a vertex $v^* \in W_0$. Let $\{w_1, \dots, w_k\}$ be the neighbours of v^*. Let $\{v_{k+1}, v_{k+2}, \dots, v_n\}$ be the neighbours of w_1 distinct from v^*. Choose any injective function, f, that maps $\{v^*\} \cup \{w_1, \dots, w_k\} \cup \{v_{k+1}, v_{k+2}, \dots, v_n\}$ to the vertices of $F_k(K_{1,n})$ such that

- $f(N(v^*)) = f(\{w_1, \dots, w_k\}) = N(f(v^*))$ and
- $f(N(w_1) \setminus \{v^*\}) = f(\{v_{k+1}, v_{k+2}, \dots, v_n\}) = N(f(w_1)) \setminus \{f(v^*)\}$.

By Lemma 2.15, we can extend f to an isomorphism ψ from F to $F_k(K_{1,k})$, if and only if F and $F_k(K_{1,n})$ are isomorphic. Assume that F and $F_k(K_{1,n})$ are isomorphic.

By Lemma 2.15, iterating over all possible choices for f, generates all isomorphisms, ψ, from F to $F_k(K_{1,n})$. This allows us to bound the size of $\text{Iso}(F, F_k(K_{1,n}))$ by counting the number of possible choices for f. If $k = (n+1)/2$ we have that $f(v^*) \in V_0$ or $f(v^*) \in V_1$. Once this choice is made, there are $\binom{n}{k}$ possible choices for $f(v^*)$. Once the value of $f(v^*)$ is fixed there are $k!$ possible choices for $\{f(w_1), \dots, f(w_k)\}$. Once these values are fixed, there are $(n-k)!$ possible choices for $\{f(v_{k+1}), f(v_{k+2}), \dots, f(v_n)\}$. We have that

$$|\,\text{Iso}(F, F_k(K_{1,n}))| = \begin{cases} n! & \text{if } k \neq (n+1)/2, \\ 2n! & \text{if } k = (n+1)/2. \end{cases}$$

Since $\text{Aut}(K_{1,n}) = S_n$ and $|\,\text{Aut}(F_k(K_{1,n})| = |\,\text{Iso}(F, F_k(K_{1,n}))|$, by Theorem 2.7 we have that F is uniquely reconstructible as the k-token graph of $K_{1,n}$. □

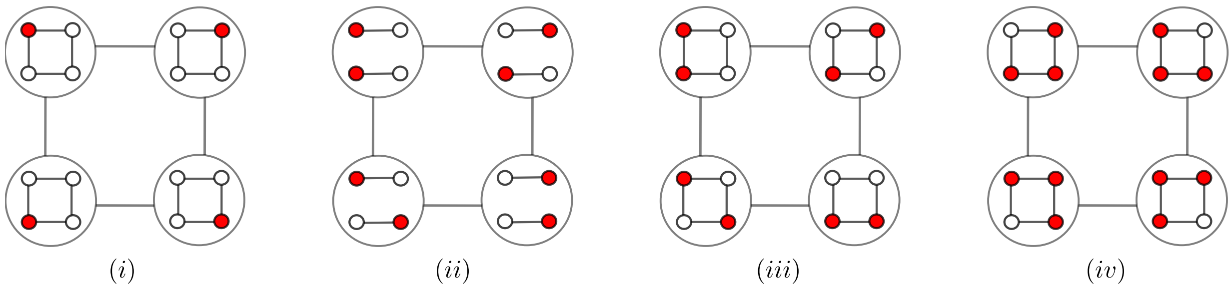


Figure 2.2: The four ways to generate an induced 4-cycle in $F_k(G)$ (tokens not shown are assumed to remain fixed).

2.4 Induced 4-cycles of $F_k(G)$ and Ladders

Let us think on the following local reconstruction problem: for an induced subgraph H of $F_k(G)$, we wonder how was generated H. It may happen that there are two or more different ways of generating such subgraph H. For example, if H is an induced 4-cycle, then it is not hard to see that H may be generated by moving one token along a 4-cycle of G and keeping the remaining $k-1$ tokens fixed on some other vertices; however, H may be also generated by moving two tokens on two disjoint edges (one token per edge) and keeping the remaining $k-2$ tokens fixed on some other vertices of G. With this in mind, in this section we study how induced 4-cycles in $F_k(G)$ can be generated. In particular we show that if G is a (C_4, D_4)-free graph, then there is only one way to generate induced 4-cycles in $F_k(G)$: by moving two tokens along two independent edges of G (one token per edge). We use this characterization to define an equivalence relationship on the edges of $F_k(G)$. This equivalence relationship is computable in polynomial time.

2.4.1 Induced 4-cycles of $F_k(G)$

We start by showing how are generated the induced 4-cycles of $F_k(G)$.

Proposition 2.17. *Every induced 4-cycle of a k-token graph is generated in one of the four ways depicted in Figure 2.2.*

Proof. Let G be a graph. Let $\mathcal{C} := (A, B, C, D)$ be an induced 4-cycle of $F_k(G)$. Let

- $A \triangle B := \{a_1, b_1\}$ with $a_1 \in A, b_1 \in B$;
- $B \triangle C := \{b_2, c_1\}$ with $b_2 \in B, c_1 \in C$; and
- $C \triangle D := \{c_2, d_1\}$ with $c_2 \in C, d_1 \in D$.

We proceed by case analysis.

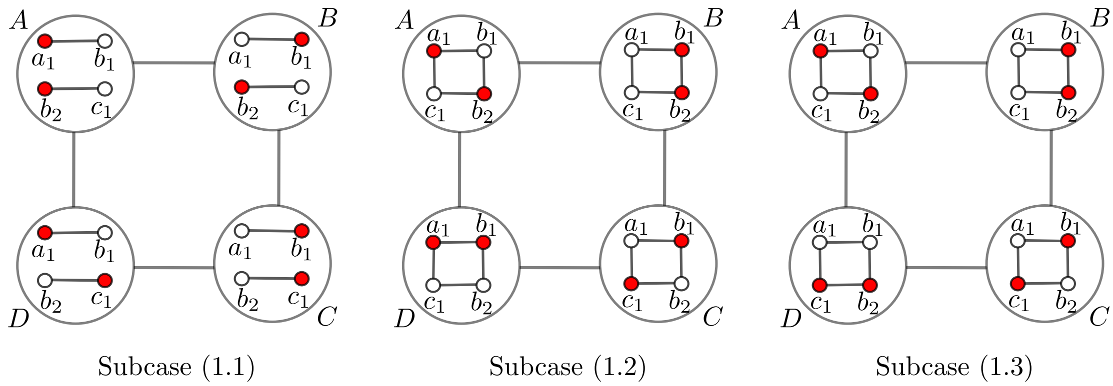


Figure 2.3: All three possibilities in Case (1).

(1) Suppose that $\{a_1, b_1\} \cap \{b_2, c_1\} = \emptyset$. This implies that $A \setminus C = \{a_1, b_2\}$ and $C \setminus A = \{b_1, c_1\}$, so $A \triangle C = \{a_1, b_2, b_1, c_1\}$. These vertices correspond to the following token configurations: with the k-subset A fixed, B is obtained from A by moving the token at vertex a_1 to vertex b_1, and then C is obtained from B by moving the token at vertex b_2 to vertex c_1. Since D is adjacent to A, we must have that $C \triangle D \subset A \triangle C$. Then, D is obtained from C by moving the token at c_2 to d_1, where $c_2 \in \{b_1, c_1\}$ and $d_1 \in \{a_1, b_2\}$; however, let us note here that if $c_2 = c_1$ and $d_1 = b_2$, then $D = B$, a contradiction. Thus, there are the following three possibilities:

(1.1) Suppose $c_2 = b_1$ and $d_1 = a_1$, so D is obtained from C by moving the token at vertex b_1 to vertex a_1. Then $C \triangle D = \{a_1, b_1\}$, and so $A \triangle D = \{b_2, c_1\}$. See Subcase (1.1) of Figure 2.3. Then, $\mathcal{C}$ is generated as in (ii) of Figure 2.2.

(1.2) Suppose $c_2 = c_1$ and $d_1 = a_1$, so D is obtained from C by moving the token at vertex c_1 to vertex a_1. Then $C \triangle D = \{a_1, c_1\}$, and so $A \triangle D = \{b_1, b_2\}$. See Subcase (1.2) of Figure 2.3. In this case, $\mathcal{C}$ is generated as in (iii) of Figure 2.2.

(1.3) Suppose $c_2 = b_1$ and $d_1 = b_2$, so D is obtained from C by moving the token at vertex b_1 to vertex b_2. Then, $C \triangle D = \{b_1, b_2\}$, and so $A \triangle D = \{c_1, a_1\}$. See Subcase (1.3) in Figure 2.3. Thus, $\mathcal{C}$ is generated as in (iii) of Figure 2.2.

(2) Suppose that $\{a_1, b_1\} \cap \{b_2, c_1\} \neq \emptyset$. Thus, $b_1 = b_2$ or $a_1 = c_1$.

(2.1) Suppose that $b_1 = b_2$.

(2.1.1) Suppose that $c_1 = c_2$, so, $d_1 \neq a_1$. In this case, D is obtained from C by moving the token at vertex c_1 to vertex d_1, so $C \triangle D = \{c_1, d_1\}$ and $A \triangle D = \{a_1, d_1, \}$. See Subcase (2.1.1) of Figure 2.4. Therefore, $\mathcal{C}$ is generated as in (i) of Figure 2.2.

(2.1.2) Suppose that $c_1 \neq c_2$. If $d_1 \neq a_1$ then A and D are not adjacent since $A \triangle D = \{a_1, c_1, c_2, d_1\}$ in this case. Therefore, $d_1 = a_1$. This implies D is obtained from C by moving the token at vertex c_2 to vertex a_1, and so, $C \triangle D = \{c_2, a_1\}$ and $A \triangle D = \{c_1, c_2\}$. See Subcase (2.1.2) of Figure 2.4. Then, $\mathcal{C}$ is generated as in (iii) of Figure 2.2.

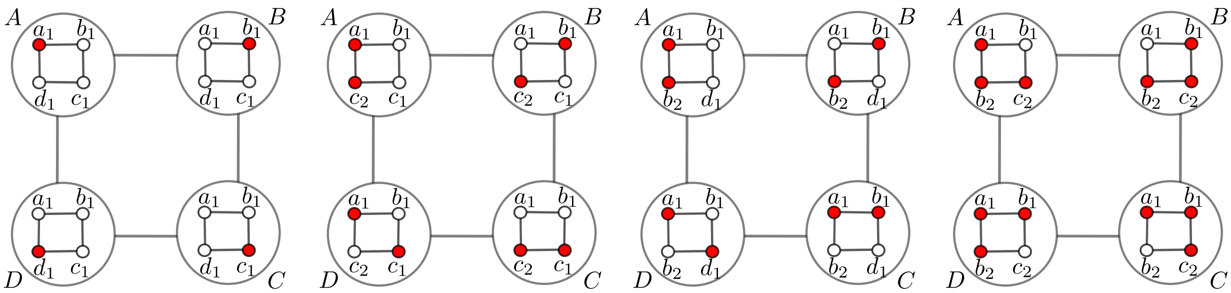


Figure 2.4: All four possibilities in Case (2).

(**2.2**) Suppose that $a_1 = c_1$. Here, let us note that if $c_2 = a_1$, then either $d_1 = b_2$ or $d_1 \neq b_2$; however, in the former case we would have $D = B$, and the later case, we would have that A and D are not adjacent, these both cases are a contradiction. Then, we may assume that $c_2 \neq a_1$.

(**2.2.1**) Suppose that $c_2 = b_1$. Then D is obtained from C by moving the token at b_1 to vertex d_1, where $d_1 \notin \{a_1, b_1, b_2\}$, so $C \triangle D = \{b_1, d_1\}$ and then $A \triangle D = \{d_1, b_2\}$. See Subcase (2.2.1) of Figure 2.4. Thus, $\mathcal{C}$ is generated as in (iii) of Figure 2.2.

(**2.2.2**) Suppose that $c_2 \notin \{a_1, b_1\}$. Note that $d_1 = b_2$, as otherwise D would not be adjacent to A. Then, D is obtained from C by moving the token at vertex c_2 to vertex b_2, so $C \triangle D = \{c_2, b_2\}$ and $A \triangle D = \{b_1, c_2\}$. See Subcase (2.2.2) of Figure 2.4. Therefore, $\mathcal{C}$ is generated as in (iv) of Figure 2.2.

□

We have the following consequence for the (C_4, D_4)-free graphs.

Corollary 2.18. *If G is a (C_4, D_4)-free graph, then every induced 4-cycle of $F_k(G)$ is generated as in (ii) of Figure 2.2.*

It may be the case that F can be reconstructed (even uniquely) as the k-token graph of two non-isomorphic graphs. The following lemma shows that if one of them does not contain induced 4-cycles as subgraphs, then the other also does not contain induced 4-cycles as subgraphs.

Lemma 2.19. *Let G be a (C_4, D_4)-free graph and let F be a graph isomorphic to $F_k(G)$. If (G', φ) is any k-token reconstruction of F then G' is also a (C_4, D_4)-free graph.*

Proof. We start by showing the following property of $F_k(G)$.

Let $ABCD$ be an induced 4-cycle of $F_k(G)$ and let $X \in V(F_k(G)) \setminus \{A, B, C, D\}$ be a vertex adjacent to a pair of non-consecutive vertices of the cycle $ABCD$. Then X is adjacent to all vertices A, B, C and D. (P1)

Let A, B, C, D and X as in (P1). Suppose that X is adjacent to A and C. Since G is (C_4, D_4)-free, the 4-cycle $ABCD$ must be generated as in (ii) of Figure 2: by moving two tokens on two disjoint edges (a_1, b_1) and (a_2, b_2) of G, while the other $k-2$ tokens remain fixed on a subset S of $V(G) \setminus \{a_1, a_2, b_1, b_2\}$. Without loss of generality assume that

$$A = S \cup \{a_1, a_2\}, \quad B = S \cup \{b_1, a_2\}, \quad C = S \cup \{b_1, b_2\}, \quad D = S \cup \{a_1, b_2\}.$$

Consider now the vertex X. Let us note that X must be obtained from C by moving a token at one of $\{b_1, b_2\}$ to a vertex in $\{a_1, a_2\}$, as otherwise we would have $|X \triangle A| > 2$, and so X and A cannot be adjacent, a contradiction. Clearly, X cannot be obtained from C by moving the token at b_2 to a_2, since in such case we would have $X = B$, a contradiction. Similarly, X cannot be obtained from C by moving the token at b_1 to a_1. Thus, either X is obtained from C by moving the token at b_1 to a_2, or by moving the token at b_2 to a_1, but these two cases are analogous. Without loss of generality let us assume that X is obtained from C by moving the token at b_1 to a_2, and so, $X = S \cup \{a_2, b_2\}$ and b_1 is adjacent to a_2. Since X is adjacent to A, it follows that a_1 is adjacent to b_2, and since G is a (C_4, D_4)-free graph, the vertex set $\{a_1, a_2, b_1, b_2\}$ must induce a complete graph in G. This fact implies that X is also adjacent to B and D, and so (P1) holds. See Figure 2.5 (left).

Suppose that G' is not a (C_4, D_4)-free graph. We are going to show that $F_k(G')$ does not hold property (P1). Let $uvwz$ be a 4-cycle in G', with at most one chord, let us assume that v and z are not adjacent. Let $S' \subseteq V(G') \setminus \{u, v, w, z\}$, with $|S| = k-2$, and consider the vertices

$$A' = S' \cup \{u, v\},\ B' = S' \cup \{u, w\},\ C' = S' \cup \{u, z\},\ D' = S' \cup \{v, z\} \text{ and } X' = S' \cup \{z, w\}.$$

Note that $A'B'C'D'$ induces a 4-cycle in $F_k(G')$ and the vertex X' is adjacent to B' and D', however, X' cannot be adjacent to A', and so (P1) does not hold for $F_k(G')$—a contradiction. See Figure 2.5 (right).

Since $F_k(G) \simeq F \simeq F_k(G')$, we must have that G' is a (C_4, D_4)-free graph, as claimed. $\square$

2.4.2 Ladders, Cartesian Products and Line Graphs

A *ladder* is a graph isomorphic to the cartesian product of K_2 and a path P_m of order $m \geq 1$. Let x and y be the two vertices of K_2; let $v_1, \ldots, v_m$ be the vertices of P_m. For $m \geq 3$, we call the edges $(x, v_i)(y, v_i)$ the *rungs* of the ladder. In the case the of $K_2 \square P_2$ the rungs may be either one of the two disjoint pairs of edges. Two edges e and f in F are said to be *connected by a ladder* if there exists an induced subgraph in F isomorphic to a ladder such that e and f are rungs of this ladder. Being connected by a ladder is an equivalence relation on the edges of F. We refer to its equivalence classes as *ladders classes*. We denote the ladder class of e with $R[e]$. The ladder classes of F are easily computed in

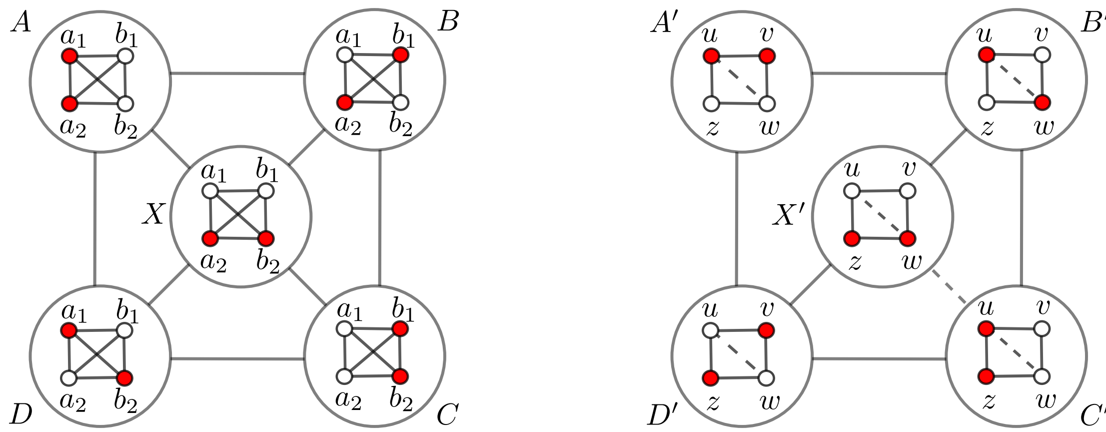


Figure 2.5: Vertex set $\{A, B, C, D, X\}$ in $F_k(G)$ (left), and vertex set $\{A', B', C', D', X'\}$ in $F_k(G')$ (right).

polynomial time as follows. Construct a graph F' whose vertices are the edges of F; two of which are adjacent if they are disjoint edges in an induced 4-cycle of F. The ladder classes of F correspond to the components of F'. In the case when F is the k-token graph of a (C_4, D_4)-free graph, we have the following.

Proposition 2.20. *Let G be a (C_4, D_4)-free graph and let $AB, A'B'$ be two edges of $F_k(G)$ in the same ladder class. Then*

$$A \triangle B = A' \triangle B';$$

thus, AB and $A'B'$ correspond to moving a token along the same edge of G.

Proof. Let $H \simeq K_2 \square P_m$ be a ladder of $F_k(G)$ such that AB is the first rung of H and $A'B'$ is the last rung of H. Since G is a (C_4, D_4)-free graph, every induced 4-cycle of $F_k(G)$ is generated as in *(ii)* of Figure 2.2. If $m = 2$ then the result follows from this observation. Suppose that $m > 2$ and that the result follows for smaller values of m. Let $A''B''$ be the rung of H previous to $A'B'$. By induction $A \triangle B = A'' \triangle B''$ and by the previous argument $A'' \triangle B'' = A \triangle B$; the result follows. □

Although Proposition 2.20 implies that, when G is (C_4, D_4)-free, every edge in a given ladder class of $F_k(G)$ corresponds to moving a token along the same edge of G, two edges in different ladder classes may correspond to moving a token along the same edge ab of G. For example, consider the graph G (that is a path graph on six vertices) depicted in Figure 2.6 and its 2-token graph. The pair of blue edges and the pair of green edges do not belong to a same ladder class of $F_k(G)$, however, all these edges correspond to moving a token along the edge 34 of G. The next lemma shows that this does not happen when $G \setminus \{a, b\}$ is connected.

Lemma 2.21. *Let G be a (C_4, D_4)-free graph. Let $e := ab$ be an edge of G such that $G \setminus \{a, b\}$ is connected. Then the set of edges of $F_k(G)$ that correspond to moving a token along e form a ladder class.*

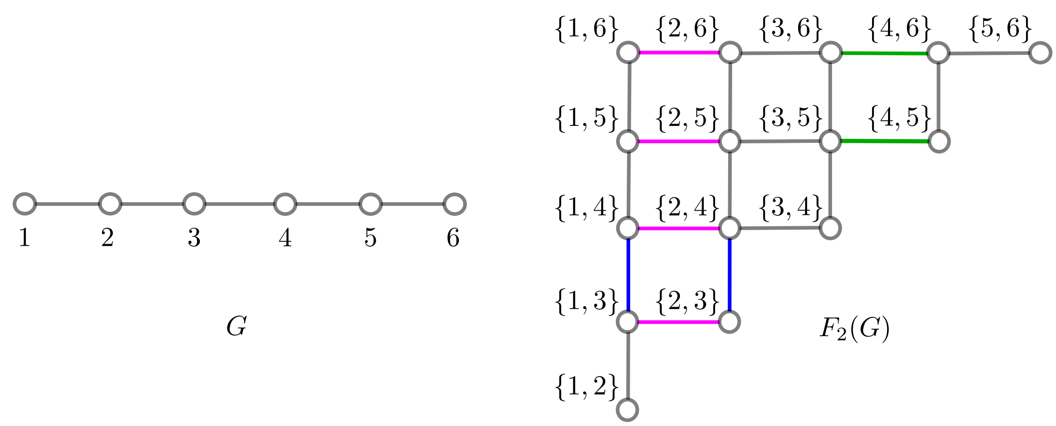


Figure 2.6: The set of pink edges belong to a same ladder class of $F_2(G)$ and corresponds to a token moving along the edge 12 of G. On the other hand, the pair of blue edges and the pair of green edges do not belong to a same ladder class of $F_2(G)$, however, all these edges correspond to moving a token along the same edge 34 of G.

Proof. Let A_1B_1 and A_2B_2 edges of $F_k(G)$ such that $A_1 \triangle B_1 = A_2 \triangle B_2 = \{a, b\}$. Without loss of generality assume that $a \in A_1$, $a \in A_2$, $b \in B_1$ and $b \in B_2$. Let $A_1' := A_1 \setminus \{a, b\}$, $B_1' := B_1 \setminus \{a, b\}$, $A_2' := A_2 \setminus \{a, b\}$ and $B_2' := B_2 \setminus \{a, b\}$. Note that A_1', B_1', A_2', B_2' are vertices of $F_{k-1}(G \setminus \{a, b\})$ Since $G \setminus \{a, b\}$ is connected then so is $F_{k-1}(G \setminus \{a, b\})$ [21]. Let $(A_1' =: C_1, C_2, \dots, C_m := A_2')$ be a path from A_1' to A_2' in $F_{k-1}(G \setminus \{a, b\})$. The set of vertices

$$\{C_i \cup \{a\} : 1 \leq i \leq m\} \cup \{C_i \cup \{b\} : 1 \leq i \leq m\}$$

induce a ladder that connects A_1B_1 to A_2B_2 in $F_k(G)$. □

Note that if G is a 3-connected (C_4, D_4)-free graph, then the edges of G and the ladder classes of $F_k(G)$ are in a one to one correspondence. By Proposition 2.17, the edges corresponding to two ladder classes R_1 and R_2 are incident to a same vertex if and only if no edge of R_1 is contained in an induced 4-cycle of $F_k(G)$ simultaneously with an edge of R_2. In particular this implies that we can recover the line graph $L(G)$ of G from its ladder graph in polynomial time when G is 3-connected. We have the following corollary.

Corollary 2.22. *Let G be a 3-connected (C_4, D_4)-free graph; let F be a graph isomorphic to $F_k(G)$. Given only F we can compute in polynomial time a graph J isomorphic to G.*

Let us think on the following situation. Consider a graph G and its k-token graph $F_k(G)$. Suppose that there exists a partition $\{V_1, V_2, \dots, V_r\}$ of $V(G)$, where each subset V_i induces a connected subgraph G_i with $|V_i| \geq 2$, for each $i \in [r]$ and $r \leq k$. Also consider some positive integers $k_1, k_2, \dots, k_r$ such that $k_1 + k_2 + \cdots + k_r = k$. Let H be the subgraph of $F_k(G)$ generated by moving k_i tokens on G_i, for each $i \in [r]$. It is not hard to see that H is isomorphic to the composite graph $F_{k_1}(G_1) \,\square\, F_{k_2}(G_2) \,\square\, \dots \,\square\, F_{k_r}(G_r)$. If this would hold in general for any composite subgraph H of $F_k(G)$, to reconstruct the graph G, it would be enough to find a suitable composite subgraph H of $F_k(G)$. Unfortunately, this does not hold in general (see an example in Figure 2.7), but as we will see in the following result, certain large composite subgraphs of $F_k(G)$ are generated in this way when G is a connected (C_4, D_4)-free graph. We remark that this class of large composite subgraphs of $F_k(G)$ is a key step in our algorithm to reconstruct the graph G when G is a connected (C_4, D_4)-free graph.

Theorem 2.23. *Let G be a connected (C_4, D_4)-free graph. Let H be a subgraph of $F_k(G)$ such that H is maximal with the property of being isomorphic to a graph $H' = H'_1 \square \cdots \square H'_r$, where each H'_i is connected and with at least two vertices. Then there exists a partition $V_1, \ldots, V_r$ of $V(G)$, and integers $k_1, \ldots, k_r$ with $k = k_1 + \cdots + k_r$, such that the following holds: H is generated by moving k_i tokens on $G_i := G[V_i]$ and every H'_i is isomorphic to $F_{k_i}(G_i)$.*

Proof. Let f be an isomorphism from H' to H. Fix an index $1 \leq i \leq r$. Let u_1u_2 and v_1v_2 be two edges of H' such that

$$u_1(i) = x = v_1(i) \text{ and } u_2(i) = y = v_2(i)$$

for some pair of adjacent vertices x, y in H'_i. We first show that

$$f(u_1)f(u_2) \text{ and } f(v_1)f(v_2) \text{ are generated by moving a token along the same edge of } G. \tag{4}$$

Let $(u_1 = w_1, \ldots, w_m = v_1)$ be a shortest path from u_1 to v_1 in H' such that for all $1 \leq j \leq m$

$$w_j(i) = x.$$

Let $(u_2 = w'_1, \ldots, w'_m = v_2)$ be the path in H' such that for all $1 \leq j \leq m$ and all $1 \leq l \leq r$

$$w'_j(l) := \begin{cases} y & \text{if } l = i, \\ w_j(l) & \text{if } l \neq i. \end{cases}$$

Note that the set of vertices

$$\{f(w_j) : 1 \leq j \leq m\} \cup \{f(w'_j) : 1 \leq j \leq m\}$$

induces a ladder in H. By Proposition 2.20, $f(u_1)f(u_2)$ and $f(v_1)f(v_2)$ are generated by moving a token along the same edge of G. This proves (4).

We now define the sets V_i's. Fix a vertex $v^* \in H'$. Let H_i be the subgraph of H induced by the set of vertices

$$\{f(u) : u \in V(H') \text{ and } u(j) = v^*(j) \text{ for all } j \neq i\}.$$

Clearly, $H_i \simeq H'_i$. Let

$$V_i := \{x \in V(G) : \text{ there exist } A, B \in V(H_i) \text{ such that } x \in A \text{ and } x \notin B\}.$$

By (4) and the fact that H_i is connected, it follows that V_i does not depend on the choice of v^*.

Let us now show that the V_i are pairwise disjoint. Suppose that for some distinct V_i and V_j there exists a vertex $x \in V_i \cap V_j$. Since H_i is connected, there exist adjacent vertices A_1 and B_1 of H_i, such that $x \in A_1$ and $x \notin B_1$; let y_1 be the vertex of V_i such that B_1 is obtained from A_1 by moving the token at x to y_1. Since H_j is connected there exists adjacent vertices A_2 and B_2 of H_j such that $x \in A_2$ and $x \notin B_2$; let y_2 be the vertex of V_j such that B_2 is obtained from A_2 by moving the token at x to y_2. Note that $f^{-1}(A_1)f^{-1}(B_1)$ is an edge of H' and $f^{-1}(A_1)(i)f^{-1}(B_1)(i)$ is an edge of H'_i. Similarly, $f^{-1}(A_2)f^{-1}(B_2)$ is an edge of H' and $f^{-1}(A_2)(j)f^{-1}(B_2)(j)$ is an edge of H'_j. Let w_1, w_2, w_3, w_4 be vertices of H' defined as follows:

- For all $1 \leq l \leq r$ and $l \neq i, j$ we have
$$w_1(l) = w_2(l) = w_3(l) = w_4(l) = v^*(l).$$
- For i, we have
$$w_1(i) = f^{-1}(A_1)(i),\ w_2(i) = f^{-1}(A_1)(i),\ w_3(i) = f^{-1}(B_1)(i),\ w_4(i) = f^{-1}(B_1)(i).$$
- For j, we have
$$w_1(j) = f^{-1}(A_2)(j),\ w_2(j) = f^{-1}(B_2)(j),\ w_3(j) = f^{-1}(B_2)(j),\ w_4(j) = f^{-1}(A_2)(j).$$

Note that (w_1, w_2, w_3, w_4) is an induced 4-cycle of H'. By Proposition 2.17, $f(w_1)f(w_2)$ and $f(w_1)f(w_4)$ are generated each by moving a token along disjoint edges of G. However, by (4) these edges are xy_1 and xy_2, respectively—a contradiction.

Let A be a vertex of H_i, we define $k_i := |A \cap V_i|$. Let B a vertex of H_i distinct from A. Let $(A =: A_1, A_2, \ldots, A_m := B)$ be a path from A to B in H_i. Note that for every $1 \leq l < m$, $A_l \triangle A_{l+1} \subset V_i$. Therefore, $|A \cap V_i| = |B \cap V_i|$. Thus, k_i does not depend on our choice of A. For every $1 \leq i \leq r$, let G_i be the subgraph of G induced by V_i. Let H'' be the subgraph of F generated by moving k_i tokens on each G_i. Note that $H'' \simeq F_{k_1}(G_1) \square \cdots \square F_{k_r}(G_r)$. Since V_i does not depend on the choice of v^*, we have that H is a subgraph of H''. The maximality of H implies that $H'' = H$, $H_i \simeq F_{k_i}(G_i)$, $k = k_1 + \cdots + k_r$ and that $V(G) = V_1 \cup \cdots \cup V_r$. This completes the proof. □

We remark that, in Theorem 2.23, the condition of G being (C_4, D_4)-free graph is necessary. For example, in Figure 2.7 is depicted a graph G, that is not a (C_4, D_4)-free graph, and its 2-token graph $F_2(G)$ with the blue subgraph holding the same hypothesis as H in Theorem 2.23; however, the blue subgraph is not generated by moving k_i tokens on $G_i := G[V_i]$, for any partition $\{V_1, \ldots, V_r\}$ of $V(G)$ and integers $k_1, \ldots, k_r$ with $k = k_1 + \cdots + k_r$.

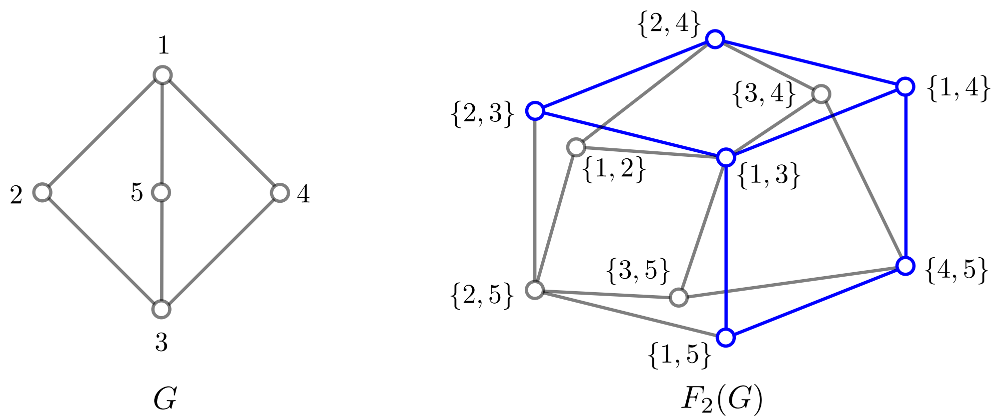


Figure 2.7: An example showing that the hypothesis of being (C_4, D_4)-free graph is necessary in Theorem 2.23. The blue subgraph is maximal with the property of being isomorphic to a composite graph; however, there is no partition $\{V_1, \ldots, V_r\}$ of $V(G)$ nor integers $k_1, \ldots, k_r$ with $k = k_1 + \cdots + k_r$ for which the blue subgraph could be generated by moving k_i tokens on $G[V_i]$.

We have the following corollary to Theorem 2.23.

Corollary 2.24. *If G is a connected (C_4, D_4)-free graph, then $F_k(G)$ is a prime graph.*

2.5 Reconstructing G

Throughout this section let:

- G be a connected (C_4, D_4)-free graph;
- F be a graph isomorphic to $F_k(G)$; and
- φ be a fixed isomorphism from F to $F_k(G)$.

In this section we present a polynomial time algorithm that given only F (but not φ, $F_k(G)$ nor G) constructs a graph isomorphic to G.

Our general strategy is as follows. We run an algorithm, which we call ProductSubgraph, on F. The first step of ProductSubgraph is to find a vertex A of F with the following property. The number of independent edges of G incident to exactly one vertex of $\varphi(A)$ is maximum. Afterwards, ProductSubgraph finds a a certain subgraph H of F, that is maximal with the property of being isomorphic to a Cartesian product $H_1 \square \cdots \square H_r$ of connected graphs H_i, each with at least two vertices. ProductSubgraph also finds these H_i. By Theorem 2.23 we know that there exist induced disjoint subgraphs $G_1, \ldots, G_r$ of G, and integers $k_1, \ldots, k_r$ that sum up to k, such that $V(G) = \bigcup_{i=1}^{r} V(G_i)$

and $H_i \simeq F_{k_i}(G_i)$. The structure of H_i is such that we can construct in polynomial time a graph isomorphic to each G_i. Finally, we reconstruct the adjacencies between these graphs.

The information stored in the ladder relationship of the edges of $F_k(G)$ allows us to locally reconstruct small parts of G. Let A be a vertex of $F_k(G)$; let

$$E_A := \{A \triangle B : B \in N(A)\}.$$

Thus, E_A is the set of edges of G with exactly one vertex of A as one of their endpoints. Let $G_{\varphi(A)}$ be the subgraph of G whose vertices are the endpoints of the edges in E_A and its edge set is E_A.

Let AB and AC be two edges of $F_k(G)$; let e_1 and e_2 be the edges of G such that AB and AC correspond to moving a token along e_1 and e_2, respectively. Since G is (C_4, D_4)-free and by Proposition 2.17, we have that AB and AC are in a common induced 4-cycle of $F_k(G)$ if and only if e_1 and e_2 are disjoint. By checking whether each pair of edges incident to A are contained in a 4-cycle (in $F_k(G)$) we can reconstruct the incidence relationships in E_A. Thus, given a vertex B of F we can construct, in polynomial time, a graph isomorphic to the line graph $L(G_{\varphi(B)})$ of $G_{\varphi(B)}$. As we mentioned in Section 1.4, for graphs with more than three vertices, there is a polynomial time algorithm that can reconstruct a graph from its line graph [44, 36]. Since triangles in $F_k(G)$ are generated by moving one or two tokens in a triangle of G [21], we have the following result.

Lemma 2.25. *Given only F we can construct in polynomial time a set of graphs*

$$\{J_A : A \in V(F)\},$$

where each J_A is isomorphic to $G_{\varphi(A)}$.

□

ProductSubgraph has two subroutines: Initialize and Extend. Initialize does the following. In line 1 it constructs the set of graphs J_A described in Lemma 2.25. In lines 2-5 for every vertex A of F it computes a maximum matching M_A of J_A; this can be done in polynomial time [37]. In line 5, a vertex $A \in F$ is chosen so that $|M_A|$ is maximum. Assuming $k \leq n/2$, this matching corresponds to a matching of G of maximum cardinality with the property of having at most k edges. The 1-token graphs of these edges are the starting H_i's. Afterwards, ProductSubgraph iteratively calls Extend for each i in turn. Extend attempts to extend H_i into a larger token graph of some (unknown) subgraph G_i of G. The initial choice of A is what enable us to reconstruct the G_i from their H_i. At the end of its execution ProductSubgraph outputs a subgraph H of F, graphs $H_1, \dots, H_r$ and an isomorphism π from H to $H_1 \square \cdots \square H_r$.

The following lemma provides structural properties of the output of ProductSubgraph; along the way, its proof also analyses ProductSubgraph, Initialize and Extend in detail.

Procedure Initialize

Construct a set of graphs $\{J_A : A \in V(F)\}$ where each J_A is isomorphic to $G_{\varphi(A)}$;
for $A \in V(F)$ **do**
    Compute a maximum cardinality matching M_A of J_A;
end
Find $A \in V(F)$ such that M_A is of maximum cardinality among these matchings;
Let $e_1, \dots, e_r$ be the edges incident to A in F corresponding to the edges of M_A;
Find the r-cube, $Q_r \subset F$ containing A as a vertex and $e_1, \dots, e_r$ as edges;
$H = Q_r$;
for $i \leftarrow 1$ **to** r **do**
    Initialize two new vertices x_i and y_i and a new graph H_i;
    $V(H_i) \leftarrow \{x_i, y_i\}$;
    $E(H_i) \leftarrow \{x_i y_i\}$;
end
for $B \in Q_r$ **do**
    Compute a shortest path P in Q_r from A to B;
    for $i \leftarrow 1$ **to** r **do**
        if P *contains an edge in* $R[e_i]$ **then**
            $\pi(B)(i) \leftarrow y_i$;
        else
            $\pi(B)(i) \leftarrow x_i$;
        end
    end
end

Lemma 2.26. *There exist disjoint induced subgraphs $G_1, \dots, G_r$ of G, and positive integers $k_1, \dots, k_r$ such that the following holds.*

(1) $k = k_1 + \cdots + k_r$ *and* $V(G) = V(G_1) \cup \cdots \cup V(G_r)$.

(2) *For every pair of vertices $A_1, A_2 \in H$ and index $1 \leq i \leq r$ we have that $\pi(A_1)(i) = \pi(A_2)(i)$ if and only if $\varphi(A_1) \cap V(G_i) = \varphi(A_2) \cap V(G_i)$.*

(3) *For every index $1 \leq i \leq r$, and vertex $u \in V(H_i)$, pick any vertex $A \in H$ such that $u = \pi(A)(i)$; let φ_i be the function that maps u to $\varphi(A) \cap V(G_i)$; then φ_i is an isomorphism from H_i to $F_{k_i}(G_i)$.*

(4) *For every $A \in V(H)$,*

$$\varphi(A) = \bigcup_{i=1}^{r} \varphi_i(\pi(A)(i)).$$

That is, the following diagram commutes.

$$\begin{tikzcd} H \arrow[r, "\varphi"] \arrow[d, "\pi"] & F_k(G) \\ H_1 \square \cdots \square H_r \arrow[ur, "\bigcup_{i=1}^{r} \varphi_i(\cdot)"'] & \end{tikzcd}$$

Procedure EXTEND(i)

Let A_1 be any vertex of H;
Let A_2 be the neighbour of A_1 in H such that $\pi(A_1)(i) \neq \pi(A_2)(i)$;
$Q = \text{Queue}()$;
$Q.\text{Insert}(A_1)$;
$Q.\text{Insert}(A_2)$;
while *Q not empty* **do**
    $A = Q.\text{Dequeue}()$;
    for *every edge AB of F that is not an edge of H* **do**
        if *every $C \in V(H)$, such that $\pi(C)(i) == \pi(A)(i)$, is incident to an edge in $R[AB]$* **then**
            if $B \notin H$ **then**
                Initialize a new vertex y;
                Add the vertex y to H_i;
                for *every $X \in V(H)$, such that $\pi(X)(i) == \pi(A)(i)$* **do**
                    Let Y be the neighbour of X in F such that XY is in $R[AB]$;
                    Add the vertex Y to H;
                    $\pi(Y) = \pi(X)$;
                    $\pi(Y)(i) = y$;
                end
                $Q.\text{Insert}(B)$;
            end
            $x = \pi(A)(i)$;
            $y = \pi(B)(i)$;
            Add the edge xy to H_i;
            for *every $X \in V(H)$, such that $\pi(X)(i) == \pi(A)(i)$* **do**
                Let Y be the neighbour of X in H such that XY is in $R[AB]$;
                Add the edge XY to H;
            end
        end
    end
end

Algorithm 1: PRODUCTSUBGRAPH

Input: A graph $F \simeq F_k(G)$ where G is a connected (C_4, D_4)-free graph.
Output: A subgraph H of F, graphs $H_1, \ldots, H_r$, and an isomorphism π from H to $H_1 \square \ldots \square H_r$.

Compute the set R of ladder classes of $E(F)$;
`Initialize` (); `// Initializes` H `and` $H_1, \ldots, H_r$
for $i \leftarrow 1$ **to** r **do**
    `Extend` (i);
end

Proof. $H, H_1, \dots, H_r$ and π are initialized when Initialize is called in line 2 of ProductSubgraph. Afterwards, these graphs and π are updated throughout the execution of ProductSubgraph. In what follows we show that throughout the execution of ProductSubgraph there exist disjoint subgraphs $G_1, \dots, G_r$ of G, and integers $k_1, \dots, k_r$ whose sum is at most k, such that at key steps of the execution of ProductSubgraph, (2) and the following properties hold.

(3′) For every index $1 \le i \le r$, and vertex $u \in V(H_i)$, pick any vertex $A \in H$ such that $u = \pi(A)(i)$; let φ_i be the function that maps u to $\varphi(A) \cap V(G_i)$; then φ_i is an isomorphism from H_i to a *subgraph* of $F_{k_i}(G_i)$.

(4′) For every $A \in V(H)$

$$\varphi(A) = \left(\bigcup_{i=1}^{r} \varphi_i(\pi(A)(i)) \right) \cup \left(\varphi(A) \setminus \bigcup_{i=1}^{r} V(G_i) \right).$$

Afterwards, we show that (1), (3) and (4) hold at the end of the execution of ProductSubgraph. We also show that at the end of the execution of ProductSubgraph that the k_i sum up to k and that G_i are induced subgraphs of G; this proves the lemma.

Consider the execution of Initialize. Let A be as in line 5 of Initialize. Since M_A is a matching of J_A, its edges are in correspondence with $r := |M_A|$ independent edges in G such that there is exactly one token of $\varphi(A)$ in each edge. Moving these tokens on their respective edges produces an r-cube in $F_k(G)$. Therefore, the r-cube, Q_r, of line 7 exists. Q_r can be computed as follows. Let $e_1, \dots, e_r$ be the edges of F incident to A that correspond to the edges of M_A (line 6 of initialize). The vertices of Q_r are all the vertices of F that are reachable from A by a path with all its edges contained in $R[e_1] \cup \cdots \cup R[e_r]$. Thus, Q_r can be found by computing the subgraph of F with edge set $R[e_1] \cup \cdots \cup R[e_r]$ and then finding the component containing A. In line 8, H is set to be Q_r. The H_i are constructed in lines $9-12$; each H_i consists of two adjacent vertices x_i and y_i. Let $e'_1, \dots, e'_r$ be the edges of G such that $\varphi(e_i)$ corresponds to moving the token along e'_i; let G_i be the subgraph of G consisting of the edge e'_i and let $k_i = 1$. In lines $14 - 22$, π is constructed so that (2), (3′) and (4′) hold.

We now consider the i-th call to Extend in line 4 of ProductSubgraph. Before proceeding, we briefly explain the intuition behind Extend. In line 1, Extend picks an arbitrary vertex A_1 of H as a representative. This is done to fix the value of the coordinates distinct from i of the tuples $\pi(B)$ with $B \in V(H)$. In lines 4 and 5 representatives of the currently only two vertices of H_i are added to a queue Q. Lines $6 - 30$ extends H_i in a way similar to the *Breadth First Search* algorithm. In the process H and π are updated.

Assume that (2), (3′) and (4′) hold before the i-th call to Extend. Throughout the execution of Extend we have the following invariant.

Every vertex X in Q satisfies that $\pi(X)(j) = \pi(A_1)(j)$ for all $j \neq i$. $(*)$

This is certainly the case before the first execution of the **while** in line 6, since Q contains only the vertices A_1 and A_2. We show that (2), (3′), (4′) and (∗) hold at the end of each execution of the **for** of line 8.

Let AB be the edge in line 8 and let $e := uv$ be the edge of G such that $\varphi(B)$ is obtained from $\varphi(A)$ by moving a token along e. We show that

the condition of line 9 is satisfied if and only if one of u and v is in G_i, while the other is not in any G_j, with $j \neq i$. (†)

Suppose that one of u and v is in G_i while the other is not in any G_j with $j \neq i$. Let $C \in V(H)$ with $\pi(C)(i) = \pi(A)(i)$. Let $(A = C_1, \ldots, C_m = C)$ be a shortest path in H from A to C. Note that $\pi(C_l)(i) = \pi(A)(i)$ for all $1 \leq l \leq m$. Since (2) holds we have that for every $1 \leq l \leq m$ there exists a vertex D_l such that $\varphi(D_l)$ is obtained from $\varphi(C_l)$ by sliding a token along e. Thus, the set of vertices

$$\{C_l : 1 \leq l \leq m\} \cup \{D_l : 1 \leq l \leq m\}$$

induce a ladder from AB to CD_m. Therefore the condition of line 9 is satisfied.

Suppose that u and v are not in $\cup_{j=1}^{r} V(G_i)$. Then $\{e, e_1, \ldots, e_r\}$ is a matching of size $r+1$ of J_A, where A is as in line 5 of Initialize; this is a contradiction to the fact that M_A is maximum. Therefore, at least one of u and v is in $\cup_{j=1}^{r} V(G_i)$. Suppose that one of u and v is in G_j for some $j \neq i$. Without loss of generality suppose it is u. Then there exist vertices C_1 and C_2 of H with $\pi(C_1)(i) = \pi(C_2)(i) = \pi(A)(i)$, such that in $\varphi(C_1)$ there is a token at u, and in $\varphi(C_2)$ there is no token at u. Depending on whether there is a token at v in $\varphi(A)$, for one of $\varphi(C_1)$ and $\varphi(C_2)$ either e contains two tokens at its endpoints or no endpoint of e contains a token. In either case, there is no token move possible along e. Therefore, there exists a vertex $C \in H$ with $\pi(C)(i) = \pi(A)(i)$ that is not incident to an edge in $R[AB]$. Thus, the condition of line 9 does not hold. Therefore, (†) holds.

Suppose that B is not a vertex of H. If $v \notin V(G_i)$, update $V(G_i)$ to $V(G_i) \cup \{v\}$, and $E(G_i)$ to $E(G_i) \cup \{uv\}$. If $\varphi(B)$ is obtained from moving a token from v to u, then this token has not been moved before. In this case update k_i to $k_i + 1$. Otherwise, if $\varphi(B)$ is obtained from moving a token from u to v, then k_i remains unchanged. In line 12 a new vertex y is added to H_i. Consider lines $13 - 15$. For every vertex $X \in H$ with $\pi(X)(i) = \pi(A)(i)$ let Y be its neighbour such that $XY \in R[AB]$; we add Y to $V(H)$. In lines 16 and 17, $\pi(Y)$ is defined so that $\pi(Y)(i) := y$ and $\pi(Y)(j) := \pi(X)(j)$ for all $j \neq i$. Thus (2) is satisfied after the execution of line 18. Since $\varphi(Y)$ is obtained from $\varphi(X)$ by sliding a token along e we have that (4′) holds after the execution of line 18. In line 19, B is inserted to Q, and (∗) still holds. Suppose that B is not necessarily a vertex of H. Let X and Y be as in lines 24 and 25. Since $\varphi(Y)$ is obtained from $\varphi(Y)$ by sliding a token along uv, we have that (3′) holds after the execution of line 27.

Suppose that the i-th execution of Extend has ended. Let uv be an edge of G_i. Let X be any vertex of H such that $\varphi(X)$ contains a token at u and no token at v. Let $Y \in F$

be such that $\varphi(Y)$ is obtained from $\varphi(X)$ by sliding a token along uv. Note that Y is also in H. At some point during the execution of Extend, in line 23 the edge $\pi(A)(i)\pi(B)(i)$ was added to H_i. Therefore, we have that

(3″) For every vertex $u \in V(H_i)$, pick any vertex $A \in H$ such that $u = \pi(A)(i)$; let φ_i be the function that maps u to $\varphi(A) \cap V(G_i)$; then φ_i is an isomorphism from H_i to $F_{k_i}(G_i)$.

Assume that the execution of ProductSubgraph has ended. Since (3″) holds for every $1 \leq i \leq r$ we have that (3) holds. Let $G' = \bigcup_{i=1}^{r} G_i$. Suppose that $G \setminus G' \neq \emptyset$. Let uv be a $G' - G \setminus G'$ edge. Let G_i be such that $u \in G_i$. Let A_1 and A_2 be vertices of H such that in $\varphi(A_1)$ there is a token at u and in $\varphi(A_2)$ there no is a token at u. Note that either there is a token at v in both $\varphi(A_1)$ and $\varphi(A_2)$ or there is no token at v in neither of $\varphi(A_1)$ and $\varphi(A_2)$. For exactly one of $\varphi(A_1)$ and $\varphi(A_2)$ we have that there is exactly one token at the endpoints of uv. Let $A := A_j$ be such that in $\varphi(A_j)$ there is exactly one token at the endpoints of uv. Let $B \in V(F)$ be such that $\varphi(B)$ is obtained from $\varphi(A)$ by sliding the token along uv. Since $v \notin G_i$ we have that $B \notin H_i$. At some point during the execution of line 7 of Extend(i) A is removed from Q. Afterwards, eventually, in line 8, AB is considered. AB satisfies the condition of line 9; thus B is added to H_i—a contradiction. Thus, $V(G) = V(G_1) \cup \cdots \cup V(G_r)$. This implies that H is maximal in F with the property of being isomorphic to the cartesian product of connected graphs with at least two vertices. By Theorem 2.23 we have that $k = k_1 + \cdots + k_r$ and that the G_i are induced subgraphs of G. In particular (1) holds. Since (4′) holds, this implies that (4) holds. The result follows. □

2.5.1 Reconstructing the G_i

Suppose that ProductSubgraph has been executed; let $G_1, \ldots, G_r$ and $k_1, \ldots, k_r$ be as in Lemma 2.26. In this section we show how to construct graphs isomorphic to the G_i's. We classify each H_i into the following four classes.

1. H_i is an edge.
2. H_i is a triangle.
3. H_i is isomorphic to the token graph of a star of at least three vertices. By Lemma 2.16, there are unique integers l and m, with $l \leq (m+1)/2$ such that $H_i \simeq F_l(K_{1,m})$. There are three more possibilities in this case:

   3a. $G_i \simeq F_l(K_{1,m})$, $k_i = 1$ or $k_i = |F_l(K_{1,m})| - 1$, and $1 < l < m$; or

   3b. $G_i \simeq K_{1,m}$, $1 < k_i < |G_i| - 1$, and $k_i = l$ or $k_i = m + 1 - l$.

   3c. $G_i \simeq K_{1,m}$ and $k_i = 1$ or $k_i = m$.

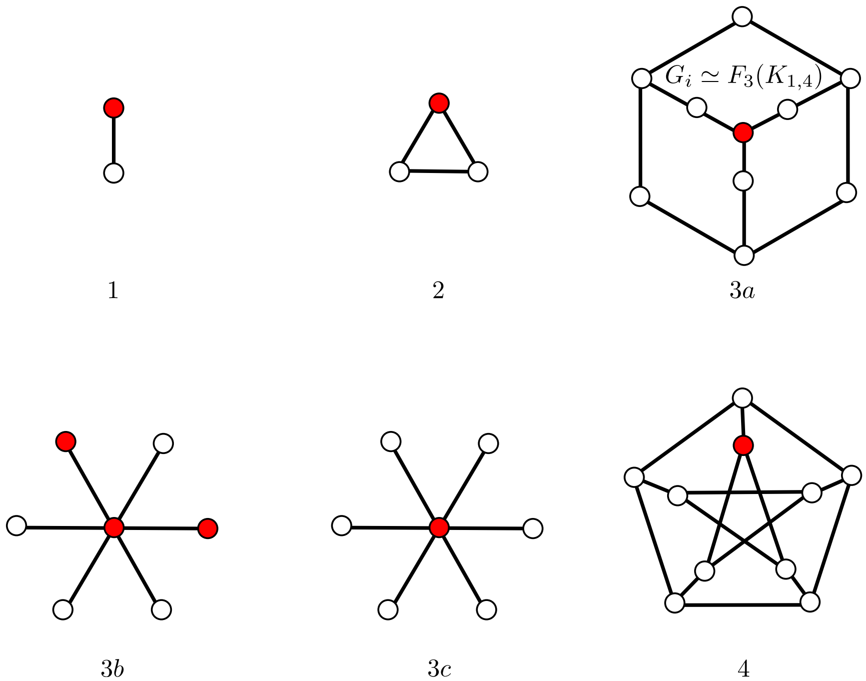


Figure 2.8: A sample vertex of $F_{k_i}(G_i)$ for an H_i of each possible class.

4. H_i is not a triangle nor isomorphic to the token graph of a star.

In Figure 2.8 is depicted a sample vertex of H_i, for each possible H_i. We now show how to determine the class of each H_i in polynomial time. The following lemma is useful for restricting the possible for the values of the k_i.

Lemma 2.27. *If some G_i contains two disjoint edges then all k_j are equal to 1 or all k_j are equal to $|G_j| - 1$.*

Proof. Consider the vertex A and the edges $e_1, \dots, e_r$ in lines 5 and 6 of Initialize, respectively. The edges $\varphi(e_1), \dots, \varphi(e_r)$ of $F_k(G)$ correspond to $e'_1, \dots, e'_r$ disjoint edges in G, each with exactly one token of $\varphi(A)$ at one of their endpoints. For every $1 \leq i \leq r$, we have that e_i is in G_i. Let e^*_1 and e^*_2 be two disjoint edges of G_i. By the maximality of M_A, in $\varphi(A)$ at least one of e^*_1 and e^*_2 contains either: no token, or two tokens at its endpoints. Without loss of generality assume it is e^*_1. This implies that $e^*_1 \neq e'_i$.

For a contradiction suppose that some k_j is different from 1 and $|G_j| - 1$. This implies that in $\varphi(A)$, G_j contains both a vertex $u \notin e'_j$ without a token, and vertex $v \notin e'_j$ with a token. If e^*_1 contains no token of $\varphi(A)$, then let $\varphi(A')$ be the token configuration that is produced from $\varphi(A)$ by removing the token at v and placing it at e^*_1. If e^*_1 contains two tokens of $\varphi(A)$, then let $\varphi(A')$ be the token configuration that is produced from $\varphi(A)$ by removing one token from e^*_1 and placing it at u. We have that $e^*_1, e'_1, \dots, e'_r$ are a set of disjoint edges each with exactly one token of $\varphi(A')$. This implies that $|M'_A| = |M_A| + 1$, which contradicts our choice of A. ☐

In an analogous way to the proof of Lemma 2.27, it can be shown the following result.

Lemma 2.28. *Suppose that there exists three disjoint edges in $G_i \cup G_j$. Then $k_i = k_j = 1$, or $k_i = |G_i| - 1$ and $k_j = |G_j| - 1$.*

We now proceed to show how to determine the class of each H_i.

Lemma 2.29. *Given F, but neither n or k, we can determine in polynomial time the class of every H_i.*

Proof. By Lemma 2.16, we can determine in polynomial time whether each H_i is of class 1, 2, $3c$ or 4. We show how to distinguish between the classes $3a$ and $3b$. By Lemma 2.27 there cannot simultaneously exists an H_i of class $3a$ and an H_j of class $3b$. Assume that at least one H_i is of class $3a$ or $3b$ as otherwise we are done. Suppose that $r = 1$; since we are assuming that $1 < k < |G| - 1$, we have that H_1 is of class $3b$ and we are done in this case. Assume that $r > 1$.

We claim that

all the H_i of class $3a$ or $3b$, are of class $3a$ if and only if F contains three edge disjoint graphs F_1, F_2 and M with the following properties: $(*)$

1. *F_1 is an induced subgraph of H;*
2. *there exists an H_i of class $3a$ or $3b$, and vertices $u \in H_i$ and $v \in H_j (j \neq i)$, such that the set of vertices of F_1 is of the form*
$$\{A \in V(H) : \pi(A)(i) \neq u \text{ and } \pi(A)(j) = v\};$$
3. *F_2 is disjoint from H;*
4. *M is a matching from the vertices of F_1 to the vertices of F_2;*
5. *all the edges in M are in the same ladder class;*
6. *the map that sends every vertex in F_1 to its matched vertex in M is an isomorphism from F_1 to F_2.*

Let F_1, F_2 and M be as above. Since all the edges in M are in the same ladder class the set of edges of $\varphi(M)$ corresponds to moving a token along the same edge xy of G. This implies that every token configuration in $\varphi(F_1)$ either: contains a token in x and no token at y, or every every token configuration in $\varphi(F_1)$ contains a token in y and no token at

x. By (2) of Lemma 2.26 there exist token configurations $B_1 \in F_{k_i}(G_i)$ and $B_2 \in F_{k_j}(G_j)$ such that

$$\varphi(V(F_1)) = \{C \in \varphi(V(H)) : C \cap V(G_i) \neq B_1 \text{ and } C \cap V(G_j) = B_2\}.$$

Thus, either $x \in G_i$ and $y \in G_j$, or $x \in G_j$ and $y \in G_i$. Without loss of generality assume it is the former. If H_i is of type $3b$ then there exists token configurations C_1 and C_2 of $F_{k_i}(G_i)$ distinct from B_1 such that $x \in C_1$ and $x \notin C_2$. This contradicts the fact that in every token configuration of $\varphi(F_1)$ either there is a token at x or there is no token at x. Therefore, if H contains subgraphs F_1, F_2 and M as above then every H_i of class $3a$ or $3b$ is of class $3a$.

Conversely, suppose that every H_i of class $3a$ or $3b$, is of class $3a$. Since $r > 1$ and G is connected there exists a pair of indices i and j, such that H_i is of class $3a$ and there exists an edge $xy \in G$ with $x \in G_i$ and $y \in G_j$. By Lemma 2.27 either all k_i are equal to 1 or all k_i are equal to $|G_i| - 1$. If all the k_i are equal to 1 then let F_1' be the subgraph of $F_k(G)$ induced by the set of token configurations

$$\{B \in \varphi(H) : x \notin B \text{ and } y \in B\}.$$

If all k_i are equal to $|G_i| - 1$ then let F_1' be the subgraph of $F_k(G)$ induced by the set of token configurations

$$\{B \in \varphi(H) : x \in B \text{ and } y \notin B\}.$$

Let $F_1 := \varphi^{-1}(F_1')$. By (2) of Lemma 2.26 and the fact that every k_i is equal to 1 or to $|G_i| - 1$, the vertex set F_1 is of the form

$$\{A \in H : \pi(A)(i) \neq u \text{ and } \pi(A)(j) = v\},$$

for some pair of vertices $u \in H_i$ and $v \in H_j$. Thus F_1 satisfies (1) and (2). Let F_2' be the subgraph of $F_k(G)$ induced by the set of vertices

$$\{C \in F_k(G) : C \text{ is obtained from a vertex } B \in F_1' \text{ by sliding the token along } xy\}.$$

Let $F_2 := \varphi^{-1}(F_2')$. Since xy is not an edge of $\bigcup_{i=1}^{r} G_i$, F_2 is disjoint from H. Thus, F_2 satisfies (3). Let

$$M' := \{C_1C_2 \in E(F_1', F_2') : \varphi(C_1) \triangle \varphi(C_2) = \{x, y\}\}.$$

Let $M := \varphi^{-1}(M')$. M' is a matching from F_1' to F_2'; thus, M satisfies (4). By construction of F_2', the map that sends every vertex in F_1' to its matched vertex in M' is an isomorphism from F_1' to F_2'. Therefore, M satisfies (6). It is not hard to show that H_i is 2-connected; this, in turn implies that F_1' is 2-connected. Thus, all the edges in M' are in the same ladder class, and M satisfies (5).

The existence of F_1, F_2 and M can be determined in polynomial time as follows. First we iterate over all possible candidates for F by considering all subgraphs induced by a set of vertices satisfying (2); there are a polynomial number of these sets and each can be constructed in polynomial time. Afterwards, we iterate over each ladder class of F and compute the subset of edges, M, in this ladder class such that exactly one of its endpoints is a vertex of F_1. We compute the graph F_2 induced by the endpoints of these edges that are not in F_1. Finally, we check whether M and F_2 satisfy (3) – (6). If the desired F_1, F_2 and M exist, they are found by this algorithm. □

Having determined the class of each H_i we can now reconstruct the G_i. For every $1 \leq i \leq r$ we construct a graph J_i isomorphic to G_i as follows. If H_i is not of class $3b$ we set J_i to be a copy of H_i. If H_i is of class $3b$, we compute m and $l \leq (m+1)/2$ such that $H_i \simeq F_l(K_{1,m})$; and set J_i to be a copy of $K_{1,m}$. Let $J := \bigcup_{i=1}^{r} J_i$; note that J is isomorphic to $\bigcup_{i=1}^{r} G_i$.

2.5.2 Reconstructing the adjacencies between the G_i's

To reconstruct G all that remains to be done is to reconstruct the adjacencies between the G_i's. This information is encoded in the adjacencies between H and $F \setminus H$. We start by labelling each H_i as a token graph of J_i.

First note that each H_i is uniquely reconstructible as the k_i-token graph of J_i: when H_i is not of class $3b$ this is straightforward; and when H_i is of class $3b$ it follows from Lemmas 2.14 and 2.16. We show that there are at most two possible values, l_i and $\bar{l}_i$, for each k_i. If H_i is of class 1, then $k_i = 1$; in this case we set $l_i := 1$. If H_i is not of class $3b$ nor 1, then by Lemma 2.27, we have that $k_i = 1$ or $k_i = |J_i| - 1$; in this case we set $l_i := 1$, and $\bar{l}_i := |J_i| - 1$. If H_i is of class $3b$, then by Lemma 2.14, there exists unique integers m and $l \leq (m+1)/2$ such that $H_i \simeq F_l(K_{1,m})$; we set $l_i := l$, and $\bar{l}_i := m + 1 - l$ in this case. Having defined the l_i and $\bar{l}_i$, for each H_i that is not of class 1, we construct in polynomial time an isomorphism $\psi_i : H_i \to F_{l_i}(J_i)$. This is straightforward when H_i is not of class $3b$; when H_i is of class $3b$ it can be done in polynomial time by Lemma 2.16. For each H_i that is not of class 1 we construct an additional isomorphism $\overline{\psi_i} : H_i \to F_{\bar{l}_i}(J_i)$ by letting $\overline{\psi_i} := \mathfrak{c} \circ \psi_i$.

Let φ_i be the isomorphism from H_i to $F_{k_i}(G_i)$ given by (3) of Lemma 2.26. Using φ_i we define an isomorphism ϕ_i' from $F_{k_i}(J_i)$ to $F_{k_i}(G_i)$ as follows. Suppose that $l_i \neq \bar{l}_i$. Let

$$\phi_i' := \begin{cases} \varphi_i \circ {\psi_i}^{-1} & \text{if } k_i = l_i; \\ \varphi_i \circ \overline{\psi_i}^{-1} & \text{if } k_i = \bar{l}_i. \end{cases}$$

Suppose that $l_i = \bar{l}_i$. By 3) of Theorem 2.7, there exists a unique $\iota^{-1}(\varphi_i \circ {\psi_i}^{-1}) \in \operatorname{Iso}(J_i, G_i)$ such that

$$\varphi_i \circ {\psi_i}^{-1} = \iota(\iota^{-1}(\varphi_i \circ {\psi_i}^{-1})) \ \text{ or } \ \varphi_i \circ {\psi_i}^{-1} = \mathfrak{c} \circ \iota(\iota^{-1}(\varphi_i \circ {\psi_i}^{-1})).$$

In the first case let

$$\phi_i' := \varphi_i \circ {\psi_i}^{-1};$$

in the second case let

$$\phi_i' := \varphi_i \circ \overline{\psi_i}^{-1}.$$

Having defined ϕ_i', by 3) of Theorem 2.7 we can define an isomorphism from J_i to G_i by letting

$$\phi_i := \iota^{-1}(\phi_i').$$

We have the following diagram.

$$\begin{tikzcd}
F_{l_i}(J_i) \arrow[dr, "\psi_i^{-1}"'] \arrow[drr, dashed, bend left, "\phi_i'"] & & & & \\
& H_i \arrow[r, "\varphi_i"] & F_{k_i}(G_i) & J_i \arrow[r, "\phi_i := \iota^{-1}(\phi_i')"] & G_i \\
F_{\bar{l}_i}(J_i) \arrow[uu, bend left, "\iota"] \arrow[ur, "\overline{\psi_i}^{-1}"] \arrow[urr, dashed, bend right, "\phi_i'"'] & & & &
\end{tikzcd}$$

In what follows we always use the same letter to denote corresponding vertices of J_i and G_i. We use a prime to distinguish the vertex in J_i. So that if $u' \in J_i$ then $u := \phi_i(u') \in G_i$.

Let $e \in E(H, F \setminus H)$. Note that there exists indices $1 \leq i < j \leq r$ such that $\varphi(e)$ corresponds to moving a token along a $G_i - G_j$ edge. Let $\mathrm{idx}_H(e) := \{i, j\}$ be the set of these indices; and let $E(H - F \setminus H)_{ij}$ be the set of edges $e \in E(H, F \setminus H)$ such that $\mathrm{idx}_H(e) = \{i, j\}$.

Lemma 2.30. *For every $H - F \setminus H$ edge, e, we can compute $\mathrm{idx}_H(e)$ in polynomial time.*

Proof. Let AB be an $H - F \setminus H$ edge, with $A \in H$ and $B \in F \setminus H$. For every pair $1 \leq i < j \leq r$ we check whether every vertex in the set

$$\{C \in H : \pi(C)(i) = \pi(A)(i) \text{ and } \pi(C)(j) = \pi(A)(j)\}$$

is incident to an edge in the same ladder class as AB. The pair where this is the case is the pair of indices we are looking for. □

2.5.2.1 Labelling the $H - F \setminus H$ edges

Let $e \in E(H, F \setminus H)_{ij}$. Let $x' := \mathrm{endpoint}_J(e)(i)$ and $y' := \mathrm{endpoint}_J(e)(j)$ be the vertices such that $x' \in J_i$, $y' \in J_j$ and $\varphi(e)$ corresponds to moving a token along the edge xy. The aim of this subsection is to compute $\mathrm{endpoint}_J(e)(i)$ when H_i is not of class 1. For this purpose we define the following auxiliary graphs. Let A be a vertex of F.

- Let $\mathbf{Move}(A,i)$ be the subgraph of F induced by all the vertices $B \in F$ such that
$$\varphi(B) \cap G_j = \varphi(A) \cap G_j \text{ for all } j \neq i.$$
Thus, $\varphi(\mathbf{Move}(A,i))$ is the subgraph of $F_k(G)$ induced by all the token configurations that can be reached from $\varphi(A)$ by moving the tokens at G_i while leaving the tokens at the other G_j fixed.

- Let $\mathbf{Move}(A)$ be the subgraph of F induced by all the vertices $B \in F$ such that
$$|\varphi(B) \cap G_i| = |\varphi(A) \cap G_i| \text{ for all } 1 \leq i \leq r.$$
Thus, $\varphi(\mathbf{Move}(A))$ is the subgraph of $F_k(G)$ induced by all the token configurations that can be reached from $\varphi(A)$ by token moves that do not involve moving tokens between different G_i's.

Note that if A is a vertex of H, then
$$\mathbf{Move}(A,i) \simeq H_i \text{ and } \mathbf{Move}(A) \simeq H.$$
In particular, in this case, $\mathbf{Move}(A,i)$ is the subgraph of H induced by the set of vertices
$$\{B \in H : \pi(B)(j) = \pi(A)(j) \text{ for all } j \neq i\}.$$
Thus, when A is a vertex of H we can compute $\mathbf{Move}(A,i)$ in polynomial time.

Let $AB := e$, with $A \in H$ and $B \in F \setminus H$.

- Let $\mathbf{FixEdge}(e,i)$ be the component, that contains A, of the subgraph of F induced by the set of vertices
$$\{C \in \mathbf{Move}(A,i) : C \text{ is incident to an edge in the ladder class of } e\}.$$
Thus, $\varphi(\mathbf{FixEdge}(e,i))$ is the subgraph of $F_k(G)$ induced by the token configurations in $\varphi(\mathbf{Move}(A,i))$ that are reachable from $\varphi(A)$ by a path in $\varphi(\mathbf{Move}(A,i))$, such that at every token move of the path no token has been moved from or placed at the endpoints of $\varphi(e)$.

- Let $\mathbf{NFixEdge}(e,i)$ be the subgraph of $\mathbf{Move}(A,i) \setminus \mathbf{FixEdge}(e,i)$ induced by neighbours of $\mathbf{FixEdge}(e,i)$ in $\mathbf{Move}(A,i) \setminus \mathbf{FixEdge}(e,i)$.

$\mathbf{FixEdge}(e,i)$ and $\mathbf{NFixEdge}(e,i)$ help us to compute x'.

Observation 2.31.

a) if A_1A_2 is an edge of $\mathbf{FixEdge}(e,i)$, *then*
$$x' \notin \psi_i(A_1) \triangle \psi_i(A_2) = \overline{\psi_i}(A_1) \triangle \overline{\psi_i}(A_2); \text{ and}$$

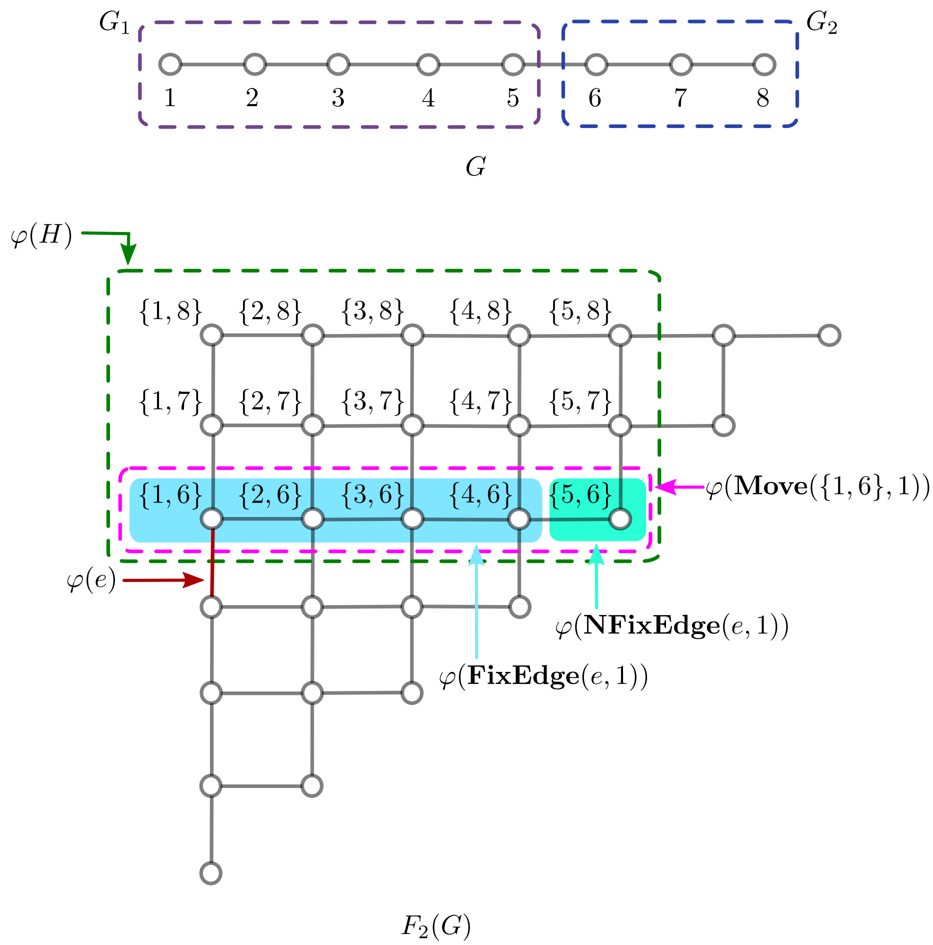


Figure 2.9: An example of subsets $\varphi(\mathbf{Move}(A,i))$, $\varphi(\mathbf{FixEdge}(e,i))$ and $\varphi(\mathbf{NFixEdge}(e,i))$ in the 2-token graph $F_2(G)$ of a path graph G on eight vertices.

b) if A_1A_2 *is a* $\mathbf{FixEdge}(e,i) - \mathbf{NFixEdge}(e,i)$ *edge then*

$$x' \in \psi_i(A_1) \triangle \psi_i(A_2) = \overline{\psi_i}(A_1) \triangle \overline{\psi_i}(A_2).$$

We have the following result.

Lemma 2.32. *Let* $e \in E(H, F \setminus H)$ *with* $i \in \mathrm{idx}_H(e)$*. Suppose that* $|\mathbf{FixEdge}(e,i)| > 1$ *or* $|\mathbf{NFixEdge}(e,i))| > 1$*. Then we can compute* $\mathrm{endpoint}_J(e)(i)$ *in polynomial time.*

Proof. Let $AB := e$ with $A \in H$. If H_i is of class 1 then $|\mathbf{FixEdge}(e,i)| = 1$ and $|\mathbf{NFixEdge}(e,i)| = 1$. Thus, H_i is not of class 1.

Suppose that H_i is not of class $3b$. We have that $k_i = 1$ or $k_i = |G_i| - 1$. Let v' the only vertex in J_i such that

$$\{v'\} = \psi_i(A) = V(J_i) \setminus \overline{\psi_i}(A).$$

Let C be a vertex in $\textbf{NFixEdge}(e, i)$ adjacent to A. Let w' be the only vertex in J_i such that

$$\{w'\} = \psi_i(C) = V(J_i) \setminus \overline{\psi_i}(C).$$

By b) of Observation 2.31 we have $\text{endpoint}_J(e)(i) \in \{v', w'\}$.

- Suppose that $|\textbf{FixEdge}(e, i)| > 1$; let $D_1 \in \textbf{FixEdge}(e, i)$ such that D_1 is adjacent to A, we have $v' \in \psi(A) \triangle \psi(D_1) = \overline{\psi}(A) \triangle \overline{\psi}(D_1)$, and so, by a) of Observation 2.31 we have $\text{endpoint}_J(e)(i) \neq v'$, which implies that $\text{endpoint}_J(e)(i) = w'$.
- Suppose that $|\textbf{FixEdge}(e, i)| = 1$; then we have that $|\textbf{NFixEdge}(e, i))| > 1$; let $D_2 \in \textbf{NFixEdge}(e, i)$ be a vertex adjacent to A with $D_2 \neq C$. By b) of Observation 2.31 we have
$$\text{endpoint}_J(e)(i) \in \psi(A) \triangle \psi(C) = \overline{\psi}(A) \triangle \overline{\psi}(C)$$
and
$$\text{endpoint}_J(e)(i) \in \psi(A) \triangle \psi(D_2) = \overline{\psi}(A) \triangle \overline{\psi}(D_2),$$
implying that $\text{endpoint}_J(e)(i) = v'$.

Suppose that H_i is of class $3b$. Thus, J_i is a star. Let v' be the center of J_i. If $|\textbf{FixEdge}(e, i)| = 1$ then $|\textbf{NFixEdge}(e, i)| > 1$ and $\text{endpoint}_J(e)(i) = v'$. Suppose that $|\textbf{FixEdge}(e, i)| > 1$ then $\text{endpoint}_J(e)(i) \neq v'$. Let CD be a $\textbf{FixEdge}(e, i)$ – $\textbf{NFixEdge}(e, i)$ edge. We have that $\text{endpoint}_J(e)(i)$ is the vertex in

$$\psi_i(C) \triangle \psi_i(D) = \overline{\psi_i}(C) \triangle \overline{\psi_i}(D)$$

distinct from v'. □

By the previous result, we can determine $\text{endpoint}_J(e)(i)$ whenever $|\textbf{FixEdge}(e, i)| > 1$ or $|\textbf{NFixEdge}(e, i)| > 1$. In the following result we describe precisely when this does not hold, that is, when $|\textbf{FixEdge}(e, i)| = 1$ and $|\textbf{NFixEdge}(e, i)| = 1$.

Lemma 2.33. *Let $e := AB \in E(H, F \setminus H)$, with $i \in \text{idx}(e)$, and suppose that H_i is not of class 1. Then, $|\textbf{FixEdge}(e, i)| = 1$ and $|\textbf{NFixEdge}(e, i)| = 1$ if and only if the following holds:*

(*i*) *$k_i = 1$ or $k_i = |G_i| - 1$,*

(*ii*) *if x' is the vertex in J_i with*
$$\{x'\} := \psi_i(\pi(A)(i)) = V(J_i) \setminus \overline{\psi_i}(\pi(A)(i)),$$
then x' is a vertex of degree one in J_i with neighbour v', and

(*iii*) $\text{endpoint}_J(e)(i) \in \{x', v'\}$.

Proof. Let us show first the forward implication. To derive a contradiction, suppose that $1 < k_i < |G_i| - 1$. Then, $G_i \simeq K_{1,m}$, $k_i = l$ or $k_i = m + 1 - l$, and $H_i \simeq F_{k_i}(K_{1,m})$. If $\mathrm{endpoint}_J(e)(i)$ is a leaf of $K_{1,m}$ then $|\mathbf{FixEdge}(e,i)| > 1$, and if $\mathrm{endpoint}_J(e)(i)$ is the center of $K_{1,m}$ then $|\mathbf{NFixEdge}(e,i)| > 1$; these both cases are a contradiction. Thus, $k_i = 1$ or $k_i = |G_i| - 1$ and so (i) holds.

Let $x' \in J_i$ as in (ii). If x' is of degree greater than one, then either $|\mathbf{FixEdge}(e,i)| > 1$ or $|\mathbf{NFixEdge}(e,i)| > 1$. Thus, x' is of degree one in J_i, and (ii) holds. Let v' be its only neighbour. Again, if $\mathrm{endpoint}_J(e)(i) \notin \{x', v'\}$ then either $|\mathbf{FixEdge}(e,i)| > 1$ or $|\mathbf{NFixEdge}(e,i)| > 1$— a contradiction. Thus, $\mathrm{endpoint}_J(e)(i) \in \{x', v'\}$ and (iii) holds.

The converse implication is straightforward. □

Consider a $G_i - G_j$ edge, say uv. Note that, to reconstruct the remaining adjacencies of G (throughout the graph J) it is enough to recognize only one $H - F \setminus H$ edge $e := AB$, for which $\varphi(e)$ is generated by sliding a token along the edge uv. Although our aim is to determine $\mathrm{endpoint}_J(e)(i)$ for all the edges $e \in E(H, F \setminus H)_{ij}$, in the following result we show that for each $G_i - G_j$ edge uv, with $u \in G_i$ and G_i not of class 1, we can recognize in polynomial time an edge $e^* \in E(H, F \setminus H)_{ij}$ with $\mathrm{endpoint}_J(e^*)(i) = u'$.

Corollary 2.34. *Suppose that H_i is not of class 1. Then for every vertex $u \in G_i$ such that u is adjacent to a vertex v of G_j, there exists $e^* \in E(H, F \setminus H)_{ij}$ such that* $\mathrm{endpoint}_J(e^*)(i) =$ *u' and for which we can compute* $\mathrm{endpoint}_J(e^*)(i)$ *in polynomial time.*

Proof. By Lemmas 2.33 and 2.32, we may assume that $k_i = 1$ or $k_i = |G_i| - 1$ and u is of degree one in G_i, as otherwise every edge $e \in E(H, F \setminus H)_{ij}$ such that $\varphi(e)$ is generated by sliding a token along the edge uv can be used as our desired e^*. Let w be the only neighbour of u in G_i. Note that since H_i is not of class 1, w is of degree greater than one in G_i. Let $A \in H$ be such that: if $k_i = 1$, then in $\varphi(A)$ there is a token at each of w and v; if $k_i = |G_i| - 1$, then in $\varphi(A)$ there is no token at w nor v. Let $e^* := AB$ such that $\varphi(e^*)$ is obtained by sliding the token along the edge uv. We have that $|\mathbf{FixEdge}(e^*, i)| > 1$ and by Lemma 2.32 we can compute $\mathrm{endpoint}_J(e^*)(i) = u'$. □

For every $1 \leq i \leq r$ such that H_i is not of class 1, and every $\psi_i' \in \{\psi_i, \overline{\psi_i}\}$, let

$$\overline{\psi_i'} := \begin{cases} \psi_i & \text{if } \psi_i' = \overline{\psi_i}, \\ \overline{\psi_i} & \text{if } \psi_i' = \psi_i. \end{cases}$$

Consider an edge $e := AB \in E(H, F \setminus H)_{ij}$, and let uv be the $G_i - G_j$ edge such that $\varphi(e)$ is generated by moving a token on uv, where $u \in G_i$ and $v \in G_j$. For each of i and j there are two possible automorphisms: $\{\psi_i, \overline{\psi_i}\}$ and $\{\psi_j, \overline{\psi_j}\}$, with corresponding values $\{l_i, \bar{l}_i\}$ and $\{l_j, \bar{l}_j\}$. We need the following.

- If there is a token at u' in $\psi_i\big(\pi(A)(i)\big)$, we choose $\psi'_j \in \{\psi_j, \overline{\psi_j}\}$ such that there is no token at v' in $\psi'_j\big(\pi(A)(j)\big)$; and,
- if there is no token at u' in $\psi_i\big(\pi(A)(i)\big)$, we choose $\psi'_j \in \{\psi_j, \overline{\psi_j}\}$ such that there is a token at v' in $\psi'_j\big(\pi(A)(j)\big)$.

Once we have matched the isomorphisms ψ_i and ψ'_j in this way, we match the isomorphisms $\overline{\psi_i}$ and $\overline{\psi'_j}$. In the following result we show that these two pairs $\{\psi_i, \psi'_j\}$ and $\{\overline{\psi_i}, \overline{\psi'_j}\}$ agree for every edge in $E(H, F \setminus H)_{ij}$.

Lemma 2.35. *Let $1 \leq i, j \leq r$ such that H_i and H_j are not of class 1. Suppose that for some $e \in E(H, F \setminus H)_{ij}$, we have computed both $\mathrm{endpoint}_J(e)(i)$ and $\mathrm{endpoint}_J(e)(j)$. Then we can compute in polynomial time a $\psi'_j \in \{\psi_j, \overline{\psi_j}\}$ with the following property. For every edge $AB \in E(H, F \setminus H)_{ij}$, there is exactly one token at $\{\mathrm{endpoint}_J(AB)(i), \mathrm{endpoint}_J(AB)(j)\}$ in each of*

$$\psi_i(\pi(A)(i)) \cup \psi'_j(\pi(A)(j)) \textit{ and } \overline{\psi_i}(\pi(A)(i)) \cup \overline{\psi'_j}(\pi(A)(j)).$$

Proof. Let $AB := e$, $x' := \mathrm{endpoint}_J(e)(i)$ and $y' := \mathrm{endpoint}_J(e)(j)$. Since $\varphi(B)$ is obtained from $\varphi(A)$ by sliding a token along the edge xy, we have that in $\varphi(A)$ there is exactly one token at one of $\{x, y\}$. By definition of $\overline{\psi_j}$ there is a token at y' in $\psi_j(\pi(A)(j))$ if and only if there is no token at y' in $\overline{\psi_j}(\pi(A)(j))$. Choose $\psi'_j \in \{\psi_j, \overline{\psi_j}\}$ so that

there is a token at y' in $\psi'_j(\pi(A)(j)$ if and only there is no token at x' in $\psi_i(\pi(A)(i))$. (5)

Let $CD \in E(H, F \setminus H)_{ij}$, $v' := \mathrm{endpoint}_J(CD)(i)$ and $w' := \mathrm{endpoint}_J(CD)(j)$. Recall that

$$\phi'_i = \varphi_i \circ {\psi_i}^{-1} \text{ or } \phi'_i = \varphi_i \circ \overline{\psi_i}^{-1}.$$

Suppose $\phi'_i = \varphi_i \circ \psi_i^{-1}$. Thus, the isomorphism ϕ_i from J_i to G_i is given by $\phi_i = \iota^{-1}(\varphi_i \circ \psi_i^{-1})$. By (5) we have that $\phi_j = \iota^{-1}(\varphi_j \circ {\psi'_j}^{-1})$ This implies that:

- there is a token at v in $\varphi(C)$ if and only if there is a token at v' in $\psi_i(\pi(C)(i))$; and
- there is a token at w in $\varphi(C)$ if and only if there is a token at w' in $\psi'_j(\pi(C)(j))$.

Therefore, there is exactly one token at $\{v', w'\}$ in each of

$$\psi_i(\pi(C)(i)) \cup \psi'_j(\pi(C)(j)) \text{ and } \overline{\psi_i}(\pi(C)(i)) \cup \overline{\psi'_j}(\pi(C)(j)).$$

Suppose $\phi'_i = \varphi_i \circ \overline{\psi_i}^{-1}$. Thus, the isomorphism ϕ_i from J_i to G_i is given by $\phi_i = \iota^{-1}(\varphi_i \circ \overline{\psi_i}^{-1})$. By (5) we have that $\phi_j = \iota^{-1}(\varphi_j \circ \overline{\psi'_j}^{-1})$ This implies that:

- there is a token at v in $\varphi(C)$ if and only if there is no token at v' in $\psi_i(\pi(C)(i))$; and
- there is a token at w in $\varphi(C)$ if and only if there is no token at w' in $\psi'_j(\pi(C)(j))$.

Therefore, there is exactly one token at $\{v', w'\}$ in each of

$$\psi_i(\pi(C)(i)) \cup \psi'_j(\pi(C)(j)) \text{ and } \overline{\psi_i}(\pi(C)(i)) \cup \overline{\psi'_j}(\pi(C)(j)).$$

$\square$

When ψ_i and ψ'_j are as in Lemma 2.35, we say that ψ_i is *compatible* with ψ'_j, and that $\overline{\psi_i}$ is *compatible* with $\overline{\psi'_j}$.

As we will see in the following result, in the case when $k_i = k_j = 1$ or $k_i = |G_i| - 1$ and $k_j = |G_j| - 1$ we have some information that allow us to compute $\text{endpoint}_J(e)(i)$, regardless the class of H_j. If in addition, H_j is not of type 1, using similar arguments we can compute $\text{endpoint}_J(e)(j)$ as well.

Lemma 2.36. *Let $e := AB \in E(H, F \setminus H)_{ij}$ such that H_i is not of class 1. If we know that it must be the case that either $k_i = k_j = 1$, or $k_i = |G_i| - 1$ and $k_j = |G_j| - 1$ then we can compute* $\text{endpoint}_J(e)(i)$ *in polynomial time.*

Proof. Let x' be the vertex of J_i such that

$$\{x'\} = \psi_i(\pi(A)(i)) = V(J_i) \setminus \overline{\psi_i}(\pi(A)(i)).$$

By Lemma 2.32 we may assume that $|\mathbf{FixEdge}(e, i)| = 1$ and $|\mathbf{NFixEdge}(e, i)| = 1$, as otherwise we are done; and by Lemma 2.33 we know that x' is of degree one and $\text{endpoint}_J(e)(i) \in \{x', v'\}$, where v' is the only neighbour of x' in J_i.

Let y' be the vertex of J_j such that

$$\{y'\} = \psi_j(\pi(A)(j)) = V(J_j) \setminus \overline{\psi_j}(\pi(A)(j)).$$

If we have computed $\text{endpoint}_J(e)(j)$ we have that: if $\text{endpoint}_J(e)(j) = y'$ then $\text{endpoint}_J(e)(i) = v'$; and if $\text{endpoint}_J(e)(j) \neq y'$ then $\text{endpoint}_J(e)(i) = x'$. As before, we may assume that $|\mathbf{FixEdge}(e, j)| = 1$ and $|\mathbf{NFixEdge}(e, j)| = 1$, as otherwise we are done, and so y' is of degree one in J_j and $\text{endpoint}_J(e)(j) \in \{y', w'\}$, where w' is the only neighbour of y' in J_j.

Let A' be the vertex of H such that

$$\psi_i(\pi(A'))(i) = V(J_i) \setminus \overline{\psi_i}(\pi(A')(i)) = \{v'\}$$

and

$$\psi_j(\pi(A'))(j) = V(J_j) \setminus \overline{\psi_j}(\pi(A')(j)) = \{w'\}.$$

Let

$$S := \{B' \in V(F \setminus H) : \mathrm{idx}_H(A'B') = \{i, j\}\}.$$

Let $B' \in S$. Since v' is of degree greater than one in J_i, it follows that

$$|\mathbf{NFixEdge}(A'B', i)| > 1 \quad \text{or} \quad |\mathbf{FixEdge}(A'B', i)| > 1.$$

By Lemma 2.32 we can determine $\mathrm{endpoint}_J(A'B')(i)$. By a similar argument, if H_j is not of class 1 we can also determine $\mathrm{endpoint}_J(A'B')(j)$.

Suppose that H_j is not of class 1. In this case we can determine if x is adjacent to w and if y is adjacent to v by means of the edges of type $A'B'$, for $B' \in S$. If x is adjacent to w but v is not adjacent to y, then $\mathrm{endpoint}_J(e)(i) = x'$. If y is adjacent to v but x is not adjacent to w, then $\mathrm{endpoint}_J(e)(i) = v'$. Assume that x is adjacent to w and that v is adjacent to y. We can determine the vertex $B' \in S$ such that $\varphi(B')$ is obtained from $\varphi(A')$ by sliding the token along the edge xw, and the vertex $B'' \in S$ such that $\varphi(B'')$ is obtained from $\varphi(A')$ by sliding the token along the edge yv. Note that $B = B'$ or $B = B''$. If $B = B'$ then $\mathrm{endpoint}_J(e)(i) = v'$; and if $B = B''$ then $\mathrm{endpoint}_J(e)(i) = x'$.

Suppose now that H_j is of class 1. Suppose that there exists a vertex $B' \in S$ such that $\mathrm{endpoint}_J(A'B')(i) = v'$. If $B = B'$, then $\mathrm{endpoint}_J(e)(i) = x'$; otherwise, $\mathrm{endpoint}_J(e)(i) = v'$. Suppose that no such vertex B' exists. If $k_i = 1$, then v is not adjacent to y, and $\mathrm{endpoint}_J(e)(i) = x'$. If $k_i = |G_i| - 1$, then v is not adjacent to w, and $\mathrm{endpoint}_J(e)(i) = x'$. In either case we have that $\mathrm{endpoint}_J(e)(i) = x'$. □

Now, we are ready to show how to compute $\mathrm{endpoint}_J(e)(i)$ in the general case.

Lemma 2.37. *Let $e \in E(H, F \setminus H)_{ij}$ such that H_i is not of class 1. Then we can compute $\mathrm{endpoint}_J(e)(i)$ in polynomial time.*

Proof. Let $AB := e$ such that $A \in H$ and $B \in F \setminus H$. If either $|\mathbf{FixEdge}(e, i)| > 1$ or $|\mathbf{NFixEdge}(e, i)| > 1$, then by Lemma 2.32 we can compute $\mathrm{endpoint}_J(e)(i)$ in polynomial time. Assume then that $|\mathbf{FixEdge}(e, i)| = 1$ and $|\mathbf{NFixEdge}(e, i)| = 1$. By Lemma 2.33 we have that $k_i = 1$ or $k_i = |G_i| - 1$, that there is a vertex $x' \in J_i$ of degree one with

$$\{x'\} = \psi_i(\pi(A)(i)) = V(J_i) \setminus \overline{\psi_i}(\pi(A)(i)),$$

and that $\mathrm{endpoint}_J(e)(i) \in \{x', y'\}$, where v' is the only neighbour of x' in J_i.

Suppose that $k_j = 1$ or $k_j = |G_j| - 1$. Then either $k_i = k_j = 1$ or $k_i = |G_i| - 1$ and $k_j = |G_j| - 1$. Then, by Lemma 2.36, we can compute $\mathrm{endpoint}_J(e)(i)$ in polynomial time. Assume then that $1 < k_j < |G_j| - 1$, so $H_j \simeq F_l(K_{1,m})$ for some values m and l, $G_j \simeq K_{1,m}$ and $k_i = l$ or $k_i = \bar{l}$. By Corollary 2.34 we can recognize an edge $e^* \in E(H, F \setminus H)_{ij}$ such that $\varphi(e^*)$ is generated by sliding a token along the edge $\mathrm{endpoint}_J(e)(i)\,\mathrm{endpoint}_J(e)(j)$; then, by Lemma 2.35 we can determine ψ'_j and $\overline{\psi'_j}$ such that ψ_i is compatible with ψ'_j, and $\overline{\psi_i}$ is compatible with $\overline{\psi'_j}$. On the other hand, by Lemma 2.33 we can determine $\mathrm{endpoint}_J(e)(j)$. Thus,

- if $\text{endpoint}_J(e)(j) \in \psi'_j(\pi(A)(j))$ then $\text{endpoint}_J(e)(i) = v'$, and
- if $\text{endpoint}_J(e)(j) \notin \psi'_j(\pi(A)(j))$ then $\text{endpoint}_J(e)(i) = x'$.

This completes the proof. □

2.5.2.2 Labelling according to token movement direction

Consider an edge $e \in E(H, F \setminus H)_{ij}$. As the reader can imagine, there are two possibilities: the edge $\varphi(e)$ corresponds to moving a token either from G_i to G_j or from G_j to G_i. We denote these two possibilities with the tuples $(e, i \to j)$ and $(e, j \to i)$, respectively. If $\varphi(e)$ corresponds to moving a token from G_i to G_j we say that $(e, i \to j)$ *agrees with* φ. For every H_i of class 1, let $V(G_i) = \{x_i, \bar{x}_i\}$ and $V(J_i) = \{x'_i, \bar{x}'_i\}$, such that $\phi_i(x'_i) = x_i$ and $\phi_i(\bar{x}'_i) = \bar{x}_i$. For every vertex $u' \in J_i$, let

$$\bar{u}' := \begin{cases} x'_i & \text{if } u' = \bar{x}'_i, \text{ and} \\ \bar{x}'_i & \text{if } u' = x'_i; \end{cases}$$

and for every $u \in G_i$, let

$$\bar{u} := \begin{cases} x_i & \text{if } u = \bar{x}_i, \text{ and} \\ \bar{x}_i & \text{if } u = x_i; \end{cases}$$

For convenience, for every vertex $v' \in J_j$, where H_j is not of class 1, we define

$$\bar{v}' = v' \text{ and } \bar{v} = v.$$

Lemma 2.38. *For all* $1 \leq i < j \leq r$ *let*

$$D'_{ij} := \{(e, i \to j) : e \in E(H, F \setminus H)_{ij}\} \cup \{(e, j \to i) : e \in E(H, F \setminus H)_{ij}\}.$$

In polynomial time we can find a partition of the set

$$\mathcal{D} := \bigcup_{1 \leq i < j \leq r} D'_{ij}$$

into two sets $\overrightarrow{\mathcal{D}}$ *and* $\overleftarrow{\mathcal{D}}$*, such that the following holds. Either for all edges* $e \in E(H, F \setminus H)$ *there is a tuple in* $\overrightarrow{\mathcal{D}}$ *that agrees with* φ*, or for all edges* $e \in E(H, F \setminus H)$ *there is a tuple in* $\overleftarrow{\mathcal{D}}$ *that agrees with* φ*.*

Proof. Let $1 \leq i < j \leq r$ be such that $E(G_i, G_j) \neq \emptyset$. We first show that in polynomial time we can find a partition of D'_{ij} into two sets D_{ij} and $\overline{D_{ij}}$ such that the following holds. Either for all edges $e \in E(H, F \setminus H)_{ij}$ there is a tuple in D_{ij} that agrees with φ, or for all edges $e \in E(H, F \setminus H)_{ij}$ there is a tuple in $\overline{D_{ij}}$ that agrees with φ. For convenience we define $D_{ji} := D_{ij}$ and $\overline{D_{ji}} := \overline{D_{ij}}$.

Let $e := AB \in E(H, F \setminus H)_{ij}$. We are going to construct such sets D_{ij} and $\overline{D_{ij}}$.

- Suppose that H_i or H_j are not of class 1.

  Without loss of generality assume that H_i is not of class 1. We use Lemma 2.37 to compute $\mathrm{endpoint}_J(e)(i)$. If $\mathrm{endpoint}_J(e)(i) \in \psi_i(\pi(A)(i))$, then append $(e, i \to j)$ to D_{ij} and append $(e, j \to i)$ to $\overline{D_{ij}}$; if $\mathrm{endpoint}_J(e)(i) \notin \psi_i(\pi(A)(i))$, then append $(e, i \to j)$ to $\overline{D_{ij}}$ and $(e, j \to i)$ to D_{ij}. Note that, either for all edges $e \in E(H, F \setminus H)_{ij}$ there is a tuple in D_{ij} that agrees with φ, or for all edges $e \in E(H, F \setminus H)_{ij}$ there is a tuple in $\overline{D_{ij}}$ that agrees with φ.

- Suppose that both H_i and H_j are of class 1.

  Fix a vertex $A^* \in H$ and let H' be the subgraph of H induced by the vertices $C \in H$ such that $\pi(C)(l) = \pi(A^*)(l)$ for all $l \neq i, j$. Note that $\varphi(H')$ is the graph generated by moving a token at each of G_i and G_j, while fixing the tokens at the other G_l. Since G_i and G_j are edges, H' is an induced 4-cycle of F. The endpoint in $F \setminus H$ of every edge in $E(H', F \setminus H)_{ij}$ must be one of two vertices ${B_1}^*$ and ${B_2}^*$, where $\varphi({B_1}^*)$ and $\varphi({B_2}^*)$ correspond to having two tokens in either G_i or in G_j. Let $P := (A = A_1, \ldots, A_m = A')$ be a path from A to a vertex $A' \in H'$ such that for all $1 \leq s \leq m$, we have that $\pi(A_s)(i) = \pi(A)(i)$ and $\pi(A_s)(j) = \pi(A)(j)$. Thus, $\varphi(P)$ corresponds to a sequence of tokens moves that leaves the tokens at G_i and G_j fixed and arrives at a vertex of $\varphi(H')$. Note that there exists exactly one edge $A'B' \in E(H, F \setminus H)_{ij}$, such that $A'B'$ and e are in the same ladder class. If $B' = {B_1}^*$ then append $(e, i \to j)$ to D_{ij} and $(e, j \to i)$ to $\overline{D_{ij}}$; if $B' = {B_2}^*$ then append $(e, i \to j)$ to $\overline{D_{ij}}$ and $(e, j \to i)$ to D_{ij}. Note that, either for all edges $e \in E(H, F \setminus H)_{ij}$ there is a tuple in D_{ij} that agrees with φ, or for all edges $e \in E(H, F \setminus H)_{ij}$ there is a tuple in D'_{ij} that agrees with φ.

Suppose that we have defined all such D_{ij} and $\overline{D_{ij}}$. For $D \in \{D_{ij}, \overline{D_{ij}}\}$, we define

$$\overline{D} := \begin{cases} \overline{D_{ij}} & \text{if } D = D_{ij}, \text{ and} \\ D_{ij} & \text{if } D = \overline{D_{ij}}. \end{cases}$$

Let $1 \leq i, j, l \leq r$ and let $D_1 \in \{D_{ij}, \overline{D_{ij}}\}$ and $D_2 \in \{D_{jl}, \overline{D_{jl}}\}$. We say that D_1 and D_2 are *adjacent* if $E(G_i, G_j) \neq \emptyset$ and $E(G_j, G_l) \neq \emptyset$. We say that two adjacent subsets D_1 and D_2 are *compatible* if, in addition, the following holds. Either for all edges $e \in E(H, F \setminus H)_{ij} \cup E(H, F \setminus H)_{jl}$ there is a tuple in $D_1 \cup D_2$ that agrees with φ, or for all edges $e \in E(H, F \setminus H)_{ij} \cup E(H, F \setminus H)_{jl}$ there is a tuple in $\overline{D_1} \cup \overline{D_2}$ that agrees with φ. Note that if D_1 and D_2 are compatible, then $\overline{D_1}$ and $\overline{D_2}$ are compatible. Moreover, there is exactly one of D_{jl} and $\overline{D_{jl}}$ that is compatible to D_{ij}.

We determine in polynomial time whether D_1 and D_2 are compatible as follows.

- Suppose that H_j is not of class 1.

  Let $e_1 := A_1B_1 \in E(H, F \setminus H)_{ij}$ and $e_2 := A_2B_2 \in E(H, F \setminus H)_{jl}$ such that $(e_1, i \to j) \in D_1$ and $(e_2, j \to l) \in D_2$. We use Lemma 2.37 to compute $\mathrm{endpoint}_J(e_1)(j)$ and endpoint$_J(e_2)(j)$ in polynomial time. Let $\psi'_j \in \{\psi_j, \overline{\psi_j}\}$ be such that there is no token at endpoint$_J(e_1)(j)$ in $\psi'_j(\pi(A_1)(j))$. D_1 and D_2 are compatible if and only if there is a token at endpoint$_J(e_2)(j)$ in $\psi'_j(\pi(A_2)(j))$.

- Suppose that H_j is of class 1.

  Let $e_1 := AB_1 \in E(H, F \setminus H)_{ij}$ and $e_2 := AB_2 \in E(H, F \setminus H)_{jl}$ such that $(e_1, i \to j) \in D_1$ and $(e_2, j \to l) \in D_2$. Let u' be the only vertex in $\psi_j(\pi(A)(j))$. If $(e_1, i \to j)$ and $(e_2, j \to l)$ agree with φ, then $\varphi(e_1)$ corresponds to moving a token from a vertex in G_i to $\bar{u}$; and $\varphi(e_2)$ corresponds to moving a token from u to a vertex if G_l. D_1 and D_2 are compatible if and only if e_1 and e_2 are contained in an induced 4-cycle of F.

We now extend the definition of compatible pairs to non necessarily adjacent pairs. Let $1 \le i, j, l, s \le r$ be indices such that $E(G_i, G_j) \neq \emptyset$ and $E(G_l, G_s) \neq \emptyset$. Let $D_1 \in \{D_{ij}, D'_{ij}\}$ and $D_2 \in \{D_{ls}, D'_{ls}\}$. We say that D_1 and D_2 are *compatible* if there exists a sequence $D_1 = C_1, C_2, \dots, C_m = D_2$, such that for all $1 < t \le m$, C_t and C_{t+1} is a compatible adjacent pair. Having computed all compatible adjacent pairs we compute all compatible pairs.

We finish the proof by computing $\overrightarrow{\mathcal{D}}$ and $\overleftarrow{\mathcal{D}}$. Without loss of generality assume that $E(G_1, G_2) \neq \emptyset$. Let

$$\overrightarrow{\mathcal{D}} := \bigcup_{\substack{1 \le i < j \le r \\ E(G_i, G_j) \neq \emptyset}} \{D : D \in \{D_{ij}, \overline{D_{ij}}\} \text{ and } D \text{ is compatible with } D_{12}\}.$$

Similarly, let

$$\overleftarrow{\mathcal{D}} := \bigcup_{\substack{1 \le i < j \le r \\ E(G_i, G_j) \neq \emptyset}} \{D : D \in \{D_{ij}, \overline{D_{ij}}\} \text{ and } D \text{ is compatible with } \overline{D_{12}}\}.$$

□

If for every edge $e \in E(H, F \setminus H)$ there is a tuple in $\overrightarrow{\mathcal{D}}$ that agrees with φ then we say that $\overrightarrow{\mathcal{D}}$ *agrees* with φ. Otherwise, we say that $\overleftarrow{\mathcal{D}}$ *agrees* with φ.

2.5.3 Constructing a graph isomorphic to G

We construct, in polynomial time two graphs $\overrightarrow{J}$ and $\overleftarrow{J}$. Let

$$V(\overrightarrow{J}) := V(\overleftarrow{J}) := \bigcup_{1 \le i \le r} V(J_i);$$

$E(\overrightarrow{J})$ and $E(\overleftarrow{J})$ both contain

$$\bigcup_{1\leq i\leq r} E(J_i).$$

For every $(e, i \to j) \in \overrightarrow{D}$ we add an additional edge to $\overrightarrow{J}$ and $\overleftarrow{J}$ as follows. Let $AB \in E(H, F \setminus H)_{ij}$ such that $AB = e$. Let

$$x' := \begin{cases} \text{endpoint}_J(e)(i) & \text{if } H_i \text{ is not of class 1;} \\ \text{the only vertex in } \psi_i(\pi(A)(i)) & \text{if } H_i \text{ is of class 1.} \end{cases}$$

Let

$$y' := \begin{cases} \text{endpoint}_J(e)(j) & \text{if } H_j \text{ is not of class 1;} \\ \text{the only vertex in } V(J_j) \setminus \psi_j(\pi(A)(j)) & \text{if } H_j \text{ is of class 1.} \end{cases}$$

We add the edge $x'\overline{y'}$ to $\overrightarrow{J}$ and the edge $\overline{x'}y'$ to $\overleftarrow{J}$. Let

$$J := \begin{cases} \overrightarrow{J} & \text{if } \overrightarrow{D} \text{ agrees with } \varphi, \\ \overleftarrow{J} & \text{if } \overleftarrow{D} \text{ agrees with } \varphi. \end{cases}$$

Let $\phi_\varphi : V(J) \to V(G)$ be the map defined by

$$\phi_\varphi(u') = u,$$

for all $u' \in J$. By Lemma 2.38 and the constructions of $\overrightarrow{J}$ and $\overleftarrow{J}$ we have the following. If φ agrees with $\overrightarrow{\mathcal{D}}$, then ϕ_φ is an isomorphism from $J = \overrightarrow{J}$ to G; and if φ agrees with $\overleftarrow{\mathcal{D}}$, then ϕ_φ is an isomorphism from $J = \overleftarrow{J}$ to G.

Let swap : $V(J) \cup V(G) \to V(J) \cup V(G)$ be the map defined by

$$\text{swap}(u') := \bar{u}'.$$

When restricted to $V(J)$, swap is an isomorphism from $\overrightarrow{J}$ to $\overleftarrow{J}$. We have proved Theorem 2.1:

Theorem 2.1. *Let G be a connected (C_4, D_4)-free graph. Given only a graph isomorphic to $F_k(G)$, we can compute in polynomial time a graph isomorphic to G.*

Add all the edges of $\overrightarrow{J}$ to J, so that throughout the remainder of this chapter we assume that $J = \overrightarrow{J}$.

2.6 F is Uniquely k-reconstructible

When studying the reconstruction problem of token graphs, we realized that this problem is highly related to the automorphism group of token graphs in the following sense. By some computational experimentation we conjectured that, for any connected (C_4, D_4)-free graph G, the only possible automorphisms of $F_k(G)$ could be either induced automorphisms, the complement automorphism or a combination of these two types. Since then, we have been wondering how a reconstruction problem can be associated to an algebraic parameter, and at this point, we believe that such property of the automorphism group of token graphs is the main reason we are able to reconstruct G from $F_k(G)$, because, roughly speaking, there is a "unique way" to label the vertices of $F_k(G)$. Motivated by this *unique* labelling, we introduced the notion of a graph F being *uniquely k-reconstructible.*

So far, given a connected graph F that is isomorphic to a token graph, we have shown that there is a unique graph G (up to isomorphisms) such that F is isomorphic to $F_k(G)$, and so, G is k-token reconstructible from its k-token graph. Our next step is to assign labels to every vertex in F so that this assignation corresponds to an isomorphism from F to $F_k(G)$. In this section, we are going to do this for connected graphs (this corresponds to our Theorem 2.2):

Theorem 2.2. *Let G be a connected (C_4, D_4)-free graph. Given only a graph F, isomorphic to $F_k(G)$, we can compute in polynomial time a k-token reconstruction of F.*

We recall the definition of equivalent k-token reconstructions. Let (G, ψ) and (G, φ) be two k-token reconstructions of a graph F. We say that (G, φ) and (G, ψ) are *equivalent* k-token reconstructions of F if there exists an automorphism $\mathfrak{s}(\varphi, \psi)$ of G such that

$$\psi = \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi \quad \text{or} \quad \psi = \mathfrak{c} \circ \iota(\mathfrak{s}(\varphi, \psi)) \circ \varphi.$$

We say that F is *uniquely reconstructible as the k-token graph of G* (or simply uniquely k-reconstructible) if any two k-token reconstructions of F as the k-token graph of G are equivalent.

For the class of connected (C_4, D_4)-free graphs we prove the following:

Theorem 2.3. *Let G be a connected (C_4, D_4)-free graph. Then $F_k(G)$ is uniquely reconstructible as the k-token graph of G.*

In the remainder of this section, we always assume that $F \simeq F_k(G)$, where G is a connected (C_4, D_4)-free graph.

For every $1 \leq i \leq r$ such that H_i is not of class 1, we relabel (if necessary) ψ_i and $\overline{\psi_i}$, so that ψ_i is compatible with the assumption that $\overrightarrow{D}$ agrees with φ. Note that this also implies relabelling l_i and $\overline{l_i}$. Let

$$k' := \sum_{i=1}^{r} l_i.$$

Without loss of generality we always assume that $k' \leq n/2$. Recall that

$$J = \begin{cases} \overrightarrow{J} & \text{if } \overrightarrow{D} \text{ agrees with } \varphi, \\ \overleftarrow{J} & \text{if } \overleftarrow{D} \text{ agrees with } \varphi. \end{cases}$$

Let $\phi(J, \varphi) := \phi_\varphi$ be the automorphism from J to G defined in Subsection 2.5.3.

In what follows we compute in polynomial time an isomorphism ψ from F to $F_{k'}(J)$ so that (J, ψ) is a k'-token reconstruction of F. For this purpose, we extend our theoretical framework developed in the previous sections. For every $1 \leq i \leq r$ let

$$\widehat{H_i} := \bigcup_{s=0}^{\min\{k', |J_i|\}} F_s(J_i).$$

Note that $H_i \subset \widehat{H_r}$, for each $i \in [r]$. Let $\widehat{F}$ be the subgraph of $\widehat{H_1} \square \cdots \square \widehat{H_r}$ induced by the vertices $\widehat{A} \in \widehat{H_1} \square \cdots \square \widehat{H_r}$ such that

$$\sum_{i=1}^{r} |\widehat{A}(i)| = k'.$$

Let $\widehat{u} : V(\widehat{F}) \to V(F_{k'}(J))$ be the map defined by

$$\widehat{u}(\widehat{A}) := \bigcup_{i=1}^{r} \widehat{A}(i),$$

for all $\widehat{A} \in V(\widehat{F})$. In the remainder of this section we prove the following.

Theorem 2.39. *We can compute in polynomial time a map $\widehat{\pi} : V(F) \to \widehat{H_1} \square \cdots \square \widehat{H_r}$ such that*

a) $\psi := \widehat{u} \circ \widehat{\pi}$ is an isomorphism from F to $F_{k'}(J)$; and

b)

$$\varphi = \begin{cases} \iota(\phi(J, \varphi)) \circ \psi & \text{if } \varphi \text{ agrees with } \overrightarrow{\mathcal{D}}; \\ \mathfrak{c} \circ \iota(\phi(J, \varphi)) \circ \psi & \text{if } \varphi \text{ agrees with } \overleftarrow{\mathcal{D}}. \end{cases}$$

That is, the following diagram commutes.

$$\begin{tikzcd}
F \arrow[rr, "\varphi"] \arrow[d, "\widehat{\pi}"'] \arrow[drr, "\psi"] & & F_k(G) \\
\widehat{H_1} \square \cdots \square \widehat{H_r} \arrow[rr, "\widehat{u}"] & & F_{k'}(J) \arrow[u, "\iota(\phi(J,\varphi)) \text{ or } \mathfrak{c} \circ \iota(\phi(J,\varphi))"']
\end{tikzcd}$$

$\square$

As we show next, Theorem 2.39 readily implies that $F_k(G)$ is uniquely k-reconstructible.

Corollary 2.40. *$F_k(G)$ is uniquely k-reconstructible.*

Proof. Let $g \in \mathrm{Iso}(F_{k'}(J), F_k(G))$. Then $g \circ \psi \in \mathrm{Iso}(F, F_k(G))$. By Theorem 2.39 we have that

$$g \circ \psi = \iota(\phi(J, g \circ \psi)) \circ \psi \text{ or } g \circ \psi = \mathfrak{c} \circ \iota(\phi(J, g \circ \psi)) \circ \psi.$$

Thus,

$$g = \iota(\phi(J, g \circ \psi)) \text{ or } g = \mathfrak{c} \circ \iota(\phi(J, g \circ \psi)).$$

By 3) of Theorem 2.7 we have that $F_k(G)$ is uniquely k-reconstructible as the k-token graph of G. $\square$

Let A be a vertex of F. We say that we can *define* $\widehat{\pi}$ on A, if we can find in polynomial time a vertex $\widehat{\pi}(A)$ of $\widehat{H_1} \square \cdots \square \widehat{H_r}$ such that the following holds.

$$\varphi(A) = \iota(\phi(J, \varphi)) \circ \psi(A) \quad \text{or} \quad \varphi(A) = \mathfrak{c} \circ \iota(\phi(J, \varphi)) \circ \psi(A).$$

We begin by defining $\widehat{\pi}$ on the vertices of H. For every vertex $A \in H$, let

$$\widehat{\pi}(A) := (\psi_1(\pi(A)(1), \dots, \psi_r(\pi(A)(r))).$$

For such vertex A note that $\widehat{u}(\widehat{\pi}(A)) = \psi(A)$.

Suppose that φ agrees with $\overrightarrow{\mathcal{D}}$. Then

$$\begin{aligned}
\iota(\phi(J, \varphi)) \circ \psi(A) &= \iota(\phi(J, \varphi)) \circ \widehat{u} \circ \widehat{\pi}(A) \\
&= \iota(\phi(J, \varphi)) \circ \widehat{u}(\psi_1(\pi(A)(1), \dots, \psi_r(\pi(A)(r))) \\
&= \iota(\phi(J, \varphi)) \left(\bigcup_{i=1}^{r} \psi_i(\pi(A)(i)) \right) \\
&= \bigcup_{i=1}^{r} \phi_i' \circ \psi_i(\pi(A)(i)) \\
&= \bigcup_{i=1}^{r} \varphi_i \circ \psi_i^{-1} \circ \psi_i(\pi(A)(i)) \\
&= \bigcup_{i=1}^{r} \varphi_i(\pi(A)(i)) \\
&= \varphi(A).
\end{aligned}$$

Suppose that φ agrees with $\overleftarrow{\mathcal{D}}$. Then

$$\begin{aligned}
\mathfrak{c} \circ \iota(\phi(J,\varphi)) \circ \psi(A) &= \mathfrak{c} \circ \iota(\phi(J,\varphi)) \circ \widehat{u} \circ \widehat{\pi}(A) \\
&= \mathfrak{c} \circ \iota(\phi(J,\varphi)) \circ \widehat{u}(\psi_1(\pi(A)(1), \dots, \psi_r(\pi(A)(r))) \\
&= \mathfrak{c} \circ \iota(\phi(J,\varphi)) \left(\bigcup_{i=1}^{r} \psi_i(\pi(A)(i)) \right) \\
&= \mathfrak{c} \left(\bigcup_{i=1}^{r} \phi_i' \circ \psi_i(\pi(A)(i)) \right) \\
&= \mathfrak{c} \circ \left(\bigcup_{i=1}^{r} \varphi_i \circ \mathfrak{c} \circ \psi_i^{-1} \circ \psi_i(\pi(A)(i)) \right) \\
&= \bigcup_{i=1}^{r} \varphi_i(\pi(A)(i)) \\
&= \varphi(A).
\end{aligned}$$

Let A be a vertex of F. We consider again the subgraphs (of F) $\mathbf{Move}(A,i)$ and $\mathbf{Move}(A)$ defined in Subsection 2.5.2.1. Besides, we define the following subgraphs of F.

- If φ agrees with $\overrightarrow{\mathcal{D}}$, then let $s := |\varphi(A) \cap G_i|$; otherwise, let $s := |G_i \setminus \varphi(A)|$. Let $\mathbf{Split}(s,i)$ be the subgraph of F induced by all the vertices $B \in F$ such that
$$|\varphi(B) \cap G_i| = |\varphi(A) \cap G_i|.$$
Thus, if φ agrees with $\overrightarrow{\mathcal{D}}$, then $\varphi(\mathbf{Split}(s,i))$ is the subgraph of $F_k(G)$ induced by all the token configurations in which there are s tokens at G_i and $k-s$ tokens at $G \setminus G_i$; and if φ agrees with $\overleftarrow{\mathcal{D}}$, then $\varphi(\mathbf{Split}(s,i))$ is the subgraph of $F_k(G)$ induced by all the token configurations in which there are $|G_i|-s$ tokens at G_i and $k-|\varphi(A)\cap G_i|+s$ tokens at $G \setminus G_i$.
- Let $\mathbf{Fix}(A,i)$ be the subgraph of $\mathbf{Split}(s,i)$ induced by all the vertices $B \in \mathbf{Split}(s,i)$ such that
$$\varphi(B) \cap G_i = \varphi(A) \cap G_i.$$

Before proceeding we mention that we have shown the following lemma with the aid of a computer.

Lemma 2.41. *If $n \leq 6$ then F is uniquely k-reconstructible as the k-token graph of G.*

□

In what follows we may assume that $n > 6$. From now on, we make use of the Boolean combinations defined in Subsection 2.2.2.

Lemma 2.42. *Let F' be an induced subgraph of F. Suppose that:*

(1) *We can determine in polynomial time which vertices of F belong to F'.*

(2) *There exists a connected induced subgraph G' of G, such that $\varphi(F')$ is generated by moving $k' \leq k$ tokens on the vertices of G' while leaving $k-k'$ tokens fixed at the vertices of a nonempty subset T of $V(G \setminus G')$.*

(3) *We can determine in polynomial time the subgraph J^* of J such that $\phi(J,\varphi)(J^*) = G'$; and the set $T^* \in V(J)$ such that*

$$T^* = \phi(J,\varphi)^{-1}(T) \text{ if } \varphi \text{ agrees with } \overrightarrow{\mathcal{D}},$$

and

$$T^* = V(J) \setminus \left(V(J^*) \cup \phi(J,\varphi)^{-1}(T)\right) \text{ if } \varphi \text{ agrees with } \overleftarrow{\mathcal{D}}.$$

(4) *Let W be the set of vertices of F' for which we have defined ψ. For every vertex $u \in J^*$ we can compute in polynomial time a Boolean combination $\Gamma(u)$ of elements in $\{\psi(A) : A \in W\}$, such that*

$$\{u\} = V(J^*) \cap \Gamma(u).$$

Moreover, suppose that F' is uniquely k'-reconstructible as the k'-token graph of J^*. Then we can define $\widehat{\pi}$ on every vertex of F'.

Proof. Let $\varphi' : F' \to F_{k'}(G')$ be the map that sends every vertex $X \in F'$ to

$$\varphi'(X) := \varphi(X) \cap V(G').$$

Note that φ' is an isomorphism from F' to $F_{k'}(G')$. Since F' is uniquely k'-reconstructible, by Theorem 2.39, we can compute in polynomial time a graph J' and an isomorphism $\psi' : F' \to F_{k'}(J')$ for which there exists an isomorphism $\phi(J',\varphi') : J' \to G'$ such that

$$\varphi' = \iota\left(\phi\left(J',\varphi'\right)\right) \circ \psi' \ \text{ or } \ \varphi' = \mathfrak{c} \circ \iota\left(\phi\left(J',\varphi'\right)\right) \circ \psi'.$$

We construct an isomorphism $g : J^* \to J'$ as follows. Let u be a vertex of J^* and let $\Gamma(u)$ as in (4). Let $\Gamma'(u)$ be the Boolean combination that results from replacing each term $\psi(A)$ in $\Gamma(u)$ with $\psi'(A)$; and let $\overline{\Gamma}'(u)$ be the Boolean combination that results from replacing each term $\psi(A)$ in $\Gamma(u)$ with $J' \setminus \psi'(A)$. By Corollary 2.40, $F_{k'}(G')$ is uniquely reconstructible as the k'-token graph of G'. Thus, by Proposition 2.13 and 3) of Theorem 2.7, we have that

$$\{g(u)\} = \Gamma'(u) \text{ or } \{g(u)\} = \overline{\Gamma}'(u),$$

for some vertex $g(u)$ of J'. Therefore, if

$$|\Gamma'(u)| = 1, \tag{6}$$

then $g(u)$ is the only vertex in $\Gamma'(u)$. Otherwise, $g(u)$ is the only vertex in $\overline{\Gamma}'(u)$.

Note that (6) either holds for all $u \in V(J^*)$ or for none of them. If (6) holds for all $u \in V(J^*)$, then we say that ψ' *agrees* with ψ; otherwise, we say that ψ' *disagrees* with ψ. Moreover, we have the following. Suppose that φ agrees with $\overrightarrow{\mathcal{D}}$. Then

$$\varphi' = \iota\left(\phi\left(J', \varphi'\right)\right) \circ \psi' \text{ if } \psi' \text{ agrees with } \psi$$

and

$$\varphi' = \mathfrak{c} \circ \iota\left(\phi\left(J', \varphi'\right)\right) \circ \psi' \text{ if } \psi' \text{ disagrees with } \psi.$$

Suppose that φ agrees with $\overleftarrow{\mathcal{D}}$. Then

$$\varphi' = \mathfrak{c} \circ \iota\left(\phi\left(J', \varphi'\right)\right) \circ \psi' \text{ if } \psi' \text{ agrees with } \psi$$

and

$$\varphi' = \iota\left(\phi\left(J', \varphi'\right)\right) \circ \psi' \text{ if } \psi' \text{ disagrees with } \psi.$$

By 3) of Theorem 2.7, g is an isomorphism from J^* to J'. The map

$$g' := \iota(\phi(J', \varphi'))^{-1} \circ \iota\left(\phi\left(J, \varphi\right)\right) : V(F_{k'}(J^*)) \to V(F_{k'}(J'))$$

is an isomorphism from $F_{k'}(J^*)$ to $F_{k'}(J')$. By Theorem 2.7, this map is the image under ι of some isomorphism from J^* to J'. By construction of g, we have

$$\iota(g) = g'.$$

We have the following diagram.

$$\begin{tikzcd}
F' \arrow[r, "\varphi'"] \arrow[dr, bend right, "\psi'"'] & F_{k'}(G') & F_{k'}(J^*) \arrow[l, "\iota(\phi(J,\varphi))"'] \arrow[dl, bend left, "g' = \iota(g)"] \\
& F_{k'}(J') \arrow[u, "\substack{\iota(\phi(J',\varphi')) \\ \text{or} \\ \mathfrak{c}\circ\iota(\phi(J',\varphi'))}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G' & J^* \arrow[l, "\phi(J,\varphi)"'] \arrow[dl, "g"] \\
J' \arrow[u, "\phi'(J',\varphi')"] &
\end{tikzcd}$$

We now define $\widehat{\pi}$ on the vertices of F'. For every vertex $X \in F'$ and every $1 \leq i \leq r$, let

$$\widehat{\pi}(X)(i) = \begin{cases} \left(\iota(g^{-1})\left(\psi'(X)\right) \cup T^*\right) \cap V(J_i), & \text{if } \psi' \text{ agrees with } \psi; \\ \left(\mathfrak{c} \circ \iota(g^{-1})\left(\psi'(X)\right) \cup T^*\right) \cap V(J_i), & \text{if } \psi' \text{ disagrees with } \psi. \end{cases}$$

We update ψ accordingly, so that

$$\psi(X) := \widehat{u} \circ \widehat{\pi}(X).$$

Let $A \in W$. Suppose that φ agrees with $\overrightarrow{\mathcal{D}}$. We have that

$$\begin{aligned}\varphi(X) \cap (G \setminus G') &= \varphi(A) \cap (G \setminus G') \\ &= \iota\left(\phi\left(J, \varphi\right)\right) \circ \psi(A) \cap (G \setminus G') \\ &= \iota\left(\phi\left(J, \varphi\right)\right) \circ \psi(X) \cap (G \setminus G').\end{aligned}$$

Suppose that ψ agrees with ψ'. Then, for all $1 \leq i \leq r$, we have that

$$\begin{aligned}\varphi(X) \cap (G' \cap G_i) &= \varphi'(X) \cap G_i \\ &= \big(\iota(\phi(J', \varphi')) \circ \psi'(X)\big) \cap G_i \\ &= \iota(\phi(J', \varphi'))(\iota(g)(\widehat{\pi}(X)(i) \setminus T^*)) \\ &= \iota(\phi(J', \varphi'))(\iota(\phi(J', \varphi'))^{-1} \circ \iota(\phi(J, \varphi)(\widehat{\pi}(X)(i) \setminus T^*)) \\ &= \iota(\phi(J, \varphi))(\widehat{\pi}(X)(i) \setminus T^*)) \\ &= \big(\iota(\phi(J, \varphi))(\widehat{\pi}(X)(i))\big) \cap G' \\ &= (\iota(\phi(J, \varphi)) \circ \widehat{\pi}(X) \cap G_i) \cap G' \\ &= \iota(\phi(J, \varphi)) \circ \widehat{\pi}(X) \cap (G_i \cap G')\end{aligned}$$

Suppose that ψ disagrees with ψ'. Then, for all $1 \leq i \leq r$, we have that

$$\begin{aligned}\varphi(X) \cap (G' \cap G_i) &= \varphi'(X) \cap G_i \\ &= \mathfrak{c} \circ \iota(\phi(J', \varphi')) \circ \psi'(X) \cap G_i \\ &= \mathfrak{c} \circ \iota(\phi(J', \varphi'))(\mathfrak{c} \circ \iota(g)(\widehat{\pi}(X)(i) \setminus T^*)) \\ &= \iota(\phi(J', \varphi'))(\iota(\phi(J', \varphi'))^{-1} \circ \iota(\phi(J, \varphi)(\widehat{\pi}(X)(i) \setminus T^*)) \\ &= \iota(\phi(J, \varphi))(\widehat{\pi}(X)(i) \setminus T^*)) \\ &= \big(\iota(\phi(J, \varphi))(\widehat{\pi}(X)(i))\big) \cap G' \\ &= (\iota(\phi(J, \varphi)) \circ \widehat{\pi}(X) \cap G_i) \cap G' \\ &= \iota(\phi(J, \varphi)) \circ \widehat{\pi}(X) \cap (G_i \cap G')\end{aligned}$$

Thus,

$$\varphi(X) = \iota\left(\phi\left(J, \varphi\right)\right) \circ \psi(X).$$

Suppose that φ agrees with $\overleftarrow{\mathcal{D}}$. We have that

$$\begin{aligned}\varphi(X) \cap (G \setminus G') &= \varphi(A) \cap (G \setminus G') \\ &= \mathfrak{c} \circ \iota\left(\phi\left(J, \varphi\right)\right) \circ \psi(A) \cap (G \setminus G') \\ &= \mathfrak{c} \circ \iota\left(\phi\left(J, \varphi\right)\right) \circ \psi(X) \cap (G \setminus G').\end{aligned}$$

Suppose that ψ agrees with ψ'. Then, for all $1 \leq i \leq r$, we have that

$$\begin{aligned}
\varphi(X) \cap (G' \cap G_i) &= \varphi'(X) \cap G_i \\
&= \mathfrak{c} \circ \iota(\phi(J', \varphi')) \circ \psi'(X) \cap G_i \\
&= \mathfrak{c} \circ \iota(\phi(J', \varphi'))(\iota(g)(\widehat{\pi}(X)(i) \setminus T^*)) \\
&= \mathfrak{c} \circ \iota(\phi(J', \varphi'))(\iota(\phi(J', \varphi'))^{-1} \circ \iota(\phi(J, \varphi)(\widehat{\pi}(X)(i) \setminus T^*)) \\
&= \mathfrak{c} \circ \iota(\phi(J, \varphi))(\widehat{\pi}(X)(i) \setminus T^*)) \\
&= \mathfrak{c} \circ \iota(\phi(J, \varphi))(\widehat{\pi}(X)(i))) \cap G' \\
&= (\mathfrak{c} \circ \iota(\phi(J, \varphi)) \circ \widehat{\pi}(X) \cap G_i) \cap G' \\
&= \mathfrak{c} \circ \iota(\phi(J, \varphi)) \circ \widehat{\pi}(X) \cap (G_i \cap G')
\end{aligned}$$

Suppose that ψ disagrees with ψ'. Then, for all $1 \leq i \leq r$, we have that

$$\begin{aligned}
\varphi(X) \cap (G' \cap G_i) &= \varphi'(X) \cap G_i \\
&= \iota(\phi(J', \varphi')) \circ \psi'(X) \cap G_i \\
&= \iota(\phi(J', \varphi'))(\mathfrak{c} \circ \iota(g)(\widehat{\pi}(X)(i) \setminus T^*)) \\
&= \mathfrak{c} \circ \iota(\phi(J', \varphi'))(\iota(\phi(J', \varphi'))^{-1} \circ \iota(\phi(J, \varphi)(\widehat{\pi}(X)(i) \setminus T^*)) \\
&= \mathfrak{c} \circ \iota(\phi(J, \varphi))(\widehat{\pi}(X)(i) \setminus T^*)) \\
&= \mathfrak{c} \circ \iota(\phi(J, \varphi))(\widehat{\pi}(X)(i))) \cap G' \\
&= (\mathfrak{c} \circ \iota(\phi(J, \varphi)) \circ \widehat{\pi}(X) \cap G_i) \cap G' \\
&= \mathfrak{c} \circ \iota(\phi(J, \varphi)) \circ \widehat{\pi}(X) \cap (G_i \cap G')
\end{aligned}$$

Thus,

$$\varphi(X) = \mathfrak{c} \circ \iota(\phi\,(J, \varphi)) \circ \psi(X).$$

This completes the proof. □

Next, we show how to define $\widehat{\pi}$ on the vertices in $\mathbf{Split}(s, i)$ under certain assumptions.

Lemma 2.43. *Let $1 \leq i \leq r$ such that $J \setminus J_i$ is connected. Suppose that there exists an integer $0 < s < |J_i|$ that satisfies the following.*

(a) *There exists a vertex $A \in F$ for which we have defined $\widehat{\pi}$ on all the vertices in $\mathbf{Move}(A, i)$ and such that $|\widehat{\pi}(A)(i)| = s$.*

(b) *Let W be the set of vertices of $\mathbf{Split}(s, i)$ for which we have defined ψ. For every vertex $u \in J \setminus J_i$, we can compute in polynomial time a Boolean combination $\Gamma(u)$ of elements in $\{\psi(B) : B \in W\}$, such that*

$$\{u\} = V(J \setminus J_i) \cap \Gamma(u).$$

Then we can define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(s, i)$.

Proof. Note that

$$V(\mathbf{Split}(s,i)) = \bigcup_{B \in \mathbf{Move}(A,i)} V(\mathbf{Fix}(B,i)).$$

Then, it is sufficient to show that we can define $\widehat{\pi}$ on all the vertices of $\mathbf{Fix}(B,i)$, for any $B \in \mathbf{Move}(A,i)$.

Fix a vertex $B \in \mathbf{Move}(A,i)$. Let $F' := \mathbf{Fix}(B,i)$, $J^* := J \setminus J_i$ and $T^* := \widehat{\pi}(B)(i)$. Then, we have conditions (2) and (3) of Lemma 2.42.

Let $(B = B_1, B_2, \ldots, B_l)$ be a walk in $\mathbf{Move}(A,i)$ containing each vertex in $\mathbf{Move}(A,i)$. Since $0 < s < |J_i|$, we have $|\mathbf{Move}(A,i)| > 1$ and so $l > 1$. Let E_A be the set of edges in $\mathbf{Move}(A,i)$ incident to A. Let U be the set of vertices $C \in F$ for which there exists a path $(B = C_1, C_2, \ldots, C_m = C)$ satisfying the following:

(i) No edge C_iC_{i+1} is in the same ladder class as an edge in E_B.

(ii) For every C_i there is a walk $C_i = D_1, D_2, \ldots, D_l$ in F such that each D_jD_{j+1} is in the same ladder as A_jA_{j+1}.

Conditions (i) and (ii) together imply that in the path $(\varphi(C_1), \varphi(C_2), \ldots, \varphi(C_m))$ no token has been placed at G_i or moved from $\varphi(B) \cap G_i$, so the set U is precisely $F' = \mathbf{Fix}(B,i)$; then, we have condition (1) of Lemma 2.42. By (b) we also have (4) of Lemma 2.42. Thus, by Lemma 2.42 we can define $\widehat{\pi}$ on every vertex of $F' = \mathbf{Fix}(B,i)$, as we wanted. □

In the following result we show that having defined $\widehat{\pi}$ on the vertices of $\mathbf{Split}(s,i)$, we can define $\widehat{\pi}$ on $\mathbf{Split}(t,j)$, for certain values of t and certain subgraphs J_i and J_j.

Corollary 2.44. *Let J_i and J_j be such that $J \setminus J_i$ and $J \setminus J_j$ are both connected. Let s be an integer such that*

$$1 \leq s \leq |J_i| - 1 \text{ and } 2 \leq k' - s \leq n - |J_i| - 2.$$

Suppose that we have defined $\widehat{\pi}$ on every vertex of $\mathbf{Split}(s,i)$. Let t be an integer such that

$$1 \leq t \leq |J_j| - 1 \text{ and } 1 \leq k' - (s+t) \leq n - |J_i| - |J_j| - 1.$$

Then we can define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(t,j)$.

Proof. Let us first note that the values s and t are such that, at the same time, we can move s tokens on J_i, t tokens on J_j and $k'-(s+t)$ tokens on $J \setminus (J_i \cup J_j)$, where $s, t, k'-(s+t) \geq 1$.

Since $t \leq k' - s$, there exists a vertex $A \in \mathbf{Split}(s,i)$ such that $|\widehat{\pi}(A)(j)| = t$. On the other hand, $\mathbf{Move}(A,j)$ is a subgraph of $\mathbf{Split}(s,i)$, so we have defined $\widehat{\pi}$ on every vertex of $\mathbf{Move}(A,j)$. Let $u \in V(J \setminus J_j)$. Let $W := V(\mathbf{Split}(s,i)) \cap V(\mathbf{Split}(t,j))$. Note that we have defined $\widehat{\pi}$ on every vertex of W. Let $S_u \subseteq W$ such that $u \in \psi(B)$ and

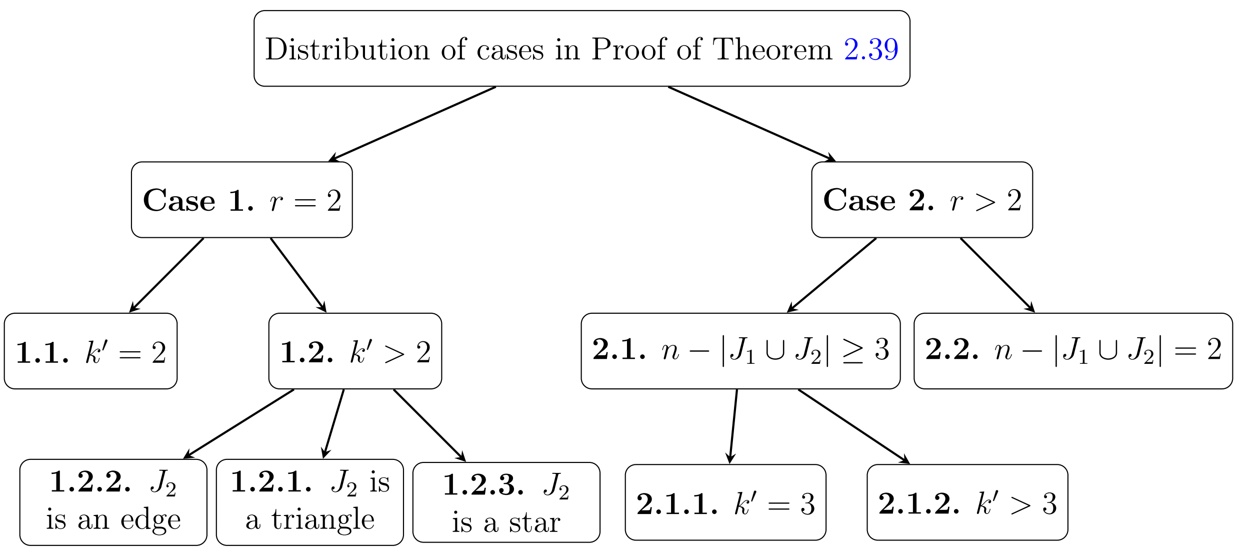
$$\Gamma(u) := \bigcap_{B \in S_u} \psi(B).$$

Since $1 \leq s \leq |J_i| - 1$ and $1 \leq k' - (s+t) \leq |J| - |J_i| - |J_2| - 1$ we have that

$$\{u\} = V(J \setminus J_j) \cap \Gamma(u).$$

By Lemma 2.43 we can define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(t,j)$. □

Now, we are ready to show Theorem 2.39. In the proof of Theorem 2.39, we follow the next distribution of cases.



Proof of Theorem 2.39. We proceed by induction on n. Suppose that Theorem 2.39 holds for smaller values of n. By Lemma 2.41, we may assume that $n \geq 7$. If $r = 1$, then there is nothing to show since $F = H$ in this case. Assume that $r \geq 2$. We consider two cases: $r = 2$ and $r > 2$.

Case 1. *Suppose that* $r = 2$.

Here we have two cases: either $k' = 2$ or $k' > 2$.

1.1. *Suppose that* $k' = 2$.

We have that $k = 2$ (resp. $k = n-2$). Note that $F_k(G) \setminus \varphi(H)$ has two components, corresponding on whether there are two (resp. $|G_1| - 2$) tokens at G_1 or there are two tokens (resp. $|G_2| - 2$) at G_2; let C_1 and C_2 be these components of $F_k(G) \setminus \varphi(H)$, respectively. Let F_1 and F_2 be the components of $F \setminus H$ such that $\varphi(F_1) = C_1$ and $\varphi(F_2) = C_2$. We prove that we can define $\widehat{\pi}$ on every vertex of F_1, by showing that it satisfies the conditions of Lemma 2.42 with $F' = F_1$; by similar arguments we can define $\widehat{\pi}$ on every vertex of F_2.

Let $uv \in E(J_1, J_2)$ and let $A \in H$ be a vertex such that $u \notin \widehat{\pi}(A)(1)$ and $v \in \widehat{\pi}(A)(2)$. By Lemma 2.37 we can find in polynomial time the vertex $B \in F \setminus H$ such that $\varphi(B)$ is obtained from $\varphi(A)$ by sliding a token along $\phi(J,\varphi)(u)\phi(J,\varphi)(v)$. Note that $\varphi(B) \in C_1$; thus, $B \in F_1$ and $F_1 = \mathbf{Move}(B, 1)$. Therefore, we have condition (2) and (3) of Lemma 2.42, where $J^* = J_1$ and $T^* = \emptyset$ in this case. Since there are no edges between F_1 and F_2, we also have condition (1) of Lemma 2.42. Let

$$S := \{X \in H : u \notin \widehat{\pi}(X)(1) \text{ and } v \in \widehat{\pi}(X)(2)\}.$$

For each $X \in S$, let $B_X \in F_1$ be the vertex such that $\varphi(B_X)$ is obtained from $\varphi(X)$ by sliding a token along $\phi(J,\varphi)(u)\phi(J,\varphi)(v)$. Let

$$S' := \{B_X : X \in S\}.$$

Define $\widehat{\pi}(B_X)(1) := \widehat{\pi}(X)(1) \cup \{u\}$ and $\widehat{\pi}(B_X)(2) := \emptyset$, so we have defined $\widehat{\pi}$ on every vertex in S'. If S' contains all the vertices in F_1 then we are done. Suppose that S' is a proper subset of F_1. This implies that

$$\{u\} = V(J_1) \cap \bigcap_{B_X \in S'} \psi(B_X).$$

Let w be a vertex of J_1 distinct from u. Let X be the vertex of H such that $\widehat{\pi}(X)(1) = w$ and $\widehat{\pi}(X)(2) = v$. Note that $B_X \in S'$, and

$$\{w\} = V(J_1) \cap (\psi(B_X) \setminus \{u\}).$$

Thus, we have condition (4) of Lemma 2.42 and we can define $\widehat{\pi}$ on all the vertices of F_1.

1.2. *Suppose that $k' > 2$.*

Assume that $2 < k < n-2$. By Lemma 2.27 J_1 and J_2 are either edges, triangles or stars. Since we are assuming that $n > 6$ then at least one of them is a star with more than three vertices. Without loss of generality assume it is J_1 and let u be its center. We proceed by cases on whether J_2 is either an edge, a triangle or a star of at least three vertices.

1.2.1. *J_2 is an edge.*

For $i = 0, 1, 2$ let F_i be the subgraph of F induced by the vertices A such that set

$$|\varphi(A) \cap G_2| = \begin{cases} i & \text{if } \varphi \text{ agrees with } \overrightarrow{\mathcal{D}}; \\ 2 - i & \text{if } \varphi \text{ agrees with } \overleftarrow{\mathcal{D}}. \end{cases}$$

Note that $H = V(F_1)$ and that $F_k(G)\setminus\varphi(H)$ has two components, F_0 and F_2. Let $xy \in E(J_1, J_2)$. Let A be a vertex of H such that $x \in \widehat{\pi}(A)(1)$ (resp. $x \notin \widehat{\pi}(A)(1)$) and $y \notin \widehat{\pi}(A)(2)$ (resp. $y \in \widehat{\pi}(A)(2)$). By Lemma 2.37 we can compute in polynomial time the vertex $B \in F$ such that $\varphi(B)$ is obtained from $\varphi(A)$ by sliding a token along the edge $\phi(J,\varphi)(x)\phi(J,\varphi)(y)$. Note that $B \in F_2$ (resp. $B \in F_0$), so this allows us to determine which vertices of F belong to F_0 and which to F_2. Note that $F_2 = \mathbf{Move}(B,1)$ (resp. $F_0 = \mathbf{Move}(B,1)$), and so we have conditions (1), (2) and (3) of Lemma 2.42 with $F' = F_2$, $J^* = J_1$ and $T^* = V(J_2)$ (resp. $F' = F_0$, $J^* = J_1$ and $T^* = \emptyset$). Let

$$S := \{X \in H : x \in \widehat{\pi}(X)(1) \text{ and } y \notin \widehat{\pi}(X)(2)\}$$

$$(\text{resp. } S := \{X \in H : x \notin \widehat{\pi}(X)(1) \text{ and } y \in \widehat{\pi}(X)(2)\})\,.$$

For every such vertex $X \in S$ let $B_X \in F_2$ ($B_X \in F_0$) be the vertex such that $\varphi(B_X)$ is obtained from $\varphi(X)$ by sliding a token along the edge $\phi(J,\varphi)(x)\phi(J,\varphi)(y)$. Let

$$S' := \{B_X : X \in S\}.$$

Define $\widehat{\pi}(B_X)(1) := \widehat{\pi}(X)(1) \setminus \{x\}$ and $\widehat{\pi}(B_X)(2) := V(J_2)$ (resp. $\widehat{\pi}(B_X)(1) := \widehat{\pi}(X)(1) \cup \{x\}$ and $\widehat{\pi}(B_X)(2) := \emptyset$). In order to use Lemma 2.42, it remains to show that condition (4) of Lemma 2.42 holds. We have

$$\{x\} = V(J_1) \setminus \bigcup_{B_X \in S'} \psi(B_X) \quad \left(\text{resp. } \{x\} = V(J_1) \cap \bigcap_{B_X \in S'} \psi(B_X)\right).$$

Let $w \in J_1 \setminus \{x\}$ and let $S'_w := \{B \in S' : w \in \widehat{\pi}(B)(1)\}$ (resp. $S'_w := \{B \in S' : w \notin \widehat{\pi}(B)(1)\}$). We have

$$\{w\} = V(J_1) \cap \bigcap_{B_X \in S'_w} \psi(B_X) \quad \left(\text{resp. } \{w\} = V(J_1) \setminus \bigcup_{B_X \in S'_w} \psi(B_X)\right).$$

Thus, by Lemma 2.42 we can define $\widehat{\pi}$ on all the vertices of F_2 (resp. F_0).

1.2.2. *J_2 is a triangle.*

By Lemma 2.28 no leave of J_1 is adjacent to a vertex of J_2, so all the $J_1 - J_2$ edges contain u as an endpoint. Moreover, since G is (C_4, D_4)-free, u cannot be adjacent to more than one vertex in J_2, so, there is only one $J_1 - J_2$ edge. Let $V(J_2) = \{v_1, v_2, v_3\}$ and without loss of generality assume that u is adjacent to v_3. Let $e \in E(H)$ such that $\varphi(e)$ is generated by sliding a token along the edge $\phi(J,\varphi)(v_1)\phi(J,\varphi)(v_2)$. Then, the ladder class containing e, say $\mathcal{C}_e$, contains all the vertices A in F such that in $\varphi(A)$ there is a token at one of $\phi(J,\varphi)(v_1)$ or $\phi(J,\varphi)(v_2)$. Let H' be the set of vertices A in F incident to some edge in the ladder class $\mathcal{C}_e$. Note that $H' \simeq H'_1 \,\square\, H'_2$, where $H'_2 = K_2$ and $H'_1 = F_{k_1}(K_{1,|J_1|})$ if $k_2 = 1$, and $H'_1 = F_{k_1+1}(K_{1,|J_1|})$ if $k_2 = 2$. Developing all our theoretic framework for H' instead of H, we may assume that J_2 is an edge, and so we are in the previous case.

1.2.3. *J_2 is a star on at least three vertices.*

Let v be the center of J_2. By Lemma 2.27 no pair of leaves of J_1 and J_2 are adjacent. Therefore, either all $J_1 - J_2$ edges contain u as an endpoint or all $J_1 - J_2$ edges contain v as an endpoint. Assume without loss of generality that all the $J_1 - J_2$ edges contain u as an endpoint. Besides, no two leaves x and y of J_2 are adjacent to u, because G is a (C_4, D_4)-free graph. Therefore, $|E(J_1, J_2)|$ is equal to one or two. Let $x \in J_2$ be a leaf such that the neighbours of u in J_2 are contained in $\{v, x\}$.

For $0 \leq i \leq \min\{k', |J_1|\}$, let F_i be the subgraph of F induced by the vertices A such that

$$|\varphi(A) \cap G_1| = \begin{cases} i & \text{if } \varphi \text{ agrees with } \overrightarrow{\mathcal{D}}; \\ |G_1| - i & \text{if } \varphi \text{ agrees with } \overleftarrow{\mathcal{D}}. \end{cases}$$

Note that $H = F_i$ for some $0 < i < \min\{k', |J_1|\}$. Suppose that we have defined $\widehat{\pi}$ on the vertices of F_l for some $0 < l < \min\{k, |J_1|\}$. We show that we can define $\widehat{\pi}$ on the vertices of F_{l+1} (resp. F_{l-1}).

Let S' be the set of vertices $X \in F_l$ such that X has a neighbour in $F \setminus F_l$ and $u \notin \widehat{\pi}(X)(1)$ (resp. $u \in \widehat{\pi}(X)(1)$). Note that $v \in \widehat{\pi}(X)(2)$ or $x \in \widehat{\pi}(X)(2)$ (resp. $v \notin \widehat{\pi}(X)(2)$ or $x \in \widehat{\pi}(X)(2)$). Let $B_X \in F \setminus F_l$ be a neighbour of X, then $B_X \in F_{l+1}$ (resp. $B_X \in F_{l-1}$) and $\varphi(B_X)$ is obtained from $\varphi(X)$ by sliding a token along one of $\phi(J, \varphi)(u)\phi(J, \varphi)(v)$ or $\phi(J, \varphi)(u)\phi(J, \varphi(x))$. We can distinguish between these two possible cases as follows.

The edge XB_X belongs to a 4-cycle of F if and only if u is adjacent to x and $\varphi(B_X)$ is obtained from $\varphi(X)$ by sliding a token along the edge $\phi(J, \varphi)(u)\phi(J, \varphi)(x)$.

This is due to the fact that the edge XB_X is contained in a 4-cycle of F if and only if there are two disjoint edges e_1 and e_2 in G and $\varphi(XB_X)$ is generated by moving a token along one of these edges. This last holds if and only if $\varphi(XB_X)$ is generated by moving a token along the edge $\phi(J, \varphi)(u)\phi(J, \varphi)(x)$.

Let

$$S := \{B_X \in F \setminus F_l : B_X \text{ is a neighbour of some } X \in S'\}.$$

Define

$$\widehat{\pi}(B_X)(1) = \widehat{\pi}(X)(1) \cup \{u\} \quad (\text{resp. } \widehat{\pi}(B_X)(1) = \widehat{\pi}(X)(1) \setminus \{u\})$$

and

$$\widehat{\pi}(B_X)(2) = \begin{cases} \widehat{\pi}(X)(2) \setminus \{x\} \text{ (resp. } \widehat{\pi}(X)(2) \cup \{x\}) & \text{if } XB_X \text{ belongs to a 4-cycle of } F, \\ \widehat{\pi}(X)(2) \setminus \{v\} \text{ (resp. } \widehat{\pi}(X)(2) \cup \{v\}) & \text{otherwise.} \end{cases}$$

Now, for a fixed vertex $B_X \in S$, let S_{B_X} be the subset of S such that for any $B \in S_{B_X}$ we have $\widehat{\pi}(B)(2) = \widehat{\pi}(B_X)(2)$. We show that we can define $\widehat{\pi}$ on every vertex in $\mathbf{Move}(B_X, 1)$ by means of Lemma 2.42. We have conditions (2) and (3) of Lemma 2.42, where $F' = \mathbf{Move}(B_X, 1)$, $J^* = J_1$ and $T^* = \widehat{\pi}(B_X)(2)$ in this case. If $l = |J_1| - 1$ (resp. $l = 1$) then $F_{l+1} = \{B_X\} = \mathbf{Move}(B_X, 1)$ (resp. $F_{l-1} = \{B_X\} = \mathbf{Move}(B_X, 1)$) and then we are done. Assume that $l < |J_1| - 1$ (resp. $l > 1$).

Next we show how to satisfy condition (1) of Lemma 2.42. Note that $S_{B_X} \subset \mathbf{Move}(B_X, 1)$ and so, we have determined every vertex $B \in \mathbf{Move}(B_X, 1)$ such that $\phi(J, \varphi)(u) \in \varphi(B)$ (resp. $\phi(J, \varphi)(u) \notin \varphi(B)$). Let A be a neighbour of B_X not in F_l. We have that $\varphi(A)$ is obtained from $\varphi(B_X)$ by either moving a token along an edge of G_1 or by moving a token along an edge of G_2. Note that the former case holds if and only if A has at least two neighbours in S_{B_X}, and in such a case $A \in \mathbf{Move}(B_X, 1)$. Moreover,

$$\mathbf{Move}(B_X, 1) = S_{B_X} \cup \{A \in F \setminus F_l : A \text{ has at least two neighbours in } S_{B_X}\}.$$

Thus, we can determine in polynomial time which vertices of F belong to $F' = \mathbf{Move}(B_X, 1)$ and so, condition (1) of Lemma 2.42 is satisfied.

Besides,

$$\{u\} = V(J_1) \cap \bigcap_{B \in S_{B_X}} \psi(B) \quad \left(\text{resp. } \{u\} = V(J_1) \setminus \bigcup_{B \in S_{B_X}} \psi(B)\right).$$

For $w \in J_1 \setminus \{u\}$ let $S_w := \{B \in S_{B_X} : w \notin \widehat{\pi}(B)(1)\}$ (resp. $S_w := \{B \in S_{B_x} : w \in \widehat{\pi}(B)(1)\}$), then

$$\{w\} = V(J_1) \setminus \bigcup_{B \in S_w} \psi(B) \quad \left(\text{resp. } \{w\} = V(J_1) \cap \bigcap_{B \in S_{B_X}} \psi(B)\right).$$

Thus, condition (4) of Lemma 2.42 holds, as we wanted, and so, by Lemma 2.42 we can define $\widehat{\pi}$ on every vertex in $\mathbf{Move}(B_X, 1)$.

Next, note that condition (a) of Lemma 2.43 holds for $A := B_X \in S$, $J_i = J_1$ and $s = l+1$ (resp. $s = l-1$). Let us now show that condition (b) of Lemma 2.43 also holds. If $k - s = 0$ (resp. $k - s = |J_2|$) then $\mathbf{Split}(A, s) = \mathbf{Move}(A, 1)$ and we are done in this case. Let us assume then that $k - s > 0$ (resp. $k - s < |J_2|$). Let $y \in J_2$. Consider the following cases.

- If y is adjacent to u, let $S_y := \{B \in S : y \notin \widehat{\pi}(B)(2)\}$ (resp. $S_y := \{B \in S : y \in \widehat{\pi}(B)(2)\}$). We have

$$\{y\} = V(J_2) \setminus \bigcup_{B \in S_y} \psi(B) \quad \left(\text{resp. } \{y\} = V(J_2) \cap \bigcap_{B \in S_y} \psi(B)\right).$$

- If y is not adjacent to u, let $S_y := \{B \in S : y \in \widehat{\pi}(B)(2)\}$ (resp. $S_y := \{B \in$

$S : y \notin \widehat{\pi}(B)(2)\}$). We have

$$\{y\} = V(J_2) \cap \bigcap_{B \in S_y} \psi(B) \quad \left(\text{resp. } \{y\} = V(J_2) \setminus \bigcup_{B \in S_y} \psi(B)\right).$$

Thus, condition (b) of Lemma 2.43 is satisfied, and then we can define $\widehat{\pi}$ on every vertex in $\mathbf{Split}(A, s) = F_{l+1}$ (resp. $\mathbf{Split}(A, s) = F_{l-1}$).

Case 2. *Suppose that $r > 2$.*

Let P be the graph whose vertex set is $\{J_1, \dots, J_r\}$ and where J_i is adjacent to J_j if and only if $E(J_i, J_j) \neq \emptyset$. Since F is connected, P is connected. Therefore, there exist at least two vertices of P, say J_1 and J_2, such that $P \setminus (J_1 \cup J_2)$ is connected. Let A be a vertex of H, and let $s := |\widehat{\pi}(A)(1)|$ and $t := |\widehat{\pi}(A)(2)|$. Note that $\mathbf{Move}(A, i) \subset H$ and so, we have defined $\widehat{\pi}$ on every vertex in $\mathbf{Move}(A, i)$. Then, condition (a) of Lemma 2.43 is satisfied. Next we show how to satisfy condition (b) of Lemma 2.43. Let $W = H$, so we have defined $\widehat{\pi}$ on every vertex in W. Consider a vertex $u \in J \setminus J_i$, and let $j \in [r]$ such that $u \in J_j$. Let $W_u := \{B \in W : u \in \widehat{\pi}(B)(j)\}$. Then,

$$\{u\} = V(J \setminus J_i) \cap \bigcap_{B \in W_u} \psi(B).$$

Thus, the following is a consequence of Lemma 2.43.

Remark 2.45. We can define $\widehat{\pi}$ on every vertex in $\mathbf{Split}(s, i)$. Similarly, we can define $\widehat{\pi}$ on every vertex in $\mathbf{Split}(t, j)$.

And then, by Corollary 2.44 we can define $\widehat{\pi}$ on all the vertices of $\mathbf{Split}(t, j)$. From now on, without loss of generality we assume $i = 1$ and $j = 2$.

2.1. *Suppose that $n - |J_1 \cup J_2| \geq 3$.*

2.1.1. *Suppose that $k' = 3$.*

Note that $r = 3$ and that $|J_3| \geq 3$. Let $uv \in E(J_1, J_3)$ and let $A \in V(H)$ such that $u \in \widehat{\pi}(A)(1)$ and $v \notin \widehat{\pi}(A)(3)$. By Lemma 2.37, we can identify in polynomial time the vertex $B \in F$ such that $\varphi(B)$ is obtained from $\varphi(A)$ by sliding a token along $\phi(J, \varphi)(u)\phi(J, \varphi)(v)$. Note that $B \in \mathbf{Split}(0, 1)$. Moreover, vertices in $\mathbf{Split}(0, 1)$ can only be adjacent to vertices in $\mathbf{Split}(0, 1)$ or in $\mathbf{Split}(1, 1)$. Therefore, $\mathbf{Split}(0, 1)$ is the component of $F \setminus \mathbf{Split}(1, 1)$ containing B. Thus, we have conditions (1), (2) and (3) of Lemma 2.42, where $F' = \mathbf{Split}(0, 1)$, $J^* = J \setminus J_1$ and $T^* = \emptyset$. For every vertex w in $J \setminus J_1$, let $S_w := \{C \in V(\mathbf{Split}(0, 1)) \cap V(\mathbf{Split}(1, 2)) : w \in \psi(C)\}$. Since $|J_3| \geq 3$ and $k' = 3$ we have that

$$\{w\} = V(J^*) \cap \bigcap_{C \in S_w} \psi(C);$$

and we have condition (4) of Lemma 2.42. Thus, we can define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(0,1)$. By similar arguments we can define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(0,2)$. Since

$$V(F) = V(\mathbf{Split}(0,1)) \cup V(\mathbf{Split}(1,1)) \cup V(\mathbf{Split}(0,2)) \cup V(\mathbf{Split}(1,2)),$$

we have defined $\widehat{\pi}$ on every vertex of F.

2.1.2. *Suppose that $k' > 3$.*

Let

$$s_{\min} := \max\{1, k' - (n - |J_1| - 2)\} \text{ and } s_{\max} := \min\{k' - 2, |J_1| - 1\}.$$

We have that

$$2 \leq r - 1 \leq |\psi(A) \setminus V(J_1)| \leq n - |J_1| - (r-1) \leq n - |J_1| - 2.$$

Since $k' - s = |\psi(A) \setminus V(J_1)|$ we have that $s_{\min} \leq s \leq s_{\max}$. Note that every s with $s_{\min} \leq s \leq s_{\max}$ satisfies that

$$1 \leq s \leq |J_1| - 1 \text{ and } 2 \leq k' - s \leq n - |J_1| - 2. \tag{7}$$

Let

$$t_{\min} := \max\{1, k' - (n - |J_2| - 2)\} \text{ and } t_{\max} := \min\{k' - 2, |J_2| - 1\}.$$

We have $t_{\min} \leq t \leq t_{\max}$. Note that every t with $t_{\min} \leq t \leq t_{\max}$ satisfies that

$$1 \leq t \leq |J_2| - 1 \text{ and } 2 \leq k' - t \leq n - |J_1| - 2. \tag{8}$$

Without loss of generality let us assume that $|J_1| \leq |J_2|$. Since $n - |J_1| - |J_2| \geq 3$, we have $|J_1| \leq \frac{n}{2} - \frac{3}{2}$. Therefore $k' - (n - |J_1| - 2) \leq \frac{n}{2}$, since $k' \leq \frac{n}{2}$ by assumption. Thus, $s_{\min} = 1$. Then, let us show the following.

Claim 2.46. *Let s (resp. t) such that $s_{\min} \leq s \leq s_{\max}$ (resp. $t_{\min} \leq t \leq t_{\max}$). Suppose that we have defined $\widehat{\pi}$ on all the vertices in $\mathbf{Split}(s,1)$ (resp. $\mathbf{Split}(t,2)$). We have the following.*

(a) If $s_{\min} < s$ (resp. $t_{\min} < t$) then we can define $\widehat{\pi}$ on all the vertices in $\mathbf{Split}(s-1,1)$ (resp. $\mathbf{Split}(t-1,2)$); and,

(b) if $s < s_{\max}$ (resp. $t < t_{\max}$) then we can define $\widehat{\pi}$ on all the vertices in $\mathbf{Split}(s+1,1)$ (resp. $\mathbf{Split}(t+1),2$).

Proof of Claim 2.46. We proof the claim for s, and note that the result for t follows similarly. We proceed by cases according to the claim.

(a) Suppose that $s_{\min} < s$.

Let

$$l = \min\{k' - s - 1, |J_2| - 1\}.$$

Since $k'-s-1 \geq 1$ and $|J_2|-1 \geq 1$, we have that $l \geq 1$. Since $l \leq k'-s-1$, we have that $k'-(s+l) \geq 1$. If $l = |J_2|-1$, then $k'-(s+l) \leq n-|J_1|-|J_2|-1$. If $l = k'-s-1$, then $k'-(s+l) = 1 < n-|J_1|-|J_2|-1$. Thus,

$$1 \leq l \leq |J_2|-1 \text{ and } 1 \leq k'-(s+l) \leq n-|J_1|-|J_2|-1. \tag{9}$$

By (7), (9) and Corollary 2.44 we can define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(l,2)$.

Since $k'-(s+l) \geq 1$ and $s > s_{\min} \geq 1$, we have that $k'-l \geq 1+s > 2$. Since $k'-(s+l) \leq n-|J_1|-|J_2|-1$, we have that $k'-l \leq n-|J_1|-|J_2|-1+s \leq n-|J_1|-2$. Thus,

$$1 \leq l \leq |J_2|-1 \text{ and } 2 \leq k'-l \leq n-|J_2|-2. \tag{10}$$

If $l = k'-s-1$, then $k'-(l+s-1) = 2 \leq n-|J_2|-|J_1|-1$. Suppose that $l = |J_2|-1$. Since $s \geq k'-(n-|J_1|-2)+1$ we have that $k'-(l+s-1) \leq n-|J_2|-|J_1|-1$. Thus,

$$1 \leq s-1 \leq |J_1|-1 \text{ and } 1 \leq k'-(l+s-1) \leq n-|J_2|-|J_1|-1. \tag{11}$$

By (10), (11) and Corollary 2.44 we can define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(s-1,1)$.

(*b*) Suppose $s < s_{\max}$.

Let

$$l = \max\{1, (k'-s)-(n-|J_1|-|J_2|-1)\}.$$

Suppose that $l = (k'-s)-(n-|J_1|-|J_2|-1)$. Since $k'-s \leq n-|J_1|-2$, we have that $l \leq |J_2|-1$. Suppose that $l = 1$. Since $s < s_{\max} \leq k'-2$ we have that $k'-(s+l) \geq 2$. If $l = (k'-s)-(n-|J_1|-|J_2|-1)$, then $k'-(s+l) = n-|J_1|-|J_2|-1 \geq 2$. Since $l \geq (k'-s)-(n-|J_1|-|J_2|-1)$, we have that $k'-(s+l) \leq n-|J_1|-|J_2|-1$. Thus,

$$1 \leq l \leq |J_2|-1 \text{ and } 2 \leq k'-(s+l) \leq n-|J_1|-|J_2|-1. \tag{12}$$

By (7), (12) and Corollary 2.44 we can define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(l,2)$.

Since $k'-(s+l) \geq 1$, we have that $k'-l \geq 1+s \geq 2$. Since $k'-(s+l) \leq n-|J_1|-|J_2|-1$, we have that $k'-l \leq n-|J_1|-|J_2|-1+s < n-|J_2|-2$. Thus,

$$1 \leq l \leq |J_2|-1 \text{ and } 2 \leq k'-l \leq n-|J_2|-2. \tag{13}$$

Since $s < s_{\max} \leq |J_2|-1$ and (12), we have that

$$1 \leq s+1 \leq |J_1|-1 \text{ and } 1 \leq k'-(l+s+1) \leq n-|J_2|-|J_1|-2. \tag{14}$$

By (13), (14) and Corollary 2.44 we can define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(s+1,1)$.

$\triangle$

Next, we show that, for some special values of $s_{\max}$ and $t_{\max}$, we can define $\widehat{\pi}$ on every vertex in $\mathbf{Split}(s_{\max}+1,1)$ or in $\mathbf{Split}(t_{\max}+1,2)$.

Claim 2.47. *If* $t_{\max} = |J_2| - 1$ *(resp.* $s_{\max} = |J_1| - 1$*) then we can define* $\widehat{\pi}$ *on every vertex of* $\mathbf{Split}(t_{\max} + 1, 2)$ *(resp.* $\mathbf{Split}(s_{\max} + 1, 2)$*).*

Proof of Claim 2.47. We show the claim for $\mathbf{Split}(t_{\max} + 1, 2)$. The result for $\mathbf{Split}(s_{\max} + 1, 1)$ is analogous.

Note that vertices in $\mathbf{Split}(t_{\max} + 1, 2)$ can only be adjacent to vertices in $\mathbf{Split}(t_{\max}, 2)$ or in $\mathbf{Split}(t_{\max} + 1, 2)$. Therefore, $\mathbf{Split}(t_{\max} + 1, 2)$ is a component of $F \setminus \mathbf{Split}(t_{\max}, 2)$. Since $n - |J_1| - |J_2| \geq 3$, we have

$$V(\mathbf{Split}(1, 1)) \cap V(\mathbf{Split}(t_{\max} + 1, 2))) \neq \emptyset.$$

Since we have defined $\widehat{\pi}$ on $V(\mathbf{Split}(1, 1))$ we can identify a vertex

$$A \in V(\mathbf{Split}(1, 1)) \cap V(\mathbf{Split}(t_{\max} + 1, 2)).$$

Therefore we can determine in polynomial time which vertices of F belong to $\mathbf{Split}(t_{\max} + 1, 2)$. Thus, we have conditions (1), (2) and (3) of Lemma 2.42, where $F' = \mathbf{Split}(t_{\max} + 1, 2)$, $J^* = J \setminus J_2$ and $T^* = V(J_2)$. For every vertex u in $J \setminus J_2$, let

$$S_u := \{B \in V(\mathbf{Split}(1, 1)) \cap V(\mathbf{Split}(t_{\max} + 1, 2)) : u \in \psi(B)\}.$$

Note that

$$\{u\} = V(J^*) \cap \left(\bigcap_{B \in S_u} \psi(B) \right);$$

and so, we have condition (4) of Lemma 2.42. Thus, we can define $\widehat{\pi}$ on all the vertices in $\mathbf{Split}(t_{\max} + 1, 2)$. △

Recall that $s_{\min} = 1$, so now let us show that we can define $\widehat{\pi}$ on the vertices of $\mathbf{Split}(0, 1)$.

Claim 2.48. *If* $s_{\min} = 1$ *(resp.* $t_{\min} = 1$*), then we can define* $\widehat{\pi}$ *on all the vertices of* $\mathbf{Split}(0, 1)$ *(resp.* $\mathbf{Split}(0, 2)$*).*

Proof of Claim 2.48. We show the claim for $\mathbf{Split}(0, 1)$. The result for $\mathbf{Split}(0, 2)$ is similar.

Note that

$$V(\mathbf{Split}(0, 1)) \cap V(\mathbf{Split}(t_{\max}, 2)) \neq \emptyset,$$

and so, we can identify a vertex $A \in V(\mathbf{Split}(0, 1)) \cap V(\mathbf{Split}(t_{\max}, 2))$. Thus, $\mathbf{Split}(0, 1)$ is the component of $F \setminus \mathbf{Split}(1, 1)$ containing the vertex A. Then, we can identify in polynomial time which vertices of F belong to $\mathbf{Split}(0, 1)$, and so, we have conditions (1), (2) and (3) of Lemma 2.42, where $F' = \mathbf{Split}(0, 1)$, $J^* = J \setminus J_1$ and $T^* = \emptyset$. It remains to show that condition (4) is also satisfied.

Note that $k' - (s_{\min} + t_{\max}) = k' - (1 + t_{\max}) \leq n - |J_1| - |J_2| - 1$, and that

$$k' - (1 + t_{\max}) = n - |J_1| - |J_2| - 1 \iff t_{\max} = |J_2| - 1.$$

Consider the following cases.

- Suppose that $t_{\max} = k' - 2 < |J_2| - 1$.

  For $u \in J \setminus J_1$ let

  $$S_u := \{B \in V(\mathbf{Split}(0,1)) \cap V(\mathbf{Split}(t_{\max},2)) : u \in \psi(B),$$

  then

  $$\{u\} = V(J^*) \cap \left(\bigcap_{B \in S_u} \psi(B) \right);$$

  and so, condition (4) of Lemma 2.42 is satisfied.

- Suppose that $t_{\max} = |J_2| - 1$.

  We have $k' = n - |J_1| - 1$. By Claim 2.47 we can define $\widehat{\pi}$ on the vertices in $\mathbf{Split}(t_{\max}+1, 2)$. Define $\widehat{\pi}$ on the vertices in $\mathbf{Split}(t_{\max}+1, 2)$. Then, we have defined $\widehat{\pi}$ on all the vertices in $V(\mathbf{Split}(t_{\max}, 2)) \cup V(\mathbf{Split}(t_{\max}+1, 2))$.

  Consider a vertex u in $J \setminus J_1$, let

  $$S_u := \begin{cases} \{B \in V(\mathbf{Split}(0,1)) \cap V(\mathbf{Split}(t_{\max},2)) : u \in \psi(B)\} & \text{if } u \in J_2, \\ \{B \in V(\mathbf{Split}(0,1)) \cap V(\mathbf{Split}(t_{\max}+1,2)) : u \in \psi(B)\} & \text{otherwise.} \end{cases}$$

  Then,

  $$\{u\} = V(J^*) \cap \left(\bigcap_{B \in S_u} \psi(B) \right);$$

  and we have condition (4) of Lemma 2.42.

Therefore, by Lemma 2.42 we can define $\widehat{\pi}$ on every vertex in $\mathbf{Split}(0,1)$, as we wanted. $\triangle$

Since $s_{\min} = 1$, define $\widehat{\pi}$ on all the vertices in $\mathbf{Split}(0,1)$. This can be done by Claim 2.48. Besides, define $\widehat{\pi}$ on every vertex in $\mathbf{Split}(s,1)$, for any s with $1 = s_{\min} \leq s \leq s_{\max}$. This last can be done by Claim 2.46. On the other hand, note that

$$V(F) = \bigcup_{s=0}^{\min\{k',|J_1|\}} V(\mathbf{Split}(s,1)).$$

Thus, it remains to define $\widehat{\pi}$ on all the vertices of $\mathbf{Split}(s,1)$, for any s with $s_{\max}+1 \leq s \leq \min\{k', |J_1|\}$. Let s as before. We distinguish two possible cases: either $t_{\min} = 1$ or $t_{\min} > 1$. Recall that $t_{\min} = \max\{1, k' - |J_2| - 2\}$.

- *Suppose that* $t_{\min} = 1$.

  In this case we have $k' - |J_2| - 2 \leq 1$. Define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(0,2)$ (this can be done by Claim 2.48). We have the following.

  - If $s_{\max} = |J_1| - 1$, we have $s = |J_1| = s_{\max} + 1$, and then by Claim 2.47 we can define $\widehat{\pi}$ on every vertex in $\mathbf{Split}(s,1)$.

– Suppose that $s_{\max} = k'-2 < |J_1|-1$, so $k' \leq |J_1|$ and then, $s \in \{k'-1, k'\}$.
If $s = k'-1$ then $V(\mathbf{Split}(s,1)) \subset V(\mathbf{Split}(0,2)) \cup V(\mathbf{Split}(1,2))$, and if $s = k'$ then $V(\mathbf{Split}(s,1)) \subset V(\mathbf{Split}(0,2))$. Since we have defined $\widehat{\pi}$ on every vertex in $V(\mathbf{Split}(0,2)) \cup V(\mathbf{Split}(1,2))$, then we have defined $\widehat{\pi}$ on every vertex in $\mathbf{Split}(s,1)$, for $s \in \{k'-1, k'\}$, as we wanted.

- *Suppose that* $t_{\min} > 1$.
Here we have $k' \geq n - |J_2|$, then $s_{\max} = |J_1| - 1$ and so $s = |J_1$. As before, by Claim 2.47, we can define $\widehat{\pi}$ on every vertex of $\mathbf{Split}(s,1)$.

2.2. *Suppose that* $n - |J_1 \cup J_2| = 2$.

We have that $r = 3$ and that J_3 is an edge. Since we are assuming that $n > 6$ at least one of J_1 and J_2 has more than two vertices. Without loss of generality assume that $|J_1| \leq |J_2|$, so $|J_2| \geq 3$. If $E(J_1, J_2) \neq \emptyset$, then $J \setminus J_3$ is connected; and we may proceed as above with J_3 playing the role of J_2, and J_2 playing the role of J_3. Assume that $E(J_1, J_2) = \emptyset$

Suppose that there exists an edge $uv_1 \in E(J_1, J_3)$ such that $J_1 \setminus u$ contains an edge w_1w_2. Let v_2 be the neighbour of v_1 in J_3, and let x_1x_2 be an edge of J_2. If $k \leq \lfloor n/2 \rfloor$, let $A \in V(F)$ be such that

$$\phi(J,\varphi)(v_1), \phi(J,\varphi)(w_1), \phi(J,\varphi)(x_1) \in \varphi(A),$$

and

$$\phi(J,\varphi)(u), \phi(J,\varphi)(v_2), \phi(J,\varphi)(w_2), \phi(J,\varphi)(x_2) \notin \varphi(A);$$

If $k > \lfloor n/2 \rfloor$, let $A \in V(F)$ be such that

$$\phi(J,\varphi)(v_1), \phi(J,\varphi)(w_1), \phi(J,\varphi)(x_1) \notin \varphi(A),$$

and

$$\phi(J,\varphi)(u), \phi(J,\varphi)(v_2), \phi(J,\varphi)(w_2), \phi(J,\varphi)(x_2) \in \varphi(A);$$

Let

$$e'_1 := \phi(J,\varphi)(w_1)\phi(J,\varphi)(w_2), \quad e'_2 := \phi(J,\varphi)(x_1)\phi(J,\varphi)(x_2),$$
$$e'_3 := \phi(J,\varphi)(v_1)\phi(J,\varphi)(v_2).$$

Note that e'_1, e'_2 and e'_3 is a matching in $G_{\varphi(A)}$. Therefore, we may use A in line 5 of INITIALIZE. Let e_1, e_2, and e_3 be the edges in F that correspond to move a token on e'_1, e'_2 and e'_3, respectively. Suppose that e_1, e_2 and e_3 are chosen in line 6 of INITIALIZE, and that the order in which they are chosen is e_3, e_1, e_2. Let J' be the graph isomorphic to G that is obtained by following all the previous construction with these choices. Let J'_1, J'_2 and J'_3 be its subgraphs such that J'_i corresponds to e_i. Let G'_i be the subgraph of G that corresponds to J'_i. Note that

$$\phi(J,\varphi)(v_1), \phi(J,\varphi)(v_2), \phi(J,\varphi)(u) \in G'_3.$$

Also note that, since $E(G_1, G_2) = \emptyset$, we have that G'_1 is a subgraph of G_1 and G'_2 is a subgraph of G_2. Therefore, $J' \setminus J'_1$ and $J' \setminus J'_2$ are connected. Since $|J'_3| \geq 3$ we may proceed as above. Thus, J_1 and J_2 are either edges or stars, and moreover, if J_i is a star, for $i \in \{1,2\}$, then only the center of J_i can be adjacent to a vertex in J_3. Also, since $n > 6$, at least one of J_1 and J_2 is a star.

Let $s_{\min} := \max\{0, k' - (|J| - |J_1|)\}$ and $s_{\max} := \min\{|J_1|, k'\}$. Note that

$$V(F) = \bigcup_{s_{\min} \leq s \leq s_{\max}} V(\mathbf{Split}(s,1)).$$

Let $s_{\min} < s^* < s_{\max}$ and let $t^* = k' - s^* - 1$. Suppose that we have defined $\widehat{\pi}$ on $\mathbf{Split}(s^*,1)$ and on $\mathbf{Split}(t^*,2)$. By Remark 2.45, this is the case for any $A \in V(H)$ and $s^* = |\widehat{\pi}(A)(1)|$ and $t^* = |\widehat{\pi}(A)(2)|$. We show that we can define $\widehat{\pi}$ on the vertices of $\mathbf{Split}(s^*-1,1) \cup \mathbf{Split}(t^*+1,2)$ (resp. $\mathbf{Split}(s^*+1,1) \cup \mathbf{Split}(t^*-1,2)$). Let $\{v_1, v_2\} := V(J_3)$. Suppose without loss of generality that v_1 is adjacent to a vertex w_1 of J_1.

Suppose that $s^* - 1 = 0$ (resp. $s^* + 1 = |J_1|$). We have that $\mathbf{Split}(s^*-1,1)$ (resp. $\mathbf{Split}(s^*+1,1)$) is a component of $F \setminus \mathbf{Split}(s^*,1)$. Now, note that $\mathbf{Split}(s^*-1,1) \cap \mathbf{Split}(t^*,2) \neq \emptyset$ (resp. $\mathbf{Split}(s^*+1,1) \cap \mathbf{Split}(t^*,2) \neq \emptyset$), and since we have already defined $\widehat{\pi}$ on the vertices in $\mathbf{Split}(t^*,2)$, we can determine which component of $F \setminus \mathbf{Split}(s^*,1)$ corresponds to $\mathbf{Split}(s^*-1,1)$ (resp. $\mathbf{Split}(s^*+1,1)$). Therefore, we have conditions (1), (2) and (3) of Lemma 2.42, where $F' = \mathbf{Split}(s^*-1,1)$ (resp. $\mathbf{Split}(s^*+1,1)$), $J^* = J \setminus J_1$ and $T^* = \emptyset$ (resp $T^* = V(J_1)$). In this case, it remains to show that condition (4) of Lemma 2.42 also holds.

Suppose that $s^*-1 > 0$ (resp. $s^*+1 < |J_1|$). Let A be a vertex in $\mathbf{Split}(s^*-1,1) \cap \mathbf{Split}(t^*,2)$ (resp. $\mathbf{Split}(s^*+1,1) \cap \mathbf{Split}(t^*,2)$). Since $\mathbf{Move}(A,1)$ is a subgraph of $\mathbf{Split}(t^*,2)$, we have defined $\widehat{\pi}$ on every vertex of $\mathbf{Move}(A,1)$. Thus, we have condition (a) of Lemma 2.43 for $s = s^*-1$ (resp. $s = s^*+1$). It remains to show that condition (b) of Lemma 2.43 holds in this case.

Suppose that $k' - (s^*-1) = |J \setminus J_1|$ (resp. $k' - (s^*+1) = 0$). If $s^*-1 = 0$ (resp. $s^*+1 = |J_1|$) then $\mathbf{Split}(s^*-1,1)$ consists of only one vertex C, for such vertex define

$$\widehat{\pi}(C)(1) := \emptyset, \widehat{\pi}(C)(2) := V(J_2) \text{ and } \widehat{\pi}(C)(3) := V(J_3),$$

$$(\text{resp. } \widehat{\pi}(C)(1) := V(J_1), \widehat{\pi}(C)(2) := \emptyset, \text{ and } \widehat{\pi}(C)(3) := \emptyset)\,.$$

If $s^*-1 > 0$ (resp. $s^*+1 < |J_1|$), then $\mathbf{Split}(s^*-1,1) = \mathbf{Move}(A,1)$, and then we are done. Assume then that $k' - (s^*-1) = |J \setminus J_1|$ (resp. $k' - (s^*+1) > 0$).

- Suppose that w_1 is not adjacent to v_2.

  Let W' be the set of all vertices $B \in \mathbf{Split}(s^*,1)$ such that

  - $w_1 \in \psi(B)(1)$ (resp. $w_1 \notin \psi(B)(1)$) and if $w_2 \in J_1$ has a neighbour in J_3, then $w_2 \notin \psi(B)$ (resp. $w_2 \in \psi(B)(3)$), and
  - $v_1 \notin \psi(B)(3)$ (resp. $v_1 \in \psi(B)(3)$).

  Let C be a neighbour of B in $F \setminus \mathbf{Split}(s^*,1)$. Since $E(J_1,J_2) = \emptyset$, we have that $\varphi(C)$ is obtained from $\varphi(B)$ by sliding a token along the edge $\phi(J,\varphi)(v_1)\phi(J,\varphi)(w_1)$.

Thus, $C \in \mathbf{Split}(s^* - 1, 1)$ (resp. $C \in \mathbf{Split}(s^* + 1, 1)$). We define $\widehat{\pi}(C)(2) := \widehat{\pi}(B)(2)$ and

$$\widehat{\pi}(C)(1) := \widehat{\pi}(B)(1) \setminus \{w_1\} \text{ and } \widehat{\pi}(C)(3) := \widehat{\pi}(B)(3) \cup \{v_1\},$$
$$(\text{resp. } \widehat{\pi}(C)(1) := \widehat{\pi}(B)(1) \cup \{w_1\} \text{ and } \widehat{\pi}(C)(3) := \widehat{\pi}(B)(1) \setminus \{v_1\}).$$

Let W be the set of all such vertices C. Note that

$$\{v_1\} = V(J \setminus J_1) \cap \bigcap_{C \in W} \psi(C) \quad \left(\text{resp. } \{v_1\} = V(J \setminus J_1) \setminus \bigcup_{C \in W} \psi(C)\right).$$

For every vertex $u \in V(J \setminus J_1)$ other than v_1, let S_u be the subset of all vertices $C \in W$ such that $u \notin \psi(C)$ (resp. $u \in \psi(C)$). Since $k' - s^* + 1 < |J \setminus J_1|$ (resp. $k' - s^* - 1 > 0$), we have that

$$\{u\} = V(J \setminus J_1) \setminus \bigcup_{C \in S_u} \psi(C) \quad \left(\text{resp. } \{u\} = V(J \setminus J_1) \cap \bigcap_{C \in S_u} \psi(C)\right).$$

If $s^* - 1 = 0$ (resp. $s^* + 1 = |J_1|$), then we have condition (4) of Lemma 2.42; and if $s^* - 1 > 0$ (resp. $s^* + 1 < |J_1|$), then we have condition (b) of Lemma 2.43. Therefore, we can define $\widehat{\pi}$ on all the vertices of $\mathbf{Split}(s^* - 1, 1)$ (resp. $\mathbf{Split}(s^* + 1, 1)$).

- Suppose that w_1 is adjacent to v_2.

  Recall that J_2 is a star on at least three vertices. Let x_1 be the center of J_2, then x_1 is the only vertex in J_2 with neighbours in J_3. Since $E(J_1, J_2) = \emptyset$, x_1 cannot be adjacent to both v_1 and v_2, as otherwise, the vertex set $\{w_1, v_1, v_2, x_1\}$ would induce a diamond graph. Without loss of generality let us assume that x_1 is adjacent to v_2. Let E_1 be the set of edges in $\mathbf{Split}(s^*, 1)$ such that $\varphi(e)$ is generated by moving a token along the edge $\phi(J, \varphi)(v_2)\phi(J, \varphi)(x_1)$, for every $e \in E_1$.

  Let W' be the set of all vertices $B \in \mathbf{Split}(s^*, 1)$ such that

  (i) $w_1 \in \psi(B)(1)$ (resp. $w_1 \notin \psi(B)(1)$) and if $w_2 \in J_1$ has a neighbour in J_3, then $w_2 \notin \psi(B)$ (resp. $w_2 \in \psi(B)(3)$),

  (ii) $v_1 \notin \psi(B)(3)$ or $v_2 \notin \psi(B)(3)$ (resp. $v_1 \in \psi(B)(3)$ or $v_2 \in \psi(B)(3)$), and

  (iii) if $v_2 \notin \psi(B)(3)$ then $x_1 \in \psi(B)(2)$ (resp. if $v_2 \in \psi(B)(3)$ then $x_1 \notin \psi(B)(2)$).

  For a vertex $B \in W'$ let $C \in F \setminus \mathbf{Split}(s^*, 1)$ be a neighbour of B, then $C \in \mathbf{Split}(s^* - 1), 1$ (resp. $C \in \mathbf{Split}(s^* + 1, 1)$). Consider the following cases.

  - If $v_2 \in \psi(B)(3)$ (resp. $v_2 \notin \psi(B)(3)$) then $v_1 \notin \psi(B)(3)$ (resp. $v_1 \in \psi(B)(3)$) and then $\varphi(C)$ is obtained from $\varphi(B)$ by sliding a token along the edge $\phi(J, \varphi)(w_1)\phi(J, \varphi)(v_1)$. In this case define $\widehat{\pi}(C)(2) := \widehat{\pi}(B)(2)$ and

$$\widehat{\pi}(C)(1) := \widehat{\pi}(B)(1) \setminus \{w_1\} \text{ and } \widehat{\pi}(C)(3) := \widehat{\pi}(B)(3) \cup \{v_1\},$$
$$(\text{resp. } \widehat{\pi}(C)(1) := \widehat{\pi}(B)(1) \cup \{w_1\} \text{ and } \widehat{\pi}(C)(3) := \widehat{\pi}(B)(3) \setminus \{v_1\}).$$

– If $v_1 \in \psi(B)(3)$ (resp. $v_1 \notin \psi(B)(3)$) then $v_2 \notin \psi(B)(3)$ (resp. $v_2 \in \psi(B)(3)$), and then $\varphi(C)$ is obtained from $\varphi(B)$ by sliding a token along the edge $\phi(J,\varphi)(w_1)\phi(J,\varphi)(v_2)$. In this case define $\widehat{\pi}(C)(2) := \widehat{\pi}(B)(2)$ and

$$\widehat{\pi}(C)(1) := \widehat{\pi}(B)(1) \setminus \{w_1\} \quad \text{and} \quad \widehat{\pi}(C)(3) := \widehat{\pi}(B)(3) \cup \{v_2\},$$

$$(\text{resp. } \widehat{\pi}(C)(1) := \widehat{\pi}(B)(1) \cup \{w_1\} \quad \text{and} \quad \widehat{\pi}(C)(3) := \widehat{\pi}(B)(3) \setminus \{v_2\})\,.$$

– Suppose that $v_1, v_2 \notin \psi(B)(3)$ (resp. $v_1, v_2 \in \psi(B)(3)$), so $x_1 \in \psi(B)(2)$ (resp. $x_1 \notin \psi(B)(2)$). Then $\varphi(C)$ is obtained from $\varphi(B)$ by sliding a token along the edge $\phi(J,\varphi)(w_1)\phi(J,\varphi)(v_1)$ or along the edge $\phi(J,\varphi)(w_1)\phi(J,\varphi)(v_2)$. Note that $\varphi(C)$ is obtained from $\varphi(B)$ by sliding a token along the edge $\phi(J,\varphi)(w_1)\phi(J,\varphi)(v_1)$ if and only if

the edge BC belongs to a 4-cycle of F together with an edge $e \in E_1$. (15)

Define $\widehat{\pi}(C)(1) := \widehat{\pi}(B)(1) \setminus \{w_1\}$ (resp. $\widehat{\pi}(C)(1) := \widehat{\pi}(B)(1) \cup \{w_1\}$), $\widehat{\pi}(C)(2) := \widehat{\pi}(B)(2)$ and

$$\widehat{\pi}(C)(3) := \begin{cases} \widehat{\pi}(B)(3) \cup \{v_1\} & \text{if (15) holds} \\ \widehat{\pi}(B)(3) \cup \{v_2\} & \text{otherwise} \end{cases}$$

$$\left(\text{resp.} \quad \widehat{\pi}(C)(3) := \begin{cases} \widehat{\pi}(B)(3) \setminus \{v_1\} & \text{if (15) holds} \\ \widehat{\pi}(B)(3) \setminus \{v_2\} & \text{otherwise} \end{cases}\right)$$

For a vertex $u \in J_2 \cup J_3$, let $S_u := \{C \in W : u \in \psi(C)\}$ (resp. $S_u := \{C \in W : u \notin \psi(C)\}$). Then,

$$\{u\} = V(J \setminus J_1) \cap \bigcap_{C \in S_u} \psi(C) \quad \left(\text{resp. } \{u\} = V(J \setminus J_1) \setminus \bigcup_{C \in S_u} \psi(C)\right).$$

If $s^* - 1 = s_{\min}$ (resp. $s^* + 1 = s_{\max}$), then we have condition (4) of Lemma 2.42; if $s^* - 1 > 0$ (resp. $s^* + 1 < s_{\max}$), then we have condition (b) of Lemma 2.43. Therefore, we can define $\widehat{\pi}$ on all the vertices of $\textbf{Split}(s^* - 1, 1)$ (resp. $\textbf{Split}(s^* + 1, 1)$).

Let $t_{\min} := \max\{0, k' - (|J| - |J_2|)\}$ and $t_{\max} := \min\{|J_2|, k'\}$. Let $t_{\min} < t^* < t_{\max}$ and let $s^* = k' - t^* - 1$. Suppose that we have defined $\widehat{\pi}$ on the vertices of $\textbf{Split}(s^*, 1)$ and $\textbf{Split}(t^*, 2)$. By similar arguments as above we can define $\widehat{\pi}$ on the vertices of $\textbf{Split}(t^* - 1, 2)$ and $\textbf{Split}(t^* + 1, 2)$. Therefore, we can define $\widehat{\pi}$ on the vertices of $\textbf{Split}(s^*, 1)$ for all $s_{\min} \le s^* \le s_{\max}$. The result follows.

□

2.7 Reconstruction of disconnected graphs

To finish this chapter we consider the case when G is disconnected. Our first result in this direction is that there exist non-isomorphic disconnected graphs G and H, and integers $k \neq l$ such that $F_k(G) \simeq F_l(H)$. Moreover, G and H have the property that are (C_4, D_4)-free graphs. See Figure 2.10, for an example. This example was found by Trujillo-Negrete in her Master's Thesis [51].

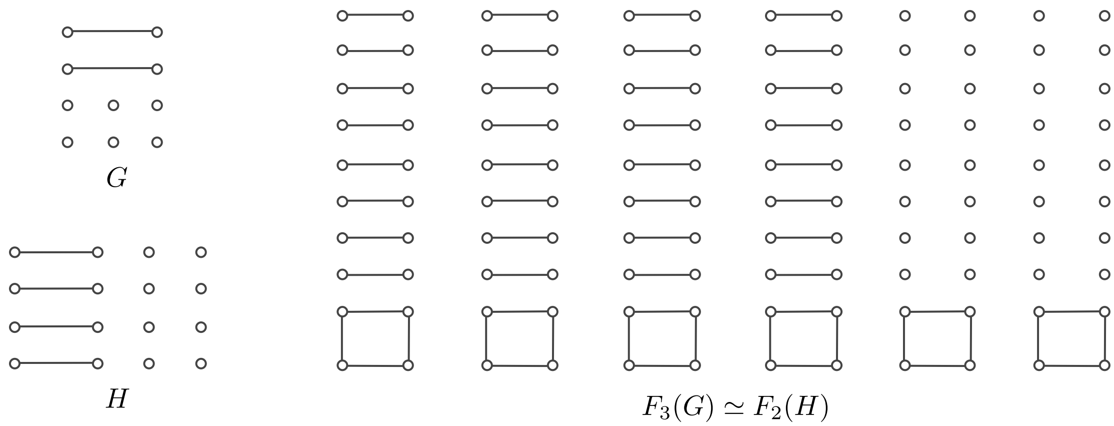


Figure 2.10: Two non-isomorphic graphs G and H for which $F_3(G)$ is isomorphic to $F_2(H)$.

On the positive side we have the following.

Theorem 2.5. *Let G and H be two (C_4, D_4)-free graphs. If $F_k(G)$ and $F_k(H)$ are isomorphic for some k, then G and H are isomorphic.*

Proof. We proceed as follows. Suppose we are given a graph F and an integer k such that F is the k-token graph of a (C_4, D_4)-free graph. We show that there is a unique G (up to isomorphism) such that $F \simeq F_k(G)$. Since $F_k(G)$ is connected if and only if G is connected [21], we may assume that G is disconnected, as otherwise we are done by Theorem 2.1. Since $|F_k(G)| = \binom{|G|}{k}$, $n := |G|$ is determined. We may assume that $k \leq n/2$. Let $G_1, \dots, G_r$ be the components of G. Note that for each component C of $F_k(G)$ there exist integers $k_1, \dots, k_r$, with $0 \leq k_i \leq |G_i|$ and $k = k_1 + \cdots + k_r$, such that C is generated by moving k_i tokens on G_i. Moreover, we have that

$$C \simeq F_{k_1}(G_1) \,\square\, \cdots \,\square\, F_{k_r}(G_r),$$

where $F_{k_i}(G_i) \simeq K_1$ if $k_i = 0$. Note that since G_i is connected, by Corollary 2.24, we have that if $0 < k_i < |G_i|$, then $F_{k_i}(G_i)$ is a prime graph. Given C, there is a unique cartesian decomposition (up to the order of the factors) such that

$$C \simeq F_1 \square \cdots \square F_{r'},$$

and every F_i is a non-trivial prime graph [47, 52]. This decomposition can be found in linear time [30]. We compute the cartesian decompositions of all components of F. Let C^* be the component with the largest number, r^*, of factors; and let $F_1 \square \cdots \square F_{r^*}$ be this decomposition. We proceed by cases depending on the value of r^*.

- $\boldsymbol{r^* < k}$.

  Note that G has exactly r^* non trivial components. Let $G_1, \ldots, G_{r^*}$ be these components. By Theorem 2.1 we can reconstruct these components in polynomial time. Finally, the number of isolated vertices of G is given by

$$n - \sum_{i=1}^{r^*} |G_i|.$$

- $\boldsymbol{r^* = k}$.

  Suppose that C^* is the only component of F having k factors in its decomposition. This implies that G has exactly k non-trivial components; and we may proceed as in the previous case. Suppose now that there are at least two components of F having k factors in their decomposition. Thus, G has more than k non-trivial components. Let $\mathcal{C}_F$ be the set of components of F with k factors in its decomposition, and let $\mathcal{C}_G$ be the set of non-trivial components of G. Let $q(F) := |\mathcal{C}_F|$ and $q(G) := |\mathcal{C}_G|$. Since $q(F) = \binom{q(G)}{k}$, we can determine the value $q(G)$. Moreover, each $G_i \in \mathcal{C}_G$ is counted in exactly $\binom{q(G)-1}{k-1}$ components of $\mathcal{C}_F$.

  For every $C \in \mathcal{C}_F$, we use Theorem 2.1 to compute a set of graphs $H'_1, \ldots, H'_k$ such that $C \simeq H_1 \square \cdots \square H'_k$. Let $\mathcal{S}$ be the set of all such graphs. By testing for graph isomorphism we obtain a set of tuples $\{(G'_1, t_1), \ldots, (G'_s, t_s)\}$, such that: the G'_i are pairwise non-isomorphic; for every $H_i \in \mathcal{S}$ there exists a graph G'_j such that $H_i \simeq G'_j$; and there are exactly t_j graphs in $\mathcal{S}$ isomorphic to G'_j.

  Note that each G_i gives way to $\binom{q(G)-1}{k-1}$ graphs in $\mathcal{S}$. Therefore, for every G'_i there are exactly $t_i / \binom{q(G)-1}{k-1}$ components of $\mathcal{C}_G$ isomorphic to G'_i. Thus we can determine the graphs in $\mathcal{C}_G$ up to isomorphism. Finally, the number of isolated vertices of G is given by

$$n - \sum_{G' \in \mathcal{S}} |G'|.$$

□

We point out that, in contrast with the connected case, we are unable to reconstruct G in polynomial time. The bottleneck of the algorithm implied in the proof of Theorem 2.5 is the Graph Isomorphism Problem. Besides, in general, when G is disconnected we cannot reconstruct G uniquely, even if G is (C_4, D_4)-free, the reason is that the number $|\operatorname{Aut}({}_k(G))|$ may be arbitrarily large. This happens, for example, in the graph $F_3(G) \simeq F_2(H)$ of Figure 2.10, since this graph contains isomorphic components.

Chapter 3

Automorphisms of token graphs

The study of combinatorial and algebraic properties of token graphs has followed, regularly, the next approach:

Given a graph invariant η, what can be said about $\eta(F_k(G))$ in terms of G and $\eta(G)$?

Following this approach, in this chapter we consider the automorphism group of graphs. As we mentioned in Section 1.4, the set of all automorphisms of G forms a group under function composition, this group is called the *automorphism group of* G and is denoted by $\mathrm{Aut}(G)$. The automorphism group of a graph G characterizes its symmetries. Determining the automorphism group of a graph is closely related to determining whether two graphs are isomorphic. Sometimes, it is easy to find some automorphisms of a graph, but it may be quite difficult to determine all the automorphisms of the graph.

Let us recall some results on the automorphism group of token graphs. As we saw in Section 1.4.1, the function $\iota : \mathrm{Aut}(G) \to \mathrm{Aut}(F_k(G))$ maps automorphisms of G to automorphisms of $F_k(G)$, where $\iota(\psi)$ is the function that maps every $A \in V(F_k(G))$ to

$$\iota(\psi)(A) := \{\psi(v) : v \in A\}.$$

We call $\iota(\psi)$ *the automorphism induced by* ψ. On the other hand, the function $\mathfrak{c} : F_k(G) \to F_{n-k}(G)$ that sends every vertex $A \in F_k(G)$ to its complement

$$\mathfrak{c}(A) := V(G) \setminus A$$

is an isomorphism from $F_k(G)$ to $F_{n-k}(G)$, so, if $k = n/2$ then this mapping is an automorphism of $F_k(G)$, which we call the *complement automorphism of* $F_k(G)$. As we saw in Section 1.4.1,

$$\mathrm{Aut}(G) \leq \mathrm{Aut}(F_k(G)) \quad \text{when } k \neq n/2, \tag{1}$$

$$\mathrm{Aut}(G) \times \mathbb{Z}_2 \leq \mathrm{Aut}(F_k(G)) \quad \text{when } k = n/2. \tag{2}$$

The inclusions (1) and (2) may be proper. For example, using SageMath [50] and GAP [26] softwares it can be shown that

$$\mathrm{Aut}(K_{2,3}) = \mathbb{Z}_2 \times S_3 < \mathbb{Z}_2 \times S_4 = \mathrm{Aut}(F_2(K_{2,3}))$$

and

$$\mathrm{Aut}(C_4) \times \mathbb{Z}_2 = D_4 \times \mathbb{Z}_2 < S_4 \times \mathbb{Z}_2 = \mathrm{Aut}(F_2(C_4)).$$

Following the line of research of Chapter 2 and using the characterization given in Theorem 2.7, in this chapter we say that $F_k(G)$ is *uniquely reconstructible as the k-token graph of G* if $\mathrm{Aut}(G) \simeq \mathrm{Aut}(F_k(G))$ when $k \neq n/2$, and if $\mathbb{Z}_2 \times \mathrm{Aut}(G) \simeq \mathrm{Aut}(F_k(G))$ when $k = n/2$. To our knowledge, the families of graphs for which $\mathrm{Aut}(F_k(G))$ has been studied are:

- $F_k(K_n)$ (which is isomorphic to the Johnson graph $J(n, k)$), for each admissible k, see, e.g., [25, 33, 42];
- $F_k(P_n)$, for $2 \leq k < n/2$, see [29];
- $F_2(G)$, where G is a cycle, a star, a fan or a wheel graph, see [29];
- $F_k(G)$, where G is a connected $(C_4, \mathit{diamond})$-free graph and k holds $2 \leq k \leq |G| - 2$ (this was done in Chapter 2 of this thesis).

For all these families, $F_k(G)$ is uniquely reconstructible as the k-token graph of G.

Knowing the existence of graphs G and values k for which $F_k(G)$ is not uniquely reconstructible as the k-token graph of G, a natural problem then is to characterize the graphs G and values k for which $F_k(G)$ is not uniquely reconstructible as the k-token graph of G. Motivated by this problem, in this chapter we study the automorphism group of the k-token graph of two families of graphs: the complete bipartite graphs and Cartesian product of graphs. In these two families, in some cases we have that $F_k(G)$ is uniquely reconstructible as the k-token graph of G, and in others it is not; surprisingly, we will see that sometimes this depends only on G, and others, for the same graph G, this depends on the value of k.

For the complete bipartite graphs we show the following result.

Theorem 3.1. *Let $1 \leq m \leq n$ and $1 \leq k \leq m + n - 1$. Then $\mathrm{Aut}(F_k(K_{m,n}))$ is uniquely reconstructible as the k-token graph of $K_{m,n}$ if and only if $m \neq 2$. Moreover,*

$$|\mathrm{Aut}(F_k(K_{2,n}))| = \begin{cases} 2^{\binom{n}{k-1}-1} |\mathrm{Aut}(K_{2,n})| & \text{if } k \neq \frac{n+2}{2}, \text{ and} \\ 2^{\binom{n}{k-1}} |\mathrm{Aut}(K_{2,n})| & \text{if } k = \frac{n+2}{2}. \end{cases}$$

For the Cartesian product of graphs we show the following.

Theorem 3.2. *Let G be a connected graph with prime factor decomposition $G = G_1 \,\square\, \dots \,\square\, G_r$, where $r > 1$, $n = |G|$ and $2 \leq k \leq n/2$. Then*

$$|\operatorname{Aut}(F_k(G))| \geq \begin{cases} 2^{r-1}\,|\operatorname{Aut}(G)| & \textit{if } k = 2, \\ 2\,|\operatorname{Aut}(G)| & \textit{if } k = \frac{n}{2}, \\ |\operatorname{Aut}(G)| & \textit{if } 2 < k < \frac{n}{2}. \end{cases}$$

Moreover, this lower bound is tight.

This chapter is based on a joint work in progress with Irene Parada and Ruy Fabila-Monroy [20].

3.1 Preliminaries

From inclusions (1) and (2), it follows trivially that

$$|Aut(F_k(G))| \geq \begin{cases} |Aut(G)| & \text{if } k \neq n/2, \text{ and} \\ 2\,|Aut(G)| & \text{if } k = n/2. \end{cases} \tag{16}$$

This lower bound is tight when $F_k(G)$ is uniquely reconstructible as the k-token graph of G. Notice that $F_k(G)$ is uniquely reconstructible as the k-token graph of G if it consists only of the induced automorphisms of G when $k < n/2$, and $\operatorname{Aut}(F_k(G))$ consists only of induced automorphisms of G, the complement automorphism and a combination of these two types when $k = n/2$.

Let us now mention some definitions, notation and basic results on Group Theory, complete bipartite graphs and Cartesian product of graphs, which will be helpful to show Theorems 3.1 and 3.2.

Let Γ be a group and Δ a subgroup of Γ. For a fixed element $a \in \Gamma$ we define two subsets of Γ:

- The *left coset* of Δ determined by a is the set $a\Delta = \{ah : h \in \Delta\}$.
- The *right coset* of Δ determined by a is the set $\Delta a = \{ha : h \in \Delta\}$.

The left cosets of Δ form a partition of Γ, and similarly for the right cosets. Moreover, $|a\Delta| = |\Delta a| = |\Delta|$. The *index* of Δ in Γ, denoted by $[\Gamma : \Delta]$, is the number of left cosets (or right cosets) of Δ in Γ.

Theorem (Lagrange)**.** *If Δ is a subgroup of a finite group Γ, then $[\Gamma : \Delta] = |\Gamma|/|\Delta|$, and in particular, $|\Delta|$ divides $|\Gamma|$.*

A group Γ is a semidirect product of a subgroup N by a subgroup H if the following conditions are satisfied:

(i) $\Gamma = NH$;

(ii) N is a normal subgroup of Γ; and

(iii) $N \cap H = \{\mathrm{id}\}$.

We write $\Gamma = N \rtimes H$.

For the complete bipartite graph $K_{m,n}$, it is known that

$$\mathrm{Aut}(K_{m,n}) = \begin{cases} S_m \times S_n & \text{if } m \neq n, \\ S_m \times S_n \rtimes S_2 & \text{if } m = n. \end{cases}$$

so, $|\,\mathrm{Aut}(K_{m,n})| = m!\,n!$ when $m \neq n$, and $|\,\mathrm{Aut}(K_{m,n})| = 2\,m!\,n!$ when $m = n$.

Let G be a composite graph. The following two results on the Cartesian product of graphs can be found in [28].

Theorem (Sabidussi-Vizing)**.** *Every connected graph has a unique representation as a product of prime graphs, up to isomorphism and the order of the factors.*

Any automorphism of G can be described as follows.

Theorem (Theorem 6.10 in [28])**.** *Suppose ϕ is an automorphism of a connected graph G with prime factor decomposition $G = G_1 \,\Box\, G_2 \,\Box\, \dots \,\Box\, G_r$. Then there is a permutation π of $\{1, 2, \dots, r\}$ and isomorphisms $\phi_i : G_{\pi(i)} \to G_i$ for which*

$$\phi\big((x_1, x_2, \dots, x_r)\big) = \big(\phi_1(x_{\pi(1)}), \phi_2(x_{\pi(2)}), \dots, \phi_r(x_{\pi(r)})\big).$$

3.2 Complete bipartite graphs

The aim of this section is to determine $|\,\mathrm{Aut}(F_k(K_{m,n}))|$, for each admissible k, m and n. We proceed by cases depending on the values of m and n. For $F_2(K_{2,2})$ we have $|\,\mathrm{Aut}(F_2(K_{2,2}))| = 48$, so from now on, assume that $\{k, m, n\} \neq \{2, 2, 2\}$. We point out that for $m \neq 1$, the graph $K_{m,n}$ has many induced 4-cycles, so this family of graphs was not considered in Chapter 2. The case $m = 1$ was considered in Chapter 2, but for completeness we also consider it in this chapter.

Let $\Gamma := \mathrm{Aut}(F_k(K_{m,n}))$ and let Δ be the subgroup of $\mathrm{Aut}(F_k(K_{2,n}))$ generated by the induced automorphisms if $k \neq (n+2)/2$, and by the induced automorphisms and the complement automorphism if $k = (n+2)/2$.

Let $\{X, Y\}$ be the bipartition of $V(K_{m,n})$, with $m := |X|$, $n := |Y|$ and $1 \leq m \leq n$. Let $X := \{x_1, \ldots, x_m\}$ and $Y := \{y_1, \ldots, y_n\}$. Let $r := \min\{m, k\}$. For $i \in \{0, \ldots, r\}$ let

$$H_i := \{A \in F_k(K_{m,n}) : |A \cap X| = i\}.$$

H_i corresponds to the vertices of $F_k(K_{m,n})$ with exactly i tokens in X and $k - i$ tokens in Y. We have $|H_i| = \binom{m}{i}\binom{n}{k-i}$ and $\deg(A) = i(n - k + i) + (k - i)(m - i)$ for every vertex $A \in H_i$. Let us remark that the subsets $H_0, H_1, \ldots, H_r$ are pairwise disjoint. Sets $\{H_0, H_1, \ldots, H_r\}$ can be seen as a partition of $V(F_k(K_{m,n}))$, where there are edges between H_i and H_j if and only if $|i - j| = 1$; also, each H_i is an independent set (see Figure 3.1).

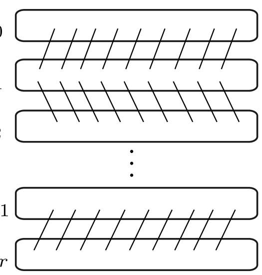


Figure 3.1: $\{H_0, H_1, \ldots, H_r\}$ is a partition of $V(F_k(K_{m,n}))$.

In [21] the authors showed that a graph G is bipartite if and only if $F_k(G)$ is bipartite. Then, $F_k(K_{m,n})$ is bipartite with bipartition $\{\mathcal{B}, \mathcal{R}\}$, where

$$\mathcal{B} := \bigcup_{i \text{ even}} H_i$$

and

$$\mathcal{R} := \bigcup_{i \text{ odd}} H_i.$$

The aim of this chapter is to show Theorem 3.1:

Theorem 3.1. *Let $1 \leq m \leq n$ and $1 \leq k \leq m + n - 1$. Then $\operatorname{Aut}(F_k(K_{m,n}))$ is uniquely reconstructible as the k-token graph of $K_{m,n}$ if and only if $m \neq 2$. Moreover,*

$$|\operatorname{Aut}(F_k(K_{2,n}))| = \begin{cases} 2^{\binom{n}{k-1}-1}|\operatorname{Aut}(K_{2,n})| & \text{if } k \neq \frac{n+2}{2}, \text{ and} \\ 2^{\binom{n}{k-1}}|\operatorname{Aut}(K_{2,n})| & \text{if } k = \frac{n+2}{2}. \end{cases}$$

□

Note that for the case $m \neq 2$, the lower bounds are given by (1) and (2), so it is enough to show the upper bounds when $m \neq 2$.

3.2.1 Lower bound of Theorem 3.1

Assume that $m = 2$, so $n > 2$. Here we have $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.

Let $A_1, \ldots, A_p, B_1, \ldots, B_p$ be the vertices in H_1, with $p = \binom{n}{k-1}$, and $A_i \setminus B_i = \{x_1\}$ and $B_i \setminus A_i = \{x_2\}$, for $i \in [p]$.

For $i \in [p]$, let $\psi_i : V(F_k(K_{2,n})) \longrightarrow V(F_k(K_{2,n}))$ be the function that swaps the labels of vertices A_i and B_i, and keeps the labels of the remaining vertices fixed. It is not hard to see that

Observation 3.3. *A_i and B_i have the same neighbours, for each $i \in [p]$. Therefore, ψ_i is an automorphism of $F_k(K_{2,n})$. Moreover, for $i \neq j$ we have $\psi_i \neq \psi_j$ and $\psi_i \circ \psi_j = \psi_j \circ \psi_i$.*

Let us now generalize the definition of ψ_i. For a subset $S = \{i_1, i_2, \ldots, i_s\} \subset [p]$, we define the function $\psi_S : V(F_k(K_{2,n})) \to V(F_k(K_{2,n}))$ as

$$\psi_S := \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_s},$$

so we have

$$\psi_S(A) := \begin{cases} B_i & \text{if } A = A_i \text{ and } i \in S, \\ A_i & \text{if } A = B_i \text{ and } i \in S, \\ A & \text{otherwise.} \end{cases}$$

In other words, ψ_S swaps the labels of A_i and B_i, for $i \in S$, and keeps the remaining labels fixed. Let us note that it does not matter the order of functions $\psi_{i_1}, \ldots, \psi_{i_s}$ in the definition of ψ_S, since they are pairwise commutative.

Let us denote by φ the automorphism of $K_{m,n}$ that swaps the labels of x_1 and x_2 and keeps the labels of the remaining vertices fixed, and by id the identity automorphism. As before, for an automorphism ϕ of $K_{m,n}$, $\iota(\phi)$ denotes the automorphism of $F_k(K_{m,n})$ induced by ϕ, and if $k = \frac{m+n}{2}$, $\mathfrak{c}$ denotes the complement automorphism. In Figure 3.2 is depicted all the automorphisms ψ_S of $F_2(K_{2,3})$.

The following observation follows from the definition of ψ_S and $\iota(\varphi)$.

Observation 3.4. *Let $S, S^*, S' \subset [p]$ with $S \neq S^*$ and $S' = [p] \setminus S$. Then,*

(1) $(\psi_S)^2 = \psi_S \circ \psi_S = \mathrm{id}$*;*

(2) $\psi_S \neq \psi_{S^*}$*; and*

(3) $\psi_S = \iota(\varphi) \circ \psi_{S'} = \psi_{S'} \circ \iota(\varphi)$.

Next we show a special property about ψ_S.

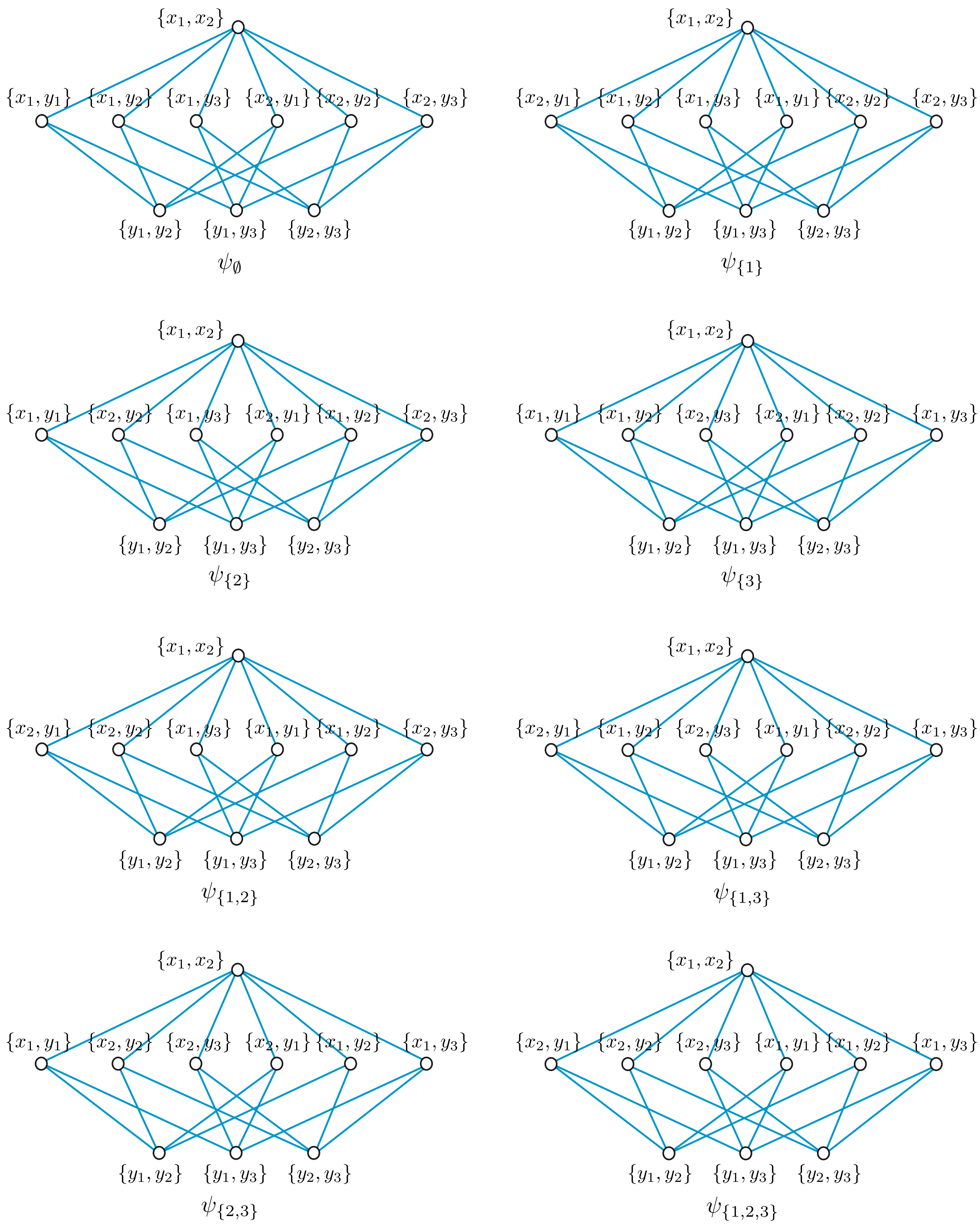


Figure 3.2: All the possible automorphisms ψ_S of $F_2(K_{2,3})$.

Proposition 3.5. *The mapping ψ_S is an automorphism of $F_k(K_{m,n})$. Moreover, $\psi_S \in \Delta$ if and only if $S = \emptyset$ or $S = [p]$; in such cases we have $\psi_S = \iota(\mathrm{id})$ if $S = \emptyset$, and $\psi_S = \iota(\varphi)$ if $S = [p]$.*

Proof. Let $S = \{i_1, i_2, \ldots, i_s\} \subseteq [p]$. Since ψ_{i_j} is an automorphism of $F_k(K_{2,n})$, for any $j \in [s]$, and $\psi_S = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_s}$, it follows that ψ_S is an automorphism of $F_k(K_{2,n})$.

It can be shown that

- If $S = \emptyset$, then ψ_S fixes each vertex in $F_k(K_{m,n})$, and therefore, $\psi_s = \iota(\mathrm{id}) \in \Delta$.
- If $S = [p]$, then ψ_S swaps the labels of the vertices in $\{A_i, B_i\}$, for each $i \in S = [p]$, and keeps the labels of the remaining vertices fixed. Note that $\iota(\varphi)$ does the same as ψ_S, so, $\psi_S = \iota(\varphi) \in \Delta$.

Assume $\emptyset \neq S \neq [p]$. There are $i, j \in [p]$ such that ψ_S swaps the labels of A_i and B_i, and fixes the labels of vertices A_j and B_j. In particular we have that $\psi_S \neq \iota(\varphi)$, because we know that $\iota(\varphi)$ swaps the labels of A_ℓ and B_ℓ, for any $\ell \in [p]$. Consider an automorphism ϕ of $K_{m,n}$, with $\mathrm{id} \neq \phi \neq \varphi$. We want to show that $\psi_S \neq \iota(\phi)$, and if $k = (n+2)/2$, we also want to show that $\psi_S \neq \mathfrak{c} \circ \iota(\phi)$. Suppose $k = (n+2)/2$. Note that ψ_S fixes any vertex $C \in H_0 \cup H_2$, while $\mathfrak{c} \circ \iota(\phi)$ maps H_0 to H_2 (because $\iota(\phi)$ maps H_0 to H_0 and $\mathfrak{c}$ maps H_0 to H_2). These two observations together imply that $\psi_S \neq \mathfrak{c} \circ \iota(\phi)$. Now, assume that k is not necessarily equal to $(n+2)/2$. If $\iota(\phi)(C) = C$ for any $C \in H_0 \cup H_2$, we would have $\phi = \mathrm{id}$, contradicting the choice of ϕ. Thus, $\psi_S \notin \Delta$. This completes the proof. □

Now we are ready to show the lower bound for the case $m = 2$.

Lemma 3.6.

$$|\operatorname{Aut}(F_k(K_{2,n}))| \geq \begin{cases} 2^{\binom{n}{k-1}-1}|\operatorname{Aut}(K_{2,n})| & \text{if } k \neq \frac{m+n}{2}, \text{ and} \\ 2^{\binom{n}{k-1}}|\operatorname{Aut}(K_{2,n})| & \text{if } k = \frac{m+n}{2}. \end{cases}$$

Proof. Suppose $k \neq \frac{n+2}{2}$. Let Γ and Δ be as before, so in this case, $|\Delta| = 2\,n!$. Let us give a lower bound for the index $[\Gamma : \Delta]$. Let $S, S' \subset [p]$ with $S \neq S'$. We claim that the cosets $\psi_S\Delta$ and $\psi_{S'}\Delta$ are equal if and only if $S' = [p] \setminus S$.

For the forward implication, if $\psi_S\Delta = \psi_{S'}\Delta$ then $\psi_S \in \psi_{S'}\Delta$ and so $\psi_S = \psi_{S'}\iota(\phi)$, for some $\iota(\phi) \in \Delta$. Note that ψ_S, as well as $\psi_{S'}$, fixes any vertex in $H_0 \cup H_2$, we must have that $\iota(\phi)$ fixes also any vertex in $H_0 \cup H_2$. So, either $\phi = \mathrm{id}$ or $\phi = \varphi$. If $\phi = \mathrm{id}$ then $S = S'$, a contradiction. If $\phi = \varphi$ then $S = [p] \setminus S'$, as claimed. For the converse, by Statement (3) of Observation 3.4 we have $\psi_S = \psi_{S'} \circ \iota(\varphi)$ and so $\psi_S \in \psi_{S'}\Delta$ implying that $\psi_S\Delta = \psi_{S'}\Delta$.

Since $|\{S : S \subset [p]\}| = 2^{\binom{n}{k-1}}$, there are at least $2^{\binom{n}{k-1}}/2 = 2^{\binom{n}{k-1}-1}$ distinct cosets of type $\psi_S\Delta$, and then by the Lagrange's Theorem we have

$$|\Gamma| = [\Gamma : \Delta]\,|\Delta| \geq 2^{\binom{n}{k-1}-1}\,|\Delta| = 2^{\binom{n}{k-1}-1}\,|\operatorname{Aut}(K_{2,n})|.$$

In the case $k = \frac{n+2}{2}$, Δ is the subgroup of Γ consisting of induced automorphisms, the complement automorphism and any combination of these two types; here we have $|\Delta| = 2\,|\operatorname{Aut}(K_{m,n})|$. Since $\psi_S \neq \mathfrak{c}$, for any $S \subset [p]$, then, as in the previous case, there are $2^{\binom{n}{k-1}-1}$ distinct cosets of Δ, and so

$$|\Gamma| = [\Gamma : \Delta]\,|\Delta| \geq 2^{\binom{n}{k-1}-1}\,|\Delta| = 2^{\binom{n}{k-1}}\,|\operatorname{Aut}(K_{2,n})|.$$

□

3.2.2 Upper bound of Theorem 3.1

As we mentioned before, we can split the graph $F_k(K_{m,n})$ on the layers $H_0, \ldots, H_r$. The local behavior of an automorphism ψ (of $F_k(K_{m,n})$) plays an important role on the proof of the upper bound, specially, on the layers H_0 and H_1. Our strategy to show the upper bound for Theorem 3.1 consists of the following four steps:

- Step 1: For an arbitrary automorphism ψ of $F_k(K_{m,n})$, first we describe the behavior of layers $H_0, \ldots, H_r$ under ψ.
- Step 2: Then, we compute the number of ways in which we can label the vertices in $H_0 \cup H_1$ under ψ. Let $\eta(k, m, n)$ be this value.
- Step 3: We show that once we know $\psi\big|_{H_0 \cup H_1}$, the labels of the vertices in $H_2 \cup \ldots \cup H_r$ under ψ are uniquely determined.
- Step 4: Here we compute the value $\eta(k, m, n)$ and show that this value is an upper bound for $|\operatorname{Aut}(F_k(K_{m,n}))|$.

In what follows, ψ denotes an arbitrary, but fixed, automorphism of $F_k(K_{m,n})$.

3.2.2.1 Step 1.

Let us describe the behavior of sets $H_0, \ldots, H_r$ under ψ. Recall that

$$H_i = \{A \in F_k(K_{m,n}) : |A \cap X| = i\}.$$

Proposition 3.7. *There is a permutation τ of $\{0, 1, \ldots, r\}$ such that $\psi\big|_{H_i} = H_{\tau(i)}$ for each $i \in [r]$. Moreover,*

- *if $k = \frac{m+n}{2}$ or $m = n$ then either $\tau(i) = i$ (the identity map) or $\tau(i) = r - i$;*
- *if $k \neq \frac{m+n}{2}$ and $m \neq n$, then $\tau(i) = i$.*

Proof. First of all, let us show that $\psi\big|_{H_0} = H_0$ or $\psi\big|_{H_0} = H_r$. Let $A \in H_0$, we have $\deg(A) = km$. Let $B := \psi(A)$, so we must have $\deg(B) = km$. Let $i \in \{0, 1, \ldots, r\}$ such that $B \in H_i$, then $\deg(B) = i(n-k+i)+(k-i)(m-i) = km$, and after some calculations we have $i = 0$ or $i = \frac{m-n+2k}{2}$.

For a contradiction suppose that $i \in \{1, \ldots, r-1\}$, so

$$i = \frac{m-n+2k}{2}. \tag{17}$$

In this case, B has a neighbour in H_{i-1} and a neighbour in H_{i+1}. Let B_1 and B_2 be such vertices, respectively. Using Equation 17 and omitting some calculations we have

$$\deg(B_1) = (i-1)(n-k+(i-1)) + (k-(i-1))(m-(i-1)) = km+n-m-2k+2,$$

$$\deg(B_2) = (i+1)(n-k+(i+1)) + (k-(i+1))(m-(i+1)) = km-n+m+2k+2.$$

Since A has neighbours only in H_1, then all the km neighbours of A have degree equal to $(n-k+1)+(k-1)(m-1)$. Then, we must have $\deg(B_1) = \deg(B_2)$, which implies that $n - m = 2k$ and replacing this equality in Equation 17 we would have $i = 0$, contrary to our hypothesis that $i > 0$.

On the other hand, it may happen that $i = r$, and in such a case we have the following.

- If $r = m$ then $k = \frac{m+n}{2}$ and $i = m$; and
- if $r = k$ then $m = n$ and $i = k$.

Thus, $\psi(A) \in H_i$, where either $i = 0$ or $i = r$.

Next, let us show that either $\psi\big|_{H_0} = H_0$ or $\psi\big|_{H_0} = H_r$ in the case when $k = \frac{m+n}{2}$ or $k < m = n$, and $\psi\big|_{H_0} = H_0$, otherwise. To see this, it is enough to show that any two vertices $A, B \in H_0$, with $|A \triangle B| = 2$ and so $d(A, B) = 2$, satisfies that either $\psi(A), \psi(B) \in H_0$ or $\psi(A), \psi(B) \in H_r$ (in the corresponding cases), because if $|A \triangle B| > 2$, then it is enough to apply the previous argument for a sequence $A_0A_1 \ldots A_t$ with $A_0 = A$, $A_t = B$, $A_j \in H_0$ and $|A_j \triangle A_{j+1}| = 2$ for each $j \in \{0, 1, \ldots, t-1\}$. We distinguish the following cases depending on r.

- $\diamond$ $\boldsymbol{r = 1.}$

  If $\psi(A) \in H_0$ and $\psi(B) \in H_1$ then $d(\psi(A), \psi(B))$ is odd, and in particular, $d(\psi(A), \psi(B)) \neq 2$. Thus, either $\psi(A), \psi(B) \in H_0$ or $\psi(A), \psi(B) \in H_1$, as we wanted.

- $\diamond$ $\boldsymbol{r = 2.}$

  Since $|A \triangle B| = 2$, the vertices A and B have m common neighbours (in H_1). To derive a contradiction, suppose that $\psi(A) \in H_0$ and $\psi(B) \in H_r$. We must have $d(\psi(A), \psi(B)) = 2$, so $|\psi(A) \triangle \psi(B)| = 4$ and $\psi(A) \setminus \psi(B) \subset Y$ and $\psi(B) \setminus \psi(A) \subset X$, this implies that $\psi(A)$ and $\psi(B)$ have four common neighbours (in H_1). Then, if $m \neq 4$, we have a contradiction. Suppose then that $m = 4$, so $r = \min\{m, k\} = k = 2$ and then we must have

  $$8 = 4k = \deg(\psi(A)) = \deg(\psi(B)) = 2n$$

  which implies that $n = m = 4$. Recall that $|A \cap B| = k - 1 = 1$. Let $\{t_1, t_2, t_3, t_4\} = [4]$ such that $A = \{y_{t_1}, y_{t_2}\}$ and $B = \{y_{t_1}, y_{t_3}\}$, and let $C := \{y_{t_1}, y_{t_4}\}$. Note that A, B and C have four common neighbours in H_1, and so, $\psi(A), \psi(B), \psi(C)$ must have four common neighbours in $\psi\big|_{H_1}$, too. Either $\psi(C) \in H_0$ or $\psi(C) \in H_2$; however, in these both cases we have that $\psi(A), \psi(B)$ and $\psi(C)$ have less than four common neighbours in $H_1 = \psi\big|_{H_1}$, a contradiction.

- $\diamond$ $\boldsymbol{r > 2.}$

  It may not happen that $\psi(A) \in H_0$ and $\psi(B) \in H_r$ because we would have $d(\psi(A), \psi(B)) \geq r > 2$, a contradiction since ψ is an automorphism and $d(A, B) = 2$.

Thus, $\psi\big|_{H_0} = H_0$ or $\psi\big|_{H_0} = H_r$ in the case when $k = \frac{m+n}{2}$ or $k < m = n$, and $\psi\big|_{H_0} = H_0$, otherwise.

The vertices in H_i are the vertices which are at distance i from H_0, and the vertices in H_{r-i} are the vertices which are at distance i from H_r, these two observations together imply the following (see Figure 3.3):

$$\psi\big|_{H_i} = \begin{cases} H_i & \text{if } k = \frac{m+n}{2} \text{ or } m = n, \text{ and } \psi\big|_{H_0} = H_0, \\ H_{r-i} & \text{if } k = \frac{m+n}{2} \text{ or } m = n, \text{ and } \psi\big|_{H_0} = H_r, \\ H_i & \text{if } k \neq \frac{m+n}{2} \text{ and } m \neq n. \end{cases}$$

Finally, define the permutation τ (in each case) in the obvious way. $\square$

From now on, τ denotes the permutation given by Proposition 3.7.

3.2.2.2 Step 2.

First, we focus on the layer H_0, and we proceed to characterize the local behavior of ψ on this layer. The following is a simple observation.

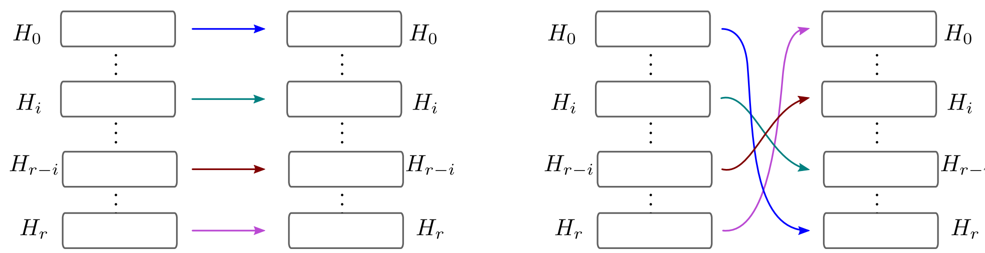


Figure 3.3: Possibilities of sets $H_0, \dots, H_r$ under ψ

Observation 3.8. *If $k = m = n$ then $|H_0| = |H_r| = 1$ and so,*

$$\psi\big(\{y_1, \dots, y_n\}\big) = \begin{cases} \{y_1, \dots, y_n\} & \text{if } \tau(i) = i, \\ \{x_1, \dots, x_m\} & \text{if } \tau(i) = r - i, \end{cases}$$

where $\{y_1, \dots, y_n\}$ (resp. $\{x_1, \dots, x_n\}$) is the only vertex in H_0 (resp. H_r).

For the case when $k < m = n$ or $m < n$, let us define the following subsets.

For $s \in [r]$ and $i \in [n]$ let

$$\begin{aligned} X(s,i) &:= \{A \in H_s : x_i \in A\} \\ X'(s,i) &:= \{A \in H_s : x_i \notin A\} \\ Y(s,i) &:= \{A \in H_s : y_i \in A\} \\ Y'(s,i) &:= \{A \in H_s : y_i \notin A\}. \end{aligned}$$

Proposition 3.9. *If $k < m = n$ or $m < n$, then there is a permutation σ of $[n]$ such that*

$$\psi\big|_{Y(0,i)} = \begin{cases} Y(0, \sigma(i)) & \text{if } \psi\big|_{H_0} = H_0, \\ X(r, \sigma(i)) & \text{if } \psi\big|_{H_0} = H_r \text{ and } k < m = n, \\ Y'(r, \sigma(i)) & \text{if } \psi\big|_{H_0} = H_r,\ m < n \text{ and } k = \frac{m+n}{2}. \end{cases}$$

Proof. Note that $k < n$. We proceed by cases.

Case 1: $\psi\big|_{H_0} = H_0$.

For a subset $S \subset Y$ with $1 \leq |S| \leq k - 1$, let

$$P_S := \{A \in H_0 : S \subset A\}.$$

Note that for $S = \{y_j\}$ we have $P_S = Y(0, j)$, for any $j \in [n]$. We claim that ψ sends P_S to $P_{S'}$, for some $S' \subset Y$ such that $|S'| = |S|$. We proceed by induction on $k - |S|$.

- Suppose $k - |S| = 1$, so $|S| = k - 1$.

Let $A_1, \dots, A_q$ be the vertices in P_S, where $q = n - (k-1)$. Let $B_1, \dots, B_q \in H_0$ such that $B_j = \psi(A_j)$, for $j \in [q]$. Since $|A_\ell \cap A_t| = |S| = k-1$ for any distinct $\ell, t \in [q]$, we have $d(A_\ell, A_t) = 2$. Since ψ is an automorphism of $F_k(K_{m,n})$, we must have $d(B_\ell, B_t) = 2$, and so $|B_\ell \cap B_t| = k-1$, for any two indices $\ell, t \in [p]$. Let $S' := \bigcap_{j \in [q]} B_j$, then it is enough to show that $|S'| = k-1$.

- If $k = n-1$, then $q = 2$ and so $S' = B_1 \cap B_2$ with $|S'| = k-1$.
- Suppose $k < n-1$. The vertices $A_1, \dots, A_q$ have m common neighbours in H_1, the vertices $S \cup \{x_1\}, \dots, S \cup \{x_m\}$. To derive a contradiction suppose that $|S'| < k-1$. Then, there are three indices $j, \ell, t \in [q]$ such that $B_j \cap B_\ell \neq B_\ell \cap B_t$, and this implies that the vertices B_j, B_ℓ and B_t have no common neighbours in $H_1 = \psi\big|_{H_1}$, which gives a contradiction since ψ is an automorphism. Thus, $|S'| = k-1$ as we wanted.

Then, the induction starts.

- Suppose that $k - |S| = k - i > 1$ and so $|S| = i < k-1$, and assume that the claim holds for any S^* such that $k - |S^*| < k - i$, so $|S^*| > i$.

Let $S_1, \dots, S_p \subset Y$ such that $S \subset S_j$ and $|S_j| = |S| + 1 = i + 1$, for $j \in [p]$. Note that $S = \bigcap_{j \in [p]} S_j$ and $P_S = \bigcup_{j \in [p]} P_{S_j}$. By the induction hypothesis, there are subsets $S'_1, \dots, S'_p$ of Y such that ψ sends P_{S_j} to $P_{S'_j}$, and $|S'_j| = |S_j| = i+1$, for $j \in [p]$. Let $S' := \bigcap_{j \in [p]} S'_j$. We claim that $|S'| = i = |S|$.

We have $|S_\ell \triangle S_t| = 2$, for any $\ell, t \in [p]$ with $\ell \neq t$, so we can match (uniquely) the vertices in $P_{S_\ell} \setminus P_{S_t}$ to the vertices in $P_{S_t} \setminus P_{S_\ell}$ as follows: for a vertex A in $P_{S_\ell} \setminus P_{S_t}$ there is a unique vertex B in $P_{S_t} \setminus P_{S_\ell}$ such that $|A \triangle B| = 2$ and so $d(A, B) = 2$, and vice versa. If $|S'_\ell \triangle S'_t| > 2$, then for a vertex $A \in P_{S'_\ell} \setminus P_{S'_t}$ such that $A \cap (S'_t \setminus S'_\ell) \neq \emptyset$, we have:

- $d(A, B) > 2$ for any $B \in P_{S'_t} \setminus P_{S'_\ell}$, or
- $d(A, B_1) = d(A, B_2) = 2$ for some two distinct vertices $B_1, B_2 \in P_{S'_t} \setminus P_{S'_\ell}$.

this gives a contradiction since ψ is an automorphism. Thus, $|S'_\ell \triangle S'_t| = 2$ and so $|S'_\ell \cap S'_t| = i$, for any distinct $\ell, t \in [p]$.

Next, we claim that $S'_\ell \cap S'_t \subset S'_j$, for any $j \in [p]$. Suppose not. Recall that $i \leq k-2$ by assumption and that $|S'_j \triangle S'_\ell| = |S'_j \triangle S'_t| = 2$. We have $|S'_j \cup S'_\ell \cup S'_t| = i + 2$ and $|S_j \cup S_\ell \cup S_t| = i + 3$. Then

$$|P_{S'_j} \cap P_{S'_\ell} \cap P_{S'_t}| = \binom{n-(i+2)}{k-(i+2)} = \binom{n-(i+2)}{n-k}$$

while

$$|P_{S_j} \cap P_{S_\ell} \cap P_{S_t}| = \binom{n-(i+3)}{k-(i+3)} = \binom{n-(i+3)}{n-k}.$$

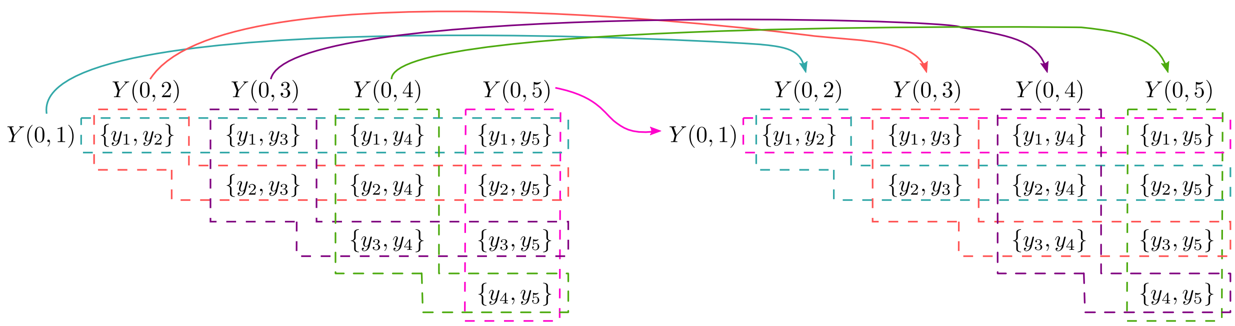


Figure 3.4: An example of σ on sets $Y(0,1), \dots, Y(0,n)$, where $n = 5$ and $\psi\big|_{H_0} = H_0$.

By the Pascal's Rule we know that $\binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b}$. Take $a = n - (i+3)$ and $b = n - k$. Since ψ is an automorphism of $F_k(K_{m,n})$ and ψ sends $(P_{S_j} \cap P_{S_\ell} \cap P_{S_t})$ to $(P_{S'_j} \cap P_{S'_\ell} \cap P_{S'_t})$, we must have $|P_{S'_j} \cap P_{S'_\ell} \cap P_{S'_t}| = |P_{S_j} \cap P_{S_\ell} \cap P_{S_t}|$, this implies that $\binom{a}{b-1} = \binom{n-(i+3)}{n-k-1} = 0$, and so $i > k-2$, which contradicts our assumption that $i \leq k-2$.

Thus, $S'_\ell \cap S'_t \subset S'_j$, and this implies that $|S'| = |S| = i$. Finally, $P_{S'} = \bigcup_{j \in [p]} P_{S'_j}$ and so, ψ sends $P_S = \bigcup_{j \in [p]} P_{S_j}$ to $P_{S'} = \bigcup_{j \in [p]} P_{S'_j}$, as claimed. In particular, ψ sends $P_i = Y(0,i)$ to some $P_j = Y(0,j)$. Let σ be the permutation of $[n]$ such that ψ sends $P_i = Y(0,i)$ to $P_{\sigma(i)} = Y(0,\sigma(i))$, this is our desired permutation. See an example in Figure 3.4.

Case 2: $\psi\big|_{H_0} = H_r$ and $k < m = n$.

Note that the subsets $X(r,1), \dots, X(r,m)$ have the same properties as the subsets $Y(0,1), \dots, Y(0,n)$, so applying similar arguments to those used in Case 1, we conclude the desired assertion.

Case 3: $\psi\big|_{H_0} = H_r$, $m < n$ and $k = \frac{m+n}{2}$.

In this case consider the isomorphism between $F_k(K_{m,n})$ and $F_{m+n-k}(K_{m,n})$ given by the complement isomorphism, which is indeed, an automorphism since $k = \frac{m+n}{2}$. For this reason, the subsets $Y'(r,1), \dots, Y'(r,n)$ have the same properties as the subsets $Y(0,1), \dots, Y(0,n)$, so in a similar way to Case 1 we get the corresponding assertion. $\square$

From now on, σ denotes the permutation given by Proposition 3.9 in the case when $k < m = n$ or $m < n$.

Next, we show that τ and σ are sufficient to determine the labels of the vertices in H_0 under ψ.

Corollary 3.10. *Suppose that $k < m = n$ or $m < n$. Given τ and σ, $\psi(A)$ is uniquely determined, for each $A \in H_0$.*

Proof. By Proposition 3.7 we know that either $\psi\big|_{H_0} = H_0$ or $\psi\big|_{H_0} = H_r$ when $k < m = n$ or $k = \frac{m+n}{2}$, and $\psi\big|_{H_0} = H_0$ otherwise, and these facts are determined by τ.

Let $A := \{y_{t_1}, \ldots, y_{t_k}\} \in H_0$. Suppose $\psi\big|_{H_0} = H_0$. Since $A \in \bigcap_{i\in[k]} Y(0, t_i)$, Proposition 3.9 implies that $\psi(A) \in \bigcap_{i\in[k]} Y(0, \sigma(t_i))$. Similarly, if $\psi\big|_{H_0} = H_r$, then $\psi(A) \in \bigcap_{i\in[k]} X(r, \sigma(t_i))$ when $k < m = n$, and $\psi(A) \in \bigcap_{i\in[k]} Y'(r, \sigma(t_i))$ when $m < n$ and $k = \frac{m+n}{2}$. Thus,

$$\psi(A) = \begin{cases} \{y_{\sigma(t_1)}, \ldots, y_{\sigma(t_k)}\} & \text{if } \psi\big|_{H_0} = H_0, \\ \{x_{\sigma(t_1)}, \ldots, x_{\sigma(t_k)}\} & \text{if } \psi\big|_{H_0} = H_r \text{ and } k < m = n, \\ V(K_{m,n}) \setminus \{y_{\sigma(t_1)}, \ldots, y_{\sigma(t_k)}\} & \text{if } \psi\big|_{H_0} = H_r,\ m < n \text{ and } k = \frac{m+n}{2}. \end{cases}$$

□

So far, we know the behavior of H_0 under ψ, let us now turn our attention to the vertex set H_1. Note that $\{X(1,1), \ldots, X(1,m)\}$ is a partition of H_1, and if $m = n$ (resp. $k = \frac{m+n}{2}$) $\{Y(r-1,1), \ldots, Y(r-1,n)\}$ (resp. $\{X'(r-1,1), \ldots, X'(r-1,1)\}$) is a partition of H_{r-1}.

In the following result we characterize the behavior of ψ on the sets $X(1,1), \ldots, X(1,m)$. Recall that we are assuming that $\{m,n\} \neq \{2,2\}$. We distinguish four cases:

- Case 1: $m = 1$.
- Case 2: $m = 2$.
- Case 3: $k = m = n > 2$.
- Case 4: $m > 2$ and either $k < m = n$ or $m < n$.

We study these cases separately, let us first consider the former case.

Proposition 3.11. *Suppose $m = 1$ and $n \geq 3$. Given τ and σ, $\psi(A)$ is uniquely determined, for each $A \in H_1$.*

Proof. Note that for any two vertices A and B in H_1, we have $N(A) \neq N(B)$, where $N(A), N(B) \subset H_0$, which implies that each vertex $A \in H_0$ is distinguished by its neighbours. Since τ and σ determine uniquely the labels of the vertices in H_0 under ψ, then $\psi(A)$ is uniquely determined for any $A \in H_1$. □

Next, we consider the second case: $m = 2$. Let $\{A_1, \ldots, A_p, B_1, \ldots, B_p\}$ be the set of vertices in H_1, where $p := \binom{n}{k-1}$, $A_i \setminus B_i = \{x_1\}$ and $B_i \setminus A_i := \{x_2\}$, for each $i \in [p]$.

Proposition 3.12. *Suppose $2 = m < n$. There exists a permutation α of $[p]$ such that ψ sends the pair $\{A_i, B_i\}$ to the pair $\{A_{\alpha(i)}, B_{\alpha(i)}\}$, where α is uniquely determined by τ and σ. Moreover, there exists a function $f : [p] \longrightarrow \{0, 1\}$ such that*

- *if $f(i) = 0$ then $\psi(A_i) = A_{\alpha(i)}$ and $\psi(B_i) = B_{\alpha(i)}$; and*
- *if $f(i) = 1$ then $\psi(A_i) = B_{\alpha(i)}$ and $\psi(B_i) = A_{\alpha(i)}$.*

Then, $\psi(A)$ is uniquely determined by τ, σ, α and f, for each $A \in H_1$.

Proof. Let $i, j \in [p]$ with $i \neq j$. Note that $N(A_i) = N(B_i)$, while $N(A_i) \neq N(A_j)$, $N(B_i) \neq N(B_j)$ and $N(A_i) \neq N(B_j)$. Note that $\psi\big|_{H_1} = H_1$.

Let $C_{i_1}, \ldots, C_{i_k} \in H_0$ be the neighbours of both A_i and B_i in H_0. Since ψ is an automorphism of $F_k(K_{2,n})$ and $\psi\big|_{H_1} = H_1$, there is $\ell \in [p]$ such that A_ℓ and B_ℓ have the common neighbours $\psi(C_{i_1}), \ldots, \psi(C_{i_k})$ in $\psi\big|_{H_0}$, and so ψ must send the pair $\{A_i, B_i\}$ to the pair $\{A_\ell, B_\ell\}$. Since the labels $\psi(C_{i_1}), \ldots, \psi(C_{i_k})$ are uniquely determined by the permutations τ and σ by Corollary 3.10, the index ℓ is uniquely determined. Let α be the permutation of $[p]$ such that ψ sends the pair $\{A_i, B_i\}$ to the pair $\{A_{\alpha(i)}, B_{\alpha(i)}\}$.

Finally, let us define the desired function f. Let $i \in [p]$. Since A_i and B_i (resp. $A_{\alpha(i)}$ and $B_{\alpha(i)}$) have the same neighbours, it may happen that $\psi(A_i) = A_{\alpha(i)}$ and $\psi(B_i) = B_{\alpha(i)}$, or $\psi(A_i) = B_{\alpha(i)}$ and $\psi(B_i) = A_{\alpha(i)}$. Then, in the former case let $f(i) := 0$, and in the latter case let $f(i) := 1$. Finally, note that $\psi(A)$ is uniquely determined by τ, σ, α and f, for each $A \in H_1$. □

Let us now consider the fourth case: $m > 2$ and either $k < m = n$ or $m < n$.

Proposition 3.13. *Suppose $k < m = n$ or $m < n$ with $m > 2$. Then there is a permutation γ of $[m]$ such that for each $i \in [m]$ we have*

$$\psi\big|_{X(1,i)} = \begin{cases} X(1, \gamma(i)) & \text{if } \psi\big|_{H_1} = H_1, \\ Y(r-1, \gamma(i)) & \text{if } \psi\big|_{H_1} = H_{r-1} \text{ and } k < m = n, \\ X'(r-1, \gamma(i)) & \text{if } \psi\big|_{H_1} = H_{r-1},\ m < n \text{ and } k = \frac{m+n}{2}. \end{cases}$$

Proof. Let $i \in [m]$ and let $A, B \in X(1, i)$. We are going to show that both $\psi(A)$ and $\psi(B)$ belong to either $X(1, j)$, $Y(r-1, j)$ or $X'(r-1, j)$, for some $j \in [m]$ and depending on each case. We may assume that $|A \triangle B| = 2$, as otherwise it is enough to consider a sequence of vertices $A_0, A_1, \ldots, A_t \subset X(1, i)$ such that $A_0 = A$, $A_t = B$ and $|A_{\ell-1} \triangle A_\ell| = 2$ for

$\ell \in [t]$. Since A and B have one common neighbour in H_0 and $m-1$ common neighbours in H_2, we must have that $\psi(A)$ and $\psi(B)$ have one common neighbour in $\psi\big|_{H_0}$ and $m-1$ common neighbours in $\psi\big|_{H_2}$. We proceed by cases.

Case 1: $\psi\big|_{H_1} = H_1$.

To derive a contradiction suppose that there are two distinct indices $\ell, t \in [m]$ such that $\psi(A) \in X(1,\ell)$ and $\psi(B) \in X(1,t)$. In this case, Proposition 3.7 implies that $\psi\big|_{H_0} = H_0$ and $\psi\big|_{H_2} = H_2$. Then,

- if $|\psi(A) \triangle \psi(B)| = 2$, then $\psi(A)$ and $\psi(B)$ have $k-1$ common neighbours in $H_2 = \psi\big|_{H_2}$ and $n-(k-1)$ common neighbours in $H_0 = \psi\big|_{H_0}$. Then, we must have $m-1 = k-1$ and $1 = n-(k-1)$, which implies that $k = m = n$, contrary to the hypothesis that either $k < m = n$ or $m < n$;
- if $|\psi(A) \triangle \psi(B)| > 2$, then $\psi(A)$ and $\psi(B)$ have at most on common neighbour in $H_2 = \psi\big|_{H_2}$, and since $m > 2$, this gives a contradiction because A and B have $m-1 > 1$ common neighbours in H_2.

Case 2: $\psi\big|_{H_1} = H_{r-1}$ and $k < m = n$.

Proposition 3.7 implies that $\psi\big|_{H_0} = H_r$ and $\psi\big|_{H_2} = H_{r-2}$. In this case, note that the sets $Y(r-1,1), \ldots, Y(r-1,n)$ have the same properties as $X(1,1), \ldots, X(1,m)$, then if $\psi(A) \in Y(r-1,\ell)$ and $\psi(B) \in Y(r-1,t)$, for $\ell, t \in [n]$ and $\ell \neq t$, in a similar way to Case 1 we get a contradiction.

Case 3: $\psi\big|_{H_1} = H_{r-1}$, $m < n$ and $k = \frac{m+n}{2}$.

Here, Proposition 3.7 implies that $\psi\big|_{H_0} = H_r$ and $\psi\big|_{H_2} = H_{r-2}$. In this case the isomorphism between $F_k(K_{m,n})$ and $F_{m+n-k}(K_{m,n})$ implies that the sets $X'(r-1,1), \ldots, X'(r-1,m)$ have the same properties as $X(1,1), \ldots, X(1,m)$. Then, applying a similar argument to Case 1 we can show the desired assertion.

Let γ be the permutation of $[m]$ such that ψ sends $X(1,i)$ to one of $X(1,\gamma(i))$, $Y(r-1,\gamma(i))$ or $X'(r-1,\gamma(i))$ (depending on the corresponding cases), for any $i \in [m]$. This completes the proof. $\square$

Finally, we consider the third case: $k = m = n > 2$. Note that the permutation σ does not apply for this case, but τ does.

Proposition 3.14. *Suppose $k = m = n > 2$. Then, given τ, there exist $\rho \in \{0,1\}$ and permutations σ' and γ' of $[m]$ which determine uniquely the labels of the vertices in H_1 under ψ.*

Proof. We have either $\tau(i) = i$ or $\tau(i) = r - i$. Suppose that $\tau(i) = i$, so $\psi\big|_{H_i} = H_i$ for each $i \in [r]$.

Consider the subsets $X(1,1),\dots,X(1,m)$ and $Y'(1,1),\dots,Y'(1,m)$. First, let us show that there is $\rho \in \{0,1\}$ such that

(a) if $\rho = 0$ then ψ sends the subsets in $\{X(1,1),\dots,X(1,m)\}$ to $\{X(1,1),\dots,X(1,m)\}$ and the subsets in $\{Y'(1,1),\dots,Y'(1,m)\}$ to the subsets in $\{Y'(1,1),\dots,Y'(1,m)\}$; and

(b) if $\rho = 1$ then ψ sends the subsets in $\{X(1,1),\dots,X(1,m)\}$ to $\{Y'(1,1),\dots,Y'(1,m)\}$ and the subsets in $\{Y'(1,1),\dots,Y'(1,m)\}$ to the subsets in $\{X(1,1),\dots,X(1,m)\}$.

Let $i \in [m]$. Note that $d(A,B) = 2$, for any two vertices A and B in $X(1,i)$, so, either $\psi(A),\psi(B) \in X(1,\ell)$, for some $\ell \in [m]$, or $\psi(A),\psi(B) \in Y'(1,t)$, for some $t \in [m]$, as otherwise we would have that $|\psi(A)\triangle\psi(B)| > 2$ and so $d(\psi(A),\psi(B)) > 2$, a contradiction. Suppose $\psi(A),\psi(B) \in X(1,\ell)$. Let $C \in X(1,i) \setminus \{A,B\}$, note that if $\psi(C) \notin X(1,\ell)$, then either $d(\psi(A),\psi(C)) > 2$ or $d(\psi(B),\psi(C)) > 2$, a contradiction since $d(A,C) = 2$, $d(B,C) = 2$ and ψ is an automorphism of $F_k(K_{m,n})$. Next, for $j \in [m] \setminus \{i\}$, we have that $X(1,i) \cap X(1,j) = \emptyset$, so we must have that $\psi\big|_{X(1,i)} \cap \psi\big|_{X(1,j)} = \emptyset$. Then, if ψ sends $X(1,j)$ to some $Y'(1,t)$, we would have

$$\psi\big|_{X(1,i)} \cap \psi\big|_{X(1,j)} = X(1,\ell) \cap Y'(1,t) \neq \emptyset,$$

a contradiction. Thus, ψ sends $X(1,j)$ to some $X(1,t)$, for some $t \in [m]$. Generalizing this approach we have that ψ sends the subsets in $\{X(1,1),\dots,X(1,m)\}$ to the subsets in $\{X(1,1),\dots,X(1,m)\}$ and the subsets in $\{Y'(1,1),\dots,Y'(1,m)\}$ to $\{Y'(1,1),\dots,Y'(1,m)\}$, so in this case let $\rho = 0$. Similarly, if $\psi(A),\psi(B) \in Y'(1,t)$, for some $t \in [m]$, then we have that ψ sends the subsets in $\{X(1,1),\dots,X(1,m)\}$ to the subsets in $\{Y'(1,1),\dots,Y'(1,m)\}$ and the subsets in $\{Y'(1,1),\dots,Y'(1,m)\}$ to the subsets in $\{X(1,1),\dots,X(1,m)\}$, here let $\rho = 1$.

Suppose that (a) holds. Then there are permutations σ' and γ' of $[m]$ such that for $i,j \in [m]$, $\psi\big|_{X(1,i)} = X(1,\sigma'(i))$ and $\psi\big|_{Y'(1,j)} = Y'(1,\gamma'(j))$. Now we claim that given these permutations σ' and γ', $\psi(A)$ is uniquely determined, for any $A \in H_1$. Indeed, let $A = (Y \setminus \{y_j\}) \cup \{x_i\} \in H_0$, then $A \in X(1,i) \cap Y'(1,j)$, and so $\psi(A) \in X(1,\sigma'(i)) \cap Y'(1,\gamma'(j))$, which implies that $\psi(A) = (Y \setminus \{y_{\gamma'(j)}\} \cup \{x_{\sigma'(i)}\})$, and so $\psi(A)$ is uniquely determined, as claimed.

Similarly, if (b) holds then we have permutations σ' and γ' of $[m]$ such that for $i,j \in [m]$, $\psi\big|_{X(1,i)} = Y'(1,\sigma'(i))$ and $\psi\big|_{Y'(1,j)} = X(1,\gamma'(j))$, and given these permutations, $\psi(A)$ is uniquely determined for each $A \in H_1$.

The case in which $\tau(i) = r - i$ can be handled in a similar manner. First, we can show that there is $\rho \in \{0,1\}$ such that

(a) if $\rho = 0$ then ψ sends the subsets in $\{X(1,1),\dots,X(1,m)\}$ to the subsets in $\{X'(r-$

$1,1),\ldots,X'(r-1,m)\}$ and the subsets in $\{Y'(1,1),\ldots,Y'(1,m)\}$ to the subsets in $\{Y(r-1,1),\ldots,Y(r-1,m)\}$; and

(b) if $\rho=1$ then ψ sends the subsets in $\{X(1,1),\ldots,X(1,m)\}$ to the subsets in $\{Y(r-1,1),\ldots,Y(r-1,m)\}$ and the subsets in $\{Y'(1,1),\ldots,Y'(1,m)\}$ to the subsets in $\{X'(r-1,1),\ldots,X'(r-1,m)\}$;

and in each one of these cases, there are permutations σ' and γ' which determine uniquely the labels of the vertices in H_1 under ψ. This completes the proof. □

3.2.2.3 Step 3.

Here, we proceed to show that once we know $\psi\big|_{H_0\cup H_1}$, the labels of the vertices in $H_2\cup\ldots\cup H_r$ under ψ are uniquely determined. To see this, in the following result we show that any vertex in H_i, for $i\geq 2$, is distinguishable by its neighbours belonging to H_{i-1}.

Proposition 3.15. *Let $i\in\{2,\ldots,r\}$, let A_1 and A_2 be two vertices in H_i with set of neighbours N_1 and N_2 in H_{i-1}, respectively. Then $N_1\neq N_2$.*

Proof. We proceed by cases.

- $(A_1\triangle A_2)\cap Y\neq\emptyset$.

  Let $y_{t_1}\in A_1\setminus A_2$ and $y_{t_2}\in A_2\setminus A_1$, for some $t_1,t_2\in[n]$ with $t_1\neq t_2$. Note that for any $B_1\in N_1$ we have $y_{t_1}\in B_1$. On the other hand, there is a vertex $B_2\in N_2$ such that $y_{t_1}\notin B_2$ (it is enough to consider a vertex $B_2\in N_2$ which is obtained by moving a token on a vertex $x'\in A_2\cap X$ to some vertex $y'\notin A_2$, where $y'\in Y\setminus\{y_{t_1}\}$). Thus, $N_1\neq N_2$.

- $(A_1\triangle A_2)\cap Y=\emptyset$.

  In this case we have $(A_1\triangle A_2)\cap X\neq\emptyset$. Let $x_{t_1}\in A_1\setminus A_2$ and $x_{t_2}\in A_2\setminus A_1$, for some $t_1,t_2\in[m]$ with $t_1\neq t_2$. Since $i\geq 2$, there is a vertex $x_t\in A_1$, where $t\neq t_1$, so there is a vertex $B_1\in N_1$ such that $x_{t_1}\in B_1$. On the other hand, there is no vertex $B_2\in N_2$ such that $x_{t_1}\in B_2$. Therefore, $N_1\neq N_2$.

□

Corollary 3.16. *Let ψ be any automorphism of $F_k(K_{m,n})$. If it is known $\psi(A)$ for each $A\in H_1$, then $\psi(B)$ is uniquely determined, for any $B\in H_2\cup\ldots\cup H_r$.*

3.2.2.4 Step 4.

Now we proceed to show the upper bound for $F_k(K_{m,n})$. Recall that $\eta(k,m,n)$ is the number of ways in which the vertices in $H_0 \cup H_1$ can be labelled by ψ. As we mentioned before, our strategy is to compute this number $\eta(k,m,n)$ and to show that $\eta(k,m,n) \geq |\operatorname{Aut}(F_k(K_{m,n}))|$.

Lemma 3.17. *Let m, n and k be integers with $\{m,n\} \neq \{2,2\}$, $1 \leq m \leq n$ and $2 \leq k \leq \frac{m+n}{2}$. Then*

$$|\operatorname{Aut}(F_k(K_{m,n}))| \leq \begin{cases} |\operatorname{Aut}(K_{m,n})| & \textit{if } m \neq 2 \textit{ and } k < \frac{m+n}{2}, \\ 2\,|\operatorname{Aut}(K_{m,n})| & \textit{if } m \neq 2 \textit{ and } k = \frac{m+n}{2}, \\ 2^{\binom{n}{k-1}-1}|\operatorname{Aut}(K_{2,n})| & \textit{if } m = 2 \textit{ and } k < \frac{m+n}{2}, \textit{ and} \\ 2^{\binom{n}{k-1}}|\operatorname{Aut}(K_{2,n})| & \textit{if } m = 2 \textit{ and } k = \frac{m+n}{2}. \end{cases}$$

Proof. By Corollary 3.16, once we know $\psi\big|_{H_1}$, we can determine uniquely $\psi(A)$ for each $A \in H_2 \cup \ldots \cup H_r$, this implies that $|\operatorname{Aut}(F_k(K_{m,n}))| \leq \eta(k,m,n)$. So, we proceed to compute $\eta(k,m,n)$ by cases.

- $\boldsymbol{m = 1.}$

  By Corollary 3.10 and Proposition 3.11 we know that τ and σ determine $\psi(A)$ for each $A \in H_0 \cup H_1$. Then, if $k < \frac{m+1}{2}$, there is one possibility for τ, and if $k = \frac{m+1}{2}$, there are two possibilities for τ; and in both cases there are $n!$ possibilities for σ. Thus,

$$|\operatorname{Aut}(F_k(K_{1,n}))| \leq \eta(k,1,n) = \begin{cases} n! = |\operatorname{Aut}(K_{1,n})| & \text{if } k < \frac{n+1}{2}, \\ 2\,n! = 2\,|\operatorname{Aut}(K_{1,n})| & \text{if } k = \frac{n+1}{2}. \end{cases}$$

- $\boldsymbol{m = 2.}$

  By Corollary 3.10 and Proposition 3.12, the permutations τ, σ and α, and the function f determine $\psi(A)$ for each $A \in H_0 \cup H_1$. If $k < \frac{n+2}{2}$ there is one possibility for τ and if $k = \frac{n+2}{2}$ there are two possibilities for τ. Moreover, there are $n!$ possibilities for σ and $2^{\binom{n}{k-1}}$ possibilities for f, while α is uniquely determined by τ and σ. Then,

$$\left|\operatorname{Aut}(F_k(K_{2,n}))\right| \leq \eta(k,2,n) = \begin{cases} 2^{\binom{n}{k-1}}\,n! = 2^{\binom{n}{k-1}-1}\left|\operatorname{Aut}(K_{2,n})\right| & \text{if } k < \frac{n+2}{2}, \\ 2^{\binom{n}{k-1}+1}\,n! = 2^{\binom{n}{k-1}}\left|\operatorname{Aut}(K_{2,n})\right| & \text{if } k = \frac{n+2}{2}. \end{cases}$$

- $\boldsymbol{k = m = n.}$

  By Observation 3.8 and Proposition 3.14, the permutations τ, σ' and γ' and the number ρ determine uniquely $\psi(A)$ for any $A \in H_0 \cup H_1$. Here note that $k = \frac{m+n}{2}$.

Since there are two possibilities for τ, $m!$ possibilities for σ', $n!$ possibilities for γ' and two possibilities for ρ, we have

$$\big|\operatorname{Aut}(F_k(K_{m,n}))\big| \leq \eta(k,m,n) = 4\,m!\,n! = 2\big|\operatorname{Aut}(K_{m,n})\big|.$$

- $\boldsymbol{k < m = n.}$

  Note that $k < \frac{m+n}{2}$. By Corollary 3.10 and Proposition 3.13, the permutations τ, σ and γ determine uniquely $\psi(A)$ for each $A \in H_0 \cup H_1$, where there are two possibilities for τ, $n!$ possibilities for σ and $m!$ possibilities for γ, so

$$\big|\operatorname{Aut}(F_k(K_{m,n}))\big| \leq \eta(k,m,n) = 2\,m!\,n! = \big|\operatorname{Aut}(K_{m,n})\big|.$$

- $\boldsymbol{2 < m < n.}$

  As before, Corollary 3.10 and Proposition 3.13, τ, σ and γ determine uniquely $\psi(A)$ for each $A \in H_0 \cup H_1$. If $k < \frac{m+n}{2}$ there are one possibility for τ, and if $k = \frac{m+n}{2}$ there are two possibilities for τ. On the other hand, there are $n!$ possibilities for σ and $m!$ possibilities for γ, which imply that

$$\big|\operatorname{Aut}(F_k(K_{m,n}))\big| \leq \eta(k,m,n) = \begin{cases} m!\,n! = \big|\operatorname{Aut}(K_{m,n})\big| & \text{if } k < \frac{m+n}{2}, \\ 2\,m!\,n! = 2\,\big|\operatorname{Aut}(K_{m,n})\big| & \text{if } k = \frac{m+n}{2}. \end{cases}$$

This completes the proof. □

3.3 Cartesian product of graphs

In this section we focus on the Cartesian product of non-trivial graphs. We point out that the Cartesian product of graphs has many induced 4-cycles, and it may have also diamond graphs. So, this family of graphs was not considered in Chapter 2.

The aim of this section is to show Theorem 3.2. We recall it next.

Theorem 3.2. *Let G be a connected graph with prime factor decomposition $G = G_1 \,\square\, \ldots \,\square\, G_r$, where $r > 1$, $n = |G|$ and $2 \leq k \leq n/2$. Then*

$$|\operatorname{Aut}(F_k(G))| \geq \begin{cases} 2^{r-1}\,|\operatorname{Aut}(G)| & \textit{if } k = 2, \\ 2\,|\operatorname{Aut}(G)| & \textit{if } k = \frac{n}{2}, \\ |\operatorname{Aut}(G)| & \textit{if } 2 < k < \frac{n}{2}. \end{cases}$$

Moreover, this lower bound is tight.

3.3.1 Proof of Theorem 3.2

Note that if $2 < k \leq n/2$, the lower bound is given by Inequalities 1 and 2. So, we may assume that $k = 2$. For the cube graph Q_2 we have $|\operatorname{Aut}(F_2(Q_2))| = 48$, so let us assume that $|G| > 4$. Note that the complement automorphism does not apply here.

We now define some automorphisms of $F_2(G)$. Recall that a vertex $a \in G$ is thought as $a = (a_1, a_2, \ldots, a_r)$, where $a_i \in G_i$ for $i \in [r]$.

For $i \in [r]$, let

$$\psi_i\big(\big\{(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_r), (b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_r)\big\}\big) := \big\{(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_r), (b_1, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots, b_r)\big\}.$$

First, let us show that ψ_i is, indeed, an automorphism of $F_2(G)$.

Proposition 3.18. *The function ψ_i is an automorphism of $F_2(G)$.*

Proof. Let us assume that $i = 1$, the case $i > 1$ is similar.

Let $A, B \in F_2(G)$. We are going to show that A and B are adjacent if and only if $\psi_1(A)$ and $\psi_1(B)$ are adjacent. Suppose that A is adjacent to B, then $A = \{x, y\}$ and $B = \{y, z\}$, for some $x, y, z \in G$, where $x = (x_1, x_2, \ldots, x_r)$, $y = (y_1, y_2, \ldots, y_r)$ and $z = (z_1, z_2, \ldots, z_r)$, and x and z are adjacent. Let $j \in [r]$ such that x_j is adjacent to z_j in G_j, and $x_t = z_t$, for $t \in [r] \setminus \{j\}$. Then,

$$\psi_1(A) = \big\{(y_1, x_2, \ldots, x_r), (x_1, y_2, \ldots, y_r)\big\},$$

and

$$\psi_1(B) = \big\{(z_1, y_2, \ldots, y_r), (y_1, z_2, \ldots, z_r)\big\}.$$

We distinguish the following two cases.

- Suppose $j = 1$.

  Then $(y_1, x_2, \ldots, x_r) = (y_1, z_2, \ldots, z_r)$, and vertex $(x_1, y_2, \ldots, y_r)$ is adjacent to vertex $(z_1, y_2, \ldots, y_r)$, which implies that $\psi_1(A)$ is adjacent to $\psi_1(B)$.

- Suppose $j > 1$.

  Then $(x_1, y_2, \ldots, y_r) = (z_1, y_2, \ldots, y_r)$, and vertex $(y_1, x_2, \ldots, x_r)$ is adjacent to vertex $(y_1, z_2, \ldots, z_r)$, implying that $\psi_1(A)$ is adjacent to $\psi_1(B)$.

Suppose now that $\psi_1(A)$ is adjacent to $\psi_1(B)$. Applying the previous argument, we have that $\psi_1(\psi_1(A))$ is adjacent to $\psi_1(\psi_1(B))$. Note that $\psi_1 \circ \psi_1 = \mathrm{id}$, which implies that $\psi_1(\psi_1(A)) = A$ is adjacent to $\psi_1(\psi_1(B)) = B$, as we wanted. $\square$

For $i \neq j$ we have $\psi_i \neq \psi_j$ and $\psi_i \circ \psi_j = \psi_j \circ \psi_i$. For $S = \{i_1, i_2, \ldots, i_t\} \subseteq [r]$, let

$$\psi_S := \psi_{i_1} \circ \psi_{i_2} \circ \ldots \circ \psi_{i_t}.$$

Let us note that it does not matter the order of functions $\psi_{i_1}, \ldots, \psi_{i_t}$ in the definition of ψ_S, since they are pairwise commutative.

Observation 3.19. *The function ψ_S is an automorphism of $F_2(G)$, where $\psi_S^{-1} = \psi_S$. Moreover, for $S_1, S_2 \subseteq [r]$, with $S_1 \neq S_2$, we have $\psi_{S_1} = \psi_{S_2}$ if and only if $S_1 = [r] \setminus S_2$.*

For every vertex $x \in G$, consider the vertex set $\kappa_G(x, 2)$ defined in Chapter 2 by

$$\kappa_G(x, 2) = \{A \in F_2(G) : x \in A\}.$$

For an induced automorphism $\iota(\sigma)$ of $F_2(G)$ we have the following: for each vertex $x \in G$ there is a vertex $y \in G$ such that $\iota(\sigma)$ sends the subset $\kappa_G(x, 2)$ to $\kappa_G(y, 2)$.

The next step is to determine when ψ_S is an induced automorphism of $F_2(G)$.

Proposition 3.20. *The automorphism ψ_S is an induced automorphism of $F_2(G)$ if and only if $S = \emptyset$ or $S = [r]$.*

Proof. If $S = \emptyset$ or $S = [r]$ then $\psi_S = \mathrm{id}$, and so it is an induced automorphism. Suppose now that $\emptyset \neq S \neq [r]$.

Without loss of generality let us assume that $S = \{1, \ldots, t\}$, where $t < r$ (otherwise change the order of $G_1, \ldots, G_r$). Consider the vertices

$$\begin{aligned} A_1 &= \{(a_1, a_2, \ldots, a_{r-1}, a_r), (b_1, a_2, \ldots, a_{r-1}, a_r)\}, \\ A_2 &= \{(a_1, a_2, \ldots, a_{r-1}, a_r), (a_1, a_2, \ldots, a_{r-1}, b_r)\}, \\ A_3 &= \{(a_1, a_2, \ldots, a_{r-1}, a_r), (b_1, a_2, \ldots, a_{r-1}, b_r)\}, \end{aligned}$$

with $a_1 \neq b_1$ and $a_r \neq b_r$. Then we have

$$\begin{aligned} \psi_S(A_1) &= \{(b_1, a_2, \ldots, a_{r-1}, a_r), (a_1, a_2, \ldots, a_{r-1}, a_r)\}, \\ \psi_S(A_2) &= \{(a_1, a_2, \ldots, a_{r-1}, a_r), (a_1, a_2, \ldots, a_{r-1}, b_r)\}, \\ \psi_S(A_3) &= \{(b_1, a_2, \ldots, a_{r-1}, a_r), (a_1, a_2, \ldots, a_{r-1}, b_r)\}. \end{aligned}$$

Then, for the vertex $a := (a_1, a_2, \ldots, a_{r-1}, a_r)$ there is no vertex $x \in G$ such that f sends the subset $\kappa_G(a, 2)$ to $\kappa_G(x, 2)$, because the vertices $\psi_S(A_1), \psi_S(A_2), \psi_S(A_3)$ do not belong to any $\kappa_G(x, 2)$, this implies that ψ_S cannot be an induced automorphism, as claimed. $\square$

Next, we show an special property about the commutativity of ψ_S in $\mathrm{Aut}(F_2(G))$.

Proposition 3.21. *Let $\iota(\phi)$ be an induced automorphism of $F_2(G)$, then ψ_S and $\iota(\phi)$ commute, that is,*

$$\psi_S \circ \iota(\phi) = \iota(\phi) \circ \psi_S,$$

for any $S \subseteq [r]$.

Proof. Let us first show that ψ_i holds the claim, for any $i \in [r]$. Without loss of generality assume $i = 1$, so we are going to show that $\psi_1 \circ \iota(\phi) = \iota(\phi) \circ \psi_1$. Let $A = \{x, y\} \in G$, with $x := (x_1, \ldots, x_r)$ and $y := (y_1, \ldots, y_r)$. Then

$$\begin{aligned}
(\psi_1 \circ \iota(\phi))(A) &= \psi_1\big(\iota(\phi)(\{(x_1, x_2, \ldots, x_r), (y_1, y_2, \ldots, y_r)\})\big) \\
&= \psi_1\big(\{\phi(x_1, x_2, \ldots, x_r), \phi(y_1, y_2, \ldots, y_r)\}\big) \\
&= \psi_1\Big(\big\{\big(\phi_1(x_{\pi(1)}), \phi_2(x_{\pi(2)}), \ldots, \phi_r(x_{\pi(r)})\big), \big(\phi_1(y_{\pi(1)}), \phi_2(y_{\pi(2)}), \ldots, \phi_r(y_{\pi(r)})\big)\big\}\Big) \\
&= \big\{\big(\phi_1(y_{\pi(1)}), \phi_2(x_{\pi(2)}), \ldots, \phi_r(x_{\pi(r)})\big), \big(\phi_1(x_{\pi(1)}), \phi_2(y_{\pi(2)}), \ldots, \phi_r(y_{\pi(r)})\big)\big\} \\
&= \iota(\phi)\Big(\big\{(y_1, x_2, \ldots, x_r), (x_1, y_2, \ldots, y_r)\big\}\Big) \\
&= \iota(\phi)\Big(\psi_1\big(\{(x_1, x_2, \ldots, x_r), (y_1, y_2, \ldots, y_r)\}\big)\Big) \\
&= (\iota(\phi) \circ \psi_1)(A)
\end{aligned}$$

Then, for $S = \{i_1, \ldots, i_t\}$ and $\iota(\phi)$ an induced automorphism of $F_2(G)$ we have

$$\psi_S \circ \iota(\phi) = \psi_{i_1} \circ \ldots \circ \psi_{i_t} \circ \iota(\phi) = \iota(\phi) \circ \psi_{i_1} \circ \ldots \circ \psi_{i_t} = \iota(\phi) \circ \psi_S.$$

□

Now, we proceed to show Theorem 3.2:

Theorem 3.2. *Let G be a connected graph with prime factor decomposition $G = G_1 \square \ldots \square G_r$, where $r > 1$, $n = |G|$ and $2 \leq k \leq n/2$. Then*

$$|\operatorname{Aut}(F_k(G))| \geq \begin{cases} 2^{r-1}\,|\operatorname{Aut}(G)| & \text{if } k = 2, \\ 2\,|\operatorname{Aut}(G)| & \text{if } k = \frac{n}{2}, \\ |\operatorname{Aut}(G)| & \text{if } 2 < k < \frac{n}{2}. \end{cases}$$

Moreover, this lower bound is tight.

Proof. As we mentioned before, the cases $2 < k < \frac{n}{2}$ and $k = \frac{n}{2}$ follow from Inequalities (1) and (2), respectively. Assume then that $k = 2$.

As in the previous section, we denote by Γ the group $\operatorname{Aut}(F_2(G))$ and by Δ the subgroup of Γ consisting of the induced automorphisms. We are going to give a lower bound for the index $[\Gamma : \Delta]$. Consider two subsets $S_1, S_2 \subseteq [r]$ with $S_1 \neq S_2$. We claim that $\psi_{S_1}\Delta = \psi_{S_2}\Delta$ if and only if $S_1 = [r] \setminus S_2$.

Suppose $\psi_{S_1}\Delta = \psi_{S_2}\Delta$, so $\psi_{S_1} \in \psi_{S_2}\Delta$, which implies that $\psi_{S_1} = \psi_{S_2} \circ \iota(\phi)$, for some $\iota(\phi) \in \Delta$. Let $S := S_1 \triangle S_2$, then we have

$$\psi_S = \psi_{S_1} \circ \psi_{S_2} = \psi_{S_2} \circ \psi_{S_1} = (\psi_{S_2})^{-1} \circ \psi_{S_1}$$

because $\psi_{S_2} = (\psi_{S_2})^{-1}$ by Observation 3.19, and so

$$\psi_S = \psi_{S_2} \circ \psi_{S_1} = (\psi_{S_2})^{-1} \circ \psi_{S_1} = \iota(\phi) \in \Delta.$$

By Proposition 3.20 we have then that $S = \emptyset$ or $S = [r]$. However, if $S = \emptyset$ then $S_1 = S_2$, contrary to our assumption that $S_1 \neq S_2$, so $S = [r]$ and then $S_1 = [r] \setminus S_2$, as we wanted.

For the converse, by Observation 3.19 we know that $\psi_{S_1} = \psi_{S_2}$ and so clearly $\psi_{S_1}\Delta = \psi_{S_2}\Delta$.

Since $|\{S : S \subseteq [r]\}| = 2^r$ and $\psi_{S_1}\Delta = \psi_{S_2}\Delta$ if and only if $S_1 = [r] \setminus S_2$, it follows that $[\Gamma : \Delta] \geq 2^{r-1}$, and then by the Lagrange's Theorem we have

$$|\operatorname{Aut}(F_2(G))| \geq 2^{r-1}\,|\operatorname{Aut}(G)|.$$

This completes the proof. $\square$

3.3.2 Theorem 3.2 is tight

Suppose $k = 2$. We are going to show that the lower bound given in Theorem 3.2 is tight. For this, we show that cube graph Q_r attains the lower bound for any $r \geq 3$, that is, the 2-token graph of a cube graph Q_r satisfies that

$$|\operatorname{Aut}(F_2(Q_r))| = 2^{r-1}(2^r\, r!) = 2^{r-1}|\operatorname{Aut}(Q_r)|.$$

For $r = 3$ we have $|\operatorname{Aut}(F_2(Q_r))| = 192 = 2^2(48) = 2^{r-1}|\operatorname{Aut}(Q_r)|$, so from now on let us assume $r \geq 4$. Let ψ be an automorphism of $F_2(Q_r)$.

The cube graph Q_r can be defined inductively as $Q_r := K_2 \square Q_{r-1}$, where Q_1 is the graph consisting of one edge. So, Q_r can be thought as two copies of Q_{r-1} joined by a perfect matching. For a vertex x of Q_r and $i \in [r]$, let $x(i)$ be its i-th coordinate, where $x(i) \in \{0, 1\}$. In Figure 1.4 are depicted the r-cubes for $r \in \{1, 2, 3, 4\}$.

In the cube graph Q_r, every vertex has degree r, so, for a vertex $A = \{x, y\} \in F_2(Q_r)$ we have

$$\deg(A) = \begin{cases} 2r - 2 & \text{if } x \text{ and } y \text{ are adjacent,} \\ 2r & \text{if } x \text{ and } y \text{ are not adjacent.} \end{cases}$$

Let us now define some subsets of $F_2(Q_r)$. For $i \in [r]$ let

$$\mathcal{R}_i := \{\{x, y\} \in F_2(Q_r) : x(i) = y(i) = 0\},$$
$$\mathcal{S}_i := \{\{x, y\} \in F_2(Q_r) : x(i) = y(i) = 1\},$$
$$\mathcal{T}_i := \{\{x, y\} \in F_2(Q_r) : x(i) \neq y(i)\}.$$

Consider the following interpretation of sets $\mathcal{R}_i$, $\mathcal{S}_i$ and $\mathcal{T}_i$: let Q and Q' be the two copies of Q_{r-1} such that when are joined by a perfect matching we obtain the cube Q_r, assume that $x(i) = 0$ for any $x \in Q$, and $y(i) = 1$ for any $y \in Q'$, then

- $\mathcal{R}_i$ corresponds to the 2-token configurations with the two tokens placed at vertices of Q,
- $\mathcal{S}_i$ corresponds to the 2-token configurations with the two tokens placed at vertices of Q', and
- $\mathcal{T}_i$ corresponds to the 2-token configurations with one token placed at a vertex of Q and the other token placed at a vertex of Q'.

Then,

(A1) $\{\mathcal{R}_i, \mathcal{S}_i, \mathcal{T}_i\}$ is a partition of $F_2(Q_r)$,

(A2) the subgraph induced by $\mathcal{R}_i$ (resp. $\mathcal{S}_i$) is isomorphic to $F_2(Q_{r-1})$, and the subgraph induced by $\mathcal{T}_i$ is isomorphic to the Cartesian product $Q_{r-1} \square Q_{r-1} = Q_{2(r-1)}$.

Let

$$\mathcal{H}_1 := \{A \in F_2(Q_r) : \deg(A) = 2r - 2\},$$
$$\mathcal{H}_2 := \{A \in F_2(Q_r) : \deg(A) = 2r\}.$$

Then,

(A3) ψ sends $\mathcal{H}_i$ to $\mathcal{H}_i$, for $i \in \{1, 2\}$.

Given two vertices $x, y \in G$, let $\Lambda(x, y) := \{i \in [r] : x(i) \neq y(i)\}$ and $\lambda(x, y) := |\Lambda(x, y)|$. Note that

(A4) The vertices x and y are adjacent if and only if $\lambda(x, y) = 1$.

For a vertex $A = \{x, y\} \in F_2(Q_r)$, let $d(A, \mathcal{H}_1)$ be the distance from A to some vertex in $\mathcal{H}_1$, where $d(A, \mathcal{H}_1) = 0$ if $A \in \mathcal{H}_1$. For a vertex $\{u, v\} \in F_2(Q_r)$, we have $d(\{u, v\}, \mathcal{H}_1) = \lambda(u, v) - 1$.

For $i \in [r]$, let
$$\mathcal{T}_i^* := \{A = \{x, y\} \in \mathcal{T}_i : \Lambda(x, y) = \{i\}\}.$$

The following result shows the local behavior of $\mathcal{T}_r^*$ under ψ.

Lemma 3.22. *There is $t \in [r]$ such that ψ sends $\mathcal{T}_r^*$ to $\mathcal{T}_t^*$.*

Proof. Consider a vertex $A = \{x, y\}$ with $\Lambda(x, y) = \{r\}$. Since $A \in \mathcal{H}_1$, by (A3) we have $\psi(A) \in \mathcal{H}_1$, and then for some $t \in [r]$, there are $w, z \in G$ such that $\Lambda(w, z) = \{t\}$ and $\psi(A) = \{w, z\} \in \mathcal{T}_t$. With this t fixed, we are going to show that for any vertex $B \in \mathcal{T}_r^*$ at distance two from A, $\psi(B) \in \mathcal{T}_t^*$.

Note that x and y are adjacent, as well as w and z. Let $N(x) := \{x_1, \ldots, x_{r-1}, y\}$, $N(y) := \{y_1, \ldots, y_{r-1}, x\}$, $N(w) := \{w_1, \ldots, w_{r-1}, z\}$ and $N(z) := \{z_1, \ldots, z_{r-1}, w\}$ be the neighborhoods in Q_r of x, y, w and z, respectively. Since x is adjacent to y, there is a perfect matching joining the subsets $N(x) \setminus \{y\}$ and $N(y) \setminus \{x\}$, so we may assume that x_i is adjacent to y_i, for $i \in [r-1]$. Similarly, there is a perfect matching joining the subsets $N(w) \setminus \{z\}$ and $N(z) \setminus \{w\}$, and we may assume that w_j is adjacent to z_j, for any $j \in [r-1]$.

Let
$$M_A := \{\{x_1, y_1\}, \ldots, \{x_{r-1}, y_{r-1}\}, \{x_1, x\}, \ldots, \{x_{r-1}, x\}, \{y_1, y\}, \ldots, \{y_{r-1}, y\}\}$$
and
$$M_{\psi(A)} := \{\{w_1, z_1\}, \ldots, \{w_{r-1}, z_{r-1}\}, \{w_1, w\}, \ldots, \{w_{r-1}, w\}, \{z_1, z\}, \ldots, \{z_{r-1}, z\}\}.$$
Note that M_A (resp. $M_{\psi(A)}$) is the set of vertices in $\mathcal{H}_1$ which are at distance two from A (resp. $\psi(A)$). Then, ψ sends M_A to $M_{\psi(A)}$. Notice that $\{x_i, y_i\}$ is at distance two from exactly two vertices in M_A, for any $i \in [r-1]$, and these vertices are $\{x_i, x\}$ and $\{y_i, y\}$. On the other hand, $\{x_i, x\}$ is at distance two from $\{y_i, y\}$ and from $\{x_j, x\}$, for each $1 \leq j \leq r-1$. Since $r \geq 4$, it follows that the vertex $\{x_i, x\}$ is at distance two from at least three vertices in M_A. Similarly, the vertex $\{y_i, y\}$ is at distance two from at least three vertices in M_A. Then,
$$L_A = \big\{\{x_1, y_1\}, \ldots, \{x_{r-1}, y_{r-1}\}\big\}$$
is the set of vertices in $F_2(Q_r)$ at distance two from exactly two vertices in M_A, and similarly,
$$L_{\psi(A)} = \big\{\{w_1, z_1\}, \ldots, \{w_{r-1}, z_{r-1}\}\big\}$$
is the set of vertices in $F_2(Q_r)$ at distance two from exactly two vertices in $M_{\psi(A)}$. Thus, ψ sends L_A to $L_{\psi(A)}$. Note that L_A (resp. $L_{\psi(A)}$) is the set of vertices in $\mathcal{T}_r^*$ (resp. $\mathcal{T}_t^*$) at distance two from A (resp. $\psi(A)$).

Applying the previous argument as often as necessary we show that ψ sends $\mathcal{T}_r^*$ to $\mathcal{T}_t^*$. $\square$

Next we focus on the behavior of $\mathcal{R}_i$, $\mathcal{S}_i$ and $\mathcal{T}_i$ under ψ.

Lemma 3.23. *There is $t \in [r]$ and $\alpha \in \{0,1\}$ such that $\psi\big|_{\mathcal{T}_r} = \mathcal{T}_t$, and*

- *if $\alpha = 0$ then $\psi\big|_{\mathcal{R}_r} = \mathcal{R}_t$ and $\psi\big|_{\mathcal{S}_r} = \mathcal{S}_t$; and*
- *if $\alpha = 1$ then $\psi\big|_{\mathcal{R}_r} = \mathcal{S}_t$ and $\psi\big|_{\mathcal{S}_r} = \mathcal{R}_t$.*

Proof. Consider a vertex $A = \{x,u\} \in \mathcal{T}_r$ and let $\psi(A) = \{v,w\}$. We proceed by induction on $\ell := d(A, \mathcal{H}_1)$. For $\ell = 0$ we have $A \in \mathcal{T}_r^*$, and then by Lemma 3.22, there is $t \in [r]$ such that $\psi(A) \in \mathcal{T}_t^* \subset \mathcal{T}_t$. Moreover, by Lemma 3.22 we know that ψ maps $\mathcal{T}_r^*$ to $\mathcal{T}_t^*$, so the induction starts. Assume $\ell > 0$.

We must have $d(A, \mathcal{H}_1) = d(\psi(A), \mathcal{H}_1)$ because $\psi\big|_{\mathcal{H}_1} = \mathcal{H}_1$, and we also have $\lambda(x,u) = d(A,\mathcal{H}_1) + 1 = \ell + 1$. Let $B = \{x,y\} \in \mathcal{T}_r$ be a neighbor of A with $d(B,\mathcal{H}_1) = \ell - 1$, so $\lambda(x,y) = \ell$. Since A and B are adjacent in $F_2(Q_r)$, $\psi(A)$ and $\psi(B)$ are also adjacent in $F_2(Q_r)$. Without loss of generality let us assume that $w \in \psi(B)$, and let $z \in Q_r$ such that $\psi(B) = \{w,z\}$, so v and z are adjacent in Q_r. Since $d(B,\mathcal{H}_1) = \ell - 1$, by the induction hypothesis we have $\psi(B) \in \mathcal{T}_t$, so $w(t) \neq z(t)$. Let $i_1, \dots, i_\ell \in [r]$ such that $\Lambda(w,z) = \{i_1, \dots, i_\ell\}$, assuming that $i_1 = t$. Consider the following cases.

- If $v(t) = z(t)$ then $v(t) \neq w(t)$, and so $\psi(A) = \{v,w\} \in \mathcal{T}_t$ and we are done.
- Suppose now that $v(t) \neq z(t)$, then $v(t) = w(t)$. Since v and z are adjacent, we have $v(j) = z(j)$ for each $j \in [r] \setminus \{t\}$, which implies that $\Lambda(v,w) = \{i_2, \dots, i_\ell\}$, and so
$$\ell - 1 = d\big(\psi(A), \psi\big|_{\mathcal{H}_1}\big) \neq d(A, \mathcal{H}_1) = \ell,$$
a contradiction.

Thus, $\psi\big|_{\mathcal{T}_r} = \mathcal{T}_t$.

Next, note that $F_2(Q_r) - \mathcal{T}_r$ has two components, $\mathcal{S}_r$ and $\mathcal{R}_r$, and similarly, $F_2(Q_r) - \mathcal{T}_t$ has two components $\mathcal{R}_t$ and $\mathcal{S}_t$, then either: $\psi\big|_{\mathcal{R}_r} = \mathcal{R}_t$ and $\psi\big|_{\mathcal{S}_r} = \mathcal{S}_t$, or $\psi\big|_{\mathcal{R}_r} = \mathcal{S}_t$ and $\psi\big|_{\mathcal{S}_r} = \mathcal{R}_t$. Define $\alpha = 0$ in the former case and $\alpha = 1$ in the later case. This completes the proof. □

We next show that the labels of vertices in $\mathcal{R}_r \cup \mathcal{S}_r$ under ψ are strongly related.

Lemma 3.24. *Let t and α as in Lemma 3.23. Given t, α and $\psi\big|_{\mathcal{R}_r}$, the labels of the vertices in $\mathcal{S}_r$ under ψ are uniquely determined.*

Proof. Suppose that $\alpha = 0$. Then ψ sends $\mathcal{R}_r$ to $\mathcal{R}_t$. Recall that $\mathcal{R}_r \simeq F_2(Q_{r-1}) \simeq \mathcal{R}_t$, and that the vertices in $\mathcal{R}_r$ can be matched uniquely to the vertices in $\mathcal{S}_r$ as follows: for a vertex $A \in \mathcal{R}_r$ there is a unique vertex $B \in \mathcal{S}_r$ such that $d(A, B) = 2$, and vice versa. This property implies that given t, α and $\psi\big|_{\mathcal{R}_r}$, the labels of the vertices in $\mathcal{S}_r$ under ψ are also uniquely determined.

The case $\alpha = 1$ can be handled in a similar manner. □

Recall that $\{\mathcal{R}_r, \mathcal{S}_r, \mathcal{T}_r\}$ is a partition of $V(F_2(Q_r))$, and so far, given α, t and $\psi\big|_{\mathcal{R}_r}$ we know the labels of the vertices in $\mathcal{R}_r \cup \mathcal{S}_r$ under ψ. Now, we consider the vertices in $\mathcal{T}_r$.

Let $A_1, \ldots, A_s$ be the vertices in $\mathcal{R}_r$, with $A_i = \{x_i, y_i\}$ and $x_i \neq y_i$, and let $C_1, \ldots, C_s$ be the vertices in $\mathcal{S}_r$, with $C_i := \{w_i, z_i\}$ and $w_i \neq z_i$. As we mentioned before, given the vertex $A \in \mathcal{R}_r$ there is a unique vertex C in $\mathcal{S}_r$ which is at distance two from A, so, without loss of generality let us assume that A_i and C_i are at distance two, for each $i \in [s]$. Let γ be the permutation of $[s]$ (and defined by $\psi\big|_{\mathcal{R}_r}$) such that

- if $\alpha = 0$, then $\gamma(A_i) = A_{\gamma(i)}$ and $\gamma(C_i) = C_{\gamma(i)}$, for each $i \in [s]$, and
- if $\alpha = 1$, then $\gamma(A_i) = C_{\gamma(i)}$ and $\gamma(C_i) = A_{\gamma(i)}$, for each $i \in [s]$.

Given $x \in Q_r$, let x' be the only vertex in Q_r such that $\Lambda(x, x') = \{r\}$. Note that the neighbors of both A_i and C_i belonging to $\mathcal{T}_r$ are the vertices $B_i := \{x'_i, y_i\}$ and $B'_i := \{x_i, y'_i\}$, for each $i \in [s]$, which corresponds to moving the token at x_i to x'_i, or moving the token at y_i to y'_i. Let $\mathcal{B} := \{B_1, \ldots, B_s\}$ and $\mathcal{B}' := \{B'_1, \ldots, B'_s\}$. We have the following.

Lemma 3.25. *There is $\beta \in \{0, 1\}$ such that*

- *if $\beta = 0$ then $\psi\big|_{\mathcal{B}} = \mathcal{B}$ and $\psi\big|_{\mathcal{B}'} = \mathcal{B}'$, and*
- *if $\beta = 1$ then $\psi\big|_{\mathcal{B}} = \mathcal{B}'$ and $\psi\big|_{\mathcal{B}'} = \mathcal{B}$.*

Moreover, given t, α, β and γ, the labels of the vertices in $\mathcal{T}_r$ under ψ are uniquely determined.

Proof. For $i \in [s]$, the vertices A_i and C_i have the neighbors B_i and B'_i in $\mathcal{T}_r$, and since $\{\psi(A_i), \psi(C_i)\} = \{\psi(A_{\gamma(i)}), \psi(C_{\gamma(i)})\}$, then ψ sends the vertices B_i and B'_i to the vertices $B_{\gamma(i)}$ and $B'_{\gamma(i)}$. Let us show that

(*i*) $\psi\big|_{\mathcal{B}} = \mathcal{B}$ and $\psi\big|_{\mathcal{B}'} = \mathcal{B}'$, or

(*ii*) $\psi\big|_{\mathcal{B}} = \mathcal{B}'$ and $\psi\big|_{\mathcal{B}'} = \mathcal{B}$.

Suppose that $\psi(B_i) = B_{\gamma(i)}$ and $\psi(B'_i) = B'_{\gamma(i)}$. Let us show that (*i*) holds. Consider the vertex A_j, for some $j \in [s]$. We may assume that A_i and A_j are adjacent (as otherwise it is enough to consider a sequence $X_1, \ldots, X_\ell$ contained in $\mathcal{R}_r$, where $X_1 = A_i$, $X_\ell = A_j$ and X_s is adjacent to X_{s+1}). Note that $B_iB_j, B'_iB'_j \in E(F_2(Q_r))$ while $B_iB'_j, B_jB'_i \notin E(F_2(Q_r))$. Similarly, $B_{\gamma(i)}B_{\gamma(j)}, B'_{\gamma(i)}B'_{\gamma(j)} \in E(F_2(Q_r))$ but $B_{\gamma(i)}B'_{\gamma(j)}, B_{\gamma(j)}B'_{\gamma(i)} \notin E(F_2(Q_r))$. So, we must have that $\psi(B_j) = B_{\gamma(j)}$ and $\psi(B'_j) = B'_{\gamma(j)}$. Thus, (i) holds. The case when $\psi(B_i) = B'_{\gamma(i)}$ and $\psi(B'_i) = B_{\gamma(i)}$ can be handled in a similar manner to show that (ii) holds. Define $\beta = 0$ if (i) holds, and $\beta = 1$ if (ii) holds.

Let $\mathcal{D} := \mathcal{T}_r \setminus (\mathcal{B} \cup \mathcal{B}')$. It remains to label the vertices in $\mathcal{D}$ under ψ. Any vertex $D \in \mathcal{D}$ has its neighbours in $\mathcal{B} \cup \mathcal{B}'$, and moreover, for $D_1, D_2 \in \mathcal{D}$, with $D_1 \neq D_2$, the neighborhoods of D_1 and D_2 are distinct, and since we have already labeled the vertices in $\mathcal{B} \cup \mathcal{B}'$ (given α, β, t and γ), then $\psi(D)$ is uniquely determined, for each $D \in \mathcal{D}$. This completes the proof. □

We are now ready to show that cube graphs attain the lower bound of Theorem 3.2.

Corollary 3.26. *For any $r \geq 3$ we have*

$$|\operatorname{Aut}(F_2(Q_r))| \leq 2^{r-1}|\operatorname{Aut}(Q_r)| = 2^{r-1}(2^r\, r!).$$

Proof. We proceed by induction on r. The case $r = 3$ was considered before, so the induction starts. Assume $r > 3$.

As we proved in Lemmas 3.23 and 3.24, the values $t \in [r]$, $\alpha, \beta \in \{0, 1\}$ and the permutation γ determine uniquely the labels of the vertices in $F_2(Q_r)$ under ψ. We know that γ is determined by t, α and $\psi|_{\mathcal{R}_r}$, and since the subgraph induced by $\mathcal{R}_r$ is isomorphic to $F_2(Q_{r-1})$, and by the induction hypothesis, there are at most $2^{r-2}|\operatorname{Aut}(Q_{r-1})|$ possibilities for γ. Then, the number of possibilities for ψ is

$$|\operatorname{Aut}(F_2(Q_r))| \leq \underbrace{2}_{\alpha} * \underbrace{2}_{\beta} * \underbrace{r}_{t} * \underbrace{|\operatorname{Aut}(F_2(Q_{r-1}))|}_{\gamma} \leq 2^{r-1}(2^r\, r!) = 2^{r-1}|\operatorname{Aut}(Q_r)|.$$

□

Regarding the case $k > 2$, using the SageMath software [50] we obtained the following computations.

Note 3.27. *For the k-token graph of the Cartesian product $K_n \,\square\, K_2$, with $3 \leq n \leq 10$ and $3 \leq k \leq \min\{8, n\}$, we have*

$$|\operatorname{Aut}(F_k(K_n \square K_2))| = \begin{cases} |\operatorname{Aut}(K_n \square K_2)| & \text{if } k \neq |K_n \square K_2|/2, \text{ and} \\ 2\,|\operatorname{Aut}(K_n \square K_2)| & \text{if } k = |K_n \square K_2|/2. \end{cases}$$

Besides, for the k-token graph of the cube graph Q_r, with $r \in \{3, 4, 5\}$ and $k \in \{3, 4, 5\}$, we have

$$|\operatorname{Aut}(F_k(Q_r))| = \begin{cases} |\operatorname{Aut}(Q_r)| & \text{if } k \neq |Q_r|/2, \text{ and} \\ 2\,|\operatorname{Aut}(Q_r)| & \text{if } k = |Q_r|/2. \end{cases}$$

This shows that the lower bound given in Theorem 3.2 is best possible for other values of k, with $k \neq 2$.

3.4 Concluding remarks and open problems

In this chapter we studied the automorphism group of token graphs of two families of graphs: complete bipartite graphs and the Cartesian product of graphs. As we showed, the fact that $\mathrm{Aut}(K_{m,n})$ is a proper subgroup of $\mathrm{Aut}(F_k(K_{m,n}))$ depends only on the graph $K_{m,n}$ and not on the value of k. On the other hand, it seems that the fact that $\mathrm{Aut}(Q_r)$ is a proper subgroup of $\mathrm{Aut}(F_k(Q_r))$ depends only on the value of k, since for $k = 2$ $\mathrm{Aut}(Q_r)$ is always a proper subgroup of $\mathrm{Aut}(F_k(Q_r))$, and for some values of $k > 2$ the computations indicate that this does not happen. So, as we mentioned in the beginning of this chapter, an open problem is the following.

Open problem 3.28. *To characterize the graphs G and values k for which*

$$\begin{aligned} \mathrm{Aut}(G) &< \mathrm{Aut}(F_k(G)) \quad \textit{when } k \neq n/2, \\ \mathrm{Aut}(G) \times \mathbb{Z}_2 &< \mathrm{Aut}(F_k(G)) \quad \textit{when } k = n/2. \end{aligned}$$

Regarding to the Cartesian product of graphs, an open problem is to determine if the lower bound of Theorem 3.2 is tight for any value of k.

Open problem 3.29. *Given any value $k > 2$, to determine if there exists a composite graph G of order n such that*

$$\begin{aligned} \mathrm{Aut}(G) &\simeq \mathrm{Aut}(F_k(G)) \quad \textit{when } k \neq n/2, \\ \mathrm{Aut}(G) \times \mathbb{Z}_2 &\simeq \mathrm{Aut}(F_k(G)) \quad \textit{when } k = n/2. \end{aligned}$$

Let us finally remark that when considering the graph G obtained from the complete bipartite graph $K_{2,n}$, with partition $\{X, Y\}$ where $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, by adding an edge between the vertices x_1 and x_2, then all the results proven in this chapter for $F_k(K_{2,n})$ also hold for $F_k(G)$. This is due to the fact that all the automorphisms of $F_k(K_{2,n})$ are also automorphisms of $F_k(G)$.

Chapter 4

Connectivity of token graphs of trees

In this chapter we study connectivity and edge-connectivity of token graphs of trees. First, we mention some known relevant results on the connectivity and edge-connectivity of token graphs. In 2012, Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia and Wood [21] showed that G is connected if and only if its k-token graph $F_k(G)$ is connected, for any $k \in \{1, 2, \ldots, n-1\}$. Moreover, the authors showed that the connectivity of $F_k(G)$ is at least the connectivity of G. Also, in [21], the authors provided families of graphs of order n with connectivity exactly t, and whose k-token graphs have connectivity exactly $k(t-k+1)$, whenever $k \leq t$; they also conjectured that if G is t-connected and $k \leq t$, then $F_k(G)$ is at least $k(t-k+1)$-connected. In 2018, this conjecture was proven by Leaños and Trujillo-Negrete [35]. Recently, a similar lower bound was proven for edge-connectivity by Leaños and Ndjatchi [34]; they showed that if G is ℓ-edge-connected and $k \leq \ell$ then $F_k(G)$ is at least $k(\ell - k + 1)$-edge-connected. An infinite family of graphs attaining this lower bound were also given in [34].

In this chapter we study connectivity and edge-connectivity of $F_k(G)$ when G is a tree. Let us recall that connectivity, edge-connectivity and minimum degree of a graph G are denoted by $\kappa(G)$, $\lambda(G)$ and $\delta(G)$, respectively. It is well known that if G is connected then

$$\kappa(G) \leq \lambda(G) \leq \delta(G). \tag{18}$$

The main result of this chapter is the following.

Theorem 4.1. *If G is a tree of order n and $1 \leq k \leq n-1$ then*

$$\kappa(F_k(G)) = \lambda(F_k(G)) = \delta(F_k(G)).$$

We remark that while the hypothesis $k \leq \kappa(G)$ has played a central role in both results on $\kappa(F_k(G))$ stated in [21, 35], this hypothesis does not hold when G is a tree; this absence is responsible for the new difficulties in proof of Theorem 4.1.

This chapter is based on joint work with Ruy Fabila-Monroy and Jesus Leaños. The main result of this chapter is presented in the paper [22].

The rest of this chapter is organized as follows. In Section 4.1.1 we establish several ways to construct paths in $F_k(G)$ which come from the concatenation of certain paths of G. These paths of $F_k(G)$ play a central role in our constructive proof of Theorem 4.1. In Section 4.1.2 we give some basic results on the connectivity structure of $F_k(G)$ which help us to simplify significantly the proof of Theorem 4.1. In Section 4.2 we prove Theorem 4.1 and in Section 4.3 we present some concluding remarks.

4.1 Basic results

In this section we prove some general basic results on the connectivity of token graphs, and we also present several constructions of paths of $F_k(G)$.

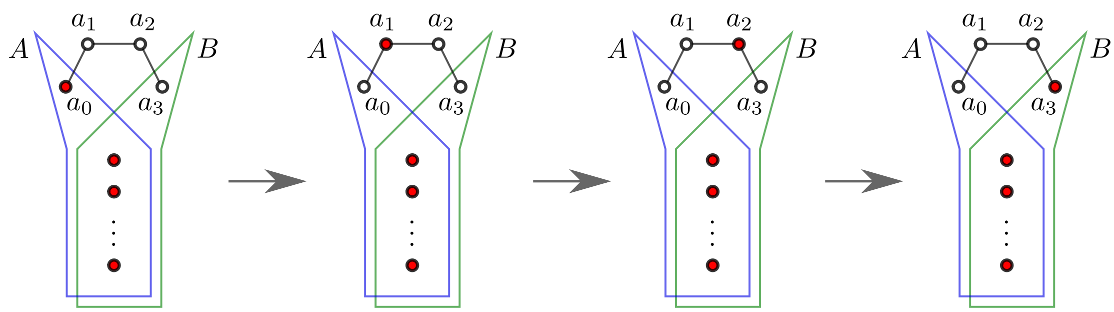
4.1.1 Constructing paths of $F_k(G)$ from paths of G

This section is devoted to constructing some paths in $F_k(G)$ using a given set of paths of G.

Let $P := a_0a_1a_2\dots a_m$ be an $a-b$ path of G ($a_0 = a$ and $a_m = b$); let $A, B \in V(F_k(G))$ such that $A \triangle B = \{a, b\}$, $P \cap A = \{a\}$ and $P \cap B = \{b\}$. A natural way of constructing an $A - B$ path $\mathcal{P}$ in $F_k(G)$ using P is by moving the token at a along P to b, while the remaining tokens are fixed at the vertices in $A \cap B$. More precisely, we start at A, then for each $i = 0, 1, \dots, m-1$, we move (in this order) the token at a_i along the edge a_ia_{i+1} to the vertex a_{i+1}. We denote this sequence of admissible token moves by

$$a_0 \to a_1 \to a_2 \cdots \to a_m.$$

Clearly, the first and last configurations of this sequence correspond to the vertices A and B of $F_k(G)$, respectively. Moreover, note that if $A_0 = A, A_m = B$, and $A_i = (A_{i-1} \setminus \{a_{i-1}\}) \cup \{a_i\}$ for $i \in [m]$, then $\mathcal{P} = AA_1A_2\dots A_{m-1}B$. We refer to $\mathcal{P}$ as the *path of $F_k(G)$ induced by P*. An example of this construction is depicted in Figure 4.1. Let $\mathcal{Q}$ be a path of $F_k(G)$ and let $\{Q_0, Q_1, \dots, Q_m\}$ be its vertex set. Since each of these Q_i's is a k-set of $V(G)$, then $q := k - |\cap_{i=0}^{m} Q_i|$ is well defined. We say that $\mathcal{Q}$ is a path *of Type q*. A path $\mathcal{Q}$ of Type q can be understood as a path in which exactly q tokens are moved along the path $\mathcal{Q}$, while the remaining $k-q$ tokens keep fixed at some vertices of G. Thus, $\mathcal{P}$ and any edge of $F_k(G)$ are examples of paths of Type 1, since only one token is moved and $k-1$ tokens are kept fixed.



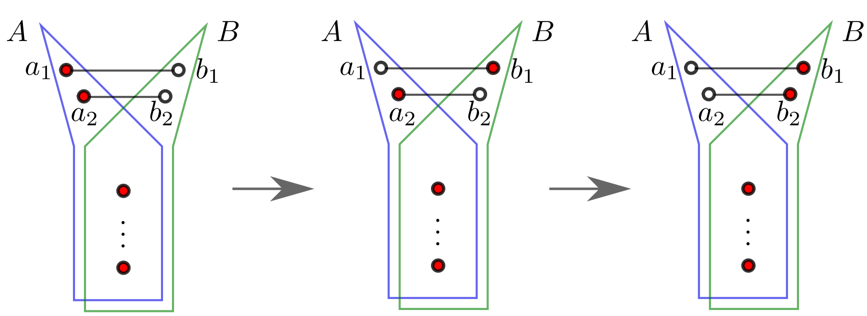
$$\mathcal{P} := a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow a_3.$$

Figure 4.1: Four configurations of G. The set of red vertices of G defining the left (respectively, right) configuration corresponds to the vertex A (respectively, B) of $F_k(G)$. These four configurations together (from left to right) define an $A-B$ path $\mathcal{P}$ of $F_k(G)$. The path $\mathcal{P}$ is induced by $P = a_0a_1a_2a_3$, because the token at a_0 is moving along P to a_3. Since only one token is moving (and so, the remaining $k-1$ tokens are fixed on the vertices in $A \cap B$), $\mathcal{P}$ is of Type 1.

We now define certain paths of Type 2. Let $e_1 = a_1b_1$ and $e_2 = a_2b_2$ be independent edges of G, and let $A, B \in F_k(G)$ such that $A \setminus B = \{a_1, a_2\}$ and $B \setminus A = \{b_1, b_2\}$. A simple way to construct an $A-B$ path $\mathcal{R}$ of Type 2 (and length 2) is by moving the token at a_1 to b_1 along e_1, and then, by moving the token at a_2 to b_2 along e_2. We denote this sequence of admissible token moves by

$$a_1 \to b_1; a_2 \to b_2.$$

Then $\mathcal{R} = A_0A_1A_2$ is an $A-B$ path in $F_k(G)$ of Type 2, where $A_0 = A$, $A_1 = (A_0 \setminus \{a_1\}) \cup \{b_1\}$, $A_2 = (A_1 \setminus \{a_2\}) \cup \{b_2\} = B$ (see Figure 4.2). We remark that $\mathcal{R}$ can be seen as the concatenation of two paths of Type 1, namely those corresponding to $a_1 \to b_1$ and $a_2 \to b_2$. As suggested above, we use a semicolon " ; " to denote the concatenation of paths of Type 1.



$$\mathcal{R} := a_1 \longrightarrow b_1; a_2 \longrightarrow b_2.$$

Figure 4.2: An $A-B$ path of Type 2.

Now, suppose that A and B are adjacent vertices in $F_k(G)$ with $A \setminus B := \{a\}$ and $B \setminus A := \{b\}$. Then ab is an edge of G. Let u and v be adjacent vertices of G such that

$u \in A \cap B$ and $v \in V(G) \setminus (A \cup B)$. As we have seen above, a way to produce an $A - B$ path $\mathcal{P}$ is simply by moving the token at a to b along the edge ab. Now we use a simple trick, involving the edges uv and ab, to produce a new $A - B$ path $\mathcal{P}_{uv}$ of $F_k(G)$ that is internally disjoint from $\mathcal{P}$. The path $\mathcal{P}_{uv}$ is constructed as follows. First we move the token at u to v along uv, and then we move the token at a to b along ab, and finally we move back the token at v to u along uv. Clearly, each of these token moves is admissible and they together define the required $\mathcal{P}_{uv}$ path, which we denote by:

$$u \to v; a \to b; v \to u.$$

We say that the vertex v is playing the role of a *distractor*, which allow us to produce a new path P_{uv} from $\mathcal{P}$ and uv. See Figure 4.3.

We now generalize the above construction. Suppose that $\mathcal{P}$ is an $A - B$ path of $F_k(G)$ and that uv is an edge of G with $u \in A \cap B$ and $v \in V(G) \setminus (A \cup B)$. If $u \in I$ and $v \notin I$ for any internal vertex I of $\mathcal{P}$, then we can get a new $A - B$ path P_{uv} from $\mathcal{P}$ and uv as follows. First move the token at u to v along uv. Then, keeping the token at v fixed, move the tokens from the vertices in $A \setminus B$ to the vertices in $B \setminus A$ according to $\mathcal{P}$, and finally move back the token at the distractor v to the initial vertex u. Note that at the end we have produced an $A - B$ path $\mathcal{P}_{uv}$ with the following property: for each inner vertex J of $\mathcal{P}_{uv}$, we have that $v \in J$ and $u \notin J$. This implies that if $u'v'$ is an edge of $G \setminus \{uv\}$ satisfying the same properties as uv with respect to $\mathcal{P}$, then the corresponding path $\mathcal{P}_{u'v'}$ is an $A - B$ path internally disjoint from both $\mathcal{P}$ and $\mathcal{P}_{uv}$. The paths produced in this way play an important role in the proof of Theorem 4.1.

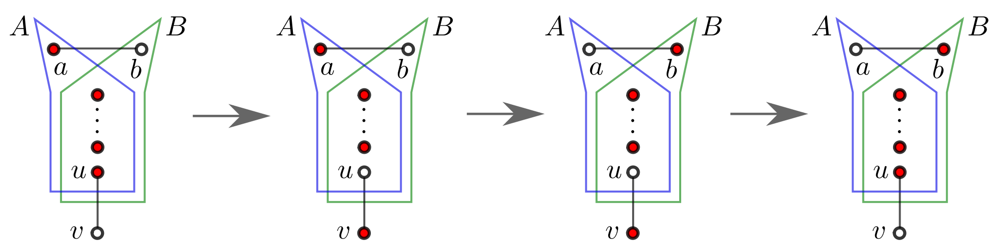


Figure 4.3: An $A - B$ path $\mathcal{P}_{uv}$ with distractor v.

4.1.2 Some general results

Let us now prove some auxiliary results that are used in the proof of Theorem 4.1. The first result is a characterization of t-connected graphs.

Proposition 4.2. *Let H be a connected graph. Then H is t-connected if and only if H has t pairwise internally disjoint $a - b$ paths, for any two vertices a and b of H such that $d_H(a, b) = 2$.*

Proof. The forward implication follows directly from Menger's Theorem. Conversely, let U be a vertex cut of H of minimum order. Let H_1 and H_2 be two distinct components of $H-U$, and let $u \in U$. Since U is a minimum cut, then u has at least a neighbour v_i in H_i, for $i=1,2$. Then $d_H(v_1,v_2)=2$. By hypothesis, H has t pairwise internally disjoint v_1-v_2 paths. Since each of these t paths intersects U, then we have that $|U| \geq t$, as required. □

The previous result suggest to focus on pairs of vertices at distance two. Now, the following result characterizes the vertices at distance two in $F_k(G)$.

Proposition 4.3. *Let X and Y be vertices of $F_k(G)$ with $d_{F_k(G)}(X,Y)=2$. Then $|X \cap Y| \in \{k-2,k-1\}$ and one of the following holds:*

(1) if $|X \cap Y| = k-2$, then G has two independent edges x_1y_1 and x_2y_2 such that $X \setminus Y = \{x_1,x_2\}$ and $Y \setminus X = \{y_1,y_2\}$;

(2) if $|X \cap Y| = k-1$, then G has two vertices x and y at distance two in G such that $X \setminus Y = \{x\}$ and $Y \setminus X = \{y\}$.

Proof. Note that $|X \triangle Y| \in \{2,4\}$. Indeed, since X and Y are distinct k-sets of $V(G)$, $|X \triangle Y|$ must be an even positive integer, and if $|X \triangle Y| \geq 6$, then we need to carry at least 3 tokens from the vertices in $X \setminus Y$ to the vertices in $Y \setminus X$, implying that $d_{F_k(G)}(X,Y) \geq 3$. Thus, $|X \triangle Y| \in \{2,4\}$, which is equivalent to $|X \cap Y| \in \{k-2,k-1\}$. See Figure 4.4. We now proceed by cases.

(1) In this case we have $|X \setminus Y| = |Y \setminus X| = 2$. Since $d_{F_k(G)}(X,Y)=2$, there is a way to carry the two tokens at the vertices of $X \setminus Y$ to the vertices of $Y \setminus X$ with exactly two admissible token moves. These two token moves corresponds to two independent edges joining vertices of $X \setminus Y$ with the vertices of $Y \setminus X$. See Figure 4.4 (i).

(2) Here, $X \setminus Y$ and $Y \setminus X$ each consist of exactly one vertex of G; say x and y, respectively. Since $d_{F_k(G)}(X,Y)=2$, then x cannot be adjacent to y in G. On the other hand, $d_{F_k(G)}(X,Y)=2$ implies the existence of an $X-Y$ path $\mathcal{P}$ produced by exactly 2 admissible token moves. Now note that $\mathcal{P}$ necessarily involves two admissible token moves $x \to v$ and $u \to y$. There are two possibilities either $x \to v$ is applied before $u \to y$ or $u \to y$ is applied before $x \to v$. Since $\mathcal{P}$ is produced by exactly 2 admissible token moves, we have that $u = v \in N_G(x) \cap N_G(y)$, and xvy is a path of length two in G, as required. The two possibilities are depicted in (ii) and (iii) of Figure 4.4.

□

Recall that the complement isomorphism $\mathfrak{c} : F_k(G) \to F_{n-k}(G)$ maps each vertex $A \in F_k(G)$ to its complement $\mathfrak{c}(A) = V(G) \setminus A$. From the definition of $\mathfrak{c}$ we can show the following.

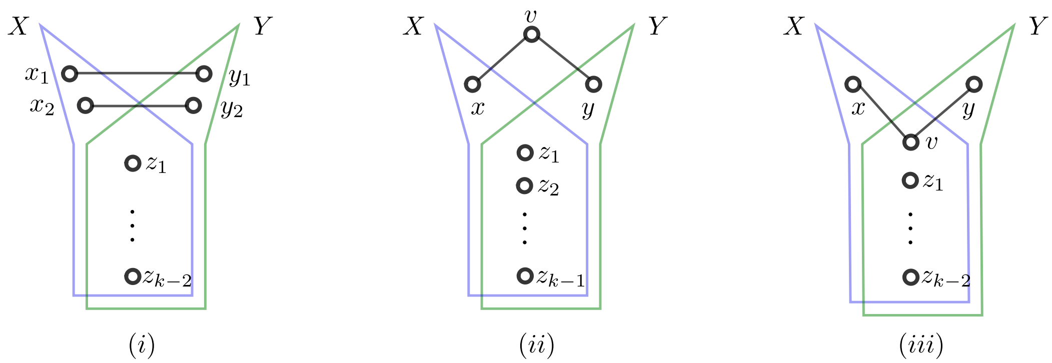


Figure 4.4: X and Y are vertices of $F_k(G)$ at distance 2. (i) $X \triangle Y = \{x_1, y_1, x_2, y_2\}$ and x_1y_1, x_2y_2 are independent edges of G. In (ii) and (iii) $X \triangle Y = \{x, y\}$ and xvy is a shortest $x - y$ path in G. The difference between the last two cases is that in (ii) $v \in V(G) \setminus (X \cup Y)$ and in (iii) $v \in X \cap Y$.

Proposition 4.4. *Let X, Y, x, y and v be as in the proof of Proposition 4.3 (2). Then exactly one of $v \notin X \cup Y$ or $v \notin \mathfrak{c}(X) \cup \mathfrak{c}(Y)$ holds.*

Proof. From Proposition 4.3 (2) we know that $\{x\} = X \setminus Y$ and $\{y\} = Y \setminus X$. Since $P = xvy$ is a path of length 2, then we have that $v \notin \{x, y\}$. This implies that exactly one of $v \in X \cap Y$ or $v \in V(G) \setminus (X \cup Y)$ holds. Since $v \in X \cap Y$ is equivalent to $v \notin \mathfrak{c}(X) \cup \mathfrak{c}(Y)$, and $v \in V(G) \setminus (X \cup Y)$ is equivalent to $v \notin X \cup Y$, we are done. □

4.2 Proof of Theorem 4.1

Throughout this section, T is a tree of order $n \geq 2$, and $k \in \{1, 2, \ldots, n-1\}$. Next, we recall Theorem 4.1.

Theorem 4.1. *If G is a tree of order n and $1 \leq k \leq n - 1$ then*

$$\kappa(F_k(G)) = \lambda(F_k(G)) = \delta(F_k(G)).$$

□

Notice that it is sufficient to show that

$$\kappa(F_k(T)) \geq \delta(F_k(T)).$$

From the definition of $F_1(G)$ it is straightforward to see that G and $F_1(G)$ are isomorphic. In this case Theorem 4.1 holds. We assume that $n \geq 4$ and $k \in \{2, \ldots, n-2\}$. By Proposition 4.2, it suffices to prove the following.

Lemma 4.5. *Let $X, Y \in V(F_k(T))$ with $d_{F_k(T)}(X, Y) = 2$. Then $F_k(T)$ has at least $\delta(F_k(T))$ pairwise internally disjoint $X - Y$ paths.*

Proof. Let $Z := X \cap Y$, $W := V(G) \setminus (X \cup Y)$ and $\delta := \delta(F_k(T))$. Informally, our general strategy to show Lemma 4.5 is as follows.

- STEP 1. First, we construct a certain number m of pairwise internally disjoint $X - Y$ paths in $F_k(T)$.
- STEP 2. If $\delta > m$, we construct the $\delta - m$ missing $X - Y$ paths.

The hypothesis $d(X, Y) = 2$ and Proposition 4.3 imply that $|Z| = k - 1$ or $|Z| = k - 2$. We analyze these cases separately.

4.2.1 CASE 1: $|Z| = k - 1$

From Proposition 4.3 (2) we know that there exist $x, y, v \in V(T)$ such that $\{x\} = X \setminus Y, \{y\} = Y \setminus X, v \notin \{x, y\}$, and $P = xvy$ is a shortest $x - y$ path of T. In view of Proposition 4.4, we can assume without any loss of generality that $v \notin X \cup Y$. Indeed, if $v \in X \cup Y$ then by Proposition 4.4 $v \notin \mathfrak{c}(X) \cup \mathfrak{c}(Y)$. Since $F_k(T)$ and $F_{n-k}(T)$ are isomorphic under the complement isomorphism $\mathfrak{c}$, then we can work with $\mathfrak{c}(X)$ and $\mathfrak{c}(Y)$ in $F_{n-k}(T)$ instead of X and Y in $F_k(T)$. We assume that X and Y are as in Figure 4.4 (ii). Let $W^\circ := W \setminus \{v\}$ and let

$$\begin{aligned} W(x) &:= \{w \in W^\circ : w \text{ is adjacent to } x\} = \{w_x^1, \dots, w_x^a\}, \\ W(y) &:= \{w \in W^\circ : w \text{ is adjacent to } y\} = \{w_y^1, \dots, w_y^d\}, \\ Z(x) &:= \{z \in Z : z \text{ is adjacent to } x\} = \{z_x^1, \dots, z_x^c\}, \\ Z(y) &:= \{z \in Z : z \text{ is adjacent to } y\} = \{z_y^1, \dots, z_y^b\}, \end{aligned}$$

where $a := |W(x)|$, $b := |Z(y)|$, $c := |Z(x)|$, and $d := |W(y)|$. See Figure 4.5.

Let us define

$$E_{Z,W} := \{zw \in E(T) : z \in Z \text{ and } w \in W\}, \text{ and } \eta := |E_{Z,W}|.$$

Since T is a tree, then $W(x), W(y), Z(x)$, and $Z(y)$ are pairwise disjoint. Then $deg(X) = a + b + \eta + 1$ and $deg(Y) = c + d + \eta + 1$. Without loss of generality we may assume that $deg(X) \leq deg(Y)$. Hence, $a + b \leq c + d$.

Let $m_x := \min\{a, c\}$, $m_y := \min\{b, d\}$, and $m := m_x + m_y + \eta + 1$.

4.2.1.1 STEP 1 of CASE 1

We produce the required m $X - Y$ paths by means of four types of constructions.

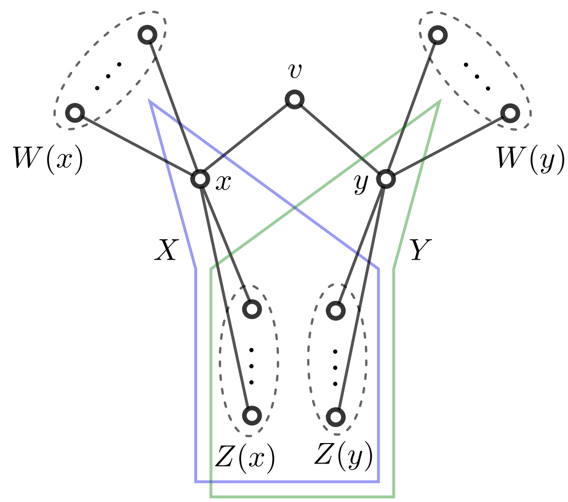


Figure 4.5: The neighbours of x and y in Case 1.

1. Using the vertex v:
$$\mathcal{P}_0 := x \to v \to y.$$
See Construction 1 of Figure 4.6. Let $\mathbb{T}_1 := \{\mathcal{P}_0\}$. Let A_0 be the (unique) inner vertex of $\mathcal{P}_0$, then

   (C1) $A_0 \cap Z = Z$ and $A_0 \cap W^\circ = \emptyset$.

2. Using the edges of $E_{Z,W}$. For each $z_iw_j \in E_{Z,W}$, let $\mathcal{P}_{i,j}$ be the $X - Y$ path defined as follows:
$$\mathcal{P}_{i,j} := \begin{cases} z_i \to w_j; x \to v \to y; w_j \to z_i & \text{if } w_j \neq v; \\ z_i \to v \to y; x \to v \to z_i & \text{if } w_j = v. \end{cases}$$
See Construction 2 of Figure 4.6 for the case $w_j \neq v$. Let $\mathbb{T}_2 := \{\mathcal{P}_{i,j} : z_iw_j \in E_{Z,W}\}$. Note that if $A_{i,j}$ is an inner vertex of $\mathcal{P}_{i,j}$, then

   (C2) $A_{i,j} \cap Z = Z \setminus \{z_i\}$.

   Moreover, depending on whether $w_j \neq v$ or $w_j = v$, then $A_{i,j}$ also satisfies the following:

   (C2.1) If $w_j \neq v$, then $A_{i,j} \cap W^\circ = \{w_j\}$.
   (C2.2) If $w_j = v$, then $A_{i,j} \cap W^\circ = \emptyset$.

   We recall that if $r = 0$, then $[r] = \emptyset$.

3. For each $s \in [m_x]$, using the vertices $w_x^s \in W(x)$ and $z_x^s \in Z(x)$. We define the path $\mathcal{P}_s$ as follows:
$$\mathcal{P}_s := x \to w_x^s; z_x^s \to x \to v \to y; w_x^s \to x \to z_x^s.$$
See Construction 3 of Figure 4.6. Let $\mathbb{T}_3 := \{\mathcal{P}_s : s \in [m_x]\}$. Again, note that if A_s is an inner vertex of $\mathcal{P}_s$, then

(C3) Either $A_s \cap Z = Z$ or $A_s \cap Z = Z \setminus \{z_x^s\}$, and either $A_s \cap W^\circ = \emptyset$ or $A_s \cap W^\circ = \{w_x^s\}$, and at least one of the following holds: $A_s \cap Z = Z \setminus \{z_x^s\}$ or $A_s \cap W^\circ = \{w_x^s\}$.

4. For each $t \in [m_y]$, using the vertices $w_y^t \in W(y)$ and $z_y^t \in Z(y)$. We define the path $\mathcal{Q}_t$ as follows:
$$\mathcal{Q}_t := z_y^t \to y \to w_y^t; x \to v \to y \to z_y^t; w_y^t \to y.$$
See Construction 4 of Figure 4.6. Let $\mathbb{T}_4 := \{\mathcal{Q}_t : t \in [m_y]\}$. Again, note that if A_t is an inner vertex of $\mathcal{Q}_t$, then

(C4) Either $A_t \cap Z = Z$ or $A_t \cap Z = Z \setminus \{z_y^t\}$, and either $A_t \cap W^\circ = \emptyset$ or $A_t \cap W^\circ = \{w_y^t\}$, and at least one of the following holds: $A_t \cap Z = Z \setminus \{z_y^t\}$ or $A_t \cap W^\circ = \{w_y^t\}$.

Let us define $\mathbb{T} := \mathbb{T}_1 \cup \mathbb{T}_2 \cup \mathbb{T}_3 \cup \mathbb{T}_4$. Since $|\mathbb{T}_1| = 1, |\mathbb{T}_2| = \eta, |\mathbb{T}_3| = m_x, |\mathbb{T}_4| = m_y$, and $m = 1 + \eta + m_x + m_y$, then in order to finish the Step 1 of Case 1, it is enough to show that the paths in $\mathbb{T}$ are pairwise internally disjoint.

Claim 4.6. *The $X - Y$ paths in $\mathbb{T}$ are pairwise internally disjoint.*

Proof of Claim 4.6. First we show separately that the paths in $\mathbb{T}_\ell$ are pairwise internally disjoint for $\ell \in \{2, 3, 4\}$.

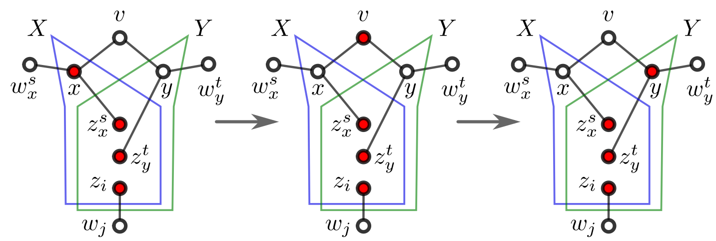
Suppose that $\boldsymbol{\ell = 2}$, and let $\mathcal{P}_{i,j}$ and $\mathcal{P}_{s,t}$ be distinct paths in $\mathbb{T}_2$. Let $A_{i,j}$ and $A_{s,t}$ be inner vertices of $\mathcal{P}_{i,j}$ and $\mathcal{P}_{s,t}$, respectively. Since $(i,j) \neq (s,t)$, then $z_i \neq z_s$ or $w_j \neq w_t$.

If $z_i \neq z_s$, then from (C2) we know that $A_{i,j} \cap Z = Z \setminus \{z_i\}$ and $A_{s,t} \cap Z = Z \setminus \{z_s\}$. Hence $z_i \in A_{s,t} \setminus A_{i,j}$, which implies that $A_{i,j} \neq A_{s,t}$.

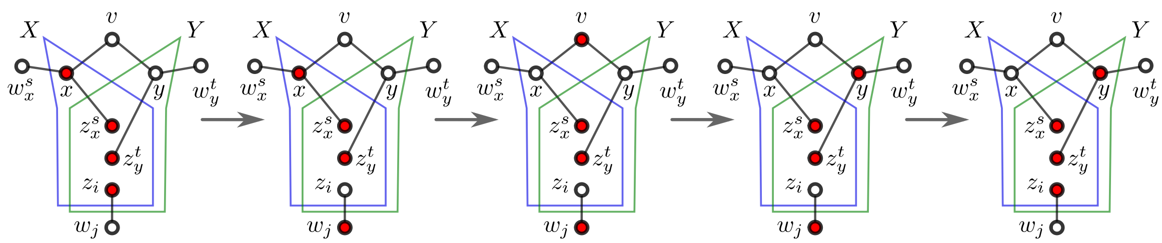
Now suppose that $w_j \neq w_t$. First suppose that $v \notin \{w_j, w_t\}$. By (C2.1) we have $A_{i,j} \cap W^\circ = \{w_j\}$, and similarly, $A_{s,t} \cap W^\circ = \{w_t\}$. Then, $A_{i,j} \cap W^\circ \neq A_{s,t} \cap W^\circ$ and so $A_{i,j} \neq A_{s,t}$. Then we may assume that $v \in \{w_j, w_t\}$. Without loss of generality suppose that $w_j = v$. We know by (C2.2) that $A_{i,j} \cap W^\circ = \emptyset$, and by (C2.1) that $A_{s,t} \cap W^\circ = \{w_t\}$, these two facts imply that $A_{i,j} \neq A_{s,t}$.

Suppose that $\boldsymbol{\ell = 3}$, and let $\mathcal{P}_s$ and $\mathcal{P}_t$ be distinct paths in $\mathbb{T}_3$. For $r \in \{s,t\}$, let A_r be an inner vertex of $\mathcal{P}_r$. From the last assertion of (C3) we know that $A_s \cap W^\circ = \{w_x^s\}$ or $A_s \cap Z = Z \setminus \{z_x^s\}$. Suppose that $A_s \cap W^\circ = \{w_x^s\}$. Since (C3) implies that $A_t \cap W^\circ = \emptyset$ or $A_t \cap W^\circ = \{w_x^t\}$, then we have $A_s \cap W^\circ \neq A_t \cap W^\circ$, and so $A_s \neq A_t$. Now suppose that $A_s \cap Z = Z \setminus \{z_x^s\}$. Again, from (C3) we know that $A_t \cap Z = Z$ or $A_t \cap Z = Z \setminus \{z_x^t\}$. Since $z_x^s \neq z_x^t$, then $A_s \cap Z \neq A_t \cap Z$, and so $A_s \neq A_t$.

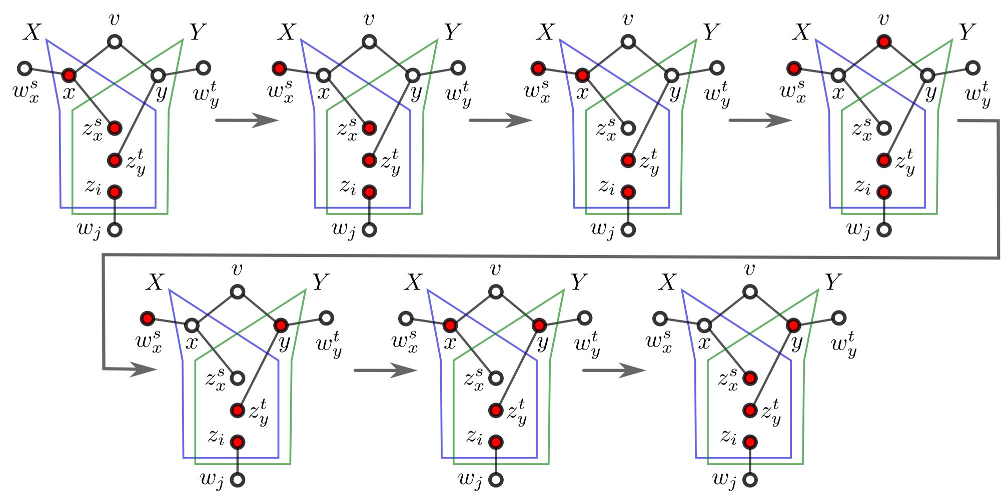
Suppose that $\boldsymbol{\ell = 4}$. This case can be handled in a totally analogous manner as the previous case.



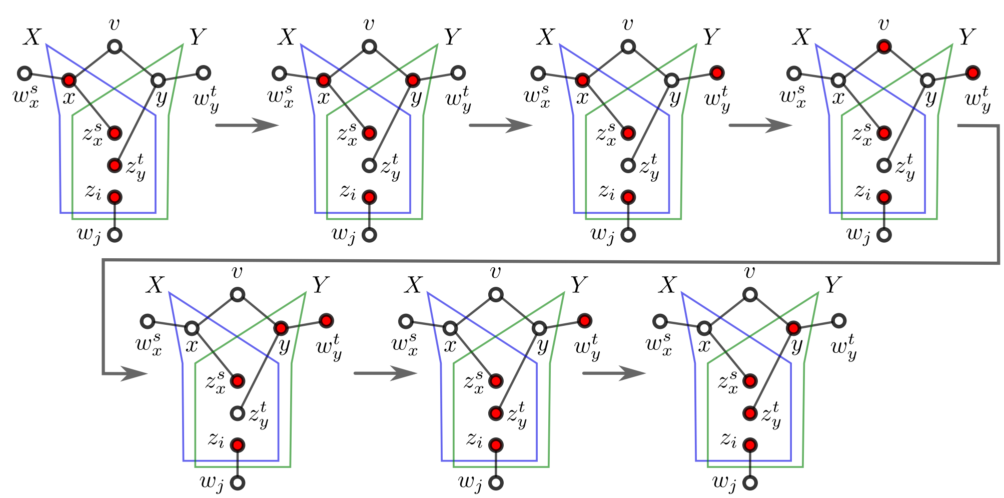
Construction 1: $\mathcal{P}_0 := x \to v \to y$



Construction 2: $\mathcal{P}_{i,j} := z_i \to w_j; x \to v \to y; w_j \to z_i$



Construction 3: $\mathcal{P}_s := x \to w_x^s; z_x^s \to x \to v \to y; w_x^s \to x \to z_x^s$



Construction 4: $\mathcal{Q}_t := x \to w_x^t; z_x^t \to x \to v \to y; w_x^t \to x \to z_x^t$

Figure 4.6: Construction of some $X - Y$ paths in Case 1.

Let $A_0, A_{i,j}, A_s$, and A_t be inner vertices of $\mathcal{P}_0 \in \mathbb{T}_1$, $\mathcal{P}_{i,j} \in \mathbb{T}_2$, $\mathcal{P}_s \in \mathbb{T}_3$, and $\mathcal{Q}_t \in \mathbb{T}_4$, respectively. It remains to show that $\mathcal{P}_0, \mathcal{P}_{i,j}, \mathcal{P}_s$, and $\mathcal{Q}_t$ are pairwise internally disjoint. We analyze separately each pair.

$\{A_0, A_{i,j}\}$: Here we have $A_0 \cap Z = Z$, while $A_{i,j} \cap Z = Z \setminus \{z_i\}$, and so $A_0 \neq A_{i,j}$.

$\{A_0, A_s\}$: By (C1) we know that $A_0 \cap Z = Z$ and that $A_0 \cap W^\circ = \emptyset$. Similarly, by the last assertion of (C3), we know that either $A_s \cap Z = Z \setminus \{z_x^s\}$ or $A_s \cap W^\circ = \{w_x^s\}$, then we have $A_0 \neq A_s$.

$\{A_0, A_t\}$: As in previous case, the last assertion of (C4) implies that either $A_t \cap Z = Z \setminus \{z_y^t\}$ or $A_t \cap W^\circ = \{w_y^t\}$. Then, since $A_0 \cap Z = Z$ and $A_0 \cap W^\circ = \emptyset$, we have $A_0 \neq A_t$.

$\{A_{i,j}, A_s\}$: First suppose that $w_j = v$. Then $z_i \neq z_x^s$, as otherwise the vertex set $\{x, z_i, v\}$ forms a cycle, contradicting that T is a tree. Since $A_{i,j} \cap Z = Z \setminus \{z_i\}$, and either $A_s \cap Z = Z$ or $A_s \cap Z = Z \setminus \{z_x^s\}$, then $A_{i,j} \cap Z \neq A_s \cap Z$, as required.

Suppose now that $w_j \neq v$. By (C3) we know that $A_s \cap Z = Z$ or $A_s \cap Z = Z \setminus \{z_x^s\}$. If $A_s \cap Z = Z$, then $A_{i,j} \cap Z = Z \setminus \{z_i\}$ implies that $A_s \neq A_{i,j}$. Thus we may assume that $A_s \cap Z = Z \setminus \{z_x^s\}$. If $z_i \neq z_x^s$, then $Z \setminus \{z_x^s\} = A_s \cap Z \neq A_{i,j} \cap Z = Z \setminus \{z_i\}$, as desired. Then we can assume that $z_x^s = z_i$. This implies that $w_x^s \neq w_j$, as otherwise $\{z_i, x, w_j\}$ forms a cycle. By (C2.1) we know that $A_{i,j} \cap W^\circ = \{w_j\}$, and by (C3) we have that either $A_s \cap W^\circ = \emptyset$ or $A_s \cap W^\circ = \{w_x^s\}$. Since $w_x^s \neq w_j$, then $A_{i,j} \cap W^\circ \neq A_s \cap W^\circ$, as required.

$\{A_{i,j}, A_t\}$: Again, this case can be handled in a totally analogous manner as previous case.

$\{A_s, A_t\}$: Since $Z(x), Z(y), W(x)$, and $W(y)$ are pairwise disjoint, then $z_x^s \neq z_y^t$ and $w_x^s \neq w_y^t$. From these inequalities and (C3)-(C4) we have that either $A_s \cap Z \neq A_t \cap Z$ or $A_s \cap W^\circ \neq A_t \cap W^\circ$, and so $A_s \neq A_t$.

This completes the proof of Claim 4.6. $\triangle$

4.2.1.2 Step 2 of Case 1

We start by showing that $\delta - m \leq 2$.

Claim 4.7. *Let δ, m, m_x, m_y, and η be as above. Then,*

$$\delta \leq \begin{cases} m_x + m_y + \eta + 1 = m & \text{if } a \leq c \text{ and } b \leq d, \text{ or } a > c, \\ m_x + m_y + \eta + 2 = m + 1 & \text{if } b = d + 1, \\ m_x + m_y + \eta + 3 = m + 2 & \text{if } b \geq d + 2. \end{cases}$$

Proof of Claim 4.7. First we note that if $a \leq c$ and $b \leq d$, then

$$\delta \leq deg(X) = a + b + \eta + 1 = m_x + m_y + \eta + 1 = m,$$

as claimed.

Suppose that $a > c$. Since $a + b \leq c + d$, then $b < d$. Let $U := W \cup \{x, y\}$. Since $T[U]$ is a forest, then it contains at least a vertex $u \in U \setminus \{v\}$ such that $deg_{T[U]}(u) \leq 1$. Note that $u \notin \{x, y\}$, because $deg_{T[U]}(x) = a + 1 \geq 2$ and $deg_{T[U]}(y) = d + 1 \geq 2$. Let $X' := (X \setminus \{x\}) \cup \{u\}$, so

$$\delta \leq deg(X') \leq b + c + \eta + deg_{T[U]}(u) \leq m_x + m_y + \eta + 1 = m,$$

as claimed.

Suppose that $b = d + 1$. Since $a + b \leq c + d$, then $a < c$. In this case we have that

$$\delta \leq deg(X) = a + b + \eta + 1 = a + (d + 1) + \eta + 1 = m_x + m_y + \eta + 2 = m + 1.$$

Finally, suppose that $b \geq d+2$. Since $a+b \leq c+d$, then $c \geq a+2$. Let $U := X \cup Y$. Since $T[U]$ is a forest, then it contains at least a vertex $u \in U$ such that $deg_{T[U]}(u) \leq 1$. Note that $u \notin \{x, y\}$, because $deg_{T[U]}(x) \geq c \geq 2$ and $deg_{T[U]}(y) \geq b \geq 2$. Let $X' = (X \setminus \{u\}) \cup \{y\}$, then

$$\delta \leq deg(X') \leq (a + 1) + (d + 1) + \eta + deg_{T[U]}(u) \leq m_x + m_y + \eta + 3 = m + 2.$$

This completes the proof of Claim 4.7. △

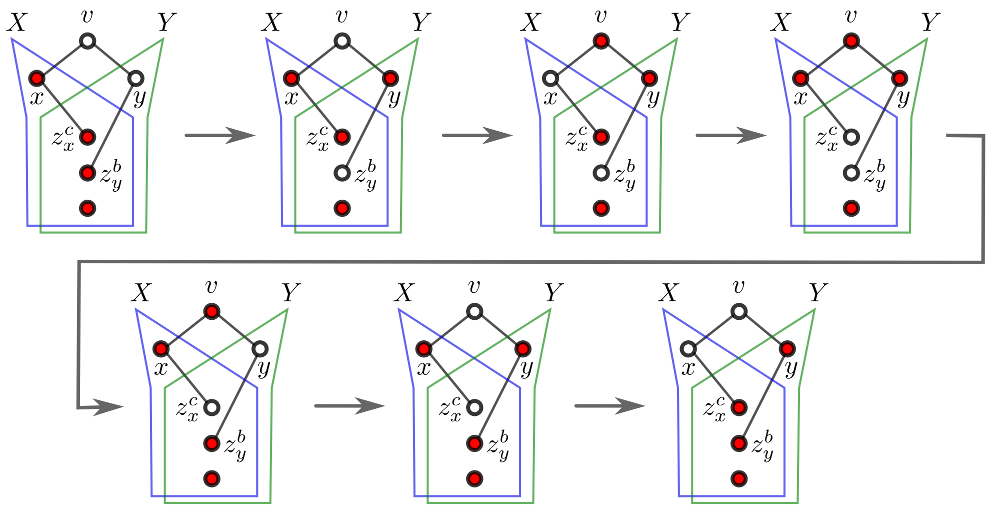
Claim 4.7 shows that almost all $X - Y$ paths claimed by Lemma 4.5 are provided by $\mathbb{T}$, when $|Z| = k - 1$. We finish the proof of Case 1 with the construction of the remaining $\delta - m$ $X - Y$ paths.

Claim 4.8. *If $|Z| = k - 1$, then $F_k(T)$ has at least δ $X - Y$ pairwise internally disjoint paths.*

Proof of Claim 4.8. We have already constructed m $X - Y$ pairwise internally disjoint paths, namely the elements of $\mathbb{T}$. Then, it remains to show the existence of $\delta - m$ additional $X - Y$ paths with similar properties. Since if $\delta \leq m$ then there is nothing to prove, we assume that $\delta > m$. From this and Claim 4.7 it follows that $b \geq d + 1$. Moreover, since $a + b \leq c + d$, then $c \geq a + 1$. Hence, $a = \min\{a, c\}$ and $d = \min\{b, d\}$.

Suppose first that $b = d + 1$. By Claim 4.7 we have that $\delta \leq m + 1$. Thus, it is enough to construct a new $X - Y$ path internally disjoint to each path in $\mathbb{T}$. Since $b = d + 1 > d = \min\{b, d\}$ and $c \geq a + 1 > a = \min\{a, c\}$, then the vertices z_y^b and z_x^c were not used in the construction of the paths of $\mathbb{T}_3 \cup \mathbb{T}_4$. We construct the required path $\mathcal{P}$ as follows:

$$\mathcal{P} := z_y^b \to y; x \to v; z_x^c \to x; y \to z_y^b; v \to y; x \to z_x^c.$$



$\mathcal{P} := z_y^b \to y; x \to v; z_x^c \to x; y \to z_y^b; v \to y; x \to z_x^c$

Figure 4.7: Construction of the $X - Y$ path $\mathcal{P}$.

See this construction in Figure 4.7. Let A be an inner vertex of $\mathcal{P}$. From the definition of $\mathcal{P}$ it follows that

(C5) Either $A \cap Z = Z \setminus \{z_y^b\}$ or $A \cap Z = Z \setminus \{z_x^c\}$ or $A \cap Z = Z \setminus \{z_x^b, z_y^c\}$, and that $A \cap W^\circ = \emptyset$.

Now we show that $\mathcal{P}$ is internally disjoint to any path in $\mathbb{T}$. Let $A_0, A_{i,j}, A_s$, and A_t be inner vertices of $\mathcal{P}_0 \in \mathbb{T}_1$, $\mathcal{P}_{i,j} \in \mathbb{T}_2$, $\mathcal{P}_s \in \mathbb{T}_3$, and $\mathcal{Q}_t \in \mathbb{T}_4$, respectively.

We analyze these cases separately.

$\{A_0, A\}$: By (C1) and (C5) we know that $A_0 \cap Z = Z$ and $A \cap Z \neq Z$, respectively, and so $A \neq A_0$.

$\{A_{i,j}, A\}$: If $w_j \neq v$, then $A_{i,j} \cap W^\circ = \{w_j\}$, and then $A \cap W^\circ \neq A_{i,j} \cap W^\circ$, which implies that $A \neq A_{i,j}$.
Now suppose that $w_j = v$. Then $z_i \notin \{z_y^b, z_x^c\}$, as otherwise T has a cycle. Then, by (C2) and (C5) we have that $A_{i,j} \cap Z \neq A \cap Z$, and so $A \neq A_{i,j}$.

$\{A_s, A\}$: Note that $z_x^s \neq z_x^c$, because $s \leq a < c$. Similarly, $z_x^s \neq z_y^b$, because $Z(x) \cap Z(y) = \emptyset$. Then, (C3) and (C5) implies that $A_s \cap Z \neq A \cap Z$, and so $A \neq A_s$.

$\{A_t, A\}$: We proceed as in previous case. Since $t \leq d < b$, then $z_y^t \neq z_y^b$, and $z_y^t \neq z_x^c$ because $Z(x) \cap Z(y) = \emptyset$. Then, (C4) and (C5) implies that $A_t \cap Z \neq A \cap Z$, and so $A \neq A_t$.

Finally, suppose that $b \geq d + 2$. By Claim 4.7 we have that $\delta \leq m + 2$. Thus, it is enough to construct two $X-Y$ paths, say $\mathcal{P}$ and $\mathcal{P}'$, such that $\{\mathcal{P},\mathcal{P}'\}\cup\mathbb{T}$ is a set of pairwise internally disjoint paths.

Since $b\geq d+2$ and $a+b\leq c+d$, then $c\geq a+2$. Now we use z_y^b, z_y^{b-1}, z_x^c, and z_x^{c-1} to construct $\mathcal{P}$ and $\mathcal{P}'$ as follows.

$$\mathcal{P} := z_y^b \to y; x\to v; z_x^c\to x; y\to z_y^b; v\to y; x\to z_x^c, \text{ and}$$
$$\mathcal{P}' := z_y^{b-1} \to y; x\to v; z_x^{c-1}\to x; y\to z_y^{b-1}; v\to y; x\to z_x^{c-1}.$$

Note that a similar argument to the one used above (for the case $b=d+1$) can be applied to show that $\mathcal{P}$ and $\mathcal{P}'$ are internally disjoint of each path in $\mathbb{T}$. Hence all that remains to be checked is that $\mathcal{P}$ and $\mathcal{P}'$ are internally disjoint.

Let A and A' be inner vertices of $\mathcal{P}$ and $\mathcal{P}'$, respectively. From the definition of $\mathcal{P}$ (respectively, $\mathcal{P}'$) we know that either $A\cap Z=Z\setminus\{z_y^b\}$, $A\cap Z=Z\setminus\{z_x^c\}$, or $A\cap Z=Z\setminus\{z_y^b,z_x^c\}$ (respectively, $A'\cap Z=Z\setminus\{z_y^{b-1}\}$, $A'\cap Z=Z\setminus\{z_x^{c-1}\}$, or $A'\cap Z=Z\setminus\{z_y^{b-1},z_x^{c-1}\}$). Since $\{z_y^b,z_x^c\}\cap\{z_y^{b-1},z_x^{c-1}\}=\emptyset$, then in all the arising cases, we always have $A\neq A'$, as required. This completes the proof of Claim 4.8, and hence the proof of CASE 1. $\triangle$

4.2.2 CASE 2: $|Z|=k-2$

From Proposition 4.3 (1) we know that T has two independent edges x_1y_1 and x_2y_2 such that $X\setminus Y=\{x_1,x_2\}$ and $Y\setminus X=\{y_1,y_2\}$. Then, we can assume that X and Y are as in Figure 4.4 (i). Similarly as in CASE 1, for $i\in\{1,2\}$, let us define

$$\begin{aligned}
W(x_i) &:= \{w\in W : w \text{ is adjacent to } x_i\} = \{w_{x_i}^1,\ldots,w_{x_i}^{a_i}\},\\
W(y_i) &:= \{w\in W : w \text{ is adjacent to } y_i\} = \{w_{y_i}^1,\ldots,w_{y_i}^{d_i}\},\\
Z(x_i) &:= \{z\in Z : z \text{ is adjacent to } x_i\} = \{z_{x_i}^1,\ldots,z_{x_i}^{c_i}\},\\
Z(y_i) &:= \{z\in Z : z \text{ is adjacent to } y_i\} = \{z_{y_i}^1,\ldots,z_{y_i}^{b_i}\},
\end{aligned}$$

where $a_i:=|W(x_i)|$, $b_i:=|Z(y_i)|$, $c_i:=|Z(x_i)|$, and $d_i:=|W(y_i)|$.

The next observation follows easily from the involved definitions and the fact that T is a tree.

Observation 4.9. *Let $i\in\{1,2\}$. Then $W(x_i)\cap W(y_i)=\emptyset$ and $Z(x_i)\cap Z(y_i)=\emptyset$, and at most one of the following occurs: $|W(x_1)\cap W(x_2)|=1$, $|W(y_1)\cap W(y_2)|=1$, $|Z(x_1)\cap Z(x_2)|=1$, or $|Z(y_1)\cap Z(y_2)|=1$.*

Let us define

$$E_{Z,W} := \{z_iw_j\in E(G) : z_i\in Z \text{ and } w_j\in W\}, \text{ and let } \eta := |E_{Z,W}|.$$

Then

$$deg(X) = \begin{cases} a_1 + a_2 + b_1 + b_2 + \eta + 2 & \text{if } x_1y_2 \notin E(T) \text{ and } x_2y_1 \notin E(T), \\ a_1 + a_2 + b_1 + b_2 + \eta + 3 & \text{otherwise.} \end{cases}$$

and,

$$deg(Y) = \begin{cases} c_1 + c_2 + d_1 + d_2 + \eta + 2 & \text{if } x_1y_2 \notin E(T) \text{ and } x_2y_1 \notin E(T), \\ c_1 + c_2 + d_1 + d_2 + \eta + 3 & \text{otherwise.} \end{cases}$$

Note that the term "+3" in $deg(X)$ and $deg(Y)$ means that T has 3 edges with an end in $\{x_1, x_2\}$ and the other end in $\{y_1, y_2\}$. Then it is impossible to have $deg(X) = a_1 + a_2 + b_1 + b_2 + \eta + 2$ and $deg(Y) = c_1 + c_2 + d_1 + d_2 + \eta + 3$ simultaneously. Similarly, $deg(X) = a_1 + a_2 + b_1 + b_2 + \eta + 3$ and $deg(Y) = c_1 + c_2 + d_1 + d_2 + \eta + 2$ cannot occur simultaneously.

Without loss of generality we assume that $deg(X) \leq deg(Y)$. This assumption together with the assertions of previous paragraph imply that $a_1+a_2+b_1+b_2 \leq c_1+c_2+d_1+d_2$. For $i \in \{1, 2\}$, let $m_{x_i} := \min\{a_i, c_i\}$, $m_{y_i} := \min\{b_i, d_i\}$, and $m := m_{x_1}+m_{x_2}+m_{y_1}+m_{y_2}+\eta+2$.

4.2.2.1 Step 1 of Case 2

We proceed similarly as in Case 1. In particular, we often use slight adaptation of many arguments given in Case 1. We start by producing m $X - Y$ paths by means of six types of constructions.

1. Let us define $\mathcal{P}_{x_1}$ and $\mathcal{P}_{x_2}$ as follows:

$$\mathcal{P}_{x_1} := x_1 \to y_1; x_2 \to y_2$$

$$\mathcal{P}_{x_2} := x_2 \to y_2; x_1 \to y_1.$$

   Paths $\mathcal{P}_{x_1}$ and $\mathcal{P}_{x_2}$ are depicted in Constructions 1(a) and 1(b) of Figure 4.8, respectively. Let $\mathbb{L}_1 := \{\mathcal{P}_{x_1}, \mathcal{P}_{x_2}\}$. Let $\mathcal{P} \in \mathbb{L}_1$, and let A be an inner vertex of $\mathcal{P}$. Then

   (D1) $A \cap Z = Z$ and $A \cap W = \emptyset$.

2. For each edge $z_iw_j \in E_{Z,W}$, let

$$\mathcal{P}_{i,j} := z_i \to w_j; x_1 \to y_1; x_2 \to y_2; w_j \to z_i.$$

   Path $\mathcal{P}_{i,j}$ is depicted in Construction 2 of Figure 4.8. Let $\mathbb{L}_2 := \{\mathcal{P}_{i,j} : z_iw_j \in E_{Z,W}\}$. Let $\mathcal{P}_{i,j} \in \mathbb{L}_2$, and let $A_{i,j}$ be an inner vertex of $\mathcal{P}_{i,j}$. Then

(D2) $A_{i,j} \cap Z = Z \setminus \{z_i\}$ and $A_{i,j} \cap W = \{w_j\}$.

3. For each $s \in [m_{x_1}]$, we define the path $\mathcal{P}_s$ as follows:

$$\mathcal{P}_s := x_1 \to w^s_{x_1}; z^s_{x_1} \to x_1 \to y_1; x_2 \to y_2; w^s_{x_1} \to x_1 \to z^s_{x_1}.$$

In Construction 3 of Figure 4.8 is depicted the path $\mathcal{P}_s$. Let $\mathbb{L}_3 := \{\mathcal{P}_s : s \in [m_{x_1}]\}$. Let $\mathcal{P}_s \in \mathbb{L}_3$, and let A_s be an inner vertex of $\mathcal{P}_s$. Then

(D3) Either $A_s \cap Z = Z$ or $A_s \cap Z = Z \setminus \{z^s_{x_1}\}$, and either $A_s \cap W = \emptyset$ or $A_s \cap W = \{w^s_{x_1}\}$, and at least one of the following holds: $A_s \cap W = \{w^s_{x_1}\}$ or $A_s \cap Z = Z \setminus \{z^s_{x_1}\}$.

4. For each $t \in [m_{y_1}]$, we define the path $\mathcal{Q}_t$ as follows:

$$\mathcal{Q}_t := z^t_{y_1} \to y_1 \to w^t_{y_1}; x_2 \to y_2; x_1 \to y_1 \to z^t_{y_1}; w^t_{y_1} \to y_1.$$

See an example of path $\mathcal{Q}_t$ in Construction 4 of Figure 4.9. Let $\mathbb{L}_4 := \{\mathcal{Q}_t : t \in [m_{y_1}]\}$. Let $\mathcal{Q}_t \in \mathbb{L}_4$, and let A_t be an inner vertex of $\mathcal{Q}_t$. Then

(D4) Either $A_t \cap Z = Z \setminus \{z^t_{y_1}\}$ or $A_t \cap Z = Z$, and either $A_t \cap W = \emptyset$ or $A_t \cap W = \{w^t_{y_1}\}$, and at least one of the following holds: $A_t \cap Z = Z \setminus \{z^t_{y_1}\}$ or $A_t \cap W = \{w^t_{y_1}\}$.

5. For each $q \in [m_{x_2}]$, we define $\mathcal{P}^*_q$ as follows:

$$\mathcal{P}^*_q := x_2 \to w^q_{x_2}; z^q_{x_2} \to x_2 \to y_2; x_1 \to y_1; w^q_{x_2} \to x_2 \to z^q_{x_2}.$$

Path $\mathcal{P}^*_q$ is depicted in Construction 5 of Figure 4.9. Let $\mathbb{L}^*_3 := \{\mathcal{P}^*_q : q \in [m_{x_2}]\}$. Let $\mathcal{P}^*_q \in \mathbb{L}^*_3$, and let A^*_q be an inner vertex of $\mathcal{P}^*_q$. Then

(D3*) Either $A^*_q \cap Z = Z$ or $A^*_q \cap Z = Z \setminus \{z^q_{x_2}\}$, and either $A^*_q \cap W = \emptyset$ or $A^*_q \cap W = \{w^q_{x_2}\}$, and at least one of the following holds: $A^*_q \cap W = \{w^q_{x_2}\}$ or $A^*_q \cap Z = Z \setminus \{z^q_{x_2}\}$.

6. For each $r \in [m_{y_2}]$, we define $\mathcal{Q}^*_r$ as follows:

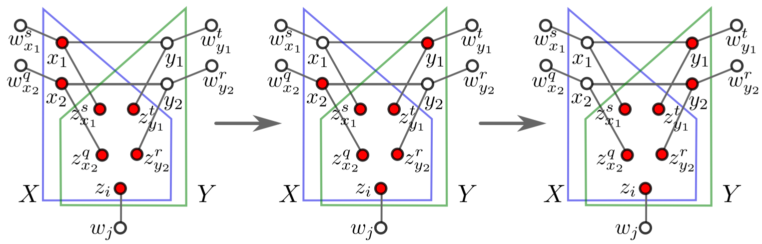
$$\mathcal{Q}^*_r := z^r_{y_2} \to y_2 \to w^r_{y_2}; x_1 \to y_1; x_2 \to y_2 \to z^r_{y_2}; w^r_{y_2} \to y_2.$$

An example of path $\mathcal{Q}^*_r$ is depicted in Construction 6 of Figure 4.9. Let $\mathbb{L}^*_4 := \{Q^*_r : r \in [m_{y_2}]\}$. Let $\mathcal{Q}^*_r \in \mathbb{L}^*_4$, and let A^*_r be an inner vertex of $\mathcal{Q}^*_r$, then

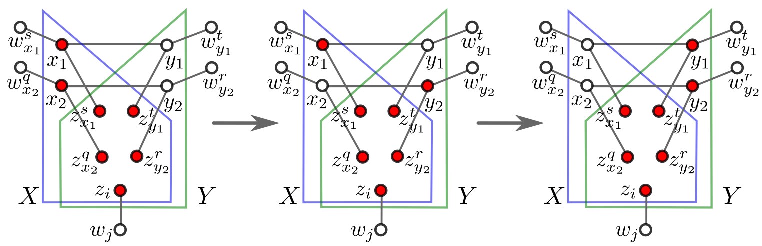
(D4*) Either $A^*_r \cap Z = Z \setminus \{z^r_{y_2}\}$ or $A^*_r \cap Z = Z$, and either $A^*_r \cap W = \emptyset$ or $A^*_r \cap W = \{w^r_{y_2}\}$, and at least one of the following holds: $A^*_r \cap Z = Z \setminus \{z^r_{y_2}\}$ or $A^*_r \cap W = \{w^r_{y_2}\}$.

Let $\mathbb{L} := \mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_3 \cup \mathbb{L}_4 \cup \mathbb{L}^*_3 \cup \mathbb{L}^*_4$. Since $|\mathbb{L}_1| = 2, |\mathbb{L}_2| = \eta, |\mathbb{L}_3| = m_{x_1}, |\mathbb{L}_4| = m_{y_1}, |\mathbb{L}^*_3| = m_{x_2}, |\mathbb{L}^*_4| = m_{y_2}$, and $m = 2 + \eta + m_{x_1} + m_{y_1} + m_{x_2} + m_{y_2}$, then in order to finish the Step 1 of Case 2, it is enough to show that the paths in $\mathbb{L}$ are pairwise internally disjoint.

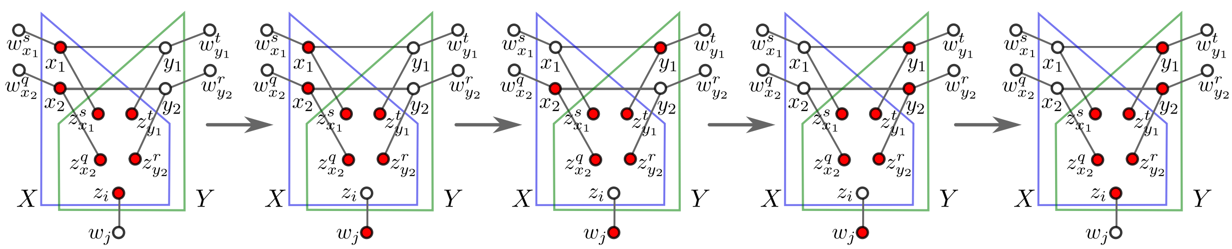
Claim 4.10. *The $X - Y$ paths in $\mathbb{L}$ are pairwise internally disjoint.*



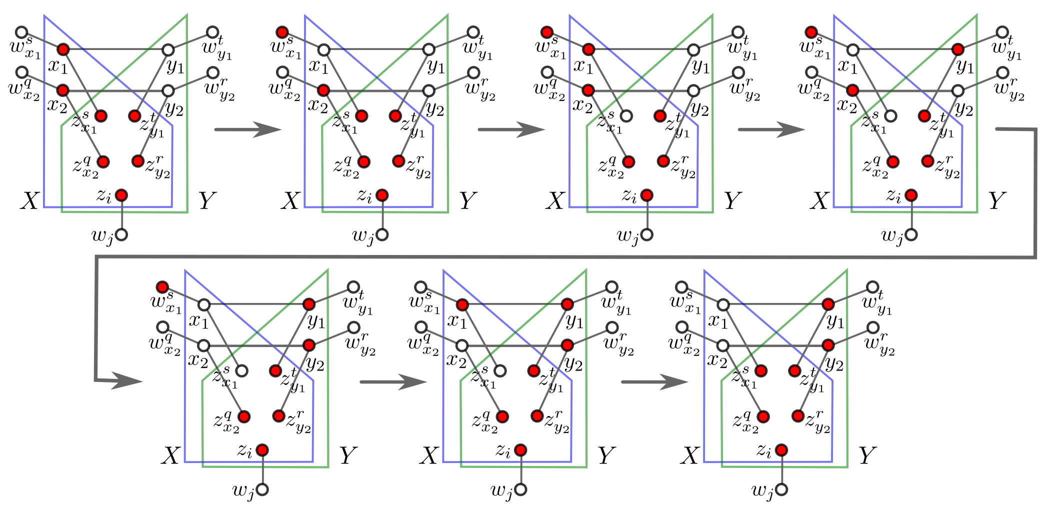
Construction 1(a): $\mathcal{P}_{x_1} := x_1 \to y_1; x_2 \to y_2$



Construction 1(b): $\mathcal{P}_{x_2} := x_2 \to y_2; x_1 \to y_1$

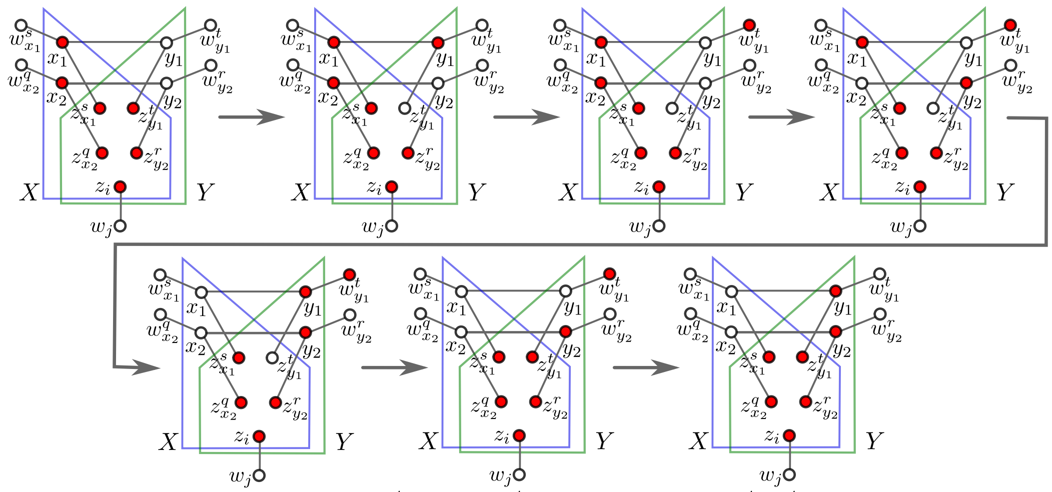


Construction 2: $\mathcal{P}_{i,j} := z_i \to w_j; x_1 \to y_1; x_2 \to y_2; w_j \to z_i$.

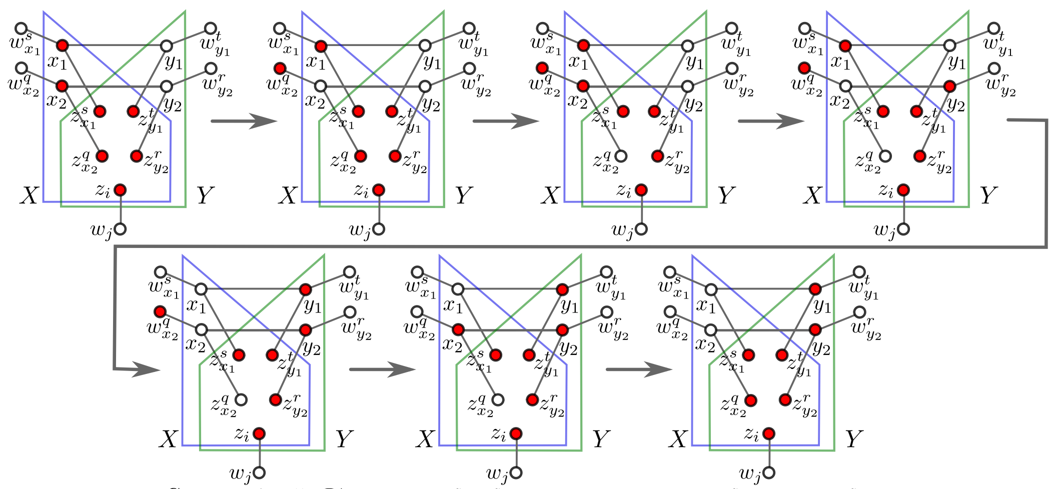


Construction 3: $\mathcal{P}_s := s := x_1 \to w^s_{x_1}; z^s_{x_1} \to x_1 \to y_1; x_2 \to y_2; w^s_{x_1} \to x_1 \to z^s_{x_1}$

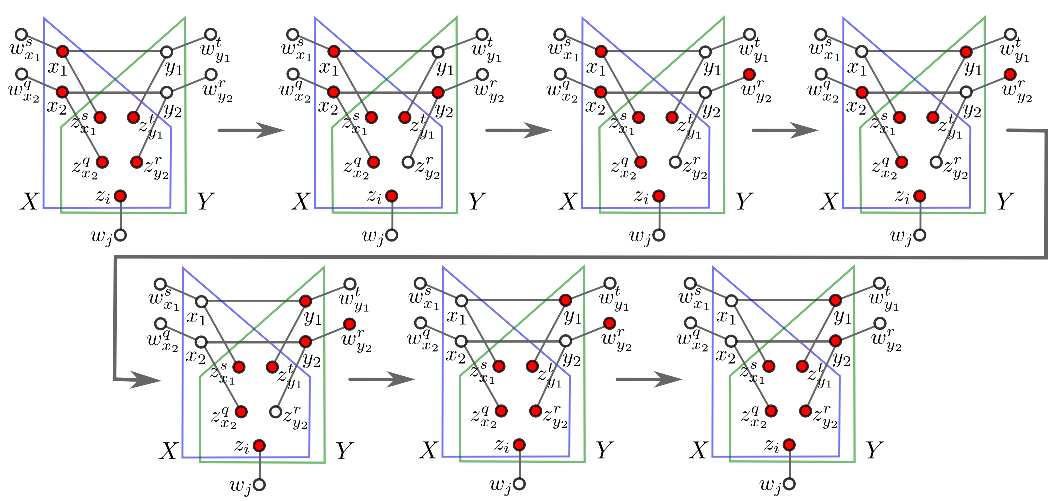
Figure 4.8: Construction of some $X - Y$ paths in Case 2.



Construction 4: $\mathcal{Q}_t := z^t_{y_1} \to y_1 \to w^t_{y_1}; x_2 \to y_2; x_1 \to y_1 \to z^t_{y_1}; w^t_{y_1} \to y_1$



Construction 5: $\mathcal{P}^*_q := x_2 \to w^q_{x_2}; z^q_{x_2} \to x_2 \to y_2; x_1 \to y_1; w^q_{x_2} \to x_2 \to z^q_{x_2}$



Construction 6: $\mathcal{Q}^*_r := z^r_{y_2} \to y_2 \to w^r_{y_2}; x_1 \to y_1; x_2 \to y_2 \to z^r_{y_2}; w^r_{y_2} \to y_2$

Figure 4.9: Construction of some $X - Y$ paths in Case 2.

Proof of Claim 4.10. We start by noting that, in some sense, the four ways in which the paths of $\mathbb{T}$ were constructed in Step 1 of Case 1 have been "repeated" in the construction of the paths of $\mathbb{L}$. This close relationship between $\mathbb{T}$ and $\mathbb{L}$ is the main ingredient in the proof of Claim 4.10.

Before moving on any further, let us verify that the two paths of $\mathbb{L}_1$ are internally disjoint. Let A_1 and A_2 be the inner vertices of $\mathcal{P}_{x_1}$ and $\mathcal{P}_{x_2}$, respectively. Then $A_1 = (X \setminus \{x_1\}) \cup \{y_1\}$ and $A_2 = (X \setminus \{x_2\}) \cup \{y_2\}$, and so $A_1 \neq A_2$.

The analogies between the paths of $\mathbb{T}$ and $\mathbb{L}$ are given by the interactions that the inner vertices of the $X - Y$ paths have with Z and W° in the Case 1 and with Z and W in the Case 2. More formally, let $\mathcal{T} \in \mathbb{T}$ and $\mathcal{L} \in \mathbb{L}$. We say that $\mathcal{T}$ and $\mathcal{L}$ are *analogous*, if $A \cap W^\circ = B \cap W$ and $A \cap Z = B \cap Z$, for any A and B inner vertices of $\mathcal{T}$ and $\mathcal{L}$, respectively. For $\mathbb{T}' \subseteq \mathbb{T}$ and $\mathbb{L}' \subseteq \mathbb{L}$ we write $\mathbb{T}' \sim \mathbb{L}'$ to mean that any path of $\mathbb{T}'$ is analogous to any path of $\mathbb{L}'$. For instance, note that $\mathbb{T}_1 \sim \mathbb{L}_1$. Indeed, let $\mathcal{P}_0 \in \mathbb{T}_1$ and $\mathcal{P}_{x_i} \in \mathbb{L}_1$, and let A_0 and A be inner vertices of $\mathcal{P}_0$ and $\mathcal{P}_{x_i}$, respectively. From (C1) we know that $A_0 \cap Z = Z$, and from (D1) we have that $A \cap Z = Z$. Similarly, from (C1) it follows that $A_0 \cap W^\circ = \emptyset$, and from (D1) that $A \cap W = \emptyset$. Analogously, we can verify that:

- (C1) and (D1) imply that $\mathbb{T}_1 \sim \mathbb{L}_1$. For completeness of this list, we include this case here again.
- (C2), (C2.1) and (D2) imply that $\mathbb{T}'_2 \sim \mathbb{L}_2$, where $\mathbb{T}'_2$ is the subset of paths in $\mathbb{T}_2$ with $w_j \neq v$.
- (C3) and (D3) imply that $\mathbb{T}_3 \sim \mathbb{L}_3$.
- (C3) and (D3*) imply that $\mathbb{T}_3 \sim \mathbb{L}^*_3$.
- (C4) and (D4) imply that $\mathbb{T}_4 \sim \mathbb{L}_4$.
- (C4) and (D4*) imply that $\mathbb{T}_4 \sim \mathbb{L}^*_4$.

We recall that the strategy in the proof of Claim 4.6 was the following. Given two inner vertices A and B belonging to distinct paths of $\mathbb{T}$, we always conclude that $A \neq B$ by showing that at least one of $A \cap W^\circ \neq B \cap W^\circ$ or $A \cap Z \neq B \cap Z$ holds. From this fact, the definition of $\sim$, and the above list, it is not hard to see that analogous arguments as those used in the proof of Claim 4.6 imply that the $X - Y$ paths belonging to $\mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}_3 \cup \mathbb{L}_4$ (resp. $\mathbb{L}_1 \cup \mathbb{L}_2 \cup \mathbb{L}^*_3 \cup \mathbb{L}^*_4$) are pairwise internally disjoint. Thus, it remains to show that the paths in $\mathbb{L}_3$ (resp. $\mathbb{L}_4$) are pairwise internally disjoint from the paths in $\mathbb{L}^*_3 \cup \mathbb{L}^*_4$.

Let A_s, A_t, A^*_q, and A^*_r be inner vertices of $\mathcal{P}_s \in \mathbb{L}_3$, $\mathcal{Q}_t \in \mathbb{L}_4$, $\mathcal{P}^*_q \in \mathbb{L}^*_3$, and $\mathcal{Q}^*_r \in \mathbb{L}^*_4$, respectively. We analyze these cases separately.

$\{A_s, A^*_q\}$: By Observation 4.9, either $z^s_{x_1} \neq z^q_{x_2}$ or $w^s_{x_1} \neq w^q_{x_2}$.

Suppose that $z^s_{x_1} \neq z^q_{x_2}$. If $A_s \cap Z = Z \setminus \{z^s_{x_1}\}$ or $A^*_q \cap Z = Z \setminus \{z^q_{x_2}\}$, then $A_s \cap Z \neq A^*_q \cap Z$, as required. Suppose then that $A_s \cap Z = Z = A^*_q \cap Z$. From the definitions of $\mathcal{P}_s$ and $\mathcal{P}^*_q$ we know that $A_s = (X \setminus \{x_1\}) \cup \{w^s_{x_1}\}$ and $A^*_q = (X \setminus \{x_2\}) \cup \{w^q_{x_2}\}$, and so $A_s \neq A^*_q$.

Suppose now that $w^s_{x_1} \neq w^q_{x_2}$. If $A_s \cap W = \{w^s_{x_1}\}$ or $A^*_q \cap W = \{w^q_{x_2}\}$, then $A_s \cap W \neq A^*_q \cap W$. Suppose then that $A_s \cap W = \emptyset = A^*_q \cap W$. Again, from the definitions of $\mathcal{P}_s$ and $\mathcal{P}^*_q$ we have that $A_s = (Y \setminus \{z^s_{x_1}\}) \cup \{x_1\}$ and $A^*_q = (Y \setminus \{z^q_{x_2}\}) \cup \{x_2\}$, and so $A_s \neq A^*_q$.

$\{A_s, A^*_r\}$: Again, by Observation 4.9, we have that either $z^s_{x_1} \neq z^r_{y_2}$ or $w^s_{x_1} \neq w^r_{y_2}$.

Suppose that $z^s_{x_1} \neq z^r_{y_2}$. If $A_s \cap Z = Z \setminus \{z^s_{x_1}\}$ or $A^*_r \cap Z = Z \setminus \{z^r_{y_2}\}$, then (D3) and (D4*) imply $A_s \cap Z \neq A^*_r \cap Z$, as required. Suppose then that $A_s \cap Z = Z = A^*_r \cap Z$. From the definitions of $\mathcal{P}_s$ and $\mathcal{Q}^*_r$ it follows that $A_s = (X \setminus \{x_1\}) \cup \{w^s_{x_1}\}$ and $A^*_r = (Y \setminus \{y_2\}) \cup \{w^r_{y_2}\}$, and so $y_1 \in A^*_r \setminus A_s$, which implies that $A_s \neq A^*_r$.

Now suppose that $w^s_{x_1} \neq w^r_{y_2}$. If $A_s \cap W = \{w^s_{x_1}\}$ or $A^*_r \cap W = \{w^r_{y_2}\}$, then (D3) and (D4*) imply that $A_s \cap W \neq A^*_r \cap W$. Suppose then that $A_s \cap W = \emptyset = A^*_r \cap W$. Again, from the definitions of $\mathcal{P}_s$ and $\mathcal{Q}^*_r$ we have that $A_s = (Y \setminus \{z^s_{x_1}\}) \cup \{x_1\}$ and $A^*_r = (X \setminus \{z^r_{y_2}\}) \cup \{y_2\}$, and so $y_1 \in A_s \setminus A^*_r$, which implies that $A_s \neq A^*_r$.

$\{A_t, A^*_q\}$: This case can be handled in the same manner as case $\{A_s, A^*_r\}$.

$\{A_t, A^*_r\}$: Again, this case can be handled in the same manner as case $\{A_s, A^*_q\}$.

$\triangle$

4.2.2.2 Step 2 of Case 2

We recall that $\deg(X) \leq \deg(Y)$ imply that

$$a_1 + a_2 + b_1 + b_2 \leq c_1 + c_2 + d_1 + d_2. \tag{19}$$

We now proceed to show that $\delta - m \leq 1$.

Claim 4.11. *Let $\delta, m, m_{x_1}, m_{y_1}, m_{x_2}, m_{y_2}$, and η as above. Then, $\delta - m \leq 1$.*

Proof of Claim 4.11. We analyze several cases separately, depending on the order relations between the elements of the sets $\{a_i, c_i\}$ and $\{b_i, d_i\}$, for $i, j \in \{1, 2\}$. The possible cases are the following:

(1)	$a_1 > c_1$, $a_2 > c_2$, $b_1 > d_1$ and $b_2 > d_2$	(9)	$a_1 \leq c_1$, $a_2 > c_2$, $b_1 > d_1$ and $b_2 > d_2$
(2)	$a_1 > c_1$, $a_2 > c_2$, $b_1 > d_1$ and $b_2 \leq d_2$	(10)	$a_1 \leq c_1$, $a_2 > c_2$, $b_1 > d_1$ and $b_2 \leq d_2$
(3)	$a_1 > c_1$, $a_2 > c_2$, $b_1 \leq d_1$ and $b_2 > d_2$	(11)	$a_1 \leq c_1$, $a_2 > c_2$, $b_1 \leq d_1$ and $b_2 > d_2$
(4)	$a_1 > c_1$, $a_2 > c_2$, $b_1 \leq d_1$ and $b_2 \leq d_2$	(12)	$a_1 \leq c_1$, $a_2 > c_2$, $b_1 \leq d_1$ and $b_2 \leq d_2$
(5)	$a_1 > c_1$, $a_2 \leq c_2$, $b_1 > d_1$ and $b_2 > d_2$	(13)	$a_1 \leq c_1$, $a_2 \leq c_2$, $b_1 > d_1$ and $b_2 > d_2$
(6)	$a_1 > c_1$, $a_2 \leq c_2$, $b_1 > d_1$ and $b_2 \leq d_2$	(14)	$a_1 \leq c_1$, $a_2 \leq c_2$, $b_1 > d_1$ and $b_2 \leq d_2$
(7)	$a_1 > c_1$, $a_2 \leq c_2$, $b_1 \leq d_1$ and $b_2 > d_2$	(15)	$a_1 \leq c_1$, $a_2 \leq c_2$, $b_1 \leq d_1$ and $b_2 > d_2$
(8)	$a_1 > c_1$, $a_2 \leq c_2$, $b_1 \leq d_1$ and $b_2 \leq d_2$	(16)	$a_1 \leq c_1$, $a_2 \leq c_2$, $b_1 \leq d_1$ and $b_2 \leq d_2$

As a first observation, the case (1) is impossible because of Inequality (19). Let us next show that it is enough to consider only six cases: (2), (4), (6), (7), (8) and (16), because the rest of cases are similar to one of these cases.

In the cases (3), (9)–(12) and (15) interchange the labels of the elements in each of the following sets: $\{x_1, x_2\}$ and $\{y_1, y_2\}$. These interchanges automatically produce the interchange of the values in each of the following sets $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ and $\{d_1, d_2\}$. By performing these relabelings, we can see that: case (3) is similar to case (2), case (9) is similar to case (5), case (10) is similar to case (7), case (11) is similar to case (6), case (12) is similar to case (8), and case (15) is similar to case (14). Thus, we may restrict our analysis to the cases (2), (4)–(8), (13), (14), and (16).

In the cases (5), (13) and (14) we consider the graph $F_{n-k}(T)$ instead of $F_k(T)$ with the following relabeling. For $i \in \{1, 2\}$, let $x'_i := y_i$ and $y'_i := x_i$. Consider the vertices $X' = \phi(X) = V(T) \setminus X$ and $Y' = \phi(Y) = V(T) \setminus Y$ in $F_{n-k}(T)$. Let $Z' := W$ and $W' := Z$, and define the values a'_i, b'_i, c'_i and d'_i analogously to a_i, b_i, c_i and d_i. Then we have $a'_i = b_i$, $b'_i = a_i$, $c'_i = d_i$ and $d'_i = c_i$, and so case (5) is similar to case (2), case (13) is similar to case (4), and case (14) is similar to case (8). Then, we may assume that one of cases (2), (4), (6), (7), (8) and (16) holds.

Our strategy is as follows. In any of the analyzed cases we show that $F_k(G)$ has a vertex X_1 "close to" X whose degree is at most $m + 1$. Recall that we need to consider only the cases (2), (4), (6), (7), (8) and (16).

(2) $a_1 > c_1$, $a_2 > c_2$, $b_1 > d_1$ and $b_2 \leq d_2$.

Then $a_1 > 0$ and $a_2 > 0$. Moreover, our suppositions and (19) imply that $d_2 > b_2$. Let $U := W \cup \{x_1, x_2, y_1, y_2\}$. From $a_1 > 0, a_2 > 0$, and $d_2 > 0$ it follows that x_1, x_2, and y_2 have degree at least 2 in $T[U]$. Since $T[U]$ is a forest, then there is a vertex $u \in U \setminus \{x_1, x_2, y_1, y_2\}$ such that $deg_{T[U]}(u) \leq 1$. Let $X_1 := (X \setminus \{x_1, x_2\}) \cup \{y_1, u\}$.

(2.1) If y_1 is not adjacent to both x_2 and y_2, then

$$\begin{aligned}\delta \leq \deg(X_1) &\leq c_1 + c_2 + d_1 + b_2 + \eta + 1 + deg_{T[U]}(u) \\ &\leq m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 2 = m.\end{aligned}$$

(2.2) If y_1 is adjacent to some of x_2 or y_2, then it is adjacent to exactly one of them, because T has no cycles. Hence, in this case

$$\begin{aligned}\delta \leq \deg(X_1) &\leq c_1 + c_2 + d_1 + b_2 + \eta + 2 + deg_{T[U]}(u)\\ &= m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 3 = m + 1.\end{aligned}$$

(4) $a_1 > c_1$, $a_2 > c_2$, $b_1 \leq d_1$ and $b_2 \leq d_2$.

If $d_1 > 0$ and $d_2 > 0$, then $\deg_{T[U]}(y_i) = d_i + 1 \geq 2$ and $\deg_{T[U]}(x_i) = a_i + 1 \geq 2$, for $U := W \cup \{x_1, x_2, y_1, y_2\}$ and $i \in \{1, 2\}$. These and the fact that $T[U]$ is a forest imply the existence of two vertices $u_1, u_2 \in U \setminus \{x_1, x_2, y_1, y_2\}$ such that $\deg_{T[U]}(u_i) \leq 1$ for $i \in \{1, 2\}$. Let $X_1 := (X \setminus \{x_1, x_2\}) \cup \{u_1, u_2\}$. Then

$$\begin{aligned}\delta \leq \deg(X_1) &\leq c_1 + c_2 + b_1 + b_2 + \eta + \deg_{T[U]}(u_1) + \deg_{T[U]}(u_2)\\ &\leq m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 2 = m.\end{aligned}$$

We now suppose $d_1 > 0$ and $d_2 > 0$ does not hold. Then $d_1 = 0$ or $d_2 = 0$. By symmetry, we may assume that $d_1 = 0$. Then $b_1 = 0$, and $d_2 > 0$ by (19). Then for $U := W \cup \{x_1, x_2, y_1, y_2\}$, we have that $\deg_{T[U]}(x_i) = a_i + 1 \geq 2$ for $i \in \{1, 2\}$ and $\deg_{T[U]}(y_2) = d_2 + 1 \geq 2$. Since $T[U]$ has no cycles, then y_1 is adjacent to at most one x_2 or y_2. From this fact, $b_1 = d_1 = 0$, and $x_1y_1 \in E(T[U])$ it follows that $1 \leq \deg_{T[U]}(y_1) \leq 2$. Again, these and the fact that $T[U]$ is a forest imply the existence of two distinct vertices $u_1, u_2 \in U \setminus \{x_1, x_2, y_2\}$ such that $\deg_{T[U]}(u_i) \leq 1$ for $i \in \{1, 2\}$. Let $X_1 = (X \setminus \{x_1, x_2\}) \cup \{u_1, u_2\}$, then

$$\begin{aligned}\delta \leq \deg(X_1) &\leq c_1 + c_2 + b_1 + b_2 + \eta + \deg_{T[U]}(u_1) + \deg_{T[U]}(u_2)\\ &\leq m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 2 = m.\end{aligned}$$

(6) $a_1 > c_1$, $a_2 \leq c_2$, $b_1 > d_1$ and $b_2 \leq d_2$.

From (19) and these inequalities it follows that at least one of $c_2 > a_2$ or $d_2 > b_2$ holds. Let $X_1 := (X \setminus \{x_1\}) \cup \{y_1\}$. Since T has no cycles, then it contains at most one of x_1x_2 or y_1y_2.

(6.1) Suppose that none of x_1x_2 or y_1y_2 is in T. Then

$$\begin{aligned}\delta \leq \deg(X_1) &\leq c_1 + a_2 + d_1 + b_2 + \eta + 2\\ &= m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 2 = m.\end{aligned}$$

(6.2) Suppose that exactly one of x_1x_2 or y_1y_2 is in T. Then

$$\begin{aligned}\delta \leq \deg(X_1) &\leq c_1 + a_2 + d_1 + b_2 + \eta + 3\\ &= m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 3 = m + 1.\end{aligned}$$

(7) $a_1 > c_1$, $a_2 \leq c_2$, $b_1 \leq d_1$ and $b_2 > d_2$.

Let $X_1 := (X \setminus \{x_1\}) \cup \{y_2\}$. Again, since T has no cycles, then there is at most one edge in T with one endvertex in $\{x_1, y_1\}$ and the other endvertex in $\{x_2, y_2\}$. Then

$$\begin{aligned}\delta \leq \deg(X_1) &\leq c_1 + a_2 + b_1 + d_2 + \eta + 1 \\ &= m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 1 < m.\end{aligned}$$

(8) $a_1 > c_1$, $a_2 \leq c_2$, $b_1 \leq d_1$ and $b_2 \leq d_2$.

As we have mentioned above, T has at most one edge with one end in $\{x_1, y_1\}$ and the other end in $\{x_2, y_2\}$.

(8.1) Suppose that $d_2 \leq b_2 + 1$. Then $X_1 := (X \setminus \{x_1\}) \cup \{y_2\}$ satisfies the following

$$\begin{aligned}\delta \leq \deg(X_1) &\leq c_1 + a_2 + b_1 + d_2 + \eta + 1 \\ &\leq c_1 + a_2 + b_1 + (b_2 + 1) + \eta + 1 \\ &\leq m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 2 = m.\end{aligned}$$

(8.2) Suppose that $d_2 \geq b_2 + 2$. Then $a_1 > 0$ and $d_2 \geq 2$, and hence x_1 and y_2 have degree at least 2 in $T[U]$, for $U := W \cup \{x_1, y_1, y_2\}$. Since $T[U]$ is a forest, then there is a vertex $u \in U \setminus \{y_1, x_1, y_2\}$ such that $\deg_{T[U]}(u) \leq 1$. Let $X_1 := (X \setminus \{x_1\}) \cup \{u\}$.

(8.2.1) Suppose that x_2 is not adjacent to both x_1 and y_1. Then,

$$\begin{aligned}\delta \leq \deg(X_1) &\leq c_1 + a_2 + b_1 + b_2 + \eta + 2 \\ &= m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 2 = m.\end{aligned}$$

(8.2.2) Suppose that x_2 is adjacent to some of x_1 or y_1. Since there is at most one edge with one end in $\{x_1, y_1\}$ and the other end in $\{x_2, y_2\}$, then x_2 is adjacent to exactly one of x_1 or y_1. Then,

$$\begin{aligned}\delta \leq \deg(X_1) &\leq c_1 + a_2 + b_1 + b_2 + \eta + 3 \\ &= m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 3 = m + 1.\end{aligned}$$

(16) $a_1 \leq c_1$, $a_2 \leq c_2$, $b_1 \leq d_1$ and $b_2 \leq d_2$.

Since there is at most one edge with one end in $\{x_1, y_1\}$ and the other end in $\{x_2, y_2\}$, then T contains at most one of x_1y_2 or x_2y_1.

(16.1) Suppose that neither x_1y_2 nor x_2y_1 is in T. Then,

$$\begin{aligned}\delta \leq \deg(X) &\leq a_1 + a_2 + b_1 + b_2 + \eta + 2 \\ &= m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 2 = m.\end{aligned}$$

(16.2) Suppose that some of x_1y_2 or x_2y_1 is in T. Then exactly one of x_1y_2 or x_2y_1 belongs to T. By symmetry, we may assume that x_1 is adjacent to y_2. Let $X_1 := (X \setminus \{x_1\}) \cup \{y_2\}$. Then,

$$\delta \leq \deg(X_1) \leq c_1 + a_2 + b_1 + d_2 + \eta + 1$$

(16.2.1) If $a_1 = c_1$ and $b_2 = d_2$, then

$$\delta \leq \deg(X_1) \leq m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 1 \leq m.$$

(16.2.2) If $a_1 < c_1$ or $b_2 < d_2$, then

$$\begin{aligned}\delta \leq \deg(X) &\leq a_1 + a_2 + b_1 + b_2 + \eta + 3 \\ &= m_{x_1} + m_{x_2} + m_{y_1} + m_{y_2} + \eta + 3 = m + 1.\end{aligned}$$

$\triangle$

Claim 4.11 shows that almost all $X - Y$ paths claimed by Lemma 4.5 are provided by $\mathbb{L}$, when $|Z| = k-2$. We finish the proof of Case 2 with the construction of the remaining $\delta - m$ $X - Y$ paths.

Claim 4.12. *If $|Z| = k - 2$, then $F_k(T)$ has at least δ $X - Y$ pairwise internally disjoint paths.*

Proof of Claim 4.12. Consider the m $X - Y$ paths of $\mathbb{L}$. Clearly, if $m \geq \delta$, then we are done. Then by Claim 4.11 we can assume that $m + 1 = \delta$, and that some of the following four cases of the proof of Claim 4.11 holds: (2.2), (6.2), (8.2.2), or (16.2.2). In view of these facts, it is enough to exhibit a new $X - Y$ path $\mathcal{P}^\ell \notin \mathbb{L}$ with $\mathcal{P}^\ell$ internally disjoint from any path in $\mathbb{L}$. We note that in any of these four cases, T has one edge e with an endvertex in $\{x_1, y_1\}$ and the other endvertex in $\{x_2, y_2\}$. Since T has no cycles, then e is the only edge of T with this property. Then $W(x_1), W(x_2), W(y_1), W(y_2), Z(x_1), Z(x_2), Z(y_1)$, and $Z(y_2)$ are pairwise disjoint, as otherwise T has a cycle.

Our strategy is as follows. First we define a set $\mathbb{P} = \{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4\}$ consisting of four new $X - Y$ paths of $F_k(T)$. Then we show that for each of the four cases mentioned in previous paragraph, there is a path in $\mathbb{P}$ which is internally disjoint from any path of $\mathbb{L}$, providing the additional required path.

1. If $a_1 > c_1$ and $d_2 > b_2$, then we define the $X - Y$ path $\mathcal{P}^1$ as follows:

$$\mathcal{P}^1 := x_1 \to w_{x_1}^{a_1}; x_2 \to y_2 \to w_{y_2}^{d_2}; w_{x_1}^{a_1} \to x_1 \to y_1; w_{y_2}^{d_2} \to y_2.$$

From the definition of $\mathcal{P}^1$ it follows that if A^1 is an inner vertex of $\mathcal{P}^1$, then

(E1) $A^1 \cap Z = Z$, and $A^1 \cap W \in \big\{\{w_{x_1}^{a_1}\}, \{w_{y_2}^{d_2}\}, \{w_{x_1}^{a_1}, w_{y_2}^{d_2}\}\big\}$.

2. If $a_1 > c_1$ and $c_2 > a_2$, then we define the $X - Y$ path $\mathcal{P}^2$ as follows:

$$\mathcal{P}^2 := x_1 \to w_{x_1}^{a_1}; x_2 \to y_2; z_{x_2}^{c_2} \to x_2; w_{x_1}^{a_1} \to x_1 \to y_1; x_2 \to z_{x_2}^{c_2}.$$

From the definition of $\mathcal{P}^2$ it follows that if A^2 is an inner vertex of $\mathcal{P}^2$, then

(E2) Either $A^2 \cap Z = Z$ or $A^2 \cap Z = Z \setminus \{z_{x_2}^{c_2}\}$, and either $A^2 \cap W = \emptyset$ or $A^2 \cap W = \{w_{x_1}^{a_1}\}$, and at least one of the following holds: $A^2 \cap W = \{w_{x_1}^{a_1}\}$ or $A^2 \cap Z = Z \setminus \{z_{x_2}^{c_2}\}$.

3. If $c_1 > a_1$ and $x_1y_2 \in E(T)$, then we define the $X - Y$ path $\mathcal{P}^3$ as follows:

$$\mathcal{P}^3 := x_1 \to y_2; z_{x_1}^{c_1} \to x_1 \to y_1; y_2 \to x_1; x_2 \to y_2; x_1 \to z_{x_1}^{c_1}.$$

From the definition of $\mathcal{P}^3$ it follows that if A^3 is an inner vertex of $\mathcal{P}^3$, then

(E3) $A^3 \cap W = \emptyset$, and $A^3 \cap Z \in \left\{Z, Z \setminus \{z_{x_1}^{c_1}\}\right\}$.

4. If $d_2 > b_2$ and $x_1y_2 \in E(T)$, then we define the $X - Y$ path $\mathcal{P}^4$ as follows:

$$\mathcal{P}^4 := x_1 \to y_2 \to w_{y_2}^{d_2}; x_2 \to y_2 \to x_1 \to y_1; w_{y_2}^{d_2} \to y_2.$$

From the definition of $\mathcal{P}^4$ it follows that if A^4 is an inner vertex of $\mathcal{P}^4$, then

(E4) $A^4 \cap Z = Z$, and $A^4 \cap W \in \left\{\emptyset, \{w_{y_2}^{d_2}\}\right\}$.

We now proceed to show that for $i \in \{1, 2, 3, 4\}$, the $X - Y$ paths in $\{\mathcal{P}^i\} \cup \mathbb{L}$ are internally disjoint. For this, let us assume that $A^1, A^2, A^3, A^4, A, A_{i,j}, A_i, A_j, A_s^*$, and A_t^* are inner vertices of $\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4, \mathcal{P} \in \mathbb{L}_1$, $\mathcal{P}_{i,j} \in \mathbb{L}_2$, $\mathcal{P}_i \in \mathbb{L}_3$, $\mathcal{Q}_j \in \mathbb{L}_4$, $\mathcal{P}_s^* \in \mathbb{L}_3^*$, and $\mathcal{Q}_t^* \in \mathbb{L}_4^*$, respectively.

$\{\mathcal{P}^1\} \cup \mathbb{L}$: We have $A \cap W = \emptyset$ while $A^1 \cap W \neq \emptyset$, so $A^1 \neq A$. Also we have $A_{i,j} \cap Z \neq Z$ and $A^1 \cap Z = Z$, thus $A^1 \neq A_{i,j}$. Let $W_1 := \{w_{x_1}^{a_1}, w_{y_2}^{d_2}\}$ and $W_2 := (W(x_1) \cup W(x_2) \cup W(y_1) \cup W(y_2)) \setminus W_1$. Note that W_1 and W_2 are disjoint. For $A' \in \{A_i, A_j, A_s^*, A_t^*\}$ we may assume that $A' \cap W \neq \emptyset$ (as otherwise we have $A' \cap W = \emptyset \neq A^1 \cap W$, and so $A^1 \neq A'$). Then, $A^1 \cap W \subset W_1$ while $A' \cap W \subset W_2$, since $W_1 \cap W_2 = \emptyset$, it follows that $A^1 \neq A'$.

$\{\mathcal{P}^2\} \cup \mathbb{L}$: By (E2) we know that $A^2 \cap Z \in \{Z, Z \setminus \{z_{x_2}^{c_2}\}\}$.

First suppose that $A^2 \cap Z = Z$, so $A^2 \cap W = \{w_{x_1}^{a_1}\}$. Since $A \cap W = \emptyset$ we have $A^2 \neq A$. Also, since $A_{i,j} \cap Z \neq Z$, we have $A^2 \neq A_{i,j}$. Let $W_1 := \{w_{x_1}^{a_1}\}$ and $W_2 := (W(x_1) \cup W(x_2) \cup W(y_1) \cup W(y_2)) \setminus W_1$. Note that W_1 and W_2 are disjoint. For $A' \in \{A_i, A_j, A_s^*, A_t^*\}$ we may assume that $A' \cap W \neq \emptyset$ (as otherwise we have $A' \cap W = \emptyset \neq A^2 \cap W$, and so $A^2 \neq A'$). Then, as in the previous case, we have $A^2 \cap W \subset W_1$ while $A' \cap W \subset W_2$, since $W_1 \cap W_2 = \emptyset$, it follows that $A^2 \neq A'$.

Suppose now that $A^2 \cap Z = Z \setminus \{z_{x_2}^{c_2}\}$. We have $A \cap Z = Z$, so $A^2 \neq A$. Let $Z_1 := \{z_{x_2}^{c_2}\}$ and $Z_2 := (Z(x_1) \cup Z(x_2) \cup Z(y_1) \cup Z(y_2)) \setminus Z_1$. For $A' \in \{A_i, A_j, A_s^*, A_t^*\}$, if $A' \cap Z = Z$ then $A^2 \neq A'$. Suppose now that $A' \cap Z \neq Z$. Then, $A' \cap Z \subset Z_2$, while $A^2 \cap Z = Z_1$; and since $Z_1 \cap Z_2 = \emptyset$ it follows that $A^2 \neq A'$. Consider now the vertex $A_{i,j}$. Note that $z_{x_2}^{c_2} \neq z_i$ or $w_{x_1}^{a_1} \neq w_j$, as otherwise the subgraph of T induced by $z_i, w_j, x_1, x_2, y_1, y_2$, and e contains a

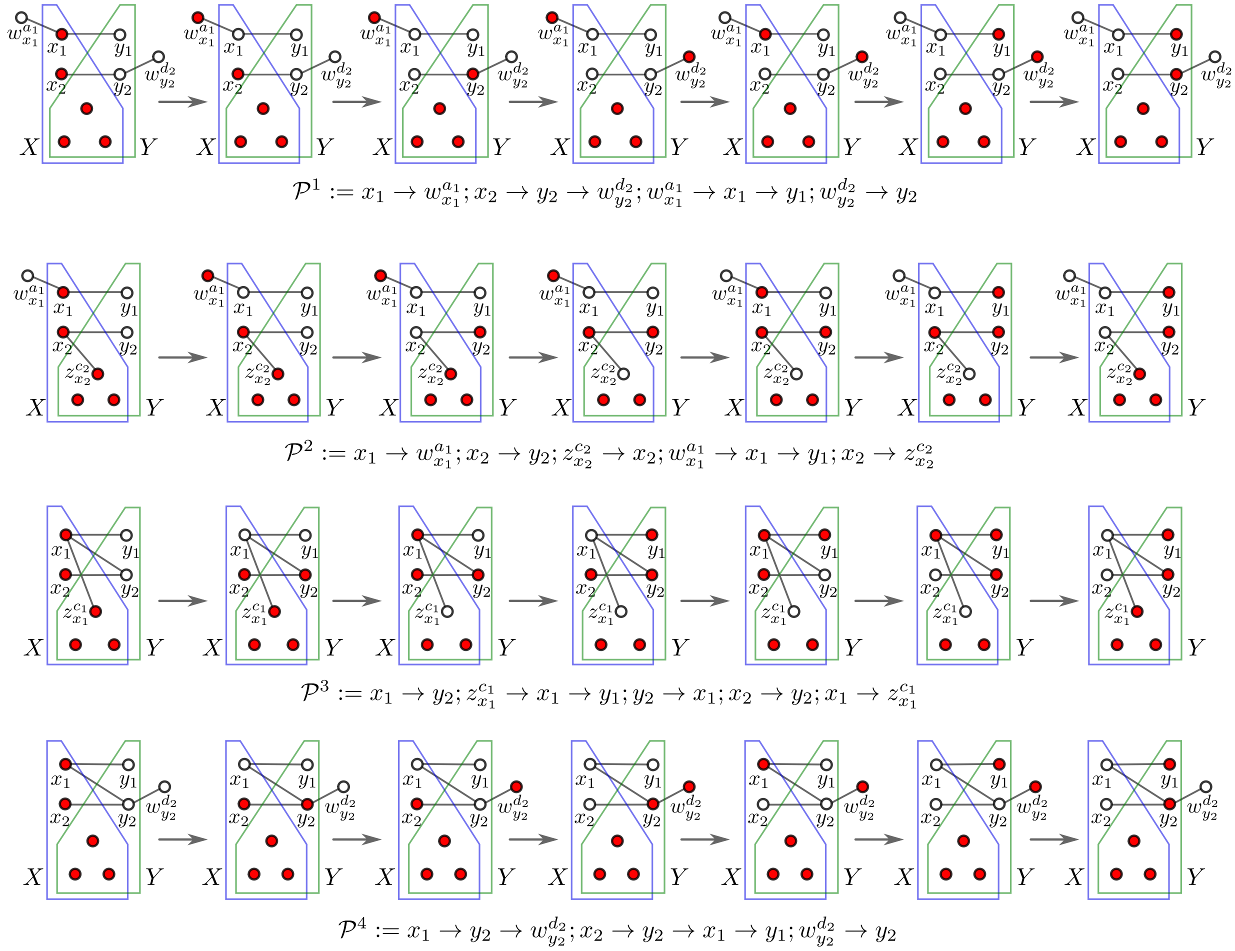


Figure 4.10: Construction of some extra $X - Y$ paths in CASE 2.

cycle. If $z_{x_2}^{c_2} \neq z_i$, then (D2) and (E2) imply $A_{i,j} \cap Z \neq A^2 \cap Z$, as required. On the other hand, if $w_{x_1}^{a_1} \neq w_j$, again (D2) and (E2) imply that $A_{i,j} \cap W \neq A^2 \cap W$, and so $A_{i,j} \neq A^2$.

$\{\mathcal{P}^3\} \cup \mathbb{L}$: By (E3) we have $A^3 \cap W = \emptyset$ and $A^3 \cap Z \in \{Z, Z \setminus \{z_{x_1}^{c_1}\}\}$.

First suppose that $A^3 \cap Z = Z$. Then $A^3 = (X \setminus \{x_1\}) \cup \{y_2\}$, and so $x_2, y_2 \in A^3$. On the other hand, for any $A' \in \{A, A_{i,j}, A_i, A_j, A_s^*, A_t^*\}$ we have that x_2 and y_2 do not belong to A' simultaneously, which implies that $A^3 \neq A'$.

Suppose now that $A^3 \cap Z = Z \setminus \{z_{x_1}^{c_1}\}$. In this case proceed in a similar way to the case $\{\mathcal{P}^2\} \cup \mathbb{L}$ when $A^2 \cap Z = Z \setminus \{z_{x_2}^{c_2}\}$.

$\{\mathcal{P}^4\} \cup \mathbb{L}$: By (E4) we have $A^4 \cap Z = Z$ and $A^4 \cap W \in \{\emptyset, \{w_{y_2}^{d_2}\}\}$. As a first observation, $A^4 \neq A_{i,j}$ because $A_{i,j} \cap Z \neq Z$.

Suppose that $A^4 \cap W = \emptyset$, then $A^4 = (X \setminus \{x_1\}) \cup \{y_2\}$, and so $x_2, y_2 \in A_4$. Similar to case $\{\mathcal{P}_3\} \cup \mathbb{L}$, for $A' \in \{A, A_i, A_j, A_s^*, A_t^*\}$ we have that x_2 and y_2 do not belong to A' simultaneously. Thus, $A^3 \neq A'$.

Suppose now that $A^4 \cap W = \{w_{y_2}^{d_2}\}$. We have $A^4 \neq A$ because $A \cap W = \emptyset$. Let $W_1 := \{w_{y_2}^{d_2}\}$ and $W_2 := (W(x_1) \cup W(x_2) \cup W(y_1) \cup W(y_2)) \setminus W_1$. Next, for $A' \in \{A_i, A_j, A_s^*, A_t^*\}$ proceed as in the case $\{\mathcal{P}^1\} \cup \mathbb{L}$ to show that $A^4 \neq A'$.

Summarizing: for $i \in \{1, 2, 3, 4\}$, we have shown that if $\mathcal{P}^i$ exists, then $\mathbb{L} \cup \{\mathcal{P}^i\}$ is a set of $\delta = m + 1$ pairwise internally disjoint $X - Y$ paths of $F_k(T)$. Now the proof of Claim 4.12 follows easily.

Note that $\mathbb{L} \cup \{\mathcal{P}^1\}$ provides the required δ paths for all those cases in which $a_1 > c_1$ and $d_2 > b_2$. Then $\mathbb{L} \cup \{\mathcal{P}^1\}$ is the required set for the Cases (2.2) and (8.2.2). Similarly, $\mathbb{L} \cup \{\mathcal{P}^2\}$ works for all those cases in which $a_1 > c_1$ and $c_2 > a_2$. From Case (6.2) we know that $a_1 > c_1$ and that at least one of $c_2 > a_2$ or $d_2 > b_2$ holds. Clearly, if $a_1 > c_1$ and $c_2 > a_2$ (respectively, $d_2 > b_2$), then $\mathbb{L} \cup \{\mathcal{P}^2\}$ (respectively, $\mathbb{L} \cup \{\mathcal{P}^1\}$) is the required set for Case (6.2). Finally, note that in Case (16.2.2) we know that $x_1y_2 \in E(T)$, and that at least one of $c_1 > a_1$ or $d_2 > b_2$ holds. If $c_1 > a_1$ (respectively, $d_2 > b_2$), then $\mathbb{L} \cup \{\mathcal{P}^3\}$ (respectively, $\mathbb{L} \cup \{\mathcal{P}^4\}$) is the required set for Case (16.2.2). $\triangle$

Clearly, the proof of Claim 4.12 finishes the proof of Lemma 4.5, which implies Theorem 4.1. $\square$

4.3 Concluding remarks and open problems

Trees and complete graphs are two families of graphs which are extremely distinct from the point of view of the connectivity and the edge-connectivity. In this chapter we have shown that if G is a tree, then $\kappa(F_k(G)) = \lambda(F_k(G)) = \delta(F_k(G))$. Surprisingly, these same equalities hold for the case of the complete graph. More precisely, from [35] and [34] we know that the connectivity and the edge-connectivity of $F_k(K_n)$ are equal to the minimum degree of $F_k(K_n)$. However, these equalities do not hold in general. For instance, consider the following families of graphs.

Example 4.1. *Consider the graph G obtained from the complete bipartite graph $K_{2,n}$ by adding an edge joining the vertices in the class containing two vertices (the graph G is depicted in Figure 4.11), and consider the k-token graph of G, where $k = \lfloor \frac{n+2}{2} \rfloor$. Here we have $\delta(F_k(G)) = n + 1$ while $\kappa(F_k(G)) = n$.*

Example 4.2. *Consider the graph H of Figure 4.11, where $m > 3$, and its 2-token graph $F_2(H)$. It is not hard to see that $\kappa(F_2(H)) = m - 1 = \lambda(F_2(H))$ and $\delta(F_2(H)) = 2(m - 2)$, and so $\kappa(F_2(H)) = \lambda(F_2(H)) < \delta(F_2(H))$.*

Using the SageMath software [50], we have computed the vertex connectivity of token graphs of connected graphs with at most ten vertices. Next, we mention some results of

Figure 4.11: The graph G is constructed from the complete bipartite graph $K_{2,n}$ by adding a new edge e. The graph H is constructed by connecting two copies of K_m by means of a new edge e'.

this computational experimentation.

- If G is a connected graph with at most ten vertices and girth greater than three, then $\kappa(F_2(G)) = \lambda(F_2(G)) = \delta(F_2(G))$.
- If G is a connected graph with at most nine vertices and girth greater than three, then $\kappa(F_3(G)) = \lambda(F_3(G)) = \delta(F_3(G))$.

Based on this computational experimentation and on some analytic approaches we have the following conjecture.

Conjecture 4.1. *Let G be a connected graph with girth at least five and let k be a positive integer with $2 \leq k \leq n-2$. Then*

$$\kappa(F_k(G)) = \delta(F_k(G)).$$

Concerning this conjecture, we remark that if G is a connected graph with girth at least five, then all the paths constructed in this chapter for the k-token graph of trees can be constructed in the k-token graph of G. This gives several ways to construct internally disjoint paths in $F_k(G)$.

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