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Around $\mathbb{F}_1$–geometry.

Monoid schemes, toric varieties and blueprints

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Centro de Investigación y Estudios Avanzados del I.P.N.
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Sobre la geometría de $\mathbb{F}_1$.

Esquemas de monoides, variedades tóricas y blueprints

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Abstract

In this thesis we study the $\mathbb{F}_1-$geometry, starting by describing the motivations that gave rise to this subject. After that we focus on the approach introduced by Deitmar in [8], about the geometry of monoids, from which we develop the notions of toric varieties over $\mathbb{F}_1$ discussed in [5], and we extend them to the non-normal case. We generalize various results, originally stated over fields (usually $\mathbb{C}$), to toric varieties over monoids. In particular we extend the results obtained in [10], about the multiplicity of toric curves over a field, to the case of toric curves over $\mathbb{F}_1$.

Finally we study the geometry of $\mathbb{F}_1$ from the approach introduced by Lorscheid in [25], which generalizes to Deitmar's, and which is based on the geometry of blueprints. We present an application of this approach to tropical geometry, which consists of treating tropical varieties as blue schemes.

Resumen

En esta tesis estudiamos la geometría de $\mathbb{F}_1$, comenzando por describir las motivaciones que dieron lugar a esta área. Posteriormente nos enfocamos en el enfoque introducido por Deitmar en [8], acerca de la geometría de monoides, desde el cual desarrollamos las nociones de variedades tóricas sobre $\mathbb{F}_1$ discutidas en [5], y las extendemos al caso no normal. Generalizamos diversos resultados, originalmente establecidos sobre campos (usualmente $\mathbb{C}$), a variedades tóricas sobre monoides. En particular extendemos los resultados obtenidos en [10], acerca de la multiplicidad de curvas tóricas sobre un campo, al caso de curvas tóricas sobre $\mathbb{F}_1$.

Finalemente estudiamos la geometría de $\mathbb{F}_1$ desde el enfoque introducido por Lorscheid en [25], el cual generaliza al de Deitmar, y el cual se basa en la geometría de blueprints. Presentamos una aplicación de este enfoque a la geometría tropical, la cual consiste en tratar a las variedades tropicales como esquemas azules.

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Contents

Chapter 1

Introduction

In this work we want to present some of the main ideas around $\mathbb{F}_1-$geometry. By $\mathbb{F}_1$ we refer to an hypothetical field with one element. Of course this meaning cannot be taken in a literal sense, however, as we shall see, the main notions of $\mathbb{F}_1$ are described in terms of the category of pointed commutative monoids. In this context, $\mathbb{F}_1$ turns out to be the initial object of such category[1], namely, the monoid $\{0, 1\}$.

The first mentions of a field with one element are due to Jacques Tits in the 1950's. The idea arises from observing that certain mathematical objects (particularly in the context of incidence geometries and algebraic groups) over finite fields $\mathbb{F}_q$ have a significant interpretations when $q = 1$ (see [21, 23]). We show some examples these ideas below.

Example 1.0.1. *To compute the number of points in the projective space $\mathbb{P}^{n-1}_{\mathbb{F}_q}$ we need to count the number of $1-$dimensional linear subspaces of the $n-$dimensional vector space over $\mathbb{F}_q$*

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

Furthermore, to compute the number of elements in the Grassmanian $Gr(k, n)(\mathbb{F}_q)$ we need to count the number of $k-$dimensional subspaces of the $n-$dimensional vector space over $\mathbb{F}_q$. To do this we first count the number of ways to obtain k linearly independent vectors $\{v_1, \cdots, v_k\}$ in $\mathbb{F}_q^n$. There are $q^n - 1$ ways to chose v_1, then, there are $q^n - q$ ways to chose v_2 and so forth. Thereafter we divide the result by the total number of ordered basis of a $k-$dimensional linear subspace, which is obtained in a similar manner. Hence we get

$$\#Gr(k, n)(\mathbb{F}_q) = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}.$$

[1] Later we will see that this category corresponds to that of $\mathbb{F}_1-$algebras.

Notice that the expression of above is given by the Gauss binomial

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \qquad \textit{where} \qquad [n]_q! = [n]_q[n-1]_q \cdots [1]_q.$$

Hence if we let $q = 1$ we obtain the following results

$$\#\mathbb{P}^{n-1}_{\mathbb{F}_1} = [n]_1 = n \qquad \textit{and} \qquad \#Gr(k,n)(\mathbb{F}_1) = \binom{n}{k}_1 = \binom{n}{k}.$$

Example 1.0.2. *Consider the general linear group $GL(n, \mathbb{F}_q)$. The cardinality of this group is obtained in a similar manner to that of the Grassmanian. Indeed*

$$\#GL(n, \mathbb{F}_q) = \prod_{i=0}^{n-1}(q^n - q^i).$$

The limit of this expression when $q \to 1$ goes to 0, however, in [23, Section 1.1] it is shown a procedure to resolve the 0 by dividing it by $(q-1)^n$. Thus the limit of the cardinality of $GL(n, \mathbb{F}_q)$ when $q \to 1$ goes to $n!$.

Furthermore, since $GL(n, \mathbb{F}_q)$ acts transitively on $Gr(k,n)(\mathbb{F}_q)$, when $q \to 1$ we may think that the vector space $\mathbb{F}_1^n$ is a set of n elements, say $A = \{1, \cdots, n\}$, thus the general linear group becomes the symmetric group S_n i.e.

$$GL(n, \mathbb{F}_q) \quad \underset{q \to 1}{\longrightarrow} \quad GL(n, \mathbb{F}_1) = S_n.$$

Finally notice that the Grassmanian $Gr(k,n)(\mathbb{F}_1)$ can be seen as all $k-$subsets of A.

The examples of above show that, indeed, there is a significant meaning of some spaces over the hypothetical field with one element. Moreover, as we mentioned earlier, Tits investigated incidence relationships among such spaces. For instance, if we consider the points of different Grassmanians $Gr(1,n)(\mathbb{F}_1)$, $Gr(2,n)(\mathbb{F}_1), \ldots, Gr(n-1,n)(\mathbb{F}_1)$ as the corresponding points, lines, ..., $(n-2)-$dimensional subspaces of the projective space $\mathbb{P}^{n-1}_{\mathbb{F}_1}$ we want to find incidence relations i.e. containment relations among those subspaces of $\mathbb{P}^{n-1}_{\mathbb{F}_1}$. Let's give an example.

Example 1.0.3. *Let $n = 3$. By Example 1.0.2 we may think that the vector space $\mathbb{F}_1^3$ is $\{1, 2, 3\}$. Therefore*

$$Gr(1,n)(\mathbb{F}_1) = \{\{1\}, \{2\}, \{3\}\} \quad \textit{and} \quad Gr(2,n)(\mathbb{F}_1) = \{\{1,2\}, \{1,3\}, \{2,3\}\}.$$

The incidence relations among the points and lines in $\mathbb{P}^2_{\mathbb{F}_1}$ are given by the contentions between the elements of $Gr(1,n)(\mathbb{F}_1)$ with the elements of $Gr(2,n)(\mathbb{F}_1)$, namely, each *point is contained in two lines, and every line contains two points. We depitc the incidence relation in the figure below, where the red vertices correspond to the lines, and the blue vertices corrrespond to the points.*

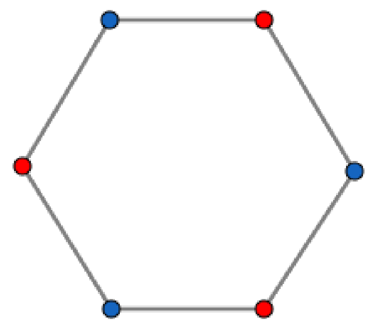

Figure 1.1: Incidence relation among the points and lines in $\mathbb{P}^2_{\mathbb{F}_1}$.

The above examples give us some expected properties of $\mathbb{F}_1-$geometry. For instance we expect that pointed sets are vector spaces over $\mathbb{F}_1$, we also expect that the projective line over $\mathbb{F}_1$ may contain two points. Moreover, since the spectrum of a field consist of only one point, we expect that $Spec(\mathbb{F}_1)$ contains only one point.

Notice that the expected properties of above cannot occur if we literally consider a field with one element by allowing that the additive and multiplicative identity be the same (i.e. if $0 = 1$) because what we will get would be the trivial ring $\{0\}$.

Despite previous observations, the idea of $\mathbb{F}_1$ did not have much interest until the late 80's and early 90's mainly due to mathematicians Alexander Smirnov and Yuri Manin, since they postulated that the $\mathbb{F}_1-$geometry would be involved in a possible proof of the Riemann hypothesis by considering $Spec(\mathbb{Z})$ as an $\mathbb{F}_1-$scheme (see [3, 23]).

In addition, Smirnov also postulated that another possible application of this geometry would lie in proving the *abc* conjecture, which tells us that for 3 positive integers a, b, c relative primes that satisfy the relation $a + b = c$, then if d denotes the product of the various prime factors of abc, then d is not much smaller than c.

However, it is not until 1999 that Cristhophe Soulé proposed the first notion of a variety over $\mathbb{F}_1$. Subsequently there were many attempts to describe $\mathbb{F}_1-$ schemes. Various theories and their generalizations have been proposed by different mathematicians. In this work we are going to study some of these theories. In particular we are going to present a refinement of the monoid scheme theory developed by Deitmar in [8], which is embedded in almost all other subsequent approaches to $\mathbb{F}_1-$schemes, which is why called the core of $\mathbb{F}_1-$geometry. One of the theories that generalize Deitmar's, and of which we will present an introduction, is the theory of blueprints and blue schemes introduced by Lorscheid in [25], which has found new applications in tropical geometry by setting tropical varieties in terms of blue schemes.

Now we give a brief description of the material of each chapter:

In Chapter 2 we introduce the necessary elements of commutative algebra for monoids. Most of the results presented here are in analogy with the usual commutative algebra for rings, however we emphasize the special properties that only apply for monoids. Likewise we present illustrative examples that will be used throughout this work.

In Chapter 3 we develop the theory of affine and non affine monoid schemes. These notions were the first attempts to describe $\mathbb{F}_1$-schemes and were introduced by Deitmar in [8]. This material constitutes the basics of $\mathbb{F}_1-$geometry. Definitions and proofs are similar to those of ring schemes. The main references for Chapters 1 and 2 are [5, 6, 8].

In Chapter 4 we develop the theory of toric varieties over $\mathbb{F}_1$ using the material from previous chapters. We start by introducing the necessary notions of multiplicative algebraic groups over $\mathbb{F}_1$ focusing on the algebraic tori. Later we will study the connection between normal toric varieties and convex polyhedral cones and fans. Subsequently we introduce divisors on toric varieties. At the end of the chapter we present some relationships between the number of generators of a numerical monoid and the multiplicity of toric curves. One of the main issues discussed in the chapter is a result due to Deitmar which tells us that essentially monoid schemes are toric varieties. The main references for this chapter are [5, 7, 9, 14, 21].

In Chapter 5 we present an introduction to the theory of blueprints and blue schemes. This theory constitutes an approach to $\mathbb{F}_1-$geometry that generalizes that of Deitmar. As we have said before, this approach has found applications in tropical geometry by setting tropical varieties in terms of blue schemes. At the end of the chapter we will make an outline of these ideas. The main references for this chapter are [17, 22, 24, 27].

The main contributions that we can mention for this work are the following.

Throughout the work, a detailed explanation to various results and examples of the theory presented are providing.

In chapter 4 we establish some results: We extend the notion of toric varieties over $\mathbb{F}_1$ discussed in [5] to the non-normal case. Furthermore, by using the theory of monoid schemes developed in Chapter 3, we have been able to generalize various results of toric varieties over $\mathbb{C}$, to the context of monoids and subsequently to rings. In particular, in Proposition 4.1.13 and Proposition 4.1.14 we obtain the characterization of affine toric varieties over $\mathbb{F}_1$. Furthermore, in Theorem 4.1.19 we show a main result of

Deitmar (see [9, Theorem 4.1]), which states that the base extension[2] of an irreducible, cancellative monoid scheme of finite type is a toric variety over $\mathbb{C}$, however, our approach of this result is stated only in terms of toric varieties over $\mathbb{F}_1$. In Section 4.5, we generalize the results obtained in [10], about the multiplicity and regularity index of toric curves over a field $\mathbb{K}$, by extending such varieties in Proposition 4.5.1 to toric curves over $\mathbb{F}_1$.

Finally, for Chapter 5, although no own results are established, a unifying introduction to the Lorscheid's theory of blueprints and blue schemes is presented. We remark this since the main results of the theory are established in many different works.

[2]The base extension to ring schemes will be considered in Section 3.3

Chapter 2

Algebraic background on monoids

The purpose of this chapter is to develop the necessary theory of commutative algebra for monoids that will be used throughout the following chapters.

2.1 Pointed commutative monoids

Definition 2.1.1. A *monoid* is a set A endowed with an associative binary operation $\cdot : A \times A \to A$ and identity element $1 \in A$ such that for all $a \in A$ we have $1 \cdot a = a \cdot 1 = a$. Moreover, A is *commutative* if for all $a, b \in A$ we also have $ab = ba$. A *basepoint*, or *zero element* is a unique element $0 \in A$ such that $0 \cdot a = a \cdot 0 = 0 \ \forall a \in A$. The monoid is *pointed* if it has a basepoint.

In the following by monoids we mean pointed commutative monoids, unless otherwise stated. In most cases we assume multiplicative notation in which case the monoid is called multiplicative. However, sometimes we use an additive notation $+$ in which case we will specify it and say that the monoid is additive. In those cases the identity is written as 0 and the basepoint is written as $-\infty$.

Definition 2.1.2. Let A be a monoid. We say that A is *finitely generated* (f.g) if there is a set $X = \{x_1, \cdots, x_n\} \subset A$ such that any element $a \in A \backslash \{0\}$ can be written as

$$a = x_1^{m_1} \cdots x_n^{m_n} \quad \text{with} \quad m_i \in \mathbb{N} \quad \text{for} \quad i = 1, \cdots, n.$$

X is called the *set of generators*.

If X is a set of generators of A, the monoid is usually denoted by $\langle X \rangle_* = \langle x_1, \cdots, x_n \rangle_*$. For instance the monoid generated by one element is $\langle x \rangle_* = \{0, 1, x, x^2, \cdots\}$.

Example 2.1.3. *Some examples of (pointed commutative) monoids:*

1. *We obtain a monoid by adding a zero to an unpointed monoid A. It is denoted by $A_* = A \cup \{0\}$.*
2. *Any commutative ring R has an underlying monoid with respect to the multiplicative operation (we just forget the addition operation).*
3. *As special case of point 1, we obtain a monoid G_* by adding a zero to any abelian group G.*

Definition 2.1.4. Let A, B be monoids. A map $f : A \to B$ is a monoid morphism if it satisfies

$$f(aa') = f(a)f(a'), \qquad f(1_A) = 1_B \quad \text{and} \quad f(0_A) = 0_B.$$

The category of pointed commutative monoids is denoted by $\mathcal{M}_*$. The initial object in this category is the monoid $\{0,1\}$, meaning that for any $A \in \mathcal{M}_*$ there exists a unique monoid morphism $\{0,1\} \to A$. The terminal object is the trivial monoid $\{0\}$, meaning that for any $A \in \mathcal{M}_*$ there exists a unique monoid morphism $A \to \{0\}$.

Remark 2.1.5. *Throughout the Chapters 1-4, by $\mathbb{F}_1$ we refer to the initial object in $\mathcal{M}_*$ i.e. the monoid $\{0,1\}$. Therefore $\mathcal{M}_*$ can be seen as the category of $\mathbb{F}_1$-algebras.*

Definition 2.1.6. A monoid A is *cancellative* if $ab = cb$ implies $a = c$ for $a, b, c \in A$ with $b \neq 0$; is *integral* if $ab = 0$ implies $a = 0$ or $b = 0$; is *torsion free* when $a^n = b^n$ implies $a = b$ for $a, b \in A$ with $a \neq 0 \neq b$. Notice that cancellative property doesn't imply torsion free, for instance consider the monoid $\{-1, 0, 1\}$. However we can see that a cancellative monoid is integral.

Definition 2.1.7. Let A be a monoid. A *submonoid* of A as a subset $Y \subset A$ such that Y is a monoid. Observe that $\mathbb{F}_1$ is a submonoid of any monoid $A \neq \{0\}$. If $X = \{x_1, \cdots, x_n\}$ is any subset of A, then the smallest submonoid of A containing X is:

$$\langle X \rangle_* = \{x_1^{m_1} \cdots x_n^{m_n}\}_* \quad \text{with} \quad m_i \in \mathbb{N} \quad \text{for} \quad i \in \{0, \cdots, n\}.$$

The elements of X are called the *system of generators* of the submonoid.

Example 2.1.8. *Some $\mathbb{F}_1$-algebras.*

1. *Consider the unpointed additive monoid $\mathbb{N}$. By adding a basepoint we obtain the monoid $\mathbb{N}_* = \mathbb{N} \cup \{-\infty\}$. This monoid can be written multiplicatively as follows:*

$$\mathbb{F}_1[T] = \{0, 1, T, T^2, \cdots\}.$$

2. *The free monoid generated by* $T_1, \cdots, T_n$ *is the monoid that consists of all the monomials* $T_1^{e_1} \cdots T_n^{e_n}$ *where all* $e_i \in \mathbb{N}$, *and a basepoint* 0. *We denote it by*

$$\mathbb{F}_1[T_1, \cdots, T_n].$$

Definition 2.1.9 (Product and coproduct)**.** Let $\{A_i\}_{i\in I}$ be a family of monoids.

1. We define the *product* of the family $\{A_i\}_{i\in I}$ as the usual cartesian product

$$\prod A_i = \{(a_i)_{i\in I} \mid a_i \in A_i\}.$$

Notice that the product give us a monoid with componentwise multiplication. Its zero is the element with zero in all components, and the identity is the element whose all components are 1. Moreover, it satisfies the universal property of products (see [32, Chapter 5]).

2. We define the *coproduct* or *smash product* of the family $\{A_i\}_{i\in I}$, as the quotient of the product $\prod A_i$ by the equivalence relation that identifies every element with a component equal to 0 with the zero element $(0)_{i\in I}$. The coproduct is denoted by $\bigwedge A_i$. For instance, is the family $\{A_i\}_{i\in I}$ consists of the monoids A and B, the smash product is

$$A \wedge B = (A \times B)/((A \times \{0\}) \cup (\{0\} \times B)).$$

Notice that smash product is also a monoid with componentwise multiplication. Its zero is the class $(0)_{i\in I}$, and the identity is the element whose all components are 1. Note that there are canonical inclusions $A_I \to \prod A_i$ defined by

$$(a \in A) \longmapsto (a_i)_{i\in I} \quad \text{with} \quad \begin{cases} a_i = a & \text{for } i = j \\ a_i = 1 & \text{otherwise} \end{cases}$$

If $\{f_i : A_i \to B\}_{i\in I}$ is a family of monoid morphisms, then, from the morphism

$$\begin{aligned} f : \bigwedge A_i &\longrightarrow B \\ (a_i)_{i\in I} &\longmapsto \prod_{i\in I} f_i(a_i) \end{aligned}$$

we can verify that the smash product satisfies the universal property of coproducts (see [32, Chapter 5]).

Definition 2.1.10 (Equalizer and coequalizer)**.** Let $f, g : A \to B$ be monoid morphisms.

1. We define the *equalizer* of f and g as the subset in A whose elements has the same image under both f and g. It's denoted it by
$$eq(f,g) = \{a \in A \mid f(a) = g(a)\}.$$

2. We define the *coequalizer* of f and g as the quotient of B by the smallest congruence generated by the relations $f(a) \sim g(a)$ for $a \in A$ (in Proposition 2.1.23 we show that this congruence always exists). The coequalizer is denoted by $coeq(f,g)$.

   Both $eq(f,g)$ and $coeq(f,g)$ satisfy their respective universal properties (see [32, Chapter 5]).

Definition 2.1.11. A *directed diagram* $\mathcal{D} = \{(A_i)_{i\in I}\}$ is a commutative diagram indexed by a directed set I, i.e. for every $i, j \in I$, there is a $k \in I$ and there are unique morphisms
$$f_i : A_i \to A_k \quad \text{and} \quad f_j : A_j \to A_k \quad \text{in} \quad \mathcal{D}.$$

Definition 2.1.12. A category $\mathfrak{C}$ is called *complete* (*cocomplete*) when every functor
$$F : \mathfrak{D} \longrightarrow \mathfrak{C},$$
with $\mathfrak{D}$ a small category, has a limit (colimit) (see [1, Section 2.8] for details).

Lemma 2.1.13. *A category $\mathfrak{C}$ is complete (cocomplete) when it contains products (coproducts) and equalizers (coequalizers).*

Proof. We refer to [1, Theorem 2.8.1] for a proof. ∎

Theorem 2.1.14 (Chu, Lorscheid, Santhanam, [6])**.** *The category $\mathcal{M}_*$ is complete and colimits of directed diagrams.*

Proof. We have seen that $\mathcal{M}_*$ contains products, coproducts, equalizers and coequalizers. Hence, by Lemma 2.1.13 $\mathcal{M}_*$ contains limits and colimits.

Now we show that $\mathcal{M}_*$ contains colimits of directed diagrams. Let $\mathcal{D} = \{A_i\}_{i\in I}$ be a commutative diagram of monoids and their morphisms indexed by a directed set I. Now, for $i \in I$ define the set
$$J(i) = \{k \in I \mid \exists f : A_i \to A_k \quad \text{in} \quad \mathcal{D}\}.$$

Which means that $J(i)$ is the cofinal directed subset of I. Now, let $\mathcal{D}(i)$ be the full subdiagram of $\mathcal{D}$ that contains precisely $\{A_i\}_{i\in J(i)}$. Then we have the following representation of the colimit of $\mathcal{D}$:

$$colim(\mathcal{D}) = \coprod \left\{ (a_j) \in \prod_{j\in J(i)} \mid \quad \text{for all} \quad f : A_j \to A_k \quad \text{in} \quad \mathcal{D}(i),\ a_k = f(a_j) \right\} / \sim$$

$$\text{where} \quad (a_j)_{j\in J(i_1)} \sim (b_j)_{i\in J(j_2)} \quad \text{if} \quad a_j = b_j \quad \text{for all} \quad j \in J(i_1) \cap J(i_2).$$

Now we have canonical morphisms

$$\varphi_i : A_i \longrightarrow colim(\mathcal{D})$$

given by

$$a_i \in A_i \longmapsto (f(a_i) \mid f : A_i \to A_k \quad \text{in} \quad \mathcal{D}(i)).$$

Now, given a family of monoid morphisms $g_i : A_i \to B$ that commute with all morphisms in $\mathcal{D}$, the map $g : colim(\mathcal{D}) \to B$ that sends an element $(a_j)_{j\in J(i_1)}$ to $g_i(a_i)$ is the unique morphism that satisfies the universal property of the colimit of $\mathcal{D}$. ■

Definition 2.1.15. Let A be a monoid. A subset $I \subset A$ is an *ideal* if I is not empty and $IA \subset I$. An ideal I is generated by a subset $Y \subset I$ if for any $x \in I$ there exists $y \in Y$ such that $x = ay$ for some $a \in A$. In addition, if Y is finite, then I is *finitely generated*.

When $\{x_1, \cdots, x_n\}$ is a set of generators of an ideal I we also denote I as $\langle x_1, \cdots, x_n\rangle$. Also note that a monoid can have infinitely many ideals even if it is finitely generated. For instance, each expression $\{x_k^i\}_{i\geq N} \cup \{0\}$ for $N \in \mathbb{N}$, defines an ideal which is usually denoted by $\langle x_k^i\rangle$.

Many of the properties of monoids behave in the same way as for rings. For instance, if we have a morphism $f : A \to B$ of monoids, and I is an ideal of B, then, we can verify that $f^{-1}(I)$ is an ideal of A since it contains the basepoint 0, and if $a \in f^{-1}(I)$ and $b \in A$, then $f(ab) = f(a)f(b) \in I$. However, we will emphasize properties that only apply for monoids.

Definition 2.1.16. Let A be a monoid. A *congruence* on A is defined as a *multiplicative equivalence relation* $\mathfrak{R}$, i.e. a equivalence relation that satisfies the following condition

$$\text{For any} \quad (a, b), (c, d) \in \mathfrak{R} \quad \text{then} \quad (a, b) \cdot (c, d) = (ac, bd) \in \mathfrak{R}.$$

Proposition 2.1.17. *Let A be a monoid, and let $\mathfrak{R}$ be a congruence on A. Then there is a well defined induced operation between the equivalence classes, namely $[a]\cdot[b] = [ab]$.*

Hence the quotient $A/\mathfrak{R}$ is a monoid with zero $[0]$ and identity $[1]$, and the canonical map $\pi : A \to A/\mathfrak{R}$ is a morphism of monoids.

Proof. The multiplication on $A/\mathfrak{R}$ doesn't depend of the choice of the class representatives since, by definition of congruence, we have $(a, b) \cdot (c, d) = (ac, bd) \in \mathfrak{R}$, and thus $A/\mathfrak{R}$ is a monoid with zero $[0]$ and identity $[1]$. ■

Example 2.1.18. *Let A be an integral monoid. Notice that*

$$\mathfrak{R} = \{(a, b) \in A \times A \mid a \neq 0 \neq b\} \cup \{(0, 0)\}$$

is a congruence and that $A/\mathfrak{R}$ is isomorphic to $\mathbb{F}_1$.

Definition 2.1.19. Let A be a monoid. The *quotient monoid* by an ideal I is defined as follows:

$$A/I = \begin{cases} [x] = [0] & x \in I \\ [x] = \{x\} & x \notin I \end{cases}$$

It follows that $A/I \cong (A \backslash I) \cup \{0\}$. Thus $\pi : A \to A/I$ is a surjective morphism of monoids. Now let $f : A \to B$ be a morphism of monoids. We define the *kernel* of f as follows:

$$ker(f) = f^{-1}(0) \subset A.$$

Let $x \in ker(f)$ and $a \in A$, then $f(xa) = 0f(a) = 0$, thus $ker(f)$ is an ideal of A. Moreover, note that $ker(f)$ induces a morphism $\overline{f} : A/ker(f) \to B$ defined by $[a] \mapsto f(a)$. However, a morphism need not be injective when the kernel is 0 as we can see in the next example.

Example 2.1.20. *Let G be the cyclic group of order n, then by adding a basepoint we obtain the monoid G_*. If $f : \mathbb{N}_* \to G_*$ is a morphism, notice that $f(x^k) = f(x^{k+n})$ for all $k \in \mathbb{N}_*$. Hence f is not injective but $ker(f) = f^{-1}(0) = 0$.*

As a consequence we can see that $A/ker(f) \ncong im(f)$, i.e. the first isomorphism theorem doesn't hold for monoids. An important implication of this is that $\mathcal{M}_*$ is not an abelian category. However, there is a general notion which will allow us to preserve the isomorphism theorem, namely *the congruence kernel* which is defined as follows:

Definition 2.1.21. Let $f : A \to B$ be a morphism of monoids. We define the *congruence kernel* of f as the following relation on A:

$$\mathfrak{R}(f) := \{(a, b) \in A \times A \mid f(a) = f(b)\}.$$

Proposition 2.1.22. *Let $f : A \to B$ be a morphism of monoids. The congruence kernel $\mathfrak{R}(f)$ is, indeed, a congruence on A.*

Proof. We show the conditions for $\mathfrak{R}(f)$:

Reflexive: $f(a) = f(a)$.

Symmetrical: $f(a) = f(b)$ implies $f(b) = f(a)$.

Transitive: $f(a) = f(b)$ and $f(b) = f(c)$ implies $f(a) = f(c)$.

Multiplicative: if $f(a) = f(b)$ and $f(c) = f(d)$ then $f(ac) = f(a)f(c) = f(b)f(d) = f(bd)$.

■

Proposition 2.1.23. *Let A be a monoid, and let $S \subset A \times A$ be a subset. Then there is a smallest congruence containing S, namely $\mathfrak{R} = \langle S \rangle$.*

Proof. The intersection of congruences is a congruence. In particular there is a smallest congruence containing S. ■

Definition 2.1.24. Let A be a monoid. An ideal $P \subsetneq A$ is prime if $ab \in P$ implies $a \in P$ or $b \in P$. We denote by $Spec(A)$ the set of all prime ideals of A.

Definition 2.1.25. Let A be a monoid. An increasing (resp. decreasing) sequence of ideals $I_0 \subset I_1 \subset \cdots \quad (I_0 \supset I_1 \supset \cdots)$ on A is called an *ascending (resp. descending) chain*. A finite chain of ideals on A of the form $I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n$ is said to has *length* n.

Definition 2.1.26. Let A be a monoid. The *Krull dimension* of A is the supremum of the lengths of all strictly ascending chains of prime ideals, and it is denoted by $dim(A)$. Now let $P \subset A$ be a prime ideal. The *height* or *codimension* of P is the supremum of the lengths of all strictly ascending chains of the form

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P.$$

The codimension of P is denoted by $codim(P)$.

Proposition 2.1.27. *Let A and B monoids, and let I be a prime ideal of B. If $f : A \to B$ is a morphism, then $f^{-1}(I)$ is a prime ideal of A.*

Proof. We have already noticed that the preimage of an ideal is an ideal. Now, if I is prime, then $B \backslash I$ is multiplicatively closed in B, then $f^{-1}(B \backslash I) = A \backslash f^{-1}(I)$ is multiplicatively closed in A, which means that $f^{-1}(I)$ is prime. ■

The next proposition and its corollary strengths differences between ideals of rings and ideals of monoids.

Proposition 2.1.28. *We denote by $A^\times$ the set of all invertible elements of a monoid A. Then $A\backslash A^\times$ is an ideal. Furthermore, it is a prime ideal.*

Proof. Let $a \in A\backslash A^\times$, and then suppose that there is $y \in A$ with $ay \in A^\times$, but then we can find $z \in A$ such that $(ay)z = a(yz) = 1$, and hence $a \in A^\times$. ■

Corollary 2.1.29. *Any monoid A, has a unique maximal ideal $A\backslash A^\times$, denoted by m_A. In particular, if G is an abelian group then G_* only has one prime ideal, namely $\{0\}$ which corresponds to the only non invertible element in G_*.*

Proposition 2.1.30. *Let A be a monoid, and I be an ideal of A, then*

1. *I is prime ideal if and only if A/I is nontrivial and integral.*
2. *I is maximal if and only if $A/I = (A/I)^\times \cup \{0\}$.*

Proof.

1. First let I be a prime ideal, and let $ab \in I$. Then, by Definition 2.1.19, if in the quotient monoid A/I we have $[ab] = [0]$ then $[a] = [0]$ or $[b] = [0]$. Hence A/I is nontrivial and integral. The converse is obtained anagously.
2. As we have noticed before, $A/I \cong (A\backslash I) \cup \{0\}$, so, when I is maximal (i.e. $I = A\backslash A^\times$), we have $A/I \cong A^\times \cup \{0\}$, thus $A/I = (A/I)^\times \cup \{[0]\}$. Now, if $[a][b] = [1]$ in A/I, and as $[a] \neq 0 \neq [b]$, passing to A/I, we have $[a] = \{a\} \notin I$ and $[b] = \{b\} \notin I$, therefore $ab = 1$ in A, i.e. units in A/I corresponds to units in A. Hence, as I is the kernel of $\cdot : A \to A/I$ and as $A/I = (A/I)^\times \cup \{[0]\}$, we have $I = A\backslash A^\times$, i.e. I is maximal.

■

Since monoids have only one maximal ideal, the concept of local morphism can be defined as follows:

Definition 2.1.31. A morphism of monoids $f : A \to B$ is called local if $f^{-1}(m_B) = m_A$.

Recall that a subset $S \subset A$ is multiplicatively closed if $1 \in S$ and $ab \in S$ for all $a, b \in S$. Now we define the *localization at S*.

Definition 2.1.32. Let $S \subset A$ be a multiplicatively closed subset. We define an equivalence relation on $A \times S$ given by

$$(a, s') \sim (a', s') \quad \text{if and only if} \quad \exists s'' \in S \quad \text{such that} \quad s''as' = s''a's.$$

The equivalence class of (a,s) is denoted by $\frac{a}{s}$. We define the *localization at S* as the set of all equivalence classes in $A \times S$, and it is denoted as follows:

$$S^{-1}A = \left\{ \frac{a}{s} \mid a \in A, s \in S \right\}.$$

Note that $S^{-1}A$ is a monoid with multiplication induced by A as follows:

$$\frac{a}{s}\frac{b}{t} = \frac{ab}{st} \quad \text{and} \quad \frac{1}{s}\frac{s}{1} = 1$$

That means that any element of S becomes a unit in $S^{-1}A$, and there is a canonical morphism

$$\varphi : A \to S^{-1}A \quad \text{given by} \quad a \mapsto \frac{a}{1}.$$

Note that if $P \subsetneq A$ is a prime ideal, then $1 \notin P$, and if $a, b \in A \backslash P$, then $ab \in A \backslash P$. In the case $S = A \backslash P$, we denote $S^{-1}A$ by A_P. Likewise, we denote the localization, at an element $f \in A$, as follows:

$$A_f = A[f^{-1}] = \left\{ \frac{a}{f^n} \mid a \in A, n \in \mathbb{N} \right\}.$$

Remark 2.1.33. *The localization operation satisfies the following* universal property *whose proof is equal to the case of localization of rings (see for instance [11, Chapter 2]):*

Let A, B be monoids, and let S be a multiplicatively closed subset of A. Then for any morphism $f : A \to B$ that maps S to units in B there is a unique morphism $\varphi' : S^{-1}A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "f"'] & S^{-1}A \arrow[d, "\varphi'"] \\ & B \end{tikzcd}$$

The next proposition shows that localization behaves different in the case of monoids, in contrast with rings.

Proposition 2.1.34 (Cortiñas, Haesemayer, Walker, Weibel, [5])**.** *If S is any multiplicatively closed of $A \backslash \{0\}$, then either $S^{-1}A = 0$ or $S^{-1}A = A_P$, for some $P \in Spec(A)$.*

Proof. First suppose that $S^{-1}A \neq 0$. This monoid has a unique maximal ideal $m_{S^{-1}A}$. Thus there is a morphism φ such that

$$\varphi : A \to S^{-1}A \qquad \text{implies} \qquad \varphi^{-1}(m_{S^{-1}A}) = P.$$

By Proposition 2.1.27 we know that P is prime ideal. Then $T = A\backslash P$ is a multiplicatively closed subset. Now let $t \in T$, then $\varphi(t)$ is a unit in $S^{-1}A$, and therefore, by the universal property of lozalization, this induces a map $A_P \to S^{-1}A$.

We claim that $S \subset T$. In fact, let $x \in S$ and suppose that $x \in P$, then $\varphi(x) \in m_{S^{-1}A}$, but $\varphi(x)$ is a unit. This proves the claim, and furthermore, notice that this induces a map $S^{-1}A \to A_P$. Finally the result follows from the universal property of localization since the composition of the induced maps should be the identity map ■

As we mentioned before, the last proposition is in general a false statement for the case of rings. For instance consider the ring $A = \mathbb{Z}$ and the multiplicative closed set $S = \{2^n \mid n \in \mathbb{N}\}$.

Proposition 2.1.35. *Let A be a monoid, and let $S \subset A\backslash\{0\}$ be a multiplicative subset. Then the proper ideals of $S^{-1}A$ are in bijection with the ideals of A contained in $A\backslash S$. In particular this bijection occurs among the prime ideals of $S^{-1}A$ and the prime ideals of A contained in $A\backslash S$.*

Proof. Suppose $I \subset A$ is an ideal with $I \cap S \neq \emptyset$. Thus the ideal $S^{-1}I$ contains a unit, then it is not a proper ideal of $S^{-1}A$. On the other hand, if $J \subset S^{-1}A$ is an ideal, i.e. $1 \notin J$, then we would write J as $S^{-1}I$ where $I = \{a \in A \mid a/1 \in J\} \subset A\backslash S$ is an ideal of A. ■

Corollary 2.1.36. *Let P be a prime ideal of A. Then, the codimension of P equals the dimension of A_P.*

Proof. Consider the multiplicative closed set $S = A\backslash P$. Thus, by Proposition 2.1.35, the chains of prime ideals in A_P are in bijection with the chains of prime ideals in A contained in P. ■

Now we give some remarks about the way to work with additive monoids. These remarks are important since many times we find monoids with additive notation in a natural way as is the case of numerical monoids which we will meet at the end of this section.

Remark 2.1.37. *Consider a cancellative monoid A with additive notation. Its group completion A_0 (see Definition 2.3.6) is the group generated formally by the equivalence* *clases on $A \times (A \backslash \{-\infty\})$ of the differences of its elements, i.e. let $a_1, a_2, b_1, b_2 \in A$ with $b_1 \neq -\infty \neq b_2$ then*

$$a_1 - b_1 = a_2 - b_2 \quad \textit{if and only if} \quad \exists c \in A \quad \textit{with} \quad a_1 + b_2 + c = a_2 + b_1 + c.$$

Note that the canonical morphism $A \to A_0$ is injective. For instance consider the map $\mathbb{N}_* \to \mathbb{Z}_*$.

Proposition 2.1.38. *Let A be a cancellative monoid. Then A is torsion free if and only if A_0 is torsion free.*

Proof. Suppose A is torsion free. Thus if there is $n \in \mathbb{N}$ such that $n(a_1 - b_1) = n(a_2 - b_2)$ in A_0, then there is $c \in A$ such that

$$n(a_1 + b_2) + c = n(a_2 + b_1) + c.$$

Therefore, since A is cancellative and torsion free, $a_1 - b_1 = a_2 - b_2$. The converse follows in the same way. ■

Remark 2.1.39. *Recall that a* lattice *N as a finitely generated torsion free abelian group. Then notice that a finitely generated monoid is both cancellative and torsion free, if and only if it is a submonoid of the pointed lattice N_*. For instance let $N_* = \mathbb{Z}_*^n$. Hence $\mathbb{N}_*$ is an finitely generated cancellative torsion free monoid.*

Therefore we obtain a characterization of those monoids that can be embedded into a pointed lattices, namely, finitely generated, cancellative and torsion free monoids. Those kind of monoids are called affine monoids*. In Chapter 4 we will see that affine monoids play an important role in the construction of toric varieties.*

The following remark show us how to go from an additive notation to a multiplicative one in the case of affine monoids.

Remark 2.1.40. *Let A be an affine monoid with additive notation and set of generators $\{a_1, \cdots, a_n\}$. Then A can be written multiplicatively as*

$$\mathbb{F}_1[A] = \mathbb{F}_1[T^{a_1}, \cdots, T^{a_n}].$$

Example 2.1.41. *Consider the monoid $\mathbb{Z}_*^n$ which can be written multiplicatively as $\mathbb{F}_1[\mathbb{Z}_*^n] = \mathbb{F}_1[(T^{e_1})^{\pm}, \cdots, (T^{e_n})^{\pm}]$, where $e_i \in \mathbb{Z}^n$ is the $i-th$ canonical vector i.e.*

$$\mathbb{F}_1[\mathbb{Z}_*^n] = \mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}].$$

Remark 2.1.42. *In what follows, by a lattice we mean a pointed lattice unless otherwise stated, and when the context is clear we simply write that lattice in additive form as N, and in multiplicative form as $\mathbb{F}_1[N]$. For instance, in the example of above we write $\mathbb{F}_1[\mathbb{Z}^n] = \mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}]$.*

To end this section we introduce a special kind of monoids which are called numerical monoids. In Chapter 5 we will use them to construct some toric varieties.

Almost all the basics on numerical monoids are given in additive notation, however, due to the above remarks, we can write numerical monoids with multiplicative notation.

Definition 2.1.43. A *numerical (additive) monoid* is a submonoid of $\mathbb{N}_*$ with finite complement in $\mathbb{N}_*$.

Lemma 2.1.44. *Let $n \in \mathbb{N}_{\geq 2}$, and let $A = \{a_1, \cdots, a_n\} \subset \mathbb{N}$. Then $\langle A \rangle_*$ is a numerical monoid if and only if $gcd(A) = 1$.*

Proof. By definition, $\mathbb{N}_* \backslash \langle A \rangle_*$ is finite, thus there is some $a \in \langle A \rangle_*$ with $a \neq -\infty$, such that $a + 1$ is also in $\langle A \rangle_*$. Hence $gcd(A) = 1$.

Conversely suppose that $gcd(A) = 1$. Therefore there exists numbers $t_1, \cdots, t_n \in \mathbb{Z}$ such that $a_1 t_1 + \cdots + a_n t_n = 1$. Then notice that the sum of the negative parts of the previous expression gives us a number whose additive inverse b is such that b and $b+1$ belongs to $\langle A \rangle_*$.

Now we show that $\langle A \rangle_*$ has finite complement. Indeed, we claim that if $n \geq (b-1)b + (b-1)$ then $n \in \langle A \rangle$. Thus let $n = qb + r$ for some integers q and r with $0 \leq r < b$. Then, since $n \geq (b-1)b + (b-1)$, it follows that $r \leq b - 1 \leq q$. Hence

$$\begin{aligned} n &= qb + r \\ &= qb + r + rb - rb \\ &= b(q - r) + (b + 1)r \in \langle A \rangle \end{aligned}$$

■

Remark 2.1.45. *In [31, Theorem 2.7] it is shown that any numerical monoid S admits a unique finite and minimal system of generators. Thus it is enough to consider a subset $A = \{a_1, \ldots, a_m\} \subset \mathbb{N}$ with $1 < a_1 < \cdots < a_m$ and $gcd(A) = 1$ to associate a numerical monoid $S = \langle A \rangle_*$, and, in most cases we may assume that A is the unique minimal set of generators of S.*

One of the historical reasons for working with numerical monoids is their close relationship with the problem of determining the nonnegative integers given by $\sum_{i=1}^{n} n_i a_i$ where $a_i \in A$ as before and $x_i \in \mathbb{N}$ arbitrary numbers (see for instance [31]).

Now consider a numerical monoid S. A number associated with the previous problem is the *Frobenius number* of S which is defined to be the greatest number contained in $\mathbb{N}_* \backslash S$, and is denoted by $F(S)$. Notice that by the definition of numerical monoid, $F(S)$ exists. This number has been used in [10] to bound the regularity index of toric curves over a field. In Chapter 5 we will extend their results to toric curves over $\mathbb{F}_1$

by interpreting the regularity index as the minimal number of generators of numerical monoids.

2.2 A-sets

The structure discussed in this subsection was introduced in [6]. We are going to define an analogue to a module over ring, but in our case we will use monoids instead of rings. We consider a pointed set M, with basepoint denoted by 0_M, and a monoid A with basepoint denoted by 0_A. We introduce the notion of A-*set*:

Definition 2.2.1. An A-*action* is a map $\theta : A \times M \to M$, satisfying the following properties for all $a, b \in A$, and $m \in M$ ($a \cdot m$ denotes $\theta(a, m)$):

i $1 \cdot m = m$

ii $0_A \cdot m = 0_M \quad \text{and} \quad a \cdot 0_M = 0_M$

iii $(ab) \cdot m = a \cdot (b \cdot m)$

An $A-set$ as a pair (M, θ), where M is a pointed set and θ is an $A-action$. However, if there is not confusion, we will simply write it as M.

In the following, if there is not confusion, we will avoid the notation $a \cdot m$ and simply write am to refer $\theta(a, m)$.

Remark 2.2.2. *If we denote by $Hom(M, M)$ the basepoint preserving self-maps $M \to M$, we can also consider an* A-set *as a pointed set M together with a multiplicative map $A \to Hom(M, M)$ that preserves 0 an 1.*

Definition 2.2.3. Let M be an $A - set$. We say that $Y \subset M$ is a A-*subset* when $ay \in Y$ for all $a \in A$ and $y \in Y$.

Example 2.2.4. *Let A be a monoid. A is itself an $A-set$. Furthermore, note that an* A-subset *$B \subset A$ is, by definition, the same as an ideal I of A.*

Example 2.2.5. *We have seen that way to obtain a monoid A from a commutative ring R, even more, in the same way, we can obtain an $A - set$ from an $R - module$. Indeed, we denote by Rings the category of commutative rings with unit, and we denote by $R - modules$ the category of modules over R, with $R \in Rings$. Thereafter we obtain functors*

$$U : Rings \longrightarrow \mathcal{M}_*$$

and

$$U' : R - modules \longrightarrow U(R) - sets.$$

That means that an $R-$module has an underlying $A-$set if we just simply forget the addition operation in the module.

Definition 2.2.6. Let M, N be *A-sets*.

1. The *product* of M and N is the usual cartesian product $M \times N$.
2. The *coproduct* or *wedge product* of M and N as
$$M \vee N = (M \amalg N)/(0_M \sim 0_N),$$
which identify the zeros of the *A-sets*.
3. We define the smash product of M and N as
$$M \wedge N = M \times N/(M \vee N),$$
which still be an $A-set$ via the action defined by $a(m.n) = (am, an)$ if $am \neq 0$ (with $m \neq 0_M$) and $an \neq 0$ (with $n \neq 0_N$)), otherwise $a(m, n) = 0$.

Definition 2.2.7. An *A-equivariant map* between *A-sets* M and N, is a map $f : M \to N$ such that $f(am) = af(m)$ for all $a \in A$ and $m \in M$. A *morphism* of *A-sets* is an $A-$equivariant map. Then, we note that in particular, a morphism of *A-sets* sends basepoints to basepoints.

If M and N are *A-sets*, we denote the set of morphisms between them as $Hom_A(M, N)$. Then, the category of *A-sets* with their morphisms is denoted by $A - Mod$, and the trivial *A-set* $0 = \{*\}$ is both, the initial and terminal object of $A - Mod$, that is, is a *zero object* in the category.

If M, N and P are *A-sets*, we can define an *biequivariant map* as a map $f : M \times N \to P$ such that $f(am, n) = af(m, n)$ and $f(m, an) = af(m, n)$ for all $a \in A$ and $(m, n) \in M \times N$. Now we are able to define the tensor products between *A-sets*.

Definition 2.2.8. Consider the equivalence relation $\sim$ on $M \times N$ generated by $(am, n) \sim (m, an)$ for $a \in A, m \in M$ and $n \in N$. Then the *tensor product* of M and N, is an *A-set* denoted by $M \otimes_A N$ together with a surjective biequivariant map $f : M \times N \to M \otimes_A N$. Note that the equivalence relation induces a bijection $M \times N/ \sim \to M \otimes_A N$. Then the tensor product satisfies the following universal property:

For every biequivariant map $\alpha : M \times N \to T$ to a third *A-set* T there is a unique equivariant map $g : M \otimes_A N \to T$ such that $\alpha = g \circ f$, i.e. such that the next diagram commutes.

$$\begin{tikzcd}
M \times N \arrow[r, "f"] \arrow[dr, "\alpha"'] & M \otimes_A N \arrow[d, "g"] \\
& T
\end{tikzcd}$$

Remark 2.2.9. *When $A = \mathbb{F}_1$ notice that the tensor product of the* A-sets *M and N is $M \otimes_{\mathbb{F}_1} N \simeq M \wedge N$ as* A-sets. *Hence, if A and B are monoids, the tensor product equals the smash product:*

$$A \otimes_{\mathbb{F}_1} B \simeq A \wedge B = (A \times B)/((A \times \{0\}) \cup (\{0\} \times B)).$$

Therefore, since any monoid A is an $\mathbb{F}_1-$algebra, we identify the monoid with their tensor product $A \otimes_{\mathbb{F}_1} \mathbb{F}_1 = A$. For instance consider the monoid generated by one element

$$\mathbb{F}_1 \otimes_{\mathbb{F}_1} \langle T \rangle = \{0, 1, T, T^2, \cdots\} = \mathbb{F}_1[T].$$

Furthermore, notice that the free monoid $\mathbb{F}_1[T_1, \cdots, T_n]$ can be seen as the tensor product

$$\mathbb{F}_1[T]^{\otimes n} = \mathbb{F}_1[T_1, \cdots, T_n] = \mathbb{F}_1[T_1] \otimes_{\mathbb{F}_1} \cdots \otimes_{\mathbb{F}_1} \mathbb{F}_1[T_n].$$

Example 2.2.10. *As example of the above remark let A be a monoid and consider the following tensor product*

$$A[T_1, \cdots T_n] = A \otimes_{\mathbb{F}_1} \mathbb{F}_1[T_1, \cdots, T_n] = A \otimes \mathbb{F}_1[T]^{\otimes n}$$

whose elements are all the monomials in $\mathbb{F}_1[T_1, \cdots T_n]$ with coefficients in A. The monoid $A \otimes \mathbb{F}_1[T]^{\otimes n}$ is called the free A-monoid *on n variables.*

We have already seen how to pass from rings to monoids. Now we show how to extend monoids to rings:

Definition 2.2.11. Let A be a monoid. We define the *base extension functor* from monoids to rings, denoted by $- \otimes_{\mathbb{F}_1} \mathbb{Z}$, as follows:

$$A_{\mathbb{Z}} = A \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[A]/\langle 1 \cdot 0_A \rangle.$$

Where $\mathbb{Z}[A]$ is the monoid ring of finite $\mathbb{Z}$-linear combinations of elements of A, i.e.

$$\mathbb{Z}[A] = \left\{ \sum n_a a \mid n_a \in \mathbb{Z}, \text{ almost all } 0 \right\}.$$

And $\langle 1 \cdot 0_A \rangle$ is the zero 0_A of A, i.e. we identify the zero of the monoid A with the zero of the ring $\mathbb{Z}[A]$.

Example 2.2.12. *Some examples of base extension:*

1. *The base extension of the trivial monoid is the trivial ring.*
2. *The base extension of $\mathbb{F}_1$ is $\mathbb{F}_1 \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}$.*

3. *The free monoid $\mathbb{F}_1[T_1, \cdots, T_n]$ has base extension the polynomial ring in n variables.*
$$\mathbb{F}_1[T_1, \cdots, T_n] \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[T_1, \cdots, T_n].$$

4. *Note that the localization of $\mathbb{F}_1[T_1, \cdots, T_n]$ at 0, is the monoid $\mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}]$, whose base extension is the* Laurent polynomials *over $\mathbb{Z}$. i.e. $\mathbb{Z}[T_1^{\pm}, \cdots, T_n^{\pm}]$.*

Remark 2.2.13. *Let $f : A \to B$ a monoid morphism. Note that, using base extension in A and B, by linearity, we can extend f to a ring homomorphism*

$$f' : A \otimes_{\mathbb{F}_1} \mathbb{Z} \longrightarrow B \otimes_{\mathbb{F}_1} \mathbb{Z}.$$

Proposition 2.2.14. *Let A be a monoid and S be a multiplicative subset. The canonical ring homomorphism $S^{-1}A_{\mathbb{Z}} \to (S^{-1}A)_{\mathbb{Z}}$, defined by linear extension of the map that sends $\frac{a}{s}$ to $\frac{a}{s}$ for $a \in A$ and $s \in S$, is an isomorphism.*

Proof. Consider the canonical ring homomorphism

$$S^{-1}A_{\mathbb{Z}} \longrightarrow (S^{-1}A)_{\mathbb{Z}}.$$

Now, we are going to construct an inverse of this map. First note that an element of $(S^{-1}A)_{\mathbb{Z}}$ can be written as

$$\sum_{i=1}^{n} m_i \frac{a_i}{s_i} = \sum_{i=1}^{n} \frac{m_i a_i \prod_{j \neq i} s_j}{\prod_{j=1}^{n} s_j} = \sum_{i=1}^{n} \frac{m_i a_i'}{s}.$$

Where

$$a_i' = a_i \prod_{j \neq i} s_j \in A \qquad \text{and} \qquad s = \prod_{j=1}^{n} s_j.$$

Then

$$\sum_{i=1}^{n} \frac{m_i a_i'}{s} \in S^{-1}A_{\mathbb{Z}}.$$

This defines the inverse map. ■

Remark 2.2.15. *If there is not confusion in the rest of the work we´ll simply denote the tensor product of two monoids A, B over $\mathbb{F}_1$ just as $A \otimes B$.*

Proposition 2.2.16. *A monoid A is finitely generated if and only if $A_{\mathbb{Z}}$ is finitely generated as a ring.*

Proof. Since A is f.g. then $\mathbb{Z}[A]$ is finitely generated as a $\mathbb{Z}-$algebra. Now suppose $A_{\mathbb{Z}}$ is finitely generated, that implies that $\mathbb{Z}[A]$ is a finitely generated by a set T, but also T generates A as a monoid. ■

2.3 Noetherian conditions and valuations

In this section we present normal monoids which will be important when we introduce affine toric varieties from cones. Moreover, in order to introduce divisors we need some theory about noetherian conditions and valuations presented here. This section is mainly based on [14].

Definition 2.3.1. An *A-set* M is called *noetherian* if it satisfies the *ascending chain condition* (ACC), namely, for every ascending chain of *A-subsets* $Y_1 \subseteq Y_2 \subseteq \cdots$ there exists $n \in \mathbb{N}$ such that $Y_i = Y_n$ for every $i \geq n$.

Remark 2.3.2. *As in the case of noetherian rings, the condition for an* A-set *to be noetherian is equivalent to say that every* A-subset $Y \subset A$ *is finitely generated and the proof follows in the same way (see for instance [16, Lemma 7.4]).*

Proposition 2.3.3. *The image of a noetherian* A-set *under an* A-set *morphism is a noetherian* A-set*.*

Proof. Let $f : M_1 \to M_2$ be an *A-set* morphism. Then $f^{-1}(f(M_1))$ is finitely generated by some elements $m_1, \cdots, m_k \in M_1$. ■

Remark 2.3.4. *A monoid A is called noetherian when is considered as an* A-set *and satisfies the ACC for ideals. Thus, by Remark 2.3.2, the condition for a monoid to be noetherian is equivalent to say that every ideal $I \subset A$ is finitely generated.*

Lemma 2.3.5. *Let A be a noetherian monoid. If $S \subset A$ is a multiplicatively closed subset, then $S^{-1}A$ is noetherian.*

Proof. It follows from Proposition 2.1.35. ■

If A is a cancellative monoid, the element 0 generates a prime ideal, then the localization $A_{(0)}$ makes sense. Then we have the following definition.

Definition 2.3.6. Let A be a cancellative monoid. We define its *group completion* or *quotient monoid* as the localization $A_{(0)}$ which is usually denoted by A_0. Indeed, notice that $A_0^{\times} = A_0 \backslash \{0\}$ is a group.

Definition 2.3.7. Let A be a monoid and let $B \subset A$ be a submonoid. An element $a \in A$ is called *integral* over B when $a^n \in B$ for some $n \geq 1$. The *integral closure* of B over A is the following set

$$\{a \in A \mid a^n \in B \text{ for some } n \geq 1\}.$$

Definition 2.3.8. We also define the *normalization* or *saturation* of a cancellative monoid A as the set

$$A^{\mathrm{nor}} = \{\alpha \in A_0 \mid \alpha^n \in A \text{ for some } n \geq 1\}.$$

We say that A is *normal*, *integrally closed* or*saturated* if it equals its integral closure in its group completion, i.e. if $A = A^{\mathrm{nor}}$.

Proposition 2.3.9. *Let A be a monoid, and $B \subset A$ be a submonoid. Suposse B is cancellative. If there is a finitely generated ideal $I \subset B$, and if there is an element $a \in A$ such that $aI \subset I$, then a is integral over B.*

Proof. Since I is finitely generated, there is a finite set of generators, say $X = \{x_0, \cdots, x_r\}$. Now, there is a function $\varphi : X \to X$ such that $ax_i \in B\varphi(x_i)$ for each $x_i \in X$ since $aI \subset I$. Note that there is an $x_j \in X$, for some j, such that $\varphi^n(x_j) = x_j$ for some $n \in \mathbb{N}$. Then, we can show inductively that $a^k x_i \in Ba\varphi^{k-1}(x_i)$ for all $k \geq 1$. Hence the numbers n, j determine an element $b \in B$ such that $a^n x_j = bx_j$, thus $a^n = b$ since A is cancellative, and therefore a is integral over B. ∎

Lemma 2.3.10. *Let A be a monoid. Let $B \subset A$ be a submonoid of A and let S be a multiplicatively closed subset of B. Then*

1. *If A is integral over B, then $S^{-1}A$ is integral over $S^{-1}B$.*
2. *If A is the integral closure of B in another monoid C, then $S^{-1}A$ is the integral closure of $S^{-1}B$ in $S^{-1}C$.*
3. *If B is normal, then $S^{-1}B$ is normal.*

Proof.

1. Let $a \in A$ such that $a^n \in B$ for some $n \in \mathbb{N}$. Thus $(\frac{a}{s})^n \in S^{-1}A$ implies $(\frac{a}{s})^n \in S^{-1}B$.
2. Suppose that $\frac{c}{1} \in S^{-1}C$ is integral over $S^{-1}B$, then $(\frac{c}{1})^n \in S^{-1}B$ for some $n \in \mathbb{N}$. Thus $(\frac{c}{1})^n = \frac{b}{s}$ which implies $c^n st = bt$ for some $t \in S$ and therefore $cst \in A$ since A is the integral closure of B, but then $\frac{c}{1} = \frac{cst}{st} \in S^{-1}A$.
3. A equals its integral closure in its group completion, then, by the last paragraph, we are done.

∎

Definition 2.3.11. Let A be a cancellative monoid. A is called a *valuation monoid* if for every non zero element a in A_0 either, $a \in A$ or $a^{-1} \in A$.

Example 2.3.12. *If* $(R, \cdot, +)$ *is a valuation ring, then by forgetting the addition we get a valuation monoid* $(R, \cdot)$.

Definition 2.3.13. Consider a valuation monoid A. Note that $A_0^\times$ is a multiplicative abelian group and $A^\times$ is a subgroup of it. The quotient group $A_0^\times/A^\times$ is called the *value group*. We define a total ordering in the value group given by $x \leq y$ if and only if $\frac{y}{x}$ is in the image of A into $A_0^\times/A^\times$ under the canonical morphism.

If we consider the value group as an additive group, we denote it by Γ. Thus recall that Γ has identity 0 and basepoint $-\infty$. Then consider the canonical map

$$v : A_0^\times/A^\times \longrightarrow \Gamma$$

called the *valuation map* which just change the multiplicative operation to the additive one.

Remark 2.3.14. *By definition of both,* Γ *and the valuation map, notice that we identify the valuation monoid* A *with the set*

$$\{a \in A_0^\times \mid v(a) \geq 0\}.$$

Likewise notice that $v(a) = 0$ *if and only if* $a \in A^\times$. *Hence the maximal ideal of* A *is*

$$\{a \in A_0^\times \mid v(a) > 0\}.$$

Proposition 2.3.15. *Valuation monoids are normal.*

Proof. Let A be a valuation monoid. Suppose $a \in A_0$, with $a \neq 0$, is such that $a^n \in A$ for some $n > 0$. If $a \in A$ we are done. Now suppose the contrary i.e. $a \notin A$, but that implies $a^{-1} \in A$ since A is a valuation monoid, then $(a^{-1})^{n-1} = a^{1-n} \in A$, therefore $(a^n)(a^{1-n}) = a \in A$, a contradiction. Hence A is normal. ∎

Definition 2.3.16. Let A be as before. We say that A is *discrete valuation* (DV) monoid if its associated value group is infinite cyclic, i.e. is isomorphic to $\mathbb{Z}$.

It's well know that a valuation ring is discrete if and only if it is noetherian (see for instance [16, Proposition 12.13]). The same fact apply for monoids, i.e. a valuation monoid is noetherian if and only if is a discrete valuation monoid. We need all of this to present the following results:

Proposition 2.3.17. *A one dimensional, noetherian, normal monoid is a discrete valuation monoid, and all DV monoids arise in this way.*

Proof. For a proof we refer to [14, Proposition 2.5]. ∎

Corollary 2.3.18. *Let A be a noetherian normal monoid. If P is a codimension one prime ideal of A, then A_P is a discrete valuation monoid.*

Proof. By Corollary 2.1.36, the dimension of A_P is one. Then by Lemmas 2.3.5 and 2.3.10 we know that A_P is normal and noetherian. Finally, the result follows from Proposition 2.3.17. ∎

The next proposition shows that noetherian normal monoids are the intersections of their localizations at codimension one prime ideals, whose proof can be found in [14, Proposition 2.8].

Proposition 2.3.19 (Flores, Weibel, [14])**.** *Let A be a noetherian normal monoid. Then*

$$A = \bigcap_{codim(P)=1} A_P.$$

Finally we will see that Hilbert Basis theorem holds for monoids, and, as a consequence finitely generated monoids are noetherian. In order to show this we need some lemmas:

Lemma 2.3.20. *Let A, B be monoids. Then, any ideal K of $A \otimes B$ is of the form*

$$K = \bigcup_{\lambda \in \Lambda} I_\lambda \otimes J_\lambda \quad \textit{for some indexing set} \quad \Lambda.$$

Where $I_\lambda \subset A$ and $J_\lambda \subset B$ are ideals, and Λ is an indexing set. Moreover, at least one of the sets of ideals $\{I_\lambda\}$ and $\{J_\lambda\}$ can be chosen to contain distinct elements.

Proof. Let $a \in A$ and $b \in B$ be elements such that $a \otimes b \in K$. Notice that aA, bB are ideals, and, by the construction of tensor product, $aA \otimes bB \subset K$. Hence

$$K = \bigcup_{a \otimes b \in K} aA \otimes bB.$$

Let $I, I' \subset A$, $J, J' \subset B$ be ideals. Then note that

$$(I \otimes J) \bigcup (I' \otimes J) = (I \bigcup I') \otimes J.$$

And

$$(I \otimes J) \bigcup (I \otimes J') = I \otimes (J \bigcup J').$$

Now for each $b \in B$ we define an ideal I_b such that we fix $I_\lambda = I_b$ for some λ and obtain some ideal J_λ as follows

$$I_b = \bigcup_{a\otimes b\in K} aA \quad \text{and let} \quad J_\lambda = \bigcup \{bB \mid I_b = I_\lambda\}.$$

Therefore $K = \bigcup_{\lambda\in\Lambda} I_\lambda \otimes J_\lambda$ and $\{I_\lambda\}$ contains distinct elements. ■

The next proposition is the analogue of the Hilbert Basis theorem for monoids:

Proposition 2.3.21. *If A, B are noetherian monoids then $A \otimes B$ is noetherian. In particular both $\mathbb{F}_1[T_1, \cdots, T_n]$ and $A[T_1, \cdots, T_n]$ are noetherian.*

Proof. Let $K \subset A \otimes B$ be an ideal. Then, by Lemma 2.3.20 we have $K = \bigcup_{\lambda\in\Lambda} I_\lambda \otimes J_\lambda$ for some indexing set Λ. Let $\lambda \in \Lambda$, then, by Remark 2.3.4, I_λ and J_λ are finitely generated. Suppose $\{a_1, \cdots, a_n\} \subset A$ and $\{b_1, \cdots, b_m\} \subset B$ are sets of generators. Hence $I_\lambda \otimes J_\lambda$ is finitely generated by $\{a_i \otimes b_j\}$.

Finally we have to show that Λ can be chosen to be finite. Thus by Lemma 2.3.20 we may assume that $\{I_\lambda\}_{\lambda\in\Lambda}$ has distinct elements. If we suppose that Λ cannot be chosen to be finite, we contradict the assumption on A to be noetherian since we obtain a increasing sequence

$$I_{\lambda_1} \subsetneq I_{\lambda_1} \cup I_{\lambda_2} \subsetneq \cdots \quad \text{with} \quad \lambda_n \in \Lambda \backslash \{\lambda_1, \cdots, \lambda_{n-1}\}.$$

Finally note that any ideal I of $\mathbb{F}_1[T]$ is generated by its element with lower degree. Hence both $\mathbb{F}_1[T]^{\otimes n}$ and $A[T_1, \cdots, T_n]$ are noetherian. ■

Corollary 2.3.22. *Let A be a finitely generated monoid, then A is noetherian. Thus, by Remark 2.3.4 every ideal of A is finitely generated.*

Proof. Let $\{a_1, \cdots, a_n\}$ be a set of generators of A. Then we define a morphism

$$\varphi : \mathbb{F}_1[T_1, \cdots, T_n] \to A \quad \text{given by} \quad T_i \mapsto a_i.$$

Note that φ is onto. Hence by Propositions 2.3.3 and 2.3.21 A is noetherian. ■

Chapter 3

The geometry of monoids

In this chapter we develop the theory of monoid schemes, which was was introduced by Deitmar in [8] and was one of the first attempts to describe $\mathbb{F}_1$-schemes[1]. The material presented here constitutes the basics of the area.

We start reviewing the spectrum of a monoid. Thereafter we introduce monoid schemes which are topological spaces that locally look like the spectrum of a monoid together with an structure sheaf. In the last section we will see how can extend the functor of Definition 2.2.11 to recover ring schemes from monoid schemes.

As in the last chapter we will try to emphasize those properties that apply in a special way to the case of monoid schemes and that do not occur in general in the case of of ring schemes.

3.1 Affine monoid schemes

Definition 3.1.1. The *spectrum* of a monoid A is the set $Spec(A)$ endowed with a topology and a structure sheaf[2]. Sometimes we denote by $|Spec(A)|$ the underlying set or topological space without the structure sheaf, although, when the context is clear, we simply write $Spec(A)$.

$Spec(A)$ is the topological space equipped with the *Zariski topology* given by closed sets, namely

$$V(I) = \{P \in Spec(A) \mid I \subset P\} \quad \text{for any ideal} \quad I \subset A.$$

We also define the *principal open subsets* of $Spec(A)$ as follows:

[1] Actually in [8] monoid schemes are called $\mathbb{F}_1$-schemes

[2] For the basic notions of sheaves that will be used in this work we refer the reader to [12, Section I.1.3] or [18, Section 2.5].

$$D(a) = \{P \in Spec(A) \mid a \notin P\} \quad \text{for any} \quad a \in A.$$

Proposition 3.1.2. *The principal open subsets form a basis of $Spec(A)$. In particular*

$$D(0) = \emptyset \quad \textit{and} \quad D(1) = Spec(A).$$

Proof. First note that $D(a) \cap D(b) = D(ab)$ since $ab \notin P$, for any two principal open subsets $D(a)$ and $D(b)$, if and only if both a and b are not contained in P.

Now let $U \subset Spec(A)$ be any open set, then $U = Spec(A) \backslash V(I)$ for some ideal I. Therefore, if X is a set of generators of I then

$$U = Spec(A) \backslash V(I) = Spec(A) \backslash \left(\bigcap_{x \in X} V(x) \right) = \bigcup_{x \in X} D(x).$$

■

If $S \subset A$ is a multiplicatively closed subset we denote by $U_S \subset Spec(A)$ the set of all primes of A that do not intersect with S. Notice that, in the particular, if $S = \{1, a, a^2, \cdots\}$ then $U_S = D(a)$. Furthermore, note that $U_S = \emptyset$ if and only if $0 \in S$.

Remark 3.1.3. *The topological concepts of connectedness and irreducibility apply in the same way as for schemes of rings (see [18, Proposition 1.15]). Let A be a monoid, then, in particular, $X = Spec(A)$ is irreducible if and only if A is integral. Moreover the following statements are equivalent:*

1. *X is irreducible.*
2. *If $U_1, U_2 \subset X$ are non empty open subsets, then $U_1 \bigcap U_2 \neq \emptyset$.*
3. *Let $U \subset X$ be a non empty open subset, then U is dense in X.*
4. *Every non empty open subset is connected.*
5. *Every non empty open subset is irreducible.*

Proposition 3.1.4. *$Spec(A)$ is quasi compact.*

Proof. Let $\{U_i\}_{i \in I}$ be a open cover of $Spec(A)$, then there exists U_j with $j \in I$, such that the maximal ideal m_A of A is contained in it. Now suppose m_A is contained in a principal open set $D(a)$, i.e. $m_A \in D(a) \subset U_i$. Thus $a \notin m_A$. Then, since m_A is maximal and therefore unique, $a \notin P$ for all $P \in Spec(A)$. Hence $P \in D(a)$. It follows that

$$D(a) = U_j = Spec(A),$$

which means that $Spec(A)$ is quasi compact. ■

Remark 3.1.5. *Since a monoid A has a unique maximal ideal $A \backslash A^{\times}$, the space $Spec(A)$ has a unique closed point i.e. m_A is the only ideal with the property that $\overline{\{m_A\}} = \{m_A\}$. Moreover, if A is finitely generated $Spec(A)$ is a finite poset of prime ideals. Indeed, let $X = \{x_1, \cdots, x_n\}$ be a set of generators of A. Now, if $P \in Spec(A)$, and $S = A \backslash P$ then S is generated by $\{x_i \mid x_i \notin P\}$.*

Hence, if $s \in S$ then $s = \prod_i x_i^{e_i}$, and then $e_i = 0$ when $x_i \in P$. Now, by counting the possible number of zeros in the e_i's we obtain an upper bound of the number of prime ideals of A, namely

$$\#|Spec(A)| \leq 2^n.$$

Hence, if A is finitely generated, the topological space has finitely many points. Then, as a consequence, the space always has finite dimension.

Proposition 3.1.6. *Let A be a f.g. monoid. Then, by considering a multiplicatively closed subset S, the localization $S^{-1}A$ is finitely generated, $U_S = D(s)$ for some $s \in A$ and $Spec(S^{-1}A)$ is open in $Spec(A)$.*

Proof. Suppose that $\{x_1, \cdots, x_n\}$ is a set of generators of A. Then, by Proposition 2.1.34 we know that $S^{-1}A = A_P$ for some prime P, thus we may assume that $S = A \backslash P$. It follows that S is generated by $\{x_i \mid x_i \notin P\}$. Now, if $s = \prod_{x_i \notin P} x_i$ then $U_S = D(s)$ and we conclude that $A_P = A_s$. Hence

$$U_S = D(s) = Spec(A_P) \subset Spec(A).$$

■

Let A be a finitely generated monoid, then, by Remark 3.1.5 we know that the space $Spec(A)$ only has finitely many points. When this happens $Spec(A)$ becomes a finite poset of prime ideals and we can represent the space as a directed graph $G = (V, E)$ whose vertices are the prime ideals. Thus, if $P, Q \in V$ and $P \subset Q$ then we obtain an edge PQ directed from P to Q.

Moreover, with the representation of the space as a directed graph we can identify both closed and open subsets of $Spec(A)$. Indeed, we identify closed subsets as upper sets i.e. sets that are closed from below, and we identify open subsets as the complements.

We give an example:

Example 3.1.7. *The monoid $\mathbb{F}_1[T_1, T_2, T_3]$ has finitely many prime ideals, each one of them generated by a subset of $\{T_1, T_2, T_3\}$. We represent the space*

$$X = Spec(\mathbb{F}_1[T_1, T_2, T_3])$$

as the directed graph depicted below.

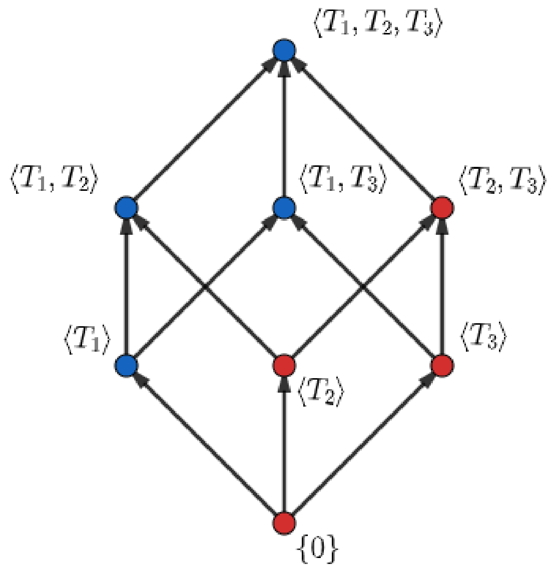

Figure 3.1: Representation of $Spec(\mathbb{F}_1[T_1, T_2, T_3])$ as a directed graph.

Now we illustrate the way to identify both open and closet subsets of X in the graph: The vertices corresponding to the upper set of $\langle T_1\rangle$ (blue vertices), correspond to the closed subset that contains $\langle T_1\rangle$, and its complement (red vertices) is the open subset whose elements do not contain T_1.

Let A be a monoid and let $X = Spec(A)$. We want to describe the structure sheaf $\mathcal{O}_X$. However, in order to make things easier, first we give an explicit description of the values of the sheaf for the principal open subsets of X which are given in terms of localization.

Definition 3.1.8. The structure sheaf of $X = Spec(A)$ for the principal open subsets is given by:

$$\mathcal{O}_X(D(s)) = A[s^{-1}] = \left\{ \frac{a}{s^n} \mid a \in A, n \in \mathbb{N} \right\}.$$

In fact, this has sense since for two open subsets $D(g) \subset D(s)$ it follows that $g = fs$ for some $f \in A$ because $D(fs) = D(f) \cap D(s)$, then, we can define the morphism:

$$\rho_{D(s),D(g)} : \mathcal{O}_X(D(s)) = A[s^{-1}] \longrightarrow A[g^{-1}] = \mathcal{O}_X(D(g)).$$

$$\text{given by} \quad \frac{a}{s^i} \longmapsto \frac{af^i}{g^i}.$$

Notice that if $s = 1$ then $D(1) = Spec(A)$, and thus $\mathcal{O}_X(Spec(A)) = A$. Hence the morphism $\rho_{D(s),D(g)}$ maps an element $a \in A$ to $\frac{a}{1} \in A[g^{-1}]$.

Finally, the structure sheaf $\mathcal{O}_X$ derives from the values obtained for the principal open subsets of X as above. The construction of the structure sheaf can be fount in [8, Section 2.1], however we give a description below.

Remark 3.1.9. *let $U \subset Spec(A)$ be a open subset. We can describe a section of $\mathcal{O}_X(U)$ as a family $t = (t_P)_{P \in U}$ with $t_P \in A_P$ for all $P \in U$ with the property that for every $P \in U$ there are $a, b \in A$ with $b \notin Q$ and*

$$t_Q = \frac{a}{b} \in A_Q \text{ for all } Q \text{ in an open subset } O_P \text{ with } P \in O_P \subset U.$$

Definition 3.1.10. We define a *monoidal space* X as a pair $X = (|X|, \mathcal{O}_X)$, where $|X|$ is a topological space, and $\mathcal{O}_X$ is a *sheaf of monoids* (the structure sheaf).

Definition 3.1.11. An *affine monoid scheme* is a monoidal space isomorphic to

$$Spec(A) = (|Spec(A)|, \mathcal{O}_{Spec(A)}) \quad \text{for some} \quad A \in \mathcal{M}_*.$$

Definition 3.1.12. A morphism between monoidal spaces $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$, is a pair $(f, f^{\#})$ such that

1. $f : |X| \longrightarrow |Y|$ is a continuous map of topological spaces.
2. $f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is a morphism of sheaves.

Where, $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is given by the *pull back* on U, that is

$$f^{\#} : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$$

for each open subset $U \subset Y$.

When the context is clear we simply denote the morphism of monoidal spaces as $f : X \to Y$.

Definition 3.1.13. Let $(X, \mathcal{O}_X)$ be a monoidal space and let $s \in X$ We define the *stalk* of the structure sheaf $\mathcal{O}_X$ at a point $x \in X$ as follows

$$\mathcal{O}_{X,x} = colim_{x \in U}(\mathcal{O}_X(U))$$

i.e. the colimit over all open neighborhoods of x. This is well defined since such colimit always exists by Theorem 2.1.14.

Remark 3.1.14. *Consider the set of open neighbourhoods as a partially ordered set by inclusion, with order relation given by $U \geq U' \iff U \subset U'$.*

Then, taking into account the restrictions maps of $\mathcal{O}_X$, we can think the notion of stalk at some point x as follows:

$$\mathcal{O}_{X,x} = \{(U,t) \mid x \in U, t \in \mathcal{O}_X(U)\}/\sim$$

Where the equivalence relation is given by $(U,t) \sim (U',t')$ if and only if there exists an open $U'' \subset U \cap U'$ with $x \in U''$ and $t|_{U''} = t'|_{U''}$. The classes of the equivalence relation are the elements of the stalk and are called the germs *of the stalk. We write $[(U,t)]$ for such an element.*

Proposition 3.1.15. *Let $X = Spec(A)$. Then for any point $P \in Spec(A)$ the stalk $\mathcal{O}_{X,P}$ is isomorphic to the localization A_P.*

Proof. By last remark consider the following morphism:

$$\varphi : \mathcal{O}_{X,P} \longrightarrow A_P \quad \text{given by} \quad [(U,t)] \longmapsto t_P.$$

Note that this is well defined since φ maps the class of family $t = (t_Q)_{Q \in U} \in \mathcal{O}_X(U)$ in the stalk at P to its element $Q = P$. The morphism φ is surjective since any element A_P can be writen as $\frac{a}{b}$ with $b \notin P$. Let $U = D(b)$. Then we have

$$\varphi\left(\left[\left(D(b), \frac{a}{b}\right)\right]\right) = \frac{a}{b}.$$

Now we show that φ is injective. Indeed, suppose

$$\varphi([(U,t)]) = \varphi([(U',t')]).$$

Then, by Remark 3.1.9, in some neighborhood O_P of P we have

$$t_Q = \frac{a}{b} \in A_Q \quad \text{and} \quad t'_Q = \frac{a'}{b'} \in A_Q \quad \text{for all} \quad Q \in O_P.$$

But then, since t_Q and t'_Q map to the same element under φ, there exists $b'' \in A \backslash P$ such that $b''b'a = b''ba'$. Therefore, since $P \in D(b'')$, the equality $\frac{a}{b} = \frac{a'}{b'}$ holds in A_Q for every $Q \in O_P \cap D(b'')$. Hence $[(U,t)] = [(U',t')]$.

■

Remark 3.1.16. *As we have seen before if $X = Spec(A)$ for some monoid A, then a point $x \in X$ is a prime ideal of A i.e. $x = P$. Moreover, since there is an isomorphism $\mathcal{O}_{X,x} \cong A_P$, and in analogy with Definition 2.1.26, we say that a point $x \in X$ has* height *or* codimension *equal to the largest* n *of the lengths of strictly descending chains of the form*

$$x = x_n \supsetneq x_{n-1} \supsetneq \cdots \supsetneq x_0.$$

Therefore, by both Corollary 2.1.36 and Proposition 3.1.15, the height of the point x *equals the dimension of the stalk* $\mathcal{O}_{X,x}$.

Definition 3.1.17. A morphism $(f, f^{\#})$ between monoidal spaces $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$ is called local if for all $x \in |X|$ the morphism $f^{\#} : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is local.

It is well know that in affine schemes for rings the morphisms corresponds to morphisms of their underlying rings (see for instance [15, Proposition 12.28]). This statement holds for affine monoid schemes and the proof proceeds in the same way as for affine ring schemes since, by Proposition 2.1.27, the inverse image of a prime ideal under a morphism of monoids is a prime ideal.

We summarize all of the above in the following proposition:

Proposition 3.1.18. *For any two monoids* A, B*, there is a bijective correspondence*

$$\{\textit{morphisms } Spec(A) \to Spec(B)\} \longleftrightarrow \{\textit{monoid morphisms } B \to A\}.$$

Thus a monoid morphism $f : B \to A$ *yields a morphism*

$$(\varphi, \varphi^{\#}) : (Spec(A), \mathcal{O}_{Spec(B)}) \longrightarrow (Spec(B), \mathcal{O}_{Spec(A)}).$$

Such that $\varphi : Spec(A) \to Spec(B)$ *is a local morphism of monoidal spaces, and conversely.*

Remark 3.1.19. *The category of affine monoid schemes is denoted by* $\mathcal{M}_* -$ *schemes. This category is dual to* $\mathcal{M}_*$*. Hence the initial object in* $\mathcal{M}_* -$ *schemes is* $Spec(0)$*, and the terminal object is* $Spec(\mathbb{F}_1)$.

Furthermore, notice that, in particular, Proposition 3.1.18 shows that there is a bijection between isomorphisms of monoids and isomorphisms of affine monoid schemes.

Before continuing with the theory, we show some examples:

Example 3.1.20 (Affine space)**.** *Motivated by the affine* $n-$*dimensional space* $\mathbb{A}^n_{\mathbb{Z}} = Spec(\mathbb{Z}[T_1, \cdots, T_n])$*, we define the affine space over* $\mathbb{F}_1$ *as the following affine monoid scheme.*

$$\mathbb{A}^n_{\mathbb{F}_1} = Spec(\mathbb{F}_1[T_1, \cdots, T_n]).$$

For instance, Figure 3.1 shows the affine space $\mathbb{A}^3_{\mathbb{F}_1}$, which was represented as a directed graph.

Example 3.1.21 (Algebraic Tori). *Consider the monoid $\mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}]$ of Laurent monomials over $\mathbb{F}_1$. We define the* algebraic torus of rank n over $\mathbb{F}_1$ *as the following affine monoid scheme:*

$$G^n_{m,\mathbb{F}_1} = Spec(\mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}]).$$

Notice that $G^n_{m,\mathbb{F}_1}$ only has one point since $\mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}]$ is a pointed group.

3.2 Monoid schemes

After having defined the building blocks of the theory we define monoid schemes.

Definition 3.2.1. A *monoid scheme* or $\mathbb{F}_1-$*scheme* in the sense of Deitmar, is a monoidal space that admits an open cover by affine monoid schemes (an affine cover). A morphism of monoid schemes is a local morphism of monoidal spaces.

Proposition 3.2.2. *The principal open subsets $D(a) \subset Spec(A)$ are isomorphic to $Spec(A_a)$. Hence affine schemes are schemes in the sense of Definition 3.2.1.*

Proof. We know that $D(a) = \{P \in Spec(A) \mid a \notin P\}$, but also note that $Spec(A_a)$ has the same underlying set. Then we show the isomorphism of their structure sheaf. In order to do this, we just show the isomorphism between $\mathcal{O}_{D(a)}(D(b))$ and $\mathcal{O}_{Spec(A_a)}(D(b))$ for all principal open subsets $D(b) \in Spec(A)$ since every open set is a union of principal opens and then, by the *sheaf axiom*[3], the isomorphism holds.

Then, since $D(a) \bigcap D(b) = D(ab)$, and, by Definition 3.2.1

$$\mathcal{O}_{Spec(A)|D(b)}(D(b)) = \mathcal{O}_{Spec(A)}(D(ab)) \cong Spec(A_{ab}).$$

And, again, by Definition 3.1.8

$$\mathcal{O}_{Spec(A_a)}(D(b)) \cong (A_a)_b.$$

Hence $\mathcal{O}_{Spec(A_a)}(D(b)) \cong \mathcal{O}_{D(a)}(D(b))$. Thus the result follows. ■

Corollary 3.2.3. *Let X be a monoid scheme, and let $U \subset X$ be an open set. Then $U = (|U|, \mathcal{O}_{X|U})$ is a monoid scheme.*

---

[3]The precise statement of the sheaf axiom can be found in [12, Section I.1.3].

Proof. Since U is open it is covered by principal open subsets. Then, by Proposition 3.2.2, U is a monoid scheme. ■

Definition 3.2.4. Let X be a monoid scheme and let $U \subset X$ be an open set. An *open monoid subscheme* of X is a pair $(U, \mathcal{O}_{X|U})$. If U is affine we say that U is an *affine open monoid subscheme*. Furthermore, note that U has an open cover by principal open subset. Hence, by Proposition 3.2.2, U is a monoid scheme.

Remark 3.2.5. *Notice that the intersection of open monoid schemes is again an open monoid scheme since the basis of the topology is given by principal open subsets which, as we have seen above, are monoid schemes. Furthermore, when the space is affine, there is a generalization of Proposition 3.2.2* [4]*:*

Let $U \subset Spec(A)$ be an affine monoid subscheme, and let $S \subset A$ the set of elements not in any prime in U, then S is multiplicatively closed. This defines the following inclusions.

$$A \hookrightarrow S^{-1}A \quad \text{and} \quad Spec(S^{-1}A) \hookrightarrow Spec(A).$$

In particular the open inclusion $U \to Spec(A)$ is given by $A \to S^{-1}A$ for some multiplicatively closed set S.

Definition 3.2.6. A monoid scheme is *integral (normal, cancellative or torsion free)* if it can be covered by affine monoid schemes that are isomorphic to the spectrum of an integral (resp. normal cancellative or torsion free) monoid. Thus, if X is an integral (resp. normal, cancellative or torsion free) monoid scheme, then any affine subset is the spectrum of an integral (resp. normal, cancellative or torsion free) monoid.

Definition 3.2.7. A point η of a monoid scheme X is called *generic* if it is contained in every non empty open set.

Remark 3.2.8. *When $X = Spec(A)$ is cancellative then 0 defines a prime ideal of A. Thus $Spec(A)$ has a unique generic point.*

Proposition 3.2.9. *A cancellative monoid scheme X is a union*

$$X = \bigcup_{\text{generic points } \eta \in X} X_\eta.$$

Where X_η is the closure of η in X. In particular, if X is connected then it has only a unique generic point.

---

[4] See [6, Section 3.1.12] for details

Proof. First consider the affine case $X = Spec(A)$ with A integral. By last Remark, X has a unique generic point.

Now let X be a cancellative monoid scheme, and let $x \in X$. Then note that any affine neighborhood $U_x = Spec(A_x)$ of x has a unique generic point η. Then, it follows that $\eta \in U_x \subset X_\eta$. Therefore $X_\eta = \bigcup U_x$ is open and closed in X.

Thus if X is connected $X = X_\eta$. Hence it has only one generic point. ■

Now we show an example of a non affine space:

Example 3.2.10 (The projective line $\mathbb{P}^1_{\mathbb{F}_1}$)**.** *(cf. [3, 3.1]) The space $\mathbb{P}^1_{\mathbb{F}_1}$ is covered by two affine spaces of the form $\mathbb{A}^1_{\mathbb{F}_1}$, namely $Spec(\mathbb{F}_1[T_0])$ and $Spec(\mathbb{F}_1[T_1])$. Then $\mathbb{P}^1_{\mathbb{F}_1}$ has two closed points denoted by x_0 and x_1, and also, a generic point η. Thus note that $\{\eta\}$ coincides with $G^1_{m,\mathbb{F}_1}$. Indeed, we can see that η is contained in every non empty open subset since we are identifying each affine space $\mathbb{A}^1_{\mathbb{F}_1}$ with the two open subsets that cover $\mathbb{P}^1_{\mathbb{F}_1}$, namely $U_0 = \{x_0, \eta\}$, $U_1 = \{x_1, \eta\}$ whose intersection is the open subset $U_{01} = \{\eta\}$. We depict the projective line over $\mathbb{F}_1$ below, where the arrows show the contention relationships among the affine pieces.*

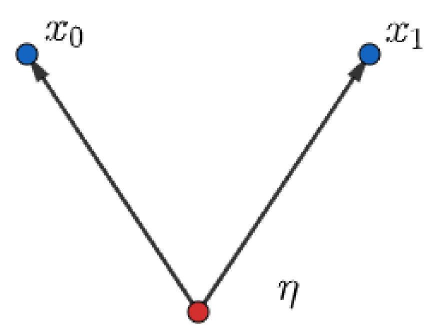

Figure 3.2: The projective line $\mathbb{P}^1_{\mathbb{F}_1}$.

Notice that the intersection $U_{01} = \{\eta\}$ can be seen as $Spec(\mathbb{F}_1[T_0, T_1]/\langle T_0T_1 = 1\rangle)$. This yields to the reestriction maps

$$\rho_{U_i,U_{01}} : \mathbb{F}_1[T_i] \longmapsto \mathbb{F}_1[T_0, T_1]/\langle T_0T_1 = 1\rangle.$$

Remark 3.2.11. *Notice that the projective line described above satisfies 1.0.1 which tells us that $\mathbb{P}^n_{\mathbb{F}_1}$ contains two points. To make sense of this consider $\mathbb{F}^2_1$ as a set with two elements, say $\{1, 2\}$, Thus, as in Example 1.0.3, the closed points of $\mathbb{P}^n_{\mathbb{F}_1}$ can be seen as the subsets $\{1\}$, $\{2\}$. Hence x_0 and x_1 corresponds to these subsets and the generic point η corresponds to the full set $\{1, 2\}$.*

Definition 3.2.12. Let S be a monoid scheme.

1. A *scheme over* S is a pair (X, f_X), with X a monoid scheme, and f_X a morphism $f_X : X \to S$.
2. A morphism between monoid schemes over S is a morphism $f : (X, f_X) \to (Y, f_Y)$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f_X"'] & & Y \arrow[dl, "f_Y"] \\ & S & \end{tikzcd}$$

We introduce the notion of the fiber product of monoid schemes.

Definition 3.2.13. Let (X, f_X), (Y, f_Y) be monoid schemes over S. A *fiber product* of X and Y over S is a monoid scheme $(X \times_S Y, f_{X \times_S Y})$ over S together with morphisms $pr_1 : X \times_S Y \to X$ and $pr_2 : X \times_S Y \to Y$ such that the following universal properety holds:

Let Z be a monoid scheme such that for any two morphisms $\psi_X : Z \to X$ and $\psi_Y : Z \to Y$ that commutes respectively with f_X and f_Y then there is a unique morphism $\psi : Z \to X \times_S Y$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\psi_X"] \arrow[ddr, bend right, "\psi_Y"'] & & \\ & X \times_S Y \arrow[r, "pr_1"'] \arrow[d, "pr_2"] & X \arrow[d, "f_X"] \\ & Y \arrow[r, "f_Y"'] & S \end{tikzcd}$$

Proposition 3.2.14. *The fiber product $X \times_S Y$ exists.*

Proof. We restrict to the affine case, since the general result follows by gluing the affine pieces. Thus let $X = Spec(A), Y = Spec(B)$ and $S = Spec(C)$ be as in Definition 3.2.12. Then, by Proposition 3.1.18 and by the universal property of tensor products (Definition 2.2.8), the monoid $A \otimes_C B$ gives the same diagram as above with all arrows reversed. Hence $Spec(A \otimes_C B)$ has the desired properties i.e.

$$X \times_S Y = Spec(A \otimes_C B).$$

■

We also have the notion of base change which plays an important role in order to pass from monoid schemes to ring schemes:

Definition 3.2.15. Let X and T be schemes over S. The second projection

$$pr_2 : X \times_S T \longrightarrow T$$

makes the fibered product $X \times_S T$ a scheme over T. This process is called *the base change* from S to T.

Remark 3.2.16. *When $S = Spec(\mathbb{F}_1)$ we simply write the fibered product of two schemes X, Y over S as $X \times Y$. Moreover, if $S = Spec(\mathbb{F}_1)$ and $T = Spec(A)$, where A is a monoid, we simply say that $pr_2 : X \times_S T \longrightarrow T$ is the base change from $\mathbb{F}_1$ to A.*

Example 3.2.17. *Let $X = Spec(\mathbb{F}_1[T_1, \cdots, T_n])$ be an affine scheme. If $Y = Spec(A)$ for some monoid A, then, by Example 2.2.10, the base change from $\mathbb{F}_1$ to A is*

$$X \times Y = Spec(A[T_1, \cdots, T_n]).$$

Example 3.2.18. *By Remark 2.2.9, the fibered product $Spec(\mathbb{F}_1[T]) \times Spec(\mathbb{F}_1[T])$ is $Spec(\mathbb{F}_1[T] \otimes \mathbb{F}_1[T]) = Spec(\mathbb{F}_1[T_1, T_2])$ i.e.*

$$\mathbb{A}^1_{\mathbb{F}_1} \times \mathbb{A}^1_{\mathbb{F}_1} = \mathbb{A}^2_{\mathbb{F}_1}.$$

Proposition 3.2.19. *The category $\mathcal{M}_* - schemes$ contains limits.*

Proof. The result follows from the existence of fibered product and Theorem 2.1.14. ∎

Definition 3.2.20. A monoid scheme X is *noetherian* if it admits a finite covering by affine monoid schemes that are isomorphic to the spectrum of a noetherian monoid. We also say that X is of *finite type* if is quasi compact and if it can be covered by affine monoid schemes that are isomorphic to the spectrum of a finitely generated monoid.

Remark 3.2.21. *Notice that finite type condition of a monoid scheme is equivalent to noetherian condition since finitely generated monoids are noetherian by Corollary 2.3.22.*

Monoid schemes of finite type have some particular properties such as the following ones:

Proposition 3.2.22. *Let X be monoid scheme of finite type. Then, each point $x \in X$ has an affine open neighborhood U such that*

$$\mathcal{O}_{X,x} \cong \mathcal{O}_X(U).$$

Proof. Let $x \in X$, and suppose x has an affine open neighborhood U with $U = Spec(A)$ with A finitely generated, which means $x = P$ for some prime ideal P of A, and thus $S = A \backslash P$ is a multiplicatively closed subset. Then note that P is the only maximal ideal in U_S. Therefore U_S is contained in all principal open subsets that contains P, but by Proposition 3.1.6, there is a principal open subset $D(s)$ such that $U_S = D(s)$ for some $s \in A$. Hence $\mathcal{O}_{X,x} \cong \mathcal{O}_X(D(s))$. ■

Proposition 3.2.23. *Let X be a monoid scheme of finite type and let $\mathcal{B}$ the set of all open affine subsets of X. Consider the map*

$$\varphi : X \longrightarrow \mathcal{B} \qquad x \longmapsto \bigcap_{x \in U_x} U_x,$$

where U_x is an open neighborhood of x. Then φ is a bijective map whose inverse φ^{-1} sends an affine open subset $U = Spec(A)$ of X to the maximal ideal m_A of A, which is a point of $U \subset X$.

Proof. By last Proposition, φ maps an element $x \in X$ to a unique principal open subset $D(s)$ in a suitable affine space of the form $Spec(A)$ such that $s \in A$ and x is a prime ideal P. Moreover P is the maximal ideal contained in U_S with $S = A \backslash P$. Finally, φ^{-1} is injective, since maximal ideals in monoids are unique. ■

We conclude this section introduced the notion of closed immersion of monoid schemes as it is defined[5] in [5, 26]. As we will see, this concept differs in the case of ring schemes.

Definition 3.2.24. A morphism $(f, f^{\#}) : X \to Y$ of monoid schemes is a *closed immersion* if $f : |X| \to |Y|$ is a homeomorphism onto its image and for every open subset $U \subset Y$, the inverse image $V = f^{-1}(U)$ is affine in X and $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective. A *closed subscheme* of Y is a monoid scheme X together with a closed immersion.

It is well know that the image of a closed immersion ring schemes $\varphi : X \to Y$ is a closed subset of Y (see for instance [18, Section 3.16]). However, this is not always true in the case of closed immersion of monoid schemes.

Example 3.2.25. *Consider the diagonal embbeding*

$$\Delta : \mathbb{A}^1_{\mathbb{F}_1} \longrightarrow \mathbb{A}^1_{\mathbb{F}_1} \times \mathbb{A}^1_{\mathbb{F}_1} = \mathbb{A}^2_{\mathbb{F}_1}$$

[5] Actually, the notion of closed immersion introduced in [26] is defined in terms of blue schemes, which are generalizations of monoid schemes. We will present blue schemes in Chapter 5.

which, by Proposition 3.1.18, corresponds to the monoid morphism

$$f_\Delta : \mathbb{F}_1[T_1, T_2] \longrightarrow \mathbb{F}_1[T] \quad \text{given by} \quad T_i \mapsto T \quad \text{for } i = 1, 2.$$

Notice that $\mathbb{F}_1[T]$ only has two prime ideals, namely, $\langle 0\rangle$ and $\langle T\rangle$ whose inverse image under f_Δ are $f_\Delta^{-1}(\langle 0\rangle) = \{0\}$ and $f_\Delta^{-1}(\langle T\rangle) = \langle T_1, T_2\rangle$. However the subset

$$\{\langle 0\rangle, \langle T_1, T_2\rangle\} \subset \mathbb{A}^2_{\mathbb{F}_1},$$

is not closed because $V(\langle 0\rangle) = \mathbb{A}^2_{\mathbb{F}_1}$.

3.3 Base extension

The results presented in this section allow us to extend the properties that we have seen for monoid schemes to ring schemes. Then, once the main results of this section have been shown, we can use the theory developed in the previous sections of this chapter and extend it to the case of ring schemes.

Remark 3.3.1. *The base extension functor $- \otimes_{\mathbb{F}_1} \mathbb{Z}$ defined in Chapter 2 leads to a functor from affine monoid schemes to affine ring schemes given by the base change from $\mathbb{F}_1$ to $\mathbb{Z}$, namely $- \times_{\mathbb{F}_1} Spec(\mathbb{Z})$. For instance, if $X = Spec(A)$ for some monoid A, its base extension to ring schemes is*

$$X_{\mathbb{Z}} = Spec(A) \times_{\mathbb{F}_1} Spec(\mathbb{Z}) = Spec(A \otimes_{\mathbb{F}_1} \mathbb{Z}) = Spec(A_{\mathbb{Z}}).$$

Example 3.3.2. *Some examples of base extension to ring schemes (or simply base extension).*

1. *The base extension of the affine line:*

$$\mathbb{A}^n_{\mathbb{Z}} = \mathbb{A}^n_{\mathbb{F}_1} \times Spec(\mathbb{Z}) = Spec(\mathbb{Z}[T_1, \cdots, T_n]).$$

2. *The base extension of the algebraic torus:*

$$G^n_{m,\mathbb{Z}} = G^n_{m,\mathbb{F}_1} \times Spec(\mathbb{Z}) = Spec(\mathbb{Z}[T_1^{\pm}, \cdots, T_n^{\pm}]).$$

Remark 3.3.3. *Consider a monoid scheme X with affine open cover $\{U_i\}_{i\in I}$ such that each element is of the form $U_i = Spec(A_i)$ for some monoid A_i. Moreover, by Remark 3.2.5 we may suppose that such an open cover is closed under intersections. Then we extend the functor $- \otimes_{\mathbb{F}_1} \mathbb{Z}$ in the following way*

$$X = \bigcup_{i\in I} U_i \longrightarrow \bigcup_{i\in I} (U_i)_{\mathbb{Z}} = X_{\mathbb{Z}}.$$

Which is glued by the gluing maps of X. Moreover, using Proposition 3.1.18 and Corollary 3.2.3 in [8, Section 2.3] it is shown that $X_{\mathbb{Z}}$ does not depend of the choice of the cover.

Proposition 3.3.4. *Let X be a monoid scheme. Then*

1. *X is of finite type if and only if $X_{\mathbb{Z}}$ is a scheme of finite type.*
2. *If X is of finite type, then only has finitely many points.*

Proof.

1. It follows from Proposition 2.2.16
2. Note that X can be covered by finitely many open subsets since is quasi compact. Then the result follows by Remark 3.1.5.

■

Recall that an open inclusion $\iota : X \to Y$, where $X = Spec(B)$ and $Y = Spec(A)$, is given by the morphism $i : A \to S^{-1}A$ for some multiplicatively closed subset $S \subset A$ by Remark 3.2.5. Then, by Proposition 2.2.14, we have a base extension $i_{\mathbb{Z}} : A_{\mathbb{Z}} \to S^{-1}A_{\mathbb{Z}}$, which induces the base extension of the open inclusion, i.e.

$$\iota_{\mathbb{Z}} : X_{\mathbb{Z}} \longrightarrow Y_{\mathbb{Z}}.$$

Which is an injective map.

Now, let X be a monoid scheme. If $U \subset X$ is an open subset, we denote by $U_{\mathbb{Z}}$ the union of all base extension of affine open subsets of U inside $X_{\mathbb{Z}}$. Now we want to construct a continuous map as the following:

$$\beta : X_{\mathbb{Z}} \longrightarrow X.$$

To do this let $x \in X_{\mathbb{Z}}$, and suppose $U_{\mathbb{Z}} \subset Spec(A_{\mathbb{Z}})$ is an affine neighborhood of x such that $U_{\mathbb{Z}}$ is the base extension of an affine open subset $U = Spec(A)$ of X. Note that x is a prime ideal $\mathfrak{P}$ in the ring $A_{\mathbb{Z}}$. Then if we intersect $\mathfrak{P}$ with A we obtain a prime ideal $P = A \cap \mathfrak{P}$ in A. Then P is a point t in some neighborhood $U \in X$.

In fact the map β is well defined and continuous by the following results:

Lemma 3.3.5. *The map $\beta : X_{\mathbb{Z}} \to X$ is well defined, i.e. if $\beta(x) = t$ as before, then this value is independent of the choice of the affine neighboorhood U of the prime ideal $P = A \cap \mathfrak{P}$.*

Proof. Suppose $V_{\mathbb{Z}}$ is another affine open neighborhood of x which we may suppose to be small enough to be contained in $U_{\mathbb{Z}}$ by replacing V with $V \cap U$. Then, by Remark 3.2.5, V is of the form $Spec(S^{-1}A)$ for some multiplicatively closed subset $S \subset A$. Then, the inclusion map $\iota : V \to U$ induces a the canonical map $i : A \to S^{-1}A$. Then for any prime ideal P of $S^{-1}A$

$$\begin{aligned} f_{\mathbb{Z}}^{-1}(P) \cap A &= \{a \in A \mid f_{\mathbb{Z}}(a) \in P\} \\ &= \{a \in A \mid f(a) \in P \cap S^{-1}A\} \\ &= f^{-1}(P \cap S^{-1}A). \end{aligned}$$

Hence the value of $\beta(x)$ is independent of the choice of U. ■

The following theorem was stated in [6, Theorem 3.2]. For completeness we present the proof

Theorem 3.3.6 (Chu, Lorscheid, Santhanam, [6]). *The map $\beta : X_{\mathbb{Z}} \to X$ is continuous. Moreover $\beta^{-1}(U) = U_{\mathbb{Z}}$ for any open subset $U \subset X$.*

Proof. Let $U \subset X$ be any open subset. Then, since $U_{\mathbb{Z}}$ is the union of all base extension of affine open subsets of U inside $X_{\mathbb{Z}}$, we show that $\beta^{-1}(U) = U_{\mathbb{Z}}$ for affine open subsets $U = Spec(A)$.

Thus, if $x \in U_{\mathbb{Z}}$ is a point, then $\beta(x) \in U$. Then if $x \notin U_{\mathbb{Z}}$ we choose another affine open neighborhood, namely $V_{\mathbb{Z}}$, then $\beta(x) \in V$ and suppose $V = Spec(B)$. Then by construction of β, x gives a prime ideal $\mathfrak{P}$ of $B_{\mathbb{Z}}$ which gives a prime ideal $P = \mathfrak{P} \cap B$ of B. Now, by Remark 3.2.5 the inclusion $\iota : V \cap U \to V$ implies that $V \cap U$ is of the form $Spec(S^{-1}B)$ for some multiplicatively closed subset $S \subset B$. Then, we know that $x \notin (V \cap U)_{\mathbb{Z}}$ which means that $P \cap S \neq \emptyset$, but, since $S \in B$ then $(\mathfrak{P} \cap A) \cap S \neq \emptyset$. Hence $\beta(\mathfrak{P}) \notin S^{-1}B$ by Proposition 2.1.35. Thus $\beta(x) \notin U \cap V$ and the result follows. ■

As a consequence of Theorem 3.3.6 we obtain the following corollary:

Corollary 3.3.7. *Let $\{U_i\}$ be any family of open subsets of X. Then*

$$\left(U_i \bigcap U_j\right)_{\mathbb{Z}} = (U_i)_{\mathbb{Z}} \bigcap (U_j)_{\mathbb{Z}} \quad \text{and} \quad \left(\bigcup U_i\right)_{\mathbb{Z}} = \bigcup (U_i)_{\mathbb{Z}}.$$

Chapter 4

Toric varieties over $\mathbb{F}_1$

The goal of this chapter is to develop the theory of toric varieties using the geometry of monoid schemes. In fact, our approach consists in construct toric varieties as a certain kind of monoid schemes in a similar manner as the way as in the case of the usual theory over $\mathbb{C}$. The references for such a theory of such toric varieties are, for example, [2, 7] and [13]. A reference for the theory of toric varieties over $\mathbb{Z}$ is, for example, [20]. It is noteworthy that in this last reference the toric varieties are constructed from the point of view of ring schemes. Throughout this chapter the notions of polyhedral geometry that are can be found in Appendix A.

We begin by giving the general description of affine toric varieties over $\mathbb{F}_1$, which is basically the spectrum of a monoid scheme with certain characteristics. After this we build the non-affine toric varieties and study their connection between them and convex polyhedral cones and fans. Later we introduce divisors, the class group and the Picard group of toric varieties over $\mathbb{F}_1$. At the end of this section we introduce the notion of Cox algebra. Finally, at the end of the chapter we present some relationships between the number of generators of a numerical monoid and the multiplicity of toric curves.

4.1 Toric varieties over $\mathbb{F}_1$

In this section we present the general notion of toric varieties $\mathbb{F}_1$, starting with the affine case.

First recall that $\mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}]$ is a $\mathbb{F}_1$-algebra with monomials given by $T^a = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ plus a 0 element, where $a = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$. Then we associate a monoid morphism

$$\begin{array}{rcl}\theta : \mathbb{Z}^n_* & \longrightarrow & \mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}]. \\ a = (\alpha_1, \cdots, \alpha_n) & \longmapsto & T^a = T_1^{\alpha_1} \cdots T_n^{\alpha_n}.\end{array}$$

Thus

$$\mathbb{Z}^n_* \cong \theta(\mathbb{Z}^n) = \mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}].$$

More generally we will use monomials associated to *lattices*. For this we need the following remark.

Remark 4.1.1. *From remark 2.1.42 recall that a lattice N is a pointed lattice N_* i.e. a pointed abelian group isomorphic to $\mathbb{Z}^n_*$ unless otherwise stated. In the same way, by $\mathbb{Z}^n$ we mean $\mathbb{Z}^n_*$ unless otherwise stated. Furthermore, if $N \cong \mathbb{Z}^n$ is a lattice, its dual lattice is*

$$M = Hom_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n.$$

Below we present the basics notions of multiplicative groups schemes over $\mathbb{F}_1$. However we will focus on the torus $G^n_{m,\mathbb{F}_1}$ of which we will describe its structure as group scheme over $\mathbb{F}_1$[1].

Definition 4.1.2. A *multiplicative group scheme* over $\mathbb{F}_1$ is a monoid scheme G together with the three morphisms:

$$\begin{array}{rrcl}\text{Multiplication} & \mu : G \times G & \longrightarrow & G \\ \text{Inversion} & \iota : G & \longrightarrow & G \\ \text{Identity} & \epsilon : Spec(\mathbb{F}_1) & \longrightarrow & G\end{array}$$

that satisfies the usual axioms of a group, meaning that the following diagrams commute

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times id"] \arrow[d, "id \times \mu"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd}$$

[1]Many of the notions of algebraic groups over $\mathbb{F}_1$ don't fit in the theory of monoid schemes, however, they do in the case of the torus. There are extension of the theory that allow us to consider models of algebraic groups in terms of blue schemes that we will meet in Chapter 5. A further discussion on algebraic groups over $\mathbb{F}_1$ can be foun in [21, 26].

$$\begin{tikzcd} G \arrow[r, "id\times\epsilon"] \arrow[dr, "id"'] & G\times G \arrow[d, "\mu"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\epsilon\times id"] \arrow[dr, "id"'] & G\times G \arrow[d, "\mu"] \\ & G \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d] & G\times G \arrow[r, "id\times\iota"] & G\times G \arrow[d, "\mu"] \\ Spec(\mathbb{F}_1) \arrow[rr, "\epsilon"] & & G \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d] & G\times G \arrow[r, "\iota\times id"] & G\times G \arrow[d, "\mu"] \\ Spec(\mathbb{F}_1) \arrow[rr, "\epsilon"] & & G \end{tikzcd}$$

Where $\Delta : G \to G \times G$ is the diagonal $x \mapsto (x, x)$ as in Example 3.2.25, and $G \to Spec(\mathbb{F}_1)$ is the unique morphism to $Spec(\mathbb{F}_1)$.

In what follows, by group scheme we mean a multiplicative group scheme over $\mathbb{F}_1$ as in Definition 4.1.2 unless otherwise stated.

Remark 4.1.3. *The torus $G^n_{m,\mathbb{F}_1}$ is a group scheme with*

1. *Multiplication* $\mu : G^n_{m,\mathbb{F}_1} \times G^n_{m,\mathbb{F}_1} \longrightarrow G^n_{m,\mathbb{F}_1}$ *given by the $\mathbb{F}_1-$algebra morphism:*
$$\begin{array}{rcl} \mu^* : \mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}] & \longrightarrow & \mathbb{F}_1[(T_1')^{\pm}, \cdots, (T_n')^{\pm}] \otimes \mathbb{F}_1[(T_1'')^{\pm}, \cdots, (T_n'')^{\pm}]. \\ T_i & \longmapsto & T_i' \otimes T_i'' \quad \textit{for} \quad i = 1, \cdots, n. \end{array}$$

2. *Inversion* $\iota : G^n_{m,\mathbb{F}_1} \longrightarrow G^n_{m,\mathbb{F}_1}$ *given by the $\mathbb{F}_1-$algebra morphism:*
$$\begin{array}{rcl} \iota^* : \mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}] & \longrightarrow & \mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}]. \\ T_i & \longmapsto & T_i^{-1} \quad \textit{for} \quad i = 1, \cdots, n. \end{array}$$

3. *Identity* $\epsilon : Spec(\mathbb{F}_1) \longrightarrow G^n_{m,\mathbb{F}_1}$ *given by the $\mathbb{F}_1-$algebra morphism:*
$$\epsilon^* : \mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}] \longrightarrow \mathbb{F}_1.$$

   That maps every non zero element to 1.

By Remark 2.2.9 recall that

$$\mathbb{F}_1[(T'_1)^{\pm}, \cdots, (T'_n)^{\pm}] \otimes \mathbb{F}_1[(T''_1)^{\pm}, \cdots, (T''_n)^{\pm}] \cong \mathbb{F}_1[(T'_1)^{\pm}, \cdots, (T'_n)^{\pm}, (T''_1)^{\pm}, \cdots, (T''_n)^{\pm}].$$

Therefore, the multiplication of the torus is given by:

$$\mu^* \quad \text{defined by} \quad T_i \longmapsto T'_i T''_i \quad \text{for} \quad i = 1, \cdots, n.$$

Moreover if A is a monoid, by Remark 2.2.9, we know that $A = A \otimes \mathbb{F}_1$. Thus the identity ϵ of the group scheme has the usual meaning. For instance

$$G^n_{m,\mathbb{F}_1} \times Spec(\mathbb{F}_1) = G^n_{m,\mathbb{F}_1}.$$

by the construction of the fiber product.

Definition 4.1.4. Consider two group schemes

1. T with multiplication μ_1 and identity ϵ_1;
2. G with multiplication μ_2 and identity ϵ_2.

A *homomorphism* of the group schemes T and G is a morphism of monoid schemes $\varphi : T \to G$ such that the following diagrams commute

$$\begin{tikzcd} T \times T \arrow[r, "\mu_1"] \arrow[d, "(\varphi,\varphi)"'] & T \arrow[d, "\varphi"] \\ G \times G \arrow[r, "\mu_2"] & G \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dd, "\varphi"] \\ Spec(\mathbb{F}_1) \arrow[ur, "\epsilon_1"] \arrow[dr, "\epsilon_2"'] & \\ & G \end{tikzcd}$$

The set of all such homomorphisms is denoted by $Hom(T, G)$

Definition 4.1.5. A *Torus* $\mathbb{T}$ is a monoid scheme isomorphic as a group scheme to $G^n_{m,\mathbb{F}_1}$ with group structure inherited from the isomorphism.

Remark 4.1.6. *A homomorphism of algebraic tori like the following:*

$$\varphi : G^n_{m,\mathbb{F}_1} \longrightarrow G^r_{m,\mathbb{F}_1}.$$

is given by the $\mathbb{F}_1-$*algebra morphism*

$$\begin{aligned} \varphi^* : \mathbb{F}_1[T_1^{\pm}, \cdots, T_r^{\pm}] &\longrightarrow \mathbb{F}_1[(T'_1)^{\pm}, \cdots, (T'_n)^{\pm}]. \\ T_i &\longmapsto (T')^{\alpha_i}. \end{aligned}$$

Where $\alpha_i \in \mathbb{Z}^n$ for $i = 1, \cdots, r$, i.e. $(T')^{\alpha_i}$ is written in multi index. Indeed, this is a homomorphism since φ^* maps units to units.

We can see that morphisms of lattices $\mathbb{Z}^r \to \mathbb{Z}^n$ are in bijection with the elements of $Hom(\mathbb{F}_1[T_1^{\pm}, \cdots, T_r^{\pm}], \mathbb{F}_1[(T_1')^{\pm}, \cdots, (T_n')^{\pm}])$, i.e. are in bijection with $Hom(G^n_{m,\mathbb{F}_1}, G^r_{m,\mathbb{F}_1})$. In particular

$$Hom(G^n_{m,\mathbb{F}_1}, G^1_{m,\mathbb{F}_1}) \cong \mathbb{Z}^n \qquad and \qquad Hom(G^1_{m,\mathbb{F}_1}, G^n_{m,\mathbb{F}_1}) \cong \mathbb{Z}^n.$$

For an arbitrary torus we have the following definition:

Definition 4.1.7. A *character* of a torus $\mathbb{T}$ is an $\mathbb{F}_1-$algebra morphism that corresponds to a homomorphism $\chi \in Hom(\mathbb{T}, G^1_{m,\mathbb{F}_1})$. A *one parameter* subgroup of a torus $\mathbb{T}$ is an $\mathbb{F}_1-$algebra morphism that corresponds to a homomorphism $\lambda \in Hom(G^1_{m,\mathbb{F}_1}, \mathbb{T})$.

Let $\mathbb{T} \cong G^n_{m,\mathbb{F}_1}$ be a torus. By Remark 4.1.6, notice that their associated characters and one parameter subgroups yields lattices

$$M = Hom(\mathbb{T}, G^1_{m,\mathbb{F}_1}) \cong \mathbb{Z}^n \qquad \text{and} \qquad N = Hom(G^1_{m,\mathbb{F}_1}, \mathbb{T}) \cong \mathbb{Z}^n.$$

which we call *character lattice* and *lattice of one parameter subgroups* respectively. Note that we identify $N \cong Hom(M, \mathbb{Z})$ and $M \cong Hom(N, \mathbb{Z})$. Sometimes we denote the torus $\mathbb{T}$ by T_N to refer to the lattice of on parameter subgroups associated with it. Likewise, by Remark 2.1.40 we denote by $\mathbb{F}_1[M]$ the underlying monoid of the torus. Notice that the base extension of the underlying monoid $\mathbb{F}_1[M] \otimes_{\mathbb{F}_1} \mathbb{Z}$ is the coordinate ring of the algebraic tori $\mathbb{T}_{\mathbb{Z}} = \mathbb{T} \times_{\mathbb{F}_1} Spec(\mathbb{Z})$.

Additionally, we can check that the multiplication $T_N \times T_N \to T_N$ of the group scheme is given by the $\mathbb{F}_1-$algebra morphism:

$$\begin{array}{rcl} \mathbb{F}_1[M] & \longrightarrow & \mathbb{F}_1[M] \otimes \mathbb{F}_1[M] \\ \chi^m & \longmapsto & \chi^m \otimes \chi^m. \end{array}$$

Example 4.1.8. *For instance, suppose $N = M = \mathbb{Z}^n$. Then $T_N = Spec(\mathbb{F}_1[\mathbb{Z}^n])$. Thus for any $m = (u_1, \cdots, u_n) \in M = \mathbb{Z}^n$ there is a character χ^m given by*

$$\begin{array}{rcl} \chi^m : \mathbb{F}_1[T_1^{\pm}] & \longrightarrow & \mathbb{F}_1[\mathbb{Z}^n] \\ \chi^m(T_1) & = & T^m = T_1^{u_1} \ldots T_n^{u_n}. \end{array}$$

In the same way, an element $n = (v_1, \cdots, v_n) \in N = \mathbb{Z}^n$ induces a one paramenter subgroup λ_n given by

$$\begin{array}{rcl} \lambda_v : \mathbb{F}_1[\mathbb{Z}^n] & \longrightarrow & \mathbb{F}_1[T_1^{\pm}] \\ \lambda_v(T_i) & = & T_1^{v_i} \quad for \quad i = 1, \cdots, n. \end{array}$$

Definition 4.1.9. A *group action* of a torus $\mathbb{T}$ over a monoid scheme X is a map $\theta : \mathbb{T} \times X \to X$ such that the following diagrams commutes.

$$\begin{tikzcd} \mathbb{T} \times \mathbb{T} \times X \arrow[r, "id\times\theta"] \arrow[d, "\mu\times id"'] & \mathbb{T} \times X \arrow[d, "\theta"] \\ \mathbb{T} \times X \arrow[r, "\theta"] & X \end{tikzcd} \qquad \begin{tikzcd} Spec(\mathbb{F}_1) \times X \arrow[r, "\epsilon\times id"] \arrow[dr, "pr_2"'] & \mathbb{T} \times X \arrow[d, "\theta"] \\ & X \end{tikzcd}$$

Now, we are ready to define toric varieties:

Definition 4.1.10. A *toric variety over* $\mathbb{F}_1$ is an irreducible monoid scheme X of finite type containing a torus T_N as an open subset such that the natural action of T_N on itself extends to an action $T_N \times X \to X$.

Notice that in our definition we don't require X being normal.

Example 4.1.11. *The affine space $\mathbb{A}^n_{\mathbb{F}_1} = Spec(\mathbb{F}_1[T_1, \cdots, T_n])$ is an affine toric variety, containing the torus $\mathbb{T} = G^n_{m,\mathbb{F}_1}$. The torus action is described by*

$$\mathbb{T} \times Spec(\mathbb{F}_1[T_1, \cdots, T_n]) \longrightarrow Spec(\mathbb{F}_1[T_1, \cdots, T_n])$$

given by the morphism

$$\begin{array}{rcl} \mathbb{F}_1[T_1, \cdots, T_n] & \longrightarrow & \mathbb{F}_1[(T_1')^{\pm}, \cdots, (T_n')^{\pm}] \otimes \mathbb{F}_1[T_1'', \cdots, T_n'']. \\ T_i & \longmapsto & T_i' \otimes T_i'' \quad \text{for} \quad i = 1, \cdots, n. \end{array}$$

Recall that when we refer to an affine monoid A we use an additive notation, however, by Remark 2.1.40, we can change this notation to a multiplicative one, namely $\mathbb{F}_1[A]$. Moreover, as we saw in Remark 2.1.39, an affine monoid can be embedded in a pointed lattice. This is the reason why these monoids provides affine toric varieties as we will see below, but before, we need a lemma whose proof can be found in [29, Proposition 2.14].

Lemma 4.1.12. *Let $\varphi : T_1 \to T_2$ be a homomorphism of tori. Then the image of T_1 under φ equals a subtorus $T' \subset T_2$.*

Proposition 4.1.13. *Let T_N be a torus with character lattice M. Consider an affine monoid A embedded in M, and suppose that $\{a_1, \cdots, a_s\}$ is a set of generators of A.*

Then $Spec(\mathbb{F}_1[A_0])$ *is a torus with character lattice generated by* $\{\pm a_1, \cdots, \pm a_s\}$ *as a monoid.*

Proof. Consider the following morphism

$$\varphi : T_N \longrightarrow G^s_{m,\mathbb{F}_1} = Spec(\mathbb{F}_1[T_1^{\pm}, \cdots, T_s^{\pm}]),$$

which by Remark 4.1.6 is entirely determined by the characters of the morphisms

$$\chi^{a_i} : T_N \longrightarrow Spec(\mathbb{F}_1[T_1^{\pm}]).$$

Then, by Lemma 4.1.12 the image of T_N under φ is a torus T. Now, we show that T has character lattice generated by $\{\pm a_1, \cdots, \pm a_s\}$ as a monoid i.e. $T = Spec(\mathbb{F}_1[A_0])$

Suppose M' is the character lattice of T, then we have a commutative diagram

$$\begin{tikzcd}
T_N \arrow[rr] \arrow[dr, "\varphi"'] & & T \arrow[dl, hook'] \\
& G^s_{m,\mathbb{F}_1} &
\end{tikzcd}$$

Which gives rise to the following commutative diagram of character lattices

$$\begin{tikzcd}
& \mathbb{Z}^s \arrow[dl, "\check{\varphi}"'] \arrow[dr, two heads] & \\
M & & M' \arrow[ll]
\end{tikzcd}$$

Where $\check{\varphi} : \mathbb{Z}^s \to M$ is defined by $e_i \to a_i$ for all $i = 1, \cdots, s$ where e_i denotes an element of the standard basis of $\mathbb{Z}^s$. Thus the image of $\mathbb{Z}^s$ under $\check{\varphi}$ is A_0. Hence, by the last diagram we deduce that M' is a character lattice generated by $\{\pm a_1, \cdots, \pm a_s\}$ as a monoid. ■

Proposition 4.1.14. *Let* A *be an affine monoid. Then* $Spec(A)$ *is a toric variety with torus* $Spec(A_0)$*.*

Proof. Let $X = Spec(A)$ and $T = Spec(A_0)$. By Proposition 4.1.13 we know that T is a torus, then we have to show that $T \hookrightarrow X$ is an embedding as an open subset, but, by Remark 3.2.5, this follows from the canonical morphism $A \to A_0$. Note that the open set is dense since A is cancellative (in particular integral).

Now, consider the torus action $\varphi : T \times T \to T$ given by the morphism

$$\begin{array}{rcl} \varphi^* : A_0 & \longrightarrow & A_0 \otimes A_0. \\ a & \longmapsto & a \otimes a. \end{array}$$

And consider the map

$$\begin{array}{rcl} \theta^* : A & \longrightarrow & A_0 \otimes A. \\ a & \longmapsto & a \otimes a, \end{array}$$

which induces a map $\theta : T \times X \to X$. Thus the following commutative diagrams show that θ extends the torus action on X :

$$\begin{tikzcd} A \arrow[r, "\theta^*"] \arrow[d, hook] & A_0 \otimes A \arrow[d, hook] \\ A_0 \arrow[r, "\varphi^*"] & A_0 \otimes A_0 \end{tikzcd}$$

Which gives rise to

$$\begin{tikzcd} X & T \times X \arrow[l, "\theta"'] \\ T \arrow[u, hook] & T \times T \arrow[l, "\varphi"'] \arrow[u, hook] \end{tikzcd}$$

Thus we can check that $\theta : T \times X \to X$ is a group action in the sense of Definition 4.1.9. ■

Remark 4.1.15. *By Proposition 4.1.14 when A is an affine monoid we obtain an affine toric variety $Spec(A)$ which also can be writen as $Spec(\mathbb{F}_1[A])$. Additionally, if M is the character lattice of the torus $Spec(A_0)$, an element $m \in M$ gives a character χ^m. Thus, by Proposition 4.1.13, the affine monoid A can be written as the following $\mathbb{F}_1-$algebra:*

$$\mathbb{F}_1[A] = \{\chi^m \mid m \in A\}.$$

Note that in our definition we don't require X to be normal, as it is shown in the next example.

Example 4.1.16. *Consider the numerical monoid $S = \langle 2, 3\rangle_* \subset \mathbb{N}_*$, which can be written multiplicatively as $\mathbb{F}_1[T^2, T^3]$. Note that S is not normal since $S = T^3/T^2 \in S^{nor}$ but $T \notin S$. We can check that S is affine, and, by Proposition 4.1.14, gives rise to the toric variety $Spec(S)$ with torus $Spec(\mathbb{F}_1[\mathbb{Z}]) = G^1_{m,\mathbb{F}_1}$.*

Example 4.1.16 shows a first connection between numerical monoids and toric varieties. A further discussion of this connection will be given in the last section of this chapter.

In what follows we consider the non affine case. For this, let X be an irreducible, cancellative, torsion free monoid scheme of finite type. Thus X has a unique generic point η. Then we have the following definition.

Definition 4.1.17. Following [14], the stalk at the generic point $\mathcal{O}_{X,\eta}$, is called the *generic monoid*, and it is denoted by A_0. The term generic monoid is due to the fact that in the affine case $X = Spec(A)$ we obtain the stalk $\mathcal{O}_{X,\eta} = A_0$.

Note that the generic monoid is the analogue of the field of rational functions of an integral ring scheme (see [18, Remark 9.29]).

Remark 4.1.18. *Let $U \subset X$ be an open subset of a monoid scheme, then there is an isomorphism*

$$\mathcal{O}_{U,\eta} \cong \mathcal{O}_{X,\eta}$$

given by the map $\mathcal{O}_{U,\eta} \to \mathcal{O}_{X,\eta}$ which sends an element $t \in \mathcal{O}_U(V) \mapsto t \in \mathcal{O}_X(V)$, and has inverse map given by $\mathcal{O}_{X,\eta} \to \mathcal{O}_{U,\eta}$ which sends an element $t \in \mathcal{O}_X(V) \mapsto t|_{V\cap U} \in \mathcal{O}_U(V \cap U)$.

Likewise, if $x \in X$ is a point contained in an open subset $U = Spec(A) \subset X$ we know that it is a prime ideal P of A. In this case

$$\mathcal{O}_{X,x} \cong \mathcal{O}_{U,x} = \mathcal{O}_{U,P} = A_P.$$

The ideas of above about toric varieties over $\mathbb{F}_1$ are due to Deitmar. Indeed, in [9, Theorem 4.1]. Deitmar shows that for a connected, torsion free, integral monoid scheme of finite type X its base extensión $X_{\mathbb{C}}$ is a toric variety (over $\mathbb{C}$). The following theorem is the analog for toric varieties over $\mathbb{F}_1$.

Theorem 4.1.19. *Let X be an irreducible, cancellative, torsion free monoid scheme of finite type. Then X is a toric variety.*

Proof. By Proposition 3.2.9 we know that X has a unique generic point η. Then let $U = Spec(A)$ be an open affine subset. By hypothesis A is an affine monoid. Thus, by Remark 4.1.18, the stalk $\mathcal{O}_{X,\eta}$ is equal to A_0. However, by Proposition 4.1.14 we know that $Spec(A)$ is an affine toric variety with torus $Spec(A_0)$ as an open subset.

Moreover, the action described in the affine case is compatible with the restriction maps of the structure sheaf. Thus the action of $Spec(A_0)$ on $Spec(A)$ extends to X. Thus X is a toric variety over $\mathbb{F}_1$. ∎

Remark 4.1.20. *Let X be an irreducible, cancellative, torsion free monoid scheme of finite type. Note that the base extension $X_{\mathbb{Z}}$ is a toric variety over $\mathbb{Z}$, and in the same way, due to Deitmar's aforementioned theorem, we obtain a toric variety over $\mathbb{C}$.*

Hence, by Theorem 4.1.19, we can see that the geometry of monoids is limited since in some sense the only varieties that are obtained from monoid schemes are the toric varieties. Therefore, as we already mentioned, toric varieties, like the ones presented here, fit into other notions of $\mathbb{F}_1$-schemes.

In the next chapter we are going to present an introduction to blueprints and its geometric counterpart blue schemes, which are generalizations of monoids and monoid schemes respectively, and which have made possible cover more types of varieties in the framework of $\mathbb{F}_1-$geometry. However, it is noteworthy that the theory of toric varieties presented here fits into the blue scheme theory.

4.2 Toric varieties associated to cones and fans

In this section we study a special kind of toric varieties, namely those that arise from the structure of cones (in the affine case) and fans. These varieties have the property of being normal. After this we will present both divisor class group and Picard group.

We start with the affine case:

Definition 4.2.1 (Affine toric variety associated to a cone)**.** Let $\sigma \in N_{\mathbb{R}}$ be a cone, and let $(S_\sigma)_*$ its associated monoid (see A.0.10). We define the *affine toric variety associated to* σ as follows:

$$X_\sigma = Spec(\mathbb{F}_1[S_\sigma]).$$

There is a relation among the elements of the monoid S_σ which we detail in the following:

Suppose that $\{a_1, \cdots, a_k\}$ is a set of generators of S_σ, where each a_i is writen as $a_i = (\alpha_{i_1}, \cdots, \alpha_{i_n})$. Let $\pi : \mathbb{Z}^k \to S_\sigma$ be the morphism that maps each canonical element e_i to the generator a_i. Then the next short exact sequence give us the relation among the elements of S_σ in terms of the generators:

$$0 \longrightarrow ker(\pi) \longrightarrow \mathbb{Z}^n \xrightarrow{\pi} S_\sigma \longrightarrow 0$$

Therefore, for some $v_j, m_j \in \mathbb{Z}_{\geq 0}$ we obtain

$$r = \sum_{j=1}^{k} v_j e_j - \sum_{j=1}^{k} m_j e_j \in ker(\pi).$$

Hence, it follows that

$$\sum_{j=1}^{k} v_j a_j = \sum_{j=1}^{k} m_j a_j.$$

Example 4.2.2. *Consider a cone $\sigma = cone(e_2, 2e_1 - e_2)$, and its dual cone $\check{\sigma} = cone(e_1, e_1 + 2e_2)$. Both σ and $\check{\sigma}$ are depicted below.*

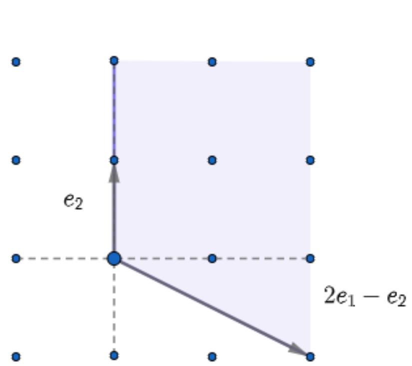

Figure 4.1: σ and its dual cone $\check{\sigma}$

The set of generators of the monoid S_σ is $\{a_1 = e_1, a_2 = e_1 + e_2, a_3 = e_1 + 2e_2\}$. Notice that the relation among the generators is given by $a_1 + a_3 = 2a_2$. Then $\mathbb{F}_1[S_\sigma] = \mathbb{F}_1[T_1, T_1T_2, T_1T_2^2]$. Hence the toric variety associated to σ is

$$X_\sigma = Spec(\mathbb{F}_1[T_1, T_1T_2, T_1T_2^2]).$$

Example 4.2.3. *Consider the cone $\sigma = cone(e_1, e_2, e_3, e_1 + e_2 - e_3)$ and its dual cone $\check{\sigma} = cone(e_1, e_2, e_1 + e_3, e_2 + e_3)$ which is depicted below.*

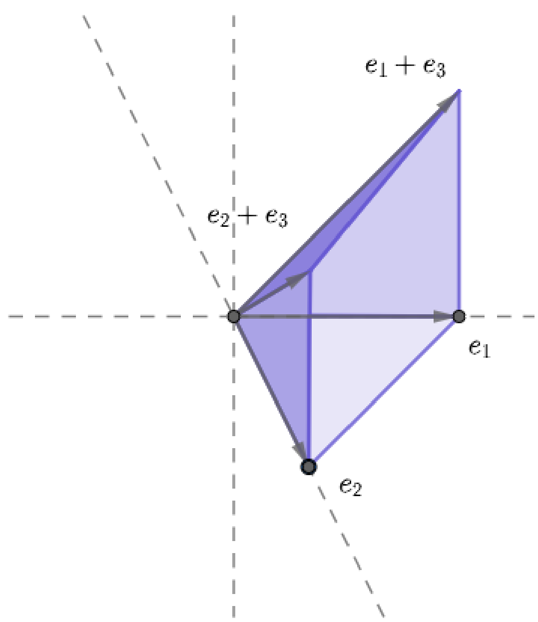

Figure 4.2: $\check{\sigma} = cone(e_1, e_2, e_1 + e_3, e_2 + e_3)$

The monoid S_σ is generated by the elements of the set $\{a_1 = e_1, a_2 = e_2, a_3 = e_1 + e_3, a_4 = e_2 + e_3\}$, thus the relation among the generators is given by $a_1 + a_4 = a_2 + a_3$. The affine toric variety associated to σ is

$$X_\sigma = Spec(\mathbb{F}_1[T_1, T_2, T_1T_3, T_2T_3]).$$

Remark 4.2.4. *Let $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. A lattice homomorphism $\varphi : N \to L$ can be extended to a homomorphism of real vector spaces $\varphi_\mathbb{R} : N_\mathbb{R} \to L_\mathbb{R}$. Then, if $\varphi_\mathbb{R}$ maps a cone σ_N in $N_\mathbb{R}$, to a cone σ_L in $L_\mathbb{R}$, the homomorphism $\varphi_\mathbb{R}$ induces a dual map*

$$\check{\varphi} : Hom_\mathbb{Z}(L, \mathbb{R}) \longrightarrow Hom_\mathbb{Z}(N, \mathbb{R}).$$

$$\textit{given by} \quad \psi \longmapsto \psi \circ \varphi \quad \textit{for all} \quad \psi \in Hom_\mathbb{Z}(L, \mathbb{R}).$$

Then we obtain a map $S_{\sigma_L} \to S_{\sigma_N}$, which induces maps

$$R_{\sigma_L} \longrightarrow R_{\sigma_N} \quad \textit{and} \quad X_{\sigma_N} \longrightarrow X_{\sigma_L}.$$

Once we have developed the affine blocks of the theory we present the toric varieties that arises from gluing them. Indeed, these toric varieties are obtained by open embbedings of affine patches given by localization of affine toric varieties whose structure is given by that of the fan.

We start by looking at the structure and relations among the affine toric varieties associated to the cones contained in a fan Δ.

Let $\sigma \in \Delta$ be a cone, and let τ be a face of σ. By Proposition A.0.11, we know that $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-\lambda)$, where $\lambda \in \check{\sigma} \cap M$ and $\tau = \sigma \cap \lambda^\perp$.

Recall that $\mathbb{F}_1[S_\sigma] = \{\chi^m \mid m \in S_\sigma\}$. Now, suppose that $A = \{a_1, \cdots, a_k\}$ is a set of generators of S_σ. Notice that S_τ is obtained from S_σ by adding a generator $-\lambda$, and then, if we assume that $\lambda = a_k$ we add $a_{k+1} = -\lambda$ which means that the relations of the generators of S_τ are the same as the generators of S_σ with an additional relation given by $a_k + a_{k+1} = 0$. Then by passing to Laurent monomials we obtain $T^{a_k}T^{a_{k+1}} = 1$. Thus there is a map

$$\mathbb{F}_1[T^{a_1}, \cdots, T^{a_k}] = \mathbb{F}_1[S_\sigma] \longrightarrow \mathbb{F}_1[S_\tau] = \mathbb{F}_1[T^{a_1}, \cdots, T^{a_k}, T^{a_{k+1}}],$$

given by the inclusion. This map can be seen as the inclusion in the localization σ at its face τ since:

$$\mathbb{F}_1[S_\tau] = S^{-1}\mathbb{F}_1[S_\sigma] \qquad \text{where} \qquad S = \{(T^{a_k})^n \mid n \in \mathbb{N}\}.$$

All of these observations are generalized in the following proposition:

Proposition 4.2.5. *Let Δ be a fan, and let σ be a cone in Δ. If τ is a face of σ i.e. $\tau = \sigma \cap \lambda^{\perp}$ with $\lambda \in \check{\sigma} \cap M$. Therefore the associated $\mathbb{F}_1-$algebra of τ is the localization*

$$\mathbb{F}_1[S_\tau] = \mathbb{F}_1[S_\sigma]_{\chi^\lambda} \qquad \textit{where} \qquad \chi^\lambda \in \mathbb{F}_1[S_\sigma].$$

Proof. In the same way as in the previous observations, the result follows from Proposition A.0.11. ∎

Remark 4.2.6. *Let σ and τ be as above. By Proposition 4.2.5 we obtain the localization $\mathbb{F}_1[S_\tau] = \mathbb{F}_1[S_\sigma]_{\chi^\lambda}$, and by Proposition 2.1.34, there is a prime ideal $P(\tau)$ of $\mathbb{F}_1[S_\sigma]$ such that*

$$S^{-1}\mathbb{F}_1[S_\sigma] = \mathbb{F}_1[S_\sigma]_{P(\tau)}.$$

In order to define toric varieties from fans consider a fan Δ, and let σ be a cone in Δ. Then, as we have seen above, if τ is a face of σ, there is an inclusion $S_\sigma \to S_\tau$, which, by Proposition 4.2.5 and Remark 3.2.5, induces an open inclusion of affine toric varieties

$$Spec(\mathbb{F}_1[S_\tau]) \longrightarrow Spec(\mathbb{F}_1[S_\sigma]).$$

All of the inclusions given as above define a directed diagram $\mathcal{D}$ (Definition 2.1.11). Thus along these inclusions we glue the affine toric varieties, which gives rise to the toric variety X_Δ. Formally we have the following definition:

Definition 4.2.7 (Toric varieties over $\mathbb{F}_1$ from fans)**.** Let Δ be a fan. We define the *toric variety over $\mathbb{F}_1$ associated to Δ* (cf. [33, Section 4.2] or [6, Section 3.1.4]) as follows:

$$X_\Delta = colim_{\sigma \in \Delta}(X_\sigma).$$

As we saw above, by Remark 4.2.6, the faces of the cones contained in a fan define localizations and define a directed diagram. Thus definition of toric varieties associated to fans is well defined. Toric varieties can be thought of as the gluing of affine pieces X_σ and $X_{\sigma'}$ along X_τ, where $\sigma, \sigma', \tau \in \Delta$ with $\tau = \sigma \cap \sigma'$.

Remark 4.2.8. *Notice that the requirements to obtain a toric variety over $\mathbb{F}_1$ associated to a fan Δ are sumarized by the structure of the fan as a follows:*

Let Δ be a fan defined over a lattice N. Let σ be a cone in Δ and let $\tau = \sigma \cap \lambda^{\perp}$ ($\lambda \in \check{\sigma} \cap M$) be a face of σ, then:

1. *We obtain inclusions $\mathbb{F}_1[S_\sigma] \subset \mathbb{F}_1[S_\tau]$ as a subsets of N.*
2. *We obtain a localization of σ at τ, namely $\mathbb{F}_1[S_\tau] = \mathbb{F}_1[S_\sigma]_{\chi^\lambda}$ where $\chi^\lambda \in \mathbb{F}_1[S_\sigma]$.*
3. *We obtain open inclusions $X_\tau \to X_\sigma$ of affine toric varieties.*

Remark 4.2.9. *Let Δ be a fan, and let σ be a cone in Δ. By Remark 4.2.6 we know that a face τ of $\sigma \in \Delta_{max}$ is a prime ideal in $\mathbb{F}_1[S_\sigma]$. Therefore, the cones of Δ corresponds to points of X_Δ. Moreover, in [5, Construction 4.2] it is shown that the points of X_Δ corresponds to the cones of Δ, and in particular codimension one points in X_Δ corresponds to the elements in $\Delta(1)$.*

Notice that the trivial cone $\{0\}$ is a face of all the elements in Δ. Its associated $\mathbb{F}_1-$algebra is $\mathbb{F}_1[M]$ where M is the character lattice. Moreover notice that $\{0\}$ corresponds to the generic point η of the toric variety X_Δ.

The torus action of a toric variety X_Δ is given by the map

$$\mathbb{F}_1[S_\sigma] \longrightarrow \mathbb{F}_1[S_{\{0\}}] \otimes \mathbb{F}_1[S_\sigma].$$

Indeed, by Definition 4.1.9 and Proposition 4.1.14 this map give us a torus action that extends the group law for T_N, namely $T_N \times T_N \to T_N$, to $T_N \times X_\sigma \to X_\sigma$.

Example 4.2.10. *The projective plane $\mathbb{P}^2_{\mathbb{F}_1}$ is a toric variety associated to a fan. Indeed, consider the fan Δ in $\mathbb{R}^2$ of the following Figure:*

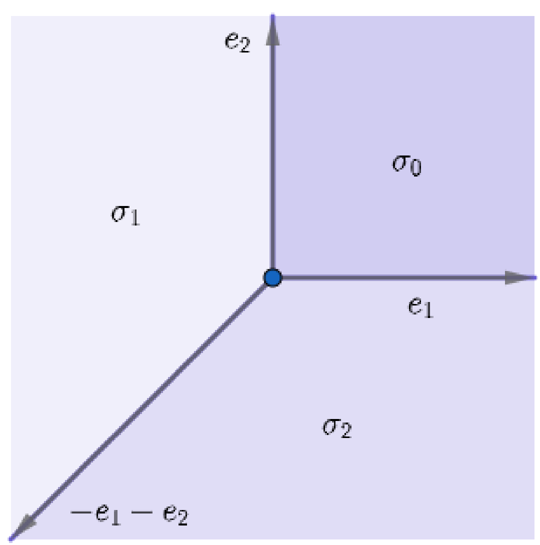

Figure 4.3: Fan Δ associated to $\mathbb{P}^2_{\mathbb{F}_1}$, and their dual cones

Then, for each cone $\sigma \in \Delta$ we associate the corresponding $\mathbb{F}_1-$ algebra and the corresponding affine toric variety. For this we need the dual cones of the cones contained in Δ. The dual cones are depicted below.

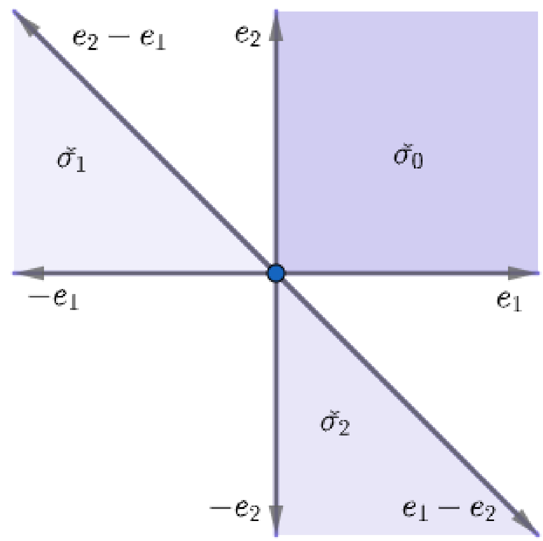

Figure 4.4: Fan Δ associated to $\mathbb{P}^2_{\mathbb{F}_1}$, and their dual cones

Then we obtain

$$\begin{aligned}
\mathbb{F}_1[S_{\sigma_0}] = \mathbb{F}_1[T_1, T_2] &\quad \textit{associated with} \quad X_{\sigma_0} = Spec(\mathbb{F}_1[T_1, T_2]),\\
\mathbb{F}_1[S_{\sigma_1}] = \mathbb{F}_1[T_1^{-1}, T_1^{-1}T_2] &\quad \textit{associated with} \quad X_{\sigma_1} = Spec(\mathbb{F}_1[T_1^{-1}, T_1^{-1}T_2]),\\
\mathbb{F}_1[S_{\sigma_2}] = \mathbb{F}_1[T_2^{-1}, T_1T_2^{-1}] &\quad \textit{associated with} \quad X_{\sigma_2} = Spec(\mathbb{F}_1[T_2^{-1}, T_1T_2^{-1}]).
\end{aligned}$$

By gluing the affine pieces we obtain $\mathbb{P}^2_{\mathbb{F}_1}$. For instance, by Remark 4.2.8, the gluing of X_{σ_0} and X_{σ_1} along $\tau = cone(e_2)$ is given by the open inclusions

$$Spec(\mathbb{F}_1[T_1, T_2]_{T_1}) \cong X_\tau \to X_{\sigma_0},$$

and

$$Spec(\mathbb{F}_1[T_1^{-1}, T_1^{-1}T_2]_{T_1^{-1}}) \cong X_\tau \to X_{\sigma_1}.$$

In other words, we identify the point $\langle T_1 \rangle$ in X_{σ_0} by the point $\langle T_1^{-1} \rangle$ in X_{σ_1}. Moreover, by Remark 4.2.9, the generic point η corresponds to the point in $X_{\{0\}} = Spec(\mathbb{F}_1[S_{\{0\}}]) \cong G^2_{m,\mathbb{F}_1}$.

In the figure below we depict the space $\mathbb{P}^2_{\mathbb{F}_1}$ with the relationships among the affine pieces, where the arrows show containment relationships each affine space as in Example 3.1.7.

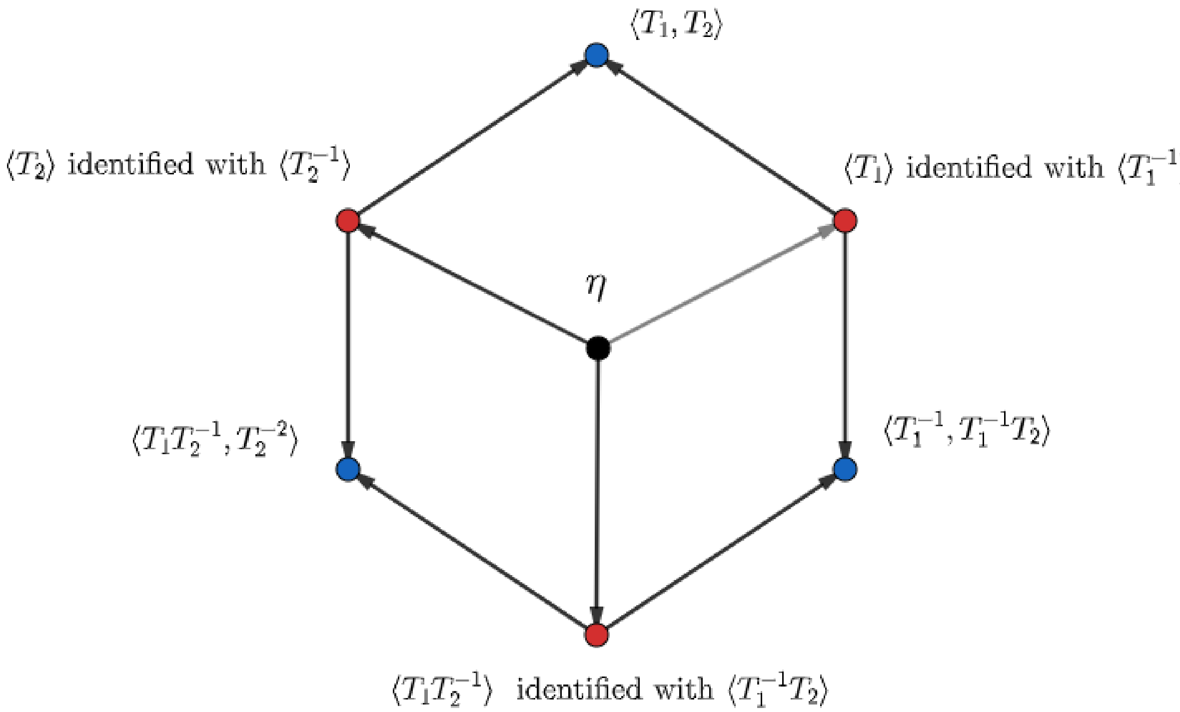

Figure 4.5: The projective plane $\mathbb{P}^2_{\mathbb{F}_1}$

Remark 4.2.11. *The projective plane $\mathbb{P}^2_{\mathbb{F}_1}$ satisfies the incidence relations of Example 1.0.3. Indeed, consider $\mathbb{F}_1^3$ as a set with three elements, say $A = \{1, 2, 3\}$. Then, the closed points $\mathbb{P}^2_{\mathbb{F}_1}$ can be seen as the subsets $\{1\}$, $\{2\}$, $\{2\}$ of A, which are represented as blue vertices of the figure above. Likewise, the red vertices correspond to the lines in $\mathbb{P}^2_{\mathbb{F}_1}$ i.e. correspond to the subsets $2-$subsets of A, and the generic point η, represented as a black vertex, corresponds to the full set A (see Figure 1.1).*

Proposition 4.2.12. *Affine toric varieties associated to cones are normal.*

Proof. Suppose that $\{v_1, \cdots, v_r\}$ is a set of generators of σ. We denote by $\tau_i \in \sigma(1)$ the cone (ray) associated to v_i. Notice that $\mathbb{F}_1[S_{\tau_i}] \cong \mathbb{F}_1[T_1, T_2^{\pm}, \cdots, T_n^{\pm}]$. Then by Proposition A.0.7 we have

$$\check{\sigma} = \bigcap_{i=1}^{r} \check{\tau}_i \qquad \text{which implies that} \qquad R_\sigma = \bigcap \mathbb{F}_1[S_\sigma].$$

This means that $\mathbb{F}_1[S_\sigma]$ is integrally closed. ∎

Remark 4.2.13. *In addition to Proposition 4.2.12, in [5, Construction 4.2.] it is shown that all normal toric varieties arises from fans.*

4.3 Divisors

The theory of divisors presented here was introduced [14], which is the main reference for this section. Our main reference for classical theory of divisors of toric varieties over $\mathbb{C}$ is [7, Chapter 5].

Our goal in this section is to present the theory of divisors for toric varieties over $\mathbb{F}_1$, and then to show that it match with the usual theory of divisors of toric varieties over $\mathbb{C}$.

Throughout this subsection, by monoid scheme we understand an irreducible, cancellative, torsion free, normal monoid scheme of finite type unless otherwise stated. Thus a monoid scheme X contains a unique generic point η.

Definition 4.3.1. A *prime divisor* in a monoid scheme X is a point $x \in X$ of codimension 1. The free abelian group generated by height one points in X is denoted by $Div(X)$. The elements of this group are called *Weil divisors*, i.e. a Weil divisor D is a finite sum

$$D = \sum_i a_i x_i.$$

where $x_i \in X$ with $codim(x_i) = 1$ and $a_i \in \mathbb{Z}$ for all i. Furtheremore, a Weil divisor $D = \sum_i a_i x_i$ is called *effective* if $a_i \geq 0$ for all i. In this case we write $D \geq 0$.

We want to work with the stalk $\mathcal{O}_{X,x}$ in order to get a valuation monoid. Thus, by Remark 4.1.18, we can restrict to the affine case $X = Spec(A)$, where A is a normal and finitely generated (and hence noetherian) monoid. Then, by Corollary 2.3.18, the stalk $\mathcal{O}_{X,x} \cong A_P$ is a discrete valuation monoid.

Remark 4.3.2. *Let x be a prime divisor on a monoid scheme X. Then, since $\mathcal{O}_{X,x}$ is a discrete valuation monoid, we obtain a valuation map*

$$v_x : A_0^\times \longrightarrow \mathbb{Z}.$$

Moreover note that $\mathcal{O}_{X,x}$ is identified with the set $\{a \in A_0^\times \mid v_x(a) \geq 0\}$.

Definition 4.3.3. Let $a \in A_0^\times$. We define the divisor of a as follows

$$div(a) = \sum_x v_x(a)x.$$

Where each x is a height one element of X. Divisors of this form are called *principal divisors.*

Remark 4.3.4. *Let $a, b \in A_0^\times$. Note that, the sum $\sum_x v_x(a)x$ is finite since X is of finite type, thus $div(a)$ belongs to $Div(X)$. Moreover, note that*

$$div(ab) = div(a) + div(b) \quad \textit{since} \quad v_x(ab) = v_x(a) + v_x(b).$$

Thus the set of all principal divisors is defined by $Div_0(X)$. Note that $Div_0(X)$ is a subgroup of $Div(X)$.

In what follows we use the notation $\mathcal{O}_X^\times$ to define the following:

$$\mathcal{O}_X^\times(U) = \{\text{invertible elements of } \mathcal{O}_X(U)\}.$$

Proposition 4.3.5. *Let X be a monoid scheme, and let $a \in A_0^\times$, then*

1. *$div(a) \geq 0$ if and only if $a \in \mathcal{O}_X(X)$, i.e. a is a global section.*
2. *$div(a) = 0$ if and only if $a \in \mathcal{O}_X^\times(X)$.*

Proof.

1. First suppose $a \in \mathcal{O}_X(X)$. Then, for every point $x \in X$ of height one, we have $a \in \mathcal{O}_{X,x}$. It follows that $v_x(a) \geq 0$ by Remark 4.3.2. On the other hand, suppose $div(a) \geq 0$, meaning that $v_x(a) \geq 0$ for all $x \in X$ of height one. Thus we can restrict to an open affine subset $U = Spec(A)$ where each point x of codimension one corresponds to a prime ideal P of codimension one. Then $a \in \mathcal{O}_{X,P}$ for all prime divisor $P \in U$. Therefore by Proposition 2.3.19

$$a \in \bigcap_{\mathrm{codim}(P)=1} \mathcal{O}_{X,P} = \bigcap_{\mathrm{codim}(P)=1} A_P = A.$$

   It follows that a is defined everywhere on U. Hence $a \in A_0^\times$ is a global section on $\mathcal{O}_X(X)$.

2. The second statement follows by the first and by Remark 2.3.14.

∎

Definition 4.3.6. We define the *divisor class group* of X as the quotient group

$$Cl(X) = Div(X)/Div_0(X).$$

In order to study local properties of Weil divisors of a monoid scheme X we restrict to a non empty open subset $U \subset X$. Thus, if $D = \sum_i a_i x_i$ is a Weil divisor, then the *restriction* of D to U is the Weil divisor on U, namely

$$D|_U = \sum_{x_i \in U} a_i x_i.$$

Definition 4.3.7. Let D be a Weil divisor on a monoid scheme X. Then D is called *Cartier* if it is *locally principal*, meaning that there is an open cover $\{U_i\}_{i\in I}$ of X such that $D|_{U_i}$ is principal in U_i for all $i \in I$. Thus, if $D|_{U_i} = div(a_i)|_{U_i}$ for $i \in I$ and $a \in A_0^{\times}$, then we call $\{U_i, a_i\}_{i\in I}$ the *local data* of D. In fact, sometimes we write $D = \{U_i, a_i\}_{i\in I}$.

Note that the set of Cartier divisors form a subgroup of $Div(X)$. We denote by $CDiv(X)$ this subgroup. Likewise note that a principal divisor is locally principal, i.e. $div(a)$ is Cartier for all $a \in A_0^{\times}$. Hence

$$Div_0(X) \subset CDiv(X) \subset Div(X).$$

Definition 4.3.8. We define the Picard group of X as the quotient group

$$Pic(X) = CDiv(X)/Div_0(X).$$

Proposition 4.3.9. *Let X be a monoid scheme. Then, there are exact sequences*

$$1 \longrightarrow \mathcal{O}_X^{\times}(X) \longrightarrow A_0^{\times} \xrightarrow{div} Div(X) \longrightarrow Cl(X) \longrightarrow 0$$

And

$$1 \longrightarrow \mathcal{O}_X^{\times}(X) \longrightarrow A_0^{\times} \xrightarrow{div} CDiv(X) \longrightarrow Pic(X) \longrightarrow 0$$

Where the map $\mathcal{O}_X^{\times}(X) \to A_0^{\times}$ is the inclusion, and $Div(X) \to Cl(X)$ sends each element to its class. Likewise in the second exact sequence the map $CDiv(X) \to Pic(X)$ sends each element to its class.

Proof. It follows directly from Proposition 4.3.5. ∎

Remark 4.3.10. *By Remark 4.2.13 we know that normal toric varieties arises from fans. Moreover, if X_Δ is a toric variety, by Remark 4.2.6, we also know that height one points in X_Δ corresponds to the elements in $\Delta(1)$. Thus, the divisors consider here correspond to the so called* toric divisors *defined in [7, Chapter 4].*

Remark 4.3.11. *Let M be the character lattice. We know that the cone $\{0\}$ corresponds to the generic point η of a toric variety X_Δ. Hence we obtain the generic monoid*

$$\mathcal{O}_{X_\Delta,\eta} \cong \mathbb{F}_1[M].$$

Therefore, by Remark 4.3.2, we obtain a valuation map

$$v_\rho : \mathbb{F}_1[M]^{\times} \longrightarrow \mathbb{Z}.$$

Where $\rho \in \Delta(1)$ represents a codimension one point $x_\rho \in X_\Delta$. Notice that a monomial described by a character is contained in the generic monoid, for instance let $\chi^m = T^m$ for some $m \in M$ as in Example 4.1.8, then $T^m \in \mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}]$.

Proposition 4.3.12. *Let X_Δ be a toric variety associated to a fan Δ, and let $m \in M \cong \mathbb{Z}^n$, then, for $\rho \in \Delta(1)$*

$$v_\rho(\chi^m) = \langle m, u_\rho \rangle.$$

Proof. The element $u_\rho \in N$ is a minimal generator of $\rho \in \Delta(1)$ and thus primitive. Then we extend it to a basis $u_\rho = e_1, e_2, \cdots, e_n$ of N. We may assume that $N = \mathbb{Z}^n$. Hence $\rho = cone(e_1) \subset \mathbb{R}^n$. Then we restrict to the affine open subset $U_\sigma \cong Spec(\mathbb{F}[T_1, T_2^\pm, \cdots, T_n^\pm])$. Therefore we obtain a DV monoid

$$\mathcal{O}_{U,x_\rho} \cong \mathbb{F}_1[T_1, \cdots, T_n]_{\langle T_1 \rangle}.$$

The valuation of an element $t \in \mathbb{F}[T_1^\pm, T_2^\pm, \cdots, T_n^\pm]$ is defined by

$$v_\rho(t) = l \in \mathbb{Z} \quad \text{when} \quad t = \left(T_1^l\right)\left(\frac{a}{b}\right).$$

with $a, b \in \mathbb{F}[T_1^\pm, T_2^\pm, \cdots, T_n^\pm] \backslash \langle T_1 \rangle$, and finally note that

$$\begin{aligned} \chi^m &= T^m \\ &= T_1^{\langle m, e_1 \rangle} \cdots T_1^{\langle m, e_n \rangle} \\ &= T_1^{\langle m, u_\rho \rangle} \cdots T_1^{\langle m, e_n \rangle}. \end{aligned}$$

where the first expression of T^m is given by the dual basis $e_1, \cdots, e_n$ of M. Therefore, by the valuation we obtain $v_\rho(\chi^m) = \langle m, u_\rho \rangle$. ∎

Corollary 4.3.13. *The divisor associated to a character χ^m, with $m \in M$, is*

$$div(\chi^m) = \sum_{\rho \in \Delta(1)} \langle m, u_\rho \rangle x_\rho.$$

Now we compute the divisor class groups of some toric varieties.

Example 4.3.14. *(cf. [14, Example 4.2.1]). Consider the toric variety $X_\sigma = (\mathbb{F}_1[S_\sigma])$ of Example 4.2.2, where $\sigma(1) = \{\rho_1 = cone(e_2), \rho_2 = cone(2e_1 - e_2)\}$, thus $u_1 = e_2$ and $u_2 = 2e_1 - e_2$. Then, since the group completion of S_σ is $\mathbb{Z}^2$, using the first exact sequence of Proposition 4.3.9 we compute the divisor class group $Cl(X_\sigma)$ as follows:*

$$0 \sim div(\chi^{e_1}) = \langle e_1, u_1 \rangle x_{\rho_1} + \langle e_1, u_2 \rangle x_{\rho_2} = 2x_{\rho_2}.$$

$$0 \sim div(\chi^{e_2}) = \langle e_2, u_1 \rangle x_{\rho_1} + \langle e_2, u_2 \rangle x_{\rho_1} = x_{\rho_2} - x_{\rho_1}.$$

Where $a \sim b$ means that a and b belongs to the same class in $Cl(X_\sigma)$. Hence, by Corollary 4.3.13 we obtain the divisor class group

$$Cl(X_\sigma) \cong \mathbb{Z}/2\mathbb{Z}.$$

4.4 Cox algebra for toric varieties

In this section we denote by $X = X_\Delta$ a normal toric variety associated to a fan Δ such that the elements in $\Delta(1)$ spans $N_\mathbb{R}$. The reason for restrict ourselves to working with this type of is due to the following proposition which is a classical result on toric varieties over $\mathbb{C}$, however, as it is pointed out in [17, Section 4.2] it does not depend of the field in which we work. Thus we refer to [7, Theorem 4.1.3] for a proof.

Proposition 4.4.1. *There is an exact sequence*

$$0 \longrightarrow M \xrightarrow{div} Div(X_\Delta) \longrightarrow Cl(X_\Delta) \longrightarrow 0$$

if and only if $\{u_\rho \mid \rho \in \Delta(1)\}$ *spans* $N_\mathbb{R}$*, where* M *is the character lattice, and the map* $Div(X_\Delta) \to Cl(X_\Delta)$ *sends each element to its class in* $Cl(X_\Delta)$*.*

By remark 4.3.12 we know that codimension one points in X_Δ corresponds to the one dimensional cones in the fan Δ. Now, notice that

$$Div(X_\Delta) = \bigoplus_{\rho \in \Delta(1)} \mathbb{Z} x_\rho.$$

To make reference of this, in what follows we denote the group $Div(X_\Delta)$ by $\mathbb{Z}^{\Delta(1)}$. Then following [17] we set

$$Cl(X_\Delta) = \mathbb{Z}^{\Delta(1)}/M.$$

In addition with the notation introduced above, when we write expressions such as $\sum_\rho$ and $\prod_\rho$, the index means that ρ ranges over all elements in $\Delta(1)$. Now we define the analogue in $\mathbb{F}_1$ of the *Cox ring* or *total coordinate ring* over $\mathbb{C}$ (see [7, Chapter 5]).

Definition 4.4.2. We define the *Cox algebra* as the $\mathbb{F}_1-$algebra over the set of one dimensional cones $\Delta(1)$:

$$Cox(X_\Delta) = \mathbb{F}_1[x_\rho \mid \rho \in \Delta(1)].$$

An important property of the Cox algebra is that it has a grading by the class group, which is obtained using the exact sequence of Proposition 4.4.1, where an element $a = (a_\rho) \in \mathbb{Z}^{\Sigma(1)}$ maps to its class group $[\sum_\rho a_\rho D_\rho] \in \mathbb{Z}^{\Delta(1)}/M$. A monomial in $Cox(X_\Delta)$ is of the form

$$x^a = \prod_\rho x_\rho^{a_\rho} \in Cox(X_\Delta).$$

Then we define the grading of x^a as follows:

$$deg(x^a) = \left[\sum_\rho a_\rho D_\rho\right] \in \mathbb{Z}^{\Delta(1)}/M.$$

Example 4.4.3. *Consider the fan Δ associated to the toric variety $\mathbb{P}^n_{\mathbb{F}_1}$. From Example 4.2.10 we know that $u_0 = -e_1 \cdots - e_n$ and $u_i = e_i$ for $i = 1, \cdots, n$ (see Figure 4.3), thus $|\Delta(1)| = n+1$, therefore, the map $M \to \mathbb{Z}^{\Delta(1)}$ can be seen as*

$$\begin{array}{rcl} f : \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^{n+1} \\ (a_1, \cdots, a_n) & \longmapsto & (-a_1 \cdots - a_n, a_1, \cdots, a_n). \end{array}$$

Furthermore, we also define the following map

$$\begin{array}{rcl} g : \mathbb{Z}^{n+1} & \longrightarrow & \mathbb{Z} \\ (b_0, \cdots, b_n) & \longmapsto & b_0 + \cdots + b_n. \end{array}$$

Using the maps defined above we obtain the next short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{f} \mathbb{Z}^{n+1} \xrightarrow{g} \mathbb{Z} \longrightarrow 0.$$

It follows that $Cl(\mathbb{P}^n_{\mathbb{F}_1}) \simeq \mathbb{Z}$. Then we can see that the Cox algebra is

$$Cox(\mathbb{P}^n_{\mathbb{F}_1}) = \mathbb{F}_1[x_0, \cdots, x_n].$$

And, from the last exact sequence by mapping $e_i \mapsto 1 \in \mathbb{Z}$ we obtain the grading of the Cox algebra, namely $deg(x_i) = 1$ for $i = 0, \cdots, n$.

Example 4.4.4. *Consider the toric variety associated to the fan Δ given by its maximal cones $= \sigma_1 = cone(e_2, -e_1)$ $\sigma_2 = cone(-e_1, 2e_1 - e_2), \sigma_3 = cone(2e_1 - e_2, e_2)$. Thus $|\Delta(1)| = 3$, and observe that $Cox(X_\Delta) = \mathbb{F}_1[x_1, x_2, x_3]$. Then, in order to obtain its grading we compute the class group.*

In the same manner as in Example 4.4.3 the map $M \to \mathbb{Z}^{\Delta(1)}$ can be seen as

$$\begin{array}{rcl} f : \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^3 \\ (a_1, a_2) & \longmapsto & (2a_1 - a_2, -a_1, a_2). \end{array}$$

And we define the map

$$\begin{array}{rcl} g : \mathbb{Z}^3 & \longrightarrow & \mathbb{Z} \\ (a_0, a_1, a_2) & \longmapsto & a_0 + 2a_1 + a_2. \end{array}$$

Then, from these maps we obtain the following exact sequence:

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^3 \xrightarrow{g} \mathbb{Z} \longrightarrow 0$$

Thus, we obtain $Cl(X_\Delta) \simeq \mathbb{Z}$. Then we obtain the grading on $Cox(X_\Delta)$ by mapping $e_i \mapsto 1 \in \mathbb{Z}$ we obtain the grading of the Cox algebra:

$$deg(x_1) = deg(x_3) = 1 \qquad and \qquad deg(x_2) = 2.$$

As it is explained in [7, Chapter 5], the reason to introduce the Cox ring was to write toric varieties (over $\mathbb{C}$) as categorical quotients imitating the case of the projective space. We will not delve into this, instead we refer to [17, Construction 4.2] to see the construction of normal toric varieties over $\mathbb{F}_1$ as a categorical quotients.

4.5 On toric curves from numerical monoids

In this section we give a first application of $\mathbb{F}_1$-geometry. It is based on [10] in which the authors studied the *Hilbert-Samuel multiplicity* at the origin of a toric curve over a field $\mathbb{K}$ using numerical monoids[2] and give bounds for the *regularity index* of the multiplicity in terms of the Frobenius number.

The goal is to generalize the results given in [10] to toric varieties over monoids (i.e. over $\mathbb{F}_1$), which in turn allows to generalize those results to toric varieties over rings using the functor given in Remark 3.3.1.

The first thing we will do is extend the notion of toric curves over a field $\mathbb{K}$ considered in *loc. cit.* to toric curves over $\mathbb{F}_1$ obtained from numerical monoids. Later we will explain how to count the minimum number of generators of a numerical monoid when possible and give some bounds otherwise. Once this is done, we will explain how to interpret this number in terms of multiplicity and regularity index.

Thus we start by describing the toric varieties that are obtained from numerical monoids.

The first example of such a monoid scheme was given in Example 4.1.16 in which we considered the numerical monoid $S = \langle 2, 3\rangle$ (or in multiplicative notation $\langle T^2, T^3\rangle$) which we can see is affine and give rise to the non normal toric variety $Spec(S)$ with torus $Spec(\mathbb{Z}_*) \cong G^1_{m,\mathbb{F}_1}$.

In general, let S be any numerical monoid with additive notation. By remark 2.1.45 we may assume that $A = \{a_1, \cdots, a_m\} \subset \mathbb{N}$ with $1 < a_1 < \cdots < a_m$ and $gcd(A) = 1$ is the unique minimal set of generators of S. It is easy to see that S is affine, and therefore, by Proposition 4.1.14, give rise to the toric variety $Spec(S)$ (or in multiplicative notation $Spec(\mathbb{F}_1[S]) = Spec(\mathbb{F}_1[T^{a_1}, \cdots, T^{a_m}])$). Then we have the following proposition.

Proposition 4.5.1. *$Spec(S)$ has the following properties:*

1. *It's a non normal toric variety with torus $Spec(\mathbb{Z}_*) \cong G^1_{m,\mathbb{F}_1}$ as an open subset, where $\mathbb{Z}_*$ is the stalk at the generic point $-\infty$.*
2. *The (Krull) dimension of S is 1.*

[2]Actually, the authors consider numerical semigroups, which are defined in the same way as in Definition 2.1.43 but they don't contain a basepoint.

Proof.

1. We have already seen that $Spec(S)$ is a toric variety. Now, notice that $a_2 - a_1 \in S^{nor}$ since $a_1(a_2-a_1) \in S$ but $a_2 - a_1 \notin S$. Hence $Spec(S)$ is non normal. Finally, notice that, since $gcd(A) = 1$, there are $t_1, \cdots, t_m \in \mathbb{Z}$ such that $\sum_{i=1}^{n} t_i a_i = 1$, and therefore $S_0 = \mathbb{Z}_*$, and the result follows by Proposition 4.1.14.

2. We show that the only prime ideals of S are $\langle -\infty \rangle$ and $\langle a_1, \cdots, a_m \rangle$. Indeed, let $a_{i_1}, \cdots, a_{i_n} \in A$ with $n < m$, and suppose that the ideal $I = \langle a_{i_1}, \cdots, a_{i_n} \rangle$ is prime, which means that $S \backslash I$ is additive closed, but $ma_j \in S$ for any $a_j \in A$ with $a_j \neq a_{i_1}, \cdots, a_{i_n}$ and for all $m \in I$, which contradicts I being a prime ideal. Hence the only prime ideals of S are $\langle -\infty \rangle$ and $\langle a_1, \cdots, a_m \rangle$, and the results follows since there is a chain $\langle -\infty \rangle \subsetneq \langle a_1, \cdots, a_m \rangle$.

   Notice that, with multiplicative notation, the unique chain of ideals looks like $\langle 0 \rangle \subsetneq \langle T^{a_1}, \cdots, T^{a_m} \rangle$.

∎

Remark 4.5.2. *By Proposition 4.5.1 we know a numerical monoid S has Krull dimension 1. This is why $Spec(S)$ is called a* toric curve*.*

Before continuing we need to introduce some vocabulary.

Let $A = \{a_1, \cdots, a_m\} \subset \mathbb{N}$ be as before. In the following we write the elements of A in a vector $a = (a_1, \cdots, a_m) \in \mathbb{N}^m$, and we denote by S^m the associated numerical monoid to make reference of the number of generators. Furthermore, by $|a|$ we mean $\sum_{i=1}^{m} a_i$, and if $b \in \mathbb{N}^m$ is another vector, we denote the usual dot product as $a \cdot b$.

We are interested in counting the minimum number of generators of the submonoids defined below:

$$S_n^m = \{0, b \cdot a \mid c, b \in \mathbb{N}^m, |b| = n, n \in \mathbb{N}_{\geq 1}\}_*.$$

Indeed, notice that each S_n^m is a submonoid of S^m, and they are related by

$$S^m = S_1^m \supset S_2^m \supset \cdots .$$

The minimum number of generators of S_n^m is denoted by $\lambda(n, m)$. Furthermore, the notation $S_n^m + a_1$ means $\{s + a_1 \mid s \in S_n^m\}_*$. Thus we obtain inclusions

$$S_n^m + a_1 \subset S_{n+1} \quad \text{for all} \quad n \geq 1.$$

In [10, Theorem 3.3] it is shown that $\lambda(n, m) = a_1$ for all $n \geq k$, for some big enough $k \in \mathbb{N}$. The smallest k such that the statement holds is the called the *regularity index* of the monoid and it is denoted by $ri(S^m)$.

Moreover, in [10, Proposition 2.1] it is shown that the simplest case to obtain an exact number for the regularity index is when $m = 2$. Indeed it is shown that $ri(S^2) =$

$a_1 - 1$, whereas for $m \geq 3$ the authors consider three cases to obtain bounds of $ri(S^m)$, namely $a_2 < F(S^m)$, $a_1 < F(S^m) < a_2$ or $F(S^m) < a_1$, where $F(S^m)$ is the Frobenius number of the monoid S^m. The bounds obtained are found in the theorem below, which summarizes [10, Theorem 3.3, Remark 3.6, Remark 3.7]. However, it should be clarified that the context in which we establish it is given in terms of monoids.

Theorem 4.5.3. *The minimum number of generators $\lambda(n, m)$ is equal to a_1 for all $n \geq k$ for some big enough $k \in \mathbb{N}$. Furthermore, if $m = 2$, the regularity index $ri(S^m)$ is equal to a_1, otherwise, if $m \geq 3$ there are three cases:*

1. *If $a_2 < F(S^m)$, let $\Delta = F(S^m) - a_1$ and let $D = a_2 - a_1$. Then, for some $L \in \mathbb{N}$, $\Delta = D + L$. In this case $ri(S^m) \leq \left\lfloor \frac{L}{D} \right\rfloor$.*
2. *If $a_1 < F(S^m) < a_2$, then $ri(S^m) = 1$.*
3. *If $F(S^m) < a_1$, then $ri(S^m) = 1$.*

Remark 4.5.4. *Let $n \in \mathbb{N}_{\geq 1}$ Consider the toric curve associated to S^m written in multiplicative notation, i.e.*

$$Spec(\mathbb{F}_1[S^m]) = Spec(\mathbb{F}_1[T^{a_1}, \cdots, T^{a_m}]).$$

Let's denote by I the ideal $\langle T^{a_1}, \cdots, T^{a_m} \rangle$. Notice that the minimal number of generators of S^m_n can be interpreted as the cardinality of the quotient I^n/I^{n+1}, i.e.

$$\lambda(n, m) = \# I^n/I^{n+1}.$$

Thus, the regularity index can be seen as the smallest $k \in \mathbb{N}$ such that the cardinality of $\# I^n/I^{n+1}$ stabilizes for $n \geq k$. Furthermore, notice that the last theorem states that $\# I^n/I^{n+1} = a_1$ when $n \geq ri(S^m)$.

Moreover, as we mentioned in Chapter 1, pointed sets can be seen as vector spaces over $\mathbb{F}_1$. In this context the cardinality $\# I^n/I^{n+1}$ equals the dimension as vector space of the quotient i.e. $\lambda(n, m) = dim_{\mathbb{F}_1}(I^n/I^{n+1})$. The Hilbert-Samuel multiplicity[3] *of the toric variety is defined as the dimension $dim_{\mathbb{F}_1}(I^n/I^{n+1})$ when $n \geq ri(S^m)$. Hence the Hilbert-Samuel multiplicity of $Spec(\mathbb{F}_1[S^m])$ is equal to a_1.*

Remark 4.5.5. *Notice that we can extend the results of above to the context of ring schemes by applying the base extension functor of Remark 3.3.1. Let $X = Spec(\mathbb{F}_1[S^m]) \times Spec(\mathbb{Z}) = Spec(\mathbb{Z}[T^{a_1}, \cdots, T^{a_m}])$. We also obtain an ideal of rings*

$$\mathfrak{m} = \langle T^{a_1}, \cdots, T^{a_m} \rangle \subset \mathbb{Z}[T^{a_1}, \cdots, T^{a_m}].$$

[3] A further discussion of the Hilbert-Samuel multiplicity can be found in [11, Chapter 12].

Also note that the dimension $\dim_{\mathbb{Z}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ *is equal to* $\lambda(n,m)$*, which means that this number only depends of the monoid* S^m*. Thus, in the same way as in the last remark, the Hilbert-Samuel multiplicity of* X *is defined as the number* $\lambda(n,m)$ *such that* $n \geq ri(S^m)$*, which, by Theorem 4.5.3 is equal to* a_1*.*

As we mentioned at the begining of this section, the observations made in both Remark 4.5.4 and Remark 4.5.5, about the multiplicity and the regularity index, were established in [10] for toric varieties over a field $\mathbb{K}$. In this way, by using the theory of monoid schemes, we have been able to generalize them to the context of monoids and subsequently to rings.

Chapter 5

The geometry of blueprints

The need for a larger theory than the monoid schemes is that the latter is not robust enough to has analogues to other types of varieties different from toric as we have highlighted in Remark 4.1.20. In this chapter we want to present a broader approach of $\mathbb{F}_1-$geometry which is based on the theory of blueprints and blue schemes introduced by Lorscheid in [25]. This approach has allowed to find applications in tropical geometry by setting tropical varieties in terms of blue schemes.

We begin by introducing semirings and giving some remarks about them. Thereafter we introduce the basic notions of blueprints, and, subsequently, we introduce their geometric counterpart, the blue schemes. After this we present the connections and applications of $\mathbb{F}_1-$geometry in terms of blue schemes to tropical geometry.

5.1 Introduction to blueprints and blue schemes

To motivate the need to expand the theory of monoid schemes to the blue schemes, consider, for instance, the special linear group $SL(2)$ as scheme over the integers

$$SL(2)_{\mathbb{Z}} = Spec(SL(2)) = Spec(\mathbb{Z}[T_1, T_2, T_3, T_4]/(T_1T_4 - T_2T_3 - 1)).$$

It makes no sense to place the ring scheme $SL(2)_{\mathbb{Z}}$ in terms of monoidal scheme since we cannot avoid the additive operation in $SL(2)_{\mathbb{Z}}$. However, as we shall see later, this variety has its analog over $\mathbb{F}_1$ in the context of blue schemes. In what follows we present the necessary elements of semirings to introduce blueprints and blue schemes.

Definition 5.1.1. A *semiring* is a triple $(R, +, \cdot)$ which satisfies the same axioms as a ring with the exception of having additive inverses for every element. However, in the rest of the chapter we only consider commutative semirings that contain zero and unitary elements. Thus, from now, by semiring we refer to commutative semiring with 0 and 1.

Notice that a triple $(R, +, \cdot)$ is a semiring if $(R, +)$ is a monoid without basepoint and unitary element 0, and $(R, \cdot)$ is a monoid with basepoint 0 and unitary element 1. A *homomorphism* of semirings R_1 and R_2 is a map $f : R_1 \to R_2$ that satisfies

$$f(x + y) = f(x) + f(y) \quad \text{and} \quad f(xy) = f(x)f(y)$$

and maps $\quad (0 \in R_1) \mapsto (0 \in R_2)$ and $(1 \in R_1) \mapsto (1 \in R_2)$.

Then, we denote by $SRings$ the category of commutative semirings with 1 and 0. The initial object in $SRing$ is the semiring of natural numbers $\mathbb{N}$ and the terminal object is the trivial semiring $\{0 = 1\}$. Thus $SRings$ can be seen as the category of $\mathbb{N}-$algebras.

Example 5.1.2. *Some semirings:*

1. *Basic examples of semirings are the non negative real numbers $\mathbb{R}_{\geq 0}$ and the polynomial semiring $R[T_1, \cdots, T_n]$ in n variables which consist in all polinomials with coefficients in a given semiring R where the addition and multiplication is the same as for polynomial rings.*

2. *The* monoid semiring *is defined as the set of finite formal sums of nonzero elements of a (multiplicative) monoid A:*

$$\mathbb{N}[A] = \left\{ \sum a_i \mid a_i \in A \backslash \{0\} \right\}.$$

The product of $\mathbb{N}[A]$ is inherited by the product of A. Moreover, note that the zero of A is identified with the zero of $\mathbb{N}[A]$ since $ab = 0$ in A implies that $1a \cdot 1b = 0$ equals the empty sum which is the neutral element of the monoid semiring.

3. *The* tropical semiring $\mathbb{T}$ *is the semiring whose elements are the non negative real numbers $\mathbb{R}_{\geq 0}$, its multiplication is the same of real numbers, but addition is defined as the maximum between two elements i.e.*

$$x + y = max\{x, y\} \quad \textit{for all} \quad x, y \in \mathbb{T}.$$

Remark 5.1.3. *Typically, the tropical semiring is described in terms of max-plus algebra with underlying set $\mathbb{R} \cup \{-\infty\}$, or in terms of min-plus algebra with underlying set $\mathbb{R} \cup \{\infty\}$, however, all the three conventions are equivalent, which means that the semirings $\mathbb{T}$, $\mathbb{R} \cup \{-\infty\}$ and $\mathbb{R} \cup \{\infty\}$, are isomorphic. For the applications to tropical geometry that we will present, the tropical semiring $\mathbb{T}$ described in the last example is the model that we will use unless otherwise stated.*

We can extend semirings to rings as follows: Given a semiring R, we can extend it to the ring $R_{\mathbb{Z}}$, which is defined as the set of formal differences of elements of the semiring i.e. $R_{\mathbb{Z}} = R \times R/\sim$ where the equivalence given by $(x, y) \sim (x', y')$ if and only if there is $z \in R$ such that $x + y' + z = x' + y + z$. The class $(x, y) \in R_{\mathbb{Z}}$ is denoted by $x - y$. Moreover, the addition and multiplication of $R_{\mathbb{Z}}$ is inherited from R as follows:

$$(x-y)+(z-t) \quad = \quad (x+z)-(y+t)$$

and

$$(x-y)\cdot(z-t) \quad = \quad (xz+yt)-(xt+yz).$$

The extension of semirings to rings is also called the base change from $\mathbb{N}-$algebras to $\mathbb{Z}-$algebras.

Example 5.1.4. *Some examples of semirings extension:*

- $\mathbb{N}_{\mathbb{Z}} = \mathbb{Z}$
- $(\mathbb{R}_{\geq 0})_{\mathbb{Z}} = \mathbb{R}$
- $R[T_1, \cdots, T_n]_{\mathbb{Z}} = R_{\mathbb{Z}}[T_1, \cdots, T_n]$ *for a given semiring R.*
- $\mathbb{T}_{\mathbb{Z}} = \{0\}$.

Definition 5.1.5. An ideal of a semiring R is a subset I such that $0 \in R$ and $x+y, tx \in R$ for all $x, y \in I$ and $t \in R$. We also have an extension of the concept of congruence in a semiring as for monoids (Definition 2.1.16): Let R be a semiring. We define a *congruence* on R as a multiplicative and additive equivalence relation $\mathfrak{R}$ which means that for $(x, y), (r, t) \in \mathfrak{R}$ then

$$(x, y) + (r, t) \in \mathfrak{R} \quad \text{and} \quad (x, y) \cdot (r, t) = (xr, yt) \in \mathfrak{R}.$$

If $(x, y) \in \mathfrak{R}$ we write $x \equiv y$.

Let $f : R_1 \to R_2$ be a morphism of semirings. Then, the concepts of kernel and congruence kernel has the same definition that for monoids i.e. the congruence kernel is the relation $\mathfrak{R}$ on R given by $\{(x, y) \in R \times R \mid f(x) = f(y)\}$ and the kernel is the ideal $ker(f) = \{x \in R_1 \mid f(x) = 0\}$.

Remark 5.1.6. *Let R be a semiring, and let $S \subset R \times R$ be a subset. In the same way as for monoids (Proposition 2.1.23), there is a smallest congruence containing S, namely $\langle S \rangle$.*

Definition 5.1.7. As in the case of a congruence on a monoid, the reason to define a congruence $\mathfrak{R}$ on a semiring R is that quotient $R/\mathfrak{R}$ is a semiring (see Proposition

2.1.17). However, in contrast with rings, it is not true that every congruence comes from an ideal nor that every ideal comes from a congruence. This lead us to focus on ideals that occurs as the kernel of a semiring morphism which are called *k-ideals* (where k is the abbreviation of kernel). Thus, in a ring, every ideal is a $k-$ideal.

Now, we are ready to define blueprints:

Definition 5.1.8. A *blueprint* is a pair $B = (B^\bullet, B^+)$ where B^+ is a semiring and $B^\bullet$ is a multiplicative subset of B^+ that contains 0 and 1 and generates B^+ as a semiring. Notice that $B^\bullet \in \mathcal{M}_*$. We call $B^\bullet$ the underlying monoid, and B^+ the ambient semiring of the blueprint.

A *morphism* of blueprints $f : B_1 \to B_2$ is a monoid morphism $f^\bullet : B_1^\bullet \to B_2^\bullet$ that extends to a semiring morphism $f^+ : B_1^+ \to B_2^+$. Note that f^+ is uniquely determined by $f^\bullet$ since $B_1^\bullet$ generates B^+ as a semiring.

We say that a is an element of B if $a \in B^\bullet$. We denote the category of blueprints by *Blpr*. The initial object in *Blpr* is $(\{0, 1\}, \mathbb{N})$ called the blueprint $\mathbb{F}_1$ and the terminal object is $(\{0\}, \{0 = 1\})$ called the trivial blueprint.

Remark 5.1.9. *The condition that $B^\bullet$ generates B^+ as a semiring is equivalent to say that B^+ is a quotient of the monoid semiring $\mathbb{N}[B^\bullet]$ by a congruence $\mathfrak{R}$ defined on it. Thus sometimes a blueprint $B = (B^\bullet, B^+)$ is denoted by $B^\bullet // \mathfrak{R}$ since $B^+ = \mathbb{N}[B^\bullet]/\mathfrak{R}$*

Remark 5.1.10. *A monoid A is associated with the blueprint $(A, \mathbb{N}[A])$. Notice that we can recover the base extension functor of A (Definition 2.2.11) by the extension of the semiring $\mathbb{N}[A]$ given by*

$$A \otimes_{\mathbb{F}_1} \mathbb{Z} = (\mathbb{N}[A])_{\mathbb{Z}}.$$

Likewise, a semiring R is associated with the blueprint (R, R). Thus, a blueprint is a simultaneous generalization of monoids and semirings. Notice that, by definition of blueprint, we obtain functors from the category of blueprints to both categories SRings and $\mathcal{M}_*$:

$$\begin{array}{rcl} (-)^+ : Blpr & \longrightarrow & SRings \\ & \textit{and} & \\ (-)^\bullet : Blpr & \longrightarrow & \mathcal{M}_*. \end{array}$$

As it is explained in [23] and [24] there are different generalization to the notion of ideals in blueprints which are used to different applications.

Definition 5.1.11. Let B be a blueprint.

1. An $m-$ideal of B is an ideal I of the underlying monoid $B^\bullet$.

2. A $k-$ideal of B is an $m-$ideal I of B such that for all $a_1, \cdots, a_n, b_1, \cdots, b_m \in I$ and $c \in B^{\bullet}$, an equality like the following

$$\sum_i a_i + c = \sum_j b_j \quad \text{in} \quad B^+,$$

implies $c \in I$.

3. An $(m/k)-$ideal P is prime if $S = B \backslash P$ is a multiplicative closed subset.

The term $m-$ideal is an abbreviation of monoid ideal, and the term $k-$ideal comes from the notion of $k-$ideal of semiring (Definition 5.1.7). Indeed, the next theorem, whose proof can be found in [22, Proposition 4.6.9] shows that the notion of $k-$ideal of blueprints is the same as the notion of $k-$ideal of semirings.

Proposition 5.1.12 (Lorscheid, [25]). *Let $B = (B^{\bullet}, B^+)$ be a blueprint. A subset I of B is an ideal if and only if there is a blueprint morphism $f : B \to C$ such that $I = f^{-1}(0)$. In fact, let I be a $k-$ideal of B and let $\mathfrak{R}(I)$ be the congruence on B^+ generated by $\{(a, 0) \mid a \in I\}$. Then I is the kernel of the morphism*

$$f_0 : B \longrightarrow B//\mathfrak{R}(I).$$

The notion of *spectrum of a blueprint* $Spec(B)$ is more sublte than that for monoids since, as we have seen above, there are different generalizations to the concept of ideals in blueprints, which lead to different affine schemes. Here, we are only going to introduce the basics of such schemes and refer to [24] to delve into the subject.

The notion of prime $k-$ideal yields to the notion of *affine blue schemes* suitable for the theory of algebraic groups (see [21] for details). In this direction $Spec(B)$ is the set of all prime $k-$ideals together with the topology generated by the principal open subsets $D(h) = \{P \in Spec(B) \mid h \notin P\}$ where $h \in B^{\bullet}$, together with an structure sheaf $\mathcal{O}_{Spec(B)}$. We do not go into details of these spaces and refer the reader to [25] for further discussion on them. Thus, in the following we just concentrate on the description of $Spec(B)$ as a topological space.

Example 5.1.13 (Special linear group over $\mathbb{F}_1$). *At the begining of this section we discussed the case the special linear group $SL(2)_{\mathbb{Z}}$. Here we present its analogue over $\mathbb{F}_1$. Consider the following blueprint*

$$\mathbb{F}_1(SL(2)) = \mathbb{F}_1[T_1, T_2, T_3, T_4]//\langle T_1T_4 \equiv T_2T_3 + 1\rangle.$$

We want to describe the topological space

$$SL(2)_{\mathbb{F}_1} = Spec(\mathbb{F}_1(SL(2))) \subset Spec(\mathbb{F}_1[T_1, T_2, T_3, T_4]) = \mathbb{A}^4_{\mathbb{F}_1}.$$

Therefore we need to test which elements of $\mathbb{A}^4_{\mathbb{F}_1}$ *(prime ideals of* $\mathbb{F}_1[T_1, T_2, T_3, T_4]$*), satisfies the last condition in the definition of* $k-$*ideal and satisfies the congruence of the quotient.*

The relation $T_1T_4 \equiv T_2T_3 + 1$ *implies that if both* T_1T_4 *and* T_2T_3 *are in the same prime ideal* P*, then, by the last condition of* $k-$*ideal,* $1 \in P$*. Thus either* T_1T_4 *or* T_2T_3 *are not in the same prime ideal, i.e. either* T_1 *and* T_4 *are not* P *or* T_2 *and* T_4 *are not in* P*.*

Hence the elements of $Spec(\mathbb{F}_1(SL(2)))$ *are* $\{0\}, \langle T_1\rangle, \langle T_2\rangle, \langle T_3\rangle, \langle T_4\rangle, \langle T_1, T_4\rangle, \langle T_2, T_3\rangle$*. We depict the space below.*

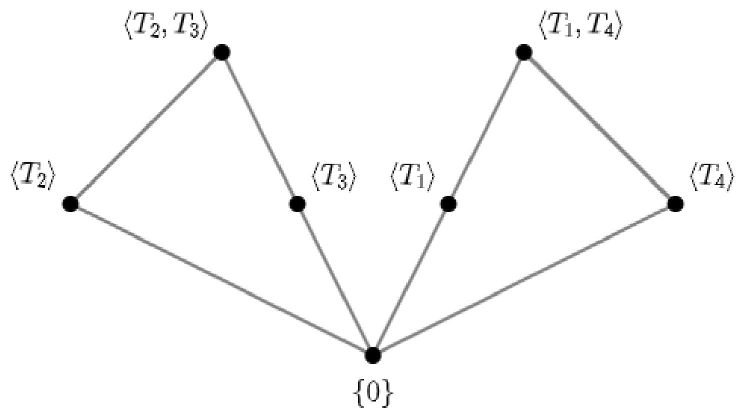

Figure 5.1: Spectrum of $\mathbb{F}_1(SL(2))$

Notice that the extension $\mathbb{F}_1(SL(2))^+_{\mathbb{Z}}$ *give us the coordinate ring of* $SL(2)_{\mathbb{Z}}$*.*

5.2 Ordered blueprints and scheme tropicalization

Most important for the purposes of this work will be to consider the case of the spectrum of prime $m-$ideals of blueprints. However, before introducing these ideas, we motivate them by presenting another recent application of $\mathbb{F}_1-$geometry to tropical geometry for which we present the basic notions that allow us to establish the connection between these areas. A general reference for tropical geometry is [28].

Definition 5.2.1. Let R be a ring. A *nonarchimedean seminorm* on R is a map $v : R \to \mathbb{R}_{\geq 0}$ such that

1. $v(0) = 0$ and $v(1) = 1$
2. $v(ab) = v(a)v(b)$
3. $v(a + b) \leq max\{v(a), v(b)\}$

When R is a field $\mathbb{K}$ we define a nontrivial *nonarchimedean absolute value* on it as a function

$$v : \mathbb{K} \longrightarrow \mathbb{R}_{\geq 0}$$

such that for all $a, b, \in \mathbb{K}$ satisfies 1) to 3) properties of above, and that additionally satisfies both $v(a) = 0$ if and only if $a = 0$, and $\overline{v(\mathbb{K})} = \mathbb{R}_{\geq 0}$. The pair $(\mathbb{K}, v)$ is called a *nonarchimedean field.*

Definition 5.2.2. Consider the tropical semiring $\mathbb{R} \cup \{\infty\}$ with min-plus convention, as in Remark 5.1.3. A *tropical polynomial* is a finite linear combination of monomials in $n-$variables $x_1, \cdots, x_n$ with coefficients in $\mathbb{T}$, such that multiplication and addition is defined as in the tropical semiring:

$$P(x_1, \cdots, x_n) = \sum_{i=1}^{m} = a_i x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}.$$

Remark 5.2.3. *Notice that each tropical polynomial induces a function*

$$\begin{array}{rcl} P : \mathbb{T}^n & \longrightarrow & \mathbb{T} \\ (x_1, \cdots, x_n) & \longmapsto & min_{i=1}^{m} \left\{ a_i + \displaystyle\sum_{j=1}^{n} i_j x_j \right\}. \end{array}$$

Definition 5.2.4. Let $(\mathbb{K}, v)$ be a nonarchimedean field. Consider an algebraic variety $X = V(f_1, \cdots, f_r) \subset (\mathbb{K}^{\times})^n$ with $f_1, \cdots, f_r \in \mathbb{K}[x_1^{\pm}, \cdots, x_n^{\pm}]$. The *tropicalization map* is defined as follows:

$$trop : (\mathbb{K}^{\times})^n \xrightarrow{(v,\cdots,v)} \mathbb{R}^n_{\geq 0} \xrightarrow{(log,\cdots,log)} \mathbb{R}^n$$

The *tropicalization* of the variety X is defined as the topological closure of the image of X in $\mathbb{R}^n$, namely

$$X^{trop} = \overline{trop(X)}.$$

Definition 5.2.5. A *tropical variety* is an equidimensional and rational polyhedral complex Σ together with a weight function

$$m : \{\mathbf{P} \in \Sigma \mid dim(\mathbf{P}) = dim(\Sigma)\} \longrightarrow \mathbb{Z}_{\geq 0}$$

such that for every polyhedron $\mathbf{P} \in \Sigma$ with $dim(\mathbf{P}) = dim(\Sigma) - 1$, the polyhedra in Σ of maximal dimension containing $\mathbf{P}$ satisfy the balancing condition modulo the affine linear span of $\mathbf{P}$, that is

$$\sum_{\mathbf{P} \subsetneq \mathbf{Q}} m(\mathbf{Q}) u_{\overline{\mathbf{Q}}} = 0.$$

Where $\overline{\mathbf{P}}$ and $\overline{\mathbf{Q}}$ are the images of the respectively polyhedra modulo the affine linear span of $\mathbf{P}$, and, by $u_{\overline{\mathbf{Q}}}$ we denote the primitive vector of $\overline{\mathbf{Q}}$ at $\overline{\mathbf{P}}$.

The way we associate the tropicalization of an algebraic variety with the concept of tropical variety is given by the following theorem (cf. [28, Theorem 3.3.5]):

Theorem 5.2.6 (Structure theorem for tropicalizations)**.** *Let $\mathbb{K}$ be a nonarchimedean field, and let $X \subset (\mathbb{K}^{\times})^n$ an equidimensional algebraic variety. Then*

1. *$X^{trop} = |\Sigma|$ for a rational and equidimensional polyhedral complex Σ.*
2. *The algebraic variety X determines a weight function*

$$m : \{\boldsymbol{P} \in \Sigma \mid dim(\boldsymbol{P}) = dim(\Sigma)\} \longrightarrow \mathbb{Z}_{\geq 0}$$

such that (Σ, m) is a tropical variety.

However there are two problems with the concept of tropical variety that motivate the use of semiring schemes.

The first problem is that the polyhedral complex Σ with $|\Sigma| = X^{trop}$ is not determined by the algebraic variety X, which means that the tropicalization of a clasical algebraic variety is not a tropical variety.

The second problem is that different polynomials define the same functions: For instance, by remark 5.2.3, we can check that the following tropical polynomials $0 + x^2$ and $(0 + x)^2$ are different as polynomial expressions, but are the same as functions.

Then, applications of $\mathbb{F}_1-$geometry to tropical geometry consist in lift tropicalization to schemes in an appropriate way.

The search for a scheme theoretic formulation of tropical geometry formally begun by Jeffrey and Noah Giansiracura in [17], in which they realize the tropicalization of an algebraic variety X as the set of $\mathbb{T}-$rational points of a semiring scheme with a structure morphism to $Spec(\mathbb{T})$. Later, in [24], Lorscheid took a further step in this direction by placing tropicalization within a framework of *ordered blueprinted spaces*.

However, the idea that we will focus in this section is one of the latest insights into tropical scheme theory, which was developed in [27]. This approach is based on the *tropical hyperfield* and on the theory of ordered blueprinted spaces which we will meet below, although we only present the main ideas that will serve to the scheme theoretic tropicalization. We refer to [24, 27] for further discussion on these ideas.

Definition 5.2.7. We previously introduced blueprints. Now we enrich them by endowing them with a partial order. We define an *ordered blueprint* as a triple $B =$

$(B^\bullet, B^+, \leq)$, where the pair $(B^\bullet, B^+)$ is a blueprint and $\leq$ is a partial order on the ambient semiring B^+ such that is additive and multiplicative i.e.

$$x \leq y \quad \text{implies} \quad x+z \leq y+z \quad \text{and} \quad xz \leq yz \quad \text{for all} \quad x, y, z \in B^+.$$

A *morphism* of ordered blueprints $f : B_1 \to B_2$ is a blueprint morphism such that $f^+ : B_1^+ \to B_2^+$ is an order preserving semiring morphism. This defines the category of ordered blueprints denoted by $OBlpr$.

The inital object in $OBlpr$ is the ordered blueprint $(\{0,1\}, \mathbb{N}, =)$ which sometimes is called *the field with one element* (see for instance [27, Example 1.3]). The terminal object in $OBlpr$ is $(\{0\}, \{0\}, =)$.

Notice that any blueprint $(B^\bullet, B^+)$ is identified with the ordered blueprint $(B^\bullet, B^+, =)$.

Remark 5.2.8. *Let $B = (B^\bullet, B^+, \leq)$ be an ordered blueprint. We say that the partial order $\leq$ is generated by a set of relations $S = \{x_i \leq y_i\}$ on the ambient semiring if $\leq$ is the smallest preorder on B^+ that contains S and that is closed under multiplication and addition.*

Definition 5.2.9. Given two morphism $B \to C$ and $B \to D$ of ordered blueprints, we define their *tensor product* as the following ordered blueprint:

$$C \otimes_B D = ((C \otimes_B D)^\bullet, (C \otimes_B D)^+, \leq).$$

Where

1. $(C \otimes_B D)^+ = C^+ \otimes_{B^+} D^+$ is the tensor product of semirings[1].
2. $(C \otimes_B D)^\bullet$ is the set of all pure tensors $c \otimes d$ for $c \in C, d \in D$ of $(C \otimes_B D)^+$.
3. $\leq$ is the partial order generated by

$$\{x \otimes 1 \leq y \otimes 1 \mid x \leq y \in C^+\} \bigcup \{1 \otimes x \leq 1 \otimes y \mid x \leq y \in D^+\}.$$

Definition 5.2.10. Let B a blueprint, and let S be a multiplicative closed subset of B. We define the *localization of B at S* as the following ordered blueprint

$$S^{-1}B = S^{-1}B^\bullet // \mathfrak{R}_s$$

where $S^{-1}B^\bullet$ is the localization of the monoid at S, and

$$\mathfrak{R}_S = \left\langle \sum \frac{a_i}{1} \equiv \sum \frac{b_j}{1} \mid \sum a_i \equiv \sum b_j \text{ in } B \right\rangle.$$

[1]Tensor product of semirings is the same that for rings. See [22, Chapter 2] for details.

Let P be a prime $m-$ideal. The localization of B at P is $B_P = S^{-1}B$ where $S = B\backslash P$. Likewise if $S = \{h^n \mid b \in B, n \in \mathbb{N}\}$ we denote the localization at S as B_h.

Now we introduce the basic definitions of hyperrings. A further discussion on hyperrings can be found in [19]

Definition 5.2.11 (Hyperoperation)**.** Let H be a nonempty set. A *hyperoperation* on H is a map

$$\boxdot : H \times H \longrightarrow \mathcal{P}(H)^*,$$

where $\mathcal{P}(H)^*$ is the set of all non empty subsets of H. For $x, y, z \in H$ we define the following subsets of H:

$$(x \boxdot y) \boxdot z = \bigcup_{w \in x \boxdot y} w \boxdot z \quad \text{and} \quad x \boxdot (y \boxdot z) = \bigcup_{w \in y \boxdot z} x \boxdot w.$$

The hyperoperation of two non empty subsets $A, B \subset H$ is denoted by

$$A \boxdot B = \left\{ \bigcup (a \boxdot b) \mid a \in A, b \in B \right\}.$$

Definition 5.2.12 (Hypergroup)**.** A *commutative canonical hypergroup* is a pair $(H, \boxdot)$, where H is a nonempty set and $\boxdot$ is a hyperoperation defined on it such that

1. $x \boxdot y = y \boxdot x$, for all $x, y \in \mathcal{R}$.
2. $(x \boxdot y) \boxdot z = x \boxdot (y \boxdot z)$, for all $x, y, z \in H$.
3. There exists a neutral element $0 \in H$ such that $0 \boxdot x = \{x\} = x \boxdot 0$.
4. For all $x \in H$ there is an element $y \in H$ such that $0 \in x \boxdot y$. It is denoted by $-x$.
5. If $x \in y \boxdot z$ then $z \in x \boxdot (-y)$.

Definition 5.2.13 (Hyperring and hyperfield)**.** A *commutative hyperring* is a triple $(\mathcal{R}, \boxplus, \cdot)$ such that $\mathcal{R}$ is a non empty subset endowed with a multiplicative operation $\cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and an hyperoperation $\boxplus : \mathcal{R} \times \mathcal{R} \to \mathcal{P}(\mathcal{R})^*$ (called hyperaddition) such that

1. The pair $(\mathcal{R}, \cdot)$ is an element of $\mathcal{M}_*$.
2. The pair $(\mathcal{R}, \boxplus)$ is a commutative canonical hypergroup.
3. $x \cdot (y \boxplus z) = x \cdot y \boxplus x \cdot z$ and $(x \boxplus y) \cdot z = x \cdot z \boxplus y \cdot z$ for all $x, y, z \in \mathcal{R}$.

An *hyperfield* is an hyperring $(\mathcal{R}, \boxplus, \cdot)$ such that the pair $(\mathcal{R}^\times, \cdot)$ is a group.

Definition 5.2.14. The *tropical hyperfield* $\mathbf{T}$ has as underlying set the non negative real numbers $\mathbb{R}_{\geq 0}$, has same multiplication as $\mathbb{T}$, and has the *hyperaddition* defined as follows:

$$a \boxplus b = \begin{cases} \{max\{a,b\}\} & \text{if } a \neq b, \\ [0,a] & \text{if } a = b \end{cases}$$

In what follows we will see how to interpret the hyperfield $\mathbf{T}$ as an ordered blueprint, and observe why the use of $\mathbf{T}$ has advantages in contrast with the tropical semiring.

Definition 5.2.15. An *algebraic blueprint* $B = (B^\bullet, B^+, \leq)$ is an ordered blueprint whose partial order $\leq$ is trivial i.e. $x \leq y$ only if $x = y$. Thus, sometimes by an algebraic blueprint we simply refer to a blueprint.

The *algebraic core* of an ordered blueprint $B = (B^\bullet, B^+, \leq)$ replaces the partial order $\leq$ with the trivial partial order, that is, $B^{core} = (B^\bullet, B^+, =)$.

Remark 5.2.16. *Notice that, by Remark 5.1.10, a monoid A is associated with the blueprint $A^{alg} = (A, \mathbb{N}[A], =)$. Likewise a semiring R is associated with the blueprint $R^{alg} = (R, R, =)$.*

Definition 5.2.17. Let $B = (B^\bullet, B^+, \leq)$ be an ordered blueprint, and let A be a monoid. The *free ordered blue B-algebra in A* is the ordered blueprint $B[A] = (B^\bullet[A], B^+[A], \leq)$, where the ambient semiring is

$$B[A]^+ = \left\{ \sum_{a \in A} x_a a \mid x_a \in B^+ \quad \text{and} \quad x_a = 0 \quad \text{for almost all } a \right\}.$$

With addition defined componentwise, and multiplication given by the linear extension of the multiplication of A. The underlying monoid is

$$B[A]^\bullet = \left\{ ca \in B[A]^+ \mid c \in B^\bullet,\ a \in A \right\}.$$

Notice that A is a submonoid of $B[A]^\bullet$ since an element $a \in A$ can be written as $\sum c_b b$ with $c_a = 1$ and $c_b = 0$ for $a \neq b$. The partial order of $B[A]^+$ is generated by the relations

$$x1 \leq y1 \quad \text{with} \quad x, y \in B^+ \quad \text{whenever} \quad x \leq y \quad \text{in} \quad B^+.$$

Example 5.2.18. *Let $B = (B^\bullet, B^+, \leq)$ be a blueprint, and let $A = \mathbb{F}_1[T_1, \cdots, T_n]$. Then, the free ordered blue $B-$algebra in A is the ordered blueprint whose ambient semiring is the polynomial semiring $B^+[T_1, \cdots, T_n]$, and its underlying monoid consist of the following set of monomials $\{cT_1^{e_1} \cdots T_n^{e_n} \mid c \in B^\bullet, e_i \in \mathbb{N}\}$ plus a 0 element. This ordered blueprint is denoted by $= B[A] = B[T_1, \cdots, T_n]$.*

Likewise, if we consider the monoid $A = \mathbb{F}_1[T_1^{\pm}, \cdots, T_n^{\pm}]$ *we obtain the ordered blueprint* $B[A] = B[T_1^{\pm}, \cdots, T_n^{\pm}]$.

Definition 5.2.19. Consider a semiring R. Its multiplicative monoid is denoted by $R^{\bullet}$. We define its *associated monomial ordered blueprint* as the orderer blueprint

$$R^{mon} = (R^{\bullet}, (R^{\bullet})^{+}, \leq),$$

where the the partial order is generated by the (left) *monomial relations*, namely, $c \leq a + b$ for which $c = a + b$ in R.

One of the reasons to consider both, algebraic blueprints and associated monomial ordered blueprints of rings, is because, through these, we can obtain the realization of hyperrings as elements of $OBlpr$ as we will see in both, Remark 5.2.20, and Remark 5.2.21.

Remark 5.2.20. *Let* $\mathcal{R}$ *be an hyperring. Let* $\mathcal{R}^{\bullet}$ *be its underlying monoid. The realization of* $\mathcal{R}$ *as an element in* $OBlpr$ *is the ordered blueprint* $(\mathcal{R}^{\bullet}, (\mathcal{R}^{\bullet})^{+}, \leq)$ *where the partial order* $\leq$ *is generated by the monomial relations*

$$c \leq a + b \quad \textit{for which} \quad c \in a \boxplus b \quad \textit{in} \quad \mathcal{R}.$$

We say that the relation $c \in a \boxplus b$ *is monomial. (Notice that this relation is not symmetric).*

We have just framed hyperrings in the context of ordered blueprints, that is, in the context of $\mathbb{F}_1-$geometry. Next we are going to highlight some properties of hyperrings that show the advantage of working with $\mathbf{T}$ instead of $\mathbb{T}$.

Using the hyperaddition of $\mathbf{T}$ we obtain a notion of additive inverses, namely for every element $a \in \mathbf{T}$, there is a unique element $b \in$ such that $0 \in a \boxplus b$ which occurs when $a = b$. Moreover, this generalizes as follows

$$0 \in a_1 \boxplus a_2 \boxplus \cdots \boxplus a_n$$

if and only if the maximum occurs twice among the a_i's. This notion of additive inverses allows us to reformulate the corner locus of a tropical polynomial p as the set of points $x \in \mathbf{T}$ such that $0 \in p(x)$. But before to show this generalization let's rewrite this in terms of ordered blueprints by introducing the tropical hyperfield as an ordered blueprint.

Remark 5.2.21 (Tropical hyperfield as ordered blueprint)**.** *First note that the multiplicative monoid of the tropical hyperfield* $\boldsymbol{T}$ *is* $\mathbb{R}_{\geq 0}^{\bullet}$ *and consider its associated blueprint* $(\mathbb{R}_{\geq 0}^{\bullet})^{alg} = (\mathbb{R}_{\geq 0}^{\bullet}, (\mathbb{R}_{\geq 0}^{\bullet})^{+}, =)$. *The realization of* $\boldsymbol{T}$ *as an ordered blueprint is obtained* *from exchanging the relation $c \in a \boxplus b$ in $\boldsymbol{T}$ by $c \leq a + b$, which means that $\boldsymbol{T}$ can be seen as the ordered blueprint*

$$\boldsymbol{T} = (\mathbb{R}_{\geq 0})^{alg} // \langle c \leq a + b \mid c \in a \boxplus b \rangle.$$

In other words, $\boldsymbol{T}$ has underlying monoid $\boldsymbol{T}^\bullet = \mathbb{R}^\bullet_{\geq 0}$, ambient semiring $\boldsymbol{T}^+ = (\mathbb{R}^\bullet_{\geq 0})^+ = \mathbb{N}[\mathbb{R}^\bullet_{\geq 0}]$, and partial order defined by all relations $c \leq a+b$ for which the maximum occurs twice among a, b, c.

Proposition 5.2.22 (Lorscheid, [27]). *Let $a, b_1, \cdots, b_n \in \boldsymbol{T}$ for $n \geq 1$. Then $a \leq \sum_{j=1}^n b_j$ in $\boldsymbol{T}$ if and only if the maximum occurs twice among $a, b_1, \cdots, b_n$.*

Proof. Note that if $n = 1$ then $a \leq b_1$ if and only if $a = b_1$, and the case $n = 2$ the results follows from the definition of partial order. Now we proceed by induction on n.

Let $n \geq 2$, and suppose that $a \leq \sum_{j=1}^n b_j$. Notice that the generators of the partial order $\leq$ are given in terms of monomial relations with only two terms on the right side. Hence, there must be a partition of $\{1, \cdots, n\}$:

$$P = \left\{ J_i \mid i \in I, \quad \text{such that} \quad |I| < n \quad \text{and} \quad \bigcup_{i \in I} J_i = \{1, \cdots, n\} \right\}.$$

such that there is a relation $a \leq \sum_{i \in I} a_i$, and relations $a_i \leq \sum_{j \in J_i} b_j$ for every $i \in I$. Then, by the inductive hypothesis, the maximum among a and a_i for all $i \in I$ occurs twice, and for each $i \in I$ the maximum among a_i and b_j for all $j \in J_i$ occurs twice.

Without loss of generality, suppose that b_j is the maximum among $a, b_1, \cdots, b_n$ for some $i \in I$ and $j \in J_i$. Thus, if $b_j = b_k$ for some $k \in J_i$ with $k \neq j$ we are done, if not, by hypothesis $a_i = a_j$, either $a_i = a$ or $a_i = a_k$ for some $k \in I$ with $i \neq k$. In the first case $a = b_j$ and thus the maximum occurs twice. For the second case note that, by hypothesis, there must be $k' \in J_k$ such that $a_k = b_{k'}$, then $b_j = a_i = a_k = b_{k'}$. Hence, in any case the maximum among $a, b_1, \cdots, b_n$ occurs twice.

Now, suppose that the maximum among $a, b_1, \cdots, b_n$ occurs twice, and again we proceed by induction on n. Note that the cases $n = 1, 2$ are trivial. Then we consider the case when the maximum occurs among $b_1, \cdots, b_n$.

Without loss of generality we assume that $b_1 = b_2 \geq b_i$ for all $i = 3, \cdots, n$. Therefore, by inductive hypothesis on n we obtain the relations

$$a \leq \sum_{i=1}^{n-1} b_j \quad \text{and} \quad b_n \leq \sum_{i=1}^{n-1} b_i.$$

Hence, if $a \geq b_n$ it follows that

$$a \leq a + b_n \leq \sum_{i=1}^{n} b_i.$$

and if $a \leq b_n$ it follows that

$$a \leq b_n + b_n \leq \sum_{i=1}^{n} b_i.$$

Thus we are done since the maximum occurs twice among $b_1, \cdots, b_n$. Now, suppose that a equals the maximum among $b_1, \cdots, b_n$. Without loss of generality assume that $a = b_1$. Thus, by the inductive hypothesis, $a \leq \sum_{i=1}^{n-1} b_i$ and $a \leq a + b_n$. Hence

$$a \leq a + b_n \leq \sum_{i=1}^{n} b_i.$$

■

A second reason to use **T** is because absolute values of a nonarchimedean field will be considered as morphism. We explain this below.

Let $(\mathbb{K}, v)$ be a nonarchimedean field. Notice that property 3) of definition of nonarchimedean field is equivalent with the monomial relation $v(a+b) \in v(a) \boxplus v(b)$ which characterizes the hyperaddition defined in **T**.

Proposition 5.2.23. *Let R be a ring and $v : R \to \mathbb{R}_{\geq 0}$ a nonarchimedean seminorm. If $a = \sum_{i=1}^{n} b_i$ in R, then the maximum between $v(a), v(b_1), \cdots, v(b_n)$ occurs twice.*

Proof. Since $v(b_i + b_j) \leq max\{v(b_i), v(b_j)\}$, then inductively

$$v(a) \leq max\{v(b_1), \cdots, v(b_n)\}.$$

Thus the maximum lies in some $v(b_j)$. Without loss of generality assume that the maximum occurs at $j = 1$, then note that $v(-1) = 1$ since $(-1)^2 = 1$, therefore $v(-b_1) = v(b_1)$. Then note that

$$\begin{aligned} v(-b_1) &= v\left(-a + \sum_{i=2}^{n} b_i\right) \\ &\leq max\{v(-a), v(b_2), \cdots, v(b_n)\} \\ &\leq max\{v(a), v(b_1), \cdots, v(b_n)\}. \end{aligned}$$

Hence the maximum $v(b_j)$ occurs twice. ■

The following theorem shows that nonarchimedeam seminorms can be interpreted as morphisms of hyperrings in terms of ordered blueprints. This result is the key piece to the approach of the scheme theoretic tropicalization which cover the main results in this direction.

Theorem 5.2.24 (Lorscheid, [27]). *Let R be a ring. Consider its associated monomial ordered blueprint R^{mon}. Then, a morphism $\boldsymbol{v} : R^{mon} \to \boldsymbol{T}$ is by definition (see Definition 5.1.8) a monoid morphism of its underlying monoids $\boldsymbol{v}^{\bullet} : R \to \mathbb{R}_{\geq 0}$. Then, the association $\boldsymbol{v} \mapsto \boldsymbol{v}^{\bullet}$ defines a bijection:*

$$\begin{array}{ccc} Hom(R^{mon}, \boldsymbol{T}) & \longleftrightarrow & \{\textit{nonarchimedean seminorms on } R\}. \\ \boldsymbol{v} : R^{mon} \longrightarrow \boldsymbol{T} & \longmapsto & \boldsymbol{v}^{\bullet} : R \longrightarrow \mathbb{R}_{\geq 0}. \end{array}$$

Proof. Given $\mathbf{v} \in Hom(R^{mon}, \mathbf{T})$, we show that $\mathbf{v}^{\bullet} : R \to \mathbb{R}_{\geq 0}$ is a nonarchimedean seminorm:

Note that the properties $\mathbf{v}^{\bullet}(0) = 0$, $\mathbf{v}^{\bullet}(1) = 1$ and $\mathbf{v}^{\bullet}(ab) = \mathbf{v}^{\bullet}(a)\mathbf{v}^{\bullet}(b)$ are trivially verified since $\mathbf{v}^{\bullet}$ is a monoid morphism. Finally we just need to check that $\mathbf{v}^{\bullet}(a+b) \leq max\{\mathbf{v}^{\bullet}(a), \mathbf{v}^{\bullet}(b)\}$ for $a, b \in R$. To do this, let $c = a+b$, then, $c \leq a+b$ in R^{mon}, which means that $\mathbf{v}^{\bullet}(c) \leq \mathbf{v}^{\bullet}(a) + \mathbf{v}^{\bullet}(b)$ in $\mathbf{T}$, therefore, by Proposition 5.2.23 the maximum among $\mathbf{v}^{\bullet}(a), \mathbf{v}^{\bullet}(b), \mathbf{v}^{\bullet}(c)$ occurs twice, in particular

$$\mathbf{v}^{\bullet}(a+b) \leq max\{\mathbf{v}^{\bullet}(a), \mathbf{v}^{\bullet}(b)\}.$$

Hence $\mathbf{v}^{\bullet} : R \to \mathbb{R}_{\geq 0}$ is a nonarchimedean seminorm.

On the other hand, let $v : R \to \mathbb{R}_{\geq 0}$ be a nonarchimedean seminorm. Consider the monomial ordered blueprint R^{mon} whose underlying monoid is R and recall that $\mathbf{T}^{\bullet} = \mathbb{R}_{\geq 0}$, thus we write v as $\mathbf{v}^{\bullet}$. We show that $\mathbf{v}^{\bullet} : R \to \mathbb{R}_{\geq 0}$ is a morphism of ordered blueprints:

Note that $\mathbf{v}^{\bullet}$ is a monoid morphism. Then, since the ambient semiring of R^{mon} is $(R^{\bullet})^{+}$, this semiring is free as in Definition 5.2.17, and thus $\mathbf{v}^{\bullet}$ extends uniquely to a semiring morphism

$$\mathbf{v}^{+} : (R^{\bullet})^{+} \longrightarrow \mathbf{T}^{+} = (\mathbb{R}^{\bullet}_{\geq 0})^{+}.$$

Finally we just need to check that $\mathbf{v}^{+}$ is order preserving.

To verify that $\mathbf{v}^{+}$ is order preserving morphism, we will check that on generators $c \leq a+b$ of the partial order $\leq$ of R^{mon}. Indeed, the relation $a \leq b_1+b_2$ means that $a = b_1+b_2$ in R and then, by Proposition 5.2.22, the maximum among $\mathbf{v}^{+}(a), \mathbf{v}^{+}(b_1), \mathbf{v}^{+}(b_2)$ occurs twice, which means, by Proposition 5.2.23, that $\mathbf{v}^{+}$ is order preserving, i.e.

$$\mathbf{v}(a) \leq \mathbf{v}(b_1) + \mathbf{v}(b_2).$$

Hence $\mathbf{v} : R^{mon} \to \mathbf{T}$ is a morphism of ordered blueprints. ∎

Finally we introduce the geometric counterpart of blueprints, namely *ordered blueprinted spaces*.

Definition 5.2.25. The *spectrum of an ordered blueprint B* is the defined as $Spec(B) = \{\text{prime } m-\text{ideals of } B\}$ with the topology generated by the principal open subsets

$$D(h) = \{P \in Spec(B) \mid h \notin P\}$$

for all $h \in B^\bullet$. Note that as a topological space $Spec(B)$ equals $Spec(B^\bullet)$.

The structure sheaf of $X = Spec(B)$ is given by the principal open subsets, namely

$$\mathcal{O}_{Spec(B)}(D(h)) = B_h.$$

Definition 5.2.26. An *ordered blueprinted space* (OBlpr-space) is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of ordered blueprints on X. An *affine ordered blue scheme* is an OBlpr-space $(Spec(B), \mathcal{O}_{Spec(B)})$.

Remark 5.2.27. *Let $X = Spec(B)$ where $B \in OBlpr$. Thus let $(X, \mathcal{O}_X)$ be a ordered blueprinted space. In [22, Section 5.5] it is shown that $OBlpr$ is complete and cocomplete. Thus the colimit $colim_{x \in U}(\mathcal{O}_X(U))$ over all open neighborhoods U of x always exists. The stalk of $\mathcal{O}_X$ at a point $x \in X$ is defined as this colimit. Moreover, in a simmilar way as in Proposition 3.1.15 it can be shown that the stalk at a prime $m-$ideal B_p is isomorphic to B_P.*

Remark 5.2.28. *We restrict ourselves to the affine case since this presentation of ideas on $\mathbb{F}_1-$schemes is introductory and to make the general case would imply to do a more exhaustive development of the algebra of blueprints, although we present the main definitions in the affine case and the results to carry out the scheme tropicalization The general notion of ordered blue scheme can be found in [24].*

Definition 5.2.29. A *morphism* of OBlpr-spaces is a continuous map $f : X \to Y$ between the topological spaces together with a morphism $\varphi^\# : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ of sheaves such that satisfies that for every $x \in X$ and $y = \varphi(x)$, the induced morphisms $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ of stalks sends non-units to non-units.

Remark 5.2.30. *(cf. [27, Section 2.2]). In the same way as Proposition 3.1.18, one can show that for two ordered blueprints B and C there are a bijective correspondence*

$$\{\textit{morphisms } Spec(B) \to Spec(C)\} \longleftrightarrow \{\textit{morphisms } C \to B\}.$$

Definition 5.2.31. Let k be an ordered blueprint. An *affine ordered blue k-scheme* is an affine ordered blue scheme $X = Spec(B)$ together with a morphism $\pi : X \to Spec(k)$

called *the structure morphism.* A morphism between affine ordered blue $k-$schemes is a morphism $f : (X, \pi_X) \to (Y, \pi_Y)$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & k & \end{tikzcd}$$

Remark 5.2.32. *By Remark 5.2.15 notice that the theory developed for monoid schemes, and in particular, for toric varieties over $\mathbb{F}_1$ is fully embbeded in terms of blueprints and blue schemes. We show some examples of spaces that were covered in terms of monoid schemes.*

Example 5.2.33 (Affine space and algebrac tori)**.** *In the same way as in Chapter 3, but in the context of ordered blueprints we define the* n-dimensional affine space *over an ordered blueprint B as the affine ordered blue scheme $\mathbb{A}^n_B = Spec(B[T_1, \cdots, T_n])$, and the* n-dimensional torus *over B as $G^n_{m,B} = Spec(B[T_1^{\pm}, \cdots, T_n^{\pm}])$.*

Remark 5.2.34. *Notice that a $B-$linear morphism $f : B[T_1, \cdots, T_n] \to B$ corresponds to an $n-$tuple $(f(T_1), \cdots, f(T_n))$ in B^n. Thus there is a canonical bijection*

$$\mathbb{A}^n_B(B) = Hom_B(B[T_1, \cdots, T_n], B) \longrightarrow B^n.$$

Likewise in the case of the $n-$dimensional torus

$$G^n_{m,B}(B) = Hom_B(B[T_1^{\pm}, \cdots, T_n^{\pm}], B) \longrightarrow (B^{\times})^n.$$

This fact applies in general for affine ordered blue schemes. Thus we have the following definition.

Definition 5.2.35. Let $X = Spec(B)$ be an affine ordered blue $\mathbf{T}-$scheme. We define *the set of $\mathbf{T}-$rational points* as the set

$$X(\mathbf{T}) = Hom_{\mathbf{T}}(B, \mathbf{T})$$

of $\mathbf{T}-$linear morphism from B to $\mathbf{T}$.

Once the necessary concepts have been introduced we are finally able to mention the concept of scheme theoretic tropicalization from the context of ordered blueprints.

Definition 5.2.36. Let $(\mathbb{K}, v)$ be a nonarchimedean field, and let $\mathbb{K}^{mon}$ be its associated monomial ordered blueprint. By Theorem 5.2.24 we associate a morphism $\mathbf{v} : \mathbb{K}^{mon} \to \mathbf{T}$ to v. Now let $Y = Spec(B)$ be an affine ordered blue $\mathbb{K}^{mon}-$scheme with $\mathbb{K}^{mon} \to B$

the structure map. We define the *scheme theoretic tropicalization of Y along* $\mathbf{v}$ as the affine ordered blue $\mathbf{T}-$scheme

$$Trop_{\mathbf{v}}(Y) = Spec(B \otimes_{\mathbb{K}^{mon}} \mathbf{T}).$$

Remark 5.2.37. *In the same way is in Chapter 2, in [22, Section 5.5] it is shown that the tensor product satisfies the compatibilities*

$$\mathbb{K}^{mon}[T_1, \cdots, T_n] \otimes_{\mathbb{K}^{mon}} \boldsymbol{T} = \boldsymbol{T}[T_1, \cdots, T_n].$$

Moreover, from Remark 5.2.30 and Definition 5.2.31, notice that tropicalization can be thought of as the set of $\boldsymbol{T}-$rational points:

$$Trop_{\boldsymbol{v}}(Y)(\boldsymbol{T}) = Hom_{\boldsymbol{T}}(B, \boldsymbol{T}).$$

Example 5.2.38. *The scheme theoretic tropicalization of the affine $n-$space over $\mathbb{K}^{mon}$ is*

$$Trop_{\boldsymbol{v}}(\mathbb{A}^n_{\mathbb{K}^{mon}}) = \mathbb{A}^n_{\boldsymbol{T}}.$$

Likewise the scheme theoretic tropicalization of the $n-$dimensional torus over $\mathbb{K}^{mon}$ is

$$Trop_{\boldsymbol{v}}(G^n_{m,\mathbb{K}^{mon}}) = G^n_{m,\boldsymbol{T}}.$$

we conclude with an example that shows the tropicalization as the set of $\mathbf{T}-$rational points.

Example 5.2.39 (Tropical line)**.** *Consider the following ordered blueprint*

$$B = \boldsymbol{T}[T_1, T_2]//\langle 0 \le T_1 + T_2 + 1\rangle.$$

Let $X = Spec(B)$. Then, note that

$$X(B) = Hom_{\boldsymbol{T}}(B, \boldsymbol{T}) = \{(a_1, a_2) \in \boldsymbol{T}^2 \mid 0 \le a_1 + a_2 + 1\}.$$

Then, since $0 \le a_1 + a_2 + 1$, by Proposition 5.2.22 the maximum among $0, a_1, a_2, 1$ occurs twice. Note that $X(B)$ is the tropical line *(cf. [28, Section 1.3]). We depict the tropical line with coordinates (a_1, a_2) in $\boldsymbol{T}$ in the figure below.*

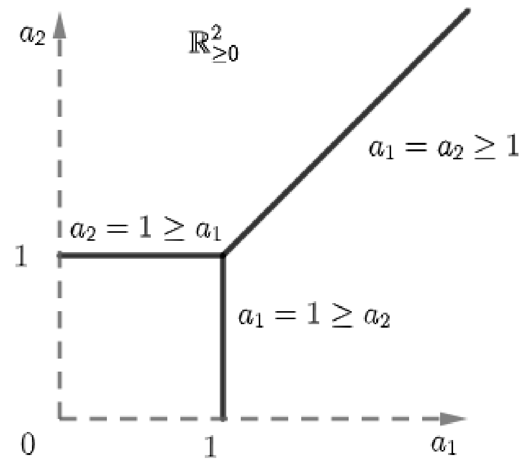

Figure 5.2: Tropical line.

Appendix A

Polyhedral geometry

In this section we introduce the basic definitions and results of polyhedral geometry which are used in Chapters 4 and 5 for toric varieties and tropical geometry respectively.

Definition A.0.1. A *polyhedron* $\mathbf{P} \subset \mathbb{R}^n$ is an intersection of finitely many halfspaces in $\mathbb{R}^n$ i.e. subsets of $\mathbb{R}^n$ of the form

$$H = \{(x_1, \cdots, x_n) \mid a_1x_1, \cdots, a_nc_n \geq b\}$$

for some $a_1, \cdots, a_n, b \in \mathbb{R}$. We say that the halfspace H is rational if $a_1, \cdots, a_n \in \mathbb{Q}$.

A *face of a polyhedron* $\mathbf{P}$ is determined by an element w of the dual space $(\mathbb{R}^n)^*$, denoted by

$$face_w(\mathbf{P}) = \{\alpha \in \mathbf{P} \mid w \cdot \alpha \geq w \cdot \beta \text{ for all } \beta \in \mathbf{P}\}.$$

A *polyhedral complex* is a finite collection Σ of polyhedra satisfying two conditions:

1. Every face of a polyhedron in Σ is in Σ
2. Let $\mathbf{P}, \mathbf{Q} \in \Sigma$, then $\mathbf{P} \cap \mathbf{Q}$ is either empty or a face of both $\mathbf{P}$ and $\mathbf{Q}$.

Definition A.0.2. Let Σ be a polyhedral complex. The support of Σ is

$$|\Sigma| = \bigcup_{\mathbf{P} \in \Sigma} \mathbf{P}.$$

The dimension of the polyhedral complex is $dim(\Sigma) = max\{dim(\mathbf{P}) \mid \mathbf{P} \in \Sigma\}$. Σ is *equidimensional* if

$$|\Sigma| = \bigcup_{dim(\mathbf{P}) = dim(\Sigma)} \mathbf{P}.$$

Finally we say that Σ is *rational* if every polyhedron $\mathbf{P}$ in Σ is the intersection of rational halfspaces.

Definition A.0.3. Let $A = \{v_1, \cdots, v_r\} \subset \mathbb{R}^n$ be a finite set. The *polyhedral cone σ generated by A* is defined as follows:

$$\sigma = cone(v_1, \cdots, v_r) = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^{r} \lambda_i v_i, \quad \lambda_i \in \mathbb{R}_{\geq 0} \right\}.$$

More generally, for the treatment of toric varieties in Chapter 4 we need the use of lattices, generalizing the spaces on $\mathbb{R}^n$.

Definition A.0.4. A *lattice* N is an abelian group isomorphic to $\mathbb{Z}^n$. The dual lattice is $M = Hom_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n$. The vector space $N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ is denoted by $N_{\mathbb{R}}$. The dual vector space $N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ is denoted by $M_{\mathbb{R}}$.

A polyhedral cone is called *convex* if it is closed under addition, and it is *strongly convex* if it does not contains a linear subspace different from $\{\vec{0}\}$. Also a cone σ is called *rational* or *lattice cone* if all of its generators belong to a lattice N. In this case $\sigma \subset N_{\mathbb{R}}$.

Definition A.0.5. Let $M_{\mathbb{R}}$ be the dual space to $N_{\mathbb{R}}$. We define the dual cone of σ as follows:

$$\check{\sigma} = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \quad \forall v \in \sigma\} \subset M_{\mathbb{R}}.$$

Proposition A.0.6 (Farkas' Theorem)**.** *Let $\sigma \subset N_{\mathbb{R}}$ be a convex polyhedral cone. Then $\check{\sigma}$ is convex polyhedral cone and $(\check{\sigma})^{\vee} = \sigma$.*

Proof. We refer to [13, Section 1.2] for a proof. ■

All the cones that we will consider from here will be polyhedral, rational and strongly convex unless otherwise stated.

Let σ be a cone, then, by Proposition A.0.6, its dual cone $\check{\sigma}$ is a convex polyhedral and rational (associated to the dual lattice M) cone. However, in general, the dual cone $\check{\sigma}$ will not be strongly convex. For instance $\sigma \subset \mathbb{R}^2$ is a cone generated by e_2, then $\check{\sigma}$ will be generated by e_1^* and $-e_1^*$, and clearly is not strongly convex since it contains the subspace $\mathbb{R}$.

Another way to describe cones by *supporting planes.* Suposse that $\{a_1, \cdots, a_r\}$ is a set of generators of $\check{\sigma}$, then each element of this set defines a half-spaces of $N_{\mathbb{R}}$ as follows:

$$H_{a_i} = \{v \in N_{\mathbb{R}} \mid \langle a_i, v \rangle \geq 0\}.$$

Therefore σ is the intersection of these half-spaces

$$\sigma = \bigcap_{i=1}^{r} H_{a_i}.$$

Proposition A.0.7. *Let σ be a cone generated by $\{a_1, \cdots, a_r\}$. Then*

$$\check{\sigma} = \bigcap_{i=1}^{r} \check{\tau}_i.$$

Where τ_i is the cone generated by a_i i.e. $\tau_i = cone(a_i)$

Definition A.0.8. A *face* of σ is an intersection of σ with a supporting plane H_λ for some $\lambda \in M$ and is denoted by $\tau = \sigma \cap H_\lambda$. Note that for $u \in \check{\sigma}$, $\sigma \cap u^\perp$ is a face of σ.

Proposition A.0.9. *Let $\tau = \sigma \cap \lambda^\perp$ be a face of σ, with $\lambda \in \check{\sigma}$, then*

$$\check{\tau} = \check{\sigma} + \mathbb{R}_{\geq 0}(-\lambda).$$

Proof. Since $\lambda \in \check{\sigma}$, it's easy to see that both $\check{\tau}$ and $\check{\sigma} + \mathbb{R}_{\geq 0}(-\lambda)$ are convex polyhedral cones, and to prove equality between them, it is enought to show it for their duals. In fact, left side is easy $(\check{\tau})^\vee = \tau$. And, for the other side we have:

$$(\check{\sigma} + \mathbb{R}_{\geq 0}(-\lambda))^\vee = \sigma \cap (-\lambda)^\vee = \sigma \cap \lambda^\perp = \tau.$$

The first equality holds since any element in right side must dot positively with left and conversely. To see the second equality take an element $v \in \sigma \cap (-\lambda)^\vee$, and then $\langle v, -\lambda\rangle \geq 0$, because $v \in (-\lambda)^\vee$, but also we have $\langle v, \lambda\rangle \geq 0$ since $v \in \sigma$, and $\lambda \in \check{\sigma}$, but that only happens if and only if $\langle v, \lambda\rangle = 0$ i.e. $v \in \sigma \cap \lambda^\perp = \tau$. ■

We can associate a monoid (without basepoint) to a cone σ, namely $S_\sigma = \check{\sigma} \cap M$, where M denotes the dual lattice. It can be multiplicatively written as $T^{\lambda+m} = T^\lambda T^m$ ($T^0 = 1$). Moreover, it can be shown that S_σ is finitely generated, and the fact arises as a consequence of the following lemma:

Lemma A.0.10 (Gordan's Lemma)**.** *Let σ be a cone, then $S_\sigma = \check{\sigma} \cap M$ is a finitely generated monoid. We call S_σ the associated monoid to σ*[1]

[1]Notice that S_σ is an unpointed monoid. By adding a basepoint we obtain an element of $\mathcal{M}_*$, namely $(S_\sigma)_*$ which is also called the associated monoid to σ.

Proof. Let $\{v_1, \cdots, v_r\} \subset M$ be a set of generators of $\check{\sigma}$, such that $v_i \in \check{\sigma} \cap M$ for all $i = 1, \cdots, r$. Then we define the set

$$K = \left\{ \sum_{i=1}^{r} t_i v_i \mid 0 \leq t_i \leq 1 \right\}.$$

Note that K is a closed and bounded subset of $M \cong \mathbb{R}^n$, i.e. K is compact. Thus, since M is discrete, it follows that $K \cap M$ is finite because M and K are closed. Then we show that $K \cap M$ generates S_σ. Let $v \in S_\sigma$ and $v = \sum_{i=1}^{r} a_i v_i$, where $a_i \geq 0 \ \forall i$, thus we write $a_i = z_i + t_i$, where $z_i \in \mathbb{Z}_{\geq 0}$ and $0 \leq t_i \leq 1$. Then we can write v as follows

$$v = \sum_{i=1}^{r} z_i v_i + \sum_{i=1}^{r} t_i v_i.$$

Note that first sum is contained in $K \cap M$ since $v_i \in K \cap M$ for all i. Then note that by definition, the second sum is in K, but also is in M since v and $\sum_{i=1}^{r} z_i v_i$ are. Hence we have shown that $K \cap M$ is a set of generators of S_σ since we have written an arbitrary v as a sum of elements of $K \cap M$. ■

Proposition A.0.11. *Let σ be a cone, and let $\tau = \sigma \cap \lambda^\perp$ be a face of σ (with $\lambda \in S_\sigma$), then*

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-\lambda).$$

Proof. Since $\check{\tau} = \check{\sigma} + \mathbb{R}_{\geq 0}(-\lambda)$, then by intersect with $M \cong \mathbb{Z}^n$ the result follows. ■

Definition A.0.12. A *fan* Δ in $N_\mathbb{R}$ is a non empty finite collection of cones such that:

1. All the cones contained in Δ are rational, polyhedral and strongly convex.
2. Let σ be a cone in Δ, and let τ be a face of σ. Then τ is a cone in σ.
3. Let σ and σ' be cones in Δ. Then $\sigma \cap \sigma'$ is a common face of σ and σ'.

Note that the trivial cone $\{0\} \subset N_\mathbb{R}$ is a common face of any other cone σ. Thus always belong to any fan Δ.

Example A.0.13. *Some examples of fans in $\mathbb{R}^2$:*

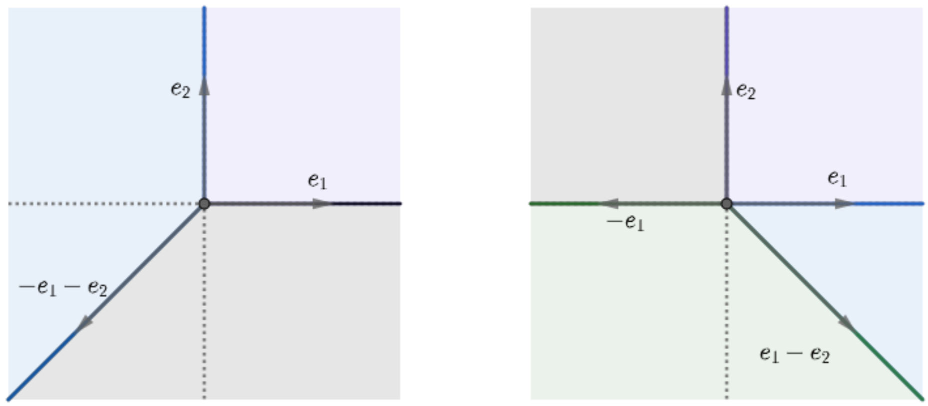

Figure A.1: Two fans in $\mathbb{R}^2$

Definition A.0.14. Let σ be a cone generated by the the finite set $A = \{v_1, \cdots, v_r\}$. The *dimension* of σ is the dimension of the smallest subspace $Span(A)$ of $N_{\mathbb{R}}$ containing σ, and it is denoted by $dim(\sigma)$.

Definition A.0.15. Let $\sigma \subset N_{\mathbb{R}}$ be a cone. A *facet* of σ is a face τ of codimension 1, i.e. $dim(\tau) = dim(\sigma) - 1$. An *edge* is a face of dimension 1. For instance, all the maximal cones contained in each of the fans of figure A.1 are of dimension 2, thus their edges are their facets.

An edge of a cone σ is usually denoted by ρ. Note that any edge ρ is a ray in $N_{\mathbb{R}}$ i.e. is a half line since σ is strongly convex. Moreover, since σ is rational, $\rho \cap N$ is an unnpointed monoid generated by a unique element $u_\rho \in \rho \cap N$, called the *ray generator* of ρ.

Thus, if σ is a cone, we denote by $\sigma(1)$ the set of all edges. Likewise, if Δ is a fan, we denote by $\Delta(r)$ the set of all $r-$dimensional cones of Δ, and by Δ_{max} the set of all maximal (dimensional) cones of Δ.

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