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“The topological complexity of polyhedral products of real projective spaces and topological groups”

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“La complejidad topológica de productos poliédricos de espacios proyectivos reales y grupos topológicos”

Tesis que presenta

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Contents

Abstract

In this thesis we compute the topological complexity of polyhedral products defined by two LS-logarithmic based families $\underline{P}$ and $\underline{G}$. The former is constructed out of real projective spaces, whereas the latter of locally compact connected CW topological groups. In the first case, the answer is given by a combinatorial formula that involves the LS category and the topological complexity of the polyhedral product factors. Such a mixed cat/$\mathbf{TC}$ phenomenon contrasts with the behavior noticed in the setting of right-angle artin groups. In the second case, the estimate is given by a combinatorial formula that involves the LS category of the polyhedral product factors. As a direct consequence, we show that the Iwase-Sakai conjecture holds true for both polyhedral products determined by the based families $\underline{P}$ and $\underline{G}$.

The proof methodology of such results involves a Fadell-Husseini flavored definition of monoidal topological complexity, which, under mild conditions, recovers the original definition given by Iwase and Sakai. Furthermore, such a new version of monoidal topological complexity represents an alternative to the slight variant given by Dranishnikov, as well as the ones provided by García-Calcines, Carrasquel-Vera, and Vandembroucq in terms of relative category.

Resumen

En esta tesis calculamos la complejidad topológica de productos poliédricos asociados a dos familias basadas y LS-logarítmicas $\underline{P}$ y $\underline{G}$. La primera está constituida por espacios proyectivos reales, mientras que la segunda por grupos topológicos conexos y localmente compactos con estructura de complejos celulares. En el primer caso, la respuesta está dada en términos de la categoría y la complejidad topológica de los factores poliédricos. Tal fenómeno mixto contrasta con el comportamiento que ocurre en el contexto de los grupos RAA (right-angle Artin groups). En el segundo caso, la respuesta está dada en términos de la categoría de los factores poliédricos. En particular, mostramos que la conjectura de Iwase y Sakai es válida para los productos poliédricos que determinan las familias basadas $\underline{P}$ y $\underline{G}$.

La prueba de dichos resultados involucra una versión tipo Fadell-Husseini de la complejidad topológica monoidal, la cual, bajo ciertas condiciones, recupera la definición original dada por Iwase y Sakai. Además, tal versión de complejidad topológica monoidal representa una alternativa a la ligera variante dada por Dranishnikov, y a la proporcionada por García-Calcines, Carrasquel-Vera y Vandembroucq en términos de categoría relativa.

1

Introduction

The topological complexity of a space X, denoted by $\mathbf{TC}(X)$, was introduced by M. Farber [19] as a way of measuring discontinuities of the process of motion planning in X, with X thought of as the configuration space of a mechanical system. In the simplest terms, $\mathbf{TC}(X)$ can be thought of as one less than the minimal number of motion planning rules required to specify the motion between any pair of initial-final configurations of the system.

In this thesis we analyze the motion planner problem in the context of polyhedral products, spaces obtained by assembling a family of based topological spaces via the combinatorial data coming from an abstract simplicial complex. More concretely, inspired by the detailed study in [23] of the higher topological complexities of polyhedral products of real dimensional spheres, we determine, in several cases, the topological complexity of any polyhedral product of real projective spaces $\underline{P}^K$; thus generalizing the equality

$$\mathbf{TC}(\mathbb{R}\mathrm{P}^n \vee \mathbb{R}\mathrm{P}^m) = \max\{\mathbf{TC}(\mathbb{R}\mathrm{P}^n), \mathbf{TC}(\mathbb{R}\mathrm{P}^m), n+m\}, \tag{1.1}$$

with $n, m \geq 1$, which follows from [15, Theorem 6] as a particular case.

As we will see in chapter 4, our estimate exhibits a mixed cat/$\mathbf{TC}$ phenomenon not present in either (1.1) or, more generally, in the setting of right-angled Artin groups (see [3] for more details). In the realm of higher topological complexities, we treat the case of polyhedral products whose factors are even dimensional real projective spaces. The main results of this chapter are included in the last section of [3], which has been accepted for publication.

On the other hand, in chapter 5 we also compute, under suitable conditions, the topological complexity of any polyhedral product whose factors are locally compact connected CW topological groups. We will point out that the answer is given by a combinatorial formula that involves the LS category of each polyhedral product factor. This result enables to provide a simple characterization of when a polyhedral product of this type admits an H-space structure.

The proof methodology of our main results involves a Fadell-Husseini version of monoidal topologial complexity ($\mathbf{TC}^{FH}$) together with its generalized counterpart ($\mathbf{TC}_g^{FH}$), which, roughly speaking, emerge naturally by imputing Dranishnikov's and García-

Calcines' points of view into Iwase-Sakai's original definition ($\mathbf{TC}^M$) (all these notions will be detailed in chapter 3).

We show that, if X is an ANR space,

$$\mathbf{TC}^{FH}(X) = \mathbf{TC}_g^{FH}(X) = \mathbf{TC}^{DM}(X) = \mathbf{TC}^{M}(X) = \mathbf{TC}_g^{M}(X), \tag{1.2}$$

where the last equality was proved by García-Calcines and the next-to-last equality is due to Dranishnikov. Namely, the former author shows that Iwase-Sakai's original definition of $\mathbf{TC}^M(X)$ can be given in terms of *arbitrary* (not necessarily open) covers of $X \times X$, whereas Dranishnikov claims that the definition of $\mathbf{TC}^M(X)$ can be relaxed in the sense that the diagonal of X does not need to be contained in each *open* domain covering $X \times X$. Therefore, our equality $\mathbf{TC}_g^{FH}(X) = \mathbf{TC}^M(X)$ shows that $\mathbf{TC}^M(X)$ also can be given in terms of *arbitrary* covers of $X \times X$ and that the diagonal of X does not need to be contained in each *arbitrary* domain covering $X \times X$. In this sense, we assemble Dranishnikov's and García-Calcines' points of view into Iwase-Sakai's original definition of $\mathbf{TC}^M(X)$.

As a by-product of the equalities (1.2), we show that the Iwase-Sakai conjecture (reviewed in chapter 3) holds true for the polyhedral products $\underline{P}^K$ and $\underline{G}^K$. Finally, this new approach to monodial topological complexity together with the main results of chapter 5 are included in [2], which has been accepted for publication.

2

Preliminaries

This chapter is devoted to reviewing basic definitions and results that will be needed in subsequent parts of the thesis. We start by defining the concepts of topological complexity, higher topological complexities, and LS category. We next introduce polyhedral products and provide some results on how to determine, under convenient hypotheses, the higher topological complexities of these spaces.

2.1 Basic definitions and results

We start by defining the sectional category of a fibration, of which topological complexity and LS category are special cases. As we shall see below, the former notion was introduced by Farber in order to pave the way for topological aspects of the motion planning problem in robotics. We refer the reader to [6, 19, 28] for further details.

Definition 2.1. The *(reduced) sectional category* of a fibration $p\colon E \to B$, denoted by $\mathbf{secat}(p)$, is defined as the smallest n for which there exists an open cover $\{U_0, \ldots, U_n\}$ of B by $n+1$ open sets, on each of which there is a continuous section $s_i\colon U_i \to E$ of p, that is, $p \circ s_i$ equals the inclusion $U_i \hookrightarrow B$. If such a n fails to exist, we set $\mathbf{secat}(p) = \infty$.

Definition 2.2. The *(reduced) topological complexity* of a path-connected space X, denoted by $\mathbf{TC}(X)$, is defined as the sectional category of the end-points evaluation map $e_2\colon X^{[0,1]} \to X \times X$. The sets U_i covering $X \times X$ and the corresponding local sections s_i of e_2 are called *local domains* and *local rules*, respectively, whereas the family $\{(U_i, s_i)\}$ is known as a *motion planner*.

The latter definition essentially captures, in a topological way, the motion planning problem discussed at the beginning of the introduction. Concretely, X can be thought of as the configuration space of a mechanical system and $X^{[0,1]}$ as the space of robot actions. With these ideas in mind, a continuous global section of the fibration e_2 takes a pair of configurations $(A, B) \in X \times X$ as an input and produces as an output a continuous path from the initial state A to the desired state B. Unfortunately, such a continuous global section exists if and only if our space X is contractible ([19, Theorem 1]), which is rather rare in practice. Farber's solution to this issue was to cover $X \times X$ with open sets and consider continuous local sections of e_2 instead of a global one. In this sense, $\mathbf{TC}(X)$ measures discontinuity of the process of motion planning in X because the sections on the overlap of two open sets are in general distinct.

On the other hand, if we need to plan the movement from A to B, passing through a certain number of intermediate states, the following definition gives a topological answer.

Definition 2.3. Let $r \in \mathbb{N}$ with $r \geq 2$. The *(reduced) rth topological complexity* of a path-connected space X, denoted by $\mathbf{TC}_r(X)$, is defined to be the sectional category of the fibration

$$\begin{array}{cccc} e_r\colon & X^{[0,1]} & \to & X^r \\ & \gamma & \mapsto & \left(\gamma\left(\frac{0}{r-1}\right), \gamma\left(\frac{1}{r-1}\right), \ldots, \gamma\left(\frac{r-1}{r-1}\right)\right). \end{array}$$

The collection $\{\mathbf{TC}_r(X)\}_{r\geq 2}$ is called the higher or sequential topological complexities of X.

Remark 2.4.

- $\mathbf{TC}_r(X)$, with $r \geq 2$, is a homotopy invariant of X.
- Notice that $\mathbf{TC}_2(X)$ is precisely $\mathbf{TC}(X)$.

Another homotopy invariant of X that, in some occasions, is easier to compute than $\mathbf{TC}(X)$ is its Lusternik-Schnirelmann category.

Definition 2.5. The *(reduced) Lusternik-Schnirelmann category* or simply the *(reduced) LS category* of X, denoted by $\mathrm{cat}(X)$, is defined as the sectional category of the fibration $e_1\colon PX \to X$, where PX denotes the space of based paths in X and e_1 takes a path to its end-point.

Remark 2.6. Observe that $\mathrm{cat}(X)$ agrees with the smallest n for which there exists an open cover $\{U_0, \ldots, U_n\}$ of X with each U_i being contractible *within* X.

In order to establish lower bounds for cat and $\mathbf{TC}_r$, we introduce the concepts of cup-length and rth zero-divisor cup-length. These notions make use of the cohomology ring of the space under consideration.

Definition 2.7. Let R be a commutative ring R with unit. The *cup-length* of X, denoted by $\mathrm{cup}_R(X)$, is the greatest integer n for which there exist n positive dimensional cohomology classes $\xi_i \in \tilde{H}^*(X; R)$ such that $\xi_1 \cdots \xi_n \neq 0$.

Definition 2.8. Let R be a commutative ring R with unit. The *rth zero-divisor cup-length* of X, denoted by $\mathrm{zcl}_r(X; R)$ for $r \geq 3$ and $\mathrm{zcl}(X; R)$ for $r = 2$, is the length of the longest nontrivial product in $\ker(\Delta_r^*\colon H^*(X^r; R) \to H^*(X; R))$, where $\Delta_r\colon X \to X^r$ is the r-fold iterated diagonal.

Bringing together the former and the latter definitions, the next theorem bounds cat and $\mathbf{TC}_r$ from below by homological methods, and from above by homotopical considerations.

Theorem 2.9. *[6, Theorem 3.9] For a path-connected space X having the homotopy type of a* CW *complex,*

$$\mathrm{cup}_R(X) \leq \mathrm{cat}(X) \leq \mathrm{hdim}(X) \quad \text{and} \quad \mathrm{zcl}_r(X;R) \leq \mathbf{TC}_r(X) \leq r \cdot \mathrm{cat}(X),$$

where $\mathrm{hdim}(X)$ *denotes the (cellular) homotopy dimension of X, i.e., the smallest dimension of* CW *complexes having the homotopy type of X.*

Example 2.10. The first three facts below can easily be proved by making use of the preceding theorem, while the proof of the last assertion can be found in [28]. In the last section, we will discuss the connection between $\mathbf{TC}(\mathbb{R}\mathrm{P}^n)$ and $\mathrm{Imm}(\mathbb{R}\mathrm{P}^n)$. Here the latter expression denotes the minimal integer k such that $\mathbb{R}\mathrm{P}^n$ admits a smooth immersion in $\mathbb{R}^k$.

1. $\mathrm{cat}(\mathbb{S}^n) = 1$ for any $n \geq 1$.
2. $\mathrm{cat}(\mathbb{R}\mathrm{P}^n) = n$ for any $n \geq 1$.
3. $\mathrm{cat}(S_g) = 2$ for any closed (orientable or not) surface S_g of genus $g \geq 1$.
4. For any $r \geq 2$, $\mathbf{TC}_r(\mathbb{S}^n) = \begin{cases} r-1, & \text{if } n \text{ is odd;} \\ r, & \text{if } n \text{ is even.} \end{cases}$

2.2 Polyhedral products

Definition 2.11. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be a family of pairs of spaces and K be an abstract simplicial complex with m vertices labeled by the set $\{1, \ldots, m\}$. For each $\sigma \in K$ set $(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^m Y_i$, where

$$Y_i = \begin{cases} A_i, & \text{if } i \in \{1, \ldots, m\} \setminus \sigma; \\ X_i, & \text{if } i \in \sigma. \end{cases}$$

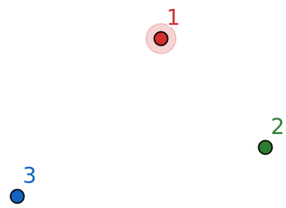
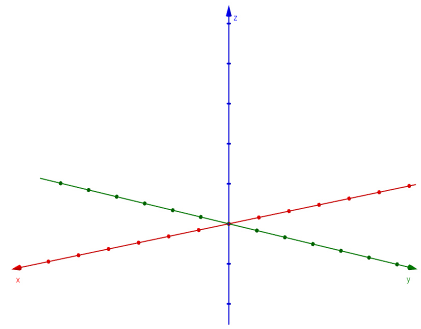
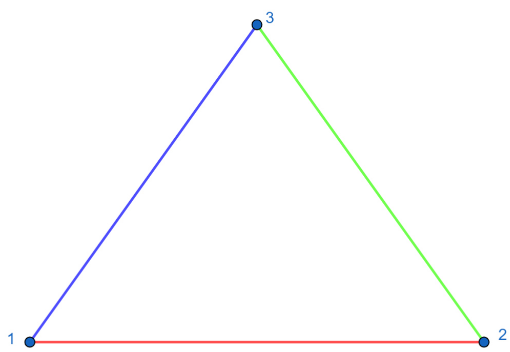
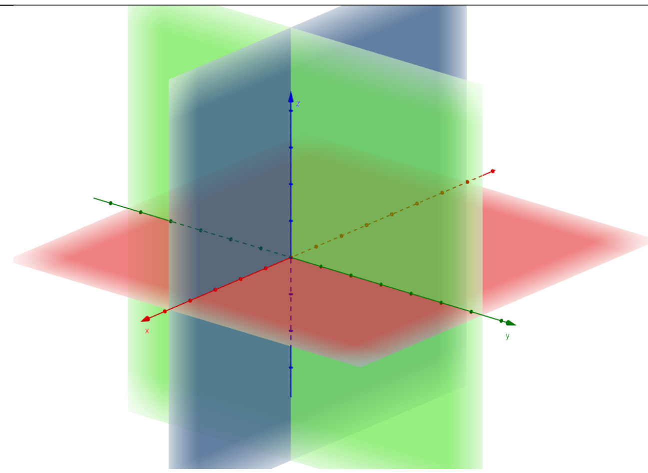
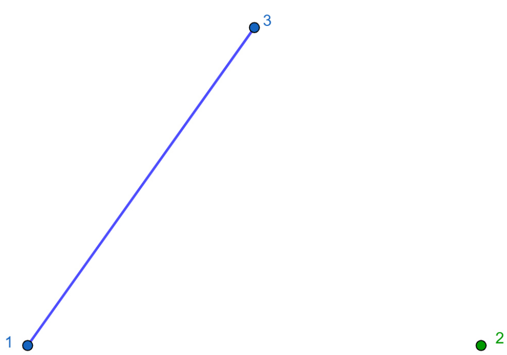
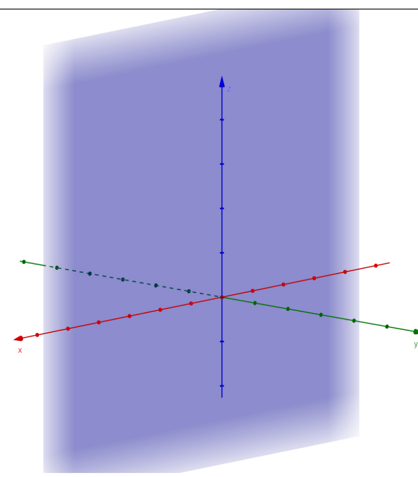
The *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is defined as

$$(\underline{X}, \underline{A})^K = \cup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^m X_i. \tag{2.1}$$

Remark 2.12. Note that $(\underline{X}, \underline{A})^{\sigma_1}$ is contained in $(\underline{X}, \underline{A})^{\sigma_2}$ provided $\sigma_1 \subseteq \sigma_2$. Therefore, it suffices to take the union over all the maximal simplices of K in (2.1), that is, simplices that are not contained in any other simplex of K.

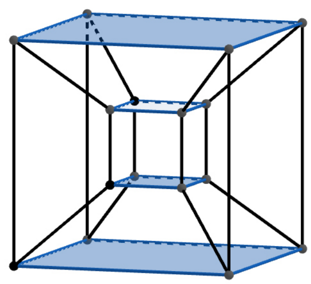
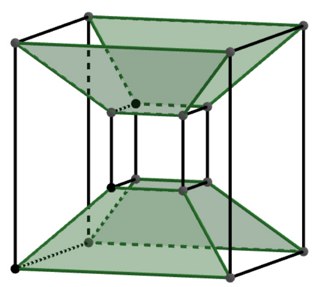
Throughout this thesis we are only interested in the case where all $A_i = *$. Thus, $(\underline{X}, *)^K$ and $(\underline{X}, *)^\sigma$ are simply denoted by $\underline{X}^K$ and $\underline{X}^\sigma$, respectively. Moreover, it is clear that, for any $\sigma \in K$, $\underline{X}^\sigma$ is a retract of $\underline{X}^K$ and $\underline{X}^\sigma \approx \prod_{i \in \sigma} X_i$.

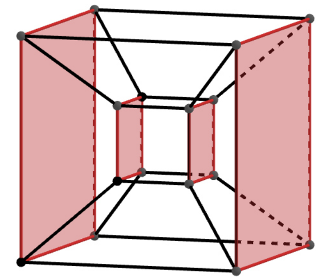
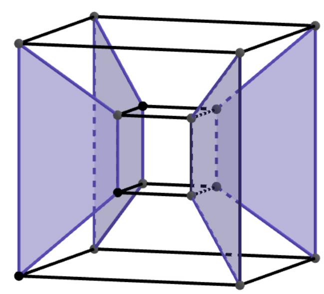
Example 2.13. Let $\underline{X} = \{(\mathbb{R}, 0)\}_{i=1}^{3}$ be a family of based spaces, where 0 is the neutral element of $\mathbb{R}$. The following table summarizes the polyhedral products we get for different choices of K.

K	Polyhedral product
	
	
	

Example 2.14 (Taken from [5]). Let $(\underline{X}, \underline{A}) = \{(\mathbb{D}^1, \mathbb{S}^0)\}$ be a family of pairs, where $\mathbb{D}^1$ denotes the closed disk of dimension 1 and $\mathbb{S}^0$ its boundary, whereas K is defined to be the boundary complex of a square on vertices $\{1, 2, 3, 4\}$. The following table illustrates how $\underline{X}^K$ turns out to be a 2-dimensional torus (we add the 1-skeleton in the first four pictures

for more clarity):

σ	$(\underline{X}, \underline{A})^{\sigma}$
$\{1,2\}$	
$\{1,4\}$	

σ	$(\underline{X}, \underline{A})^{\sigma}$
$\{2,3\}$	
$\{3,4\}$	

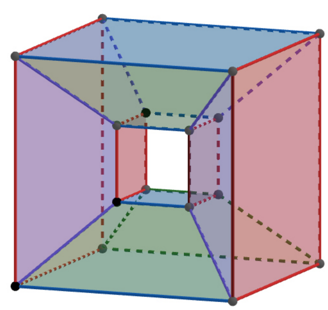


Figure 2.1: A 2-dimensional torus.

There are some approaches to explore the higher topological complexities of general polyhedral products. For example, [23, Theorem 5.3] computes, under suitable conditions, the higher topological complexities of $\underline{X}^K$ in terms of the cone length of each polyhedral product factor X_i. In fact, such a result will provide a proof of the result on sequential topological complexities of polyhedral products constructed out of even dimensional projective spaces (Theorem 4.12).

As an alternative to work with the cone length of each polyhedral product factor X_i, one can wonder what hypotheses have to be imposed to the family of based spaces $\underline{X}$ in order to guarantee that the general upper bound for $\mathbf{TC}_r(\underline{X}^K)$ given by $r \cdot \mathrm{cat}(\underline{X}^K)$ is optimal.

The first problem that has to be conquered is the computation of $\text{cat}(\underline{X}^K)$. Fortunately, this task has been completed in [3, Section 6.3], where the proof methodology involves certain properties of open covers of spaces. Indeed, with the purpose of making explicit such a result, we introduce the notions of LS-logarithmic and $\mathbf{TC}_r$-logarithmic families.

Definition 2.15. A family of based spaces $\underline{X} = \{(X_i, *)\}_{i=1}^m$ is said to be LS-*logarithmic* if

$$\text{cat}(X_{i_1} \times \cdots \times X_{i_k}) = \text{cat}(X_{i_1}) + \cdots + \text{cat}(X_{i_k})$$

holds for any strictly increasing sequence $1 \le i_1 < \cdots < i_k \le m$. Likewise, $\underline{X}$ is said to be $\mathbf{TC}_r$-*logarithmic*, for some $r \ge 2$, if

$$\mathbf{TC}_r(X_{i_1} \times \cdots \times X_{i_k}) = \mathbf{TC}_r(X_{i_1}) + \cdots + \mathbf{TC}_r(X_{i_k})$$

holds for any strictly increasing sequence $1 \le i_1 < \cdots < i_k \le m$.

Theorem 2.16. *[3, Lemma 6.7, Theorem 1.4] Let $\underline{X}^K$ be the polyhedral product associated to a family of based spaces $\underline{X} = \{(X_i, *)\}_{i=1}^m$ and an abstract simplicial complex K. We have*

$$\text{cat}(\underline{X}^K) \le \max\left\{\text{cat}(X_{i_1}) + \cdots + \text{cat}(X_{i_n}) \colon \{i_1, \ldots, i_n\} \in K\right\}.$$

Further, if the family $\{(X_i, *)\}_{i=1}^m$ is LS-*logarithmic, then the latter inequality is in fact an equality.*

Theorem 2.17. *[3, Theorem 1.5] Let $\{(X_i, *)\}_{i=1}^m$ be a collection of based spaces. If, for some $r \ge 2$,*

1. *$\mathbf{TC}_r(X_i) = r \cdot \text{cat}(X_i)$ for all $i \in \{1, \ldots, m\}$,*
2. *the collection $\{(X_i, *)\}_{i=1}^m$ is* LS-*logarithmic, and*
3. *the collection $\{(X_i, *)\}_{i=1}^m$ is $\mathbf{TC}_r$-logarithmic,*

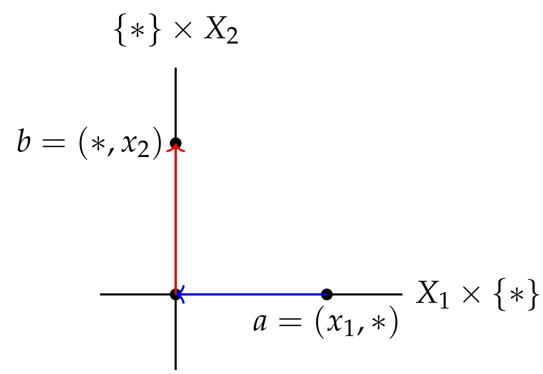
then

$$\mathbf{TC}_r(\underline{X}^K) = r \cdot \text{cat}(\underline{X}^K) = \max\left\{\mathbf{TC}_r(X_{i_1}) + \cdots + \mathbf{TC}_r(X_{i_n}) \colon \{i_1, \ldots, i_n\} \in K\right\}.$$

The latter result provides a nice alternative to compute, under suitable conditions, some higher topological complexities of general polyhedral products. Nonetheless, for $r = 2$, a real projective space does not fulfill the first hypothesis of Theorem 2.17 because $\mathbf{TC}(\mathbb{R}\mathrm{P}^n) < 2n = 2\,\text{cat}(\mathbb{R}\mathrm{P}^n)$. The same complication occurs if we consider a topological group G since $\mathbf{TC}_r(G) = \text{cat}(G^{r-1}) < r \cdot \text{cat}(G)$ for any $r \ge 2$ ([6, Proposition 3.4]).

We solve the above issues by constructing motion planners to determine the topological complexity of polyhedral products whose factors are real projective spaces and (locally compact connected CW) topological groups, separately. These projects will be carried out in full detail in chapters 4 and 5. Nevertheless, as the next example shows, there exist considerations that have to be taken into account in order to construct a motion planner for a polyhedral product $\underline{X}^K$ from motion planners for each polyhedral product factor X_i. As a warm-up of the techniques we will use in following chapters, we sketch out the case of a wedge sum of two arbitrary based spaces.

Example 2.18. Let $X_1 \vee X_2 = (X_1 \times *) \cup (* \times X_2)$ and suppose that we know how to move between points of X_1 and X_2 separately. If we need to find a path (in $X_1 \vee X_2$) connecting $a = (x_1, *)$ and $b = (*, x_2)$, the plan is clear: move from x_1 to $*$ through some path α_1 in X_1, and from $*$ to x_2 through some path α_2 in X_2. We reparametrize α_1 and α_2 in such a way that for times $t \in [0, 1/2]$ we move, via α_1, from x_1 to $*$ (reaching the latter stage when $t = 1/2$), while the path α_2 keeps its initial position at $*$ during all this period of time (see the blue arrow in the picture below). Then, for times $t \in [1/2, 1]$ we move, via α_2, from $*$ to x_2 (reaching the latter stage when $t = 1$), while α_1 remains constant at $*$ during all this period of time (see the red arrow in the picture below). This strategy is the heart of the motion planners constructed for more general polyhedral products and it will be formalized in chapter 4.



3

The Fadell-Husseini monoidal topological complexity

Having analyzed some basic results on topological complexity and polyhedral products, in this chapter we introduce the Fadell-Husseini monoidal topological complexity ($\mathbf{TC}^{FH}$). Such a notion emerges naturally by blending Dranishnikov's ($\mathbf{TC}^{DM}$) and García-Calcines' ($\mathbf{TC}^{M}_{g}$) views—reviewed after showing Proposition 3.2— into Iwase-Sakai's original definition of monoidal topological complexity ($\mathbf{TC}^{M}$).

As we will see in Theorem 3.13, which is the principal result of this chapter, all the above variants of $\mathbf{TC}^{M}$ agree when dealing with ANR spaces. In particular, this new approach to monoidal topological complexity will allow to show that the Iwase-Sakai conjecture holds true for polyhedral products constructed out of real projective spaces (chapter 4) and topological groups (chapter 5).

3.1 Monoidal topological complexity

The following notion was introduced by Iwase and Sakai in [25] and it is apparently more restrictive than Farber's topological complexity. In motion planning terms, $\mathbf{TC}^{M}(X)$ can be viewed in the same way as $\mathbf{TC}(X)$, except that the former satisfies an additional condition: if the A, B is a pair of initial-final configurations of the system with $A = B$, then the continuous motion from A to B is required to be the constant path at A. Such a requirement seems to be quite natural in actual applications.

In fact, Iwase and Sakai showed in [24] that, for a locally finite simplicial complex X, $\mathbf{TC}^{M}(X)$ differs from $\mathbf{TC}(X)$ by at most one unit, and conjectured the equality $\mathbf{TC}^{M}(X) = \mathbf{TC}(X)$. In this thesis we provide a sufficient condition to guarantee that the Iwase-Sakai conjecture $\mathbf{TC}^{M}(X) = \mathbf{TC}(X)$ occurs when dealing with an ANR space X (see Corollary 3.15).

Definition 3.1. The *monoidal topological complexity* of a path-connected space X, denoted $\mathbf{TC}^{M}(X)$, is the smallest n for which there is an open cover $\{U_0, \ldots, U_n\}$ of $X \times X$ by

$n+1$ open sets, each one containing the diagonal $\Delta X = \{(x,x) \colon x \in X\}$, and on each of which there is a continuous local section $s_i \colon U_i \to X^{[0,1]}$ of the end-points evaluation map $e_2 \colon X^{[0,1]} \to X \times X$ such that, for each $x \in X$, $s_i(x,x) = c_x$, the constant path at x. Such a section is called reserved. If the coverings fail to exist, we agree to set $\mathbf{TC}^M(X) = \infty$.

Unlike the usual topological complexity, the example exhibited in [21, p.13] shows that the monoidal topological complexity is not a homotopy invariant in general. It is known that $\mathbf{TC}^M(X)$ is a homotopy invariant if X is locally equiconnected, i.e., provided the canonical embedding $\Delta X \hookrightarrow X \times X$ is a cofibration (see [8, Theorem 12] and [21, Proposition 2.17]). An important instance of equiconnected spaces is given by an *absolute neighborhood retract* (ANR), which, throughout this thesis, means a metrizable space X satisfying the following property: every map $f \colon A \to X$, where A is a closed subset of any metrizable space Y, can be continuously extended over an open neighborhood U of A in Y.

Following Dranishnikov's observation in [14], the next proposition claims that the condition $\Delta X \subseteq U_i$ imposed to each set of the open cover $\{U_i\}_{i=0}^n$ of $X \times X$ in Definition 3.1 can be omitted in the case when X is an ANR space. Since some of the proof details in [14] are not provided, we give a complete proof.

Proposition 3.2. *If X is an* ANR *space, then $\mathbf{TC}^M(X) = \mathbf{TC}^{DM}(X)$, where the latter expression is defined to be the smallest nonnegative integer n for which there is an open cover $\{U_0, \ldots, U_n\}$ of $X \times X$, on each of which there is a continuous local section $s_i \colon U_i \to X^{[0,1]}$ of the fibration $e_2 \colon X^{[0,1]} \to X \times X$ such that $s_i(x,x) = c_x$ for all $x \in X$ with $(x,x) \in U_i$.*

Proof. Clearly $\mathbf{TC}^{DM}(X) \leq \mathbf{TC}^M(X)$. In what follows we show the opposite inequality.

Let $\{U_0, \ldots, U_n\}$ be an open cover of $X \times X$ by sets that admit continuous local sections $s_i \colon U_i \to X^{[0,1]}$ of the fibration e_2 such that $s_i(x,x) = c_x$ for all $x \in X$ with $(x,x) \in U_i$. Since $X \times X$ is a normal space, we can assure the existence of a closed cover $\{V_0, \ldots, V_n\}$ of $X \times X$ with $V_i \subseteq U_i$ for all $i \in \{0, 1, \ldots, n\}$. Then we have a continuous extension

$$\bar{s}_i \colon V_i \cup \Delta X \to X^{[0,1]}$$

of $s_i|_{V_i}$ defined by

$$\bar{s}_i(x,x') = \begin{cases} s_i(x,x'), & \text{if } (x,x') \in V_i; \\ c_x, & \text{if } (x,x') \in \Delta X. \end{cases}$$

For $i \in \{0, 1, \ldots, n\}$, let Γ_i stand for the closed subset $((V_i \cup \Delta X) \times [0,1]) \cup (X \times X \times \{0,1\})$ of $X \times X \times [0,1]$ and define a continuous function $u_i \colon \Gamma_i \to X$ by

$$u_i(x,x',t) = \begin{cases} \bar{s}_i(x,x')(t), & \text{if } (x,x',t) \in (V_i \cup \Delta X) \times [0,1]; \\ x, & \text{if } (x,x',t) \in X \times X \times \{0\}; \\ x', & \text{if } (x,x',t) \in X \times X \times \{1\}. \end{cases}$$

Note that u_i is well-defined because $\bar{s}_i$ is a continuous local section of e_2. Since X is an ANR space, there are open neighborhoods W_i of Γ_i in $X \times X \times [0,1]$ and continuous maps

$\overline{u}_i\colon W_i \to X$ with $\overline{u}_i|_{\Gamma_i} = u_i$. By the compactness of $[0,1]$, we can take an open set N_i in $X \times X$ containing $V_i \cup \Delta X$ such that $N_i \times [0,1] \subseteq W_i$.
Finally, the required reserved section $s_i'\colon N_i \to X^{[0,1]}$ of the fibration e_2 is defined by $s_i'(x,x')(t) = \overline{u}_i(x,x',t)$ for all $(x,x') \in N_i$ and $t \in [0,1]$. Indeed, by construction, s_i' is a continuous extension of $\bar{s}_i$. Therefore, the new open cover $\{N_0,\ldots,N_n\}$ of $X \times X$ fulfills the requirements of Definition 3.1. □

The latter result due to Dranishnikov shows that, if X is an ANR space, Iwase-Sakai's definition of $\mathbf{TC}^M(X)$ can be relaxed in the sense that the diagonal of X does not have to be contained in each open domain covering $X \times X$ (yet, the continuous local sections of the end-points evaluation map $e_2\colon X^{[0,1]} \to X \times X$ are still required to yield constant paths on points of the diagonal). Furthermore, in [21, Remark 2.19], García-Calcines proved that, when X is an ANR space, $\mathbf{TC}^M(X)$ can be defined in terms of general (not necessarily open) covers of $X \times X$ by following the lines in Iwase-Sakai's work, that is, by requiring that the diagonal lies in each subset covering $X \times X$, and that the corresponding local sections yield constant paths when restricted to the diagonal of X. On the other hand, Fadell and Husseini defined in [17] the relative category of a cofibered pair (X,A), denoted by $\mathrm{cat}^{FH}(X,A)$, in terms of open sets covering X. In contrast to the definition of $\mathrm{cat}(X)$, in the relative-category context exactly one of the covering open subsets is required to contain A and deform to A (rel A) within X, while the rest of the open sets are actually required to deform within X to a point.

In the next section we will introduce the Fadell-Husseini monoidal topological complexity (Definition 3.4) by imputing the definition of $\mathrm{cat}^{FH}(X,A)$ into Dranishnikov's and García-Calcines' viewpoints for $\mathbf{TC}^M(X)$. Nonetheless, before delving into this task it will necessary to touch on relative category.

3.2 *Relative category and principal results*

We start by recalling the definition of the join of two maps having the same target [13]. Let $f\colon X \to Z$ and $g\colon Y \to Z$ be maps, the join of f and g, denoted $X *_Z Y$, is defined as the homotopy pushout of the homotopy pullback of f and g:

$$\begin{tikzcd} P \arrow[rr] \arrow[dd] & & Y \arrow[dl] \arrow[dd, "g"] \\ & X *_Z Y \arrow[dr, dashed] & \\ X \arrow[ur] \arrow[rr, "f"'] & & Z, \end{tikzcd}$$

where the dashed arrow, called join map or whisker map, is given by the weak universal property of the homotopy pushout.

The previous definition enables us to set forth the main notion of this section, the relative category of a map as introduced in [13]. We need the following ingredient. Let $i_X\colon A \to X$ denote a map, the n-th Ganea map of i_X, $g_n\colon G_n(X) \to X$ ($n \geq 0$), is the join map inductively defined by the join construction

$$\begin{tikzcd}
P_{n-1} \arrow[rr] \arrow[dd] & & A \arrow[dl, "\alpha_n"'] \arrow[dd, "i_X"] \\
& G_n(X) \arrow[dr, dashed, "g_n"] & \\
G_{n-1}(X) \arrow[ur] \arrow[rr, "g_{n-1}"'] & & X,
\end{tikzcd}$$

where $g_0 := i_X$ and, if $g_{n-1}\colon G_{n-1}(X) \to X$ is already given, $G_n(X)$ is the join of i_X and g_{n-1}. Then, the *relative category* of i_X, denoted $\mathrm{relcat}(i_X)$, is defined as the least nonnegative integer n such that $g_n\colon G_n(X) \to X$ admits a homotopy section $\sigma\colon X \to G_n(X)$ satisfying $\sigma \circ i_X \simeq \alpha_n$.

Doeraene and El Haouari proved in [13] that the relative category of a map possesses a characterization in terms of the n-th sectional fat-wedge $t_n\colon T^n(i_X) \to X^{n+1}$ of i_X (see Theorem 3.3 below), which is inductively defined as follows: For $n = 0$, set $T^0(i_X) := A$ and $t_0 = i_X\colon A \to X$. If $t_{n-1}\colon T^{n-1}(i_X) \to X^n$ is already defined, then t_n is the join map making commutative, up to homotopy, the following diagram

$$\begin{tikzcd}
P_{n-1} \arrow[rr] \arrow[dd] & & X^n \times A \arrow[dl] \arrow[dd, "1_{X^n}\times i_X"] \\
& T^n(i_X) \arrow[dr, dashed, "t_n"] & \\
T^{n-1}(i_X) \times X \arrow[ur] \arrow[rr, "t_{n-1}\times 1_X"'] & & X^{n+1},
\end{tikzcd}$$

where $T^n(i_X)$ is the join of $t_{n-1} \times 1_X$ and $1_{X^n} \times i_X$.

Lastly, the next result pieces together [13, Proposition 26] and [22, Corollary 11]. We have chosen the statement in Theorem 3.3 because, on the one hand, of the simple description of the n-th sectional fat-wedge and of the formulas of all maps appearing in diagram (3.1) in the case when $i_X : A \hookrightarrow X$ is a cofibration and, on the other hand, because we are interested in the case of the diagonal inclusion in $X \times X$. Nonetheless, we remark that the characterization of relative category given by Doeraene and El Haouari in [13, Proposition 26] applies for *any* map $i_X : A \to X$.

Theorem 3.3. *Let $i_X \colon A \hookrightarrow X$ be a cofibration. We have* $\mathrm{relcat}(i_X) \leq n$ *if and only if there exits a map $f \colon X \to T^n(i_X)$ making commutative, up to homotopy, the diagram*

$$\begin{tikzcd} A \arrow[r, "\tau_n"] \arrow[d, hook, "i_X"'] & T^n(i_X) \arrow[d, hook, "t_n"] \\ X \arrow[r, "\Delta_{n+1}"'] \arrow[ur, dashed, "f"] & X^{n+1}, \end{tikzcd} \tag{3.1}$$

where Δ_{n+1} is the diagonal map, $\tau_n = \Delta_{n+1}|_A$, and $t_n : T^n(i_X) \hookrightarrow X^{n+1}$ is the inclusion of the subspace $T^n(i_X) = \{(x_0, \ldots, x_n) \in X^{n+1} \colon x_i \in A$ for some $i\}$.

Definition 3.4.

(a) The *Fadell-Husseini monoidal topological complexity* of a path connected space X, denoted by $\mathbf{TC}^{FH}(X)$, is the smallest nonnegative integer n for which there is an open cover $\{U_0, \ldots, U_n\}$ of $X \times X$ by $n+1$ subsets, on each of which there exists a continuous local section $s_i \colon U_i \to X^{[0,1]}$ of the end-points evaluation map $e_2 \colon X^{[0,1]} \to X \times X$ so that:

  (1) U_0 contains the diagonal $\Delta X = \{(x,x) \colon x \in X\}$;

  (2) $s_0(x,x) = c_x$, the constant path at x, for all $x \in X$;

  (3) $\Delta X \cap U_i = \varnothing$ for all $i \geq 1$.

(b) The *Fadell-Husseini generalized monoidal topological complexity* of a path-connected space X, denoted by $\mathbf{TC}^{FH}_g(X)$, is defined as above, except that the elements of the covers $\{U_0, \ldots, U_n\}$ are not required to be open.

We agree to set $\mathbf{TC}^{FH}(X) = \infty$ or $\mathbf{TC}^{FH}_g(X) = \infty$ if the required coverings fail to exist.

Remark 3.5. In the generalized setting, condition (3) in Definition 3.4 can be omitted without altering the value of $\mathbf{TC}^{FH}_g$, as the diagonal ΔX can be removed, if needed, from the sets $U_1, \ldots, U_n$. A similar observation applies in the non-generalized setting provided X is a Hausdorff space. Furthermore, the inequalities $\mathbf{TC}^{DM}(X) \leq \mathbf{TC}^{FH}(X) \leq \mathbf{TC}^{M}(X)$ follow directly from the definitions. In fact, these inequalities turn out to be equalities if X is an ANR, in view of Proposition 3.2.

What is more striking is the fact that the *generalized* version of $\mathbf{TC}^{FH}$ also agrees with the original definition of monoidal topological complexity when working with ANR spaces. Nevertheless the proof of such a fact is not trivial, and can be proved by following the guideline established by García-Calcines in [21] to show that $\mathbf{TC}^M(X) = \mathbf{TC}^M_g(X)$ when dealing with an ANR space X. In fact, paralleling many of his techniques, we introduce in Definitions 3.6 and 3.9 below the notions of $\mathrm{relcat}^{FH}_{op}(i_X)$ and $\mathrm{relcat}^{FH}_g(i_X)$, where $i_X : A \hookrightarrow X$ stands for a cofibration. Both concepts represent a Fadell-Husseini version of the concepts $\text{relcat}_{op}(i_X)$ and $\text{relcat}_g(i_X)$, which were widely studied in [21] in order to provide a characterization by covers of relative category in the sense of Doeraene-El Haouari. We show that if X is an ANR space and $i_X : \Delta X \hookrightarrow X \times X$ denotes the canonical cofibration, then

$$\text{relcat}_g^{FH}(i_X) = \text{relcat}_{op}^{FH}(i_X) = \text{relcat}(i_X) = \text{relcat}_{op}(i_X) = \text{relcat}_g(i_X), \tag{3.2}$$

where the last two equalities were proved in [21, Theorems 1.6 and 2.16].

Definition 3.6. Let $i_X : A \hookrightarrow X$ be a cofibration. We say that a subset U of X is *A-relatively sectional* if $A \subseteq U$ and there exists a homotopy of pairs $H\colon (U, A) \times [0, 1] \to (X, A)$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in U$. Then we define $\text{relcat}_{op}^{FH}(i_X)$ as the least nonnegative integer n such that X admits an open cover $\{U_i\}_{i=0}^n$ satisfying:

1. U_0 is A-relatively sectional;
2. for $i \geq 1$, $U_i \cap A = \emptyset$ and there are homotopies $H_i\colon U_i \times [0, 1] \to X$ with $H_i(x, 0) = x$ and $H_i(x, 1) \in A$ for all $x \in U_i$.

If such an integer does not exist, then we simply set $\text{relcat}_{op}^{FH}(i_X) = \infty$.

Remark 3.7. If each U_i is required to be A-relatively sectional in Definition 3.6, we obtain the notion of $\text{relcat}_{op}(i_X)$. The latter concept agrees with the relative category of i_X because of [21, Theorem 1.6]. In the next proposition we show that $\text{relcat}(i_X) = \text{relcat}_{op}^{FH}(i_X)$ holds as well by following similar techniques to those exposed in [21].

Proposition 3.8. *Let X be a normal space. If $i_X\colon A \hookrightarrow X$ is a cofibration, then* $\text{relcat}(i_X) = \text{relcat}_{op}^{FH}(i_X)$.

Proof. First we show that $\text{relcat}_{op}^{FH}(i_X) \leq \text{relcat}(i_X) =: n$. Since the latter coincides with $\text{relcat}_{op}(i_X)$ ([21, Theorem 1.6]), we can choose an open cover $\{U_0, \ldots, U_n\}$ of X such that, for any $i \geq 0$, U_i is A-relatively sectional. Then there are homotopies of pairs $H_i\colon (U_i, A) \times [0, 1] \to (X, A)$ such that $H_i(x, 0) = x$ and $H_i(x, 1) \in A$ for all $x \in U_i$. It is clear that setting $U_0^* := U_0$, $U_i^* := U_i \setminus A$ and restricting the homotopies H_i to U_i^* for $i \geq 1$, the two items of Definition 3.6 are fulfilled. Thereby, the desired inequality $\text{relcat}_{op}^{FH}(i_X) \leq \text{relcat}(i_X)$ follows.

Now, in order to prove $\text{relcat}(i_X) \leq \text{relcat}_{op}^{FH}(i_X)$, let $\text{relcat}_{op}^{FH}(i_X) = n$ and consider an open cover $\{U_i\}_{i=0}^n$ of X such that:

1. U_0 is A-relatively sectional, i.e., $A \subseteq U_0$ and there exists a homotopy of pairs $H_0 : (U_0, A) \times [0, 1] \to (X, A)$ with $H_0(x, 0) = x$ and $H_0(x, 1) \in A$ for all $x \in U_0$;
2. for $i \geq 1$, $U_i \cap A = \emptyset$ and there are homotopies $H_i\colon U_i \times [0, 1] \to X$ with $H_i(x, 0) = x$ and $H_i(x, 1) \in A$ for all $x \in U_i$.

Since X is a normal space, there are closed sets A_i, B_i and open sets Θ_i ($i = 0, \ldots, n$) fulfilling $A \subseteq A_0$ and $A_i \subseteq \Theta_i \subseteq B_i \subseteq U_i$ for all i, with $\{A_i\}_{i=0}^n$ covering X. Furthermore, there exist Urysohn maps $h_i \colon X \to [0,1]$ such that $h_i(A_i) = \{1\}$ and $h_i(X \setminus \Theta_i) = \{0\}$. For $i \geq 0$, let $L_i \colon (X, A) \times [0,1] \to (X, A)$ be the continuous map defined by

$$L_i(x,t) = \begin{cases} x, & \text{if } x \in X \setminus B_i; \\ H_i(x, h_i(x) \cdot t), & \text{if } x \in U_i. \end{cases}$$

Observe that L_i is well-defined because $x = H_i(x, h_i(x) \cdot t)$ for all $x \in U_i \setminus B_i$. Likewise, the facts $L_0(a,t) = H_0(a,t) \in A$ and $L_i(a,t) = a$ for $t \in [0,1]$, $a \in A$ and $i \geq 1$ (recall, $U_i \cap A = \varnothing$ for $i \geq 1$) imply that L_i restricts to a map $A \times [0,1] \to A$.
We define $L : (X, A) \times [0,1] \to (X^{n+1}, T^n(i_X))$ to be $L := (L_0, \ldots, L_n)$. Since $\{A_i\}_{i=0}^n$ covers X, we get a well-defined map $f : X \to T^n(i_X)$ by setting $f(x) := L(x,1)$. Such a map satisfies $L : \Delta_{n+1} \simeq t_n \circ f$ and $L|_{A \times [0,1]} : \tau_n \simeq f \circ i_X$. Therefore, by Theorem 3.3, $\operatorname{relcat}(i_X) \leq n$. □

We now discuss the Fadell-Husseini generalized relative category $\operatorname{relcat}_g^{FH}(i_X)$, which is determined just as in Definition 3.6, but without requiring that the covers should consist of open sets. We show that, under mild hypotheses, dropping such a condition is immaterial (Proposition 3.12 below).

Definition 3.9. Let $i_X : A \hookrightarrow X$ be a cofibration. We define $\operatorname{relcat}_g^{FH}(i_X)$ as the least nonnegative integer n such that X admits a (not necessarily open) cover $\{U_0, \ldots, U_n\}$ satisfying:

1. U_0 is A-relatively sectional;
2. for $i \geq 1$, $U_i \cap A = \varnothing$ and there are homotopies $H_i \colon U_i \times [0,1] \to X$ with $H_i(x, 0) = x$ and $H_i(x,1) \in A$ for all $x \in U_i$.

If such an integer does not exist, then we simply set $\operatorname{relcat}_g^{FH}(i_X) = \infty$.

Remark 3.10. If each subset U_i is required to be A-relative sectional in Definition 3.9, we obtain the notion of $\operatorname{relcat}_g(i_X)$. In view of [21, Theorem 2.16], the latter concept coincides with $\operatorname{relcat}(i_X)$ provided $i_X \colon A \hookrightarrow X$ is a cofibration between ANR spaces. Paralleling the proof of such a fact, we will prove that the same conclusion holds in the context of $\operatorname{relcat}_g^{FH}(i_X)$.

Before delving into the equality $\operatorname{relcat}(i_X) = \operatorname{relcat}_g^{FH}(i_X)$, we prove the following technical lemma (cf. [21, Lemma 2.14]).

Lemma 3.11. *Let $i_X \colon A \hookrightarrow X$ be a cofibration with X a normal space. Assume that*

1. *$\{U_0, \ldots, U_n\}$ is an open cover of X;*

2. *$A \subseteq U_0$ and there is a homotopy $H_0\colon U_0 \times [0,1] \to X$ so that $H_0(x,0) = x$, $H_0(x,1) \in A$, and $H_0(-,1)|_A \simeq 1_A$;*

3. *for $i \geq 1$, $U_i \cap A = \emptyset$ and there are homotopies $H_i\colon U_i \times [0,1] \to X$ with $H_i(x,0) = x$ and $H_i(x,1) \in A$ for any $x \in U_i$.*

Then $\mathrm{relcat}(i_X) \leq n$.

Proof. The first half of the argument follows the constructions in the proof of Proposition 3.8. Let A_i, B_i be closed sets and Θ_i be open sets ($i = 0, \ldots, n$), with $\{A_i\}_{i=0}^n$ covering X, such that $A \subseteq A_0$ and $A_i \subseteq \Theta_i \subseteq B_i \subseteq U_i$ for all i. Choose Urysohn maps $h_i\colon X \to [0,1]$ such that $h_i(A_i) = \{1\}$ and $h_i(X \setminus \Theta_i) = \{0\}$. For $i \geq 0$, let $L_i\colon X \times [0,1] \to X$ be defined by

$$L_i(x,t) = \begin{cases} x, & \text{if } x \in X \setminus B_i; \\ H_i(x, h_i(x) \cdot t), & \text{if } x \in U_i. \end{cases}$$

Set $L = (L_0, \ldots, L_n)\colon X \times [0,1] \to X^{n+1}$ and note that we still have $L_i|_{A\times[0,1]} : A \times [0,1] \to A$, as well as $L : \Delta_{n+1} \simeq t_n \circ f$, where $f(x) := L(x,1)\colon X \to T^n(i_X)$.
On the other hand, let $G_0 : A \times [0,1] \to A$ be the homotopy between 1_A and $H_0(-,1)|_A$, that is, $G_0(a,0) = a$ and $G_0(a,1) = H_0(a,1)$ for all $a \in A$. Define $G\colon A \times [0,1] \to A^{n+1} \subseteq T^n(i_X)$ to be $G = (G_0, L_1|_{A\times[0,1]}, \ldots, L_n|_{A\times[0,1]})$. Observe that $L_0(a,1) = H_0(a,1) = G_0(a,1)$, so $G : \tau_n \simeq f \circ i_X$. Therefore, the desired inequality $\mathrm{relcat}(i_X) \leq n$ comes from Theorem 3.3. □

Proposition 3.12. *Let $i_X\colon A \hookrightarrow X$ be a cofibration between* ANR *spaces. We have* $\mathrm{relcat}(i_X) = \mathrm{relcat}_g^{FH}(i_X)$.

Proof. Clearly $\mathrm{relcat}_g^{FH}(i_X) \leq \mathrm{relcat}_{op}^{FH}(i_X) = \mathrm{relcat}(i_X)$, where the latter relation holds in view of Proposition 3.8 (recall, X is a normal space since it is metrizable). We show the inequality $\mathrm{relcat}(i_X) \leq \mathrm{relcat}_g^{FH}(i_X)$.
Let $n := \mathrm{relcat}_g^{FH}(i_X)$ and consider a (not necessarily open) cover $\{U_i\}_{i=0}^n$ of X such that:

1. U_0 is A-relatively sectional, i.e., $A \subseteq U_0$ and there exists a homotopy of pairs $H_0 : (U_0, A) \times [0,1] \to (X, A)$ with $H_0(x,0) = x$ and $H_0(x,1) \in A$ for all $x \in U_0$;

2. for $i \geq 1$, $U_i \cap A = \emptyset$ and there are homotopies $H_i\colon U_i \times [0,1] \to X$ with $H_i(x,0) = x$ and $H_i(x,1) \in A$ for all $x \in U_i$.

The argument below for $i = 0$ is the one in the proof of [21, Theorem 2.16]. We review the details since we will then describe a slight variant in order to deal with the case of $i > 0$. Consider the following factorization of i_X through its mapping cocylinder:

$$\begin{tikzcd} A \arrow[rr, hook, "i_X"] \arrow[dr, "j"'] & & X \\ & \widehat{A}, \arrow[ur, "p"'] & \end{tikzcd}$$

where $\widehat{A} = \{(a, \beta) \in A \times X^{[0,1]} : i_X(a) = \beta(0)\}$, p is a fibration and j a homotopy equivalence. As observed in [21, Lemma 2.13], $\widehat{A}$ is also an ANR. Define $s_0 \colon U_0 \to \widehat{A}$ to be $s_0 = j \circ H_0(-, 1)$, then $p \circ s_0 \simeq U_0 \hookrightarrow X$ and $s_0|_A \simeq j$. Actually, since p is a fibration, there is no problem in assuming that $p \circ s_0 = U_0 \hookrightarrow X$. Following the proof of [21, Theorem 2.7], there exist an open neighborhood V_0 of U_0 in X and a map $\sigma_0 \colon V_0 \to \widehat{A}$ such that $p \circ \sigma_0 = V_0 \hookrightarrow X$ and $\sigma_0|_{U_0} \simeq s_0$. In particular,

$$\sigma_0|_A = (\sigma_0|_{U_0})|_A \simeq s_0|_A \simeq j. \tag{3.3}$$

If $j' : \widehat{A} \to A$ denotes a homotopy inverse of j, then

$$i_X \circ j' \circ \sigma_0 = p \circ j \circ j' \circ \sigma_0 \simeq p \circ \sigma_0 = V_0 \hookrightarrow X.$$

Hence, there exists a homotopy $G_0 : V_0 \times [0, 1] \to X$ such that $G_0(x, 0) = x$, $G_0(x, 1) = i_X \circ j' \circ \sigma_0(x) \in A$ and $G_0(-, 1)|_A = j' \circ (\sigma_0|_A) \simeq j' \circ j = 1_A$.
On the other hand, for $i \geq 1$, set $s_i := j \circ H_i(-, 1) \colon U_i \to \widehat{A}$. An examination of the proof above (omitting those steps that involve $A \subseteq U_0$) reveals that there are open neighborhoods V_i of U_i in X together with maps $\sigma_i : V_i \to \widehat{A}$ so that $p \circ \sigma_i = V_i \hookrightarrow X$ and $\sigma_i|_{U_i} \simeq s_i$. Without losing generality we may assume that $V_i \cap A = \emptyset$ for, otherwise, we simply set $V_i' := V_i \setminus A$ and $\sigma_i' = \sigma_i|_{V_i'}$. Furthermore, we have homotopies $G_i \colon V_i \times [0, 1] \to X$ such that $G_i(x, 0) = x$ and $G_i(x, 1) = i_X \circ j' \circ \sigma_i(x) \in A$ for all $x \in V_i$.
Therefore $\mathrm{relcat}(i_X) \leq n$, by Lemma 3.11. □

We bring together the above results to show the following theorem.

Theorem 3.13. *If X is an* ANR *space, then*

$$\mathbf{TC}^{FH}(X) = \mathbf{TC}^{FH}_g(X) = \mathbf{TC}^{DM}(X) = \mathbf{TC}^{M}(X) = \mathbf{TC}^{M}_g(X).$$

Proof. The last equality is due to García-Calcines ([21, Remark 2.19]), whereas the next-to-last equality is due to Dranishnikov (the proof details were completed in Proposition 3.2). Likewise, by Remark 3.5, the equality $\mathbf{TC}^{M}(X) = \mathbf{TC}^{FH}_g(X)$ is the only one requiring argumentation.

First of all, since X is an ANR space, the canonical embedding $i_X \colon \Delta X \hookrightarrow X \times X$ is a cofibration between ANR spaces and, by [21, Theorems 1.6 and 2.16], $\mathrm{relcat}(i_X) = \mathrm{relcat}_{op}(i_X) = \mathrm{relcat}_g(i_X)$. The equalities $\mathrm{relcat}(i_X) = \mathrm{relcat}^{FH}_{op}(i_X) = \mathrm{relcat}^{FH}_g(i_X)$ come from Propositions 3.8 and 3.12 (so, all equalities in (3.2) follow).

We next observe that $\mathrm{relcat}^{FH}_g(i_X) \leq \mathbf{TC}^{FH}_g(X)$. This fact comes by noticing that, if $\{U_i\}_{i=0}^n$ is a (not necessarily open) cover of $X \times X$ and $s_i \colon U_i \to X^{[0,1]}$ are the local sections of the fibration e_2 coming from Definition 3.4, then one can define homotopies $H_0 \colon (U_0, \Delta X) \times [0, 1] \to (X \times X, \Delta X)$ and $H_i \colon U_i \times [0, 1] \to X \times X$ ($i \geq 1$) as $H_i(x, y, t) = (s_i(x, y)(t), y)$ ($i \geq 0$) satisfying the two items of Definition 3.9. Likewise, it is clear that $\mathbf{TC}^{FH}_g(X) \leq \mathbf{TC}^{FH}(X)$; nevertheless, the latter expression agrees with $\mathbf{TC}^{M}(X)$ due to

Remark 3.5. Finally, $\mathbf{TC}^M(X) = \mathrm{relcat}(i_X)$ follows from [8, Theorem 12], while $\mathrm{relcat}(i_X)$ equals $\mathrm{relcat}_g^{FH}(i_X)$ by our initial discussion. In summary,

$$\mathrm{relcat}_g^{FH}(i_X) \leq \mathbf{TC}_g^{FH}(X) \leq \mathbf{TC}^{FH}(X) = \mathbf{TC}^M(X) = \mathrm{relcat}_g^{FH}(i_X),$$

which completes the proof. □

Finally, having shown the principal result of this chapter, we focus on providing a sufficient condition to guarantee that the Iwase-Sakai conjecture holds true when working with ANR spaces. The next result was pointed out by García-Calcines, who kindly accepted to review a preliminary version of [2].

Theorem 3.14. *If X is a locally equiconnected Hausdorff space such that $X \times X$ is normal, then the stasis condition (2) in Definition 3.4 can be ignored without altering the resulting numerical value of $\mathbf{TC}^{FH}(X)$. The same conclusion holds in the generalized setting if X is an* ANR *space.*

Proof. We start in the non-generalized setting, i.e., by proving that the stasis condition (2) in Definition 3.4 can be omitted without altering the numerical value of $\mathbf{TC}^{FH}(X)$. Let $\{(U_i, s_i)\}_{i=0}^n$ be a motion planner with $\Delta X \subseteq U_0$ and $\Delta X \cap U_i = \varnothing$ for all $i \geq 1$. We do not assume that the section s_0 of e_2 yields constant paths when restricted to ΔX, but we do assume that all subsets U_i are open. The task is to construct a motion planner $\{(V_i, \sigma_i)\}_{i=0}^n$ of a Fadell-Husseini type, that is, one that consists of open sets V_i satisfying $\Delta X \cap V_i = \varnothing$ for all $i \geq 1$, as well as $\Delta X \subseteq V_0$ with $\sigma_0(x, x) = c_x$, the constant path at x, for all $x \in X$.

If $n = 0$, then X is in fact contractible, so that the homotopy invariance of the monoidal topological complexity for locally equiconnected spaces ([21, Proposition 2.17]) gives the required motion planner of a Fadell-Husseini type. We can therefore assume $n \geq 1$. By [16, Theorem II.1], there is an open neighborhood W of ΔX in $X \times X$ and a local section $\lambda : W \to X^{[0,1]}$ of the end-points evaluation map e_2 satisfying $\lambda(x, x) = c_x$ for all $x \in X$. Furthermore, by the normality assumption, there is an open cover $\{W_i\}_{i=0}^n$ of $X \times X$ such that $W_i \subseteq \overline{W_i} \subseteq U_i$ for all $i \geq 0$. Consider the open neighborhood N of ΔX given by

$$N = W \cap U_0 \cap \left((X \times X) \setminus \overline{W_1}\right) \cap \cdots \cap \left((X \times X) \setminus \overline{W_n}\right).$$

Using once more the normality of $X \times X$, take an open set M in $X \times X$ with $\Delta X \subseteq M \subseteq \overline{M} \subseteq N$. Let $V_0 = (U_0 \setminus \overline{M}) \sqcup M$ and define the reserved section $\sigma_0 : V_0 \to X^{[0,1]}$ of e_2 by

$$\sigma_0(x, x') = \begin{cases} s_0(x, x'), & \text{if } (x, x') \in U_0 \setminus \overline{M}; \\ \lambda(x, x'), & \text{if } (x, x') \in M. \end{cases}$$

Lastly, for $1 \leq i \leq n$, set $V_i = W_i \sqcup (N \setminus \Delta X)$ and define the local sections $\sigma_i : V_i \to X^{[0,1]}$ of e_2 by

$$\sigma_i(x, x') = \begin{cases} s_i(x, x'), & \text{if } (x, x') \in W_i; \\ \lambda(x, x'), & \text{if } (x, x') \in N \setminus \Delta X. \end{cases}$$

Then $\{(V_i,\sigma_i)\}_{i=0}^n$ is the required motion planner of Fadell-Husseini type. It remains to show that $\{V_0,V_1,\dots,V_n\}$ covers $X\times X$. Indeed, since $W_i\subseteq V_i$ for $i\geq 1$, the covering assertion follows by observing that $W_0\setminus(V_1\cup\cdots\cup V_n)\subseteq V_0$:

$$\begin{aligned}
&W_0\setminus\,(V_1\cup\cdots\cup V_n)\subseteq U_0\setminus(V_1\cup\cdots\cup V_n)\\
&=\big((U_0\setminus\overline{M})\cup(\overline{M}\setminus M)\cup M\big)\setminus(V_1\cup\cdots\cup V_n)\\
&=\big(V_0\cup(\overline{M}\setminus M)\big)\setminus(V_1\cup\cdots\cup V_n)\\
&=(V_0\setminus(V_1\cup\cdots\cup V_n))\cup\big(\overline{M}\setminus(M\cup V_1\cup\cdots\cup V_n)\big)\,,
\end{aligned}$$

where

$$\begin{aligned}
&\overline{M}\setminus(M\cup V_1\cup\cdots\cup V_n)\subseteq N\setminus(\Delta X\cup V_1\cup\cdots\cup V_n)\\
&=(N\setminus\Delta X)\setminus(V_1\cup\cdots\cup V_n)=\varnothing,
\end{aligned}$$

as $n\geq 1$.

We now sketch the argument for the generalized case. Let $\{(U_i,s_i)\}_{i=0}^n$ be a generalized motion planner consisting of a cover $\{U_i\}_{i=0}^n$ of $X\times X$ by not necessarily open subsets U_i such that $\Delta X\subseteq U_0$ and $\Delta X\cap U_i=\varnothing$ for all $i\geq 1$, and of sections $s_i\colon U_i\to X^{[0,1]}$ of e_2. Again, without assuming that s_0 is a reserved section of e_2, the task is to assure the existence of a corresponding generalized motion planner, one of whose rules is defined on the whole diagonal via constant paths. Following the proof of [21, Theorem 2.7], we can construct a new motion planner $\{(V_i,\sigma_i)\}_{i=0}^n$ so that, for all $i\geq 0$, V_i is an open subset of $X\times X$, $U_i\subseteq V_i$ (so $\Delta X\subseteq V_0$), and $\sigma_i|_{U_i}\simeq s_i$. Furthermore, without loss of generality we can assume $\Delta X\cap V_i=\varnothing$ for all $i\geq 1$. Then, by the argument in the non-generalized case, we can fix the situation so to have in addition $\sigma_0(x,x)=c_x$, the constant path at x, for all $x\in X$, which completes the argument. $\square$

Corollary 3.15. *The equality $\mathbf{TC}(X)=\mathbf{TC}^M(X)$ in the Iwase-Sakai conjecture holds true for any* ANR *space X for which there is a (not necessarily monoidal) motion planner with $\mathbf{TC}(X)+1$ (not necessarily open) local domains one of which contains the diagonal $\Delta(X)$ (cf. [24, Corollary 3]).*

4

Real projective spaces

In this chapter we compute, under suitable conditions, $\mathbf{TC}(\underline{P}^K)$, where $\underline{P}^K$ denotes the polyhedral product determined by an abstract simplicial complex K and a based family $\underline{P} = \{(\mathbb{R}\mathrm{P}^{n_i}, *)\}_{i=1}^m$ of real projective spaces. As a particular case, we show that the Iwase-Sakai conjecture holds true for $\underline{P}^K$ (Theorem 4.7).

The proof of such a result is treated in sections 4.2.1 and 4.2.2. We first deal with the lower bound for $\mathbf{TC}(\underline{P}^K)$ by considering the mod-2 cohomology ring of $\underline{P}^K$, then we carry out the project of constructing a motion planner leading to an upper bound for $\mathbf{TC}^M(\underline{P}^K)$. As we shall see in section 4.2.2, the engine that powers this construction, and produces the mixed cat/**TC** behavior in $\mathbf{TC}(\underline{P}^K)$, is precisely the concept of strong axial map. For this reason, we start by proving some crucial results about the connection between the latter notion and the topological complexity of a real projective space.

4.1 Axial and nonsingular maps

Historically, axial and nonsingular maps were widely studied because of their connection with the immersion problem of real projective spaces. Later, Farber, Tabachnikov, and Yuzvinsky show in [20] that this classical and tough problem is closely related to the computation of $\mathbf{TC}(\mathbb{R}\mathrm{P}^n)$. In fact, for our purposes it was necessary to go back to Sanderson [29] to guarantee that we can involve strong axial maps in the impressive connection between $\mathbf{TC}(\mathbb{R}\mathrm{P}^n)$ and $\mathrm{Imm}(\mathbb{R}\mathrm{P}^n)$.

Most of the ideas and facts exposed here are inspired by the ones given in [20].

Definition 4.1. Let $n, k \in \mathbb{N}$. A continuous map $f\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ is said to be *nonsingular* if it satisfies the following conditions:

(a) $f(\lambda x, \mu y) = \lambda\mu f(x,y)$ for all $x, y \in \mathbb{R}^{n+1}$, $\lambda, \mu \in \mathbb{R}$;

(b) $f(x,y) = 0$ implies that either $x = 0$ or $y = 0$.

Additionally, if $f(x,*) = x$ and $f(*,x) = x$ for all $x \in \mathbb{R}^{n+1}$, we say that f is a *strong nonsingular map*. Here, $*$ denotes the basepoint $(1, 0, \ldots, 0)$ of $\mathbb{R}^{n+1}$. Furthermore, we use the canonical embedding $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{k+1}$ to identify $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ with $(x_1, \ldots, x_{n+1}, 0, \ldots, 0) \in \mathbb{R}^{k+1}$.

It can be shown that a nonsingular map as above exists only for $k \geq n$, and that $n = k$ is possible only for $n \in \{1, 3, 7\}$ by using the operations of the complex, quaternion, and octonion numbers, respectively. Indeed, in these cases, the map $f\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by $f(x, y) = xy$ produces a strong nonsingular map, whereas $f(x, y) = x\overline{y}$ yields a nonsingular map whose first coordinate map is positive on the diagonal, that is,

$$f(x, x) = (\lambda_x, 0, \ldots, 0),$$

with $\lambda_x \geq 0$ for all $x \in \mathbb{R}^{n+1}$ and $\lambda_x = 0$ if and only if $x = 0$.

The principal aim of this section is to guarantee, for $n \neq 1, 3, 7$, the existence of a strong nonsingular map whose first coordinate map is positive on the diagonal. As we have already seen in the previous paragraph, both requirements fail to exist for the same map in the cases $n \in \{1, 3, 7\}$.

Definition 4.2. Let n, k be natural numbers. A continuous map

$$g\colon \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$$

is called *axial of type (n, k)* if its restrictions to $\mathbb{RP}^n \times *$ and $* \times \mathbb{RP}^n$ are homotopic to the equatorial inclusion map $\mathbb{RP}^n \to \mathbb{RP}^k$ (so $n \leq k$ is forced). Additionally, if $g(A, *) = A$ and $g(*, A) = A$ for any $A \in \mathbb{RP}^n$, we say that g is a *strong axial map*. Here, $*$ is the basepoint of $\mathbb{RP}^n$ given by the equivalence class of $(1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$.

Remark 4.3. A nonsingular map $f\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ induces and is induced by an axial map $g\colon \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$. In this sense, f is strong if and only if g is strong (see the proof of Proposition 4.6 below).

Sanderson proved in [29, Theorem 2.1] that if $\mathbb{RP}^n$ admits a smooth immersion in $\mathbb{R}^k$, then there is a strong axial map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$ (note that $n < k$ is forced). Conversely, if $n < k$ and there exists an axial map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$ (not necessarily strong) then $\mathbb{RP}^n \looparrowright \mathbb{R}^k$ (see [1] for details). On the other hand, it was shown in [20, Theorem 7.1] that $\mathbf{TC}(\mathbb{RP}^n) = \mathrm{Imm}(\mathbb{RP}^n)$ for $n \neq 1, 3, 7$ and $\mathbf{TC}(\mathbb{RP}^n) = n$ for $n \in \{1, 3, 7\}$. In summary, for $n \neq 1, 3, 7$,

$$\mathbf{TC}(\mathbb{RP}^n) = \min\{k\colon \text{there exists a strong axial map } \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k \text{ and } n < k\}. \quad (4.1)$$

We now focus on showing the existence of a strong nonsingular map

$$f\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{\mathbf{TC}(\mathbb{RP}^n)+1}$$

such that its first coordinate map is positive on the diagonal for $n \neq 1, 3, 7$. This result will be a consequence of the following two lemmas.

Lemma 4.4. *Let $\alpha\colon \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$ be an axial map with $n < k$. The restriction of α to the diagonal of $\mathbb{RP}^n$ is null-homotopic.*

Proof. For $i \in \{1,2\}$, consider the commutative diagram

$$\begin{tikzcd} \mathbb{RP}^n \arrow[r, "\Delta"] \arrow[dr, equal] & \mathbb{RP}^n \times \mathbb{RP}^n \arrow[d, "\pi_i"] \\ & \mathbb{RP}^n, \end{tikzcd}$$

where $\Delta(A) = (A, A)$ and $\pi_i\colon \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^n$ is the canonical projection onto the ith factor.

Let $x \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ be the generator of $H^1(\mathbb{RP}^n; \mathbb{Z}_2)$, and define x_i to be the pullback of x under π_i. By the Künneth formula, $H^1(\mathbb{RP}^n \times \mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by x_1 and x_2, and hence $\alpha^*(x) = x_1 + x_2$ since α is axial. Then

$$\begin{aligned} \Delta^*\alpha^*(x) &= \Delta^*(x_1 + x_2) \\ &= \Delta^*(x_1) + \Delta^*(x_2) \\ &= 2x \\ &= 0, \end{aligned}$$

that is, the composition $\alpha \circ \Delta$ is trivial in cohomology. Furthermore, considering that $n < k$, the cellular approximation theorem asserts that the inclusion map $j\colon \mathbb{RP}^k \hookrightarrow \mathbb{RP}^\infty$ induces a one-to-one correspondence between homotopy classes

$$[\mathbb{RP}^n, \mathbb{RP}^k] \xrightarrow{\psi} [\mathbb{RP}^n, \mathbb{RP}^\infty].$$

By Brown's representability theorem, the abelian group $[\mathbb{RP}^n, \mathbb{RP}^\infty]$ is isomorphic to $H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ under the map $\tau\colon [\mathbb{RP}^n, \mathbb{RP}^\infty] \to H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ given by $\tau([f]) = f^*(u)$, where $u \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$ stands for the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$. Such facts imply that

$$\begin{aligned} \tau \circ \psi([\alpha \circ \Delta]) &= \Delta^*\alpha^* j^*(u) \\ &= \Delta^*\alpha^*(x) \\ &= 0, \end{aligned}$$

and therefore $\alpha \circ \Delta$ is null-homotopic. □

Lemma 4.5. *Let $n \in \mathbb{N}$ with $n \neq 1, 3, 7$. There exists a strong axial map $\alpha\colon \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^{\mathbf{TC}(\mathbb{RP}^n)}$ such that $\alpha(A, A) = *$ for all $A \in \mathbb{RP}^n$.*

Proof. Throughout the proof $\mathbf{TC}(\mathbb{RP}^n)$ is simply denoted by k.

Let $\beta\colon \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$ be a strong axial map given by (4.1). If we set $E = \mathbb{RP}^n \vee \mathbb{RP}^n$ and D stands for the diagonal in $\mathbb{RP}^n \times \mathbb{RP}^n$, the restriction $\beta|_D$ is null-homotopic by Lemma 4.4. In fact, $\beta|_D$ is based null-homotopic in view of [12, Corollary 6.58]. Let $H_D\colon D \times [0,1] \to \mathbb{RP}^k$ be such a based null-homotopy for $\beta|_D$, that is,

- $H_D((A,A),0) = \beta(A,A)$ for all $(A,A) \in D$,
- $H_D((A,A),1) = *$ for all $(A,A) \in D$, and
- $H_D((*,*),t) = *$ for all $t \in [0,1]$.

Let $H_E\colon E \times [0,1] \to \mathbb{RP}^k$ be the constant homotopy defined by $H(e,t) = \beta(e)$ for all $(e,t) \in E \times [0,1]$. Since $D \cap E = \{(*,*)\}$, both homotopies H_D and H_E can be pieced together to obtain a new homotopy $H\colon (D \cup E) \times [0,1] \to \mathbb{RP}^k$ satisfying

(*a*) $H(z,0) = \beta(z)$ for all $z \in D \cup E$,

(*b*) $H((A,*),1) = A = H((*,A),1)$ for all $A \in \mathbb{RP}^n$, and

(*c*) $H((A,A),1) = *$ for all $(A,A) \in D$.

In other words, the map $H(\_\_,1)\colon D \cup E \to \mathbb{RP}^k$ fulfills the properties we need. So, in the rest of the proof we extend H to all $\mathbb{RP}^n \times \mathbb{RP}^n \times [0,1]$.
Consider the commutative diagram

$$\begin{tikzcd}
D \cup E \arrow[r, hook, "\iota_0"] \arrow[d, hook] & (D \cup E) \times [0,1] \arrow[d, hook] \arrow[ddr, bend left, "H"] & \\
\mathbb{RP}^n \times \mathbb{RP}^n \arrow[r, hook, "\iota_0"'] \arrow[drr, bend right, "\beta"'] & (\mathbb{RP}^n \times \mathbb{RP}^n) \times [0,1] \arrow[dr, dashed, "G"] & \\
 & & \mathbb{RP}^k,
\end{tikzcd} \tag{4.2}$$

where ι_0 is the closed embedding $t \mapsto (t,0)$. Since $D \cup E \hookrightarrow \mathbb{RP}^n \times \mathbb{RP}^n$ is a cofibration, there exists a homotopy $G\colon (\mathbb{RP}^n \times \mathbb{RP}^n) \times [0,1] \to \mathbb{RP}^k$ making commute diagram (4.2). Finally, we define $\alpha(A,B) = G((A,B),1)$ for all $(A,B) \in \mathbb{RP}^n \times \mathbb{RP}^n$. By construction, α is a strong axial map such that $\alpha(A,A) = *$ for all $A \in \mathbb{RP}^n$. □

Proposition 4.6. *Let $n \in \mathbb{N}$ with $n \neq 1,3,7$. There exists a strong nonsingular map*

$$f\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{\mathbf{TC}(\mathbb{RP}^n)+1}$$

such that $f(x,x) = \lambda_x \cdot *$, where $* = (1,0,\dots,0) \in \mathbb{R}^{\mathbf{TC}(\mathbb{RP}^n)+1}$, $\lambda_x \geq 0$ for all $x \in \mathbb{R}^{n+1}$ and $\lambda_x = 0$ if and only if $x = 0$.

Proof. Set $k = \mathbf{TC}(\mathbb{RP}^n)$. By Lemma 4.5, there exists a strong axial map $\alpha\colon \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$ such that $\alpha(A,A) = *$ for all $A \in \mathbb{RP}^n$. Likewise, in view of the Lifting criterion, there exists a based map $g\colon (\mathbb{S}^n \times \mathbb{S}^n, (*,*)) \to (\mathbb{S}^k, *)$ fitting in a commutative diagram

$$\begin{tikzcd} \mathbb{S}^n \times \mathbb{S}^n \arrow[r, "g"] \arrow[d, "p\times p"'] & \mathbb{S}^k \arrow[d, "p"] \\ \mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n \arrow[r, "\alpha"'] & \mathbb{R}\mathrm{P}^k . \end{tikzcd}$$

Here, $p\colon \mathbb{S}^\ell \to \mathbb{R}\mathrm{P}^\ell$ stands for the universal covering of $\mathbb{R}\mathrm{P}^\ell$.
The crux of the proof is that the condition $g(*,*) = *$ implies that both restrictions $g|_{*\times\mathbb{S}^n}$ and $g|_{\mathbb{S}^n\times *}$ are the equatorial inclusion of $\mathbb{S}^n$ into $\mathbb{S}^k$, while g is constant in the diagonal of $\mathbb{S}^n$ by the same reason. Therefore, the required strong nonsingular map $f\colon \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\to\mathbb{R}^{k+1}$ is given by

$$f(\lambda x, \mu y) = \lambda\mu g(x,y)$$

for all $x, y \in \mathbb{S}^n$ and $\lambda, \mu \geq 0$. □

4.2 Polyhedral products of projective spaces

Theorems 4.7 and 4.12 are the main results of this section. The first one computes $\mathbf{TC}(\underline{P}^K)$ and, in particular, shows that the Iwase-Sakai conjecture is valid for $\underline{P}^K$ under suitable conditions. The second one explores some sequential topological complexities of $\underline{P}^K$ when the based family $\underline{G}$ is constructed out of even dimensional real projective spaces. The proof of the first fact will be deferred until we have analyzed some interesting consequences.

Theorem 4.7. *Let $\underline{P}^K$ denote the polyhedral product determined by an abstract simplicial complex K and a based family $\underline{P} = \{(\mathbb{R}\mathrm{P}^{n_i}, *)\}_{i=1}^m$ of real projective spaces. If $\mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i}) = \mathrm{zcl}(\mathbb{R}\mathrm{P}^{n_i};\mathbb{Z}_2)$ for all $i \in \{1,\ldots,m\}$, then*

$$\mathbf{TC}(\underline{P}^K) = \mathbf{TC}^M(\underline{P}^K) = \max\Big\{\sum_{i\in\sigma_1\triangle\sigma_2} n_i + \sum_{i\in\sigma_1\cap\sigma_2}\mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})\colon \sigma_1,\sigma_2\in K\Big\},$$

where $\sigma_1\triangle\sigma_2 = (\sigma_1\setminus\sigma_2)\cup(\sigma_2\setminus\sigma_1)$ is the symmetric difference of σ_1 and σ_2

Remark 4.8. From [7] we know that the equality $\mathbf{TC}(\mathbb{R}\mathrm{P}^n) = \mathrm{zcl}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2)$ holds precisely in the following cases:

1. $n \in \{1,3,7\}$, i.e., $\mathbb{R}\mathrm{P}^n$ is parallelizable;
2. $n = 2^a + b$ with $a \geq 1$ and $b \in \{0,1\}$;
3. $n = 6$.

Corollary 4.9. *If $\mathbb{R}\mathrm{P}^{n_i}$ is parallelizable for all $i \in \{1,\ldots,m\}$, then*

$$\mathbf{TC}(\underline{P}^K) = \mathbf{TC}^M(\underline{P}^K) = \max\Big\{\sum_{i\in\sigma_1\cup\sigma_2} n_i\colon \sigma_1,\sigma_2\in K\Big\}.$$

Example 4.10. As we indicated at the beginning of the chapter, there are instances of Theorem 4.7 where the topological complexity exhibits a mixed cat/**TC** phenomenon. For example, take $K = \partial\Delta^2$, where Δ^2 is the standard 2-simplex, and $n_1 = n_2 = n_3 = 2^a$ with $a \geq 0$. Since $\mathbf{TC}(\mathbb{R}\mathrm{P}^{2^a}) = 2^{a+1} - 1$, we get the following data

σ_1	σ_2	$\sum_{i\in\sigma_1\triangle\sigma_2} n_i$	$\sum_{i\in\sigma_1\cap\sigma_2} \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})$
$\{1,2\}$	$\{1,2\}$	0	$2^{a+2} - 2$
$\{1,2\}$	$\{1,3\}$	2^{a+1}	$2^{a+1} - 1$
$\{1,2\}$	$\{2,3\}$	2^{a+1}	$2^{a+1} - 1$
$\{1,3\}$	$\{1,3\}$	0	$2^{a+2} - 2$
$\{1,3\}$	$\{2,3\}$	2^{a+1}	$2^{a+1} - 1$
$\{2,3\}$	$\{2,3\}$	0	$2^{a+2} - 2$

.

By Theorem 4.7, $\mathbf{TC}(\underline{P}^K) = \mathbf{TC}^M(\underline{P}^K) = 2^{a+2} - 1$, a maximum realized by maximal simplices σ_1 and σ_2 of K with $|\sigma_1\triangle\sigma_2| = 2$ and $|\sigma_1 \cap \sigma_2| = 1$.

Theorem 4.7 does not seem to have a counterpart in the context of $\mathbf{TC}_r$; however, it was shown in [7] that, if n is even and r is large enough, $\mathbf{TC}_r(\mathbb{R}\mathrm{P}^n) = r\cdot \mathrm{cat}(\mathbb{R}\mathrm{P}^n) = \mathrm{zcl}_r(\mathbb{R}\mathrm{P}^n) = rn$. We will see in Theorem 4.12 that such a behavior spreads in the context of polyhedral products whose factors are even dimensional real projective spaces. Before stating such a result, we need set forth some terminology.

Definition 4.11. [11, Definition 1.1] Let n be a natural number. If $n = \sum_j \epsilon_j 2^j$ with $\epsilon_j \in \{0,1\}$ is the binary representation of n, we define $Z_i(n)$[1] and $S(n)$[2] by setting

$$Z_i(n) = \sum_{j=0}^{i} (1-\epsilon_j)2^j$$

and

$$S(n) = \{i \colon \epsilon_i = \epsilon_{i-1} = 1 \text{ and } \epsilon_{i+1} = 0\}.$$

Theorem 4.12. *If all n_i are even, then*

$$\begin{aligned}\mathbf{TC}_r(\underline{P}^K) &= r\cdot\max\{n_{i_1} + \cdots + n_{i_\ell} \colon \{i_1,\ldots,i_\ell\} \in K\}\\ &= r\cdot \mathrm{cat}(\underline{P}^K)\end{aligned}$$

for $r \geq \max\{M(n_1),\ldots,M(n_m)\}$, where

$$M(n_i) := \max\left\{3, \left\lceil \frac{2^{k+1}-1}{Z_k(n_i)} \right\rceil : k \in S(n_i)\right\}$$

for all $i \in \{1,\ldots,m\}$.

[1] $Z_i(n)$ is the sum of 2-powers less than or equal to 2^i corresponding to the 0's in the binary representation of n.

[2] The elements of $S(n)$ are those that begin (from the left) a sequence of two or more consecutive 1's in the binary expansion of n.

Remark 4.13. Our first proof of Theorem 4.12 was obtained by using a slight variation of [23, Theorem 5.3], where the topological complexity of a general polyhedral product is calculated in terms of the cone length of their polyhedral product factors. Since we have not defined such a notion throughout this thesis, and for the sake of readability, we have chosen an alternative proof based on Theorem 2.17.

Proof of Theorem 4.12. In what follows we verify the hypotheses of Theorem 2.17. It is clear that the family $\underline{P}$ is LS-logarithmic, while the condition

$$\mathbf{TC}_r(\mathbb{R}\mathrm{P}^{n_i}) = r\cdot \mathrm{cat}(\mathbb{R}\mathrm{P}^{n_i}) = \mathrm{zcl}_r(\mathbb{R}\mathrm{P}^{n_i}) = rn_i$$

follows from [11, Corollary 3.4] and the assumption on r. So, it remains to check the $\mathbf{TC}_r$-logarithmicity of the family $\underline{P}$. The proof methodology of such a fact is standard: note that

$$\begin{aligned} r\left(\sum_{j=1}^{k} n_{i_j}\right) = \sum_{j=1}^{k} \mathrm{zcl}_r(\mathbb{R}\mathrm{P}^{n_{i_j}};\mathbb{Z}_2) \leq \mathrm{zcl}_r\left(\prod_{j=1}^{k}\mathbb{R}\mathrm{P}^{n_{i_j}};\mathbb{Z}_2\right) &\leq \mathbf{TC}_r\left(\prod_{j=1}^{k}\mathbb{R}\mathrm{P}^{n_{i_j}}\right) \\ &\leq r\left(\sum_{j=1}^{k} n_{i_j}\right), \end{aligned}$$

where the first inequality follows from [9, Lemma 2.1], and the second and the third ones come from Theorem 2.9.
In summary, for $r \geq \max\{M(n_1),\ldots,M(n_m)\}$,

$$\mathbf{TC}_r(\underline{P}^K) = r\cdot \mathrm{cat}(\underline{P}^K) = \max\left\{\sum_{i\in\sigma}\mathbf{TC}_r(\mathbb{R}\mathrm{P}^{n_i})\colon \sigma\in K\right\},$$

which completes the proof. $\square$

We now delve into the proof of Theorem 4.7. We first treat the lower bound for $\mathbf{TC}(\underline{P}^K)$:

Proposition 4.14. *Let $\underline{P}^K$ be as in Theorem 4.7. $\mathbf{TC}(\underline{P}^K)$ is bounded from below by*[3]

$$\max\left\{\sum_{i\in\sigma_1\triangle\sigma_2} n_i + \sum_{i\in\sigma_1\cap\sigma_2}\mathrm{zcl}(\mathbb{R}\mathrm{P}^{n_i};\mathbb{Z}_2)\colon \sigma_1,\sigma_2\in K\right\}.$$

Since $\mathbf{TC}(\underline{P}^K) \leq \mathbf{TC}^M(\underline{P}^K)$, the proof of Theorem 4.7 will be complete once we prove:

Proposition 4.15. *Let $\underline{P}^K$ be as in Theorem 4.7. $\mathbf{TC}^M(\underline{P}^K)$ is estimated from above by*

$$\mathcal{N}^{(n_1,\ldots,n_m)}(K) = \max\left\{\sum_{i\in\sigma_1\triangle\sigma_2} n_i + \sum_{i\in\sigma_1\cap\sigma_2}\mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})\colon \sigma_1,\sigma_2\in K\right\}.$$

[3] Notice that $n \leq \mathrm{zcl}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \leq \mathbf{TC}(\mathbb{R}\mathrm{P}^n)$ for all $n \geq 1$. Therefore, both maxima in Propositions 4.14 and 4.15 are achieved by simplices that are not contained in any other simplex of K.

4.2.1 Proof of Proposition 4.14

The principal ingredient required is the mod-2 cohomology ring of $\underline{P}^K$. We state it in the following proposition as an easy application of [4, Theorem 2.35].

Proposition 4.16. *The mod-2 cohomology ring of $\underline{P}^K$ is given by*

$$H^*(\underline{P}^K;\mathbb{Z}_2) \cong \bigotimes_{i=1}^{m} H^*(\mathbb{R}\mathrm{P}^{n_i};\mathbb{Z}_2)/I(K),$$

where $I(K)$ is the generalized Stanley-Reisner ideal, that is, $I(K)$ is generated by all elements $x_{r_1} \otimes x_{r_2} \otimes \cdots \otimes x_{r_t}$, with $x_{r_i} \in \overline{H}^*(\mathbb{R}\mathrm{P}^{n_{r_i}};\mathbb{Z}_2)$ and the simplex $\{r_1,\ldots,r_t\} \notin K$.

For each $i \in \{1,\ldots,m\}$, let $\pi_i\colon \underline{P}^K \to \mathbb{R}\mathrm{P}^{n_i}$ be the canonical projection onto the ith coordinate. If $v_i \in H^1(\mathbb{R}\mathrm{P}^{n_i};\mathbb{Z}_2)$ stands for the generator of $H^1(\mathbb{R}\mathrm{P}^{n_i};\mathbb{Z}_2)$, we define u_i to be the pullback of v_i under π_i. Hence, the mod-2 cohomology ring of $\underline{P}^K$ can be rewritten as follows:

$$H^*(\underline{P}^K;\mathbb{Z}_2) \cong \mathbb{Z}_2[u_1,\ldots,u_m]/\left(\{u_1^{n_1+1},\ldots,u_m^{n_m+1}, u_{r_1}\cdots u_{r_t}\colon \{r_1,\ldots,r_t\} \notin K\}\right).$$

In particular, a graded basis for $H^*(\underline{P}^K;\mathbb{Z}_2)$ consists of the monomials

$$u_1^{e_1}\cdots u_m^{e_m}$$

having $0 \le e_i \le n_i$ and $\{i\colon e_i > 0\} \in K$.

Proof of Proposition 4.14. Let σ_1 and σ_2 be two simplices of K. Assume $\sigma_1 \setminus \sigma_2 = \{i_1,\ldots,i_r\}$, $\sigma_2 \setminus \sigma_1 = \{j_1,\ldots,j_s\}$, and $\sigma_1 \cap \sigma_2 = \{k_1,\ldots,k_w\}$. By Theorem 2.9, it suffices to show that the product, in $[H^*(\underline{P}^K;\mathbb{Z}_2)]^{\otimes 2}$,

$$\left(\prod_{i\in\sigma_1\triangle\sigma_2}(u_i\otimes 1 + 1\otimes u_i)^{n_i}\right)\left(\prod_{i\in\sigma_1\cap\sigma_2}(u_i\otimes 1 + 1\otimes u_i)^{\mathrm{zcl}(\mathbb{R}\mathrm{P}^{n_i};\mathbb{Z}_2)}\right) \neq 0. \tag{4.3}$$

The left-hand side factor in (4.3) agrees with

$$\begin{aligned}
&\left(\sum_{\ell_1=0}^{n_{i_1}}\binom{n_{i_1}}{\ell_1} u_{i_1}^{n_{i_1}-\ell_1}\otimes u_{i_1}^{\ell_1}\right)\cdots\left(\sum_{\ell_r=0}^{n_{i_r}}\binom{n_{i_r}}{\ell_r} u_{i_r}^{n_{i_r}-\ell_r}\otimes u_{i_r}^{\ell_r}\right)\\
&\quad\cdot\left(\sum_{q_1=0}^{n_{j_1}}\binom{n_{j_1}}{q_1} u_{j_1}^{n_{j_1}-q_1}\otimes u_{j_1}^{q_1}\right)\cdots\left(\sum_{q_s=0}^{n_{j_s}}\binom{n_{j_s}}{q_s} u_{j_s}^{n_{j_s}-q_s}\otimes u_{j_s}^{q_s}\right)\\
&\quad=\left(\sum_{\substack{0\le\ell_t\le n_{i_t}\\1\le t\le r}}\binom{n_{i_1}}{\ell_1}\cdots\binom{n_{i_r}}{\ell_r} u_{i_1}^{n_{i_1}-\ell_1}\cdots u_{i_r}^{n_{i_r}-\ell_r}\otimes u_{i_1}^{\ell_1}\cdots u_{i_r}^{\ell_r}\right) &(4.4)\\
&\qquad\cdot\left(\sum_{\substack{0\le q_t\le n_{j_t}\\1\le t\le s}}\binom{n_{j_1}}{q_1}\cdots\binom{n_{j_s}}{q_s} u_{j_1}^{n_{j_1}-q_1}\cdots u_{j_s}^{n_{j_s}-q_s}\otimes u_{j_1}^{q_1}\cdots u_{j_s}^{q_s}\right). &(4.5)
\end{aligned}$$

On the other hand, recall that $\operatorname{zcl}(\mathbb{R}\mathrm{P}^{n_i};\mathbb{Z}_2) = 2^{\theta_i} - 1$ with $2^{\theta_i - 1} \leq n_i < 2^{\theta_i}$, and in particular $n_i \leq 2^{\theta_i} - 1 < 2n_i$. With this in mind, the right-hand side factor in (4.3) equals

$$\left(\sum_{h_1 = 2^{\theta_{k_1}} - 1 - n_{k_1}}^{n_{k_1}} u_{k_1}^{2^{\theta_{k_1}} - 1 - h_1} \otimes u_{k_1}^{h_1} \right) \cdots \left(\sum_{h_w = 2^{\theta_{k_w}} - 1 - n_{k_w}}^{n_{k_w}} u_{k_w}^{2^{\theta_{k_w}} - 1 - h_w} \otimes u_{k_w}^{h_w} \right)$$
$$= \left(\sum_{\substack{2^{\theta_{k_t}} - 1 - n_{k_t} \leq h_t \leq n_{k_t} \\ 1 \leq t \leq w}} u_{k_1}^{2^{\theta_{k_1}} - 1 - h_1} \cdots u_{k_w}^{2^{\theta_{k_w}} - 1 - h_w} \otimes u_{k_1}^{h_1} \cdots u_{k_w}^{h_w} \right). \tag{4.6}$$

The result follows by observing that, in the product of (4.4), (4.5), and (4.6), the basis element

$$u_{i_1}^{n_{i_1}} \cdots u_{i_r}^{n_{i_r}} u_{k_1}^{n_{k_1}} \cdots u_{k_w}^{n_{k_w}} \otimes u_{j_1}^{n_{j_1}} \cdots u_{j_s}^{n_{j_s}} u_{k_1}^{2^{\theta_{k_1}} - 1 - n_{k_1}} \cdots u_{k_w}^{2^{\theta_{k_w}} - 1 - n_{k_w}}$$

arises only from the product of:

- $u_{i_1}^{n_{i_1}} \cdots u_{i_r}^{n_{i_r}} \otimes 1$ in (4.4), with $\ell_t = 0$ for all $t \in \{1, \ldots, r\}$;
- $1 \otimes u_{j_1}^{n_{j_1}} \cdots u_{j_s}^{n_{j_s}}$ in (4.5), with $q_t = n_{j_t}$ for all $t \in \{1, \ldots, s\}$;
- $u_{k_1}^{n_{k_1}} \cdots u_{k_w}^{n_{k_w}} \otimes u_{k_1}^{2^{\theta_{k_1}} - 1 - n_{k_1}} \cdots u_{k_w}^{2^{\theta_{k_w}} - 1 - n_{k_w}}$ in (4.6), with $h_t = 2^{\theta_{k_t}} - 1 - n_{k_t}$ for all $t \in \{1, \ldots, w\}$.

□

4.2.2 *Proof of Proposition 4.15*

Notice that the n-dimensional compact smooth manifold $\mathbb{R}\mathrm{P}^n$ is an ANR space. Consequently, $\underline{P}^\sigma$ is an ANR for each $\sigma \in K$, and therefore $\underline{P}^K = \cup_{\sigma \in K} \underline{P}^\sigma$ is an ANR as well. Furthermore, in view of Theorem 3.13, it suffices to show that

$$\mathbf{TC}_g^{FH}(\underline{P}^K) \leq \mathcal{N}^{(n_1, \ldots, n_m)}(K) = \max \left\{ \sum_{i \in \sigma_1 \triangle \sigma_2} n_i + \sum_{i \in \sigma_1 \cap \sigma_2} \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i}) \colon \sigma_1, \sigma_2 \in K \right\}.$$

In order to attain this result, we will construct a (not necessarily open) cover of $\underline{P}^K \times \underline{P}^K$ together with continuous local sections of the fibration $e_2 \colon (\underline{P}^K)^{[0,1]} \longrightarrow \underline{P}^K \times \underline{P}^K$ fulfilling the conditions of Definition 3.4 (b). The main ingredients we need are the local domains and rules used in [20] to determine the topological complexity of $\mathbb{R}\mathrm{P}^n$. For the sake of readability we provide such details.

For each $i \in \{1, \ldots, m\}$, we fix once and for all a nonsingular map

$$f_i \colon \mathbb{R}^{n_i + 1} \times \mathbb{R}^{n_i + 1} \longrightarrow \mathbb{R}^{\mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i}) + 1}$$

such that

$$f_i(x,x) = (\lambda_x, 0, \dots, 0) \tag{4.7}$$

for all $x \in \mathbb{R}^{n_i+1}$ with $\lambda_x \geq 0$ as well as $\lambda_x > 0$ for $x \neq 0$. Further, if $n_i \neq 1, 3, 7$ we require f_i to be strong:

$$f_i(*, x) = x = f_i(x, *) \tag{4.8}$$

for all $x \in \mathbb{R}^{n_i+1}$ (Proposition 4.6). Also, for each $k \in \{0, 1, \dots, \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})\}$, let f_{ik} be the $(k+1)$st coordinate map of f_i. Set

$$V_{i0} = \{(A, B) \in \mathbb{R}\mathrm{P}^{n_i} \times \mathbb{R}\mathrm{P}^{n_i} \colon f_{i0}(a, b) \neq 0 \text{ for some points } a \in A \text{ and } b \in B\}$$

and, for $k \in \{1, \dots, \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})\}$,

$$V_{ik} = \{(A, B) \in \mathbb{R}\mathrm{P}^{n_i} \times \mathbb{R}\mathrm{P}^{n_i} \colon A \neq B \text{ and } f_{ik}(a, b) \neq 0 \text{ for some points } a \in A \text{ and } b \in B\}.$$

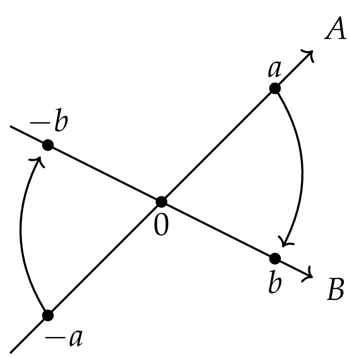
Note that the diagonal of $\mathbb{R}\mathrm{P}^{n_i}$ is contained in V_{i0} in view of (4.7), and hence

$$\{V_{i0}, V_{i1}, \dots, V_{i\,\mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})}\}$$

is an open cover of $\mathbb{R}\mathrm{P}^{n_i} \times \mathbb{R}\mathrm{P}^{n_i}$.

On each set V_{ik} there is a continuous local section $\lambda_{ik} \colon V_{ik} \to (\mathbb{R}\mathrm{P}^{n_i})^{[0,1]}$ of the fibration $e_2 \colon (\mathbb{R}\mathrm{P}^{n_i})^{[0,1]} \to \mathbb{R}\mathrm{P}^{n_i} \times \mathbb{R}\mathrm{P}^{n_i}$ defined as follows:

- If $A = B$, we choose the constant path at A.
- If $A \neq B$, we take unit vectors $a \in A$ and $b \in B$ such that $f_{ik}(a, b) > 0$. Instead of such vectors we could have chosen $-a$ and $-b$; nevertheless, both pairs a, b and $-a, -b$ determine the same orientation of the plane spanned by A and B. In this case, we rotate with constant velocity A toward B in the plane spanned by A and B in the positive direction determined by the orientation:



Clearly λ_{ik} is continuous on V_{ik} if $k \in \{1, \dots, \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})\}$. The continuity of λ_{i0} on V_{i0} follows from (4.7) and the fact that $\lambda_{i0}(A, A)$ is the constant path for all $A \in \mathbb{R}\mathrm{P}^{n_i}$.

The open sets V_{ik} might not be disjoint, but this can easily be fixed by redefining

$$U_{ik} = V_{ik} \setminus (V_{i0} \cup \dots \cup V_{i(k-1)})$$

for each $k \in \{1, \dots, \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})\}$ and $i \in \{1, \dots, m\}$, so $U_{i0} = V_{i0}$. We are interested in the number of zeroes produced by taking an element in the sets U_{ik}. Specifically, we say that a pair of lines (A, B) in $\mathbb{R}^{n_i+1}$ produces k initial zeroes if $(A, B) \in U_{ik}$, with $k \in \{0, 1, \dots, \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})\}$. The justification of the latter convention comes from the observation: $(A, B) \in U_{ik}$ if and only if

$$f_{i0}(a, b) = \cdots = f_{i(k-1)}(a, b) = 0 \neq f_{ik}(a, b)$$

for some (and therefore any) vectors $a \in A$ and $b \in B$. The number of zeroes produced by (A, B) is denoted $Z(A, B)$.

An element (A_1, A_2) of $\underline{P}^K \times \underline{P}^K$ is thought of as a matrix of size $m \times 2$, i.e.,

$$(A_1, A_2) = \begin{pmatrix} A_{11} & A_{12} \\ \vdots & \vdots \\ A_{m1} & A_{m2} \end{pmatrix},$$

where each column belongs to $\underline{P}^K$, say $(A_{11}, \dots, A_{m1}) \in \underline{P}^{\sigma_1}$ and $(A_{12}, \dots, A_{m2}) \in \underline{P}^{\sigma_2}$ for some $\sigma_1, \sigma_2 \in K$. We know that each row (A_{i1}, A_{i2}) of the matrix (A_1, A_2) lies in a unique set U_{ik} with $k = Z(A_{i1}, A_{i2})$. Hence, the number of zeroes determined by (A_1, A_2), denoted $Z(A_1, A_2)$, is defined to be the sum of zeroes produced by the rows (A_{i1}, A_{i2}) of (A_1, A_2), that is,

$$Z(A_1, A_2) := \sum_{i=1}^{m} Z(A_{i1}, A_{i2}).$$

As a first approximation, it is clear that $Z(A_1, A_2) \leq \sum_{i \in \sigma_1 \cup \sigma_2} \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})$; nevertheless, the previous upper bound for $Z(A_1, A_2)$ can be streamlined because the number of zeroes generated by the rows of type either $(A_{i1}, *)$ or $(*, A_{i2})$ of (A_1, A_2) needs to be considered more carefully in view of (4.8). In fact, note that if $i \in \sigma_1 \setminus \sigma_2$, then $Z(A_{i1}, A_{i2}) = Z(A_{i1}, *) \leq n_i$ since f_i fulfills (4.8) when $n_i \neq 1, 3, 7$ (recall that $n_i = \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})$ if $n_i = 1, 3, 7$). In like manner, if $i \in \sigma_2 \setminus \sigma_1$, then $Z(A_{i1}, A_{i2}) = Z(*, A_{i2}) \leq n_i$. In summary,

$$Z(A_1, A_2) \leq \sum_{i \in \sigma_1 \triangle \sigma_2} n_i + \sum_{i \in \sigma_1 \cap \sigma_2} \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i}) \leq \mathcal{N}^{(n_1, \dots, n_m)}(K),$$

and hence we have proved:

Proposition 4.17. *The sets $W_j = \{(A_1, A_2) \in \underline{P}^K \times \underline{P}^K \colon Z(A_1, A_2) = j\}$, with j belonging to $\{0, 1, \dots, \mathcal{N}^{(n_1, \dots, n_m)}(K)\}$, form a pairwise disjoint cover of $\underline{P}^K \times \underline{P}^K$.*

The proof of Proposition 4.15 will be complete once a local rule is constructed on each W_j. This is attained by splitting W_j into topological disjoint subsets (see Proposition 4.18 below), and then defining a local section of the fibration e_2 on each one of them.

A partition of j, with $j \in \{0, 1, \dots, \mathcal{N}^{(n_1, \dots, n_m)}(K)\}$, is an ordered tuple $(j_1, \dots, j_m)$ of nonnegative integers such that $j = j_1 + \cdots + j_m$ and $0 \leq j_i \leq \mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})$ for each $i \in \{1, \dots, m\}$. For such a partition of j, set

$$W_{(j_1, \dots, j_m)} = \{(A_1, A_2) \in \underline{P}^K \times \underline{P}^K \colon Z(A_{i1}, A_{i2}) = j_i \text{ for each } i \in \{1, \dots, m\}\}.$$

It is straightforward to see that

$$W_j = \bigsqcup_{(j_1,\ldots,j_m)} W_{(j_1,\ldots,j_m)}, \tag{4.9}$$

where the index runs over all partitions of j. We next show that (4.9) is a topological union, that is, W_j has the weak topology determined by the several $W_{(j_1,\ldots,j_m)}$.

Proposition 4.18. *Let $j \in \{0,1,\ldots,\mathcal{N}^{(n_1,\ldots,n_m)}(K)\}$. If $(j_1,\ldots,j_m)$ and $(r_1,\ldots,r_m)$ are two different partitions of j, then*

$$\overline{W_{(j_1,\ldots,j_m)}} \cap W_{(r_1,\ldots,r_m)} = \varnothing = W_{(j_1,\ldots,j_m)} \cap \overline{W_{(r_1,\ldots,r_m)}}.$$

Proof. Since $(j_1,\ldots,j_m) \neq (r_1,\ldots,r_m)$, there is a natural number $\ell \in \{1,\ldots,m\}$ with $j_\ell \neq r_\ell$, say $j_\ell < r_\ell$, while the equality $j_1 + \cdots + j_m = j = r_1 + \cdots + r_m$ forces the existence of another natural number $q \in \{1,\ldots,m\}$ such that $r_q < j_q$.
For elements $(A_1,A_2) \in W_{(j_1,\ldots,j_m)}$ and $(B_1,B_2) \in W_{(r_1,\ldots,r_m)}$ we have

$$(A_{\ell 1},A_{\ell 2}) \in U_{\ell j_\ell} \text{ and } (B_{\ell 1},B_{\ell 2}) \in U_{\ell r_\ell},$$

then $f_{\ell j_\ell}(a_{\ell 1},a_{\ell 2}) \neq 0$ for all $a_{\ell 1} \in A_{\ell 1}, a_{\ell 2} \in A_{\ell 2}$ and $f_{\ell j_\ell}(b_{\ell 1},b_{\ell 2}) = 0$ for all vectors $b_{\ell 1} \in B_{\ell 1}, b_{\ell 2} \in B_{\ell 2}$ since $j_\ell < r_\ell$. It is clear that the latter condition is inherited by elements of $\overline{W_{(r_1,\ldots,r_m)}}$, so the second equality of our proposition follows.
In the same way as we proceeded in the case $j_\ell < r_\ell$, the statement $\overline{W_{(j_1,\ldots,j_m)}} \cap W_{(r_1,\ldots,r_m)} = \varnothing$ is proved by using the condition $r_q < j_q$. □

In the rest of the section we construct a continuous local section of the end-points evaluation map e_2 on each $W_{(j_1,\ldots,j_m)}$ by adapting the techniques from [23]. As we noted in Example 2.18, we have to be careful in order to guarantee that the path connecting two points of $\underline{P}^K$ lies in $\underline{P}^K$, so many of our efforts are devoted to carrying out this task successfully.

For $i \in \{1,\ldots,m\}$, consider the canonical Riemannian structure in the unit n_i-sphere. Since the antipodal involution $\mathbb{S}^{n_i} \to \mathbb{S}^{n_i}$ is a fixed-point free properly discontinuous isometry, $\mathbb{R}\mathrm{P}^{n_i}$ inherits a canonical quotient Riemannian structure g_i. Let $L_i(\gamma)$ denote the resulting length of a smooth curve γ in $\mathbb{R}\mathrm{P}^{n_i}$, and let

$$d_i(x,y) = \inf\{L_i(\gamma)\colon \gamma \text{ is a geodesic on } \mathbb{R}\mathrm{P}^{n_i} \text{ from } x \text{ to } y\}$$

be the associated metric. Note that the curves $\lambda_{ik}(A,B)$ set forth at the middle of page 36 are geodesic on $\mathbb{R}\mathrm{P}^{n_i}$. Without loss of generality we can assume that each g_i is normalized so that any geodesic γ on $\mathbb{R}\mathrm{P}^{n_i}$ satisfies $L_i(\gamma) \leq 1/2$.

Now, we reparametrize the initial navigational instructions λ_{ik} in the following way: for $k \in \{0,1,\ldots,\mathbf{TC}(\mathbb{R}\mathrm{P}^{n_i})\}$, consider the section $\tau_{ik}\colon U_{ik} \to (\mathbb{R}\mathrm{P}^{n_i})^{[0,1]}$ of the end-points evaluation map $e_2\colon (\mathbb{R}\mathrm{P}^{n_i})^{[0,1]} \to \mathbb{R}\mathrm{P}^{n_i} \times \mathbb{R}\mathrm{P}^{n_i}$ where, for $(A_1,A_2) \in U_{ik}$,

$$\tau_{ik}(A_1,A_2)(t) = \begin{cases} A_1, & \text{if } d_{i1}+d_{i2} = 0; \\ \lambda_{ik}(A_1,A_2)(\frac{t}{d_{i1}+d_{i2}}), & \text{if } 0 \leq t \leq (d_{i1}+d_{i2}) \neq 0; \\ A_2, & \text{if } 0 \neq (d_{i1}+d_{i2}) \leq t \leq 1; \end{cases}$$

and $d_{ij} = d_i(A_j, *)$, $j = 1, 2$.
The path τ_{ik} is clearly continuous on the open subset of U_{ik} determined by the condition $d_{i1} + d_{i2} \neq 0$. The latter open subset of U_{ik} equals in fact U_{ik} unless $k = 0$, so that τ_{ik} is continuous on the whole U_{ik} for $k \in \{1, \ldots, \mathbf{TC}(\mathbb{RP}^{n_i})\}$. The continuity of τ_{i0} on U_{i0} follows from the continuity of λ_{i0} and the fact that $\lambda_{i0}(A, A)$ is the constant path for all $A \in \mathbb{RP}^{n_i}$.

Remark 4.19. Note that τ_{ik} is a reparametrization of λ_{ik} that makes the path λ_{ik} reach its end point at time $d_{i1} + d_{i2}$. In particular $\tau_{ik}(A_1, *)$ reaches $*$ at time d_{i1}.

Let $\varphi\colon \underline{P}^K \times \underline{P}^K \to (\prod_{i=1}^m \mathbb{RP}^{n_i})^{[0,1]}$ be the map defined by

$$\varphi(A_1, A_2) = (\varphi_1(A_{11}, A_{12}), \ldots, \varphi_m(A_{m1}, A_{m2})),$$

whose ith coordinate $\varphi_i(A_{i1}, A_{i2})$ is the path in $\mathbb{RP}^{n_i}$, from A_{i1} to A_{i2}, given by

$$\varphi_i(A_{i1}, A_{i2})(t) = \begin{cases} A_{i1}, & \text{if } 0 \leq t \leq t_{A_{i1}}; \\ \mu(A_{i1}, A_{i2})(t - t_{A_{i1}}), & \text{if } t_{A_{i1}} \leq t \leq 1. \end{cases} \tag{4.10}$$

Here, $t_{A_{i1}} = 1/2 - d_i(A_{i1}, *)$ and

$$\mu(A_{i1}, A_{i2}) = \begin{cases} \tau_{i0}(A_{i1}, A_{i2}), & \text{if } (A_{i1}, A_{i2}) \in U_{i0}; \\ \vdots & \vdots \\ \tau_{i\,\mathbf{TC}(\mathbb{RP}^{n_i})}(A_{i1}, A_{i2}), & \text{if } (A_{i1}, A_{i2}) \in U_{i\,\mathbf{TC}(\mathbb{RP}^{n_i})}. \end{cases} \tag{4.11}$$

By construction, the map φ is clearly a section of

$$e_2\colon (\prod_{i=1}^m \mathbb{RP}^{n_i})^{[0,1]} \to (\prod_{i=1}^m \mathbb{RP}^{n_i})^2.$$

Although φ is not a continuous global section of e_2, its restriction to each $W_{(j_1,\ldots,j_m)}$, where $(j_1, \ldots, j_m)$ is a partition of $j \in \{0, 1, \ldots, \mathcal{N}^{(n_1,\ldots,n_m)}(K)\}$, is continuous since formulas (4.11) can be rewritten as

$$\mu = \begin{cases} \tau_{i0}, & \text{if } j_i = 0; \\ \vdots & \vdots \\ \tau_{i\,\mathbf{TC}(\mathbb{RP}^{n_i})}, & \text{if } j_i = \mathbf{TC}(\mathbb{RP}^{n_i}). \end{cases}$$

Remark 4.20. In preparation for the proof of our final result (Proposition 4.23), we unravel formulas (4.10) by providing a complete description at the level of each polyhedral product factor.
Suppose $(A_{i1}, A_{i2}) \in U_{ik}$ with $k = Z(A_{i1}, A_{i2})$, the path $\varphi_i(A_{i1}, A_{i2})$ is described as follows:

- if $0 \leq t \leq 1/2 - d_i(A_{i1}, *)$, then stay at A_{i1};

- if $1/2 - d_i(A_{i1}, *) \le t \le 1/2 + d_i(A_{i2}, *)$, then move from A_{i1} to A_{i2} at constant speed;
- if $1/2 + d_i(A_{i2}, *) \le t \le 1$, then stay at A_{i2}.

Having spelled out formulas (4.11), the next examples explain the following critical situations: the motion from the base point toward an arbitrary element of $\mathbb{R}\mathrm{P}^{n_i}$, and vice versa. These cases illustrate in full the proof methodology in Proposition 4.23.

Example 4.21. Suppose $A_{i1} = *$ and let A_{i2} be any line through the origin in $\mathbb{R}^{n_i+1}$. Assume $(A_{i1}, A_{i2}) \in U_{ik}$ with $k = Z(A_{i1}, A_{i2})$. In this case, $t_{A_{i1}} = 1/2 - d_i(A_{i1}, *) = 1/2 - 0 = 1/2$. By Remark 4.20, the navigational instruction at level $\mathbb{R}\mathrm{P}^{n_i}$ is described as follows:

- if $0 \le t \le \frac{1}{2}$, then we stay at $*$;
- if $\frac{1}{2} \le t \le \frac{1}{2} + d_i(A_{i2}, *)$, then we move, by the corresponding local rule, from $*$ to A_{i2} at constant speed;
- if $\frac{1}{2} + d_i(A_{i2}, *) \le t \le 1$, then we stay at A_{i2}.

Example 4.22. In Example 4.21 we moved from the basepoint $*$ to A_{i2}. In this example we explain the opposite situation. Assume $A_{i2} = *$ and let A_{i1} be any line through the origin in $\mathbb{R}^{n_i+1}$. Suppose $(A_{i1}, A_{i2}) \in U_{ik}$ with $k = Z(A_{i1}, A_{i2})$. In this case, the path $\varphi_i(A_{i1}, A_{i2})$ is described as follows:

- if $0 \le t \le \frac{1}{2} - d_i(A_{i1}, *)$, then we stay at A_{i1};
- if $\frac{1}{2} - d_i(A_{i1}, *) \le t \le 1/2 + d_i(A_{i2}, *) = 1/2$, then we move from A_{i1} to $*$ at constant speed by making use of the corresponding local rule;
- if $\frac{1}{2} \le t \le 1$, then we stay at $*$.

Recall that φ was defined from $\underline{P}^K \times \underline{P}^K$ to $(\prod_{i=1}^m \mathbb{R}\mathrm{P}^{n_i})^{[0,1]}$; however, we next show that $\varphi(\underline{P}^K \times \underline{P}^K) \subseteq (\underline{P}^K)^{[0,1]}$, thus completing the proof of Proposition 4.15.

Proposition 4.23. *The image of φ is contained in $(\underline{P}^K)^{[0,1]}$.*

Proof. Let $(A_1, A_2) \in \underline{P}^K \times \underline{P}^K$, we need to prove that $\varphi(A_1, A_2)([0,1]) \subseteq \underline{P}^K$. Assume $(A_{11}, \ldots, A_{m1}) \in \underline{P}^{\sigma_1}$ and $(A_{12}, \ldots, A_{m2}) \in \underline{P}^{\sigma_2}$ with $\sigma_1, \sigma_2 \in K$. By Example 4.21, for all $i \notin \sigma_1$, $A_{i1} = *$ keeps its position through time $t \le 1/2$, so that $\varphi(A_1, A_2)([0, 1/2]) \subseteq \underline{P}^{\sigma_1} \subseteq \underline{P}^K$. Likewise, Example 4.22 shows that, for $i \notin \sigma_2$, the path $\varphi_i(A_{i1}, A_{i2})$ has reached its final position $A_{i2} = *$ at time $1/2$, so that $\varphi(A_1, A_2)([1/2, 1]) \subseteq \underline{P}^{\sigma_2} \subseteq \underline{P}^K$, and the proof is complete. $\square$

5

Topological groups

In chapter 4 we detailed how to construct a motion planner for $\underline{P}^K$ from motion planners of each polyhedral product factor $\mathbb{RP}^{n_i}$. Namely, the latter motion planners possess the following key properties:

- $\mathbb{RP}^{n_i} \times \mathbb{RP}^{n_i}$ has an open covering such that the diagonal of $\mathbb{RP}^{n_i}$ lies in only one open subset.

- The path connecting two equal lines in $\mathbb{R}^{n_i+1}$ is the constant path, hence the proposed section φ for the fibration $e_2 : (\underline{P}^K)^{[0,1]} \to \underline{P}^K \times \underline{P}^K$ is continuous by restricting it to a suitable subspace of $\underline{P}^K \times \underline{P}^K$.

This chapter arose from the observation that the preceding two items are fulfilled by topological groups, and therefore the construction of a motion planner for polyhedral products whose factors are locally compact connected CW topological groups comes for free. Because of this, many of our principal results are stated without proof.

This chapter is organized as follows. We first note that the proof of the equalities $\mathbf{TC}(G) = \mathrm{cat}(G) = \mathbf{TC}^M(G)$ given in [14, Lemma 2.7], where G is a connected Lie group, can be adapted to show that the Iwase-Sakai conjecture holds true for a locally compact connected CW topological group. We spell out the details since they will be useful for constructing an explicit motion planner leading to an upper bound for $\mathbf{TC}^M(\underline{G}^K)$, where $\underline{G}^K$ stands for the polyhedral product determined by an abstract simplicial complex K and a based family $\underline{G} = \{(G_i, e_i)\}_{i=1}^m$ of locally compact connected CW topological groups. Here e_i denotes the neutral element of G_i.

In particular, we show that the Iwase-Sakai conjecture holds true for $\underline{G}^K$ (Theorem 5.2) when dealing with an LS-logarithmic family $\underline{G}$, thus generalizing the equalities $\mathrm{cat}(G) = \mathbf{TC}(G) = \mathbf{TC}^M(G)$ noticed by Dranishnikov to the realm of polyhedral products.

5.1 Polyhedral products of topological groups

In this section we compute, under suitable conditions, the topological complexity of polyhedral products associated to an abstract simplicial complex K and a based family $\underline{G} = \{(G_i, e_i)\}_{i=1}^m$ of locally compact connected CW topological groups, where e_i stands for the neutral element of G_i. Indeed, following the guideline of the previous chapter, we state our principal result and its proof will be deferred until we analyze some interesting consequences.

We start by showing that the Iwase-Sakai conjecture holds for a locally compact connected CW topological group.

Proposition 5.1. *If G is a locally compact connected* CW *topological group, then* $\mathbf{TC}(G) = \mathbf{TC}^M(G) = \mathrm{cat}(G)$.

Proof. Since $\mathbf{TC}(G) \leq \mathbf{TC}^M(G)$, with the former agreeing with $\mathrm{cat}(G)$ (see [18, Lemma 8.2], where the same proof works for topological groups), it suffices to show that $\mathbf{TC}^M(G) \leq \mathrm{cat}(G)$. Furthermore, since a locally compact connected CW complex G is an ANR space (see Appendix II of [26]), we only need to show that $\mathbf{TC}^{FH}(G) \leq \mathrm{cat}(G)$, in view of Remark 3.5.
Let $n := \mathrm{cat}(G)$ and choose an open cover $\{N_0, \ldots, N_n\}$ of G together with homotopies $H_i \colon N_i \times [0,1] \to G$ satisfying $H_i(a,0) = a$ and $H_i(a,1) = e$, $a \in N_i$, for all $i \in \{0,1,\ldots,n\}$ (here e denotes the neutral element of G). We can assume that $e \notin N_i$ for all $i > 0$ and $H_0(e,t) = e$ for all $t \in [0,1]$, where the latter requirement follows from [10, Lemma 1.25] and the fact that $\{e\} \hookrightarrow G$ is a cofibration (recall, CW complexes have non-degenerate base points).
For each $i \in \{0,1,\ldots,n\}$, set $V_i := \{(a,b) \in G \times G \colon b^{-1}a \in N_i\}$. On each V_i of the open cover $\{V_0, \ldots, V_n\}$ of $G \times G$ there exists a continuous reserved section $s_i \colon V_i \to G^{[0,1]}$ of e_2 defined by $s_i(a,b)(t) = bH_i(b^{-1}a,t)$, $t \in [0,1]$. Note that $\Delta G \cap V_i = \emptyset$ for all $i \in \{1,\ldots,n\}$ and $s_0(a,a)(t) = aH_0(a^{-1}a,t) = aH_0(e,t) = ae = a$ with $(a,a) \in \Delta G$ and $t \in [0,1]$. Therefore $\mathbf{TC}^{FH}(G) \leq \mathrm{cat}(G)$, thus completing the proof. □

The next result generalizes the equalities given by Proposition 5.1 to the setting of polyhedral products defined by LS-logarithmic families of locally compact connected CW topological groups. Explicitly:

Theorem 5.2. *Let $\underline{G}^K$ be the polyhedral product determined by an abstract simplicial complex K and a based family $\underline{G} = \{(G_i, e_i)\}_{i=1}^m$ of locally compact connected* CW *topological groups, where e_i denotes the neutral element of G_i. If the family $\underline{G}$ is* LS*-logarithmic, i.e., if the equality*

$$\mathrm{cat}(G_{i_1} \times \cdots \times G_{i_k}) = \mathrm{cat}(G_{i_1}) + \cdots + \mathrm{cat}(G_{i_k})$$

holds true for any strictly increasing sequence $1 \leq i_1 < \cdots < i_k \leq m$, then

$$\mathbf{TC}(\underline{G}^K) = \mathbf{TC}^M(\underline{G}^K) = \mathcal{C}(G_1, \ldots, G_m; K) := \max\left\{ \sum_{i \in \sigma_1 \cup \sigma_2} \mathrm{cat}(G_i) \colon \sigma_1, \sigma_2 \in K \right\}.[1]$$

[1]Note that the maximum is realized by maximal simplices of K.

Remark 5.3. We point out that the LS category of $\underline{G}^K$ is also given in terms of the LS category of the polyhedral product factors. Explicitly, under the hypotheses of Theorem 5.2, Theorem 2.16 gives

$$\operatorname{cat}(\underline{G}^K) = \max\left\{\sum_{i\in\sigma}\operatorname{cat}(G_i)\colon \sigma\in K\right\}. \tag{5.1}$$

Additionally, Theorem 5.2 is analogous to the equality case of Theorem 2.16, but their hypotheses and conclusion contrast with those of Theorem 2.17 (for $r=2$). Indeed, a fact with the flavor of Theorem 5.2 is Corollary 4.9; in this sense, the former generalizes the behavior noticed for the topological complexity of polyhedral products whose factors are parallelizable real projective spaces.

Example 5.4. The family $\underline{G} = \{(U(n), e)\}_{i=1}^m$, where $U(n)$ denotes the n-th unitary group, fulfills the requirements of Theorem 5.2. The LS-logarithmicity hypothesis comes from [27, Example 3.3], while $\operatorname{cat}(U(n)) = n$ is guaranteed by [30, Theorem 1]. Theorem 5.2 thus gives

$$\begin{aligned}\mathbf{TC}(\underline{G}^K) = \mathbf{TC}^M(\underline{G}^K) &= \max\left\{\sum_{i\in\sigma_1\cup\sigma_2}\operatorname{cat}(U(n))\colon \sigma_1,\sigma_2\in K\right\}\\ &= \max\left\{\sum_{i\in\sigma_1\cup\sigma_2} n\colon \sigma_1,\sigma_2\in K\right\} = n\cdot\max\{|\sigma_1\cup\sigma_2|\colon \sigma_1,\sigma_2\in K\}.\end{aligned}$$

Setting $n=1$, we recover the result obtained in [23, Theorem 2.7] (for $r=2$ and all spheres being 1-dimensional). In fact, Theorem 5.2 also determines the topological complexity and the monoidal topological complexity of polyhedral products whose factors are unitary groups or special unitary groups of possibly different dimensions. In such a case, the LS-logarithmicity hypothesis is guaranteed by [27, Example 3.3].

Corollary 5.5. *Let $\underline{G}$ be an* LS*-logarithmic based family as the one in Theorem 5.2. If no G_i is contractible, then $\underline{G}^K$ admits an H-space structure if and only if K is the standard $(m-1)$-simplex.*

Proof. If K is the standard $(m-1)$-simplex, then $\underline{G}^K = G_1\times\cdots\times G_m$ is a topological group, and hence it is an H-space. On the other hand, suppose that $\underline{G}^K$ admits an H-space structure. Being connected and cellular, $\underline{G}^K$ satisfies

$$\max\left\{\sum_{i\in\sigma_1\cup\sigma_2}\operatorname{cat}(G_i)\colon \sigma_1,\sigma_2\in K\right\} = \mathbf{TC}(\underline{G}^K) = \operatorname{cat}(\underline{G}^K) = \max\left\{\sum_{i\in\sigma}\operatorname{cat}(G_i)\colon \sigma\in K\right\},$$

where the first equality comes from Theorem 5.2, the second one follows from [27, Theorem 1], and the third one is guaranteed by (5.1). Finally, bearing in mind that both maxima above agree and $\operatorname{cat}(G_i)\geq 1$ for all $i\in\{1,\ldots,m\}$, we conclude that K is the standard $(m-1)$-simplex. $\square$

We now delve into the proof of Theorem 5.2, starting with the following auxiliary result:

Proposition 5.6. *Let $\underline{G}$ be as in Theorem 5.2. If $\underline{G}$ is* LS*-logarithmic, then*

$$\mathbf{TC}(\underline{G}^K) \geq \max\left\{ \sum_{i\in\sigma_1\cup\sigma_2} \mathrm{cat}(G_i) \colon \sigma_1, \sigma_2 \in K \right\}.$$

Proof. From [3, Corollary 6.15] we get $\mathbf{TC}(\underline{G}^K) \geq \mathrm{cat}(\underline{G}^{\sigma_1} \times \underline{G}^{\sigma_2})$ for any disjoint simplices $\sigma_1, \sigma_2 \in K$. The result follows in view of the LS-logarithmicity hypothesis. □

Since $\mathbf{TC}(\underline{G}^K) \leq \mathbf{TC}^M(\underline{G}^K)$, the proof of Theorem 5.2 will be complete once we prove:

Proposition 5.7. *Let $\underline{G}$ be as in Theorem 5.2. Then $\mathbf{TC}^M(\underline{G}^K) \leq \mathcal{C}(G_1, \ldots, G_m; K)$.*

Remark 5.8. The previous upper bound for $\mathbf{TC}^M(\underline{G}^K)$ exhibits a similar behavior to the first statement of Theorem 2.16. As we pointed out in the paragraph preceding Definition 2.15, the proof of such a result given in [3] involves certain properties of open covers of spaces. In our case, as we shall see below, the proof of Proposition 5.7 is based on Proposition 5.1 and the tight control of the monoidal topological complexity of each polyhedral product factor.

5.1.1 *Proof of Proposition 5.7*

As we remarked in the proof of Proposition 5.1, locally compact connected CW complexes are ANR spaces. Consequently, $\underline{G}^\sigma$ is an ANR for each $\sigma \in K$, and therefore $\underline{G}^K = \cup_{\sigma\in K}\underline{G}^\sigma$ is an ANR as well. Furthermore, in view of Theorem 3.13, it suffices to show that $\mathbf{TC}_g^{FH}(\underline{G}^K) \leq \mathcal{C}(G_1, \ldots, G_m; K)$.

By following similar ideas to those of section 4.2.2, we construct a general (not necessarily open) cover of $\underline{G}^K \times \underline{G}^K$ by sets that admit continuous local sections of the fibration $e_2 \colon (\underline{G}^K)^{[0,1]} \to \underline{G}^K \times \underline{G}^K$.

For each $i \in \{1, \ldots, m\}$, let $\{V_{i0}, \ldots, V_{ic_i}\}$ be an open cover of $G_i \times G_i$ together with reserved sections $\lambda_{ik} \colon V_{ik} \to G_i^{[0,1]}$ of the end-points evaluation map $e_2 \colon G_i^{[0,1]} \to G_i \times G_i$. Here, c_i stands for the LS category of the corresponding polyhedral product factor G_i. As shown in the proof of Proposition 5.1, we can assume that the diagonal of G_i is contained in V_{i0} and $\Delta G_i \cap V_{ik} = \varnothing$ for all $k \in \{1, \ldots, c_i\}$.

The open sets V_{ik} might not be disjoint; however, this requirement can be achieved by redefining

$$U_{ik} = V_{ik} \setminus (V_{i0} \cup \cdots \cup V_{i(k-1)})$$

for each $k \in \{0, 1, \ldots, c_i\}$ and $i \in \{1, \ldots, m\}$ (so $U_{i0} = V_{i0}$). Instead of counting zeroes produced by taking an element in the sets U_{ik}, just as we did in the case of real projective spaces, we equivalently count the number of closed conditions. Specifically, we say that a pair (a, b) in $G_i \times G_i$ produces k closed conditions if $(a, b) \in U_{ik}$, with $k \in \{0, 1, \ldots, c_i\}$. The number of closed conditions produced by (a, b) is denoted by $C(a, b)$.

Recall, we regard an element (a_1, a_2) of $\underline{G}^K \times \underline{G}^K$ as a matrix of size $m \times 2$, i.e.,

$$(a_1, a_2) = \begin{pmatrix} a_{11} & a_{12} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{pmatrix},$$

where each column belongs to $\underline{G}^K$, say $(a_{11}, \ldots, a_{m1}) \in \underline{G}^{\sigma_1}$ and $(a_{12}, \ldots, a_{m2}) \in \underline{G}^{\sigma_2}$, with $\sigma_1, \sigma_2 \in K$. We know that each row (a_{i1}, a_{i2}) of the matrix (a_1, a_2) lies in a unique set U_{ik} for $k = C(a_{i1}, a_{i2}) \in \{0, 1, \ldots, c_i\}$, so the number of closed conditions determined by (a_1, a_2), denoted $C(a_1, a_2)$, is defined to be the sum of closed conditions produced by the rows (a_{i1}, a_{i2}) of (a_1, a_2), that is,

$$C(a_1, a_2) := \sum_{i=1}^{m} C(a_{i1}, a_{i2}).$$

It is clear that $C(a_1, a_2) \leq \sum_{i \in \sigma_1 \cup \sigma_2} c_i \leq \mathcal{C}(G_1, \ldots, G_m; K)$, and hence we have proved:

Proposition 5.9. *The sets $W_j = \{(a_1, a_2) \in \underline{G}^K \times \underline{G}^K \colon C(a_1, a_2) = j\}$, with j belonging to $\{0, 1, \ldots, \mathcal{C}(G_1, \ldots, G_m; K)\}$, form a pairwise disjoint cover of $\underline{G}^K \times \underline{G}^K$.*

The proof of Proposition 5.7 will be complete once a local rule is constructed on each W_j. Further, since the proof of Proposition 4.18 applies word for word to show that

$$W_j = \bigsqcup_{(j_1, \ldots, j_m)} W_{(j_1, \ldots, j_m)}$$

is a topological disjoint union, where the index runs over all partitions of j and

$$W_{(j_1, \ldots, j_m)} := \{(a_1, a_2) \in \underline{G}^K \times \underline{G}^K \colon C(a_{i1}, a_{i2}) = j_i \text{ for each } i \in \{1, \ldots, m\}\},$$

it suffices to construct a continuous local section of the fibration e_2 on each $W_{(j_1, \ldots, j_m)}$. Such a task is performed in the rest of the section and the proof methodology is similar to that developed in the last part of section 4.2.2. For this reason, we only provide the most fundamental aspects: the definition of the (not necessarily continuous) global section $\varphi : \underline{G}^K \times \underline{G}^K \to (\prod_{i=1}^{m} G_i)^{[0,1]}$ of e_2, and the key observation that the image of φ lands in the adequate subspace $(\underline{G}^K)^{[0,1]}$, and that φ is continuous when is restricted to $W_{(j_1, \ldots, j_m)}$.

For $i \in \{1, \ldots, m\}$, let d_i denote a metric on G_i. Since d_i is always equivalent to a bounded metric, we can assume that the diameter of G_i, defined by

$$\delta(G_i) = \sup\{d_i(a, b) \colon a, b \in G_i\},$$

is finite. Likewise, there is no problem in assuming that each diameter $\delta(G_i)$ is positive.

Now, we reparametrize the initial navigational instructions λ_{ik} in the following way: For $k \in \{0, 1 \ldots, c_i\}$, consider the section $\tau_{ik} \colon U_{ik} \to G_i^{[0,1]}$ of the end-points evaluation map

$e_2\colon G_i^{[0,1]} \to G_i \times G_i$ where, for $(a_1, a_2) \in U_{ik}$,

$$\tau_{ik}(a_1,a_2)(t) = \begin{cases} a_1, & \text{if } d_{i1}+d_{i2}=0; \\ \lambda_{ik}(a_1,a_2)\left(\frac{2\delta(G_i)t}{d_{i1}+d_{i2}}\right), & \text{if } 0 \le t \le \frac{d_{i1}+d_{i2}}{2\delta(G_i)} \neq 0; \\ a_2, & \text{if } 0 \neq \frac{d_{i1}+d_{i2}}{2\delta(G_i)} \le t \le 1; \end{cases} \tag{5.2}$$

and $d_{ij} = d_i(a_j, e_i)$, $j = 1, 2$. Recall, e_i denotes the neutral element of G_i. The path τ_{ik} is clearly continuous on the open subset of U_{ik} determined by the condition $d_{i1} + d_{i2} \neq 0$. The latter open subset of U_{ik} equals in fact U_{ik} unless $k = 0$, so that τ_{ik} is continuous on the whole U_{ik} for $k \in \{1, \ldots, c_i\}$. The continuity of τ_{i0} on U_{i0} follows from the continuity of the reserved section λ_{i0}.

Define the (not necessarily continuous) section

$$\varphi\colon \underline{G}^K \times \underline{G}^K \to (\prod_{i=1}^{m} G_i)^{[0,1]}$$

of the fibration e_2 to be $\varphi(a_1, a_2) = (\varphi_1(a_{11}, a_{12}), \ldots, \varphi_m(a_{m1}, a_{m2}))$, whose ith coordinate $\varphi_i(a_{i1}, a_{i2})$ is the path in G_i, from a_{i1} to a_{i2}, given by

$$\varphi_i(a_{i1},a_{i2})(t) = \begin{cases} a_{i1}, & \text{if } 0 \le t \le t_{a_{i1}}; \\ \mu(a_{i1},a_{i2})(t - t_{a_{i1}}), & \text{if } t_{a_{i1}} \le t \le 1. \end{cases} \tag{5.3}$$

Here, $t_{a_{i1}} = \frac{1}{2} - \frac{d_i(a_{i1},e_i)}{2\delta(G_i)}$ and

$$\mu(a_{i1},a_{i2}) = \begin{cases} \tau_{i0}(a_{i1},a_{i2}), & \text{if } (a_{i1},a_{i2}) \in U_{i0}; \\ \vdots & \vdots \\ \tau_{ic_i}(a_{i1},a_{i2}), & \text{if } (a_{i1},a_{i2}) \in U_{ic_i}. \end{cases} \tag{5.4}$$

By construction, the map φ is clearly a section of the fibration

$$e_2\colon (\prod_{i=1}^{m} G_i)^{[0,1]} \to \prod_{i=1}^{m} G_i \times \prod_{i=1}^{m} G_i.$$

Although φ fails to be a continuous global section of e_2, its restriction to each $W_{(j_1,\ldots,j_m)}$, where $(j_1, \ldots, j_m)$ is a partition of $j \in \{0, 1, \ldots, \mathcal{C}(G_1, \ldots, G_m; K)\}$, is continuous since formulas (5.4) can be rewritten as

$$\mu = \begin{cases} \tau_{i0}, & \text{if } j_i = 0; \\ \vdots & \vdots \\ \tau_{ic_i}, & \text{if } j_i = c_i. \end{cases}$$

Remark 5.10. From Remark 4.20 we learned that formulas (5.3) can be spelled out in order to understand better the motion provided by φ at the level of each polyhedral product factor G_i. Concretely, if $(a_{i1}, a_{i2}) \in U_{ik}$ for some $k \in \{0, 1, \ldots, c_i\}$, the path $\varphi_i(a_{i1}, a_{i2})$ is described as follows:

- if $0 \leq t \leq \frac{1}{2} - \frac{d_i(a_{i1}, e_i)}{2\delta(G_i)}$, then stay at a_{i1};
- if $\frac{1}{2} - \frac{d_i(a_{i1}, e_i)}{2\delta(G_i)} \leq t \leq \frac{1}{2} + \frac{d_i(a_{i2}, e_i)}{2\delta(G_i)}$, then move from a_{i1} to a_{i2} at constant speed via τ_{ik};
- if $\frac{1}{2} + \frac{d_i(a_{i2}, e_i)}{2\delta(G_i)} \leq t \leq 1$, then stay at a_{i2}.

Keeping in mind the proof of Proposition 4.23 and the previous observation, one can show that the map φ lands in the adequate place, that is, $\varphi(\underline{G}^K \times \underline{G}^K) \subseteq (\underline{G}^K)^{[0,1]}$; thus completing the proof of Proposition 5.7.

6

Conclusions

In this thesis we constructed motion planners to estimate from above the monoidal topological complexity of polyhedral products whose factors are real projective spaces and locally compact connected CW topological groups, separately. Furthermore, under suitable conditions, such upper bounds for $\mathbf{TC}^M(\underline{P}^K)$ and $\mathbf{TC}^M(\underline{G}^K)$ turned out to be optimal in a number of situations. In particular, we showed that the Iwase-Sakai conjecture holds true for the polyhedral products $\underline{P}^K$ and $\underline{G}^K$.

The engine that enabled to prove both results is a Fadell-Husseini point of view of monoidal topological complexity. We showed that if X is an ANR space, then $\mathbf{TC}^{FH}(X)$ and its generalized counterpart $\mathbf{TC}_g^{FH}(X)$ recover the original definition provided by Iwase and Sakai ($\mathbf{TC}^M(X)$), as well as the ones given by Dranishnikov ($\mathbf{TC}^{DM}(X)$) and García-Calcines ($\mathbf{TC}_g^M(X)$). Likewise, we provided a sufficient condition to guarantee the equality $\mathbf{TC}^M(X) = \mathbf{TC}(X)$ (Iwase-Sakai's conjecture) when dealing with ANR spaces.

On the other hand, our approach to constructing motion planners to determine $\mathbf{TC}(\underline{P}^K)$ does not seem to have a counterpart in the realm of $\mathbf{TC}_r$ for $r \geq 3$. Even though Theorem 4.12 provides a partial result, we do not know how to deal with the general case, that is, without assuming that each polyhedral product factor be an even dimensional real projective space and r be large enough. We think that this difficulty arises because we do not have local domains and rules to determine $\mathbf{TC}_r(\mathbb{R}\mathrm{P}^n)$ for $r \geq 3$ and $n \geq 2$. In fact, it was conjectured in [7, Conjecture 4.1] that such a motion planner algorithm can be constructed from a certain $(\mathbb{Z}_2)^{\times(r-1)}$-equivariant map; nonetheless, to the best of our knowledge, this problem remains open.

In the case of motion planners determining $\mathbf{TC}_r(\underline{G}^K)$ for $r \geq 3$, it is well-known that, for a topological group G, the equality $\mathbf{TC}_r(G) = \mathrm{cat}(G^{r-1})$ ($r \geq 2$) comes from an optimal motion planner ([6, Proposition 3.4]), i.e., there exist $\mathrm{cat}(G^{r-1}) + 1$ open sets covering G^r together with local sections of the fibration e_r. Indeed, a typical section s of e_r provides a motion between any points $g_1, \ldots, g_r$ of G by a "pivotal" strategy: from g_1 to g_2, from g_1 to g_3, and so on. However, there is no mention if the motion given by s from g_1 to $g_\ell = g_1$, for some $\ell \geq 2$, is the constant map at g_1. If such a requisite does not hold, formula (5.2) fails to be a *continuous* reparametrization of s, and therefore the techniques developed in the last section of chapter 5 cannot be implemented in the context of $\mathbf{TC}_r(\underline{G}^K)$ with $r \geq 3$.

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