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Homotopical Properties of Non-k-Equal Spaces

Thesis

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Propiedades homotópicas de espacios de no k iguales

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Abstract

The objective of this thesis is to present recent developments of the homotopy theory of non-k-equal spaces, a generalization of configuration spaces where collisions of k coordinates are avoided in n tuples of $\mathbb{R}^d$.
In particular, recent results about the Lusternik-Schnirelmann category, topological complexity, and sequential topological complexity for these spaces are developed. The invariants mentioned above are computed in full for $d = 1$, while for $d > 1$, bounds for k and n where these invariants have been determined are provided.
An essential part of the description of the cohomology of non-k-equal spaces is the use of a generalized Poincare duality for non-compact manifolds. For this reason, some necessary results to establish this duality are briefly exposed and, as an application, some non-trivial Massey products are computed geometrically.

Resumen

El objetivo de esta tesis es exponer desarrollos recientes de la teoría de homotopía de los espacios de no k iguales, una generalización de los espacios de configuraciones donde se evitan colisiones de k coordenadas en n tuplas de $\mathbb{R}^d$.
En particular se desarrollan resultados recientes acerca de la categoría de Lusternik-Schnirelmann, complejidad topológica y complejidad topológica secuencial para estos espacios. Los invariantes anteriormente mencionados se determinan totalmente para $d = 1$ mientras que para $d > 1$ se proporcionan cotas para k y n donde se han determinado esos invariantes.
Parte esencial para la descripción de la cohomología de los espacios de no k iguales es el uso de la dualidad de Poincaré generalizada para variedades no compactas. Por tal motivo, se exponen brevemente algunos resultados necesarios para establecer esa dualidad y, como una aplicación, algunos productos de Massey no triviales son calculados geométricamente.

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Contents

Introduction

The non-k-equal manifold $M_d^{(k)}(n)$—so named in Baryshnikov preprint [1]— is defined as the complement in $\left(\mathbb{R}^d\right)^n$ of the diagonal-subspace arrangement, $A_d^{(k)}(n)$, formed by the union of subspaces

$$A_I = \left\{(x_1,\ldots,x_n) \in \left(\mathbb{R}^d\right)^n \mid x_{i_1} = \cdots = x_{i_k}\right\},$$

where $I = \{i_1,\ldots,i_k\}$ runs through all cardinality-k subsets of the segment $\mathbf{n} = \{1,2,\ldots,n\}$. For the smallest possible value $k = 2$, $M_d^{(k)}(n)$ yields the classical and extensively studied configuration space of n distinct ordered points in $\mathbb{R}^d$. On the other extreme, $M_d^{(n)}(n) \simeq \mathbb{S}^{dn-d-1}$ whereas $M_d^{(k)}(n) = \left(\mathbb{R}^d\right)^n$ for $k > n$. So, the present thesis will only deal with the cases where $3 \leq k < n$.

The study of the topology of k-equal arrangements $A_d^{(k)}(n)$ and its complement began with Björner and his coauthors in [3–5] where they studied the k-equal manifold arrangement and its complement as a generalization of the widely studied configuration spaces ($k = 2$, $d = 1$) and the pure braid space ($k = 2$, $d = 2$). Björner and Lovász found in [3, 4] a connection between the algorithmic problem of determining bounds for the complexity of the linear decision tree of the k-equal problem—given n real numbers x_1, …, x_n, decide if some k of them are equal—and the Euler characteristic of $A_d^{(k)}(n)$ or $M_d^{(k)}(n)$. Also, in [5] Björner and Welker gave a recursive formula for computing the Betti numbers of $M_d^{(k)}(n)$ using the Goresky-MacPherson combinatorial formula for the cohomology of complements of subspace arrangements and obtained the ranks where the cohomology groups of these spaces do not vanish.

Nevertheless the multiplicative structure of the cohomology ring of $M_d^{(k)}(n)$ could not be recovered using the Goresky-MacPherson formula and therefore some efforts were made to compute the cohomology ring of these manifolds. The first successful attempt was made by Yuzvinsky in [35]. Yuzvinsky obtained a description of the rational cohomology ring of $M_d^{(k)}(n)$ for $d = 2$ (the complement of the complex k-arrangement) by means of the De Concini-Procesi differential graded algebra.

After that, Baryshnikov determined the cohomology ring of $M_d^{(k)}(n)$ for $d = 1$ "geometrically" by taking geometric intersections of submanifolds representing cohomology classes and this idea was used by Dobrinskaya and Turchin in [8] to produce the cohomology ring of $M_d^{(k)}(n)$ for $d > 1$.

The main interest of the author's work along the Ph. D. program was to determine the Lusternik–Schnirelmann category (LS category in what follows) and (higher) topological complexity of non-k-equal manifolds. This work began in a joint work with Jesús González and Christopher Roque-Márquez in [15] where these invariants where calculated for the non-k-equal manifold $M_d^{(k)}(n)$ with $d = 1$. The main result states that the Lusternik–Schnirelmann category is $\operatorname{cat}\left(M_1^{(k)}(n)\right) = \lfloor n/k \rfloor$, the topological complexity is $\operatorname{TC}\left(M_1^{(k)}(n)\right) = 2\lfloor n/k \rfloor$ and the higher topological complexity is $\operatorname{TC}_s\left(M_1^{(k)}(n)\right) = s\lfloor n/k \rfloor$. The determination of these invariants is examined in Chapter 2.

The case $d > 1$ was combinatorially more challenging, and unfortunately, the tools used in [15] could not give a complete description of the invariants studied. Nevertheless the invariants are determined for some configuration of values of d, k and n and in the general case the bounds in the following theorem were established by Jesús González and the author in [14]. In the following, $\lfloor \ell \rfloor$ (respectively $\lceil \ell \rceil$) stands for the greatest (smallest) integer less than (greater than) or equal to the real number ℓ.

Theorem. *Let $a = d(k-1) - 1$ and $b = n - k\left\lfloor \frac{n}{k} \right\rfloor$, then*

$$\left\lfloor \frac{n}{k} \right\rfloor \leq \operatorname{cat}(M_d^{(k)}(n)) \leq \left\lfloor \frac{n}{k} \right\rfloor + \left\lceil \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} - 1 \right\rceil,$$

$$2\left\lfloor \frac{n}{k} \right\rfloor \leq \operatorname{TC}(M_d^{(k)}(n)) \leq 2\left(\left\lfloor \frac{n}{k} \right\rfloor + \left\lceil \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} - 1 \right\rceil\right),$$

$$s\left\lfloor \frac{n}{k} \right\rfloor \leq \operatorname{TC}_s(M_d^{(k)}(n)) \leq s\left(\left\lfloor \frac{n}{k} \right\rfloor + \left\lceil \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} - 1 \right\rceil\right).$$

The proof of this theorem is given in Chapter 3.

Finally, an essential tool for establishing the results mentioned before was the cohomology ring of non-k-equal spaces. Therefore, the cohomology ring of $M_1^{(k)}(n)$ given by Baryshnikov in [1] and the cohomology ring of $M_d^{(k)}(n)$ for $d > 1$ given by Dobrinskaya and Turchin in [8] are briefly treated in Chapter 1 as well as some classical theorems giving bounds for the LS category and (higher) topological complexity. Also, as a consequence of the geometric nature of the cohomology ring, some non trivial Massey products are computed in Chapter 4.

1 | Preliminaries

This chapter introduces the definitions of Lusternik Schnirelmann category and (higher) topological complexity along with the classical theorems giving bounds for these invariants. The essence of these theorems is to use the cohomological description of a space to produce adequate non zero products of a certain length. Therefore, a review of the cohomology rings for $M_1^{(k)}(n)$ and $M_d^{(k)}(n)$ for $d > 1$ is given in Sections 1.2 and 1.3.

1.1 LS category and (higher) topological complexity

For a space X, the LS category, $\mathrm{cat}(X)$, the topological complexity, $\mathrm{TC}(X)$, and the higher topological complexity, $\mathrm{TC}_s(X)$ are homotopy invariants of X and special cases of the notion of sectional category (or Schwarz genus) of a fibration.

Definition 1.1. *The (reduced) sectional category of a fibration $p : E \to B$,* $\mathrm{secat}(p)$*, is defined as the smallest non-negative integer k so that there exists an open covering of the base $B = U_0 \cup U_1 \cup \cdots \cup U_k$ such that the fibration p admits a continuous section on each U_ℓ.*[1].

$$\begin{array}{ccc} & & E \\ & \overset{s_\ell}{\nearrow} & \downarrow p \\ U_\ell & \xhookrightarrow[i]{} & B \end{array}$$

As a special case, we obtain the (reduced) LS category of a space X, $\mathrm{cat}(X)$, defined as the sectional category of the fibration $e_1 \colon P_0(X) \to X$, where $P_0(X)$ is the space of based paths γ on X (i.e. $\gamma(0) = *$) and e_1 is the evaluation map given by $e_1(\gamma) = \gamma(1)$. On the other hand, the (reduced) topological complexity of a space X, $\mathrm{TC}(X)$, is defined as the sectional category of the fibration $e_{0,1} \colon P(X) \to X \times X$, where $P(X)$ is the space of free paths on X

[1]Schwarz' original (unreduced) definition is recovered as $\mathrm{genus}(p) = \mathrm{secat}(p) + 1$.

and $e_{0,1}$ is the double evaluation map given by $e_{0,1}(\gamma) = (\gamma(0), \gamma(1))$. The open[2] sets U_i covering $X \times X$ so that $e_{0,1}$ admits a continuous section on each U_i are called *local domains*, and the corresponding local sections are called *local rules*. The system of local domains and local rules is called a motion planner for X. A motion planner is said to be optimal if it has $\mathrm{TC}(X)+1$ local rules.

Similarly the *s-th topological complexity* of a space X, $\mathrm{TC}_s(X)$, is the sectional category of the fibration

$$e_s^X = e_s : X^{J_s} \to X^s, \quad e_s(\gamma) = (\gamma(1_1), \ldots, \gamma(1_s))$$

where J_s is the wedge of s closed intervals $[0,1]$ (each having $0 \in [0,1]$ as the base point), and 1_i stands for the 1 in the i^{th} interval.

As explained by Farber in his seminal work [9, 11], topological complexity gives a homotopical framework for studying the motion planning problem in robotics. Indeed, $\mathrm{TC}(X)$ gives a measure of the complexity of motion-planning an autonomous system with state-space X and which should perform robustly within a noisy environment. Similarly higher topological complexity is a natural generalization of topological complexity where the motion planning does not only depend on a couple of initial-final states of a robot, but in a sequence of prescribed intermediate stages that the robot should reach through the motion, see [2, 30].

Most of the existing methods to estimate the topological complexity of a given space are cohomological in nature and are based on some form of obstruction theory. One of the most simple and successful such methods is:

Theorem 1.2 ([9, Theorem 7] and [2, Theorem 3.9])**.** *Let X be a c-connected space having the homotopy type of a CW complex, then*

$$\mathrm{cl}(X) \leq \mathrm{cat}(X) \leq \frac{\mathrm{hdim}(X)}{c+1},$$

$$\mathrm{zcl}(X) \leq \mathrm{TC}(X) \leq \frac{2\,\mathrm{hdim}(X)}{c+1},$$

$$\mathrm{zcl}_s(X) \leq \mathrm{TC}_s(X) \leq \frac{s\,\mathrm{hdim}(X)}{c+1}.$$

The notation $\mathrm{hdim}(X)$ stands for the (cellular) homotopy dimension of X, i.e. the minimal dimension of CW complexes having the homotopy type of X. On the other hand, the cup-

[2]For practical purposes, the openness condition on local domains can be replaced (without altering the resulting numerical value of $\mathrm{TC}(X)$) by the requirement that local domains are pairwise disjoint Euclidean neighborhood retracts (ENR) see [10, Theorem 13.1].

length of X, $\mathrm{cl}(X)$, the zero-divisor cup-length of X, $\mathrm{zcl}(X)$, and the s-th zero-divisors, $\mathrm{zcl}_s(X)$, are defined in purely cohomological terms.

- $\mathrm{cl}(X)$ is the largest non-negative integer ℓ such that there are coefficients systems $A_1,\ldots,A_\ell$ over X and corresponding positive-dimensional classes $c_j \in H^*(X;A_j)$ so that the product $c_1\cdots c_\ell \in H^*(X;\bigotimes_i A_i)$ is non-zero.

- Likewise, $\mathrm{zcl}(X)$ is the largest non-negative integer ℓ such that there are coefficients systems $A_1,\ldots,A_\ell$ over $X\times X$ and corresponding classes $z_j \in H^*(X\times X;A_j)$, each with trivial restriction under the diagonal inclusion $\Delta\colon X \hookrightarrow X\times X$, and so that the product $z_1\cdots z_\ell \in H^*(X\times X;\bigotimes_i A_i)$ is non-zero. Each such class z_i is called a zero-divisor for X.

- Finally, $\mathrm{zcl}_s(X)$ is the largest non-negative integer ℓ such that there are coefficients systems $A_1,\ldots,A_\ell$ over X^s and corresponding classes $z_j \in H^*(X^s;A_j)$, each with trivial restriction under the iterated diagonal map $\Delta_s\colon X \hookrightarrow X^s$, and so that the product $z_1\cdots z_\ell \in H^*(X^s;\bigotimes_i A_i)$ is non-zero. Each such class z_i is called a s-th zero-divisor for X.

Throughout this work, we will only be concerned with simple coefficients in $\mathbb{Z}_2$ or $\mathbb{Z}$, and will omit reference of coefficients in writing a cohomology group $H^*(X)$. In these terms, $\Delta^*\colon H^*(X^s)\to H^*(X)$ is given by cup-multiplication, which explains the name "zero-divisors".

1.2 Baryshnikov cohomological description

The objective of this section is to present the description of the cohomology ring $H^*(M_1^{(k)}(n))$ given by Baryshnikov in the preprint [1] and later stated by Dobrinskaya and Turchin in [8, sec. 4]. The essential combinatorial objects to consider are string preorders encoded in terms of the following definition.

Definition 1.3. *A string preorder is an arrangement of alternating () and []-blocks of the form*

$$(I_0)\,[J_1]\,(I_1)\,[J_2]\cdots(I_{\ell-1})\,[J_\ell]\,(I_\ell)$$

where the sets I_0, J_1, …, J_ℓ, I_ℓ are mutually disjoint and their union is the set $\mathbf{n}=\{1,2,\ldots,n\}$. Such a string preorder determines a submanifold in $\mathbb{R}^n$ defined by the following conditions:

- *$x_{k_1}=x_{k_2}$ if k_1, $k_2\in J_m$ for some $m=1,\ldots,\ell$,*

- $x_i \leq x_j$ *if* $i \in I_m$ *and* $j \in J_{m+1}$ *for some* $m = 0, \dots, \ell - 1$,
- $x_j \leq x_i$ *if* $j \in J_m$ *and* $i \in I_m$.

Hence, the first condition says that []-blocks encode collided coordinates and the second and third conditions ensure that the coordinates are ordered according to the corresponding sets from left to right.[3]

Example 1.4. *In* $\mathbb{R}^8$ *we can consider, for example, the string preorder*

$$(\{1\})[\{2,3,4\}](\emptyset)[\{5,6\}](\{7,8\}),$$

and this preorder has the associated submanifold

$$\{(x_1, x_2, \dots, x_8) \,|\, x_1 \leq x_2 = x_3 = x_4, x_5 = x_6 \leq x_7, x_8\}.$$

As a convention, the braces in the sets will be omitted so to get a cleaner notation. Hence the string preorder of Example 1.4 is simply

$$(1)[2,3,4]()[5,6](7,8).$$

String preorders give a convenient way of writing submanifolds of $\mathbb{R}^n$ where the coordinates are subjected to order restrictions. In particular, we are interested in all those string preorders generating cohomology classes by means of duality. The basic building blocks for cohomology classes will be all those string preorders with only one []-block of cardinality $k-1$. From this point of view, k is fixed and corresponds to the collision restriction in $M_d^{(k)}(n)$.

Definition 1.5. *A string preorder is said to be* k-elementary *or just elementary for short, if it has the form* $(I)[J](K)$ *with* $|J| = k-1$.

Example 1.6. *(1)[2,3,4](5,6,7,8) and (1,2,7,4,5)[6,3,8]( ) are elementary string preorders in* $M_1^{(4)}(8)$.

Note that, as a consequence of the definition and our assumption $k < n$, any elementary string preorder must have at least one non-empty ()-block. Furthermore, the boundary of the associated submanifold lies in the k-equal arrangement, so the locally finite chain given by the sum of the top simplices in a triangulation of this submanifold determines a homology

[3] Note that in [1], the increasing order for the coordinates is given from right to left, but both conventions lead to the same cohomological description.

class in Borel-Moore homology and therefore, by duality, that class renders a cohomology class in $H^{k-2}(M_1^{(k)}(n))$—where $k-2$ is the codimension of any elementary string preorder. See Section 4.1 for further details about the duality isomorphism.
Now, the additive relations for the cohomology classes induced by elementary string preorders appear as a consequence of the boundary of string preorders with only one []-block of cardinality $k-2$. But for a consistent management of orientations, a standard orientation for string preorders needs to be chosen.

Definition 1.7. *The orientation of each submanifold corresponding to a string preorder* $(I)[J](K)$ *is the (co)orientation obtained by first considering an ordered basis for the normal space—obtained from the []-block whose elements are listed in the natural order—and then taking the induced orientation on its tangent space relative to the standard orientation of* $\mathbb{R}^n$*, assuming that the elements in each ()-block are also naturally oriented.*

Example 1.8. *The string preorders* $S_1 = (2,3)[1,4,5](6,7)$*,* $S_2 = (1,3)[2,4,5](6,7)$*,* $S_3 = (1,2)[3,4,5](6,7)$*,* $S_4 = (1,2,3)[4,5,6](7)$ *and* $S_5 = (1,2,3)[4,5,7](6)$ *are elementary in* $M_1^{(4)}(7)$ *and, as explained below, they constitute the boundary of another string preorder, and considering their orientations, satisfy the additive relation* $S_1 + S_2 + S_3 - S_4 - S_5 = 0$.

The string preorders in Example 1.8 have only one []-block with 3 elements, hence they are elementary string preorders and it is easy to verify they are boundary submanifolds of $S = (1,2,3)[4,5](6,7)$. In order to obtain the oriented boundary of S we must determine the orientation for each manifold corresponding to the string preorders, which we still label as $S, S_1, \dots, S_5$ for convenience.
First, note that $T_xS = [4,5] = \{x \in \mathbb{R}^7 \,|\, x_4 = x_5\}$ and this is the level set of the function $f_{[4,5]} : \mathbb{R}^7 \to \mathbb{R}$ given by $f_{[4,5]}(x) = x_5 - x_4$. Hence, $T_xS = \ker df_{[4,5]} = \langle e_1, e_2, e_3, e_4 + e_5, e_6, e_7\rangle$, and the normal space is $\langle e_5 - \mathrm{proj}_{e_4+e_5} e_5\rangle$, where $\mathrm{proj}_v u$ denotes the projection of vector u onto vector v. Now, note that

$$\langle e_5 - \mathrm{proj}_{e_4+e_5} e_5, e_1, e_2, e_3, e_4 + e_5, e_6, e_7\rangle = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7\rangle = \mathcal{E}_7$$

is the standard orientation of $\mathbb{R}^7$. Therefore, the orientation frame for T_xS is $\langle e_1, e_2, e_3, e_4 + e_5, e_6, e_7\rangle$.
Now, to find the orientation frame for $T_xS_1 = [1,4,5]$ consider the function $f_{[1,4,5]} : \mathbb{R}^7 \to \mathbb{R}^2$ given by $f_{[1,4,5]}(x) = (x_4 - x_1, x_5 - x_1)$, then $T_xS_1 = \ker df_{[1,4,5]} = \langle e_2, e_3, e_1 + e_4 + e_5, e_6, e_7\rangle$ and the normal space is $\langle e_4 - \mathrm{proj}_{e_1+e_4+e_5}, e_5 - \mathrm{proj}_{e_1+e_4+e_5}\rangle$, this time we also have

$$\langle e_4 - \mathrm{proj}_{e_1+e_4+e_5}, e_5 - \mathrm{proj}_{e_1+e_4+e_5}, e_2, e_3, e_1 + e_4 + e_5, e_6, e_7\rangle = \mathcal{E}_7$$

so the orientation frame for S_1 is $\langle e_2, e_3, e_1+e_4+e_5, e_6, e_7\rangle$. The process for finding the orientation frame for the remaining manifolds is similar, and in summary, we have the following orientations:

$$
\begin{aligned}
O\big((1,2,3)[4,5](6,7)\big) &= \langle e_1, e_2, e_3, e_4+e_5, e_6, e_7\rangle,\\
O\big((2,3)[1,4,5](6,7)\big) &= \langle e_2, e_3, e_1+e_4+e_5, e_6, e_7\rangle,\\
O\big((1,3)[2,4,5](6,7)\big) &= -\langle e_1, e_3, e_2+e_4+e_5, e_6, e_7\rangle,\\
O\big((1,2)[3,4,5](6,7)\big) &= \langle e_1, e_2, e_3+e_4+e_5, e_6, e_7\rangle,\\
O\big((1,2,3)[4,5,6](7)\big) &= \langle e_1, e_2, e_3, e_4+e_5+e_6, e_7\rangle,\\
O\big((1,2,3)[4,5,7](6)\big) &= -\langle e_1, e_2, e_3, e_4+e_5+e_7, e_6\rangle.
\end{aligned}
$$

Note that, in general, the sign could be obtained by finding the sign of the corresponding permutation on **n** sending each coordinate index to its position according to Definition 1.7. For example, the string preorder $(1,2,3)[4,5,7](6)$ has $\langle e_1, e_2, e_3, e_4+e_5+e_7, e_6\rangle$ as a basis and the sign is given by the permutation

$$
\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 1 & 7 & 2 \end{pmatrix}.
$$

Since σ can be expressed as the product of 5 transpositions its sign is negative.

Now, the oriented boundary S_1 of S is obtained as follows:

S_1 is a boundary of S that can be described as the inverse image of $\mathbb{R}_+$ under the function $g_1 : (2,3)[4,5](6,7) \to \mathbb{R}$ given by $g_1(x) = x_4 - x_1$. Hence, we have

$$
\begin{aligned}
T_x(2,3)[4,5](6,7) = [4,5]&\\
&= \langle e_1, e_2, e_3, e_4+e_5, e_6, e_7\rangle\\
&= \langle e_1, e_2, e_3, e_1+e_4+e_5, e_6, e_7\rangle\\
&= \langle e_1 - \operatorname{proj}_{e_1+e_4+e_5} e_1, e_2, e_3, e_1+e_4+e_5, e_6, e_7\rangle
\end{aligned}
$$

and the vector $e_1 - \operatorname{proj}_{e_1+e_4+e_5} e_1$ is an outward-pointing normal vector. Therefore S_1 has the induced boundary orientation given by the frame $\langle e_2, e_3, e_1+e_4+e_5, e_6, e_7\rangle$ which agrees with the orientation frame for that manifold. Therefore S_1 appears as a boundary term of S with positive sign.

Similarly we can deduce the boundary orientation for S_2 by considering the function $g_2 : (13)[4,5](6,7) \to \mathbb{R}$ given by $g_2(x) = x_4 - x_2$. This time we have

$$\begin{aligned} T_x(1,3)[4,5](6,7) = [4,5] \\ &= \langle e_1, e_2, e_3, e_4 + e_5, e_6, e_7 \rangle \\ &= \langle e_1, e_2, e_3, e_2 + e_4 + e_5, e_6, e_7 \rangle \\ &= -\langle e_2 - \mathrm{proj}_{e_2+e_4+e_5} e_2, e_1, e_3, e_2 + e_4 + e_5, e_6, e_7 \rangle \end{aligned}$$

and, since $e_2 - \mathrm{proj}_{e_2+e_4+e_5} e_2$ is an outward-pointing normal vector, the boundary orientation for S_2 is given by the frame $-\langle e_1, e_3, e_2 + e_4 + e_5, e_6, e_7 \rangle$ which agrees with the orientation of S_2. Therefore S_2 appears as a boundary term of S with positive sign. By continuing with the process it is easy to verify that S_3 appears as a boundary term with positive sign, S_4 appears with negative sign, and S_5 with negative sign.

Generalizing the example above, we can obtain the boundary formula of a string preorder with only one []-block with $k-2$ elements.[4]

Theorem 1.9. *Let $(I)[J](K)$ be a string preorder where $|J| = k-2$. The boundary formula for $(I)[J](K)$ is*

$$\partial(I)[J](K) = \sum_{\iota \in I} (-1)^{g(\iota)} (I - \iota)[J + \iota](K) + \sum_{\kappa \in K} (-1)^{g(\kappa)+1} (I)[J + \kappa](K + \kappa),$$

where $g(x)$ is the number of elements in J greater than x.

Proof. The orientation frame for $S = (I)[J](K)$ can be expressed as

$$\mathcal{O}\big((I)[J](K)\big) = \mathrm{sgn}(\sigma_S) \left\langle e_I, \sum_{j \in J} e_j, e_K \right\rangle,$$

where σ_S is the permutation on $\mathbf{n}$ sending each coordinate index to its position according to Definition 1.7, and e_I and e_K are the sets of elementary vectors indicated by the naturally ordered elements in I and K respectively.

Let us determine the sign of the boundary term S_ι corresponding to moving an element ι from set I to set J. Let $p(\iota)$ be the position of element ι in the string preorder $(I)[J](K)$.

[4] This is a minor correction of the boundary formula of [1, pg. 3] concerning the signs of the boundary terms and does not affect the multiplicative description of $M_1^{(k)}(n)$.

Hence, working with the ordered tangent space for S we have

$$\begin{aligned} T_x S &= \operatorname{sgn}(\sigma_S)\left\langle e_I, e_\iota + \sum_{j\in J} e_j, e_K \right\rangle \\ &= \operatorname{sgn}(\sigma_S)(-1)^{p(\iota)-1}\left\langle e_\iota - \operatorname{proj}_{e_\iota+\sum e_K} e_\iota, e_{I-\{\iota\}}, e_\iota + \sum_{j\in J} e_j, e_K \right\rangle \end{aligned}$$

and since $e_\iota - \operatorname{proj}_{e_\iota+\sum e_K} e_\iota$ is an outward-pointing normal vector, the boundary orientation for S_ι is given by the frame

$$\operatorname{sgn}(\sigma_S)(-1)^{p(\iota)-1}\left\langle e_{I-\{\iota\}}, e_\iota + \sum_{j\in J} e_j, e_K \right\rangle$$

but the orientation imposed on S_ι according to Definition 1.7 is

$$\operatorname{sgn}(\sigma_{S_\iota})\left\langle e_{I-\{\iota\}}, e_\iota + \sum_{j\in J} e_j, e_K \right\rangle$$

so the sign appearing in the boundary formula for term S_ι should be

$$(-1)^{p(\iota)-1}\operatorname{sgn}(\sigma_S)\operatorname{sgn}(\sigma_{S_\iota})$$

But note that σ_{S_ι} can be recovered from σ_S by pre-multiplying with a permutation fixing the values in K as well as all those values in I greater than ι and the values in J smaller than ι. Specifically, the permutation is given by the cycle $(b(\iota), \iota, a(\iota))$ where $b(\iota)$ are the elements in I smaller than ι and $a(\iota)$ are the elements in J greater than ι. Therefore, the total sign contribution of $\operatorname{sgn}(\sigma_S)\operatorname{sgn}(\sigma_{S_\iota})$ is equal to $(-1)^{p(\iota)+g(\iota)-1}$. Hence, the sign appearing in the boundary formula should be $(-1)^{g(\iota)}$.

Similarly, we can determine the sign of the boundary term S_κ corresponding to moving an element κ from set K to set J. From the ordered frame for S we have

$$\begin{aligned} T_x S &= \operatorname{sgn}(\sigma_S)\left\langle e_I, e_\kappa + \sum_{j\in J} e_j, e_K \right\rangle \\ &= \operatorname{sgn}(\sigma_S)(-1)^{p(\kappa)-1-(k-3)}\left\langle e_\kappa - \operatorname{proj}_{e_\kappa+\sum e_K} e_\kappa, e_I, e_\kappa + \sum_{j\in J} e_j, e_{K-\{\kappa\}} \right\rangle \end{aligned}$$

and since $e_\kappa - \text{proj}_{e_\kappa + \sum e_K} e_\kappa$ is an inward-pointing normal vector, the boundary orientation for S_κ is given by the frame

$$\text{sgn}(\sigma_S)(-1)^{p(\kappa)+k+1}\left\langle e_I, e_\kappa + \sum_{j\in J} e_j, e_{K-\{\kappa\}}\right\rangle$$

but the orientation imposed on S_κ according to Definition 1.7 is

$$\text{sgn}(\sigma_{S_\kappa})\left\langle e_I, e_\kappa + \sum_{j\in J} e_j, e_{K-\{\kappa\}}\right\rangle$$

so the sign appearing in the boundary formula for term S_κ should be

$$(-1)^{p(\kappa)+k+1}\,\text{sgn}(\sigma_S)\,\text{sgn}(\sigma_{S_\kappa}).$$

Again, we can recover σ_{S_κ} by pre-multiplying with an adequate permutation fixing all values in K greater than κ. Let $r(\kappa)$ be the position of κ inside the ()-block K and $\ell(\kappa)$ be the number of elements in J less than κ, we have two cases:

- $\ell(\kappa) = 0$. Hence, κ is the minimum in the new block $[J+\kappa]$ and shifts the positions of the elements in J and I producing the cycle (J, I), also, shifts the positions of κ's from κ_1 to $\kappa_{r(\kappa)-1}$ producing the cycle $(\kappa_1, \ldots, \kappa_{r(\kappa)})$. Therefore $\text{sgn}(\sigma_S)\,\text{sgn}(\sigma_{S_\kappa})$ gives a sign contribution of $(-1)^{p(\kappa)-2} = (-1)^{p(\kappa)}$.

- $\ell(\kappa) > 0$. Hence, the minimum in the new block $[J+\kappa]$ is the same as before in $[J]$. The elements in J less than κ except $\min J$ remain fixed, and the elements starting from $j_{\ell(\kappa)+1}$ are shifted as well as the elements in I. Note that the last element of I now takes the value of $\min J$ in σ and $\min J$ takes the value of κ_1 in σ, also the elements $\kappa_1, \ldots, \kappa_{r(\kappa)}$ are shifted and $\kappa_{r(\kappa)}$ takes the value of $j_{\ell(\kappa)+1}$ in σ, giving the cycle $(\min J, \kappa_1, \ldots, \kappa_{r(\kappa)}, j_{\ell(\kappa)+1}, \ldots, j_{k-2}, I)$. Therefore $\text{sgn}(\sigma_S)\,\text{sgn}(\sigma_{S_\kappa})$ gives a sign contribution of $(-1)^{p(\kappa)-\ell(\kappa)}$.

In any case, the sign $\text{sgn}(\sigma)\,\text{sgn}(\sigma_\kappa)$ is $(-1)^{p(\kappa)-\ell(\kappa)}$ and since $\ell(\kappa)+g(\kappa) = k-2$ the boundary element S_κ appears with sign $(-1)^{g(\kappa)+1}$. □

Example 1.10. *According to Theorem 1.9 we have the following boundary in $M_1^{(4)}(7)$.*

$$\begin{aligned}\partial(2,5,7)[3,6](1,4) = &+(5,7)[2,3,6](1,4) - (2,7)[3,5,6](1,4) + (2,5)[3,6,7](1,4)\\ &- (2,5,7)[1,3,6](4) + (2,5,7)[3,4,6](1).\end{aligned}$$

So far, we have identified the elementary string preorders as generators for Borel-Moore homology in dimension $k-2$ subject to the additive relations obtained in Theorem 1.9. After dualization, the multiplicative structure of the cohomology ring is determined by the transverse intersection—intersection product—of the corresponding submanifolds. Therefore transverse intersection of manifolds associated with elementary string preorders—or, more specifically their corresponding string preorders—are the basic elements generating the cohomology ring in dimensions which are multiples of $k-2$.
For string preorders $(I)[J](K)$ and $(I')[J'](K')$ (possibly non-elementary), there are three cases:

(i) If $I \cup J \subseteq I'$, then $\left(I\right)\left[J\right]\left(K\right) \pitchfork \left(I'\right)\left[J'\right]\left(K'\right) = \left(I\right)\left[J\right]\left(K \cap I'\right)\left[J'\right]\left(K'\right)$.

(ii) Similarly, if $I' \cup J' \subseteq I$, $\left(I\right)\left[J\right]\left(K\right) \pitchfork \left(I'\right)\left[J'\right]\left(K'\right) = \left(I'\right)\left[J'\right]\left(K' \cap I\right)\left[J\right]\left(K\right)$.

(iii) If $I \cup J \nsubseteq I'$ and $I' \cup J' \nsubseteq I$, there are two cases

- If $|J \cup J' \cup (I \cap K') \cup (I' \cap K)| = |J| + |J'| - 1 > |J|$ then the transverse intersection is $\left(I\right)\left[J\right]\left(K\right) \cap \left(I'\right)\left[J'\right]\left(K'\right) = \left(I \cap I'\right)\left[J \cup J' \cup (I \cap K') \cup (I' \cap K)\right]\left(K \cap K'\right)$ but it lies in the k-equal arrangement giving a trivial (co)homology class.
- Otherwise, after applying a displacement, the transverse intersection is empty, giving again the trivial cohomology class.

Definition 1.11. *A string preorder is said to be:*

(a) admissible*, if it has the form* $(I_0)[J_1](I_1)[J_2]\cdots[J_d](I_d)$ *with* $\operatorname{card}(J_i) = k-1$ *for all* $i = 1, \ldots, d$*. In such a case, the admissible string preorder is said to have* dimension $d(k-2)$.

(b) basic*, if it is specified by a string* $(I_0)[J_1](I_1)[J_2]\cdots[J_d](I_d)$ *satisfying* $\operatorname{card}(J_i) = k-1$ *and* $\max(J_i \cup I_i) \in I_i$*, for all* $i = 1, \ldots, d$ *(the maximal element of* $J_i \cup I_i$ *is taken with respect to the standard order of integers).*

Example 1.12. *An admissible (basic) preorder* $(I_0)[J_1](I_1)[J_2]\cdots[J_d](I_d)$ *of dimension* $d(k-2)$ *factors as* $\varepsilon_1 \pitchfork \cdots \pitchfork \varepsilon_d$*, where*

$$\varepsilon_i = \left(I_0 \cup J_1 \cup I_1 \cup \cdots \cup J_{i-1} \cup I_{i-1}\right)\left[J_i\right]\left(I_i \cup J_{i+1} \cup I_{i+1} \cup \cdots \cup J_d \cup I_d\right)$$

is an elementary (basic) preorder of dimension $k-2$.

Therefore admissible string preorders encode transverse intersections of elementary strings, and from those, basic string preorders of dimension $d(k-2)$ form an additive basis for the cohomology group of $M_1^{(k)}(n)$ in dimension $d(k-2)$. The proof of the fact that basic string preorders indeed form a basis for the $d(k-2)$-cohomology group is given by Baryshnikov [1, Theorem 2] by showing that the pairing matrix between the proposed basic string preorders and a homological basis for $M_1^{(k)}(n)$ has a triangular matrix with diagonal elements ± 1. Finally we are ready to state Baryshnikov's description of the ring $H^*(M_1^{(k)}(n))$.

Theorem 1.13 (Baryshnikov [1, Theorem 1], Dobrinskaya-Turchin [8, Section 4])**.** *For $k \geq 3$, the cohomology ring $H^*(M_1^{(k)}(n))$ is isomorphic to the (anti)commutative free exterior algebra generated in dimension $k-2$ by the elementary preorders subject to the following relations:*

1. $\sum_{\iota \in I}(-1)^{g(\iota)}(I-\iota)[J+\iota](K) = \sum_{\kappa \in K}(-1)^{g(\kappa)}(I)[J+\kappa](K+\kappa)$ *whenever* $\mathbf{n}$ *can be written as a disjoint union* $\mathbf{n} = I \coprod J \coprod K$ *with* $\operatorname{card}(J) = k-2$.
2. $(I)[J](K) \cdot (I')[J'](K') = 0$, *for elementary preorders* $(I)[J](K)$ *and* $(I')[J'](K')$ *whose intersection has a* [ ]*-block of cardinality larger than* $k-1$.

Example 1.14 ([5, Table 1] and [28, Example 4.6])**.** *Consider the non-3-equal arrangement $M_1^{(3)}(7)$. Examples of elementary preorders in dimension 1 are: $()[1,2](3,4,5,6,7)$, $(1,2)[3,4](5,6,7)$, $(3,6)[1,4](5,2,7)$, etc. Note that we can count the number of elementary preorders in dimension 1 by first selecting at least 3 elements from the set $\mathbf{7}$, this will constitute the blocks $[J_1](I_1)$ but the maximum is fixed in I_1 so it remains to choose 2 elements from the remaining. Therefore, the total number of elementary string preorders is*

$$\sum_{i=3}^{7}\binom{7}{i}\binom{i-1}{2} = 351.$$

Now, it remains to examine possible cup products of elementary terms but note that it is not possible to multiply more than 2 elementary strings to obtain a new string preorder because a string preorder of the form $(I_0)[J_1](I_1)[J_2](I_2)[J_3](I_3)$ will require at least 9 numbers. Hence cup products appear only in dimension 2 and are of the form $(I_0)[I_1](I_1)[J_2](I_2)$. Using a similar combinatorial argument as in dimension 1, we have the following number of basic string preorders in dimension 2:

$$\binom{7}{3,4}\binom{2}{2}\binom{3}{2}+\binom{7}{4,3}\binom{3}{2}\binom{2}{2}+\binom{7}{1,3,3}\binom{2}{2}\binom{2}{2} = 350,$$

where $\binom{n_1+n_2+\cdots+n_\ell}{n_1,n_2,\ldots,n_\ell}$ *denotes the multinomial coefficient* $\frac{(n_1+n_2+\cdots+n_\ell)!}{n_1!\,n_2!\cdots n_\ell!}$.

If it happens that after multiplying the resulting string preorder is not written in a basic form, we can apply the boundary relation of Theorem 1.9 to express the product in terms of basic elements. For example

$$(2)[5,6](1,3,4,7) \smile (2,5,6,7)[1,3](4) = (2)[5,6](1)[3,4](7)$$

is not a basic element, but the first term appears in the boundary of $(2)[5](1,3,4,6,7)$*, so, using Theorem 1.9, we have*

$$\begin{aligned}(2)[5,6](1,3,4,7) \;=\; &-\;()[2,5](1,3,4,6,7) \;+\; (2)[1,5](3,4,6,7) \;+\; (2)[3,5](1,4,6,7)\\ &+\;(2)[4,5](1,3,6,7) \;-\; (2)[5,7](1,3,4,6).\end{aligned}$$

Therefore,

$$(2)[5,6](1)[3,4](7) \;=\; (2)[1,5](6)[3,4](7) \;-\; ()[2,5](1,6)[3,4](7)$$

expresses the product in terms of basic elements of dimension 2*.*

1.3 Dobrinskaya-Turchin cohomological description

In this section, we recall the combinatorial description of the cohomology ring of $M_d^{(k)}(n)$ given in [8]. Similar to the objects considered in Section 1.2, the cohomological description of $M_d^{(k)}(n)$ for $d \geq 2$ is encoded by combinatorial objects called admissible k-forests.

Definition 1.15. *A k-forest on* $\mathbf{n}$ *(or simply a k-forest) is an acyclic simple graph with two types of vertices, square ones and round ones, each containing a certain subset of* $\mathbf{n}$*. A square vertex must contain $k-1$ elements of* $\mathbf{n}$*, and cannot be an isolated vertex; in fact the set of immediate neighbors of a square vertex must contain a round vertex. A round vertex must contain a single element of* $\mathbf{n}$*, and must be either an isolated vertex or have valency 1, in which case it must be connected to a square vertex. We require that the subsets of integers inside the various vertices of a k-forest form a disjoint partition of* $\mathbf{n}$*. Square vertices are declared to have degree $d(k-2)$, while edges are declared to have degree $d-1$. The degree of a k-forest is then defined as the sum of the degrees of its square vertices and edges. An orientation for a k-forest consists of three ingredients:*

(a) An orientation for each edge;

(b) A total ordering for the elements inside each square vertex;

(c) A total ordering for the orientation set*, i. e., the set consisting of all edges and all square vertices.*

Example 1.16. *Figure 1.1 is an oriented* 5*-forest of degree* 28 *(with $d = 3$ and $n = 14$). In such a picture, we agree that the ordering of elements inside a square vertex is spelled out by listing the elements from left to right.*

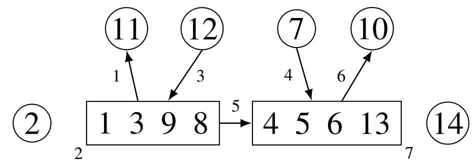


Figure 1.1: Small numbers attached to square vertices and edges indicate the ordering in the orientation set.

Theorem 1.17 ([8, Theorem 6.1])**.** *Let $d \geq 2$, $k \geq 3$ and $n \geq 1$. The cohomology $H^*(M_d^{(k)}(n))$ is free and generated by oriented k-forests on* **n** *subject to the relations listed below.*

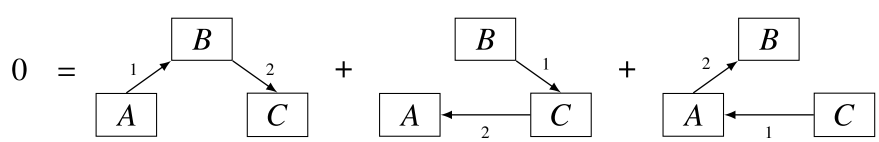
1. *Orientation relations:*

   (i) Permuting the order of the orientation set introduces the Koszul sign induced by the permutation (with respect to the degrees of the elements of the orientation set).

   (ii) A permutation $\sigma \in \Sigma_{k-1}$ of the elements inside a square vertex introduces the sign $\epsilon(\sigma)^d$, where $\epsilon(\sigma)$ stands for the sign of σ.

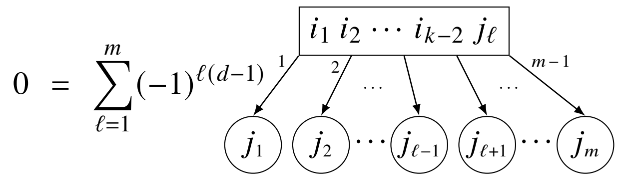
   (iii) Reversing the orientation of an edge introduces the sign $(-1)^d$.

2. *Three-term relations:*



These three pictures are local in the sense that we have three oriented k-forests that are identical except for the disposition of oriented edges connecting vertices A, B and C, whose orderings in the corresponding orientation sets are indicated by the attached numbers.

3. *Generalized Jacobi relations:*



These m pictures are again local. Moreover, in each of the global pictures, the square vertex cannot be connected to other (non shown) round vertices.

In the orientation set, the interchange of a square vertex and an edge produces a positive Koszul sign because $d(k-2)(d-1)$ is always even. Thus, the ordering in the orientation set is really a pair of orderings, one for square vertices and another for edges.
A connected component of a k-forest that reduces to an isolated round vertex is said to be trivial. Orientation and three-term relations can be used to write any oriented k-forest as an R-linear combination of oriented "linear" k-forests, i.e., k-forests whose non-trivial connected components are trees with square vertices lying along an embedded arc, as in Figure 1.2. In turn, orientation and generalized Jacobi relations can be used to express a

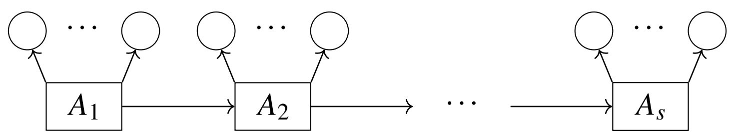
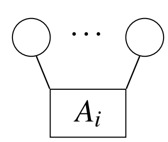


Figure 1.2: A non-trivial component of a linear k-forest.

linear k-forest as an R-linear combination of "ordered" linear k-forests, i.e., those whose non-trivial connected components satisfy that the largest of the integers inside round vertices attached to any given square vertex A_i is larger than any of the integers inside A_i. Orientation relations can then be used to choose explicit signs on ordered linear k-forests:

Definition 1.18. *An ordered linear k-forest is called* basic *provided its non-trivial components satisfy the following conditions, where we use the notation in Figure 1.2:*

- *Edge orientations are as indicated in Figure 1.2.*
- *In the orientation set,* $A_1 < A_2 < \cdots < A_s$.
- *For a portion of the form*



the elements inside the square vertex appear in their natural order. Likewise, the ordering (in the orientation set) of the edges attaching round vertices to the square vertex agrees with the natural order of the integers inside the round vertices. Furthermore, if $i > 1$, then the edge from A_{i-1} to A_i is smaller than all edges connecting A_i to round vertices. Likewise, if $i < s$, then the edge from A_i to A_{i+1} is larger than all edges connecting A_i to round vertices.

- *The minimal $m \in \mathbf{n}$ inside the vertices of the linear tree component C in Figure 1.2 appears either inside A_1 or inside a round vertex attached to A_1. Furthermore, if m' is the corresponding minimal element in another linear tree component C' of the k-forest, and $m < m'$, then orientation elements associated to C are smaller than orientation elements associated to C'.*

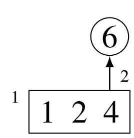
Theorem 1.17 (Continued)**.** *Basic k-forests yield a graded basis for the cohomology of $M_d^{(k)}(n)$.*

For the purposes of Chapter 4, we need to recall how a given k-forest relates to its corresponding dual fundamental class in the Borel-Moore homology of $M_d^{(k)}(n)$.
Let $p_1 : \mathbb{R}^d \to \mathbb{R}^{d-1}$ denote the projection to the last $d-1$ coordinates, $p_1(x) = (x^{(2)}, \ldots, x^{(d)})$, where $x = (x^{(1)}, x^{(2)}, \ldots, x^{(d)})$. A k-forest T determines the cell c_T consisting of all tuples $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ satisfying the following (in)equalities:

- If i and j in $\mathbf{n}$ lie in the same square vertex, then $x_i = x_j$.
- If two vertices A and B of T are connected by an edge oriented from A to B, then for all $i \in A$, $j \in B$, one has $x_i^{(1)} \leq x_j^{(1)}$ and $p_1(x_i) = p_1(x_j)$.

We then choose a locally finite triangulation of c_T and consider the corresponding locally finite chain of $(\mathbb{R}^d)^n$ with boundary in $A_d^{(k)}(n)$. As an abuse of notation, we think of the cell c_T as both the actual chain and the corresponding submanifold. The ingredients (b) and (c) in an orientation of T determine a coorientation of c_T as illustrated in Example 1.19. Note that, if i and j lie in the same connected component of T, then the condition $p_1(x_i) = p_1(x_j)$ holds true for all points in the component.

Example 1.19. *For example, with the notation and conventions above, the oriented 4-forest T in $M_3^{(4)}(7) \subset (\mathbb{R}^3)^7$ given by*

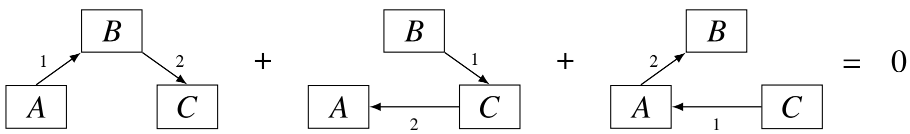


corresponds to the locally finite chain c_T consisting of tuples $(x_1,\ldots,x_7)$ in $(\mathbb{R}^3)^7$ such that $x_1 = x_2 = x_4$, $p_1(x_1) = p_1(x_6)$ and $x_1^{(1)} \leq x_6^{(1)}$. The tangent space to c_T is simply the set of points satisfying $x_1 = x_2 = x_4$ and $p_1(x_1) = p_1(x_6)$ so we can describe a frame for the tangent space by means of the kernel of the differential map $\pi_T\colon (\mathbb{R}^3)^7 \to \mathbb{R}^{\deg(T)} = (\mathbb{R}^3)^2 \times \mathbb{R}^2$ with components $\pi_\square\colon (\mathbb{R}^3)^7 \to (\mathbb{R}^3)^2$ and $\pi_\bigcirc\colon (\mathbb{R}^3)^7 \to \mathbb{R}^2$ given by

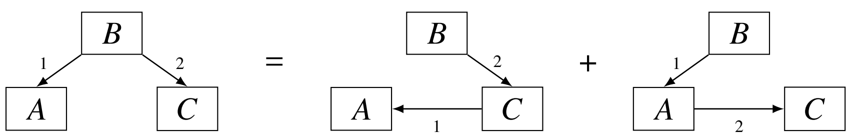
$$\pi_\square(x_1,\ldots x_7) = (x_2 - x_1, x_4 - x_1) \;\; \text{and} \;\; \pi_\bigcirc(x_1,\ldots,x_7) = p_1(x_6 - x_1).$$

Hence we can coorient it by taking the natural order of the vectors generating its normal space in a similar way as it was done in Section 1.2.

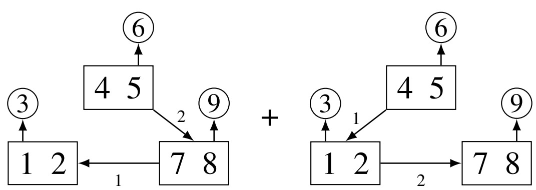
In such a setting, sums correspond to unions of chains, while signs arise from a consistent management of chain orientations. For example, the three-term relation



is a rearrangement, under the sign conventions, of the element corresponding to the union of two locally finite chains:



Example 1.20. *To illustrate the phenomenon, consider the oriented locally finite chain in $M_d^3(9)$ corresponding to the sum*



which is the union of the cooriented chains c_1 and c_2 with common defining inequalities

$$x_1^{(1)} = x_2^{(1)} \leq x_3^{(1)}, \;\; x_4^{(1)} = x_5^{(1)} \leq x_6^{(1)} \;\; \text{and} \;\; x_7^{(1)} = x_8^{(1)} \leq x_9^{(1)}, \tag{1.1}$$

together with the requirement that all x_i-coordinates have the same projection under p_1. The additional defining inequalities in c_1 are

$$\underbrace{x_4^{(1)} \leq x_7^{(1)}}_{2} \quad \textit{and} \quad \underbrace{x_7^{(1)} \leq x_1^{(1)}}_{1},$$

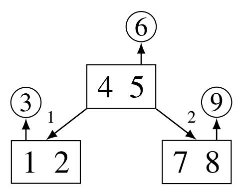
while the additional defining inequalities in c_2 are

$$\underbrace{x_4^{(1)} \leq x_1^{(1)}}_{1} \quad \textit{and} \quad \underbrace{x_1^{(1)} \leq x_7^{(1)}}_{2}.$$

The union of these conditions can be stated as

$$\underbrace{x_4^{(1)} \leq x_1^{(1)}}_{1} \quad \textit{and} \quad \underbrace{x_4^{(1)} \leq x_7^{(1)}}_{2}$$

which, together with (1.1), define the cooriented chain associated to



.

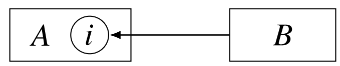
Similarly, the generalized Jacobi relation arises as the boundary of a cell described by a forest one of whose square vertices has $k-2$ (rather than $k-1$) elements.
Finally, since each of these k-forests represents a cell whose boundary lies in the k-equal arrangement, its locally finite chain represents the (Borel-Moore) fundamental class of the submanifold of $M_d^{(k)}(n)$ determined by the interior of c_T. Furthermore, cohomology cup-products are readable as intersection products in Borel-Moore homology. The resulting product structure is spelled out next.

Theorem 1.21 ([8, Theorem 7.1])**.** *For n, k, d as in Theorem 1.17, let T_1, $T_2 \in H^*(M_d^{(k)}(n))$ be two oriented k-forests. The cup product of T_1 and T_2 is zero if either of the following three conditions holds:*

(1) There exist a square vertex A in T_1 and a square vertex B in T_2 such that $A \cap B \neq \emptyset$.

In case that no square vertex of T_1 intersects a square vertex of T_2, we define the superposition $T_1 \cup T_2$ as the oriented graph obtained by superposition of the vertices of

T_1 and T_2 with the convention that if some integer $i \in \mathbf{n}$ lies in a round vertex in, say, T_2 as well as in a square vertex A in T_1, then i appears in $T_1 \cup T_2$ inside the corresponding square vertex A, and if there were some oriented edge in T_2 between the round vertex containing i and some square vertex B, then a corresponding oriented edge between vertices A and B in $T_1 \cup T_2$ would have to be added:

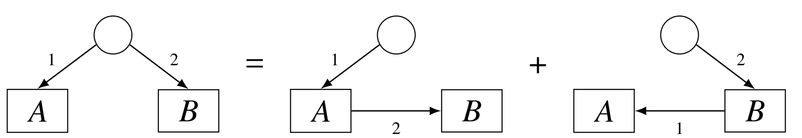


(*This of course might lead to multiple oriented edges between square vertices in $T_1 \cup T_2$, as well as to round vertices having two square vertices as immediate neighboring vertices.*)

(2) *$T_1 \cup T_2$ has unoriented cycles (for instance if two square vertices of $T_1 \cup T_2$ are joined by multiple edges).*

(3) *$T_1 \cup T_2$ has a square vertex with no round vertex attached.*

Otherwise $T_1 \cdot T_2 = T_1 \cup T_2$, the superposition of the k-forests, with orientation set given by the concatenation of the orientation sets of the factors, and with the convention that, if $T_1 \cup T_2$ is not a k-forest (in the sense of Definition 1.15), so that $T_1 \cup T_2$ has one or several round vertices of valency 2*, then we use repeatedly the following form of the three-term relation to write $T_1 \cup T_2$ as a sum of k-forests:*



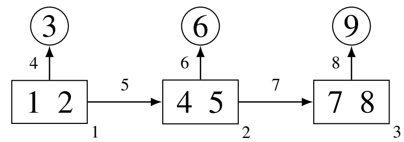
(R)

As above, pictures are local.

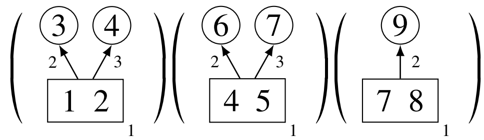
Remark 1.22. *Items (2) and (3) in Theorem 1.21 might have to be used in the iterative process of applying relation (R) to write $T_1 \cup T_2$ as a sum of k-forests. For instance, if the pictures in (R) are in fact global (omitting isolated round vertices), then the two summands on the right of (R) would vanish in view of item (3) in Theorem 1.21.*

Relevant for us is the fact that $H^*(M_d^{(k)}(n))$ is multiplicatively generated by basic oriented k-forests having a single square vertex; such a generator will be said to be *elementary*. Explicitly, a basic oriented k-forest is, up to sign, the product of its connected components.

In turn, each such connected component is, up to sign, a product of elementary oriented k-forests. For example, the basic oriented 3-forest

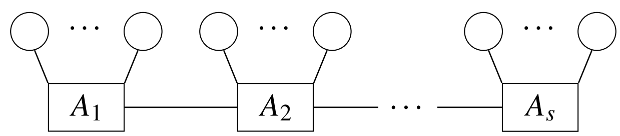


is the product



where we have omitted to write isolated round vertices.

In some arguments of Chapter 3 we will consider k-forests with $\mathbb{Z}_2$-coefficients in order to avoid sign and orientation conventions. In those cases, a non-trivial connected component of a basic k-forest is simply a linear undirected tree



where one of the integers in round vertices attached to each A_i is larger than any of the vertices inside A_i, and where the smallest of the integers in the vertices of the component lies either in A_1 or A_s or in a round vertex attached to A_1 or to A_s. Such a $\mathbb{Z}_2$-equipped basic k-forest lifts canonically to a $\mathbb{Z}$-oriented basic k-forest that satisfies the sign conventions in Definition 1.18.

Example 1.23. *Yuzvinsky described the non-trivial groups $H^*(M_2^{(3)}(6))$ in [35, page 1944] as*

$H^*(M_2^{(3)}(6))$	*rank*
$H^3(M_2^{(3)}(6))$	*20*
$H^4(M_2^{(3)}(6))$	*45*
$H^5(M_2^{(3)}(6))$	*36*
$H_1^6(M_2^{(3)}(6))$	*10*
$H_2^6(M_2^{(3)}(6))$	*10*
$H^7(M_2^{(3)}(6))$	*10*

where $H^6 = H^6_1 \oplus H^6_2$. This and Yuzvinsky's product description, which is captured by the two multiplication maps $H^3(M_2^{(3)}(6)) \otimes H^3(M_2^{(3)}(6)) \twoheadrightarrow H^6_2(M_2^{(3)}(6))$ and $H^3(M_2^{(3)}(6)) \otimes H^4(M_2^{(3)}(6)) \twoheadrightarrow H^7(M_2^{(3)}(6))$, are transparent in terms of forests:

- *The smallest positive dimension where the cohomology is non trivial is given by the minimal dimension of an elementary 3-forest, namely a square vertex with one round vertex attached, which has dimension 3. As three numbers determine an elementary 3-forest, there are $\binom{6}{3} = 20$ basis elements in $H^3(M_2^{(3)}(6))$. Here and below, non-explicit $\mathbb{Z}$-orientations and orderings are taken as explained above.*

- *The next cohomological dimension is generated by elementary 3-forests with one square and two round vertices attached. In this case we can first select 4 numbers to fill in the square and round vertices. From those numbers, the greatest value is forced to be in a round vertex so it only remains to determine the value of the other round vertex. Therefore there are $\binom{6}{4}\binom{3}{1} = 45$ basis elements in $H^4(M_2^{(3)}(6))$.*

- *The rank of $H^5(M_2^{(3)}(6))$ is obtained similarly, in this case there are $\binom{6}{5}\binom{4}{2} = 36$ basis elements.*

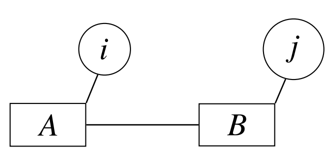
- *Dimension 6 is the first case where products appear. Here we have two types of basis elements:*

  - *Basic 3-forests with two square vertices and a single round vertex attached to each square vertex. There are $\binom{6}{3}/2 = 10$ such basis elements.*

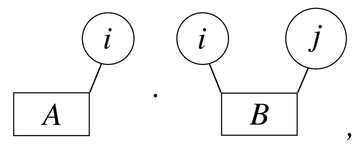
    The group $H^6_2(M_2^{(3)}(6))$ is generated by such basic 3-forests, each of which is the product (up to a sign) of two elementary 3-forests of dimension 3 (the superposition of its two components).

  - *Elementary 3-forests with one square vertex connected to four round vertices. There are $\binom{5}{2} = 10$ such basis elements, all of them being linearly independent modulo product-decomposable elements. This corresponds to the $H^6_1(M_2^{(3)}(6))$ summand.*

- *Ignoring orientation matters, basic 3-forests in $H^7(M_2^{(3)}(6))$ are necessarily of the form*



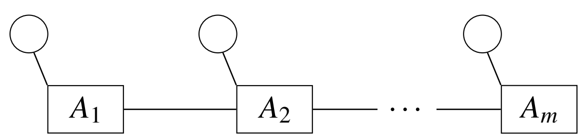
and we can assume without loss of generality that $i < j$. Such a basis element is the product of two elementary 3-forests, one of dimension 3 and one of dimension 4, namely



where isolated round vertices in both factors have been omitted. Note there are $\binom{6}{3}/2 = 10$ *basis elements in dimension* 7.

- *Finally, it is not possible to construct generators of dimension greater than 7, because we can add neither more vertices nor more edges.*

Elements of maximal dimension in Example 1.23 are k-forests having a single linear tree component of the form



More generally:

Lemma 1.24. *Let n, k, d be as in Theorem 1.17. Elements of maximal dimension in the cohomology ring $H^*(M_d^{(k)}(n))$ are given by linear combinations of basic k-forests having a single component which is an ordered linear tree with $\left\lfloor \frac{n}{k} \right\rfloor$ square vertices and $n-(k-1)\left\lfloor \frac{n}{k} \right\rfloor$ non-isolated round vertices.*

Proof. The mod-2 reduction map $H^*\left(M_d^{(k)}(n);\mathbb{Z}\right) \to H^*\left(M_d^{(k)}(n);\mathbb{Z}_2\right)$ yields an isomorphism after tensoring with $\mathbb{Z}_2$, so it suffices to prove the lemma for $\mathbb{Z}_2$ coefficients. Consequently, we can ignore all orientation and sign conventions. In addition, it suffices to check the stated characterization for basic k-forests of maximal dimension. We start by noticing that such a basic k-forest f cannot have isolated round vertices (for any such vertex can be attached to some square vertex of f to produce a basic k-forest of larger dimension), and must have a single linear tree component (otherwise a basis element of larger dimension can be constructed by adding edges that concatenate the components of f). Let $A_1, \ldots, A_m$ denote the square vertices of f, and b_i stand for the number of round vertices attached to A_i, so that

$$n = m(k-1) + \sum_{i=1}^{m} b_i \qquad (1.2)$$

as there are no isolated round vertices. We claim that

$$0 \leq \sum_{i=1}^{m} (b_i - 1) < k. \tag{1.3}$$

The first inequality is obvious as each b_i is positive. If the second inequality fails, then k of the round vertices, except for the greatest round vertex attached to each square vertex, can be detached from its corresponding A_i, to yield a smaller-dimensional k-tree f'. The integers corresponding to the detached round vertices can then be assembled into a new elementary basic k-forest that can further be concatenated to f' to yield a basic k-forest f''. By construction, $\deg(f') = \deg(f) - k(d-1)$, while $\deg(f'') = \deg(f') + d(k-2) + 2(d-1)$, which yields $\deg(f'') > \deg(f)$, as $k \geq 3$, contradicting the maximality of f. This proves (1.3). The conclusion of the lemma now follows from (1.2) and (1.3): $n = mk + b$, where $b := \sum_{i=1}^{m}(b_i - 1)$ is in fact the residue in the division of n by k (so that $m = \lfloor \frac{n}{k} \rfloor$). □

Remark 1.25. *As illustrated by Example 1.23, any subset of* $\mathbf{n}$ *with* k *elements determines (up to a sign) a cohomology class of minimal dimension (i.e. dimension* a *in the notation of Corollary 1.26): an elementary* k*-forest with a single attached round vertex (and some prescribed orientations). More generally, choosing* mk *elements of* $\mathbf{n}$*, and partitioning these elements into* m *subsets of cardinality* k*, say* $P_1 \sqcup P_2 \sqcup \cdots \sqcup P_m$*, we can form a basic* k*-forest of dimension* ma *which, in addition, factors (up to a sign) as a product of* m *elementary minimal-dimension* k*-forests, namely those determined by each* P_i*. This observation will be the basis to construct, in Chapter* 3*, a number of relevant cohomology classes in cartesian products of* $M_d^{(k)}(n)$*.*

Corollary 1.26. *Let* n, k, d *be as in Theorem 1.17. The largest (respectively lowest) positive dimension where the cohomology of* $M_d^{(k)}(n)$ *is non-zero equals* $ma + (d-1)(m+b-1)$ *(respectively* a*), where* $m = \lfloor \frac{n}{k} \rfloor$*,* $a = d(k-1) - 1$ *and* $b = n - mk$ *(so that* $0 \leq b < k$*).*

Proof. The first observation in Remark 1.25 below yields the assertion about the bottom non-trivial dimension. Lemma 1.24 yields the assertion about the top non-trivial dimension. □

2 | Lusternik-Schnirelmann Category and Topological Complexity for the Real Case

This chapter develops the results published in [15] related to the LS category and (higher) topological complexity of non-k-equal arrangements $M_1^{(k)}(n)$. The invariants are fully determined by means of Theorem 1.2. The upper bound is easily obtained since Severs and White established in [31, Theorems 1.1 and 1.2] that $M_1^{(k)}(n)$ admits a minimal cellular model, i.e., it has the homotopy type of a cell complex having as many cells in each dimension d as the rank of the cohomology group $H^d(M_1^{(k)}(n))$—since $H_d(M_1^{(k)}(n))$ is torsion-free. Hence, using that the minimum and maximum dimensions for cohomology are $k-2$ and $(k-2)\left\lfloor \frac{n}{k} \right\rfloor$ respectively ([31, Theorem 1.2], [5, Theorem 1.1] or Section 2.1 below), we have the following lemma.

Lemma 2.1. *$M_1^{(k)}(n)$ is a $(k-3)$-connected space having* $\mathrm{hdim}(M_1^{(k)}(n)) = (k-2)\left\lfloor \frac{n}{k} \right\rfloor$.

Corollary 2.2. *The Lusternik-Schnirelmann and (higher) topological complexity for $M_1^{(k)}(n)$ satisfy the following inequalities*

$$\mathrm{cl}(M_1^{(k)}(n)) \leq \mathrm{cat}(M_1^{(k)}(n)) \leq \left\lfloor \frac{n}{k} \right\rfloor,$$

$$\mathrm{zcl}(M_1^{(k)}(n)) \leq \mathrm{TC}(M_1^{(k)}(n)) \leq 2\left\lfloor \frac{n}{k} \right\rfloor,$$

$$\mathrm{zcl}_s(M_1^{(k)}(n)) \leq \mathrm{TC}_s(M_1^{(k)}(n)) \leq s\left\lfloor \frac{n}{k} \right\rfloor.$$

Therefore, in order to prove that $\mathrm{cat}(M_1^{(k)}(n)) = \left\lfloor \frac{n}{k} \right\rfloor$ and $\mathrm{TC}_s(M_1^{(k)}(n)) = s\left\lfloor \frac{n}{k} \right\rfloor$—identifying TC_2 with TC—it is enough to exhibit a non trivial cup product or s-th zero-divisor of appropriate length.

2.1 Cup length and zero-divisors cup length

In Example 1.14 it was shown that, in $M_1^{(3)}(7)$, only cup products in dimension 2 exist since any basic string preorder of dimension 3 requires $9 > 7$ numbers, and so $2 = \lfloor \frac{7}{3} \rfloor$ is the maximum number of factors that can be multiplied to obtain a non zero cohomology class. This fact can be easily generalized by noticing that in the definition of a basic string preorder

$$(I_0)[J_1](I_1)[J_2]\cdots[J_d](I_d)$$

the requirement $\max J_i \cup I_i \in I_i$ implies that each (I)-block, except possibly the first one, has at least one element, hence, the basic string preorder has at least a sequence of $[J_k](I_k)$ blocks containing a total of k elements, so the least possible number of values required to produce such a basic element is dk. Therefore, in $M_1^{(k)}(n)$, the maximum value for d is $\lfloor \frac{n}{k} \rfloor$ and an example of such basic element is

$$[1,\ldots,k-1]\,(k)\,[k+1,\ldots,2k-1]\,(2k)\cdots[(q-1)k+1,\ldots,qk-1]\,(qk) \tag{2.1}$$

where $q = \lfloor \frac{n}{k} \rfloor$. This proves the following lemma.

Lemma 2.3. $\mathrm{cl}(M_1^{(k)}(n)) = \lfloor \frac{n}{k} \rfloor$.

Corollary 2.4. $\mathrm{cat}(M_1^{(k)})(n) = \lfloor \frac{n}{k} \rfloor$

In order to address the topological complexity using zero divisors, we introduce a few key elements in $H^*(M_1^{(k)}(n))$ and in $H^*(M_1^{(k)}(n))^{\otimes 2}$. In what follows cohomology groups will be taken with $\mathbb{Z}_2$-coefficients, a restriction that is not essential but allows us to simplify calculations.

Definition 2.5. *For a positive integer m satisfying $m+k \leq n+2$, consider the elements $x_m, x'_m \in H^{k-2}(M_1^{(k)}(n))$ given by*

$$x_m = (1,\ldots,m-2,m-1)\,[m,m+1,\ldots,m+k-2]\,(m+k-1,\ldots,n)\,,$$

$$x'_m = (1,\ldots,m-2,m)\,[m-1,m+1,\ldots,m+k-2]\,(m+k-1,\ldots,n)\,,$$

where x'_m is defined only for $m \geq 2$.

Each of the corresponding zero-divisors $y_m = x_m \otimes 1 + 1 \otimes x_m$ for $M_1^{(k)}(n)$ is central in what follows, with the elements x'_m playing a subtle role.

Remark 2.6. *If $m+k \le n+1$ the elements x_m and x'_m end with a ()-block with the number $m+k-1$ and therefore they are elementary basic elements in $M_1^{(k)}(n)$.*

Example 2.7. *As illustrated in Example 1.12, the product of the elements x_i can be easily expressed in terms of a string preorder as follows*

$$\begin{aligned}&\prod_{j=1}^{i} x_{(j-1)k+2} = x_2 x_{k+2} \cdots x_{(i-1)k+2}\\ &= (1)[2,\ldots,k](k+1)[k+2,\ldots,2k](2k+1)\cdots[(i-1)k+2,\ldots,ik](ik+1,\ldots,n)\end{aligned} \tag{2.2}$$

moreover, if $ik+1 \le n$ the product is a basis element in $H^*(M_1^{(k)}(n))$.
Likewise, if $\widetilde{x}_{(j-1)k+1}$ stands for either $x_{(j-1)k+1}$ or $x'_{(j-1)k+1}$ (the latter one being a possibility only for $j \ge 2$), then

$$\begin{aligned}&\prod_{j=1}^{i} \widetilde{x}_{(j-1)k+1} = \widetilde{x}_1 \widetilde{x}_{k+1} \cdots \widetilde{x}_{(i-1)k+1}\\ &= [1,\ldots,k-1](k)[k+1,\ldots,2k-1]\cdots((i-1)k)[(i-1)k+1,\ldots,ik-1](ik,\ldots,n),\end{aligned} \tag{2.3}$$

where curved arrows indicate pairs of elements that might have to be switched (depending on the actual term $\widetilde{x}_{(j-1)k+1}$ under consideration), is a basis element in $H^*(M_1^{(k)}(n))$ provided $ik \le n$.

Example 2.8. *Since $3 \le k < n$, both x_1 and x_2 are Baryshnikov basis elements in $H^*(M_1^{(k)}(n))$, and since $x_1 \ne x_2$, we obviously have*

$$y_1 y_2 = (x_1 \otimes 1 + 1 \otimes x_1)(x_2 \otimes 1 + 1 \otimes x_2) = \cdots + x_1 \otimes x_2 + x_2 \otimes x_1 + \cdots \ne 0. \tag{2.4}$$

So $2 \le \mathrm{zcl}(M_1^{(k)}(n))$, which determines the topological complexity of $M_1^{(k)}(n)$ for $2k > n$ in view of Corollary 2.2.

The proof for $n \ge 2k$ requires a major generalization of the simple calculation in (2.4). The product indicated in (2.5) below will play the role of the product $y_1 y_2$ on the left-hand side of (2.4). Most importantly, the tensor factors x_1 and x_2 in the two highlighted summands on the right-hand side of (2.4) will be replaced by products of the form (2.2), and by certain products of the form (2.3), some of which are made explicit as follows:

$$p_{i,1} = \begin{cases} x_1\left(\prod_{j=1}^{a-1} x_{(2j-1)k+1}x'_{2jk+1}\right)x_{(2a-1)k+1}, & \text{if } i = 2a \geq 2; \\ x_1\left(\prod_{j=1}^{a} x_{(2j-1)k+1}x'_{2jk+1}\right), & \text{if } i = 2a+1 \geq 3, \end{cases}$$

$$p_{i,2} = \begin{cases} x_1\left(\prod_{j=1}^{a-1} x'_{(2j-1)k+1}x_{2jk+1}\right)x'_{(2a-1)k+1}, & \text{if } i = 2a \geq 2; \\ x_1\left(\prod_{j=1}^{a} x'_{(2j-1)k+1}x_{2jk+1}\right), & \text{if } i = 2a+1 \geq 3. \end{cases}$$

Theorem 2.9. *If the integers i, k, n satisfy $2 \leq i$, and $ik \leq n$, then the product*

$$\prod_{j=1}^{i} y_{(j-1)k+1}y_{(j-1)k+2} \in H^*(M_1^{(k)}(n))^{\otimes 2} \tag{2.5}$$

is non-zero. Explicitly:

1. *If $ik+1 \leq n$, then the expression of (2.5) as a linear combination of Baryshnikov tensor basis elements for $H^*(M_1^{(k)}(n))^{\otimes 2}$ has the following element as a summand*

$$\prod_{j=1}^{i} x_{(j-1)k+1} \otimes \prod_{j=1}^{i} x_{(j-1)k+2}.$$

2. *If $ki = n$, then the expression of (2.5) as a linear combination of Baryshnikov tensor basis elements for $H^*(M_1^{(k)}(n))^{\otimes 2}$ uses the Baryshnikov basis element $p_{i,1} \otimes p_{i,2}$.*

The proof of the theorem is done throughout the remainder of this section. Note that the hypothesis $i \geq 2$ is relevant only for the second half of Theorem 2.9. The actual exceptional case that has to be avoided is $n = k$, for which y_1y_2 is forced to vanish.
The validity of $2\left\lfloor \frac{n}{k} \right\rfloor \leq \mathrm{zcl}(M_1^{(k)}(n))$ for $n \geq 2k$ (i.e. the cases that remain to be considered) follows from Theorem 2.9 below by taking $i = \lfloor n/k \rfloor$. So, the rest of the section is devoted to the proof of Theorem 2.9.

Lemma 2.10. *The following relations hold in $H^*(M_1^{(k)}(n))$:*

1. $x_2 x_{k+1} = x_1 x_{k+1}$*, for* $n \geq 2k-1$.

2. $x_{n-2k+4} x_{n-k+2} = x_{n-2k+3} x_{n-k+2} = 0$*, for* $n \geq 2k-2$.

3. $x_{n-2k+2} x_{n-k+2} = x_{n-2k+2} x_{n-k+1}$*, for* $n \geq 2k-1$.

4. $x_{n-2k+1} x_{n-k+2} = x_{n-2k+1} x_{n-k+1} + x_{n-2k+1} x'_{n-k+1}$*, for* $n \geq 2k$.

5. $x_r x_{r+k} x_{r+2k-1} = x_r x_{r+k-1} x_{r+2k-1}$*, for* $n \geq r+3k-3$ *and* $r \geq 1$.

6. $x_r x_{r+k+1} x_{r+2k} = x_r x_{r+k} x_{r+2k} + x_r x'_{r+k} x_{r+2k}$*, for* $n \geq r+3k-2$ *and* $r \geq 1$.

Remark 2.11. *The numeric restrictions on* k*,* n *and* r *ensure that each of the factors* x_m *in the six items above is an element of* $H^*(M_1^{(k)}(n))$.

Proof of Lemma 2.10. All these equalities follow from Theorem 1.13 and the discussion preceding Definition 1.11. We give full details for completeness.

- Assume $n \geq 2k-2$. Taking $A = \{1, \ldots, n-k+1\}$, $B = \{n-k+2, \ldots, n-1\}$, $C = \{n\}$ and $\mathbb{Z}_2$ coefficients in Theorem 1.13.1, we get

$$\begin{aligned} x_{n-k+2} &= (1, \ldots, n-k+1)[n-k+2, \ldots, n] \\ &= \sum_{i=1}^{n-k+1} (1, \ldots, \widehat{i}, \ldots, n-k+1)[i, n-k+2, \ldots, n-1](n). \end{aligned} \tag{2.6}$$

  As explained in the paragraph preceding Definition 1.11, all terms in the summation in (2.6) vanish when multiplied by

$$x_{n-2k+3} = (1, \ldots, n-2k+2)[n-2k+3, \ldots, n-k+1](n-k+2, \ldots, n).$$

  This yields $x_{n-2k+3} x_{n-k+2} = 0$, also the equality $x_{n-2k+4} x_{n-k+2} = 0$ follows directly. This proves item 2.

- Assume $n \geq 2k-1$. Terms with $i \leq n-k$ in the summation in (2.6) vanish when multiplied by

$$x_{n-2k+2} = (1, \ldots, n-2k+1)[n-2k+2, \ldots, n-k](n-k+1, \ldots, n).$$

  This yields $x_{n-2k+2} x_{n-k+2} = x_{n-2k+2} x_{n-k+1}$, proving item 3.

- Assume $n \geq 2k$. Terms with $i < n-k$ in the summation in (2.6) vanish when multiplied by
$$x_{n-2k+1} = (1,\ldots,n-2k)[n-2k+1,\ldots,n-k-1](n-k,\ldots,n).$$
This yields $x_{n-2k+1}x_{n-k+2} = x_{n-2k+1}x_{n-k+1} + x_{n-2k+1}x'_{n-k+1}$, proving item 4.

- Assume $n \geq r+3k-2$ and $r \geq 1$. Take $A = \{1,\ldots,r+k-1\}$, $B = \{r+k,\ldots,r+2k-3\}$ and $C = \{r+2k-2,\ldots,n\}$ in Theorem 1.13.1
$$\sum_{i=1}^{r+k-1}(1,\ldots,\widehat{i},\ldots,r+k-1)[i,r+k,\ldots,r+2k-3](r+2k-2,\ldots,n)$$
$$= \sum_{i=r+2k-2}^{n}(1,\ldots,r+k-1)[r+k,\ldots,r+2k-3,i](r+2k-2,\ldots,\widehat{i},\ldots,n).$$
Terms with $i < r+k-1$ in the first summation vanish when multiplied by $x_r = (1,\ldots,r-1)[r,\ldots,r+k-2](r+k-1,\ldots,n)$, and terms with $i > r+2k-2$ in the second summation vanish when multiplied by
$$x_{r+2k-1} = (1,\ldots,r+2k-2)[r+2k-1,\ldots,r+3k-3](r+3k-2,\ldots,n).$$
This yields the equality $x_r x_{r+k-1} x_{r+2k-1} = x_r x_{r+k} x_{r+2k-1}$, proving item 5.

- When $n \geq 2k-1$, the previous argument applies for $r = 2-k$ —by vacuity in the case of the assertion about the first summation, whose only one term is x_1. This yields $x_1 x_{k+1} = x_2 x_{k+1}$, proving item 1.

- Assume $n \geq r+3k-2$ and $r \geq 1$. Take $A = \{1,\ldots,r+k\}$, $B = \{r+k+1,\ldots,r+2k-2\}$, $C = \{r+2k-1,\ldots,n\}$, and $\mathbb{Z}_2$ coefficients in Theorem 1.13.1 to get
$$\sum_{i=1}^{r+k}(1,\ldots,\widehat{i},\ldots,r+k)[i,r+k+1,\ldots,r+2k-2](r+2k-1,\ldots,n)$$
$$= \sum_{i=r+2k-1}^{n}(1,\ldots,r+k)[r+k+1,\ldots,r+2k-2,i](r+2k-1,\ldots,\widehat{i},\ldots,n). \quad (2.7)$$
Terms with $i < r+k-1$ in the first summation vanish when multiplied by
$$x_r = (1,\ldots,r-1)[r,\ldots,r+k-2](r+k-1,\ldots,n),$$

while terms with $i > r+2k-1$ in the second summation vanish when multiplied by

$$x_{r+2k} = (1,\ldots,r+2k-1)[r+2k,\ldots,r+3k-2](r+3k-1,\ldots,n).$$

This yields the equality $x_r x'_{r+k} x_{r+2k} + x_r x_{r+k} x_{r+2k} = x_r x_{r+k+1} x_{r+2k}$, proving item 6. □

Proof of part 1 in Theorem 2.9. Using the description of non-trivial products given in the paragraph preceding Definition 1.11, we get

$$\begin{aligned} y_{(j-1)k+1} y_{(j-1)k+2} &= (x_{(j-1)k+1} \otimes 1 + 1 \otimes x_{(j-1)k+1})(x_{(j-1)k+2} \otimes 1 + 1 \otimes x_{(j-1)k+2}) \\ &= x_{(j-1)k+1} \otimes x_{(j-1)k+2} + x_{(j-1)k+2} \otimes x_{(j-1)k+1}, \end{aligned}$$

so the product in (2.5) is

$$\begin{aligned} \prod_{j=1}^{i} y_{(j-1)k+1} y_{(j-1)k+2} &= (x_1 \otimes x_2 + x_2 \otimes x_1)(x_{k+1} \otimes x_{k+2} + x_{k+2} \otimes x_{k+1}) \cdots \\ &\qquad \cdots (x_{(i-1)k+1} \otimes x_{(i-1)k+2} + x_{(i-1)k+2} \otimes x_{(i-1)k+1}) \\ &= \sum_{\substack{\epsilon_j \in \{1,2\} \\ 1 \le j \le i}} x_{3-\epsilon_1} x_{k+3-\epsilon_2} \cdots\, x_{(i-1)k+3-\epsilon_i} \otimes x_{\epsilon_1} x_{k+\epsilon_2} \cdots\, x_{(i-1)k+\epsilon_i}. \end{aligned} \tag{2.8}$$

The basis element we care about, namely

$$\prod_{j=1}^{i} x_{(j-1)k+1} \otimes \prod_{j=1}^{i} x_{(j-1)k+2}, \tag{2.9}$$

is the summand in (2.8) with $\epsilon_j = 2$ for all j. The proof task is to argue that, when we expand the other terms of (2.8) as sums of tensor of basis elements, the tensor (2.9) does not appear. This is obvious for the summand in (2.8) with $\epsilon_j = 1$ for all j. For all other summands, the assertion will be argued by focusing on the sequence of leaps associated to the subscripts of both tensor factors of each summand in (2.8). Explicitly, the first leap in the subscripts of $x_{3-\epsilon_1} x_{k+3-\epsilon_2} \cdots\, x_{(i-1)k+3-\epsilon_i}$ is $k+3-\epsilon_2-(3-\epsilon_1) = k+\epsilon_1-\epsilon_2$, and the full sequences of leaps associated to

$$x_{3-\epsilon_1} x_{k+3-\epsilon_2} \cdots\, x_{k(i-1)+3-\epsilon_i} \quad \text{and} \quad x_{\epsilon_1} x_{k+\epsilon_2} \cdots\, x_{k(i-1)+\epsilon_i} \tag{2.10}$$

are, respectively,

$$(k+\epsilon_1-\epsilon_2, k+\epsilon_2-\epsilon_3, \ldots, k+\epsilon_{i-1}-\epsilon_i) \quad \text{and} \quad (k-\epsilon_1+\epsilon_2, k-\epsilon_2+\epsilon_3, \ldots, k-\epsilon_{i-1}+\epsilon_i). \tag{2.11}$$

Such sequences of leaps clearly satisfy:

(A) Leap values are either $k-1$, k, or $k+1$. Moreover, if all k-leaps are removed from either one of the sequences in (2.11), then the resulting sequence of leaps either is empty or, else, has leap values that alternate between $k-1$ and $k+1$: $(k-1, k+1, k-1, \ldots)$ or $(k+1, k-1, k+1, \ldots)$.

(B) The two sequences of leaps in (2.11) are coordinate-wise complementary to each other with respect to $2k$.

(C) The first leap different from k (if any) in either of the sequences of leaps (2.11) is a $(k+1)$-leap (respectively $(k-1)$-leap) provided the corresponding product in (2.10) starts with x_1 (respectively x_2).

Since the right tensor factor in (2.9), i.e. $\prod_{j=1}^{i} x_{(j-1)k+2}$, is a basic string preorder starting as $(1)[2, \ldots, k]\cdots$, the proof is complete in view of Proposition 2.12 below. □

Proposition 2.12. *Any summand in (2.8) whose associated sequences of leaps (2.11) contain at least a $(k-1)$-leap (equivalently a $(k+1)$-leap) is a linear combination of tensor basis elements $u \otimes v$ where* both *u and v are basic string preorders starting as*

$$[1, \ldots, k-1](I_1)\cdots(I_{i-1})[J_i](I_i).$$

Proof. Take a product $p = x_{k_1}x_{k_2}\cdots x_{k_i}$ in (2.10), so $k_1 \in \{1, 2\}$, with associated sequence of leaps $(\ell_1, \ldots, \ell_{i-1})$ satisfying conditions (A)–(C) above, and so that not all leap values ℓ_j are k.

Case $k_1 = 1$: p has the form

$$x_1 \cdots \underbrace{x_{kr_1+1}x_{k(r_1+1)+2}}_{(k+1)\text{-leap}} \cdots \underbrace{x_{kr_2+2}x_{k(r_2+1)+1}}_{(k-1)\text{-leap}} \cdots \underbrace{x_{kr_3+1}x_{k(r_3+1)+2}}_{(k+1)\text{-leap}} \cdots \underbrace{x_{kr_4+2}x_{k(r_4+1)+1}}_{(k-1)\text{-leap}} \cdots, \tag{2.12}$$

where we only indicate $(k-1)$-leaps and $(k+1)$-leaps. Items 5 and 6 in Lemma 2.10 allow us to replace each portion $x_{kr_j+1}x_{k(r_j+1)+2}\cdots x_{kr_{j+1}+2}x_{k(r_{j+1}+1)+1}$, having an initial $(k+1)$-leap, a final $(k-1)$-leap, and (perhaps) some intermediate k-leaps, by

$$x_{kr_j+1}(x_{k(r_j+1)+1} + x'_{k(r_j+1)+1})x_{k(r_j+2)+1}\cdots x_{kr_{j+1}+1}x_{k(r_{j+1}+1)+1},$$

which only has k-leaps. The replacing process can be iterated since the initial and final terms in the replacing portion agree with those in the replaced portion. After all replacements are made, and sums are distributed, p becomes a sum of expressions each of which is similar to the original one (2.12), except that some of the initial elements x_{kj+1} get replaced by the corresponding x'_{kj+1}, and in such a way that no $(k-1)$-leaps show up, and at most one $(k+1)$-leap shows up. But any such expression is a basis element of the required form (the latter assertion uses the hypothesis $ik+1 \leq n$ in part 1 of Theorem 2.9 —see Remark 2.13 below).

Case $k_1 = 2$: p has the form

$$x_2 \cdots \underbrace{x_{kr_1+2}x_{k(r_1+1)+1}}_{(k-1)\text{-leap}} \cdots \underbrace{x_{kr_2+1}x_{k(r_2+1)+2}}_{(k+1)\text{-leap}} \cdots \underbrace{x_{kr_3+2}x_{k(r_3+1)+1}}_{(k-1)\text{-leap}} \cdots \underbrace{x_{kr_4+1}x_{k(r_4+1)+2}}_{(k+1)\text{-leap}} \cdots,$$

Items 1 and 5 in Lemma 2.10 allow us to replace the initial portion $x_2 \cdots x_{kr_1+2}x_{k(r_1+1)+1}$ by $x_1 \cdots x_{kr_1+1}x_{k(r_1+1)+1}$. Then, the replacement process described in the previous case allows us to write p as a sum of basis elements of the required form. □

Remark 2.13. Part 2 in Theorem 2.9 will be proved using an argument similar to that in the previous proof, except that it will be necessary to deal first with an additional subtlety. Namely, note that when $ik = n$, we have

$$\begin{aligned} x_{(i-1)k+2} &= (1,\cdots,(i-1)k+1)\,[(i-1)k+2,\cdots,ik]\,(ik+1,\cdots,n) \\ &= (1,\cdots,(i-1)k+1)\,[(i-1)k+2,\cdots,n], \end{aligned}$$

which is an elementary *non-basic* element (i.e., under the main hypothesis in part 2 of Theorem 2.9). So, when analyzing a typical tensor factor $x_{\epsilon_1}x_{k+\epsilon_2}\cdots x_{(i-1)k+\epsilon_i}$ in (2.8) with $\epsilon_i = 2$, the recursive process described in the previous proof will not end up producing sums of basis elements. This issue will be resolved using item 4 in Lemma 2.10.

Let us go back to the starting point for the proof of part 2 in Theorem 2.9, i.e., the expression in (2.8) for the product $\prod_{j=1}^{i} y_{(j-1)k+1}y_{(j-1)k+2}$. As observed in Remark 2.13, we no longer work with the basis element indicated in part 1 of Theorem 2.9. Instead, the basis element we now care about is $p_{i,1} \otimes p_{i,2}$, where $ki = n$, and which arises from one of the two summands in (2.8) for which the values of the indices ϵ_j alternate between 1 and 2.

In order to simplify the argument, it is convenient to note that all y_j, and therefore their product $\prod_{j=1}^{i} y_{(j-1)k+1}y_{(j-1)k+2}$, are invariant under the involution induced by the map that switches coordinates in $M_1^{(k)}(n) \times M_1^{(k)}(n)$. We show the following (equivalent, by the symmetry just noted, but slightly simpler-to-prove) version of part 2 in Theorem 2.9:

Theorem 2.14. *For $i \geq 2$, $k \geq 3$ and $n = ki$, both $p_{i,1} \otimes p_{i,2}$ and $p_{i,2} \otimes p_{i,1}$ are used in the expression of the product (2.5) as a linear combination of Baryshnikov tensor basis elements for $H^*(M_1^{(k)}(n))^{\otimes 2}$.*

Proof. We provide full proof details when $i = 2a$ is even; the parallel argument for i odd is left as an exercise for the reader. In order to simplify notation, we let $r_1 \cdot r_2 \cdots r_t$ and $r_1 \cdot r_2 \cdots r_t | s_1 \cdot s_2 \cdots s_t$ stand for $x_{r_1} x_{r_2} \cdots x_{r_t}$ and $x_{r_1} x_{r_2} \cdots x_{r_t} \otimes x_{s_1} x_{s_2} \cdots x_{s_t}$, respectively. With this notation, (2.8) becomes

$$\begin{aligned}
&(1|2+2|1)\,((k+1)|(k+2)+(k+2)|(k+1))\,\cdots \\
&\qquad\cdots\,(((2a-1)k+1)|((2a-1)k+2)+((2a-1)k+2)|((2a-1)k+1)) \\
&=\sum_{\substack{\epsilon_j\in\{1,2\}\\ 1\leq j\leq i}}(3-\epsilon_1)(k+3-\epsilon_2)\cdots((2a-1)k+3-\epsilon_{2a})\,|\,(\epsilon_1)(k+\epsilon_2)\cdots((2a-1)k+\epsilon_{2a}).
\end{aligned} \tag{2.13}$$

The summand with $(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2a}) = (1, 2, 1, \ldots, 2)$ is

$$\begin{aligned}
&2\cdot(k+1)\cdot(2k+2)\cdot(3k+1)\cdots((2a-2)k+2)\cdot((2a-1)k+1) \\
&\qquad |\, 1\cdot(k+2)\cdot(2k+1)\cdot(3k+2)\cdots((2a-2)k+1)\cdot((2a-1)k+2),
\end{aligned} \tag{2.14}$$

whose associated sequences of leaps are

$$(k-1, k+1, k-1, \ldots, k-1) \quad \text{and} \quad (k+1, k-1, k+1, \ldots, k+1). \tag{2.15}$$

Using the replacing process explained in the previous proof, it is clear that the expression of

$$2\cdot(k+1)\cdot(2k+2)\cdot(3k+1)\cdots((2a-2)k+2)\cdot((2a-1)k+1)$$

in terms of Baryshnikov basis elements uses $p_{2a,1}$, but not $p_{2a,2}$. Likewise, the replacing process and item 4 in Lemma 2.10 imply that the expression of

$$1\cdot(k+2)\cdot(2k+1)\cdot(3k+2)\cdots((2a-2)k+1)\cdot((2a-1)k+2)$$

in terms of Baryshnikov basis uses $p_{2a,2}$. Therefore the expression of (2.14) in terms of Baryshnikov (tensor) basis elements uses $p_{2a,1} \otimes p_{2a,2}$ without using $p_{2a,2} \otimes p_{2a,1}$. Further, the symmetry coming from the involution induced by the switching map on $M_1^{(k)}(n)^{\times 2}$ implies that the expression in terms of Baryshnikov basis of the summand in (2.13) with $(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2a}) = (2, 1, 2 \ldots, 1)$ uses $p_{2a,2} \otimes p_{2a,1}$ without using $p_{2a,1} \otimes p_{2a,2}$.

It remains to prove that neither $p_{2a,1} \otimes p_{2a,2}$ nor $p_{2a,2} \otimes p_{2a,1}$ are used in the expression in terms of basis elements of any summand in (2.13) whose associated sequences of leaps is different from those in (2.15). By symmetry, it suffices to consider the case of a summand

$$(3-\epsilon_1)(k+3-\epsilon_2)\cdots((2a-1)k+3-\epsilon_{2a})\,|\,(\epsilon_1)(k+\epsilon_2)\cdots((2a-1)k+\epsilon_{2a}) \tag{2.16}$$

with $\epsilon_1 = 1$. Let $\lambda \in \{k-1, k, k+1\}$ ($\rho \in \{k+1, k, k-1\}$) stand for the value of the last leap in the tensor factor on the left (right) of (2.16). Recall $\lambda + \rho = 2k$.

Case $\lambda = \rho = k$: The ending portion of one of the two tensor factors in (2.16) is forced to be

$$\cdots((2a-2)k+1)\cdot((2a-1)k+1).$$

The replacing process shows that such a factor cannot give rise to $p_{2a,1}$ or $p_{2a,2}$ in its expression in terms of Baryshnikov basis.

Case $(\lambda, \rho) = (k-1, k+1)$: The equalities $\epsilon_{2a-1} = 1$ and $\epsilon_{2a} = 2$ are now forced. Letting j' stand for x'_j, and ignoring Baryshnikov basis elements different from $p_{2a,1}$ and $p_{2a,2}$, the right factor in (2.16) then becomes

$$1\cdot(k+\epsilon_2)\cdots((2a-2)k+1)((2a-1)k+2) = 1\cdot(k+\epsilon_2)\cdots((2a-2)k+1)((2a-1)k+1)',$$

in view of the replacing process and item 4 in Lemma 2.10. Further, the replacing process makes it clear that the expression of the latter element in terms of Baryshnikov basis elements does not use $p_{2a,1}$, and that it uses $p_{2a,2}$ only if the sequence of leaps associated to the right tensor factor in (2.16) is the second sequence in (2.15).

Case $(\lambda, \rho) = (k+1, k-1)$: The equalities $\epsilon_{2a-1} = 2$ and $\epsilon_{2a} = 1$ are now forced. Ignoring Baryshnikov basis elements different from $p_{2a,1}$ and $p_{2a,2}$, the left factor in (2.16) becomes

$$2\cdot(k+3-\epsilon_2)\cdots((2a-2)k+1)((2a-1)k+2) = 2\cdot(k+3-\epsilon_2)\cdots((2a-2)k+1)((2a-1)k+1)',$$

where the latter expression further evolves under the replacing process (still ignoring Baryshnikov basis elements different from $p_{2a,1}$ and $p_{2a,2}$) to either zero or to

$$2\cdot(k+1)\cdot(2k+1)\cdot(3k+1)'\cdots((2a-2)k+1)((2a-1)k+1)'. \tag{2.17}$$

Note the factor "$(k+1)$", rather than a (primed) "$(k+1)'$", due to the initial "2" in (2.17). In any case, a final application of item 1 in Lemma 2.10 shows that (2.17) vanishes modulo Baryshnikov basis elements different from $p_{2a,1}$ and $p_{2a,2}$. □

Finally, the equality $\mathrm{TC}_s(M_1^{(k)}(n)) = s\left\lfloor \frac{n}{k} \right\rfloor$ will follow once we exhibit a non-zero product of $s\lfloor n/k \rfloor$ "s-th zero-divisors" for $M_1^{(k)}(n)$, i.e., of elements in the kernel of the iterated cup product $H^*(M_1^{(k)}(n))^{\otimes s} \to H^*(M_1^{(k)}(n))$.

Let $i = \lfloor n/k \rfloor$, $q \in \{1, \ldots, s-1\}$, and consider the s-th zero-divisors

$$z_{m,q} = 1 \otimes \cdots \otimes 1 \otimes \underbrace{x_m}_{q\text{-th}} \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \cdots \otimes 1 \otimes x_m \in H^*(M_1^{(k)}(n))^{\otimes s},$$

whenever $m + k \leq n + 2$. For instance

$$\prod_{j=1}^{i} z_{(j-1)k+1,s-1} z_{(j-1)k+2,s-1} = 1 \otimes \cdots \otimes 1 \otimes \prod_{j=1}^{i} y_{(j-1)k+1} \cdot y_{(j-1)k+2}$$

and, for $q \leq s-2$,

$$z_{m,q} \prod_{j=1}^{i} z_{(j-1)k+1,s-1} z_{(j-1)k+2,s-1} = 1 \otimes \cdots \otimes 1 \otimes \underbrace{x_m}_{q-\text{th}} \otimes 1 \otimes \cdots \otimes 1 \otimes \prod_{j=1}^{i} y_{(j-1)k+1} \cdot y_{(j-1)k+2}$$
$$+ 1 \otimes \cdots \otimes 1 \otimes \left((1 \otimes x_m) \cdot \prod_{j=1}^{i} y_{(j-1)k+1} y_{(j-1)k+2} \right).$$

The second summand in the latter expression vanishes in view of Lemma 2.1 (by dimensional considerations or, alternatively, by cat-considerations). Consequently

$$\prod_{j=1}^{i} z_{(j-1)k+1,1} \cdot \prod_{j=1}^{i} z_{(j-1)k+1,2} \cdots \prod_{j=1}^{i} z_{(j-1)k+1,s-2} \cdot \prod_{j=1}^{i} z_{(j-1)k+1,s-1} z_{(j-1)k+2,s-1}$$

$$= \left(\prod_{j=1}^{i} x_{(j-1)k+1} \right) \otimes \cdots \otimes \left(\prod_{j=1}^{i} x_{(j-1)k+1} \right) \otimes \prod_{j=1}^{i} y_{(j-1)k+1} y_{(j-1)k+2},$$

which is non-zero because the first $s-2$ tensor factors in the latter expression are Baryshnikov basis elements, whereas the last tensor factor is non-zero by Theorem 2.9.

Theorem 2.15. *Summarizing, the Lusternik-Schnirelmann and (higher) topological complexity for $M_1^{(k)}(n)$ are*

$$\operatorname{cat}(M_1^{(k)}(n)) = \left\lfloor \frac{n}{k} \right\rfloor, \quad \operatorname{TC}(M_1^{(k)}(n)) = 2\left\lfloor \frac{n}{k} \right\rfloor, \quad \operatorname{TC}_s(M_1^{(k)}(n)) = s\left\lfloor \frac{n}{k} \right\rfloor.$$

3 | Lusternik-Schnirelmann Category and Topological Complexity for the non real case

This chapter describes the work done in [14] related to LS category and TC_s of non-k-equal manifolds for $d \geq 2$.

First we deduce the upper bound given by Theorem 1.2 in a similar way to how it was done in Chapter 2 for $M_1^{(k)}(n)$. This time the key point is that for $d \geq 2$, $M_d^{(k)}(n)$ is simply connected [22, Theorem 1.2] and has torsion-free $\mathbb{Z}$-homology [8, Proposition 3.9]. Hence, by Hurewicz' Theorem, we get

$$\mathrm{conn}(M_d^{(k)}(n)) + 1 = a$$

where $a = d(k-1) - 1$ is the lowest positive dimension where $H^*(M_d^{(k)}(n))$ is non-zero by Corollary 1.26.

Also, Corollary 1.26 identifies the greatest positive dimension where $H^*(M_d^{(k)}(n))$ is non-zero determining the cellular homotopy dimension of $M_d^{(k)}(n)$ by means of [19, Proposition 4C.1],

$$\mathrm{hdim}(M_d^{(k)}(n)) = ma + (d-1)(m+b-1), \text{ with } m = \lfloor \tfrac{n}{k} \rfloor. \tag{3.1}$$

Corollary 3.1. *The LS category and (higher) topological complexity for $M_d^{(k)}(n)$ satisfy the following inequalities*

$$\mathrm{cl}(M_d^{(k)}(n)) \leq \mathrm{cat}(M_d^{(k)}(n)) \leq \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{\left(\lfloor \frac{n}{k} \rfloor + b - 1\right)(d-1)}{a} \right\rfloor,$$

$$\mathrm{zcl}(M_d^{(k)}(n)) \le \mathrm{TC}(M_d^{(k)}(n)) \le 2\left(\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} \right\rfloor\right),$$

$$\mathrm{zcl}_s(M_d^{(k)}(n)) \le \mathrm{TC}_s(M_d^{(k)}(n)) \le s\left(\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} \right\rfloor\right).$$

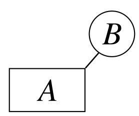
In the following section the values of $\mathrm{cl}(M_d^{(k)}(n))$, and $\mathrm{zcl}_s(M_d^{(k)}(n))$ for $s \ge 2$ will be determined.

3.1 Cup-length and zero-divisors cup-length

If the ring under consideration needs to be specified we will use the more explicit notation zcl_s^R (or zcl^R, cl^R).

Lemma 3.2. *For $s \ge 1$, $\mathrm{zcl}_s(M_d^{(k)}(n)) = s\left\lfloor \frac{n}{k} \right\rfloor$.*

Proof. Recall we assume $k < n$, in particular $m := \left\lfloor \frac{n}{k} \right\rfloor \ge 1$. We start by working with $\mathbb{Z}_2$-coefficients. For $k \le i \le n$, let x_i be the elementary k-forest of the form



(3.2)

where $A = \{i-k+1, i-k+2, \ldots, i-1\}$, $B = i$ and the remaining indices of $\mathbf{n}$ lie on isolated round vertices. Similarly, let $\widetilde{x}_i$ be the elementary k-forest of the form (3.2) where now $A = \{1, i-k+2, \ldots, i-1\}$ and $B = i$. Consider in addition the products

$$x_k \cdot x_{2k} \cdot \cdots \cdot x_{(m-1)k} \cdot x_{mk} \qquad \text{and} \qquad x_{k+1} \cdot x_{2k+1} \cdot \cdots \cdot x_{(m-1)k+1} \cdot \widetilde{x}_{mk}. \tag{3.3}$$

Note that $x_k = \widetilde{x}_k$, however x_i and $\widetilde{x}_i$ are different basis elements for $i > k$. Likewise, the products in (3.3) are different basic k-forests except for $m = 1$, in which case both coincide with $\widetilde{x}_k = x_k$. By Remark 1.25, both products in (3.3) are basic k-forests and, thus, are non-zero. This yields in particular the inequality

$$\left\lfloor \frac{n}{k} \right\rfloor \le \mathrm{cl}^{\mathbb{Z}_2}(M_d^{(k)}(n)). \tag{3.4}$$

On the other hand, the mod-2 reduction map $\mathbb{Z} \to \mathbb{Z}_2$ induces a ring epimorphism, thus, we get $\mathrm{cl}^{\mathbb{Z}_2}(M_d^{(k)}(n)) \leq \mathrm{cl}^{\mathbb{Z}}(M_d^{(k)}(n))$. Therefore, the case $s = 1$ will be proved once we show

$$\mathrm{cl}^{\mathbb{Z}}(M_d^{(k)}(n)) \leq \left\lfloor \frac{n}{k} \right\rfloor.$$

Switching to $\mathbb{Z}$-coefficients, we need to show that the product of any set of $m+1$ elementary oriented k-forests vanishes. In turn it suffices to show that there are no oriented k-forests with $m+1$ square vertices. But any such k-forest would have, in addition to the integers inside the $m+1$ square vertices, at least one integer attached to each square vertex, making a total of at least $(m+1)k$ integers inside $\mathbf{n}$. This is impossible for $m = \lfloor \frac{n}{k} \rfloor$.

Next we bound from below the zero-divisors cup-length $\mathrm{zcl}_2(M_d^{(k)}(n))$. Working with $\mathbb{Z}_2$-coefficients, we can consider the zero-divisors in $H^*(M_d^{(k)}(n)) \otimes H^*(M_d^{(k)}(n))$ given as

$$\begin{aligned}
y_{i,1} &= 1 \otimes x_{ik+1} + x_{ik+1} \otimes 1, \quad \text{for } 1 \leq i < m,\\
y_{m,1} &= \begin{cases} 1 \otimes x_{k+1} + x_{k+1} \otimes 1, & \text{if } m = 1 \text{ (recall } k < n);\\ 1 \otimes \widetilde{x}_{mk} + \widetilde{x}_{mk} \otimes 1, & \text{if } m > 1, \end{cases}\\
y_{i,2} &= 1 \otimes x_{ik} + x_{ik} \otimes 1, \quad \text{for } 1 \leq i \leq m.
\end{aligned}$$

If $m = 1$, $y_{1,1}y_{1,2} = x_{k+1} \otimes x_k + x_k \otimes x_{k+1} \neq 0$, showing $2m \leq \mathrm{zcl}^{\mathbb{Z}_2}(M_d^{(k)}(n))$. For $m > 1$ observe that the square vertex in x_{ik} intersects the square vertex in x_{ik+1} (as $k \geq 3$) and so their product is zero. Consequently $y_{i,1}y_{i,2} = x_{ik} \otimes x_{ik+1} + x_{ik+1} \otimes x_{ik}$ for $i < m$. Likewise, $y_{m,1}y_{m,2} = \widetilde{x}_{mk} \otimes x_{mk} + x_{mk} \otimes \widetilde{x}_{mk}$. Note also that each product $x_{ik+1}x_{(i+1)k}$ vanishes (cf. Remark 1.22), as well as the product $x_k\widetilde{x}_{mk}$, so we have

$$\prod_{i=1}^{m} y_{i,1}y_{i,2} = \left(\left(\prod_{i=1}^{m-1} x_{ik+1}\right)\widetilde{x}_{mk}\right) \otimes \prod_{i=1}^{m} x_{ik} + \prod_{i=1}^{m} x_{ik} \otimes \left(\left(\prod_{i=1}^{m-1} x_{ik+1}\right)\widetilde{x}_{mk}\right),$$

which is the (symmetric) sum of the tensor product of the basis elements in (3.3). This gives again $2m \leq \mathrm{zcl}^{\mathbb{Z}_2}(M_d^{(k)}(n))$. Furthermore, the surjectivity argument used in the case of cup-length allows us to assemble $\mathbb{Z}$-zero-divisors (of the form $1 \otimes z - z \otimes 1$, rather than $1 \otimes z + z \otimes 1$) giving $2m \leq \mathrm{zcl}^{\mathbb{Z}}\left(M_d^{(k)}(n)\right)$. The fact that both inequalities are sharp will follow once we observe that, actually, the product of any $2m+1$ positive-dimensional basic tensors $b_i = u_i \otimes v_i$ in $H^*(M_d^{(k)}(n)) \otimes H^*(M_d^{(k)}(n))$ vanishes (with either $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients). To see the latter fact, just note that, in the product

$$\prod_{i=1}^{2m+1} b_i = (u_1 \cdots u_{2m+1}) \otimes (v_1 \cdots v_{2m+1}),$$

one of the factors $u_1 \cdots u_{2m+1}$ or $v_1 \cdots v_{2m+1}$ vanishes as it is the product of at least $m+1$ positive-dimensional cohomology classes.

The last argument generalizes easily to yield $\mathrm{zcl}_s^R(M_d^{(k)}(n)) = sm$: Working with $R = \mathbb{Z}_2$, consider the s-th zero divisors in $H^*(M_d^{(k)}(n))^{\otimes s}$:

$$z_{i,1} = 1 \otimes x_{ik+1} \otimes 1 \otimes \cdots \otimes 1 + x_{ik+1} \otimes 1 \otimes \cdots \otimes 1, \quad \text{for } 1 \leq i < m,$$

$$z_{m,1} = \begin{cases} 1 \otimes x_{k+1} \otimes 1 \otimes \cdots \otimes 1 + x_{k+1} \otimes 1 \otimes \cdots \otimes 1, & \text{for } m = 1; \\ 1 \otimes \widetilde{x}_{mk} \otimes 1 \otimes \cdots \otimes 1 + \widetilde{x}_{mk} \otimes 1 \otimes \cdots \otimes 1, & \text{for } m > 1, \end{cases}$$

$$z_{i,j} = 1 \otimes \cdots \otimes 1 \otimes \underbrace{x_{ik}}_{j-\text{th}} \otimes 1 \otimes \cdots \otimes 1 + x_{ik} \otimes 1 \otimes \cdots \otimes 1, \quad \text{for } 1 \leq i \leq m \text{ and } 2 \leq j \leq s.$$

Direct calculation yields $\prod_{i=1}^{m} \prod_{j=1}^{s} z_{i,j} \neq 0$. For instance, if $m > 1$, we have

$$\begin{aligned} \prod_{j=1}^{s} z_{i,j} &= x_{ik+1} \otimes x_{ik} \otimes x_{ik} \otimes \cdots \otimes x_{ik} + x_{ik} \otimes x_{ik+1} \otimes x_{ik} \otimes \cdots \otimes x_{ik} \\ &= (y_{i,1} y_{i,2}) \otimes x_{ik} \otimes \cdots \otimes x_{ik} \end{aligned}$$

for $i < m$, while

$$\begin{aligned} \prod_{j=1}^{s} z_{m,j} &= \widetilde{x}_{mk} \otimes x_{mk} \otimes \cdots \otimes x_{mk} + x_{mk} \otimes \widetilde{x}_{mk} \otimes x_{mk} \otimes \cdots \otimes x_{mk} \\ &= (y_{m,1} y_{m,2}) \otimes x_{mk} \otimes \cdots \otimes x_{mk}. \end{aligned}$$

So

$$\prod_{i=1}^{m} \prod_{j=1}^{s} z_{i,j} = \left(\prod_{i=1}^{m} y_{i,1} y_{i,2} \right) \otimes \prod_{i=1}^{m} x_{ik} \otimes \cdots \otimes \prod_{i=1}^{m} x_{ik} \neq 0.$$

Therefore $sm \leq \mathrm{zcl}_s^R(M_d^{(k)}(n))$ for $R = \mathbb{Z}_2$ and, as above, for $R = \mathbb{Z}$. The latter inequality is sharp by considerations completely parallel to those in the case $s = 2$. □

Corollary 3.3. *The LS category and* TC_s *for* $M_d^{(k)}(n)$ *is bounded by*

$$\left\lfloor \frac{n}{k} \right\rfloor \leq \mathrm{cat}(M_d^{(k)}(n)) \leq \left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} \right\rfloor,$$

$$2\left\lfloor \frac{n}{k} \right\rfloor \leq \mathrm{TC}(M_d^{(k)}(n)) \leq 2\left(\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} \right\rfloor\right),$$

$$s\left\lfloor \frac{n}{k} \right\rfloor \leq \mathrm{TC}_s(M_d^{(k)}(n)) \leq s\left(\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} \right\rfloor\right).$$

where $a = d(k-1) - 1$ *and* $b = n - k\left\lfloor \frac{n}{k} \right\rfloor$ *(so* $0 \leq b < k$*).*

Corollary 3.4. *If* $n - (k-1)\left\lfloor \frac{n}{k} \right\rfloor < \frac{dk-2}{d-1}$,

$$\mathrm{cat}(M_d^{(k)}(n)) = \left\lfloor \frac{n}{k} \right\rfloor,$$

$$\mathrm{TC}(M_d^{(k)}(n)) = 2\left\lfloor \frac{n}{k} \right\rfloor,$$

$$\mathrm{TC}_s(M_d^{(k)}(n)) = s\left\lfloor \frac{n}{k} \right\rfloor.$$

Proof. With the notation in Corollary 3.3 and its proof, the equality $\mathrm{TC}_s\left(M_d^{(k)}(n)\right) = sm$ holds provided $(m + b - 1)(d-1) < a$ or, equivalently, $m + b < \frac{dk-2}{d-1}$. □

3.2 A fine tuning using Obstruction Theory

The objective of this section is to determine the LS category and TC_s also for the values satisfying the equality in the hypothesis of Corollary 3.4.

Theorem 3.5. *If* $n-(k-1)\left\lfloor \frac{n}{k} \right\rfloor \leq \frac{dk-2}{d-1}$,

$$\mathrm{cat}(M_d^{(k)}(n)) = \left\lfloor \frac{n}{k} \right\rfloor,$$

$$\mathrm{TC}(M_d^{(k)}(n)) = 2\left\lfloor \frac{n}{k} \right\rfloor,$$

$$\mathrm{TC}_s(M_d^{(k)}(n)) = s\left\lfloor \frac{n}{k} \right\rfloor.$$

where $a = d(k-1)-1$ *and* $b = n-k\left\lfloor \frac{n}{k} \right\rfloor$ *(so* $0 \leq b < k$*).*

In fact, we improve by s units the upper bound in Corollary 3.3 for all cases where $(\left\lfloor \frac{n}{k} \right\rfloor + b-1)(d-1)$ is divisible by a (see Theorem 3.8 below).
The following fact is standard, see for instance [13, Theorem 3.1].

Theorem 3.6. *Let* $p : E \to B$ *be a fibration with fiber* F *whose base* B *is a CW complex. Assume* p *admits a section* ϕ *over the s-skeleton* $B^{(s)}$ *of* B *for some* $s \geq 1$*. If* F *is s-simple and the obstruction cocycle to the existence of an extension of* ϕ *to* $B^{(s+1)}$ *lies in the cohomology class*

$$\eta \in H^{s+1}\big(B; \{\pi_s(F)\}\big),$$

then $p(\ell)$ *(the* $(\ell+1)$*-th fiberwise join power of* p*) admits a section over* $B^{(s+1)(\ell+1)-1}$ *whose obstruction cocycle to extending to* $B^{(s+1)(\ell+1)}$ *belongs to the cohomology class*

$$\eta^{\ell+1} \in H^{(s+1)(\ell+1)}\big(B; \{\pi_{s\ell+s+\ell}(F^{*(\ell+1)})\}\big).$$

The class $\eta^{\ell+1}$ in Theorem 3.6 stands for the image of the $(\ell+1)$-fold cup-power of η under the $\pi_1(B)$-morphism of coefficients

$$\pi_s(F)^{\otimes(\ell+1)} \to \pi_{s\ell+s+\ell}(F^{*(\ell+1)}) \tag{3.5}$$

given by iterated join of homotopy classes. We use Theorem 3.6 when B is simply connected, so that all cohomology groups above have trivial systems of coefficients, and when F is $(s-1)$-connected, so that (3.5) is an isomorphism, and $\eta^{\ell+1}$ is really the $(\ell+1)$-st cup-power of η. In addition, our connectivity hypothesis on F implies that η and $\eta^{\ell+1}$ are the primary obstructions for sectioning p and $p(\ell)$, respectively, and thus they are well defined (no indeterminacy). Lastly, since the pull-back $p^*(p)$ admits a tautological section, we have $p^*(\eta) = 0$ *a fortiori*.

As in the previous section, we denote by m and b the quotient and remainder, respectively, of the division of n by k. The role of p in Theorem 3.6 will be played by the based path-space fibration

$$\Omega M_d^{(k)}(n) \longrightarrow P_0(M_d^{(k)}(n)) \xrightarrow{e_1} M_d^{(k)}(n). \tag{3.6}$$

We analyze the obstructions for having $\operatorname{cat}\left(M_d^{(k)}(n)\right) = \operatorname{secat}(e_1) \leq m+i-1$, where i is a positive integer, or, equivalently, for having $\operatorname{secat}(e_1(m+i-1)) = 0$, where as in Theorem 3.6

$$\mathop{*}_{m+i} \left(\Omega M_d^{(k)}(n)\right) \longrightarrow J_{m+i-1}\left(P_0(M_d^{(k)}(n))\right) \xrightarrow{e_1(m+i-1)} M_d^{(k)}(n)$$

stands for the $(m+i)$-fold fiberwise join-power of e_1 (so $\ell = m+i-1$ in Theorem 3.6). Since $\Omega M_d^{(k)}(n)$ is $(a-2)$-connected, there are no obstructions for the existence of a section ϕ over the $(a-1)$-skeleton of $M_d^{(k)}(n)$ (so $s = a-1$ in Theorem 3.6). Therefore, if $\eta \in H^a\left(M_d^{(k)}(n); \pi_{a-1}\left(\Omega M_d^{(k)}(n)\right)\right)$ stands for the primary obstruction for sectioning e_1, then the primary obstruction for sectioning $e_1(m+i-1)$ is the $(m+i)$-st cup-power

$$\begin{aligned}\eta^{m+i} &\in H^{a(m+i)}\left(M_d^{(k)}(n); \pi_{a(m+i)-1}\left(\mathop{*}_{m+i}\left(\Omega M_d^{(k)}(n)\right)\right)\right)\\ &= H^{a(m+i)}\left(M_d^{(k)}(n); \left(\pi_{a-1}\left(\Omega M_d^{(k)}(n)\right)\right)^{\otimes(m+i)}\right).\end{aligned}$$

In view of (3.1), all potential obstructions for sectioning $e_1(m+i-1)$ lie in trivial groups when $a(m+i) > ma+(d-1)(m+b-1)$. For $i=1$, this of course yields a direct obstruction-theoretic argument for the inequality $\operatorname{cat}(M_d^{(k)}(n)) \leq m$ in Corollary 3.4. Yet, we need the cup-length arguments in the previous section in order to deal with the case where the primary obstruction η^{m+i} does not lie in a trivial group. Actually, we prove next the triviality of the $(m+1)$-st cup-power of any element in $H^a\left(M_d^{(k)}(n); \pi_{a-1}\left(\Omega M_d^{(k)}(n)\right)\right)$.

Lemma 3.7. *Recall $m = \lfloor \frac{n}{k} \rfloor$ and $a = dk-d-1$. Any element*

$$\eta \in H^a\left(M_d^{(k)}(n); \pi_{a-1}\left(\Omega M_d^{(k)}(n)\right)\right)$$

has trivial $(m+1)$-st cup-power.

Proof. Hurewicz' theorem and the considerations in Section 1.3 (see particularly Theorem 1.17 and Remark 1.25) show that the coefficient group $\pi_{a-1}\left(\Omega M_d^{(k)}(n)\right)$ is free abelian of rank $\binom{n}{k}$. So, in terms of the decomposition $H^a\left(M_d^{(k)}(n); \bigoplus_{\binom{n}{k}} \mathbb{Z}\right) = \bigoplus_{\binom{n}{k}} H^a\left(M_d^{(k)}(n); \mathbb{Z}\right)$, we write $\eta = \sum_{\binom{n}{k}} \eta_j$. The naturality of cup-product on coefficients yields $\eta^{m+1} = \left(\sum \eta_j\right)^{m+1} = \sum \eta_{j_1} \cdots \eta_{j_{m+1}}$ where each summand $\eta_{j_1} \cdots \eta_{j_{m+1}}$ stands for the image of the cup-product $\eta_{j_1} \cup \cdots \cup \eta_{j_{m+1}} \in H^{a(m+1)}\left(M_d^{(k)}(n); \mathbb{Z}\right)$ under the map induced on coefficients by

$$\mathbb{Z} = \mathbb{Z} \otimes \cdots \otimes \mathbb{Z} \xrightarrow{\iota_{j_1} \otimes \cdots \otimes \iota_{j_{m+1}}} \left(\bigoplus_{\binom{n}{k}} \mathbb{Z}\right)^{\otimes(m+1)}.$$

Here $\iota_r : \mathbb{Z} \hookrightarrow \bigoplus_{\binom{n}{k}} \mathbb{Z}$ stands for the inclusion into the r-th summand. The triviality of η^{m+1} then follows from that of each $\eta_{j_1} \cup \cdots \cup \eta_{j_{m+1}}$ which, in turn, follows from the case $s = 1$ in Lemma 3.2. □

Returning to the discussion prior to Lemma 3.7, we next prove a strengthening of Corollary 3.3, from which Theorem 3.5 follows as an immediate consequence.

Theorem 3.8. *For $s \geq 1$,*

$$s \left\lfloor \frac{n}{k} \right\rfloor \leq \mathrm{TC}_s(M_d^{(k)}(n)) \leq s\left(\left\lfloor \frac{n}{k} \right\rfloor + \left\lceil \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} - 1 \right\rceil\right),$$

where $a = d(k-1) - 1$, $b = n - k\left\lfloor \frac{n}{k} \right\rfloor$ and $\lceil \ell \rceil$ stands for the smallest integer greater than or equal to the real number ℓ.

Proof. Recall $m = \lfloor n/k \rfloor$. We only need to focus on the cases not covered by Corollary 3.3, i.e., those satisfying

$$ai = (d-1)(m+b-1) \tag{3.7}$$

for some positive integer i. Further, in such a case, the well known estimate $\mathrm{TC}_s \leq s \cdot \mathrm{cat}$ implies that it suffices to prove

$$\mathrm{cat}(M_d^{(k)}(n)) \leq m + i - 1. \tag{3.8}$$

Lemma 3.7 and its preparatory discussion give the vanishing of the primary obstruction for (3.8), i.e., for sectioning $e_1(m+i-1)$, whereas the rest of the higher obstructions lie in trivial groups in view of (3.1) and (3.7). □

For fixed $k \geq 3$, the function $f_k(d) = \frac{dk-2}{d-1}$ is decreasing, so that Theorem 3.5 applies for more values of n when $d = 2$. The following assertion identifies the first complete interval of values of n where Theorem 3.5 holds for $d = 2$.

Corollary 3.9. *If* $3 \leq k < n \leq k^2 + k - 2$ *and* $s \geq 1$, *then*

$$\mathrm{TC}_s(M_2^{(k)}(n)) = s \left\lfloor \frac{n}{k} \right\rfloor.$$

Representative cases where Theorem 3.5 determines $\mathrm{cat}(M_d^{(k)}(n))$ for $d = 2, 3, 5, 10$, are organized in Tables 3.1 to 3.5. The value of TC_s can then be read off by multiplying by s. For example, in Table 3.1, $\mathrm{TC}_s(M_2^{(8)}(40)) = 5s$. Shading tones in the table are intended to simplify the task of identifying cases with a common value of $\lfloor \frac{n}{k} \rfloor$. Actual tabulated numbers indicate the values of $\mathrm{cat}(M_d^{(k)}(n))$ coming from Theorem 3.5. Instances where the equality $\mathrm{cat}(M_d^{(k)}(n)) = \lfloor \frac{n}{k} \rfloor$ is not established by Theorem 3.5 are indicated with a question mark. The general structure of the table is simple: column k is divided into vertical blocks of size k (except for the very first block, whose size is $k - 1$) sharing a common value for $\lfloor \frac{n}{k} \rfloor$. In the top blocks, the common value is the answer for $\mathrm{cat}\big(M_d^{(k)}(n)\big)$, while lower blocks start having instances (the first one holding for $n = k^2 + k - 1$) where condition $\lfloor \frac{n}{k} \rfloor + b \leq \frac{dk-2}{d-1}$ in Theorem 3.5 fails.

In principle, the obstruction techniques used in this section for the base path evaluation map (3.6) could be used directly with the fibrations defining the higher topological complexities TC_s. It is interesting to remark that such a strategy does not seem to lead to any improved TC_s upper bounds for the manifolds $M_d^{(k)}(n)$; instead, it suggests the possibility that the gap in Corollary 3.3 would have to be resolved by improving the lower bound. In view of the results in [16] non-trivial Massey products holding in non-formal spaces $M_d^{(k)}(n)$ might be a way to formalize the suggested phenomenon.

Table 3.1: Lusternik-Schnirelmann category values for $M_2^{(k)}(n)$.

$n\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$k+1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+2$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+3$	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+4$	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+5$	?	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+6$	3	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+7$	?	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+8$	?	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+9$	?	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+10$	?	3	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+11$	?	?	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+12$	?	4	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+13$	?	4	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
$k+14$	?	?	?	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1
$k+15$	?	?	4	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1
$k+16$	?	5	4	3	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
$k+17$	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1
$k+18$	?	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1
$k+19$	?	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
$k+20$	?	?	5	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
$k+21$	?	?	5	4	4	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1
$k+22$	?	?	?	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1
$k+23$	?	?	?	?	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	1	1
$k+24$	?	?	?	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	1
$k+25$	?	?	6	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2
$k+26$	?	?	?	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2
$k+27$	?	?	?	5	?	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2
$k+28$	?	?	?	?	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+29$	?	?	?	?	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+30$	?	?	?	6	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+31$	?	?	?	6	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+32$	?	?	?	6	5	5	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+33$	?	?	?	?	?	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+34$	?	?	?	?	?	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+35$	?	?	?	?	6	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+36$	?	?	?	7	6	5	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2
$k+37$	?	?	?	7	6	5	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2
$k+38$	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2	2
$k+39$	?	?	?	?	?	?	5	4	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2
$k+40$	?	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2
$k+41$	?	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2
$k+42$	?	?	?	8	7	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2	2
$k+43$	?	?	?	?	7	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2	2
$k+44$	?	?	?	?	7	6	?	5	5	4	4	4	3	3	3	3	3	3	3	3	2	2	2
$k+45$	?	?	?	?	?	6	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2
$k+46$	?	?	?	?	?	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2	2
$k+47$	?	?	?	?	?	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2	2
$k+48$	?	?	?	?	?	7	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2
$k+49$	?	?	?	?	8	7	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2
$k+50$	?	?	?	?	8	7	6	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	3
$k+51$	?	?	?	?	?	7	6	6	5	5	4	4	4	4	4	3	3	3	3	3	3	3	3
$k+52$	?	?	?	?	?	7	?	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3
$k+53$	?	?	?	?	?	?	?	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3
$k+54$	?	?	?	?	?	?	7	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3	3
$k+55$	?	?	?	?	?	?	7	6	6	5	5	4	4	4	4	4	3	3	3	3	3	3	3
$k+56$	?	?	?	?	9	8	7	6	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3
$k+57$	?	?	?	?	?	8	7	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+58$	?	?	?	?	?	8	7	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+59$	?	?	?	?	?	8	7	?	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+60$	?	?	?	?	?	?	?	7	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3

Table 3.2: Lusternik-Schnirelmann category values for $M_3^{(k)}(n)$.

$n \backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$k+1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+2$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+3$	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+4$	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+5$	?	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+6$	3	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+7$	?	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+8$	?	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+9$	?	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+10$	?	3	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+11$	?	?	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+12$	?	4	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+13$	?	4	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
$k+14$	?	?	?	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1
$k+15$	?	?	4	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1
$k+16$	?	5	4	3	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
$k+17$	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1
$k+18$	?	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1
$k+19$	?	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
$k+20$	?	?	5	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
$k+21$	?	?	5	4	4	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1
$k+22$	?	?	?	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1
$k+23$	?	?	?	?	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	1	1
$k+24$	?	?	?	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	1
$k+25$	?	?	6	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2
$k+26$	?	?	?	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2
$k+27$	?	?	?	5	?	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2
$k+28$	?	?	?	?	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+29$	?	?	?	?	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+30$	?	?	?	6	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+31$	?	?	?	6	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+32$	?	?	?	6	5	5	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+33$	?	?	?	?	?	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+34$	?	?	?	?	?	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+35$	?	?	?	?	6	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+36$	?	?	?	7	6	5	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2
$k+37$	?	?	?	7	6	5	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2
$k+38$	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2	2
$k+39$	?	?	?	?	?	?	5	4	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2
$k+40$	?	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2
$k+41$	?	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2
$k+42$	?	?	?	8	7	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2	2
$k+43$	?	?	?	?	7	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2	2
$k+44$	?	?	?	?	7	6	?	5	5	4	4	4	3	3	3	3	3	3	3	3	2	2	2
$k+45$	?	?	?	?	?	6	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2
$k+46$	?	?	?	?	?	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2	2
$k+47$	?	?	?	?	?	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2	2
$k+48$	?	?	?	?	?	7	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2
$k+49$	?	?	?	?	8	7	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2
$k+50$	?	?	?	?	8	7	6	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	3
$k+51$	?	?	?	?	?	7	6	6	5	5	4	4	4	4	4	3	3	3	3	3	3	3	3
$k+52$	?	?	?	?	?	7	?	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3
$k+53$	?	?	?	?	?	?	?	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3
$k+54$	?	?	?	?	?	?	7	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3	3
$k+55$	?	?	?	?	?	?	7	6	6	5	5	4	4	4	4	4	3	3	3	3	3	3	3
$k+56$	?	?	?	?	9	8	7	6	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3
$k+57$	?	?	?	?	?	8	7	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+58$	?	?	?	?	?	8	7	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+59$	?	?	?	?	?	8	7	?	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+60$	?	?	?	?	?	?	?	7	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3

Table 3.3: Lusternik-Schnirelmann category values for $M_4^{(k)}(n)$.

$n\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$k+1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+2$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+3$	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+4$	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+5$	?	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+6$	3	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+7$	?	?	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+8$	?	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+9$	?	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+10$	?	?	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+11$	?	?	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+12$	?	4	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+13$	?	?	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
$k+14$	?	?	?	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1
$k+15$	?	?	4	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1
$k+16$	?	?	4	3	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
$k+17$	?	?	4	?	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1
$k+18$	?	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1
$k+19$	?	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
$k+20$	?	?	5	4	?	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
$k+21$	?	?	5	4	4	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1
$k+22$	?	?	?	?	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1
$k+23$	?	?	?	?	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	1	1
$k+24$	?	?	?	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	1
$k+25$	?	?	6	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2
$k+26$	?	?	?	5	?	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2
$k+27$	?	?	?	?	?	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2
$k+28$	?	?	?	?	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+29$	?	?	?	?	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+30$	?	?	?	6	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+31$	?	?	?	6	5	?	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+32$	?	?	?	?	?	5	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+33$	?	?	?	?	?	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+34$	?	?	?	?	?	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+35$	?	?	?	?	6	5	?	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+36$	?	?	?	7	6	5	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2
$k+37$	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2
$k+38$	?	?	?	?	?	?	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2	2
$k+39$	?	?	?	?	?	?	5	?	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2
$k+40$	?	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2
$k+41$	?	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2
$k+42$	?	?	?	?	7	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2	2
$k+43$	?	?	?	?	7	6	?	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2	2
$k+44$	?	?	?	?	?	6	?	5	5	4	4	4	3	3	3	3	3	3	3	3	2	2	2
$k+45$	?	?	?	?	?	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2
$k+46$	?	?	?	?	?	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2	2
$k+47$	?	?	?	?	?	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2	2
$k+48$	?	?	?	?	?	7	6	?	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2
$k+49$	?	?	?	?	8	7	6	?	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2
$k+50$	?	?	?	?	?	7	6	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	3
$k+51$	?	?	?	?	?	7	?	6	5	5	4	4	4	4	4	3	3	3	3	3	3	3	3
$k+52$	?	?	?	?	?	?	?	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3
$k+53$	?	?	?	?	?	?	?	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3
$k+54$	?	?	?	?	?	?	7	6	?	5	5	4	4	4	4	4	3	3	3	3	3	3	3
$k+55$	?	?	?	?	?	?	7	6	6	5	5	4	4	4	4	4	3	3	3	3	3	3	3
$k+56$	?	?	?	?	?	8	7	6	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3
$k+57$	?	?	?	?	?	8	7	?	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+58$	?	?	?	?	?	8	7	?	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+59$	?	?	?	?	?	?	?	?	6	?	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+60$	?	?	?	?	?	?	?	7	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3

Table 3.4: Lusternik-Schnirelmann category values for $M_5^{(k)}(n)$.

$n \backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$k+1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+2$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+3$	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+4$	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+5$	?	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+6$	3	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+7$	?	?	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+8$	?	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+9$	?	3	?	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+10$	?	?	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+11$	?	?	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+12$	?	4	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+13$	?	?	?	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
$k+14$	?	?	?	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1
$k+15$	?	?	4	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1
$k+16$	?	?	4	3	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
$k+17$	?	?	?	?	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1
$k+18$	?	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1
$k+19$	?	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
$k+20$	?	?	5	4	?	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
$k+21$	?	?	?	4	4	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1
$k+22$	?	?	?	?	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1
$k+23$	?	?	?	?	4	?	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	1	1
$k+24$	?	?	?	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	1
$k+25$	?	?	?	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2
$k+26$	?	?	?	5	?	4	?	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2
$k+27$	?	?	?	?	?	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2
$k+28$	?	?	?	?	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+29$	?	?	?	?	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+30$	?	?	?	6	5	?	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+31$	?	?	?	6	5	?	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+32$	?	?	?	?	?	5	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+33$	?	?	?	?	?	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+34$	?	?	?	?	?	5	?	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+35$	?	?	?	?	6	5	?	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+36$	?	?	?	7	6	5	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2
$k+37$	?	?	?	?	6	?	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2
$k+38$	?	?	?	?	?	?	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2	2
$k+39$	?	?	?	?	?	?	5	?	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2
$k+40$	?	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2
$k+41$	?	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2
$k+42$	?	?	?	?	7	6	?	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2	2
$k+43$	?	?	?	?	7	6	?	5	?	4	4	4	3	3	3	3	3	3	3	2	2	2	2
$k+44$	?	?	?	?	?	?	?	5	5	4	4	4	3	3	3	3	3	3	3	3	2	2	2
$k+45$	?	?	?	?	?	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2
$k+46$	?	?	?	?	?	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2	2
$k+47$	?	?	?	?	?	?	6	5	5	?	4	4	4	3	3	3	3	3	3	3	3	2	2
$k+48$	?	?	?	?	?	7	6	?	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2
$k+49$	?	?	?	?	8	7	6	?	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2
$k+50$	?	?	?	?	?	7	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	3	3
$k+51$	?	?	?	?	?	?	?	6	5	5	?	4	4	4	4	3	3	3	3	3	3	3	3
$k+52$	?	?	?	?	?	?	?	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3	3
$k+53$	?	?	?	?	?	?	?	6	?	5	5	4	4	4	4	3	3	3	3	3	3	3	3
$k+54$	?	?	?	?	?	?	7	6	?	5	5	4	4	4	4	4	3	3	3	3	3	3	3
$k+55$	?	?	?	?	?	?	7	6	6	5	5	4	4	4	4	4	3	3	3	3	3	3	3
$k+56$	?	?	?	?	?	8	7	6	6	5	5	5	4	4	4	4	3	3	3	3	3	3	3
$k+57$	?	?	?	?	?	8	7	?	6	5	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+58$	?	?	?	?	?	?	?	?	6	?	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+59$	?	?	?	?	?	?	?	?	6	?	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+60$	?	?	?	?	?	?	?	7	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3

Table 3.5: Lusternik-Schnirelmann category values for $M_{10}^{(k)}(n)$.

$n\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$k+1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+2$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+3$	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+4$	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+5$	?	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+6$	3	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+7$	?	?	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+8$	?	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+9$	?	3	?	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+10$	?	?	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+11$	?	?	3	?	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+12$	?	4	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+13$	?	?	?	3	?	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
$k+14$	?	?	?	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1
$k+15$	?	?	4	3	3	?	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1
$k+16$	?	?	4	?	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
$k+17$	?	?	?	?	3	3	?	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1
$k+18$	?	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1
$k+19$	?	?	?	4	?	3	3	?	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1
$k+20$	?	?	5	4	?	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
$k+21$	?	?	?	?	4	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1
$k+22$	?	?	?	?	4	?	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1	1	1
$k+23$	?	?	?	?	4	?	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	1	1
$k+24$	?	?	?	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	1
$k+25$	?	?	?	5	?	4	?	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2
$k+26$	?	?	?	?	?	4	?	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2
$k+27$	?	?	?	?	?	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2
$k+28$	?	?	?	?	5	4	4	?	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+29$	?	?	?	?	5	?	4	?	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+30$	?	?	?	6	5	?	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+31$	?	?	?	?	?	?	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+32$	?	?	?	?	?	5	4	4	?	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+33$	?	?	?	?	?	5	?	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+34$	?	?	?	?	?	5	?	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+35$	?	?	?	?	6	5	?	4	4	?	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+36$	?	?	?	?	6	?	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2
$k+37$	?	?	?	?	?	?	5	?	4	4	3	3	3	3	3	3	2	2	2	2	2	2	2
$k+38$	?	?	?	?	?	?	5	?	4	4	?	3	3	3	3	3	3	2	2	2	2	2	2
$k+39$	?	?	?	?	?	?	5	?	4	4	4	3	3	3	3	3	3	2	2	2	2	2	2
$k+40$	?	?	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2
$k+41$	?	?	?	?	?	6	?	5	4	4	4	?	3	3	3	3	3	3	2	2	2	2	2
$k+42$	?	?	?	?	7	6	?	5	?	4	4	4	3	3	3	3	3	3	3	2	2	2	2
$k+43$	?	?	?	?	?	?	?	5	?	4	4	4	3	3	3	3	3	3	3	2	2	2	2
$k+44$	?	?	?	?	?	?	?	5	5	4	4	4	?	3	3	3	3	3	3	3	2	2	2
$k+45$	?	?	?	?	?	?	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2	2	2
$k+46$	?	?	?	?	?	?	6	?	5	?	4	4	4	3	3	3	3	3	3	3	3	2	2
$k+47$	?	?	?	?	?	?	6	?	5	?	4	4	4	?	3	3	3	3	3	3	3	2	2
$k+48$	?	?	?	?	?	7	6	?	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2
$k+49$	?	?	?	?	?	7	?	?	5	5	4	4	4	4	3	3	3	3	3	3	3	3	2
$k+50$	?	?	?	?	?	?	?	6	5	5	?	4	4	4	?	3	3	3	3	3	3	3	3
$k+51$	?	?	?	?	?	?	?	6	5	5	?	4	4	4	4	3	3	3	3	3	3	3	3
$k+52$	?	?	?	?	?	?	?	6	?	5	5	4	4	4	4	3	3	3	3	3	3	3	3
$k+53$	?	?	?	?	?	?	?	6	?	5	5	4	4	4	4	?	3	3	3	3	3	3	3
$k+54$	?	?	?	?	?	?	7	6	?	5	5	?	4	4	4	4	3	3	3	3	3	3	3
$k+55$	?	?	?	?	?	?	7	?	6	5	5	?	4	4	4	4	3	3	3	3	3	3	3
$k+56$	?	?	?	?	?	8	7	?	6	5	5	5	4	4	4	4	?	3	3	3	3	3	3
$k+57$	?	?	?	?	?	?	?	?	6	?	5	5	4	4	4	4	4	3	3	3	3	3	3
$k+58$	?	?	?	?	?	?	?	?	6	?	5	5	?	4	4	4	4	3	3	3	3	3	3
$k+59$	?	?	?	?	?	?	?	?	6	?	5	5	?	4	4	4	4	3	3	3	3	3	3
$k+60$	?	?	?	?	?	?	?	7	6	6	5	5	5	4	4	4	4	4	3	3	3	3	3

4 | Massey products

In [27], Miller obtained partial results on the structure of Massey products on non-k-equal manifolds with $d = 2$. He showed that all p-order Massey products on $M_2^{(k)}(n)$ vanish provided either one of the following conditions holds:

$$3 \leq p < k \qquad \text{(in view of [27, Theorem 1.2]).} \tag{4.1}$$

$$n \leq k(k-1) \quad \text{(in view of [27, Corollary 1.3]).} \tag{4.2}$$

Both of these restrictions are optimal; neither the upper bound for p in (4.1) nor the upper bound for n in (4.2) can be improved for general k. Indeed, Miller proved in addition that

$$M_2^{(3)}(n) \text{ admits non-trivial triple Massey products when } n > 6. \tag{4.3}$$

Remark 4.1. *In rational-formality terms, the above picture is particularly pleasant for* $k = 3$. *Indeed, work of Halperin and Stasheff ([18, Corollary 5.16]) implies that* $M_d^{(k)}(n)$ *is formal when* n *is sufficiently small, namely, when*

$$n(d-1) + m(k-2) \leq 3dk - 2(d+3). \tag{4.4}$$

Here and below m *stands for the integral part of* n/k. *Thus* $M_2^{(3)}(n)$ *is formal for* $n \leq 6$, *but non-formal otherwise in view of (4.3). However, for* $k > 3$, *condition (4.2) is less restrictive than the* $(d = 2)$*-case of condition (4.4). As a result, for* $k > 3$ *fixed, there are manifolds* $M_2^{(k)}(n)$ *which, despite of not supporting non-trivial Massey products, are not known to be formal.*

Miller's paper ends by conjecturing that

$$M_2^{(k)}(n) \text{ admits non-trivial Massey products provided } n > k(k-1). \tag{4.5}$$

Presumably, the non-trivial Massey products conjectured by Miller to hold in $M_2^{(k)}(n)$ would be of order precisely k.
This chapter is devoted to provide a slight generalization of Miller's result (4.3) by proving the following theorem.

Theorem 4.2. *For $d \geq 2$, and $n > 6$, $M_d^{(3)}(n)$ admits a non-trivial ternary Massey product.*

The relevance of Theorem 4.2, is that it would leave, for $k = 3$ and $d \geq 2$, a finite number of spaces $M_d^{(k)}(n)$ with undecided formality.
Miller's cup- and Massey-product computations are based on Yuzvinsky's DGA structure on the relative atomic complex for $M_d^{(k)}(n)$ introduced by Vassiliev ([33, 34]). With such an approach, much effort is required to show non-triviality of an appropriate cohomology class and, as a consequence, the extent of results in [27] was somehow limited. We circumvent the problem by following a more direct route. Namely, as originally noted in [25], in many cases Poincaré duality and intersection theory (using Borel-Moore homology in our non-compact case) can be used to evaluate Massey products. Actually, Baryshnikov [1] and Dobrinskaya and Turchin [8] used Poincaré duality to give a fully workable description of the cohomology ring of $M_d^{(k)}(n)$. We extend the approach in order to evaluate Massey products. The idea was already suggested on page 263 of [8].

4.1 The Duality Isomorphism

The cohomological descriptions of $M_1^{(k)}(n)$ and $M_d^{(k)}(n)$ developed in Sections 1.2 and 1.3 are grounded on a Poincaré Duality Isomorphism briefly indicated in this section. Consider the following ingredients that can be found in [7, Section V.11], [21, Sections II.9, IX.3, IX.4 and IX.5], [12, Section 19.1] and [32, Theorem 10.4].
For a locally compact space Z, there is a (sheaf theoretic supported) cap product $\frown\colon H_a^{\mathrm{BM}}(Z) \otimes H^b(Z) \to H_{a-b}^{\mathrm{BM}}(Z)$ that has several properties including:

1. $f_*(a' \frown f^*\xi) = f_*a' \frown \xi$, for any proper map $f\colon Z' \to Z$ and classes $a' \in H_*^{\mathrm{BM}}(Z')$ and $\xi \in H^*(Z)$.

2. $(a \frown \xi) \frown \eta = a \frown (\xi \smile \eta)$, for classes $a \in H_*^{\mathrm{BM}}(Z)$ and $\xi, \eta \in H^*(Z)$.

3. For an oriented n-dimensional (Hausdorff paracompact) manifold N, cap product with the fundamental class $[N] \in H_n^{\mathrm{BM}}(N)$ yields a duality isomorphism

$$D\colon H^*(N) \to H_{n-*}^{\mathrm{BM}}(N).$$

4. For an oriented properly embedded submanifold $V \subset N$ of codimension k, the orientation class $\mathfrak{o}_V^N \in H^k(N)$ of V in N, i.e., the restriction of the (normal) Thom class $\mathfrak{u}_V^N \in H^k(N, N-V)$ of V in N, satisfies

$$D(\mathfrak{o}_V^N) = [V]_N,$$

the image of $[V]$ in Borel-Moore homology of N under the inclusion $V \hookrightarrow N$.

This information suffices to prove (just as in [6, Theorem 11.9]) that the intersection pairing at the bottom of the commutative square

$$\begin{array}{ccc} H^{n-p}(N) \otimes H^{n-q}(N) & \xrightarrow{\smile} & H^{2n-p-q}(N) \\ \downarrow {\scriptstyle \mathrm{twist}\circ(D\otimes D)} & & \downarrow {\scriptstyle D} \\ H_p^{\mathrm{BM}}(N) \otimes H_q^{\mathrm{BM}}(N) & \xrightarrow{\bullet} & H_{p+q-n}^{\mathrm{BM}}(N), \end{array} \tag{4.6}$$

i.e., the product given by $\beta \bullet \alpha = D(D^{-1}\alpha \smile D^{-1}\beta)$, allows us to compute cup products in geometrical terms:

Theorem 4.3. *Let N be as in item (3) above. If X and Y are properly embedded oriented submanifolds of N with transverse intersection, then*

$$[X]_N \bullet [Y]_N = [X \cap Y]_N.$$

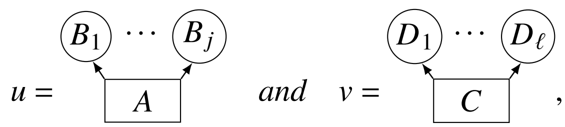
Theorem 4.3 and diagram (4.6) lead to Theorem 1.21, a description of the cohomology ring of $M_d^{(k)}(n)$ for $d \geq 2$ in terms of the basis of k-forests.
Here it should be noted that a cup product of admissible forests $T \cdot T'$ vanishes in any of the following two situations related to their corresponding locally finite chains as described in the paragraph before Example 1.19 in Section 1.3:

- the intersection $c_T \cap c_{T'}$ is empty—for instance, if the General Position Lemma has to be applied to one of the submanifolds in order to assure the transverse-intersection hypothesis in Theorem 4.3.

- the transverse intersection $c_T \cap c_{T'}$ is not empty but lies in the k-equal arrangement, $A_d^{(k)}(n)$.

Those situations are precisely reflected in Theorem 1.21 and furthermore, we obtain that a product of elementary terms is zero only if the numbers appearing in the essential picture of both terms—without considering all those round vertices not attached to any square vertex—intersect in at least one number, see Theorem 1.21 (1)–(3). Hence, the following observation will be useful in the rest of the chapter.

Remark 4.4. *Let u and v be elementary k forests*



and suppose that $u \cdot v = 0$. Then the numbers appearing in u and v intersect in a set with $m > 0$ elements.

1. *If $m > 1$ we have two cases*
   - *(a) $A \cap C \neq \emptyset$. The intersection of the corresponding chains c_u and c_v is not transverse but, as explained in the proof of (1) in [8, Theorem 7.1], after applying a translation we obtain an empty transverse intersection.*
   - *(b) $A \cap C = \emptyset$. Then the superposition of the trees encoding c_u and c_v has cycles and, as explained in the proof of (2) in [8, Theorem 7.1], the factors u and v can be represented up to a sign by submanifolds c_u and c_v with actual empty intersection.*
2. *If $m = 1$ we have three situations:*
   - *(a) $A \cap C \neq \emptyset$. The intersection is transverse but it lies in the k-equal arrangement, thus, the product is represented by the empty chain.*
   - *(b) $j = 1$ and $B_1 \in C$. In this case, after the superposition, the square vertex with set A has no round vertex attached so, as noted in [8, Remark 6.2], the case $m = 1$ in the Jacobi relation implies that the product is represented by a Borel-Moore boundary. A similar situation holds when $\ell = 1$ and $D_1 \in A$.*
   - *(c) $j = \ell = 1$ and $B_1 = D_1$. In this case, the three-terms relation can be used to see that the product is represented as a sum of terms as those considered on item 2 (b) above. So the product is again represented by a sum of Borel-Moore boundaries.*

4.2 Massey products

In his seminal work [25], Massey introduced a geometric method to compute his higher order cohomology operations by using Poincaré duality so to replace cup products by intersection products. Since then, variants of the technique have been used in knot theory to compute higher order linking numbers ([17, 20, 29]). Outside low dimensional topology, intersection theory has been successfully used to evaluate Massey products in classical configuration spaces ([24, 26]). The basic (folklore) observation is the following. For oriented submanifolds K, L and M of an oriented n-dimensional manifold N, let κ, λ and μ be the Poincaré duals of the fundamental classes $[K]_N, [L]_N, [M]_N \in H_*^{\mathrm{BM}}(N)$. Assume bounding transverse intersections $K \pitchfork L = \partial X$ and $L \pitchfork M = \partial Y$ for oriented submanifolds X and Y with $X \pitchfork M$ and $K \pitchfork Y$. Then the Poincaré dual of the fundamental class

$$\left[X \cap M - (-1)^{\dim K} K \cap Y\right]_N$$

lies in the triple Massey product $\langle \kappa, \lambda, \mu \rangle$. In our case, $N = M_d^{(k)}(n)$ and the relevant submanifolds are given by the cells c_T associated to k-forests.

The rules listed below are clearly valid and will be used without notice in the rest of the paper.

1. The submanifolds X and Y used in the description of the Massey product $\langle \kappa, \lambda, \mu \rangle$ given above will fail to be admissible k-forests but they can be described using the terminology of k-forests.

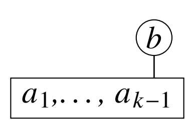
2. If we remove a number from a square vertex, it might happen that the square vertex remains with only one value inside. In that case we will consider it as a round vertex instead and will apply the usual rules for rewriting a final product in terms of admissible k-forests.

3. If either the transverse intersection $K \pitchfork L$ or $L \pitchfork M$ is empty in $M_d^{(k)}(n)$, the Massey product is simply the cohomology class corresponding to $K \pitchfork Y$ or $X \pitchfork M$ respectively.

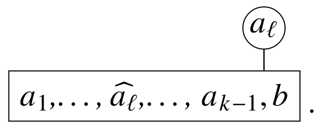
In what follows orientations will be avoided by working with $\mathbb{Z}_2$ coefficients.

The following result, which is an immediate consequence of the Jacobi relation, allows us to rewrite the representative of any elementary k-forest with only one round vertex.

Remark 4.5. *The cohomology class of any k-elementary term of the form*

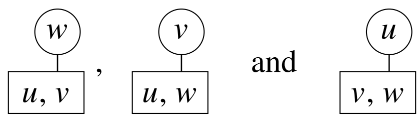


is equal to the cohomology class of the term



Indeed, both terms appear as the boundary of the $(k-2)$*-forest with numbers* $a_1,\ldots,\widehat{a_\ell},\ldots,$ a_{k-1} *in its square vertex and two round vertices containing* a_ℓ *and* b*.*

For instance, for $k = 3$, the three forests



represent the same class, which will be denoted simply by

$$\boxed{\{u,v,w\}}\,.$$

The following result, which is a partial generalization (to Euclidean dimensions $d \geq 2$) of [27, Theorem 1.2], indicates that many triple Massey products in $M_d^{(k)}(n)$ are trivial, i.e., as sets, those products agree with their indeterminacy.

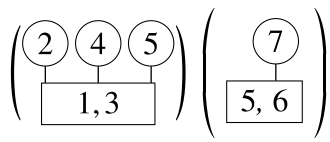
Theorem 4.6. *Every defined triple Massey product of* k*-elementary terms is trivial.*

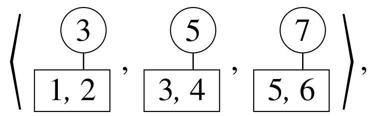
Proof. Let us suppose that $\langle \kappa, \lambda, \mu \rangle$ is a defined Massey product where κ, λ, and μ are k-elementary terms. By the geometric description of Massey products at the beginning of this subsection, it suffices to argue that we can take representatives c_κ, c_λ and c_μ in such a way that both transverse intersections $c_\kappa \pitchfork c_\lambda$ and $c_\lambda \pitchfork c_\mu$ are empty. According to Remark 4.4, that is already the case if the consecutive terms coincide in more than one number or in exactly one number in their corresponding square vertices, so it is enough to indicate how to modify the representatives when the intersection of consecutive k-forests falls in the cases 2 (b) or 2 (c) of Remark 4.4.

If the product of $\kappa \cdot \lambda$ falls either in case 2 (b) or 2 (c) of Remark 4.4, we can modify the corresponding k-forests by using Remark 4.5 to force that the common number occurs in their square vertices, in order to have the needed empty transverse intersection. Let us call the new representatives κ' and λ'.

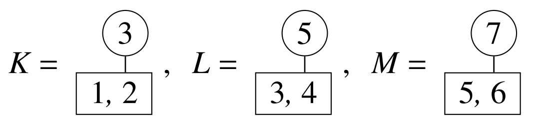
We can proceed similarly with the pair λ' and μ; the only additional requirement to meet is that, if we have to replace λ' again, we must do it in a way that the common number between κ' and λ' still remains in the square vertex. This is possible since $k \geq 3$. □

The next example illustrates the type of calculations we will use in order to detect non-trivial triple Massey products in $M_d^{(3)}(n)$. From this point on $k = 3$.

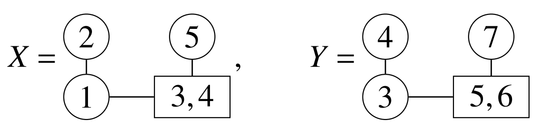
Example 4.7. *The* 3*-admissible forest*  *lies in the indeterminacy of*



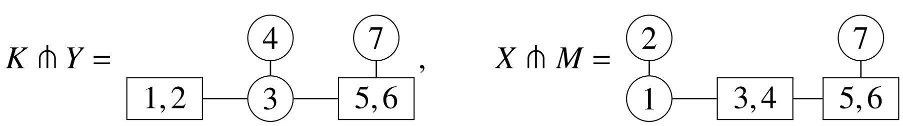
so, by Theorem 4.6, the 3*-admissible forest lies also in the actual Massey product. We next give an explicit geometric argument for the latter assertion. Let us label the representing manifolds as follows*



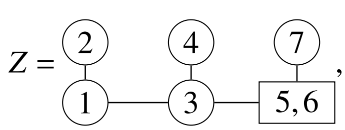
The submanifolds X and Y required to compute the Massey product are



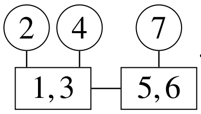
Hence, we obtain the following intersections



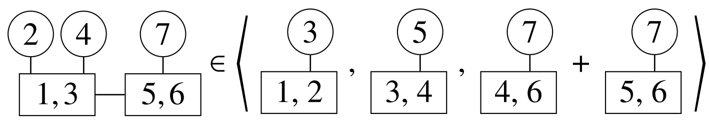
and, consequently, their union represents an element of the triple Massey product.
In order to identify the resulting class, note that X and Y appear in the boundary of the forest



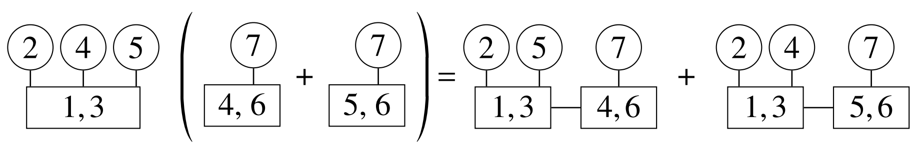
thus the required class is given by the rest of the boundary:

$$K \pitchfork Y + X \pitchfork M =$$  .

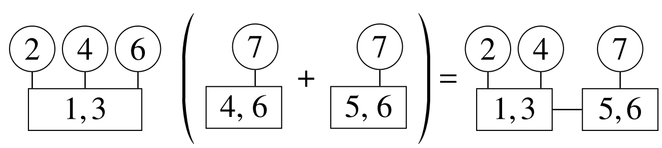
The previous geometric argument allows us to identify concrete elements representing general Massey products. Yet, if we want to identify non-trivial products, we need to be able to rule out indeterminacy equalities as the one noted at the beginning of Example 4.7. A naive way to achieve such a goal is through slight perturbations of the first and third factors in a Massey product. For instance, using the techniques in the example, it is easy to check that

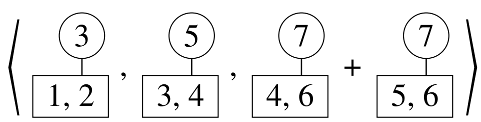


which partially achieves the goal as now we have

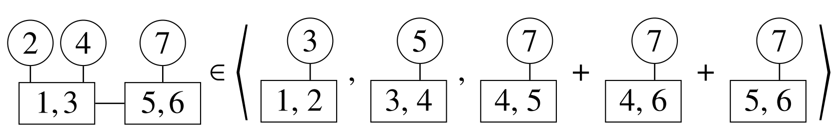


However, we still have the equality

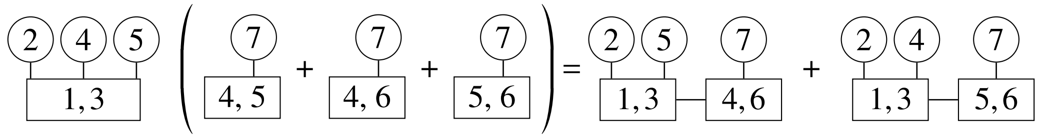


which shows that  is trivial.

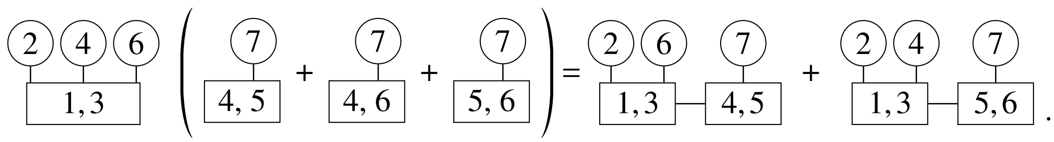
The latter indeterminacy equality can be fixed (without altering the first fix) by another direct computation giving



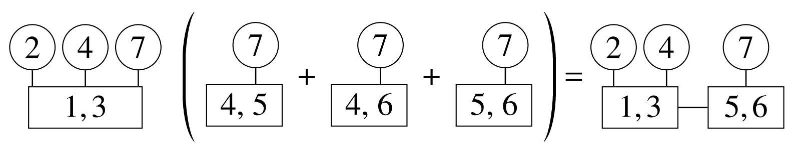
as now



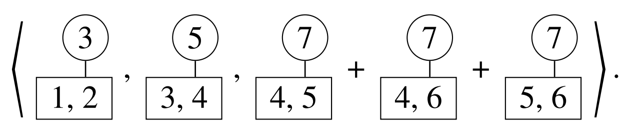
and



Unfortunately we also have

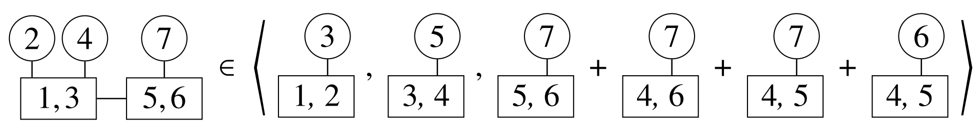


which yields the triviality of the product



But the next natural try leads to the desired goal:

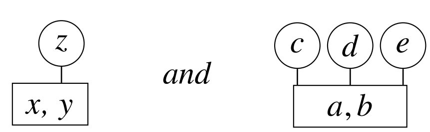
Theorem 4.8. *For $n \geq 7$ and $d \geq 2$, the Massey product in $M_d^{(3)}(n)$*



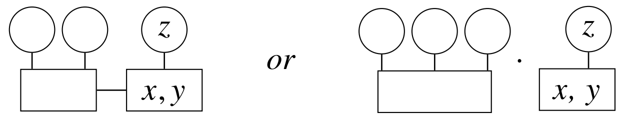
is not trivial.

In preparation for the proof, we observe the following fact whose verification is an elementary exercise using the cup-product description in Section 1.3.

Lemma 4.9. *The product of two basis elements*

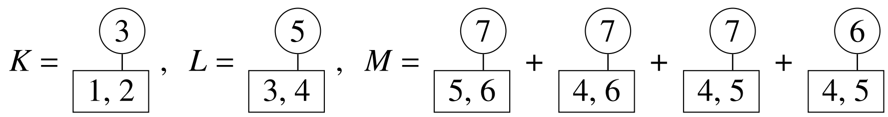


in $H^*(M_d^{(3)}(n))$ *is either zero or a sum of basic elements of one of the two forms*

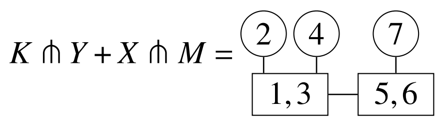


and, in either case, the set of numbers inside the vertices of each of these summands is precisely $\{a,b,c,d,e,x,y,z\}$.

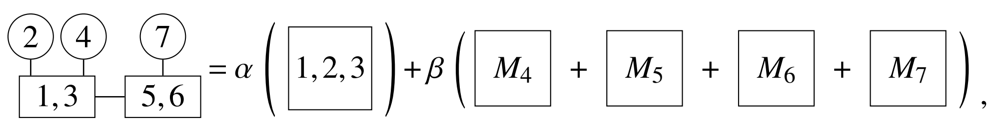
Proof of Theorem 4.8. Let us label the representing manifolds as



Since representatives for products of consecutive terms are the same as in Example 4.7, we have that



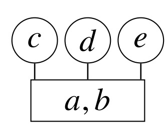
represents a cohomology class in this Massey product. The non-triviality of the product will be established once we rule out any possible solution to the equation



(4.7)

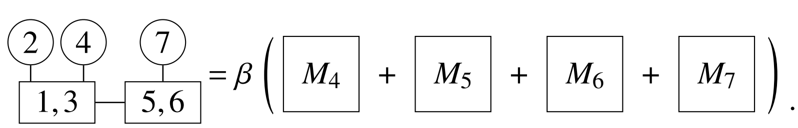
where $M_\ell = \{4,5,6,7\} \setminus \{\ell\}$ for $\ell \in \{4,5,6,7\}$—so $M_\ell \neq M_{\ell'}$ if $\ell \neq \ell'$.

Notice that, by dimensional reasons, both α and β should be sums of basis elements of the general form



(4.8)

for some numbers a, b, c, d and e such that $\{a,b,c,d,e\} \subset \mathbf{n}$. Lemma 4.9 then shows that the first product on the right-hand side of any expression (4.7) must vanish. Consequently, we only need to rule out solutions to the simpler equation



(4.9)

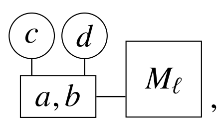
In such a setting, we can assume that all basic summands (4.8) in β having zero product with $[M]$ have been deleted. Furthermore, by Lemma 4.9, we have the vanishing of the product of $[M]$ with the sum of all summands (4.8) of β for which $\{a,b,c,d,e\}$ contains an integer $i \geq 8$. Consequently, we assume also that all those summands have been deleted from β. We then proceed to analyze the product with $[M]$ of any of the remaining terms τ in (4.8), which is then forced to have

$$\{1,2,3\} \subset \{a,b,c,d,e\} \subset \{1,2,3,4,5,6,7\}. \tag{4.10}$$

To begin with, there must be a (not necessarily unique) $\ell \in \{4,5,6,7\}$ with

$$\tau \cdot \boxed{M_\ell} \neq 0.$$

In particular, $\{a,b\} \cap M_\ell = \varnothing$ and $|\{c,d,e\} \cap M_\ell| = 1$, in view of (4.10). Say $e \in M_\ell$, so $\{c,d\} \cap M_\ell = \varnothing$ and the product takes the (perhaps non-basic) form

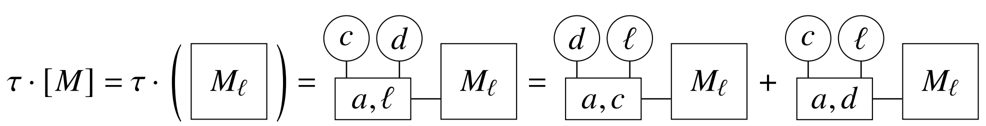


where $\{1,2,3\} \subset \{a,b,c,d\} \subset \{1,2,\ldots,7\} \setminus M_\ell$, i.e.,

$$\{a,b,c,d\} = \{1,2,3,\ell\}. \tag{4.11}$$

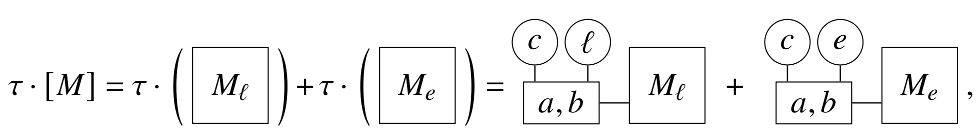
The above information allows us to evaluate in full the product $\tau \cdot [M]$:

- If $\ell \in \{a,b\}$, say $b = \ell$, we have



  which is a sum of two basis elements, in view of (4.11).

- If $\ell \in \{c,d\}$, say $d = \ell$, we have

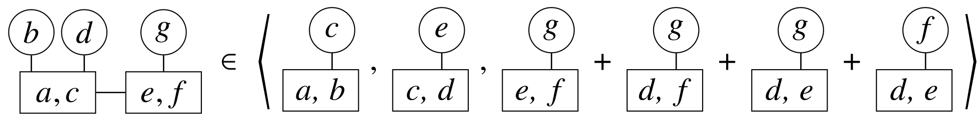


  a sum of two basis elements, again in view of (4.11).

This shows that the product $\beta \cdot [M]$ is a sum of an even number of basic elements, therefore ruling out any possible solution to (4.9). □

The argument in the proof above applies word for word to yield the more general statement:

Theorem 4.10. *For $n \geq 7$, $d \geq 2$ and seven pairwise distinct numbers a, b, c, d, e, f, g in* **n** *with* $\max\{a,b\} \leq c$, $\max\{c,d\} \leq e$ *and* $\max\{e,f\} \leq g$, *the Massey product in $M_d^{(3)}(n)$*



is not trivial.

Corollary 4.11. *For $n \geq 7$ and $d \geq 2$, $M_d^{(3)}(n)$ is non formal.*

Conclusions

No k-equal manifolds constitute one of the few known examples where Poincaré duality—in the non-compact case—allows us to recover totally the cohomology ring of these spaces and, even more, assess Massey products in this geometric fashion, showing a fascinating interplay between geometry, topology and algebra.

Also, these manifolds play a central role in several research topics, like k-equal immersions as described in [8], or the fact that $M_1^{(3)}(n)$ is the $\mathbb{R}$-analogue of the classifying Artin pure braid group as shown in [23]. Worth mentioning also is the fact that the Betti numbers of $M_1^{(k)}(n)$ bound the computational complexity of the k-equal problem as was shown in [5].

Computing the LS category and (higher) topological complexity of these manifolds is a highly non-trivial problem with potential applications to motion planning problems in robotics. We hope that the developments in this work can be successfully applied or generalized in the future. For instance, an interesting open problem that would benefit from our contribution is the design of reasonably efficient motion planning algorithms of automated guided particles that are allowed to interact (collision) among them in an organized and controlled way.

Our results show that, unlike the case of ordinary configuration spaces on $\mathbb{R}^d$, the parity of the dimension of the ambient Euclidean space does not seem to be a decisive parameter for the actual value of the topological complexity of collision-controlled motion planning of particles in $\mathbb{R}^d$. While the 1-dimensional case is as difficult as it can get, there is the (intuition-compatible) possibility that the higher dimensional situation exhibits lower TC values that could actually be independent of (the parity of) d.

On the purely theoretical side, our methods and contributions on the structure of Massey products on Euclidean no k-equal manifolds reveal a deep and fruitful connection between geometric aspects and homotopical properties of these spaces.

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