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T H E S I S

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Abstract

In this work, we solve three problems related to operator algebras over reproducing kernel Hilbert spaces (RKHS), namely: we compute the reproducing kernel of the polyanalytic Bergman space on the unit ball, we propose a scheme to describe the structure of W*-algebras of invariant operators acting over RKHS, and we get the description of the W*-algebra of radial operators in the polyanalytic Bergman space over the unit disk.

In the first chapter, we make a general study of homogeneous polyanalytic functions of total order m over an n-dimensional domain, and then we prove that they can be written as polynomials in the conjugate variables $\overline{z_1}, \ldots, \overline{z_n}$ with analytic coefficients. For such functions, we obtain a weighted mean-value property and generalize some ideas by Pessoa to obtain an explicit formula for the reproducing kernel of the m-analytic Bergman space over the unit ball. Then, using a unitary weighted change of variables, we transform this kernel into the reproducing kernel of the m-analytic weighted Bergman space over the n-dimensional Siegel domain.

In the second chapter, which is the main part of this work, we propose a general scheme to describe the centralizer of unitary representations of abelian groups acting on a RKHS over a "tube" type domain. That is, for a locally compact abelian group G and a measure space Y, we consider a closed subspace H of $L^2(G \times Y)$ which is a RKHS by its own. By computing the G-Fourier transform of the reproducing kernel, we provide a criterion to determine whether the W*-algebra of translation invariant operators acting on H is commutative or not. For the commutative case, we construct a unitary "diagonalizer" operator that turns all translation invariant operators into multiplication operators with bounded symbols. We emphasize the role of the reproducing kernel in these results. We show explicitly how this scheme covers many of the results obtained by Vasilevski and other authors.

Finally, we give a complete description of the W*-algebra of radial operators acting on the polyanalytic Bergman space over the unit disk, which is a particular case of the space studied in the first chapter. Here, we construct an explicit isomorphism between radial operators and some bounded sequences of matrices. For this case, we show that Toeplitz operators with bounded generating symbols are not weakly dense in the algebra of all bounded operators.

Resumen

En este trabajo resolvemos tres problemas relacionados con el estudio de álgebras de operadores sobre espacios de Hilbert con núcleo reproductor (EHNR), a saber: calculamos el núcleo reproductor del espacio polianalítico de Bergman sobre la bola unitaria, proponemos un esquema para describir la estructura de algunas clases de operadores invariantes sobre EHNR, y obtenemos la descripción del álgebra W* de operadores radiales que actúan sobre el espacio polianalítico de Bergman sobre el disco unitario.

En el primer capítulo, hacemos un estudio general de las funciones polianalíticas homogéneas de orden total m que actúan sobre dominios de dimensión n. Después probamos que dichas funciones pueden expresarse como polinomios en las variables conjugadas $\overline{z_1}, \ldots, \overline{z_n}$ con coeficientes analíticos. Para tales funciones, mostramos una propiedad del valor medio con peso y generalizamos algunas ideas de Pessoa para obtener una forma explícita del núcleo reproductor del espacio polianalítico de Bergman sobre la bola unitaria. Luego, usando un cambio de variable unitario con peso, convertimos dicho núcleo reproductor en aquel del espacio polianalítico de Bergman sobre el dominio de Siegel.

En la parte principal de este trabajo, proponemos un esquema general para describir el centralizador o conmutante de represenrtaciones unitarias de algunos grupos que actúan sobre EHNR en dominios "tubulares". Esto es, que para un grupo abeliano localmente compacto G y un espacio de medida Y, consideramos un EHNR H subespacio de $L^2(G \times Y)$. A través del cálculo de la transformada de Fourier del núcleo reproductor sobre la primera coordenada, enunciamos un criterio para determinar si es o no conmutativa el álgebra W* que forman todos los operadores invariantes bajo traslaciones. Para el caso en el que lo es, construimos un operador unitario "diagonalizador" que convierte los operadores invariantes en operadores de multiplicación. Enfatizamos cada vez el rol del núcleo reproductor en estos resultados. Mostramos, además, que este esquema generaliza varios trabajos obtenidos por Vasilevski y otros investigadores.

Finalmente, damos una descripción completa del álgebra W* de los operadores radiales que actúan en el espacio polianalítico de Bergman sobre el disco unitario (el cual es un caso particular en la primera parte). En este capítulo construimos un isomorfismo isométrico entre operadores radiales y algunas sucesiones de matrices. Para este caso, probamos que los operadores de Toepliz con símbolos radiales no son débilmente densos en el álgebra de todos los operadores acotados.

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Introduction

In 1999, Vasilevski [79] found a way to describe the structure of the C*-algebra generated by Toeplitz operators with vertical symbols acting in the classical analytic Bergman space. After this idea, many similar constructions have taken place around by other authors, replacing three main characteristics of the original problem: the kind of symbols, the underlying function space, and its domain. Many surprising results have been achieved, but, above all, some geometric properties of the domain have been noticed to be related to algebraic properties of some classes of Toeplitz operators under consideration. This can be seen as showing a new face of an old problem: determining the structure of operator algebras.

In the main part of this work, we extract some of the essential components of the large list of examples induced by the one performed by Vasilevski, and then we generalize the outcome as much as we can.

One essential feature present in every example is the existence of a reproducing kernel (RK) in the Hilbert space structure of the function space H, where Toeplitz operators are defined to act in. Another common feature is the "tubular" shape of the underlying domain, which every time turns to be (or to be unitarily equivalent to) a Cartesian product of a locally compact abelian group G and a measure space Y. So we combine both of these common characteristics to rewrite some of the classic results (and a couple of new ones) in a new fashion by turning the action group into translations and considering the image of the reproducing kernel under the "first-coordinate" Fourier transform. We deal not only with Teoplitz operators with invariant symbols, but invariant operators under the action of the group G on H. The natural structure for such operators (our object of study) is the von Neumann algebra (or W* algebra), which is proven to be decomposable into a direct integral of bounded operator algebras over the fiber spaces of H.

We provide a criterion to determine whether this von Neumann algebra is commutative or not, and in the affirmative case, we construct an explicit operator that simultaneously diagonalizes all operators belonging to this algebra.

The scheme developed in this work is applicable to a list of suitable RKHS's (see Examples in Section 2.9). It is used to unify many of the results produced in separated investigations. Additionally, we find two more examples for this list, but unfortunately, we haven't make them fit in this scheme, and we let this task for a future project. Namely, we construct the RK of the polyanalytic weighted Bergman space over the unit disk and the unit ball by computing an explicit orthonormal basis in terms of generalized Jacobi polynomials, which is been used to prove a mean value property of homogeneously polyanalytic functions.

The results of this work are published in two articles [8, 53] and one preprint [38]. They were presented in "International Workshop Operator Algebras, Toeplitz Operators and Related Topics" in Boca de Río Veracruz in 2018, "International Workshshop on Operator Theory and its Applications" in Lisbon in 2019, and some seminars, such as "Seminario de Operadores de Toeplitz" in CINVESTAV, and "Harmonic Analysis Seminar" in Louisiana State University in the summer of 2021.

Here is a more specific description of each chapter in this work.

Homogeneous polyanalytic kernels

Bergman [12] comprehensively studied spaces of square-integrable analytic functions on one-dimensional domains, considering them as reproducing kernel Hilbert spaces (RKHS). For some multidimensional generalizations, see [24, 83, 87]. Polyanalytic functions have been attracted attention of many mathematicians since the beginning of the 20th century. For some of their properties, applications, and history, see, for example [1, 2, 7, 25, 36, 80].

Koshelev [51] proved that every integrable m-analytic function f on $\mathbb{D}$ fulfills an analog of the mean value property:

$$f(0) = \frac{1}{\pi} \int_{\mathbb{D}} f(z)\, P(|z|^2)\, \mathrm{d}\mu(z),$$

where P is a certain polynomial of degree $m-1$ with explicitly computed coefficients. Furthermore, he proved that the corresponding space $\mathcal{A}^2_m(\mathbb{D})$ is a RKHS and gave an explicit formula for the RK at the arbitrary point z_0 of the disk, using the Möbius transormation φ_{z_0} that interchanges z_0 with the origin. Due to the format of the journal, his explanation was extremely short: "although the class of polyanalytic functions is not invariant relative to fractional-linear transformations, this device is still usefull thanks to the presence of $K_n(z, z_0)$ under the integral sign". Pessoa [59] identified P with a certain shifted Jacobi polynomial and explained very clearly, how to translate the reproducing property from the origin to an arbitrary point z_0 of the disk. Namely, he found a correcting factor that restores the polyanalyticity and converts the composition operator $f \mapsto f \circ \varphi_{z_0}$ into a unitary operator in $\mathcal{A}_m^2(\mathbb{D})$. He also computed [58] the RK of the space $\mathcal{A}_m^2(\mathbb{H}_1)$ of m-analytic functions on the upper halfplane $\mathbb{H}_1$ in $\mathbb{C}$. Hachadi and Youssfi [35] studied polyanalytic functions on the disk and on the entire complex plane, provided with radial measures. In particular, they computed the RK of $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$, where $\mathrm{d}\mu_\alpha(z) = \frac{1}{\pi}(1 - |z|^2)^\alpha \,\mathrm{d}\mu(z)$.

In Chapter 1, we compute explicitly the reproducing kernel of $\mathcal{A}_m^2(\mathbb{B}_n, \mu_\alpha)$, which is, the homogeneous m-analytic weighted Bergman space over the n-dimensional unit ball, by proving a reproducing formula at the origin or Mean-value property for such functions and then constructing a unitary operator in $\mathcal{A}_m^2(\mathbb{B}_n, \mu_\alpha)$ in order to "reset" this property from origin to an arbitrary point inside $\mathbb{B}_n$.

The Mean-value property shown in this work is a consequence of a reproducing property of generalized Jacobi polynomials, and the unitary operator mentioned above is a natural generalization of the previous one constructed by Pessoa [59].

Among these results, we give a general approach to the study of weighted changes of variables in the heomogeneous polyanalytic weighted Bergman space over more general domains, and compute with these tools, the RK of $\mathcal{A}_m^2(\mathbb{H}_n, \mu_\alpha)$, where $\mathbb{H}_n$ is the n-dimensional Siegel domain, which is a multimensional generalization of the complex upper halfplane. These results are published in [53].

There are several multidimensional results about polyanalytic spaces and kernels in other settings. Askour, Intissar, and Mouayn [6] computed the RK of the space of polyanalytic functions on $\mathbb{C}^n$, square-integrable with respect to the Gaussian weight (i.e., the polyanalytic Bargmann–Segal–Fock space). If $k \in \mathbb{N}^n$ and (Ω, ν) is a direct product of one-dimensional domains with some weights (for example, $\Omega = \mathbb{C}^n$ or $\Omega = \mathbb{D}^n$), then the RK of $\mathcal{A}_k(\Omega, \nu)$ can be obtained as the tensor product of the corresponding reproducing kernels on one-dimensional domains [35]. Ramírez Ortega and Sánchez Nungaray [67] defined some polyanalytic-type spaces on the Siegel domain $\mathbb{H}_n$ by other systems of differential equations, involving non-constant coefficients.

After submitting [53] to the journal, we found a preprint [86] by Youssfi, in which he has found another interesting way to get some of the results presented in this work. Namely, he also computed the RK of the weighted polyanalytic Bergman space over the unit ball, and his formula coincides (after normalizing the measure) with our formula (1.45). Moreover, he found the RK of the "homogeneously polyanalytic" Fock space.

Translation invariant operators

It is well known and easy to see that the radial Toeplitz operators on the Bergman space $\mathcal{A}^2(\mathbb{D})$ are diagonal in the monomial basis and therefore generate a commutative C*-algebra. In 1999, Vasilevski [79] found another non-trivial commutative C*-algebra of operators on the Bergman space. Namely, he considered "vertical" Toeplitz operators (i.e. invariant under horizontal translations), acting in the Bergman space $\mathcal{A}^2(\mathbb{H}_1)$ of analytic functions over the upper halfplane $\mathbb{H}_1$, and constructed a unitary operator $\widetilde{R}\colon \mathcal{A}^2(\mathbb{H}_1) \to L^2(\mathbb{R}_+)$ that simultaneously "diagonalizes" all vertical Toeplitz operators, converting them into multiplication operators. After that, many mathematicians obtained similar results for other groups of transformations, other spaces of functions, and other domains [28–30, 45, 47, 54, 65, 82]. Grudsky, Quiroga, and Vasilevski [31] performed a complete study of non-trivial commutative C*-algebras of Toeplitz operators on the weighted Bergman spaces over the unit disk. Dawson, Ólafsson, and Quiroga-Barranco [15, 16] showed that in the case of group-invariant operators acting in the weighted Bergman spaces of analytic functions over multidimensional domains, some of the previous results follow naturally from the general theory of unitary representations of C*-algebras. Quiroga-Barranco and Sánchez-Nungaray [61] studied commutative C*-algebras of Toeplitz operators in the weighted Bergman spaces over the unit ball using moment maps of the abelian subgroups of the biholomorphism group.

Here we propose another scheme to study group-invariant operators in reproducing kernel Hilbert spaces (RKHS). We are inspired by the following general idea. If G is a locally compact group acting on a measure space D such that the translations are unitary operators in $L^2(D)$, and H is a RKHS over D invariant under these translations, then it is natural to expect that the W*-algebra of translation-invariant operators can be described in terms of the Fourier transform (along the orbits of the group action) of the reproducing kernel. In this work, we apply this idea to the particular case when G is a locally compact abelian group (LCAG) and the domain D is a "tube" $G \times Y$. Our scheme is a natural generalization and developement of Vasilevski [79], [82, Section 3.1], see Example 2.9.1.

In Sections 2.2 and 2.3, we prove two simple general results about W*-algebras: an analog of the Stone–Weierstrass theorem and a criterion of commutativity of a direct integral.

In Section 2.4, we recall some properties of the Fourier transform and consider the unitary representation of the group G on the space $L^2(G \times Y)$ defined by

$$(\rho_{G\times Y}(a)f)(u,v) \coloneqq f(u-a,v) \qquad (a \in G,\ u \in G,\ v \in Y). \tag{0.1}$$

Using the Fourier transform with respect to the first argument, $F \otimes I_{L^2(Y)}$, we describe the W*-algebra $(\rho_{G\times Y})'$ of bounded linear operators on $L^2(G \times Y)$, commuting with the horizontal translations $\rho_{G\times Y}(a)$.

The main ideas of Sections 2.2–2.4 are well known, but we recall them in a convenient form and state explicitly some results that we have been unable to find in the literature.

In Section 2.5 we suppose that H is a closed subspace of $L^2(G \times Y)$, invariant under $\rho_{G\times Y}(a)$ for every a in G. Let $\rho_H(a)$ be the compression of $\rho_{G\times Y}(a)$ onto H. Then ρ_H is a unitary representation of G in H. Our principal object of study is the W*-algebra $\mathcal{V}$ of translation-invariant bounded linear operators acting in H, i.e., the centralizer of the representation ρ_H:

$$\mathcal{V} \coloneqq (\rho_H)' = \{A \in \mathcal{B}(H)\colon \quad \forall a \in G \quad \rho_H(a)A = A\rho_H(a)\}. \tag{0.2}$$

We show that the space $\widehat{H} \coloneqq (F \otimes I)(H)$ decomposes into the direct integral of fibers $\widehat{H}_\xi \subseteq L^2(Y)$, and the W*-algebra $\mathcal{V}$ is spatially isomorphic to the direct integral of the factors $\mathcal{B}(\widehat{H}_\xi)$:

$$\widehat{H} = \int_\Omega^\oplus \widehat{H}_\xi \, \mathrm{d}\widehat{\nu}(\xi), \qquad \Phi\mathcal{V}\Phi^* = \int_\Omega^\oplus \mathcal{B}(\widehat{H}_\xi)\, \mathrm{d}\widehat{\nu}(\xi).$$

Here $\widehat{G}$ is the dual group of G, $\widehat{\nu}$ is the Haar measure on $\widehat{G}$ associated to ν, $\Phi\colon H \to \widehat{H}$ is the compression of $F \otimes I$, and Ω is defined as the set of all "frequencies" ξ in $\widehat{G}$ corresponding to non-zero fibers $\widehat{H}_\xi$.

In particular, we conclude that $\mathcal{V}$ is commutative if and only if $\dim \widehat{H}_\xi = 1$ for $\widehat{\nu}$-almost all ξ in Ω. This condition is close to the multiplicity-free condition from [15, 16].

In Section 2.6 we assume that H is a RKHS and denote by $(K_{x,y})_{x\in G,y\in Y}$ the reproducing kernel of H. The translation-invariance of H is equivalent to the following property of the reproducing kernel:

$$K_{x,y}(u,v) = K_{0,y}(u-x,v) \qquad (x,u \in G,\ y,v \in Y).$$

We define L as the Fourier transform of K along the action of the group:

$$L_{\xi,y}(v) \coloneqq (\Phi K_{0,y})(\xi,y) = \int_G \overline{\xi(u)}\, K_{0,y}(u,v)\,\mathrm{d}\nu(u) \qquad (\xi \in \widehat{G},\ y,v \in Y). \tag{0.3}$$

Under some additional assumptions, we show that each fiber $\widehat{H}_\xi$ is a RKHS, and its reproducing kernel is $(L_{\xi,y})_{y\in Y}$. As a consequence, we establish a constructive criterion for commutativity of $\mathcal{V}$, in terms of L.

In Section 2.7 we consider the commutative case (when $\dim(\widehat{H}_\xi) = 1$ for all ξ in Ω) and construct a unitary operator R that simultaneously diagonalizes all operators belonging to $\mathcal{V}$. In particular, we diagonalize Toeplitz operators with translation-invariant generating symbols.

In Section 2.8 we consider the non-commutative case with finite-dimensional fibers and construct a unitary operator R that transforms elements of $\mathcal{V}$ into matrix families.

Finally, in Section 2.9 we apply this scheme to various examples.

Our scheme may be viewed as an application of the von Neumann theory to RKHS over domains of the form $G \times Y$. As an advantage of this work, we deal with general RKHS and general LCAG, without requiring any analytic or differential structure on the domain. We reduce the study of the algebra $\mathcal{V}$ to the computation of one Fourier integral (0.3).

The scheme proposed in this work unifies many of the currently known results on translation-invariant operators acting in RKHS, but it is not universal. For example, it cannot be applied to radial operators on RKHS over the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ with $n > 1$, because the corresponding unitary group $U(n)$ is not commutative, and $\mathbb{B}_n$ does not decompose into a product of the form $U(n) \times Y$.

Radial operators in the poly-Bergman space

As the concrete case of $n = 1$ in Chapter 1, let $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$ be the space of (homogeneous) m-analytic functions square integrable with respect to readial measure μ_α in $\mathbb{D}$. We denote by $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$ the orthogonal complement of $\mathcal{A}^2_{m-1}(\mathbb{D},\mu_\alpha)$ in $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$.

For every τ in the unit circle $\mathbb{T} := \{z \in \mathbb{C}\colon\ |z| = 1\}$, let $\rho_m^{(\alpha)}(\tau)$ be the rotation operator acting in $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$ by the rule

$$(\rho_m^{(\alpha)}(\tau)f)(z) := f(\tau^{-1}z).$$

The family $\rho_m^{(\alpha)}$ is a unitary representation of the group $\mathbb{T}$ in the space $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$, which in turn, is invariant under rotations (see Proposition (3.6.8)). We denote by $\mathcal{R}_m^{(\alpha)}$ its commutant, i.e., the von Neumann algebra that consists of all bounded linear operators acting in $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$ that commute with $\rho_m^{(\alpha)}(\tau)$ for every τ in $\mathbb{T}$. In other words, the elements of $\mathcal{R}_m^{(\alpha)}$ are the operators intertwining the representation $\rho_m^{(\alpha)}$. The elements of $\mathcal{R}_m^{(\alpha)}$ are called *radial* operators in $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$.

In a similar manner, we denote by $\rho_{(m)}^{(\alpha)}(\tau)$ the rotation operators acting in $\mathcal{A}_{(m)}^2(\mathbb{D}, \mu_\alpha)$ and by $\mathcal{R}_{(m)}^{(\alpha)}$ the von Neumann algebra of radial operators in $\mathcal{A}_{(m)}^2(\mathbb{D}, \mu_\alpha)$. We also consider the rotation operators $\rho^{(\alpha)}(\tau)$ in $L^2(\mathbb{D}, \mu_\alpha)$ and the corresponding algebra $\mathcal{R}^{(\alpha)}$ of radial operators.

In Section 3.2, we recall various equivalent formulas for the disk polynomials that can be obtained by orthogonalizing the monomials in z and $\overline{z}$. Using this orthonormal basis $(b_{p,q}^{(\alpha)})_{p,q\in\mathbb{N}_0}$ we descompose $L^2(\mathbb{D}, \mu_\alpha)$ into the orthogonal sum of subspaces $\mathcal{W}_\xi^{(\alpha)}$ corresponding to different frequences ξ, with $\xi \in \mathbb{Z}$.

In Section 3.3 we give an elementary proof of the weighted mean value property of polyanalytic functions and show the boundedness of the evaluation functionals for the spaces of polyanalytic functions over general domains in $\mathbb{C}$. In the unweighted case, this mean value property was proven by Koshelev [51] and Pessoa [59]. In the weighted case, it was found by Hachadi and Youssfi [35] and used by them to compute the reproducing kernel of $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$.

In Section 3.4, extending results by Ramazanov [63, 64] to the weighted case, we verify that the family $(b_{p,q}^{(\alpha)})_{p\geq 0, 0\leq q<m}$ is an orthonormal basis of $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$. Using this fact, we decompose $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$ into subspaces $\mathcal{W}_\xi^{(\alpha)} \cap \mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$.

In Section 3.5 we prove that the set of all Toeplitz operators with bounded generating symbols is not weakly dense in $\mathcal{B}(\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha))$. This simple result was surprising for us.

In Section 3.6 we decompose the von Neumann algebras $\mathcal{R}^{(\alpha)}$, $\mathcal{R}_m^{(\alpha)}$, and $\mathcal{R}_{(m)}^{(\alpha)}$, into direct sums of factors. In particular, Theorems 3.6.9 and 3.6.10 imply that the algebra $\mathcal{R}_m^{(\alpha)}$ is noncommutative for $m \geq 2$, whereas $\mathcal{R}_{(m)}^{(\alpha)}$ is commutative for every m in $\mathbb{N}$.

In Section 3.7, we find explicit representations of the radial Toeplitz operators acting in the spaces $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$ and $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$. The results of Sections 3.6 and 3.7 are similar to [57]. The main difference is that the orthonormal bases are given by other formulas.

1 Polyanalytic Kernels in the Unit Ball

1.1 Scope

The aim of this chapter is to write an explicit formula for the reproducing kernel of the homogeneous m-analytyc wheighted Bergman space over the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ and over the n-dimensional Siegel domain $\mathbb{H}_n$. All these results were published in [53].

Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and take $n \in \mathbb{N}$ for the rest. We employ the usual notation for the multi-indices and the notation $|\cdot|$ for the norm in $\mathbb{C}^n$, see [70, Section 1.1]. Given an open set Ω in $\mathbb{C}^n$, a multi-index $k = (k_1, \ldots, k_n)$ in $\mathbb{N}_0^n$ and a function of the class $C^{|k|}(\Omega)$, we denote by $\overline{D}^k f$ the Wirtinger derivative of f of the order k (such derivatives were previously used by Poincaré, Pompeiu, and Kolossov). In a more classical notation,

$$\overline{D}^k f(z) \coloneqq \frac{\partial^{|k|}}{\partial^{k_1}\overline{z_1}\cdots\partial^{k_n}\overline{z_n}} f(z) \qquad (z \in \Omega).$$

Let $\mathcal{A}(\Omega)$ be the class of all analytic functions on Ω. It is defined by the system of equations

$$\overline{D}^{(1,0,\ldots,0)} f = 0, \qquad \overline{D}^{(0,1,\ldots,0)} f = 0, \qquad \ldots, \qquad \overline{D}^{(0,0,\ldots,1)} f = 0. \tag{1.1}$$

Given an open subset Ω of $\mathbb{C}^n$ and a multi-index $k = (k_1, \ldots, k_n)$ in $\mathbb{N}^n$, k-analytic functions on Ω are defined [7, Section 6.4] as functions that can be represented in the form

$$f(z) = \sum_{j_1,\ldots,j_n=0}^{k_1-1,\ldots,k_n-1} g_j(z)\,\overline{z}^j, \tag{1.2}$$

where all functions g_j are analytic. We denote by $\mathcal{A}_k(\Omega)$ the class of all functions of the form (1.2). For simply connected domains Ω, such functions can also be characterized as smooth solutions of the system of differential equations

$$\overline{D}^{(k_1,0,\ldots,0)} f = 0, \qquad \overline{D}^{(0,k_2,\ldots,0)} f = 0, \qquad \ldots, \qquad \overline{D}^{(0,0,\ldots,k_n)} f = 0. \tag{1.3}$$

Instead of considering polyanalytic functions of a given multi-order k, we prefer to work with the following classes of "homogeneously polyanalytic" functions.

Definition 1.1.1. Let Ω be an open set in $\mathbb{C}^n$ and $m \in \mathbb{N}$. We say that $f\colon \Omega \to \mathbb{C}$ is *homogeneously polyanalytic of total order* m or just m*-analytic*, if f belongs to the class $C^m(\Omega)$ and $\overline{D}^k f = 0$ for every k in $\mathbb{N}_0^n$ with $|k| = m$. We denote by $\mathcal{A}_m(\Omega)$ the set of all such functions.

The multi-indices k with $|k| = m$ can be associated with m-multisubsets of the set $\{1, \ldots, n\}$, and the number of such multi-indices is $\binom{n+m-1}{m}$. For example, the class $\mathcal{A}_1(\Omega) = \mathcal{A}(\Omega)$ is defined by n differential equations (1.1).

Definition 1.1.2. Let Ω be an open set in $\mathbb{C}^n$ and $m \in \mathbb{N}$. We denote by $\widetilde{\mathcal{A}}_m(\Omega)$ the set of all functions $f\colon \Omega \to \mathbb{C}$ that can be written in the form

$$f(z) = \sum_{|j|<m} h_j(z)\overline{z}^j, \tag{1.4}$$

where $h_j \in \mathcal{A}(\Omega)$ for all $j \in \mathbb{N}_0^n$ with $|j| < m$.

This class of functions has been studied by Daghighi [14], who has proved some uniqueness theorems. He used the term "polyanalytic functions of absolute order q", where q plays the role of our m.

In Section 1.2 we prove that $\widetilde{\mathcal{A}}_m(\Omega) = \mathcal{A}_m(\Omega)$. Obviously, $\mathcal{A}_m(\Omega)$ is a complex vector space. In Proposition 1.2.7 we show that the space $\mathcal{A}_m(\Omega)$, with m in $\mathbb{N}$, is invariant under linear changes of variables (of course, the domain can change). In a contrast, the spaces $\mathcal{A}_k$, with $n \geq 2$ and $k \in \mathbb{N}^n$, $k \neq (1, 1, \ldots, 1)$, are not invariant under linear changes of variables; see Proposition 1.2.8. If $m \in \mathbb{N}$, then $\mathcal{A}_m(\Omega) \subseteq \mathcal{A}_{(m,\ldots,m)}(\Omega)$, and some results about k-analytic functions ($k \in \mathbb{N}^n$) can be applied to $\mathcal{A}_m(\Omega)$. On the other hand, if k in $\mathbb{N}^n$, then $\mathcal{A}_k(\Omega) \subseteq \mathcal{A}_{|k|+1-n}(\Omega)$.

From now on, we denote by μ the Lebesgue measure on $\mathbb{C}^n$.

Definition 1.1.3. Let Ω be an open set in $\mathbb{C}^n$, $m \in \mathbb{N}$, $W\colon \Omega \to (0, +\infty)$ be a continuous function, and $\mathrm{d}\nu = W\,\mathrm{d}\mu$. We denote by $\mathcal{A}^2_m(\Omega, \nu)$ the set of all functions $f \in \mathcal{A}_m(\Omega)$ that are square-integrable with respect to ν. We consider this space with the inner product inherited from $L^2(\Omega, \nu)$. Furthermore, we denote by $\mathcal{A}^2_{(m)}(\Omega, \nu)$ the orthogonal complement of $\mathcal{A}^2_{m-1}(\Omega, \nu)$ in $\mathcal{A}_m(\Omega, \nu)$. Here $\mathcal{A}^2_0(\Omega, \nu) \coloneqq \{0\}$.

Section 1.3 contains a weighted mean-value property for integrable functions belonging to $\mathcal{A}_m(\Omega)$. As a consequence of this property, $\mathcal{A}^2_m(\Omega,\nu)$ is a RKHS. In Section 1.4 we show how the RK transforms under a weighted change of variables. In Section 1.5 we use the previous tools to compute the RK of $\mathcal{A}^2_m(\mathbb{B}_n,\mu_\alpha)$, where $\mathbb{B}_n$ is the unit ball in $\mathbb{C}^n$ and μ_α is the standard radial measure on $\mathbb{B}_n$, see (1.27). Finally, in Section 1.6 we compute the RK of $\mathcal{A}^2_m(\mathbb{H}_n,\nu_\alpha)$, where $\mathbb{H}_n$ is the standard Siegel domain in $\mathbb{C}^n$ and ν_α is a weighted Lebesgue measure (see (1.51) and (1.52)).

There are many recent studies of Toeplitz operators, acting in polyanalytic Bergman spaces over one-dimensional domains [13, 46, 56, 57, 68, 82]. We hope that this work can serve as a basis for some multidimensional generalizations, see Remarks 1.5.10, 1.5.11, and 1.6.17.

1.2 Homogeneously polyanalytic functions

Polyanalytic functions naturally arise in some physical models (plane elasticity theory, Landau levels) and in some methods of signal processing, see [1, 2, 4, 36, 42, 43]. Koshelev [51] computed the reproducing kernel of the m-analytic Bergman space $\mathcal{A}^2_m(\mathbb{D})$ on the unit disk. In [7], Balk explained fundamental properties of polyanalytic functions. Dzhuraev [19] related polyanalytic projections with singular integral operators. Vasilevski [80, 81] studied polyanalytic Bergman spaces on the upper halfplane and polyanalytic Fock spaces using the Fourier transform. Ramazanov [63, 64] constructed an orthonormal basis in $\mathcal{A}^2_m(\mathbb{D})$ and studied various properties of $\mathcal{A}^2_m(\mathbb{D})$. In fact, the elements of this basis are well-known disk polynomials studied by Koornwinder [49], Wünsche [84], and other authors. Pessoa [59] related Koshelev's formula with Jacobi polynomials and gave a very clear proof of this formula. He also obtained similar results for some other one-dimensional domains. Hachadi and Youssfi [35] developed a general scheme for computing the reproducing kernels of the spaces of polyanalytic functions on radial plane domains (disks or the whole plane) with radial measures.

There are general investigations about bounded linear operators in reproducing kernel Hilbert spaces (RKHS), especially about Toeplitz operators in Bergman or Fock spaces [9, 85, 88], but the complete description of the spectral properties is found only for some special classes of operators, in particular, for Toeplitz operators with generating symbols invariant under some group actions, see Vasilevski [82], Grudsky, Quiroga-Barranco, and

Vasilevski [31], Dawson, Ólafsson, and Quiroga-Barranco [15]. The simplest class of this type consists of Toeplitz operators with bounded radial generating symbols. Various properties of these operators (boundedness, compactness, and eigenvalues) have been studied by many authors, see [33, 50, 60, 89]. The C*-algebra generated by such operators, acting in the Bergman space, was explicitly described in [10, 32, 40, 76]. Loaiza and Lozano [55] obtained similar results for radial Toeplitz operators in harmonic Bergman spaces. Maximenko and Tellería-Romero [57] studied radial operators in the polyanalytic Fock space.

Hutník, Hutníková, Ramírez-Mora, Ramírez-Ortega, Sánchez-Nungaray, Loaiza, and other authors [46, 48, 56, 66, 68, 71] studied vertical and angular Toeplitz operators in polyanalytic and true-polyanalytic Bergman spaces. In particular, vertical Toeplitz operators in the m-analytic Bergman space over the upper half-plane are represented in [68] as $m \times m$ matrices whose entries are continuous functions on $(0, +\infty)$, with some additional properties at 0 and $+\infty$.

Rozenblum and Vasilevski [69] investigated Toeplitz operators with distributional symbols and showed that Toeplitz operators in true-polyanalytic spaces Bergman or Fock spaces are equivalent to some Toeplitz operators with distributional symbols in the analytic Bergman or Fock spaces.

Let Ω be an open set in $\mathbb{C}^n$ and $m \in \mathbb{N}$. In this section we show that $\mathcal{A}_m(\Omega) = \widetilde{\mathcal{A}}_m(\Omega)$ and mention some other properties of $\mathcal{A}_m(\Omega)$.

Lemma 1.2.1. *Let $f \in \mathcal{A}_m(\Omega)$. Then the following function is analytic:*

$$g(z) \coloneqq \sum_{\substack{k\in\mathbb{N}_0^n\\ |k|<m}} \frac{(-1)^{|k|}}{k!} (\overline{D}^k f)(z)\, \overline{z}^k.$$

Proof. Let $p \in \{1, \ldots, n\}$ and e_p be the p-th canonical vector in $\mathbb{N}_0^n$, i.e., $e_p \coloneqq (\delta_{p,s})_{s=1}^n$, where δ is the Kronecker's delta. We have to show that $\overline{D}^{e_p} g = 0$. By the product rule,

$$(\overline{D}^{e_p} g)(z) = S_1(z) + S_2(z) + S_3(z) + S_4(z),$$

where

$$S_1(z) = \sum_{|k|<m-1} \tfrac{(-1)^{|k|}}{k!} (\overline{D}^{k+e_p} f)(z)\overline{z}^k, \qquad S_2(z) = \sum_{|k|=m-1} \tfrac{(-1)^{|k|}}{k!} (\overline{D}^{k+e_p} f)(z)\overline{z}^k,$$

$$S_3(z) = \sum_{\substack{|k|<m \\ k_p=0}} \tfrac{(-1)^{|k|}}{k!} (\overline{D}^k f)(z)\overline{D}^{e_p}(\overline{z}^k), \qquad S_4(z) = \sum_{\substack{|k|<m \\ k_p>0}} \tfrac{(-1)^{|k|}}{(k-e_p)!} (\overline{D}^k f)(z)\overline{z}^{k-e_p}.$$

We have that $S_2(z) = 0$, because $f \in \mathcal{A}_m(\Omega)$ and $|k + e_p| = m$ in the sum defining S_2. Also $S_3(z) = 0$, because $\overline{D}^{e_p}\overline{z}^k = 0$ when $k_p = 0$. Finally, with the change of variable $j = k - e_p$, we rewrite $S_4(z)$ as

$$S_4(z) = -\sum_{|j|<m-1} \frac{(-1)^{|j|}}{j!} (\overline{D}^{r+e_p} f)(z)\,\overline{z}^j.$$

Therefore, $(\overline{D}^{e_p} g)(z) = S_1(z) + S_4(z) = 0$. □

Lemma 1.2.2. *Let $f \in \widetilde{\mathcal{A}}_p(\Omega)$ and $g \in \widetilde{\mathcal{A}}_q(\Omega)$. Then $fg \in \widetilde{\mathcal{A}}_{p+q-1}(\Omega)$.*

Proof. This lemma follows from the elementary observation that if $j \in \mathbb{N}_0^n$ and $k \in \mathbb{N}_0^n$, with $|j| < p$ and $|k| < q$, then $\overline{z}^j\,\overline{z}^k = \overline{z}^{j+k}$ and $|j + k| = |j| + |k| < p + q - 1$. □

Theorem 1.2.3. *Let Ω be an open set in $\mathbb{C}^n$ and $m \in \mathbb{N}$. Then $\mathcal{A}_m(\Omega) = \widetilde{\mathcal{A}}_m(\Omega)$.*

Proof. It is well known that $\mathcal{A}(\Omega) = \widetilde{\mathcal{A}}(\Omega)$. Let $m > 1$. It is obvious that $\widetilde{\mathcal{A}}_m(\Omega) \subseteq \mathcal{A}_m(\Omega)$. We show, by induction on m, that $\mathcal{A}_m(\Omega) \subseteq \widetilde{\mathcal{A}}_m(\Omega)$. Suppose $\mathcal{A}_p(\Omega) \subseteq \widetilde{\mathcal{A}}_p(\Omega)$ for every $p < m$ and let $f \in \mathcal{A}_m(\Omega)$. Define g as in Lemma 1.2.1, then observe that

$$f(z) = -\sum_{0<|k|<m} \frac{(-1)^{|k|}}{k!} (\overline{D}^k f)(z)\,\overline{z}^k + g(z).$$

For every k with $0 < |k| < m$, we have $\overline{z}^k \in \widetilde{\mathcal{A}}_{|k|+1}(\Omega)$ and $\overline{D}^k f \in \mathcal{A}_{m-|k|}(\Omega) \subseteq \widetilde{\mathcal{A}}_{m-|k|}(\Omega)$; the last inclusion holds by the induction hypothesis. Finally, apply Lemma 1.2.2. □

Corollary 1.2.4. *Let $f \in \mathcal{A}_m(\Omega)$ and $a \in \Omega$. Then there exists a family of functions $(h_k)_{|k|<m}$ in $\mathcal{A}(\Omega)$, such that for every z in Ω,*

$$f(z) = \sum_{\substack{k\in\mathbb{N}_0^n \\ |k|<m}} h_k(z)\,(\overline{z} - \overline{a})^k. \tag{1.5}$$

Proof. First, we write f as (1.4). Then, expanding $\overline{z}^j = (\overline{z} - \overline{a} + \overline{a})^j$ into multi-powers of $\overline{z} - \overline{a}$ and regrouping the summands, we obtain a sum of the form (1.5). $\square$

Corollary 1.2.5. *Let $f \in \mathcal{A}_m(\Omega)$, $a \in \Omega$, and $r > 0$ such that $a + r\mathbb{B}_n \subseteq \Omega$. Then there exists a family $(\beta)_{j,k\in\mathbb{N}_0^n,|k|<m}$ of complex numbers such that for every z in $a + r\mathbb{B}_n$,*

$$f(z) = \sum_{j\in\mathbb{N}_0^n} \sum_{\substack{k\in\mathbb{N}_0^n \\ |k|<m}} \beta_{j,k}(z-a)^j(\overline{z}-\overline{a})^k. \tag{1.6}$$

Moreover, this series converges uniformly on every compact subset of $\mathbb{B}_n$.

Proof. It is well known that every holomorphic function on $\mathbb{B}_n$, decomposes on $\mathbb{B}_n$ into a power series, converging on $\mathbb{B}_n$ and uniformly converging on compact subsets of $\mathbb{B}_n$. Applying this fact to each h_j from Corollary 1.2.4, we obtain (1.6). $\square$

Let us mention a version of the uniqueness property for m-analytic functions.

Proposition 1.2.6. *Let Ω be a connected open set in $\mathbb{C}^n$, Ω_1 be an open subset of Ω, and $f \in \mathcal{A}_m(\Omega)$ such that $f(z) = 0$ for every z in Ω_1. Then $f(z) = 0$ for every z in Ω.*

Proof. For k in $\mathbb{N}^n$, the uniqueness property of k-analytic functions is proven in [7, Section 6.4]. The uniqueness property for m-analytic functions is a corollary of this fact, since $\mathcal{A}_m(\Omega) \subseteq \mathcal{A}_{(m,\ldots,m)}(\Omega)$. $\square$

To finish this section, we will show that the class $\mathcal{A}_m$ with m in $\mathbb{N}$ is closed under linear changes of variables, while the classes $\mathcal{A}_k$ with $k \in \mathbb{N}^n$ are generally not.

Proposition 1.2.7. *Let M be an invertible $n \times n$ complex matrix and f in $\mathcal{A}_m(\Omega)$. Define $g\colon M\Omega \to \mathbb{C}$ by $g(z) \coloneqq f(M^{-1}z)$. Then $g \in \mathcal{A}_m(M\Omega)$.*

Proof. Theorem 1.2.3 allows us to work with $\widetilde{\mathcal{A}}_m$ instead of $\mathcal{A}_m$. Let f be as in Definition 1.1.2. Then

$$g(z) = \sum_{|j|<m} h_j(M^{-1}z)\,\overline{(M^{-1}z)}^{\,j}.$$

The functions $z \mapsto h_j(M^{-1}z)$ are analytic. Let $M^{-1} = [c_{r,s}]_{r,s=1}^n$. Then

$$\overline{(M^{-1}z)}^j = \prod_{r=1}^n \left(\sum_{s=1}^n \overline{c_{r,s}}\,\overline{z_s} \right)^{j_r}.$$

The last expression is a homogeneous polynomial in $\overline{z_1}, \ldots, \overline{z_n}$ of total degree $|j|$, which is strictly less than m (the same conclusion can also be obtained by Lemma 1.2.2). Therefore $g \in \widetilde{\mathcal{A}}_m(M\Omega)$. □

Proposition 1.2.8. *Let $n \geq 2$, Ω be an open subset of $\mathbb{C}^n$, $k \in \mathbb{N}^n$, $k \neq (1,1,\ldots,1)$. Then there exists a function f in $\mathcal{A}_k(\Omega)$ and an invertible matrix M in $\mathbb{C}^{n\times n}$ such that the function $g\colon M\Omega \to \mathbb{C}$, defined by $g(z) \coloneqq f(M^{-1}z)$, does not belong to $\mathcal{A}_k(M\Omega)$.*

Proof. To simplify the notation, we suppose that $k_1 > 1$. The general case is analogous. Define M in such a manner that

$$M^{-1}z = (z_1 + z_2, z_2 - z_1, z_3, \ldots, z_n).$$

Consider $f\colon \Omega \to \mathbb{C}$, $f(z) \coloneqq \overline{z_1}^{k_1-1}\,\overline{z_2}^{k_2-1}$. Then

$$g(z) = (\overline{z_1} + \overline{z_2})^{k_1-1}\,(\overline{z_2} - \overline{z_1})^{k_2-1}.$$

In the expansion of the last polynomial, one of the terms is $\overline{z_2}^{k_1+k_2-2}$. Since $k_1 + k_2 - 2 > k_2 - 1$, we obtain $g \notin \mathcal{A}_k(M\Omega)$, though $f \in \mathcal{A}_k(\Omega)$. □

1.3 Weighted mean value property

In this section we prove that the value of a m-analytic function at the center of the unit ball $\mathbb{B}_n$ can be expressed as the integral of this function over the ball, with a certain real radial weight (Theorem 1.3.3). Similar results in the one-dimensional case were proved in [35, 51, 59].

Jacobi polynomials and their reproducing property

Some integrals over the unit ball, written in the spherical coordinates, can be reduced to integrals over the unit interval $(0,1)$ with weights of power type at the boundary points 0 and 1. Thereby Jacobi polynomials naturally appear.

For every ξ and η in $\mathbb{R}$, the (generalized) Jacobi polynomial of degree m is defined by Rodrigues formula:

$$P_m^{(\xi,\eta)}(x) := \frac{(-1)^m}{2^m\, m!}\,(1-x)^{-\xi}(1+x)^{-\eta}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\Big((1-x)^{m+\xi}(1+x)^{m+\eta}\Big). \tag{1.7}$$

This definition and the general Leibniz rule imply its expansion into powers of $x-1$ and $x+1$:

$$P_m^{(\xi,\eta)}(x) = \sum_{s=0}^{m}\binom{m+\xi}{m-s}\binom{m+\eta}{s}\left(\frac{x-1}{2}\right)^{s}\left(\frac{x+1}{2}\right)^{m-s}. \tag{1.8}$$

Formula (1.8) yields a symmetry relation, the values at the points 1 and -1, and a formula for the derivative:

$$P_m^{(\xi,\eta)}(-x) = (-1)^m P_m^{(\eta,\xi)}(x), \tag{1.9}$$

$$P_m^{(\xi,\eta)}(1) = \binom{m+\xi}{m}, \qquad P_m^{(\xi,\eta)}(-1) = (-1)^m\binom{m+\eta}{m}, \tag{1.10}$$

$$\big(P_m^{(\xi,\eta)}\big)'(x) = \frac{\xi+\eta+m+1}{2}\,P_{m-1}^{(\xi+1,\eta+1)}(x). \tag{1.11}$$

With the above properties, it is easy to compute the derivatives of $P_m^{(\xi,\eta)}$ at the point 1. Now Taylor's formula yields another two explicit expansions for $P_m^{(\xi,\eta)}$:

$$\begin{aligned} P_m^{(\xi,\eta)}(x) &= \sum_{s=0}^{m}\binom{\xi+\eta+m+s}{s}\binom{\xi+m}{m-s}\left(\frac{x-1}{2}\right)^{s} && (1.12)\\ &= \sum_{s=0}^{m}(-1)^s\binom{\xi+\eta+m+s}{s}\binom{\eta+m}{m-s}\left(\frac{x+1}{2}\right)^{s}. && (1.13) \end{aligned}$$

If $\xi > -1$ and $\eta > -1$, then (1.12) can be rewritten as

$$P_m^{(\xi,\eta)}(x) = \frac{\Gamma(\xi+m+1)}{m!\,\Gamma(\xi+\eta+m+1)}\sum_{s=0}^{m}\binom{m}{s}\frac{\Gamma(\xi+\eta+m+s+1)}{\Gamma(\xi+s+1)}\left(\frac{x-1}{2}\right)^{s}. \tag{1.14}$$

For $\xi > -1$ and $\eta > -1$, we equip $(-1,1)$ with the weight $(1-x)^\xi(1+x)^\eta$, then denote by $\langle\cdot,\cdot\rangle_{(-1,1),\xi,\eta}$ the corresponding inner product:

$$\langle f, g\rangle_{(-1,1),\xi,\eta} := \int_{-1}^{1} f(x)\overline{g(x)}\,(1-x)^\xi(1+x)^\eta\,\mathrm{d}x.$$

Then $L^2((-1,1),(1-x)^\xi(1+x)^\eta)$ is a Hilbert space, and the set $\mathcal{P}$ of Jacobi polynomials is a dense subset. Using (1.7) and integrating by parts, for every f in $\mathcal{P}$ we get

$$\langle f, P_m^{(\xi,\eta)}\rangle_{(-1,1),\xi,\eta} = \frac{1}{2m}\langle f', P_{m-1}^{(\xi+1,\eta+1)}\rangle_{(-1,1),\xi+1,\eta+1}. \tag{1.15}$$

Applying (1.15) and induction, it is easy to prove that the sequence $(P_m^{(\xi,\eta)})_{m=0}^\infty$ is an orthogonal basis of $L^2((-1,1),(1-x)^\xi(1+x)^\eta)$, that is, for every polynomial h of degree less than m,

$$\int_{-1}^{1} h(x)P_m^{(\xi,\eta)}(x)(1-x)^\xi(1+x)^\eta\mathrm{d}x = 0. \tag{1.16}$$

Furthermore,

$$\langle P_\ell^{(\xi,\eta)}, P_m^{(\xi,\eta)}\rangle_{(-1,1),\xi,\eta} = \frac{2^{\xi+\eta+1}\,\Gamma(m+\xi+1)\Gamma(m+\eta+1)}{(2m+\xi+\eta+1)\Gamma(m+\xi+\eta+1)\,m!}\delta_{\ell,m}. \tag{1.17}$$

Formulas (1.7) and (1.16), and induction allow us to compute the following integral for $\eta > 0$:

$$\int_{-1}^{1} P_m^{(\xi,\eta+1)}(x)\,(1-x)^\xi(1+x)^\eta\,\mathrm{d}x = 2^{\xi+\eta+1}(-1)^m\,\mathrm{B}(\xi+m+1,\eta+1), \tag{1.18}$$

where B is the classical Beta function.

Definition 1.3.1. Let $m \in \mathbb{N}$ and $\xi,\eta > -1$. We denote by $R_m^{(\xi,\eta)}$ the following polynomial:

$$R_m^{(\xi,\eta)}(t) := \frac{(-1)^m\,\mathrm{B}(\xi+1,\eta+1)}{\mathrm{B}(\xi+m+1,\eta+1)}\,P_m^{(\xi,\eta+1)}(2t-1). \tag{1.19}$$

Equivalently, by the symmetry relation for Jacobi polynomials, we have

$$R_m^{(\xi,\eta)}(t) = \frac{\mathrm{B}(\xi+1,\eta+1)}{\mathrm{B}(\xi+m+1,\eta+1)}\,P_m^{(\eta+1,\xi)}(1-2t). \tag{1.20}$$

Combining (1.19) with (1.13) or (1.20) with (1.12), we get the following explicit formulas for $R_m^{(\xi,\eta)}$:

$$R_m^{(\xi,\eta)}(t) = \frac{\Gamma(\xi+1)\,\Gamma(\eta+m+2)}{\Gamma(\xi+\eta+2)\,\Gamma(\xi+m+1)}\sum_{s=0}^{m}\frac{(-1)^s\,\Gamma(\xi+\eta+m+s+2)}{s!\,(m-s)!\,\Gamma(\eta+s+2)}\,t^s \tag{1.21}$$

$$= \frac{\Gamma(\xi+1)\,\Gamma(\eta+m+2)}{\Gamma(\xi+\eta+2)\,(\xi+m)\,m!}\sum_{s=0}^{m}\frac{(-1)^s\binom{m}{s}}{\mathrm{B}(\xi+m,\eta+s+2)}\,t^s. \tag{1.22}$$

The next simple result was proven in [8] using the orthogonality of the Jacobi polynomials and formula (1.18). Previously, Hachadi and Youssfi [35, formula (5.7)] gave another proof for the case $\eta = 0$.

Proposition 1.3.2. *Let $m \in \mathbb{N}$ and $\xi, \eta > -1$. Then for every polynomial h with complex coefficients and $\deg(h) \le m$,*

$$\frac{1}{\mathrm{B}(\xi+1,\eta+1)} \int_0^1 h(t) R_m^{(\xi,\eta)}(t)\,(1-t)^{\xi} t^{\eta}\,\mathrm{d}t = h(0). \tag{1.23}$$

The polynomials of degree $\le m$, considered as square-integrable functions on the interval $(0,1)$ with the normalized weight $\frac{1}{\mathrm{B}(\xi+1,\eta+1)}(1-t)^{\xi}t^{\eta}$, form a RKHS. Formula (1.23) means that $R_m^{(\xi,\eta)}$ is the RK of this space at the point 0.

As a particular case of (1.23), for every k in $\mathbb{N}_0$ with $k \le m$,

$$\frac{1}{\mathrm{B}(\xi+1,\eta+1)} \int_0^1 R_m^{(\xi,\eta)}(t)(1-t)^{\xi} t^{\eta+k}\,\mathrm{d}t = \delta_{k,0}. \tag{1.24}$$

Weighted mean value property of homogeneously polyanalytic functions

We denote by μ the Lebesgue measure on $\mathbb{C}^n$, by $\mathbb{S}_n$ the unit sphere in $\mathbb{C}^n$, and by $\mu_{\mathbb{S}_n}$ the (non-normalized) area measure on $\mathbb{S}_n$. It is well known [70, Section 1.4] that

$$\mu(\mathbb{B}_n) = \frac{\pi^n}{n!}, \qquad \mu_{\mathbb{S}_n}(\mathbb{S}_n) = \frac{2\pi^n}{(n-1)!},$$

and

$$\int_{\mathbb{S}_n} \zeta^j \overline{\zeta}^k \mathrm{d}\mu_{\mathbb{S}_n}(\zeta) = \frac{2\pi^n j!}{(n-1+|j|)!} \cdot \delta_{j,k} \qquad (j,k \in \mathbb{N}_0^n). \tag{1.25}$$

Given an integrable function f on $\mathbb{B}_n$, its integral over $\mathbb{B}_n$ can be written as

$$\int_{\mathbb{B}_n} f\,\mathrm{d}\mu = \int_0^1 r^{2n-1} \left(\int_{\mathbb{S}_n} f(r\zeta)\,\mathrm{d}\mu_{\mathbb{S}_n}(\zeta) \right) \mathrm{d}r. \tag{1.26}$$

For $\alpha > -1$, we denote by μ_α the Lebesgue measure on $\mathbb{B}_n$ with the standard radial weight:

$$\mathrm{d}\mu_\alpha(z) = c_\alpha (1-|z|^2)^\alpha\,\mathrm{d}\mu(z). \tag{1.27}$$

The normalizing constant c_α is chosen so that $\mu_\alpha(\mathbb{B}_n) = 1$:

$$c_\alpha := \frac{\Gamma(n+\alpha+1)}{\pi^n\,\Gamma(\alpha+1)}. \tag{1.28}$$

Theorem 1.3.3. *Let $f \in \mathcal{A}_m(\mathbb{B}_n)$ such that $f \in L^1(\mathbb{B}_n, \mu_\alpha)$. Then*

$$f(0) = \int_{\mathbb{B}_n} f(z) R_{m-1}^{(\alpha,n-1)}(|z|^2)\,\mathrm{d}\mu_\alpha(z). \tag{1.29}$$

Proof. We represent f in the form (1.6) with $a = 0$, then make the change of variables $z = r\zeta$ with $0 \le r < 1$, $\zeta \in \mathbb{S}_n$:

$$f(z) = \sum_{j\in\mathbb{N}_0^n} \sum_{\substack{k\in\mathbb{N}_0^n\\|k|<m}} \beta_{j,k} r^{|j|+|k|} \zeta^j \overline{\zeta}^k. \tag{1.30}$$

For every s in $(0,1)$, let I_s be the integral similar to the right-hand side of (1.29), but over the ball $s\mathbb{B}_n$:

$$I_s := \int_{s\mathbb{B}_n} f(z) R_{m-1}^{(\alpha,n-1)}(|z|^2)\,\mathrm{d}\mu_\alpha(z).$$

Since the series (1.30) converges uniformly over r in $[0,s]$ and ζ in $\mathbb{S}_n$, it can be interchanged with the integral over $s\mathbb{B}_n$. Then we apply (1.26) and (1.25):

$$\begin{aligned}
I_s &= c_\alpha \sum_{j\in\mathbb{N}_0^n} \sum_{\substack{k\in\mathbb{N}_0^n\\|k|<m}} \beta_{j,k} \int_0^s r^{2n-1+|j|+|k|} R_{m-1}^{(\alpha,n-1)}(r^2)(1-r^2)^\alpha \left(\int_{\mathbb{S}_n} \zeta^j \overline{\zeta}^k \,\mathrm{d}\mu_{\mathbb{S}_n}(\zeta) \right) \mathrm{d}r \\
&= c_\alpha \sum_{\substack{k\in\mathbb{N}_0^n\\|k|<m}} \beta_{k,k} \cdot \frac{\pi^n\, k!}{(n-1+|k|)!} \int_0^s R_{m-1}^{(\alpha,n-1)}(t)(1-t)^\alpha t^{n-1+|k|}\,\mathrm{d}t.
\end{aligned}$$

The condition $f \in L^1(\mathbb{B}_n, \mu_\alpha)$ implies that $I_s \to I_1$, as $s \to 1$. Passing to this limit and using (1.24), we finally obtain

$$\begin{aligned}
I_1 &= \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \sum_{\substack{k\in\mathbb{N}_0^n\\|k|<m}} \beta_{k,k} \cdot \frac{k!}{(n-1+|k|)!} \int_0^1 R_{m-1}^{(\alpha,n-1)}(t)(1-t)^\alpha t^{n-1+|k|}\,\mathrm{d}t \\
&= \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)} \sum_{|k|<m} \beta_{k,k} \frac{k!}{(n-1+|k|)!} \cdot \delta_{k,0}\,\mathrm{B}(\alpha+1,n) = \beta_{0,0} = f(0). \qquad \square
\end{aligned}$$

Here is an analog of (1.29) for an arbitrary ball and for $\alpha = 0$.

Corollary 1.3.4. *Let Ω be an open subset of $\mathbb{C}^n$, $f \in \mathcal{A}_m(\Omega)$, $a \in \Omega$, and $r > 0$ such that $a + r\mathbb{B}_n \subseteq \Omega$. Suppose that $f \in L^1(a + r\mathbb{B}_n, \mu)$. Then*

$$f(a) = \frac{n!}{\pi^n} \frac{1}{r^{2n}} \int_{a+r\mathbb{B}_n} f(z) R_{m-1}^{(0,n-1)} \left(\frac{|z-a|^2}{r^2} \right) \mathrm{d}\mu(z). \tag{1.31}$$

Bergman spaces of homogeneously polyanalytic functions

In the rest of this section, we suppose that Ω, m, W, ν are like in Definition 1.1.3. Using (1.31), it is easy to prove the upcoming Lemma 1.3.5 and Proposition 1.3.6. See similar proofs for the one-dimensional case in [8, Lemma 4.3, Proposition 4.4].

Lemma 1.3.5. *Let K be a compact subset of Ω. There exists a number $C_{m,W,K} > 0$ such that for every f in $\mathcal{A}_m^2(\Omega, \nu)$ and every z in K,*

$$|f(z)| \leq C_{m,W,K} \|f\|_{\mathcal{A}_m^2(\Omega,\nu)}. \tag{1.32}$$

Proposition 1.3.6. *$\mathcal{A}_m^2(\Omega, \nu)$ is a RKHS.*

As a corollary, the spaces $\mathcal{A}_{(m)}^2(\Omega, \nu)$ are also RKHS.

Proposition 1.3.7. *In the conditions of Definition 1.1.3, suppose additionally that Ω is bounded and ν is finite. Then*

$$L^2(\Omega, \nu) = \bigoplus_{m=1}^{\infty} \mathcal{A}_{(m)}^2(\Omega, \nu). \tag{1.33}$$

Proof. This is a simple consequence of three facts: 1) the continuous functions with compact supports form a dense subset of $L^2(\Omega, \nu)$; 2) by the Stone–Weierstrass theorem, every continuous function on the closure of Ω can be uniformly approximated by polynomials in $z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n}$; and 3) the norm of $L^2(\Omega, \nu)$ can be estimated from above by a constant multiple of the maximum-norm. □

In the one-dimensional case, the "true-m-analytic" spaces $\mathcal{A}^2_{(m)}$ were studied by Ramazanov [63] and Vasilevski [80–82]. According to [81] (see also another proof in [57]), the decomposition (1.33) holds for the poly-Fock space $\mathcal{A}^2_m(\mathbb{C}, \mathrm{e}^{-|z|^2}\,\mathrm{d}\mu)$. On the other hand, if Ω is the upper halfplane $\mathbb{H}_1$ with the Lebesgue measure, then $L^2(\mathbb{H}_1)$ decomposes into the orthogonal sum of the spaces $\mathcal{A}^2_{(m)}(\mathbb{H}_1)$ and their conjugates (see [80] or [82, Theorem 3.3.5]), and (1.33) fails. It is natural to ask if Proposition 1.3.7 remains true if $\nu(\Omega) < +\infty$, without assuming Ω to be bounded.

1.4 Pushforward reproducing kernel

In this section we show how to transform a RK using a weighted change of variables. First, we deal with abstract positive kernels [5], then we consider reproducing kernels in Hilbert spaces.

Let X be a non-empty set. We denote by $\mathbb{C}^X$ the complex vector space of all functions $X \to \mathbb{C}$ with pointwise operations. A family $(K_x)_{x\in X}$ with values in $\mathbb{C}^X$ is called a *positive kernel* on X if for every m in $\mathbb{N}$, every $x_1, \ldots, x_m$ in X and every $\alpha_1, \ldots, \alpha_m$ in $\mathbb{C}$,

$$\sum_{r,s=1}^{m} \alpha_r \overline{\alpha_s} K_{x_r}(x_s) \geq 0.$$

Proposition 1.4.1. *Let X, Y be non-empty sets, $\psi\colon Y \to X$ and $J\colon Y \to \mathbb{C}$ be some functions, and $(K_x)_{x\in X}$ be a positive kernel on X. Then the family $(L_u)_{u\in Y}$, defined by*

$$L_u(v) \coloneqq \overline{J(u)}\, J(v) K_{\psi(u)}(\psi(v)),$$

is a positive kernel on Y.

Proof. Let $m \in \mathbb{N}$, $u_1, \ldots, u_m \in Y$, $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$. For every s in $\{1, \ldots, m\}$ put $x_s \coloneqq \psi(u_s)$ and $\beta_s \coloneqq \overline{J(u_s)}\, \alpha_s$. Then

$$\sum_{r,s=1}^{m} \alpha_r \overline{\alpha_s} L_{u_r}(u_s) = \sum_{r,s=1}^{m} \beta_r \overline{\beta_s} K_{x_r}(x_s) \geq 0. \qquad \square$$

Let X be a non-empty set. We say that H is a Hilbert space of functions on X if H is a vector subspace of $\mathbb{C}^X$, provided with an inner product and complete with respect to the corresponding norm. Furthermore, if $x \in X$, $\mathcal{K} \in H$ and $\langle f, \mathcal{K}\rangle = f(x)$ for every f in H, then we say that $\mathcal{K}$ is a reproducing kernel of H at the point x. In case of existence, this function is unique.

Proposition 1.4.2. *Let X, Y be non-empty sets, $\psi\colon Y \to X$ and $J\colon Y \to \mathbb{C}$ be some functions, H_1 be a Hilbert space of functions over X, H_2 be a Hilbert space of functions over Y, and*

$$(Uf)(z) \coloneqq J(z) f(\psi(z))$$

be a well-defined unitary operator mapping H_1 onto H_2. Suppose that $u \in Y$ and $\mathcal{K}$ be the reproducing kernel of H_1 at the point $\psi(u)$. Then the function $\mathcal{L}\colon Y \to \mathbb{C}$, defined by the following rule, is the reproducing kernel of H_2 at the point u:

$$\mathcal{L}(v) \coloneqq \overline{J(u)}\, J(v)\, \mathcal{K}(\psi(v)).$$

Proof. Let $g \in H_2$ and $f = U^{-1} g$. Then

$$g(u) = J(u) f(\psi(u)) = J(u) \langle f, \mathcal{K}\rangle_{H_1} = \langle g, \overline{J(u)}\, U\mathcal{K}\rangle_{H_2}.$$

Defining $\mathcal{L}$ by $\mathcal{L}(v) = \overline{J(u)}(U\mathcal{K})(v) = \overline{J(u)} J(v) \mathcal{K}(\psi(v))$, we get the RK of H_2 at u. □

Proposition 1.4.3. *Let X, Y be non-empty sets, $\psi\colon Y \to X$ and $J\colon Y \to \mathbb{C}$ be some functions, H_1 be a Hilbert space of functions over X with reproducing kernel $(K_x)_{x \in X}$, H_2 be a Hilbert space of functions over Y, and*

$$(Uf)(z) \coloneqq J(z) f(\psi(z))$$

be a well-defined unitary operator mapping H_1 onto H_2. Then H_2 is a RKHS, and its reproducing kernel $(L_u)_{u \in Y}$ is given by

$$L_u(v) = \overline{J(u)}\, J(v)\, K_{\psi(u)}(\psi(v)).$$

Proof. Apply Proposition 1.4.2 at every point u of Y. □

As a simple application of this scheme, let us express the Berezin transform in H_2 via the Berezin transform in H_1. Given a Hilbert space H, we denote by $\mathcal{B}(H)$ the C*-algebra of all bounded linear operators acting in H. Given a set X, we denote by $B(X)$ the Banach space of all bounded functions on X, with the supremum norm. If H is a RKHS over X and its RK satisfies $\|K_x\|_H \neq 0$ for every x in X, then the *Berezin transform* $\operatorname{Ber}_H \colon \mathcal{B}(H) \to B(X)$ is defined by

$$\operatorname{Ber}_H(A)(x) := \frac{\langle AK_x, K_x\rangle_H}{\langle K_x, K_x\rangle_H} \qquad (A \in \mathcal{B}(H),\ x \in X).$$

Proposition 1.4.4. *In the conditions of Proposition 1.4.3, suppose that* $\|K_x\|_{H_1} \neq 0$ *for every* x *in* X *and* $J(u) \neq 0$ *for every* u *in* Y. *Then*

$$\operatorname{Ber}_{H_2}(A)(u) = \operatorname{Ber}_{H_1}(U^*AU)(\psi(u)) \qquad (A \in \mathcal{B}(H_2),\ u \in Y).$$

Proof. As we have seen in Proposition 1.4.2, $L_u(v) = \overline{J(u)}(UK_{\psi(u)})(v)$. Therefore,

$$\begin{aligned}\operatorname{Ber}_{H_2}(A)(u) &= \frac{\langle AL_u, L_u\rangle_{H_2}}{\|L_u\|^2} = \frac{|J(u)|^2\langle AUK_{\psi(u)}, UK_{\psi(u)}\rangle_{H_2}}{|J(u)|^2\|UK_{\psi(u)}\|^2} \\ &= \frac{\langle U^*AUK_{\psi(u)}, K_{\psi(u)}\rangle_{H_1}}{\|K_{\psi(u)}\|^2} = \operatorname{Ber}_{H_1}(U^*AU)(\psi(u)).\end{aligned}$$ □

Corollary 1.4.5. *In the conditions of Proposition 1.4.4, suppose that* Ber_{H_1} *is injective. Then* Ber_{H_2} *is also injective. Moreover, if* ψ *is a bijection, than the injectivity of* Ber_{H_1} *is equivalent to the injectivity of* Ber_{H_2}.

1.5 Reproducing kernel on the unit ball

In this section we consider the domain $\Omega = \mathbb{B}_n$ with the standard radial measure μ_α, given by (1.27). Using the weighted mean value property and appropriate unitary operators, we compute the RK of $\mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha)$.

On the unit ball biholomorphisms

For a fixed a in $\mathbb{B}_n \setminus \{0\}$, we denote by φ_a the function $\mathbb{B}_n \to \mathbb{B}_n$, defined by

$$\varphi_a(z) := \frac{a - \frac{\langle z,a\rangle}{\langle a,a\rangle}\, a - \sqrt{1-|a|^2}\left(z - \frac{\langle z,a\rangle}{\langle a,a\rangle} a\right)}{1 - \langle z, a\rangle}. \tag{1.34}$$

For $a = 0$, $\varphi_a(z) := z$. It is well known [70, Theorem 2.2.2] that for every a in $\mathbb{B}_n$, φ_a is a biholomorphism of $\mathbb{B}_n$, $\varphi_a(\varphi_a(z)) = z$ for every z in $\mathbb{B}_n$, $\varphi_a(0) = a$, $\varphi_a(a) = 0$, and

$$1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{(1 - \langle a, a\rangle)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)}. \tag{1.35}$$

Here are particular cases of (1.35), with $w = z$ and $w = 0$, respectively:

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle|^2}, \tag{1.36}$$

$$1 - \langle \varphi_a(z), a\rangle = \frac{1 - |a|^2}{1 - \langle z, a\rangle}. \tag{1.37}$$

The real Jacobian of φ_a is [87, Lemma 1.7]

$$(J_{\mathbb{R}}\varphi_a)(z) = \left(\frac{1 - |a|^2}{|1 - \langle z, a\rangle|^2}\right)^{n+1}. \tag{1.38}$$

We denote by $\rho_{\mathbb{B}_n}(z, w)$ the expression $|\varphi_z(w)|$, known as the *pseudohyperbolic distance* between z and w, see [87, Corollary 1.22] or [18]. Formula (1.36) provides a simple recipe to compute $\rho_{\mathbb{B}_n}(z, w)$.

A factor to preserve the polyanalyticity

Definition 1.5.1. Given a in $\mathbb{B}_n$, we define $p_{m,a} \colon \mathbb{B}_n \to \mathbb{C}$ by

$$p_{m,a}(z) := \left(\frac{1 - \langle a, z\rangle}{1 - \langle z, a\rangle}\right)^{m-1}.$$

In the one-dimensional case, the function $p_{m,a}$ was introduced and studied by Pessoa [59]. As it is shown in the proof of Lemma 1.5.3, the main purpose of $p_{m,a}$ is to eliminate the denominators in the multi-powers of $\overline{\varphi_a(z)}$.

Lemma 1.5.2. *For every* a, z *in* $\mathbb{B}_n$,

$$|p_{m,a}(z)| = 1, \tag{1.39}$$

$$p_{m,a}(\varphi_a(z))p_{m,a}(z) = 1. \tag{1.40}$$

Proof. Formula (1.39) follows directly from the definition of $p_{m,a}$. Identity (1.40) is easy to verify using (1.37). □

Lemma 1.5.3. *Let* $a \in \mathbb{B}_n$ *and* $f \in \mathcal{A}_m(\mathbb{B}_n)$. *Then* $(f \circ \varphi_a) \cdot p_{m,a} \in \mathcal{A}_m(\mathbb{B}_n)$.

Proof. Let f be of the form (1.4). Denote by $N_a(z)$ the numerator of (1.34); it is a polynomial of degree 1 in $z_1, \ldots, z_n$. Then,

$$\begin{aligned} f(\varphi_a(z))p_{m,\alpha}(z) &= \left(\frac{1 - \langle a, z\rangle}{1 - \langle z, a\rangle}\right)^{m-1} \sum_{|j|<m} h_j(\varphi_a(z)) \frac{\overline{N_a(z)}^j}{(1 - \langle a, z\rangle)^{|j|}} \\ &= \sum_{|j|<m} \frac{h_j(\varphi_a(z))}{(1 - \langle z, a\rangle)^{m-1}} \overline{N_a(z)}^j (1 - \langle a, z\rangle)^{m-1-|j|}. \end{aligned}$$

The quotients in the last sum are analytic functions of z. The multi-power $\overline{N_a(z)}^j$ is a polynomial in $\overline{z_1}, \ldots, \overline{z_n}$ of total degree $|j|$, and the expression $(1 - \langle a, z\rangle)^{m-1-|j|}$ is a polynomial in $\overline{z_1}, \ldots, \overline{z_n}$ of total degree $m - 1 - |j|$. Therefore, the whole sum is a polynomial in $\overline{z_1}, \ldots, \overline{z_m}$ of total degree at most $m-1$, with some analytic coefficients. □

A factor to preserve the norm

Remark 1.5.4. In the upcoming formula for $g_{\alpha,a}$ and in some other formulas, we work with (non necesarily integer) powers of complex numbers. Given t in $\mathbb{C} \setminus \{0\}$ and β in $\mathbb{C}$, we define t^β as $\exp(\beta \log(t))$, where $\log(t) = \log_{\mathbb{R}} |t| + \mathrm{i} \arg(t)$, $\log_{\mathbb{R}} |t|$ is the real logarithm of $|t|$, and $\arg(t)$ is the principal argument of t, belonging to $(-\pi, \pi]$.

Given a in $\mathbb{B}_n$, we denote by $g_{\alpha,a}$ the following function $\mathbb{B}_n \to \mathbb{C}$:

$$g_{\alpha,a}(z) \coloneqq \frac{(1 - |a|^2)^{\frac{n+1+\alpha}{2}}}{(1 - \langle z, a\rangle)^{n+1+\alpha}}. \tag{1.41}$$

This function and their properties stated below appear in Vukotić [83]. See also [87, Proposition 1.13] or [22, formula (2.4)]. By (1.37),

$$g_{\alpha,a}(\varphi_a(z))g_{\alpha,a}(z) = 1. \tag{1.42}$$

By (1.42), (1.36), and (1.38),

$$|g_{\alpha,a}(\varphi_a(w))|^2(J_{\mathbb{R}}\varphi_a)(w)(1 - |\varphi_a(w)|^2)^\alpha = (1 - |w|^2)^\alpha. \tag{1.43}$$

Using (1.43) and the change of variables $w = \varphi_a(z)$, one easily shows that for every f in $f \in L^2(\mathbb{B}_n, \mu_\alpha)$,

$$\|(f \circ \varphi_a) \cdot g_{\alpha,a}\|_{L^2(\mathbb{B}_n,\mu_\alpha)} = \|f\|_{L^2(\mathbb{B}_n,\mu_\alpha)}. \tag{1.44}$$

A weighted shift operator preserving $\mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha)$

Definition 1.5.5. Given a in $\mathbb{B}_n$, we define $U_a\colon \mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha) \to \mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha)$ by

$$(U_a f)(z) \coloneqq f(\varphi_a(z))p_{m,a}(z)g_{\alpha,a}(z).$$

Proposition 1.5.6. *Let $a \in \mathbb{B}^n$. Then U_a is a unitary operator in $\mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha)$, and $U_a^2 = I$.*

Proof. Given f in $\mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha)$, Lemma 1.5.3 assures that $U_a f \in \mathcal{A}_m(\mathbb{B}_n)$. Formula (1.44), combined with (1.39), implies that U_a is an isometry. Finally, (1.40) and (1.42) yield the involutive property $U_a^2 = I$. □

Computation of the RK on the unit ball

Recall that $R^{(\alpha,\beta)}_m$ is defined by (1.19) and $\rho_{\mathbb{B}_n}(z, w)$ denotes $|\varphi_z(w)|$.

Theorem 1.5.7. *Let $n, m \in \mathbb{N}$ and $\alpha > -1$. Then for every z in $\mathbb{B}_n$, the following function K_z is the reproducing kernel of $\mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha)$ at the point z:*

$$K_z(w) = \frac{(1 - \langle z, w\rangle)^{m-1}}{(1 - \langle w, z\rangle)^{n+m+\alpha}}\, R^{(\alpha,n-1)}_{m-1}(\rho_{\mathbb{B}_n}(z, w)^2). \tag{1.45}$$

Proof. For $z=0$, the function defined by the right-hand side of (1.45) simplifies to

$$K_0(w)=R_{m-1}^{(\alpha,n-1)}(|w|^2).$$

Theorem 1.3.3 means that K_0 is indeed the RK at the point 0. Now, for z in $\mathbb{B}_n$, we apply Proposition 1.4.2 with $H_1=H_2=\mathcal{A}_m^2(\mathbb{B}_n)$, φ_z instead of ψ, and $J_z:=p_{m,z}g_{\alpha,z}$. Since $\varphi_z(z)=0$, we obtain

$$K_z(w)=\overline{J_z(z)}J_z(w)K_0(\varphi_z(w)). \tag{1.46}$$

It is easy to see that $J_z(z)=(1-|z|^2)^{-\frac{n+1+\alpha}{2}}$. So, after some simplifications, we arrive at (1.45):

$$\begin{aligned}K_z(w)&=\frac{1}{(1-|z|^2)^{\frac{n+1+\alpha}{2}}}\left(\frac{1-\langle z,w\rangle}{1-\langle w,z\rangle}\right)^{m-1}\frac{(1-|z|^2)^{\frac{n+1+\alpha}{2}}}{(1-\langle w,z\rangle)^{n+1+\alpha}}R_{m-1}^{(\alpha,n-1)}(|\varphi_z(w)|^2)\\&=\frac{(1-\langle z,w\rangle)^{m-1}}{(1-\langle w,z\rangle)^{n+m+\alpha}}\,R_{m-1}^{(\alpha,n-1)}(\rho_{\mathbb{B}_n}(z,w)^2).\end{aligned}$$ □

Formula (1.45) is a natural generalization of the previous results: [51, 58] for $n=1$ and $\alpha=0$, [35] for $n=1$ and $\alpha>-1$, and [87, Theorem 2.7] for $m=1$.

Corollary 1.5.8. *Let* $n,m\in\mathbb{N}$ *and* $\alpha>-1$. *Then for every* z *in* $\mathbb{B}_n$*,*

$$\|K_z\|^2_{\mathcal{A}_m^2(\mathbb{B}_n,\mu_\alpha)}=K_z(z)=\binom{n+m-1}{n}\frac{\mathrm{B}(\alpha+1,n)}{\mathrm{B}(\alpha+m,n)}\,\frac{1}{(1-|z|^2)^{n+\alpha+1}}. \tag{1.47}$$

Remark 1.5.9. We get other formulas, equivalent to (1.45), using (1.36) and (1.12):

$$K_z(w)=\frac{(1-\langle z,w\rangle)^{m-1}}{(1-\langle w,z\rangle)^{n+m+\alpha}}\,\frac{(-1)^{m-1}\,\mathrm{B}(\alpha+1,n)}{\mathrm{B}(\alpha+m,n)}P_{m-1}^{(\alpha,n)}(2\rho_{\mathbb{B}_n}(z,w)^2-1) \tag{1.48}$$

$$=\frac{(1-\langle z,w\rangle)^{m-1}}{(1-\langle w,z\rangle)^{n+m+\alpha}}\,\frac{(-1)^{m-1}\,\mathrm{B}(\alpha+1,n)}{\mathrm{B}(\alpha+m,n)}\,P_{m-1}^{(\alpha,n)}\left(1-\frac{2(1-|z|^2)(1-|w|^2)}{|1-\langle w,z\rangle|^2}\right) \tag{1.49}$$

$$\begin{aligned}&=\frac{(1-\langle z,w\rangle)^{m-1}}{(1-\langle w,z\rangle)^{n+m+\alpha}}\,\frac{(-1)^{m-1}\,\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)\,(m-1)!}\times\\&\qquad\times\sum_{s=0}^{m-1}(-1)^s\binom{m-1}{s}\frac{\Gamma(\alpha+m+n+s)}{\Gamma(\alpha+s+1)}\left(\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle w,z\rangle|^2}\right)^s.\end{aligned} \tag{1.50}$$

Remark 1.5.10. If M is a unitary n by n matrix, then the RK computed in Theorem 1.5.7 is invariant under the simultaneous action of M in both arguments:

$$K_{Mz}(Mw)=K_z(w)\qquad(z,w\in\mathbb{B}^n).$$

Therefore, by [57, Proposition 4.1], the space $\mathcal{A}^2_m(\mathbb{B}^n, \mu_\alpha)$ is invariant under the action of the rotation operator

$$(R_M f)(z) := f(M^{-1}z).$$

This follows also directly from Proposition 1.2.7. Notice that the unitary matrices include permutation matrices, diagonal matrices with unimodular complex entries, and real rotations in any two coordinates.

Remark 1.5.11. Generalizing ideas of this section, it is possible to construct a unitary weighted shift operator U_φ acting in $\mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha)$, for every biholomorphism φ of $\mathbb{B}_n$.

The next result was published by Engliš [21, Section 2] for RKHS of harmonic functions. We reformulate it for our situation and recall the idea of the proof.

Proposition 1.5.12. *Let $H = \mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha)$, with $n \geq 1$ and $m \geq 2$. Then Ber_H is not injective.*

Proof. The functions $f(z) = z_1$ and $g(z) = \overline{z_1}$ are linearly independent elements of H. Therefore, the operator $Sh = \langle h, f\rangle_H f - \langle h, g\rangle_H g$ is not zero, but the Berezin transform maps it into the zero function. □

1.6 Reproducing kernel on the Siegel domain

Let $n, m \in \mathbb{N}$ and $\alpha > -1$. In this section we compute the RK of the space $\mathcal{A}^2_m(\mathbb{H}_n, \nu_\alpha)$, where $\mathbb{H}_n$ is the standard Siegel domain (which can be considered as an unbounded realization of the unit ball) and ν_α is a usual weighted measure on $\mathbb{H}_n$:

$$\mathbb{H}_n := \{\xi = (\xi', \xi_n) \in \mathbb{C}^{n-1} \times \mathbb{C}\colon\ \operatorname{Im}(\xi_n) - |\xi'|^2 > 0\}, \tag{1.51}$$

$$\mathrm{d}\nu_\alpha(\xi) := \frac{c_\alpha}{4}(\operatorname{Im}(\xi_n) - |\xi'|^2)^\alpha\, \mathrm{d}\mu(\xi). \tag{1.52}$$

For this purpose, we will construct a unitary operator $V\colon \mathcal{A}^2_m(\mathbb{B}_n, \mu_\alpha) \to \mathcal{A}^2_m(\mathbb{H}_n, \nu_\alpha)$, using some recipes from [62, Section 2] and an analog of the Pessoa factor which helps to preserve the polyanalyticity.

Cayley transform

Following [62, Section 2], we employ the biholomorphism $\omega\colon \mathbb{B}_n \to \mathbb{H}_n$ defined by

$$\omega(z) \coloneqq \left(\mathrm{i}\,\frac{z_1}{1+z_n},\ldots,\mathrm{i}\,\frac{z_{n-1}}{1+z_n},\mathrm{i}\,\frac{1-z_n}{1+z_n}\right).$$

Its inverse $\psi\colon \mathbb{H}_n \to \mathbb{B}_n$ is given by

$$\psi(\xi) \coloneqq \left(-\frac{2\,\mathrm{i}\,\xi_1}{1-\mathrm{i}\,\xi_n},\ldots,-\frac{2\,\mathrm{i}\,\xi_{n-1}}{1-\mathrm{i}\,\xi_n},\frac{1+\mathrm{i}\,\xi_n}{1-\mathrm{i}\,\xi_n}\right).$$

By a direct computation,

$$1-\langle\psi(\xi),\psi(\eta)\rangle = 4\,\frac{\frac{\xi_n-\overline{\eta_n}}{2\,\mathrm{i}}-\langle\xi',\eta'\rangle}{(1-\mathrm{i}\,\xi_n)(1+\mathrm{i}\,\overline{\eta_n})}. \tag{1.53}$$

In particular,

$$1-|\psi(\xi)|^2 = 4\,\frac{\operatorname{Im}(\xi_n)-|\xi'|^2}{|1-\mathrm{i}\,\xi_n|^2}. \tag{1.54}$$

The complex Jacobian matrices of ψ and ω are triangular, and their determinants are easy to compute:

$$(J_{\mathbb{C}}\omega)(z) = -\frac{2\,\mathrm{i}^n}{(1+z_n)^{n+1}},\qquad (J_{\mathbb{C}}\psi)(\xi) = -\frac{(-2\,\mathrm{i})^n}{(1-\mathrm{i}\,\xi_n)^{n+1}}. \tag{1.55}$$

Therefore, the real Jacobians of ω and ψ are

$$(J_{\mathbb{R}}\omega)(z) = \frac{4}{|1+z_n|^{2(n+1)}},\qquad (J_{\mathbb{R}}\psi)(\xi) = \frac{4^n}{|1-\mathrm{i}\,\xi_n|^{2(n+1)}}. \tag{1.56}$$

Pseudohyperbolic distance on the Siegel domain

Definition 1.6.1. Define a distance on $\mathbb{H}_n$ by

$$\rho_{\mathbb{H}_n}(\xi,\eta) \coloneqq \rho_{\mathbb{B}_n}(\psi(\xi),\psi(\eta)). \tag{1.57}$$

The following proposition provides an efficient formula to compute $\rho_{\mathbb{H}_n}(\xi,\eta)$.

Proposition 1.6.2. *For every ξ,η in $\mathbb{H}_n$,*

$$1-\rho_{\mathbb{H}_n}(\xi,\eta)^2=\frac{(\operatorname{Im}(\xi_n)-|\xi'|^2)(\operatorname{Im}(\eta_n)-|\eta'|^2)}{\left|\dfrac{\xi_n-\overline{\eta_n}}{2\,\mathrm{i}}-\langle\xi',\eta'\rangle\right|^2}. \tag{1.58}$$

Proof. Substitute $\psi(\xi)$ and $\psi(\eta)$ instead of z and w in (1.36):

$$1-\rho_{\mathbb{H}_n}(\xi,\eta)^2=1-\rho_{\mathbb{B}_n}(\psi(\xi),\psi(\eta))^2=\frac{(1-|\psi(\xi)|^2)(1-|\psi(\eta)|^2)}{|1-\langle\psi(\xi),\psi(\eta)\rangle|^2}.$$

Applying (1.53) and (1.54) we obtain (1.58). □

Remark 1.6.3. For $n=1$, formulas (1.57) and (1.58) simplify to

$$\rho_{\mathbb{H}_1}(\xi,\eta)=\frac{|\xi-\eta|}{|\overline{\xi}-\eta|},\qquad 1-\rho_{\mathbb{H}_1}(\xi,\eta)^2=\frac{4\operatorname{Im}(\xi)\operatorname{Im}(\eta)}{|\overline{\xi}-\eta|^2}. \tag{1.59}$$

A factor to preserve the norm when passing from $\mathbb{H}_n$ to $\mathbb{B}_n$

The material of this subsection is equivalent to some computations from [62, Section 2]. Define $h_\alpha\colon\mathbb{H}_n\to\mathbb{C}$ by

$$h_\alpha(\xi)\coloneqq\left(\frac{2}{1-\mathrm{i}\,\xi_n}\right)^{n+\alpha+1}. \tag{1.60}$$

Lemma 1.6.4. *For every ξ in $\mathbb{H}_n$,*

$$|h_\alpha(\xi)|^2=\frac{4(1-|\psi(\xi)|^2)^\alpha(J_{\mathbb{R}}\psi)(\xi)}{(\operatorname{Im}(\xi_n)-|\xi'|^2)^\alpha}. \tag{1.61}$$

For any z in $\mathbb{B}_n$,

$$\frac{1}{4}|h_\alpha(\omega(z))|^2\left(\frac{1-|z|^2}{|1+z_n|^2}\right)^\alpha(J_{\mathbb{R}}\omega)(z)=(1-|z|^2)^\alpha. \tag{1.62}$$

Proof. Formula (1.61) is obtained by (1.54) and (1.56). Then (1.62) follows from (1.61) and the well-known formula for the Jacobian of the inverse function. □

Lemma 1.6.5. *Let $u\in L^2(\mathbb{B}_n,\mu_\alpha)$. Then*

$$\|(u\circ\psi)\cdot h_\alpha\|_{L^2(\mathbb{H}_n,\nu_\alpha)}=\|u\|_{L^2(\mathbb{B}_n,\mu_\alpha)}. \tag{1.63}$$

Proof. First, using (1.54), we observe that the change of variable $z = \psi(\zeta)$ transforms the weight function in the following way:

$$(\mathrm{Im}(\zeta_n) - |\zeta'|^2)^\alpha = \left(\frac{1-|z|^2}{|1+z_n|^2}\right)^\alpha.$$

Apply this change of variables in the integral:

$$\begin{aligned}\|(u\circ\psi)\cdot h_\alpha\|^2_{L^2(\mathbb{H}_n,\nu_\alpha)} &= \frac{c_\alpha}{4}\int_{\mathbb{H}^n} |u(\psi(\zeta))|^2 |h_\alpha(\zeta)|^2 (\mathrm{Im}(\zeta_n)-|\zeta'|^2)^\alpha \mathrm{d}\mu(\zeta)\\ &= c_\alpha \int_{\mathbb{B}^n} |u(z)|^2 |h_\alpha(\omega(z))|^2 \left(\frac{1-|z|^2}{|1+z_n|^2}\right)^\alpha (J_{\mathbb{R}}\omega)(z)\,\mathrm{d}\mu(z)\\ &= c_\alpha \int_{\mathbb{B}^n} |u(z)|^2 (1-|z|^2)^\alpha \mathrm{d}\mu(z) = \|u\|^2_{L^2(\mathbb{B}_n,\mu_\alpha)}.\end{aligned}$$

□

A factor to preserve the polyanalyticity when passing from the unit ball to the Siegel domain

Definition 1.6.6. Define $q_m \colon \mathbb{H}_n \to \mathbb{C}$,

$$q_m(\xi) \coloneqq \left(\frac{1+\mathrm{i}\,\overline{\xi_n}}{1-\mathrm{i}\,\xi_n}\right)^{m-1}.$$

Lemma 1.6.7. *Let $f \in \mathcal{A}_m(\mathbb{B}_n)$. Then $(f\circ\psi)\cdot q_m \in \mathcal{A}_m(\mathbb{H}_n)$.*

Proof. This proof is similar to the proof of Lemma 1.5.3. The main idea is that the factor $(1+\mathrm{i}\,\overline{\xi_n})^{m-1}$, appearing in the numerator of $q_m(\xi)$, cancels the denominators of the expressions $\overline{\psi(\xi)}^{\,j}$, where $|j| < m$. We represent f in the form (1.4), compose with ψ, and multiply by q_m:

$$\begin{aligned}u(\xi) \coloneqq (f\circ\psi)(\xi) q_m(\xi) &= \sum_{|j|\le m} h_j(\psi(\xi)) \left(\prod_{s=1}^{n-1} \frac{(2\,\mathrm{i}\,\overline{\xi_s})^{j_s}}{(1+\mathrm{i}\,\overline{\xi_n})^{j_s}}\right) \frac{(1-\mathrm{i}\,\overline{\xi_n})^{j_n}}{(1+\mathrm{i}\,\overline{\xi_n})^{j_n}} \frac{(1+\mathrm{i}\,\overline{\xi_n})^{m-1}}{(1-\mathrm{i}\,\xi_n)^{m-1}}\\ &= \sum_{|j|\le m} \frac{h_j(\psi(\xi))}{(1-\mathrm{i}\,\xi_n)^{m-1}} \left(\prod_{s=1}^{n-1} (2\,\mathrm{i}\,\overline{\xi_s})^{j_s}\right) (1-\mathrm{i}\,\overline{\xi_n})^{j_n} (1+\mathrm{i}\,\overline{\xi_n})^{m-|j|-1}.\end{aligned}$$

For each j, the corresponding summand is the product of an analytic function by a polynomial in $\overline{\xi_1},\dots,\overline{\xi_n}$ of total degree $m-1$. □

Remark 1.6.8. Another way to prove Lemma 1.6.7, computing $\overline{D}^j u$, seems to be more complicated. We will show it only for $n=2$ and $m=2$. In this case,

$$u(\xi)=\frac{1+\mathrm{i}\,\overline{\xi_2}}{1-\mathrm{i}\,\xi_2}\,f\left(-\frac{2\,\mathrm{i}\,\xi_1}{1-\mathrm{i}\,\xi_2},\frac{1+\mathrm{i}\,\xi_2}{1-\mathrm{i}\,\xi_2}\right),$$

By the well-known chain rule and product rule for Wirtinger derivatives,

$$\begin{aligned}
(\overline{D}^{(1,0)}u)(\xi)&=\frac{2\,\mathrm{i}(\overline{D}^{(1,0)}f)(\psi(\xi))}{1-\mathrm{i}\,\xi_2},\\
(\overline{D}^{(0,1)}u)(\xi)&=\frac{\mathrm{i}\,f(\psi(\xi))}{1-\mathrm{i}\,\xi_2}+\frac{2\left(\left(\overline{\xi_1}\,\overline{D}^{(1,0)}-\mathrm{i}\,\overline{D}^{(0,1)}\right)f\right)(\psi(\xi))}{(1-\mathrm{i}\,\xi_2)(1+\mathrm{i}\,\overline{\xi_2})},\\
(\overline{D}^{(2,0)}u)(\xi)&=\frac{-4(\overline{D}^{(2,0)}f)(\psi(\xi))}{(1-\mathrm{i}\,\xi_2)(1+\mathrm{i}\,\overline{\xi_2})},\\
(\overline{D}^{(1,1)}u)(\xi)&=\frac{4\left(\left(\mathrm{i}\,\overline{\xi_1}\,\overline{D}^{(2,0)}+\overline{D}^{(1,1)}\right)f\right)(\psi(\xi))}{(1-\mathrm{i}\,\xi_2)(1+\mathrm{i}\,\overline{\xi_2})^2},\\
(\overline{D}^{(0,2)}u)(\xi)&=\frac{4\left(\left(\overline{\xi_2}^{\,2}\,\overline{D}^{(2,0)}-2\,\mathrm{i}\,\overline{\xi_1}\,\overline{D}^{(1,1)}-\overline{D}^{(0,2)}\right)f\right)(\psi(\xi))}{(1-\mathrm{i}\,\xi_2)(1+\mathrm{i}\,\overline{\xi_2})^3}.
\end{aligned}$$

Since $f\in\mathcal{A}_2(\mathbb{B}_n)$, we conclude that $u\in\mathcal{A}_2(\mathbb{H}_n)$.

A weighted change of variables which unitarily maps $\mathcal{A}^2_m(\mathbb{B}_n,\mu_\alpha)$ onto $\mathcal{A}^2_m(\mathbb{H}_n,\nu_\alpha)$

Definition 1.6.9. Define $V\colon\mathcal{A}^2_m(\mathbb{B}_n,\mu_\alpha)\to\mathcal{A}^2_m(\mathbb{H}_n,\nu_\alpha)$ by $Vu:=(u\circ\psi)\cdot h_\alpha\cdot q_m$, i.e.,

$$(Vu)(\xi):=u(\psi(\xi))h_\alpha(\xi)q_m(\xi).$$

Proposition 1.6.10. *V is a well-defined unitary operator $\mathcal{A}^2_m(\mathbb{B}_n,\mu_\alpha)\to\mathcal{A}^2_m(\mathbb{H}_n,\nu_\alpha)$.*

Proof. Lemma 1.6.7 assures that $Vu\in\mathcal{A}_m(\mathbb{H}_n)$ for every u in $\mathcal{A}^2_m(\mathbb{B}_n,\mu_\alpha)$. Lemma 1.6.5, combined with the identity $|q_m(\xi)|=1$, provides the isometric property of V. It is easy to verify that the adjoint operator V^* acts by

$$(V^*f)(z)=\frac{f(\omega(z))}{h_\alpha(\omega(z))q_m(\omega(z))},\tag{1.64}$$

and that V^* is the inverse operator to V. □

Computation of the RK on the Siegel domain

We define t^β via the principal argument of t, see Remark 1.5.4. The formulas $(tu)^\beta = t^\beta u^\beta$ and $(t/u)^\beta$ are not always true. Let us recall some sufficient conditions for these formulas to be true.

Lemma 1.6.11. *Let $t, u \in \mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$.*

1. *If $\operatorname{Re}(t) > 0$ and $\operatorname{Re}(u) > 0$, then $(tu)^\beta = t^\beta u^\beta$.*
2. *If $\operatorname{Re}(t) > 0$ and $\operatorname{Re}(t/u) > 0$, then $(t/u)^\beta = t^\beta / u^\beta$.*

Proof. 1. The assumptions on t and u imply that $\arg(tu) = \arg(t) + \arg(u)$.
2. Follows from part 1 applied to t and u/t. □

Lemma 1.6.12. *Let $\xi, \eta \in \mathbb{H}_n$ and $\beta \geq 0$. Then*

$$(1 - \langle \psi(\xi), \psi(\eta) \rangle)^\beta = \frac{4^\beta \left(\frac{\xi_n - \overline{\eta_n}}{2\,\mathrm{i}} - \langle \xi', \eta' \rangle \right)^\beta}{(1 - \mathrm{i}\,\xi_n)^\beta \, (1 + \mathrm{i}\,\overline{\eta_n})^\beta}.$$

Proof. Due to (1.53), $1 - \langle \psi(\xi), \psi(\eta) \rangle = t/(uv)$, where

$$t \coloneqq 4 \left(\frac{\xi_n - \overline{\eta_n}}{2\,\mathrm{i}} - \langle \xi', \eta' \rangle \right), \qquad u \coloneqq 1 - \mathrm{i}\,\xi_n, \qquad v \coloneqq 1 + \mathrm{i}\,\overline{\eta_n}.$$

Since $|\psi(\xi)| < 1$ and $|\psi(\eta)| < 1$, we obtain

$$\operatorname{Re}(t/(uv)) = \operatorname{Re}(1 - \langle \psi(\xi), \psi(\eta) \rangle) > 0.$$

Furthermore, $\operatorname{Re}(u) = 1 + \operatorname{Im}(\xi_n) > 0$ and $\operatorname{Re}(v) = 1 + \operatorname{Im}(\eta_n) > 0$. So, by Lemma 1.6.11,

$$\left(\frac{t}{uv} \right)^\beta = \frac{t^\beta}{(uv)^\beta} = \frac{t^\beta}{u^\beta v^\beta}.$$ □

Theorem 1.6.13. *Let $n, m \in \mathbb{N}$ and $\alpha > -1$. Then for every ξ in $\mathbb{H}_n$, the following function $\widetilde{K}_\xi$ is the reproducing kernel of $\mathcal{A}^2_m(\mathbb{H}_n, \nu_\alpha)$ at the point ξ:*

$$\widetilde{K}_\xi(\eta) = \frac{\left(\frac{\xi_n - \overline{\eta_n}}{2\,\mathrm{i}} - \langle \xi', \eta' \rangle \right)^{m-1}}{\left(\frac{\eta_n - \overline{\xi_n}}{2\,\mathrm{i}} - \langle \eta', \xi' \rangle \right)^{n+m+\alpha}} \, R^{(\alpha, n-1)}_{m-1}(\rho_{\mathbb{H}_n}(\xi, \eta)^2). \tag{1.65}$$

Proof. Due to Proposition 1.6.10, we can apply Proposition 1.4.1 with $H_1 = \mathcal{A}_m(\mathbb{B}_n, \mu_\alpha)$, $H_2 = \mathcal{A}_m(\mathbb{H}_n, \nu_\alpha)$, and $J(\xi) := h_\alpha(\xi) q_m(\xi)$. So, for every ξ in $\mathbb{H}_n$, the next function is the RK of $\mathcal{A}_m(\mathbb{H}_n, \nu_\alpha)$ associated to the point ξ:

$$\widetilde{K}_\xi(\eta) = \overline{h_\alpha(\xi) q_m(\xi)} h_\alpha(\eta) q_m(\eta) K_{\psi(\xi)}(\psi(\eta)).$$

Substitute formula (1.45) for K:

$$\widetilde{K}_\xi(\eta) = \overline{h_\alpha(\xi) q_m(\xi)} h_\alpha(\eta) q_m(\eta)\ \frac{(1 - \langle \psi(\xi), \psi(\eta)\rangle)^{m-1}}{(1 - \langle \psi(\eta), \psi(\xi)\rangle)^{n+m+\alpha}}\ R_{m-1}^{(\alpha,n-1)}(\rho_{\mathbb{H}_n}(\xi,\eta)^2).$$

Then, substitute the definitions of h_α, q_m and use Lemma 1.6.12:

$$\begin{aligned}\widetilde{K}_\xi(\eta) = R_{m-1}^{(\alpha,n-1)}(\rho_{\mathbb{H}_n}(\xi,\eta)^2)\ &\frac{2^{n+\alpha+1}(1 + \mathrm{i}\,\overline{\eta_n})^{m-1}}{(1 - \mathrm{i}\,\eta_n)^{n+m+\alpha}}\ \frac{2^{n+\alpha+1}\,(1 - \mathrm{i}\,\xi_n)^{m-1}}{(1 + \mathrm{i}\,\overline{\xi_n})^{n+m+\alpha}} \\ &\times \frac{4^{m-1}\left(\frac{\xi_n - \overline{\eta_n}}{2\,\mathrm{i}} - \langle \xi', \eta'\rangle\right)^{m-1}}{(1 - \mathrm{i}\,\xi_n)^{m-1}\,(1 + \mathrm{i}\,\overline{\eta_n})^{m-1}}\ \frac{(1 + \mathrm{i}\,\overline{\xi_n})^{n+m+\alpha}\,(1 - \mathrm{i}\,\eta_n)^{n+m+\alpha}}{4^{n+m+\alpha}\left(\frac{\eta_n - \overline{\xi_n}}{2\,\mathrm{i}} - \langle \eta', \xi'\rangle\right)^{n+m+\alpha}}.\end{aligned}$$

Simplifying this expression we obtain the right-hand side of (1.65). □

Corollary 1.6.14. *Let* $n, m \in \mathbb{N}$ *and* $\alpha > -1$. *Then for every* ξ *in* $\mathbb{H}_n$,

$$\|\widetilde{K}_\xi\|^2_{\mathcal{A}^2_m(\mathbb{H}_n,\nu_\alpha)} = \widetilde{K}_\xi(\xi) = \binom{n+m-1}{n}\,\frac{\mathrm{B}(\alpha+1,n)}{\mathrm{B}(\alpha+m,n)}\,\frac{1}{(\mathrm{Im}(\xi_n) - |\xi'|^2)^{\alpha+n+1}}. \tag{1.66}$$

Remark 1.6.15. Analogously to the case of the unit ball, using (1.58) and (1.12), we get the following formulas equivalent to (1.65):

$$\widetilde{K}_\xi(\eta) = \frac{\left(\frac{\xi_n - \overline{\eta_n}}{2\,\mathrm{i}} - \langle \xi', \eta'\rangle\right)^{m-1}}{\left(\frac{\eta_n - \overline{\xi_n}}{2\,\mathrm{i}} - \langle \eta', \xi'\rangle\right)^{n+m+\alpha}}\ \frac{(-1)^{m-1}\,\mathrm{B}(\alpha+1,n)}{\mathrm{B}(\alpha+m,n)} P_{m-1}^{(\alpha,n)}(2\rho_{\mathbb{H}_n}(\xi,\eta)^2 - 1) \tag{1.67}$$

$$\begin{aligned} &= \frac{\left(\frac{\xi_n - \overline{\eta_n}}{2\,\mathrm{i}} - \langle \xi', \eta'\rangle\right)^{m-1}}{\left(\frac{\eta_n - \overline{\xi_n}}{2\,\mathrm{i}} - \langle \eta', \xi'\rangle\right)^{n+m+\alpha}}\ \frac{(-1)^{m-1}\,\mathrm{B}(\alpha+1,n)}{\mathrm{B}(\alpha+m,n)} \times \\ &\quad \times P_{m-1}^{(\alpha,n)}\left(1 - \frac{2(\mathrm{Im}(\xi_n) - |\xi'|^2)(\mathrm{Im}(\eta_n) - |\eta'|^2)}{\left|\frac{\xi_n - \overline{\eta_n}}{2\,\mathrm{i}} - \langle \xi', \eta'\rangle\right|^2}\right) \end{aligned} \tag{1.68}$$

$$\begin{aligned} &= \frac{\left(\frac{\xi_n - \overline{\eta_n}}{2\,\mathrm{i}} - \langle \xi', \eta'\rangle\right)^{m-1}}{\left(\frac{\eta_n - \overline{\xi_n}}{2\,\mathrm{i}} - \langle \eta', \xi'\rangle\right)^{n+m+\alpha}}\ \frac{(-1)^{m-1}\,\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)\,(m-1)!} \times \\ &\quad \times \sum_{s=0}^{m-1} (-1)^s \binom{m-1}{s} \frac{\Gamma(\alpha+m+n+s)}{\Gamma(\alpha+s+1)} \left(\frac{(\mathrm{Im}(\xi_n) - |\xi'|^2)(\mathrm{Im}(\eta_n) - |\eta'|^2)}{\left|\frac{\xi_n - \overline{\eta_n}}{2\,\mathrm{i}} - \langle \xi', \eta'\rangle\right|^2}\right)^s. \end{aligned} \tag{1.69}$$

Remark 1.6.16. In the case $n = 1$, i.e., for the upper halfplane $\mathbb{H}_1$, formula (1.65) simplifies to

$$\widetilde{K}_\xi(\eta) = \frac{\left(\frac{\xi-\overline{\eta}}{2\,\mathrm{i}}\right)^{m-1}}{\left(\frac{\eta-\bar{\xi}}{2\,\mathrm{i}}\right)^{m+\alpha+1}} R_{m-1}^{(\alpha,0)}\left(\frac{|\xi-\eta|^2}{|\bar{\xi}-\eta|^2}\right) \qquad (\xi,\eta\in\mathbb{H}_1). \tag{1.70}$$

In particular, for $\alpha = 0$, this expression coincides with formula [58, Corollary 2.5] obtained by another method.

Remark 1.6.17. Generalizing ideas of this work, it is possible to associate a unitary operator (namely, a certain weighted shift) in $\mathcal{A}_m^2(\mathbb{H}_n,\nu_\alpha)$ to every biholomorphism of the Siegel domain $\mathbb{H}_n$. In particular, using (1.65), we have verified that the space $\mathcal{A}_m(\mathbb{H}_n,\nu_\alpha)$ is invariant under the unweighted changes of variables, corresponding to the quasi-parabolic, nilpotent, and quasi-nilpotent groups from [62, Section 3].

2 Translation-Invariant Operators in Reproducing Kernel Hilbert Spaces

2.1 Scope

This chapter contains the main part of this work, which is a general scheme to describe the structure of some operator algebras. This scheme combine many ideas contained in the papers by Vasilevski's et al, with the theory of W*-algebras and group-representation. We propose a more general approach to the description of commutative operator algebras acting in RKHS over "tube" type domains. Many already known results are covered by the scheme, see Section 2.9. Also a new case is proven to fit this scheme by the end of this chapter: The radial basis function kernel on the complex domain. These results can be found in the Preprint [38].

2.2 An analog of the Stone–Weierstrass theorem for subalgebras of L^∞

In this section we recall some facts about commutative W*-algebras. The main result, Theorem 2.2.2, is an analog of the classic Stone–Weierstrass theorem adapted for W*-subalgebras of $L^\infty(X,\mu)$. We use an information about W*-algebras from Dixmier [17], Sakai [74], and Takesaki [77].

Given a Hilbert space H, we denote by $\mathcal{B}(H)$ the W*-algebra of bounded linear operators acting on H and by WOT the weak operator topology in $\mathcal{B}(H)$. Given a subset $\mathcal{S}$ of $\mathcal{B}(H)$, we denote by $\mathcal{S}'$ the *centralizer* or *commutant* of $\mathcal{S}$ in $\mathcal{B}(H)$, that is the set of all bounded operators that commute with every operator in $\mathcal{S}$. Given a subset $\mathcal{S}$ of $\mathcal{B}(H)$, we denote by $W^*(\mathcal{S})$ the von Neumann algebra generated by $\mathcal{S}$. It is known that $W^*(\mathcal{S}) = (\mathcal{S}')'$.

In this section, X is a locally compact space and μ is a Radon measure whose support is X. For simplicity, we additionally suppose that X is a σ-compact metric space. We denote by τ_X or just by τ the weak-$*$ topology in $L^\infty(X,\mu)$. Recall that if $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(X,\mu)$ converging pointwise to a function b, then $a_n \xrightarrow{\tau} b$, i.e., $\int_X a_n f \,\mathrm{d}\mu \to \int_X bf\,\mathrm{d}\mu$ for every f in $L^1(X,\mu)$. Indeed, if $C < +\infty$ and $\|a_n\|_\infty \le C$ for every n, then the dominated convergence theorem can be applied with the “dominant function” $C|f|$.

Given b in $L^\infty(X,\mu)$, let $M_b\colon L^2(X,\mu) \to L^2(X,\mu)$ be the multiplication operator by b:

$$(M_b f)(x) \coloneqq b(x) f(x).$$

We denote by $\mathcal{M}_X$ the set of all such multiplication operators:

$$\mathcal{M}_X \coloneqq \{M_b\colon\ b \in L^\infty(X,\mu)\}.$$

It is well known and easy to see that $\mathcal{M}_X$ is a commutative W*-subalgebra of $\mathcal{B}(L^2(X,\mu))$. The function $b \mapsto M_b$ is an isometric isomorphism between the W*-algebras $L^\infty(X,\mu)$ and $\mathcal{M}_X$. In particular, $M_{b_1}M_{b_2} = M_{b_1 b_2} = M_{b_2}M_{b_1}$, $\|M_b\| = \|b\|_\infty$, and the spectrum of M_b is the essential range of b. The τ-convergence of a net in $L^\infty(X,\mu)$ is equivalent to the WOT-convergence of the corresponding multiplication operators. It can be shown that

$$\mathcal{M}_X' = \mathcal{M}_X. \tag{2.1}$$

The following proposition is well known. It can be proven by applying Luzin’s theorem [27, Theorem 7.10] and the Tietze extension theorem, or by using techniques of C*- and W*-algebras [77, proof of Theorem 3.1.2].

Proposition 2.2.1. *Let Y be a compact Hausdorff space with a Radon measure μ_Y. Then* $\operatorname{clos}_{\tau_Y}(C(Y)) = L^\infty(Y,\mu_Y)$.

The next result is a generalization of Proposition 2.2.1 to spaces with infinite measure. Notice that $\mathcal{A}$ is not supposed to be closed or dense in the norm topology of $C_b(X)$.

Theorem 2.2.2. *Let X be a locally compact and σ-compact metric space, μ be a Radon measure on X, and $\mathcal{A}$ be a self-adjoint unital subalgebra of $C_b(X)$ separating points of X. Then* $\operatorname{clos}_\tau(\mathcal{A}) = L^\infty(X,\mu)$.

Proof. We denote $\operatorname{clos}_\tau(\mathcal{A})$ by $\mathcal{W}$. Let $(K_n)_{n\in\mathbb{N}}$ be an increasing compact covering of X. In the steps 1–4 of the proof, Y is an arbitrary compact subset of X. We denote by μ_Y the restriction of μ, and by $\mathcal{A}_Y$ and $\mathcal{W}_Y$ the "restrictions" of the algebras $\mathcal{A}$ and $\mathcal{W}$, respectively:

$$\mathcal{A}_Y \coloneqq \{f|_Y\colon\ f \in \mathcal{A}\}, \qquad \mathcal{W}_Y \coloneqq \{f|_Y\colon\ f \in \mathcal{W}\}.$$

Step 1. $\mathcal{A}_Y$ is a self-adjoint unital subalgebra of $C(Y)$ that separates points of Y. So, by the Stone–Weierstrass theorem, $\mathcal{A}_Y$ is dense in $C(Y)$ with respect to the uniform topology.

Step 2. We will prove that $1_Y \in \mathcal{W}$. For every n in $\mathbb{N}$, put

$$Z_n \coloneqq \{x \in K_n\colon\ d(x, Y) \geq 1/n\}.$$

Using Urysohn's lemma, choose $f_n \in C(K_n, [0,1])$ such that

$$f_n(y) = 1 \quad (y \in Y \cap K_n), \qquad f_n(x) = 0 \quad (x \in Z_n).$$

By Step 1, applied to K_n instead of Y, we find g_n in $\mathcal{A}$ such that $\|g_n|_{K_n} - f_n\| < 1/n$. Put

$$h_n(x) \coloneqq \min\{|g_n(x)|, 1\} = \frac{|g_n(x)| + 1 - ||g_n(x)| - 1|}{2}.$$

Then h_n belong to the unital C*-algebra generated by g_n; in particular, $h_n \in \mathcal{W}$. It is easy to verify that the sequence $(h_n)_{n\in\mathbb{N}}$ is bounded in the uniform norm and converges pointwise to 1_Y. Therefore $h_n \xrightarrow{\tau} 1_Y$ and $1_Y \in \mathcal{W}$.

Step 3. We will prove that $\mathcal{W}_Y$ is a τ_Y-closed subset of $L^\infty(Y, \mu_Y)$. Let $(f_j)_{j\in J}$ be a net in $\mathcal{W}_Y$ that τ_Y-converges to $g \in L^\infty(Y, \mu_Y)$. Choose $u_j \in \mathcal{W}$ such that $u_j|_Y = f_j$, put $v_j = u_j 1_Y$, and denote by h the extension by zero of the function g to the domain X. Then, by Step 2, $v_j \in \mathcal{W}$. The assumption $f_j \xrightarrow{\tau_Y} g$ implies that $v_j \xrightarrow{\tau_X} h$. Therefore $h \in \mathcal{W}$ and $g = h|_Y \in \mathcal{W}_Y$.

Step 4. We will prove that $\mathcal{W}_Y = L^\infty(Y, \mu_Y)$. Combining Step 1 with Proposition 2.2.1 we see that $\operatorname{clos}_{\tau_Y}(\mathcal{A}_Y) = L^\infty(Y, \mu_Y)$. Since $\operatorname{clos}_{\tau_Y}(\mathcal{A}_Y) \subseteq \operatorname{clos}_{\tau_Y}(\mathcal{W}_Y) = \mathcal{W}_Y$, we conclude that $\mathcal{W}_Y = L^\infty(Y, \mu_Y)$.

Step 5. Let $f \in L^\infty(X, \mu)$. For every n in $\mathbb{N}$, applying the result of Step 4 to the compact $Y = K_n$, find g_n in $\mathcal{W}$ such that $g_n|_{K_n} = f|_{K_n}$. By Step 2, $g_n 1_{K_n} \in \mathcal{W}$, i.e., $f 1_{K_n} \in \mathcal{W}$. The sequence $(f 1_{K_n})_{n\in\mathbb{N}}$ is bounded in the uniform norm and converges pointwise to f. Therefore it converges to f in the topology τ_X, and $f \in \mathcal{W}$. □

2.3 Criterion for commutativity of a direct integral of W*-algebras

For the definition and some properties of direct integrals see, for example, Dixmier [17, Part II, Chapters 1–3], Folland [26, Section 7.4] and Takesaki [77, Section 4.8]. In this section, we assume that (Ω, μ) is a σ-finite measure space and $(H_\xi)_{\xi\in\Omega}$ is a measurable field of non-zero separable Hilbert spaces. By definition, this concept requires the existence of a "fundamental sequence of measurable vector fields" $(g_j)_{j\in\mathbb{N}}$ such that for every ξ in Ω, the sequence $(g_j(\xi))_{j\in\mathbb{N}}$ is complete in H_ξ, and for every j, k in $\mathbb{N}$, the function $\xi \mapsto \langle g_j(\xi), g_k(\xi)\rangle_{H_\xi}$ is measurable.

The following fact about the existence of a "measurable field of orthonormal bases" uses the Gram–Schmidt orthogonalization; see detailed proofs in [17, Part II, Chapter 1, Section 2, Lemma 1], [26, Proposition 7.19] or [77, Lemma 8.12].

Proposition 2.3.1. *Let $(H_\xi)_{\xi\in\Omega}$, $(g_j)_{j\in\mathbb{N}}$ be a measurable field of non-zero separable Hilbert spaces, with dimensions $d_\xi \coloneqq \dim(H_\xi) \in \mathbb{N}\cup\{\infty\}$. Then $\{\xi\in\Omega\colon\ d_\xi = m\}$ is measurable for every m in $\mathbb{N}\cup\{\infty\}$. Moreover, there exists a sequence $(b_j)_{j\in\mathbb{N}}$ of vector fields with the following properties:*

- *for each $\xi \in \Omega$, $(b_j(\xi))_{\xi=1}^{d_\xi}$ is an orthonormal basis for H_ξ, and $b_j(\xi) = 0$ for $j > \dim(H_\xi)$;*
- *for each j in $\mathbb{N}$, there is a measurable partition of Ω, $\Omega = \cup_{k=1}^{\infty}\Omega_{j,k}$, such that on each $\Omega_{j,k}$, $b_j(\xi)$ is a finite linear combination of the family $(g_k(\xi))_{k\in\mathbb{N}}$, with coefficients depending measurably on ξ.*

We consider the following direct integral of W*-algebras:

$$\mathcal{A} \coloneqq \int_\Omega^{\oplus} \mathcal{B}(H_\xi)\,\mathrm{d}\mu(\xi). \tag{2.2}$$

Recall that if $S\in\mathcal{A}$ and

$$S = \int_\Omega^{\oplus} S(\xi)\,\mathrm{d}\mu(\xi),$$

then the norm of S coincides with the essential supremum of the function $\xi\mapsto\|S(\xi)\|_{\mathcal{B}(H_\xi)}$:

$$\|S\| \coloneqq \operatorname*{ess\,sup}_{\xi,\mu}\|S(\xi)\|_{\mathcal{B}(H_\xi)}.$$

In particular, this means that $S = 0$ if and only if the equality $S(\xi) = 0$ holds for μ-almost all points ξ.

Proposition 2.3.2. *Algebra $\mathcal{A}$ defined by (2.2) is commutative if, and only if,* $\mu(\Omega_2) = 0$, *where*

$$\Omega_2 \coloneqq \{\xi \in \Omega \colon\ \dim(H_\xi) \geq 2\}.$$

Proof. Let $\Omega_1 \coloneqq \{\xi \in \Omega \colon\ \dim(H_\xi) = 1\}$. For every ξ in Ω_1, we have $\dim(H_\xi) = 1$, and $\mathcal{B}(H_\xi)$ is commutative.

1. Suppose that $\mu(\Omega_2) = 0$. Given S_1, S_2 in $\mathcal{A}$, the operators $(S_1 S_2)(\xi)$ and $(S_2 S_1)(\xi)$ coincide for every ξ in Ω_1, which implies that $S_1 S_2 = S_2 S_1$. So, in this case, $\mathcal{A}$ is commutative.

2. Suppose that $\mu(\Omega_2) > 0$. We are going to prove that $\mathcal{A}$ is not commutative. Let $(b_j)_{j\in\mathbb{N}}$ be a sequence like in Proposition 2.3.1. In particular, for every ξ in Ω_2, the vectors $b_1(\xi)$ and $b_2(\xi)$ are orthonormal. Given

$$f = (f(\xi))_{\xi\in\Omega} \in \int_\Omega^\oplus H_\xi \, \mathrm{d}\mu(\xi),$$

we define $S_1 f$ and $S_2 f$ by

$$(S_1 f)(\xi) \coloneqq \begin{cases} \langle f(\xi), b_1(\xi)\rangle b_2(\xi), & \xi \in \Omega_2, \\ 0, & \xi \in \Omega_1; \end{cases}$$

$$(S_2 f)(\xi) \coloneqq \begin{cases} \langle f(\xi), b_2(\xi)\rangle b_1(\xi), & \xi \in \Omega_2, \\ 0, & \xi \in \Omega_1. \end{cases}$$

It is easy to see that $S_1, S_2 \in \mathcal{A}$. For every ξ in Ω_2, the restrictions of the operators $S_1(\xi)$ and $S_2(\xi)$ to $\operatorname{span}(b_1(\xi), b_2(\xi))$ have the following matrices with respect to the orthonormal basis $b_1(\xi), b_2(\xi)$:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In particular, $\|(S_1 S_2 - S_2 S_1)(\xi)\|_{\mathcal{B}(H_\xi)} = 1$ for every ξ in Ω_2, and $S_1 S_2 \neq S_2 S_1$. □

2.4 Translation-invariant operators in $L^2(G \times Y)$

Let us recall some well-known concepts and facts related to the translation operators and to the Fourier transform on LCAG [26, 41]. In this section, we accept the following assumption about G and Y.

Assumption 1. *Let G be a locally compact abelian group (LCAG) with a Haar measure ν, and Y be a measure space with a measure λ. We suppose that G is σ-compact and metrizable, λ is σ-finite, and the spaces $L^2(G, \mu)$ and $L^2(Y, \lambda)$ are separable. The cartesian product $G \times Y$ is considered with the product measure $\nu \times \lambda$.*

We denote by $\widehat{G}$ the dual group of G. The conditions on G imply that $\widehat{G}$ is also σ-compact and metrizable; see, for example, [41, (24.48)]. Let ν be a Haar measure on G. The Fourier transform of a function f in $L^1(G)$ is defined by

$$(F_1 f)(\xi) := \int_G \overline{\xi(x)}\, f(x)\, \mathrm{d}\nu(x) \qquad (\xi \in \widehat{G}).$$

Let $\widehat{\nu}$ be the dual Haar measure on $\widehat{G}$, such that $\|F_1 f\|_{L^2(\widehat{G},\widehat{\nu})} = \|f\|_{L^2(G)}$ for every f in $L^1(G) \cap L^2(G)$. We write $L^p(\widehat{G})$ instead of $L^p(\widehat{G}, \widehat{\nu})$ and denote by F the Fourier–Plancherel transform which coincides with F_1 on $L^1(G) \cap L^2(G)$

Given a in G, we denote by $\rho_G(a)$ the translation operator acting in $L^2(G)$ by the rule

$$(\rho_G(a) f)(x) := f(x - a).$$

Given a in G, we denote by E_a the function $\widehat{G} \to \mathbb{C}$ defined by $E_a(\xi) := \xi(a)$. Let $\rho_{\widehat{G}}(a)$ be the operator of multiplication by E_{-a}:

$$\rho_{\widehat{G}}(a) := M_{E_{-a}}. \tag{2.3}$$

It is well known and easy to see that $(\rho_G, L^2(G))$ and $(\rho_{\widehat{G}}, L^2(\widehat{G}))$ are (strongly continuous) unitary representations of G, and the Fourier–Plancherel transform intertwines them:

$$F \rho_G(a) F^* = \rho_{\widehat{G}}(a). \tag{2.4}$$

We shortly denote by $\rho'_{\widehat{G}}$ the centralizer of the set $\{\rho_{\widehat{G}}(a) \colon\ a \in G\}$. A similar notation is used through this work also for other unitary representations.

The following proposition describes the operators acting in $\mathcal{B}(L^2(\widehat{G}))$ and commuting with the multiplications by characters of $\widehat{G}$.

Proposition 2.4.1. $W^*(\{E_a \colon a \in G\}) = L^\infty(\widehat{G})$*, and* $\rho'_{\widehat{G}} = \mathcal{M}_{\widehat{G}}$.

Proof. The first statement follows from Theorem 2.2.2 and the fact that the set $\{E_{-a} \colon a \in G\}$ separates the points of $\widehat{G}$ (see, for example, [41, Theorem (22.17)]). The second statement is a consequence of formula (2.1). $\square$

An operator A of the class $\mathcal{B}(L^2(G))$ is called a *multiplier* of $L^2(G)$ if A commutes with $\rho_G(a)$ for every a in G. The next proposition, being an equivalent form of Proposition 2.4.1, means that the Fourier–Plancherel transform converts every multiplier of $L^2(G)$ into a multiplication operator in $L^2(\widehat{G})$. See Larsen [52, proof of Theorem 4.1.1] for a more constructive proof.

Proposition 2.4.2. $F\rho'_G F^* = \mathcal{M}_{\widehat{G}}$.

Here $F\rho'_G F^*$ is a short notation for $\{FAF^* \in \mathcal{B}(L^2(G)) \colon \ \forall a \in G \quad \rho_G(a)A = A\rho_G(a)\}$.

Corollary 2.4.3. *Let* Ω *be a measurable subset of* $\widehat{G}$ *and let* $A \in \mathcal{B}(L^2(\Omega))$*. Suppose that* A *commutes with the multiplications by all characters of* $\widehat{G}$ *restricted to* Ω*:*

$$\forall a \in G \qquad AM_{E_a|\Omega} = M_{E_a|\Omega}A.$$

Then $A \in \mathcal{M}_\Omega$*, i.e., there exists* b *in* $L^\infty(\Omega)$ *such that* $A = M_b$.

Proof. Define $B \colon L^2(\widehat{G}) \to L^2(\widehat{G})$ by the following rule:

$$(Bf)(\xi) := \begin{cases} (Af|_\Omega)(\xi), & \xi \in \Omega; \\ 0, & \xi \notin \Omega. \end{cases}$$

It is easy to see that $BM_{E_a} = M_{E_a}B$ for every a in G. By Proposition 2.4.1, there exists $b_1 \in L^\infty(\widehat{G})$ such that $B = M_{b_1}$. Put $b = b_1|_\Omega$. Then $A = M_b$. $\square$

Now we pass to the domains $G \times Y$ and $\widehat{G} \times Y$, the spaces $L^2(G \times Y)$ and $L^2(\widehat{G} \times Y)$, and the natural unitary representations of G in these spaces. It is well known [17, Part II, Chapter 1, Section 8, Proposition 11 and its Corollary] that

$$L^2(\widehat{G} \times Y) = L^2(\widehat{G}) \otimes L^2(Y) = \int_{\widehat{G}}^{\oplus} L^2(Y) \, \mathrm{d}\widehat{\nu}(\xi). \tag{2.5}$$

Set $\rho_{G\times Y}(a) \coloneqq \rho_G(a) \otimes I_{L^2(Y)}$ for each $a \in G$. More explicitly, $\rho_{G\times Y}$ is defined by (0.1). Then, $\rho_{G\times Y}$ is a unitary representation of G in $L^2(G \times Y)$. We are going to understand the structure of the centralizer $\rho'_{G\times Y}$. The crucial role here is played by the operator $F \otimes I_{L^2(Y)} : L^2(G \times Y) \to L^2(\widehat{G} \times Y)$, which we call "the Fourier transform with respect to the first coordinate" and denote shortly by $F \otimes I$.

For each a in G, we set $\rho_{\widehat{G}\times Y}(a) \coloneqq \rho_{\widehat{G}}(a) \otimes I_{L^2(Y)}$, i.e.,

$$(\rho_{\widehat{G}\times Y}(a)g)(\xi, y) = E_{-a}(\xi)g(\xi, y) \qquad (a \in G,\ \xi \in \widehat{G},\ y \in Y). \tag{2.6}$$

Then $\rho_{\widehat{G}\times Y}$ is a unitary representation of G in $L^2(\widehat{G} \times Y)$. Formula (2.4) implies that $F \otimes I$ intertwines $\rho_{G\times Y}$ with $\rho_{\widehat{G}\times Y}$:

$$(F \otimes I)\rho_{G\times Y}(a)(F \otimes I)^* = \rho_{\widehat{G}\times Y}(a). \tag{2.7}$$

Lemma 2.4.4. *Let H_1 and H_2 be separable Hilbert spaces, $(A_j)_{j\in J}$ be a net in $\mathcal{B}(H_1)$, and $B \in \mathcal{B}(H_1)$. Then $(A_j \otimes I_{H_2})_{j\in J}$ weakly converges to $B \otimes I_{H_2}$ if and only if $(A_j)_{j\in J}$ weakly converges to B.*

Proof. Given f, g in H_1 and u, v in H_2,

$$\begin{aligned} \langle (A_j \otimes I_{H_2})f \otimes u, g \otimes v\rangle_{H_1\otimes H_2} &= \langle A_j f, g\rangle_{H_1} \langle u, v\rangle_{H_2}, \\ \langle (B \otimes I)f \otimes u, g \otimes v\rangle_{H_1\otimes H_2} &= \langle Bf, g\rangle_{H_1} \langle u, v\rangle_{H_2}. \end{aligned}$$

These identities yield immediately the sufficiency part. For the necessity part, we take u and v to be the same normalized vector in H_2. □

Lemma 2.4.5. *Let H_1 and H_2 be separable Hilbert spaces, and S be a selfadjoint subset of $\mathcal{B}(H_1)$. Then*

$$W^*(\{A \otimes I_{H_2} \colon A \in S\}) = W^*(S) \otimes (\mathbb{C}I_{H_2}).$$

Proof. Let R be the unital algebra generated by S, and $P = \{A \otimes I_{H_2} \colon A \in R\}$. Then, obviously, P is the unital algebra generated by $\{A \otimes I_{H_2} \colon A \in S\}$. Furthermore,

$$\begin{aligned} W^*(\{A \otimes I_{H_2} \colon A \in S\}) &= \operatorname{clos}_{\mathrm{WOT}}(P) = \{B \otimes I_{H_2} \colon B \in \operatorname{clos}_{\mathrm{WOT}}(R)\} \\ &= \operatorname{clos}_{\mathrm{WOT}}(R) \otimes (\mathbb{C}I_{H_2}) = W^*(S) \otimes (\mathbb{C}I_{H_2}). \end{aligned}$$

The second equality in this chain follows from Lemma 2.4.4. □

Proposition 2.4.6.

$$(F \otimes I)\rho'_{G\times Y}(F \otimes I)^* = \int_{\widehat{G}}^{\oplus} \mathcal{B}(L^2(Y))\,\mathrm{d}\widehat{\nu}. \tag{2.8}$$

Equivalently,

$$(F \otimes I)\rho'_{G\times Y}(F \otimes I)^* = \mathcal{M}_{\widehat{G}} \otimes \mathcal{B}(L^2(Y)). \tag{2.9}$$

Proof. Since $F \otimes I$ is a unitary operator and $(F \otimes I)\rho_{G\times Y}(F \otimes I)^* = \rho_{\widehat{G}\times Y}$, we have

$$(F \otimes I)(\rho_{G\times Y})'(F \otimes I)^* = (\rho_{\widehat{G}\times Y})'.$$

Furthermore, by Lemma 2.4.5,

$$W^*(\rho_{\widehat{G}\times Y}) = W^*(\{\rho_{\widehat{G}}(a) \otimes I_{L^2(Y)} \colon a \in G\}) = W^*(\rho_{\widehat{G}}) \otimes (\mathbb{C}I_{L^2(Y)}).$$

Now we apply the fact [74, Theorem 2.8.1], [77, Theorem 5.9] that the centralizer of the tensorial product is the tensor product of the corresponding centralizers, and use Proposition 2.4.1:

$$\rho'_{\widehat{G}\times Y} = W^*(\rho_{\widehat{G}\times Y})' = W^*(\rho_{\widehat{G}})' \otimes \mathcal{B}(L^2(Y)) = \mathcal{M}_{\widehat{G}} \otimes \mathcal{B}(L^2(Y)).$$

We have proven (2.9). Furthermore, it is well known (see a more general result in [77, Corollary 8.30]) that

$$\mathcal{M}_{\widehat{G}} = \int_{\widehat{G}}^{\oplus} \mathbb{C}\,\mathrm{d}\widehat{\nu}.$$

Now, using the "distributive relation" between the direct integral and the tensor product of von Neumann algebras [17, Part II, Chapter 3, Section 4, Proposition 4], we obtain (2.9):

$$\rho'_{\widehat{G}\times Y} = \mathcal{M}_{\widehat{G}} \otimes \mathcal{B}(L^2(Y)) = \left(\int_{\widehat{G}} \mathbb{C}\,\mathrm{d}\widehat{\nu}\right) \otimes \mathcal{B}(L^2(Y)) = \int_{\widehat{G}} \mathcal{B}(L^2(Y))\,\mathrm{d}\widehat{\nu}. \qquad \square$$

The next corollary gives a constructive recipe for the decomposition (2.8).

Corollary 2.4.7. *Let $S \in \rho'_{G\times Y}$. For every ξ in $\widehat{G}$, define $A_\xi \colon L^2(Y) \to L^2(Y)$ by*

$$(A_\xi h)(v) = \frac{(F \otimes I)S(f \otimes h)(\xi, v)}{(F_1 f)(\xi)} \qquad (h \in L^2(Y)), \tag{2.10}$$

where f is any function of the class $L^1(G) \cap L^2(G)$ such that its Fourier transform $F_1 f$ does not vanish, and $(f \otimes h)(u, v) := f(u)h(v)$. Then

$$(F \otimes I)S(F \otimes I)^* = \int_{\widehat{G}}^{\oplus} A_\xi\,\mathrm{d}\widehat{\nu}(\xi). \tag{2.11}$$

Proof. The existence of a family $(A_\xi)_{\xi\in\widehat{G}}$ satisfying (2.11) follows from Proposition 2.4.6. We are going to prove (2.10). Let $f \in L^1(G) \cap L^2(G)$ such that $F_1 f$ does not vanish, and let $h \in L^2(Y)$. Put $g := F_1 f = Ff$. Then $g \otimes h = (F \otimes I)(f \otimes h)$, and

$$\begin{aligned}((F \otimes I)S(f \otimes h))(\xi, v) &= ((F \otimes I)S(F \otimes I)^*)(g \otimes h))(\xi, v)\\ &= (A_\xi(g \otimes h)(\xi, \cdot))(v) = (A_\xi(g(\xi)h))(v) = (F_1 f)(\xi) \cdot (A_\xi h)(v).\end{aligned}$$

Dividing by $(F_1 f)(\xi)$ we get (2.10). □

Corollary 2.4.7 (and thereby Proposition 2.4.6) can be proved with a more direct and elementary reasoning, similarly to Larsen [52, proof of Theorem 4.1.1].

2.5 Translation-invariant operators in Hilbert spaces

In this section, we make the following assumption.

Assumption 2. *Additionally to Assumption 1, let H be a closed subspace of $L^2(G \times Y)$, and $P\colon L^2(G \times Y) \to L^2(G \times Y)$ be the orthogonal projection with $H = P(L^2(G \times Y))$. We suppose that H is an invariant subspace of the representation $\rho_{G\times Y}$. Equivalently, P commutes with $\rho_{G\times Y}(a)$ for all a in G.*

Recall that the unitary representation ρ_H and its centralizer $\mathcal{V} := \rho_H'$ were defined in the Introduction, see (0.2). Using the general tools from previous sections, in this section we easily obtain a decomposition of $\mathcal{V}$.

Let $\widehat{H} := (F \otimes I)(H)$ and let $\widehat{P}$ be the orthogonal projection acting in $L^2(\widehat{G} \times Y)$ such that $\widehat{P}(L^2(\widehat{G} \times Y)) = \widehat{H}$. Equivalently,

$$\widehat{P} = (F \otimes I)P(F \otimes I)^*.$$

Proposition 2.5.1. *There exists a family of orthogonal projections $(\widehat{P}_\xi)_{\xi\in\widehat{G}}$ acting in $L^2(Y)$ such that*

$$\widehat{P} = \int_{\widehat{G}}^{\oplus} \widehat{P}_\xi \, \mathrm{d}\widehat{\nu}(\xi). \tag{2.12}$$

Proof. Since $P \in \rho'_{G\times Y}$, by Proposition 2.4.6, there exists a family $(\widehat{P}_\xi)_{\xi\in\widehat{G}}$ in $L^2(Y)$ such that $(F \otimes I)P(F \otimes I)^*$ decomposes into the direct integral (2.12). We have that $\widehat{P}^2 = \widehat{P}$ and $\widehat{P}^* = \widehat{P}$. By well-known properties of the direct integral [26, formula (7.24)],

$$\int_{\widehat{G}}^{\oplus} \widehat{P}_\xi^2 \,\mathrm{d}\widehat{\nu}(\xi) = \int_{\widehat{G}}^{\oplus} \widehat{P}_\xi \,\mathrm{d}\widehat{\nu}(\xi), \qquad \int_{\widehat{G}}^{\oplus} \widehat{P}_\xi^* \,\mathrm{d}\widehat{\nu}(\xi) = \int_{\widehat{G}}^{\oplus} \widehat{P}_\xi \,\mathrm{d}\widehat{\nu}(\xi).$$

Therefore, the equalities $\widehat{P}_\xi^2 = \widehat{P}_\xi$ and $\widehat{P}_\xi^* = \widehat{P}_\xi$ are fulfilled for almost every ξ in $\widehat{G}$. After modifying $\widehat{P}_\xi$ on a set of zero measure, we assure these properties for all ξ in $\widehat{G}$. □

Remark 2.5.2. Formula (2.10) yields an explicit expression for $\widehat{P}_\xi$:

$$(\widehat{P}_\xi h)(v) = \frac{((F \otimes I)P(f \otimes h))(\xi, v)}{(F_1 f)(\xi)} \qquad (h \in L^2(Y)), \tag{2.13}$$

where f is any function of the class $L^1(G,\nu) \cap L^2(G,\nu)$ such that its Fourier transform $F_1 f$ does not vanish.

In the rest of this section, we fix a family $(\widehat{P}_\xi)_{\xi\in\widehat{G}}$ as in Proposition 2.5.1. For each ξ in $\widehat{G}$, we denote by $\widehat{H}_\xi$ the image of the operator $\widehat{P}_\xi$ and by d_ξ its dimension:

$$\widehat{H}_\xi := \widehat{P}_\xi(L^2(Y)), \qquad d_\xi := \dim(\widehat{H}_\xi). \tag{2.14}$$

Furthermore, we denote by Ω the set of the frequencies corresponding to the non-trivial fibers:

$$\Omega := \{\xi \in \widehat{G} \colon\ d_\xi > 0\}. \tag{2.15}$$

Proposition 2.5.3. *$(\widehat{H}_\xi)_{\xi\in\Omega}$ is a measurable field of Hilbert spaces. Moreover, there exists a sequence of measurable vector fields $(q_j)_{j\in\mathbb{N}}$ with the following properties:*

(i) $(q_{j,\xi})_{j=1}^{d_\xi}$ is an orthonormal basis for $\widehat{H}_\xi$, and $q_{j,\xi} = 0$ for $j > \dim(\widehat{H}_\xi)$,
(ii) for each j in $\mathbb{N}$, the function $\Omega \times Y \to \mathbb{C}$, $(\xi, v) \mapsto q_{j,\xi}(v)$, is measurable.

Proof. Given an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ in $L^2(Y)$, we set

$$g_{j,\xi} := \widehat{P}_\xi e_j.$$

Then $(g_{j,\xi})_{j\in\mathbb{N}}$ is complete in $\widehat{H}_\xi$ for each ξ. Due to (2.13), the functions $(\xi, v) \mapsto g_{j,\xi}(v)$ are measurable on $\Omega \times Y$.

Applying Proposition 2.3.1 we get a family $(q_{j,\xi})_{j\in\mathbb{N},\xi\in\Omega}$ with desired properties. Indeed, if $\Omega_{j,k}$ are as in Proposition 2.3.1, then $(\xi,v)\mapsto q_{j,\xi}(v)$ is measurable on $A_{j,k}\times Y$ being a finite linear combination of measurable functions $(\xi,v)\mapsto g_{k,\xi}(v)$.

We notice that the measurability in this sense (as functions defined on $\Omega\times Y$) is stronger then the measurability which appears in the definition of a measurable field of Hilbert spaces. □

Proposition 2.5.4. *$\widehat{H}$ is the direct integral of the spaces $\widehat{H}_\xi$:*

$$\widehat{H}=\int_\Omega^\oplus \widehat{H}_\xi\,\mathrm{d}\widehat{\nu}(\xi). \tag{2.16}$$

Proof. If $g\in\widehat{H}$ and $g_\xi\coloneqq g(\xi,\cdot)$ for every ξ, then $\widehat{P}_\xi g_\xi=g_\xi$ for almost every ξ. After modifying g on a set of measure zero, if needed, we assume that $\widehat{P}_\xi g_\xi=g_\xi$ for all ξ in Ω and $g_\xi=0$ for every ξ in $\widehat{G}\setminus\Omega$. So, the family $(g_\xi)_{\xi\in\Omega}$ belongs to the direct integral in the right-hand side of (2.16).

Conversely, given a vector field $(g_\xi)_{\xi\in\Omega}$ belonging to the right-hand side of (2.16), we trivially extend $g_\xi=0$ for ξ in $\widehat{G}\setminus\Omega$ and obtain a function g of the class $L^2(\widehat{G}\times Y)$ such that $\widehat{P}g=g$. □

Let $\Phi\colon H\to\widehat{H}$ be defined by $\Phi(f)\coloneqq(F\otimes I)(f)$. In other words, Φ is the compression of $F\otimes I$ to the domain H and codomain $\widehat{H}$.

Theorem 2.5.5. *With Assumption 2,*

$$\Phi\mathcal{V}\Phi^*=\int_\Omega^\oplus \mathcal{B}(\widehat{H}_\xi)\,\mathrm{d}\widehat{\nu}(\xi). \tag{2.17}$$

Proof. We will explain the inclusion $\subseteq$ only. Let $S\in\mathcal{V}$. Define $A\in\mathcal{B}(L^2(G\times Y))$ by $Af\coloneqq SPf$. Since S takes values in H, we obtain $PA=PAP=AP$. Furthermore, Assumption 2 implies that $P\in\rho'_{G\times Y}$ and therefore $A\in\rho'_{G\times Y}$. By Proposition 2.4.6, there exists a family $(B_\xi)_{\xi\in\widehat{G}}$ in $\mathcal{B}(L^2(Y))$ such that

$$(F\otimes I)A(F\otimes I)^*=\int_{\widehat{G}}^\oplus B_\xi\,\mathrm{d}\widehat{\nu}(\xi).$$

Since A commutes with P, we conclude that $(F\otimes I)A(F\otimes I)^*$ commutes with $\widehat{P}$. By (2.12), for almost all ξ we obtain that B_ξ commutes with $\widehat{P}_\xi$, i.e., $\widehat{H}_\xi$ is an invariant subspace of B_ξ. Let D_ξ be the compression of B_ξ to $\widehat{H}_\xi$. Then for every g in $\widehat{H}$ and almost every ξ in Ω,

$$(\Phi S\Phi^* g)(\xi,\cdot) = ((F\otimes I)A(F\otimes I)^* g)(\xi,\cdot) = B_\xi g(\xi,\cdot) = D_\xi g(\xi,\cdot).$$

For ξ in $\widehat{G}\setminus\Omega$, the space $\widehat{H}_\xi$ is trivial, and we omit these values of ξ. So,

$$\Phi S\Phi^* = \int_\Omega D_\xi \,\mathrm{d}\widehat{\nu}(\xi). \qquad \square$$

Proposition 2.5.6. *$\mathcal{V}$ is commutative if and only if $d_\xi = 1$ for $\widehat{\nu}$-almost every point ξ of Ω.*

Proof. Follows from Proposition 2.3.2 and Theorem 2.5.5. $\square$

2.6 Translation-invariant operators in RKHS

In this section, we consider the case when H is a RKHS over $G\times Y$. We freely use some basic properties of RKHS. See, for example, Aronszajn [5] or Agler and McCarthy [3].

First, we give a simple criterion for $\rho_{G\times Y}$-invariance of H in terms of the reproducing kernel. This is a particular case of [57, Proposition 4.1].

Proposition 2.6.1. *Let G and Y satisfy Assumption 1, and let H be a RKHS over $G\times Y$, with reproducing kernel $(K_{x,y})_{(x,y)\in G\times Y}$. Then the following conditions are equivalent.*

(a) $\rho_{G\times Y}(H)\subseteq H$ for every a in G.
(b) $P\rho_{G\times Y}(a) = \rho_{G\times Y}(a)P$ for every a in G, where P is the orthogonal projection on $L^2(G\times Y)$ such that $P(L^2(G\times Y)) = H$.
(c) For every x,u in G and every y,v in Y,

$$K_{x,y}(u,v) = K_{0,y}(u-x,v). \tag{2.18}$$

(d) For every a,x in G and every y in Y,

$$\rho_{G\times Y}(a)K_{x,y} = K_{a+x,y}. \tag{2.19}$$

In the rest of this section, we make the following assumption.

Assumption 3. *Additionally to Assumption 2, suppose that H is a RKHS over $G \times Y$, and the reproducing kernel $(K_{x,y})_{(x,y)\in G\times Y}$ satisfies*

$$\forall y \in Y \qquad \sup_{v\in Y} \int_G |K_{0,y}(u,v)| \,\mathrm{d}\nu(u) < +\infty. \tag{2.20}$$

For every ξ in $\widehat{G}$ and every y, v in Y, we define $L_{\xi,y}(v)$ by (0.3). In particular, (2.20) implies that the integral in (0.3) exists in the Lebesgue sense, and for every y, v in Y the function $\xi \mapsto L_{\xi,y}(v)$ is continuous.

The goal of this section is to provide more constructive descriptions of the projections $\widehat{P}_\xi$ and spaces $\widehat{H}_\xi$ than in Section 2.5.

Using Proposition 2.6.1 and the Hermitian property of K we can write P as

$$(Pf)(x,y) = \int_Y \int_G f(u,v) K_{0,v}(x-u,y) \,\mathrm{d}\nu(u) \,\mathrm{d}\lambda(v). \tag{2.21}$$

The inner integral in the right-hand side of (2.21) is a convolution. The following lemma can be viewed as an application of the convolution theorem to this inner integral. The technical assumptions on (G,μ), (Y,λ), and K allow us to interchange the order of integration.

Lemma 2.6.2. *Let $f \in L^1(G\times Y) \cap L^2(G\times Y)$. Then for every ξ in $\widehat{G}$ and every y in Y,*

$$((F\otimes I)Pf)(\xi,y) = \int_Y ((F\otimes I)f)(\xi,v)\, \overline{L_{\xi,y}(v)} \,\mathrm{d}\lambda(v). \tag{2.22}$$

Equivalently,

$$((F\otimes I)Pf)(\xi,y) = \langle ((F\otimes I)f)(\xi,\cdot), L_{\xi,y}\rangle_{L^2(Y)}. \tag{2.23}$$

Proof. Step 1. We denote by C_y the supremum in (2.20). Let us estimate from above the following triple integral:

$$J := \int_Y \int_G \int_G |f(u,v)|\, |K_{x,y}(u,v)| \,\mathrm{d}\nu(x) \,\mathrm{d}\nu(u) \,\mathrm{d}\lambda(v).$$

We write $K_{x,y}(u,v)$ as $K_{0,y}(u-x,y)$, make the change of variables $t = u - x$ (where u is a fixed parameter), apply Tonelli's theorem and assumption (2.20):

$$\begin{aligned} J &= \int_Y \int_G \int_G |f(u,v)|\, |K_{0,y}(t,v)| \,\mathrm{d}\nu(t) \,\mathrm{d}\nu(u) \,\mathrm{d}\lambda(v) \\ &= \int_G \int_Y |f(u,v)| \left(\int_G |K_{0,y}(t,v)| \,\mathrm{d}\nu(t) \right) \mathrm{d}\lambda(v) \,\mathrm{d}\nu(u) \\ &\le C_y \int_G \int_Y |f(u,v)| \,\mathrm{d}\lambda(v) \,\mathrm{d}\nu(u) = C_y \|f\|_{L^1(G\times Y)} < +\infty. \end{aligned}$$

Step 2. Due to Step 1, we can apply Fubini's theorem to the following integrals.

$$\begin{aligned}((F\otimes I)Pf)(\xi,y) &= \int_G\int_G\int_Y \overline{\xi(x)}f(u,v)\overline{K_{x,y}(u,v)}\,\mathrm{d}\lambda(v)\,\mathrm{d}\nu(u)\,\mathrm{d}\nu(x)\\ &= \int_Y\int_G \overline{\xi(u)}f(u,v)\overline{\left(\int_G \overline{\xi(u-x)}K_{0,v}(u-x,y)\,\mathrm{d}\nu(x)\right)}\mathrm{d}\nu(u)\,\mathrm{d}\lambda(v)\\ &= \int_Y ((F\otimes I)f)(\xi,v)\overline{L_{\xi,y}(v)}\,\mathrm{d}\lambda(v).\end{aligned}$$

□

Lemma 2.6.3. *For every y,v in Y and every ξ in $\widehat{G}$,*

$$L_{\xi,y}(v) = \langle L_{\xi,y}, L_{\xi,v}\rangle_{L^2(Y)}. \tag{2.24}$$

Proof. Follows from Lemma 2.6.2 applied to $f = K_{0,y}$. □

The following general fact can be seen as a corollary from Moore–Aronszajn theorem. We have not found the explicit statement of this fact in the bibliography. In many applications, $\mathcal{H}_1$ is a space of square-integrable *functions*, rather than their equivalence classes.

Proposition 2.6.4 (On RKHS generated by a reproducing family in a complete space with pre-inner product)**.** *Let X is a set and $\mathcal{H}_1$ be a space of functions $X\to\mathbb{C}$ with a pre-inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$, not necessarily strictly positive. We suppose that $\mathcal{H}_1$ is complete with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$. Let $(\mathcal{K}_x)_{x\in X}$ be a family in $\mathcal{H}_1$ such that*

$$\forall x,y\in X \qquad \mathcal{K}_x(y) = \langle \mathcal{K}_x, \mathcal{K}_y\rangle_{\mathcal{H}_1}. \tag{2.25}$$

Let

$$\mathcal{H}_2 \coloneqq \{f\in\mathcal{H}_1\colon\quad \forall x\in X\quad f(x) = \langle f,\mathcal{K}_x\rangle\}.$$

Then $\mathcal{H}_2$ is a RKHS and $(\mathcal{K}_x)_{x\in X}$ is the reproducing kernel of $\mathcal{H}_2$. The rule

$$(\mathcal{P}f)(x) \coloneqq \langle f,\mathcal{K}_x\rangle_{\mathcal{H}_1}, \tag{2.26}$$

defines an orthogonal projection in $\mathcal{H}_1$, and $\mathcal{P}(\mathcal{H}_1) = \mathcal{H}_2$.

Proof. The main challenge is to prove that $\mathcal{P}f \in \mathcal{H}_1$ for every f in $\mathcal{H}_1$. We will get this fact indirectly, using the existence of an orthogonal projection onto a closed subspace of a Hilbert space. Since $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ is not necessarily strictly positive, we have to pass from elements of $\mathcal{H}_1$ to equivalent classes and return back.

Let $\mathcal{H}_0 := \{f \in \mathcal{H}_1 \colon \ \langle f, f \rangle_{\mathcal{H}_1} = 0\}$. Then $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}_1$ and $\mathcal{H}_1/\mathcal{H}_0$ is a Hilbert space (with a strictly positive inner product). We denote by π_1 the canonical projection $\mathcal{H}_1 \to \mathcal{H}_1/\mathcal{H}_0$.

Condition (2.25) easily implies that $(\mathcal{K}_x)_{x \in X}$ is a positive definite kernel. Let $\mathcal{H}_3$ be the span of $\{\mathcal{K}_x \colon \ x \in X\}$ and $\mathcal{H}_4$ be the RKHS constructed in the Moore–Aronszajn theorem. Due to (2.25), the inner product in $\mathcal{H}_4$ is inherited from $\mathcal{H}_1$. The elements of $\mathcal{H}_4$ are pointwise limits of Cauchy sequences in $\mathcal{H}_3$. At this point, we know that $\mathcal{H}_4 \subseteq \mathcal{H}_2$.

Since $\pi_1(\mathcal{H}_4)$ is a closed subset of $\mathcal{H}_1/\mathcal{H}_0$, there exists an orthogonal projection $\mathcal{P}_1$ in $\mathcal{H}_1/\mathcal{H}_0$ such that $\mathcal{P}_1(\mathcal{H}_1/\mathcal{H}_0) = \pi_1(\mathcal{H}_4)$. Given f in $\mathcal{H}_1$, let $g \in \mathcal{H}_4$ be such a function that $\mathcal{P}_1(\pi_1(f)) = \pi_1(g)$. Then, for every x in X,

$$\begin{aligned} g(x) &= \langle g, \mathcal{K}_x \rangle_{\mathcal{H}_1} = \langle \pi_1(g), \pi_1(\mathcal{K}_x) \rangle_{\mathcal{H}_1/\mathcal{H}_0} = \langle \mathcal{P}_1(\pi_1(f)), \pi_1(\mathcal{K}_x) \rangle_{\mathcal{H}_1/\mathcal{H}_0} \\ &= \langle \pi_1(f), \mathcal{P}_1(\pi_1(\mathcal{K}_x)) \rangle_{\mathcal{H}_1/\mathcal{H}_0} = \langle \pi_1(f), \pi_1(\mathcal{K}_x) \rangle_{\mathcal{H}_1/\mathcal{H}_0} = \langle f, \mathcal{K}_x \rangle_{\mathcal{H}_1} = (\mathcal{P}f)(x). \end{aligned} \tag{2.27}$$

Thereby we get $\mathcal{P}f = g \in \mathcal{H}_4$. So, $\mathcal{P}$ is a well-defined function $\mathcal{H}_1 \to \mathcal{H}_1$. Computation (2.27) means that $\pi_1 \circ \mathcal{P} = \mathcal{P}_1 \circ \pi_1$. Since $\mathcal{P}_1$ is a bounded selfadjoint linear operator and π_1 is a linear isometry, we easily conclude that $\mathcal{P}$ is a bounded autoadjoint linear operator.

If $f \in \mathcal{H}_2$ and $g \in \mathcal{H}_4$ such that $\pi_1(g) = \mathcal{P}_1(\pi_1(f))$, then the definition of $\mathcal{H}_2$ and the reproducing property in $\mathcal{H}_4$ imply that $f = g$. Hence, $\mathcal{H}_4 = \mathcal{H}_2$. Finally, we can conclude that $\mathcal{P}(\mathcal{H}_1) = \mathcal{H}_2$ and $\mathcal{P}^2 = \mathcal{P}$. □

For every ξ in $\widehat{G}$, we define $\widehat{P}_\xi \colon L^2(Y) \to L^2(Y)$ by

$$(\widehat{P}_\xi h)(y) := \langle h, L_{\xi,y} \rangle_{L^2(Y)} = \int_Y h(v) \overline{L_{\xi,y}(v)} \, \mathrm{d}\lambda(v). \tag{2.28}$$

Then, we denote by $\widehat{H}_\xi$ the image of $\widehat{P}_\xi$:

$$\widehat{H}_\xi := \widehat{P}_\xi(L^2(Y)). \tag{2.29}$$

We will prove that (2.28) is equivalent to the definition of $\widehat{P}_\xi$ in Section 2.5.

Theorem 2.6.5. *Let Assumption 3 hold. For every ξ in $\widehat{G}$, $\widehat{P}_\xi$ is an orthogonal projection in $L^2(Y)$ and $\widehat{H}_\xi$ is a RKHS with reproducing kernel $(L_{\xi,y})_{y\in Y}$. For each ξ in $\widehat{G}$,*

$$\widehat{H}_\xi = \operatorname{clos}_{L^2(Y)}(\operatorname{span}(\{L_{\xi,y}\colon y\in Y\})), \tag{2.30}$$

where the closure is understood as the set of the pointwise limits of Cauchy sequences. Moreover,

$$\widehat{P} = \int_{\widehat{G}} \widehat{P}_\xi \,\mathrm{d}\widehat{\nu}(\xi), \tag{2.31}$$

i.e., for every g in $L^2(\widehat{G}\times Y)$,

$$(\widehat{P}g)(\xi,y) = \langle g(\xi,\cdot), L_{\xi,y}\rangle_{L^2(Y)} = \int_Y g(\xi,v)\,\overline{L_{\xi,y}(v)}\,\mathrm{d}\lambda(v) \qquad (\xi\in\widehat{G},\ y\in Y). \tag{2.32}$$

Proof. The first statements follow from Lemma 2.6.3 and Proposition 2.6.4. Let A be the operator in $L^2(\widehat{G}\times Y)$ defined by the the right-hand side of (2.31) or (2.32). For each $\xi\in\widehat{G}$, $\|\widehat{P}_\xi\|\le 1$. This easily implies that A is a bounded linear operator with $\|A\|\le 1$.

By Lemma 2.6.2, the equality $(F\otimes I)Pf = A(F\otimes I)f$ holds for every f in the intersection $L^2(G\times Y)\cap L^1(G\times Y)$, which is a dense subset of $L^2(G\times Y)$. Since $(F\otimes I)P$ and $A(F\otimes I)$ are bounded linear operators, we conclude that the equality $(F\otimes I)P = A(F\otimes I)$ holds on the whole space $L^2(G\times Y)$. Hence, $\widehat{P} = (F\otimes I)P(F\otimes I)^* = A$. □

Corollary 2.6.6. *Let $g\in\widehat{H}$. Then for almost all ξ in $\widehat{G}$ and almost all y in Y,*

$$g(\xi,y) = \langle g(\xi,\cdot), L_{\xi,y}\rangle_{L^2(Y)} = \int_Y g(\xi,v)\overline{L_{\xi,y}(v)}\,\mathrm{d}\lambda(v). \tag{2.33}$$

Remark 2.6.7. The integral in (2.33) is taken over Y, not over the $\widehat{G}\times Y$. In general, $\widehat{H}$ does not have to be a RKHS.

Proposition 2.6.8. *For every ξ in $\widehat{G}$,*

$$\dim(\widehat{H}_\xi) = \int_Y L_{\xi,y}(y)\,\mathrm{d}\lambda(y) = \int_Y \|L_{\xi,y}\|^2_{L^2(Y)}\,\mathrm{d}\lambda(y). \tag{2.34}$$

Proof. This is a general formula for the dimension of the image of the orthogonal projection defined as an integral operator. Let us outline the proof in our settings. Recall that $(q_{j,\xi})_{j=1}^{d_\xi}$ is an orthonormal basis for $\widehat{H}_\xi$. Therefore, $L_{\xi,y}(y) = \sum_{j=1}^{d_\xi}|q_{j,\xi}(y)|^2$ and

$$\int_Y L_{\xi,y}(y)\,\mathrm{d}\lambda(y) = \sum_{j=1}^{d_\xi}\int_Y |q_{j,\xi}(y)|^2\,\mathrm{d}\lambda(y) = \sum_{j=1}^{d_\xi} 1 = d_\xi. \qquad \square$$

As a consequence of Proposition 2.6.8, we get a constructive description of Ω:

$$\Omega = \left\{ \xi \in \widehat{G} \colon \quad \int_Y L_{\xi,y}(y)\,\mathrm{d}\lambda(y) > 0 \right\}. \tag{2.35}$$

Theorem 2.6.9. *With Assumption 3, the following conditions are equivalent.*

(a) $\mathcal{V}$ *is commutative.*
(b) For every ξ *in* Ω, $\quad \dim(\widehat{H}_\xi) = 1$.
(c) For every ξ *in* Ω,
$$\int_Y L_{\xi,y}(y)\,\mathrm{d}\lambda(y) = 1. \tag{2.36}$$
(d) For every ξ *in* Ω *and every* y, v *in* Y,
$$|L_{\xi,y}(v)|^2 = L_{\xi,y}(y) L_{\xi,v}(v). \tag{2.37}$$
(e) There exists a family $(q_\xi)_{\xi\in\Omega}$ *in* $L^2(Y)$ *such that the function* $(\xi, v) \mapsto q_\xi(v)$ *is measurable, the function* q_ξ *forms an orthonormal basis of* $\widehat{H}_\xi$, *and*
$$L_{\xi,y}(v) = \overline{q_\xi(y)} q_\xi(v) \qquad (\xi \in \Omega,\ y, v \in Y). \tag{2.38}$$

Proof. The major part of the proof follows from Propositions 2.5.6 and 2.6.8. We will comment only a few missing ideas. If (2.36) holds for almost every ξ, then, by continuity of $L_{\xi,y}(v)$ with respect to ξ, it holds for every ξ.

Condition (d) means that the Schwarz inequality for $L_{\xi,y}$ and $L_{\xi,v}$ reduces to an equality, i.e., the functions $L_{\xi,y}$ and $L_{\xi,v}$ are linear dependent. Since y and v are arbitrary elements of Y and $\{L_{\xi,y} \colon y \in Y\}$ is a total subset of $\widehat{H}_\xi$, (d) implies (b).

If (b) holds, then we apply Proposition 2.5.3 with $d_\xi = 1$ and obtain a family $(q_\xi)_{\xi\in\Omega}$ such that $(\xi, v) \mapsto q_\xi(v)$ is measurable and q_ξ is an orthonormal basis of $\widehat{H}_\xi$. The reproducing kernel of $\widehat{H}_\xi$ expresses through this orthonormal basis by (2.38). □

Remark 2.6.10. In the context of the last part of the proof, for every ξ in Ω, there exists z in Y and τ in $\mathbb{C}$ (both depending on ξ) such that $\|L_{\xi,z}\| \neq 0$, $|\tau| = 1$, and

$$q_\xi = \tau \frac{L_{\xi,z}}{\|L_{\xi,z}\|}.$$

This means that q is essentially determined by L. In many examples, a decomposition of the form (2.38) with a measurable function q is obvious.

Remark 2.6.11. Let us emphasize additional properties that obtain $\widehat{P}_\xi$ and $\widehat{H}_\xi$ when passing from Assumption 2 to Assumption 3.

1. Now $\widehat{P}_\xi$ and $\widehat{H}_\xi$ are uniquely defined *for every* ξ, instead of almost everywhere.
2. $\widehat{P}_\xi$ and $\widehat{H}_\xi$ have *simple explicit expressions* in terms of $(L_{\xi,y})_{y\in Y}$.
3. The elements of $\widehat{H}_\xi$, in contrast to $L^2(Y)$, can be treaten as *functions*, instead of classes of equivalence.
4. We have *simple formulas* (2.34) and (2.35) to compute Ω and the dimensions of $\widehat{H}_\xi$.
5. Theorem 2.6.9 is a *constructive criterion* for the commutativity of $\mathcal{V}$.

2.7 Diagonalization in the commutative case

In this section we set Assumption 2 to be fulfilled and, additionally, $d_\xi = \dim(\widehat{H}_\xi) = 1$ for every ξ in Ω. In this case, Proposition 2.5.4 implies that there exists a family of functions $(q_\xi)_{\xi\in\Omega}$ with the following properties:

(i) $\widehat{H}_\xi = \mathbb{C}q_\xi$ and $\|q_\xi\|_{L^2(Y)} = 1$ for every ξ in Ω;
(ii) the function $\Omega \times Y \to \mathbb{C}$, $(\xi, v) \mapsto q_\xi(v)$, is measurable.

For each ξ in Ω, the function q_ξ is uniquely defined, up to a constant of absolute value 1.

In particular, if H is a RKHS satisfying Assumption 3 and equivalent conditions from Theorem 2.6.9, then q_ξ is usually easy to find from L_ξ, see Remark 2.6.10.

Identifying $\widehat{H}_\xi$ and $\mathcal{B}(\widehat{H}_\xi)$ with $\mathbb{C}$, in this section we will simplify the descomposition from Theorem 2.5.5 and construct a unitary operator $R\colon H \to L^2(\Omega)$ such that $R\mathcal{V}R^* = \mathcal{M}_\Omega$. Our treatment generalizes ideas from Vasilevski [82].

Define $N\colon \widehat{H} \to L^2(\Omega)$ by

$$(Ng)(\xi) := \langle g(\xi,\cdot), q_\xi\rangle_{L^2(Y)} = \int_Y \overline{q_\xi(v)}\, g(\xi, v)\, \mathrm{d}\lambda(v). \tag{2.39}$$

Proposition 2.7.1. *N is a unitary operator, and its inverse $N^*\colon L^2(\Omega) \to \widehat{H}$ acts by the following rule:*

$$(N^*h)(\xi, y) = \begin{cases} q_\xi(y)h(\xi), & \xi \in \Omega; \\ 0, & \xi \in \widehat{G} \setminus \Omega. \end{cases} \tag{2.40}$$

Proof. 1. Let $g \in \widehat{H}$. For every ξ in Ω, by Proposition 2.5.4 we have $g(\xi,\cdot) \in \widehat{H}_\xi$. Since $\widehat{H}_\xi = \mathbb{C}q_\xi$ and $\|q_\xi\|^2_{L^2(Y)} = 1$, we obtain $\|g(\xi,\cdot)\|_{L^2(Y)} = |(Ng)(\xi)|$. Hence, N is isometric:

$$\|Ng\|^2_{L^2(\Omega)} = \int_\Omega |(Ng)(\xi)|^2 \,\mathrm{d}\widehat{\nu}(\xi) = \int_\Omega \|g(\xi,\cdot)\|^2_{L^2(Y)} \,\mathrm{d}\widehat{\nu}(\xi) = \|g\|^2_{\widehat{H}}.$$

2. Let Z be the operator defined by the right-hand side of (2.40). Proposition 2.5.4 assures that Zh indeed belongs to $\widehat{H}$ and Z is well-defined. A simple direct computation yields $NZh = h$, which completes the proof. □

Define $R\colon H \to L^2(\Omega)$ by the following rule:

$$R \coloneqq N\Phi, \tag{2.41}$$

i.e.,

$$(Rf)(\xi) = \int_Y ((F \otimes I)f)(\xi, v)\overline{q_\xi(v)} \,\mathrm{d}\lambda(v). \tag{2.42}$$

Remark 2.7.2. The idea of the operator R is similar to the ideas of some lossless audio- and video-codecs: it is a kind of a Fourier transform followed by a “general compression”.

Proposition 2.7.3. *R is a unitary operator from H onto $L^2(\Omega)$.*

Proof. Indeed, R is the composition of two unitary operators. □

Proposition 2.7.4. *Let $y \in Y$ and $\xi \in \Omega$. Then*

$$(RK_{0,y})(\xi) = \overline{q_\xi(y)}. \tag{2.43}$$

Proof. $(RK_{0,y})(\xi) = \langle (\Phi K_{0,y})(\xi,\cdot), q_\xi\rangle_{L^2(Y)} = \langle L_{\xi,y}, q_\xi\rangle_{L^2(Y)} = \overline{q_\xi(y)}$. □

Remark 2.7.5. Additionally to the operators $N\colon \widehat{H} \to L^2(\Omega)$ and $R\colon H \to L^2(\Omega)$, one can define in a similar way their extended versions $\widetilde{N}\colon L^2(\widehat{G} \times Y) \to L^2(\Omega)$ and $\widetilde{R}\colon L^2(G \times Y) \to L^2(\Omega)$. Then

$$\widetilde{N}^*\widetilde{N} = \widehat{P}, \qquad \widetilde{N}\widetilde{N}^* = I_{L^2(\Omega)}, \qquad \widetilde{N}^*(L^2(\Omega)) = \widehat{H},$$

$$\widetilde{R}^*\widetilde{R} = P, \qquad \widetilde{R}\widetilde{R}^* = I_{L^2(\Omega)}, \qquad \widetilde{R}^*(L^2(\Omega)) = H.$$

Recall that E_a is defined by $E_a(\xi) = \xi(a)$, where $a \in G$ and $\xi \in \widehat{G}$.

Proposition 2.7.6. *Let $a \in G$. Then*

$$R\rho_H(a)R^* = M_{E_{-a}|\Omega}. \tag{2.44}$$

Proof. Let $h \in L^2(\Omega)$. Substituting the definitions and using (2.7) we easily get

$$R\rho_H(a)R^*h = N\Phi\rho_H(a)\Phi^*N^*h = N(F\otimes I)\rho_{G\times Y}(a)(F\otimes I)^*N^*h = N(\rho_{\widehat{G}}(a)\otimes I)N^*h.$$

Therefore, for every ξ in Ω,

$$(R\rho_H(a)R^*h)(\xi) = \langle E_{-a}(\xi)q_\xi h(\xi), q_\xi\rangle_{L^2(Y)} = E_{-a}(\xi)h(\xi).$$

□

Theorem 2.7.7. *Define $\Lambda\colon L^\infty(\Omega) \to \mathcal{V}$ by $\Lambda(\sigma) := R^*M_\sigma R$. Then Λ is an isometric isomorphism of W*-algebras. In particular, for every S in $\mathcal{V}$, the product RSR^* is a multiplication operator in $L^2(\Omega)$.*

Proof. The algebraic properties of Λ and the isometric property of Λ follow easily from well-known properties of multiplication operators and from the fact that R is a unitary operator.

We have to show that Λ is surjective. Let $S \in \mathcal{V}$ and $B := RSR^*$. By Proposition 2.7.6, for each a in G we have

$$BM_{E_a|\Omega} = (RSR^*)(R\rho(-a)R^*) = RS\rho_H(-a)R^* = R\rho_H(-a)SR^* = M_{E_a|\Omega}B,$$

i.e., B commutes with E_a. By Corollary 2.4.3, we conclude that $B \in \mathcal{M}_\Omega$. □

In particular, Theorem 2.7.7 means that the W*-algebras $\mathcal{V}$ and $\mathcal{M}_\Omega$ are spatially isomorphic. Figure 2.1 shows a commutative diagram corresponding to the formula $S = \Lambda(\sigma) = R^*M_\sigma R$ from Theorem 2.7.7, jointly with some auxiliary objects.

Given S in $\mathcal{V}$, we say that $\sigma := \Lambda^{-1}(S)$ is the *spectral function* of the operator S. The next corollary provides is an explicit formula for σ.

Corollary 2.7.8. *Let $S \in \mathcal{V}$. Then for every ξ in Ω,*

$$(\Lambda^{-1}(S))(\xi) = \frac{(RSK_{0,y})(\xi)}{\overline{q_\xi(y)}}, \tag{2.45}$$

where y is an arbitrary point of Y so that $q_\xi(y) \neq 0$.

$$
\begin{tikzcd}
H < L^2(G\times Y) \arrow[r, "\Phi"] \arrow[d, "S\in\rho'_H"'] \arrow[rr, bend left, "R"] & \widehat{H} < L^2(\widehat{G}\times Y) \arrow[r, "N"] \arrow[d, "\Phi S\Phi^*"'] & L^2(\Omega) \arrow[d, "M_\sigma\in\mathcal{M}_\Omega"] \arrow[ll, bend left, "R^*", shift left=3ex, phantom] \\
H < L^2(G\times Y) & \widehat{H} < L^2(\widehat{G}\times Y) \arrow[l, "\Phi^*"] & L^2(\Omega) \arrow[l, "N^*"] \arrow[ll, bend left, "R^*"]
\end{tikzcd}
$$

Figure 2.1: Operators participating in Theorem 2.7.7.

Proof. Let $\sigma = \Lambda^{-1}(S)$, i.e., $RS = M_\sigma R$. Furthermore, let $\xi \in \Omega$ and $y \in Y$ such that $q_\xi(y) \neq 0$. Using (2.43) we obtain

$$(RSK_{0,y})(\xi) = (M_\sigma RK_{0,y})(\xi) = \sigma(\xi)\overline{q_\xi(y)}.$$

Dividing over $\overline{q_\xi(y)}$ we get (2.45). $\square$

Corollary 2.7.9. *Let $S \in \mathcal{V}$ and $\sigma = \Lambda^{-1}(S)$. Then $\|S\| = \|\sigma\|_\infty$, and the spectrum of S is the essential range of σ.*

Berezin transform of a translation-invariant operator in terms of its spectral function

Proposition 2.7.10. *Let Assumption 3 holds, $S \in \mathcal{V}$, and $\sigma = \Lambda^{-1}(S)$. Then*

$$\operatorname{Ber}(S)(x,y) = \frac{\int_\Omega \sigma(\xi)|q_\xi(y)|^2 \,\mathrm{d}\widehat{\nu}(\xi)}{\int_\Omega |q_\xi(y)|^2 \,\mathrm{d}\widehat{\nu}(\xi)} \qquad (x\in G,\ y\in Y). \tag{2.46}$$

In particular, $\operatorname{Ber}(S)(x,y)$ *does not depend on* x.

Proof. Recall that the Berezin transform $\operatorname{Ber}(S)$ of S is defined by

$$\operatorname{Ber}(S)(x,y) := \frac{\langle SK_{x,y}, K_{x,y}\rangle}{\langle K_{x,y}, K_{x,y}\rangle} \qquad (x\in G,\ y\in Y).$$

Now we apply (2.19) and the hypothesis that S commutes with $\rho_H(x)$:

$$\begin{aligned}\operatorname{Ber}(S)(x,y) &= \frac{1}{\|K_{x,y}\|^2}\langle SK_{x,y}, K_{x,y}\rangle = \frac{1}{\|\rho_H(x)K_{0,y}\|^2}\langle S\rho_H(x)K_{0,y}, \rho_H(x)K_{0,y}\rangle \\ &= \frac{1}{\|K_{0,y}\|^2}\langle \rho_H(x)SK_{0,y}, \rho_H(x)K_{0,y}\rangle = \frac{1}{\|K_{0,y}\|^2}\langle SK_{0,y}, K_{0,y}\rangle \\ &= \frac{1}{\|K_{0,y}\|^2}\langle R^*M_\sigma RK_{0,y}, K_{0,y}\rangle = \frac{1}{\|K_{0,y}\|^2}\langle M_\sigma RK_{0,y}, RK_{0,y}\rangle.\end{aligned}$$

Substituting (2.43) we get (2.46). □

Spectral functions of Toeplitz operators with translation-invariant generating symbols

Given $\varphi \in L^\infty(G \times Y)$, we denote by T_φ the *Toeplitz operator with generating symbol* φ, acting in H by

$$T_\varphi(f) := P(\varphi f) = PM_\varphi f.$$

In the following proposition we compute the spectral function of T_φ, supposing that φ depends only on the Y-component.

Proposition 2.7.11. *Let $\psi \in L^\infty(Y)$. Define $\varphi \in L^\infty(G \times Y)$ by $\varphi(u,v) := \psi(v)$. Then $T_\varphi \in \mathcal{V}$ and $T_\varphi = \Lambda(\gamma_\psi)$, where $\gamma_\psi\colon \Omega \to \mathbb{C}$ is defined by*

$$\gamma_\psi(\xi) := \int_Y \psi(v)|q_\xi(v)|^2 \,\mathrm{d}\lambda(v). \tag{2.47}$$

First proof. It is easy to see that T_φ commutes with the horizontal translations $\rho_H(a)$, $a \in G$. Since $\varphi(u,v)$ does not depend on u, the operator M_φ commutes with $F \otimes I$, and

$$(F \otimes I)PM_\varphi = \widehat{P}(F \otimes I)M_\varphi = \widehat{P}M_\varphi(F \otimes I). \tag{2.48}$$

Let $\xi \in \Omega$ and $y \in Y$ such that $q_\xi(y) \neq 0$. Using (2.48) we simplify $RSK_{0,y}$:

$$\begin{aligned}(RSK_{0,y})(\xi) &= (N(F \otimes I)PM_\varphi K_{0,y})(\xi) = (N\widehat{P}M_\varphi(F \otimes I)K_{0,y})(\xi) = \langle \widehat{P}_\xi M_\varphi L_{\xi,y}, q_\xi\rangle_{L^2(Y)} \\ &= \langle M_\varphi L_{\xi,y}, \widehat{P}_\xi q_\xi\rangle_{L^2(Y)} = \langle M_\varphi \overline{q_\xi(y)} q_\xi, q_\xi\rangle_{L^2(Y)} = \overline{q_\xi(y)} \int_Y \psi(v)|q_\xi(v)|^2 \,\mathrm{d}\lambda(v).\end{aligned}$$

With the help of (2.45) we conclude that $\Lambda^{-1}(T_\varphi) = \gamma_\psi$. □

Second proof. Let us verify directly that $RT_\varphi R^* = M_{\gamma_\psi}$. Given h in $L^2(\Omega)$, we simplify $RT_\varphi R^* h$ applying (2.48):

$$RT_\varphi R^* h = N(F \otimes I)PM_\varphi(F \otimes I)^* N^* h = N\widehat{P}M_\varphi N^* h. \tag{2.49}$$

If $\xi \in \Omega$ and v in Y, then $(M_\varphi N^* h)(\xi, v) = h(\xi)\psi(v)q_\xi(v)$. Therefore,

$$(\widehat{P}M_\varphi N^* h)(\xi, v) = h(\xi)(\widehat{P}_\xi(\psi\, q_\xi))(v),$$

and

$$\begin{aligned}(RT_\varphi R^* h)(\xi) &= \langle(\widehat{P}M_\varphi N^* h)(\xi,\cdot), q_\xi\rangle_{L^2(Y)} = h(\xi)\,\langle\widehat{P}_\xi(\psi q_\xi), q_\xi\rangle_{L^2(Y)} \\ &= h(\xi)\,\langle\psi q_\xi, q_\xi\rangle_{L^2(Y)} = h(\xi)\gamma_\psi(\xi).\end{aligned}$$

□

We denote by $\mathcal{VT}_0$ the set of all Toeplitz operators of the form T_φ, where φ is as in Proposition 2.7.11, and by $\mathcal{G}_0$ the set of the spectral functions of such Toeplitz operators:

$$\mathcal{G}_0 \coloneqq \{\gamma_\psi\colon\ \psi \in L^\infty(Y)\}. \tag{2.50}$$

Let $\mathcal{VT}$ and $\mathcal{G}$ be the C*-algebras generated by $\mathcal{VT}_0$ and $\mathcal{G}_0$, respectively.

Corollary 2.7.12. *The C*-algebra $\mathcal{VT}$ is the image of the C*-algebra $\mathcal{G}$ with respect to the isometric isomorphism Λ. The C*-algebra $\mathcal{VT}$ is weakly dense in $\mathcal{V}$ if and only if the C*-algebra $\mathcal{G}$ is dense in $L^\infty(\Omega)$ with respect to the weak-* topology τ_Ω.*

Proof. Λ is an isometrical isomorphism $L^\infty(\Omega) \to \mathcal{V}$, and is restriction to $\mathcal{G}$ is an isometrical isomorphism from $\mathcal{G}$ onto $\mathcal{VT}$. Moreover, Λ maps the weak-* topology of $L^\infty(\Omega)$ onto the weak operator topology in $\mathcal{V}$. Therefore, $\mathcal{VT}$ is weakly dense in $\mathcal{V}$ if and only if $\mathcal{G}$ is dense in $(L^\infty(\Omega), \tau_\Omega)$. □

Corollary 2.7.12 provides us with a tool to study the C*-algebra $\mathcal{VT}$ generated by Toeplitz operators with translation-invariant generating symbols. A natural problem is to find the C*-algebra generated by all Toeplitz operators with bounded symbols (not necesarily translation-invariant), acting in a RKHS H. Various characterizations of this Toeplitz algebra have been found for the Bergman and Segal–Bargmann–Fock spaces, see Xia [85], Bauer and Fulsche [9], and Hagger [34]. Much earlier, Engliš [20] proved that Toeplitz operators acting in the Bergman space $L^2_{\text{hol}}(\mathbb{D})$ are weakly dense in $\mathcal{B}(L^2_{\text{hol}}(\mathbb{D}))$.

2.8 Non-commutative case with finite-dimensional fibers

This section is a generalization of the previous one. In this section we require Assumption 2 and additionally suppose that $d_\xi \coloneqq \dim(\widehat{H}_\xi)$ is finite for every ξ in Ω. Let $(q_{j,\xi})_{j\in\mathbb{N},\xi\in\Omega}$ be a measurable basis family for the spaces $\widehat{H}_\xi$, like in Proposition 2.5.3. For each ξ in Ω, we denote by Q_ξ the column-vector-function

$$Q_\xi(v) \coloneqq \big[q_{j,\xi}(v)\big]_{j=1}^{d_\xi} \qquad (v \in Y).$$

Its conjugate transpose is the row-vector-function

$$Q_\xi^*(v) = \Big(\big[\,\overline{q_{j,\xi}(v)}\,\big]_{j=1}^{d_\xi}\Big)^\top = \big[\,\overline{q_{1,\xi}(v)}, \ldots, \overline{q_{d_\xi,\xi}(v)}\,\big]_{j=1}^{d_\xi}.$$

Since $\widehat{H}_\xi$ is finite-dimensional, it is a RKHS over Y, and its reproducing kernel $(L_{\xi,y})_{y\in Y}$ can be expressed via the orthonormal basis $q_{1,\xi}, \ldots, q_{d_\xi,\xi}$ of $\widehat{H}_\xi$:

$$L_{\xi,y}(v) = \sum_{j=1}^{d_\xi} \overline{q_{j,\xi}(y)}\, q_{j,\xi}(v) = Q_\xi^*(y) Q_\xi(v). \tag{2.51}$$

When Assumption 3 holds, L can be computed in terms of K by (0.3), and in some examples one can find functions $q_{j,\xi}$ decomposing L like in (2.51).

This section has many similarities with the previous one, thereby we omit detailed proofs.

We denote by $\mathcal{X}$ the following direct integral of Hilbert spaces $\mathbb{C}^{d_\xi}$:

$$\mathcal{X} \coloneqq \int_\Omega^\oplus \mathbb{C}^{d_\xi}\, \mathrm{d}\widehat{\nu}(\xi).$$

The elements of $\mathcal{X}$ are classes of equivalence of vector sequences, component-wise measurable on $\{\xi \in \Omega\colon\ d_\xi = m\}$ for every m, and square-integrable. We define $N\colon \widehat{H} \to \mathcal{X}$ by

$$(Ng)(\xi) \coloneqq \Big[\langle g(\xi,\cdot), q_{j,\xi}\rangle_{\widehat{H}_\xi}\Big]_{j=1}^{d_\xi} = \left[\int_Y \overline{q_{j,\xi}(v)} g(\xi,v)\,\mathrm{d}\lambda(v)\right]_{j=1}^{d_\xi} = \int_Y \overline{Q_\xi(v)}\, g(\xi,v)\,\mathrm{d}\lambda(v). \tag{2.52}$$

Proposition 2.8.1. *N is a unitary operator from $\widehat{H}$ onto $\mathcal{X}$. Its inverse $N^*\colon \mathcal{X} \to \widehat{H}$ acts by the following rule:*

$$(N^*h)(\xi, y) = \sum_{j=1}^{d_\xi} q_{j,\xi}(y) h_j(\xi) = Q_\xi^\top(y) h(\xi) \qquad (h \in \mathcal{X},\ \xi \in \widehat{G},\ y \in Y). \tag{2.53}$$

Proof. We apply Proposition 2.5.4 and use the isomorphism between $\widehat{H}_\xi$ and $\mathbb{C}^\xi$ induced by the orthonormal basis $(q_{j,\xi})_{j=1}^{d_\xi}$. □

In particular, formula (2.53) tells us that $(N^*h)(\xi, y) = 0$ for ξ in $\widehat{G} \setminus \Omega$, because the corresponding sum in (2.53) is empty.

We define $R\colon H \to \mathcal{X}$ as the composition $R \coloneqq N\Phi$.

Proposition 2.8.2. *R is a unitary operator from H onto $\mathcal{X}$.*

Proposition 2.8.3. *Let $\xi \in \Omega$ and $y \in Y$. Then*

$$(RK_{0,y})(\xi) = \left[\,\overline{q_{j,\xi}(y)}\,\right]_{j=1}^{d_\xi} = \overline{Q_\xi(y)}. \tag{2.54}$$

We denote by $\mathcal{Z}$ the following direct integral of matrix algebras:

$$\mathcal{Z} \coloneqq \int_\Omega^\oplus \mathbb{C}^{d_\xi \times d_\xi}\, \mathrm{d}\widehat{\nu}(\xi). \tag{2.55}$$

Given a matrix family $\sigma = (\sigma(\xi))_{\xi\in\Omega}$ in $\mathcal{Z}$, let M_σ be the "multiplication operator" acting in $\mathcal{X}$ by

$$(M_\sigma h)(\xi) \coloneqq \sigma(\xi) h(\xi).$$

Finally, we define $\Lambda\colon \mathcal{Z} \to \mathcal{V}$ by $\Lambda(\sigma) \coloneqq R^* M_\sigma R$.

Theorem 2.8.4 (from shift-invariant operators to matrix families)**.** *Λ is an isometric isomorphism of the W*-algebras $\mathcal{Z}$ and $\mathcal{V}$.*

Idea of the proof. Follows from Theorem 2.5.5, converting each $\mathcal{B}(\widehat{H}_\xi)$ into $\mathbb{C}^{d_\xi \times d_\xi}$. □

Corollary 2.8.5. *Let $S \in \mathcal{V}$. Then for every ξ in Ω,*

$$(\Lambda^{-1}(S))(\xi) = \left[(RSK_{0,y_1})(\xi), \ldots, (RSK_{0,y_{d_\xi}})(\xi)\right] \left[\overline{Q_\xi(y_1)}, \ldots, \overline{Q_\xi(y_{d_\xi})}\right]^{-1}, \tag{2.56}$$

where $y_1, \ldots, y_{d_\xi}$ are chosen in Y such that the vectors $Q_\xi(y_1), \ldots, Q_\xi(y_{d_\xi})$ are linearly independent.

Proof. Let $\sigma \coloneqq \Lambda^{-1}(S)$, i.e., $RS = M_\sigma R$. Then, by (2.54),

$$(RSK_{0,y})(\xi) = (M_\sigma RK_{0,y})(\xi) = \sigma(\xi)\overline{Q_\xi(y)},$$

Apply the above equality to the points $y_1, \ldots, y_{d_\xi}$, then join the resulting columns:

$$\big[(RSK_{0,y_1})(\xi), \ldots, (RSK_{0,y_{d_\xi}})(\xi)\big] = \sigma(\xi)\big[\overline{Q_\xi(y_1)}, \ldots, \overline{Q_\xi(y_{d_\xi})}\big].$$

Solving this matrix equation for $\sigma(\xi)$ we get (2.56). ☐

Proposition 2.8.6 (Berezin transform of a translation-invariant operator)**.** *Let Assumption 3 holds, $S \in \mathcal{V}$, and let $\sigma \in \mathcal{Z}$ such that $S = \Lambda(\sigma)$. Then*

$$\operatorname{Ber}(S)(x,y) = \frac{\int_\Omega \sigma(\xi) L_{\xi,y}(y)\,\mathrm{d}\widehat{\nu}(\xi)}{\int_\Omega L_{\xi,y}(y)\,\mathrm{d}\widehat{\nu}(\xi)} \qquad (x \in G,\ y \in Y). \tag{2.57}$$

In particular, $\operatorname{Ber}(S)(x,y)$ *does not depend on* x.

Proof. Similar to the proof of Proposition 2.7.10, but applying (2.54). ☐

Proposition 2.8.7 (matrix families corresponding to Toeplitz operators with translation-invariant generating symbols)**.** *Let $\psi \in L^\infty(Y)$. Define $\varphi \in L^\infty(G \times Y)$ by $\varphi(x,y) = \psi(y)$. Then $T_\varphi = \Lambda(\gamma_\psi)$, where*

$$\gamma_\psi(\xi) \coloneqq \int_Y \psi(v)\overline{Q_\xi(v)}Q_\xi^*(v)\,\mathrm{d}\lambda(v) = \left[\int_Y \psi(v)\overline{q_{j,\xi}(v)}q_{k,\xi}(v)\,\mathrm{d}\lambda(v)\right]_{j,k=1}^{d_\xi}. \tag{2.58}$$

Proof. We will verify that $RT_\varphi R^* = M_{\gamma_\psi}$. Same as in the proof of Proposition 2.7.11, we get (2.48). If $\xi \in \Omega$ and v in Y, then

$$(M_\varphi N^* h)(\xi, v) = \psi(v)Q_\xi^\top(v)h(\xi) = \sum_{j=1}^{d_\xi} h_j(\xi)q_{j,\xi}(v)\psi(v)$$

Therefore,

$$(\widehat{P}M_\varphi N^* h)(\xi,\cdot) = \widehat{P}_\xi\big((M_\varphi N^* h)(\xi,\cdot)\big) = \sum_{k=1}^{d_\xi} h_k(\xi)\widehat{P}_\xi(q_{k,\xi}\psi),$$

and

$$(RT_\varphi R^* h)(\xi) = \big[\langle(\widehat{P}M_\varphi N^* h)(\xi,\cdot), q_{j,\xi}\rangle_{L^2(Y)}\big]_{j=1}^{d_\xi} = \left[\sum_{k=1}^{d_\xi} h_k(\xi)\langle\widehat{P}_\xi(q_{k,\xi}\psi), q_{j,\xi}\rangle_{L^2(Y)}\right]_{j=1}^{d_\xi}$$

$$= \left[\sum_{k=1}^{d_\xi}\langle\psi q_{k,\xi}, q_{j,\xi}\rangle_{L^2(Y)} h_k(\xi)\right]_{j=1}^{d_\xi} = \gamma_\psi(\xi)h(\xi).$$ ☐

2.9 Examples

To keep this work to a reasonable length, we restrict ourself to a series of just 9 simple examples, mostly with one-dimensional domains G and Y. Example 2.9.11 is probably new. In the other examples, the spectral functions γ_σ of Toeplitz operators are already known. Nevertheless, the description of the whole W*-algebra $\mathcal{V}$ is new for some of these "old examples". We notice that the C*-algebra $\mathcal{VT}$ from Examples 2.9.2 and 2.9.8 is not weakly dense in $\mathcal{V}$.

We use the following notation: μ_n is the Lebesgue measure on $\mathbb{R}^n$ or a subset of $\mathbb{R}^n$; $\mathbb{R}_+ \coloneqq (0,+\infty)$, $\mathbb{N}_0 \coloneqq \{0,1,2,\ldots\}$, $\mathbb{T} \coloneqq \{\tau \in \mathbb{C}\colon\ |\tau| = 1\}$, 1_A is the characteristic function of A; its domain is clear from the context.

In this section, given a LCAG G, we denote by $\widehat{G}$ a LCAG topologically isomorphic to the dual group of G, and we use some pairing $E\colon G \times \widehat{G} \to \mathbb{T}$. This means that $\xi \mapsto E(\cdot,\xi)$ is a topological isomorphism between $\widehat{G}$ and the dual group of G. We select the Haar measures $\nu, \widehat{\nu}$ on $G, \widehat{G}$ in such a manner that the Fourier–Plancherel operator F is unitary. For example, if $G = \mathbb{R}$, then we put $\widehat{G} = \mathbb{R}$. One possible pairing is $E(x,\xi) = \mathrm{e}^{\mathrm{i}\,x\xi}$ with the measures $\nu = \widehat{\nu} = \frac{1}{\sqrt{2\pi}}\mu_1$; another one is $E(x,\xi) = \mathrm{e}^{2\pi\,\mathrm{i}\,x\xi}$, with $\nu = \widehat{\nu} = \mu_1$.

For each example we have verified assumption (2.20), but we have omitted the corresponding computation, for the sake of brevity.

Example 2.9.1 (vertical operators in the holomorphic Bergman space over the upper half-plane)**.** Let $\Pi \coloneqq \mathbb{R} \times \mathbb{R}_+$ and $H = L^2_{\text{hol}}(\Pi)$. In this example, $G = \widehat{G} = \mathbb{R}$, $Y = \mathbb{R}_+$, $\nu = \widehat{\nu} = \frac{1}{\sqrt{2\pi}}\mu_1$, $E(x,\xi) = \mathrm{e}^{\mathrm{i}\,x\xi}$, $\lambda = \sqrt{2\pi}\mu_1$, $\nu \times \lambda = \mu_2$, It is well known that H is a Hilbert space with reproducing kernel

$$K_z(w) = -\frac{1}{\pi(w-\overline{z})^2}.$$

Identifying z with (x,y) and w with (u,v), we rewrite the reproducing kernel as

$$K_{x,y}(u,v) = -\frac{1}{\pi((u-x)+\mathrm{i}(v+y))^2}.$$

The space H is invariant under horizontal translations. A simple computation with residues shows that

$$L_{\xi,y}(v) = \sqrt{\frac{2}{\pi}}\,\xi\ \mathrm{e}^{-\xi(y+v)}\ 1_{\mathbb{R}_+}(\xi).$$

So, in this example $\mathcal{V}$ is commutative, $\Omega = \mathbb{R}_+$, and

$$q_\xi(v) = \left(\frac{2}{\pi}\right)^{1/4} \sqrt{\xi}\ \mathrm{e}^{-\xi v}\ 1_{\mathbb{R}_+}(\xi).$$

Using (2.47) we compute the spectral functions of vertical Toeplitz operators:

$$\gamma_\sigma(\xi) = 2\xi \int_{\mathbb{R}_+} \sigma(v)\,\mathrm{e}^{-2\xi v}\ \mathrm{d}v \qquad (\xi > 0).$$

This formula coincides with Vasilevski [79, Theorem 3.1] and [82, Theorem 5.2.1], see also Grudsky, Karapetyants, and Vasilevski [28]. The C*-algebra $\mathcal{G}$ in this example consists of all bounded functions on $\mathbb{R}_+$, uniformly continuous with respect to the $\log$-distance, see [37, 39].

Example 2.9.2 (vertical operators in the harmonic Bergman space over the upper half–plane)**.** Let G, Y, ν, λ, and E be the same as in Example 2.9.1, but $H := L^2_{\text{harm}}(\Pi)$ be the Bergman space of harmonic functions on Π. Using Riesz theorem about the Hardy spaces of harmonic functions, one can show that $L^2_{\text{harm}}(\Pi) = L^2_{\text{hol}}(\Pi) \oplus \overline{L^2_{\text{hol}}(\Pi)}$. Therefore, H is a RKHS with reproducing kernel

$$K_z(w) = -\frac{1}{\pi(w-\overline{z})^2} - \frac{1}{\pi(\overline{w}-z)^2}.$$

Identifying z with (x,y) and w with (u,v), we obtain

$$K_{x,y}(u,v) = -\frac{1}{\pi((u-x)+\mathrm{i}(v+y))^2} - \frac{1}{\pi((u-x)-\mathrm{i}(v+y))^2}.$$

Now

$$L_{\xi,y}(v) = \sqrt{\frac{2}{\pi}}\,|\xi|\ \mathrm{e}^{-|\xi|(y+v)} \qquad (\xi \in \mathbb{R}).$$

We conclude that in this example $\mathcal{V}$ is commutative, $\Omega = \mathbb{R} \setminus \{0\}$,

$$q_\xi(v) = \left(\frac{2}{\pi}\right)^{1/4} \sqrt{|\xi|}\ \mathrm{e}^{-2|\xi|v},$$

and

$$\gamma_\sigma(\xi) = \gamma_\sigma(|\xi|) = 2|\xi| \int_{\mathbb{R}_+} \sigma(v)\,\mathrm{e}^{-2|\xi|v}\ \mathrm{d}v.$$

Thereby we reproduce a result by Loaiza and Lozano [54, Theorem 4.16]. In this example, the spectral functions γ_σ are even. The C*-algebra $\mathcal{G}$ generated by $\mathcal{G}_0$ coincides with the closure of $\mathcal{G}_0$ in the norm topology and consists of all even function on $\mathbb{R} \setminus \{0\}$

whose restrictions to $\mathbb{R}_+$ are uniformly continuous with respect to the $\log$-distance. By Theorem 2.2.2, the W*-algebra generated by $\mathcal{G}_0$ is the class of all essentially bounded even functions on $\mathbb{R}$, which is a proper subset of $L^\infty(\mathbb{R})$. So, $\mathcal{G}$ is not τ_Ω-dense in $L^\infty(\Omega)$. By Corollary 2.7.12, this means that $\mathcal{VT}$ is not weakly dense in $\mathcal{V}$.

Example 2.9.3 (vertical operators in the Bergman space of true-polyanalytic functions over the upper half-plane)**.** Let G, Y, ν, $\widehat{\nu}$, λ, E be the same as in Example 2.9.1. For a fixed m in $\mathbb{N}$, we consider the space $H \coloneqq L^2_{(m)\text{-hol}}(\Pi)$ of all square-integrable m-true-polyanalytic functions on the upper half-plane Π. Applying the Fourier transform to the differential equation defining H, Vasilevski computed [82, Section 3.4] the operator $(F \otimes I)P(F \otimes I)^*$ which we denote by $\widehat{P}$. Namely, he proved that $\widehat{P}$ acts by (2.32), with

$$L_{\xi,y}(v) = 1_{\mathbb{R}_+}(\xi)\sqrt{\frac{2}{\pi}}\,\xi\,\mathrm{e}^{-\xi(y+v)}\,L_{m-1}(2\xi y)L_{m-1}(2\xi v), \tag{2.59}$$

where L_k is the Laguerre polynomial of degree k. This means that $\Omega = \mathbb{R}_+$,

$$q_\xi(v) = \sqrt{2\xi}\,\mathrm{e}^{-\xi v}\,L_{m-1}(2\xi v)\,1_{\mathbb{R}_+}(\xi) \qquad (\xi \in \mathbb{R},\ v > 0),$$

and

$$\gamma_\sigma(\xi) = 2\xi \int_{\mathbb{R}_+} \sigma(v)\,\mathrm{e}^{-2\xi v}(L_{m-1}(2\xi v))^2\,\mathrm{d}v. \tag{2.60}$$

Formula (2.60) was found by Hutník [44, Theorem 3.2] and by Ramírez-Ortega and Sánchez-Nungaray [65, Theorem 3.2]. The C*-algebra $\mathcal{G}$ for this example coincides with the C*-algebra $\mathcal{G}$ from Example 2.9.1, see [48]. Vasilevski noticed [82, Theorem 3.4.1] that the reproducing kernel of $L^2_{(m)\text{-hol}}(\Pi)$ can be obtained by applying $(F \otimes I)^*$ to L given by (2.59). Using explicit expressions for the Laguerre polynomials one obtains

$$K_z(w) = -\frac{1}{(w-\overline{z})^2}\sum_{j,k=0}^{m-1}(-1)^{j+k}\frac{(m-1)!\,(j+k+1)!}{(j!\,k!)^2\,(m-1-j)!\,(m-1-k)!}\,\frac{(w-\overline{w})^j\,(z-\overline{z})^k}{(w-\overline{z})^{j+k}}. \tag{2.61}$$

Example 2.9.4 (vertical operators in the Bergman space of polyanalytic functions over the upper half-plane)**.** Here G and Y are the same as in Example 2.9.3, and $H = L^2_{n\text{-hol}}(\Pi)$ is the space of square-integrable n-analytic functions on Π. The decomposition $H = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, where $\mathcal{H}_m$ is the space from Example 2.9.3, implies that

$$L_{\xi,y}(v) = 1_{\mathbb{R}_+}(\xi)\sqrt{\frac{2}{\pi}}\,\xi\,\mathrm{e}^{-\xi(y+v)}\sum_{m=1}^{n} L_{m-1}(2\xi y)L_{m-1}(2\xi v). \tag{2.62}$$

It would be interesting to prove (2.62) directly, applying the Fourier transform to the reproducing kernel (1.70) of $\mathcal{A}^2_m(\mathbb{H}_1,\nu_\alpha)$, computed in [58] and [53] in terms of Jacobi polynomials:

$$K_z^H(w) = \frac{n\,(-1)^n}{\pi}\frac{(z-\overline{w})^{n-1}}{(w-\overline{z})^{n+1}}\,P_{n-1}^{(0,1)}\left(2\,\frac{|w-z|^2}{|w-\overline{z}|^2}-1\right). \tag{2.63}$$

The orthogonality of the Laguerre polynomials implies that (2.62) is a particular case of (2.51), with $\Omega=\mathbb{R}_+$, $d_\xi=n$, and

$$q_{j,\xi}(v) = (2/\pi)^{1/4}\,\sqrt{\xi}\,\mathrm{e}^{-\xi v}\,L_{j-1}(2\xi v) \qquad (j=1,\ldots,n,\ \xi>0,\ v>0).$$

Thereby, the W*-algebra $\mathcal{V}$ in this example is spatially isomorphic to the direct integral of matrix algebras,

$$\mathcal{V}\cong\int_{\mathbb{R}_+}^{\oplus}\mathbb{C}^{n\times n}\,\mathrm{d}\widehat{\nu}(\xi)\cong L^\infty(\mathbb{R}_+,\mathbb{C}^{n\times n}).$$

Ramírez-Ortega and Sánchez–Nungaray [65, Theorem 4.7] found a complete description of a certain non-commutative C*-subalgebra of $\mathcal{V}\mathcal{T}$.

Example 2.9.5 (translation-invariant operators in wavelet spaces over the positive affine group)**.** Let ψ be a wavelet of the class $L^2(\mathbb{R})$ satisfying the admissibility condition:

$$\int_{\mathbb{R}_+}|(F\psi)(t\xi)|^2\,\frac{\mathrm{d}t}{t}=1 \qquad (\xi\in\mathbb{R}\setminus\{0\}), \qquad (F\psi)(0)=0. \tag{2.64}$$

Put $G=\mathbb{R}$, $\nu=\widehat{\nu}=\mu_1$, $E(x,\xi)=\mathrm{e}^{2\pi\,\mathrm{i}\,x\xi}$, $Y=\mathbb{R}_+$, $\mathrm{d}\lambda(y)=\frac{\mathrm{d}y}{y^2}$. Notice that $G\times Y$ can be identified with the positive affine group. For every (x,y) in $G\times Y$, put

$$\psi_{x,y}(t)=\frac{1}{\sqrt{y}}\psi\left(\frac{t-x}{y}\right).$$

Define $W_\psi\colon L^2(\mathbb{R})\to L^2(G\times Y)$ by

$$(W_\psi f)(x,y)\coloneqq\langle f,\psi_{x,y}\rangle_{L^2(\mathbb{R})}.$$

The wavelet space H associated with ψ can be defined as $W_\psi(L^2(\mathbb{R}))$. It is a RKHS over $G\times Y$, with reproducing kernel

$$K_{x,y}(u,v)=\langle\psi_{u,v},\psi_{x,y}\rangle_{L^2(\mathbb{R})}=\langle\psi_{u-x,v},\psi_{0,y}\rangle_{L^2(\mathbb{R})}.$$

Then

$$L_{\xi,y}(v)=\sqrt{yv}\,(F\psi)(y\xi)\,\overline{(F\psi)(v\xi)}.$$

So, in this example $\Omega = \mathbb{R}$, $\mathcal{V}$ is commutative, and

$$q_\xi(v) = \sqrt{v}\,\overline{(F\psi)(v\xi)}.$$

The property $\|q_\xi\|_{L^2(Y)} = 1$ follows from (2.64). The spectral functions are given by

$$\gamma_\sigma(\xi) = \int_{\mathbb{R}_+} \sigma(v)\,|(F\psi)(v\xi)|^2\,\frac{\mathrm{d}v}{v}.$$

This formula was found by Hutník and Hutníková [45].

Let us mention without further details another similar example, studied by Hutníková and Miśková [47]: translation-invariant operators in the space related to the continuous Stockwell transform.

In some examples, it is convenient to transform the domain of the functions and the RKHS. The next simple proposition provides a recipe to compute the reproducing kernel after a change of variables followed by the multiplication by some weight.

Proposition 2.9.6. *Let D_1 and D_2 be some non-empty sets, $\mathcal{H}_1$ be a RKHS over D_1, with reproducing kernel $(K_z^{\mathcal{H}_1})_{z\in D_1}$, and $\mathcal{H}_2$ be a complex vector space of functions over D_2, with a pre-inner product. Suppose that A is a linear isometry from $\mathcal{H}_1$ to $\mathcal{H}_2$, acting by the rule*

$$(Af)(z) = p(z)f(\varphi(z)) \qquad (z \in D_2,\ f \in \mathcal{H}_1),$$

where $\varphi\colon D_2 \to D_1$ and $p\colon D_2 \to \mathbb{C}$. Then $H := A(\mathcal{H}_1)$ is a RKHS over D_2, and the reproducing kernel in H can be computed by

$$K_z^H(w) = \overline{p(z)}K_{\varphi(z)}^{\mathcal{H}_1}(\varphi(w))p(w). \tag{2.65}$$

Proof. Since A is a linear isometry and $\mathcal{H}_1$ is a Hilbert space, H is also a Hilbert space. The rest of the proof is the same as in [53, Proposition 4.3]. A similar construction is explained in [3, Section 2.6]. □

Example 2.9.7 (radial operators in the analytic Bergman space over the unit disk)**.** Let $\mathcal{H}_1 = L^2_{\mathrm{hol}}(\mathbb{D})$ be the Bergman space of analytic functions over the unit disk $\mathbb{D}$ provided with the plane Lebesgue measure μ_2. It is well known that the reproducing kernel of $\mathcal{H}_1$ is

$$K_z^{\mathcal{H}_1}(w) = \frac{1}{\pi(1-\overline{z}w)^2}.$$

Let G be the group $\mathbb{R}/(2\pi\mathbb{Z})$ with the normalized Haar measure ν (we identify G with $[0,2\pi)$), $\widehat{G}=\mathbb{Z}$ with the counting measure $\widehat{\nu}$, $E(u+2\pi\mathbb{Z},\xi)=\mathrm{e}^{\mathrm{i}\,u\xi}$ for $u\in\mathbb{R}$ and ξ in $\mathbb{Z}$, and Y be the interval $[0,1)$ with the measure $\mathrm{d}\lambda(v)=v\,\mathrm{d}v$. Define $\varphi\colon G\times Y\to\mathbb{D}$ and $p\colon G\times Y\to\mathbb{C}$ by

$$\varphi(u,v)=v\,\mathrm{e}^{\mathrm{i}\,u},\qquad p(u,v)=\sqrt{2\pi}.$$

Let $\mathcal{H}_2=L^2(G\times Y,\nu\otimes\lambda)$. The operator A, defined as Proposition 2.9.6, is a linear isometry:

$$\|Af\|_{\mathcal{H}_2}^2=\int_0^1\int_0^{2\pi}|f(v\,\mathrm{e}^{\mathrm{i}\,u})|^2\,v\,\mathrm{d}u\,\mathrm{d}v=\int_{\mathbb{D}}|f(z)|^2\mathrm{d}\mu_2(z)=\|f\|_{\mathcal{H}_1}^2.$$

Hence, A converts $\mathcal{H}_1$ into a certain RKHS H over $G\times Y$, with reproducing kernel

$$K_{x,y}(u,v)=\overline{p(x,y)}\,K_{\varphi(x,y)}^{\mathcal{H}_1}(\varphi(u,v))\,p(u,v)=\frac{2}{(1-yv\,\mathrm{e}^{\mathrm{i}(u-x)})^2}.$$

Obviously, A intertwines the rotation operators acting in $\mathcal{H}_1$ into “horizontal translations” acting in H. Now we notice that the function $K_{0,y}(\cdot\,,v)$ decomposes into the Fourier series

$$K_{0,y}(u,v)=\sum_{\xi=0}^{\infty}2(\xi+1)\,(yv)^{\xi}\;\mathrm{e}^{\mathrm{i}\,\xi u},$$

which means that its Fourier coefficients are

$$L_{\xi,y}(v)=2(\xi+1)(yv)^{\xi}\,1_{\mathbb{N}_0}(\xi).$$

Thus, in this example, $\Omega=\mathbb{N}_0$ and $q_\xi(v)=\sqrt{2(\xi+1)}\,v^{\xi}$. The W*-algebra of radial operators in $\mathcal{H}_1$ is commutative, and the sequence of the eigenvalues of a radial Toeplitz operator is computed by

$$\gamma_\sigma(\xi)=2(\xi+1)\int_0^1\sigma(v)\,v^{2\xi+1}\,\mathrm{d}v=(\xi+1)\int_0^1\sigma(\sqrt{r})\,r^{\xi}\,\mathrm{d}r\qquad(\xi\in\mathbb{N}_0).$$

These results are well known and easily obtained from the fact that the radial operators are diagonal in the monomial basis $(\sqrt{(\xi+1)/\pi}\,z^{\xi})_{\xi=0}^{\infty}$. Our treatment of this example is close to [82, Chapters 4, 6] and [30], where $L^2(\mathbb{D},\mu_2)$ is decomposed into $L^2(\mathbb{R}/(2\pi\mathbb{Z}))\otimes L^2([0,1),r\,\mathrm{d}r)$, and the Fourier transform over $\mathbb{R}/(2\pi\mathbb{Z})$ is applied to the equation defining $\mathcal{H}_1$. The C*-algebra $\mathcal{VT}$ for this example was described in [32] using Suárez [76].

Radial operators in the Segal–Bargmann–Fock space on $\mathbb{C}$ can be studied similarly to Example 2.9.7. Moreover, Example 2.9.7 is easily generalized to the case of separately radial operators acting on the Bergman space over the unit ball in $\mathbb{C}^n$. In that case $G=(\mathbb{R}/(2\pi\mathbb{Z}))^n$ and $\Omega=\mathbb{N}_0^n$.

Example 2.9.8 (radial operators in the harmonic Bergman space over the unit disk)**.** For $\mathcal{H}_1 = L^2_{\text{harm}}(\mathbb{D})$,

$$K_z^{\mathcal{H}_1}(w) = \frac{1}{\pi(1-\overline{z}w)^2} + \frac{1}{\pi(1-\overline{w}z)^2} - 1.$$

Similarly to Example 2.9.7, after passing to the polar coordinates and computing the Fourier coefficients, we have

$$L_{\xi,y}(v) = 2(|\xi|+1)(yv)^{|\xi|} \qquad (\xi \in \mathbb{Z},\ y, v \in [0,1)).$$

The W*-algebra of radial operators in $L^2_{\text{harm}}(\mathbb{D})$ is commutative, $\Omega = \mathbb{Z}$, $q_\xi(v) = \sqrt{2(|\xi|+1)}\, v^{|\xi|}$, and

$$\gamma_\sigma(\xi) = (|\xi|+1) \int_0^1 \sigma(\sqrt{r})\, r^{|\xi|}\, \mathrm{d}r. \tag{2.66}$$

Formula (2.66) was previously obtained by Loaiza and Lozano [54, Theorem 3.4].

Similarly to Example 2.9.2, the symmetry of formula (2.66) with respect to the sign of ξ implies that $\mathcal{G}$ is a subclass of bounded symmetric sequences. By Corollary 2.7.12, the C*-algebra generated by Toeplitz operators with radial symbols is not weakly dense in the W*-algebra of all bounded radial operators on $L^2_{\text{harm}}(\mathbb{D})$.

Remark 2.9.9. Since the radialization transform of bounded linear operators in $L^2_{\text{harm}}(\mathbb{D})$ is continuous in WOT and converts Toeplitz operators into radial Toeplitz operators, the last paragraph of Example 2.9.8 implies that the set of all Toeplitz operators is not weakly dense in $\mathcal{B}(L^2_{\text{harm}}(\mathbb{D}))$. This result was proven more directly in [8]. In contrast, the weak density of Toeplitz operators $\mathcal{B}(L^2_{\text{hol}}(\mathbb{D}))$ has already been proven by Engliš [20].

Example 2.9.10 (angular operators in the analytic Bergman space over the upper half–plane)**.** Let $\mathcal{H}_1 = L^2_{\text{hol}}(\Pi)$. We say that an operator A of the class $\mathcal{B}(\mathcal{H}_1)$ is *angular* if A commutes with all dilations D_h $(h > 0)$, where D_h is given by

$$(D_h f)(w) = h^{-1} f(h^{-1} w).$$

Let $G = \mathbb{R}$, $Y = (0, \pi)$, $\nu = \widehat{\nu} = \frac{1}{\sqrt{2\pi}}\mu_1$, $E(x,\xi) = \mathrm{e}^{\mathrm{i}\, x\xi}$, and λ be the Lebesgue measure on $(0,\pi)$. Define $\varphi\colon G \times Y \to \Pi$, $p\colon G \times Y \to \mathbb{C}$, and $A\colon \mathcal{H}_1 \to L^2(G \times Y)$ by

$$\varphi(u,v) \coloneqq \mathrm{e}^{u+\mathrm{i}\, v}, \quad p(u,v) \coloneqq (2\pi)^{1/4}\, \mathrm{e}^{u+\mathrm{i}\, v}, \quad (Af)(u,v) = (2\pi)^{1/4}\, \mathrm{e}^{u+\mathrm{i}\, v}\, f(\mathrm{e}^{u+\mathrm{i}\, v}).$$

It is easy to see that A is a linear isometry, so we can apply Proposition 2.9.6. The space $H \coloneqq A(\mathcal{H}_1)$ has reproducing kernel

$$K_{x,y}(u,v) = -\sqrt{\frac{2}{\pi}}\,\frac{\mathrm{e}^{x-\mathrm{i}\,y}\ \mathrm{e}^{u+\mathrm{i}\,v}}{\left(\mathrm{e}^{u+\mathrm{i}\,v}-\mathrm{e}^{x-\mathrm{i}\,y}\right)^2} = -\sqrt{\frac{2}{\pi}}\,\frac{1}{4\left(\sinh\frac{u-x+\mathrm{i}(v+y)}{2}\right)^2}.$$

The linear isometry A intertwines the dilations, acting in $\mathcal{H}_1$, with the horizontal translations, acting in H:

$$AD_{\mathrm{e}^a}A^* = \rho_H(a).$$

Hence, the algebra of angular operators in $\mathcal{H}_1$ is converted into $\mathcal{V}$. For $\xi > 0$, integral (0.3) can be computed via the residues at the points $u_k \coloneqq -\,\mathrm{i}(v+y) - 2\pi\,\mathrm{i}\,k$, $k \in \mathbb{N}_0$:

$$\begin{aligned} L_{\xi,y}(v) &= -\frac{1}{4\pi}\int_{\mathbb{R}}\frac{\mathrm{e}^{-\mathrm{i}\,\xi u}\ \mathrm{d}u}{\left(\sinh\frac{u+\mathrm{i}(v+y)}{2}\right)^2} = \frac{\mathrm{i}}{2}\sum_{k=0}^{\infty}\operatorname*{res}_{u=u_k}\frac{\mathrm{e}^{-\mathrm{i}\,\xi u}}{\left(\sinh\frac{u+\mathrm{i}(v+y)}{2}\right)^2} \\ &= \frac{\mathrm{i}}{2}\sum_{k=0}^{\infty}\left(-4\,\mathrm{i}\,\xi\,\mathrm{e}^{-\xi(v+y)}\ \mathrm{e}^{-2k\pi\xi}\right) = \frac{2\xi\ \mathrm{e}^{-\xi(y+v)}}{1-\mathrm{e}^{-2\pi\xi}}. \end{aligned}$$

For $\xi < 0$, the integral expresses through the residues at the points u_k with $k < 0$, but the final formula for $L_{\xi,y}(v)$ is the same. We conclude that $\mathcal{V}$ is commutative, $\Omega = \mathbb{R}$, and

$$q_\xi(v) = \sqrt{\frac{2\xi}{1-\mathrm{e}^{-2\pi\xi}}}\ \mathrm{e}^{-\xi v}.$$

The spectral functions of angular Toeplitz operators can be computed by

$$\gamma_\sigma(\xi) = \frac{2\xi}{1-\mathrm{e}^{-2\pi\xi}}\int_0^{\pi}\sigma(v)\ \mathrm{e}^{-2\xi v}\ \mathrm{d}v.$$

This formula coincides with [82, Theorem 7.2.1], see also [29]. The C*-algebra $\mathcal{G}$ for this example is found by Esmeral, Maximenko, and Vasilevski [23].

Example 2.9.11 (the radial basis function kernel on the complex domain)**.** The following reproducing kernel and its restriction to $\mathbb{R}^n$ are extensively used in machine learning:

$$K_z(w) = \exp\left(-\alpha^2\sum_{j=1}^{n}(z_j-\overline{w_j})^2\right) \qquad (z,w \in \mathbb{C}^n).$$

Here α is a fixed positive number. Steinwart, Hush, and Scovel [75] proved that the corresponding RKHS is $H = \{f \in \operatorname{Hol}(\mathbb{C}^n)\colon\ \|f\|_{\mathrm{RBFK}} < +\infty\}$, where

$$\|f\|_{\mathrm{RBFK}} \coloneqq \left(\frac{2^n\alpha^{2n}}{\pi^n}\int_{\mathbb{C}^n}|f(z)|^2\exp\left(-4\alpha^2\sum_{j=1}^{n}\operatorname{Im}(z_j)^2\right)\mathrm{d}\mu_{2n}(z)\right)^{1/2}.$$

We identify the domain $\mathbb{C}^n$ with $G \times Y$, where $G = Y = \mathbb{R}^n$. The measures and the pairing are

$$\nu = \widehat{\nu} = \mu_n, \qquad \mathrm{d}\lambda(v) = \frac{2^n \alpha^{2n}}{\pi^n} \exp\left(-4\alpha^2 \|v\|^2\right), \qquad E(x,y) = \exp\left(2\pi\,\mathrm{i}\langle x, y\rangle\right).$$

Then the kernel takes the form

$$K_{x,y}(u,v) = \exp\left(-\alpha^2 \sum_{j=1}^{n} ((u_j - x_j)^2 - (v_j + y_j)^2 + 2\,\mathrm{i}(u_j - x_j)(v_j + y_j))\right).$$

The computation of $L_{\xi,y}(v)$ can be reduced to the Gaussian integral and results in

$$L_{\xi,y}(v) = \left(\frac{\sqrt{\pi}}{\alpha}\right)^n \exp\left(-\sum_{j=1}^{n} \left(2\pi(v_j + y_j)\xi_j + \frac{\pi^2 \xi_j^2}{\alpha^2}\right)\right).$$

In this example, $\Omega = \mathbb{R}^n$, $\mathcal{V}$ is commutative, and

$$q_\xi(v) = \left(\frac{\sqrt{\pi}}{\alpha}\right)^{n/2} \exp\left(-\sum_{j=1}^{n} \left(2\pi v_j \xi_j + \frac{\pi^2 \xi_j^2}{2\alpha^2}\right)\right).$$

Remark 2.9.12. For each example, we tested the equality $((F \otimes I)K_{0,y})(\xi, v) = L_{\xi,y}(v)$ numerically in Sagemath. In Example 2.9.5, we used the Mexican hat wavelet.

3 Radial operators in the poly-Bergman space

3.1 Scope

In this section, we describe the von-Neumann algebra of radial operators acting in the polyanalytic weighted Bergman space over the unit disk. This is the case when $n = 1$ in Chapter 1. Unfortunately, although this example may fit in the scheme described in Chapter 2, we have been unable to compute the Fourier transform of the reproducing kernel, and, instead, we used the canonical orthonormal basis to provide the Fourier decomposition of the space following ideas by Čučković [13]. All these calculations were published in [8].

Jacobi polynomials for the unit interval

The function $t \mapsto 2t - 1$ is a bijection from $(0, 1)$ onto $(-1, 1)$. Denote by $Q_m^{(\alpha,\beta)}$ the "shifted Jacobi polynomial" obtained from $P_m^{(\alpha,\beta)}$ by composing it with this change of variables:

$$Q_m^{(\alpha,\beta)}(t) \coloneqq P_m^{(\alpha,\beta)}(2t-1).$$

The properties of $Q_m^{(\alpha,\beta)}$ follow easily from the properties of $P_m^{(\alpha,\beta)}$. In particular, here are analogs of (1.7), (1.13), and (1.14):

$$Q_m^{(\alpha,\beta)}(t) = \frac{(-1)^m}{m!}\,(1-t)^{-\alpha}t^{-\beta}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\Big((1-t)^{m+\alpha}t^{m+\beta}\Big), \tag{3.1}$$

$$Q_m^{(\alpha,\beta)}(t) = \sum_{k=0}^{m}\binom{\alpha+\beta+m+k}{k}\binom{\beta+m}{m-k}(-1)^{m-k}t^k. \tag{3.2}$$

$$Q_m^{(\alpha,\beta)}(t) = \frac{\Gamma(m+\beta+1)}{m!\,\Gamma(m+\alpha+\beta+1)}\sum_{k=0}^{m}\binom{m}{k}\frac{\Gamma(\alpha+\beta+m+k+1)}{\Gamma(\beta+k+1)}\,(-1)^{m-k}t^k. \tag{3.3}$$

The sequence $(Q_m^{(\alpha,\beta)})_{m=0}^{\infty}$ is orthogonal on $(0,1)$ weight as $(1-t)^\alpha t^\beta$, and

$$\int_0^1 Q_m^{(\alpha,\beta)}(t)Q_n^{(\alpha,\beta)}(t)(1-t)^\alpha t^\beta \mathrm{d}t = \delta_{m,n}\frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)\,m!}. \tag{3.4}$$

Also, here are analogs of (1.16) and (1.18):

$$\int_0^1 h(t)Q_m^{(\alpha,\beta)}(t)(1-t)^\alpha t^\beta \mathrm{d}t = 0, \tag{3.5}$$

$$\int_0^1 Q_m^{(\alpha,\beta+1)}(t)(1-t)^\alpha t^\beta \mathrm{d}t = (-1)^m\,\mathrm{B}(\alpha+m+1,\beta+1). \tag{3.6}$$

Substituting in (3.1) t by tu and applying the chain rule, we get

$$\frac{\partial^m}{\partial t^m}\Big((1-tu)^{m+\alpha}t^{m+\beta}\Big) = m!\,(1-tu)^\alpha\,t^\beta\,Q_m^{(\alpha,\beta)}(tu). \tag{3.7}$$

Inspired by (3.4) we define the Jacobi function $\mathcal{J}_m^{(\alpha,\beta)}$ on $(0,1)$ as

$$\mathcal{J}_m^{(\alpha,\beta)}(t) \coloneqq c_m^{(\alpha,\beta)}(1-t)^{\alpha/2}t^{\beta/2}Q_m^{(\alpha,\beta)}(t), \tag{3.8}$$

where

$$c_m^{(\alpha,\beta)} \coloneqq \sqrt{\frac{(2m+\alpha+\beta+1)\,\Gamma(m+\alpha+\beta+1)\,m!}{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}}. \tag{3.9}$$

Then

$$\int_0^1 \mathcal{J}_m^{(\alpha,\beta)}(t)\mathcal{J}_n^{(\alpha,\beta)}(t)\,\mathrm{d}t = \delta_{m,n}. \tag{3.10}$$

Reproducing property for the polynomials on the unit interval

Given m in $\mathbb{N}_0$ and $\alpha,\beta>-1$, we denote by $R_m^{(\alpha,\beta)}$ the polynomial

$$R_m^{(\alpha,\beta)}(t) \coloneqq \frac{(-1)^m\;\mathrm{B}(\alpha+1,\beta+1)}{\mathrm{B}(\alpha+m+1,\beta+1)}\,Q_m^{(\alpha,\beta+1)}(t). \tag{3.11}$$

The following formula is analogous to (1.23) in Chapter 1.

Proposition 3.1.1. *Let $m\in\mathbb{N}_0$ and $\alpha,\beta>-1$. Then for every polynomial h with $\deg(h)\le m$,*

$$\frac{1}{\mathrm{B}(\alpha+1,\beta+1)}\int_0^1 h(t)R_m^{(\alpha,\beta)}(t)\,(1-t)^\alpha t^\beta\,\mathrm{d}t = h(0). \tag{3.12}$$

Proof. See Proposition (1.3.2). □

As a particular case of (3.12), for $\beta = 0$ and $k \le m$,

$$\frac{1}{\alpha+1}\int_0^1 t^k R_m^{(\alpha,0)}(t)(1-t)^\alpha \mathrm{d}t = \delta_{k,0}. \tag{3.13}$$

Formula (3.13) was proven in [35] in other way.

3.2 Orthonormal basis and Fourier decomposition of $L^2(\mathbb{D}, \mu_\alpha)$

For each $p, q \in \mathbb{N}_0$, denote by $m_{p,q}$ the monomial function

$$m_{p,q}(z) := z^p \overline{z}^q.$$

The inner product of two monomial functions is

$$\langle m_{p,q}, m_{j,k}\rangle = (\alpha+1)\,\mathrm{B}\left(p+j+1, \alpha+1\right)\ \delta_{p-q,j-k}. \tag{3.14}$$

In particular, this means that the family $(m_{p,q})_{p,q\in\mathbb{N}_0}$ is not orthogonal.

In this section, we recall various equivalent formulas for an orthonormal basis in $L^2(\mathbb{D}, \mu_\alpha)$, that can be obtained by orthonormalizing $(m_{p,q})_{p,q\in\mathbb{N}_0}$, and whose elements are known as *Jacobi polynomials in z and $\overline{z}$*, see Koornwinder [49], or *disk polynomials*, see Wünsche [84], among others. These polynomials, in the unweighted case, were also rediscovered in [51], [63], and [59], in the context of polyanalytic functions. We work with a normalized version of the disk polynomials and define them by

$$b_{p,q}^{(\alpha)}(z) := (-1)^{p+q}\, \widetilde{c}_{p,q}^{(\alpha)}\, (1-z\overline{z})^{-\alpha}\, \frac{\partial^q}{\partial z^q}\, \frac{\partial^p}{\partial \overline{z}^p}\Big((1-z\,\overline{z})^{p+q+\alpha}\Big), \tag{3.15}$$

where

$$\widetilde{c}_{p,q}^{(\alpha)} = \sqrt{\frac{(\alpha+p+q+1)\Gamma(\alpha+p+1)\Gamma(\alpha+q+1)}{(\alpha+1)p!\,q!\,\Gamma(\alpha+p+q+1)^2}}. \tag{3.16}$$

Since $\dfrac{\partial}{\partial z}(1-z\overline{z}) = -\overline{z}$ and $\dfrac{\partial}{\partial \overline{z}}(1-z\overline{z}) = -z$, the expression in (3.15) can be rewritten in other equivalent forms:

$$b_{p,q}^{(\alpha)}(z) = (-1)^q \sqrt{\frac{(\alpha+p+q+1)\Gamma(\alpha+p+1)}{(\alpha+1)p!\,q!\,\Gamma(\alpha+q+1)}}(1-z\overline{z})^{-\alpha}\,\frac{\partial^q}{\partial z^q}\Big(z^p(1-z\,\overline{z})^{\alpha+q}\Big), \tag{3.17}$$

$$b_{p,q}^{(\alpha)}(z) = (-1)^p \sqrt{\frac{(\alpha+p+q+1)\Gamma(\alpha+q+1)}{(\alpha+1)p!\,q!\,\Gamma(\alpha+p+1)}}(1-z\overline{z})^{-\alpha}\,\frac{\partial^p}{\partial \overline{z}^p}\Big(\overline{z}^q(1-z\,\overline{z})^{\alpha+p}\Big). \tag{3.18}$$

By (3.7), $b_{p,q}^{(\alpha)}$ can be expressed via the shifted Jacobi polynomials:

$$b_{p,q}^{(\alpha)}(z) = \begin{cases} \sqrt{\dfrac{(\alpha+p+q+1)\Gamma(\alpha+p+1)q!}{(\alpha+1)\,\Gamma(\alpha+q+1)p!}}\, z^{p-q}Q_q^{(\alpha,p-q)}(|z|^2), & \text{if } p \geq q; \\ \sqrt{\dfrac{(\alpha+p+q+1)\Gamma(\alpha+q+1)p!}{(\alpha+1)\,\Gamma(\alpha+p+1)q!}}\, \overline{z}^{q-p}Q_p^{(\alpha,q-p)}(|z|^2), & \text{if } p < q. \end{cases} \tag{3.19}$$

The two cases in (3.19) can be joined and written in terms of (3.8) and (3.9):

$$b_{p,q}^{(\alpha)}(r\tau) = \frac{c_{\min\{p,q\}}^{(\alpha,|p-q|)}}{\sqrt{\alpha+1}} r^{|p-q|}\tau^{p-q}Q_{\min\{p,q\}}^{(\alpha,|p-q|)}(r^2) \qquad (r \geq 0,\ \tau \in \mathbb{T}), \tag{3.20}$$

$$b_{p,q}^{(\alpha)}(r\tau) = \frac{\tau^{p-q}(1-r^2)^{-\alpha/2}}{\sqrt{\alpha+1}}\mathcal{J}_{\min\{p,q\}}^{(\alpha,|p-q|)}(r^2). \tag{3.21}$$

Notice that

$$c_{\min\{p,q\}}^{(\alpha,|p-q|)} = \sqrt{\frac{(\alpha+p+q+1)(\min\{p,q\})!\,\Gamma(\alpha+\max\{p,q\}+1)}{(\max\{p,q\})!\,\Gamma(\alpha+\min\{p,q\}+1)}}.$$

The family $(b_{p,q}^{(\alpha)})_{p,q\in\mathbb{N}_0}$ has the following conjugate symmetric property:

$$\overline{b_{p,q}^{(\alpha)}(z)} = b_{q,p}^{(\alpha)}(z). \tag{3.22}$$

Applying (3.3) in the right-hand side of (3.19) we obtain

$$\begin{aligned} b_{p,q}^{(\alpha)}(z) = {} & \sqrt{\frac{(\alpha+p+q+1)p!\,q!}{(\alpha+1)\,\Gamma(\alpha+p+1)\Gamma(\alpha+q+1)}}\times \\ & \times \sum_{k=0}^{\min\{p,q\}} (-1)^k \frac{\Gamma(\alpha+p+q+1-k)}{k!\,(p-k)!\,(q-k)!} z^{p-k}\overline{z}^{q-k}, \end{aligned} \tag{3.23}$$

In particular, (3.23) implies that $b_{p,q}^{(\alpha)}$ is a polynomial in z and $\overline{z}$ whose leading term (when $k=0$) is a positive multiple of the monomials $m_{p-k,q-k}$.

Let $\mathcal{P}$ be the set of all polynomials functions in z and $\overline{z}$, i.e., the linear span of the monomials

$$\mathcal{P} := \operatorname{span}\{m_{p,q}\colon\, p,q\in\mathbb{N}_0\}.$$

For every $\xi\in\mathbb{Z}$ and every $s\in\mathbb{N}$, denote by $\mathcal{W}_{\xi,s}^{(\alpha)}$ the subspace of $\mathcal{P}$ generated by $m_{p,q}$ with $p-q=\xi$ and $\min\{p,q\}<s$:

$$\mathcal{W}_{\xi,s}^{(\alpha)} := \operatorname{span}\{m_{p,q}\colon\;\, p-q=\xi,\ \min\{p,q\}<s\}. \tag{3.24}$$

The vector space $\mathcal{W}_{\xi,s}^{(\alpha)}$ does not depend on α, but we endow it with the inner product from $L^2(\mathbb{D},\mu_\alpha)$. Obviously, $\dim(\mathcal{W}_{\xi,s}^{(\alpha)})=s$. Let us show that

$$\mathcal{W}_{\xi,s}^{(\alpha)} = \operatorname{span}\{b_{p,q}^{(\alpha)}\colon\;\, p-q=\xi,\ \min\{p,q\}<s\}. \tag{3.25}$$

Indeed, by (3.23),

$$\begin{aligned} m_{p,q} &= \sqrt{\frac{(\alpha+1)\Gamma(\alpha+p+1)\Gamma(\alpha+q+1)p!\,q!}{\Gamma(\alpha+p+q+2)}}\, b_{p,q}^{(\alpha)} \\ &\quad - \frac{p!\,q!}{\Gamma(\alpha+p+q+1)} \sum_{\nu=1}^{\min\{p,q\}} (-1)^\nu \frac{\Gamma(\alpha+p+q+1-\nu)}{\nu!\,(p-\nu)!\,(q-\nu)!}\, m_{p-\nu,q-\nu}. \end{aligned} \tag{3.26}$$

Proceeding by induction on s, we see that the monomials $m_{p,q}$ are linear combinations of $b_{p-s,q-s}^{(\alpha)}$ with $0\le s\le\min\{p,q\}$. So, formula (3.25) means that the first s elements in the diagonal ξ of the table $(b_{p,q}^{(\alpha)})_{p,q=0}^\infty$ generate the same subspace as the first s elements of the diagonal ξ in the table $(m_{p,q})_{p,q=0}^\infty$. For example,

$$\begin{aligned} \mathcal{W}_{-2,3}^{(\alpha)} &= \operatorname{span}\{m_{0,2},m_{1,3},m_{2,4}\} = \operatorname{span}\{b_{0,2}^{(\alpha)},b_{1,3}^{(\alpha)},b_{2,4}^{(\alpha)}\}, \\ \mathcal{W}_{1,4}^{(\alpha)} &= \operatorname{span}\{m_{1,0},m_{2,1},m_{3,2},m_{4,3}\} = \operatorname{span}\{b_{1,0}^{(\alpha)},b_{2,1}^{(\alpha)},b_{3,2}^{(\alpha)},b_{4,3}^{(\alpha)}\}. \end{aligned}$$

In the following tables we show generators of $\mathcal{W}_{1,4}^{(\alpha)}$ (light blue) and $\mathcal{W}_{-2,3}^{(\alpha)}$ (pink).

$$\begin{array}{ccccccc} m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & m_{0,4} & m_{0,5} & \ddots \\ m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} & \ddots \\ m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} & \ddots \\ m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m_{3,5} & \ddots \\ m_{4,0} & m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & m_{4,5} & \ddots \\ m_{5,0} & m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array} \qquad \begin{array}{ccccccc} b_{0,0}^{(\alpha)} & b_{0,1}^{(\alpha)} & b_{0,2}^{(\alpha)} & b_{0,3}^{(\alpha)} & b_{0,4}^{(\alpha)} & b_{0,5}^{(\alpha)} & \ddots \\ b_{1,0}^{(\alpha)} & b_{1,1}^{(\alpha)} & b_{1,2}^{(\alpha)} & b_{1,3}^{(\alpha)} & b_{1,4}^{(\alpha)} & b_{1,5}^{(\alpha)} & \ddots \\ b_{2,0}^{(\alpha)} & b_{2,1}^{(\alpha)} & b_{2,2}^{(\alpha)} & b_{2,3}^{(\alpha)} & b_{2,4}^{(\alpha)} & b_{2,5}^{(\alpha)} & \ddots \\ b_{3,0}^{(\alpha)} & b_{3,1}^{(\alpha)} & b_{3,2}^{(\alpha)} & b_{3,3}^{(\alpha)} & b_{3,4}^{(\alpha)} & b_{3,5}^{(\alpha)} & \ddots \\ b_{4,0}^{(\alpha)} & b_{4,1}^{(\alpha)} & b_{4,2}^{(\alpha)} & b_{4,3}^{(\alpha)} & b_{4,4}^{(\alpha)} & b_{4,5}^{(\alpha)} & \ddots \\ b_{5,0}^{(\alpha)} & b_{5,1}^{(\alpha)} & b_{5,2}^{(\alpha)} & b_{5,3}^{(\alpha)} & b_{5,4}^{(\alpha)} & b_{5,5}^{(\alpha)} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}$$

As a consequence, $\mathcal{P}=\bigcup_{\xi\in\mathbb{Z}}\bigcup_{s\in\mathbb{N}}\mathcal{W}_{\xi,s}^{(\alpha)} = \operatorname{span}\{b_{p,q}^{(\alpha)}\colon\, p,q\in\mathbb{N}_0\}$.

Proposition 3.2.1. *The family* $(b_{p,q}^{(\alpha)})_{p,q\in\mathbb{N}_0}$ *is an orthonormal basis of* $L^2(\mathbb{D},\mu_\alpha)$.

Proof. The orthonormal property follows straightforwardly from (3.21) and (3.10):

$$\begin{aligned}\langle b_{p,q}^{(\alpha)}, b_{j,k}^{(\alpha)}\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(p-q-j+k)\theta}\,\mathrm{d}\theta \int_0^1 \mathcal{J}_{\min\{p,q\}}^{(\alpha,|p-q|)}(t)\,\mathcal{J}_{\min\{j,k\}}^{(\alpha,|j-k|)}(t)\,\mathrm{d}t \\ &= \delta_{p-q,j-k}\cdot\delta_{\min\{p,q\},\min\{j,k\}} = \delta_{p,j}\cdot\delta_{q,k}.\end{aligned}$$

By the Stone–Weierstrass theorem, $\mathcal{P}$ is dense in $C(\operatorname{clos}(\mathbb{D}))$. In turn, by Luzin's theorem, the set $C(\operatorname{clos}(\mathbb{D}))|_{\mathbb{D}}$ is dense in $L^2(\mathbb{D},\mu_\alpha)$ and for every $f\in C(\operatorname{clos}(\mathbb{D}))$ we have $\|f\|\le \max_{z\in\operatorname{clos}(\mathbb{D})}|f(z)|$. Now it is easy to see that the set $\mathcal{P}$ is dense in $L^2(\mathbb{D},\mu_\alpha)$, that is, the set of all linear combinations of elements of the family is dense in $L^2(\mathbb{D},\mu_\alpha)$. For that reason, $(b_{p,q}^{(\alpha)})_{p,q\in\mathbb{N}_0}$ is a complete orthonormal family. $\square$

Corollary 3.2.2. *Let* $\xi\in\mathbb{Z}$ *and* $s\in\mathbb{N}$. *Then* $(b_{q+\xi,q}^{(\alpha)})_{q=\max\{0,-\xi\}}^{\max\{s-1,s-\xi-1\}}$ *is an orthonormal basis of* $\mathcal{W}_{\xi,s}^{(\alpha)}$.

Remark 3.2.3. By Proposition 3.2.1 and formula (3.26),

$$\langle m_{\xi+k,k}, b_{\xi+q,q}^{(\alpha)}\rangle = \begin{cases}\sqrt{\frac{(\alpha+1)\Gamma(\alpha+p+1)\Gamma(\alpha+q+1)p!\,q!}{\Gamma(\alpha+p+q+2)\Gamma(\alpha+p+q+1)}}, & k=q;\\ 0, & \max\{0,-\xi\}\le k<q.\end{cases} \tag{3.27}$$

The table of basic functions can be expressed as follows:

$$\begin{array}{lll} b_{0,0}^{(\alpha)}(z) = h_{0,0}^{(\alpha)}(|z|^2), & b_{0,1}^{(\alpha)}(z) = \overline{z}\,h_{0,1}^{(\alpha)}(|z|^2), & b_{0,2}^{(\alpha)}(z) = \overline{z}^2\,h_{0,2}^{(\alpha)}(|z|^2),\\ b_{1,0}^{(\alpha)}(z) = z\,h_{1,0}^{(\alpha)}(|z|^2), & b_{1,1}^{(\alpha)}(z) = h_{1,1}^{(\alpha)}(|z|^2), & b_{1,2}^{(\alpha)}(z) = \overline{z}\,h_{1,2}^{(\alpha)}(|z|^2),\\ b_{2,0}^{(\alpha)}(z) = z^2\,h_{2,0}^{(\alpha)}(|z|^2), & b_{2,1}^{(\alpha)}(z) = z\,h_{2,1}^{(\alpha)}(|z|^2), & b_{2,2}^{(\alpha)}(z) = h_{2,2}^{(\alpha)}(|z|^2),\end{array}$$

where $h_{p,q}^{(\alpha)}(t) := \frac{c_{\min\{p,q\}}^{(\alpha,|p-q|)}}{\sqrt{\alpha+1}} Q_{\min\{p,q\}}^{(\alpha,|p-q|)}(t)$. Below we show explicitly some elements of this basis.

$$b_{0,0}^{(\alpha)}(z) = 1,\qquad b_{1,0}^{(\alpha)}(z) = \sqrt{\alpha+2}\,z,\qquad b_{2,0}^{(\alpha)}(z) = \sqrt{\frac{(\alpha+3)(\alpha+2)}{2}}\,z^2,$$

$$b_{1,1}^{(\alpha)}(z) = \sqrt{\frac{\alpha+3}{\alpha+1}}\big((\alpha+2)z\overline{z}-1\big),\qquad b_{2,1}^{(\alpha)}(z) = \sqrt{\frac{2(\alpha+3)(\alpha+2)}{\alpha+1}}\left(\frac{\alpha+3}{2}z^2\overline{z}-z\right),$$

$$b_{2,2}^{(\alpha)}(z) = \sqrt{\frac{\alpha+5}{\alpha+1}}\left(\frac{(\alpha+4)(\alpha+3)}{2}z^2\overline{z}^2 - 2(\alpha+3)z\overline{z}+\frac{1}{2}\right).$$

Now, for every ξ in $\mathbb{Z}$ we introduce the subspace $\mathcal{W}_\xi^{(\alpha)}$ associated to the “frequency” ξ or, equivalently, to the diagonal ξ in the tables $(m_{p,q})_{p,q\in\mathbb{N}_0}$ and $(b_{p,q}^{(\alpha)})_{p,q\in\mathbb{Z}}$:

$$\mathcal{W}_\xi^{(\alpha)} := \operatorname{clos}(\operatorname{span}\{m_{p,q}\colon p-q=\xi\}) = \operatorname{clos}\left(\bigcup_{s\in\mathbb{N}} \mathcal{W}_{\xi,s}^{(\alpha)}\right). \tag{3.28}$$

Corollary 3.2.4. *The sequence $(b_{\xi+q,q}^{(\alpha)})_{q=\max\{0,-\xi\}}^{\infty}$ is an orthonormal basis of $\mathcal{W}_\xi^{(\alpha)}$.*

The space $\mathcal{W}_\xi^{(\alpha)}$ can be naturally identified with L^2 over $(0,1)$, providing $(0,1)$ with various weights.

Proposition 3.2.5. *Each one of the following linear operators is an isometric isomorphism of Hilbert spaces:*

1) $L^2((0,1),(\alpha+1)\,(1-t)^\alpha \mathrm{d}t) \to \mathcal{W}_\xi^{(\alpha)}$, $\quad h \mapsto f$,

$$f(r\tau) := \tau^\xi h(r^2), \quad \textbf{\textit{i.e.,}} \quad f(z) := \operatorname{sgn}^\xi(z) h(z\overline{z}), \tag{3.29}$$

where $z\in\mathbb{D}$, $0\le r<1$, $\tau\in\mathbb{T}$;

2) $L^2((0,1),(\alpha+1)\,t^{|\xi|}(1-t)^\alpha) \to \mathcal{W}_\xi^{(\alpha)}$, $\quad h \mapsto f$,

$$f(r\tau) := \tau^\xi r^{|\xi|} h(r^2), \quad \textbf{\textit{i.e.,}} \quad f(z) := \begin{cases} z^\xi h(z\overline{z}), & \xi \ge 0, \\ \overline{z}^\xi h(z\overline{z}), & \xi < 0; \end{cases} \tag{3.30}$$

3) $L^2((0,1)) \to \mathcal{W}_\xi^{(\alpha)}$, $\quad h \mapsto f$,

$$f(r\tau) := \tau^\xi \frac{(1-r^2)^{-\alpha/2}}{\sqrt{\alpha+1}} h(r^2), \quad \textbf{\textit{i.e.,}} \quad f(z) := \operatorname{sgn}^\xi(z) \frac{(1-z\overline{z})^{-\alpha/2}}{\sqrt{\alpha+1}} h(z\overline{z}). \tag{3.31}$$

Proof. In each case, the isometric property is verified directly using polar coordinates, and the surjective property is justified with the help of the orthonormal basis of $\mathcal{W}_\xi^{(\alpha)}$ (Corollary 3.2.4). The function $\operatorname{sgn}\colon \mathbb{C}\to\mathbb{C}$ is defined by $\operatorname{sgn}(z) := z/|z|$ for $z\ne 0$ and $\operatorname{sgn}(0) := 0$. □

Corollary 3.2.6. *The space $L^2(\mathbb{D}, \mathrm{d}\mu_\alpha)$ is the orthogonal sum of the subspaces $\mathcal{W}_\xi^{(\alpha)}$:*

$$L^2(\mathbb{D}, \mathrm{d}\mu_\alpha) = \bigoplus_{\xi\in\mathbb{Z}} \mathcal{W}_\xi^{(\alpha)}. \tag{3.32}$$

The result of Corollary 3.2.6 can be seen as the *Fourier decomposition* of the space $L^2(\mathbb{D}, \mathrm{d}\mu_\alpha)$, where each space $\mathcal{W}_\xi^{(\alpha)}$ corresponds to the *frequency* ξ.

Here we show the generators of $\mathcal{W}_0^{(\alpha)}$ (pink) and $\mathcal{W}_{-1}^{(\alpha)}$ (light blue):

$$\begin{matrix} m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & \ddots \\ m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & \ddots \\ m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & \ddots \\ m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{matrix} \qquad \begin{matrix} b_{0,0}^{(\alpha)} & b_{0,1}^{(\alpha)} & b_{0,2}^{(\alpha)} & b_{0,3}^{(\alpha)} & \ddots \\ b_{1,0}^{(\alpha)} & b_{1,1}^{(\alpha)} & b_{1,2}^{(\alpha)} & b_{1,3}^{(\alpha)} & \ddots \\ b_{2,0}^{(\alpha)} & b_{2,1}^{(\alpha)} & b_{2,2}^{(\alpha)} & b_{2,3}^{(\alpha)} & \ddots \\ b_{3,0}^{(\alpha)} & b_{3,1}^{(\alpha)} & b_{3,2}^{(\alpha)} & b_{3,3}^{(\alpha)} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{matrix}$$

3.3 Weighted mean value property of polyanalytic functions

It is known [7, Section 1.1] that any m-analytic function can be expressed as a "polynomial" of degree $m-1$ in the variable $\overline{z}$ with 1-analytic coefficients, that is, for any $f \in \mathcal{A}_m(\mathbb{D})$, there exist analytic functions $g_0,\ g_1, \ldots, g_{m-1}$ in $\mathbb{D}$ such that

$$f(z) = \sum_{k=0}^{m-1} g_k(z)\overline{z}^k \qquad (z \in \mathbb{D}).$$

Replacing every g_k by its Taylor series, we get another classic form of m-analytic functions: there exist coeficients $\lambda_{j,k}$ in $\mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{m-1} \sum_{j=0}^{\infty} \lambda_{j,k} z^j \overline{z}^k \qquad (z \in \mathbb{D}). \tag{3.33}$$

The following weighted mean value property was proved by Hachadi y Youssfi [35] using a slightly different method. The mean value property for solutions of more general elliptic equations is studied in [78].

Proposition 3.3.1. *Let* $f \in \mathcal{A}_m(\mathbb{D})$ *such that*

$$\int_{\mathbb{D}} |f(z)|\,(1-|z|^2)^\alpha\,\mathrm{d}\mu(z) < +\infty.$$

Then

$$f(0) = \frac{\alpha+1}{\pi} \int_{\mathbb{D}} f(z) R_{m-1}^{(\alpha,0)}(|z|^2)\,(1-|z|^2)^\alpha\,\mathrm{d}\mu(z). \tag{3.34}$$

Formula (3.34) is equivalent to formula (1.29) in Chapter 1.

Proof. See Proposition 3.3.1. □

For $\alpha = 0$, Proposition 3.3.1 reduces to the following mean value property that appeared in [51] and [59].

Corollary 3.3.2. *Let $z \in \mathbb{C}$, $r > 0$, and $f \in \mathcal{A}_m(z + r\mathbb{D})$ such that*

$$\int_{z+r\mathbb{D}} |f(w)| \, \mathrm{d}\mu(w) < +\infty.$$

Then

$$f(z) = \frac{\alpha+1}{\pi r^2} \int_{z+r\mathbb{D}} f(w) R_{m-1}^{(\alpha,0)}\left(\frac{|w-z|^2}{r^2}\right) \mathrm{d}\mu(w). \tag{3.35}$$

Proof. Denote by φ the linear change of variables $\varphi(w) \coloneqq rw + z$. If $f \in \mathcal{A}_m(z + r\mathbb{D})$, then $f \circ \varphi \in \mathcal{A}_m(\mathbb{D})$. Applying (3.34) to $f \circ \varphi$, we obtain (3.35). □

Weighted Bergman spaces of polyanalytic functions on general complex domains

Given m in $\mathbb{N}$, an open subset Ω of $\mathbb{C}$ and a continuous function $W\colon \Omega \to (0, +\infty)$, we denote by $\mathcal{A}_m^2(\Omega, W)$ the space of m-analytic functions belonging to $L^2(\Omega, W)$ and provided with the norm of $L^2(\Omega, W)$. The mean value property (3.35) implies that the evaluation functionals in $\mathcal{A}_m^2(\Omega, W)$ are bounded (moreover, they are uniformly bounded on compacts), and $\mathcal{A}_m^2(\Omega, W)$ is a RKHS.

Lemma 3.3.3. *Let K be a compact subset of Ω. There exists a number $C_{m,W,K} > 0$ such that for every f in $\mathcal{A}_m^2(\Omega, W)$ and every z in K,*

$$|f(z)| \le C_{m,W,K} \|f\|_{\mathcal{A}_m^2(\Omega,W)}. \tag{3.36}$$

Proof. Let r_1 be the distance from K to $\mathbb{C} \setminus \Omega$. Since K is compact and $\mathbb{C} \setminus \Omega$ is closed, $r_1 > 0$. Put $r \coloneqq \min\{r_1/2, 1\}$, $K_1 \coloneqq \{w \in \mathbb{C}\colon\ d(w, K) \le r\}$,

$$C_1 \coloneqq \left(\max_{0 \le t \le 1} |R_{m-1}^{(\alpha,0)}(t)|\right)\left(\max_{w \in K_1} \frac{1}{\sqrt{W(w)}}\right).$$

For every z in K, we estimate $|f(z)|$ from above applying (3.35) and Schwarz inequality:

$$\begin{aligned}|f(z)| &\leq \frac{1}{\pi r^2}\int_{z+r\mathbb{D}} |f(w)| \left| R_{m-1}^{(\alpha,0)}\left(\frac{|w-z|^2}{r^2}\right)\right| \mathrm{d}\mu(w)\\ &\leq \frac{C_1}{\pi r^2}\int_{z+r\mathbb{D}} |f(w)| \sqrt{W(w)}\,\mathrm{d}\mu(w)\\ &\leq \frac{C_1}{\pi r^2}\left(\int_{z+r\mathbb{D}} |f(w)|^2 W(w)\,\mathrm{d}\mu(w)\right)^{1/2}\left(\int_{z+r\mathbb{D}} 1\,\mathrm{d}\mu(w)\right)^{1/2}\\ &\leq \frac{C_1}{\sqrt{\pi r^2}}\,\|f\|_{\mathcal{A}_m^2(\Omega,W)}.\end{aligned}$$

So, (3.36) is fulfilled with $C_{m,W,K} = \frac{C_1}{\sqrt{\pi r^2}}$. □

Proposition 3.3.4. *$\mathcal{A}_m^2(\Omega, W)$ is a RKHS.*

Proof. Given a Cauchy sequence in $\mathcal{A}_m^2(\Omega, W)$, for every compact K it converges uniformly on K by Lemma 3.3.3. The pointwise limit of this sequence is also polyanalytic by [7, Corollary 1.8], and it coincides a.e. with the limit in $L^2(\Omega, W)$. Lemma 3.3.3 also assures the boundedness of the evalution functionals and thereby the existence of the reproducing kernel. See similar proofs in [57, Proposition 3.3]. □

Denote by $\mathcal{A}_{(m)}^2(\Omega, W)$ the orthogonal complement of $\mathcal{A}_{m-1}^2(\Omega, W)$ in $\mathcal{A}_m^2(\Omega, W)$.

Corollary 3.3.5. *$\mathcal{A}_{(m)}^2(\Omega, W)$ is a RKHS.*

3.4 Weighted Bergman spaces of polyanalytic functions on the unit disk

In the rest of the chapter, we suppose that $m \in \mathbb{N}$ and $\alpha > -1$. Given z in $\mathbb{D}$, denote by $K_{m,z}^{(\alpha)}$ the reproducing kernel of $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$ at the point z and by $K_{(m),z}^{(\alpha)}$ the reproducing kernel of $\mathcal{A}_{(m)}^2(\mathbb{D}, \mu_\alpha)$ at the point z. Hachadi and Youssfi [35] computed the reproducing kernel of $\mathcal{A}_m^2(\mathbb{D}, \mu_\alpha)$:

$$K_{m,z}^{(\alpha)}(w) = \frac{(1-\overline{w}z)^{m-1}}{(1-\overline{z}w)^{m+1}} R_{m-1}^{(\alpha,0)}\left(\left|\frac{z-w}{1-\overline{z}w}\right|^2\right). \tag{3.37}$$

Their method uses (3.34) and a generalization of the unitary operator constructed by Pessoa [59]. Formula (3.37) implies an exact expression for the norm of $K_{m,z}^{(\alpha)}$, which is also the norm of the evaluation functional at the point z:

$$\|K_{m,z}^{(\alpha)}\| = \sqrt{(m+\alpha)\binom{m+\alpha-1}{m-1}}\,\frac{1}{1-|z|^2}. \tag{3.38}$$

Obviously, the reproducing kernel of $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$ can be written as

$$K_{(m),z}^{(\alpha)}(w) = K_{m,z}^{(\alpha)}(w) - K_{m-1,z}^{(\alpha)}(w). \tag{3.39}$$

Unfortunately, we are unable to obtain a simpler formula for $K_{(m),z}^{(\alpha)}$.

Orthonormal basis in $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$

Proposition 3.4.1. *The family $(b_{p,q}^{(\alpha)})_{p\in\mathbb{N}_0,q<m}$ is an orthonormal basis of $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$.*

Proof. It is clear that the family is contained in $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$, and by Proposition 3.2.1 is orthonormal. Using ideas of Ramazanov [63, proof of Theorem 2] we will show the total property. Suppose that $f \in \mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$ and $\langle f, b_{p,q}^{(\alpha)}\rangle = 0$ for every p in $\mathbb{N}_0$ and $q < m$. For $r > 0$, using expansion (3.33) and the orthogonality of the Fourier basis on $\mathbb{T}$, we easily obtain

$$\int_{r\mathbb{D}} f\,\overline{b_{p,q}^{(\alpha)}}\,\mathrm{d}\mu_\alpha = \sum_{k=0}^{m-1} \lambda_{k+p-q,k} \int_{r\mathbb{D}} m_{k+p-q,k}\overline{b_{p,q}^{(\alpha)}}\,\mathrm{d}\mu_\alpha.$$

The dominated convergence theorem allows us to pass to integrals over $\mathbb{D}$, because $f\,\overline{b_{p,q}^{(\alpha)}}$ and $m_{k+p-q,k}\,\overline{b_{p,q}^{(\alpha)}}$ belong to $L^1(\mathbb{D},\mu_\alpha)$. Now the assumption $f \perp b_{p,q}^{(\alpha)} = 0$ yields

$$\sum_{k=0}^{m-1} \langle m_{k+p-q,k}, b_{p,q}^{(\alpha)}\rangle \lambda_{k+p-q,k} = 0 \qquad (p\in\mathbb{N}_0,\ 0 \le q < m). \tag{3.40}$$

For a fixed ξ in $\mathbb{Z}$ with $\xi > -m$, put $s = \min\{m, m+\xi\}$. The vector $[\lambda_{k+\xi,k}]_{k=\max\{0,-\xi\}}^{m-1}$ satisfies the homogeneous linear system (3.40) with the $s\times s$ matrix

$$\left[\langle m_{\xi+k,k}, b_{\xi+q,q}^{(\alpha)}\rangle\right]_{q,k=\max\{0,-\xi\}}^{m-1}.$$

By (3.27), this is an upper triangular matrix with nonzero diagonal entries, hence the unique solution of (3.40) is zero. $\square$

Corollary 3.4.2. *The sequence $(b_{p,m-1}^{(\alpha)})_{p\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$.*

We denote by $P_m^{(\alpha)}$ and $P_{(m)}^{(\alpha)}$ the orthogonal projections acting in $L^2(\mathbb{D},\mu_\alpha)$, whose images are $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$ and $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$, respectively. They can be computed in terms of the corresponding reproducing kernels:

$$(P_m^{(\alpha)}f)(z) = \langle f, K_{m,z}^{(\alpha)}\rangle, \qquad (P_{(m)}^{(\alpha)}f)(z) = \langle f, K_{(m),z}^{(\alpha)}\rangle.$$

For example, $(b_{p,q}^{(\alpha)})_{p\in\mathbb{N}_0,q<4}$ is an orthonormal basis of $\mathcal{A}^2_4(\mathbb{D},\mu_\alpha)$, and $(b_{p,3}^{(\alpha)})_{p\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{A}^2_{(4)}(\mathbb{D},\mu_\alpha)$.

$$\begin{array}{cccccc}
b_{0,0}^{(\alpha)} & b_{0,1}^{(\alpha)} & b_{0,2}^{(\alpha)} & b_{0,3}^{(\alpha)} & b_{0,4}^{(\alpha)} & \cdots \\
b_{1,0}^{(\alpha)} & b_{1,1}^{(\alpha)} & b_{1,2}^{(\alpha)} & b_{1,3}^{(\alpha)} & b_{1,4}^{(\alpha)} & \cdots \\
b_{2,0}^{(\alpha)} & b_{2,1}^{(\alpha)} & b_{2,2}^{(\alpha)} & b_{2,3}^{(\alpha)} & b_{2,4}^{(\alpha)} & \cdots \\
b_{3,0}^{(\alpha)} & b_{3,1}^{(\alpha)} & b_{3,2}^{(\alpha)} & b_{3,3}^{(\alpha)} & b_{3,4}^{(\alpha)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\qquad\qquad
\begin{array}{cccccc}
b_{0,0}^{(\alpha)} & b_{0,1}^{(\alpha)} & b_{0,2}^{(\alpha)} & b_{0,3}^{(\alpha)} & b_{0,4}^{(\alpha)} & \cdots \\
b_{1,0}^{(\alpha)} & b_{1,1}^{(\alpha)} & b_{1,2}^{(\alpha)} & b_{1,3}^{(\alpha)} & b_{1,4}^{(\alpha)} & \cdots \\
b_{2,0}^{(\alpha)} & b_{2,1}^{(\alpha)} & b_{2,2}^{(\alpha)} & b_{2,3}^{(\alpha)} & b_{2,4}^{(\alpha)} & \cdots \\
b_{3,0}^{(\alpha)} & b_{3,1}^{(\alpha)} & b_{3,2}^{(\alpha)} & b_{3,3}^{(\alpha)} & b_{3,4}^{(\alpha)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Decomposition of $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$ into subspaces corresponding to different "frequences"

We will use the following elementary fact about orthonormal bases in Hilbert spaces. In the next proposition we treat orthonormal bases like sets rather than families.

Proposition 3.4.3. *Let H_1 be a Hilbert space and $\mathcal{B}_1 \subseteq H_1$ be an orthonormal basis of H_1. Suppose that $\mathcal{B}_2$ and $\mathcal{B}_3$ are some subsets of $\mathcal{B}_1$. Denote by H_2 and H_3 the closed subspaces of H_1 generated by $\mathcal{B}_2$ and $\mathcal{B}_3$, respectively. Then $\mathcal{B}_2\cap\mathcal{B}_3$ is an orthonormal basis of $H_2\cap H_3$.*

Applying Proposition 3.4.3 to the Hilbert space $L^2(\mathbb{D},\mu_\alpha)$ and thinking in terms of orthonormal bases (see Propositions 3.2.1, 3.4.1, and Corollaries 3.2.2, 3.2.4), we easily find the intersection of $\mathcal{W}_\xi^{(\alpha)}$ and $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$:

$$\mathcal{W}_\xi^{(\alpha)} \cap \mathcal{A}^2_m(\mathbb{D},\mu_\alpha) = \begin{cases} \mathcal{W}^{(\alpha)}_{\xi,\min\{m,m+\xi\}}, & \xi \geq -m+1; \\ \{0\}, & \xi < -m+1. \end{cases} \tag{3.41}$$

Here is a description of the subspaces $\mathcal{W}_{\xi,m}^{(\alpha)}$ in terms of the polar coordinates.

Proposition 3.4.4. *For every ξ in $\mathbb{Z}$ and every s in $\mathbb{N}$, the space $\mathcal{W}_{\xi,s}^{(\alpha)}$ consists of all functions of the form*

$$f(r\tau) = \tau^{\xi} r^{|\xi|} Q(r^2) \qquad (r \geq 0,\ \tau \in \mathbb{T}),$$

where Q is a polynomial of degree $\leq s-1$. Moreover,

$$\|f\| = \|Q\|_{L^2([0,1),(\alpha+1)(1-t)^{\alpha} t^{|\xi|}\,\mathrm{d}t)}.$$

Proof. The result follows directly by Proposition (3.2.5) and formula (3.30). □

The decomposition of $\mathcal{A}_m^2(\mathbb{D},\mu_\alpha)$ into a direct sum of the "truncated frequency subspaces" shown below follows from Proposition 3.4.1 and Corollary 3.2.2, and plays a crucial role in the study of radial operators. It can be seen as the "Fourier series decomposition" of $\mathcal{A}_m^2(\mathbb{D},\mu_\alpha)$.

Proposition 3.4.5.

$$\mathcal{A}_m^2(\mathbb{D},\mu_\alpha) = \bigoplus_{\xi=-m+1}^{\infty} \mathcal{W}_{\xi,\min\{m,m+\xi\}}^{(\alpha)}. \tag{3.42}$$

Let us illustrate Proposition 3.4.5 for $m=3$ with a table (we have marked in different shades of blue the basic functions that generate each truncated diagonal):

$$\begin{array}{ccccc}
b_{0,0}^{(\alpha)} & b_{0,1}^{(\alpha)} & b_{0,2}^{(\alpha)} & b_{0,3}^{(\alpha)} & \cdots \\
b_{1,0}^{(\alpha)} & b_{1,1}^{(\alpha)} & b_{1,2}^{(\alpha)} & b_{1,3}^{(\alpha)} & \cdots \\
b_{2,0}^{(\alpha)} & b_{2,1}^{(\alpha)} & b_{2,2}^{(\alpha)} & b_{2,3}^{(\alpha)} & \cdots \\
b_{3,0}^{(\alpha)} & b_{3,1}^{(\alpha)} & b_{3,2}^{(\alpha)} & b_{3,3}^{(\alpha)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Define $U_m^{(\alpha)} \colon \mathcal{A}_m^2(\mathbb{D},\mu_\alpha) \to \bigoplus_{\xi=-m+1}^{\infty} \mathbb{C}^{\min\{m,m+\xi\}}$,

$$(U_m^{(\alpha)} f)_{\xi,q} \coloneqq \langle f, b_{q+\xi,q}^{(\alpha)} \rangle \qquad (\xi \geq -m+1,\ \max\{0,-\xi\} \leq q \leq m-1). \tag{3.43}$$

Here, for $-m+1 \leq \xi < 0$, the componentes of vectors in $\mathbb{C}^{m+\xi}$ are enumerated from $-\xi$ to $m-1$.

Proposition 3.4.6. *The operator $U_m^{(\alpha)}$ is an isometric isomorphism of Hilbert spaces.*

Proof. Follows from Proposition 3.4.1 or, even easier, from Proposition 3.4.5 and the fact that $(b_{q+\xi,q}^{(\alpha)})_{q=\max\{0,-\xi\}}^{m-1}$ is an orthonormal basis of $\mathcal{W}_{\xi,\min\{m,m+\xi\}}^{(\alpha)}$ (see Corollary 3.2.2). □

An analog of the next fact for the unweighted poly-Bergman space was proved by Vasilevski [82, Section 4.2]. We obtain it as a corollary from Proposition 3.2.1 and Corollary 3.4.2.

Corollary 3.4.7. *The space $L^2(\mathbb{D},\mu_\alpha)$ is the orthogonal sum of the subspaces $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$, $m\in\mathbb{N}$:*

$$L^2(\mathbb{D},\mu_\alpha)=\bigoplus_{m\in\mathbb{N}}\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha).$$

3.5 The set of Toeplitz operators is not weakly dense

Given a Hilbert space H, we denote by $\mathcal{B}(H)$ the algebra of all bounded operators acting in H. If H is a RKHS and $S\in\mathcal{B}(H)$, then the *Berezin transform* of S is defined by

$$\operatorname{Ber}_H(S)(z)\coloneqq\frac{\langle SK_z,K_z\rangle_H}{\langle K_z,K_z\rangle_H},\qquad\text{i.e.,}\qquad \operatorname{Ber}_H(S)(z)=\frac{(SK_z)(z)}{K_z(z)}.$$

The Berezin transform can be considered as a bounded linear operator $\mathcal{B}(H)\to L^\infty(\Omega)$. Stroethoff proved [72] that Ber_H is injective for various RKHS of analytic functions, in particular, for $H=\mathcal{A}^2_1(\mathbb{D})$. Engliš noticed [21, Section 2] that Ber_H is not injective for various RKHS of harmonic functions. The idea of Engliš can be applied without any changes to various spaces of polyanalytic and polyharmonic functions. For clarity of presentation, we state the result of Engliš for $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$, $m\geq 2$, and repeat his proof.

Proposition 3.5.1. *Let $H=\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$ with $m\geq 2$. Then the Berezin transform Ber_H is not injective.*

Proof. Let $f\in H$ such that $\overline{f}\in H$ and the functions $f,\overline{f}$ are linearly independent. For example, $f(z)\coloneqq z$. Following the idea from [21, Section 2], consider the operator

$$Sh\coloneqq\langle h,f\rangle_H f-\langle h,\overline{f}\rangle_H\,\overline{f}.$$

Then $S\neq 0$, but $\langle SK_z,K_z\rangle_H=|f(z)|^2-|f(z)|^2=0$ for every z in $\mathbb{D}$. So, $\operatorname{Ber}_H(S)$ is the zero constant. $\square$

Given a function g in $L^\infty(\mathbb{D})$, let M_g be the multiplication operator defined on $L^2(\mathbb{D}, \mu_\alpha)$ by $M_g f := gf$. If H is a closed subspace of $L^2(\mathbb{D}, \mu_\alpha)$, then the *Toeplitz operator* $T_{H,g}$ is defined on H by

$$T_{H,g}(f) := P_H(gf) = P_H M_g f.$$

For $H = \mathcal{A}^2_m(\mathbb{D}, \mu_\alpha)$ and $H = \mathcal{A}^2_{(m)}(\mathbb{D}, \mu_\alpha)$, we write just $T^{(\alpha)}_{m,g}$ and $T^{(\alpha)}_{(m),g}$, respectively. The proof of the following fact is the same as the proof of [57, Proposition 3.18] or the proof of [11, Theorem 4].

Proposition 3.5.2. *If $g \in L^\infty(\mathbb{D})$ and $T^{(\alpha)}_{m,g} = 0$, then $g = 0$ a.e. In other words, the function $g \mapsto T^{(\alpha)}_{m,g}$, acting from $L^\infty(\mathbb{D})$ to $\mathcal{B}(\mathcal{A}^2_m(\mathbb{D}, \mu_\alpha))$, is injective.*

Inspired by the idea of Engliš explained in the proof of Proposition 3.5.1, we will prove that the set of Toeplitz operators is not weakly dense in $\mathcal{B}(\mathcal{A}^2_m(\mathbb{D}, \mu_\alpha))$ with $m \geq 2$. First, let us prove an auxiliary fact from linear algebra: bounded quadratic forms separate linearly independent vectors.

Lemma 3.5.3. *Let H be a Hilbert space and f, g be two linearly independent vectors in H. Then there exists S in $\mathcal{B}(H)$ such that*

$$\langle Sf, f\rangle_H \neq \langle Sg, g\rangle_H.$$

Proof. Without lost of generality, we will suppose that $\|f\|_H = 1$. Decompose g into the linear combination $g = \lambda_1 f + \lambda_2 h$, with $\lambda_1, \lambda_2 \in \mathbb{C}$, $\|h\|_H = 1$, $h \perp f$. More explicitly,

$$\lambda_1 := \langle g, f\rangle_H, \qquad w := g - \lambda_1 f, \qquad \lambda_2 := \|w\|_H, \qquad h := \frac{1}{\lambda_2} w.$$

Define S as the orthogonal projection onto h:

$$Sv := \langle v, h\rangle_H h \qquad (v \in H).$$

Then $Sf = 0$ and $Sg = \lambda_2 h$, therefore $\langle Sf, f\rangle_H = 0$ and $\langle Sg, g\rangle_H = \lambda_2^2 > 0$. $\square$

Theorem 3.5.4. *Let $H = \mathcal{A}^2_m(\mathbb{D}, \mu_\alpha)$ with $m \geq 2$. Then the set of the Toeplitz operators with bounded symbols is not weakly dense in $\mathcal{B}(H)$.*

Proof. Let $f \in H$ such that $\overline{f} \in H$ and the functions $f, \overline{f}$ are linearly independent. For example, $f(z) := z$. The set

$$W := \{S \in \mathcal{B}(H) \colon \ \langle Sf, f\rangle_H = \langle S\overline{f}, \overline{f}\rangle_H\}$$

is a weakly closed subspace of $\mathcal{B}(H)$. By Lemma 3.5.3, $W \neq \mathcal{B}(H)$. On the other hand, for every a in $L^\infty(\mathbb{D})$

$$\langle T_{m,a}^{(\alpha)} f, f\rangle_H = \int_X a\,|f|^2 \,\mathrm{d}\mu_\alpha = \langle T_{m,a}^{(\alpha)}\,\overline{f},\,\overline{f}\rangle_H,$$

i.e., $\{T_{m,a}^{(\alpha)} \colon \ a \in L^\infty(\mathbb{D})\} \subseteq W$. □

Remark 3.5.5. An analog of Theorem 3.5.4 is true for the space of μ_α-square integrable m-harmonic functions on $\mathbb{D}$, with $m \geq 1$.

3.6 Von Neumann algebras of radial operators

Set of operators diagonalized by a family of subspaces

The theory of von Neumann algebras and their decompositions is well developed. For our purposes, it is sufficient to use the following elementary scheme from [57]. This scheme is similar to ideas from [32, 60, 89].

Definition 3.6.1. Let H be a Hilbert space, $\mathcal{U}$ be a self-adjoint subset of $\mathcal{B}(H)$, and $(W_j)_{j\in J}$ be a finite or countable family of nonzero closed subspaces of H such that $H = \bigoplus_{j\in J} W_j$. We say that this family *diagonalizes* $\mathcal{U}$ if the following two conditions are satisfied.

1. For each j in J and each U in $\mathcal{U}$, there exists $\lambda_{U,j}$ in $\mathbb{C}$ such that $W_j \subseteq \ker(\lambda_{U,j} I - U)$, i.e., $U(v) = \lambda_{U,j} v$ for every v in W_j.
2. For every j, k in J with $j \neq k$, there exists U in $\mathcal{U}$ such that $\lambda_{U,j} \neq \lambda_{U,k}$.

Proposition 3.6.2. *Let H, $\mathcal{U}$, and $(W_j)_{j\in J}$ be as in Definition 3.6.1. Denote by $\mathcal{A}$ the commutant of $\mathcal{U}$. Then $\mathcal{A}$ consists of all bounded linear operators that act invariantly on each of the subspaces W_j, with $j \in J$:*

$$\mathcal{A} = \{S \in \mathcal{B}(H) \colon \quad \forall j \in J \quad S(W_j) \subseteq W_j\}. \tag{3.44}$$

Furthermore, $\mathcal{A}$ is isometrically isomorphic to $\bigoplus_{j\in J} \mathcal{B}(W_j)$, and the von Neumann algebra generated by $\mathcal{U}$ is isometrically isomorphic to $\bigoplus_{j\in J} \mathbb{C} I_{W_j}$.

Example 3.6.3. Let $j_1, \ldots, j_m \in J$, $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$, and $u_{j_k}, v_{j_k} \in W_{j_k}$ for every k in $\{1, \ldots, m\}$. Then the operator $S\colon H \to H$ defined by

$$Sf \coloneqq \sum_{k=1}^{m} \lambda_k \langle f, u_{j_k} \rangle v_{j_k}, \tag{3.45}$$

belongs to $\mathcal{A}$. Moreover, every operator of finite rank, belonging to $\mathcal{A}$, can be written in this form. See the proof of [57, Corollary 5.7] for a similar situation.

Proposition 3.6.4. *Let $H, \mathcal{U}$, and $(W_j)_{j\in J}$ be as in Definition 3.6.1, and H_1 be a closed subspace of H invariant under $\mathcal{U}$. For every U in $\mathcal{U}$, denote by $U|_{H_1}^{H_1}$ the compression of U onto the invariant subspace H_1, and put*

$$\mathcal{U}_1 \coloneqq \left\{ U|_{H_1}^{H_1}\colon\ U \in \mathcal{U} \right\}, \qquad J_1 \coloneqq \{ j \in J\colon\ W_j \cap H_1 \neq \{0\} \}.$$

Then

$$H_1 = \bigoplus_{j\in J_1} (W_j \cap H_1), \tag{3.46}$$

and the family $(W_j \cap H_1)_{j\in J}$ diagonalizes $\mathcal{U}_1$.

Example 3.6.5. The operators of finite rank, commuting with $U|_{H_1}^{H_1}$ for every U in $\mathcal{U}$, are of the form (3.45), but with $u_{j_k}, v_{j_k} \in W_{j_k} \cap H_1$.

Radial operators in $L^2(\mathbb{D}, \mu_\alpha)$

For each τ in $\mathbb{T}$, we denote by $\rho^{(\alpha)}(\tau)$ the rotation operator acting in $L^2(\mathbb{D}, \mu_\alpha)$ by the rule

$$(\rho^{(\alpha)}(\tau) f)(z) \coloneqq f(\tau^{-1} z). \tag{3.47}$$

It is easy to see that $\rho^{(\alpha)}(\tau_1\tau_2) = \rho^{(\alpha)}(\tau_1)\rho^{(\alpha)}(\tau_2)$, the operators $\rho^{(\alpha)}(\tau)$ are unitary, and for every f in $L^2(\mathbb{D}, \mu_\alpha)$ the mapping $\tau \mapsto \rho^{(\alpha)}(\tau) f$ is continuous (this is easy to check first for the case when f is a continuous function with compact support). So, $(\rho^{(\alpha)}, L^2(\mathbb{D}, \mu_\alpha))$ is a unitary representation of the group $\mathbb{T}$. The operators commuting with $\rho^{(\alpha)}(\tau)$ for every τ in $\mathbb{T}$ are called *radial operators*. We denote the set of all radial operators in $L^2(\mathbb{D}, \mu_\alpha)$ by $\mathcal{R}^{(\alpha)}$:

$$\mathcal{R}^{(\alpha)} \coloneqq \{\rho^{(\alpha)}(\tau)\colon\ \tau \in \mathbb{T}\}' = \{S \in \mathcal{B}(L^2(\mathbb{D}, \mu_\alpha))\colon\quad \forall \tau \in \mathbb{T} \quad \rho^{(\alpha)}(\tau) S = S \rho^{(\alpha)}(\tau)\}.$$

Since $\{\rho^{(\alpha)}(\tau)\colon \tau\in\mathbb{T}\}$ is a selfadjoint subset of $\mathcal{B}(L^2(\mathbb{D},\mu_\alpha))$, its commutant $\mathcal{R}^{(\alpha)}$ is a von Neumann algebra [74].

Recall that the subspaces $\mathcal{W}_\xi^{(\alpha)}$ are defined by (3.28).

Lemma 3.6.6. *The family $(\mathcal{W}_\xi^{(\alpha)})_{\xi\in\mathbb{Z}}$ diagonalizes the collection $\{\rho^{(\alpha)}(\tau)\colon\ \tau\in\mathbb{T}\}$ in the sense of Definition 3.6.1.*

Proof. 1. Let $\tau\in\mathbb{T}$. For every $p,q\in\mathbb{Z}$ with $p-q=\xi$, formula (3.20) implies

$$c \tag{3.48}$$

i.e., $b_{p,q}^{(\alpha)}\in\ker(\tau^{-\xi}I-\rho^{(\alpha)}(\tau))$. By Corollary 3.2.4, the functions $b_{p,q}^{(\alpha)}$ with $p-q=\xi$ form an orthonormal basis of $\mathcal{W}_\xi^{(\alpha)}$. So,

$$\mathcal{W}_\xi^{(\alpha)}\subseteq\ker(\tau^{-\xi}I-\rho^{(\alpha)}(\tau)). \tag{3.49}$$

2. Let $\xi_1,\xi_2\in\mathbb{Z}$ and $\xi_1\neq\xi_2$. Put $\tau=\exp\frac{\mathrm{i}\,\pi}{\xi_1-\xi_2}$. Then $\tau^{-\xi_1}\neq\tau^{-\xi_2}$. □

Proposition 3.6.7. *The von Neumann algebra $\mathcal{R}^{(\alpha)}$ consists of all operators that act invariantly on $\mathcal{W}_\xi^{(\alpha)}$ for every ξ in $\mathbb{Z}$, and is isometrically isomorphic to $\bigoplus_{\xi\in\mathbb{Z}}\mathcal{B}(\mathcal{W}_\xi^{(\alpha)})$.*

Proof. Follows from Proposition 3.6.2 and Lemma 3.6.6. □

The *radialization transform* $\operatorname{Rad}^{(\alpha)}\colon\mathcal{B}(L^2(\mathbb{D},\mu_\alpha))\to\mathcal{B}(L^2(\mathbb{D},\mu_\alpha))$, introduced by Zorboska [89], acts by the rule

$$\operatorname{Rad}^{(\alpha)}(S) := \int_{\mathbb{T}}\rho(\tau)S\rho(\tau^{-1})\,\mathrm{d}\mu_{\mathbb{T}}(\tau),$$

where $\mu_{\mathbb{T}}$ is the normalized Haar measure on $\mathbb{T}$, and the integral is understood in the weak sense. The condition $S\in\mathcal{R}^{(\alpha)}$ is equivalent to $\operatorname{Rad}^{(\alpha)}(S)=S$.

Radial operators in $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$

Proposition 3.6.8. *The space $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$ is invariant under every rotation $\rho^{(\alpha)}(\tau)$, $\tau\in\mathbb{T}$.*

First proof. The reproducing kernel of $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$, given by (3.37), is invariant under simultaneous rotations in both arguments:

$$K^{(\alpha)}_{m,\tau z}(\tau w) = K^{(\alpha)}_{m,z}(w) \qquad (z,w\in\mathbb{D},\ \tau\in\mathbb{T}). \tag{3.50}$$

According to [57, Proposition 4], this implies the invariance of the subspace. □

Second proof. By (3.47), the elements of the orthonormal basis $(b^{(\alpha)}_{p,q})_{p\in\mathbb{N}_0,0\le q<n}$ are eigenfunctions of $\rho^{(\alpha)}$. □

For every τ in $\mathbb{T}$, we denote by $\rho^{(\alpha)}_m(\tau)$ the compression of $\rho^{(\alpha)}(\tau)$ onto the space $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$. In other words, the operator $\rho^{(\alpha)}_m(\tau)$ acts in $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$ and is defined by (3.47). So, $(\rho^{(\alpha)}_m,\mathcal{A}^2_m(\mathbb{D},\mu_\alpha))$ is a unitary representation of $\mathbb{T}$. We denote by $\mathcal{R}^{(\alpha)}_m$ the commutant of this representation, i.e., the von Neumann algebra of all bounded linear radial operators acting in $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$.

Denote by $\mathfrak{M}_m$ the following direct sum of matrix algebras:

$$\mathfrak{M}_m := \bigoplus_{\xi=-m+1}^{\infty}\mathcal{M}_{\min\{m,m+\xi\}} = \left(\bigoplus_{\xi=-m+1}^{-1}\mathcal{M}_{m+\xi}\right)\oplus\left(\bigoplus_{\xi=0}^{\infty}\mathcal{M}_m\right).$$

For example,

$$\mathfrak{M}_3 = \underbrace{\mathcal{M}_1}_{\xi=-2}\oplus\underbrace{\mathcal{M}_2}_{\xi=-1}\oplus\underbrace{\mathcal{M}_3}_{\xi=0}\oplus\underbrace{\mathcal{M}_3}_{\xi=1}\oplus\underbrace{\mathcal{M}_3}_{\xi=2}\oplus\dots.$$

According to the definition of the direct sum (see [74, Definition 1.1.5]), $\mathfrak{M}_m$ consists of all matrix sequences of the form $A=(A_\xi)_{\xi=-m+1}^{\infty}$, where $A_\xi\in\mathcal{M}_{m+\xi}$ if $\xi<0$, $A_\xi\in\mathcal{M}_m$ if $\xi\ge 0$, and

$$\sup_{\xi\ge -m+1}\|A_\xi\|<+\infty.$$

Being a direct sum of W*-algebras, $\mathfrak{M}_m$ is a W*-algebra. We identify the elements of $\mathfrak{M}_m$ with the bounded linear operators acting in $\bigoplus_{\xi=-m+1}^{\infty}\mathbb{C}^{\min\{m,m+\xi\}}$. Now we are ready to describe the structure of $\mathcal{R}^{(\alpha)}_m$. Recall that $U^{(\alpha)}_m$ is given by (3.43).

Theorem 3.6.9. *Let $n \in \mathbb{N}$. Then $\mathcal{R}_m^{(\alpha)}$ consists of all operators belonging to $\mathcal{B}(\mathcal{A}_m^2(\mathbb{D},\mu_\alpha))$ that act invariantly on each subspace $\mathcal{W}_{\xi,\min\{m,m+\xi\}}^{(\alpha)}$, for $\xi \ge -m+1$. Furthermore,*

$$\mathcal{R}_m^{(\alpha)} \cong \bigoplus_{\xi=-m+1}^{\infty} \mathcal{B}(\mathcal{W}_{\xi,\min\{m,m+\xi\}}^{(\alpha)}),$$

and $\mathcal{R}_m^{(\alpha)}$ is spatially isomorphic to $\mathfrak{M}_m$:

$$U_m^{(\alpha)} \mathcal{R}_m^{(\alpha)} (U_m^{(\alpha)})^* = \mathfrak{M}_m.$$

Proof. We apply the scheme from Propositions 3.6.2, 3.6.4,

$$W_j = \mathcal{W}_\xi^{(\alpha)}, \qquad \mathcal{U} = \{\rho^{(\alpha)}(\tau)\colon\ \tau \in \mathbb{T}\},$$

and $H_1 = \mathcal{A}_m^2(\mathbb{D},\mu_\alpha)$. By (3.41), we obtain

$$J_1 = \{\xi \in \mathbb{Z}\colon\ \xi \ge -m+1\}, \qquad \mathcal{A}_m^2(\mathbb{D},\mu_\alpha) \cap \mathcal{W}_\xi^{(\alpha)} = \mathcal{W}_{\xi,\min\{m,m+\xi\}}^{(\alpha)}.$$

So, the W*-algebra $\mathcal{R}_m^{(\alpha)}$ is isometrically isomorphic to the direct sum of $\mathcal{B}(\mathcal{W}_{\xi,\min\{m,m+\xi\}}^{(\alpha)})$, with $\xi \ge -m+1$. Using the orthonormal basis $(b_{\xi+k,k}^{(\alpha)})_{k=\max\{0,-\xi\}}^{m-1}$ of $\mathcal{W}_{\xi,\min\{m,m+\xi\}}^{(\alpha)}$, we represent linear operators on this space as matrices. Define $\Phi_m^{(\alpha)}\colon \mathcal{R}_m^{(\alpha)} \to \mathfrak{M}_m$ by

$$\Phi_m^{(\alpha)}(S) := \left(\left[\left\langle S b_{\xi+k,k}^{(\alpha)}, b_{\xi+j,j}^{(\alpha)} \right\rangle \right]_{j,k=\max\{0,-\xi\}}^{m-1} \right)_{\xi=-m+1}^{\infty}. \tag{3.51}$$

In other words, $\Phi_m^{(\alpha)}(S) = U_m^{(\alpha)} S (U_m^{(\alpha)})^*$, i.e., $\Phi_m^{(\alpha)}$ is an isometrical isomorphism of W*-algebras induced by the unitary operator $U_m^{(\alpha)}$. □

Radial operators of finite rank, acting in $\mathcal{A}_m^2(\mathbb{D},\mu_\alpha)$, can be constructed as in Examples 3.6.3 and 3.6.5.

It is easy to verify (see a more general result in [57, Corollary 4.3]) that if $\mathcal{A}_m^2 = \mathcal{A}_m^2(\mathbb{D},\mu_\alpha)$ and $S \in \mathcal{R}_m^{(\alpha)}$, then $\operatorname{Ber}_{\mathcal{A}_m^2}(S)$ is a radial function. For $m=1$, the Berezin transform $\operatorname{Ber}_{\mathcal{A}_1^2}$ is injective. So, if $S \in \mathcal{B}(\mathcal{A}_1^2(\mathbb{D},\mu_\alpha))$ and the function $\operatorname{Ber}_{\mathcal{A}_1^2}(S)$ is radial, then the operator S is radial.

Radial operators in $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$

Let $n\in\mathbb{N}$. The space $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$ is invariant under the rotation $\rho^{(\alpha)}(\tau)$ for all τ in $\mathbb{T}$. The proof is similar to the proof of Proposition 3.6.8. Denote the compression of $\rho^{(\alpha)}(\tau)$ onto $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$ by $\rho_{(m)}(\tau)$. Let $\mathcal{R}^{(\alpha)}_{(m)}$ be the von Neumann algebra of all radial operators in $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$.

Theorem 3.6.10. *$\mathcal{R}^{(\alpha)}_{(m)}$ consists of all operators belonging to $\mathcal{B}(\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha))$ that are diagonal with respect to the orthonormal basis $(b^{(\alpha)}_{p,m-1})_{p=0}^\infty$. Furthermore,*

$$\mathcal{R}^{(\alpha)}_{(m)}\cong\ell^\infty(\mathbb{N}_0).$$

Proof. Corollaries 3.2.4 and 3.4.2 give

$$\mathcal{W}^{(\alpha)}_\xi\cap\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)=\begin{cases}\mathbb{C}b^{(\alpha)}_{\xi+m-1,m-1}, & \xi\geq -m+1,\\ \{0\}, & \xi<-m+1.\end{cases}\tag{3.52}$$

By Propositions 3.6.2, 3.6.4 and formula (3.52), $\mathcal{R}^{(\alpha)}_{(m)}$ consists of the operators that act invariantly on $\mathbb{C}b^{(\alpha)}_{\xi+m-1,m-1}$, $\xi\geq -m+1$, i.e., are diagonal with respect to the basis $(b^{(\alpha)}_{p,m-1})_{p=0}^\infty$. Therefore the function $\Phi^{(\alpha)}_{(m)}\colon\mathcal{R}^{(\alpha)}_{(m)}\to\ell^\infty(\mathbb{N}_0)$, defined by

$$\Phi^{(\alpha)}_{(m)}(S)=\big(\langle Sb^{(\alpha)}_{p,m-1},b^{(\alpha)}_{p,m-1}\rangle\big)_{p=0}^\infty,\tag{3.53}$$

is an isometric isomorphism. □

3.7 Radial Toeplitz operators in polyanalytic Bergman spaces

This section is similar to [57, Section 6], but here we use Jacobi polynomials instead of the generalized Laguerre polynomials.

Radial functions

Given g in $L^\infty(\mathbb{D})$, define $\operatorname{rad}(g)\colon \mathbb{D} \to \mathbb{C}$ by

$$\operatorname{rad}(g)(z) \coloneqq \int_{\mathbb{T}} g(\tau z)\,\mathrm{d}\mu_{\mathbb{T}}(\tau). \tag{3.54}$$

Given a in $L^\infty([0,1))$, define $\widetilde{a}\colon \mathbb{D} \to \mathbb{C}$ by

$$\widetilde{a}(z) \coloneqq a(|z|) \qquad (z \in \mathbb{D}).$$

The proof of the following criterion is a simple exercise.

Proposition 3.7.1. *Given g in $L^\infty(\mathbb{D})$, the following conditions are equivalent:*

(a) for every τ in $\mathbb{T}$, the equality $g(\tau z) = g(z)$ is true for a.e. z in $\mathbb{D}$;
(b) for every τ in $\mathbb{T}$, the equality $\rho^{(\alpha)}(\tau)g = g$ is true a.e.;
(c) $\operatorname{rad}(g) = g$ a.e.;
(d) there exists a in $L^\infty([0,1))$ such that $g = \widetilde{a}$ a.e.

Radial multiplication operators in $L^2(\mathbb{D}, \mu_\alpha)$

The following result is easily obtained by direct calculation

Proposition 3.7.2. *Let $g \in L^\infty(\mathbb{D})$. Then $\operatorname{Rad}^{(\alpha)}(M_g) = M^{(\alpha)}_{\operatorname{rad}(g)}$.*

Given a in $L^\infty([0,1))$, we define the numbers $\beta_{a,\alpha,\xi,j,k}$ by

$$\beta_{a,\alpha,\xi,j,k} \coloneqq \int_0^1 a(\sqrt{t})\mathcal{J}^{(\alpha,|\xi|)}_{\min\{j,j+\xi\}}(t)\mathcal{J}^{(\alpha,|\xi|)}_{\min\{k,k+\xi\}}(t)\,\mathrm{d}t, \tag{3.55}$$

i.e.,

$$\beta_{a,\alpha,\xi,j,k} \coloneqq c^{(\alpha,|\xi|)}_{\min\{q+\xi,q\}}c^{(\alpha,|\xi|)}_{\min\{k+\xi,k\}} \int_0^1 a(\sqrt{t})Q^{(\alpha,|\xi|)}_{\min\{q,q+\xi\}}(t)Q^{(\alpha,|\xi|)}_{\min\{k,k+\xi\}}(t)\,(1-t)^\alpha\, t^{|\xi|}\,\mathrm{d}t. \tag{3.56}$$

Proposition 3.7.3. *Let $a \in L^\infty([0,1))$. Then $M_{\widetilde{a}} \in \mathcal{R}^{(\alpha)}$, and*

$$\langle M_{\widetilde{a}} b^{(\alpha)}_{p,q}, b^{(\alpha)}_{j,k}\rangle = \langle \widetilde{a} b^{(\alpha)}_{p,q}, b^{(\alpha)}_{j,k}\rangle = \delta_{p-q,j-k}\beta_{a,\alpha,p-q,q,k}. \tag{3.57}$$

Proof. Since $\widetilde{a}$ is invariant under rotations, it follows directly from definitions that $M^{(\alpha)}_{\widetilde{a}}$ commutes with $\rho^{(\alpha)}(\tau)$ for every τ. This is a particular case of [57, Lemma 4.4]. Formula (3.57) is obtained directly using polar coordinates. □

Radial Toeplitz operators in $\mathcal{A}^2_m(\mathbb{D},\mu_\alpha)$

Proposition 3.7.4. *Let $g \in L^\infty(\mathbb{D})$. Then $T^{(\alpha)}_{m,g}$ is radial if and only if the function g is radial.*

Proof. Follows from Proposition 3.5.2 and [57, Corollaries 4.6, 4.7]. □

Recall that $\Phi^{(\alpha)}_m : \mathcal{R}^{(\alpha)}_m \to \mathfrak{M}_m$ is defined by (3.51).

Given a in $L^\infty([0,1))$, denote by $\gamma^{(\alpha)}_m(a)$ the sequence of matrices $[\gamma^{(\alpha)}_m(a)_\xi]_{\xi=-m+1}^\infty$, where $\gamma^{(\alpha)}_m(a)_\xi \in \mathcal{M}_{\min\{m+\xi,m\}}$ is given by

$$\gamma^{(\alpha)}_m(a)_\xi \coloneqq \left[\beta_{a,\alpha,\xi,j,k}\right]_{j,k=\max\{0,-\xi\}}^{m-1}. \tag{3.58}$$

Proposition 3.7.5. *Let $a \in L^\infty([0,1))$. Then $T^{(\alpha)}_{m,\widetilde{a}} \in \mathcal{R}^{(\alpha)}_m$ and $\Phi_m(T^{(\alpha)}_{m,\widetilde{a}}) = \gamma^{(\alpha)}_m(a)$.*

Proof. Apply Propositions 3.7.3 and 3.7.4. □

Radial Toeplitz operators in $\mathcal{A}^2_{(m)}(\mathbb{D},\mu_\alpha)$

Proposition 3.7.6. *Let $a \in L^\infty([0,1))$. Then $T^{(\alpha)}_{(m),\widetilde{a}} \in \mathcal{R}^{(\alpha)}_{(m)}$, the operator $T^{(\alpha)}_{(m),\widetilde{a}}$ is diagonal with respect to the orthonormal basis $(b^{(\alpha)}_{p,m-1})_{p=0}^\infty$, and the corresponding eigenvalues can be computed by*

$$\lambda_{a,\alpha,m}(p) = \int_0^1 a(\sqrt{t})\,\left(\mathcal{J}^{(\alpha,|p-m+1|)}_{\min\{p,m-1\}}(t)\right)^2 \mathrm{d}t \qquad (p \in \mathbb{N}_0). \tag{3.59}$$

Proof. From Proposition 3.7.4 we get $T^{(\alpha)}_{(m),\widetilde{a}} \in \mathcal{R}^{(\alpha)}_{(m)}$. Due to Proposition 3.7.3 and Theorem 3.6.10,

$$\lambda_{a,\alpha,m}(p) = (\Phi_{(m)}(T^{(\alpha)}_{(m),\widetilde{a}}))_p = \langle T^{(\alpha)}_{(m),\widetilde{a}} b^{(\alpha)}_{p,m-1}, b^{(\alpha)}_{p,m-1}\rangle = \beta_{a,p-m+1,m-1,m-1}.$$ □

Colofón

Beauty is the first test: there is no permanent place
in the world for ugly mathematics. G. H. Hardy

La belleza es el primer examen, dice Hardy, concepto que a propósito confunde con simpleza. La simpleza es el primer examen: las matemáticas feas no tienen un lugar permanente porque no son simples. La simpleza, la claridad, la evidencia: lo natural es lo bello. La matemática resuelve, sí, pero también simplifica. La matemática depura porque exprimiendo las cosas, éstas rebelan sus entrañas.

La matemática es una suerte de alma que posee la esencia de las cosas una vez que es creada. En un acto de posesión demoniaca, brinca de la imaginación humana al corazón de las cosas. Allí las posee de veras porque las comanda. La matemática describe, sí, pero también predice. Las cosas obedecen cuando las reglas son claras. Por eso la fealdad es la ambigüedad, lo intrincado, lo inextricable.

Dice Kuntzmann (1969) que todos, quizá sin que hayamos reflexionado sobre ello, estamos seguros de dos cosas: no se puede prescindir de las matemáticas y no se puede hacer trampa con ellas. No si se siguen bien sus reglas. A través de la lupa de las matemáticas no hay resultado ambiguo ni misterio que no sea rebelado a su debido tiempo. Por mucho o poco que nos agraden, podemos estar seguros de que en ellas yace una verdad inapelable. Todos confiamos en ellas porque la experiencia nos dice que funcionan y nos hacen sentir seguros. Ernesto Sábato, a propósito de esto, dice en *Uno y el universo* que

> *Existe una opinión generalizada según la cual la matemática es la ciencia más difícil cuando en realidad es la más simple de todas. La causa de esta paradoja reside en el hecho de que, precisamente por su simplicidad, los razonamientos matemáticos equivocados quedan a la vista* [...] *El resultado es que cualquier tonto se cree en condiciones de discutir sobre política y arte –y en verdad lo hace–, mientras que mira la matemática desde una respetuosa distancia.*

La matemática es creada ad hoc, simulando el ejercicio de algo, así posee: imitando. Y como virus que se implanta, también la matemática se propaga. Ha sido varias veces descubierta en lugares inopinados. Se hace a la medida de lo que vemos, pero una vez en marcha, nos habla de cosas inesperadas.

La matemática es un invento (el mejor invento): es un bisturí que corta un problema complejo en partes pequeñas y simples. Por eso estudiamos el triángulo (que tiene muy pocos lados) y el círculo (que está demasiado redondo) porque las figuras son rompecabezas de triángulos y círculos. De ahí su belleza simple: el análisis. Sin embargo, el mundo funciona sin que nosotros sepamos muchas veces cómo. Hemos creado un lenguaje que describe lo que hay, lo que ya está, lo que ya funciona. Adecuamos los signos y las palabras para representar las cosas, pero son las cosas mismas las que hablan de sí en este lenguaje, de lo que son y de sus relaciones. Entonces la matemática es un descubrimiento (el más sorprendente). De ahí su simple belleza: la síntesis.

La investigación en matemáticas funciona más o menos como una mina: alguien con mucha experiencia intuye que aquí habrá oro. El oro de la matemática es la simplificación, que se obtiene estableciendo vínculos con ideas más sencillas y mejor comprendidas. Cuando el milagro del oro se confirma, todo un equipo se encarga de escarbar al rededor. La experimentación en matemáticas es así: se cambian algunas condiciones y se observa si se obtuvo el mismo resultado. Se cambian características de un problema resuelto y se observa si la solución se puede trasplantar o adaptar al nuevo problema.

Siempre que se resuelve un problema hay que preguntarse qué otros problemas se resuelven del mismo modo. Es decir, hay que intentar resolver muchos problemas con las herramientas que ya inventamos. ¿Cuáles características del problema permiten que cierta solución funcione? ¿Cuáles otros problemas tienen esas mismas características esenciales? ¿De qué nos hablan esas características esenciales? Una vez que se ha escarbado lo suficiente en la mina, ya es posible determinar hacia dónde va la veta, es decir, se puede mirar desde un plano más general y proponer una hipótesis estructural. Así el oro ya no es milagro: *milagro* que es parecido a *inexplicable* (matemáticas feas). Las cosas que se simplifican se vuelven prototipos. Observando las características de la mina (una vez explorada), podremos reconocer otras más fácilmente. Ese es el modus operandi de la matemática, que es por demás, el método científico: observa un milagro, trata de reproducirlo, y cuando hayas creado una miscelánea de ejemplos, será más fácil identificar las características esenciales que producen el milagro. Entonces formula una teoría que haga encajar el milagro en el mosaico de las matemáticas.

Esta elaborada tarea de análisis y síntesis es en realidad el proceso mediante el cual aprendemos: primero advertimos un milagro, un hecho fuera el alcance de nuestra comprensión. La curiosidad conduce a la manipulación: hacemos el examen de belleza, es decir, buscamos qué partes son simples, comprensibles, aceptables. Luego estudiamos cómo se relacionan las partes entre sí y finalmente juntamos todo: construimos los puentes entre lo que ya sabemos y lo nuevo. Resalto que para aprender hace falta tener los ojos bien abiertos y una genuina sorpresa. Hace falta ser curioso, preguntón, incómodo pero sobre todo ingenioso, pues nunca se cuenta con herramientas para explorar lo desconocido, hay que usar las que ya se tiene. Es importante saber usar esas herramientas, pero siempre se puede aprender sobre la marcha: así ocurre siempre. El deber del alumno es ser curioso. El deber del profesor es fomentar la curiosidad y propiciar la confianza de meter las manos: el ejercicio de (intentar) responder nuestras propias preguntas nos acerca al aprendizaje significativo. El profesor debe fomentar la autonomía. Los programas deben presentar a la matemática más como una herramienta para explorar que para resolver.

La matemática en la actualidad florece no sólo como lenguaje universal de las ciencias naturales o por su aplicación en las finanzas o la computación, también en su enseñanza, en sus fundamentos, en su propio lenguaje y sus objetos de estudio. Esta ciencia ha crecido aceleradamente las últimas décadas. Prueba de ello es que, hasta 1950, a nivel mundial se producían menos de 10 mil artículos de investigación al año. Actualmente la producción anual rebasa los 90 mil. Es natural preguntarse: ¿Todos estos artículos son relevantes para la matemática? No individualmente (salvo algunos, quizás) sino todos en colectivo: mientras exista este flujo de información, esta comunicación permanente, habrá la discusión, el análisis y la síntesis que gesta las ideas nuevas. Evidentemente la matemática no es una ciencia terminada, como suelen creer los estudiantes. Está siempre sometida a un replanteamiento, pues las nuevas ideas y sus conexiones develan nuevas caras de las ideas anteriores. Es esta búsqueda y replanteamiento el sino de las matemáticas.

Este crecimiento exagerado es debido en mucho al uso de las computadoras, pues permiten una manipulación cuantitativa veloz, precisa y abundante: justo lo que se requiere para un análisis cualitativo más profundo. Los cálculos numéricos ahorran tiempo y permiten desechar rápidamente hipótesis equivocadas o confirmar conjeturas acertadas. Las demostraciones analíticas son actualmente el paso siguiente de las comprobaciones numéricas. Éstas últimas alumbran el arduo camino de la justificación teórica. Para desarrollar la intuición hace falta tener experiencia, dice mi asesor, por eso hay que dar más peso al carácter estadístico de la matemática en la educación, porque la inferencia es producto de la síntesis. Las computadoras facilitan ese análisis estadístico pues es justamente el rumbo de la informática: la minería de datos. La enseñanza de las matemáticas debe incluir cuanto antes esta herramienta. La matemática debe simplificar lo que ahora es complejo, vincular las ideas principales con todas las áreas del conocimiento y unificar el lenguaje para propiciar la transdisciplinariedad. Su enseñanza debe plantearla como una herramienta de exploración.

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