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Transmutation operators: construction and application in boundary value and inverse spectral problems

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Presented by

M. Sc. Víctor Alfonso Vicente Benítez

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Thesis advisor:

Dr. Vladyslav Kravchenko Cherkasski

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Operadores de transmutación: construcción y aplicación a problemas de valor en la frontera y espectrales inversos

TESIS

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Ph. D. Thesis

Department of mathematics

CINVESTAV, Querétaro

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Dedicated to the memory of my father,
Víctor Vicente Santiago (1959-2021),
thank you for your love, your teachings
and all the moments we spent together.

Abstract

In this thesis, some new applications of the transmutation operator theory to the solution of direct and inverse spectral problems for the Sturm-Liouville equation in impedance form (SLEIF), and to the construction of complete systems for the radial Schrödinger equation, are presented.

In the first part of this thesis, an integro-differential transmutation operator for the SLEIF is constructed. The properties of the integral tranmsutation kernel are proved, together with the boundedness and invertibility of the transmutation operator in appropriate functional spaces. A representation of the transmutation kernel as a Fourier series in terms of Legendre polynomials is obtained, and as a corollary a new representation for the solutions of the SLEIF in terms of Neumann series of Bessel functions.

It is shown that the transmutation kernel satisfies a Gelfand-Levitan equation for the Sturm-Liouville spectral problem for the SLEIF on a finite interval. Substitution of the Fourier-Legendre series of the kernel in the Gelfand-Levitan equation reduces the inverse spectral problem to solution of an infinite system of linear algebraic equations. Moreover, we show that to recover the impedance function, only the first component of the solution vector is required. From the truncation of the system of equations, we obtain an algorithm to solve the inverse spectral problem. The stability and effectiveness of the method are illustrated by several numerical examples.

For the SLEIF on the half-line, a Levin-type integral representation for the Jost solution and its derivative is constructed. Using such integral representation, the Jost solution and its derivative can be represented in the form of power series in the unit disk, whose coefficients can be computed by a simple recursive procedure. The series representations lead to an explicit representation for spectral data and analytic method for their

computation.

In the second part of this work, we study a transmutation operator for the radial Schrödinger equation in a star-shaped domain. New properties of the transmutation operator, as its boundedness and invertibility on the spaces of C^2-functions and the harmonic Bergman space are established, together with a representation of the transmutation kernel as a Fourier series of Jacobi polynomials. With the aid of such Forier-Jacobi series, a new complete system of solutions, called the formal powers, of the radial Schrödinger equation is constructed. The completeness of the formal powers in the space of classical solutions and the Bergman space is shown. Similarly, using a new Runge property for strongly elliptic equations, we prove the completeness of the formal powers in the space of weak solutions with respect to the L_2, H^1 and H^2-norms.

Resumen

En esta tesis se presentan dos nuevas aplicaciones de la teoría de operadores de transmutación, a la solución de problemas espectrales directos e inversos para la ecuación de Sturm-Liouville en forma de impedancia (SLEIF, por sus siglas en inglés), y a la construcción de sistemas completos de soluciones para la ecuación de Schrödinger radial.

En la primera parte de esta tesis se construye un operador de transmutación integro-diferencial para la SLEIF. Se prueban las propiedades del kernel integral de transmutación, junto con la continuidad e invertibilidad del operador de transmutación en espacios funcionales apropiados. Se obtiene una representación para el kernel de transmutación en términos de los polinomios de Legendre, y como un corolario una nueva representación para las soluciones de la SLEIF en términos de series de Neumann de funciones de Bessel.

Demostramos que el kernel de transmutación satisface una ecuación de Gelfand-Levitan para el problema espectral de Sturm-Lioville de la SLEIF en un intervalo finito. La sustitución de la serie de Fourier-Legendre del kernel en la ecuación de Gelfand-Levitan reduce el problema espectral inverso a la solución de un sistema infinito de ecuaciones lineales algebraicas. Más aún, mostramos que para recuperar la función de impedancia solo se necesita la primer componente del vector solución. A partir de truncar el sistema de ecuaciones, obtenemos un algoritmo para resolver el problema espectral inverso. Ilustramos la estabilidad y la efectividad del método a través de varios ejemplos numéricos.

Para la SLEIF en el semi-eje positivo, construimos una representación integral de tipo Levin para la solución de Jost y su derivada. Usando tal representación integral, la solución de Jost y su derivada pueden ser representadas en la forma de series de potencias en el disco unitario, cuyos coeficientes pueden calcularse mediande un sencillo procedimiento recursivo. Estas representaciones en serie nos llevan a una representación

explícita de los datos espectrales y a un método analítico para calcularlos.

En la segunda parte de este trabajo estudiamos un operador de transmutación para la ecuación de Schrödinger radial en un dominio estrellado. Se establecen nuevas propiedades para el operador de transmutación, como su continuidad e invertibilidad en los espacios de funciones de clase C^2 y el espacio de Bergman armónico, junto con una representación para el kernel de transmutación como una serie de Fourier de polinomios de Jacobi. Con la ayuda de la serie de Fourier-Jacobi, construimos un nuevo sistema completo de soluciones, llamados potencias formales, para la ecuación de Schrödinger radial. Demostramos que las potencias formales son completas en el espacio de soluciones clásicas y en el espacio de Bergman. De manera similar, usando una nueva propiedad de Runge para ecuaciones fuertemente elípticas, demostramos que las potencias formales son completas en el espacio de soluciones débiles con respecto a las normas L_2, H^1 y H^2.

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This thesis is dedicated to the memory of my father, Víctor Vicente Santiago. Dad, I know you couldn't see the culmination of my studies, but I know you would have wanted me to finish this project, I appreciate all the support, love and advice you gave me. Thanks for everything, dad.

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Contents

Introduction

The aim of this dissertation is to present a series of applications of the transmutation operator method to direct and inverse spectral problems for the Sturm-Liouville equation in impedance form (SLEIF), as well as to the construction of complete systems of solutions and solutions of boundary value problems for partial differential equations (PDE) with variable coefficients possessing certain symmetry.

Roughly speaking, a transmutation (or transformation) operator is a linear operator $\mathbf{T}$ defined on a topological vector space E, which relates two linear operators $\mathbf{A}$ and $\mathbf{B}$ defined on a subspace $E_1 \subset E$ by the following relation

$$\mathbf{AT} = \mathbf{TB} \quad \text{in } E_1.$$

Operator $\mathbf{T}$ is usually required to be continuous and invertible with continuous inverse. One of the most important applications for transmutation operators is to relate a "simple" differential equation whose solutions are known (e.g. the ordinary differential equation $v'' + \lambda v = 0$ with $\lambda \in \mathbb{C}$, or the Laplace's equation) with another more complicated equation, such as the Schrödinger equation. The concept of transmutation operator and the idea of using it to relate two linear differential operators was proposed by J. Delsarte in 1938 in the work [40]. In 1948, A. Ya. Povzner proved in [118] that a transmutation operator that relates the *one-dimensional Schrödinger equation*

$$-y'' + q(x)y = \lambda y, \qquad \text{with } \lambda \in \mathbb{C}, \tag{1}$$

to equation $v'' + \lambda v = 0$, can be written in the form of a Volterra integral operator of the second type. The study of such transmutation operators was developed throughout various publications, such as [2, 11, 25, 41, 52, 66, 86, 105, 111, 125, 128, 129]. One of the applications of the transmutation operator for the Schrödinger equation is its use in the solution of direct and inverse spectral problems. With the work [6] of V. A. Ambartsumyan, the theory of inverse Sturm-Liouville problems began to develop rapidly and took an important place in general spectral theory, with numerous applications in mathematical physics. In [55], Gelfand and Levitan proved that the integral kernel of the transmutation operator can be used to provide a solution to the inverse problem of recovering the Schrödinger operator from the spectral function of the problem, because the integral kernel satisfies an integral equation which receives the name of the *Gelfand-Levitan equation*. With this, the inverse spectral problem reduces to the solution of a Fredholm integral equation of the second kind. However, computationally solving this integral equation presents several challenges which has led to different techniques for the approximation of the transmutation kernel [11, 32, 89, 90]. The use of transmutation operators for direct spectral problems has been developed in numerous publications (see, e.g., [11, 26, 30, 48, 103, 105, 110]). One of the recent discoveries about the transmutation operator

$$\mathbf{T}u(x) := u(x) + \int_{-x}^{x} K(x,t)u(t)dt, \tag{2}$$

that transmutes the one-dimensional Schrödinger equation, is the fact that it is possible to know explicitly the action of the operator $\mathbf{T}$ over the powers $\{x^k\}_{k=0}^{\infty}$, without the need to build the transmutation kernel $K(x,t)$ [24]. The functions $\varphi^{(k)}(x) = \mathbf{T}[x^k]$ are known as *formal powers* and form a complete system in the spaces $C[-\ell,\ell]$ and $L_2(-\ell,\ell)$, $\ell > 0$ (see [80]). Formal powers are used to represent solutions of Eq. (1) and to solve direct spectral problems [83, 88], as well as to analytically approximate the transmutation kernel $K(x,t)$ [89, 90]. In [81], V. V. Kravchenko, S. M. Torba and L. M. Navarro proposed to write the transmutation kernel as a Fourier series of Legendre polynomials

$$K(x,t) = \sum_{n=0}^{\infty} \frac{a_n(x)}{x} P_n\left(\frac{t}{x}\right), \tag{3}$$

where the coefficients can be computed by a simple recursive integration procedure. Substitution of this series in the representations of the solutions of Eq. (1) gives us a new representation in the form of a Neumann series of Bessel functions (NSBF). It has been demonstrated that the NSBF representation is an efficient method to compute the eigenvalues of Sturm-Liouville problems associated with Eq. (1) [38, 81, 93]. In relation to the inverse spectral problem, in [76] it was shown by substituting the Fourier-Legendre series (3) in the Gelfand-Levitan equation that the inverse problem is reduced to solution of a system of linear algebraic equations, whose unknowns are the coefficients $\{a_n(x)\}_{n=0}^{\infty}$. Moreover, only the first coefficient $a_0(x)$ is required to recover the potential q. This procedure has been improved in [78, 92], and applied to inverse problems on the half-line [39] and inverse scattering problems [69, 85].

One of the first results obtained in this thesis is the solution of the inverse problem for the Schrödinger equation with Dirichlet-Dirichlet conditions in the interval $[0, \pi]$. For this problem, the asymptotics of the eigenvalues are known (see, e.g., [52, Ch. I]). However, in order to derive a Gelfand-Levitan equation for the inverse problem, the asymptotics for the normalizing constants are deduced. Once the corresponding Gelfand-Levitan equation has been derived, we use the Fourier-Legendre series representation of the transmutation kernel (which in this case corresponds to the operator that transmutes the sine function) to derive an infinite system of algebraic equations whose unknowns are the Fourier-Legendre coefficients. Only the first coefficient needs to be retrieved to compute the potential $q(x)$. From this system, a method to solve the inverse problem is deduced. This consists of solving the truncated system, calculating the first coefficient of the solution vector and with this approximating the potential. We also show that for a certain number of equations, the truncated system is uniquely solvable and the solution is stable. This allows us to derive a solution method for the inverse problem, which consists only of solving a system of linear equations and using the first component of the solution vector to calculate the potential. We present some examples (performed in Matlab) of the calculation of the potential.

A second goal of this thesis is to develop the theory of the transmutation operator for the *Sturm-Liouville equation in impedance form* (SLEIF)

$$\frac{d}{dx}\left(a^2(x)\frac{du}{dx}\right) + \lambda a^2(x)u = 0, \quad \text{for } -b < x < b,\ b > 0, \tag{4}$$

analogous to that developed for the Schrödinger equation [20, 24, 87, 88, 89, 111]. The coefficient a in Eq. (4) is a function (in general complex-valued) that does not vanish in the whole segment $[-b, b]$ and it is called the *impedance function*. Spectral problems related to (4) appear, e.g., in classical vibration theory [61, 137], wave propagation theory

[27, 142] and geophysics [26, 28, 124].

One way to proceed with the study of transmutation operators for Eq. (4) consists in transforming (4) into the Schrödinger equation by means of the Liouville transformation [47, 64, 79, 93, 109, 144, 145]. However, in this case it is necessary to assume that $a \in W^{2,1}(-b,b)$. In the case $a \in W^{1,p}(-b,b)$, $1 \leqslant p \leqslant \infty$, it is possible to transform (4) into a 2×2 linear system, and obtain a representation for its solutions by means of a Volterra integral operator [5, 7, 31]. In [26], an integral representation for the solution $C(\sqrt{\lambda}, x)$ of (4) that satisfies the conditions $C(\sqrt{\lambda}, 0) = 1$, $C'(\sqrt{\lambda}, 0) = 0$, was proposed. However, the analytical properties of the operator and the transmutation kernel were not investigated. In this work we propose a direct representation for the solutions of Eq. (4), in which the transmutation operator appears in the form of an integro-differential operator. This representation is obtained for impedance functions $a \in W^{1,\infty}(-b,b)$. We show that the kernel of the transmutation operator exists as the solution of a Goursat problem for a hyperbolic equation. For $a \in C^1[-b,b]$, the transmutation operator turns out to be bounded and invertible in $C^1[-b,b]$ and hence it is a transmutation operator for the pair of operators $\mathbf{L}_a = -\frac{1}{a^2(x)}\frac{d}{dx}a^2(x)\frac{d}{dx}$ and $\frac{d^2}{dx^2}$, in the sense of Definition 1 of [24]. Following [81], a representation of the transmutation kernel as a Fourier series in terms of Legendre polynomials is obtained, and as a corollary a representation for the solutions of equation (4) in terms of a NSBF.

The inverse problem for Eq. (4) consists in recovering the impedance function from the spectral data which are the eigenvalues of the Sturm-Liouville problem and the norming constants. Inverse spectral problems related to (4) appear, e.g., in classical vibration theory [61, 137], wave propagation theory [142] and geophysics [26]. One way to proceed with the study of the inverse problem consists in transforming (4) into the Schrödinger equation by means of the Liouville transformation, and as mentioned above, condition $a \in W^{2,1}(0,\pi)$ is required. In the case $a \in W^{1,p}(0,\pi)$, $1 \leqslant p \leqslant \infty$, it has been established that it is possible to recover the impedance function from two spectra, corresponding to the problem with Neumann-Neumann and Neumann-Dirichlet conditions, respectively (see [5, 7]). In [31], the asymptotics for the eigenvalues and norming constants for the problem with Dirichlet-Dirichlet conditions are established. In [54], a numerical method (based on finite diferences) was proposed to solve the inverse problem with Dirichlet-

Dirichlet conditions.

We propose a direct method for solving the inverse problem for (4) with Neumann-Neumann conditions. We derive a Gelfand-Levitan equation analogous to the case of the Schrödinger equation. Using the Fourier-Legendre representation of the transmutation kernel we reduce the inverse problem to an infinite system of linear algebraic equations. Moreover, we show that it is possible to recover the impedance function from the first component of the solution vector, thus extending the method developed for the Schrödinger equation. We emphasize that it is not necessary to convert the Sturm-Liouville problem for (4) into a problem for the Schrödinger equation, the impedance function is recovered directly from the first component of the solution vector. The developed approach leads to a simple and direct numerical method for solving inverse spectral problems for (4) which is illustrated by several numerical tests. It is interesting to mention that the matrix of the system obtained for this problem is exactly the same as the one obtained for the problem with the Dirichlet-Dirichlet condition for the Schrödinger equation.

Another contribution of this thesis work is to develop a series representation for the *Jost solution* of the SLEIF. We recall that the Jost solution of the SLEIF on the half-line $\mathbb{R}^+ := (0, \infty)$, is the solution $e(\rho, x)$, $\rho^2 = \lambda$ with $\operatorname{Im} \rho \geqslant 0$ satisfying the asymptotic relations $e^{(k)}(\rho, x) = (i\rho)^k e^{i\rho x}(1 + o(1))$, $x \to \infty$ for $k = 0, 1$. It is well known that the Jost solution of the Schrödinger equation admits an integral representation, sometimes called Levin's representation, which involves the inverse Fourier transform of a continuous kernel $A(x,t)$ satisfying $A(x, \cdot) \in L_2(x, \infty)$ for all $x \in \mathbb{R}^+$ (see [48, 52, 105]). Levin's representation is one of the most actively used tools in studying corresponding direct and inverse spectral and scattering problems [30, 38, 48, 77, 78, 111, 129]. Levin representations have been studied for different types of Sturm-Liouville equations on the half-line (see, e.g., [64, 109, 144, 145]), but in all cases it is necessary to transform the Sturm-Liouville equation to the Schrödinger equation, through the Liouville transformation. An advantage of the impedance equation is that Liouville's transformation is reduced to a unitary multiplication operator, as we will see in Chapter 3. We obtain a Levin representation for the Jost solution without resort to the Liouville transformation and study the regularity of the corresponding integral kernel $A(x,t)$. Applying the idea from [77] (see also [78]) we employ a Fourier-Laguerre series expansion of the kernel $A(x,t)$ to obtain a new analytic representation for the Jost solution in the form of a power series with respect to the parameter $z = \frac{\frac{1}{2}+i\rho}{\frac{1}{2}-i\rho}$. The series converges in the unit disk $|z| < 1$. Moreover, similarly to [38] (see also [78]) for the coefficients of the series a recurrent integration procedure is derived which represents a convenient tool for their practical computation. Additionally we study the Birkhoff solution of (4), which is complementary to the Jost solution, and give its characterization.

Based on the representations for the solutions of the SLEIF we obtain an explicit representation for the direct spectral data of the Sturm-Liouville problem with the boundary condition

$$u'(0) - hu(0) = 0, \qquad \text{where } \ h \in \mathbb{R}. \tag{5}$$

Under the assumptions $p'(x) < 0$ for all $x \in \mathbb{R}^+$ and $h = 0$ this spectral problem for (4) was studied in [26]. We characterize the continuous spectrum and show that if $h \geqslant 0$, the problem has no eigenvalue, while in the case $h < 0$ there can exist a finite number of the eigenvalues. Applying results of the general theory of Sturm-Liouville operators ([3, 15, 17]) together with the properties of the Jost and Birkhoff solutions, we obtain an explicit representation of the spectral measure in terms of the so-called normalizing constants, and the characteristic equation of the problem. Using the representations obtained for the Jost solution and its derivative, an analytic method is obtained to calculate the eigenvalues, the normalizing constants and the spectral function. Such methods based on the analytical representations of Jost-type solutions have shown their efficiency for solving direct and inverse spectral problems in the case of the Schrödinger equation (see [38, 78]).

As an application, we show how to solve an inverse problem for the wave equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial v(t,x)}{\partial x}\right) = p(x)\frac{\partial^2 v(t,x)}{\partial t^2}, \quad (t,x) \in \mathbb{R} \times (0,\infty).$$

The direct problem consists in finding a function (or distribution) $v(x,t)$ such that $v(t,x) = 0$ for $t < 0$ and $v'(t,0) = \delta(t)$, where δ is the Dirac delta distribution. The inverse problem is about reconstructing the principal coefficient $p(x)$ from the function (or distribution) $g(t) = v(0,t)$. These types of problems appears in geophysics [26, 27, 28, 124], and under certain conditions on the coefficient p it is possible to reduce the inverse problem to a spectral problem for the SLEIF. In this case it is known that the inverse problem is reduced to a Gelfand-Levitan equation [26]. Following [39], we employ the Fourier-Legendre series representation of the transmutation kernel to reduce the problem to an infinite system of linear algebraic equations. As in the problem for a finite interval, it is enough to calculate the first coefficient of the solution vector to recover the potential. Unlike the case of the Schrödinger equation, the asymptotic behavior of the kernel that appears in the Gelfand-Levitan equation is not exactly known. Using a relation between the spectral function and the function $g(t)$ (see [26]), we show that if $g \in W^{3,1}(0,\infty)$, $g(0) = -1$, $g''(0) = 0$, then from a certain number of equations, the truncated system has a unique solution, and this is stable.

The last goal of this thesis concerns the application of the transmutation operator methods to PDE. Specifically, we study the properties of a transmutation operator for the radial Schrödinger equation

$$-\triangle_d u(x) + q(|x|)u(x) = 0, \quad x \in \Omega, \tag{6}$$

where $\Omega \subset \mathbb{R}^d$ ($d \geqslant 2$) is a bounded star-shaped domain and $q \in C^1(\overline{\Omega})$ depends only on the radial component $r = |x|$. In [14], S. Bergman showed for the case $d = 2$ and the potential q being an analytic function of the radial component $r = |x|$, that any solution u of (6) can be written in the form

$$u(x) = H(x) + \int_0^1 \sigma G(r, 1-\sigma^2) H(\sigma^2 x) d\sigma$$

where $H(x) = \int_{-1}^1 h\left(\frac{x}{2}[1-t^2]\right) \frac{dt}{(1-t^2)^{\frac{1}{2}}}$ is the *Bergman transform* of a harmonic function h, and G is an analytic function of r. In [56], R. Gilbert showed that for any solution u there exists a unique harmonic function h such that

$$u(x) = \mathbf{T}h(x) = h(x) + \int_o^1 \sigma^{d-1} G(r, 1-\sigma^2) h(\sigma^2 x) d\sigma. \tag{7}$$

The representation (7) can be generalized for $d \geqslant 3$ and C^1-potentials (see [57, 58, 59] and [11, Ch. V]). The kernel G satisfies some initial value problem for a hyperbolic PDE (see [57]). When the potential q is analytic, the kernel G is an analytic function of the radial component (see [14, 56, 57]). In [138], I. Vekua constructed the operator (7) explicitly for the Helmholtz equation and showed its invertibility. In a general context, the invertibility of the operator (7) was shown in [57, 59]. The operator (8.2) is usually called a *transformation operator* ([11]). It was applied, for example, to solving the Dirichlet problem on an admissible domain (see [32, 58, 59]), as well as to studying properties of generalized sub-harmonic functions [59].

We establish some new properties of the operator $\mathbf{T}$. The operator $\mathbf{T}$ was mistakenly thought to satisfy the transmutation relation $(\triangle_d - q(r))\mathbf{T} = \mathbf{T}\triangle_d$ (see [59]). We show that the correct transmutation relation for the operator $\mathbf{T}$ is

$$r^2 \left(\triangle_d - q(r)\right) \mathbf{T} = \mathbf{T} r^2 \triangle_d$$

(valid on C^2-functions), together with the continuity of $\mathbf{T}$ and of its inverse on the space $C(\Omega)$. We show as well that the operator is bounded and invertible from the Bergman space of harmonic functions $b_2(\Omega)$ onto the Bergman space of solutions of (6) and obtain some bounds for the operator norm. The established properties of the operator $\mathbf{T}$ allow us to obtain a complete system of solutions of (6). The idea to use the transmutation operators for obtaining complete systems of solutions of PDEs was developed, for example, in [11, 14, 32]. The difficulty of this approach always consisted in the necessity of constructing the corresponding integral transmutation kernel by solving a Goursat type problem for a corresponding PDE [11, 32, 57, 58]. Here extending further the approach from [23, 24, 74, 82] we obtain a complete system of solutions, without the need of constructing the kernel. Namely, following the idea of [81], the kernel G is expanded into a Fourier series with respect to an appropriate system of orthogonal polynomials, and with its aid a series representation for the solutions of (6) is obtained. In particular, a complete system of solutions (that we call *formal powers*) is obtained as a result of transmutation of homogeneous harmonic polynomials. The completeness of the formal powers in the sense of the uniform convergence on compact subsets of Ω is shown. When the domain is an open ball centered at the origin, the formal powers represent an orthogonal basis for the Bergman space of solutions.

It was shown that formal powers are not only a complete system in $C(\Omega)$, but also for weak solutions. For this, some results were developed on the approximation of solutions of an elliptic PDE. The idea is to use the property that guarantees that a solution u in a domain D_1 can be approximated by means of another solution v defined in a larger domain D_2 containing D_1. This is closely related to the *Runge property* (RP), which originally referred to the uniform approximation of holomorphic functions on compact subsets of simply connected domains by means of polynomials (see, e.g., [121], Th. 13.11.) or to the approximation of harmonic functions by harmonic polynomials in simply connected domains in the plane (a corollary of the property for holomorphic functions) or in higher dimensions (see [117]). In this work we consider the approximation of weak solutions of a strongly elliptic equation in the Sobolev space $H^1(D)$, where D is a bounded domain. Typically the RP theorem establishes the following.

Suppose that $D_1 \Subset D_2$ are bounded domains (with some properties that we shall specify). Given $u \in H^1(D_1)$ a weak solution of a strongly elliptic equation and $\epsilon > 0$, there exists a solution $v \in H^1(D_2)$ such that

$$\|u - v|_{D_1}\|_{L_2(D_1)} < \epsilon.$$

One of the questions is finding appropriate conditions on D_1 and D_2. The first version of the result holds when $D_1 \Subset D_2$ are both *simply connected domains with smooth boundaries.* It was established in [102] (for H^2 solutions) and [108, 115]. As it is known, for a strongly elliptic PDE the RP is closely related to the *uniqueness of the Cauchy problem* (UCP). In Lax's paper [102] the equivalence between the RP and the UCP is established. Moreover, there are known results on the equivalences between the RP, the UCP and the *unique continuation property of the solutions.* In [32, Ch. III] it was proved for elliptic operators with analytic coefficients.

Our main interest is a RP corresponding to the H^1-norm for Lipschitz domains. One of the recent results proved by A. Rüland and M. Salo in [120] is a version of the RP for a strongly elliptic operator in divergence form, whose principal coefficient is a Lipschitz function, defined in Lipschitz domains $D_1 \Subset D_2$ with $D_2 \setminus \overline{D_1}$ being connected. The result establishes the approximation in the L_2-norm of a subfamily of solutions whose boundary values lie in a non-empty open subset of ∂D_2. The idea of the proof is based on the UCP, which is valid for this class of operators and Lipschitz domains (see [4]).

We give a proof of the RP in Lipschitz domains with respect to the H^1-norm, based on [102] and using certain ideas from [120] and [4]. A corresponding definition of a complete system of solutions is proposed for the space of classical solutions and for the Bergman space of square integrable solutions. With the aid of auxiliary results on the completeness of classical solutions in the class of weak solutions, we show that under certain conditions on the domains $\omega \Subset D$, the weak solutions in ω can be approximated by complete systems of classical solutions in the L_2, H^1 and H^2-norms. We apply these results to the radial Schrödinger equation and the Schrödinger equation with a separable potential (with respect to the Cartesian variables) considered in simply connected domains of the plane.

The thesis is structured as follows. Chapter 1 summarizes some results on linear operators, approximation of linear equations in Hilbert spaces, Sobolev spaces, orthogonal polynomials, transmutation operators and Sturm-Liouville problems, which will be used throughout the text.

In Chapter 2, a Gelfand-Levitan equation for the spectral problem of the Schrödinger equation with the Dirichlet-Dirichlet conditions is derived. Using the Fourier-Legendre series representation for the transmutation kernel associated with the generalized sine solution, the Gelfand-Levitan equation reduces to an infinite system of linear algebraic equations, with which the inverse problem is solved. Furthermore, to recover the potential, only the first coefficient of the Fourier representation is needed. The system of equations will be used for the study of inverse problems of the SLEIF.

Chapter 3 develops the theory of transmutation operators for Eq. (4). The existence of the transmutation operator as an integro-differential operator is proved, the transmutation kernel is constructed and its analytic properties are established. The transmutation operator is shown to be bounded, invertible, and the explicit form of the inverse operator is constructed. In addition, a Fourier-Legendre series representation for the transmutation kernel is derived, from which a NSBF representation of the solutions of Eq. (4) is obtained.

In Chapter 4 we derive the Gelfand-Levitan equation for spectral problem for Eq. (4) with the Neumann-Neumann conditions. Using the Fourier-Legendre representation for the transmutation kernel, the Gelfand-Levitan equation is reduced to a system of linear algebraic equations. Again, it is proved that only the first coefficient of the Fourier-Legendre expansion is needed to recover the impedance function. An algorithm to solve the inverse problem is deduced and its effectiveness is shown through several numerical examples.

In Chapter 5, a Levin integral representation for the Jost solution of the SLEIF is presented. Such representation has a continuous kernel $A(x,t)$, that admits a Fourier-Laguerre series representation. From this series, it follows that the Jost solution (and its derivative) can be written as a power series with respect to the parameter $z = \frac{\frac{1}{2}+i\rho}{\frac{1}{2}-i\rho}$. The series converges in the unit disk $|z| < 1$ and leads to an explicit representation for spectral data for the problem with the boundary condition (5), and analytic method for their computation.

In Chapter 6 we employ the techniques developed in Chapters 4 and 5 to solve a wave propagation problem. The inverse problem associated with the wave equation can be transformed into an inverse spectral problem for the SLEIF, for which a Gelfand-Levitan equation exists. Again, using the Fourier-Legendre series representation of the transmutation kernel, the inverse problem reduces to solution of an infinite system of linear algebraic equations, where only the first coefficient of the series is required to recover the impedance function.

Chapter 7 presents results aimed at approximation of solutions of a strongly elliptic equation considered in a bounded domain. We give a proof of the RP in Lipschitz domains with respect to the H^1-norm. A corresponding definition of a complete system of solutions is proposed for the space of classical solutions and for the Bergman space of square integrable solutions. With the aid of auxiliary results on the completeness of classical solutions in the class of weak solutions, we show that under certain conditions on the domains, the weak solutions can be approximated by complete systems of classical solutions in the L_2, H^1 and H^2-norms. We apply these results to the Schrödinger equation with a separable potential (with respect to the Cartesian variables) considered in simply connected domains of the plane.

Finally, Chapter 8 presents new properties of a transmutation operator for the radial Schödinger equation in a star-shaped domain. It is shown that the operator is bounded and invertible on the spaces of continuous functions and on the Bergman space, and that the integral kernel admits a Fourier-Jacobi series representation. From such a representation, a complete system of solutions of the radial equation is constructed and its completeness is proved in the space of classical solutions and in the Bergman space of solutions. Using the results of Chapter 7, it is proved that the system is also complete on the space of weak solutions in the L_2, H^1 and H^2-norms.

The results presented in this dissertation are published in the articles [95, 96, 97, 98, 99]. These results were presented at the following congresses and seminars.

1. International Workshop on TRANSMUTATION OPERATORS AND RELATED TOPICS. IWTORT 2019. Cinvestav, Querétaro, México. September 17-18,2019. Talk: *Transmutation operators and complete systems of solutions for the d-dimensional radial Schrödinger equation.*

2. QUANTUM FEST: International Conference on Quantum Phenomena, Quantum Control and Quantum Optics. Cinvestav, Zacatenco, México. October 28- November 1, 2019. Talk: *Complete orthogonal systems of solutions for the d-dimensional radial Schrödinger equation.*

3. OTAMP Conference: Operator Theory, Analysis and Mathematical Physics 2020. UNAM, Ciudad de México, México. January 8-14, 2020. Talk: *Transmutation operators and complete systems of solutions for the d-dimensional radial Schrödinger equation.*

4. CONGRESO VIRTUAL DE LA SOCIEDAD MATEMÁTICA MEXICANA 2020. México. October 18-22, 2021. Virtual poster session: *Operadores de transmutación y la propiedad de Runge para la ecuación de Schödinger radial.*

5. Seminar of Analysis and Mathematical Physics, Southern Federal University, Rostov on Don, Russia. May 24, 2019.

6. Seminario de Análisis y Ecuaciones Diferenciales, Departamento de Matemáticas, CINVESTAV Querétaro, May 2022.

Chapter 1

Preliminaries

Throughout the text, we use the following notations: $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{C}^{\pm} := \{z \in \mathbb{C} \,|\, \pm \operatorname{Im} z > 0\}$, $\mathbb{D} := \{z \in \mathbb{C} \,|\, |z| < 1\}$. Given $U \subset \mathbb{R}^d$ an open set and m a positive Borel measure in U, we denote the corresponding L_p space, $1 \leqslant p \leqslant +\infty$, by $L_p(U; dm)$. When m is the Lebesgue measure we simply denote $L_p(U)$. In the case that m be absolutely continuous with respect to the positive Borel measure λ, we denote $dm = f d\lambda$ where f is the Radon-Nikodym derivative of m with respect to λ. If $\mathcal{I}$ is a set, let us denote the corresponding L_p space with the counting measure by $\ell_p(\mathcal{I})$. In the case $\mathcal{I} = \mathbb{N}_0$ we simply denote ℓ_p. For a compact interval $[a, b] \subset \mathbb{R}$, the class of absolutely continuous functions on $[a, b]$ is denoted by $AC[a, b]$. If $J \subset \mathbb{R}$ is a infinite interval, $AC_{loc}(\overline{J})$ denotes the class of measurable functions $f : J \to \mathbb{C}$ such that $f \in AC[\alpha, \beta]$ for all $[\alpha, \beta] \subset \overline{J}$. If $G \subset \mathbb{C}$ is an open set we denote the class of holomorphic functions on G by $\operatorname{Hol}(G)$.

Suppose that $I \subset \mathbb{R}$ is an interval (possibly infinite) and let x_0 be a limit point of I (with the possibility $x_0 = \infty$). Let $f, g : I \to \mathbb{C}$ functions. We use the following asymptotic relations [49, Sec. 1]:

1. $f(x) = O(g(x))$, when $x \to x_0$, means that there exists a constant $C > 0$ and a neighborhood U of x_0 such that $|f(x)| \leqslant C|g(x)|$ for all $x \in I \cap U$.

2. $f(x) = o(g(x))$, when $x \to x_0$, means that for all $\varepsilon > 0$ there exists a neighborhood U of x_0 such that $|f(x)| \leqslant \varepsilon|g(x)|$ for all $x \in I \cap U$. If there exists a neighborhood U' of x_0 such that $g(x) \neq 0$ for all $x \in U'$, then $f(x) = o(g(x))$, when $x \to x_0$, is equivalent to saying $\displaystyle\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.

Similar notations are used for sequences.

1.1 Abstract linear equations

Throughout the text, all the vector spaces under consideration will be complex. Let X and Y be Banach spaces. For $M \subset X$, the closure of M in the norm of X is denoted by $\overline{M}^X$. The closed unit ball in X is denoted by $\overline{\mathbb{B}}_X$. Given $\mathbf{A} : \mathscr{D}(\mathbf{A}) \subset X \to Y$ a linear operator, $\mathscr{D}(\mathbf{A})$ denotes the domain of $\mathbf{A}$, $\mathscr{R}(\mathbf{A})$ the range and $\ker \mathbf{A}$ its kernel. For bounded operators we assume that $\mathscr{D}(\mathbf{A}) = X$. The Banach space of all bounded linear operators from X into Y is denoted by $\mathcal{B}(X, Y)$, and the subspace of compact operators by $\mathcal{K}(X, Y)$. The identity operator on X is denoted by $\mathbf{I}_X$ (if there is no danger of confussion, we simply denote it by $\mathbf{I}$). When $X = Y$, we denote $\mathcal{B}(X, X) = \mathcal{B}(X)$ and $\mathcal{K}(X, X) = \mathcal{K}(X)$. In this case, $\mathcal{G}(X)$ denotes the set of bijective operators $\mathbf{A} \in \mathcal{B}(X)$. By the open mapping theorem, if $\mathbf{A} \in \mathcal{G}(X)$ then $\mathbf{A}^{-1} \in \mathcal{G}(X)$ and $\mathcal{G}(X)$ is a group ([35, Th. 6.5.4]).

The dual space of X is denoted by $X^\star$. For $W \subset X$ we define the *annihilator* of W as the set $W^a := \{f \in X^\star \,|\, \forall x \in W \ f(x) = 0\}$. On the other hand, if $M \subset X^\star$ the *pre-annihilator* of M is the set $M_a := \{x \in X \,|\, \forall f \in M \ f(x) = 0\}$. Both sets are closed subspaces of X and $X^\star$ respectively. It is not difficult to see that $(W^a)_a = W$ iff W is a closed subspace. For our purposes, the following lemma will be necessary.

Lemma 1. *If M is a finite dimensional subspace of $X^\star$, then $(M_a)^a = M$*

Proof. The containment $W \subset (W_a)^a$ is clear. For the other containment, consider the weak star topology $w^\star$ in $X^\star$. Since $(X^\star, w^\star)$ is a locally convex Hausdorff space [122, Sec. 3.14] and M is finite dimensional, M is closed in $(X^\star, w^\star)$ [122, Th. 1.21]. Since $(M_a)^a$ is the closure of M in $(X^\star, w^\star)$ [122, Th. 4.7 (b)], hence $(M_a)^a = M$. **Q.E.D.**

We recall that an operator $\mathbf{A} \in \mathcal{B}(X, Y)$ is called a *Fredholm operator* if the range $\mathscr{R}(\mathbf{A})$ is closed in Y and $\ker \mathbf{A}$ and $Y/\mathscr{R}(\mathbf{A})$ are finite dimensional. In this case the *index* of $\mathbf{A}$ is defined as $\operatorname{Ind}(\mathbf{A}) := \dim \ker \mathbf{A} - \dim Y/\mathscr{R}(\mathbf{A})$. For example, bijective operators $\mathbf{A} \in \mathcal{B}(X, Y)$ are Fredholm with $\operatorname{Ind}(\mathbf{A}) = 0$. If $\mathbf{A}$ is Fredholm and $\mathbf{K} \in \mathcal{K}(X, Y)$, then $\mathbf{A}+\mathbf{K}$ is Fredholm with $\operatorname{Ind}(\mathbf{A} + \mathbf{K}) = \operatorname{Ind}(\mathbf{A})$ [107, Th. 2.26]. The well known *Fredholm* *alternative theorem* establishes that if $\mathbf{A}$ is a Fredholm operator with $\mathrm{Ind}(\mathbf{A}) = 0$, hence $\mathbf{A}$ is injective iff is surjective [107, Th. 2.27 (i)]. In either case, the operator is a bijection. As a consequence, given $\mathbf{K} \in \mathcal{K}(X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, the operator $\mathbf{I}_X - \lambda\mathbf{K}$ is injective iff it is surjective.

1.1.1 Classification of the spectrum of an operator

Let $\mathbf{A} : \mathscr{D}(\mathbf{A}) \subset X \to X$ be a linear operator (possibly unbounded). The *resolvent* of $\mathbf{A}$ is the set $\rho(\mathbf{A})$ consisting of all values $\lambda \in \mathbb{C}$ satisfying the following conditions.

1. $\ker(\mathbf{A} - \lambda\mathbf{I}_X) = \{0\}$.

2. $\overline{\mathscr{R}(\mathbf{A} - \lambda\mathbf{I}_X)}^X = X$.

3. The *resolvent operator* $\mathbf{R}_\lambda := (\mathbf{A} - \lambda\mathbf{I}_X)^{-1} : \mathscr{R}(\mathbf{A} - \lambda\mathbf{I}_X) \to \mathscr{D}(\mathbf{A})$ is bounded (that is, $\displaystyle\sup_{y \in \mathscr{R}(\mathbf{A}-\lambda\mathbf{I}_X)\setminus\{0\}} \frac{\|\mathbf{R}_\lambda y\|_X}{\|y\|_X} < \infty$).

The elements of $\rho(\mathbf{A})$ are called the *regular values* of $\mathbf{A}$. The *spectrum* of $\mathbf{A}$ is the set $\sigma(\mathbf{A}) := \mathbb{C} \setminus \rho(\mathbf{A})$. The spectrum $\sigma(\mathbf{A})$ admits the following decomposition.

1. *The point spectrum* $\sigma_p(\mathbf{A})$ consists of those values $\lambda \in \mathbb{C}$ for which there exists $x \in \mathscr{D}(\mathbf{A}) \setminus \{0\}$ such that $\mathbf{A}x = \lambda x$. A value $\lambda \in \sigma_p(\mathbf{A})$ is called an *eigenvalue* of $\mathbf{A}$ and a vector $x \in \ker(\mathbf{A} - \lambda\mathbf{I}_X)$ is the corresponding *eigenvector* (the term *eigenfunction* is used when $\mathbf{A}$ is a differential operator on a space of functions). The *multiplicity* of $\lambda \in \sigma_p(\mathbf{A})$ is the value $\dim\ker(\mathbf{A} - \lambda\mathbf{I}_X)$. If the multiplicity is 1 we say that λ is *simple.*

2. *The continuous spectrum* $\sigma_c(\mathbf{A})$ consists of those values $\lambda \in \mathbb{C}$ such that $\ker(\mathbf{A} - \lambda\mathbf{I}_X) = \{0\}$, $\overline{\mathscr{R}(\mathbf{A} - \lambda\mathbf{I}_X)}^X = X$, but $\mathbf{R}_\lambda$ is unbounded.

3. *The residual spectrum* $\sigma_r(\mathbf{A})$ is the set of values $\lambda \in \mathbb{C}$ such that $\ker(\mathbf{A} - \lambda\mathbf{I}_X) = \{0\}$ but $\overline{\mathscr{R}(\mathbf{A} - \lambda\mathbf{I}_X)}^X \neq X$.

Thus, $\sigma(\mathbf{A}) = \sigma_p(\mathbf{A}) \cup \sigma_c(\mathbf{A}) \cup \sigma_r(\mathbf{A})$ and the union is disjoint. Additionally, we can define the *discrete spectrum* $\sigma_d(\mathbf{A})$ as the set of isolated eigenvalues of finite multiplicity and the *essential spectrum* $\sigma_{ess}(\mathbf{A}) := \sigma(\mathbf{A}) \setminus \sigma_d(\mathbf{A})$.

If $\mathbf{K} \in \mathcal{K}(X)$, then $\sigma(\mathbf{K}) \setminus \{0\} = \sigma_d(\mathbf{K}) \setminus \{0\}$ [35, Th. 10.6.3] and the Fredholm alternative can be reformulated as follows: if $\lambda \neq 0$, then $\lambda \notin \sigma_d(\mathbf{K})$ iff $\mathbf{K} - \lambda \mathbf{I}_X$ is a bijection.

If $\mathcal{H}$ is a Hilbert space and $\mathbf{A} : \mathscr{D}(\mathbf{A}) \subset \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator (see [17, Sec. 3.2.1] for the definition of the adjoint of an unbounded operator on a Hilbert space), then $\sigma(\mathbf{A}) \subset \mathbb{R}$ [126, Cor. 3.14] and $\sigma_r(\mathbf{A}) = \emptyset$ [100, pp. 546]. Eigenvectors corresponding to different eigenvalues are mutually orthogonal [126, Lemma 3.4].

1.1.2 Linear equations involving sesquilinear forms

We recall that a *sesquilinear form* on $X \times Y$ is a map $\Psi : X \times Y \to \mathbb{C}$ such that Ψ is *conjugate-linear* in the first slot and linear in the second one. We say that Ψ is bounded if there exists a constant $M > 0$ such that $|\Psi(x, y)| \leqslant M\|x\|_X\|y\|_Y$ for all $(x, y) \in X \times Y$.

Suppose that X is a reflexive space. A *conjugation* on X is a map $X \ni x \mapsto x^* \in X$ such that $(\lambda x + y)^* = \overline{\lambda} x^* + y^*$ and $\|x^*\|_X = \|x\|_X$ for all $x, y \in X$, $\lambda \in \mathbb{C}$. For example, if $X = L_2(U)$, $u^* = \overline{u}$ for $u \in L_2(U)$ is a conjugation. We can define a conjugation on $X^\star$ as follows. For $f \in X^\star$ define $f^*(x) := \overline{f(x^*)}$. It is easy to verify that this is a conjugation on X^*. In consequence, $X^\star \times X \ni (f, x) \mapsto (f, x)_X := f^*(x) \in \mathbb{C}$ is a sesquilinear form. Similarly, we denote $(x, f)_{X^\star} := \overline{(f, x)_X}$. If W (M) is a subspace of X $(X^\star)$, we define $W^* := \{x^* \,|\, x \in W\}$ $(M^* := \{f^* \,|\, f \in M\})$ and the following equalities hold:

$$(W^*)^a = (W^a)^*, \quad (M^*)_a = (M_a)^*. \tag{1.1}$$

If $\mathbf{A} \in \mathcal{B}(X, X^\star)$, the *adjoint* of $\mathbf{A}$ is the unique operator $\mathbf{A}^\star \in \mathcal{B}(X^\star, X)$ satisfying $(\mathbf{A}x, y)_X = (x, \mathbf{A}^\star y)_{X^\star}$ for all $x, y \in X$. Actually, $\mathbf{A}^\star x = \left(\mathbf{A}^T\left((J_X x)^*\right)\right)^*$ where $\mathbf{A}^T \in \mathcal{B}(X^{\star\star}, X^\star)$ is the transpose of $\mathbf{A}$ $(\mathbf{A}^T\varphi = \varphi \circ \mathbf{A}$ for $\varphi \in X^{\star\star})$ and $J_X : X \hookrightarrow X^{\star\star}$ is the canonical injection $(J_X x)f = f(x)$ for $x \in X$, $f \in X^\star$. For this class of operators, we establish the following version of the Fredholm alternative.

Theorem 2 ([107], Th. 2.27)**.** *If* $\mathbf{A} \in \mathcal{B}(X, X^\star)$ *is a Fredholm operator with* $\operatorname{Ind}(\mathbf{A}) = 0$, *then* $\mathbf{A}^\star$ *is Fredholm with* $\operatorname{Ind}(\mathbf{A}^\star) = 0$ *and* $\dim\ker\mathbf{A} = \dim\ker\mathbf{A}^\star$. *Also, only one of the following statements is true.*

1. *If* $\ker \mathbf{A} = \{0\}$, *hence* $\mathbf{A}$ *and* $\mathbf{A}^\star$ *are bijections and* $\mathbf{A}^{-1} \in \mathcal{B}(X^\star, X)$, $(\mathbf{A}^\star)^{-1} \in \mathcal{B}(X, X^\star)$. *In particular, for every* $f \in X^\star$ *the equation* $\mathbf{A}x = f$ *has a unique solution* $x \in X$.

2. *If* $\dim \ker \mathbf{A} = p > 0$, *then the equation* $\mathbf{A}x = f$ *has a solution iff* $(f, w_j)_X = 0$ *where* $\{w_1, \cdots, w_p\}$ *is a basis for* $\ker \mathbf{A}^\star$.

Let $\Phi : X \times X \to \mathbb{C}$ be a bounded sesquilinear form. Define the operator $\mathbf{A}_\Phi : X \to X^\star$ whose action is given by $(\mathbf{A}_\Phi x, y)_X := \Phi(x, y)$ for $x, y \in X$. Then $\mathbf{A}^\star \in \mathcal{B}(X, X^\star)$ with $\|\mathbf{A}_\Phi\|_{\mathcal{B}(X,X^\star)} \leqslant \sup\limits_{(x,y)\in X^2\setminus\{(0,0)\}} \dfrac{|\Phi(x,y)|}{\|x\|\|y\|}$. The adjoint of Φ is the bounded sesquilinear form $\Phi^\star : X \times X \to \mathbb{C}$ given by $\Phi^\star(x, y) = \overline{\Phi(y, x)}$. Is not difficult to see that $\mathbf{A}_{\Phi^\star} = (\mathbf{A}_\Phi)^\star$. If $\Phi = \Phi^\star$ we say that Φ is *Hermitian* and $\mathbf{A}_\Phi$ is *self-adjoint*.

Suppose that $\mathcal{H}$ is a Hilbert space with a conjugation. Using the Riesz representation theorem we identify $\mathcal{H}$ with $\mathcal{H}^\star$ by the map $\imath : \mathcal{H} \hookrightarrow \mathcal{H}^\star$ given by $(\imath u, v)_{\mathcal{H}} := \langle v, u\rangle_{\mathcal{H}}$ (the angular parenthesis denotes the inner product in $\mathcal{H}$). One can verify that this is a linear isometry. Now suppose that $V \subset \mathcal{H}$ is a dense subspace with a norm $\|\cdot\|_V$ which makes it into a Hilbert space and such that the natural embedding $V \hookrightarrow \mathcal{H}$ is continuous and the conjugation on $\mathcal{H}$ is also a conjugation on V with the norm $\|\cdot\|_V$. We consider the triple $V \hookrightarrow \mathcal{H} \hookrightarrow V^\star$ with the embedding $\imath_1 : \mathcal{H} \hookrightarrow V^\star$ given by $\imath_1 u = \imath u|_V$. The space $\mathcal{H}$ is called a *pivot* space for V. We say that a bounded sesquilinear form $\Phi : V \times V \to \mathbb{C}$ is *coercive* on V with respect to the pivot space $\mathcal{H}$ if there exists constants $C_1, C_2 > 0$ such that

$$\forall v \in V \quad \operatorname{Re}\Phi(v, v) \geqslant C_1\|v\|_V^2 - C_2\|v\|_{\mathcal{H}}^2.$$

Theorem 3 ([107], Th. 2.34)**.** *Suppose that* Φ *is a bounded sesquilinear form on* V, *coercive with respect to the pivot space* $\mathcal{H}$. *If the natural embedding* $V \hookrightarrow \mathcal{H}$ *is compact, then the operator* $\mathbf{A}_\Phi$ *is Fredholm with* $\operatorname{Ind}(\mathbf{A}_\Phi) = 0$.

1.1.3 Representation and matrix approximation of equations with Hilbert-Schmidt operators.

In this subsection we assume that $\mathcal{H}$ is a separable Hilbert space. Fix an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$. Given $\mathbf{K} \in \mathcal{B}(\mathcal{H})$ we define the infinite matrix

$$A^{\mathbf{K}} = (a^{\mathbf{K}}_{m,n})_{m,n\in\mathbb{N}_0} \quad \text{with entries} \quad a^{\mathbf{K}}_{m,n} := \langle \mathbf{K}e_n, e_m\rangle_{\mathcal{H}} \quad \text{for} \quad m,n \in \mathbb{N}_0. \tag{1.2}$$

Note the relations between the adjoints $\left(A^{\mathbf{K}}\right)^* = A^{\mathbf{K}^*}$. Hence $A^{\mathbf{K}}$ is an Hermitian matrix iff $\mathbf{K}$ is self-adjoint. We say that $\mathbf{K}$ is a *Hilbert-Schmidt* operator, if it satisfies the condition

$$\sum_{n=0}^{\infty} \|\mathbf{K}e_n\|^2_{\mathcal{H}} < \infty.$$

It is possible to see that this condition does not depend on the choice of the basis (see, e.g., [46, pp. 195]). By Parseval's identity and Tonelly's theorem we have the equality $\sum_{n=0}^{\infty} \|\mathbf{K}e_n\|^2_{\mathcal{H}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |a^{\mathbf{K}}_{m,n}|^2$. Hence $A^{\mathbf{K}} \in \ell_2\left(\mathbb{N}_0^2\right)$. We denote the class of Hilbert-Schmidt operators by $\mathcal{B}_2(\mathcal{H})$. Note that $\mathbf{K} \in \mathcal{B}_2(\mathcal{H})$ iff $A^{\mathbf{K}} \in \ell_2\left(\mathbb{N}_0^2\right)$. We remember that the map

$$\mathcal{H} \ni x \mapsto \widehat{x} \in \ell_2 \ \text{ given by } \widehat{x} := \{\langle x, e_n\rangle_{\mathcal{H}}\}_{n\in\mathbb{N}_0},$$

is an isomorphism between Hilbert spaces, i.e., $\langle x, y\rangle_{\mathcal{H}} = \langle \widehat{x}, \widehat{y}\rangle_{\ell_2}$ (see [3, pp. 16]). For $M \in \mathbb{N}_0$ consider the operator

$$\mathbf{K}_M x := \sum_{m=0}^{M}\left(\sum_{n=0}^{M} a_{m,n}\widehat{x}_n\right) e_m.$$

Thus, $\mathbf{K}_M$ is bounded with finite range and

$$\begin{aligned}
\|\mathbf{K}x - \mathbf{K}_M x\|^2_{\mathcal{H}} &\leqslant \left(\sum_{m=M+1}^{\infty}\sum_{n=0}^{\infty} |a_{m,n}|^2 + \sum_{m=0}^{M}\sum_{n=M+1}^{\infty} |a_{m,n}|^2\right)\|x\|^2_{\mathcal{H}} \\
&\leqslant \left(\sum_{m=N+1}^{\infty}\sum_{n=0}^{\infty} |a_{m,n}|^2 + \sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty} |a_{m,n}|^2\right)\|x\|^2_{\mathcal{H}},
\end{aligned}$$

which implies that $\|\mathbf{K} - \mathbf{K}_M\|_{\mathcal{B}(\mathcal{H})} \leqslant \left(\sum_{m=N+1}^{\infty}\sum_{n=0}^{\infty} |a_{m,n}|^2 + \sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty} |a_{m,n}|^2\right)^{\frac{1}{2}} \to 0,$ $M \to \infty$. Hence $\mathbf{K} \to \mathbf{K}_M$, $M \to \infty$, in $\mathcal{B}(\mathcal{H})$. In particular $\mathbf{K} \in \mathcal{K}(\mathcal{H})$ [35, Th. 5.6.4].

Let $\lambda \in \mathbb{C} \setminus \{0\}$. Suppose that $\frac{1}{\lambda} \notin \sigma_p(\mathbf{K})$. We are interested in approximating the equation

$$x - \lambda \mathbf{K} x = y \quad \text{where } \ y \in \mathcal{H}. \tag{1.3}$$

Remark 4. *Taking the inner product of (1.3) with e_m for each $m \in \mathbb{N}_0$ we obtain that (1.3) is equivalent to the infinite system of linear algebraic equations*

$$\widehat{x}_m - \lambda \sum_{n=0}^{\infty} a^{\mathbf{K}}_{m,n} \widehat{x}_m = \widehat{y}_m \quad \textit{for } \ m \in \mathbb{N}_0. \tag{1.4}$$

Since $A^{\mathbf{K}} \in \ell_2(\mathbb{N}_0^2)$, the series in (1.4) converges absolutely. The system (1.4) has a unique solution $\widehat{x} \in \ell_2$ iff Eq. (1.3) has a unique solution $x \in \mathcal{H}$.

Set $E_M = \operatorname{Span}\{e_m\}_{m=0}^{M}$ and let $\mathbf{P}_M : \mathcal{H} \to E_M$ be the orthogonal projection onto E_M. Note that $\mathbf{K}_M \in \mathcal{B}(E_M)$ and we can take $A^{\mathbf{K}_M} \in \mathbb{C}^{(M+1)}$. We look for a solution $x^{(M)} \in E_M$ for the *approximated equation*

$$x^{(M)} - \lambda \mathbf{K}_M x^{(M)} = \mathbf{P}_M y. \tag{1.5}$$

Eq. (1.5) is equivalent to the finite system of linear algebraic equations

$$(I_M - \lambda A^{\mathbf{K}_M}) \vec{x}^{(M)} = \vec{y}^{(M)}, \tag{1.6}$$

where $I_M \in \mathbb{C}^{(M+1)\times(M+1)}$ is the identity matrix and $\vec{x}^{(M)}, \vec{y}^{(M)} \in \mathbb{C}^{M+1}$ with $\vec{x}^{(M)} = \left(\widehat{x^{(M)}}_m\right)_{m=0}^{M}$, $\vec{y}^{(M)} = (\widehat{y}_m)_{m=0}^{M}$. Eq. (1.5) has a unique solution $x^{(M)} \in E_M$ iff the system (1.6) has a unique solution $\vec{x}^{(M)} \in \mathbb{C}^{(M+1)}$.

Theorem 5. *Let $\mathbf{K} \in \mathcal{B}_2(\mathcal{H})$ and $\lambda \in \mathbb{C} \setminus \{0\}$ with $\frac{1}{\lambda} \notin \sigma_p(\mathbf{K})$. Given $y \in \mathcal{H}$, there exits $M_0 \in \mathbb{N}$ large enough such that for all $M \geqslant M_0$, Eq. (1.5) has a unique solution $x^{(M)} \in E_M$ (and then also system (1.6) has a unique solution $\vec{x}^{(M)} \in \mathbb{C}^{M+1}$). Moreover, if $x \in \mathcal{H}$ is the unique solution of (1.3) then*

$$\|x^{(M)} - x\|_{\mathcal{H}} \to 0, \ M \to \infty. \tag{1.7}$$

If we consider $\vec{x}^{(M)}$ as an element of ℓ_2, then

$$\|\vec{x}^{(M)} - \widehat{x}\|_{\ell_2} \to 0, \ M \to \infty. \tag{1.8}$$

Proof. This is a consequence of the theory developed in [68, Ch. XIV], in specific Theorems 14.1-14.3. **Q.E.D.**

Now we analyze the stability of the solution of the truncated system (1.6). Following [112, Sec. 9], consider a *non-exact system*

$$\left(I_M + A^{\mathbf{K}_M} + \Gamma_M\right)\vec{v} = \vec{y}^{(M)} + \vec{\delta}_M,$$

where $\Gamma_M \in \mathbb{C}^{(M+1)\times(M+1)}$ represents errors in the coefficients $a_{k,j}^{\mathbf{K}_M}$ and $\vec{\delta}_M \in \mathbb{C}^{(M+1)}$ errors in the coefficients $y_k^{(M)}$. Let $\vec{u}_M$ be the solution of the exact truncated system (1.6) and $\vec{v}_M$ the solution of the non-exact one. The solution is called *stable* if there exist positive constants c_1, c_2, r independent of M, such that for $\|\Gamma_M\| \leqslant r$ and arbitrary $\vec{\delta}_M$ the non-exact system is solvable and the following inequality holds

$$\|\vec{u}_M - \vec{v}_M\| \leqslant c_1\|\Gamma_M\| + c_2\|\vec{\delta}_M\|.$$

Proposition 6. *The approximate solution $\vec{u}_M$ of the truncated system (1.6) is stable.*

Proof. Consider the Bubnov-Galerkin procedure [112, Sec. 14] applied to the equation (1.3) with respect to the orthonormal basis $\{e_m\}_{m=0}^{\infty}$. One can verify that this leads to the truncated system (1.6). Since the system $\{e_m\}_{m=0}^{\infty}$ is strongly minimal [112, Sec. 2], it follows from [112, Th. 14.1 and Th. 14.2] that the solution of the truncated system is stable. **Q.E.D.**

Remark 7. *Let $J \subset \mathbb{R}$ be an open interval and m a regular positive Borel measure in J. In this case $L_2(J; dm)$ is a separable Hilbert space (see [35, Prop. 4.4.2]) and then $L_2(J; dm)$ possesses a countable orthonormal basis $\{\phi_n(x)\}_{n=0}^{\infty}$. Let $K \in L_2(J \times J; d(m \times m))$ and define the linear Fredholm integral operator with kernel K by*

$$\mathbf{T}_K f(x) := \int_J K(x,t) f(t) dm(t) \quad \textit{for} \;\; f \in L_2(J; dm). \tag{1.9}$$

It is known that $\mathbf{T}_K \in \mathcal{B}_2\left(L_2(J; dm)\right)$ and

$$a_{m,n}^{\mathbf{T}_K} = \int_J \int_J k(x,t)\phi_n(t)\overline{\phi_m(x)} dm(x) dm(t) \quad \forall m, n \in \mathbb{N}_0, \tag{1.10}$$

and $\left\|a^{\mathbf{T}_K}\right\|_{\ell_2\left(\mathbb{N}_0^2\right)} = \|K\|_{L_2(J\times J; d(m\times m))}$ (see [12, Th. 3.39]). The operator $\mathbf{T}_K$ is self-adjoint iff $K(x,t) = \overline{K(t,x)}$ a.e. $x, t \in J$. For this integral operator we can apply

Theorem 5 to the linear Fredholm integral equation of the second kind

$$f(x) - \lambda \int_J K(x,t)f(t)dm(t) = g(x) \quad \text{where } g \in L_2(J; dm(t)). \tag{1.11}$$

In Section 1.3 we see some useful basis of orthogonal polynomials.

1.2 Some facts related with the Sobolev spaces

In this section, $d \in \mathbb{N}$ denotes the dimension of the Euclidean space in which the spaces of functions to be treated will be defined.

Definition 8. *Given $U, V \subset \mathbb{R}^d$ open sets, we say that V is* **strictly interior** *to U and we denote $V \Subset U$, if V is bounded and $\overline{V} \subset U$.*

Throughout this section, U denotes an open subset of $\mathbb{R}^d$. Given $1 \leqslant p \leqslant +\infty$, we denote by $L^p_{loc}(U)$ the class of all measurable functions $f : U \to \mathbb{C}$ for which $f\big|_V \in L_p(V)$ for all $V \Subset U$. The set of all infinitely differentiable functions with compact support in U is denoted by $C_0^\infty(U)$. In a similar way $C_0^\infty(\overline{U}) := \left\{ \varphi\big|_{\overline{U}} \,\middle|\, \varphi \in C_0^\infty(\mathbb{R}^d) \right\}$. Let $\alpha \in \mathbb{N}_0^d$ be a multi-index of order $|\alpha| := \sum_{k=1}^d \alpha_k > 0$. We use the notation $\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}$. Let $u \in L^1_{loc}(U)$. A function $v_\alpha \in L^1_{loc}(U)$ is said to be the α^{th}-weak derivative of u if

$$\int_U u(x)\partial^\alpha \varphi(x)dx = (-1)^{|\alpha|} \int_U v_\alpha(x)\varphi(x)dx \quad \forall \varphi \in C_0^\infty(U). \tag{1.12}$$

If $\alpha = 0$, we define $\partial^\alpha u = u$. The α^{th}-weak derivative of a function $u \in L^1_{loc}(U)$ is uniquely determinated a.e. in U [101, Sec. 5.2.1].

For this reason, we use the standard notation for the α^{th}-weak derivative $\partial^\alpha u = v_\alpha$. When $u \in C^k(U)$, the weak derivatives coincides with the classical derivatives. We denote the *gradient* of u by $\nabla u(x) := \left(\frac{\partial u(x)}{\partial x_j} \right)_{j=1}^d$.

Let $k \in \mathbb{N}$. The **Sobolev space** of order k, denoted by $W^{k,p}(U)$, consists of all functions $u \in L_p(U)$ such that for each multi-index $\alpha \in \mathbb{N}_0^d$ of order k, $\partial^\alpha u$ exists and belongs to $L_p(U)$. The space $W^{k,p}(U)$ is endowed with the norm

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|^p_{L_p(U)} \right)^{\frac{1}{p}}, & \text{if } 1 \leqslant p < \infty, \\ \sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|_{L_\infty(U)}, & \text{if } p = \infty. \end{cases}$$

In the case $p = 2$ we denote $H^k(U) := W^{k,2}(U)$ and we introduce the inner product

$$\langle u, v\rangle_{H^1(U)} := \sum_{|\alpha|\leqslant\alpha} \int_U \partial^\alpha u(x)\overline{\partial^\alpha v(x)}dx,$$

that induce the norm $\|\cdot\|_{H^1(U)}$. The Sobolev space $W^{k,p}(U)$ is a Banach space and $H^k(U)$ is a separable Hilbert space [101, Th. 2, Sec. 5.2.3]. We use the convention $W^{0,p}(U) := L_p(U)$. For second order derivatives, we use the notation $u_{x_j x_k} := \frac{\partial^2 u}{\partial x_k \partial x_j}$. We define the local Sobolev space $W^{k,p}_{loc}(U)$ as the set of functions $u \in L^1_{loc}(U)$ such that $u\big|_V \in W^{k,p}(V)$, for all $V \Subset U$. Again, when $p = 2$ we denote $H^k_{loc}(U) := W^{k,2}_{loc}(U)$. Another important space that we use is $H^1_0(U) := \overline{C^\infty_0(U)}^{H^1}$. We have that $H^1_0(U)$ is a closed subspace of $H^1(U)$ (both spaces are equal iif $U = \mathbb{R}^d$) and then $H^1_0(U)$ is a Hilbert space.

By a *domain* $\Omega \subset \mathbb{R}^d$ we understand a connected open subset (usually, the symbol Ω is most common to denote domains instead the letters U or V). We assume that Ω is bounded and we denote $\Gamma = \partial\Omega$. Let us denote by $\mathrm{Iso}(\mathbb{R}^d)$ the set of all isometries $T : \mathbb{R}^d \to \mathbb{R}^d$ (an element of $\mathrm{Iso}(\mathbb{R}^d)$ is also called a *rigid movement*). Actually, $T \in \mathrm{Iso}(\mathbb{R}^d)$ iff there exists $Q \in O(\mathbb{R}^d)$ (where $O(\mathbb{R}^d)$ denotes the set of $d \times d$ orthogonal matrix) and $y \in \mathbb{R}^d$ such that $Tx = Qx + y$ for all $x \in \mathbb{R}^d$ (see [133, Prop. 1.58]). Remember that a function $\varphi : \widehat{V}' \to \mathbb{R}$ is Lipschitz, if $\sup\limits_{\substack{x,y\in\widehat{V}'\\ x\neq y}} \dfrac{|\varphi(x) - \varphi(y)|}{|x - y|} < +\infty$.

Definition 9. *A domain Ω is said to be a* **Lipschitz** *domain, if its boundary Γ is compact and for any $x \in \Gamma$ there exists a neighbourhood V of x in $\mathbb{R}^d$ and $T \in Iso(\mathbb{R}^d)$ such that:*

1. $\widehat{V} := T(V) = \left\{(y_1, \cdots, y_d) \in \mathbb{R}^d \,\big|\, -a_j < y_j < a_j,\ j = \overline{1,d}\right\}$ *for some* $\{a_j\}_{j=1}^d \subset \mathbb{R}_+$.

2. *There exists a Lipschitz function* $\varphi : \widehat{V}' \to \mathbb{R}$, *where* $\widehat{V}' = \left\{y' = (y_1, \cdots, y_{d-1}) \in \mathbb{R}^{d-1} \,\big|\, -a_j < y_j < a_j,\ j = \overline{1,d-1}\right\}$, *satisfying*

   (i) $|\varphi(y')| \leqslant \frac{a_d}{2} \quad \forall y' \in \widehat{V}'$.

   (ii) $T(\Omega \cap V) = \left\{y = (y', y_d) \in \widehat{V} \,\big|\, y_d < \varphi(y')\right\}$.

   (iii) $T(\Gamma \cap V) = \left\{y = (y', y_d) \in \widehat{V} \,\big|\, y_d = \varphi(y')\right\}$.

If for all x, the function $\varphi \in C^k(\widehat{V}')$, we say that Ω is a C^k-domain. Ω is said to be a smooth domain if $\varphi \in C^\infty(\widehat{V}')$.

Lipschitz domains are more flexible than C^k domains, since they include domains with corners, such as rectangles. The importance of this type of domains with certain regularity in their boundaries, is that in them the concept of *boundary value* can be extended to the class of functions in the Sobolev spaces H^k.

Theorem 10 ([107], pp. 102-106)**.** *Suppose that Ω is a Lipschitz domain. Then there exists a bounded linear operator $\mathrm{tr}_\Gamma : H^1(\Omega) \to L_2(\Gamma)$ such that $\mathrm{tr}_\Gamma(u) = u\big|_\Gamma$, if $u \in C^\infty(\overline{\Omega})$, and satisfying $\ker(\mathrm{tr}_\Gamma) = H_0^1(\Omega)$. The operator tr_Γ is called the* **trace operator***.*

Definition 11. *If Ω is a Lipschitz domain, we define the Sobolev space of order $\frac{1}{2}$ in Γ as $H^{\frac{1}{2}}(\Gamma) := \mathrm{tr}_\Gamma(H^1(\Omega))$.*

The space $H^{\frac{1}{2}}(\Gamma)$ is endowed with the norm $\|\varphi\|_{H^{\frac{1}{2}}(\Gamma)} := \inf_{w \in \mathrm{tr}_\Gamma^{-1}(\varphi)} \|w\|_{H^1(\Omega)}$. By the first isomorphism theorem $H^{\frac{1}{2}}(\Gamma)$ is a Banach space.

Proposition 12 ([107], pp. 102)**.** *Let Ω be a bounded Lipschitz domain. The trace operator $\mathrm{tr}_\Gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ is bounded and admits a bounded right-inverse $\mathcal{E} : H^{\frac{1}{2}}(\Gamma) \to H^1(\Omega)$ (called an extension operator).*

The following proposition summarizes some important properties of inclusion and density.

Proposition 13. *If $\Omega \subset \mathbb{R}^d$ is a bounded domain, then the following statements hold:*

1. *For $k, l \in \mathbb{N}_0$, if $k < l$ then the natural embedding $H^l(\Omega) \hookrightarrow H^k(\Omega)$ is compact.*
2. *If Ω is a Lipschitz domain, $\overline{C^\infty(\overline{\Omega})}^{H^k} = H^k(\Omega)$.*

The first assertion can be found in [107, Th. 3.27] and the second one in [107, Th. 3.29]. Additionally, we define the following Sobolev spaces of negative order as the dual spaces of the Sobolev spaces of positive order:

1. $H^{-1}(\Omega) := (H_0^1(\Omega))^\star$.
2. $\tilde{H}^{-1}(\Omega) := (H^1(\Omega))^\star$.

3. If Ω is a bounded Lipschitz domain, we define $H^{-\frac{1}{2}}(\Gamma) := \left(H^{\frac{1}{2}}(\Gamma)\right)^{\star}$.

It is worth to mention that this Sobolev spaces can be defined in an equivalent form using the Schwarz distribution's theory, see, e.g., [107, Ch. III, pp. 77-92].

Finally, we see certain regularity conditions in the Sobolev spaces when the domains are Lipschitz or C^k type. Given a compact $K \subset \mathbb{R}^d$, we denote by $C^{0,1}(K)$ the class of Lipschitz functions in K. For Lipschitz domains we have the following characterization:

Theorem 14 ([101], pp. 279)**.** *Let Ω be a bounded Lipschitz domain. If $u \in C^{0,1}(\overline{\Omega})$, then $u \in W^{1,\infty}(\Omega)$. Conversely, if $u \in W^{1,\infty}(\Omega)$, there exists $\tilde{u} \in C^{0,1}(\overline{\Omega})$ such that $u = \tilde{u}$ a.e. in Ω.*

The last statement can be reformulated as follows: if $u \in W^{1,\infty}(\Omega)$, by Theorem 14 exists $\tilde{u} \in C^{0,1}(\overline{\Omega})$ with $u = \tilde{u}$ a.e. in Ω, and the continuity implies that such $\tilde{u}$ is unique. Then we have a map $W^{1,\infty}(\Omega) \ni u \mapsto \tilde{u} \in C^{0,1}(\overline{\Omega})$ and Theorem 14 establishes that this is a bijection. In this sense, we can write $W^{1,\infty}(\Omega) = C^{0,1}(\overline{\Omega})$.

In a general context, we look for relations of the type $H^k(\Omega) \subset C^N(\Omega)$. This means that every function in $H^k(\Omega)$ has a representative element in $C^N(\Omega)$. The principal properties that we need are presented in the next theorem.

Theorem 15. *Let Ω be a bounded domain and $l \in \mathbb{N}$.*

(i) $H_{loc}^{l+1+\left[\frac{d}{2}\right]}(\Omega) \subset C^l(\Omega)$.

(i) If Ω is of class $C^{l+1+\left[\frac{d}{2}\right]}$, then $H^{l+1+\left[\frac{d}{2}\right]}(\Omega) \subset C^l(\overline{\Omega})$.

(The expression $[\frac{d}{2}]$ means the entire part of $\frac{d}{2}$). The proof of these statements can be found in [113, Ch. III, pp. 154-155].

Remark 16. *In the case $d = 1$, if $J \subset \mathbb{R}$ is an open interval, we have the characterization*

$$W^{k,p}(J) := \{u \in L_p(J) \,|\, u \in C^{k-1}(\overline{J}), u^{(k-1)} \in AC_{loc}(\overline{J}) \text{ and } u^{(k)} \in L_p(J)\}$$

(see [19, Ch. 8]).

1.3 Orthogonal basis of polynomials

In this section, we discuss some orthogonal families of polynomials to be used throughout the text.

1.3.1 Jacobi polynomials

Let us consider a weight function of the form $w_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ where $\alpha, \beta > -1$, and consider the *Jacobi polynomials* $\{P_n^{(\alpha,\beta)}(t)\}_{n=0}^\infty$, given by

$$P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{d^n}{dt^n}\left[(1-t)^\alpha(1+t)^\beta(1-t^2)^n\right].$$

It is well known that $\{P_n^{(\alpha,\beta)}(t)\}_{n=0}^\infty$ is an orthogonal basis of $L_2((-1,1); w_{\alpha,\beta}(t)dt)$ that satisfies the relation

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(t)P_m^{(\alpha,\beta)}(t)(1-t)^\alpha(1+t)^\beta dt = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\delta_{n,m}$$

(see [131, Ch. VII]). Given $d \geqslant 2$, one can obtain an orthogonal basis for $L_2((0,1); w_d(t)dt)$, with $w_d(t) = t^{\frac{d}{2}-1}$ by means of the translation $x = 2t-1$. This leads to the *shifted Jacobi polynomials*

$$\widehat{P}_n^{(d)}(t) := P_n^{(0,\frac{d}{2}-1)}(2t-1) \tag{1.13}$$

that are an orthogonal basis in $L_2((0,1); w_d(t)dt)$ (see [42]) with norm

$$\|\widehat{P}_n^{(d)}\|_{L_2((0,1);w_d(t)dt)} = \sqrt{\frac{2}{4n+d}} \quad \text{for } n \in \mathbb{N}_0. \tag{1.14}$$

Now we see two special cases of the Jacobi Polynomials. When $\alpha = \beta = 0$, we obtain the *Legendre Polynomials*, that we denote by $\{P_n(t)\}_{n=0}^\infty$. These are an orthogonal basis for $L_2(-1,1)$ with norm

$$\|P_n\|_{L_2(-1,1)} = \sqrt{\frac{2}{2n+1}} \quad \text{for } n \in \mathbb{N}_0. \tag{1.15}$$

The Legendre polynomial $P_n(t)$ is odd (even) if n is odd (even) and $P_n(1) = 1$ for all $n \in \mathbb{N}_0$. In the interval $(0,\ell)$, $\ell > 0$, we have that $\left\{P_{2n}\left(\frac{t}{\ell}\right)\right\}_{n=0}^\infty$ and $\left\{P_{2n+1}\left(\frac{t}{\ell}\right)\right\}_{n=0}^\infty$ are orthogonal basis for $L_2(0,\ell)$ with norms

$$\|P_{2n+\delta}\|^2_{L_2(0,\ell)} = \frac{\ell}{4n+2\delta+1} \quad \text{for } n \in \mathbb{N}_0,\ \delta = 0, 1. \tag{1.16}$$

Usually, one uses $\left\{P_{2n+1}\left(\frac{t}{\ell}\right)\right\}_{n=0}^\infty$ to represent odd functions.

For $\lambda > -\frac{1}{2}$, the *Gegenbauer* polynomials are defined as[1] $C_n^{(\lambda)}(t) := \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n}P_n^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(t)$ (see [131]). This polynomials will be useful in the study of harmonic functions.

[1] $(t)_n := t(t+1)\cdots(t+n-1)$.

Set $\mathbb{R}^+ := (0, \infty)$. The *Laguerre polynomials* $\{L_n(t)\}_{n=0}^{\infty}$ are defined by

$$L_0(t) := 1, \quad L_n(t) := \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}) \ \text{ for } \ n \geqslant 1.$$

The Laguerre polynomials $\{L_n(t)\}_{n=0}^{\infty}$ are an orthonormal basis for $L_2(\mathbb{R}^+; e^{-t}dt)$. These satisfy the condition at the origin $L_n(0) = 1$ for $n \in \mathbb{N}_0$ (see [131, Ch. VI]).

1.3.2 Spherical harmonics

Let $d \in \mathbb{N}$, $d > 1$. The open ball of radius $R > 0$ centered at the origin is denoted by $B_R^d(0)$. The d-dimensional unit ball is denoted by $\mathbb{B}^d := B_1^d(0)$ and $\mathbb{S}^{d-1} := \partial\mathbb{B}^d$. Given an open set $U \subset \mathbb{R}^d$, we denote by $\operatorname{Har}(U)$ the set of the harmonic functions on U, i.e., the functions $u \in C^2(U)$ such that $\triangle_d u = 0$ where $\triangle_d := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the d-Laplacian. In spherical coordinates $r = |x|$ and $x' := \frac{x}{|x|}$, the Laplacian $\triangle_d$ can be written as

$$\triangle_d = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\triangle_{\mathbb{S}^{d-1}} = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_{\mathbb{S}^{d-1}}, \tag{1.17}$$

where $\triangle_{\mathbb{S}^{d-1}}$ is the *spherical Laplacian* (see [45, Prop. 2.5], or [37, Lemma 1.4.1]). The action of this operator does not affect the radial component of functions that can be written in separate variables.

The set of all homogeneous harmonic polynomials of degree m will be denoted by $\mathcal{H}_m(\mathbb{R}^d)$. In the case $d = 2$, the set $\mathcal{H}_m(\mathbb{C})$ is generated by the analytic and antianalytic powers $\{z^m, \overline{z}^m\}$, and $\dim \mathcal{H}_m(\mathbb{C}) = 2$, for any $m > 0$, $\dim \mathcal{H}_0(\mathbb{C}) = 1$. In the general case we have

$$d_m := \dim \mathcal{H}_m(\mathbb{R}^d) = \begin{cases} 1, & \text{if } m = 0, \\ d, & \text{if } m = 1, \\ \binom{d+m-1}{d-1} - \binom{d+m-3}{d-1}, & \text{if } m \geqslant 2, \end{cases} \tag{1.18}$$

(see [8, Prop. 5.8]). We recall the following facts.

Theorem 17 ([8], pp. 100)**.** *If $u \in \operatorname{Har}(B_r^d(a))$, then there exists a sequence of polynomials $p_m \in \mathcal{H}_m(\mathbb{R}^d)$ such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a) \quad \text{for } x \in B_r^d(a),$$

and the series converges absolutely and uniformly on compact subsets of $B_r^d(a)$.

Theorem 18 (Runge's approximation theorem [117]). *Let $D \subset \mathbb{R}^d$ be a domain, and $K \subset D$ be a compact such that $\mathbb{R}^d \setminus K$ is connected. Then for any $h \in \mathrm{Har}(D)$, there exists a sequence $\{p_n\}$, with $p_n \in \mathrm{Span}\left(\bigcup_{m=0}^{n} \mathcal{H}_m(\mathbb{R}^d)\right)$ such that p_n converges uniformly to h in K.*

Definition 19 ([8]). *A spherical harmonic of degree m is the restriction to $\mathbb{S}^{d-1}$ of an element of $\mathcal{H}_m(\mathbb{R}^d)$. The collection of all spherical harmonics of degree m will be denoted by $\mathcal{H}_m(\mathbb{S}^{d-1})$.*

Note that every $p \in \mathcal{H}_m(\mathbb{R}^d)$ can be written as $p(x) = |x|p'\left(\frac{x}{|x|}\right)$, with $p' = p\big|_{\mathbb{S}^{d-1}}$ and this representation is unique. Conversely, every spherical harmonic $p' \in \mathcal{H}_m(\mathbb{S}^{d-1})$ defines an element of $\mathcal{H}_m(\mathbb{R}^d)$. Hence, the map $\mathcal{H}_m(\mathbb{R}^d) \ni p \mapsto p\big|_{\mathbb{S}^{d-1}} \in \mathcal{H}_m(\mathbb{S}^{d-1})$ provides an identification of the complex vector space $\mathcal{H}_m(\mathbb{R}^d)$ with $\mathcal{H}_m(\mathbb{S}^{d-1})$. The following statement contains some main properties of the spherical harmonics.

Proposition 20.

1. *$\dim \mathcal{H}_m(\mathbb{S}^{d-1}) = d_m$.*
2. *If $m \neq n$, and $p \in \mathcal{H}_m(\mathbb{S}^{d-1}), q \in \mathcal{H}_m(\mathbb{S}^{d-1})$, then*

$$\langle p, q\rangle_{L_2(\mathbb{S}^{d-1})} := \int_{\mathbb{S}^{d-1}} p(\zeta)\overline{q(\zeta)}d\sigma_\zeta = 0,$$

   where σ is the surface area measure. In consequence $\mathcal{H}_m(\mathbb{S}^{d-1}) \perp \mathcal{H}_n(\mathbb{S}^{d-1})$ if $n \neq m$.
3. *The spherical harmonics are the eigenfunctions of the spherical Laplacian. More precisely, if $m \in \mathbb{N}_0$ and $p \in \mathcal{H}_m(\mathbb{S}^{d-1})$, then*

$$\triangle_{\mathbb{S}^{d-1}} p = -m(m+d-2)p. \tag{1.19}$$

4. *$\overline{\mathrm{Span}\left(\bigcup_{m=0}^{\infty} \mathcal{H}_m(\mathbb{S}^{d-1})\right)} = C(\mathbb{S}^{d-1})$ (in the uniform norm).*
5. *$L_2(\mathbb{S}^{d-1}) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(\mathbb{S}^{d-1})$.*

Property 1 is a consequence of the identification of $\mathcal{H}_m(\mathbb{S}^{d-1})$ with $\mathcal{H}_m(\mathbb{R}^d)$. Properties 2,4,5 can be found in Chapter 5 of [8], and property 3 in [37, Th. 1.4.5].

If $U \subset \mathbb{R}^d$ is open, the *harmonic Bergman space* in U is defined as

$$b_2(U) := \left\{ u \in \mathrm{Har}(U) \,\middle|\, \int_U |u(x)|^2 dx < \infty \right\}.$$

This is a Hilbert space with the L_2-norm ([8, Prop. 8.3]). In the case $U = B_R^d(0)$, $R > 0$, writing in spherical coordinates the inner product and using property 2, one obtains that the restrictions of the spaces $\mathcal{H}_m(\mathbb{R}^d)$ and $\mathcal{H}_n(\mathbb{R}^d)$ are mutually orthogonal for $n \neq m$. For $m \in \mathbb{N}$ fixed, one can choose an orthogonal basis $\{Y_1^{(m)}, \cdots, Y_{d_m}^{(m)}\}$ of $\mathcal{H}_m(\mathbb{S}^{d-1})$. Then any harmonic polynomial $p \in \mathcal{H}(\mathbb{R}^d)$ can be written in the form

$$p(x) = r^m \sum_{k=0}^{d_m} b_k^{(m)} Y_k^{(m)}(x'), \quad \text{where } b_k^{(m)} := \langle p\big|_{\mathbb{S}^{d-1}}, Y_k^{(m)} \rangle_{L_2(\mathbb{S}^{d-1})},\ k = \overline{1, d_m}.$$

Theorem 21. *For any $m \in \mathbb{N}_0$, fix a basis $\{Y_1^{(m)}, \cdots, Y_{d_m}^{(m)}\}$ of $\mathcal{H}_m(\mathbb{S}^{d-1})$. Then $\left\{ \left\{ r^m Y_k^{(m)}(x') \right\}_{k=1}^{d_m} \right\}_{m=0}^{\infty}$ is an orthogonal basis for the Bergman space $b_2(B_R^d(0))$.*

Proof. It follows from the density of $\bigoplus_{m=0}^{\infty} \left\{ p\big|_{B_R^d(0)} \,\Big|\, p \in \mathcal{H}_m(\mathbb{R}^d) \right\}$ in $b_2(B_R(0))$ (see [8, Lemma 8.8]) and the previous remark. **Q.E.D.**

1.4 Transmutation operators

1.4.1 Basic definitions and main examples

Now we introduce the main tool of this thesis work: the transmutation operators. The following definition, the most suitable for our purposes, is a modification of the definitions presented in [88, 105].

Definition 22 ([88])**.** *Let E be a topological vector space, $E_1, E_2 \subset E$ linear subspaces (not necessarily closed) and $\mathbf{A} : E_1 \to E$, $\mathbf{B} : E_2 \to E$ linear operators. A linear invertible operator $\mathbf{T} : E \to E$, such that $\mathbf{T}(E_2) \subset E_1$, is said to be a **transmutation operator** for the pair of operators $\mathbf{A}, \mathbf{B}$, if the following conditions are fulfilled:*

1. *Both the operator $\mathbf{T}$ and its inverse $\mathbf{T}^{-1}$ are continuous in E.*
2. *The following operator equality is valid*

$$\mathbf{AT} = \mathbf{TB} \quad \text{in } E_2. \tag{1.20}$$

When $E_1 = E_2$, Definition 22 is the same as the one presented in [88]. The basic idea of the transmutation operator is to "transmute" the solutions of a simpler equation $\mathbf{B}v = \lambda v$, into those of a more complicated one $\mathbf{A}u = \lambda u$.

Remark 23. *Let* $\lambda \in \mathbb{C}$*. Suppose that* $v \in E_2$ *satisfies* $\mathbf{B}v = \lambda v$*. Take* $u = \mathbf{T}v$*. Since* $\mathbf{T}(E_2) \subset E_1$*,* $u \in E_1$ *and by property 2 we obtain*

$$\mathbf{A}u = \mathbf{AT}v = \mathbf{TB}v = \lambda \mathbf{T}v = \lambda u.$$

Hence $\mathbf{A}u = \lambda u$*. In particular* $\mathbf{T}(\ker \mathbf{B}) \subset \ker \mathbf{A}$*.*

Remark 24. *Suppose that* $\mathbf{T}(E_2) = E_1$*. If* $u \in \ker \mathbf{A}$*, then* $u = \mathbf{T}v$ *for some* $v \in E_2$*. Thus,*

$$0 = \mathbf{A}u = \mathbf{AT}v = \mathbf{TB}v.$$

Since $\mathbf{T}$ *is invertible,* $v \in \ker \mathbf{B}$*. Therefore* $\mathbf{T}(\ker \mathbf{B}) = \ker \mathbf{A}$*.*

One of the applications of transmutation operator theory is to describe the operators that transmute $\mathbf{D}^2$, where $\mathbf{D} := \frac{d}{dx}$, to the *one-dimensional Schrödinger operator* given by

$$\mathbf{S} := -\mathbf{D}^2 + q(x),$$

where q is a function of the class L_2 called the *potential* of the operator $\mathbf{S}$. In this case we consider $E = L_2(-b,b)$, $b > 0$ and $E_1 = E_2 = H^2(-b,b)$. According to Remark 23, a transmutation operator would allow us to transmute the solutions of equation $v'' + \lambda v = 0$ into solutions of the Schrödiner equation

$$-y'' + q(x)y = \lambda y, \tag{1.21}$$

where $\lambda \in \mathbb{C}$ is called the *spectral parameter* of Eq. (1.21). It is known that the transmutation operator has the form of a Volterra operator of the second kind with a continuous kernel that does not depend on the spectral parameter (see, e.g., [52, 105, 111]). Actually, it is possible to obtain a parameterized family of transmutation operators. The following proposition is a summary of results obtained in [24, 87, 88, 89].

Theorem 25. *Suppose that* $q \in L_2(-b,b)$*. Given* $h \in \mathbb{C}$*, there exists a function* $K_h \in C(\overline{\Omega}) \cap W^{1,1}(\Omega)$*, where* $\Omega = (-b,b) \times (-b,b)$*, satisfying the Goursat conditions*

$$K_h(x,x) = \frac{h}{2} + \frac{1}{2}\int_0^x q(s)ds, \quad K_h(x,-x) = \frac{h}{2}, \tag{1.22}$$

and such that the integral Volterra operator

$$\mathbf{T}_h u(x) := u(x) + \int_{-x}^{x} K_h(x,t)u(t)dt \tag{1.23}$$

satisfies $\mathbf{T}_h \in \mathcal{G}(L_2(-b,b))$, $\mathbf{T}_h(H^2(-b,b)) = H^2(-b,b)$ *and the following relation holds*

$$\left(\frac{d^2}{dx^2} - q(x)\right)\mathbf{T}_h u(x) = \mathbf{T}u''(x), \quad \forall u \in H^2(-b,b). \tag{1.24}$$

That is, $\mathbf{T}_h$ *is a transmutation operator for the pair* $\mathbf{S}$, $-\mathbf{D}^2 : H^2(-b,b) \to L_2(-b,b)$. *Moreover, if* $q \in C^p[-b,b]$ *for some* $p \in \mathbb{N}_0$, *then* $K_h \in C^{p+1}(\overline{\Omega})$, $\mathbf{T}_h \in \mathcal{G}(C[-b,b])$ *and it is a transmutation operator for* $\mathbf{S}, -\mathbf{D}^2 : C^2[-b,b] \to C[-b,b]$.

Of crucial importance is the fact that the integral kernel $K_h(x,t)$ is independent of the spectral parameter λ of (1.21). Of such transmutation operators, we highlight the following. Suppose that $q \in C[-b,b]$. Take $f \in C^2(-b,b) \cap C^1[-b,b]$ a solution of $f'' - q(x)f = 0$ in $(-b,b)$, such that f does not vanish in the whole interval $[-b,b]$ and satisfies the normalizing condition $f(0) = 1$ (such solution always exists, see [83]). Taking $h = f'(0)$ and employing Theorem 25 we obtain a transmutation operator $\mathbf{T}_f$ whose kernel will be denoted by $K_f(x,t)$. $\mathbf{T}_f$ and $K_f(x,t)$ are called the *canonical* transmutation operator and integral transmutation kernel, respectively.

Theorem 26 ([24])**.** *Define the following sequence of recursive integrals* $\{X^{(k)}(x)\}_{k=0}^{\infty}$, $\{\widetilde{X}^{(k)}(x)\}_{k=0}^{\infty}$ *by* $X^{(0)} \equiv \widetilde{X}^{(0)} \equiv 1$, *and for* $k \geqslant 1$

$$X^{(k)}(x) := k\int_0^x X^{(k-1)}(s)(f^2(s))^{(-1)^k}ds, \quad \widetilde{X}^{(k)}(x) := \int_0^x \widetilde{X}^{(k-1)}(f^2(s))^{(-1)^{k-1}}ds.$$

Define the family of functions $\left\{\varphi_f^{(k)}(x)\right\}_{k=0}^{\infty}$ *by*

$$\varphi_f^{(k)}(x) := \begin{cases} f(x)X^{(k)}(x), & \text{if } k \text{ is odd,} \\ f(x)\widetilde{X}^{(k)}, & \text{if } k \text{ is even,} \end{cases} \quad \text{for } k \in \mathbb{N}_0. \tag{1.25}$$

Then the following relations hold:

$$\mathbf{T}_f[x^k] = \varphi_f^{(k)}(x) \quad \forall k \in \mathbb{N}_0. \tag{1.26}$$

The functions $\left\{\varphi_f^{(k)}(x)\right\}_{k=0}^{\infty}$ are called the *formal powers* associated to f. They have the property that $\overline{\text{Span}\left\{\varphi_f^{(k)}(x)\right\}_{k=0}^{\infty}}^{C[-b,b]} = C[-b,b]$ (see [80]).

Given $h \in \mathbb{C}$, define the kernels

$$K_h^C(x,t) := K_h(x,t) + K_h(x,-t), \quad K^S(x,-t) := K_h(x,t) - K_h^S(x,-t), \tag{1.27}$$

and the functions

$$c_h(\rho,x) := \mathbf{T}^C[\cos(\rho x)] := \cos(\rho x) + \int_0^x K_h^C(x,t)\cos(\rho t)dt, \tag{1.28}$$

$$s(\rho,x) := \mathbf{T}^S\left[\frac{\sin(\rho x)}{x}\right] := \frac{\sin(\rho x)}{\rho} + \int_0^x K_h^S(x,t)\frac{\sin(\rho t)}{\rho}dt, \tag{1.29}$$

where $\rho \in \mathbb{C}$. Hence $c_h(\rho,x)$ and $s(\rho,x)$ are solutions of (1.21) in $(-b,b)$ with $\lambda = \rho^2$, satisfying the initial conditions

$$c_h(\rho,0) = 1, \quad c_h'(\rho,0) = h, \tag{1.30}$$

$$s(\rho,0) = 0, \quad s'(\rho,0) = 1. \tag{1.31}$$

For $x \in [-b,b]$ fixed, $c_h^{(k)}(\cdot,x), s^{(k)}(\cdot,x) \in \text{Hol}(\mathbb{C})$ for $k = 0,1$, (see, e.g., [83]).

1.4.2 Applications of transmutation operators to the spectral inverse problem for the Schrödinger equation

Consider the Schrödinger equation

$$-y'' + q(x)y = \lambda y, \quad 0 < x < \pi, \tag{1.32}$$

with a real-valued potential $q \in L_2(0,\pi)$ (if originally the equation is considered on another finite interval (a,b), then by a simple change of the independent variable $t = \frac{x-a}{b-a}\pi$ it can always be transferred to $(0,\pi)$). Together with Eq. (1.32) we consider the boundary conditions

$$Uy := y'(0) - hy(0) = 0 \quad \text{and} \quad Vy = y'(\pi) + Hy(\pi) = 0, \tag{1.33}$$

with $h, H \in \mathbb{R}$.

Let us associate to the problem (1.32), (1.33) an unbounded operator in the Hilbert space $L_2(0,\pi)$. Let $\mathbf{S}^{h,H} : \mathscr{D}\left(\mathbf{S}^{h,H}\right) \subset L_2(0,\pi) \to L_2(0,\pi)$ be the operator whose action is given by $\mathbf{S}y$ and with the domain $\mathscr{D}\left(\mathbf{S}^{h,H}\right) := \{y \in H^2(0,\pi) \,|\, y \text{ satisfies } (1.33)\}$.

It is well known that the operator $\mathbf{S}^{h,H}$ is self-adjoint in $L_2(0,2\pi)$ and $\sigma\left(\mathbf{S}^{h,H}\right) = \sigma_d\left(\mathbf{S}^{h,H}\right)$ (see [114, Th. 2.7.4]). The operator $\mathbf{S}^{h,H}$ possesses a infinite number of eigenvalues $\sigma_d\left(\mathbf{S}^{h,H}\right) = \{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{R}$ satisfying

$$\lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty, \quad n \to \infty,$$

and each eigenvalue λ_n is simple (see Chapter 1 of [52] and [106]). For $\lambda \in \mathbb{C}$, we take $\rho \in \overline{\mathbb{C}^+}$ such that $\lambda = \rho^2$ and denote $\varphi(\rho, x) := c_h(\rho, x)$. Then $\varphi(\rho, x)$ satisfies the first boundary condition and $V\varphi(\rho, x) = 0$ iff $\varphi(\rho, x)$ is an eigenfunction iff $\rho^2 \in \sigma_d\left(\mathbf{S}^{h,H}\right)$. Then we have the sequence $\{\rho_n\}_{n=0}^{\infty} \subset \mathbb{R} \cup i\mathbb{R}$ such that $\rho_n^2 = \lambda_n$ for all $n \in \mathbb{N}_0$. For each $n \in \mathbb{N}_0$, the eigenspace associated to λ_n is generated for $\varphi(\rho_n, x)$ and this eigenfunction is real valued (see [52, Ch. I]). The eigenfunction $\varphi(\rho_n, x)$ is said to be *normalized at the origin* and we define the *norming constant*

$$\alpha_n := \int_0^{\pi} \varphi^2(\rho_n, x)dx. \tag{1.34}$$

Thus, the set $\left\{\frac{\varphi(\rho_n,x)}{\sqrt{\alpha_n}}\right\}_{n=0}^{\infty}$ is an orthonormal basis for $L_2(0, \pi)$ (see [52, Th. 1.2.1]). The sequence of pairs $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ is called the *spectral data* of the problem (1.32), (1.33). The *direct spectral problem* associated to (1.32), (1.33) consists of finding the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ and the corresponding eigenfunctions $\{\varphi(\rho_n, x)\}_{n=0}^{\infty}$. The *inverse spectral problem* associated to the Schrödinger equation (1.32) is the following:

Inverse problem: Given an increasing sequence of real numbers $\{\lambda_n\}_{n=0}^{\infty}$ that tends to infinity and a sequence of positive numbers $\{\alpha_n\}_{n=0}^{\infty}$, find a real valued function $q \in L_2(0, \pi)$ and constants $h, H \in \mathbb{R}$ such that $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ be the spectral data of (1.32),(1.33).

The following theorem establishes the necessary and sufficient conditions for a set $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ to be the spectral data of a Sturm-Liouville problem.

Theorem 27 ([52])**.** *The set $\{\lambda_n, \alpha_n\}_{n=0}^{\infty}$ is the spectral data of a problem (1.32), (1.33) iff the following asymptotics hold:*

$$\rho_n = n + \frac{\omega}{\pi n} + \frac{k_n}{n}, \quad \{k_n\} \in \ell_2, \tag{1.35}$$

$$\alpha_n = \frac{\pi}{2} + \frac{K_n}{n}, \quad \{K_n\} \in \ell_2, \tag{1.36}$$

where $\omega = h + H + \frac{1}{2}\int_0^{\pi} q(s)ds.$

Denote $G(x, t) = K_h^C(x, t)$, in such a way that

$$\varphi(\rho, x) = \cos(\rho x) + \int_0^x G(x, t)\cos(\rho t)dt. \tag{1.37}$$

According to (1.22) and (1.27) we have

$$G(x,x) = h + \frac{1}{2}\int_0^x q(s)ds, \quad \text{and } \; G(0,0) = h. \tag{1.38}$$

Suppose that $\{\rho_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ satisfy the asymptotic relations (1.35) and (1.36), respectively. Define the function

$$A(x) := \frac{\cos(\rho_0 x)}{\alpha_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{\cos(\rho_n x)}{\alpha_n} - 2\frac{\cos(nx)}{\pi}\right). \tag{1.39}$$

The following result improves the properties of A obtained in [52, Lemma 1.5.4].

Lemma 28. *The function $A \in H^1(-\pi, 2\pi)$.*

Proof. Denote $\delta_n := \rho_n - n$ and note that

$$\frac{\cos(\rho_n x)}{\alpha_n} - 2\frac{\cos(nx)}{\pi} = \frac{2}{\pi}\left(\cos(\rho_n x) - \cos(nx)\right) + \left(\frac{1}{\alpha_n} - \frac{2}{\pi}\right)\cos(\rho_n x)$$

and

$$\begin{aligned}
\cos(\rho_n x) - \cos(nx) &= \cos(nx + \delta_n x) - \cos(nx)\\
&= \cos(nx)\cos(\delta_n x) - \sin(nx)\sin(\delta_n x) - \cos(nx)\\
&= \cos(nx)\left(\cos(\delta_n x) - 1\right) - \left(\sin(\delta_n x) \pm \delta_n x\right)\sin(nx)\\
&= 2\sin^2\left(\frac{\delta_n x}{2}\right)\cos(nx) - \delta_n x\sin(nx) - \left(\sin(\delta_n x) - \delta_n x\right)\sin(nx)
\end{aligned}$$

Substituting this expressions in (1.39) we have

$$A(x) = \frac{\cos(\rho_0 x)}{\alpha_0} - \frac{1}{\pi} + A_1(x) - \frac{2\omega x}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}, \tag{1.40}$$

where

$$\begin{aligned}
A_1(x) = &\sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n} - \frac{2}{\pi}\right)\cos(\rho_n x) + \frac{4}{\pi}\sum_{n=1}^{\infty}\sin^2\left(\frac{\delta_n x}{2}\right)\cos(nx)\\
&- \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\sin(\delta_n x) - \delta_n x\right)\sin(nx) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{k_n}{n}\sin(nx).
\end{aligned}$$

Since $\{\frac{k_n}{n}\}, \{\frac{K_n}{n}\} \in \ell_1$, $\frac{1}{\alpha_n} - \frac{2}{\pi} = O\left(\frac{K_n}{n}\right)$ and the rest of the sums have terms of order $O\left(\frac{1}{n^2}\right)$, then the series of A_1 converges absolutely and uniformly on $[-2\pi, 2\pi]$. Differentiating the terms of the series we have

$$\begin{aligned}
\left(\left(\frac{1}{\alpha_n}-\frac{2}{\pi}\right)\cos(\rho_n x)\right)' =& O\left(K_n\right)\sin(\rho_n x) = O\left(K_n\right)\sin(nx) + O\left(\frac{K_n}{n}\right),\\
\left(2\sin^2\left(\frac{\delta_n x}{2}\right)\cos(nx)\right)' =& \frac{1}{2}\sin(\delta_n x)\cos(nx) - n\sin^2\left(\frac{\delta_n x}{2}\right)\sin(nx)\\
=& O\left(\frac{1}{n}\right)\cos(nx) + O\left(\frac{1}{n}\right)\sin(nx),\\
((\sin(\delta_n x)-\delta_n x)\sin(nx))' =& (\delta_n\cos(\delta_n x)-\delta_n)\sin(nx) + n(\sin(\delta_n x)-\delta_n x)\cos(nx)\\
=& O\left(\frac{1}{n^3}\right)\sin(nx) + O\left(\frac{1}{n^2}\right)\cos(nx).
\end{aligned}$$

Since $\left\{\frac{1}{2\pi}\sin(nx)\right\}_{n=1}^{\infty}$ and $\left\{\frac{1}{2\pi}\cos(nx)\right\}_{n=1}^{\infty}$ are orthonormal sequences in $L_2(-2\pi,2\pi)$, then the series of derivatives of A_1 converges in $L_2(-2\pi,2\pi)$, and then $A_1 \in H^1(-2\pi,2\pi)$. In particular $A_1 \in H^1(-\pi,2\pi)$.

On the other hand, the series $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$ converges in $L_2(-2\pi,2\pi)$ (and hence in $L_2(-\pi,2\pi)$). Furthermore, for $-\pi < x < 2\pi$, $-\frac{2\omega x}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} = -\frac{\omega|x|(\pi-|x|)}{\pi^2}$ and this function belongs to $H^1(-\pi,2\pi)$. In this way $A \in H^1(-\pi,2\pi)$. **Q.E.D.**

Let us define $F(x,t)=\frac{A(x-t)-A(x+t)}{2}$, for $0 \leqslant x,t \leqslant \pi$, it is

$$F(x,t) = \frac{\sin(\kappa_n x)\sin(\kappa_n t)}{\alpha_n} - \frac{\sin(nx)\sin(nt)}{\pi} + \sum_{n=1}^{\infty}\left(\frac{\sin(\kappa_n x)\sin(\kappa_n t)}{\alpha_n} - 2\frac{\sin(nx)\sin(nt)}{\pi}\right). \tag{1.41}$$

By Lemma 28, $F(x,t)$ admits a continuous representative for $0 \leqslant x,t \leqslant \pi$ and $\frac{d}{dx}F(x,x) \in L_2(0,\pi)$. The following theorem is a compilation of results shown in [52, Sec. 1.5].

Theorem 29. *For every $x \in (0,\pi)$, the kernel $G(x,t)$ satisfies the Fredholm integral equation*

$$G(x,t) + F(x,t) + \int_0^x G(x,s)F(s,t)ds = 0, \quad 0 < t < x. \tag{1.42}$$

Reciprocally, if $\{\rho_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ are sequences satisfying the asymptotic relations (1.35) and (1.36), then for every $x \in (0,\pi]$ fixed, Eq. (1.42) possesses a unique solution $G(x,\cdot) \in L_2(0,x)$. Moreover, $G(x,x) \in H^1(0,\pi)$, and if we take

$$q(x) = \frac{d}{dx}G(x,x), \quad h = G(0,0), \quad H = \omega - h - \frac{1}{2}\int_0^{\pi} q(s)ds, \tag{1.43}$$

hence $\{\rho_n^2,\alpha_n\}_{n=0}^{\infty}$ are the direct spectral data of the Sturm-Liouville problem (1.32), (1.33).

Eq. (29) is called the *Gelfand-Levitan equation.* The Gelfand-Levitan equation was derived in [55] and reduces the inverse Sturm-Liouville problem to the solution of a Fredholm integral equation of the second kind.

Remark 30. *The Gelfand-Levitan equation can be written in the form*

$$G(x,t) = -\mathbf{T}^C[F](x,t),$$

and hence the function $F(x,t)$ is the preimage of $G(x,t)$ under the transmutation operator $\mathbf{T}^C$.

For every $x \in (0,\pi]$ fixed, Eq. (29) is a Fredholm integral equation, then the methods developed in Subsection 1.1.3 are applicable. Specifically, according to Remark 7, it is enough to develop the kernel $G(x,t)$ as a Fourier series for a certain orthonormal basis. Choosing the orthogonal basis of $L_2(0,x)$ given by the Legendre polynomials $\left\{P_{2n}\left(\frac{t}{x}\right)\right\}_{n=0}^{\infty}$, the kernel $G(x,t)$ admits the Fourier-Legendre series representation

$$G(x,t) = \sum_{n=0}^{\infty} \frac{g_n(x)}{x} P_{2n}\left(\frac{t}{x}\right). \tag{1.44}$$

The idea of representing the kernel $G(x,t)$ as a Fourier-Legendre series was developed in [81], and its application to the solution of the inverse spectral problem in [76]. This technique has been developed for different types of problems associated with the Schrödinger equation [39, 78, 92]. In fact, the following relations hold

$$q(x) = \frac{g_0''(x)}{g_0(x)+1}, \quad h = g_0'(0). \tag{1.45}$$

(See [78], Th. 9.2 and Cor. 9.1). In this way, to solve the inverse problem it is only necessary to compute the first coefficient $g_0(x)$, and then recover q and h using the formulas (1.45). H is recovered by means of (1.43).

1.4.3 The Sturm-Liouville equation in impedance form

Let $I \subset \mathbb{R}$ be an open interval (possibly infinite). The *Sturm-Liouville equation in impedance form* (SLEIF, for short) is the second-order linear ordinary differential equation

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + \lambda p(x)u = 0, \quad x \in I,\ \lambda \in \mathbb{C}. \tag{1.46}$$

The function $p \in AC_{loc}(\overline{I})$ is called the *principal coefficient* and we assume that $p(x) \neq 0$ for all $x \in \overline{I}$. The complex number λ is called the *spectral parameter*. Eq. (1.46) can be written as

$$-u'' + q(x)u' = \lambda u, \tag{1.47}$$

where

$$q(x) := -\frac{p'(x)}{p(x)}. \tag{1.48}$$

The function q is called the *potential* of Eq. (5.1). By the properties of p, the potential $q \in L^1_{loc}(I)$. Choosing $x_0 \in \overline{I}$ we have the relation

$$p(x) = p(x_0) \exp\left(-\int_{x_0}^{x} q(s)\right). \tag{1.49}$$

Note that if $P = \alpha p$, for some $\alpha \in \mathbb{C}$, the coefficient P generates the same equation and potential that p. For this reason, in certain problems it is convenient to use a normalized coefficient p at the point $x_0 \in \overline{I}$. Such normalization is given by $p(x_0) = 1$.

The *impedance function* of Eq. (1.46) (associated to the normalized coefficient p) is given by

$$a(x) := \sqrt{p(x)} = \exp\left(-\frac{1}{2}\int_{x_0}^{x} q(s)ds\right), \quad \text{from where } \; q(x) = -2\frac{a'(x)}{a(x)}. \tag{1.50}$$

Depending on the problem under consideration, it is sometimes more convenient to work with a instead of p.

Note that $p, \frac{1}{p} \in L^1_{loc}(I)$, hence Eq. (1.46) possesses two linearly independent solutions $u_1, u_2 \in W^{2,1}_{loc}(I)$ such that $\{u_1, u_2\}$ is a fundamental set of solutions for (1.46) (any solution of (1.46) can be expressed as a linear combination of u_1 and u_2, see [146, Th. 2.2.1] and [12, Cor. 4.1.2]). Denote the differential operator

$$\mathbf{L} := -\frac{1}{p(x)}\mathbf{D}p(x)\mathbf{D} = -\mathbf{D}^2 - q(x)\mathbf{D}.$$

For $u, v \in W^{2,1}_{loc}(I)$ we denote the *p-Wronskian* $W_p[u,v](x) := \begin{vmatrix} u(x) & v(x) \\ p(x)u'(x) & p(x)v'(x) \end{vmatrix}$. Note that

$$\frac{d}{dx}W_p[u,v](x) = \frac{d}{dx}(upv' - pu'v) = (pv')'u + pu'v' - (pu')'v - pu'v' = (pv')'u - (pu')'v,$$

from where we obtain the *Lagrange identity*

$$\frac{d}{dx}W_p[u,v](x) = p(x)v(x)\mathbf{L}u(x) - p(x)u(x)\mathbf{L}v(x). \tag{1.51}$$

In particular, if $u, v \in W^{2,1}_{loc}(I)$ are solutions of (5.1), $W_p[u,v]$ is constant. The set $\{u, v\}$ is a fundamental set of solutions iff $W_p[u,v] \neq 0$.

Note 31. *It is possible to formulate the concept of a weak solution for Eq. (1.46) (see [19, Sec. 8.4]). However, in [16, Prop. 2] it was proved that every weak solution belongs to space* $W^{2,1}_{loc}(I)$ *and satisfies the equation a.e.* $x \in I$.

Chapter 2

Gelfand-Levitan equation and solution of the inverse Dirichlet problem for the one dimensional Schrödinger equation

Let $q \in L_2(0,\pi)$ be real-valued. In this chapter we consider the Schrödinger equation (1.32) with the *Dirichlet conditions*

$$u(0) = u(\pi) = 0. \tag{2.1}$$

We obtain a Gelfand-Levitan equation similar to Eq. (29). Once the equation is deduced, the methods developed in Subsection 1.1.3 are applied to obtain an algorithm for solving the inverse spectral problem.

2.1 Properties of the Sturm-Liouville problem with Dirichlet condition

Let us associate to the problem (1.32), (2.1) an unbounded operator in the Hilbert space $L_2(0,\pi)$. Let $\mathbf{S}_D : \mathscr{D}(\mathbf{S}_D) \subset L_2(0,\pi) \to L_2(0,\pi)$ be the operator whose action is given by $\mathbf{S}u$ and with the domain $\mathscr{D}(\mathbf{S}_D) = H^2(0,b) \cap H_0^1(0,b)$. The operator $\mathbf{S}_D$ is densely defined and self-adjoint in $L_2(0,\pi)$, with purely discrete spectrum $\sigma_d(\mathbf{L}_D) = \{\lambda_n\}_{n=1}^{\infty}$ satisfying

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty, \ n \to \infty, \tag{2.2}$$

(see [114, Th. 2.7.4] or [52, Ch. I]). The eigenvalues can be characterized in the following way. Consider the solution $S(\rho, x)$ of (1.32) satisfying the initial conditions (1.31). Then $\lambda \in \sigma_d(\mathbf{L}_D)$ iff $\lambda = \rho^2$, where ρ is a root of the characteristic equation

$$S(\rho, \pi) = 0. \tag{2.3}$$

A corresponding eigenfunction of $\lambda_n = \rho_n^2$ is $S(\rho_n, x)$. However, it is convenient to use the eigenfunctions $\widehat{S}(\rho_n, x)$, where $\widehat{S}(\rho, x) := \rho S(\rho, x)$. Thus, the eigenspace associated to λ_n is generated by $\widehat{S}(\rho_n, x)$ and we define the *norming constant*

$$\alpha_n := \int_0^{\pi} \left|\widehat{S}(\rho_n, x)\right|^2 dx. \tag{2.4}$$

In this way, $\left\{\frac{1}{\sqrt{\alpha_n}}\widehat{S}(\rho_n, x)\right\}_{n=1}^{\infty}$ is an orthonormal basis for $L_2(0,\pi)$ (see [114, Th. 2.7.4]). Furthermore, if $\phi \in AC[0,\pi]$, then

$$\phi(x) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\widehat{S}(\rho_n, x)}{\alpha_n} \int_0^{\pi} \phi(t)\widehat{S}(\rho_n, t)dt, \tag{2.5}$$

uniformly on $x \in [0,\pi]$ (see [52, Th. 1.2.1]). For example, when $q \equiv 0$, $\lambda_n = n^2$, $\widehat{S}(n, x) = \sin(nx)$ and $\alpha_n = \frac{\pi}{2}$ for all $n \in \mathbb{N}$.

Inverse problem. Given an increasing bounded from bellow sequence $\{\lambda_n\}_{n=1}^{\infty}$ that tends to infinity and a sequence of positive numbers $\{\alpha_n\}_{n=1}^{\infty}$, find a real-valued function $q \in L_2(0,\pi)$ such that $\{\lambda_n\}_{n=1}^{\infty}$ be the spectrum of the Dirichlet problem (1.32),(2.1) and $\{\alpha_n\}_{n=1}^{\infty}$ its sequence of norming constants.

The sequence of pairs $\{\lambda_n, \alpha_n\}_{n=1}^{\infty}$ is called the *spectral data* of the Dirichlet problem (1.32),(2.1).

2.2 Gelfand-Levitan equation

A necessary condition for the sequence $\{\lambda_n\}_{n=1}^{\infty}$ to be the eigenvalues of a Dirichlet problem, is that the sequence $\{\rho_n\}_{n=1}^{\infty}$ with $\rho_n := \sqrt{\lambda_n}$ (choosing the branch of the square root with $\arg \lambda_n \in (-\pi, \pi]$) satisfies the asymptotic relation

$$\rho_n = n + \frac{\omega}{\pi n} + \frac{k_n}{n} \quad \text{where } \{k_n\} \in \ell_2 \text{ and } \omega = \frac{1}{2}\int_0^\pi q(s)ds \tag{2.6}$$

(see [52, pp. 13]). A similar condition for the norming constants is established.

Theorem 32. *The norming constants* $\{\alpha_n\}_{n=1}^\infty$ *satisfy the asymptotic relation*

$$\alpha_n = \frac{\pi}{2} + \frac{K_n}{n} \quad \textit{where } \{K_n\} \in \ell_2. \tag{2.7}$$

Proof. The solution $\widehat{S}(\rho, x)$ satisfies the integral equation

$$\widehat{S}(\rho, x) = \sin(\rho x) + \int_0^x \frac{\sin(\rho(x-t))}{\rho} q(t)\widehat{S}(\rho, t)dt \tag{2.8}$$

and the asymptotic relation

$$\widehat{S}(\rho, x) = \sin(\rho x) + O\left(\frac{e^{x|\operatorname{Im}\rho|}}{|\rho|}\right), \quad |\rho| \to \infty. \tag{2.9}$$

(see [52, pp. 10-15]). Substitution of the asymptotic relation (2.9) into (2.8) and the relation $\frac{\sin(z)}{z} = O\left(\frac{e^{|\operatorname{Im} z|}}{|z|}\right)$ lead us to

$$\begin{aligned}
\widehat{S}(\rho, x) &= \sin(\rho x) + \int_0^x \frac{\sin(\rho(x-t))}{\rho} q(t)\sin(\rho t)dt + \int_0^x \frac{\sin(\rho(x-t))}{\rho} q(t) O\left(\frac{e^{t|\operatorname{Im}\rho|}}{|\rho|}\right) dt \\
&= \sin(\rho x) + \int_0^x \frac{\cos(\rho(x-2t)) - \cos(\rho x)}{2\rho} q(t)dt \\
&\quad + \int_0^x q(t) O\left(\frac{e^{(x-t)|\operatorname{Im}\rho|}}{|\rho|}\right) O\left(\frac{e^{t|\operatorname{Im}\rho|}}{|\rho|}\right) dt \\
&= \sin(\rho x) + \frac{\cos(\rho x)}{2\rho} Q(x) + \frac{1}{2\rho}\int_0^x \cos(\rho(x-2t))q(t) + O\left(\frac{e^{x|\operatorname{Im}\rho|}}{|\rho|^2}\right),
\end{aligned}$$

with $Q(x) = \int_0^x q(t)dt$. In particular, for ρ_n we have the asymptotics

$$\widehat{S}(\rho_n, x) = \sin(\rho_n x) + \frac{\cos(\rho_n x)}{2\rho_n} Q(x) + \frac{1}{2\rho_n}\int_0^x \cos(\rho_n(x-2t))q(t)dt + O\left(\frac{1}{n^2}\right), \tag{2.10}$$

because $\rho_n = O(n)$, $n \to \infty$ by (2.6). Writing (2.6) as $\rho_n = n + \frac{\varepsilon_n}{n}$ with $\varepsilon_n := \frac{\omega}{\pi} + k_n$, we deduce the following relations:

$$\begin{aligned}
\cos(\rho_n x) &= \cos(nx)\cos\left(\frac{\varepsilon_n}{n}x\right) - \sin(nx)\sin\left(\frac{\varepsilon_n}{n}x\right) \\
&= \cos(nx) + \cos(nx)\left(\cos\left(\frac{\varepsilon_n}{n}x\right) - 1\right) - \sin(nx)\sin\left(\frac{\varepsilon_n}{n}x\right) \\
&= \cos(nx) + O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{n}\right),
\end{aligned}$$

and in a similar way $\sin(nx) = \sin(nx) + \cos(nx)\sin\left(\frac{\varepsilon_n}{n}x\right) + O\left(\frac{1}{n^2}\right)$.

Substituting these asymptotics in (2.10) we obtain

$$\widehat{S}(\rho_n, x) = \sin(nx) + \frac{\xi_n(x)}{n},$$

where

$$\xi_n(x) = n\cos(nx)\sin\left(\frac{\varepsilon_n}{n}x\right) + \frac{n\cos(nx)}{2\rho_n}Q(x) + \frac{n}{2\rho_n}\int_0^x \cos(n(x-2t))q(t)dt + O\left(\frac{1}{n}\right). \tag{2.11}$$

Note that the sequence of functions $\{\xi_n(x)\}$ is uniformly bounded in the whole segment $[0,\pi]$. Now, we deduce that

$$\begin{aligned}
\alpha_n &= \int_0^\pi \left|\widehat{S}(\rho_n, x)\right|^2 dx = \int_0^\pi \sin^2(nx)dx + \operatorname{Re}\left(\frac{2}{n}\int_0^\pi \sin(nx)\xi_n(x)\right) + \frac{1}{n^2}\int_0^\pi |\xi_n(x)|^2 dx \\
&= \frac{\pi}{2} + \operatorname{Re}\left(\frac{2}{n}\int_0^\pi \sin(nx)\xi_n(x)\right) + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Then $\alpha_n = \frac{\pi}{2} + \frac{K_n}{n}$, where

$$\begin{aligned}
K_n &= \operatorname{Re}\left(2\int_0^\pi \sin(nx)\xi_n(x)\right) + O\left(\frac{1}{n}\right) \\
&= \operatorname{Re}\left(2n\int_0^\pi \sin(nx)\cos(nx)\sin\left(\frac{\varepsilon_n}{n}x\right)dx + \frac{2n}{\rho_n}\int_0^\pi \sin(nx)\cos(nx)Q(t)dt\right. \\
&\quad \left.+ \frac{2n}{\rho_n}\int_0^\pi \sin(nx)\left[\int_0^x \cos(n(x-2t))q(t)dt\right]dx\right) + O\left(\frac{1}{n}\right).
\end{aligned}$$

Developing the integrals involved we obtain

$$\begin{aligned}
2n\int_0^\pi \sin(nx)\cos(nx)\sin\left(\frac{\varepsilon_n}{n}x\right)dx &= n\int_0^\pi \sin(2nx)\sin\left(\frac{\varepsilon_n}{n}x\right)dx \\
&= n\left(\frac{\sin\left(\left(2n-\frac{\varepsilon_n}{n}\right)\pi\right)}{2n-\frac{\varepsilon_n}{n}} - \frac{\sin\left(\left(2n+\frac{\varepsilon_n}{n}\right)\pi\right)}{2n+\frac{\varepsilon_n}{n}}\right) \\
&= O(1)\left(2\sin\left(\frac{\varepsilon_n}{n}\pi\right)\right) = O\left(\frac{1}{n}\right), \\
2\int_0^\pi \sin(nx)\cos(nx)Q(x)dx &= \int_0^\pi \sin(2nx)Q(x)dx = \langle \sin(2n\cdot), Q\rangle_{L_2(0,\pi)},
\end{aligned}$$

and the sequence $\{\langle \sin(2n\cdot), Q\rangle_{L_2(0,\pi)}\} \in \ell_2$ because $\{\sin(2nx)\}_{n=0}^\infty$ is an orthonormal sequence in $L_2(0,\pi)$. Finally, taking $I=\int_0^\pi \sin(nx)\left[\int_0^x \cos(n(x-2t))q(t)dt\right]dx$ we have

$$
\begin{aligned}
I &= -\frac{\cos(n\pi)}{n}\int_0^\pi \cos(n(\pi-2t))q(t)dt+\int_0^\pi \frac{\cos(nx)}{n}\left[q(x)+n\int_0^x \sin(n(x-2t))q(t)dt\right]dx\\
&= \int_0^\pi \cos(nx)\left[\int_0^x \sin(n(x-2t))q(t)dt\right]dx+O\left(\frac{1}{n}\right)\\
&= -\int_0^\pi \sin(nx)\left[\int_0^x \cos(n(x-2t))q(t)dt\right]dx+O\left(\frac{1}{n}\right),
\end{aligned}
$$

hence $I=O\left(\frac{1}{n}\right)$, and since $\frac{n}{\kappa_n}=O(1)$, we conclude that $\{K_n\}\in\ell_2$. **Q.E.D.**

Define the function

$$A_D(x):=\sum_{n=1}^{\infty}\left(\frac{\cos(\rho_n x)}{\alpha_n}-2\frac{\cos(nx)}{\pi}\right) \tag{2.12}$$

According to Lemma 28, $A_D\in H^1(-\pi,2\pi)$. Let us define $F_D(x,t)=\frac{A_D(x-t)-A_D(x+t)}{2}$ for $0\leqslant x,t\leqslant\pi$, it is

$$F_D(x,t)=\sum_{n=1}^{\infty}\left(\frac{\sin(\rho_n x)\sin(\rho_n t)}{\alpha_n}-2\frac{\sin(nx)\sin(nt)}{\pi}\right). \tag{2.13}$$

By Lemma 28, $F_D(x,t)$ admits a continuous representative for $0\leqslant x,t\leqslant\pi$ and $\frac{d}{dx}F(x,x)\in L_2(0,\pi)$. Note that $F_D(x,t)=F_D(t,x)$. For $N\in\mathbb{N}$ we denote

$$F_{D,N}(x,t)=\sum_{n=1}^{N}\left(\frac{\sin(\rho_n x)\sin(\rho_n t)}{\alpha_n}-2\frac{\sin(nx)\sin(nt)}{\pi}\right).$$

It is clear that $F_{D,N}(x,t)$ converges weakly in t to $F_D(x,t)$, i.e., for all $\phi\in AC[0,\pi]$, $\int_0^\pi \phi(t)F_N(x,t)dt\to\int_0^\pi\phi(t)F(x,t)dt$ for every $x\in(0,\pi]$.

By (1.29), we can write $\widehat{S}(\rho,x)$ as

$$\widehat{S}(\rho,x)=\sin(\rho x)+\int_0^x G(x,t)\sin(\rho t)dt, \tag{2.14}$$

where $G(x,t)=K^S(x,t)$. We deduce a Gelfand-Levitan equation for the transmutation kernel $G(x,t)$.

Theorem 33. *For every $x\in(0,\pi]$ fixed, the kernel $G(x,t)$ satisfies the Gelfand-Levitan equation*

$$G(x,t)+F_D(x,t)+\int_0^x G(x,s)F_D(s,t)ds=0,\quad 0<t<x. \tag{2.15}$$

Proof. Consider (2.14) as a Volterra integral equation. Since the inverse of a Volterra integral operator is also a Volterra integral operator, we can write

$$\sin(\rho x) = \widehat{S}(\rho, x) + \int_0^x H(x,t)\widehat{S}(\rho, t)dt, \tag{2.16}$$

where $H(x,t)$ is a continuos function (actually, since G is absolutely continuous in both variables, H also). Then for each $N \in \mathbb{N}_0$ we have the equalities

$$\begin{aligned}
\frac{\widehat{S}(\rho_n, x)\sin(\rho_n t)}{\alpha_n} &= \frac{\sin(\rho_n x)\sin(\rho_n t)}{\alpha_n} + \int_0^x G(x,s)\frac{\sin(\rho_n s)\sin(\rho_n t)}{\alpha_n}ds,\\
\frac{\widehat{S}(\rho_n, x)\sin(\rho_n t)}{\alpha_n} &= \frac{\widehat{S}(\rho_n x)\widehat{S}(\rho_n t)}{\alpha_n} + \int_0^t H(t,s)\frac{\widehat{S}(\rho_n, s)\widehat{S}(\rho_n, x)}{\alpha_n}ds.
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
\frac{\widehat{S}(\rho_n, x)\widehat{S}(\rho_n, t)}{\alpha_n} - \frac{2\sin(nx)\sin(nt)}{\pi} &= \frac{\widehat{S}(\rho_n, x)\sin(\rho_n t)}{\alpha_n} - \int_0^t H(t,s)\frac{\widehat{S}(\rho_n, s)\widehat{S}(\rho_n, x)}{\alpha_n}ds\\
&\quad - \frac{2\sin(nx)\sin(nt)}{\pi}\\
&= \frac{\sin(\rho_n x)\sin(\rho_n t)}{\alpha_n} - \frac{2\sin(nx)\sin(nt)}{\pi}\\
&\quad + \int_0^x G(x,s)\frac{\sin(\rho_n s)\sin(\rho_n t)}{\alpha_n}ds\\
&\quad - \int_0^t H(t,s)\frac{S(\kappa_n, s)S(\rho_n, x)}{\alpha_n}ds\\
&\quad \pm \int_0^x G(x,s)\frac{2\sin(ns)\sin(nt)}{\pi}ds
\end{aligned}$$

If we denote $\Phi_N(x,t) := \sum_{n=1}^{N}\left(\frac{\widehat{S}(\rho_n, x)\widehat{S}(\rho_n, t)}{\alpha_n} - \frac{2\sin(nx)\sin(nt)}{\pi}\right)$, hence

$$\Phi_N(x,t) = F_{D,N}(x,t) + \sum_{k=2}^{4} I_{k,N}(x,t), \tag{2.17}$$

where

$$\begin{aligned}
I_{2,N}(x,t) &:= \sum_{n=1}^{N}\int_0^x G(x,s)\frac{2\sin(ns)\sin(nt)}{\pi}ds,\\
I_{3,N}(x,t) &:= \int_0^x G(x,s)F_{D,N}(s,t)ds,\\
I_{4,N}(x,t) &:= -\sum_{n=1}^{N}\int_0^t H(t,s)\frac{\widehat{S}(\rho_n, s)S(\rho_n, x)}{\alpha_n}ds.
\end{aligned}$$

Take $\phi \in AC[0,\pi]$. Multipliying (2.17) by ϕ and using (2.5) we obtain the following limits uniformly for $x \in (0,\pi]$:

$$\lim_{N\to\infty}\int_0^\pi \phi(t)\Phi_N(x,t)dt = 0,$$

$$\lim_{N\to\infty}\int_0^\pi \phi(t)F_{D,N}(x,t)dt = \int_0^\pi \phi(t)F_D(x,t)dt,$$

$$\begin{aligned}\lim_{N\to\infty}\int_0^\pi \phi(t)I_{N,2}(x,t)dt &= \lim_{N\to\infty}\int_0^\pi \phi(t)\left(\sum_{n=1}^N \frac{2\sin(nt)}{\pi}\int_0^x G(x,s)\sin(ns)ds\right)dt\\ &= \lim_{N\to\infty}\int_0^x G(x,s)\left(\sum_{n=1}^N \frac{2\sin(ns)}{\pi}\int_0^\pi \phi(t)\sin(nt)dt\right)ds\\ &= \int_0^x G(x,s)\phi(s)ds.\end{aligned}$$

$$\begin{aligned}\lim_{N\to\infty}\int_0^\pi \phi(t)I_{3,N}(x,t)dt &= \lim_{N\to\infty}\int_0^\pi \phi(t)\left(\int_0^x G(x,s)F_{D,N}(s,t)ds\right)dt\\ &= \lim_{N\to\infty}\int_0^x G(x,s)\left(\int_0^\pi \phi(t)F_{D,N}(s,t)ds\right)dt\\ &= \int_0^x G(x,s)\left(\int_0^\pi \phi(t)F_D(s,t)ds\right)dt = \int_0^\pi \phi(t)\left(\int_0^x G(x,s)F_D(s,t)ds\right)dt\end{aligned}$$

$$\begin{aligned}\lim_{N\to\infty}\int_0^\pi \phi(t)I_{4,N}(x,t)dt &= -\lim_{N\to\infty}\sum_{n=1}^N \frac{\widehat{S}(\rho_n,x)}{\alpha_n}\int_0^\pi \phi(t)\left(\int_0^t H(t,s)\widehat{S}(\rho_n,s)ds\right)dt\\ &= -\lim_{N\to\infty}\sum_{n=1}^N \frac{\widehat{S}(\rho_n,x)}{\alpha_n}\int_0^\pi\int_s^\pi \phi(t)H(t,s)\widehat{S}(\rho_n,s)dtds\\ &= -\lim_{N\to\infty}\sum_{n=1}^N \frac{\widehat{S}(\rho_n,x)}{\alpha_n}\int_0^\pi \widehat{S}(\rho_n,s)\left(\int_s^\pi \phi(t)H(t,s)dt\right)ds\\ &= \int_x^\pi H(t,x)\phi(t)dt.\end{aligned}$$

Extend $G(x,t) = H(x,t) = 0$ for $t < x$. Then

$$\int_0^\pi \left\{G(x,t) + F_D(x,t) + \int_0^x G(x,s)F_D(s,t)ds - H(x,t)\right\}\phi(t)dt = 0.$$

Due to the arbitrariness of $\phi \in AC[0,\pi]$, we conclude that

$$G(x,t) + F_D(x,t) + \int_0^x G(x,s)F_D(s,t)ds - H(x,t) = 0 \quad \text{for } x,t \in [0,\pi].$$

In particular, for $t < x$ we obtain (2.15). **Q.E.D.**

Remark 34. *Note that for $x \in (0,\pi]$ fixed, Eq. (2.15) is a Fredholm integral equation of the second kind. Since A_D admits a continuous representative in $[-\pi,2\pi]$, the kernel*

$F \in L_2((0,x)\times(0,x))$. Furthermore, for $x \in (0,\pi]$, Eq. (2.15) has a unique solution. By the Freedholm alternative, it is enough to show that for $g \in L_2(0,x)$ the equation

$$g(t) + \int_0^x F(s,t)g(s)ds = 0,$$

has only the trivial solution. Indeed, multiplying by $g(s)$ and integrating we obtain

$$\int_0^x g^2(t)ds + \int_0^x \int_0^x F(s,t)g(s)g(t)dsdt = 0.$$

Considering the extension $\tilde{g}(t) = g(t)\chi_{(0,\pi)}(t)$ (where $\chi_{(0,\pi)}$ is the characteristic equation of the interval $(0,\pi)$) we obtain the equality

$$\int_0^\pi \tilde{g}^2(t)ds + \sum_{n=1}^\infty \left(\frac{1}{\alpha_n} \left(\int_0^\pi \tilde{g}(s)\sin(\rho_n s)ds \right)^2 - \frac{2}{\pi} \left(\int_0^\pi \tilde{g}(s)\sin(ns)ds \right)^2 \right) = 0.$$

Since $\left\{ \frac{\sin(ns)}{\sqrt{\frac{\pi}{2}}} \right\}_{n=1}^\infty$ is an orthonormal basis in $L_2(0,\pi)$, by the Parseval identity $\int_0^\pi g^2(s)ds = \sum_{n=1}^\infty \frac{2}{\pi}\left(\int_0^\pi \tilde{g}(s)\sin(ns)ds\right)^2$, thus $\sum_{n=0}^\infty \frac{1}{\alpha_n}\left(\int_0^\pi \tilde{g}(s)\sin(\rho_n s)ds\right)^2 = 0$. Since $\alpha_n > 0$ for all $n \in \mathbb{N}$, we conclude that

$$\int_0^\pi \tilde{g}(s)\sin(\rho_n s)ds = 0 \qquad \forall n \in \mathbb{N}.$$

Under the conditions (2.6) on the sequence $\{\rho_n\}_{n=1}^\infty$ the system $\{\sin(\rho_n)\}_{n=1}^\infty$ is complete in $L_2(0,\pi)$ (see [143, pp. 122]). Hence $\tilde{g}(s) = 0$ a.e. in $(0,\pi)$, and the equation has only the trivial solution.

2.3 Solution of the inverse problem

Fix $x \in (0,\pi]$. Since $G(x,\cdot) \in L_2(0,x)$, it admits a Fourier-Legendre series

$$G(x,t) = \sum_{n=0}^\infty \frac{b_n(x)}{x} P_{2n+1}\left(\frac{t}{x}\right). \tag{2.18}$$

The choice of the odd polynomials is due to condition $G(x,0) = 0$. The series (2.18) converges with respect to the variable t in $L_2(0,x)$. Furthermore, for $q \in C[0,\pi]$, the series converges uniformly for $0 < x \leqslant \pi$, $t \in [0,x]$ (see [81, Th. 3.3])

Remark 35. *The potential $q(x)$ can be recovered from the first Fourier-Legendre coefficient $b_0(x)$ by the formula*

$$q(x) = \frac{(xb_0(x))''}{x(b_0(x)+3)}. \tag{2.19}$$

Indeed, multiplying (2.18) by $P_1\left(\frac{t}{x}\right)$, integrating form 0 to x with respect to t and using the $L_2(0,x)$-convergence of series (2.18), we obtain

$$b_0(x) = 3\int_0^x G(x,t)P_1\left(\frac{t}{x}\right)dt = \frac{3}{x}\left(\mathbf{T}_S[x]-x\right). \tag{2.20}$$

Note that $S(\rho,x) = \mathbf{T}^S\left[\frac{\sin(\rho x)}{\rho}\right]$ and $\frac{\sin(\rho x)}{\rho}\Big|_{\rho=0} = x$, then we have the relation $\mathbf{T}^S[x] = S(0,x)$. In this way

$$b_0(x) = \frac{3}{x}\left(S(0,x)-x\right), \qquad or \qquad S(0,x) = x\left(\frac{b_0}{3}+1\right).$$

Applying the Schrödinger operator $\mathbf{S}$ to both sides of the equality we obtain

$$0 = \frac{1}{3}\mathbf{S}[xb_0+3] = -(xb_0)'' + q(x)(xb_0+3),$$

from where we obtain (2.19).

2.3.1 A infinite system for the the Fourier-Legendre coefficients

Following [71], we improve the convergence of series (2.12) in the following way. In the proof of Lemma 28 we see that $A_D(x) = A_1(x) - \frac{2\omega x}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$, where A_1 converges absolutely and uniformly on $[-\pi, 2\pi]$. Then, using that $-\frac{2\omega x}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} = -\frac{\omega|x|(\pi-|x|)}{\pi^2}$, we can write a in the form

$$A_D(x) = \sum_{n=1}^{\infty}\left(\frac{\cos(\rho_n x)}{\alpha_n} - \frac{2\cos(nx)}{\pi} + \frac{2\omega x}{\pi^2}\frac{\sin(nx)}{n}\right) - \frac{\omega}{\pi^2}|x|(\pi-|x|) \tag{2.21}$$

The series in (2.21) converges absolutely and uniformly on $x \in [-\pi, 2\pi]$. Hence, for $0 \leqslant x, t \leqslant \pi$ we have

$$
\begin{aligned}
F_D(x,t) =& \frac{A_D(x-t) - A_D(x+t)}{2} \\
=& -\frac{\omega}{2\pi^2}\left(|x-t|(\pi - |x-t|) - |x+t|(\pi - |x+t|)\right) \\
&+ \sum_{n=1}^{\infty}\left[\frac{\cos(\rho_n(x-t)) - \cos(\rho_n(x+t))}{2\alpha_n} - \frac{\cos(n(x-t)) - \cos(n(x+t))}{\pi}\right. \\
&\qquad \left. + \left(\frac{\omega(x-t)}{\pi^2}\frac{\sin(n(x-t))}{n} - \frac{\omega(x+t)}{\pi^2}\frac{\sin(n(x+t))}{n}\right)\right] \\
=& \frac{\omega}{2\pi^2}\left(\pi(|x+t| - |x-t|) + |x-t|^2 - |x+t|^2\right) \\
&+ \sum_{n=1}^{\infty}\left[\frac{\sin(\rho_n x)\sin(\rho_n t)}{\alpha_n} - \frac{2\sin(nx)\sin(nt)}{\pi}\right. \\
&\qquad \left. - \frac{2\omega}{\pi^2 n}\left(x\cos(nx)\sin(nt) + \sin(nx)x\cos(nx)\right)\right]
\end{aligned}
$$

Since $x, t \geqslant 0$, then $|x+t| - |x-t| = x + t - |x-t| = 2\min(x,t)$, hence we obtain the representation

$$
\begin{aligned}
F_D(x,t) =& \frac{\omega}{\pi^2}(\pi\min(x,t) - 2xt) + \sum_{n=1}^{\infty}\left[\frac{\sin(\kappa_n x)\sin(\kappa_n t)}{\alpha_n} - \frac{2\sin(nx)\sin(nt)}{\pi}\right. \\
&\qquad \left. - \frac{2\omega}{\pi^2 n}\left(x\cos(nx)\sin(nt) + \sin(nx)x\cos(nx)\right)\right]
\end{aligned} \tag{2.22}
$$

The series in (2.22) converges absolutely and uniformly for $0 \leqslant x, t \leqslant \pi$. Now we use the ideas developed in subsection 1.2 to obtain an infinite system of linear algebraic equations for the Fourier-Legendre coefficients $\{b_n(x)\}_{n=0}^{\infty}$, in a similar way as for the Neumann problem (see [76, 78, 92]).

Theorem 36. *For every $x \in (0, \pi]$ fixed, the coefficients $\{b_n(x)\}_{n=0}^{\infty}$ of the series expansion (2.18) satisfy the infinite system of linear algebraic equations*

$$
\frac{b_m(x)}{(4m+3)x} + \sum_{n=0}^{\infty} A_{m,n}(x) b_n(x) = B_m(x), \qquad \textit{for } m \in \mathbb{N}_0, \tag{2.23}
$$

where

$$
\begin{aligned}
A_{m,n}(x) = &- \frac{2\omega x^2 \delta_{(n,0)}\delta_{(m,0)}}{9\pi^2} - \frac{\omega x}{8\pi}\left[\frac{\delta_{(n+1,m)}}{\left(2n+\frac{3}{2}\right)_3} - \frac{2\delta_{(n,m)}}{(2n+1)_3} + \frac{\delta_{(n-1,m)}}{\left(2n-\frac{1}{2}\right)_3}\right] \\
&+ (-1)^{m+n}\sum_{k=1}^{\infty}\Bigg[\left(\frac{j_{2n+1}(\rho_k x) j_{2m+1}(\rho_k x)}{\alpha_k} - \frac{2 j_{2n+1}(kx) j_{2m+1}(kx)}{\pi}\right) \\
&- \frac{2\omega}{\pi^2 k}\Bigg(\left(\frac{2n+1}{k} j_{2n+1}(kx) - x j_{2n+2}(kx)\right) j_{2m+1}(kx) \\
&+ j_{2n+1}(kx)\left(\frac{2m+1}{k} j_{2m+1}(kx) - x j_{2m+2}(kx)\right)\Bigg)\Bigg],
\end{aligned} \tag{2.24}
$$

and

$$
\begin{aligned}
B_m(x) = &\frac{\omega x}{3\pi^2}(2x-\pi)\delta_{(m,0)} + (-1)^m \sum_{k=1}^{\infty}\Bigg[\frac{2\sin(kx) j_{2m+1}(kx)}{\pi} - \frac{\sin(\rho_k x) j_{2m+1}(\rho_k x)}{\alpha_k} \\
&+ \frac{2\omega}{\pi^2 k}\left(\sin(kx)\left(\frac{2m+1}{k} j_{2m+1}(kx) - x j_{2m+2}(kx)\right) + x\cos(kx) j_{2m+1}(kx)\right)\Bigg].
\end{aligned} \tag{2.25}
$$

Proof. Fix $x \in (0, \pi]$. Thus, the Gelfand-Levitan equation (2.15) is a Fredholm integral equation of the second kind. By Remak 7 and (1.4), (2.15), it is equivalent to the infinite system of algebraic equations

$$
\xi_j(x) + \sum_{k=0}^{\infty} c_{j,k}(x)\xi_k(x) = y_j(x), \tag{2.26}
$$

where, according to (1.10) and (1.16), we have

$$
\begin{aligned}
\xi_j &= \frac{b_j(x)}{\sqrt{4j+3}\sqrt{x}}, \\
c_{j,k} &= \sqrt{4j+3}\sqrt{4k+3}\,x \int_0^{\pi}\int_0^{\pi} F_D(s,t) P_{2k+1}\left(\frac{s}{x}\right) P_{2j+1}\left(\frac{t}{x}\right) ds dt, \\
y_j &= -\sqrt{4j+3}\sqrt{x}\int_0^{\pi} F_D(x,t) P_{2j+1}\left(\frac{t}{x}\right) dt.
\end{aligned}
$$

If we define

$$
\begin{aligned}
A_{m,n}(x) &:= \int_0^{\pi}\int_0^{\pi} F_D(s,t) P_{2n+1}\left(\frac{s}{x}\right) P_{2m+1}\left(\frac{t}{x}\right)\frac{ds dt}{x^2}, \\
B_m(x) &:= -\int_0^{\pi} F_D(x,t) P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x},
\end{aligned}
$$

hence the infinite system (2.23) is equivalent to the normalized one (2.26). By Remark (4), the series in (2.23) converge pointwise. Now we compute the matrix $A_{m,n}(x)$ and the right-hand side $B_m(x)$. We denote $\mathcal{T}_n(x,t) := \int_0^x F_D(s,t)P_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x}$. Thus,

$$\begin{aligned}\mathcal{T}_n(x,t) =& \int_0^x F_D(s,t)P_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x}\\ =&\frac{\omega}{\pi^2}\left(\pi\int_0^x \min\{s,t\}P_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x} - 2t\int_0^x sP_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x}\right)\\ &+\sum_{k=1}^{\infty}\left[\left(\frac{\sin(\rho_k t)}{\alpha_k}\int_0^x P_{2n+1}\left(\frac{s}{x}\right)\sin(\rho_k s)\frac{ds}{x} - \frac{2\sin(kt)}{\pi}\int_0^x P_{2n+1}\left(\frac{s}{x}\right)\sin(ks)\frac{ds}{x}\right)\right.\\ &\left.-\frac{2\omega}{\pi^2 k}\left(\sin(kt)\int_0^x P_{2n+1}\left(\frac{s}{x}\right)s\cos(ks)\frac{ds}{x} + t\cos(kt)\int_0^x P_{2n+1}\left(\frac{s}{x}\right)\sin(ks)\frac{ds}{x}\right)\right].\end{aligned}$$

The exchange of the order of summation and integration is due to the uniform convergence of the series in both variables $s,t\in[0,\pi]$. Using the formula 2.17.7 from ([119, pp. 433])

$$\int_0^x P_{2n+1}\left(\frac{s}{x}\right)\sin(zs)\frac{ds}{x} = (-1)^m j_{2n+1}(zx), \tag{2.27}$$

differentiating it with respect to z

$$\int_0^x P_{2n+1}\left(\frac{s}{x}\right)s\cos(zs)\frac{ds}{x} = (-1)^n x j_{2n+1}'(zx) = (-1)^n\left(\frac{2n+1}{z}j_{2n+1}(zx) - xj_{2n+2}(zx)\right),$$

and using that $\int_0^x sP_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x} = \delta_{(n,0)}\frac{x}{3}$ we obtain

$$\begin{aligned}\mathcal{T}_n(x,t) =&\frac{\omega}{\pi^2}\left(\pi\int_0^x \min\{s,t\}P_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x} - \frac{2xt}{3}\delta_{(n,0)}\right)\\ &+\sum_{k=1}^{\infty}(-1)^n\left[\left(\frac{\sin(\rho_k t)j_{2n+1}(\rho_k x)}{\alpha_k} - \frac{2\sin(kt)j_{2n+1}(kx)}{\pi}\right)\right.\\ &\left.-\frac{2\omega}{\pi^2 k}\left(\sin(kt)\left(\frac{2n+1}{k}j_{2n+1}(kx) - xj_{2n+2}(kx)\right) + t\cos(kt)j_{2n+1}(kx)\right)\right].\end{aligned}$$

The integral $\int_0^x \min\{s,t\}P_{2n+1}\left(\frac{s}{x}\right)ds$ can be computed as follows. Let $n>0$. Then

$$\int_0^x \min\{s,t\}P_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x} = \int_0^t sP_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x} + t\int_t^x P_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x}.$$

Using the relations

$$P_m\left(\frac{s}{x}\right) = \frac{x}{2m+1}\frac{d}{ds}\left(P_{m+1}\left(\frac{s}{x}\right) - P_{m-1}\left(\frac{s}{x}\right)\right),$$

$$sP_m\left(\frac{s}{x}\right) = x\left(\frac{m+1}{2m+1}P_{m+1}\left(\frac{s}{x}\right) + \frac{m}{2m+1}P_{m-1}\left(\frac{s}{x}\right)\right),$$

we obtain

$$\begin{aligned}
\int_0^t \frac{s}{x}P_{2n+1}\left(\frac{s}{x}\right) ds = & \int_0^x \left(\frac{2n+2}{4n+3}P_{2n+2}\left(\frac{s}{x}\right) + \frac{2n+1}{4n+3}P_{2n}\left(\frac{s}{x}\right)\right) ds \\
=& \frac{2n+2}{4n+3}\left[\frac{x}{4n+5}\left(P_{2n+3}\left(\frac{s}{x}\right) - P_{2n+1}\left(\frac{s}{x}\right)\right)\Big|_0^t\right] \\
&+ \frac{2n+1}{4n+3}\left[\frac{x}{4n+1}\left(P_{2m+1}\left(\frac{s}{x}\right) - P_{2n-1}\left(\frac{s}{x}\right)\right)\Big|_0^t\right] \\
=& x\left[\frac{2n+2}{(4n+3)(4n+5)}P_{2n+3}\left(\frac{t}{x}\right) + \frac{P_{2n+1}\left(\frac{t}{x}\right)}{(4n+1)(4n+5)}\right. \\
&\left. - \frac{2n+1}{(4n+1)(4n+3)}P_{2n-1}\left(\frac{t}{x}\right)\right]
\end{aligned}$$

(here we use the fact that $P_{2m+1}(0) = 0$). On the other hand,

$$\begin{aligned}
t\int_t^x P_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x} =& \frac{t}{4n+3}\left[P_{2n+2}\left(\frac{s}{x}\right) - P_{2n}\left(\frac{s}{x}\right)\right]\Big|_t^x \\
=& \frac{1}{4n+3}\left(tP_{2n}\left(\frac{t}{x}\right) - tP_{2n+2}\left(\frac{t}{x}\right)\right) \\
=& \frac{x}{4n+3}\left\{\left[\frac{2n+1}{4n+1}P_{2n+1}\left(\frac{t}{x}\right) + \frac{2n}{4n+1}P_{2n-1}\left(\frac{t}{x}\right)\right]\right. \\
&\left. - \left[\frac{2n+3}{4n+5}P_{2n+3}\left(\frac{t}{x}\right) + \frac{2n+2}{4n+5}P_{2n+1}\left(\frac{t}{x}\right)\right]\right\} \\
=& x\left[-\frac{2n+3}{(4n+3)(4n+5)}P_{2n+3}\left(\frac{t}{x}\right) + \frac{P_{2n+1}\left(\frac{t}{x}\right)}{(4n+1)(4n+5)}\right. \\
&\left. + \frac{2n}{(4n+1)(4n+3)}P_{2n-1}\left(\frac{t}{x}\right)\right]
\end{aligned}$$

(here we use the equality $P_{2m}(1) = 1$). Thus,

$$\int_0^x \min\{s,t\}P_{2n+1}\left(\frac{s}{x}\right)\frac{ds}{x} = x\left[-\frac{P_{2n+3}\left(\frac{t}{x}\right)}{(4n+3)(4n+5)} + \frac{2P_{2n+1}\left(\frac{t}{x}\right)}{(4n+1)(4n+5)} - \frac{P_{2n-1}\left(\frac{t}{x}\right)}{(4n+1)(4n+3)}\right]. \tag{2.28}$$

Defining $P_{-1} \equiv 0$, one can verify that formula (2.28) is valid for $n = 0$ as well. Hence we obtain the formula

$$\mathcal{T}_n(x,\xi) = -\frac{2\omega x t}{3\pi^2}\delta_{(n,0)} + \frac{\omega x}{\pi}\left[-\frac{P_{2n+3}\left(\frac{t}{x}\right)}{(4n+3)(4n+5)} + \frac{2P_{2n+1}\left(\frac{t}{x}\right)}{(4n+1)(4n+5)} - \frac{P_{2n-1}\left(\frac{t}{x}\right)}{(4n+1)(4n+3)}\right] \tag{2.29}$$

$$+ (-1)^n \sum_{k=1}^{\infty}\left[\left(\frac{\sin(\kappa_k t) j_{2n+1}(\kappa_k x)}{\alpha_k} - \frac{2\sin(kt) j_{2n+1}(kx)}{\pi}\right)\right.$$

$$\left. - \frac{2\omega}{\pi^2 k}\left(\sin(kt)\left(\frac{2n+1}{k} j_{2n+1}(kx) - x j_{2n+2}(kx)\right) + t\cos(kt) j_{2n+1}(kx)\right)\right].$$

Thus, we have

$$
\begin{aligned}
A_{m,n}(x) =& \int_0^x \mathcal{T}_n(x,t) P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x}\\
=& -\frac{2\omega x \delta_{(n,0)}}{3\pi^2}\int_0^x t P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x}\\
&+\frac{\omega x}{\pi}\Bigg[-\frac{\int_0^x P_{2n+3}\left(\frac{t}{x}\right)P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x}}{(4n+3)(4n+5)}\\
&+\frac{2\int_0^x P_{2n+1}\left(\frac{t}{x}\right)P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x}}{(4n+1)(4n+5)}-\frac{\int_0^x P_{2n-1}\left(\frac{t}{x}\right)P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x}}{(4n+1)(4n+3)}\Bigg]\\
&+(-1)^n\sum_{k=1}^{\infty}\Bigg[\Bigg(\frac{j_{2n+1}(\rho_k x)\int_0^x \sin(\rho_k t)P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x}}{\alpha_k}\\
&-\frac{2j_{2n+1}(kx)\int_0^x \sin(kt)P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x}}{\pi}\Bigg)\\
&-\frac{2\omega}{\pi^2 k}\Bigg(\left(\frac{2n+1}{k}j_{2n+1}(kx)-xj_{2n+2}(kx)\right)\int_0^x \sin(k\xi)P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x}\\
&+j_{2n+1}(kx)\int_0^x t\cos(kt)P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x}\Bigg)\Bigg]\\
=& -\frac{2\omega x^2\delta_{(n,0)}\delta_{(m,0)}}{9\pi^2}+\frac{\omega x}{\pi}\Bigg[-\frac{\langle P_{2n+3},P_{2m+1}\rangle_{L_2(0,1)}}{(4n+3)(4n+5)}\\
&+\frac{2\langle P_{2n+1},P_{2m+1}\rangle_{L_2(0,1)}}{(4n+1)(4n+5)}-\frac{\langle P_{2n-1},P_{2m+1}\rangle_{L_2(0,1)}}{(4n+1)(4n+3)}\Bigg]\\
&+(-1)^{m+n}\sum_{k=1}^{\infty}\Bigg[\left(\frac{j_{2n+1}(\rho_k x)j_{2m+2}(\rho_k x)}{\alpha_k}-\frac{2j_{2n+1}(kx)j_{2m+1}(kx)}{\pi}\right)\\
&-\frac{2\omega}{\pi^2 k}\Bigg(\left(\frac{2n+1}{k}j_{2n+1}(kx)-xj_{2n+2}(kx)\right)j_{2m+1}(kx)\\
&+j_{2n+1}(kx)\left(\frac{2m+1}{k}j_{2m+1}(kx)-xj_{2m+2}(kx)\right)\Bigg)\Bigg].
\end{aligned}
$$

Expanding the inner products we obtain formula (2.24) for $A_{m,n}(x)$, $n,m\in\mathbb{N}_0$.

On the right-hand side, $B_m(x) := -\int_0^x F_D(x,t)P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x} = -\mathcal{T}_m(x,x)$, it is

$$B_m(x) = -\frac{\omega}{\pi^2}\left(\pi\int_0^x \min\{x,t\}P_{2n+1}\left(\frac{t}{x}\right)\frac{dt}{x} - \frac{2x^2}{3}\delta_{(m,0)}\right)$$
$$-\sum_{k=1}^{\infty}(-1)^m\left[\left(\frac{\sin(\rho_k x)j_{2n+1}(\rho_k x)}{\alpha_k} - \frac{2\sin(kx)j_{2n+1}(kx)}{\pi}\right)\right.$$
$$\left.-\frac{2\omega}{\pi^2 k}\left(\sin(kx)\left(\frac{2n+1}{k}j_{2n+1}(kx) - xj_{2n+2}(kx)\right) + x\cos(kx)j_{2n+1}(kx)\right)\right].$$

Since $t < x$, then $\int_0^x \min\{x,t\}P_{2m+1}\left(\frac{t}{x}\right)\frac{dt}{x} = \int_0^x tP_{2n+1}\left(\frac{t}{x}\right)\frac{dt}{x} = \frac{x\delta_{(m,0)}}{3}$, from where we obtain (2.25). **Q.E.D.**

We emphasize that to recover the potential q it is enough to compute the coefficient b_0 and use relation (2.19).

For the numerical solution of system (2.23) it is natural to consider its reduced version, i.e.,

$$\frac{b_m(x)}{(4m+3)x} + \sum_{n=0}^{M} A_{m,n}(x)b_n(x) = B_m(x), \qquad \text{for } m = \overline{0, M}. \tag{2.30}$$

The following theorem is a consequence of Remark 7, Theorem 5 and Proposition 6.

Theorem 37. *Fix $x \in (0,\pi]$. For M large enough the truncated system (2.30) has a unique solution $(b_0^{(M)}(x), \cdots, b_m^{(M)}(x))$ and*

$$\sum_{m=0}^{M}\frac{|b_m(x) - b_m^{(M)}(x)|^2}{4m+3} + \sum_{m=M+1}^{\infty}\frac{|b_m(x)|^2}{4m+3} \to 0, \quad M \to \infty. \tag{2.31}$$

In particular, it follows that

$$b_0^{(M)}(x) \to b_0(x), \quad M \to \infty. \tag{2.32}$$

Furthermore, the approximate solution $U_M = \left\{\frac{b_k^{(M)}(x)}{\sqrt{4k+3}}\right\}_{k=0}^{M}$ of the normalized system (2.26) is stable.

2.4 Numerical algorithm and examples

2.4.1 General algorithm

Given a finite set of spectral data $\{\rho_n, \alpha_n\}_{n=0}^{N_1}$ with $\rho_n \in \mathbb{R}$, $\rho_n^2 \neq \rho_m^2$ for $m \neq n$ and $\alpha_n > 0$ for $n = \overline{0, N_1}$, we propose the following method for recovering the potential q, using the normalized version (2.26) of system (2.23).

1. Choose $M \in \mathbb{N}$. For a set of points $\{x_l\}$ from $(0, \pi]$, compute the approximate values of the following functions for $n, m = \overline{0, M}$,

$$\widetilde{B}_m(x) = (-1)^m \sqrt{(4m+1)x} \sum_{k=1}^{N_1} \left[\frac{\sin(\rho_k x) j_{2m+1}(\rho_k x)}{\alpha_k} - \frac{2\sin(kx) j_{2m+1}(kx)}{\pi} \right.$$
$$\left. + \frac{2\omega}{\pi^2 k} \left(\sin(kx) \left(\frac{2m+1}{k} j_{2m+1}(kx) - x j_{2m+2}(kx) \right) + x\cos(kx) j_{2m+1(kx)} \right) \right],$$

$$\widetilde{A}_{m,n} = (-1)^{n+m} \sqrt{(4m+1)(4n+1)} x \sum_{k=1}^{N_1} \left[\left(\frac{2 j_{2n+1}(kx) j_{2m+1}(kx)}{\pi} \right. \right.$$
$$\left. - \frac{j_{2n+1}(\rho_k x) j_{2m+1}(\rho_k x)}{\alpha_k} \right)$$
$$+ \frac{2\omega}{\pi^2 k} \left(\left(\frac{2n+1}{k} j_{2n+1}(kx) - x j_{2n+2}(kx) \right) j_{2m+1}(kx) \right.$$
$$\left. \left. + j_{2n+1}(kx) \left(\frac{2m+1}{k} j_{2m+1}(kx) - x j_{2m+2}(kx) \right) \right) \right],$$

$$\widehat{B}_{m,n} = \begin{cases} \widetilde{B}_{m,n} + \frac{\omega x \sqrt{3x}}{3\pi^2}(\pi - 2x), & \text{if } m = n = 0, \\ \widetilde{B}_{m,n}, & \text{otherwise,} \end{cases}$$

$$\widehat{A}_{m,n} = \begin{cases} \widetilde{A}_{m,n} + \frac{2\omega x^3}{3\pi^2} - \frac{\omega x^2}{\pi} \cdot \frac{2}{5}, & \text{if } m = n = 0, \\ \widetilde{A}_{m,n} + \frac{\omega x^2}{\pi} \cdot \frac{1}{(\sqrt{4n+3})(4n+5)(\sqrt{4n+7})}, & \text{if } m = n+1, \\ \widetilde{A}_{m,n} - \frac{\omega x^2}{\pi} \cdot \frac{2}{(4n+1)(4n+5)}, & \text{if } m = n, m > 0 \\ \widetilde{A}_{m,n} + \frac{\omega x^2}{\pi} \cdot \frac{1}{(\sqrt{4n-1})(4n+1)(\sqrt{4n+3})}, & \text{if } m = n-1, \\ \widetilde{A}_{m,n}, & \text{otherwise.} \end{cases}$$

2. Solve the system

$$\begin{bmatrix} \widehat{C}_{0,0} & \widehat{C}_{0,1} & \cdots & \widehat{C}_{0,M} \\ \widehat{C}_{1,0} & \widehat{C}_{1,1} & \cdots & \widehat{C}_{1,M} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{C}_{M,0} & \widehat{C}_{M,1} & \cdots & \widehat{C}_{M,M} \end{bmatrix} \begin{bmatrix} \widehat{\xi}_0 \\ \widehat{\xi}_1 \\ \vdots \\ \widehat{\xi}_M \end{bmatrix} = \begin{bmatrix} \widehat{B}_0 \\ \widehat{B}_1 \\ \vdots \\ \widehat{B}_M \end{bmatrix},$$

where $\widehat{C}_{m,n} = \widehat{A}_{m,n} - \frac{\delta_{(n,m)}}{4m+3}$.

3. Compute $\widehat{b}_0(x)$ from $\widehat{\xi}_0(x)$ by $b_0(x) = \sqrt{3x}\widehat{\xi}_0(x)$.

4. Compute q in $\{x_l\}$ from $\widehat{b}_0$ using (2.19).

2.4.2 Numerical examples

We illustrate the performance of the algorithm. All the computations were performed in Matlab R2021a. The potential $q(x)$ was recovered using the first coefficient β_0, which was approximated by a spline using the routine `spapi` and differentiated twice using the routine `fnder`.

In many formulas we need the values of the spherical Bessel functions $j_k(t)$ for a list of indices $k = \overline{0, M}$ for the same argument. A considerable speedup is achieved by applying the backwards recursion formula

$$j_{n-1}(t) = \frac{2n+1}{t} j_n(t) - j_{n+1}(t)$$

(see [9, 60]). So only $j_{M-1}(t)$ and $j_M(t)$ have to be computed using the Matlab function `besselj`. Next, we show 3 examples, when the potential q is an smooth function, a continuous $H^1(0,\pi)$-function and a discontinuous function.

Example 38. *Consider the potential* $q(x) = \sin(2x)$, *for* $0 \leqslant x \leqslant \pi$. *We compute* 200 *"exact" spectral data* $\{\rho_n, \alpha_n\}_{n=1}^{200}$ *applying the method proposed in [81]. We take* $M = 4$ *and the number of spectral data* $N_s \in \{11, 50, 100, 200\}$. *In the right side of Figure 4.1 whe see the graphs of the absolute error* $|q - q_{N_s}|$. *On the left, the graphs of* q *and* q_{N_s} *with* $N_s = 200$. *For* $N_s = 200$, *we obtain that* $\|q - q_{N_s}\|_{L_1(0,\pi)} = 0.00351$.

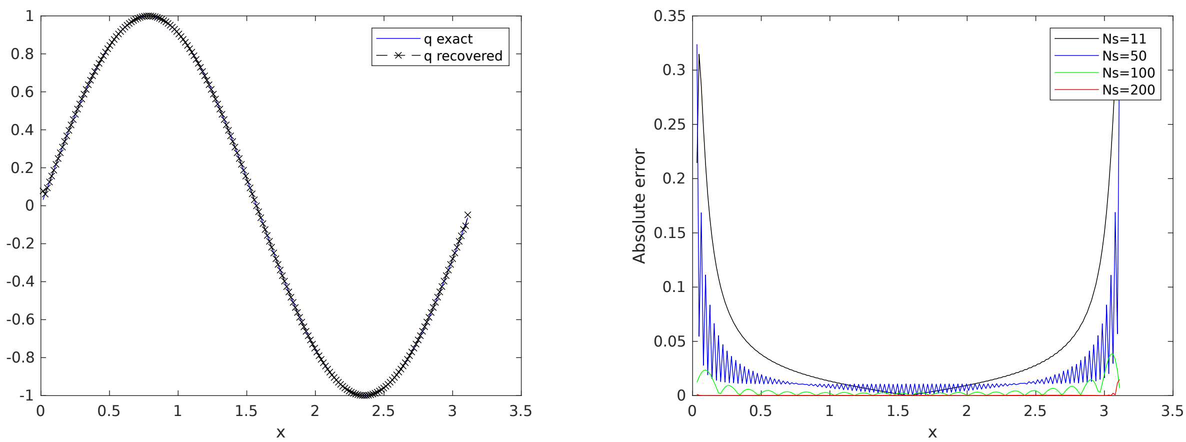

Figure 2.1: On the left, the graphs of the exact potential $q(x) = \sin(2x)$ and the recovered potential for $N_s = 200$. On the right, the graphs of the absolute error $|q_{N_s} - q|$ for $N_s = 11, 50, 100, 200$.

Example 39. *In this case, we take the "saw-tooth" potential*

$$q(x) = \int_0^x \operatorname{sgn}\left(\sin\left(\frac{10t}{4-t}\right)\right) dt. \tag{2.33}$$

In this case $q \in H^1(0,\pi)$. Again, we take $M = 4$, but in this case we complete the spectral data with an additional sequence of N_A "asymptotic spectral data" $\{n + \frac{\omega}{\pi n}, \frac{\pi}{2}\}_{n=N_s+1}^{N_A+N_s}$ to recover the potential q_{N_A}. In the left side of Figure 2.2 we see the potential q and the recovered potential q_{N_A} with $N_s = 20$ exact data and $N_A = 500$ asymptotic spectral data. On the right side we see the absolute error $|q - q_{N_A}|$. In this case we obtain $\|q - q_{N_A}\|_{L_1(0,\pi)} = 0.095897$. In the left side of Figure 2.3 we see the potential q and the recovered potential q_{N_A} with $N_s = 200$ exact data and $N_A = 1000$ asymptotic spectral data. On the right side we see the absolute error $|q - q_{N_A}|$. In this case $\|q - q_{N_A}\|_{L_1(0,\pi)} = 0.064168$.

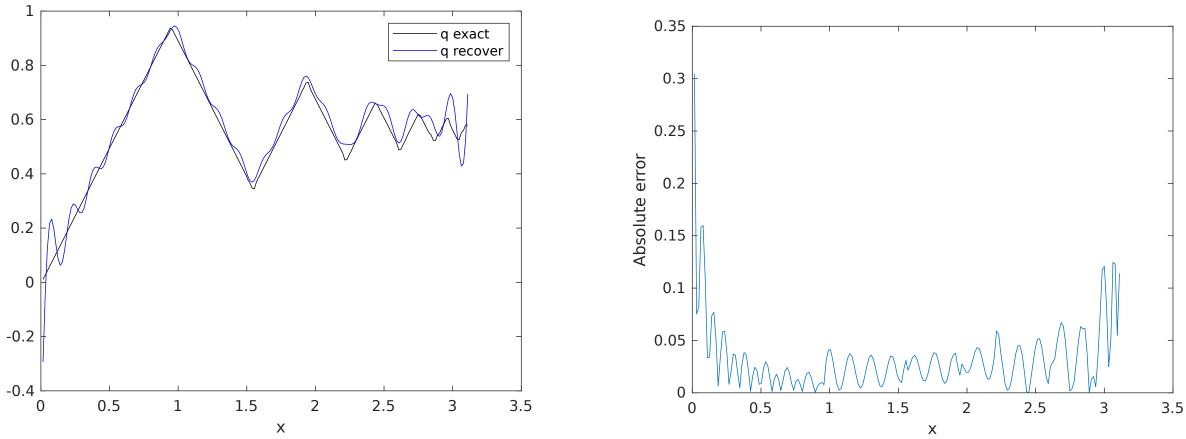

Figure 2.2: On the left, the graphs of the exact potential (2.33) and the recovered potential for $N_s = 20$ and $N_A = 500$. On the right, the graph of the absolute error $|q_{N_A} - q|$.

Example 40. *Finally, we consider the discontinuous potential*

$$q(x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{\pi}{8}\right] \cup \left[\frac{3\pi}{8}, \frac{3\pi}{5}\right], \\ -\frac{12x}{\pi} + \frac{3}{2}, & \text{if } x \in \left(\frac{\pi}{8}, \frac{\pi}{4}\right], \\ \frac{12x}{\pi} - \frac{9}{2}, & \text{if } x \in \left(\frac{\pi}{4}, \frac{3\pi}{8}\right), \\ 4, & \text{if } x \in \left(\frac{3\pi}{5}, \frac{4\pi}{5}\right), \\ 2, & \text{if } x \in \left[\frac{4\pi}{5}, \pi\right]. \end{cases} \tag{2.34}$$

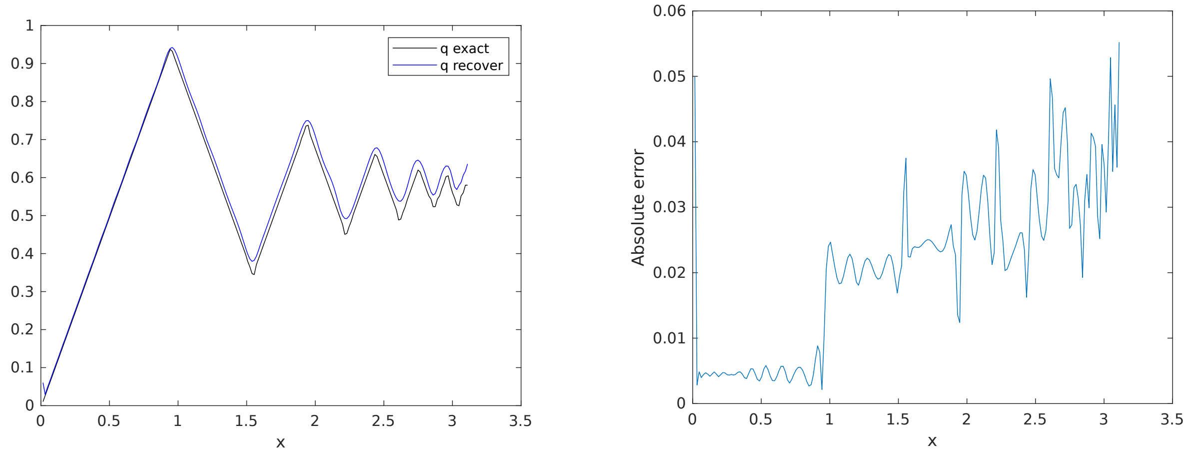

Figure 2.3: On the left, the graphs of the exact potential (2.33) and the recovered potential for $N_s = 200$ and $N_A = 1000$. On the right, the graph of the absolute error $|q_{N_A} - q|$.

We take $M = 4$. In the left side of Figure 2.4, the potential q and the recovered potential q_A, with $N_s = 40$ exact spectral data and $N_A = 500$ asymptotic spectral data. On the right side, the absolute error $|q - q_{N_A}|$. In this case $\|q - q_{N_A}\|_{L_1(0,\pi)} = 0.31111$. In the left side of Figure 2.5, the potential q and the recovered potential q_A, with $N_s = 200$ exact spectral data and $N_A = 1000$ asymptotic data. On the right side, the absolute error $|q - q_{N_A}|$. In this case $\|q - q_{N_A}\|_{L_1(0,\pi)} = 0.27707$.

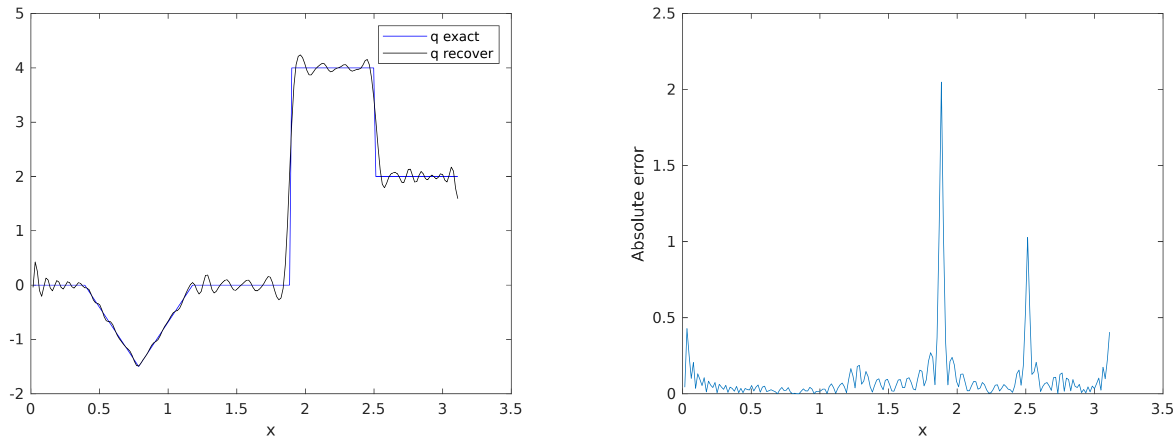

Figure 2.4: On the left, the graphs of the exact potential (2.34) and the recovered potential for $N_s = 40$ and $N_A = 500$. On the right, the graph of the absolute error $|q_{N_A} - q|$.

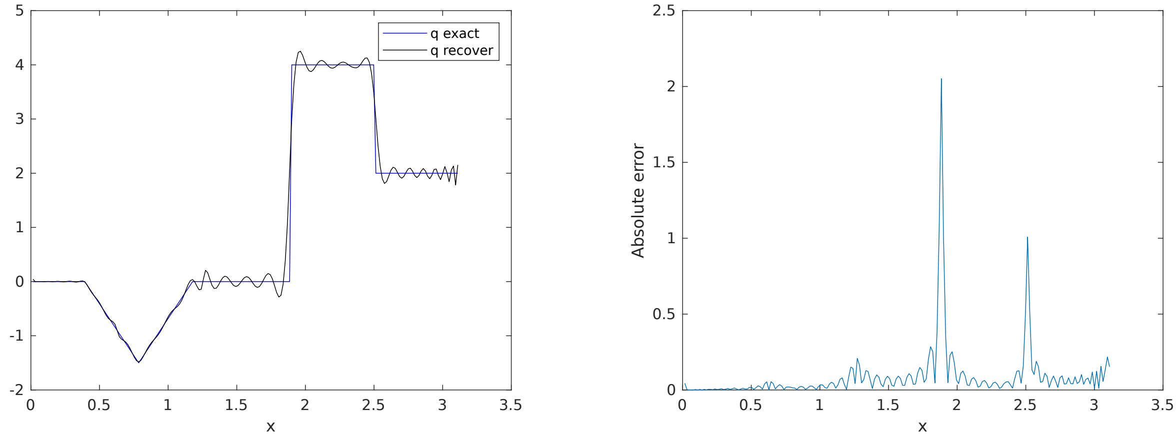

Figure 2.5: On the left, the graphs of the exact potential (2.34) and the recovered potential for $N_s = 200$ and $N_A = 1000$. On the right, the graph of the absolute error $|q_{N_A} - q|$.

Chapter 3

Construction and analytical properties of transmutation operators for the SLEIF

The aim of this chapter is to develop a transmutation operator theory (analogously to the case of the Schrödinger equation) for the SLEIF (1.46). The construction of integro-differential transmutation operators is presented. Their analytical properties of boundedness and invertiblity in appropriate functional spaces are studied. A Fourier-Legendre series for the integral transmutation kernel in terms of the Legendre polynomials is obtained together with a representation of the solutions of (1.46) as NSBF.

3.1 Integral representations for solutions and transmutation kernels

3.1.1 Properties of the Sturm-Liouville equation in impedance form

Consider Eq. (1.46) with the impedance function

$$-\frac{d}{dx}\left(a^2(x)\frac{du}{dx}\right) = \rho^2 a^2(x), \quad -b < x < b \tag{3.1}$$

where $\rho \in \mathbb{C}$ and $b > 0$. Suppose that $a \in W^{1,\infty}(-b, b)$. Due to (1.50) $q(x) = -2\frac{a'(x)}{a(x)} \in L_\infty(-b, b)$ and eq. (3.1) can be written as

$$-u'' + q(x)u = \rho^2 u, \quad -b < x < b. \tag{3.2}$$

Without loss of generality we assume that $a(0) = 1$. Thus, choosing $x_0 = 0$ in (1.50) we obtain $a(x) := e^{-\frac{1}{2}\int_0^x q(s)ds}$, The dependence of q on a is denoted by q_a. Note that when q is real valued, $a > 0$ in $[-b, b]$. We denote the differential operator

$$\mathbf{L}_a := \frac{1}{a^2(x)}\mathbf{D}a^2(x)\mathbf{D} = \mathbf{D}^2 - q(x)\mathbf{D} \quad \text{with} \ \ \mathbf{D} = \frac{d}{dx}.$$

3.1.2 Integral representations and transmutation kernels

We look for a fundamental set of solutions $\{C(\rho, x), S_1(\rho, x)\}$ for (3.1), satisfying the initial conditions

$$C(\rho, 0) = S_1'(\rho, 0) = 1, \quad C'(\rho, 0) = S_1(\rho, 0) = 0. \tag{3.3}$$

By Lagrange identity (1.51), $W_p[C, S_1](x) = W_p[C, S_1](0) = 1$. The solutions $C(\rho, x)$, $S_1(\rho, x)$ must satisfy the integral equations

$$C(\rho, x) = \cos(\rho x) + \int_0^x \frac{\sin(\rho(x-t))}{\rho} q(t)C'(\rho, t)dt, \tag{3.4}$$

$$S_1(\rho, x) = \frac{\sin(\rho x)}{\rho} + \int_0^x \frac{\sin(\rho(x-t))}{\rho} q(t)S_1'(\rho, t)dt. \tag{3.5}$$

Let $x \in [0, b]$. The solution $C(\rho, x)$ is an entire function of exponential type satisfying the estimate

$$|C(\rho, x) - \cos(\rho x)| \leqslant C\frac{|\rho|x}{1+|\rho|x} e^{|\operatorname{Im}\rho|x} e^{\int_0^x |q(s)|ds} \quad \forall \rho \in \mathbb{C}, \tag{3.6}$$

where $C > 0$ is a constant independent of ρ and x (see [26, Th. 2.2]). Similar properties of the solution $S_1(\rho, x)$ are established by solving the integral equation (3.5) by the method of successive approximations, choosing $s_0(\rho, x) = \frac{\sin(\rho x)}{\rho}$ and $s_n(\rho, x) = \int_0^x \frac{\sin(\rho(x-t))}{\rho} q(t)s_{n-1}'(\rho, t)dt$ for $n \geqslant 1$. Then $S_1(\rho, x) = \sum_{n=0}^\infty s_n(\rho, x)$. In order to show the convergence of this series we need the following inequality

$$|\sin(\rho x)| \leqslant c\frac{|\rho|x e^{x|\operatorname{Im}\rho|}}{1+|\rho|x} \quad \forall \rho \in \mathbb{C}, \ x \geqslant 0 \tag{3.7}$$

with $c = \max\left\{\max_{z\in\mathbb{D}}(1+|z|)e^{-|\operatorname{Im} z|}\left|\frac{\sin(z)}{z}\right|, 2\right\}$.

Proposition 41. *Let $q \in L_1(0,b)$. Then the solution $S_1(\rho,x)$ of the integral equation (3.5) is an entire function of the spectral parameter ρ, and*

$$|S_1(\rho,x)| \leqslant \frac{cx}{1+|\rho|x} e^{x|\operatorname{Im}\rho|} \exp\left(c\int_0^x |q(s)|ds\right) \tag{3.8}$$

for all $\rho \in \mathbb{C}$, $x \in [0,b]$. For $q \in L_\infty(0,b)$, the function $S_1(\rho,x)$ is a solution of (3.2) belonging to $H^2(0,b)$.

Proof. Let us see by induction in $n \geqslant 1$ that the following estimates hold for $\rho \in \mathbb{C}$, $x \geqslant 0$

$$|s_n(\rho,x)| \leqslant \frac{c^{n+1} x e^{x|\operatorname{Im}\rho|}}{1+|\rho|x} \frac{\tilde{Q}(x)}{n!}, \tag{3.9}$$

$$|s_n'(\rho,x)| \leqslant c^{n+1} e^{x|\operatorname{Im}\rho|} \frac{\tilde{Q}(x)}{n!}, \tag{3.10}$$

where $\tilde{Q}(x) = \int_0^x |q(s)|ds$.

For $n=1$,

$$\begin{aligned}
|s_1(\rho,x)| &\leqslant \int_0^x \left|\frac{\sin(\rho(x-s))}{\rho}\right| |q(s)|\, |\cos(\rho x)|\, ds \\
\text{(By (5.9))} &\leqslant \frac{ce^{x|\operatorname{Im}\rho|}}{|\rho|} \int_0^x \frac{|\rho|(x-s)e^{-s|\operatorname{Im}\rho|}}{1+|\rho|(x-s)} |q(s)||\cos(\rho s)| ds.
\end{aligned}$$

Note that $|\cos(z)| = \left|\sin\left(z+\frac{\pi}{2}\right)\right| \leqslant ce^{|\operatorname{Im} z|} \frac{\left|z+\frac{\pi}{2}\right|}{1+\left|z+\frac{\pi}{2}\right|} \leqslant ce^{|\operatorname{Im} z|}$, hence

$$|s_1(\rho,x)| \leqslant c^2 e^{x|\operatorname{Im}\rho|} \int_0^x \frac{x}{1+|\rho|x} |q(s)| ds = \frac{c^2 x e^{x|\operatorname{Im}\rho|}}{1+|\rho|x} \tilde{Q}(x),$$

and

$$\begin{aligned}
|s_1'(\rho,x)| &\leqslant \int_0^x |\cos(\rho(x-s))||q(s)||\cos(\rho s)| ds \\
&\leqslant c^2 \int_0^x e^{(x-s)|\operatorname{Im}\rho|} |q(s)| e^{s\operatorname{Im}\rho} ds = c^2 e^{x|\operatorname{Im}\rho|} \tilde{Q}(x).
\end{aligned}$$

Thus (3.9) and (3.10) are valid for $n=1$. Assuming their validity for n we establish it for $n+1$:

$$\begin{aligned}
|s_{n+1}(\rho,x)| &\leqslant \int_0^x \frac{1}{|\rho|} \frac{c|\rho|(x-s)e^{(x-s)|\operatorname{Im}\rho|}}{1+|\rho|(x-s)} |q(s)||s_n'(\rho,s)| ds \\
&\leqslant c^{n+1} \frac{e^{x|\rho|}}{n!} \int_0^x \frac{x}{1+|\rho|x} |q(s)| \tilde{Q}^n(s) ds \\
&= c^{n+1} \frac{xe^{|\operatorname{Im}\rho|}}{1+|\rho|x} \frac{\tilde{Q}^{n+1}(x)}{(n+1)!},
\end{aligned}$$

and

$$\begin{aligned}|s'_{n+1}(\rho,x)| &\leqslant \int_0^x |\cos(\rho(x-s))||q(s)||S'_n(\rho,s)|ds\\ &\leqslant c^{n+1}\int_0^x e^{(x-s)|\operatorname{Im}\rho|}|q(s)|e^{s|\operatorname{Im}\rho|}\frac{\tilde{Q}^n(s)}{n!}ds\\ &= c^{n+1}e^{x|\operatorname{Im}\rho|}\frac{\tilde{Q}^{n+1}(x)}{(n+1)!}.\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}|s_n(\rho,x)| \leqslant \frac{cxe^{x|\operatorname{Im}\rho|}}{1+|\rho|x}\sum_{n=0}^{\infty}\frac{c^{n+1}\tilde{Q}(x)}{n!} = \frac{cxe^{x|\operatorname{Im}\rho|}}{1+|\rho|x}e^{c\tilde{Q}(x)},$$

and the series $S_1(\rho,x) = \sum_{n=0}^{\infty} s_n(\rho,x)$ converges absolutely and uniformly on $[0,b]$ and for ρ such that $|\operatorname{Im}\rho| \leqslant M$, $M > 0$. Similarly

$$\sum_{n=1}^{\infty}|s'_n(\rho,x)| \leqslant ce^{x|\operatorname{Im}\rho|}e^{c\tilde{Q}(x)},$$

and the series of the derivative converges absolutely and uniformly. Thus the solution $S_1(\rho,x)$ of the integral equation exists, $S_1(\rho,\cdot) \in AC[0,b]$, $S_1(\cdot,x) \in Hol(\mathbb{C})$ and satisfies (3.8). Furthermore, one can verify that $S_1(\rho,x)$ satisfies (3.2) a.e. in $(0,b)$. Thus, $S_1, S'_1 \in AC[0,b]$ and if $q \in L_\infty(0,b)$, then $S''_1 = q(x)S'_1 - \rho^2 S_1 \in L_2(0,b)$. **Q.E.D.**

An important corollary of (3.8) is the existence of the transmutation operator. Indeed, define $\varphi(\rho,x) := \frac{\sin(\rho x)}{\rho} - S_1(\rho,x)$. For $x \in (0,b]$ fixed, $\varphi(\cdot,x) \in Hol(\mathbb{C})$ and $|\varphi(\rho,x)| \leqslant \frac{cx}{1+|\rho|x}e^{c\|q\|_{L_1(0,b)}}e^{x|\operatorname{Im}\rho|}$, i.e., $\varphi(\cdot,x)$ is an entire function of exponential type, and for $\rho \in \mathbb{R}$ we have

$$\int_{\mathbb{R}}|\varphi(\rho,x)|^2 d\rho \leqslant cxe^{c\|q\|_{L_1(0,b)}}\int_{\mathbb{R}}\frac{d\rho}{(1+|\rho|x)^2} < \infty.$$

Then the Paley-Wiener theorem (see, e.g., [121, Th. 19.3]) implies that $\varphi(\rho,x) = \int_{-x}^{x} K_1(x,t)e^{i\rho t}dt$, for some function $K_1(x,t)$ satisfying $K_1(x,\cdot) \in L_2(-x,x)$. Thus,

$$S_1(\rho,x) = \frac{\sin(\rho x)}{\rho} - \int_{-x}^{x} K_1(x,t)e^{i\rho t}dt. \tag{3.11}$$

Note that for ρ fixed, $S_1(-\rho,x)$ satisfies (3.2) and (3.3). By uniqueness of the solution $S_1(\rho,x) = S_1(-\rho,x)$ and substituting in (3.11) we have

$$S_1(\rho,x) = \frac{S_1(\rho,x)+S_1(-\rho,x)}{2} = \frac{\sin(\rho x)}{\rho} - \int_{-x}^{x} K_1(x,t)\left(\frac{e^{i\rho t}+e^{-i\rho t}}{2}\right)dt,$$

from which we obtain the integral representation

$$S_1(\rho, x) = \frac{\sin(\rho x)}{\rho} - \int_{-x}^{x} K_1(x,t)\cos(\rho t)dt. \tag{3.12}$$

We deduce a formal differential equation for the kernel K_1. Deriving $S_1(\rho, x)$ with respect to x we obtain

$$S_1'(\rho, x) = \cos(\rho x) - K_1(x,x)\cos(\rho x) - K_1(x,-x)\cos(\rho x) - \int_{-x}^{x} \frac{\partial K_1(x,t)}{\partial x}\cos(\rho t)dt,$$

$$\begin{aligned} S_1''(\rho, x) = &-\rho\sin(\rho x) - \frac{d}{dx}K_1(x,x)\cos(\rho x) + K_1(x,x)\rho\cos(\rho x) - \frac{d}{dx}K_1(x,-x)\cos(\rho x) \\ &+ K_1(x,-x)\rho\sin(\rho x) - \frac{\partial K_1(x,x)}{\partial x}\cos(\rho x) - \frac{\partial K_1(x,-x)}{\partial x}\cos(\rho x) \\ &- \int_{-x}^{x} \frac{\partial^2 K_1(x,t)}{\partial x^2}\cos(\rho t)dt. \end{aligned}$$

From which we have

$$\begin{aligned} -S_1''(\rho, x) + q(x)S_1'(\rho, x) = \rho\sin(\rho x) &+ \int_{-x}^{x}\left(\frac{\partial^2}{\partial x^2} - q(x)\frac{\partial}{\partial x}\right)K_1(x,t)\cos(\rho t)dt \\ &+ \left\{\frac{\partial K_1(x,x)}{\partial x} + \frac{d}{dx}K_1(x,x) - q(x)\left[K_1(x,x) - 1\right]\right\}\cos(\rho x) \\ &+ \left\{\frac{\partial K_1(x,-x)}{\partial x} + \frac{d}{dx}K_1(x,-x) - q(x)K_1(x,-x)\right\}\cos(\rho x) \\ &- \left(K_1(x,x) + K_1(x,-x)\right)\rho\sin(\rho x) \end{aligned}$$

Integration by parts on the left hand side of (3.12) leads to the equality

$$\begin{aligned} S_1(\rho, x) = &\frac{\sin(\rho x)}{\rho} - \left(K_1(x,x) + K_1(x,-x)\right)\rho\sin(\rho x) \\ &+ \left(-\frac{\partial K_1(x,x)}{\partial t} + \frac{\partial K_1(x,-x)}{\partial t}\right)\frac{\cos(\rho x)}{\rho} + \int_{-x}^{x}\frac{\partial^2 K_1(x,t)}{\partial t^2}\cos(\rho t)dt. \end{aligned}$$

Substituting both equalities in (3.1), and using that $\frac{d}{dx}K(x,\pm x) = \frac{\partial K(x,\pm x)}{\partial x} \pm \frac{\partial K(x,\pm x)}{\partial t}$ we obtain

$$\begin{aligned} &\left\{2\frac{d}{dx}K_1(x,x) - q(x)\left[K_1(x,x) - 1\right]\right\}\cos(\rho x) + \left\{2\frac{d}{dx}K_1(x,-x) - q(x)K_1(x,-x)\right\}\cos(\rho x) \\ &\qquad + \int_{-x}^{x}\left(\frac{\partial^2}{\partial x^2} - q(x)\frac{\partial}{\partial x} - \frac{\partial^2}{\partial t^2}\right)K_1(x,t)\cos(\rho t)dt = 0. \end{aligned}$$

Hence the kernel K must satisfy (at least formally) the hyperbolic equation

$$\frac{\partial^2 K_1(x,t)}{\partial x^2} - q(x)\frac{\partial K_1(x,t)}{\partial x} = \frac{\partial^2 K_1(x,t)}{\partial t^2}, \quad |t| < x < b. \tag{3.13}$$

with the Goursat conditions

$$\frac{d}{dx}K_1(x,x) = \frac{1}{2}q(x)\left[K_1(x,x)-1\right], \quad \frac{d}{dx}K_1(x,-x) = \frac{1}{2}q(x)K_1(x,-x), \tag{3.14}$$

so that

$$K_1(x,x) = 1 + Ae^{\frac{1}{2}\int_0^x q(s)ds}, \quad K_1(x,-x) = Be^{\frac{1}{2}\int_0^x q(s)ds} \tag{3.15}$$

for some constants $A, B \in \mathbb{C}$. For the continuity of the solution the equality $B = 1 + A$ is necessary. On the other hand, by the initial condition $S_1'(\rho, 0) = 0$, we have $K_1(0,0) = 0$ and hence

$$K_1(x,x) = 1 - e^{\frac{1}{2}\int_0^x q(s)ds}, \quad K_1(x,-x) = 0. \tag{3.16}$$

Considering the function $\psi(\rho,x) = i\frac{C(\rho,x)-\cos(\rho x)}{\rho}$, due to (3.6), the Paley-Wiener theorem and the oddity of ψ in ρ, we obtain that $C(\rho,x)$ admits the representation

$$C(\rho,x) = \cos(\rho x) + \int_{-x}^{x} K_2(x,t)\rho\sin(\rho t)dt, \tag{3.17}$$

where $K_2(x,\cdot) \in L_2(-x,x)$ for all $x \in (0,b]$. Repeating the same procedure as for the solution $S_1(\rho,x)$ we obtain that $K_2(x,t)$ satisfies equation (3.13) with the Goursat conditions (3.16). If the solution of this Goursat problem exists, it must be unique, and $K_1(x,t) = K_2(x,t)$. We denote this kernel by $K(x,t)$. Thus,

$$C(\rho,x) = \mathbf{T}\left[\cos(\rho x)\right], \quad S_1(\rho,x) = \mathbf{T}\left[\frac{\sin(\rho x)}{\rho}\right], \tag{3.18}$$

where

$$\mathbf{T}u(x) = u(x) + \int_{-x}^{x} K(x,t)u'(t)dt. \tag{3.19}$$

We call $K(x,t)$ the *transmutation kernel.* Its crucial feature is its independence of the spectral parameter ρ.

3.1.3 Existence of the transmutation kernel

Let $x \in [-b,b]$. We seek to extend representations (3.18) to this interval. For this the kernel has to be defined in the domain $\mathcal{R} = \{(x,t) \in \mathbb{R}^2 \,|\, |x| \leqslant b, |t| \leqslant |x|\}$. However, for convenience we consider the Goursat problem (3.13), (3.15) in the whole rectangle $\overline{\Omega} = [-b,b] \times [-b,b]$. The change of the variables $\xi = \frac{x+t}{2}$, $\zeta = \frac{x-t}{2}$ transforms (3.13) to the equation

$$\frac{\partial^2 H(\xi,\zeta)}{\partial\zeta\partial\xi} = \frac{1}{2}q(\xi+\zeta)\left(\frac{\partial H(\xi,\zeta)}{\partial\xi} + \frac{\partial H(\xi,\zeta)}{\partial\zeta}\right), \tag{3.20}$$

considered in the closed rectangle $\mathcal{Q}$ with vertices $(\pm b, 0)$ and $(0, \pm b)$ (see Figure 3.1). Here $H(\xi,\zeta) = K(\xi+\zeta, \xi-\zeta)$.

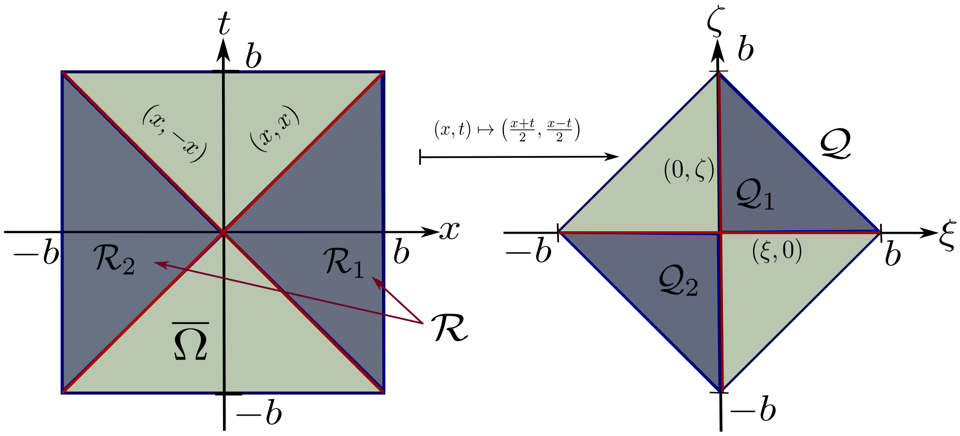

Figure 3.1: Domains $\mathcal{R}$ and $\mathcal{Q}$.

The Goursat conditions take the form

$$H(\xi,0) = 1 + Ae^{\frac{1}{2}\int_0^\xi q(s)ds}, \quad H(\zeta,0) = Be^{\frac{1}{2}\int_0^\zeta q(s)ds} \quad \text{with } B = 1 + A. \tag{3.21}$$

The problem (3.20),(3.21) can be written as the integral equation

$$H(\xi,\zeta) = Ae^{\frac{1}{2}\int_0^\xi q(s)ds} + Be^{\frac{1}{2}\int_0^\zeta q(s)ds} - A + \frac{1}{2}\int_0^\xi \int_0^\zeta q(\alpha+\beta)\left(\frac{\partial H(\alpha,\beta)}{\partial \alpha} + \frac{\partial H(\alpha,\beta)}{\partial \beta}\right) d\beta d\alpha. \tag{3.22}$$

Let us see that (3.22) has a unique solution $H \in C^1(\mathcal{Q})$ for $q \in C[-b,b]$. First, we endow $C^1(\mathcal{Q})$ with a Banach space structure as follows. For $u \in C^1(\mathcal{Q})$ we define

$$\mathfrak{D}u(\xi,\zeta) = u_\xi(\xi,\zeta) + u_\zeta(\xi,\zeta), \quad \|\mathfrak{D}u\| := \max_{(\xi,\zeta)\in\mathcal{Q}} |u_\xi(\xi,\zeta)| + \max_{(\xi,\zeta)\in\mathcal{Q}} |u_\zeta(\xi,\zeta)|$$

and $\|u\|_{C^1(\mathcal{Q})} := \|u\|_{C(\mathcal{Q})} + \|\mathfrak{D}u\|$. With this norm $C^1(\mathcal{Q})$ is a Banach space and a sequence $\{u_k\}$ converges to u in $C^1(\mathcal{Q})$ iff $u_k \overset{\mathcal{Q}}{\rightrightarrows} u$, $\partial_\xi u_k, \partial_\zeta u_k \overset{\mathcal{Q}}{\rightrightarrows} \partial_\xi u, \partial_\zeta u$, $k \to \infty$.

Consider the operator $\mathbf{A} : C^1(\mathcal{Q}) \to C^1(\mathcal{Q})$ defined by

$$\mathbf{A}u(\xi,\zeta) := \frac{1}{2}\int_0^\xi \int_0^\zeta q(\alpha+\beta)\mathfrak{D}u(\alpha,\beta) d\beta d\alpha. \tag{3.23}$$

For $u \in C^1(\mathcal{Q})$ and $(\xi,\zeta) \in \mathcal{Q}$ we have

$$\begin{aligned} |\mathbf{A}u(\xi,\zeta)| &= \frac{1}{2}\left|\int_0^{|\xi|}\int_0^{|\zeta|} q\left(\operatorname{sgn}(\xi)\alpha + \operatorname{sgn}(\zeta)\beta\right)\mathfrak{D}u\left(\operatorname{sgn}(\xi)\alpha, \operatorname{sgn}(\zeta)\beta\right) d\beta d\alpha\right| \\ &\leqslant \frac{1}{2}\|q\|_{C[-b,b]}\int_0^{|\xi|}\int_0^{|\zeta|} \left|\mathfrak{D}u\left(\operatorname{sgn}(\xi)\alpha, \operatorname{sgn}(\zeta)\beta\right)\right| d\beta d\alpha, \end{aligned} \tag{3.24}$$

$$\begin{aligned} |\mathfrak{D}\mathbf{A}u(\xi,\zeta)| &\leqslant \frac{1}{2}\left(\left|\int_0^{|\zeta|} q\left(\xi + \operatorname{sgn}(\zeta)\beta\right)\mathfrak{D}u\left(\xi, \operatorname{sgn}(\zeta)\beta\right) d\beta\right|\right. \\ &\quad \left. + \left|\int_0^{|\xi|} q\left(\operatorname{sgn}(\xi)\alpha + \zeta\right)\mathfrak{D}u\left(\operatorname{sgn}(\xi)\alpha, \zeta\right) d\alpha\right|\right) \\ &\leqslant \frac{1}{2}\|q\|_{C[-b,b]}\left(\int_0^{|\zeta|}\left|\mathfrak{D}u\left(\xi, \operatorname{sgn}(\zeta)\beta\right)\right| d\beta + \int_0^{|\xi|}\left|\mathfrak{D}u\left(\operatorname{sgn}(\xi)\alpha, \zeta\right)\right| d\alpha\right). \end{aligned} \tag{3.25}$$

Inequalities (3.24), (3.25) and $|\xi|+|\zeta| \leqslant b$ lead us to $\|\mathbf{A}u\|_{C^1(\mathcal{Q})} \leqslant \left(\frac{b^2}{2}+b\right)\|q\|_{C[-b,b]}\|u\|_{C^1(\mathcal{Q})}$. Thus, $\mathbf{A} \in \mathcal{B}\left(C^1(\mathcal{Q})\right)$. Note that equation (3.22) can be written as $\left(\mathbf{I}-\mathbf{A}\right)u = \phi$ with $\phi(\xi,\zeta) = Ae^{\frac{1}{2}\int_0^{\xi} q(s)ds} + Be^{\frac{1}{2}\int_0^{\zeta} q(s)ds} - A$. With the aid of the method of successive approximations, by taking $\phi_0 := \phi$ and $\phi_{k+1} := \mathbf{A}\phi_k$ we prove that this equation is uniquely solvable for any $\phi \in C^1(\mathcal{Q})$.

Lemma 42. *Given $\phi \in C^1(\mathcal{Q})$, the functions $\{\phi_k\}_{k=0}^{\infty}$ of the method of successive approximations satisfy the following estimates for $k \geqslant 1$*

1. $|\phi_k(\xi,\zeta)| \leqslant \frac{1}{2}\|q\|^k_{C[-b,b]}\|\mathfrak{D}\phi_0\|\frac{(|\xi|+|\zeta|)^{k+1}}{(k+1)!} \quad \forall(\xi,\zeta) \in \mathcal{Q}$,
2. $|\mathfrak{D}\phi_k(\xi,\zeta)| \leqslant \frac{1}{2}\|q\|^k_{C[-b,b]}\|\mathfrak{D}\phi_0\|\frac{(|\xi|+|\zeta|)^{k}}{k!} \quad \forall(\xi,\zeta) \in \mathcal{Q}$.

Proof. For $k=1$, we use (3.24) to obtain

$$|\phi_1(\xi,\zeta)| \leqslant \frac{1}{2}\|q\|_{C[-b,b]}\int_0^{|\xi|}\int_0^{|\zeta|}|\mathfrak{D}\phi_0(\operatorname{sgn}(\xi)\alpha, \operatorname{sgn}(\zeta)\beta)|d\beta d\alpha \leqslant \frac{1}{2}\|q\|_{C[-b,b]}\|\mathfrak{D}\phi_0\||\xi||\zeta|,$$

and since $|\xi||\zeta| \leqslant \frac{(|\xi|+|\zeta|)^2}{2}$, we obtain $|\phi_0| \leqslant \|q\|_{C[-b,b]}\|\mathfrak{D}\phi_0\|\frac{(|\xi|+|\zeta|)^2}{2}$. For the derivatives, inequality (3.25) leads us to

$$\begin{aligned} |\mathfrak{D}\phi_1(\xi,\zeta)| &\leqslant \frac{1}{2}\|q\|_{C[-b,b]}\left[\int_0^{|\zeta|}|\mathfrak{D}\phi_0(\xi, \operatorname{sgn}(\zeta)\beta)|d\beta + \int_0^{|\xi|}|\mathfrak{D}\phi_0(\operatorname{sgn}(\xi)\alpha, \zeta)|d\alpha\right] \\ &\leqslant \frac{1}{2}\|q\|_{C[-b,b]}\|\mathfrak{D}\phi_0\|(|\xi|+|\zeta|). \end{aligned}$$

This establishes the inequalities for $k = 1$. Now suppose the estimates hold for $k \in \mathbb{N}$. Let us check their validity for $k+1$. First note that

$$\int_0^{|\xi|} (\alpha + |\zeta|)^k d\alpha \leqslant \frac{(|\xi| + |\zeta|)^{k+1}}{k+1} \quad \text{and} \quad \int_0^{|\xi|} \int_0^{|\zeta|} (\alpha + \beta)^k d\beta d\alpha \leqslant \frac{(|\xi| + |\zeta|)^{k+2}}{(k+1)(k+2)} \tag{3.26}$$

and

$$\begin{aligned} |\phi_{k+1}| &\leqslant \frac{1}{2}\|q\|_{C[-b,b]} \int_0^{|\xi|} \int_0^{|\zeta|} |\mathfrak{D}\phi_k(\xi\, \alpha, \zeta\, \beta)| d\beta d\alpha \\ &\leqslant \frac{1}{2}\|q\|^{k+1}_{C[-b,b]} \|\mathfrak{D}\phi_0\| \int_0^{|\xi|} \int_0^{|\zeta|} \frac{(|\alpha| + |\beta|)^k}{k!} d\beta d\alpha \\ \text{(by (3.26))} &\leqslant \frac{1}{2}\|q\|^{k+1}_{C[-b,b]} \|\mathfrak{D}\phi_0\| \frac{(|\alpha| + |\beta|)^{k+2}}{(k+2)!} \end{aligned}$$

For the derivatives we have

$$|\mathfrak{D}\phi_{k+1}| \leqslant \frac{1}{2}\|q\|_{C[-b,b]} \left[\int_0^{|\zeta|} |\mathfrak{D}\phi_k(\xi, \operatorname{sgn}(\zeta)\beta)| d\beta + \int_0^{|\xi|} |\mathfrak{D}\phi_k(\operatorname{sgn}(\xi)\alpha, \zeta)| d\alpha \right].$$

By the induction hypothesis we obtain

$$\begin{aligned} |\mathfrak{D}\phi_{k+1}| &\leqslant \frac{1}{4}\|q\|^{k+1}_{C[-b,b]} \|\mathfrak{D}\phi_0\| \left[\int_0^{\zeta} \frac{(|\xi| + |\beta|)^k}{k!} d\beta + \int_0^{|\xi|} \frac{(|\alpha| + |\zeta|)^k}{k!} d\alpha \right] \\ \text{(by (3.26))} &\leqslant \frac{1}{2}\|q\|^{k+1}_{C[-b,b]} \frac{(|\xi| + |\zeta|)^{k+1}}{(k+1)!} \end{aligned}$$

This concludes the induction and proves the estimates for all $k \in \mathbb{N}$. **Q.E.D.**

Theorem 43. *For any $\phi \in C^1(\mathcal{Q})$ the equation*

$$(\mathbf{I} - \mathbf{A})\, u = \phi, \tag{3.27}$$

possesses a unique solution $u \in C^1(\mathcal{Q})$.

Proof. We define the series $u = \sum_{k=0}^{\infty} \phi_k$ with $\{\phi_k\}_{k=0}^{\infty}$ being the sequence of successive approximations. By Lemma 42(1) we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} |\phi_k| &\leqslant \|\phi_0\|_{C(\mathcal{Q})} + \|\mathfrak{D}\phi_0\| \frac{(|\xi| + |\zeta|)}{2} \sum_{k=1}^{\infty} \frac{\left(\|q\|_{C[-b,b]} \cdot (|\xi| + |\zeta|)\right)^k}{(k+1)!} \\ &\leqslant \|\phi_0\|_{C(\mathcal{Q})} + \frac{b}{2}\|\mathfrak{D}\phi_0\| \exp\left(b\|q\|_{C[-b,b]}\right). \end{aligned}$$

Hence by the Weierstrass M test the series of u converges uniformly and absolutely on $\mathcal{Q}$. For the derivatives note that Lemma 42(2) implies that $\|\mathfrak{D}\phi_k\| \leqslant \|\mathfrak{D}\phi_0\| \frac{\left(b\|q\|_{C[-b,b]}\right)^k}{2k!}$, and thus,

$$\sum_{k=0}^{\infty} \left|\frac{\partial \phi_k}{\partial \xi}\right| \leqslant \|\mathfrak{D}\phi_0\| + \|\mathfrak{D}\phi_0\| \sum_{k=1}^{\infty} \frac{\left(b\|q\|_{C[-b,b]}\right)^k}{2k!} \leqslant \|\mathfrak{D}\phi_0\| \exp\left(b\|q\|_{C[-b,b]}\right).$$

Again, by the Weierstrass M test the series of u_ξ converges uniformly and absolutely on $\mathcal{Q}$, and similarly for u_ζ. Hence the series converges in $C^1(\mathcal{Q})$ and $u \in C^1(\mathcal{Q})$. Finally, using the continuity of $\mathbf{A}$ we have

$$\mathbf{A}u = \mathbf{A}\left(\sum_{k=0}^{\infty} \phi_k\right) = \sum_{k=0}^{\infty} \mathbf{A}\phi_k = \sum_{k=0}^{\infty} \phi_{k+1} = \sum_{k=1}^{\infty} \phi_k = u - \phi_0.$$

$\therefore$ u is a solution for $(\mathbf{I} - \mathbf{A})\, u = \phi$.

For the uniqueness, suppose that there exists a solution $v \in C^1(\mathcal{Q})$ for $(\mathbf{I} - \mathbf{A})\, v = 0$. Thus v is a fixed point for $\mathbf{A}$, and applying Lemma 42(1) we have that $\|v\|_{C(\mathcal{Q})} = \|\mathbf{A}^k v\|_{C(\mathcal{Q})} \leqslant \frac{b(b\|q\|_{C[-b,b]})^k}{2(k+1)!} \to 0$, $k \to \infty$, then $v = 0$. Thus, Eq. (3.27) must have a unique solution. **Q.E.D.**

Proposition 44. *If $q \in C[-b, b]$, then the kernel K satisfying the Goursat conditions (3.15) belongs to $C^1(\overline{\Omega})$. Moreover, if $q \in C^1[-b, b]$, then $K \in C^2(\overline{\Omega})$ and satisfies (3.13).*

Proof. Define $H(\xi, \zeta) = K(\xi + \zeta, \xi - \zeta)$. Then H must satisfy (3.27) with $\phi(\xi, \zeta) = Ae^{\frac{1}{2}\int_0^\xi q(a)ds} + Be^{\frac{1}{2}\int_0^\zeta q(a)ds} - A$. Since $q \in C[-b, b]$, then $\phi \in C^1(\mathcal{Q})$, and by Theorem 43 we have $H \in C^1(\mathcal{Q})$. Hence $K \in C^1(\overline{\Omega})$.

Now suppose that $q \in C^1[-b, b]$. Then taking partial derivatives in (3.22) we have $H_{\xi\zeta}(\xi, \zeta) = \frac{1}{2} q(\xi + \zeta)\mathfrak{D}H(\xi, \zeta)$, and hence $H_{\xi\zeta} \in C(\mathcal{Q})$. On the other hand

$$H_{\xi\xi}(\xi, \zeta) = \frac{1}{2}\int_0^\zeta q'(\xi + \beta)\mathfrak{D}H(\xi, \beta)d\beta + \frac{1}{2}\int_0^\zeta \left[H_{\xi\xi}(\xi, \beta) + H_{\alpha\xi}(\xi, \alpha)\right] d\beta.$$

In this particular case $\phi_0 \in C^2(\mathcal{Q})$, and deriving $\mathbf{A}\phi_k$ we see that $\phi_k \in C^2(\mathcal{Q})$ for all $k \in \mathbb{N}$. Actually we have

$$\begin{aligned}\frac{\partial^2 \phi_{k+1}(\xi, \zeta)}{\partial \xi^2} &= \frac{1}{2}\int_0^\zeta q'(\xi + \beta)\mathfrak{D}\phi_k(\xi, \beta)d\beta + \frac{1}{2}\int_0^\zeta q(\xi + \beta)\left[\frac{\partial^2 \phi_k(\xi, \beta)}{\partial \xi^2} + \frac{\partial^2 \phi_k(\xi, \alpha)}{\partial \xi \partial \beta}\right] d\beta \\ &= \frac{1}{2}\int_0^\zeta q'(\xi + \beta)\mathfrak{D}\phi_k(\xi, \beta)d\beta + \frac{1}{2}\int_0^\zeta q(\xi + \beta)\mathfrak{D}\left[\frac{\partial \phi_k}{\partial \xi}\right](\xi, \beta)d\beta.\end{aligned}$$

To the first integral we apply Lemma 42(2) with q' instead of q, and in the second integral we change ϕ_0 by $\frac{\partial \phi_0}{\partial \xi}$. In this way we have the estimate

$$\left|\frac{\partial^2 \phi_{k+1}(\xi,\zeta)}{\partial \xi^2}\right| \leqslant \frac{1}{2}\left(\|q'\|^{k+1}_{C[-b,b]}\|\mathfrak{D}\phi_0\| + \|q\|^{k+1}_{C[-b,b]}\left\|\mathfrak{D}\left(\frac{\partial \phi_0}{\partial \xi}\right)\right\|\right)\frac{(|\xi|+|\zeta|)^{k+1}}{(k+1)!},$$

from which we obtain that the series $\sum_{k=1}^{\infty}\left|\frac{\partial^2\phi_k}{\partial\xi^2}\right|$ converge absolutely and uniformly on $\mathcal{Q}$. Then $H_{\xi\xi} \in C(\mathcal{Q})$. In a similar way $H_{\zeta\zeta} \in C(\mathcal{Q})$, and hence $H \in C^2(\mathcal{Q})$. Thus $K \in C^2(\overline{\Omega})$, and rewriting (3.20) with the change of variables $(\xi,\zeta) \mapsto (\xi+\zeta, \xi-\zeta)$ we obtain that K satisfies (3.13). **Q.E.D.**

We conclude this section with an important approximation theorem.

Theorem 45. *Let $\{q_n\} \subset C[-b,b]$ such that $q_n \to q$, $n \to \infty$ in $C[-b,b]$. Then the solutions H_n of (3.22) with potential q_n (B fixed) converge in $C^1(\mathcal{Q})$ to the solution H of (3.22) with potential q. The corresponding solution K_n converges to K in $C^1(\overline{\Omega})$.*

Proof. Fix $N \in \mathbb{N}$. Denote by $\mathbf{A}_{q_N}, \mathbf{A}_q$ the corresponding operators (3.23) for q_N, q respectively, and let $\phi_N(\xi,\zeta) = Ae^{Q_N(\xi)} + Be^{Q_N(\zeta)} - A$, $\phi(\xi,\zeta) = Ae^{Q(\xi)} + Be^{Q(\zeta)} - A$, with $Q_N(x) = \frac{1}{2}\int_0^x q_N(s)ds$, $Q(x) = \frac{1}{2}\int_0^x q(s)ds$. If $H_N, H \in C^1(\mathcal{Q})$ are the unique solutions of $(\mathbf{I} - \mathbf{A}_{q_N})H_N = \phi_N$, $(\mathbf{I}-\mathbf{A}_q)H = \phi$, define $M_N = H_N - H$ and note that $M_N - \mathbf{A}_{q_N}H_N + \mathbf{A}_q H = \phi_N - \phi$, and

$$\begin{aligned}\mathbf{A}_{q_N}H_N - \mathbf{A}_q H &= \frac{1}{2}\int_0^{\xi}\int_0^{\zeta}\left(q_N(\alpha+\beta)\mathfrak{D}H_N(\alpha,\beta) - q(\alpha+\beta)\mathfrak{D}H(\alpha,\beta)\right)d\beta d\alpha\\ &= \frac{1}{2}\int_0^{\xi}\int_0^{\zeta} q_N(\alpha+\beta)\mathfrak{D}(H_N-H)d\beta d\alpha + \frac{1}{2}\int_0^{\xi}\int_0^{\zeta}(q_N-q)(\alpha+\beta)\mathfrak{D}H d\beta d\alpha\\ &= \mathbf{A}_{q_N}M_N + \mathbf{A}_{q_N-q}H.\end{aligned}$$

Then M_N is the unique solution of $(\mathbf{I}-\mathbf{A}_{q_N})M_N = \tilde{\phi}_N$, where $\tilde{\phi}_N = \phi_N - \phi + \mathbf{A}_{q_N-q}H$. According to Theorem 43 the solution M_N satisfies the estimate

$$\|M_N\|_{C^1(\mathcal{Q})} \leqslant \|\tilde{\phi}_N\|_{C(\mathcal{Q})} + \left(1+\frac{b}{2}\right)\|\mathfrak{D}\tilde{\phi}_N\|e^{b\|q_N\|_{C[-b,b]}}.$$

Since $q_N \to q$ in $C[-b,b]$, then $C_1 := \sup_{N\in\mathbb{N}}\|q_N\|_{C[-b,b]} < \infty$ and

$$\|M_N\|_{C^1(\mathcal{Q})} \leqslant \left(1+\left(1+\frac{b}{2}\right)e^{bC_1}\right)\left\{\|\phi_N-\phi\|_{C^1(\mathcal{Q})} + \|\mathbf{A}_{q_N-q}H\|_{C^1(\mathcal{Q})}\right\}e^{bC_1}.$$

We know that operator $\mathbf{A}_{q_N-q}$ satisfies $\|\mathbf{A}_{q_N-q}H\|_{C^1(\mathcal{Q})} \leqslant \frac{1}{2}\left(\frac{b^2}{2}+b\right)\|q_N - q\|_{C^1[-b,b]}\|H\|_{C^1(\mathcal{Q})}$, hence $\|\mathbf{A}_{q_N-q}H\|_{C^1(\mathcal{Q})} \to 0,\ N\to\infty$. On the other hand, note that

$$\|\phi_N - \phi\|_{C^1(\mathcal{Q})} \leqslant (A+B)\left(\|e^{Q_N} - e^{Q}\|_{C^1[-b,b]} + \|q_N e^{Q_N} - q e^{Q}\|_{C^1[-b,b]}\right).$$

Using the inequality $\|e^Q\|_{C[-b,b]} \leqslant e^{\|Q\|_{C[-b,b]}}$ and the uniform convergence we obtain $\|e^{Q_N} - e^{Q}\|_{C[-b,b]} \to 0$, and $\|q_N e^{Q_N} - q e^{Q}\|_{C[-b,b]} \to 0$, $N \to \infty$. $\therefore$ $\|M_N\|_{C^1(\mathcal{Q})} \to 0,\ N\to\infty$, i.e., $H_N \to H$ in $C^1(\mathcal{Q})$. Since the change of variables $(x,t) \mapsto \left(\frac{x+t}{2}, \frac{x-t}{2}\right)$ is an isometry between $C^1(\overline{\Omega})$ and $C^1(\mathcal{Q})$, we have that $K_N \to K$ in $C^1(\overline{\Omega})$. **Q.E.D.**

Remark 46. *In $\mathcal{Q}$ consider the subdomains $\mathcal{Q}_1 = \{(\xi,\zeta)\in\mathbb{R}^2, \mid \xi,\zeta \geqslant 0, \xi+\zeta \leqslant b\}$ and $\mathcal{Q}_2 = \{(\xi,\zeta)\in\mathbb{R}^2, \mid \xi,\zeta \leqslant 0, \xi+\zeta \geqslant -b\}$ (see Figure 3.1). The existence of the solution H in $\mathcal{Q}_1$ depends only on values of q in $[0,b]$. Moreover, the change of variables $(x,t) \mapsto \left(\frac{x+t}{2}, \frac{x-t}{2}\right)$ transforms $\mathcal{R}_1 = \{(x,t)\in\mathbb{R}^2 \mid 0 \leqslant x \leqslant b, |t| \leqslant x\}$ to $\mathcal{Q}_1$, hence the kernel K in $\mathcal{R}_1$ depends only on values of q in $[0,b]$. Similarly, the existence of H in $\mathcal{Q}_2$ depends only on values of q in $[-b,0]$.*

Remark 47. *If $q \in L_\infty(-b,b)$ we can consider Eq. (3.22) in the Sobolev space $W^{1,\infty}(\Omega)$. In this case $\mathbf{A} \in \mathcal{B}(W^{1,\infty}(\mathcal{Q}^\circ))$, where $\mathcal{Q}^\circ$ is the interior of $\mathcal{Q}$. The proof of Theorem 43 based on successive approximations works then by replacing the C norms with L_∞ norms. Since $\mathcal{Q}^\circ$ is a bounded Lipschitz domain, H is Lipschitz continuous in $\mathcal{Q}$ (see Theorem 14). Thus K is Lipschitz continuous in $\overline{\Omega}$. In the same way Theorem 45 holds if $q_n \to q$ in $L_\infty(-b,b)$.*

Remark 48. *If $q \in C[-b,b]$, take a sequence $\{q_n\} \subset C^1[-b,b]$ such that $q_n \to q$ in $C[-b,b]$. By Proposition 44, $K_n \in C^2(\overline{\Omega})$ and satisfies Eq. (3.13) with potential q_n. This equation can be written in divergence form as $\frac{\partial}{\partial x}\left(p_n(x)\frac{\partial K_n}{\partial x}\right) = p_n(x)\frac{\partial^2 K_n}{\partial t^2}$. Multiplying this equation by $\varphi \in C_0^\infty(\Omega)$ and using integration by parts we have*

$$\iint_\Omega \left\{p_n(x)\frac{\partial K_n(x,t)}{\partial x}\frac{\partial \varphi(x,t)}{\partial x} - p_n(x)\frac{\partial K_n(x,t)}{\partial t}\frac{\partial \varphi(x,t)}{\partial t}\right\} dxdt = 0.$$

Uniform convergence of q_n implies that $p_n \overset{[-b,b]}{\rightrightarrows} p$, and by Theorem 45, $\frac{\partial K_n}{\partial x}, \frac{\partial K_n}{\partial t} \overset{\overline{\Omega}}{\rightrightarrows} \frac{\partial K}{\partial x}, \frac{\partial K}{\partial t}$, from which we obtain

$$\iint_\Omega \left\{p(x)\frac{\partial K(x,t)}{\partial x}\frac{\partial \varphi(x,t)}{\partial x} - p(x)\frac{\partial K(x,t)}{\partial t}\frac{\partial \varphi(x,t)}{\partial t}\right\} dxdt = 0 \qquad \forall \varphi \in C_0^\infty(\Omega).$$

The integral form is continuous in the H^1-norm, hence the equality is valid for $\varphi \in H_0^1(\Omega)$. In conclusion, if $q \in C[-b, b]$, then K is a weak solution of (3.13) in Ω.

3.2 Analytical properties of transmutation operators

Throughout this section we assume that $q \in C[-b, b]$. Let $h \in \mathbb{C}$. Denote by $H_h(\xi, \zeta)$ the unique solution in $C^1(\mathcal{Q})$ of equation (3.22) with $B = \frac{1-h}{2}$. Denote $K_a^h(x,t) := H_h\left(\frac{x+t}{2}, \frac{x-t}{2}\right)$. Then $K_a^h(0,0) = \frac{1-h}{2}$. By $\mathbf{T}_a^h$ we denote the operator

$$\mathbf{T}_a^h u(x) = u(x) - \int_{-x}^{x} K_a^h(x,t)u'(t)dt. \tag{3.28}$$

Note that $\mathbf{T}_a^h : C^1[-b, b] \to C^1[-b, b]$, and

$$\left(\mathbf{T}_a^h u\right)(0) = u(0), \quad \left(\mathbf{T}_a^h u\right)'(0) = u'(0)\left(1 - 2K_a^h(0,0)\right) = hu'(0). \tag{3.29}$$

In particular, $\mathbf{T}_a^1$ preserves the initial conditions and transforms $\cos(\rho, x)$ and $\frac{\sin(\rho x)}{\rho}$ into $C(\rho, x)$, $S_1(\rho, x)$. Denoting the operator

$$\mathbf{K}_a^h u(x) := \int_{-x}^{x} K_a^h(x,t)u'(t)dt,$$

we see that $\mathbf{K}_a^h : C^1[-b, b] \to C^1[-b, b]$ and $\mathbf{T}_a^h = \mathbf{I} - \mathbf{K}_a^h$.

3.2.1 Transmutation property

Consider $C^1[-b, b]$ as a Banach space with the norm $\|u\|_{C^1[-b,b]} := \|u\|_{C[-b,b]} + \|u'\|_{C[-b,b]}$. For $u \in C^1[-b, b]$ we have

$$\begin{aligned}
|\mathbf{K}_a^h u(x)| &\leqslant 2b\|K_a^h\|_{C(\overline{\Omega})}\|u'\|_{C[-b,b]},\\
|\left(\mathbf{K}_a^h u\right)'(x)| &\leqslant |K_a^h(x,x)||u'(x)| + |K_a^h(x,-x)||u'(-x)| + 2b\left\|\frac{\partial K_a^h}{\partial x}\right\|_{C(\overline{\Omega})}\|u'\|_{C[-b,b]}\\
&\leqslant 2\left(\left\|K_a^h\right\|_{C(\overline{\Omega})} + b\left\|\frac{\partial K_a^h}{\partial x}\right\|_{C(\overline{\Omega})}\right)\|u'\|_{C[-b,b]}.
\end{aligned}$$

Hence $\|\mathbf{K}_a^h u\|_{C^1[-b,b]} \leqslant 2(1+b)\|K_a^h\|_{C^1(\overline{\Omega})}\|u\|_{C^1[-b,b]}$, and $\mathbf{K}_h \in \mathcal{B}\left(C^1[-b, b]\right)$ (as well as $\mathbf{T}_h$).

Proposition 49. *Let* $\{q_n\} \subset C[-b,b]$ *and* $\{\mathbf{K}_{a,n}^h\}$ *be the sequence of corresponding operators. If* $q_n \to q$ *in* $C[-b,b]$*, then* $\mathbf{K}_{a,n}^h \to \mathbf{K}_a^h$ *in* $\mathcal{B}\left(C^1[-b,b]\right)$*. In particular,* $\mathbf{T}_{a,n}^h \to \mathbf{T}_a^h$ *in* $\mathcal{B}\left(C^1[-b,b]\right)$*.*

Proof. Take $u \in C^1[-b,b]$. Consider

$$|\mathbf{K}_{a,n}^h u(x) - \mathbf{K}_a^h u(x)| \leqslant 2b\|K_{a,n}^h - K_a^h\|_{C(\overline{\Omega})}\|u'\|_{C[-b,b]},$$

and

$$\begin{aligned}|(\mathbf{K}_{a,n}^h u)'(x) - (\mathbf{K}_a^h u)'(x)| &\leqslant |K_{a,n}^h(x,x) - K_a^h(x,x)||u'(x)| + |K_{a,n}^h(x.-x) - K_a^h(x,x)||u'(x)| \\ &\quad + 2b\left\|\frac{\partial K_{a,n}^h}{\partial x} - \frac{\partial K_h}{\partial x}\right\|_{C(\overline{\Omega})}\|u'\|_{C[-b,b]} \\ &\leqslant 2\left(\|K_{a,n} - K_a^h\|_{C(\overline{\Omega})} + b\left\|\frac{\partial K_{a,n}^h}{\partial x} - \frac{\partial K_a^h}{\partial x}\right\|_{C(\overline{\Omega})}\right)\|u'\|_{C[-b,b]}.\end{aligned}$$

Hence $\|\mathbf{K}_{a,n}^h u - \mathbf{K}_a^h u\|_{C^1[-b,b]} \leqslant 2(1+b)\|K_{a,n}^h - K_a^h\|_{C^1(\overline{\Omega})}\|u\|_{C^1[-b,b]}$. Thus,

$$\|\mathbf{K}_{a,n}^h - \mathbf{K}_a^h\|_{\mathcal{B}(C^1[-b,b])} \leqslant 2(1+b)\|K_{a,n}^h - K_a^h\|_{C^1(\overline{\Omega})}.$$

By Theorem 45, the right hand side tends to zero when $n \to \infty$. Hence $\mathbf{K}_{a,n}^h \to \mathbf{K}_a^h$ in $\mathcal{B}\left(C^1[-b,b]\right)$. **Q.E.D.**

We obtain the following transmutation property for the operators $\mathbf{L}_a = \frac{1}{a^2(x)}\mathbf{D}a^2(x)\mathbf{D} = \mathbf{D}^2 - q(x)\mathbf{D}$ and $\mathbf{D}^2$.

Theorem 50. *For all* $h \in \mathbb{C}$ *the equality is valid*

$$\mathbf{L}_a\mathbf{T}_a^h u(x) = \mathbf{T}_a^h\mathbf{D}^2 u(x) \quad \forall u \in C^3[-b,b]. \tag{3.30}$$

Proof. We consider two cases.

Case 1: $q \in C^1[-b,b]$. By Proposition 44, $K_a^h \in C^2(\mathcal{R})$ and satisfies the Goursat problem (3.13)-(3.15). For $u \in C^3[-b,b]$ we have

$$\frac{d}{dx}\mathbf{T}_a^h u(x) = u'(x) - K_a^h(x,x)u'(x) - K_a^h(x,-x)u'(-x) - \int_{-x}^{x} \frac{\partial K_a^h(x,t)}{\partial x} u'(t)dt,$$

$$\frac{d^2}{dx^2}\mathbf{T}_a^h u(x) = u''(x) - \frac{d}{dx}K_a^h(x,x)u'(x) - K_a^h(x,x)u''(x) - \frac{d}{dx}K_a^h(x,-x)u'(-x) + K_a^h(x,-x)u''(-x) - \frac{\partial K_a^h(x,x)}{\partial x}u'(x) - \frac{\partial K_a^h(x,-x)}{\partial x}u'(-x) - \int_{-x}^{x} \frac{\partial^2 K_a^h(x,t)}{\partial x^2} u'(t)dt.$$

Thus,

$$\mathbf{L}_a\mathbf{T}_a^h u(x) = u''(x) - \left\{\frac{d}{dx}K_a^h(x,x) + \frac{\partial K_a^h(x,x)}{\partial x} - q(x)[K_a^h(x,x)-1]\right\}u'(x) - \left\{\frac{d}{dx}K_h(x,-x) + \frac{\partial K_a^h(x,-x)}{\partial x} - q(x)K_a^h(x,-x)\right\}u'(-x) - K_a^h(x,x)u''(x) + K_a^h(x,-x)u''(-x) - \int_{-x}^{x}\left(\frac{\partial^2}{\partial x^2} - q(x)\frac{\partial}{\partial x}\right)K_a^h(x,t)u'(t)dt.$$

On the other hand integration by parts in $\mathbf{T}_a^h u''(x) = u''(x) - \int_{-x}^{x} K_a^h(x,t)u'''(t)dt$ gives

$$\int_{-x}^{x} K_a^h(x,t)u'''(t)dt = K_a^h(x,x)u''(x) - K_a^h(x,-x)u''(-x) - \frac{\partial K_a^h(x,x)}{\partial t}u'(x) + \frac{\partial K_a^h(x,-x)}{\partial t}u'(-x) + \int_{-x}^{x} \frac{\partial^2 K_a^h(x,t)}{\partial t^2}u'(t)dt.$$

Hence

$$\begin{aligned}\mathbf{L}_a\mathbf{T}_a^h u(x) - \mathbf{T}_a^h\mathbf{D}^2 u(x) = &- \int_{-x}^{x}\left(\frac{\partial^2}{\partial x^2} - q(x)\frac{\partial}{\partial x} - \frac{\partial_t^2}{\partial t^2}\right)K_a^h(x,t)u'(t)dt \\ &- \left\{\frac{d}{dx}K_a^h(x,x) + \frac{\partial K_a^h(x,x)}{\partial x} + \frac{\partial K_a^h(x,-x)}{\partial t} - q(x)[K_a^h(x,x)-1]\right\}u'(x) \\ &- \left\{\frac{d}{dx}K_a^h(x,-x) + \frac{\partial K_a^h(x,-x)}{\partial x} - \frac{\partial K_a^h(x,-x)}{\partial t} - q(x)K_a^h(x.-x)\right\}u'(-x) \\ = &- \left\{2\frac{d}{dx}K_a^h(x,x) - q(x)[K_a^h(x,x)-1]\right\}u'(x) \\ &- \left\{2\frac{d}{dx}K_a^h(x,-x) - q(x)K_a^h(x,-x)\right\}u'(-x) = 0,\end{aligned}$$

due to (3.13)-(3.15). Due to the arbitrariness of $u \in C^3[-b,b]$ we obtain (160).

Case 2: $q \in C[-b,b]$. Take a sequence $\{q_n\} \subset C^1[-b,b]$ such that $q_n \to q$, $n \to \infty$ in $C[-b,b]$. Let $\mathbf{T}_{a,n}^h$ be the corresponding operators. Given $u \in C^3[-b,b]$, by the first part,

$\mathbf{L}_a\mathbf{T}^h_{a,n}u = \mathbf{T}^h_{a,n}\mathbf{D}^2u$ or more explicitly,

$$\left(\mathbf{T}^h_{a,n}u(x)\right)'' = q_n(x)\left(\mathbf{T}^h_{a,n}u(x)\right)' + \mathbf{T}^h_{a,n}u''(x).$$

Due to Proposition 49, $\mathbf{T}^h_{a,n} \to \mathbf{T}^h_a$ in $\mathcal{B}\left(C^1[-b,b]\right)$. In particular, $\mathbf{T}^h_{a,n}u''(x) \overset{[-b,b]}{\rightrightarrows} \mathbf{T}^h_a u''(x)$, $q_n(x)\left(\mathbf{T}^h_{a,n}u(x)\right)' \overset{[-b,b]}{\rightrightarrows} q(x)\left(\mathbf{T}^h_a u(x)\right)'$, $n \to \infty$. Set $y_n = \left(\mathbf{T}^h_{a,n}u\right)'$, and $v = q\left(\mathbf{T}^h_a u\right)' + \mathbf{T}^h_a u''$. Then $y_n \overset{[-b,b]}{\rightrightarrows} \left(\mathbf{T}^h_a u\right)'$ and $y_n' \overset{[-b,b]}{\rightrightarrows} v$, $n \to \infty$. Hence $\left(\mathbf{T}^h_a u\right)'' = v = q\left(\mathbf{T}^h_a u\right)' + \mathbf{T}^h_a u''$, which is (160). **Q.E.D.**

According to Remark 23, the operator $\mathbf{T}^h_a$ transforms solutions of the elementary equation $v'' + \rho^2 v = 0$ into solutions of (3.2). Due to (3.29) the function $S_h(\rho,x) = \mathbf{T}^h_a\left[\frac{\sin(\rho x)}{\rho}\right]$ is the solution of (3.2) satisfying the initial conditions $S_h(\rho,0) = 0$, $S_h'(\rho,0) = h$. Contrary to this dependence on h, the function $v = \mathbf{T}^h_a\left[\cos(\rho x)\right]$ is the solution satisfying $v(0) = 1$, $v'(0) = 0$, i.e., it is independent of h, and hence $C(\rho,x) = \mathbf{T}^h_a\left[\cos(\rho x)\right]$. Note that $\mathbf{T}^h_a$ maps the solutions of $v'' + \rho^2 v = 0$ *onto* the solutions of (3.2).

In the case $h = 1$ we denote $\mathbf{T}^1_a$ by $\mathbf{T}_a$ and call it the *canonical transmutation operator*. The corresponding canonical kernel is denoted by $K_a(x,t)$.

Proposition 51. *The canonical transmutation operator is injective, and if $q \in C^1[-b,b]$, then $\mathbf{T}_a \in \mathcal{G}\left(C^1[-b,b]\right)$.*

Proof. Suppose there exists $u \in C^1[-b,b]$ such that $\mathbf{T}_a u = 0$. Integration by parts gives

$$(1 - K_a(x,x))u(x) + \int_{-x}^{x} \frac{\partial K_a(x,t)}{\partial t}u(t)dt = 0$$

(because $K_a(x,-x) = 0$). Also $K_a(x,x) = 1 - e^{\frac{1}{2}\int_0^x q(s)ds} = 1 - a^{-1}(x)$, and the last equation can be written in the form

$$0 = u(x) + \int_{-x}^{x} \tilde{K}_a(x,t)u(t)dt,$$

where $\tilde{K}_a(x,t) = a(x)\frac{\partial K_a(x,t)}{\partial t}$. This is a homogeneus Volterra integral equation of the second kind with a continuous kernel. Hence it does not admit non-trivial solutions (see, e.g., [72, Ch. X]). Hence $\mathbf{T}_a$ is injective.

Now suppose that $q \in C^1[-b,b]$. Given $y \in C^1[-b,b]$, consider the equation

$$a(x)y(x) = u(x) + \int_{-x}^{x} \tilde{K}_a(x,t)u(t)dt.$$

It has a unique solution $u \in C[-b,b]$ (see [72, Ch. X]).

Since $K_a \in C^2(\overline{\Omega})$, then $u(x) = a(x)\left(y(x) - \int_{-x}^{x} \frac{\partial K_1(x,t)}{\partial t} u(t)dt\right) \in C^1[-b,b]$. Applying integration by parts we recover the equation $y = \mathbf{T}_a u$. Then $\mathbf{T}_a$ is a surjective map and hence a bijection, i.e., $\mathbf{T}_a \in \mathcal{G}\left(C^1[-b,b]\right)$. **Q.E.D.**

Remark 52. *In the proof of Proposition 51 we obtain that the operator*

$$\mathbf{V}_a u(x) = \frac{u(x)}{a(x)} + \int_{-x}^{x} \frac{\partial K_a(x,t)}{\partial t} u(t)dt \tag{3.31}$$

belongs to $\mathcal{G}\left(C[-b,b]\right)$*. Additionally* $\mathbf{T}_a u = \mathbf{V}_a u$ *for all* $u \in C^1[-b,b]$*. In the case* $q \in C^1[-b,b]$ *we have* $\mathbf{V}_a \in \mathcal{B}\left(C^1[-b,b]\right)$*.*

3.2.2 Mapping property

Here we establish how the transmutation operators $\mathbf{T}_a^h$ act on non-negative integer powers of the independent variable. Let $g \in C[-b,b]$ be a nonvanishing function (in general, complex valued). We define the following recursive integrals: $\tilde{Y}^{(0)} \equiv Y^{(0)} \equiv 1$, and for $k \geqslant 1$:

$$\tilde{Y}^{(k)}(x) := k \int_0^x \tilde{Y}^{(k-1)}(s)\left(g^2(s)a^2(s)\right)^{(-1)^{k-1}} ds, \tag{3.32}$$

$$Y^{(k)}(x) := k \int_0^x \tilde{Y}^{(k-1)}(s)\left(g^2(s)a^2(s)\right)^{(-1)^{k}} ds. \tag{3.33}$$

The functions $\{\varphi_{a,g}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\varphi_{a,g}^{(k)}(x) := \begin{cases} g(x)Y^{(k)}(x), & \text{for } k \text{ odd },\\ g(x)\tilde{Y}^{(k)}(x), & \text{for } k \text{ even}. \end{cases} \tag{3.34}$$

for $k \in \mathbb{N} \cup \{0\}$ are called *formal powers* associated to g.

Theorem 53 (SPPS Method)**.** *Suppose that* $g \in C^1[-b,b] \cap C^2(-b,b)$ *is a nonvanishing in* $[-b,b]$ *solution of* $\mathbf{L}_a g = 0$*. Then the general solution of Eq. (3.1) has the form*

$$u = c_1 u_1 + c_2 u_2,$$

where $c_1, c_2 \in \mathbb{C}$ *and*

$$u_1(x) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k \varphi_{a,g}^{(2k)}(x)}{(2k)!}, \quad u_2(x) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k \varphi_{a,g}^{(2k+1)}(x)}{(2k+1)!}. \tag{3.35}$$

The solutions u_1, u_2 satisfy the initial conditions

$$\begin{aligned} u_1(0) &= g(0), \quad u_1'(0) = g'(0), \\ u_2(0) &= 0, \qquad u_2'(0) = \tfrac{1}{g(0)}. \end{aligned} \tag{3.36}$$

Proof. The proof consists in applying Theorem 1 from [83] to Eq. (3.1). **Q.E.D.**

A possible choice of the solution g can be $g_1 \equiv 1$. In this case we use the notation $\{\varphi_a^{(k)}\}_{k=0}^{\infty}$ and call these functions the *canonical formal powers*. Another linearly independent solution is given by $y(x) = \int_0^x \frac{dt}{p(t)}$. In order to obtain a nonvanishing solution one can choose $g_2(x) = A + By(x)$ for some appropriate constants A, B. For our purposes we take $g_h(x) = \frac{1}{h} + y(x)$, where $h \neq 0$, and $h \neq -y(x)$ for all $x \in [-b, b]$.

Theorem 54. *The following relations hold*

1. $\forall k \in \mathbb{N} \cup \{0\} \quad \mathbf{T}_a[x^k] = \varphi_a^{(k)}(x)$,
2. $\forall k \in \mathbb{N} \cup \{0\} \quad \mathbf{T}_a^h[x^k] = \begin{cases} \varphi_{a,g_h}^{(k)}(x), & \text{if } k \text{ is odd}, \\ h\varphi_{a,g_h}^{(k)}(x) - \frac{\varphi_{a,g_h}^{(k+1)}(x)}{k+1}, & \text{if } k \text{ is even}. \end{cases}$

Proof. 1. We apply the SPPS method with the solution g_1. Using the initial conditions (3.36) we have that $u_1(x) = C(\rho, x)$, $u_2(x) = S_1(\rho, x)$. Then $u_1(x) = \mathbf{T}_a[\cos(\rho x)]$. Since the Taylor series $\cos(\rho x) = \sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k} x^{2k}}{(2k)!}$ converges in $C^1[-b, b]$ we have

$$u_1(x) = \mathbf{T}_a \left[\sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k} x^{2k}}{(2k)!} \right] = \sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k} \mathbf{T}_a[x^{2k}]}{(2k)!}.$$

On the other hand, the SPPS series for the equation $\mathbf{L}_a u = \rho^2 u$ is $u_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k} \varphi_a^{(2k)}(x)}{(2k)!}$. Comparing both series as the Taylor series of an entire function of ρ we conclude that $\mathbf{T}_a[x^{2k}] = \varphi_a^{(2k)}$. For the odd powers note that $\frac{\sin(\rho x)}{\rho} = \sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k} x^{2k+1}}{(2k+1)!}$. Hence comparing with the SPPS series for u_2 we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k} \varphi_a^{(2k+1)}(x)}{(2k+1)!} = u_2(x) = \mathbf{T}_a \left[\frac{\sin(\rho x)}{\rho} \right] = \sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k} \mathbf{T}_a[x^{2k+1}]}{(2k+1)!}.$$

Thus, $\mathbf{T}_a[x^{2k+1}] = \varphi_a^{(2k+1)}(x)$.

2. In this case, we apply the SPPS method to the equation $\mathbf{L}_a u = \rho^2 u$ with g_h. Then the solutions u_1, u_2 satisfy $u_1(0) = \frac{1}{h}$, $u_1'(0) = 1$ and $u_2(0) = 0, u_2(0) = h$. Hence $u_2(x) = S_h(\rho, x) = \mathbf{T}_h\left[\frac{\sin(\rho x)}{\rho}\right]$. Proceeding as in the case 1 we obtain $\mathbf{T}_a^h[x^{2k+1}] = \varphi_{a,g_h}^{(2k+1)}(x)$.

   On the other hand $u_1 = \frac{1}{h}\left(C(\rho,x) + S_h(\rho,x)\right)$. Hence

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k\rho^{2k}\varphi_{a,g_h}^{(2k)}(x)}{(2k)!} =& u_2(x) = \frac{1}{h}\mathbf{T}_a^h\left[\cos(\rho x) + \frac{\sin(\rho x)}{\rho}\right] \\ =& \frac{1}{h}\left(\sum_{k=0}^{\infty}\frac{(-1)^k\rho^{2k}\mathbf{T}_a^h[x^{2k}]}{(2k)!} + \sum_{k=0}^{\infty}\frac{(-1)^k\rho^{2k}\mathbf{T}_a^h[x^{2k+1}]}{(2k+1)!}\right).\end{aligned}$$

   Comparing both series as Taylor series of an entire function of ρ, we conclude that $\frac{\varphi_{a,g_h}^{(2k)}(x)}{(2k)!} = \frac{1}{h}\left(\frac{\mathbf{T}_a^h[x^{2k}]}{(2k)!} + \frac{\mathbf{T}_a^h[x^{2k+1}]}{(2k+1)!}\right)$. Hence $h\varphi_{a,g_h}^{(2k)}(x) = \mathbf{T}_a^h[x^{2k}] + \frac{\varphi_{a,g_h}^{(2k+1)}(x)}{2k+1}$, as required.

Q.E.D.

Remark 55. *Due to Theorem 160 we have the relations*

$$\mathbf{L}_a\varphi_a^{(k)} = \mathbf{L}_a\mathbf{T}_a[x^k] = \mathbf{T}_a\left[\mathbf{D}^2 x^k\right].$$

Hence for all $k \in \mathbb{N}_0$

$$\mathbf{L}_a\varphi_a^{(k)} = \begin{cases} 0, & \text{if } k = 0, 1, \\ k(k-1)\varphi_a^{(k-2)}, & \text{if } k \geqslant 2. \end{cases} \tag{3.37}$$

We say that the canonical formal powers form an $\mathbf{L}_a$-basis [20, 24].

It is easy to prove that the system of canonical formal powers is complete in $C[-b,b]$. Additionally, if $q \in C^1[-b,b]$, it is complete in $C^1[-b,b]$.

3.2.3 Even and odd operators

Denote by $\mathbf{P}^{\pm}$ the *even* and *odd* projections, defined by

$$\mathbf{P}^+u(x) := \frac{u(x)+u(-x)}{2}, \quad \mathbf{P}^-u(x) := \frac{u(x)-u(-x)}{2}, \quad \text{for } u \in C[-b,b].$$

Obviously, $\mathbf{P}^{\pm} \in \mathcal{B}(C[-b,b])$ and $(\mathbf{P}^{\pm})^2 = \mathbf{P}^{\pm}$, $\mathbf{P}^{\pm}\mathbf{P}^{\mp} = 0$, $\mathbf{P}^+ + \mathbf{P}^- = \mathbf{I}_{C[-b,b]}$. $\mathbf{P}^+(C[-b,b])$ $(\mathbf{P}^-(C[-b,b]))$ is the set of even (odd) functions. Note that $D : \mathbf{P}^{\pm}(C^1[-b,b]) \to \mathbf{P}^{\mp}(C[-b,b])$.

Definition 56. *Let $h \in \mathbb{C}$. Define the kernels in $\overline{\Omega}$.*

(i) $K^C_{a,h}(x,t) := K^h_a(x,t) - K^h_a(x,-t)$ *(the cosine kernel).*

(ii) $K^S_{a,h}(x,t) := K^h_a(x,t) + K^h_a(x,-t)$ *(the sine kernel).*

The corresponding "sine" and "cosine" operators are defined on $C^1[-b,b]$ *by*

(a) $\mathbf{T}^S_{a,h}u(x) = u(x) - \int_0^x K^S_{a,h}(x,t)u'(t)dt,$

(b) $\mathbf{T}^C_{a,h}u(x) = u(x) - \int_0^x K^C_{a,h}(x,t)u'(t)dt.$

Clearly $\mathbf{T}^S_{a,h}, \mathbf{T}^C_{a,h} \in \mathcal{B}\left(C^1[-b,b]\right)$. Actually $\mathbf{T}^S_{a,h}$ and $\mathbf{T}^C_{a,h}$ are well defined and bounded on $C^1[0,b]$.

Proposition 57. $\mathbf{T}^h_a = \mathbf{T}^C_{a,h}\mathbf{P}^+ + \mathbf{T}^S_{a,h}\mathbf{P}^-$ *on* $C^1[-b,b]$.

Proof. Since $\mathbf{T}^h_a = \mathbf{T}^h_a\mathbf{P}^+ + \mathbf{T}^h_a\mathbf{P}^-$, it is sufficient to show that $\mathbf{T}^h_a\mathbf{P}^+ = \mathbf{T}^C_{a,h}\mathbf{P}^+$ and $\mathbf{T}^h_a\mathbf{P}^- = \mathbf{T}^S_{a,h}\mathbf{P}^-$. We show the first equality, the proof of the second is analogous. Take $u \in C^1[-b,b]$ and set $v = \mathbf{P}^+u$. Thus $v \in \mathbf{P}^+\left(C^1[-b,b]\right)$, $v' \in \mathbf{P}^-\left(C[-b,b]\right)$, and

$$\begin{aligned}\mathbf{T}^h_a v(x) =& v(x) - \int_{-x}^x K^h_a(x,t)v'(t)dt = v(x) + \int_{-x}^x K^h_a(x,t)v'(-t)dt\\ =& v(x) + \int_{-x}^x K^h_a(x,-t)v'(t)dt.\end{aligned}$$

Then

$$\mathbf{T}^h_a v(x) = v(x) - \frac{1}{2}\int_{-x}^x \left(K^h_a(x,t) - K^h_a(x,-t)\right)v'(t)dt = v(x) - \frac{1}{2}\int_{-x}^x K^C_{a,h}(x,t)v'(t)dt.$$

Since $K^C_{a,h}(x,t)$ is an odd function in t, the integrand is even and

$$\mathbf{T}^h_a v(x) = v(x) - \frac{1}{2}\int_{-x}^x K^C_{a,h}(x,t)v'(t)dt = v(x) - \int_0^x K^C_{a,h}(x,t)v'(t)dt,$$

i.e., $\mathbf{T}^h_a\mathbf{P}^+u(x) = \mathbf{T}^C_{a,h}\mathbf{P}^+u(x)$. **Q.E.D.**

In particular,

$$C(\rho,x) = \mathbf{T}^C_{a,h}[\cos(\rho x)], \quad S_h(\rho,x) = \mathbf{T}^S_{a,h}\left[\frac{\sin(\rho x)}{\rho}\right].$$

Remark 58. *(i) Let $q \in C[0,b]$. According to Remark 46, it is sufficient to construct the kernel K_a^h in $\mathcal{R}_1$, then $K_{a,h}^C$ and $K_{a,h}^S$ can be constructed knowing the values of q only in $[0,b]$. It is not difficult to see that if $q \in C^1[0,b]$, the sine an cosine kernels satisfy the hyperbolic equation (3.13) in the domain $\mathcal{R}_3 := \{(x,t) \in \mathbb{R}^2 \,|\, x \in [0,b],\, 0 \leqslant t \leqslant x\}$. In this case the characteristics are $\{(x,x)\,|\, x \in [0,b]\}$, $\{(x,0)\,|\,x \in [0,b]\}$. Along the first characteristic we have*

1. $K_{a,h}^C(x,x) = K_a^h(x,x) - K_a^h(x,-x) = 1 - \frac{h+1}{2} e^{\frac{1}{2}\int_0^x q(s)ds} - \frac{1-h}{2} e^{\frac{1}{2}\int_0^x q(s)ds} = 1 - e^{\frac{1}{2}\int_0^x q(s)ds} = 1 - a^{-1}(x)$
2. $K_{a,h}^S(x,x) = K_a^h(x,x) + K_a^h(x,-x) = 1 - \frac{h+1}{2} e^{\frac{1}{2}\int_0^x q(s)ds} + \frac{1-h}{2} e^{\frac{1}{2}\int_0^x q(s)ds} = 1 - he^{\frac{1}{2}\int_0^x q(s)ds} = 1 - ha^{-1}(x)$.

In particular both kernels satisfy the Goursat condition

$$\frac{d}{dx} G(x,x) = \frac{1}{2} q(x) \left(G(x,x) - 1\right). \tag{3.38}$$

For the second characteristic we note that

$$K_{a,h}^C(x,0) = K_a^h(x,0) - K_a^h(x,0) = 0, \tag{3.39}$$

$$\partial_t K_{a,h}^S(x,0) = \left(\frac{\partial K_a^h(x,t)}{\partial t} - \frac{\partial K_a^h(x,-t)}{\partial t} \right)\Bigg|_{t=0} = 0. \tag{3.40}$$

(ii) Let $\{q_n\} \subset C^1[0.b]$ converge to q in $C[0,b]$. Due to Theorem 45 the corresponding kernels $K_{a,h,n}^C$, $K_{a,h,n}^S$ and operators $\mathbf{T}_{a,h,n}^C$, $\mathbf{T}_{a,h,n}^S$ converge to $K_{a,h}^C$, $K_{a,h}^S$ and $\mathbf{T}_{a,h}^C$, $\mathbf{T}_{a,h}^S$ in $C^1(\mathcal{R}_3)$ and $\mathcal{B}\left(C[0,b]\right)$, respectively.

(iii) Additionally, both operators preserve the value at $x = 0$, and

$$\left(\mathbf{T}_{a,h}^C u\right)'(0) = u'(0), \quad \left(\mathbf{T}_{a,h}^S u\right)'(0) = hu'(0)$$

In this sense the kernel $K_{a,h}^C(x,t)$ is independent of h.

Again, when $h = 1$ we use the notation $K_a^C(x,t)$, $K_a^S(x,t)$, $\mathbf{T}_a^C$, $\mathbf{T}_a^S$, and call these objects the canonical sine and cosine kernels and operators, respectively.

Remark 59. *Suppose $G \in C^2(\mathcal{R}_3)$ satisfies Eq. (3.13) with the Goursat condition (3.38). Defining the operator*

$$\mathbf{T}u(x) = u(x) - \int_0^x G(x,t)u'(t)dt,$$

deriving with respect to x and applying the same procedure as in the proof of Theorem 50 we obtain the relation

$$\mathbf{L}_a\mathbf{T}u(x) - \mathbf{T}u''(x) = G(x,0)u''(0) - \frac{\partial G(x,0)}{\partial t}u'(0).$$

Proposition 60. *Denote $C^3_{0,j}[0,b] := \{u \in C^3[0,b] \,|\, u^{(j)}(0) = 0\}$, $j = 1,2$. Then*

$$\mathbf{L}_a\mathbf{T}^C_{a,h} = \mathbf{T}^C_{a,h}\mathbf{D}^2 \text{ in } C^3_{0,1}[0,b]. \tag{3.41}$$

$$\mathbf{L}_a\mathbf{T}^S_{a,h} = \mathbf{T}^S_{a,h}\mathbf{D}^2 \text{ in } C^3_{0,2}[0,b]. \tag{3.42}$$

In particular (3.41) is valid in $\mathbf{P}^+(C^3[-b,b])$ and (3.42) in $\mathbf{P}^-(C^3[-b,b])$.

Proof. Assume $q \in C^1[0,b]$ (if $q \in C[0,b]$, we apply Remark 58(ii) and a procedure analogous to Theorem 50). According to Remark 59, if $v \in C^3_{0,1}[0,b]$,

$$\mathbf{L}_a\mathbf{T}^C_{a,h}v - \mathbf{T}^C_{a,h}\mathbf{D}^2v = K^C_{a,h}(x,0)v''(0) - \frac{\partial K^C_{a,h}(x,0)}{\partial t}v'(0).$$

In this case $v'(0) = 0$ and by (3.39), $\mathbf{L}_a\mathbf{T}^C_{a,h}v - \mathbf{T}^C_{a,h}\mathbf{D}^2v = 0$. On the other hand, if $v \in C^3_{0,2}[0,b]$,

$$\mathbf{L}_a\mathbf{T}^S_{a,h}v - \mathbf{T}^S_{a,h}\mathbf{D}^2v = K^S_{a,h}(x,0)v''(0) - \frac{\partial K^S_{a,h}(x,0)}{\partial t}v'(0) = 0,$$

because $v''(0) = 0$ and (3.40).

The validity in $\mathbf{P}^\pm(C[-b,b])$ is due to the embeddings $\mathbf{P}^+(C^3[-b,b]) \hookrightarrow C^3_{0,1}[0,b]$, $\mathbf{P}^-(C^3[-b,b]) \hookrightarrow C^3_{0,2}[0,b]$. **Q.E.D.**

Proposition 61. *For all $k \in \mathbb{N} \cup \{0\}$ the following relations hold.*

$$\mathbf{T}^C_a[x^{2k}] = \varphi^{(2k)}_a(x), \tag{3.43}$$

$$\mathbf{T}^S_a[x^{2k+1}] = \varphi^{(2k+1)}_a(x). \tag{3.44}$$

Proof. Consider

$$\mathbf{T}^C_a[x^{2k}] = \mathbf{T}_a\mathbf{P}^+[x^{2k}] = \mathbf{T}_a[x^{2k}] = \varphi^{(2k)}_a(x),$$

and

$$\mathbf{T}^S_a[x^{2k+1}] = \mathbf{T}_a\mathbf{P}^-[x^{2k+1}] = \mathbf{T}_a[x^{2k+1}] = \varphi^{(2k+1)}_a(x).$$

Q.E.D.

3.2.4 Liouville transformation

Direct computation shows that the operator $\mathbf{L}_a$ admits the factorization

$$\mathbf{L}_a u = \left\{ \frac{1}{a} \left(\mathbf{D} + \frac{a'}{a} \right) \left(\mathbf{D} - \frac{a'}{a} \right) a \right\} u \qquad \forall u \in C^2[-b,b]. \tag{3.45}$$

Note that if $u \in C^1[-b,b]$, then $\left(\mathbf{D} - \frac{a'}{a}\right) u = a \left(\frac{u}{a}\right)'$, and $\left(\mathbf{D} + \frac{a'}{a}\right) u = \frac{1}{a}(au)'$. Thus we can write

$$\mathbf{L}_a u = \frac{1}{a} \left(\frac{1}{a} \mathbf{D} a \right) \left(a \mathbf{D} \frac{1}{a} \right) au \qquad \forall u \in C^2[-b,b]. \tag{3.46}$$

If $a \in C^2[-b,b]$ (i.e., $q \in C^1[-b,b]$), the following factorization holds

$$\mathbf{S}_a u := \left(\mathbf{D}^2 - Q_a(x) \right) u = \left(\mathbf{D} + \frac{a'}{a} \right) \left(\mathbf{D} - \frac{a'}{a} \right) u \qquad \forall u \in C^2[-b,b], \tag{3.47}$$

where $Q_a := \frac{a''}{a}$ (see [75, Th. 25]). Comparing (3.45) with (3.47) we have the relation

$$\mathbf{L}_a u = \frac{1}{a} \left(\mathbf{D}^2 - Q_a(x) \right) au \quad \forall u \in C^2[-b,b].$$

Consider the operator $\mathbf{R}_a : C[-b,b] \to C[-b,b]$ given by $\mathbf{R}_a u(x) := a(x)u(x)$. It is bounded on $C[-b,b]$ with the norm $\|\mathbf{R}_a\|_{\mathcal{B}(C[-b,b])} = \|a\|_{C[-b,b]}$. Its inverse is $\mathbf{R}_a^{-1} u(x) = \frac{u(x)}{a(x)}$. According to (3.47), if $u \in C^2[-b,b]$ is a solution of (3.1), then $v = \mathbf{R}_a u$ is a solution of the Schrödinger equation $-v'' + Q_a(x)v = \lambda v$. Actually $\mathbf{R}_a$ is the *Liouville transformation* of equation (3.1) (see [79]). This transformation is applicable for a general equation of the form $-\mathbf{D}\left(P(x)\mathbf{D}y\right) + Q(x)y = -\lambda R(x)y$, with P, R positive in $[-b,b]$ and $P, R \in C^2[-b,b]$. In our case $P = R = a^2$, and the relation holds

$$\mathbf{S}_a \mathbf{R}_a u = \mathbf{R}_a \mathbf{L}_a u \quad \forall u \in C^2[-b,b] \tag{3.48}$$

(see [79, Prop. 2.7]). Equality (3.48) establishes that $\mathbf{R}_a$ is a transmutation operator for the pair $\mathbf{S}_a, \mathbf{L}_a$, in the sense of Definition 22.

Note that according to (3.48), $a = \mathbf{R}_a 1$ is a nonvanishing solution of $\mathbf{S}_a u = 0$ in $[-b,b]$. Following the terminology of Theorem 26, we denote by $\{\psi_a^{(k)}\}_{k=0}^{\infty}$ the canonical formal powers associated with $\mathbf{S}_a$, defined by (1.25). The canonical transmutation operator of $\mathbf{S}_a$ associated to the nonvanishing solution a and given by (1.23) is denoted by $\widehat{\mathbf{V}}_a$, and the corresponding canonical kernel by $\widehat{K}_a$. Using the Liouville transformation it is possible to obtain a transmutation operator for the pair $\mathbf{L}_a, \mathbf{D}^2 : C^2[-b,b] \to C[-b,b]$.

Theorem 62. *The following relation holds*

$$\mathbf{L}_a\left(\mathbf{R}_a^{-1}\widehat{\mathbf{V}}_a\right) = \left(\mathbf{R}_a^{-1}\widehat{\mathbf{V}}_a\right)\mathbf{D}^2 \quad in\ C^2[-b,b]. \tag{3.49}$$

Additionally the formal powers $\{\varphi_a^{(k)}\}_{k=0}^{\infty}$ *and* $\{\psi_a^{(k)}\}_{k=0}^{\infty}$ *satisfy the relation*

$$\psi_a^{(k)}(x) = \mathbf{R}_a\varphi_a^{(k)} \quad \forall k \in \mathbb{N}\cup\{0\}. \tag{3.50}$$

Proof. The first assertion can be found in [79, Th. 4.3], and (3.50) is an application of [79, Th. 3.5] to our particular case. **Q.E.D.**

Hence $\mathbf{R}_a^{-1}\widehat{\mathbf{V}}_a$ is a transmutation operator for $\mathbf{L}_a, \mathbf{D}^2$. In contrast to this, $\mathbf{T}_a$ is not a transmutation operator in the sense of Definition 22 . However it satisfies condition 2 in $C^3[-b,b]$.

Theorem 63. *The following relations hold.*

1. $\widehat{\mathbf{V}}_a = \mathbf{R}_a\mathbf{V}_a$ *in* $C[-b,b]$.
2. $\dfrac{\partial K_a(x,t)}{\partial t} = \dfrac{1}{a(x)}\widehat{K}_a(x,t) \quad \forall (x,t)\in\mathcal{R}.$

Proof. First note that according to (3.50), for all $k\in\mathbb{N}_0$ we have

$$\mathbf{R}_a\mathbf{V}_a[x^k] = \mathbf{R}_a\varphi_a^{(k)}(x) = \psi_a^{(k)}(x) = \widehat{\mathbf{V}}_a[x^k].$$

Thus $\mathbf{R}_a\mathbf{V}_a$ and $\widehat{\mathbf{V}}_a$ coincide on the dense set $\operatorname{Span}\{x^k\}_{k=0}^{\infty}$. Hence $\mathbf{R}_a\mathbf{V}_a = \widehat{\mathbf{V}}_a$ on $C[-b,b]$, or equivalently $\mathbf{V}_a = \mathbf{R}_a^{-1}\widehat{\mathbf{V}}_a$. Expanding this equality we have

$$\begin{aligned}\mathbf{V}_a u(x) =& \frac{u(x)}{a(x)} + \int_{-x}^{x}\frac{\partial_t K_a(x,t)}{\partial t}u(t)dt,\\ \mathbf{R}_a^{-1}\widehat{\mathbf{V}}_a u(x) =& \frac{1}{a(x)}\left(u(x) + \int_{-x}^{x}\widehat{K}_a(x,t)u(t)dt\right).\end{aligned}$$

Thus $\displaystyle\int_{-x}^{x}\frac{\partial K_a(x,t)}{\partial t}u(t)dt = \int_{-x}^{x}\frac{1}{a(x)}\widehat{K}_a(x,t)u(t)dt$, for all $u\in C[-b,b]$ and for all $x\in[-b,b]$. For a fixed $x\in[-b,b]$ the integrals are equal for all $u\in C[-x,x]$. Hence $\frac{\partial K_a(x,t)}{\partial t} = \frac{1}{a(x)}\widehat{K}_a(x,t)$, for all $|t|\leqslant|x|$. Due to the arbitrariness of $x\in[-b,b]$ the equality is valid in $\mathcal{R}$. **Q.E.D.**

Remark 64. *If $a \in W^{2,\infty}(-b,b)$, as noticed in Remark 47, $K_a \in W^{2,\infty}(\Omega)$. In this case $Q_a \in L_\infty(-b,b)$, the kernel $\widehat{K}_a \in W^{1,\infty}(\Omega)$ and the operator $\widehat{\mathbf{V}}_a$ is a transmutation operator for $\mathbf{S}_a$,$\mathbf{D}^2$ in the class $W^{2,\infty}(-b,b)$ (Theorem 25). Repeating the same procedure we can obtain that relation 2 of Theorem 63 is still valid.*

Example 65. *Consider the equation $-y'' + cy' = \lambda y$, with $c \in \mathbb{R} \setminus \{0\}$. In this case $q = c$, $p = e^{-cx}$, $a = e^{-\frac{1}{2}cx}$, and the potential of the corresponding Schrödinger equation is given by $Q_a = \frac{c^2}{4}$. Thus we obtain the Schrödinger equation $-u'' + \frac{1}{4}c^2 u = \lambda u$. Denote the Schrödinger operator by $\mathbf{S}_a := \mathbf{D}^2 + C$, with $C = -\frac{1}{4}c^2$. Then the transmutation kernel $\widehat{K}_a$ satisfying $\widehat{\mathbf{V}}_a[1] = a = e^{-\frac{c}{2}x}$, $\widehat{K}_a(x,x) = -\frac{1}{2}a'(0) + \frac{1}{2}\int_0^x Q_a(s)ds = \frac{c}{4} - \frac{1}{8}c^2 x$, $\widehat{K}_a(x,-x) = \frac{1}{4}c$ is given by*

$$\widehat{K}_a(x,t) = \frac{1}{4}c + G(x,t) + \frac{1}{4}c\int_t^x (G(x,t) - G(x,-t))dt,$$

where G is a solution of $\frac{\partial^2 G}{\partial x^2} + CG = \frac{\partial^2 G}{\partial t^2}$ satisfying the Goursat conditions $G(x,x) = -\frac{Cx}{2}$, $G(x,x) = 0$ (see [24]). The solution of this problem is given by

$$G(x,t) = -\frac{1}{2}\frac{\sqrt{C(x^2-t^2)}J_1(\sqrt{C(x^2-t^2)})}{x-t}$$

(see [88, Example 4.3]). Moreover, it is known that $G(x,t) - G(x,-t) = -\frac{t\sqrt{C(x^2-t^2)}J_1(\sqrt{C(x^2-t^2)})}{x^2-t^2}$. Using $J_0'(z) = J_1(z)$ we have

$$-\int_t^x \frac{s\sqrt{C(x^2-s^2)}J_1(\sqrt{C(x^2-s^2)})}{x^2-s^2}ds = \int_t^x \frac{d}{ds}\left(J_0(\sqrt{C(x^2-s^2)})\right)ds$$
$$= 1 - J_0(\sqrt{C(x^2-t^2)}).$$

Then

$$\widehat{K}_a(x,t) = \frac{c+1}{4} - \frac{1}{2}\frac{\sqrt{C(x^2-t^2)}J_1(\sqrt{C(x^2-t^2)})}{x-t} + J_0(\sqrt{C(x^2-t^2)}).$$

Hence the transmutation kernel $K_1(x,t)$ has the form

$$K_a(x,t) = \frac{(c+1)}{8c}\left(e^{\frac{c}{2}t} - 1\right) + \int_0^t e^{\frac{cs}{2}}\left(J_0(\sqrt{C(x^2-s^2)}) - \frac{1}{2}\frac{\sqrt{C(x^2-s^2)}J_1(\sqrt{C(x^2-s^2)})}{x-s}\right)ds.$$

Remark 66. *Considering operators corresponding to $\frac{1}{a}$ instead of a we obtain the factorization*

$$\mathbf{L}_{\frac{1}{a}}u = a\left(a\mathbf{D}\frac{1}{a}\right)\left(\frac{1}{a}\mathbf{D}a\right)\frac{u}{a} = a\left(\mathbf{D} - \frac{a'}{a}\right)\left(\mathbf{D} + \frac{a'}{a}\right)\frac{u}{a} \quad \forall u \in C^2[-b,b].$$

Note that in this case $\mathbf{L}_{\frac{1}{a}} = \mathbf{D}^2 + q_a(x)$*, then* $q_{\frac{1}{a}} = -q_a$*. If* $a \in C^2[-b,b]$*, the last factorization can be written as*

$$\mathbf{L}_{\frac{1}{a}} = \mathbf{R}_a \mathbf{S}_{\frac{1}{a}} \mathbf{R}_a^{-1} \quad in \;\; C^2[-b,b]$$

where

$$\mathbf{S}_{\frac{1}{a}} = \left(\mathbf{D} - \frac{a'}{a}\right)\left(\mathbf{D} + \frac{a'}{a}\right) = \mathbf{D}^2 - Q_{\frac{1}{a}}(x), \quad and \;\; Q_{\frac{1}{a}} = 2\left(\frac{a'}{a}\right)^2 - Q_a = a\left(\frac{1}{a}\right)''.$$

The operator $\mathbf{S}_{\frac{1}{a}}$ *is known as the Darboux associated operator of* $\mathbf{S}_a$*. Generalizing this concept, the operator* $\mathbf{L}_{\frac{1}{a}}$ *will be called the Darboux associated operator of* $\mathbf{L}_a$*. The corresponding Darboux associated canonical kernel will be denoted by* $K_{\frac{1}{a}}(x,t)$*.*

3.2.5 Existence of the inverse operator

Let $a \in C^2[-b,b]$. By Proposition 51 $\mathbf{T}_a^{-1}$ exists, and the relation holds

$$\mathbf{D}^2 \mathbf{T}_a^{-1} v = \mathbf{T}_a^{-1} \mathbf{L}_a v \quad \forall v \in C^3[-b,b]. \tag{3.51}$$

Consider an operator of the form $\mathbf{Q}v(x) = v(x) - \int_{-x}^{x} G(x,t)v'(t)dt$ with a kernel $G \in C^2(\mathcal{R})$. Suppose that $\mathbf{Q}$ satisfies relation (3.51). Deriving with respect to x we obtain

$$\begin{aligned}
\frac{d}{dx}\mathbf{Q}v(x) =& v'(x) - G(x,x)v'(x) - G(x,-x)v'(-x) - \int_{-x}^{x} \frac{\partial G(x,t)}{\partial x} v'(t)dt,\\
\frac{d^2}{dx^2}\mathbf{Q}v(x) =& v''(x) - \frac{d}{dx}G(x,x)v'(x) - G(x,x)v''(x) - \frac{d}{dx}G(x,-x)v'(x) + G(x,x)v''(-x)\\
&- \frac{\partial G(x,x)}{\partial x}v'(x) - \frac{\partial G(x,-x)}{\partial x}v'(-x) - \int_{-x}^{x} \frac{\partial^2 G(x,t)}{\partial x^2} v'(t)dt.
\end{aligned}$$

On the other hand

$$\mathbf{Q}\mathbf{L}_a v(x) = v''(x) - q_a(x)v'(x) - \int_{-x}^{x} G(x,t)v'''(t)dt + \int_{-x}^{x} G(x,t)\left(q_a(t)v'(t)\right)' dt.$$

Integration by parts as in the proof of Theorem 50 leads to

$$\begin{aligned}
\int_{-x}^{x} G(x,t)v'''(t)dt =& G(x,x)v''(x) - G(x,-x)v''(-x) - \frac{\partial G(x,x)}{\partial t}v'(x) + \frac{\partial G(x,-x)}{\partial t}v'(-x)\\
&+ \int_{-x}^{x} \frac{\partial^2 G(x,t)}{\partial t^2} v'(t)dt.
\end{aligned}$$

On the other hand

$$\int_{-x}^{x} G(x,t)\left(q_a(t)v'(t)\right)' dt = G(x,x)q_a(x)v''(x) - G(x,-x)q_a(-x)v(-x) - \int_{-x}^{x} q_a(t)\frac{\partial G(x,t)}{\partial t}v'(t)dt.$$

Hence

$$\begin{aligned}\mathbf{QL}_a v(x) =& v''(x) - q_a(x)v'(x) - G(x,x)v''(x) + G(x,-x)v''(-x) + \frac{\partial G(x,x)}{\partial t}v'(x)\\ &- \frac{\partial G(x,-x)}{\partial t}v'(-x) + G(x,x)q_a(x)v'(x) - G(x,-x)q_a(-x)v'(-x)\\ &- \int_{-x}^{x}\left(\frac{\partial^2}{\partial t^2} + q_a(t)\frac{\partial}{\partial t}\right)G(x,t)v'(t)dt.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{D}^2\mathbf{Q}v(x) - \mathbf{QL}_a v(x) =& \int_{-x}^{x}\left(\frac{\partial^2}{\partial t^2} + q_a(t)\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)G(x,t)v'(t)dt\\ &- \left\{\frac{d}{dx}G(x,x) + \frac{\partial G(x,x)}{\partial x} + \frac{\partial G(x,x)}{\partial t} + q_a(x)(G(x,x)-1)\right\}v'(x)\\ &- \left\{\frac{d}{dx}G(x,x) + \frac{\partial G(x,x)}{\partial x} - \frac{\partial G(x,x)}{\partial t} + q_a(x)(G(x,x)-1)\right\}v'(-x).\end{aligned}$$

In particular, relation (3.51) holds if G satisfies the equation

$$\frac{\partial^2 G(x,t)}{\partial x^2} = \frac{\partial^2 G(x,t)}{\partial t^2} + q_a(t)\frac{\partial G(x,t)}{\partial t} \quad \text{in } \mathcal{R}, \tag{3.52}$$

with the Goursat conditions

$$\frac{d}{dx}G(x,x) = -\frac{1}{2}q_a(x)(G(x,x)-1), \quad G(x,-x) = 0. \tag{3.53}$$

With the continuity condition $G(0,0) = 0$ we have

$$G(x,x) = 1 - e^{-\frac{1}{2}\int_0^x q_a(s)ds} = 1 - e^{\frac{1}{2}\int_0^x q_{\frac{1}{a}}(s)ds} = 1 - a(x).$$

If we consider the kernel $K_{\frac{1}{a}}$ of the Darboux associated operator in the domain $\overline{\Omega}\setminus\mathcal{R}$, then the function $G(x,t) := K_{\frac{1}{a}}(t,x)$ belongs to $C^2(\mathcal{R})$ and satisfies equation (3.13) with the Goursat conditions (3.53).

Now we define the operator

$$\mathbf{Q}_a v(x) := v(x) - \int_{-x}^{x} K_{\frac{1}{a}}(t,x)v'(t)dt, \tag{3.54}$$

in such a way that it satisfies relation (3.51). Note that $\mathbf{Q}_a$ is well defined and belongs to $\mathcal{B}(C^1[-b,b])$ even if $a \in C^1[-b,b]$. Furthermore, if $q_n \to q$, $n \to \infty$ in $C[-b,b]$ (that implies that the corresponding sequence $\{a_n\}$ converges to a in $C^1[-b,b]$), then $K_{\frac{1}{a},n} \to K_{\frac{1}{a}}$, in $C^1(\overline{\Omega})$ and $\mathbf{Q}_{a,n} \to \mathbf{Q}_a$ in $\mathcal{B}(C^1[-b,b])$.

Theorem 67. *Let $a \in C^1[-b,b]$, then $\mathbf{T}_a \in \mathcal{G}\left(C^1[-b,b]\right)$ and $\mathbf{T}_a^{-1} = \mathbf{Q}_a$.*

Proof. First suppose $a \in C^2[-b,b]$. Then $\mathbf{Q}_a$ satisfies relation (3.51). Let $v \in C^1[-b,b]$, then

$$\begin{aligned}(\mathbf{Q}_a v)(0) =& v(0),\\ (\mathbf{Q}_a v)'(0) =& \left(1 - K_{\frac{1}{a}}(0,0)\right) v'(0) = v'(0).\end{aligned}$$

Thus $\mathbf{Q}_a$ preserves the conditions at $x = 0$. Let us define $\psi_k(x) := \mathbf{Q}_a \varphi_a^{(k)}(x)$ for $k \in \mathbb{N}_0$. Note that $\psi_0(x) = 1$, and ψ_1 satisfies $\mathbf{D}^2\psi_1 = 0$ with the conditions $\psi_1(0) = \varphi_a^{(1)}(0) = 0$, $(\psi_1)'(0) = (\varphi_a^{(1)})'(0) = 1$, which implies that $\psi_1(x) = x$. For $k \geqslant 2$ we have

$$\mathbf{D}^2\psi_k = \mathbf{D}^2\mathbf{Q}_a\varphi_a^{(k)} = \mathbf{Q}_a\mathbf{L}_a\varphi_a^{(k)} = k(k-1)\mathbf{Q}_a\varphi_a^{(k-2)} = k(k-1)\psi_{k-2}.$$

Thus $\{\psi_k\}_{k=0}^{\infty}$ is a $\mathbf{D}^2$-basis. According to [20, Remark 9], ψ_k can be constructed for $k \geqslant 2$ by the formula

$$\begin{aligned}\psi_k(x) =& k(k-1)\int_0^x \frac{\psi_0(s)\psi_1(s) - \psi_0(x)\psi(s)}{W(\psi_0,\psi_1)}\psi_{k-2}(s)ds\\ =& k(k-1)\int_0^x (x-s)\psi_{k-2}(s)ds = k(k-1)\int_0^x\int_0^t \psi_{k-2}(t)dsdt.\end{aligned}$$

By induction we can see that $\psi_k(x) = x^k$. Then $\mathbf{Q}_a\varphi_a^{(k)} = x^k$ for all $k \in \mathbb{N}_0$, i.e.,

$$\mathbf{Q}_a\mathbf{T}_a x^k = x^k \quad \forall k \in \mathbb{N}_0.$$

Hence $\mathbf{Q}_a\mathbf{T}_a = \mathbf{1}_{C^1[-b,b]}$ on the dense set $\operatorname{Span}\{x^k\}_{k=0}^{\infty}$. Since both operators are continuous we conclude that $\mathbf{Q}_a\mathbf{T}_a = \mathbf{1}_{C^1[-b,b]}$. By Proposition 51, $\mathbf{T}_a \in \mathcal{G}\left(C^1[-b,b]\right)$. Hence $\mathbf{T}_a^{-1} = \mathbf{Q}_a$.

Now suppose that $a \in C^1[-b,b]$. Take a sequence $\{q_n\} \subset C^1[-b,b]$ such that $q_n \to q$, $n \to \infty$ in $C^1[-b,b]$. Hence $a_n \to a$, $n \to \infty$ in $C^1[-b,b]$, and $\mathbf{T}_{a_n}^{-1} = \mathbf{Q}_{a_n} \to \mathbf{Q}_a$, $n \to \infty$, in $\mathcal{B}\left(C^1[-b,b]\right)$.

Thus, $\mathbf{T}_a\mathbf{Q}_a = \lim_{n\to\infty}\mathbf{T}_{a_n}\mathbf{Q}_{a_n} = \mathbf{I}_{C^1[-b,b]}$. Similarly $\mathbf{Q}_a\mathbf{T}_a = \mathbf{I}_{C^1[-b,b]}$. Hence $\mathbf{T}_a \in \mathcal{G}\left(C^1[-b,b]\right)$ and $\mathbf{T}_a^{-1} = \mathbf{Q}_a$. **Q.E.D.**

Remark 68. *In contrast to the case of the Schrödinger operator, where the kernel of the inverse of the transmutation operator is directly related to the kernel of the initial problem (see [87, Th. 10]), in this case the kernel of the inverse operator comes from the associated Darboux operator.*

3.3 Fourier-Legendre series expansion of the transmutation kernel

In this section we propose a Fourier-Legendre series representation for the kernel K_h, study some of its properties and derive formulas for its coefficients. Following [81] we employ the fact that for $x \in (0, b]$ fixed the kernel $K_a^h(x, \cdot) \in L_2(-x, x)$, so it can be expanded into a series in terms of Legendre polynomials $\{P_n(t)\}_{n=0}^{\infty}$. More precisely the following proposition is valid.

Proposition 69. *For $x \in (0, b]$ fixed the transmutation kernel $K_h(x, t)$ admits the Fourier-Legendre series representation*

$$K_a^h(x,t) = \sum_{n=0}^{\infty} \frac{\alpha_n^h(x)}{x} P_n\left(\frac{t}{x}\right), \tag{3.55}$$

where the coefficients are defined by

$$\alpha_n^h(x) = \left(n + \frac{1}{2}\right) \int_{-x}^{x} K_a^h(x,t) P_n\left(\frac{t}{x}\right) dt \quad \text{for } n \in \mathbb{N}_0. \tag{3.56}$$

The series converges with respect to t in the norm of $L_2(-x, x)$.

Proof. Proposition 44 implies that $K_a^h(x, \cdot) \in L_2(-x, x)$. Since $\left\{P_n\left(\frac{t}{x}\right)\right\}$ is an orthogonal basis for $L_2(-x, x)$, $K_a^h(x, \cdot)$ has a series of the form $K_a^h(x, t) = \sum_{n=0}^{\infty} b_n(x) P_n\left(\frac{t}{x}\right)$, and the series converges with respect to t in $L_2(-x, x)$. For convenience we choose $b_n(x) = \frac{\alpha_n^h(x)}{x}$ and obtain (3.55). Equality (3.56) is obtained by multiplying $K_a^h(x, t)$ by $P_n\left(\frac{t}{x}\right)$ and integrating

$$\begin{aligned}\int_{-x}^{x} K_a^h(x,t) P_n\left(\frac{t}{x}\right) dt &= \int_{-x}^{x} \left(\sum_{m=0}^{\infty} \frac{\alpha_n^h(x)}{x} P_m\left(\frac{t}{x}\right)\right) P_n\left(\frac{t}{x}\right) dt \\ &= \sum_{m=0}^{\infty} \frac{\alpha_n^h(x)}{x} \int_{-x}^{x} P_m\left(\frac{t}{x}\right) P_n\left(\frac{t}{x}\right) dt \\ &= \sum_{m=0}^{\infty} \alpha_n^h(x) \int_{-1}^{1} P_m(u) P_n(u) du = \alpha_n^h(x) \frac{2}{2n+1}.\end{aligned}$$

The exchange of the order of integration and summation is justified since the integral is a bounded functional in $L_2(-x, x)$, and the last equality is due to $\|P_n\|_{L_2(-1,1)} = \frac{2}{2n+1}$. Thus we obtain (3.56). **Q.E.D.**

Corollary 70. *The sine and cosine kernels admit the Fourier-Legendre series representations*

$$K_{a,h}^{C}(x,t)=\sum_{n=0}^{\infty}\frac{\beta_n^h(x)}{x}P_{2n+1}\left(\frac{t}{x}\right) \tag{3.57}$$

and

$$K_{a,h}^{S}(x,t)=\sum_{n=0}^{\infty}\frac{\gamma_n^h(x)}{x}P_{2n}\left(\frac{t}{x}\right). \tag{3.58}$$

The coefficients are given by

$$\beta_n^h(x)=(4n+3)\int_0^x K_h^C(x,t)P_{2n+1}\left(\frac{t}{x}\right)dt, \tag{3.59}$$

$$\gamma_n^h(x)=(4n+1)\int_0^x K_h^S(x,t)P_{2n}\left(\frac{t}{x}\right)dt. \tag{3.60}$$

Proof. Using (56) and (3.55) we obtain

$$K_{a,h}^{C}(x,t)=K_a^h(x,t)-K_a^h(x,-t)=\sum_{n=0}^{\infty}\frac{\alpha_n^h(x)}{x}\left(P_n\left(\frac{t}{x}\right)-P_n\left(-\frac{t}{x}\right)\right).$$

Since $P_n(-t)=(-1)^nP_n(t)$, $K_{a,h}^{C}(x,t)=\sum_{n=0}^{\infty}2\frac{\alpha_{2n+1}^h(x)}{x}P_{2n+1}\left(\frac{t}{x}\right)$. Denoting $\beta_n^h(x):=2\alpha_{2n+1}(x)$ we obtain (3.57), and (3.59) is obtained from

$$\begin{aligned}\beta_n^h(x)=&2\left((2n+1)+\frac{1}{2}\right)\int_{-x}^{x}K_a^h(x,t)P_{2n+1}\left(\frac{t}{x}\right)dt\\=&(4n+3)\left\{\int_0^x K_a^h(x,t)P_{2n+1}\left(\frac{t}{x}\right)dt+\int_{-x}^{0}K_a^h(x,t)P_{2n+1}\left(\frac{t}{x}\right)dt\right\}\\=&(4n+3)\left\{\int_0^x K_a^h(x,t)P_{2n+1}\left(\frac{t}{x}\right)dt+\int_0^x K_a^h(x,-t)P_{2n+1}\left(-\frac{t}{x}\right)dt\right\}\\=&(4n+3)\int_0^x K_{a,h}^C(x,t)P_{2n+1}\left(\frac{t}{x}\right)dt.\end{aligned}$$

Formulas (3.58) and (3.60) are obtained analogously. **Q.E.D.**

Remark 71. *Note that* $K_{a,h}^C(x,x)=\sum_{n=0}^{\infty}\frac{\beta_n^h(x)}{x}P_{2n+1}(1)=\sum_{n=0}^{\infty}\frac{\beta_n^h(x)}{x}$*, because* $P_k(1)=1$ *for all* $k\in\mathbb{N}_0$*. Since* $K_{a,h}^C(x,x)=1-a^{-1}(x)$*, we obtain the equality*

$$a^{-1}(x)=1-\sum_{n=0}^{\infty}\frac{\beta_n^h(x)}{x}. \tag{3.61}$$

Remark 72. *Representations for* $K_{a,h}^C(x,t)$ *and* $K_h^S(x,t)$ *can be obtained in another way by using the fact that* $K_{a,h}^C(x,\cdot),K_{a,h}^S(x,\cdot)\in L_2(0,x)$ *and* $\left\{P_{2n+1}\left(\frac{t}{x}\right)\right\}_{n=0}^{\infty}$, $\left\{P_{2n}\left(\frac{t}{x}\right)\right\}_{n=0}^{\infty}$

are orthogonal bases in $L_2(0,x)$. The fact that $K^C_{a,h}$ is represented by a series of odd Legendre polynomials is in agreement with the property $K^C_{a,h}(x,0)=0$.

Furthermore, if $q\in L_1(0,b)$, the kernels $K^C_{a,h}(x,t)$ and $K^S_{a,h}(x,t)$ exist and satisfy $K^C_{a,h}(x,\cdot)$, $K^S_{a,h}(x,\cdot)\in L_2(0,x)$, as seen in Section 2. Hence the representations (3.57) and (3.58) remain valid.

By $K^h_{a,N}(x,t) := \sum_{n=0}^{N}\frac{\alpha_n^h(x)}{x}P_n\left(\frac{t}{x}\right)$ we denote the N-th partial sum of (3.55).

Lemma 73. *Let $p\in\mathbb{N}$ and $f\in C^{p+1}[-1,1]$. There exists a constant $c_p>0$ such that*

$$\|f-f_N\|_{C[-1,1]} \leqslant \frac{c_p}{N^{p+\frac{1}{2}}}\cdot\|f^{(p+1)}\|_{C[-1,1]},\quad \forall N>p, \tag{3.62}$$

where $f_N(x) := \sum_{n=0}^{N} a_nP_n(x)$ is an N-th partial sum of the Fourier-Legendre series of f. The constant c_p does not depend on f.

Proof. See the proof of Theorem 4.10 from [131] with Theorem 5.21 of [135]. **Q.E.D.**

From this we obtain that (3.55) converges uniformly to K_h and an estimate for the remainder $K^h_a - K^h_{a,N}$.

Proposition 74. *Let $q\in C^p[0,b]$, $p\in\{0,1\}$. Denote $M_p = \left\|\frac{\partial^{p+1}K^h_a}{\partial t^{p+1}}\right\|_{C(\mathcal{R}_1)}$. Then there exists a constant $c_p>0$ such that for all $x\in(0,b]$ and $t\in[-x,x]$ the inequality holds*

$$|K^h_a(x,t)-K^h_{a,N}(x,t)| \leqslant \frac{c_pM_px^{p+1}}{N^{p+\frac{1}{2}}}\quad \forall N>p. \tag{3.63}$$

Proof. Proposition 44 establishes that $K^h_a\in C^{p+1}(\mathcal{R}_1)$. Fix $x\in(0,b]$ and define $g(t) := K^h_a(x,xt)$, $g_N(t) := K^h_{a,N}(x,xt)$. Then $g\in C^{p+1}[-1,1]$ and $g^{(p+1)}(t) = x^{p+1}\frac{\partial^{p+1}K^h_a(x,y)}{\partial y}\Big|_{y=tx}$. Since g_N is a partial sum of a Fourier-Legendre series for g, by Lemma 73,

$$\|g-g_N\|_{C[-1,1]} \leqslant \frac{c_p}{N^{p+1}}\|g^{(p+1)}\|_{C[-1,1]}$$

for all $N>p$. Note that $\|g-g_N\|_{C[-1,1]} = \max_{t\in[-x,x]}|K^h_a(x,t)-K^h_{a,N}(x,t)|$, and $\|g^{(p+1)}\|_{C[-1,1]} = \max_{t\in[-x,x]}\left|x^{p+1}\frac{\partial^{p+1}K^h_a(x,t)}{\partial t^{p+1}}\right| \leqslant x^{p+1}M_p$. From this we obtain (3.63). Since c_p does not depend on x, the estimate is valid for all $x\in(0,b]$. **Q.E.D.**

Similar estimates are valid for $K^C_{a,h}$ and $K^S_{a,h}$.

For every $n \in \mathbb{N} \cup \{0\}$ we write the Legendre polynomial $P_n(x)$ in the form $P_n(x) = \sum_{k=0}^{n} l_{k,n} x^k$. Note that if n is even, $l_{k,n} = 0$ for odd k, and $P_{2n}(x) = \sum_{k=0}^{n} \tilde{l}_{k,n} x^{2k}$ with $\tilde{l}_{k,n} = l_{2k,2n}$. Similarly $P_{2n+1}(x) = \sum_{k=0}^{n} \hat{l}_{k,n} x^{2k+1}$ with $\hat{l}_{k,n} = l_{2k+1,2n+1}$. With this notation we write an explicit formula for the coefficients of (3.55) for the canonical kernel.

Proposition 75. *The coefficients $\{\alpha_n^1(x)\}_{n=0}^{\infty}$ of the Fourier-Legendre expansion of $K_1(x,t)$ are given by*

$$\alpha_n^1(x) = \left(n + \frac{1}{2}\right) \sum_{k=0}^{n} \frac{l_{k,n}}{k+1} \left(\frac{x^{k+1} - \varphi_a^{(k+1)}(x)}{x^k} \right) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.64}$$

The coefficients of the cosine and sine kernels satisfy the following relations for all $n \in \mathbb{N} \cup \{0\}$

$$\beta_n^1(x) = (4n+3) \sum_{k=0}^{n} \frac{\tilde{l}_{k,n}}{2k+2} \left(\frac{x^{2k+2} - \varphi_a^{(2k+2)}(x)}{x^{2k+1}} \right), \tag{3.65}$$

$$\gamma_n^1(x) = (4n+1) \sum_{k=0}^{n} \frac{\hat{l}_{k,n}}{2k+1} \left(\frac{x^{2k+1} - \varphi_a^{(2k+1)}(x)}{x^{2k}} \right). \tag{3.66}$$

Proof. According to (3.56)

$$\alpha_n^1(x) = \left(n + \frac{1}{2}\right) \int_{-x}^{x} K_a(x,t) P_n\left(\frac{t}{x}\right) dt = \left(n + \frac{1}{2}\right) \sum_{k=0}^{n} \frac{l_{k,n}}{x^k} \int_{-x}^{x} K_a(x,t) t^k dt.$$

Note that

$$\int_{-x}^{x} K_a(x,t) t^k dt = \int_{-x}^{x} K_a(x,t) \frac{d}{dt}\left[\frac{t^{k+1}}{k+1}\right] = \frac{x^{k+1}}{k+1} - \mathbf{T}_a\left[\frac{t^{k+1}}{k+1}\right] = \frac{x^{k+1} - \varphi_a^{(k+1)}(x)}{k+1},$$

by Theorem (54). Substituting this in the previous formula we obtain (3.64). To obtain the formula for $\{\beta_n^1(x)\}_{n=0}^{\infty}$, we see that

$$\begin{aligned} \beta_n^1(x) =& 2\alpha_{2n+1}^1(x) = (4n+3) \sum_{k=0}^{2n+1} \frac{l_{k,2n+1}}{k+1} \left(\frac{x^{k+1} - \varphi_a^{(k+1)}(x)}{x^k} \right) \\ =& (4n+3) \sum_{k=0}^{n} \frac{\tilde{l}_{k,n}}{2k+2} \left(\frac{x^{2k+2} - \varphi_a^{(2k+2)}(x)}{x^{2k+1}} \right). \end{aligned}$$

The proof of (3.66) is analogous. **Q.E.D.**

Remark 76. *Formula (3.65) gives the following expression for $\beta_0^1(x)$:*

$$\beta_0^1(x) = \frac{3}{2} \left(\frac{x^2 - \varphi_a^{(2)}(x)}{x} \right).$$

Applying the operator $\mathbf{L}_a$ *to* $x\beta_0^1(x)$ *we have* $\mathbf{L}_a\left[x\beta_0^1(x)\right] = \frac{3}{2}\left(\mathbf{L}_a[x^2] - \mathbf{L}_a\left[\varphi_a^{(2)}(x)\right]\right)$. *By Remark 55,* $\mathbf{L}_a\left[\varphi_a^{(2)}\right] = 2$, *thus*

$$(x\beta_0^1)'' - q(x)(x\beta_0^1)' = \frac{3}{2}\left(2 - 2xq(x) - 2\right) = -3xq(x).$$

From this we obtain that the potential $q(x)$ *can be recovered from the coefficient* $\beta_0^1(x)$ *by the formula*

$$q(x) = \frac{(x\beta_0^1(x))''}{(x\beta_0^1(x))' - 3x}. \tag{3.67}$$

If q *is real valued, the denominator* $\vartheta(x) = (x\beta_0^1(x))' - 3x$ *does not vanish for* $x \in (0, b]$. *In fact from the equation* $\mathbf{L}_a\left[x\beta_0^1(x) - \frac{3}{2}x^2\right] = -3$ *we deduce that* ϑ *satisfies the first order equation* $(p(x)\vartheta(x))' = -3p(x)$, *for* $x \in (0, b)$, *and* $\vartheta(0) = 0$. *Since* $p > 0$ *in* $[0, b]$, $\vartheta(x) = -\frac{3}{p(x)}\int_0^x p(s)ds$ *does not vanish for* $x \in (0, b]$.

Theorem 77. *Let* $q \in C[0, b]$. *The solutions* $C(\rho, x)$ *and* $S_h(\rho, x)$ *admit the representations*

$$C(\rho, x) = \cos(\rho x) + \rho\sum_{n=0}^{\infty}(-1)^n\beta_n^h(x)j_{2n+1}(\rho x), \quad x \in (0, b] \tag{3.68}$$

and

$$S_h(\rho, x) = \frac{\sin(\rho x)}{\rho} - \sum_{n=0}^{\infty}(-1)^n\gamma_n^h(x)j_{2n}(\rho x), \quad x \in (0, b], \tag{3.69}$$

where j_k *stands for the spherical Bessel function of order* k, *defined as* $j_k(z) := \sqrt{\frac{\pi}{2z}}J_{k+\frac{1}{2}}(z)$ *(and* J_ν *stands for the Bessel function of the first kind of order* ν*). Both series converge uniformly with respect to* x *on* $(0, b]$ *and converge uniformly with respect to* ρ *on any compact subset of the complex* ρ*-plane. Moreover, for each* $N \in \mathbb{N}$, *the approximations*

$$C_N(\rho, x) := \cos(\rho x) + \rho\sum_{n=0}^{\left[\frac{N-1}{2}\right]}(-1)^n\beta_n^h(x)j_{2n+1}(\rho x) \tag{3.70}$$

and

$$S_{h,N}(\rho, x) := \frac{\sin(\rho x)}{\rho} - \sum_{n=0}^{\left[\frac{N}{2}\right]}(-1)^n\gamma_n^h(x)j_{2n+1}(\rho x), \tag{3.71}$$

obey the estimates

$$\left|\frac{C(\rho, x) - C_N(\rho, x)}{\rho}\right| \leqslant 2x\varepsilon_N(x), \quad |S_h(\rho, x) - S_{h,N}(\rho, x)| \leqslant 2x\varepsilon_N(x), \tag{3.72}$$

for any $\rho \in \mathbb{R} \setminus \{0\}$*, and*

$$\left|\frac{C(\rho,x)-C_N(\rho,x)}{\rho}\right| \leqslant \frac{2\varepsilon_N(x)\sinh(Cx)}{C}, \quad |S_h(\rho,x)-S_{h,N}(\rho,x)| \leqslant \frac{2\varepsilon_N(x)\sinh(Cx)}{C}, \tag{3.73}$$

for any $\rho \in \mathbb{C}\setminus\{0\}$ *belonging to the strip* $|\operatorname{Im}\rho| \leqslant C$*,* $C > 0$*, where* $\varepsilon_N(x)$ *is a sufficiently small nonnegative function such that* $|K_h(x,t) - K_{h,N}(x,t)| \leqslant \varepsilon_N(x)$*,* $\varepsilon_N(x) \to 0$ *when* $N \to \infty$*.*

Proof. We show the corresponding results for $C(\rho, x)$ (the proofs for $S_h(\rho, x)$ are analogous). Substitution of $K_h(x,t)$ in the form of the series (3.55) into the equality $C(\rho, x) = \mathbf{T}_h[\cos(\rho x)]$ leads to the equalities

$$\begin{aligned}
C(\rho,x) &= \cos(\rho x) + \rho \int_{-x}^{x} K_a^h(x,t)\sin(\rho t)dt \\
&= \cos(\rho x) + \rho \int_{-x}^{x} \left(\sum_{n=0}^{\infty} \frac{\alpha_n^h(x)}{x} P_n\left(\frac{t}{x}\right)\right)\sin(\rho t)dt \\
&= \cos(\rho x) + \rho \sum_{n=0}^{\infty} \frac{\alpha_n^h(x)}{x} \int_{-x}^{x} P_n\left(\frac{t}{x}\right)\sin(\rho t)dt \\
&= \cos(\rho x) + \rho \sum_{n=0}^{\infty} \frac{\alpha_{2n+1}^h(x)}{x} \int_{-x}^{x} P_{2n+1}\left(\frac{t}{x}\right)\sin(\rho t)dt,
\end{aligned}$$

because $\int_{-x}^{x} P_n\left(\frac{t}{x}\right)\sin(\rho t)dt = 0$ if n is even. Using formula 2.17.7 from [119, pp. 433]

$$\int_0^a \begin{Bmatrix} P_{2n+1}\left(\frac{y}{a}\right)\cdot\sin(by) \\ P_{2n}\left(\frac{y}{a}\right)\cdot\cos(by) \end{Bmatrix} dy = (-1)^n \sqrt{\frac{\pi a}{2b}} J_{2n+\delta+\frac{1}{2}}(ab), \quad \delta = \begin{Bmatrix} 1 \\ , \\ 0 \end{Bmatrix} \quad a > 0, \tag{3.74}$$

we obtain the representation

$$\begin{aligned}
C(\rho,x) &= \cos(\rho x) + 2\rho \sum_{n=0}^{\infty} \frac{\alpha_{2n+1}^h(x)}{x} \int_0^x P_{2n+1}\left(\frac{t}{x}\right)\sin(\rho t)dt \\
&= \cos(\rho x) + 2\rho \sum_{n=0}^{\infty} \frac{\alpha_{2n+1}^h(x)}{x} (-1)^n \sqrt{\frac{\pi x}{2\omega}} J_{2n+1+\frac{1}{2}}(\rho x) \\
&= \cos(\rho x) + \rho \sum_{n=0}^{\infty} (-1)^n \beta_n^h(x) j_{2n+1}(\rho x).
\end{aligned}$$

The convergence of the series with respect to ρ can be established using the fact that for each x, (3.68) is a NSBF for the entire function $C(\cdot, x)$. The radius of convergence of the Neumann series coincides with the radius of convergence of its associated power series (obtained from the SPPS representation) (see [141, pp. 524-526]), hence the series (3.68) converges uniformly on every compact subset of the complex ρ-plane.

For the estimates note that $C_N(\rho,x) = \cos(\rho x) + \rho\int_{-x}^{x} K_{a,N}^h(x,t)\sin(\rho t)dt$, hence

$$|C(\rho,x) - C_N(\rho,x)| \leqslant |\rho| \int_{-x}^{x} |K_h(x,t) - K_{h,N}(x,t)||\sin(\rho t)|dt.$$

By Proposition 74 we can take $\varepsilon_N(x) = \frac{c_0 M_0 x}{N^{\frac{3}{2}}}$. For $\rho \in \mathbb{R}\setminus\{0\}$ we have

$$|C(\rho,x) - C_N(\rho,x)| \leqslant |\rho|\varepsilon_N(x)\int_{-x}^{x} |\sin(\rho t)|dt = 2|\rho|x\varepsilon_N(x).$$

On the other hand, if $\rho \in \mathbb{C}\setminus\{0\}$ belongs to the strip $|\operatorname{Im}\rho| \leqslant C$, then

$$\begin{aligned}
|C(\rho,x) - C_N(\rho,x)| \leqslant& |\rho|\varepsilon_N(x)\int_{-x}^{x} |\sin(\rho t)|dt \\
\leqslant& |\rho|\varepsilon_N(x)\int_{-x}^{x} \frac{e^{|\operatorname{Im}\rho t|} + e^{-|\operatorname{Im}\rho t|}}{2}dt \\
=& 2|\rho|\varepsilon_N(x)\int_{0}^{x} \cosh(|\operatorname{Im}\rho t|)dt \\
=& 2|\rho|\varepsilon_N(x)\frac{\sinh(|\operatorname{Im}\rho x|)}{|\operatorname{Im}\rho|}.
\end{aligned}$$

Since the function $\frac{\sinh(\xi x)}{\xi}$ is increasing in both variables when $\xi, x \geqslant 0$, we obtain (3.73). Hence, if $x \in (0,b]$ then

$$|C(\rho,x) - C_N(\rho,x)| \leqslant 2|\rho|\max\left\{b, \frac{\sinh(Cb)}{C}\right\}\cdot\frac{c_0 M_0 b}{N^{\frac{3}{2}}}.$$

The right hand side tends to zero when $N\to\infty$, thus (3.68) converges uniformly in x. **Q.E.D.**

Remark 78. *In the case $q \in L_1(0,b)$ the representations (3.68) and (3.69) remain valid. For $C(\rho,x)$, using (3.17) and the evenness of $C(\rho,x)$ in ρ we obtain the representation $C(\rho,x) = \cos(\rho x) + \rho\int_0^x K_{a,h}^C(x,t)\sin(\rho t)dt$ with $K_{a,h}^C(x,\cdot) \in L_2(0,x)$. According to Remark 72 this kernel admits the representation (3.57). Substituting it into the integral we obtain (3.68). In this case the estimate for the remainder is obtained with the aid of the Cauchy-Bunyakovsky-Schwarz inequality. Suppose that $\rho \in \mathbb{C}\setminus\{0\}$ and denote $\tau = \operatorname{Im}\rho$. For $|\tau| \leqslant C$ we have*

$$\begin{aligned}
|C(\rho,x) - C_N(\rho,x)| \leqslant& \int_0^x |K_{a,h}^C(x,t) - K_{a,h,N}(x,t)||\rho\sin(\rho t)|dt \\
&\leqslant \left\|K_{a,h}^C(x,\cdot) - K_{a,h,N}(x,\cdot)\right\|_{L_2(0,x)}\cdot|\rho|\|\sin(\rho t)\|_{L_2(0,x)},
\end{aligned}$$

and

$$\|\sin(\rho t)\|^2_{L_2(0,x)} \leqslant \left(\frac{\|e^{i\rho t}\|_{L_2(0,x)} + \|e^{-i\rho t}\|_{L_2(0,x)}}{2}\right)^2 \leqslant \frac{1}{2}\left(\|e^{i\rho t}\|^2_{L_2(0,x)} + \|e^{-i\rho t}\|^2_{L_2(0,x)}\right)$$
$$= \int_0^x \cosh(2\tau t)dt = \frac{\sinh(2\tau x)}{2\tau}.$$

Thus,

$$|C(\rho,x) - C_N(\rho,x)| \leqslant \epsilon_N(x)|\rho|\frac{\sinh(2Cx)}{2C},$$

where $\epsilon_N(x) = \left\|K^C_{a,h}(x,\cdot) - K^C_{a,h,N}(x,\cdot)\right\|_{L_2(0,x)}$. *Similarly for* $S_h(\rho,x)$ *we obtain the estimate*

$$|S_h(\rho,x) - S_{h,N}(\rho,x)| \leqslant \epsilon_N(x)\frac{\sinh(2Cx)}{2C}.$$

Remark 79. *Representations (3.68) and (3.69) are useful for solving direct spectral problems related to (3.1). For example, finding the eigenvalues of a Dirichlet problem reduces to finding zeros of an analytic function in the form (3.68). Furthermore, expression (3.70) can be used to find the approximate eigenvalues. For a Neumann or a more general spectral problem, it is possible to find the NSBF representations for the derivatives* $C'(\rho,x)$ *and* $S'_h(\rho,x)$ *(see [81, Sec. 7] and [93], for examples of how to use the Neumann series in solving spectral problems).*

Chapter 4

Solution of the inverse problem for the SLEIF on finite intervals

In this chapter we consider the inverse problem for Eq. (1.46) in the interval $(0, \pi)$, with the Neumann-Neumann boundary conditions. The impedance function a is assumed to be positive in $[0, \pi]$. The inverse problem consists in recovering the impedance function from the spectral data which are the eigenvalues of the Sturm-Liouville problem and the norming constants defined below by (4.2). A corresponding Gelfand-Levitan integral equation is derived. A Fourier-Legendre series expansion for the transmutation operator kernel combined with the Gelfand-Levitan equation leads to a simple direct method for solving the inverse problem of recovering the impedance function from spectral data by solving a system of linear algebraic equations, such that the impedance function is recovered from the first element of the solution vector. Stability of the method is proved. Its numerical performance is illustrated by several examples.

4.1 Properties of the Neumann problem

Suppose that $p \in W^{1,\infty}(0, \pi)$ with $p(x) > 0$ for all $x \in [0, \pi]$ and $p(0) = 1$. Consider the spectral problem

$$\mathcal{N} = \begin{cases} -(p(x)y')' = \lambda p(x)y & x \in (0, \pi), \\ y'(0) = y'(\pi) = 0. \end{cases}$$

With the problem $\mathcal{N}$ the following unbounded operator in the Hilbert space $\mathcal{H}_p := L_2(0,\pi;p(x)dx)$ is associated, $\mathbf{L}_{\mathcal{N}} : \mathcal{D}(\mathbf{L}_{\mathcal{N}}) \subset \mathcal{H}_p \to \mathcal{H}_p$ with the domain

$$\mathscr{D}(\mathbf{L}_{\mathcal{N}}) := \left\{ y \in H^2(0,\pi) \,|\, y'(0) = y'(\pi) = 0 \right\},$$

and acting as $\mathbf{L}_{\mathcal{N}} y := -\frac{1}{p}(py')'$. The operator is densely defined and symmetric.

Proposition 80. *The operator $\mathbf{L}_{\mathcal{N}}$ is self-adjoint and semi-positive in $\mathcal{H}_p$, and the spectrum $\sigma(\mathbf{L}_{\mathcal{N}})$ is purely discrete. The eigenvalues are simple and satisfy*

$$0 \leqslant \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty. \tag{4.1}$$

Proof. According to [114, Th. 2.7.4] the operator $\mathbf{L}_{\mathcal{N}}$ is self-adjoint, its spectrum is purely discrete, and the eigenvalues $\sigma(\mathbf{L}_{\mathcal{N}}) = \{\lambda_n\}_{n=0}^{\infty}$ are simple and satisfy $|\lambda_n| \to \infty,\ n \to \infty$.

For $y \in \mathscr{D}(\mathbf{L}_{\mathcal{N}})$ the associated quadratic form satisfies

$$\begin{aligned}\langle \mathbf{L}_{\mathcal{N}} y, y\rangle_{\mathcal{H}} &= -\int_0^{\pi} (p(x)y'(x))'\overline{y(x)}dx \\ &= -\,p(x)y'(x)\overline{y(x)}\Big|_0^{\pi} + \int_0^{\pi} |y'(x)|^2 p(x)dx = \|y'\|_{\mathcal{H}}^2 \geqslant 0.\end{aligned}$$

Hence $\mathbf{L}_{\mathcal{N}}$ is semi-positive, thus $\sigma(\mathbf{L}_{\mathcal{N}}) \subset [0,\infty)$ [122, Th. 13.31]. Rearranging indices in the eigenvalues we obtain (4.1). **Q.E.D.**

Eigenvalues of $\mathcal{N}$ correspond to those of the operator $\mathbf{L}_{\mathcal{N}}$. It is easy to see that $\lambda = \rho^2$ is an eigenvalue iff ρ is a zero of the entire function $C'(\rho,\pi) = 0$. Moreover, since the eigefunctions are real and simple, $C(\rho,x)$ is real valued.

For each eigenvalue $\lambda_n = \rho_n^2$ we define the *norming constant*

$$\alpha_n := \int_0^{\pi} C^2(\rho_n,x)p(x)dx, \tag{4.2}$$

so that $\left\{\frac{1}{\sqrt{\alpha_n}}C(\rho_n,x)\right\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ (see [114, Th. 2.7.4]).

Remark 81. *For the Neumann problem $\mathcal{N}$, $\lambda_0 = 0$ is the first eigenvalue with the eigenfunction $C(0,x) = 1$ and the norming constant is given by $\alpha_0 = \|a\|_{L_2(0,\pi)}^2$.*

Inverse problem. Given an increasing sequence starting with zero $\{\lambda_n\}_{n=0}^{\infty}$ and a sequence of positive numbers $\{\alpha_n\}_{n=0}^{\infty}$, find a real-valued function $a \in W^{1,\infty}(0,\pi)$ such that

$\{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of the problem $\mathcal{N}$ and $\{\alpha_n\}_{n=0}^{\infty}$ its sequence of norming constants.

Suppose $a \in W^{2,\infty}(0,\pi)$. As seen in Subsection 3.2.4, the operator $\mathbf{R}_a : \mathcal{H} \to L_2(0,\pi)$ defined by $\mathbf{R}_a y(x) = a(x)y(x)$ is unitary, and if $y \in \mathcal{D}(\mathbf{L}_{\mathcal{N}})$ then $u = \mathbf{R}_a y$ satisfies the equation $\mathbf{R}_a^{-1}\mathbf{S}_a\mathbf{R}_a y = \mathbf{L}_{\mathcal{N}} y$, where $\mathbf{S}_a = -\mathbf{D}^2 + Q_a(x)$, with $Q_a = \frac{a''}{a}$ and the boundary conditions $u'(0) = a'(0)y(0) = a'(0)u(0)$ (because $a(0) = 1$), $u'(\pi) = a'(\pi)y(\pi) = \frac{a'(\pi)}{a(\pi)}u(\pi)$.

Consider the problem

$$\mathcal{S} = \begin{cases} -u'' + Q_a(x)u = \lambda u \quad x \in (0,\pi), \\ u'(0) - a'(0)u(0) = 0, \\ u'(\pi) - \frac{a'(\pi)}{a(\pi)}u(\pi) = 0. \end{cases} \tag{4.3}$$

If λ is an eigenvalue of $\mathcal{N}$ with an eigenfunction $y(\lambda,x)$ then λ is also an eigenvalue of $\mathcal{S}$ with an eigenfunction $\mathbf{R}_a y(\lambda,x)$. Reciprocally, if λ is an eigenvalue of $\mathcal{S}$ with an eigenfunction $u(\lambda,x)$, it is an eigenvalue of $\mathcal{N}$ with an eigenfunction $y(\lambda,x) = \mathbf{R}_a^{-1}u(\lambda,x)$. That is, *the eigenvalues of $\mathcal{N}$ and $\mathcal{S}$ coincide.*

Consider $\widehat{C}(\rho,x) = \mathbf{R}_a C(\rho,x)$. Then

$$-\widehat{C}'' + Q_a(x)\widehat{C} = \rho^2\widehat{C} \quad \text{and } \widehat{C}(\rho,0) = 1 \;\; \widehat{C}'(\rho,0) = a'(0). \tag{4.4}$$

As seen in subsection 1.4.2, the eigenvalues of $\mathcal{S}$ are determined by the characteristic equation $\widehat{C}'(\rho,\pi) - \frac{a'(\pi)}{a(\pi)}\widehat{C}(\rho,\pi) = 0$. The eigenspace of the eigenvalue $\lambda_n = \rho_n^2$ is generated by $\widehat{C}(\rho_n,x)$, and the corresponding norming constant is

$$\widehat{\alpha}_n = \int_0^{\pi} \left(\widehat{C}(\rho_n,x)\right)^2 dx = \int_0^{\pi} a^2(x)\left(C(\rho_n,x)\right)^2 dx = \alpha_n.$$

In summary, *the spectral data of $\mathcal{N}$* and *$\mathcal{S}$ coincide.* By Theorem 27 they satisfy the asymptotic relations

$$\rho_n = n + \frac{\omega}{n\pi} + \frac{k_n}{n}, \tag{4.5}$$

$$\alpha_n = \frac{\pi}{2} + \frac{K_n}{n}, \tag{4.6}$$

where $\omega = a'(0) - \frac{a'(\pi)}{a(\pi)} + \frac{1}{2}\int_0^{\pi} Q_a(s)ds$ and $\{k_n\}, \{K_n\} \in \ell_2$.

4.2 Gelfand-Levitan equation

Let $\widehat{K}_a(x,t)$ the canonical transmutation kernel of $\mathbf{S}_a$, and denote $\widehat{G}(x,t) := \widehat{K}_a(x,t) + \widehat{K}_a(x,-t)$, the cosine kernel. By Theorem 29, the kernel $\widehat{G}(x,t)$ satisfies the Gelfand-Levitan equation

$$\widehat{G}(x,t) + F(x,t) + \int_0^x \widehat{G}(x,s)F(s,t)ds = 0, \quad \text{for } 0 < t < x. \tag{4.7}$$

Here

$$F(x,t) := \frac{1}{\alpha_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{\cos(\rho_n x)\cos(\rho_n t)}{\alpha_n} - 2\frac{\cos(nx)\cos(nt)}{\pi}\right), \quad \text{for } 0 \leqslant t, x < \pi \tag{4.8}$$

(remember that $\rho_0 = 0$). We recall that $F(x,t) = \frac{A(x-t)-A(x+t))}{2}$, where

$$A(x,t) = \frac{1}{\alpha_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{\cos(\rho_n x)}{\alpha_n} - 2\frac{\cos(nx)}{\pi}\right)$$

(see formula (1.39)). By Lemma 28, $A \in H^1(-\pi,\pi)$ and hence $F(x,t)$ admits a continuous representative for $0 \leqslant x, t \leqslant \pi$ and $\frac{d}{dx}F(x,x) \in L_2(0,\pi)$.

The potential Q_a can be recovered from the formula $Q_a(x) = 2\frac{d}{dx}\widehat{G}(x,x)$, $h = a'(0) = \widehat{G}(0,0)$ and $H = -\frac{a'(\pi)}{a(\pi)} = \omega - h - \frac{1}{2}\int_0^{\pi} Q_a(s)ds$. However, as was shown in [76, 78, 92] and as we discuss below, there are more practical ways to recover Q_a, h and H from (4.7). After having recovered Q_a and h, the function a can be recovered by solving the Cauchy problem

$$\begin{cases} a'' - Q_a(x)a = 0 \quad x \in (0,\pi), \\ a(0) = 1, \\ a'(0) = h. \end{cases}$$

Let us show that in fact a can be recovered directly, without previous recovering Q_a, h and H. We use the notation $G(x,t) = K_a^C(x,t)$ for the canonical cosine transmutation kernel.

Theorem 82 (Gelfand-Levitan equation)**.** *For every $x \in (0,\pi]$ fixed, the kernel $G(x,\xi)$ satisfies the linear integral equation*

$$G(x,\xi) + \Omega(x,\xi) - \int_0^x G(x,s)\widetilde{\Omega}(s,\xi)ds = 0 \quad \textit{for } 0 < \xi < x, \tag{4.9}$$

where $\Omega(x,\xi) := \int_0^\xi F(x,t)dt$ *and* $\widetilde{\Omega}(x,\xi) := \frac{\partial \Omega(x,\xi)}{\partial x}$. *Furthermore, the functions* Ω *and* $\widetilde{\Omega}$ *can be represented by the series*

$$\Omega(x,\xi) = \left(\frac{1}{\alpha_0} - \frac{1}{\pi}\right)\xi + \sum_{n=1}^{\infty}\left(\frac{\cos(\rho_n x)\sin(\rho_n \xi)}{\rho_n \alpha_n} - 2\frac{\cos(nx)\sin(n\xi)}{n\pi}\right) \tag{4.10}$$

and

$$\widetilde{\Omega}(x,\xi) = \sum_{n=1}^{\infty}\left(2\frac{\sin(nx)\sin(n\xi)}{\pi} - \frac{\sin(\rho_n x)\sin(\rho_n \xi)}{\alpha_n}\right). \tag{4.11}$$

Proof. According to Theorem 63 (2), $\frac{\partial K_a(x,t)}{\partial t} = \frac{1}{a(x)}\widehat{K}_a(x,t)$, for all $(x,t) \in \mathcal{R}$, thus

$$\widehat{G}(x,t) = \widehat{K}_a(x,t) + \widehat{K}_a(x,-t) = a(x)\frac{\partial K_a(x,t)}{\partial t} + a(x)\left(\frac{\partial K_a}{\partial t}\right)(x,z)|_{z=-t}$$
$$= a(x)\left(\frac{\partial K_a(x,t)}{\partial t} - \frac{\partial}{\partial t}\left(K_a(x,-t)\right)\right) = a(x)\frac{\partial G(x,t)}{\partial t}.$$

Substituting this expression into (4.7) we obtain

$$\frac{\partial G(x,t)}{\partial t} + \frac{1}{a(x)}F(x,t) + \int_0^x \frac{\partial G(x,s)}{\partial s}F(s,t)ds = 0, \quad \text{for } 0 < t < x. \tag{4.12}$$

Choose ξ such that $0 < t < \xi < x$ and integrate (4.12) from 0 to ξ with respect to t:

$$\int_0^\xi \frac{\partial G(x,t)}{\partial t}dt + \frac{1}{a(x)}\int_0^\xi F(x,t)dt + \int_0^\xi \left[\int_0^x \frac{\partial G(x,s)}{\partial s}F(s,t)ds\right]dt = 0.$$

Since the series of F converges in the L_2 norm with respect to the variable t and due to Remark 81, we obtain the representation (4.10) for $\Omega(x,s)$. By (3.39) $G(x,0) = 0$, and since the function $F(x,t)$ is continuous in $0 \leqslant x, t \leqslant \pi$, due to the Fubini theorem we obtain

$$G(x,\xi) + \frac{1}{a(x)}\Omega(x,\xi) + \int_0^\xi \frac{\partial G(x,s)}{\partial s}\Omega(s,\xi)ds = 0, \quad \text{for } 0 < \xi < x. \tag{4.13}$$

Integrating by parts gives

$$\int_0^x \frac{\partial G(x,s)}{\partial s}\Omega(s,\xi)ds = G(x,x)\Omega(x,\xi) - \int_0^\xi G(x,s)\frac{\partial \Omega(s,\xi)}{\partial s}ds.$$

Differentiating formally in (4.10) and using Remark 81, we obtain $\frac{\partial \Omega(s,\xi)}{\partial s} = \widetilde{\Omega}(s,\xi)$, where $\widetilde{\Omega}(s,\xi)$ is given by (4.11). Note that $\widetilde{\Omega}(s,\xi) = \frac{A(s+\xi)-A(s-\xi)}{2}$, thus $\widetilde{\Omega}(s,\xi)$ is continuous and the integration by parts is justified. Finally, since $G(x,x) = 1 - \frac{1}{a(x)}$ (see Remark 58), substituting this into (4.13) we obtain (4.9). **Q.E.D.**

The Gelfand-Levitan equation (4.9) is analogous to that obtained for the spectral problem for equation (1.46) on the half axis (see [26, Theorem 4.2]).

Remark 83. *Eq. (4.9) can be written in terms of the canonical transmutation operator as*

$$G(x,s) = -\mathbf{T}_a^C \left[\Omega(x,\xi)\right], \tag{4.14}$$

and hence the function $-\Omega(x,\xi)$ *is the preimage of the function* $G(x,\xi)$ *under the action of the transmutation operator* $\mathbf{T}_a^C$*.*

Remark 84. *Note that for* $x \in (0,\pi]$ *fixed, Eq. (4.9) is a Fredholm integral equation of the second kind. By Lemma 28, the function* A *admits a continuous extension onto* $[-\pi, 2\pi]$ *and then we have* $\tilde{\Omega}(s,\xi) \in L_2\left((0,x)\times(0,x)\right)$*. Furthermore, for* $x \in (0,\pi]$ *equation (4.9) is uniquely solvable. The proof of this fact is completely analogous to that of Remark 34.*

Remark 85. *Comparing the Gelfand-Levitan equation (4.9) with Eq. (2.15) obtained in Chapter 2, we have that* $\widetilde{\Omega}(s,\xi) = -F_D(s,\xi)$*, it is, the kernel of the Gelfand-Levitan equation of the Neumann problem for the Sturm-Liouville equation is equal to that of the Dirichlet problem for the Schrödinger equation.*

4.3 Infinite system for coefficients β_n

Fix $x \in (0,\pi]$. By Corollary 70 the kernel $G(x,\xi)$ admits the Fourier-Legendre series representation

$$G(x,\xi) = \sum_{n=0}^{\infty} \frac{\beta_n(x)}{x} P_{2n+1}\left(\frac{\xi}{x}\right). \tag{4.15}$$

By Remark 71 we have

$$a^{-1}(x) = 1 - \sum_{n=0}^{\infty} \frac{\beta_n(x)}{x}. \tag{4.16}$$

Furthermore, the potential q (and hence the impedance function a) can be recovered from the first coefficient $\beta_0(x)$ by the relation

$$q(x) = \frac{(x\beta_0(x))''}{(x\beta_0(x))' - 3x}, \tag{4.17}$$

where $(x\beta_0(x))' - 3x \neq 0$ for $x \in (0,\pi]$ (see Remark 76).

Before deriving the system of equations for $\{\beta_n(x)\}_{n=0}^{\infty}$ let us improve the convergence of the series (4.11) as follows. By Remark 85, $\widetilde{\Omega}(s,\xi) = F_D(s,\xi)$, and using formula (2.22) we obtain

$$\widetilde{\Omega}(s,\xi) = -\frac{\omega}{\pi^2}\left(\pi\min\{s,\xi\} - 2s\xi\right) + \sum_{n=1}^{\infty}\left(\frac{2\sin(ns)\sin(n\xi)}{\pi} - \frac{\sin(\rho_n s)\sin(\rho_n\xi)}{\alpha_n}\right. \tag{4.18}$$
$$\left. + \frac{2\omega}{\pi^2 n}\left(s\cos(ns)\sin(n\xi) + \sin(ns)\xi\cos(n\xi)\right)\right).$$

The series in (4.18) converges absolutely and uniformly for $s,t \in [0,\pi]$. Hence $\widetilde{\Omega} \in C\left([0,\pi]\times[0,\pi]\right)$.

Similarly, since $F(s,t) = \frac{A(s+t)+A(s-t)}{2}$, by repeating the same procedure the following representation is obtained (see [71])

$$F(s,t) = \left(\frac{1}{\alpha_0} - \frac{1}{\pi}\right) - \frac{\omega}{\pi}\max\{s,t\} + \frac{\omega}{\pi^2}(s^2+t^2) \tag{4.19}$$
$$+ \sum_{n=1}\left(\frac{\cos(\rho_n s)\cos(\rho_n t)}{\alpha_n} - \frac{2\cos(ns)\cos(nt)}{\pi} + \frac{2\omega}{\pi^2 n}\left(s\sin(ns)\cos(nt) + t\sin(nt)\cos(ns)\right)\right).$$

This series converges absolutely and uniformly for $s,t \in [0,\pi]$. Since on the right-hand side of (4.13) we have $x > t$, then $\Omega(x,\xi) = \int_0^{\xi} F(x,t)dt$ is given by

$$\Omega(x,\xi) = \left(\frac{1}{\alpha_0} - \frac{1}{\pi}\right)\xi - \frac{\omega x\xi}{\pi} + \frac{\omega x^2\xi}{\pi^2} + \frac{\omega\xi^3}{3\pi^2}$$
$$+ \sum_{n=1}\left[\frac{\cos(\rho_n x)\sin(\rho_n\xi)}{\rho_n\alpha_n} - \frac{2\cos(nx)\sin(n\xi)}{n\pi}\right.$$
$$\left. + \frac{2\omega}{\pi^2 n^2}\left(x\sin(nx)\sin(n\xi) + \cos(nx)\left(\frac{\sin(n\xi)}{n} - \xi\cos(n\xi)\right)\right)\right].$$

Theorem 86. *Let $a \in W^{2,\infty}(0,\pi)$. For every $x \in (0,\pi]$ fixed, the coefficients $\{\beta_n(x)\}_{n=0}^{\infty}$ of the series expansion (3.57) satisfy the infinite system of linear algebraic equations*

$$\frac{\beta_m(x)}{(4m+3)x} + \sum_{n=0}^{\infty}\beta_n(x)A_{m,n}(x) = B_m(x), \qquad \textit{for } m \in \mathbb{N}_0, \tag{4.20}$$

where

$$
\begin{aligned}
A_{m,n}(x) = &- \frac{2\omega x^2 \delta_{(n,0)}\delta_{(m,0)}}{9\pi^2} - \frac{\omega x}{8\pi}\left[\frac{\delta_{(n+1,m)}}{\left(2n+\frac{3}{2}\right)_3} - \frac{2\delta_{(n,m)}}{(2n+1)_3} + \frac{\delta_{(n-1,m)}}{\left(2n-\frac{1}{2}\right)_3}\right] \\
&- \sum_{k=1}^{\infty}(-1)^{n+m}\left[\left(\frac{2j_{2n+1}(kx)j_{2m+2}(kx)}{\pi} - \frac{j_{2n+1}(\rho_k x)j_{2m+1}(\rho_k x)}{\alpha_k}\right)\right. \\
&+ \frac{2\omega}{\pi^2 k}\left(\left(\frac{2n+1}{k}j_{2n+1}(kx) - xj_{2n+2}(kx)\right)j_{2m+1}(kx)\right. \\
&\left.\left. + j_{2n+1}(kx)\left(\frac{2m+1}{k}j_{2m+1}(kx) - xj_{2m+2}(kx)\right)\right)\right],
\end{aligned} \tag{4.21}
$$

$$
\begin{aligned}
B_m(x) = &- \left[\left(\frac{1}{\alpha_0} - \frac{1}{\pi}\right)\frac{x}{3} - \frac{\omega x^2}{3\pi} + \frac{\omega x^3}{2\pi^2} + \frac{\omega x^3}{15\pi^2}\right]\delta_{(m,0)} - \frac{2\omega x^3}{105\pi^2}\delta_{(m,1)} \\
&- (-1)^m \sum_{k=1}^{\infty}\left[\frac{\cos(\rho_k x)j_{2m+1}(\rho_k x)}{\rho_k \alpha_k} - \frac{2\cos(kx)j_{2m+1}(kx)}{k\pi}\right. \\
&\left. + \frac{2\omega}{\pi^2 k^2}\left(x\sin(kx)j_{2m+1}(kx) + \cos(kx)\left(xj_{2m+2}(kx) - \frac{2m}{k}j_{2m+1}(kx)\right)\right)\right].
\end{aligned} \tag{4.22}
$$

Proof. Fix $x \in (0,\pi]$. Thus, the Gelfand-Levitan equation (4.9) is a Fredholm integral equation of the second kind. By Remak 7 and (1.4), (2.15), it is equivalent to the infinite system of algebraic equations

$$
\xi_j(x) + \sum_{k=0}^{\infty} c_{j,k}(x)\xi_k(x) = y_j(x), \tag{4.23}
$$

where, according to (1.10) and (1.16), we have

$$
\begin{aligned}
\xi_j &= \frac{b_j(x)}{\sqrt{4j+3}\sqrt{x}}, \\
c_{j,k} &= -\sqrt{4j+3}\sqrt{4k+3}x\int_0^{\pi}\int_0^{\pi}\widetilde{\Omega}(s,\xi)P_{2k+1}\left(\frac{s}{x}\right)P_{2j+1}\left(\frac{\xi}{x}\right)dsd\xi, \\
y_j &= -\sqrt{4j+3}\sqrt{x}\int_0^{\pi}\Omega(x,\xi)P_{2j+1}\left(\frac{\xi}{x}\right)d\xi
\end{aligned}
$$

If we define

$$
\begin{aligned}
A_{m,n}(x) &:= -\int_0^{\pi}\int_0^{\pi}\widetilde{\Omega}(s,\xi)P_{2n+1}\left(\frac{s}{x}\right)P_{2m+1}\left(\frac{\xi}{x}\right)\frac{dsd\xi}{x^2}, \\
B_m(x) &:= -\int_0^{\pi}\Omega(x,\xi)P_{2m+1}\left(\frac{\xi}{x}\right)\frac{d\xi}{x},
\end{aligned}
$$

hence the infinite system (4.20) is equivalent to the normalized one (4.23). By Remark 4, the series in (2.23) converge pointwise.

Eq. (4.21) is verified by repeating the same procedure as in the proof of Theorem 36.

For the equality (4.22) we substitute (4.19) into $B_m(x) = -\int_0^\pi \Omega(x,\xi)P_{2m+1}\left(\frac{\xi}{x}\right)\frac{d\xi}{x}$ and we use formula (2.27) and that $\xi \leqslant x$ to obtain

$$B_m(x) = \left[\left(\frac{1}{\alpha_0} - \frac{1}{\pi}\right) - \frac{\omega x}{\pi} + \frac{\omega x^2}{\pi^2}\right]\frac{x}{3}\delta_{(m,0)} + \frac{\omega}{3\pi^2}\int_0^\xi \xi^3 P_{2m+1}\left(\frac{\xi}{x}\right)\frac{d\xi}{x}$$
$$-(-1)^m\sum_{k=1}^\infty\left\{\frac{2\cos(kx)j_{2m+1}(kx)}{k\pi} - \frac{\cos(\rho_k x)j_{2m+1}(\rho_k x)}{\rho_k\alpha_k} + \frac{2\omega}{\pi^2k^2}x\sin(kx)j_{2m+1}(kx)\right.$$
$$\left.+\frac{2\omega}{\pi^2k^2}\cos(kx)\left(\frac{j_{2m+1}(kx)}{k} - \left(\frac{2m+1}{k}j_{2m+1}(kx) - xj_{2m+2}(kx)\right)\right)\right\}.$$

Since $P_3(t) = \frac{5t^3-3t}{2}$, we have $t^3 = \frac{2P_3(t)+3P_1(t)}{5}$, and

$$\int_0^\xi \xi^3 P_{2m+1}\left(\frac{\xi}{x}\right)\frac{d\xi}{x} = \frac{x^3}{5}\left(2\langle P_3, P_{2m+1}\rangle_{L^2(0,1)} + 3\langle P_1, P_{2m+1}\rangle_{L^2(0,1)}\right) = \frac{2x^3}{35}\delta_{(m,1)} + \frac{x^3}{5}\delta_{(m,0)}.$$

Substitution of this expression into the previous formula for $B_m(x)$ leads to (4.22). From this we obtain (86). **Q.E.D.**

Now we obtain that the truncated system

$$\frac{\beta_m(x)}{(4m+3)x} + \sum_{n=0}^M \beta_n(x)A_{m,n}(x) = B_m(x) \quad \text{for } m = \overline{0,M}. \tag{4.24}$$

is solvable, and the stability of the solution. The proof of the following theorem is the same of Theorem 37.

Theorem 87. *Fix $x \in (0,\pi]$. For M large enough the truncated system (4.24) has a unique solution $(\beta_0^{(M)}(x), \cdots, \beta_m^{(M)}(x))$, and*

$$\sum_{m=0}^M \frac{|\beta_m(x) - \beta_m^{(M)}(x)|^2}{4m+3} + \sum_{m=M+1}^\infty \frac{|\beta_m(x)|^2}{4m+3} \to 0, \ \ M \to \infty. \tag{4.25}$$

In particular, it follows that

$$\beta_0^{(M)}(x) \to \beta_0(x), \ \ M \to \infty. \tag{4.26}$$

Moreover, The approximate solution $U_M = \left\{\frac{\beta_k^{(M)}(x)}{\sqrt{4k+3}}\right\}_{k=0}^M$ of the normalized system (4.23) is stable.

Remark 88. *The parameter ω can be recovered using the asymptotics (4.5) from the relation*

$$\omega = \pi \lim_{n\to\infty} n(\rho_n - n)$$

(see [123, Sec. 4.2]). Note that due to (4.5), the sequence $\left\{n(\rho_n - n) - \frac{\omega}{\pi}\right\}_{n=1}^{\infty} = \{k_n\}_{n=1}^{\infty} \in \ell_2$. If we have only a finite number of spectral data $\{\rho_1, \cdots, \rho_{N_1}\}$, an approximate value of ω can be found by minimizing the ℓ_2 norm of the sequence $\left\{n(\rho_n - n) - \frac{\omega}{\pi}\right\}_{n=N_s}^{N_1}$, where N_s is an integer between 1 and N_1 chosen to skip several first eigenvalues which can differ a lot from the asymptotic formula. One can take, e.g., $N_s = \left[\frac{N_1}{2}\right]$ and obtain that

$$\omega \approx \arg\min_{\omega} \sum_{n=\left[\frac{N_1}{2}\right]}^{N} \left(n(\rho_n - n) - \frac{\omega}{\pi}\right)^2 \tag{4.27}$$

(see [92, Sec. 3.3] for a deeper discussion of this method).

Remark 89. *If $a \in W^{1,\infty}(0,\pi)$, one cannot guarantee the asymptotics (4.5). However, a similar representation for $\{\rho_n\}_{n=0}^{\infty}$ can be obtained as follows. For each $n \in \mathbb{N}_0$, ρ_n is a zero of the entire function*

$$C'(\rho,\pi) = -\frac{\rho\sin(\rho\pi)}{a(\pi)} + \rho\int_0^{\pi} \frac{\partial G(\pi,t)}{\partial x}\sin(\rho t)dt$$

From Definition 56, $\frac{\partial G(x,t)}{\partial x}$ is an odd function in t. Hence we can write

$$\int_0^x \frac{\partial G(x,t)}{\partial x}\sin(\rho t)dt = \frac{1}{2}\int_{-x}^{x} \frac{\partial G(x,t)}{\partial x}\sin(\rho t)dt = \frac{1}{4i}\left(\int_{-x}^{x} \frac{\partial G(x,t)}{\partial x}e^{i\rho t}dt - \int_{-x}^{x} \frac{\partial G(x,t)}{\partial x}e^{-i\rho t}dt\right)$$
$$= \int_{-x}^{x} \frac{1}{4i}\left(\frac{\partial G(x,t)}{\partial x} - \frac{\partial G(x,-t)}{\partial x}\right)e^{i\rho t}dt = \int_{-x}^{x} \frac{1}{2i}\frac{\partial G(x,t)}{\partial x}e^{i\rho t}dt.$$

Then $\{\rho_n\}_{n=0}^{\infty}$ are zeros of the entire function

$$F(\rho) = \sin(\rho) - \int_{-\pi}^{\pi} \frac{a(\pi)}{2i}\frac{\partial G(\pi,t)}{\partial x}e^{i\rho t}dt.$$

Since $\tilde{f}(t) = \frac{a(\pi)}{2i}\frac{\partial G(\pi,t)}{\partial x}$ belongs to $L_\infty(-\pi,\pi) \subset L_2(-\pi,\pi)$, according to [104], zeros of $F(\rho)$ are distributed as follows

$$\rho_n = n + \zeta_n, \qquad \text{with } \{\zeta_n\} \in \ell_2.$$

4.4 Numerical algorithm and examples

4.4.1 General algorithm

Given a finite set of spectral data $\{\rho_n, \alpha_n\}_{n=0}^{N_s}$ with $\rho_n \in \mathbb{R}$, $\rho_n^2 \neq \rho_m^2$ for $m \neq n$ and $\alpha_n > 0$ for $n = \overline{0, N_s}$, we propose the following method for recovering the potential $q(x)$ and the impedance function $a(x)$.

1. If the parameter ω is not given, find it using formula (4.27), i.e., take
$$\omega \approx \frac{\pi}{N_s - \left[\frac{N_s}{2}\right]} \sum_{n=\left[\frac{N_s}{2}\right]}^{N_s} n(\rho_n - n).$$

2. Choose $M \in \mathbb{N}$. For a set of points $\{x_l\}$ from $(0, \pi]$, compute the approximate values of the following functions for $n, m = \overline{0, M}$:

$$\begin{aligned}
\widehat{B}_m(x) = &- \left[\left(\frac{1}{\alpha_0} - \frac{1}{\pi}\right)\frac{x^2}{3} - \frac{\omega x^3}{3\pi} + \frac{\omega x^4}{2\pi^2} + \frac{\omega x^4}{15\pi^2}\right]\delta_{(m,0)} - \frac{2\omega x^4}{105\pi^2}\delta_{(m,1)} \\
&- (-1)^m x \sum_{k=1}^{N_s}\left[\frac{\cos(\rho_k x) j_{2m+1}(\rho_k x)}{\rho_k \alpha_k} - \frac{2\cos(kx) j_{2m+1}(kx)}{k\pi}\right. \\
&\left. + \frac{2\omega}{\pi^2 k^2}\left(x \sin(kx) j_{2m+1}(kx) + \cos(kx)\left(x j_{2m+2}(kx) - \frac{2m}{k} j_{2m+1}(kx)\right)\right)\right].
\end{aligned}$$

$$\begin{aligned}
\widetilde{A}_{m,n} = &- x \sum_{k=1}^{N_s} (-1)^{n+m}\left[\left(\frac{2 j_{2n+1}(kx) j_{2m+1}(kx)}{\pi} - \frac{j_{2n+1}(\rho_k x) j_{2m+1}(\rho_k x)}{\alpha_k}\right)\right. \\
&+ \frac{2\omega}{\pi^2 k}\left(\left(\frac{2n+1}{k} j_{2n+1}(kx) - x j_{2n+2}(kx)\right) j_{2m+1}(kx)\right. \\
&\left.\left. + j_{2n+1}(kx)\left(\frac{2m+1}{k} j_{2m+1}(kx) - x j_{2m+2}(kx)\right)\right)\right],
\end{aligned}$$

$$\widehat{A}_{m,n} = \begin{cases}
\widetilde{A}_{m,n} + \frac{2\omega x^3}{9\pi^2} + \frac{\omega x^2}{\pi} \cdot \frac{2}{15}, & \text{if } m = n = 0, \\
\widetilde{A}_{m,n} - \frac{\omega x^2}{\pi} \cdot \frac{1}{(4n+3)(4n+5)(4n+7)}, & \text{if } m = n + 1, \\
\widetilde{A}_{m,n} + \frac{\omega x^2}{\pi} \cdot \frac{2}{(4n+1)(4n+3)(4n+5)}, & \text{if } m = n, m > 0 \\
\widetilde{A}_{m,n} - \frac{\omega x^2}{\pi} \cdot \frac{1}{(4n-1)(4n+1)(4n+3)}, & \text{if } m = n - 1, \\
\widetilde{A}_{m,n}, & \text{otherwise.}
\end{cases}$$

3. Solve the system

$$\begin{bmatrix} \widehat{C}_{0,0} & \widehat{C}_{0,1} & \cdots & \widehat{C}_{0,M} \\ \widehat{C}_{1,0} & \widehat{C}_{1,1} & \cdots & \widehat{C}_{1,M} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{C}_{M,0} & \widehat{C}_{M,1} & \cdots & \widehat{C}_{M,M} \end{bmatrix} \begin{bmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_1 \\ \vdots \\ \widehat{\beta}_M \end{bmatrix} = \begin{bmatrix} \widehat{B}_0 \\ \widehat{B}_1 \\ \vdots \\ \widehat{B}_M \end{bmatrix}, \tag{4.28}$$

where $\widehat{C}_{m,n} = \widehat{A}_{m,n} + \frac{\delta_{(n,m)}}{4m+3}$.

4. Compute q at $\{x_l\}$ from $\widehat{\beta}_0$ using (4.17). Subsequently, the impedance function is calculated from q using formula $a(x) = e^{-\frac{1}{2}\int_0^x q(s)ds}$. Alternatively the impedance function can be approximated directly by using formula (4.16), i.e., $a(x_l) \approx \left(1 - \sum_{m=0}^{M} \frac{\widehat{\beta}_m(x_l)}{x_l}\right)^{-1}$. However this alternative showed to be slightly less accurate.

4.4.2 Numerical examples

We illustrate the performance of the algorithm with two examples: a smooth impedance function, and an impedance function belonging to $W^{2,\infty}(0,\pi)$ and possessing a discontinuous second derivative. All the computations were performed in Matlab R2021a. On the fourth step of the algorithm the impedance function a was recovered from the first coefficient β_0. The numerical differentiation of β_0 was performed by converting first the set of values obtained on the third step into a spline using the routine `spapi` and then differentiating it twice using the routine `fnder`.

Example 90. *Consider the impedance function*

$$a(x) = \cos\left(\frac{x}{4}\right), \quad \textit{for } 0 \leqslant x \leqslant \pi. \tag{4.29}$$

In this case the eigenvalues are given by $\lambda = z^2 - \frac{1}{16}$, *where* z *is a non-negative zero of the characteristic equation*

$$z\sin(\pi z) - \frac{1}{4}\cos(\pi z) = 0.$$

Thus, for every $n \in \mathbb{N}_0$ *the spectral data are given by*

$$\rho_n = \sqrt{z_n^2 - \frac{1}{16}} \quad \textit{and} \quad \alpha_n = \frac{\pi}{2} + \frac{\sin(2\pi z_n)}{4z_n}.$$

Moreover, $\omega = \frac{1}{4} - \frac{\pi}{32}$.

First, we consider $N_s = 5$ *pairs of spectral data and solve the system (4.28) for* $M \in \{1, 2, 3, 4\}$, *using the known value of* ω. *The impedance function recovered with* $M + 1$ *equations is denoted by* a_M. *It is recovered in 200 points* $\{x_l\} \subset (0, \pi]$ *distributed uniformly. In all the cases the maximum absolute error is of order* 5.6×10^{-4}. *For* $M = 1$: $\max_{0 \leqslant x \leqslant \pi} |a_M(x) - a(x)| = 5.681341 \times 10^{-4}$, *and for* $M = 4$: $\max_{0 \leqslant x \leqslant \pi} |a_M(x) - a(x)| = 5.674308 \times 10^{-4}$. *In all the cases the* L_1*-norm* $\|a_M - a\|_{L_1(0,\pi)}$ *is of order* 4.9×10^{-4}.

For $N_s = 10$ *the order of the absolute error is* 1.39×10^{-4}. *For* $M = 1$: $\max_{0 \leqslant x \leqslant \pi} |a_M(x) - a(x)| = 1.396005 \times 10^{-4}$ *and* $\|a_M - a\|_{L_1(0,\pi)} = 1.002899 \times 10^{-4}$, *while for* $M = 4$: $\max_{0 \leqslant x \leqslant \pi} |a_M(x) - a(x)| = 1.395702 \times 10^{-4}$ *and* $\|a_M - a\|_{L_1(0,\pi)} = 1.0028 \times 10^{-4}$. *In both cases (*$N_s = 5$ *and* $N_s = 10$*) the best accuracy is obtained for* $M = 4$.

For the second numerical test we fix $M = 4$ *and vary the number of given spectral data* $N_s \in \{2, 4, 6, 8, 10, 12\}$. *Here we again use the exact value of* ω. *The recovered impedance function is denoted by* a_{N_s}. *Both* a_{N_s} *(for* $N_s = 12$*) and the original* a *are presented on the left side of Figure 4.1. The absolute error for each* N_s *is presented on the right side of Figure 4.1, and the errors of approximation for* a_{N_s} *with respect to the maximum and* L_1 *norms are presented in Table 4.1.*

Thus, starting from a certain number of equations, increasing it further and preserving a fixed number of given spectral data does not improve considerably the accuracy. On the other hand, for a fixed number of equations, increasing the number of given spectral data leads to a better accuracy.

Further, we repeat the same calculations, but using an approximate parameter ω_a *computed by the formula (4.27). The recovered impedance function is denoted by* $a_{N_s}^{\omega_a}$. *Table 4.2 shows the maximum of* $|a_{N_s}^{\omega_a} - a|$ *in* $[0, \pi]$ *and the* L_1*-error. Of course, the error in the approximation of the parameter* ω_a *makes the convergence of the method slower. Except for the first case* $N_s = 2$, *all maximum errors are one tenth greater than when the exact parameter is used. In this case, the error in the* L_1*-norm behaves better.*

For the third experiment we take $M = 4$, $N_s \in \{5, 20\}$ *of the original spectral data and compute* $N_A = 1000$ *of asymptotic spectral data* $\{\rho_n = n + \frac{\omega}{n}, \alpha_n = \frac{\pi}{2}\}_{n=N_s+1}^{N_A}$, *using the exact parameter* ω *and the approximate parameter* ω_a. *The impedance function obtained with the aid of these combined spectral data with the exact* ω *is denoted by* a_{N_A}, *while with* *the approximate* ω_a *by* $a_{N_A}^{\omega_a}$.

For $Ns = 5$ *we have* $\max_{0\leqslant x\leqslant \pi} |a_{N_A}(x) - a(x)| = 4.126184 \times 10^{-4}$ *and* $\|a_{N_A} - a\|_{L_1(0,\pi)} = 4.182432 \times 10^{-4}$. *In the case of* $a_{N_A}^{\omega_a}$ *we have* $\max_{0\leqslant x\leqslant \pi} |a_{N_a}^{\omega_a}(x) - a(x)| = 4.2397 \times 10^{-3}$ *and* $\|a_{N_A}^{\omega_a} - a\|_{L_1(0,\pi)} = 1.1623 \times 10^{-3}$.

For $N_s = 20$ *we have* $\max_{0\leqslant x\leqslant \pi} |a_{N_A}(x) - a(x)| = 2.802095 \times 10^{-5}$ *and* $\|a_{N_A} - a\|_{L_1(0,\pi)} = 4.09982 \times 10^{-5}$. *In the case of* $a_{N_A}^{\omega_a}$ *we have* $\max_{0\leqslant x\leqslant \pi} |a_{N_A}^{\omega_a}(x) - a(x)| = 3.382623 \times 10^{-4}$ *and* $\|a_{N_A}^{\omega_a} - a\|_{L_1(0,\pi)} = 5.381838 \times 10^{-5}$.

The graphs of the absolute errors $|a_{N_A} - a|$, $|a_{N_A}^{\omega_a} - a|$ *for* $N_s = 5$ *(left) and* $N_s = 20$ *(right) are presented on Figure 4.2.*

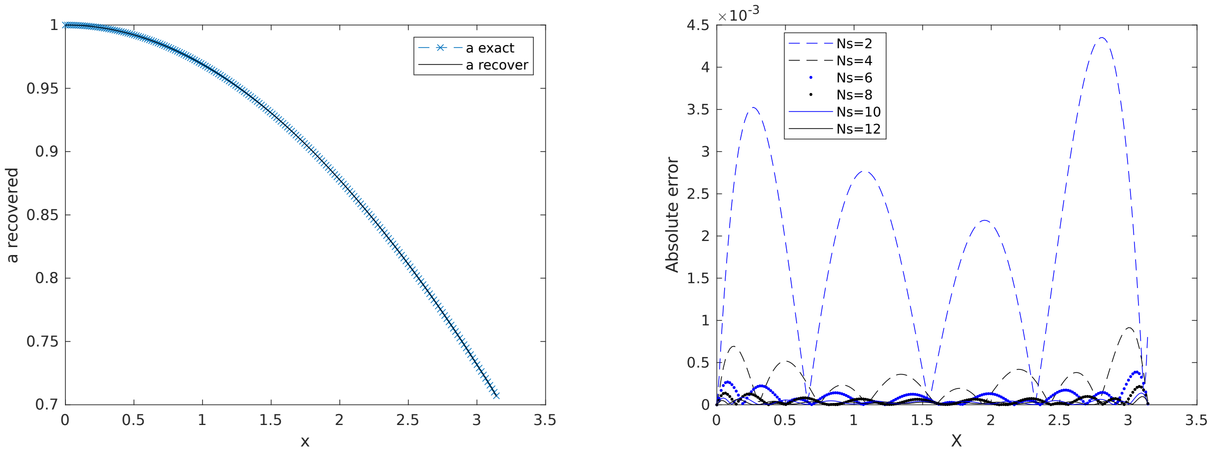

Figure 4.1: On the left, the graphs of a given by (4.29) and the recovered impedance function a_{N_s}. On the right, the absolute error $|a_{N_s} - a|$.

Ns	$\max_{0\leqslant x\leqslant \pi} \lvert a_{N_s}(x) - a(x)\rvert$	$\lVert a_{N_s} - a\rVert_{L_1(0,\pi)}$
2	4.351711×10^{-3}	6.336344×10^{-3}
4	9.141057×10^{-4}	9.0798846×10^{-4}
6	3.875865×10^{-4}	3.019686×10^{-4}
8	2.166957×10^{-4}	1.515506×10^{-4}
10	1.3957×10^{-4}	1.002807×10^{-4}
12	9.59477×10^{-5}	7.562398×10^{-5}

Table 4.1: Approximation error for a_{N_s} from the first numerical example in the maximum and L_1 norms.

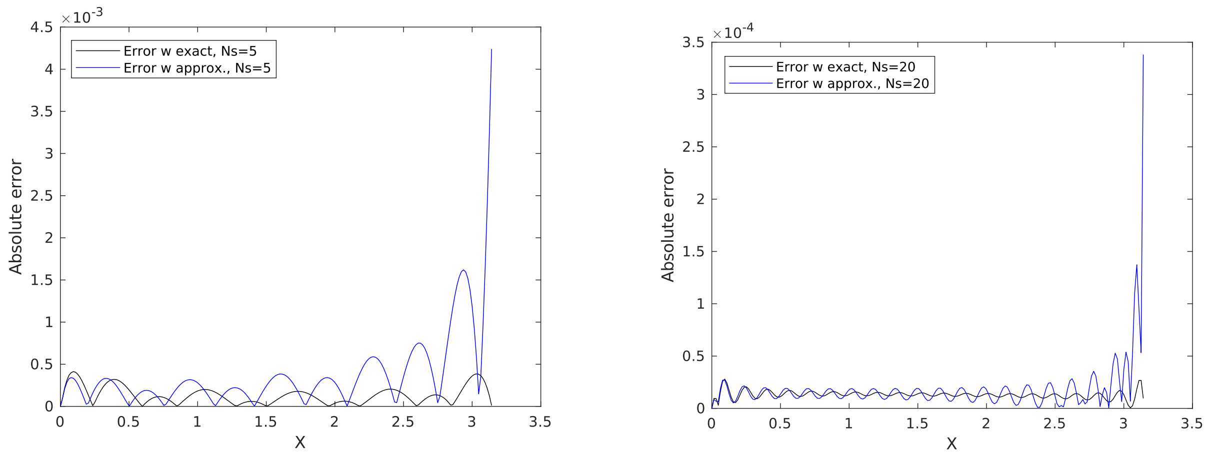

Figure 4.2: On the left, the graphs of absolute errors $|a_{N_A} - a|$ and $|a_{N_A}^{\omega_a} - a|$ for $N_s = 5$ and a given by (4.29). On the right, the absolute error for $N_s = 20$.

Ns	$\max_{0 \leqslant x \leqslant \pi} \lvert a_{N_s}^{\omega_a}(x) - a(x)\rvert$	$\lVert a_{N_s}^{\omega_a} - a\rVert_{L_1(0,\pi)}$
2	4.780866×10^{-3}	6.844988×10^{-3}
4	8.294079×10^{-3}	2.647811×10^{-3}
6	3.886554×10^{-3}	9.138956×10^{-4}
8	2.184072×10^{-3}	4.19633×10^{-4}
10	1.389329×10^{-3}	2.319543×10^{-4}
12	9.588838×10^{-4}	1.463004×10^{-4}

Table 4.2: Approximation error for $a_{N_s}^{\omega_a}$ from the first numerical example in the maximum and L_1 norms.

Example 91. *Consider an impedance function $a \in W^{2,\infty}(0,\pi)$ possessing a discontinuous second derivative*

$$a(x) = \begin{cases} 1, & \text{if } 0 \leqslant x < \frac{\pi}{3}, \\ \frac{3x^2}{\pi^2} - \frac{2x}{\pi} + \frac{4}{3}, & \text{if } \frac{\pi}{3} \leqslant x < \frac{2\pi}{3}, \\ \frac{2x}{\pi}, & \text{if } \frac{2\pi}{3} \leqslant x \leqslant \pi. \end{cases} \tag{4.30}$$

To compute the "exact" spectral data the Neumann problem was transformed into the Sturm-Liouville problem (4.3) with the potential

$$Q_a(x) = \begin{cases} \frac{18}{9x^2 - 6\pi x + 4\pi^2}, & \text{if } \frac{\pi}{3} \leqslant x < \frac{2\pi}{3}, \\ 0, & \text{otherwise}, \end{cases}$$

with $h = 0$ *and* $H = -\frac{2}{\pi}$. *Applying the method from [81] we computed* 201 *spectral data* $\{(\rho_n, \alpha_n)\}_{n=0}^{200}$ *and the parameter* ω. *For the first two experiments we use the "exact"* ω.

As in the first example, we take $N_s = 5$ *spectral data and compute* a_M *for* $M \in \{1, 2, 3, 4\}$. *On the left side of Figure 4.3 the absolute error of* a_M *is shown. For* $M = 1$, $\max_{0\leqslant x\leqslant\pi} |a_M(x) - a(x)| = 1.626015 \times 10^{-3}$ *with an* L_1*-norm error of order* 8.6613×10^{-4}. *For the other values of* M, *the error* $|a_M - a|$ *is of order* 6.204×10^{-4}, *and the* L_1*-error is of order* 7×10^{-4}.

For the next experiment we fix $M = 4$ *and vary* $N_s \in \{2, 4, 6, 8, 10, 12\}$. *We use the "exact" parameter* ω. *The right side of Figure 4.3 shows the absolute error of* a_{N_s}. *Table 4.3 presents the maximum of* $|a_{N_s} - a|$ *in* $[0, \pi]$ *and the* L_1*-error.*

Further, we repeat the same calculations, but using the approximate parameter ω_a. *The recovered potential is denoted by* $a_{N_s}^{\omega_a}$. *Table 4.4 presents the maximum of* $|a_{N_s}^{\omega_a} - a|$ *in* $[0, \pi]$ *and the* L_1*-error. Obviously, the approximation error in the parameter* ω_a *makes the convergence slower.*

Finally, for $M = 4$ *and* $N_s \in \{6, 12, 50\}$ *we compute the approximate parameter* ω_a *and add* $N_A = 1000$ *asymptotic spectral data. The graph of the recovered impedance function (for* $N_s = 6$*) compared with the exact one is depicted on the left side of Figure 4.4, and on the right side the graph of the absolute error of the recovered impedance function* $a_{N_A}^{\omega_a}$ *compared to* a *is presented. With* $N_s = 6$ *exact data we have* $\max_{0\leqslant x\leqslant\pi} |a_{N_A}^{\omega_a}(x) - a(x)| = 5.60573 \times 10^{-3}$ *and* $\|a_{N_A}^{\omega_a} - a\|_{L_1(0,\pi)} = 1.188489 \times 10^{-3}$. *With* $N_s = 12$ *exact data we have* $\max_{0\leqslant x\leqslant\pi} |a_{N_A}^{\omega_a}(x) - a(x)| = 3.002263 \times 10^{-4}$ *and* $\|a_{N_A}^{\omega_a} - a\|_{L_1(0,\pi)} = 1.2074 \times 10^{-4}$. *Finally, for* $N_s = 50$ *we obtain* $\max_{0\leqslant x\leqslant\pi} |a_{N_A}^{\omega_a}(x) - a(x)| = 5.60809 \times 10^{-5}$ *and* $\|a_{N_A}^{\omega_a} - a\|_{L_1(0,\pi)} = 2.2613124 \times 10^{-5}$. *On the right side of Figure 4.4 the graphs of the absolute error* $|a_{N_A}^{\omega_a} - a|$ *for* $N_s = 12, 50$ *are presented.*

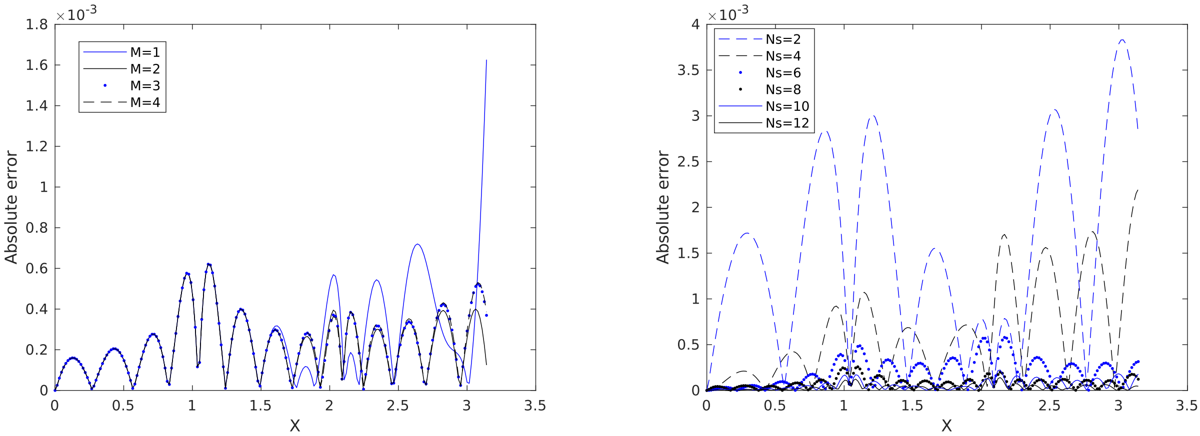

Figure 4.3: On the left, graphs of the absolute error $|a_M - a|$ for a given by (4.30). On the right, the absolute error $|a_{N_s} - a|$.

N_s	$\max_{0\leqslant x\leqslant \pi} \lvert a_{N_s}(x) - a(x)\rvert$	$\lVert a_{N_s} - a\rVert_{L_1(0,\pi)}$
2	3.83115×10^{-3}	4.895084×10^{-3}
4	2.19969×10^{-3}	2.021055×10^{-3}
6	5.792752×10^{-4}	6.081727×10^{-4}
8	2.57998×10^{-4}	2.476142×10^{-4}
10	2.227975×10^{-4}	1.883309×10^{-4}
12	1.410056×10^{-4}	1.079353×10^{-4}

Table 4.3: Approximation error for a_{N_s} from the second example, in the maximum and L_1 norms

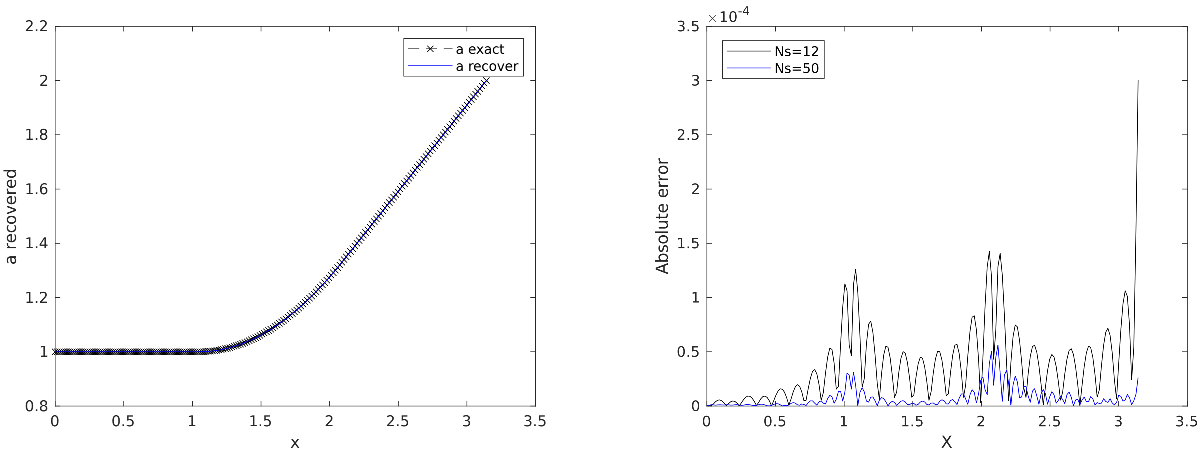

Figure 4.4: On the left, graphs of a given by (4.30) and the recovered impedance function. On the right, the absolute error $|a_{N_A}^{\omega_a} - a|$.

N_s	$\max_{0\leqslant x\leqslant\pi} \lvert a_{N_s}^{\omega_a}(x) - a(x)\rvert$	$\lVert a_{N_s}^{\omega_a} - a\rVert_{L_1(0,\pi)}$
2	2.111937×10^{-2}	7.40011×10^{-3}
4	1.93921×10^{-2}	4.630266×10^{-3}
6	5.613387×10^{-3}	1.184495×10^{-3}
8	1.036525×10^{-3}	2.98156×10^{-4}
10	1.315091×10^{-3}	2.715286×10^{-4}
12	3.012736×10^{-4}	1.211153×10^{-4}

Table 4.4: Approximation error for $a_{N_s}^{\omega_a}$ from example 2, in the maximum and L_1 norms

Chapter 5

Series representation for the Jost solution of the SLEIF

In this chapter we consider Eq. (1.46) on the half-line $\mathbb{R}^+ := (0, \infty)$, it is,

$$-\frac{1}{p(x)}\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) = \lambda u, \quad x \in \mathbb{R}^+,\ \lambda \in \mathbb{C}, \tag{5.1}$$

where $\mathbb{R}^+ := (0, \infty)$ and $p \in AC_{loc}(\overline{\mathbb{R}^+})$ is a positive bounded function. Under the condition $\frac{p'}{p} \in L_1(\mathbb{R}^+)$, it is known (see [26]) that (5.1) admits a *Jost solution*, similarly to the case of the one-dimensional Schrödinger equation [30, 48, 52, 78, 105].

A series representation for the Jost solution is obtained in the form of a power series with respect to the parameter $z = \frac{\frac{1}{2}+i\sqrt{\lambda}}{\frac{1}{2}-i\sqrt{\lambda}}$. The series converges in the unit disk $|z| < 1$ and leads to an explicit representation for spectral data and analytic method for their computation. The characterization of the spectral data for (5.1) with the boundary condition

$$u'(0) - hu(0) = 0, \quad \text{where } h \in \mathbb{R}, \tag{5.2}$$

is given.

5.1 Integral representation for the Jost solution

5.1.1 Properties of the Jost solution

We start by considering the following assumptions on the function p.

1. p is positive, bounded, and $\alpha_p := \inf_{x\geqslant 0} p(x) > 0$.

2. $\lim_{x\to\infty} p(x) = p_\infty$ exists and is finite.

3. $\lim_{x\to\infty} p'(x) = 0$.

4. $p' \in L_1(\mathbb{R}^+)$.

Denote the square root of the spectral parameter λ by ρ, $\lambda = \rho^2$, and choose it such that $\rho \in \overline{\mathbb{C}^+}$. Eq. (5.1) can be written as

$$-u'' + q(x)u' = \rho^2 u, \quad x \in \mathbb{R}^+, \tag{5.3}$$

where $q(x) = -\frac{p'(x)}{p(x)}$ (formula (1.48)). Choosing $x_0 = \infty$ in (1.48) we have the relation

$$p(x) = p_\infty \exp\left(\int_x^\infty q(s)ds\right). \tag{5.4}$$

Note that from (5.1), we can assume without lost of generality that $p_0 := p(0) = 1$. Due to conditions 1 and 4, $q \in L_1(\mathbb{R}^+)$.

It is known (see [26]), that (5.3) possesses a unique solution $e(\rho, x)$, called the *Jost solution*, that satisfies the following asymptotic conditions

$$e^{(k)}(\rho, x) = (i\rho)^k e^{i\rho x}(1 + o(1)), \quad x \to \infty; \quad k = 0, 1, \tag{5.5}$$

uniformly with respect to $\rho \in \overline{\mathbb{C}^+}$. Of course, one can obtain a solution for $\rho \in \overline{\mathbb{C}^-}$ taking $e_-(\rho, x) := e(-\rho, x)$. In the case that $\rho \in \mathbb{R} \setminus \{0\}$, the solutions $e(\rho, x)$ and $e_-(\rho, x)$ are linearly independent and satisfy $e(-\rho, x) = \overline{e(\rho, x)}$. Due to (5.5), $e(\rho, \cdot) \in H^1(\mathbb{R}^+)$, for all $\rho \in \mathbb{C}^+$. In this chapter we denote the differential operator

$$\mathbf{L} := -\frac{1}{p(x)}\mathbf{D}p(x)\mathbf{D} = -\mathbf{D}^2 + q(x)\mathbf{D}.$$

5.1.2 Levin's representation for the Jost solution

In order to obtain Levin's integral representation for the Jost solution analogous to that for solutions of the one-dimensional Schrödinger equation (see [48, Sec. 4] or [30, Ch. V]), we review the deduction of $e(\rho, x)$ given in [26]. The solution $e(\rho, x)$ must satisfy the integral equation

$$e(\rho, x) = e^{i\rho x} + \int_x^\infty \frac{\sin(\rho(s - x))}{\rho} q(s)e'(\rho, s)ds. \tag{5.6}$$

Eq. (5.6) is solved by the successive approximation method, taking $e(\rho,x)=\sum_{n=0}^{\infty}e_n(\rho,x)$, with $e_0(\rho,x):=e^{i\rho x}$, and $e_{n+1}(\rho,x):=\int_x^{\infty}\frac{\sin(\rho(s-x))}{\rho}q(s)e_n'(\rho,s)ds,\ n\geqslant 0$.

In [26, Th. 2.2.] it is shown that for $n\geqslant 1$ the function e_n and its derivative satisfy the estimates

$$|e_n(\rho,x)| \leqslant e^{-x\operatorname{Im}\rho}c^n\frac{\left(\int_x^{\infty}|q(s)|ds\right)^n}{n!}, \tag{5.7}$$

$$|e_n'(\rho,x)| \leqslant |\rho|e^{-x\operatorname{Im}\rho}c^n\frac{\left(\int_x^{\infty}|q(s)|ds\right)^n}{n!}, \tag{5.8}$$

where $c>0$ is a constant, for which the inequality holds

$$|\sin(\rho x)|\leqslant c\frac{|\rho|xe^{x|\operatorname{Im}\rho|}}{1+|\rho|x},\quad \forall\,\rho\in\mathbb{C},\ \forall\,x\in\overline{\mathbb{R}^+} \tag{5.9}$$

(actually, we can choose $c=\max\left\{\max_{z\in\mathbb{D}}(1+|z|)e^{-|\operatorname{Im}z|}\left|\frac{\sin(z)}{z}\right|,2\right\}$). In [26, Th. 2.2] it is shown that $e(\cdot,x)\in\operatorname{Hol}(\mathbb{C}^+)$ for all $x\in\overline{\mathbb{R}^+}$, and the estimate is valid

$$|e^{(k)}(\rho,x)-(i\rho)^ke^{i\rho x}|\leqslant|\rho|^ke^{-x\operatorname{Im}\rho}\exp\left(c\int_x^{\infty}|q(s)|ds\right),\ k=0,1;\ \forall\,\rho\in\overline{\mathbb{C}^+},\ \forall x\in\overline{\mathbb{R}^+}. \tag{5.10}$$

In order to obtain the integral representation, we need to improve the estimate for $e(\rho,x)-e^{i\rho x}$. For this reason, and similar to the case of the Schrödinger equation (see [52, Ch. II]), assume that $q\in L_1(\mathbb{R}^+;(1+x)dx)$. Note that $L_1(\mathbb{R}^+;(1+x)dx)=L_1(\mathbb{R}^+)\cap L_1(\mathbb{R}^+;xdx)$.

Lemma 92. *Let $q\in L_1(\mathbb{R}^+;(1+x)dx)$. For $n\geqslant 1$ the following inequality holds*

$$|e_n(\rho,x)|\leqslant\frac{|\rho|e^{-x\operatorname{Im}\rho}}{1+|\rho|x}c^n\frac{(Q(x))^n}{n!},\qquad \text{for } \rho\in\overline{\mathbb{C}^+},\ x\in\overline{\mathbb{R}^+}, \tag{5.11}$$

where $Q(x)=\int_x^{\infty}\tilde{q}(s)ds$ with $\tilde{q}(s):=\begin{cases}|q(s)|, & \text{if } 0<s<1,\\ s|q(s)|, & \text{if } s>1.\end{cases}$

Proof. Consider

$$\begin{aligned}|e_n(\rho,x)| &\leqslant \int_x^{\infty}\left|\frac{\sin(\rho(s-x))}{\rho}\right||q(s)||e_n'(\rho,s)|ds\\ &\leqslant c\int_x^{\infty}e^{(s-x)\operatorname{Im}\rho}\frac{|\rho|(s-x)}{|\rho|(1+|\rho|(s-x))}|q(s)||e_{n-1}'(\rho,s)|ds\\ &\leqslant c\int_x^{\infty}e^{(s-x)\operatorname{Im}\rho}\frac{|\rho|s}{1+|\rho|s}|q(s)|\left(\frac{e^{-s\operatorname{Im}\rho}}{(n-1)!}c^{n-1}\left(\int_s^{\infty}|q(\xi)|d\xi\right)^{n-1}\right)ds,\end{aligned}$$

where we used (5.8) and (5.9). Hence

$$\begin{aligned} |e_n(\rho,x)| &\leqslant c^n \frac{e^{-x\operatorname{Im}\rho}}{(n-1)!}\frac{|\rho|}{1+|\rho|x}\int_x^\infty s|q(s)|\left(Q(s)\right)^{n-1}ds \\ &\leqslant c^n \frac{e^{-x\operatorname{Im}\rho}}{(n-1)!}\frac{|\rho|}{1+|\rho|x}\int_x^\infty \tilde{q}(s)\left(Q(s)\right)^{n-1}ds \\ &= \frac{|\rho|e^{-x\operatorname{Im}\rho}}{1+|\rho|x}c^n\frac{(Q(x))^n}{n!}, \end{aligned}$$

where we used the fact that for $t\geqslant 0$ the function $\frac{t}{1+t}$ is increasing, and $\frac{1}{1+t}$ is non-increasing, respectively, and that $x\leqslant s$. **Q.E.D.**

Theorem 93. *Let $q\in L_1(\mathbb{R}^+;(1+x)dx)$. Then there exists a function $A(x,t)$, defined in $\Pi=\{(x,t)\in\mathbb{R}^2\,|\,0<x<t\}$, such that for any $x\in\mathbb{R}^+$ fixed, $A(x,\cdot)\in L_2(x,\infty)$ and*

$$e(\rho,x)=e^{i\rho x}+\int_x^\infty A(x,t)i\rho e^{i\rho t}dt,\quad \forall\rho\in\overline{\mathbb{C}^+}. \tag{5.12}$$

Proof. Fix $x\in\mathbb{R}^+$ and define $\psi(\rho,x)=\frac{e(\rho,x)-e^{i\rho x}}{i\rho}$ for $\rho\in\overline{\mathbb{C}^+}$. Using the estimate (5.11) we get

$$|\psi(\rho,x)|=\frac{1}{|\rho|}\sum_{n=1}^\infty|e_n(\rho,x)|\leqslant\frac{e^{-x\operatorname{Im}\rho}}{1+|\rho|x}\sum_{n=1}^\infty\frac{(cQ(x))^n}{n!}=\frac{e^{-x\operatorname{Im}\rho}}{1+|\rho|x}\exp\left(cQ(x)\right).$$

This implies that $|\psi(\rho,x)|\leqslant\frac{Me^{-x\operatorname{Im}\rho}}{1+|\rho|x}$ for all $\rho\in\overline{\mathbb{C}^+}$ with $M=\exp\left(c\|\tilde{q}\|_{L_1(\mathbb{R}^+)}\right)$. Hence for $\rho\in\mathbb{R}$ we have

$$\int_{\mathbb{R}}|\psi(\rho,x)|^2d\rho\leqslant M^2\int_{\mathbb{R}}\frac{d\rho}{(1+|\rho|x)^2}<\infty.$$

Thus, $\psi(\cdot,x)\in L_2(\mathbb{R})$, and by the Plancherel theorem its Fourier transform $A(x,t)=2\pi\int_{\mathbb{R}}\psi(\rho,x)e^{-i\rho x}d\rho$ exists, and $A(x,\cdot)\in L_2(\mathbb{R})$.

Denoting $\rho_1=\operatorname{Re}\rho$ and $\rho_2=\operatorname{Im}\rho$, we obtain

$$\int_{\mathbb{R}}|\psi(\rho_1+i\rho_2,x)|^2d\rho_1\leqslant M^2e^{-2x\rho_2}\int_{\mathbb{R}}\frac{d\rho_1}{(1+|\rho_1+i\rho_2|x)^2}=O(e^{-2x\rho_2}). \tag{5.13}$$

Since $\psi(\cdot,x)\in Hol(\mathbb{C}^+)$ (because $e(\cdot,x)\in\operatorname{Hol}(\mathbb{C}^+)$), it is known (see [136, Th. 96, pp. 129]) that condition (5.13) implies that $A(x,t)=0$, for all $t<x$. Thus $A(x,t)\in L_2(x,\infty)$ and applying the inverse Fourier transform

$$\psi(\rho,x)=\int_x^\infty A(x,t)e^{i\rho t}dt,$$

from where we obtain (5.12). **Q.E.D.**

Differentiating formally in Eq. (5.12) we get

$$e'(\rho,x)=i\rho e^{i\rho x}-A(x,x)i\rho e^{i\rho x}+\int_x^\infty A_x(x,t)i\rho e^{i\rho t}dt, \tag{5.14}$$

$$e''(\rho,x)=-\rho^2e^{i\rho x}\left(1-A(x,x)\right)-i\rho e^{i\rho x}\left(\frac{d}{dx}A(x,x)+A_x(x,x)\right)+\int_x^\infty A_{xx}(x,t)i\rho e^{i\rho t}dt.$$

Substituting (5.12) in (5.1) we have

$$\begin{aligned}\rho^2\int_x^\infty A(x,t)i\rho e^{i\rho t}dt &= \int_x^\infty \left\{-A_{xx}(x,t)+q(x)A_x(x,t)\right\}i\rho e^{i\rho t}dt-\rho^2e^{i\rho x}A(x,x)\\ &\quad +i\rho e^{i\rho x}\left[\frac{d}{dx}A(x,x)+A_x(x,x)+q(x)\left(1-A(x,x)\right)\right]\end{aligned}$$

Integrating by parts in the left hand we arrive at

$$\int_x^\infty A(x,t)i\rho e^{i\rho t}dt=-A(x,x)e^{i\rho x}+A_t(x,x)\frac{e^{i\rho x}}{i\rho}+\int_x^\infty A_{tt}(x,t)\frac{e^{i\rho t}}{i\rho}dt$$

Hence

$$i\rho e^{i\rho x}\left\{\frac{d}{dx}A(x,x)+A_x(x,x)+A_t(x,x)+q(x)\left[1-A(x,x)\right]\right\} \tag{5.15}$$

$$+\int_x^\infty \left\{-A_{xx}(x,t)+q(x)A_x(x,t)+A_{tt}(x,t)\right\}i\rho e^{i\rho t}dt = 0. \tag{5.16}$$

Using that $\frac{d}{dx}A(x,x)=A_x(x,x)+A_t(x,x)$, we conclude that the kernel A must satisfy (at least formally) the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}-q(x)\frac{\partial}{\partial x}\right)A(x,t)=\frac{\partial^2A(x,t)}{\partial t^2},\quad (x,t)\in\Pi, \tag{5.17}$$

with the Goursat condition

$$\frac{d}{dx}A(x,x)=\frac{1}{2}q(x)\left[A(x,x)-1\right]. \tag{5.18}$$

Integrating Eq. (5.18) we obtain the relation

$$A(x,x)=1-\exp\left(-\frac{1}{2}\int_x^\infty q(s)ds\right). \tag{5.19}$$

Eq. (5.17) in terms of the new variables $u=\frac{x+t}{2}$, $v=\frac{t-x}{2}$ can be written in form

$$\frac{\partial H(u,v)}{\partial u\partial v}=-\frac{1}{2}q(u-v)\left(\frac{\partial H(u,v)}{\partial u}-\frac{\partial H(u,v)}{\partial v}\right), \tag{5.20}$$

where $H(u,v) = A(u-v,u+v)$. In its turn equation (5.20) together with (5.18) is equivalent to the integral equation

$$H(u,v) = 1 - \exp\left(-\frac{1}{2}\int_u^\infty q(\xi)d\xi\right) + \frac{1}{2}\int_u^\infty \int_0^v q(\xi-\zeta)\left(H_\xi(\xi,\zeta) - H_\zeta(\xi,\zeta)\right)d\zeta d\xi. \tag{5.21}$$

Let us see that Eq. (5.21) possesses a unique solution of class C^1 in the domain $\Omega = \{(u,v)\in\mathbb{R}^2 \,|\, 0<v<u\}$. By the method of successive approximations, we propose the solution $H(u,v) = \sum_{n=0}^{\infty} H_n(u,v)$, where

$$H_0(u,v) := 1-\exp\left(-\frac{1}{2}\int_u^\infty q(\xi)d\xi\right), \quad H_{n+1}(u,v) := \frac{1}{2}\int_u^\infty \int_0^v q(\xi-\zeta)\mathfrak{D}H_n(\xi,\zeta)d\zeta d\xi, \; n\geqslant 0,$$

where $\mathfrak{D}f(u,v) := \dfrac{\partial f(u,v)}{\partial u} - \dfrac{\partial f(u,v)}{\partial v}$, for $f\in W^{1,\infty}(\Omega)$. In the same way we introduce the semi-norm $\|\mathfrak{D}f\| := \left\|\dfrac{\partial f}{\partial u}\right\|_{L_\infty(Q)} + \left\|\dfrac{\partial f}{\partial v}\right\|_{L_\infty(\Omega)}$. Hence we can take the $W^{1,\infty}(\Omega)$-norm as $\|f\|_{W^{1,\infty}(\Omega)} = \|f\|_{L_\infty(\Omega)} + \|\mathfrak{D}f\|$. We denote

$$Q_0(x) := \frac{1}{2}\int_0^x |q(s)|ds. \tag{5.22}$$

Theorem 94. *Suppose that $q\in L_1(\mathbb{R}^+;(1+x)dx)\cap L_\infty(\mathbb{R}^+)$. Then Eq. (5.21) possesses a unique solution $H\in W^{1,\infty}(\Omega)\cap C(\overline{\Omega})$. In consequence $A\in W^{1,\infty}(\Pi)\cap C(\overline{\Pi})$.*

Proof. Note that $\mathfrak{D}H_0(u,v) = \frac{1}{2}q(u)\exp\left(-\frac{1}{2}\int_u^\infty q(s)ds\right)$, then $|\mathfrak{D}H_0(u,v)| \leqslant \frac{1}{2}|q(u)|e^{Q_0(u)}$. Hence if $(u,v)\in\overline{\Omega}$ we obtain

$$\begin{aligned}|H_1(u,v)| &\leqslant \frac{1}{4}\int_u^\infty\int_0^v |q(\xi-\zeta)|\mathfrak{D}H_0(\xi,\zeta)|d\zeta d\xi \leqslant \frac{1}{4}\int_u^\infty |q(\xi)|e^{Q_0(\xi)}\int_0^v |q(\xi-\zeta)|d\zeta d\xi \\ &\leqslant \frac{1}{4}e^{Q_0(u)}\int_u^\infty |q(\xi)|\int_{\xi-v}^{\xi}|q(s)|dsd\xi \leqslant \frac{e^{Q_0(u)}}{2}\int_u^\infty |q(\xi)|Q_0(\xi-v)d\xi \\ &\leqslant e^{Q_0(u)}Q_0(u)Q_0(u-v),\end{aligned}$$

because $Q_0(u)$ is decreasing. For the derivative we show the estimate

$$|\mathfrak{D}H_n(u,v)| \leqslant 2^{n-1}\|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(u)}\frac{Q_0^n(u-v)}{n!} \quad \forall (u,v)\in\overline{\Omega},\ \forall n\in\mathbb{N}. \tag{5.23}$$

By induction in n, for $n=1$ we get

$$\begin{aligned}|\mathfrak{D}H_1(u,v)| &\leqslant \frac{1}{4}\left(\int_0^v |q(u-\zeta)||q(u)|e^{Q_0(u)}d\zeta + \int_u^\infty |q(\xi-v)||q(\xi)|e^{Q_0(\xi)}d\xi\right) \\ &\leqslant \frac{\|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(u)}}{4}\left(\int_{u-v}^u |q(s)|ds + \int_{u-v}^\infty |q(s)|ds\right) \leqslant \|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(u)}Q_0(u-v)\end{aligned}$$

Now we suppose (5.23) for n and we check it for $n+1$:

$$\begin{aligned}
|\mathfrak{D}H_{n+1}(u,v)| &\leqslant \frac{1}{2}\left(\int_0^v |q(u-\zeta)||\mathfrak{D}H_n(u,\zeta)|d\zeta + \int_u^\infty |q(\xi-v)||\mathfrak{D}H_n(\xi,v)|d\xi\right)\\
&\leqslant \frac{2^{n-1}\|q\|_{L_\infty(\mathbb{R})}}{n!}\int_0^v |q(u-\zeta)|Q_0^n(u-\zeta)e^{Q_0(u)}d\zeta\\
&\quad + \frac{2^{n-1}\|q\|_{L_\infty(\mathbb{R})}}{n!}\int_u^\infty |q(\xi-v)|Q_0^n(\xi-v)e^{Q_0(\xi)}d\xi\\
&\leqslant \frac{2^{n-1}\|q\|_{L_\infty(\mathbb{R})}e^{Q_0(u)}}{n!}\left(\int_{u-v}^u |q(s)|Q_0^n(s)ds + \int_{u-v}^\infty |q(s)|Q_0^n(s)ds\right)\\
&\leqslant \frac{2^n\|q\|_{L_\infty(\mathbb{R})}e^{Q_0(u)}}{n!}\int_{u-v}^\infty |q(s)|Q_0^n(s)ds = 2^n\|q\|_{L_\infty(\mathbb{R})}e^{Q_0(u)}\frac{Q_0^{n+1}(u-v)}{(n+1)!},
\end{aligned}$$

which establishes (5.23). Hence, for $n \geqslant 2$

$$\begin{aligned}
|H_n(u,v)| &\leqslant \frac{1}{2}\int_u^\infty\int_0^v |q(\xi-\zeta)||\mathfrak{D}H_{n-1}(\xi,\zeta)|d\zeta d\xi\\
&\leqslant \frac{2^{n-2}\|q\|_{L_\infty(\mathbb{R}^+)}}{(n-1)!}\int_u^\infty\int_0^v |q(\xi-\zeta)|e^{Q_0(\xi)}Q_0^{n-1}(\xi-\zeta)d\zeta d\xi\\
&\leqslant \frac{2^{n-2}\|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(u)}}{(n-1)!}\int_u^\infty\int_{\xi-v}^\xi |q(s)|Q_0^{n-1}(s)dsd\xi\\
&\leqslant \frac{2^{n-2}\|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(u)}}{(n-1)!}\int_u^\infty\int_{\xi-v}^\infty |q(s)|Q_0^{n-1}(s)dsd\xi\\
&= \frac{2^{n-2}\|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(u)}}{n!}\int_u^\infty Q_0^n(\xi-v)d\xi\\
&\leqslant \frac{2^{n-2}\|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(u)}}{n!}Q_0^{n-1}(u-v)Q_1(u-v),
\end{aligned}$$

where $Q_1(x) := \int_x^\infty Q_0(s)ds = \int_x^\infty (s-x)|q(s)|ds$. From this we have

$$\begin{aligned}
\sum_{n=1}^\infty |H_n(u,v)| &\leqslant e^{Q_0(u)}Q_0(u)Q_0(u-v) + \|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(u)}Q_1(u-v)\sum_{n=2}^\infty \frac{2^{n-2}Q_0^{n-1}(u-v)}{n!}\\
&\leqslant e^{Q_0(0)}Q_0(0)Q_0(u-v) + \|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(0)}Q_1(0)e^{2Q_0(0)}Q_0(u-v).
\end{aligned}$$

Then the series converges absolutely and uniformly on $\overline{\Omega}$ and $H = \sum_{n=0}^\infty H_n(u,v) \in C(\overline{\Omega})$. For the derivatives we obtain

$$\sum_{n=1}^\infty |\mathfrak{D}H_n(u,v)| \leqslant \|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(u)}\sum_{n=1}^\infty \frac{2^{n-1}Q_0^n(u-v)}{n!} \leqslant \|q\|_{L_\infty(\mathbb{R}^+)}e^{Q_0(u)}e^{2Q_0(0)}Q_0(u-v),$$

and the series converges absolutely and uniformly on $\overline{\Omega}$. In this way $H \in W^{1,\infty}(Q)$.

Furthermore, from the previous computations we have

$$|H(u,v)| \leqslant 1 - e^{-\frac{1}{2}\int_0^\infty q(s)ds} + \left(e^{Q_0(0)}Q_0(0) + \|q\|_{L_\infty(\mathbb{R}^+)} e^{3Q_0(0)}Q_1(0)\right) Q_0(u-v) \quad (5.24)$$

$$|\mathfrak{D}H(u,v)| \leqslant \frac{1}{2}|q(u)|e^{Q_0(0)} + \|q\|_{L_\infty(\mathbb{R}^+)} e^{3Q_0(u)} Q_0(u-v) \quad (5.25)$$

From (5.25) we obtain that integration in (5.21) makes sense. Then $H \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ is the unique solution of (5.6). Since the change of variables $u = \frac{x+t}{2}$, $v = \frac{t-x}{2}$ transforms Π onto Ω, we conclude that $A \in W^{1,\infty}(\Pi) \cap C(\overline{\Pi})$. **Q.E.D.**

5.2 Birkhoff solution

Analogously the case of the Schrödinger equation ([52, Ch. 2]), we look for a solution $E(\rho, x)$ of (5.3) that satisfies the following asymptotic relation

$$E^{(k)}(\rho, x) = (-i\rho)^k e^{-i\rho x}(1 + o(1)),\ x \to \infty, \quad \forall \rho \in \mathbb{C}^+ \ \text{ for } k = 0, 1. \quad (5.26)$$

Such a solution is referred to as *Birkhoff solution*. Fix $a \in \mathbb{R}^+$. Application of the method of variation of parameters to (5.3) leads to the equation

$$E(\rho, x) = e^{-i\rho x} + \frac{1}{2i\rho}\int_a^\infty e^{i\rho|x-t|} q(t) E'(\rho, t) dt, \quad \forall x \geqslant a. \quad (5.27)$$

The method of successive approximations is applicable to (5.27).

Theorem 95. *For every $a \in \mathbb{R}^+$, such that $Q_0(a) < 1$, there exists a unique solution $E^a(\rho, x)$ of (5.3) that satisfies the integral equation (5.27) for all $x \geqslant a$. The solution $E^a(\rho, x)$ satisfies (5.26), and $(E^a)^{(k)}(\cdot, x) \in \mathrm{Hol}(\mathbb{C}^+)$ for all $x \in \mathbb{R}^+$ and $k = 0, 1$.*

Proof. For applying successive approximations we define $\widetilde{E}(\rho, x) := \sum_{n=0}^{\infty} \widetilde{E}_n(\rho, x)$ with $\widetilde{E}_0(\rho, x) := e^{-i\rho x}$ and $\widetilde{E}_{n+1}(\rho, x) := \frac{1}{2i\rho}\int_a^\infty e^{i\rho|x-t|} q(t) \widetilde{E}'_n(\rho, t) dt$ for $n \geqslant 0$. Note that $\widetilde{E}'_n(\rho, x) = -i\rho e^{-i\rho x} + \frac{1}{2}\int_a^\infty e^{i\rho|x-t|} q(t) \widetilde{E}'_{n-1}(\rho, t) dt$. Let us prove by induction the inequality

$$|\widetilde{E}'_n(\rho, x)| \leqslant |\rho| e^{x \operatorname{Im} \rho} Q_0^n(a), \quad \forall x \geqslant a,\ \forall \rho \in \mathbb{C}^+,\ \forall n \in \mathbb{N}_0. \quad (5.28)$$

For $n = 0$ (5.28) holds. Suppose that (5.28) is valid for n and check it for $n+1$:

$$\begin{aligned}
|\widetilde{E}'_{n+1}(\rho,x)| &\leqslant \frac{1}{2}\left(\int_a^x e^{(t-x)\operatorname{Im}\rho}|q(t)||\widetilde{E}'_n(\rho,t)|dt + \int_x^\infty e^{(x-t)\operatorname{Im}\rho}|q(t)||\widetilde{E}'_n(\rho,t)|dt\right)\\
&\leqslant \frac{|\rho|Q_0^n(a)}{2}\left(\int_a^x e^{(t-x)\operatorname{Im}\rho}|q(t)|e^{t\operatorname{Im}\rho}dt + \int_x^\infty e^{(x-t)\operatorname{Im}\rho}|q(t)|e^{t\operatorname{Im}\rho}dt\right)\\
&\leqslant \frac{|\rho|Q_0^n(a)}{2}\int_a^\infty e^{x\operatorname{Im}\rho}|q(t)|dt = |\rho|e^{x\operatorname{Im}\rho}Q_0^{n+1}(a)
\end{aligned}$$

(because $e^{(t-x)\operatorname{Im}\rho} \leqslant 1$ and $e^{t\operatorname{Im}\rho} \leqslant e^{x\operatorname{Im}\rho}$ for $a \leqslant t \leqslant x$). This establishes (5.28).

Now we get for $n \geqslant 1$

$$\begin{aligned}
|\widetilde{E}_n(\rho,x)| &\leqslant \frac{1}{2|\rho|}\left(\int_a^x e^{(t-x)\operatorname{Im}\rho}|q(t)||\widetilde{E}'_{n-1}(\rho,t)|dt + \int_x^\infty e^{(x-t)\operatorname{Im}\rho}|q(t)||\widetilde{E}'_{n-1}(\rho,t)|dt\right)\\
&\leqslant \frac{Q_0^{n-1}(a)}{2}\left(\int_a^x e^{(t-x)\operatorname{Im}\rho}|q(t)|e^{t\operatorname{Im}\rho}dt + \int_x^\infty e^{(x-t)\operatorname{Im}\rho}|q(t)|e^{t\operatorname{Im}\rho}dt\right)\\
&\leqslant \frac{Q_0^n(a)}{2}\int_a^\infty e^{x\operatorname{Im}\rho}|q(t)|dt = e^{x\operatorname{Im}\rho}Q_0^n(a).
\end{aligned}$$

By hypothesis $Q_0(a) < 1$, hence

$$\sum_{n=0}^\infty |\widetilde{E}_n^{(k)}(\rho,x)| \leqslant |\rho|^k e^{x\operatorname{Im}\rho}\sum_{n=0}^\infty Q_0^n(a) = \frac{|\rho|^k e^{x\operatorname{Im}\rho}}{1-Q_0(a)}, \quad \text{for } \ k = 0,1.$$

Then both series converge absolutely and uniformly for $x \geqslant a$ and $\widetilde{E}(\rho,\cdot) \in C^1[a,\infty)$ is a solution of (5.27) and a strong solution of (5.3) in the interval (a,∞). Furthermore,

$$\begin{aligned}
|\widetilde{E}^{(k)}(\rho,x)(-i\rho)^{-k}e^{i\rho x} - 1| &\leqslant \frac{|\rho|^{k-1}e^{-x\operatorname{Im}\rho}}{2(1-Q_0(a))}\left(\int_a^x e^{(2t-x)\operatorname{Im}\rho}|q(t)|dt + \int_x^\infty e^{x\operatorname{Im}\rho}|q(t)|dt\right)\\
&\leqslant \frac{|\rho|^{k-1}}{2(1-Q_0(a))}\left(\int_a^x e^{2(t-x)\operatorname{Im}\rho}|q(t)|dt + \int_x^\infty |q(t)|dt\right)\\
&\leqslant \frac{|\rho|^{k-1}}{2(1-Q_0(a))}\left(e^{-x\operatorname{Im}\rho}\int_a^{\frac{x}{2}} e^{(2t-x)\operatorname{Im}\rho}|q(t)|dt + \int_{\frac{x}{2}}^\infty |q(t)|dt\right)\\
&\leqslant \frac{|\rho|^{k-1}}{2(1-Q_0(a))}\left(e^{-x\operatorname{Im}\rho}\int_a^{\frac{x}{2}} |q(t)|dt + \int_{\frac{x}{2}}^\infty |q(t)|dt\right) = o(1),
\end{aligned}$$

where we took into account that $e^{(t-x)\operatorname{Im}\rho} \leqslant 1$. Thus, we obtain (5.26).

Now we show by induction that $\widetilde{E}_n^{(k)}(\rho,x)$ is holomorphic for $\rho \in \mathbb{C}^+$. For $n = 0$ it is obvious. We suppose this for n. Consider $\phi_x(\rho,t) = e^{i\rho|x-t|}\widetilde{E}'_n(\rho,t)$ so that $\widetilde{E}_{n+1}^{(k)}(\rho,x) = \frac{1}{2i\rho^{1-k}}\psi_x(\rho)$, where $\psi_x(\rho) = \displaystyle\int_a^\infty \phi_x(\rho,t)q(t)dt$. By the induction hypothesis $\phi_x(\cdot,t) \in Hol(\mathbb{C}^+)$ for all $t \geqslant a$, and due to (5.28), $|\phi_x(\rho,t)| \leqslant Re^{xR}Q_0^n(a)$ for all $\rho \in \mathbb{C}^+ \cap B_R(0)$,

$t \geqslant a$ and $R > 0$. Take $\rho_0 \in \mathbb{C}^+$ and a sequence $\{\rho_m\} \subset \mathbb{C}^+$ such that $\rho_m \to \rho_0$. Choosing $R > 0$ such that $\{\rho_m\} \cup \{\rho_0\} \subset \mathbb{C}^+ \cap B_R(0)$ we obtain $|\phi_x(\rho_n, t) - \phi_x(\rho, t)| \leqslant 2Re^{xR} Q_0^n(a)$. Hence by dominated convergence $\psi_x(\rho_n) \to \psi_x(\rho)$, thus $\psi_x(\rho)$ is continuous for all $\rho \in \mathbb{C}^+$. Take a triangular path Γ in $\mathbb{C}^+$ of length $|\Gamma|$. We have

$$\int_\Gamma \int_a^\infty |\phi_x(\rho, t)||q(t)|dt|d\rho| \leqslant |\Gamma| \|q\|_{L_1(a,\infty)} T e^{Tx} Q_0^n(a), \qquad \text{where } T = \max_{\rho \in \Gamma} |\rho|.$$

Applying the Fubini and Cauchy theorems we obtain

$$\int_\Gamma \psi_x(\rho) d\rho = \int_\Gamma \left[\int_a^\infty \phi_x(\rho, t) q(t) dt \right] d\rho = \int_a^\infty \left[\int_\Gamma \phi_x(\rho, t) d\rho \right] q(t) dt = 0.$$

By Morera's theorem ([121, Th. 10.17]), $\psi_x \in \text{Hol}(\mathbb{C}^+)$ and hence $\widetilde{E}_{n+1}^{(k)}(\cdot, x) \in \text{Hol}(\mathbb{C}^+)$. Since the series $\widetilde{E}^{(k)}(\rho, x) = \sum_{n=0}^\infty \widetilde{E}_n^{(k)}(\rho, x)$ converges absolutely and uniformly for $\rho \in \mathbb{C}^+ \cap B_R(0)$, for each $R > 0$ we obtain that $\widetilde{E}^{(k)}(\cdot, x) \in \text{Hol}(\mathbb{C}^+)$ for all $x \geqslant a$, $k = 0, 1$.

Now set $\xi_k(\rho) := \widetilde{E}^{(k)}(\rho, a)$ for $k = 0, 1$, and take $\widetilde{e}(\rho, x)$ the unique solution of (5.3) in $(0, a)$ satisfying the initial conditions $\widetilde{e}^{(k)}(\rho, a) = \xi_k(\rho)$ for $k = 0, 1$. Since $\xi_k(\rho) \in \text{Hol}(\mathbb{C}^+)$, then $\widetilde{e}^{(k)}(\cdot, x) \in \text{Hol}(\mathbb{C}^+)$ for all $0 \leqslant x \leqslant a$ (this can be seen, for example, with the aid of the *SPPS method*, see [83]). Finally we take

$$E^a(\rho, x) = \begin{cases} \widetilde{e}(\rho, x), & \text{if } 0 \leqslant x \leqslant a \\ \widetilde{E}(\rho, x), & \text{otherwise.} \end{cases}$$

Hence $E^a(\rho, \cdot) \in C^1(\overline{\mathbb{R}^+})$ and it is easy to see that $E^a(\rho, \cdot) \in W_{loc}^{2,1}(\mathbb{R}^+)$, and hence it is a strong solution of (5.3). **Q.E.D.**

Remark 96. *The asymptotic relations (5.5) and (5.26) yield*

$$\lim_{x \to \infty} W_p(e(\rho, x), E^a(\rho, x)) = -2ip_\infty \rho.$$

Thus, $W_p(e(\rho, x), E^a(\rho, x)) = -2ip_\infty \rho$, *and* $\{e(\rho, x), E^a(\rho, x)\}$ *is a fundamental set of solutions of (5.3).*

5.3 Series representation of the Jost solution

Following [77], we introduce a functional series representation for the kernel $A(x, t)$ in terms of Laguerre polynomials.

5.3.1 Fourier-Laguerre series expansion of the kernel $A(x,t)$

Fix $x \in \mathbb{R}^+$, and denote

$$a(x,t) := A(x, x+t)e^{\frac{t}{2}}.$$

By Theorem 93, $a(x,\cdot) \in L_2(\mathbb{R}^+; e^{-t}dt)$. An orthonormal basis for $L_2(\mathbb{R}^+; e^{-t}dt)$ is given by the Laguerre polynomials $\{L_n(t)\}_{n=0}^{\infty}$ (see [131, Ch. VI]). Thus $a(x,t)$ admits a Fourier-Laguerre representation

$$a(x,t) = \sum_{n=0}^{\infty} a_n(x) L_n(t).$$

This series converges with respect to t in the norm of the space $L_2(\mathbb{R}^+; e^{-t}dt)$. Returning to the Levin kernel we obtain

$$A(x,t) = \sum_{n=0}^{\infty} a_n(x) L_n(t-x) e^{\frac{x-t}{2}}. \tag{5.29}$$

In particular, by (5.4) and by the equality $L_n(0) = 1$ for all $n \in \mathbb{N}_0$ (see [131, Ch. VI]), we have

$$\sum_{n=0}^{\infty} a_n(x) = A(x,x) = 1 - \exp\left(-\frac{1}{2}\int_x^{\infty} q(s)ds\right) = 1 - \sqrt{\frac{p_\infty}{p(x)}}. \tag{5.30}$$

With the aid of (5.29) we obtain a series representation for the Jost solution.

Proposition 97. *The Jost solution $e(\rho,x)$ admits the following series representation*

$$e(\rho,x) = e^{i\rho x}\left(1 + \frac{(z-1)}{2}\sum_{n=0}^{\infty}(-1)^n z^n a_n(x)\right), \quad \forall x \in \mathbb{R}^+,\ \rho \in \mathbb{C}^+, \tag{5.31}$$

where $z = \frac{\frac{1}{2}+i\rho}{\frac{1}{2}-i\rho}$. The series converges pointwise for all $x \in \mathbb{R}^+$, $\rho \in \mathbb{C}^+$.

Proof. Due to (5.12) and (5.29) we obtain

$$\begin{aligned} e(\rho,x) &= e^{i\rho x} + \int_x^{\infty} A(x,t) i\rho e^{i\rho t} dt = e^{i\rho x} + \int_0^{\infty} A(x,x+t) i\rho e^{i\rho(x+t)} dt \\ &= e^{i\rho x}\left\{1 + \int_0^{\infty}\left(a(x,t)e^{-\frac{t}{2}}\right) i\rho e^{i\rho t} dt\right\} \\ &= e^{i\rho x}\left\{1 + i\rho\sum_{n=0}^{\infty} a_n(x)\int_0^{\infty} L_n(t) e^{-\left(\frac{1}{2}-i\rho\right)t} dt\right\} \end{aligned}$$

(the change of the order of integration and summation is due to Parseval's identity [3, pp. 16]). According to [62, formula 7.414(2)],

$$\int_0^{\infty} L_n(t) e^{-\left(\frac{1}{2}-i\rho\right)t} dt = (-1)^n \frac{\left(\frac{1}{2}+i\rho\right)^n}{\left(\frac{1}{2}-i\rho\right)^{n+1}}.$$

Thus,

$$e(\rho,x)=e^{i\rho x}\left(1+i\rho\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}+i\rho\right)^n}{\left(\frac{1}{2}-i\rho\right)^{n+1}}a_n(x)\right).$$

Denote $z=\frac{\frac{1}{2}+i\rho}{\frac{1}{2}-i\rho}$. Then $i\rho=\frac{z-1}{2(z+1)}$ and $\frac{1}{2}-i\rho=\frac{1}{z+1}$, from which we obtain (5.31).

Q.E.D.

Remark 98. *Note that* $z=\frac{\frac{1}{2}+i\rho}{\frac{1}{2}-i\rho}$ *is a Möbius transformation of the upper half-plane* $\mathbb{C}^+$ *of the complex variable* ρ *onto the unit disk* $\mathbb{D}$. *It maps the point* $\rho=0$ *to* $z=1$, *the ray* $\rho=i\tau$, $\tau>0$ *to the interval* $(-1,1)$ *in terms of* z, *and when* τ *runs from* 0 *to* $+\infty$, z *runs from* 1 *to* -1. *When* ρ *runs from* 0 *to* $+\infty$ *along the real line,* z *runs from* 1 *to* -1 *along the upper unit semicircle (see Figure 5.1). Note that the function* $e(\rho,x)e^{-i\rho x}$ *is just*

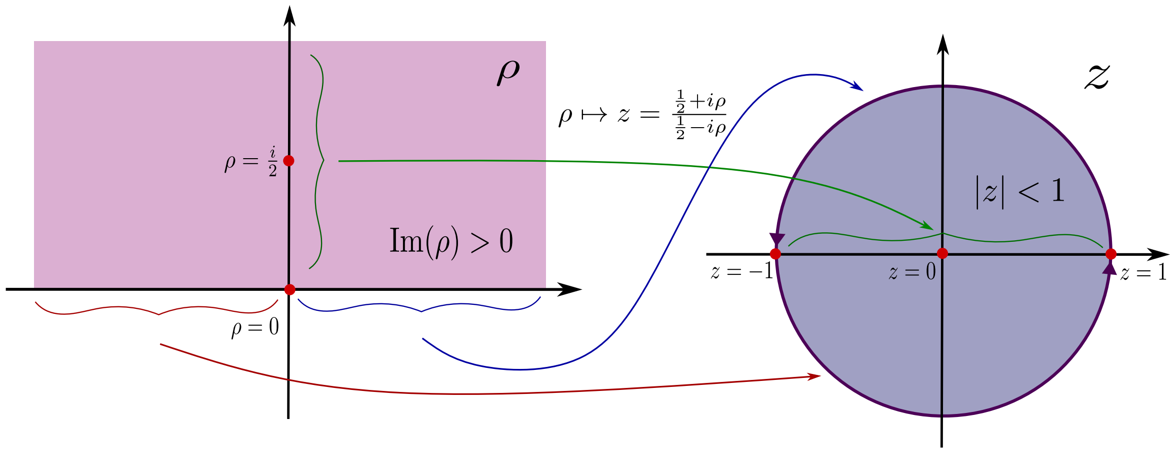

Figure 5.1: Schematic illustration of the Möbious transformation $z=\frac{\frac{1}{2}+i\rho}{\frac{1}{2}-i\rho}$ of the upper half-plane onto the unit disk.

a power series in the parameter z. *The series converges in the unit disk of the variable* z. *For* $x\in\mathbb{R}^+$ *fixed,* $\{a_n(x)\}\in\ell_2$, *because these are the Fourier coefficients of the function* $a(x,\cdot)\in L_2\left(\mathbb{R}^+;e^{-t}dt\right)$ *with respect to the orthonormal basis of Laguerre polynomials. Hence the function* $e(\rho,x)e^{-i\rho x}$ *belongs to the Hardy space* $\mathcal{H}^2(\mathbb{D})$ *as a function of* z *(see [121, Th. 17.12]).*

Let us denote by $e_N(\rho,x)$ the N-th partial sum of (5.31) (not to be confused with the notation used in section 2). Then $e_N(\rho,x)=e^{i\rho x}+\int_0^\infty a_N(x,t)e^{-\frac{t}{2}}e^{i\rho(x+t)}dt$ where

$a_N(x,t) = \sum_{n=0}^{N} a_n(x)L_n(t)$. In a similar way we define

$$A_N(x,t) := a_N(x,t-x)e^{\frac{x-t}{2}} = \sum_{n=0}^{N} a_n(x)L_n(t-x)e^{\frac{x-t}{2}}.$$

Theorem 99. *Fix $x \in \mathbb{R}^+$, and for $N \in \mathbb{N}$ define $\mathcal{R}_N(\rho,x) := \frac{e(\rho,x)-e_N(\rho,x)}{i\rho}$.*

1. *Let $\rho \in \mathbb{C}^+$. Then*
$$|\mathcal{R}_N(\rho,x)| \leqslant \varepsilon_N(x)\frac{e^{-x\operatorname{Im}\rho}}{\sqrt{2\operatorname{Im}\rho}}, \tag{5.32}$$
where $\varepsilon_N(x) = \left(\sum_{n=N+1}^{\infty} |a_n(x)|^2\right)^{\frac{1}{2}}$.

2. *For $\rho \in \mathbb{R}$*
$$\|\mathcal{R}_N(\cdot,x)\|_{L_2(\mathbb{R})} = \sqrt{2\pi}\varepsilon_N(x). \tag{5.33}$$

In particular, series (5.31) converges uniformly with respect to ρ such that $C_1 \leqslant \operatorname{Im}\rho \leqslant C_2$, with $C_1, C_2 > 0$.

Proof. 1. Suppose that $\rho \in \mathbb{C}^+$. Consider

$$\begin{aligned} |\mathcal{R}_N(\rho,x)| &= \left|\int_0^\infty \left(A(x,x+t) - A_N(x,x+t)\right) e^{i\rho(x+t)}\right| \\ &= \left|\int_0^\infty \left(a(x,t) - a_N(x,t)\right) e^{-\frac{t}{2}+i\rho(x+t)} dt\right| \\ &= \left|\left\langle e^{\frac{t}{2}+i\rho(x+t)}, \overline{a(x,t)} - \overline{a_N(x,t)}\right\rangle_{L_2(\mathbb{R}^+;e^{-t}dt)}\right| \end{aligned}$$

Using the Cauchy-Bunyakovsky-Schwarz inequality we obtain

$$|\mathcal{R}_N(\rho,x)| \leqslant \|a(x,t) - a_N(x,t)\|_{L_2(\mathbb{R}^+;e^{-t}dt)} \|e^{\frac{t}{2}+i\rho(x+t)}\|_{L_2(\mathbb{R}^+;e^{-t}dt)}.$$

Due to the Parseval identity $\|a(x,t) - a_N(x,t)\|_{L_2(\mathbb{R}^+;e^{-t}dt)} = \left(\sum_{n=N+1}^{\infty} |a_n(x)|^2\right)^{\frac{1}{2}} = \varepsilon_N(x)$, and

$$\|e^{\frac{t}{2}+i\rho(x+t)}\|_{L_2(\mathbb{R}^+;e^{-t}dt)} = \left(\int_0^\infty e^{-2(x+t)\operatorname{Im}\rho} dt\right)^{\frac{1}{2}} = \frac{e^{-x\operatorname{Im}\rho}}{\sqrt{2\operatorname{Im}\rho}},$$

from which (5.32) follows.

2. For $\rho \in \mathbb{R}$, we consider the difference

$$\mathcal{R}_N(\rho, x) = \int_x^\infty (A(x,t) - A_N(x,t))\, e^{i\rho t} dt.$$

Extending the function $A(x,t) - A_N(y,t)$ by zero for $t < x$, we obtain that $\mathcal{R}_N(\rho, x)$ is the Fourier transform of an L_2 function, then by the Parseval identity we have

$$\|\mathcal{R}_N(\cdot, x)\|_{L_2(\mathbb{R})} = \sqrt{2\pi}\|A(x,\cdot) - A_N(x,\cdot)\|_{L_2(x,\infty)},$$

and

$$\int_x^\infty |A(x,t) - A_N(x,t)|^2 dt = \int_0^\infty |a(x,t) - a_N(x,t)| e^{-t} dt = \varepsilon_N(x),$$

that establishes (5.33).

Q.E.D.

Remark 100. *The coefficient $a_0(x)$ is given by*

$$a_0(x) = 2\left(1 - e^{\frac{x}{2}} e\left(\frac{i}{2}, x\right)\right). \tag{5.34}$$

Indeed, multiplying $A(x, x+t)$ by $L_n(t)e^{-\frac{t}{2}}$ and integrating we obtain

$$\int_0^\infty A(x, x+t) L_n(t) e^{-\frac{t}{2}} dt = \sum_{m=0}^\infty a_m(x) \int_0^\infty L_n(t) L_m(t) e^{-t} dt = a_n(x).$$

In particular, for $n = 0$:

$$a_0(x) = \int_0^\infty A(x, x+t) L_0(t) e^{-\frac{t}{2}} dt = \int_0^\infty A(x, x+t) e^{-\frac{t}{2}} dt = \int_x^\infty A(x,t) e^{\frac{x-t}{2}} dt.$$

Note that $a_0(x) = e^{\frac{x}{2}} \int_x^\infty A(x,t) e^{i\left(\frac{i}{2}\right)t} dt$, and then $e\left(\frac{i}{2}, x\right) = e^{-\frac{x}{2}}\left(1 - \frac{1}{2}a_0(x)\right)$.

In the next subsection we derive a system of equations for the coefficients $\{a_n(x)\}_{n=0}^\infty$.

5.3.2 System of equations for the coefficients $\{a_n(x)\}_{n=0}^\infty$

Differentiating (5.31) we obtain

$$\begin{aligned} e'(\rho, x) &= e^{i\rho x}\left\{ i\rho\left(1 + \frac{z-1}{2}\sum_{n=0}^\infty (-1)^n z^n a_n(x)\right) + \frac{z-1}{2}\sum_{n=0}^\infty (-1)^n z^n a_n'(x)\right\}, \\ e''(\rho, x) &= e^{i\rho x}\left\{ -\rho^2\left(1 + \frac{z-1}{2}\sum_{n=0}^\infty (-1)^n z^n a_n(x)\right) + i\rho(z-1)\sum_{n=0}^\infty (-1)^n z^n a_n'(x)\right\} \\ &\quad + e^{i\rho x}\frac{z-1}{2}\sum_{n=0}^\infty (-1)^n z^n a_n''(x). \end{aligned}$$

Substituting these expressions into (5.3) leads to the equality

$$i\rho(z-1)\sum_{n=0}^{\infty}(-1)^n z^n a_n'(x)+\frac{z-1}{2}\sum_{n=0}^{\infty}(-1)^n z^n a_n''(x)=q(x)i\rho\left(1+\frac{z-1}{2}\sum_{n=0}^{\infty}(-1)^n z^n a_n(x)\right)$$
$$+\frac{z-1}{2}q(x)\sum_{n=0}^{\infty}(-1)^n z^n a_n'(x).$$

Dividing the last equality by $i\rho$ and taking into account that $i\rho=\frac{z-1}{2(z+1)}$, we obtain

$$(z-1)\sum_{n=0}^{\infty}(-1)^n z^n a_n'(x)+(z+1)\sum_{n=0}^{\infty}(-1)^n z^n a_n''(x)=q(x)\left(1+\frac{z-1}{2}\sum_{n=0}^{\infty}(-1)^n z^n a_n(x)\right)$$
$$+(z+1)q(x)\sum_{n=0}^{\infty}(-1)^n z^n a_n'(x),$$

or equivalently,

$$\sum_{n=1}^{\infty}(-1)^n z^n\left\{a_n''-a_n'-a_{n-1}'-a_{n-1}''\right\}+a_0''-a_0'=q-\frac{qa_0}{2}+qa_0'$$
$$+\sum_{n=1}^{\infty}(-1)^n z^n\left\{qa_n'-qa_{n-1}'-\frac{qa_n}{2}-\frac{qa_{n-1}}{2}\right\}.$$

Comparing both series we obtain the equations

$$\mathbf{L}a_0(x)+a_0'(x)-q(x)\frac{a_0(x)}{2} = -q(x), \tag{5.35}$$
$$\mathbf{L}a_n(x)+a_n'(x)-q(x)\frac{a_n(x)}{2} = \mathbf{L}a_{n-1}(x)-a_{n-1}'(x)+q(x)\frac{a_{n-1}(x)}{2} \quad \text{for } n\geqslant 1 \tag{5.36}$$

Due to (5.34) and (5.5), $a_0(x)\to 0$ when $x\to\infty$. Since $e\left(\frac{i}{2},x\right)$ is a solution of $\left(\mathbf{L}-\frac{1}{4}\right)e\left(\frac{i}{2},x\right)=0$, a_0 satisfies (5.35).

Proposition 101. *Set $b_n:=1-\frac{a_n}{2}$ for $n\in\mathbb{N}_0$. Then the system (5.35), (5.36) can be written as*

$$\mathbf{L}b_0+b_0'-q(x)\frac{b_0}{2} = 0, \tag{5.37}$$
$$\mathbf{L}b_n+b_n'-q(x)\frac{b_n}{2} = \sum_{k=0}^{n-1}\left(a_k'-q(x)\frac{a_k}{2}\right) \quad \textit{for } n\geqslant 1. \tag{5.38}$$

Proof. First, note that $\mathbf{L}a_n=-2\mathbf{L}b_n$ for all $n\in\mathbb{N}_0$. Hence Eq. (5.35) can be written as $\mathbf{L}b_0+b_0'-q(x)\frac{b_0}{2}=0$. Eq. (5.38) is proved by induction. For $n=1$ we have

$$\begin{aligned}-2\mathbf{L}b_1+a_1'-q(x)\frac{a_1}{2} &= \mathbf{L}a_0-a_0'+q(x)\frac{a_0}{2}\\ &= -q(x)-2\left(a_0'-q(x)\frac{a_0}{2}\right),\end{aligned}$$

where we used (5.35). Thus,

$$\mathbf{L}b_1 + b_1' - q(x)\frac{b_1}{2} = a_0' - q(x)\frac{a_0}{2}.$$

This establishes (5.38) for $n = 1$. Suppose that (5.38) is valid for n. Then for $n + 1$ we have

$$-2\mathbf{L}b_{n+1} + a_{n+1}' - q(x)\frac{a_{n+1}}{2} = \mathbf{L}a_n - a_n' + q(x)\frac{a_n}{2},$$

and hence

$$\mathbf{L}b_{n+1} + b_{n+1}' - q(x)\frac{b_{n+1}}{2} = -\frac{1}{2}\left(\mathbf{L}a_n - a_n' + q(x)\frac{a_n}{2} + q(x)\right).$$

Using the hypothesis of induction we obtain

$$-\frac{1}{2}\mathbf{L}a_n - \frac{a_n'}{2} - \frac{1}{2}q(x)\left(1 - \frac{a_n}{2}\right) = \sum_{k=0}^{n-1}\left(a_k' - q(x)\frac{a_k}{2}\right),$$

and hence

$$\mathbf{L}a_n = -q(x) - a_n' + q(x)\frac{a_n}{2} - 2\sum_{k=0}^{n-1}\left(a_k' - q(x)\frac{a_k}{2}\right).$$

Thus,

$$\mathbf{L}b_{n+1} + b_{n+1}' - q(x)\frac{b_{n+1}}{2} = \sum_{k=0}^{n}\left(a_k - q(x)\frac{a_k}{2}\right).$$

This establishes (5.38) for $n + 1$ and finishes the proof. **Q.E.D.**

Theorem 102. *For all $n \in \mathbb{N}_0$, Eq. (5.36) possesses a unique solution $a_n(x)$ satisfying the condition*

$$a_n(x) \to 0, \ x \to \infty. \tag{5.39}$$

Proof. Let us show that the system of equations (5.35) and (5.36) admits solutions satisfying (5.39) and such solutions are unique.

First, note that the left hand side of (5.37), (5.38) admits the factorization

$$\mathbf{L}u + u' - q(x)\frac{u}{2} = e^{\frac{x}{2}}\left(\mathbf{L} + \frac{1}{4}\right)\left(e^{-\frac{x}{2}}u\right). \tag{5.40}$$

Actually, (5.37) is equivalent to $\left(\mathbf{L} + \frac{1}{4}\right)u = 0$, and we know that $e\left(\frac{i}{2}, x\right)$ is its solution. With the notation $\zeta_n = e^{-\frac{x}{2}}b_n$ for $n \geqslant 1$, Eq. (5.38) takes the form

$$\left(\mathbf{L} + \frac{1}{4}\right)\zeta_n = \sigma_n, \quad n \in \mathbb{N}, \tag{5.41}$$

where $\sigma_n(x) = e^{-\frac{x}{2}} \sum_{k=0}^{n-1} \left(a_k' - q(x)\frac{a_k}{2} \right)$. A particular solution for (5.41) can be obtained by the method of variation of parameters by the formula

$$\zeta_{n,0}(x) = \int_x^\infty \begin{vmatrix} u_1(t) & u_2(t) \\ u_1(x) & u_2(x) \end{vmatrix} \frac{\sigma_n(t)}{W_p(u_1, u_2)(t)} p(t)dt,$$

where $\{u_1, u_2\}$ is a fundamental system of solutions for $\left(\mathbf{L} + \frac{1}{4}\right) u = 0$. Take a Birkhoff solution $E^a\left(\frac{i}{2}, x\right)$ and consider the normalized solution $\widehat{E}^a(\rho, x) = p_\infty^{-1} E^a(\rho, x)$. By Remark 96, $\left\{ e\left(\frac{i}{2}, x\right), \widehat{E}^a\left(\frac{i}{2}, x\right) \right\}$ is a fundamental set of solutions with $W_p\left(e\left(\frac{i}{2}, x\right), \widehat{E}^a\left(\frac{i}{2}, x\right) \right) = 1$. Hence the particular solution $\zeta_{n,0}$ is given by

$$\zeta_{n,0}(x) = \int_x^\infty \begin{vmatrix} e\left(\frac{i}{2}, t\right) & \widehat{E}^a\left(\frac{i}{2}, t\right) \\ e\left(\frac{i}{2}, x\right) & \widehat{E}^a\left(\frac{i}{2}, x\right) \end{vmatrix} p(t)\sigma_n(t)dt = \widehat{E}^a\left(\frac{i}{2}, x\right) I_{1,n}(x) - e\left(\frac{i}{2}, x\right) I_{2,n}(x),$$

where

$$I_{1,n}(x) = \int_x^\infty e\left(\frac{i}{2}, t\right) e^{-\frac{t}{2}} \left(\sum_{k=0}^{n-1} \left(a_k'(t) - q(t)\frac{a_k(t)}{2} \right) \right) p(t)dt$$

$$I_{2,n}(x) = \int_x^\infty \widehat{E}^a\left(\frac{i}{2}, t\right) e^{-\frac{t}{2}} \left(\sum_{k=0}^{n-1} \left(a_k'(t) - q(t)\frac{a_k(t)}{2} \right) \right) p(t)dt$$

Now we prove by induction that the solutions $\{a_n(x)\}_{n=0}^\infty$ satisfy (5.39). For $n = 0$ this has already been established. Assume it for $n \geqslant 0$. Then using (5.26) we have

$$\begin{aligned} I_{2,n+1}(x) &= \int_x^\infty (1 + o(1)) \sum_{k=0}^n \left(a_k'(t) - q(t)\frac{a_k(t)}{2} \right) dt \\ &= \sum_{k=0}^n \left\{ \int_x^\infty (1 + o(1))\, a_k'(t)dt - \int_x^\infty (1 + o(1))\, q(t)\frac{a_k(t)}{2} dt \right\}. \end{aligned}$$

By the induction hyphotesis, $a_k(x) = o(1)$, $x \to \infty$, thus the second integral is $o(1)$. For the first integral we have

$$\int_x^\infty a_k'(t)dt = -a_k(x) = o(1),\ x \to \infty.$$

Hence $I_{2,n+1}(x) = o(1)$, $x \to \infty$. For $I_{1,n}(x)$ we have

$$I_{1,n+1}(x) = \sum_{k=0}^n \left(\int_x^\infty e\left(\frac{i}{2}, t\right) e^{-\frac{t}{2}} a_k'(t)p(t)dt - \int_x^\infty e\left(\frac{i}{2}, t\right) e^{-\frac{t}{2}} q(t)\frac{a_k(t)}{2} p(t)dt \right).$$

The second integral is $e^{-x}o(1)$, and for the first, integration by parts yields

$$\begin{aligned}\int_x^\infty e\left(\frac{i}{2},t\right)e^{-\frac{t}{2}}a_k'(t)p(t)dt &= -e^{-\frac{x}{2}}e\left(\frac{i}{2},x\right)p(x)a_k(x)+\frac{1}{2}\int_x^\infty e\left(\frac{i}{2},t\right)e^{-\frac{t}{2}}p(t)a_k(t)dt\\ &-\int_x^\infty e'\left(\frac{i}{2},t\right)e^{-\frac{t}{2}}p(t)a_k(t)dt-\int_x^\infty e\left(\frac{i}{2},t\right)e^{-\frac{t}{2}}p'(t)a_k(t)dt\\ &= e^{-x}\left(1+o(1)\right)o(1)+\frac{1}{2}e^{-x}\left(1+o(1)\right)o(1)\\ &+e^{-x}\left(1+o(1)\right)o(1)+e^{-x}\left(1+o(1)\right)o(1)\\ &= e^{-x}o(1),\quad x\to\infty.\end{aligned}$$

Thus, $I_{1,n+1}(x)=e^{-x}o(1)$, $x\to\infty$. Substituting the asymptotics of $I_{1,n+1}$ and $I_{2,n+1}$ in $\zeta_{n,0}$ we obtain that $\zeta_{n+1,0}(x)=e^{-\frac{x}{2}}o(1)$, $x\to\infty$. Then the unique solution ζ_{n+1} of (5.41) satisfying $\zeta_{n+1}(x)=e^{-\frac{x}{2}}\left(1+o(1)\right)$, $x\to\infty$ is given by $\zeta_{n+1}(x)=\zeta_{n,0}(x)+e\left(\frac{i}{2},x\right)$, and eventually $a_{n+1}(x)=2-2b_n(x)=2-2e^{\frac{x}{2}}\zeta_n(x)=o(1)$, $x\to\infty$. **Q.E.D.**

5.3.3 Series representation for the derivative of the Jost solution

In this subsection we obtain a series expansion for $e'(\rho,x)$. For this it is necessary to establish the conditions under which $A_x(x,t)\in L_2(x,\infty)$ for all $x\in\mathbb{R}^+$.

Theorem 103. *Suppose that $p\in W^{2,\infty}(\mathbb{R}^+)$. If $p',p''\in L_1(\mathbb{R}^+;(1+x)^2dx)$, then $A_x(x,t)\in L_2(x,\infty)$ for every $x\in\mathbb{R}^+$.*

Proof. Consider $y(\rho,x)=\sqrt{p(x)}e(\rho,x)$. This is the Liouville transformation of Eq. (5.1), and in this case $y(\rho,x)$ is a solution of the Schödinger equation

$$-y''(\rho,x)+\widehat{q}(x)y(\rho,x)=\rho^2y(\rho,x),\ x\in\mathbb{R}^+,\tag{5.42}$$

with $\widehat{q}(x)=\frac{1}{\sqrt{p(x)}}\frac{d^2}{dx^2}\sqrt{p(x)}=\frac{p''(x)}{2p(x)}-\left(\frac{p'(x)}{2p(x)}\right)^2$ (see Subsection 3.2.4). Note that $\lim\limits_{x\to\infty}\dfrac{y(\rho,x)}{e^{i\rho x}}=\sqrt{p_\infty}$. On the other hand $y'(\rho,x)=\frac{p'(x)}{2\sqrt{p(x)}}y(\rho,x)+\sqrt{p(x)}y'(\rho,x)$. Since $\lim\limits_{x\to\infty}p'(x)=0$, we obtain $\lim\limits_{x\to\infty}\dfrac{y'(\rho,x)}{i\rho e^{i\rho x}}=\sqrt{p_\infty}$. Hence $y(\rho,x)=\sqrt{p_\infty}\widehat{e}(\rho,x)$, where $\widehat{e}(\rho,x)$ is the Jost solution of (5.42). By hypothesis, $p,p'\in L_2(\mathbb{R}^+;(1+x)^2dx)$ and $p'\in L_\infty(\mathbb{R}^+)$, hence

$$|\widehat{q}|\leqslant\frac{|p''|}{2}\alpha_p+\frac{|p'|}{2}\alpha_p^2\|p'\|_{L_\infty(\mathbb{R}^+)}\in L_1(\mathbb{R}^+;(1+x)^2dx)\subset L_1(\mathbb{R}^+;(1+x)dx).$$

Then it is known that $\widehat{e}(\rho,x)$ admits the Levin representation

$$\widehat{e}(\rho,x)=e^{i\rho x}+\int_x^\infty\widehat{A}(x,t)e^{i\rho t}dt,\quad\forall x\in\mathbb{R}^+,\ \rho\in\overline{\mathbb{C}^+},\tag{5.43}$$

where $\widehat{A} \in C(\overline{\Pi}) \cap W^{1,\infty}(\Pi)$ and $\widehat{A}(x,\cdot) \in L_2(x,\infty)$ (see [52, Th. 2.1.3]). Define

$$B(x,t) := \int_t^\infty \widehat{A}(x,s)ds \quad \text{for } (x,t) \in \overline{\Pi}.$$

Using the estimate $|\widehat{A}(x,t)| \leqslant M_1 \widehat{Q}_0\left(\frac{x+t}{2}\right)$, where $\widehat{Q}_0(x) := \int_x^\infty |\widehat{q}(s)|ds$ and $M_1 > 0$ depends only on $\widehat{q}$ ([52], formula 2.1.35), we obtain

$$\int_t^\infty |A(x,s)|ds \leqslant M_1 \int_t^\infty \int_{\frac{x+s}{2}}^\infty |\widehat{q}(r)|dr = 2M_1 \int_x^\infty (s-x)|\widehat{q}(s)|ds,$$

which is finite for all $x \in \mathbb{R}^+$. Hence $B(x,t)$ is continuous, bounded and

$$\int_x^\infty |B(x,t)|dt \leqslant M_1 \int_x^\infty \int_t^\infty \int_{\frac{x+s}{2}}^\infty |\widehat{q}(r)|drdsdt = 4M_1 \int_x^\infty \frac{(s-x)^2}{2}|\widehat{q}(s)|ds,$$

which is finite because $\widehat{q} \in L_1(\mathbb{R}^+;(1+x)^2dx)$. Then we can apply the integration by parts in (5.43) to obtain the relations

$$\begin{aligned}
\widehat{e}(\rho,x) &= e^{i\rho x} + B(x,x)e^{i\rho x} + \int_x^\infty B(x,t)i\rho e^{i\rho t}dt \\
&= e^{i\rho x}\left(1 + \int_x^\infty \widehat{A}(x,s)ds\right) + \int_x^\infty B(x,t)i\rho e^{i\rho t}dt \\
&= e^{i\rho x}\widehat{e}(0,x) + \int_x^\infty B(x,t)i\rho e^{i\rho t}dt = e^{i\rho x}\sqrt{\frac{p(x)}{p_\infty}} + \int_x^\infty B(x,t)i\rho e^{i\rho t}dt, \qquad (5.44)
\end{aligned}$$

where we used the equalities $\widehat{e}(0,x) = \frac{y(0,x)}{\sqrt{p_\infty}} = \sqrt{\frac{p(x)}{p_\infty}}e(0,x)$ and $e(0,x) = 1$. Since $\widehat{e}(\rho,x) = \sqrt{\frac{p(x)}{p_\infty}}e(\rho,x)$ for all $\rho \in \overline{\mathbb{C}^+}$, comparing (5.12) with (5.44) and taking the inverse Fourier transform we obtain that $B(x,t) = \sqrt{\frac{p(x)}{p_\infty}}A(x,t)$ for all $(x,t) \in \Pi$.

Now we use the estimate $|\widehat{A}_x(x,t)| \leqslant M_2(\left|\widehat{q}\left(\frac{x+t}{2}\right)\right| + \widehat{Q}_0\left(\frac{x+t}{2}\right))$ ([52], formula 2.1.37), where $M_2 > 0$ depends only on $\widehat{q}$. Since the right hand side of this inequality belongs to $L_1(\mathbb{R}^+)$ in the variable t, we have that $B_x(x,t) = \int_t^\infty A_x(x,s)ds$. Hence

$$A_x(x,t) = \sqrt{\frac{p_\infty}{p(x)}}\left(B_x(x,t) - \frac{p'(x)}{2p(x)}B(x,t)\right).$$

Under the hypothesis on p, to prove the inclusion $A_x(x,t) \in L_1(x,\infty)$ it is enough to show that $B_x(x,\cdot) \in L_1(x,\infty)$. Using the estimate for $\widehat{A}_x(x,t)$ we get

$$\begin{aligned}
\int_x^\infty |B_x(x,t)|dt &\leqslant M_2\left(\int_x^\infty \int_t^\infty \left|q\left(\frac{x+s}{2}\right)\right|ds + \int_x^\infty \int_t^\infty \int_{\frac{x+s}{2}}^\infty |\widehat{q}(r)|drdsdt\right) \\
&\leqslant 4M_2\left(\int_x^\infty (s-x)|\widehat{q}(s)|ds + \int_x^\infty \frac{(s-x)^2}{2}|\widehat{q}(s)|ds\right)
\end{aligned}$$

which is finite for all $x \in \mathbb{R}^+$. Hence $A_x(x,\cdot) \in L_1(x,\infty)$. By Theorem 94, $A_x \in L_\infty(\Pi)$, thus $A_x(x,\cdot) \in L_2(x,\infty)$ for all $x \in \mathbb{R}^+$. **Q.E.D.**

Proposition 104. *Let $p \in W^{2,\infty}(\mathbb{R}^+)$ and $p, p' \in L_1(\mathbb{R}^+;(1+x)^2dx)$. The derivative of the Jost solution admits the series representation*

$$e'(\rho,x) = \frac{z-1}{2(z+1)}e^{i\rho x}\left(\sqrt{\frac{p_\infty}{p(x)}} + (z+1)\sum_{n=0}^{\infty}(-1)^n d_n(x) z^n\right), \quad \forall x \in \mathbb{R}^+,\ \rho \in \mathbb{C}^+. \tag{5.45}$$

The series converges pointwise for all $x \in \mathbb{R}^+$ and uniformly for ρ such that $C_1 \leqslant \operatorname{Im}\rho \leqslant C_2$ with $C_1, C_2 > 0$. The coefficients $\{d_n(x)\}_{n=0}^{\infty}$ can be computed from the relations

$$d_0(x) = 1 - \sqrt{\frac{p_\infty}{p(x)}} - \frac{a_0(x)}{2} + a_0'(x), \tag{5.46}$$

$$d_n(x) = d_{n-1}(x) + a_n(x) - a_{n-1}(x) + \frac{a_n'(x) - a_{n-1}'(x)}{2} \quad \text{for } n \geqslant 1. \tag{5.47}$$

Proof. Denote $b_1(x,t) := e^{\frac{t}{2}}A_x(x,x+t)$. By Theorem 103, $b_1(x,\cdot) \in L_2(\mathbb{R}^+; e^{-t}dt)$. Hence $b_1(x,t)$ admits the Fourier-Laguerre series representation

$$b_1(x,t) = \sum_{n=0}^{\infty} d_n(x) L_n(t). \tag{5.48}$$

From (5.14) we have the following representation for the derivative of the Jost solution

$$e'(\rho,x) = i\rho e^{i\rho x}\left(\sqrt{\frac{p_\infty}{p(x)}} + \int_0^\infty A_x(x,x+t)e^{i\rho t}dt\right)$$

(here we use (5.19) and (5.4) to obtain $1 - A(x,x) = \sqrt{\frac{p_\infty}{p(x)}}$). Substituting (5.48) in the previous expression we get

$$\begin{aligned}
e'(\rho,x) &= i\rho e^{i\rho x}\left(\sqrt{\frac{p_\infty}{p(x)}} + \int_0^\infty b_1(x,t)e^{-\left(\frac{1}{2}+i\rho\right)t}dt\right) \\
&= i\rho e^{i\rho x}\left(\sqrt{\frac{p_\infty}{p(x)}} + \sum_{n=0}^{\infty} d_n(x)\int_0^\infty L_n(t)e^{-\left(\frac{1}{2}+i\rho\right)t}dt\right) \\
&= i\rho e^{i\rho x}\left(\sqrt{\frac{p_\infty}{p(x)}} + \sum_{n=0}^{\infty} d_n(x)(-1)^n\frac{\left(\frac{1}{2}+i\rho\right)^n}{\left(\frac{1}{2}-i\rho\right)^{n+1}}\right).
\end{aligned}$$

In terms of the variable z it can be written as (5.45).

On the other hand, differentiating (5.31) we have

$$e'(\rho,x) = e^{i\rho x}\left\{i\rho\left(1+\frac{z-1}{2}\sum_{n=0}^{\infty}(-1)^n z^n a_n(x)\right)+\frac{z-1}{2}\sum_{n=0}^{\infty}(-1)^n z^n a_n'(x)\right\}$$
$$= \frac{z-1}{2(z+1)}e^{i\rho x}\left(1+\frac{z-1}{2}\sum_{n=0}^{\infty}(-1)^n z^n a_n(x)+(z+1)\sum_{n=0}^{\infty}(-1)^n z^n a_n'(x)\right). \tag{5.49}$$

Comparing (5.31) with (5.49) we get

$$\sqrt{\frac{p_\infty}{p(x)}}+(z+1)\sum_{n=0}^{\infty}(-1)^n d_n(x) z^n = 1+\frac{z-1}{2}\sum_{n=0}^{\infty}(-1)^n z^n a_n(x)+(z+1)\sum_{n=0}^{\infty}(-1)^n z^n a_n'(x).$$

Thus, we arrive at the equations (5.46), (5.47). The proof of the uniform convergence of (5.45) in ρ is similar to that of Theorem 99(1). **Q.E.D.**

5.4 Representation of spectral data

5.4.1 Spectrum characterization

The eigenvalues of problem (5.1), (5.2) are those values of λ for which there exists a solution of (5.1), (5.2) belonging to the Hilbert space $\mathcal{H}_p := L_2(\mathbb{R}^+; p(x)dx)$. Note that by the conditions on p from subsection 2.1, $L_2(\mathbb{R}^+) = \mathcal{H}_p$, and their respective norms are equivalent.

Proposition 105. *The Sturm-Liouville problem (5.1), (5.2) has no non-negative eigenvalue.*

Proof. Suppose that there exists an eigenvalue $\lambda_0 > 0$, and let u_0 be the associated eigenfunction. In this case $\lambda_0 = \rho_0^2$, $\rho_0 \in \mathbb{R}$, and $\{e(\rho_0,x), e_-(\rho_0,x)\}$ is a fundamental set of solutions, so $u_0(x) = Ae(\rho_0,x)+Be_-(\rho_0,x)$. Since $e(\pm\rho_0,x) \sim e^{\pm i\rho_0 x}$, $x\to\infty$, we have that $e(\rho_0,\cdot), e_-(\rho_0,\cdot) \notin L_2(\mathbb{R}^+)$. Moreover, $|u_0(x)|^2 \sim \left(|A|^2+|B|^2+A\overline{B}e^{2i\rho_0 y}+\overline{A}Be^{-2i\rho_0 x}\right)$, $x\to\infty$. Hence $u_0 \in L_2(\mathbb{R}^+)$ if only $A=B=0$. Thus $\lambda_0>0$ cannot be an eigenvalue.

In the case $\lambda_0 = 0$, the solutions satisfying (5.2) are of the form $u_0(x) = C\left(1+h\int_0^x \frac{dt}{p(t)}\right)$, and hence $u_0 \in L_2(\mathbb{R}^+)$ if only $C=0$. Hence zero is not an eigenvalue. **Q.E.D.**

Define

$$\Delta(\rho) := e'(\rho,0) - he(\rho,0). \tag{5.50}$$

Remark 106. *(i)* $\Delta \in \operatorname{Hol}(\mathbb{C}^+)$.

(ii) For $\rho \in \mathbb{C}^+$ *the asymptotic relation (5.5) implies that* $e(\rho, \cdot) \in \mathcal{H}_p$. *Thus,* $\rho \in \mathbb{C}^+$ *is a zero of* $\Delta(\rho)$ *iff* $e(\rho, x)$ *is an eigenfunction.*

(iii) Suppose that $\rho \in \mathbb{R} \setminus \{0\}$ *is a zero of* Δ. *In this case* $e_-^{(k)}(\rho, x) = \overline{e^{(k)}(\rho, x)}$. *Hence* $\overline{\Delta(\rho)} = \Delta(-\rho)$ *and* $\Delta(-\rho) = 0$. $W_p(e, e_-)$ *is constant. On the one hand, letting* $x \to \infty$ *we obtain* $W_p(e, e_-)(x) = -2ip_\infty \rho \neq 0$, *while on the other*

$$W_p(e, e_-)(x) = p(0)W(e, e_-)(0) = (e(\rho, 0)e'(-\rho, 0) - e'(\rho, 0)e(-\rho, 0)) = e(\rho, 0)\Delta(-\rho) = 0.$$

This contradiction implies that $\Delta(\rho) \neq 0$ *for* $\rho \in \mathbb{R} \setminus \{0\}$.

We denote by $\varphi(\lambda, x)$ the unique solution of (5.1) satisfying the initial conditions

$$\varphi(\lambda, 0) = 1, \varphi'(\lambda, 0) = h. \tag{5.51}$$

Note that $\varphi(\lambda, x) = C(\rho, x) + hS_1(\rho, x)$ (see (3.3)), and $W_p[\varphi, S](x) = 1$. As a consequence of the estimates (3.6) and (3.8) of $C(\rho, x)$ and $S_1(\rho, x)$, we have the following result.

Proposition 107. *The solution* $\varphi(\lambda, x)$ *is an entire function of the spectral parameter* λ, *and for all* $\lambda = \rho^2 \in \mathbb{C}$, $x \in \overline{\mathbb{R}^+}$ *the inequality hold*

$$|\varphi(\lambda, x)| \leqslant (1 + |h|)ce^{x|\operatorname{Im}\rho|} \exp\left(c \int_0^x |q(s)|ds\right).$$

Remark 108. *Take* $\rho \in \mathbb{R} \setminus \{0\}$ *and* $\lambda = \rho^2$. *We can write* $\varphi(\lambda, x) = c(\rho)e(\rho, x) + d(\rho)e_-(\rho, x)$, *where* $c(\rho)$ *and* $d(\rho)$ *are corresponding coefficients. Since* $\varphi(\lambda, x)$ *is an even function in* ρ, $\varphi(\lambda, x) = c(-\rho)e_-(\rho, x) + d(-\rho)e(\rho, x)$. *By the uniqueness of the coefficients,* $c(\rho) = d(-\rho)$, *and we obtain the representation*

$$\varphi(\lambda, x) = c(\rho)e(\rho, x) + c(-\rho)e_-(\rho, x). \tag{5.52}$$

The function $c(\rho)$ *is called the Jost coefficient [26]. A direct computation shows that* $W_p(\varphi(\lambda, x), e(\rho, x)) = -c(-\rho)W_p(e(\rho, x), e_-(\rho, x)) = -2ip_\infty\rho$. *On the other hand* $W_p(\varphi(\lambda, x), e(\rho, x)) = \Delta(\rho)$, *from where we obtain*

$$c(-\rho) = \frac{\Delta(\rho)}{2ip_\infty}. \tag{5.53}$$

Thus, $c(-\rho)$ *admits an analytic extension onto* $\mathbb{C}^+$.

Proposition 109. *Let $\rho \in \mathbb{C}^+$.*

(i) If $\Delta(\rho) = 0$, then $\operatorname{Re}\rho = 0$.

(ii) If $\rho = i\tau$ with $\tau > 0$, then $\Delta(i\tau) = 0$ iff $\lambda = -\tau^2$ is an eigenvalue.

Proof. (i) Suppose that $\rho \in \mathbb{C}^+$ and $\Delta(\rho) = 0$. Then $e(\rho, \cdot) \in L_2(\mathbb{R}^+)$ is an eigenfunction. Note that $\mathbf{L}\overline{e(\rho, x)} = \overline{\rho}^2 p(x)\overline{e(\rho, x)}$. Thus, by the Lagrange identity (1.51) we have

$$\frac{d}{dx}W_p(e, \overline{e})(x) = \rho^2 p(x)e(\rho, x)\overline{e(\rho, x)} - \overline{\rho}^2 p(x)e(\rho, x)\overline{e(\rho, x)} = 4i \operatorname{Re}\rho \cdot \operatorname{Im}\rho \cdot p(x)|e(\rho, x)|^2.$$

Hence $4i \operatorname{Re}\rho \cdot \operatorname{Im}\rho \cdot \int_0^\infty |e(\rho, x)|^2 p(x)dx = \lim_{x\to\infty} W_p(e, \overline{e})(x) - W_p(e, \overline{e})(0)$. By (5.5),

$$W(e, \overline{e})(x) = \left(-i\overline{\rho}|e^{i\rho x}|^2 - i\rho|e^{i\rho x}|^2\right)(1 + o(1)) = \left(-2i \operatorname{Re}\rho \cdot e^{-2x \operatorname{Im}\rho}\right)(1 + o(1)) = o(1),$$

when $x \to \infty$, and

$$W(e, \overline{e})(0) = e(\rho, 0)\overline{e'(\rho, 0)} - e'(\rho, 0)\overline{e(\rho, 0)} = h|e(\rho, 0)|^2 - h|e(\rho, 0)|^2 = 0.$$

Then $4i \operatorname{Re}\rho \cdot \operatorname{Im}\rho\|e(\rho, \cdot)\|^2_{\mathcal{H}_p} = 0$ and therefore $\operatorname{Re}\rho = 0$.

(ii) If $\Delta(i\tau) = 0$, with $\tau > 0$, by Remark 106 (ii), $\lambda = -\tau^2$ is an eigenvalue.

Reciprocally, suppose that $\lambda < 0$ is an eigenvalue. Let u_0 be the corresponding eigenfunction, and write $\lambda = (i\tau)^2$, $\tau > 0$. In this case $e(i\tau, \cdot) \in \mathcal{H}_p$. Take a Birkhoff solution $E^a(i\tau, x)$. Then $u_0(x) = Ae(i\tau, x) + BE^a(i\tau, x)$. Since $u_0 \in \mathcal{H}_p$, by the asymptotic relation (5.26), we obtain that $B = 0$. Then $u_0(x) = Ae(i\tau, x)$ and $0 = u'(0) - hu(0) = A\Delta(i\tau)$.

Q.E.D.

Let us associate to the problem (5.1),(5.2) an unbounded operator in the Hilbert space $\mathcal{H}_p$. Let $\mathbf{L}^h : \mathcal{D}\left(\mathbf{L}^h\right) \subset \mathcal{H}_p \to \mathcal{H}_p$, whose action is given by $\mathbf{L}u$ and with the domain

$$\mathcal{D}\left(\mathbf{L}^h\right) = \left\{u \in \mathcal{H}_p \cap W^{2,1}_{loc}\left(\mathbb{R}^+\right) \,\middle|\, \mathbf{L}u \in \mathcal{H}_p \text{ and satisfies (5.2) } \right\}.$$

Theorem 110. *The operator $\mathbf{L}^h$ is self-adjoint.*

Proof. Consider the operator $\mathbf{L}_{max} : \mathcal{D}(\mathbf{L}_{max}) \subset \mathcal{H}_p \to \mathcal{H}_p$, whose action is given by $\mathbf{L}u$ and with the domain $\mathcal{D}(\mathbf{L}_{max}) = \left\{u \in \mathcal{H}_p \cap W^{2,1}_{loc}(\mathbb{R}^+) \,\middle|\, \mathbf{L}u \in \mathcal{H}_p\right\}$. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Note that every solution $y \in W^{2,1}_{loc}(\mathbb{R}^+)$ of (5.1) belongs to $L_2((0,a');p(x)dx)$, for some $a' > 0$. Hence $a = 0$ is in the *limit-circle case* of Eq. (5.1) ([15, Th. 6.1.6 and Cor. 6.1.7]). On the other hand, choosing $\rho \in \mathbb{C}^+$ such that $\lambda = \rho^2$, by the asymptotic relation (5.5) the Jost solution $e(\rho,\cdot) \in L_2((b',\infty);p(x)dx)$ for some $b' > 0$. Now take a Birkhoff solution $E^a(\rho,x)$. Due to the asymptotic relation (5.26), $E^a(\rho,\cdot) \notin L_2(b',\infty;p(x)dx)$ for all $b' > 0$. Thus, $b = \infty$ is in the *limit-point case* (see [15, Th. 6.1.6 and Cor. 6.1.7]). Denote $f^{[0]}(0) := W_p(f(x),\varphi(\lambda,x))(0) = hf(0) - f'(0)$, for $f \in \mathcal{D}(\mathbf{L}_{max})$. With this notation we can write

$$\mathcal{D}\left(\mathbf{L}^h\right) = \left\{u \in \mathcal{D}(\mathbf{L}_{max}) \,\middle|\, u^{[0]}(0) = 0\right\}.$$

Since $a = 0$ is in the limit-circle case and $b = \infty$ is in the limit-point case, it is known that $\mathbf{L}^h = \mathbf{L}_{max}\big|_{\mathcal{D}(\mathbf{L}^h)}$ is a self-adjoint operator in $\mathcal{H}_p$ (see [15, Prop. 6.4.9]). **Q.E.D.**

Let us denote the spectrum and the resolvent of $\mathbf{L}^h$ by $\sigma\left(\mathbf{L}^h\right)$ and $\rho\left(\mathbf{L}^h\right)$, respectively. Thus, $\sigma\left(\mathbf{L}^h\right) \subset \mathbb{R}$, and consists of the point and continuous spectrum, that we denote by $\sigma_p\left(\mathbf{L}^h\right)$ and $\sigma_c\left(\mathbf{L}^h\right)$ respectively. The discrete and essential spectrum are denoted by $\sigma_d\left(\mathbf{L}^h\right)$ and $\sigma_{ess}\left(\mathbf{L}^h\right)$, respectively.

Remark 111. *Proposition 105 implies that $\sigma_p\left(\mathbf{L}^h\right) \subset (-\infty,0)$. If $\lambda \in \sigma_p\left(\mathbf{L}^h\right)$, due to Proposition 109 (ii), $\lambda = -\tau^2$ with $\tau > 0$ such that $\Delta(i\tau) = 0$, and the corresponding eigenspace is generated by the Jost solution $e(i\tau,y)$. Hence all the eigenvalues are simple.*

Theorem 112. *1. If $h \geqslant 0$, then $\sigma\left(\mathbf{L}^h\right) = \sigma_c\left(\mathbf{L}^h\right) = \sigma_{ess}\left(\mathbf{L}^h\right)$.*

2. If $h < 0$, then $\sigma_p\left(\mathbf{L}^h\right) = \sigma_d\left(\mathbf{L}^h\right)$, and the number of the eigenvalues is finite.

Proof. 1. Suppose that there exists $\lambda \in \sigma_p\left(\mathbf{L}^h\right)$. By Remark 111, $\lambda = -\tau^2$ for some $\tau > 0$ and $e(i\tau,x)$ is an eigenfunction. Note that the asymptotics (5.5) implies $e'(i\tau,\cdot) \in \mathcal{H}_p$ and $\lim_{x\to\infty} e^{(k)}(i\tau,x) = 0$, $k = 0,1$. Integrating by parts we obtain

$$\begin{aligned}\lambda\|e(i\tau,\cdot)\|^2_{\mathcal{H}_p} &= \langle \mathbf{L}^h e(i\tau,\cdot), e(i\tau,\cdot)\rangle_{\mathcal{H}_p} = p(0)e'(i\tau,0)\overline{e(i\tau,0)} + \int_0^\infty |e'(i\tau,x)|^2 p(x)dx \\ &= h|e(i\tau,0)|^2 + \|e'(i\tau,\cdot)\|^2_{\mathcal{H}_p} \geqslant 0,\end{aligned}$$

which is a contradiction. Thus, $\sigma_p\left(\mathbf{L}^h\right)=\emptyset$ and $\sigma\left(\mathbf{L}^h\right)=\sigma_c\left(\mathbf{L}^h\right)$. In particular, $\sigma_d\left(\mathbf{L}^h\right)=\emptyset$ and $\sigma\left(\mathbf{L}^h\right)=\sigma_{ess}\left(\mathbf{L}^h\right)$.

2. By Remark 111, all the eigenvalues are simple and $\sigma_p\left(\mathbf{L}^h\right)=\{-\tau^2\,|\,\tau>0 \text{ and } \Delta(i\tau)=0\}$. Since $\Delta\not\equiv 0$ and $\Delta\in\operatorname{Hol}(\mathbb{C}^+)$, all the eigenvalues are isolated points and hence $\sigma_p\left(\mathbf{L}^h\right)=\sigma_d\left(\mathbf{L}^h\right)$. Now we prove that $\sigma_d\left(\mathbf{L}^h\right)$ has no limit point. Suppose, there exists a convergent sequence $\{\lambda_n\}\subset\sigma_d\left(\mathbf{L}^h\right)$. The only possibility is that $\lambda_n\to 0$. Write $\lambda_n=-\tau_n^2$, $\tau_n>0$. Thus, $i\tau_n\to 0$. By (5.10), $e'(i\tau_n,0)\to 0$, $n\to\infty$, and
$$e(i\tau_n,0)=1+\int_0^\infty\frac{\sin(i\tau_n s)}{i\tau_n}q(s)e'(i\tau_n,s)ds.$$
Using (5.9) and (5.10), we obtain $\left|\frac{\sin(i\tau_n s)}{i\tau_n}q(s)e'(i\tau_n,s)\right|\leqslant c\left(1+e^{c\|q\|_{L_1(\mathbb{R}^+)}}\right)|q(s)|$. Then by the dominated convergence $e(i\tau_n,0)\to 1$, $n\to\infty$. Thus, $0=\lim\limits_{n\to\infty}\Delta(i\tau_n)=h$, which is a contradiction. Therefore $\sigma_p\left(\mathbf{L}^h\right)$ has no limit point. Now choose an arbitrary $\lambda\in\sigma_d\left(\mathbf{L}^h\right)$. Repeating the reasoning from part 1 of this proof, we obtain
$$\lambda\|e(i\tau,\cdot)\|^2_{\mathcal{H}_p}\geqslant -|h||e(i\tau,0)|^2+\|e'(i\tau,\cdot)\|^2_{\mathcal{H}_p}.$$
Since $e(i\tau,\cdot)\in C(\overline{\mathbb{R}^+})$ and p is positive and bounded in $\overline{\mathbb{R}^+}$, applying the first mean value theorem for integrals in the interval $[0,1]$ we have
$$\int_0^1|e(i\tau,x)|^2p(x)dx=|e(i\tau,\xi)|^2\int_0^1 p(x)dx\quad\text{for some }\xi\in[0,1].$$
Note that $\int_0^\xi\overline{e(i\tau,x)}e'(i\tau,x)dx=\overline{e(i\tau,x)}e(i\tau,x)\big|_0^\xi-\int_0^\xi\overline{e'(i\tau,x)}e(i\tau,x)dx$, from where we obtain
$$\begin{aligned}|e(i\tau,0)|^2=&|e(i\tau,\xi)|^2-2\int_0^\xi\operatorname{Re}\left(\overline{e'(i\tau,x)}e(i\tau,x)\right)dx\\ \leqslant&\frac{|e(i\tau,\xi)|^2\int_0^1p(x)dx}{\int_0^1p(x)dx}+\int_0^1 2|e'(i\tau,x)||e(i\tau,x)|dx\\ \leqslant&\frac{\|e(i\tau,\cdot)\|^2_{\mathcal{H}_p}}{\|p\|_{L_1(0,1)}}+\int_0^1 2|e'(i\tau,x)||e(i\tau,x)|dx.\end{aligned}$$

Note that $\frac{1}{p(x)} \leqslant \frac{p(x)}{\alpha_p^2}$, hence

$$\begin{aligned}|e(i\tau,0)|^2 \leqslant& \frac{\|e(i\tau,\cdot)\|^2_{\mathcal{H}_p}}{\|p\|_{L_1(0,1)}} + \int_0^1 \left(\frac{p(x)}{|h|}|e'(i\tau,x)|^2 + \frac{|h|}{p(x)}|e(i\tau,x)|^2\right) dx \\ \leqslant& \frac{\|e(i\tau,\cdot)\|^2_{\mathcal{H}_p}}{\|p\|_{L_1(0,1)}} + \frac{1}{|h|}\int_0^1 |e'(i\tau,x)|^2 p(x)dx + \frac{|h|}{\alpha_p^2}\int_0^1 |e(i\tau,x)|^2 p(x)dx \\ \leqslant& \left(\|p\|^{-1}_{L_1(0,1)} + \frac{|h|}{\alpha_p^2}\right)\|e(i\tau,\cdot)\|^2_{\mathcal{H}_p} + \frac{1}{|h|}\|e'(i\tau,\cdot)\|^2_{\mathcal{H}_p}.\end{aligned}$$

From this we obtain the inequality

$$\lambda\|e(i\tau,\cdot)\|^2_{\mathcal{H}_p} \geqslant -\left(|h|\|p\|^{-1}_{L_1(0,1)} + \frac{|h|^2}{\alpha_p^2}\right)\|e(i\tau,\cdot)\|^2_{\mathcal{H}_p}.$$

Thus, $\lambda \geqslant -\left(|h|\|p\|^{-1}_{L_1(0,1)} + \frac{|h|^2}{\alpha_p^2}\right)$. Due to the arbitrariness of λ we conclude that $\sigma_d\left(\mathbf{L}^h\right) \subset \left[-\left(|h|\|p\|^{-1}_{L_1(0,1)} + \frac{|h|^2}{\alpha^2}\right), 0\right]$. Since $\sigma_d\left(\mathbf{L}^h\right)$ has no limit point, it must be a finite set.

Q.E.D.

Note that the mapping $\mathbb{C}^+ \ni \rho \mapsto \lambda = \rho^2 \in \mathbb{C}\setminus\overline{\mathbb{R}^+}$ is a conformal bijection with the inverse $\rho = \sqrt{\lambda}$ with the chosen branch $\arg\lambda \in (0,2\pi)$. If $\lambda > 0$, we take $\rho = \lim\limits_{\varepsilon\to 0^+}\sqrt{\lambda + i\varepsilon}$, that corresponds to the positive square root of λ. For $\lambda \notin \sigma_d\left(\mathbf{L}^h\right)$ define $\Phi(\lambda,x) := -\frac{e(\rho,x)}{\Delta(\rho)}$. Then $\Phi(\lambda,x)$ satisfies (5.1) with the conditions $\Phi'(\lambda,0) - h\Phi(\lambda,0) = -1$ and $\Phi^{(k)}(\lambda,x) = O((i\rho)^k e^{i\rho x})$, $x\to\infty$, for $k = 0,1$. Thus, $\Phi(\lambda,\cdot) \in H^1(\mathbb{R}^+)$. This function is called the *Weyl solution* of the problem (5.1), (5.2). The *Weyl function* of the problem is defined as

$$M(\lambda) := \Phi(\lambda,0) = -\frac{e(\rho,0)}{\Delta(\rho)}. \tag{5.54}$$

Note that $M(\lambda)$ is analytic in $\lambda \in \rho\left(\mathbf{L}^h\right)$. We have the relation

$$\Phi(\lambda,x) = M(\lambda)\varphi(\lambda,x) - S(\lambda,x). \tag{5.55}$$

Proposition 113. *If* $\lambda \in \mathbb{C}\setminus\left(\mathbb{R}^+ \cup \sigma_d\left(\mathbf{L}^h\right)\right)$, *then* $\lambda \in \rho\left(\mathbf{L}^h\right)$, *and the resolvent operator* $\mathbf{R}_\lambda = \left(\mathbf{L}^h - \lambda\mathbf{I}_{\mathcal{H}_p}\right)^{-1}$ *can be written in the form of the Fredholm integral operator*

$$\mathbf{R}_\lambda f(x) = \int_0^\infty G(\lambda,x,t)f(t)p(t)dt \quad \text{for } f\in\mathcal{H}_p,\ x\in\mathbb{R}^+, \tag{5.56}$$

where $G(\lambda,x,t)$ *is the Green function*

$$G(\lambda,x,t) = \begin{cases}\varphi(\lambda,x)\Phi(\lambda,t), & \text{if } x\leqslant t,\\ \varphi(\lambda,t)\Phi(\lambda,x), & \text{if } t\leqslant x.\end{cases} \tag{5.57}$$

Proof. If $\lambda \in \mathbb{C} \setminus \left(\mathbb{R}^+ \cup \sigma_d\left(\mathbf{L}^h\right)\right)$, then $\Phi(\lambda, x)$ is well defined and belongs to $H^1(\mathbb{R}^+) \cap H^2_{loc}(\mathbb{R}^+)$. On the other hand, $W_p(\varphi, \Phi) = \Phi'(\lambda, 0) - h\Phi(\lambda, 0) = -1$ and $\varphi(\lambda, x)$, $\Phi(\lambda, x)$ are linearly independent. Define the integral operator $\mathbf{R}_\lambda$ by formula (5.56). Asymptotics (5.5) implies that there exists a constant $C_1 > 0$ for which $|e(\rho, x)| \leqslant C_1 e^{-x \operatorname{Im}\rho}$ for all $x \in \overline{\mathbb{R}^+}$. Similarly, by Proposition 107 (ii), there exists a constant $C_2 > 0$ such that $|\varphi(\lambda, x)| \leqslant C_2 e^{x \operatorname{Im}\rho}$ for all $x \in \overline{\mathbb{R}^+}$. Set $P_1 = \sup\limits_{x \in \overline{\mathbb{R}^+}} p(x)$. Hence for $t \in \mathbb{R}^+$ we have

$$\begin{aligned}\int_0^\infty |G(\lambda, x, t)| p(t) dx &\leqslant \frac{P_1}{|\Delta(\rho)|}\left(\int_0^t |\varphi(\lambda, x)||e(\rho, t)| dx + \int_t^\infty |\varphi(\lambda, t)||e(\rho, x)| dx\right)\\ &\leqslant \frac{C_1 C_2 P_1}{|\Delta(\rho)|}\left(\int_0^t e^{x \operatorname{Im}\rho} e^{-t \operatorname{Im}\rho} dx + \int_t^\infty e^{t \operatorname{Im}\rho} e^{-x \operatorname{Im}\rho} dx\right)\\ &= C_1 C_2 P_1 \frac{2 - e^{-t \operatorname{Im}\rho}}{\operatorname{Im}\rho |\Delta(\rho)|} \leqslant \frac{2 C_1 C_2 P_1}{\operatorname{Im}\rho |\Delta(\rho)|} < \infty\end{aligned}$$

uniformly for $t \in \overline{\mathbb{R}^+}$. Since the Green function is symmetric, we obtain the similar estimate $\int_0^\infty |G(\lambda, x, t)| p(t) dt \leqslant \frac{2 C_1 C_2 P_1}{\operatorname{Im}\rho |\Delta(\rho)|} < \infty$ uniformly for $x \in \overline{\mathbb{R}^+}$. By the Schur test ([51, Th. 6.18]), $\mathbf{R}_\lambda \in \mathcal{B}(L_2(\mathbb{R}^+))$. Since $L_2(\mathbb{R}^+) = \mathcal{H}_p$ with equivalent norms, we obtain $\mathbf{R}_\lambda \in \mathcal{B}(\mathcal{H}_p)$.

Direct computation shows that $\mathbf{R}_\lambda f \in \mathcal{D}\left(\mathbf{L}^h\right)$ for all $f \in \mathcal{H}_p$ and $\left(\mathbf{L}^h - \lambda \mathbf{I}_{\mathcal{H}_p}\right) \mathbf{R}_\lambda f = f$. Then $\mathbf{L}^h - \lambda \mathbf{I}_{\mathcal{H}_p}$ is surjective, and since $\lambda \notin \sigma_p\left(\mathbf{L}^h\right)$, it is a bijection and $\left(\mathbf{L}^h - \lambda \mathbf{I}_{\mathcal{H}_p}\right)^{-1} = \mathbf{R}_\lambda \in \mathcal{B}\left(\mathcal{H}_p\right)$. Hence $\lambda \in \rho\left(\mathbf{L}^h\right)$. **Q.E.D.**

Theorem 114. *$\sigma_c\left(\mathbf{L}^h\right) = [0, \infty)$ for all $h \in \mathbb{R}$.*

Proof. By Proposition 113, it is only necessary to show that $[0, \infty) \subset \sigma_c\left(\mathbf{L}^h\right)$. Let $\lambda = \rho^2$, with $\rho \geqslant 0$. Take $\psi \in C_0^\infty(0, \infty)$ with $\operatorname{Supp}(\psi) \subset (1, 2)$ and $\int_0^\infty |\psi(x)|^2 dx = 1$. Define the sequence $v_k(x) = e(\rho, x)\psi\left(\frac{x}{k}\right)$ for $k \in \mathbb{N}$. The asymptotics (5.5) implies that there are $R_1, C_3 > 0$ such that $|e(\rho, x)| \geqslant C_3$ for $x \geqslant R_1$. Hence for $k \geqslant R_1$ we have

$$\|v_k\|^2_{\mathcal{H}_p} \geqslant \alpha_p \int_k^{2k} |e(\rho, x)|^2 \left|\psi\left(\frac{x}{k}\right)\right|^2 dx = \alpha_p k \int_1^2 |e(\rho, kx)|^2 |\psi(x)|^2 dx \geqslant \alpha_p k C_3^2, \quad (5.58)$$

Clearly $\{v_k\}_{k=1}^{\infty} \subset \mathcal{D}\left(\mathbf{L}^h\right)$ and

$$
\begin{aligned}
\mathbf{L}^h v_k(x) &= -\frac{1}{p(x)}\frac{d}{dx}p(x)\left(\frac{1}{k}\psi'\left(\frac{x}{k}\right)e(\rho,x)+\psi\left(\frac{x}{k}\right)e'(\rho,x)\right)\\
&= -\frac{1}{p(x)}\Big(\frac{1}{k}\frac{d}{dx}\left(p(x)\psi'\left(\frac{x}{k}\right)\right)e(\rho,x)+\frac{2}{k}p(x)\psi'\left(\frac{x}{k}\right)e'(\rho,x)+\psi\left(\frac{x}{k}\right)\frac{d}{dx}\left(p(x)e'(\rho,x)\right)\Big)\\
&= \frac{1}{k}e(\rho,x)\mathbf{L}^h\psi\left(\frac{x}{k}\right)-\frac{2}{k}e'(\rho,x)\psi'\left(\frac{x}{k}\right)+\psi\left(\frac{x}{k}\right)\mathbf{L}^h e(\rho,x)\\
&= \frac{1}{k}e(\rho,x)\mathbf{L}^h\psi\left(\frac{x}{k}\right)-\frac{2}{k}e'(\rho,x)\psi'\left(\frac{x}{k}\right)+\lambda v_k(x).
\end{aligned}
$$

By (5.5) the functions $e(\rho,x)$ and $e'(\rho,x)$ are bounded, and choosing $C_4 = \max\limits_{k=0,1}\|e^{(k)}(\rho,\cdot)\|_{L_\infty(\mathbb{R}^+)}$ and $P_1 = \sup\limits_{x\in\overline{\mathbb{R}^+}} p(x)$ we obtain

$$
\begin{aligned}
\|\left(\mathbf{L}^h-\lambda\mathbf{I}_{\mathcal{H}_p}\right)v_k\|^2_{\mathcal{H}_p} &\leqslant \frac{2}{k^2}\int_k^{2k}\left(\left|e(\rho,x)\frac{d}{dx}\left(p(x)\psi'\left(\frac{x}{k}\right)\right)\right|^2+4\left|e'(\rho,x)\psi'\left(\frac{x}{k}\right)\right|^2p(x)\right)dx\\
&\leqslant \frac{2C_4}{k^2}\int_k^{2k}\left|\frac{1}{k}p(x)\psi''\left(\frac{x}{k}\right)+p'(x)\psi'\left(\frac{x}{k}\right)\right|^2dx+\frac{8C_4P_1}{k^2}\|\psi'\|^2_{L_2(1,2)}\\
&\leqslant \frac{4C_4}{k^2}\left(\frac{1}{k^2}\int_k^{2k}\left|p(x)\psi''\left(\frac{x}{k}\right)\right|^2dx+\int_k^{2k}\left|p'(x)\psi'\left(\frac{x}{k}\right)\right|^2dx\right)\\
&\quad+\frac{8C_4P_1}{k^2}\|\psi'\|^2_{L_2(1,2)}\\
&\leqslant \frac{4C_4P_1}{k^4}\|\psi''\|^2_{L_2(1,2)}+\frac{4C_4}{k^2}\int_k^{2k}\left|p'(x)\psi'\left(\frac{x}{k}\right)\right|^2dx+\frac{8C_4P_1}{k^2}\|\psi'\|^2_{L_2(1,2)}.
\end{aligned}
$$

Since $p'(x)\to 0$, $x\to\infty$ (see subsection 2.1), there exists $R_2>0$ such that $|p'(x)|<1$ for $x\geqslant R_2$. Thus, if $k\geqslant R_2$ we have

$$
\|\left(\mathbf{L}^h-\lambda\mathbf{I}_{\mathcal{H}_p}\right)v_k\|^2_{\mathcal{H}_p}\leqslant\frac{4C_4P_1}{k^4}\|\psi''\|^2_{L_2(1,2)}+\frac{4C_4}{k^2}\|\psi'\|^2_{L_2(1,2)}+\frac{8C_4P_1}{k^2}\|\psi'\|^2_{L_2(1,2)}. \tag{5.59}
$$

Consider the normalized sequence $\hat{v}_k=\frac{v_k}{\|v_k\|_{\mathcal{H}_p}}$. If $k\geqslant\max\{R_1,R_2\}$, by (5.58) and (5.59) we have

$$
\|\left(\mathbf{L}^h-\lambda\mathbf{I}_{\mathcal{H}_p}\right)\hat{v}_k\|^2_{\mathcal{H}_p}\leqslant\frac{\max\{4C_4,8C_4P_1\}}{\alpha_pC_3^2k^3}\|\psi'\|^2_{H^1(1,2)}.
$$

Hence $\|\left(\mathbf{L}^h-\lambda\mathbf{I}_{\mathcal{H}_p}\right)\hat{v}_k\|^2_{\mathcal{H}_p}\to 0$, $k\to\infty$. By the Weyl criterion [17, Th. 4.16] and Proposition 109 (ii), $\lambda\in\sigma_c\left(\mathbf{L}^h\right)$. **Q.E.D.**

5.4.2 Spectral expansion theorem

In [26] it was shown that for the problem with the Neumann condition ($h = 0$), the following expansion theorem (in terms of the Jost coefficient) is valid for $f \in C_0^\infty(\mathbb{R}^+)$:

$$f(x) = \frac{1}{2\pi}\int_0^\infty F(\rho)C(\rho^2, x)\frac{d\rho}{p_\infty |c(\rho)|^2}, \quad \text{where} \quad F(\rho) = \int_0^\infty f(y)C(\rho^2, y)p(y)dy.$$

The spectral measure for the Neumann problem is given by $d\nu(\rho) = \frac{d\rho}{2\pi p_\infty |c(\rho)|^2}$. This result was proved under the assumption that the potential q is non-negative and $\lim_{\rho\to\infty} c(-\rho) \neq 0$. We prove the expansion theorem in terms of the characteristic function $\Delta(\rho)$ for all $h \in \mathbb{R}$ and give a representation for the spectral data analogously to the case of the Schrödinger equation ([52, Ch. II]).

In the case $h < 0$ we order the eigenvalues as follows $\lambda_N < \cdots < \lambda_0 < 0$. For $\lambda_j = -\tau_j^2$, $\tau_j > 0$ we obtain $W_p(\varphi(\lambda_j, x), e(i\tau_j, x)) = \Delta(i\tau_j) = 0$. Since $h \neq 0$, we have $e(i\tau_j, 0) \neq 0$ and hence $\varphi(\lambda_j, x) = \beta_j e(i\tau_j, x)$ is an eigenfunction with $\beta_j = \frac{1}{e(i\tau_j, 0)}$. In this case we define the *normalizing constant*

$$\alpha_j := \|\varphi(\lambda_j, \cdot)\|_{\mathcal{H}_p}^{-2} = \left(\int_0^\infty \varphi^2(\lambda_j, x)p(x)dx\right)^{-1}. \tag{5.60}$$

Proposition 115. *The Weyl function $M(\lambda)$ is meromorphic for $\lambda \in \mathbb{C}\setminus\overline{\mathbb{R}^+}$ with simple poles in $\sigma_d\left(\mathbf{L}^h\right)$ and residues*

$$\mathrm{Res}_{\lambda=\lambda_j} M(\lambda) = -\alpha_j \quad \textit{for } j = \overline{0, N}. \tag{5.61}$$

Proof. Take λ_j for some $j \in \{0, \cdots, N\}$ and λ in a neighborhood of λ_j in $\mathbb{C}\setminus\overline{\mathbb{R}^+}$. By the Lagrange identity (1.51) we have

$$\frac{d}{dx}W_p(\varphi(\lambda_j, x), e(\rho, x)) = (\lambda_j - \lambda)\, p(x)\varphi(i\tau_j, x)e(\rho, x).$$

Thus, $-\int_0^\infty \frac{d}{dx}W_p(\varphi(\lambda_j, x), e(\rho, x))dx = (\lambda - \lambda_j)\int_0^\infty \varphi(\lambda_j, x)e(\rho, x)p(x)dx.$
Since $\varphi(\lambda_j, x)$ is an eigenfunction, $\lim_{x\to\infty} W_p(\varphi(\lambda_j, x), e(\rho, x)) = 0$ and hence

$$-\int_0^\infty \frac{d}{dx}W_p(\varphi(\lambda_j, x), e(\rho, x))dx = W_p(\varphi(\lambda_j, x), e(\rho, x))(0) = \Delta(\rho).$$

Writing $\lambda_j = -\tau_j^2$ with $\tau_j > 0$, we obtain

$$\frac{\Delta(\rho) - \Delta(i\tau_j)}{\lambda - \lambda_j} = \int_0^\infty \varphi(\lambda_j, x)e(\rho, x)p(x)dx. \tag{5.62}$$

Letting $\lambda \to \lambda_j$ we obtain on the left-hand side $\frac{d}{d\lambda}\Delta(\rho)\Big|_{\lambda=\lambda_j}$. On the right-hand side we can use the estimates for $e(\rho, x)$ and $\varphi(i\tau_j, x)$ and dominated convergence to obtain that $\int_0^\infty \varphi(\lambda_j, x)e(\rho, x)p(x)dx \to \frac{1}{\beta_j}\int_0^\infty \varphi^2(\rho_j, x)p(x)dx$. Thus,

$$\frac{d}{d\lambda}\Delta(\rho)\Big|_{\lambda=\lambda_j} = \frac{1}{\alpha_j\beta_j} = \frac{e(i\tau_j, 0)}{\alpha_j}. \tag{5.63}$$

Hence λ_j is a simple zero of $\Delta(\sqrt{\lambda})$. Then

$$\operatorname{Res}_{\lambda=\lambda_j} M(\lambda) = -e(i\tau_j, 0)\left(\frac{d}{d\lambda}\Delta(\rho)\Big|_{\lambda=\lambda_j}\right)^{-1} = -\alpha_j.$$

Q.E.D.

The following proposition establishes basic properties of the Weyl function.

Proposition 116. *The Weyl function is a Nevanlinna function, that is*

(i) $\operatorname{Im} M(\lambda) > 0$ *if* $\lambda \in \mathbb{C}^+$,

(ii) $\overline{M(\overline{\lambda})} = M(\lambda)$, *for* $\lambda \in \rho\left(\mathbf{L}^h\right)$,

(iii) there exists an increasing and left-continuous function ν *such that*

$$M(\lambda) = A + B\lambda + \int_{-\infty}^{\infty} \frac{1 - t\lambda}{t - \lambda}\frac{d\nu(t)}{1 + t^2}, \tag{5.64}$$

where $A = \operatorname{Re} M(i)$, $B = \lim_{y\to+\infty} \frac{\operatorname{Im} M(iy)}{y}$ *and* $d\nu$ *denotes the integration with respect to the Lebesgue-Stieltjes measure generated by* ν. *The function* ν *is unique under the normalization* $\nu(0) = 0$.

Proof. (i) Take $\lambda \in \mathbb{C}^+$ and note that

$$\operatorname{Im} M(\lambda) = -\frac{1}{2i}\left(\frac{e(\rho, 0)}{\Delta(\rho)} - \frac{\overline{e(\rho, 0)}}{\overline{\Delta(\rho)}}\right) = \frac{-1}{2i|\Delta(\rho)|^2}\left(e(\rho, 0)\overline{e'(\rho, 0)} - e'(\rho, 0)\overline{e(\rho, 0)}\right)$$
$$= -\frac{W_p(e, \overline{e})(0)}{2i|\Delta(\rho)|^2}.$$

By the Lagrange identity (1.51), $\frac{d}{dx}W_p(e, \overline{e})(x) = (\lambda - \overline{\lambda})p(x)|e(\rho, x)|^2$. In the proof of Proposition 109 (i) we see that $\lim_{x\to\infty} W_p(e, \overline{e})(x) = 0$, then $-W_p(e, \overline{e})(0) = 2i \operatorname{Im}\lambda \|e\|^2_{\mathcal{H}_p}$. Thus,

$$\operatorname{Im} M(\lambda) = -\frac{W_p(e, \overline{e})(0)}{2i|\Delta(\rho)|^2} = \frac{\operatorname{Im}\lambda}{|\Delta(\rho)|^2}\|e\|^2_{\mathcal{H}_p} > 0.$$

(ii) If $\lambda \in \rho\left(\mathbf{L}^h\right)$, then $\mathbf{R}_\lambda^* = \mathbf{R}_{\overline{\lambda}}$ because $\mathbf{L}^h$ is self-adjoint ([3, pp. 98]). For $u, v \in C_0^\infty(\mathbb{R}^+)$ we have $\langle \mathbf{R}_\lambda u, v\rangle_{\mathcal{H}_p} = \langle u, \mathbf{R}_{\overline{\lambda}} v\rangle_{\mathcal{H}_p}$. Expanding the inner products and using the symmetry of G and Fubini's theorem we obtain

$$\int_0^\infty \int_0^\infty G(\lambda, x, t) p(x) p(t) u(t) \overline{v(x)} dt dx = \int_0^\infty \int_0^\infty \overline{G(\overline{\lambda}, x, t)} p(x) p(t) u(t) \overline{v(x)} dt dx.$$

Due to the arbitrariness of $u, v \in C_0^\infty(\mathbb{R}^+)$ this implies that $G(\lambda, x, t) = \overline{G(\overline{\lambda}, x, t)}$ a.e. $x, t \in \mathbb{R}^+$ (see [139], lemmas from pp. 72 and pp. 99). Note tha $\overline{\varphi}'' - q(x)\overline{\varphi} = \overline{\lambda}\overline{\varphi}$, and $\overline{\varphi}$ satisfies the initial conditions (5.51). Then by uniqueness of solution $\overline{\varphi(\lambda, x)} = \varphi(\overline{\lambda}, x)$, a.e. $x \in \mathbb{R}^+$. In a similar way $\overline{S(\lambda, x)} = S(\overline{\lambda}, x)$. Using (5.55) and (5.57) we obtain

$$M(\lambda)\varphi(\lambda, \min(x,t))\varphi(\lambda, \max(x,t)) - \varphi(\lambda, \min(x,t))S(\lambda, \max(x,t))$$

$$= \overline{M(\overline{\lambda})}\varphi(\lambda, \min(x,t))\varphi(\lambda, \max(x,t)) - \varphi(\lambda, \min(x,t))S(\lambda, \max(x,t))$$

a.e. $x, t \in \mathbb{R}^+$. This concludes the proof of $M(\lambda) = \overline{M(\overline{\lambda})}$.

(iii) Follows from the classical theory of Nevanlinna functions (see [12, Th. E.1.1], [3, Sec. 59] and [15, A. 2]).

Q.E.D.

The function ν is called the spectral function of $\mathbf{L}^h$.

Theorem 117. *The spectral function has the form*

$$\nu(\lambda) := \begin{cases} -\displaystyle\sum_{\{0 \leqslant j \leqslant N \,:\, \lambda \leqslant \lambda_j\}} \alpha_j, & \text{if } \lambda \leqslant 0, \\ \displaystyle\int_0^\lambda V(s) ds, & \text{if } \lambda > 0, \end{cases} \tag{5.65}$$

where

$$V(\lambda) = \frac{p_\infty \sqrt{\lambda}}{\pi |\Delta(\sqrt{\lambda})|^2}. \tag{5.66}$$

Proof. The following inversion formula for Nevanlinna functions is known (see [15, Corollary A.2.8]) for $a, b \in \mathbb{R}$:

$$\begin{aligned} \frac{\nu(b^+) + \nu(b^-)}{2} - \frac{\nu(a^+) + \nu(a^-)}{2} &= \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_a^b \operatorname{Im} M(s + i\varepsilon) ds \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_a^b (M(s + i\varepsilon) - \overline{M(s + i\varepsilon)}) ds. \end{aligned} \tag{5.67}$$

Let $(a,b) \subset (-\infty, 0)$ be a subinterval (possibly infinite). According to [15, Prop. A.2.9], if $M(\lambda)$ is holomorphic for all $\lambda \in \mathbb{C} \setminus \mathbb{R} \cup (a,b)$, this implies that ν is constant on (a,b). Hence ν is constant on the intervals $(-\infty, \lambda_N)$, $(\lambda_0, 0)$ and $(\lambda_j, \lambda_{j-1})$ for $j = \overline{1,N}$. Then ν is a step function on $(-\infty, 0)$. For $\lambda > 0$ we use the formula A.2.28 from [15]:

$$\lim_{\varepsilon \to 0^+} \varepsilon \operatorname{Im} M(\lambda + i\varepsilon) = \nu(\lambda^+) - \nu(\lambda^-).$$

From the proof of Proposition 116 (i), $\operatorname{Im} M(\lambda+i\varepsilon) = -\dfrac{W_p\left(e(\sqrt{\lambda+i\varepsilon}, x), \overline{e(\sqrt{\lambda+i\varepsilon}, x)}\right)(0)}{2i|\Delta(\sqrt{\lambda+i\varepsilon})|^2}$, and we obtain $\lim_{\varepsilon\to 0^+} \operatorname{Im} M(\lambda+i\varepsilon) = -\dfrac{W_p\left(e(\sqrt{\lambda}, x), \overline{e(\sqrt{\lambda}, x)}\right)(0)}{2i|\Delta(\sqrt{\lambda})|^2}$. Since $\sqrt{\lambda}$ is real, then $\overline{e(\sqrt{\lambda}, x)} = e_-(\sqrt{\lambda}, x)$, and $W_p(e, e_-)$ is constant. Hence

$$W_p(e, e_-)(0) = \lim_{x\to\infty} p(x)\left(e(\sqrt{t}, x)e_-'(\sqrt{\lambda}, x) - e'(\sqrt{\lambda}, x)e_-(\sqrt{\lambda}, x)\right) = -p_\infty 2i\sqrt{\lambda}$$

by (5.5). Thus $\lim_{\varepsilon\to 0^+} \operatorname{Im} M(\lambda+i\varepsilon) = \dfrac{p_\infty\sqrt{\lambda}}{|\Delta(\sqrt{\lambda})|}$, from where we obtain that $\nu(\lambda^+) = \nu(\lambda^-)$ and ν is continuous for $\lambda > 0$. Thus, using that ν is left-continuous and $\nu(0) = 0$ we have

$$\nu(\lambda) - \frac{\nu(0^+)}{2} = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_0^\lambda \operatorname{Im} M(s+i\varepsilon)ds = \int_0^\lambda \frac{p_\infty\sqrt{s}}{\pi|\Delta(\sqrt{s})|}ds$$

by dominated convergence. Now take $\lambda < 0$. If $\lambda_j < \lambda < \lambda_{j-1}$ for some $j \in \{1, \cdots, N\}$, by formula (5.67) and Proposition 116 (ii) we have

$$\nu(\lambda) = \frac{\nu(0^+)}{2} - \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_\lambda^0 (M(s+i\varepsilon) - M(s-i\varepsilon))ds.$$

Consider the rectangle R_ε with vertices at $\lambda \pm i\varepsilon$, $\pm i\varepsilon$, oriented counterclockwise. This rectangle contains the singularities $\lambda_0, \cdots, \lambda_{j-1}$. Hence by residue theorem and Proposition 115 we have

$$\begin{aligned}
-\sum_{k=0}^{j-1}\alpha_k =& \frac{1}{2\pi i}\int_{R_\varepsilon} M(\zeta)d\zeta \\
=& -\int_\lambda^0 (M(s+i\varepsilon) - M(s-i\varepsilon))ds + \int_{-\varepsilon}^{\varepsilon} (M(is) - M(t-is))ds.
\end{aligned}$$

Taking the limit when $\varepsilon \to 0^+$ we obtain

$$\nu(\lambda) = \frac{\nu(0^+)}{2} - \sum_{k=0}^{j-1}\alpha_k.$$

Applying the same procedure for $\lambda_0 < \lambda < 0$ we obtain $\nu(\lambda) = \frac{\nu(0^+)}{2}$. Then ν is continuous at $\lambda = 0$, and (5.65) is established. **Q.E.D.**

Theorem 118. *Let $f \in C_0^\infty(\mathbb{R}^+)$. For $\lambda \in \mathbb{R}$ define*

$$\hat{f}(\lambda) := \int_0^\infty f(x)\varphi(\lambda, x)p(x)dx. \tag{5.68}$$

Then $\hat{f} \in L_2(\mathbb{R}; d\nu(\lambda))$. Furthermore, the action of the map (5.68) can be extended to an isometry from $\mathcal{H}_p$ onto $L_2(\mathbb{R}; d\nu(\lambda))$ and for $f \in \mathcal{H}_p$ we have

$$f(x) = \sum_{j=0}^{N} \alpha_j \langle f, \varphi(\lambda_j, \cdot)\rangle_{\mathcal{H}_p} \varphi(\lambda_j, x) + \int_0^\infty \hat{f}(\lambda)\varphi(\lambda, x)V(\lambda)d\lambda. \tag{5.69}$$

The integral in (5.69) converges in $L_2(\mathbb{R}; d\nu(\lambda))$.

Proof. Formulas (5.57), (5.55) and Proposition 116 establish that $M(\lambda)$ is the Titchmarsh-Weyl function of $\mathbf{L}^h$ from the general theory of Sturm-Liouville operators, and it is known that (5.68) can be extended to an isometry $\mathcal{U} : \mathcal{H}_p \to L_2(\mathbb{R}, d\nu(\lambda))$. More explicitly, for $f \in \mathcal{H}_p$, $\mathcal{U}f(\lambda) = \lim\limits_{X\to\infty} \int_0^X f(x)\varphi(\lambda, x)p(x)dx$, where the limit converges in $L_2(\mathbb{R}; d\nu(\lambda))$, and the inverse is given by $\mathcal{U}^{-1}g(x) = \lim\limits_{\substack{B\to\infty \\ A\to-\infty}} \int_A^B g(\lambda)\varphi(\lambda, x)d\nu(\lambda)$, where the limit converges in $\mathcal{H}_p$ (the scheme of the proof can be found in [12, Th. 4.37] and [3, Th. 2, pp. 192]). Note that $\nu(\lambda)$ can be written as

$$\nu(\lambda) = -\sum_{\{0\leqslant j\leqslant N\,:\,\lambda\leqslant\lambda_j\}} \alpha_j + \chi_{\mathbb{R}^+}(\lambda) \int_0^\lambda V(s)ds,$$

where $\chi_{\mathbb{R}^+}$ is the characteristic function of $\mathbb{R}^+$. The jumps at the discontinuity points $\lambda_0, \cdots, \lambda_N$ are given by $\nu(\lambda_j^+) - \nu(\lambda_j^-) = \alpha_j$ for $j = \overline{0, N}$, then the Lebesgue-Stieltjes measure m_ν generated by ν is given by

$$m_\nu = \sum_{j=0}^{N} \alpha_j \delta_{\lambda_j} + \chi_{\mathbb{R}^+}(\lambda)V(\lambda)d\lambda,$$

where δ_{λ_j} is the Dirac measure at $\lambda = \lambda_j$. Hence the inverse transform is given by

$$\mathcal{U}^{-1}g(x) = \lim_{\substack{B\to\infty \\ A\to-\infty}} \int_A^B g(\lambda)\varphi(\lambda, x)d\nu(\lambda) = \sum_{j=0}^{N} \alpha_j g(\lambda_j)\varphi(\lambda_j, x) + \lim_{B\to\infty} \int_0^B g(\lambda)\varphi(\lambda, x)V(\lambda)d\lambda,$$

from where we obtain (5.69). **Q.E.D.**

The collection $\{\lambda_j, \alpha_j\}_{j=0}^N$ and the function $V(\lambda)$ are called the *spectral data* of the problem (5.1), (5.2).

5.5 Representation for spectral data

Denoting $\Delta_1(\rho) = \frac{d}{d\lambda}\Delta(\rho)$, the spectral data can be written in the form

$$\lambda_j = \rho_j^2, \quad \alpha_j = \frac{e(\rho_j, 0)}{\Delta_1(\rho_j)}, \quad V(\lambda) = \frac{p_\infty \rho}{\pi|\Delta(\rho)|^2},$$

where $\{\rho_j\}_{j=0}^N$ are zeros of $\Delta(\rho)$. Considering the change of variables $z = \frac{\frac{1}{2}+i\rho}{\frac{1}{2}-i\rho}$, according to Remark 98 the corresponding values $\{z_j\}_{j=0}^N$ lie in the interval $(-1, 1)$, $z_N < z_{N-1} < \cdots < z_0$. On the other hand, the continuous spectrum corresponds to the upper unit semicircle, where z runs counterclockwise, that is, $z = e^{i\theta}$, $0 < \theta < \pi$ (see Figure 5.2). If

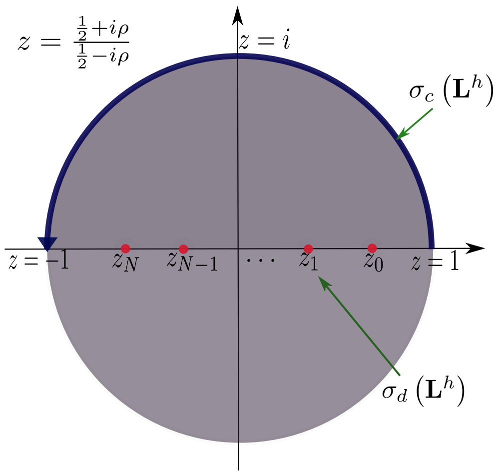

Figure 5.2: Schematic illustration of the distribution of the spectrum of $\mathbf{L}^h$ in terms of the parameter z.

$h < 0$, using representations (5.31) and (5.45), the problem to compute the eigenvalues is reduced to finding zeros in $(-1, 1)$ of the function

$$\Psi(z) := \Delta(\rho) = -h + \frac{z-1}{2(z+1)}\rho(0) + \frac{z-1}{2}\sum_{n=0}^{\infty}(-1)^n \left(d_n(0) - ha_n(0)\right) z^n, \tag{5.70}$$

and the eigenvalues are obtained as

$$\lambda_j = -\left(\frac{z_j - 1}{2(z_j+1)}\right)^2 \quad \text{for } j = \overline{0, N}. \tag{5.71}$$

On the other hand, we denote $\Psi_1(z) = \Delta_1(\rho)$ in terms of z, that is

$$\begin{aligned}\Psi_1(z) &= \frac{d\rho}{d\lambda}\frac{d}{d\rho}\Delta(\rho) = \frac{1}{2\rho}\frac{dz}{d\rho}\frac{d}{dz}\Psi(z) = -\frac{(z+1)^3}{(z-1)}\frac{d}{dz}\Psi(z) \\ &= \frac{(z+1)^3}{(z-1)}h - \frac{(z+1)}{z-1}\rho(0) - \frac{(z+1)^3}{(z-1)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{d_n(0) - ha_n(0)}{2}\right)\left((n+1)z^n - nz^{n-1}\right).\end{aligned}$$

Then

$$\alpha_j = \frac{1 + \dfrac{(z_j - 1)}{2} \displaystyle\sum_{n=0}^{\infty} (-1)^n a_n(0) z_j^n}{\Psi_1(z_j)}, \quad j = \overline{0, N}, \tag{5.72}$$

and the spectral function has the form

$$V(\lambda) = \frac{i p_\infty (1 - z)}{2\pi (z + 1) |\Psi(z)|^2} \quad \text{for } z = e^{i\theta},\ 0 < \theta < \pi. \tag{5.73}$$

Chapter 6

Inverse spectral problem for a type of wave equation

In this chapter, we solve a one-dimensional wave propagation problem. It is known that the inverse problem associated with a certain type of wave equation can be transformed into an inverse problem for the SLEIF [26]. Following [39], using the Fourier-Legendre series representation of the canonical cosine kernel, the inverse problem is reduced to solution of a system of linear algebraic equations.

6.1 Problem setting

Suppose that $p \in AC_{loc}(\overline{\mathbb{R}^+})$ satisfies conditions 1-4 of Subsection 5.1.1. Consider the wave equation

$$\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{1}{p(x)}\frac{\partial}{\partial x}\left(p(x)\frac{\partial v(x,t)}{\partial x}\right) \quad \text{for} \quad (t,x) \in \mathbb{R} \times \mathbb{R}^+. \tag{6.1}$$

Eq. (6.1) can be written as

$$\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial^2 v(x,t)}{\partial x^2} - q(x)\frac{\partial v(x,t)}{\partial x} \quad \text{for} \quad (t,x) \in \mathbb{R} \times \mathbb{R}^+, \tag{6.2}$$

where $q(x) = -\frac{p'(x)}{p(x)}$. Choosing $x_0 = 0$ in formula (1.48) we have $p(x) := p(0)e^{-\int_0^x q(s)}$. As in Chapter 3, we assume that $p(0) = 1$.

Together with equation (6.1) the following conditions are considered

$$v(t,x) = 0 \quad \text{for } t < 0, \tag{6.3}$$

$$v_y(t,0) = \delta(t), \tag{6.4}$$

where δ is de Dirac delta distribution. The *direct problem* for equation (6.1) is formulated as follows.

Direct problem 1. *Given a positive function $p \in AC_{loc}(\overline{\mathbb{R}^+})$ satisfying conditions 1-4 of Subsection 5.1.1, find the function (or distribution) $g(t) = v(t,0)$, where $v(t,x)$ is the solution of equation (6.1) satisfying conditions (6.3) and (6.4).*

The solution of the problem is known and is given in the following way (see [26] for more details). Denote by $\mathcal{F}$ the Fourier transform acting on a function $\phi \in L_1(\mathbb{R})$ by

$$\mathcal{F}(\phi)(\rho) = \hat{\phi}(\rho) := \int_{\mathbb{R}} \phi(t) e^{-it\rho} dt,$$

and extending its action onto Schwarz distributions $f \in \mathcal{S}'(\mathbb{R})$ by $(\mathcal{F}(f), \phi) := (f, \mathcal{F}(\phi))$, for $\phi \in \mathcal{S}(\mathbb{R})$. Assuming that it is possible to apply the Fourier transform to both sides of the main equation, we obtain that $\hat{v}(\rho,x) := \int_{\mathbb{R}} v(t,x)e^{-it\rho} dt$ satisfies the equation

$$-\hat{v}_{xx}(\rho,x) + q(x)\hat{v}_x(\rho,x) = \rho^2 \hat{v}(\rho,x), \quad \text{for } x \in \mathbb{R}^+. \tag{6.5}$$

Condition (6.4) turns into the equality $\hat{v}'(\rho,0) = 1$. If there exists a solution of this problem, the solution $v(t,x)$ is given (at least formally) by $v(t,x) = \mathcal{F}^{-1}(\hat{v}(\rho,x)) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{v}(\rho,y)e^{i\rho t} d\rho$. Once the solution is obtained, one can compute the function (or distribution) $g(t) = v(t,0)$. The important point is that the problem is reduced to study the Sturm-Liouville equation in impedance form (6.5).

We are interested in the inverse problem for (6.1), that can be formulated as follows.

Inverse problem 1. *Given a function (or distribution) $g(t)$, find a function p such that there exists the solution $v(t,x)$ of (6.1) satisfying (6.3) and (6.4), and the equality holds $g(t) = v(t,0)$.*

6.2 Relation with the inverse problem for the SLEIF

In [26], a procedure to reduce the inverse problem for Eq. (6.2) to the spectral inverse problem of eq. (5.1) with condition (5.2) with $h = 0$ (that is, with Neumann condition) was proved. Let $C(\rho, x)$ be the solution of (5.1) satisfying $C(\rho, 0) = 1$, $C'(\rho, 0) = 0$. We recall that the Jost coefficient $c(\rho)$ is the unique holomorphic function in $\mathbb{C}^-$ satisfying $C(\rho, x) = c(\rho)e(\rho, x) + c(-\rho)e(-\rho, x)$ for $\rho \in \mathbb{R} \setminus \{0\}$, where $e(\rho, x)$ is the Jost solution (Remark 108). The function $c(\rho)$ does not vanish for $\rho \in \mathbb{R} \setminus \{0\}$ (see [26, Lemma 2.6]). We suppose that there exists a constant $c_1 > 0$ such that $\left|\frac{1}{c(\rho)}\right| \leqslant c_1$ (this occurs, for example, if $q(x) \leqslant 0$ for all $x \in \mathbb{R}^+$, see [26, Remark 2.8]). The following result shows the relation between the solutions of problems (6.2), (6.3), (6.4) and (5.1), (5.2).

Theorem 119 ([26])**.** *The solution $v(t, x)$ of the problem (6.2), (6.3), (6.4) is given by*

$$v(t, x) - g(t) = -\frac{1}{2\pi p_\infty} \int_0^\infty [C(\rho, x) - 1] \frac{\sin(\rho t)}{\rho} \frac{d\rho}{|c(\rho)|^2}. \tag{6.6}$$

In consequence

$$g(t) = -\frac{1}{2\pi p_\infty} \int_0^\infty \frac{\sin(\rho t)}{\rho} \frac{d\rho}{|c(\rho)|^2}. \tag{6.7}$$

Denote

$$\widehat{V}(\rho) := \frac{1}{2\pi p_\infty} \frac{1}{|c(\rho)|^2}. \tag{6.8}$$

(actually, this is the spectral function $V(\lambda)$ in terms of ρ and $c(\rho)$). The main fact is the relation between the data $g(t)$ with the function $\widehat{V}(\rho)$ via the sine Fourier transform

$$\widehat{V}(\rho) = -\frac{2\rho}{\pi} \int_0^\infty g(t) \sin(\rho t) dt \tag{6.9}$$

(see [26]). Thus, the inverse problem for (6.2) is reduced to that for (5.1).

Remark 120. *If $g \in W^{k,1}(\mathbb{R}^+)$ for $k \in \mathbb{N}$, then integrating by parts and using the Riemann-Lebesgue lemma [121, Th. 9.6] we have that*

$$\int_0^\infty g(t) \sin(\rho t) dt = \frac{g(0)}{\rho} + \frac{g''(0)}{\rho^3} + \cdots + \frac{g^{(\alpha)}(0)}{\rho^{\alpha+1}} + o\left(\frac{1}{\rho^k}\right), \quad \rho \to +\infty,$$

where $\alpha = \begin{cases} k-1, & \text{if } k \text{ is odd,} \\ k-2, & \text{if } k \text{ is even} \end{cases}$. *Hence*

$$\widehat{V}(\rho) = -\frac{2}{\pi}\left(g(0) + \frac{g''(0)}{\rho^2} + \cdots + \frac{g^{(\alpha)}(0)}{\rho^\alpha}\right) + o\left(\frac{1}{\rho^{k-1}}\right), \quad \rho \to \infty. \tag{6.10}$$

6.3 Solution of the inverse problem

Let $G(x,t)$ the canonical cosine transmutation kernel. As in the case of a finite interval, it is known that $G(x,t)$ satisfies a Gelfand-Levitan equation for the Neumann problem on the half-line.

Theorem 121 ([26]). *Given the spectral function $\widehat{V}(\rho)$, the transmutation kernel $G(x,t)$ satisfies the Gelfand-Levitan equation*

$$G(x,z) + \Omega(x,z) = \int_0^x G(x,\eta)\Omega_\eta(\eta,z)d\eta, \quad z < x, \tag{6.11}$$

where

$$\Omega(x,z) = \int_0^\infty \frac{\sin(\rho z)}{\rho}\cos(\rho x)d\sigma(\rho) \quad \text{and} \quad \Omega_x(x,z) = -\int_0^\infty \sin(\rho z)\sin(\rho x)d\sigma(\rho), \tag{6.12}$$

and the measure $d\sigma(\rho)$ is defined by $d\sigma(\rho) := \left(\widehat{V}(\rho) - \frac{2}{\pi}\right)d\rho$.

Roughly speaking, the function $\Omega_x(x,z)$ is the derivative of $\Omega(x,z)$ with respect to x. Both integrals exist, at least in the weak sense. Under some additional conditions on the function $\widehat{V}(\rho)$ it can be shown that the integrals converge absolutely.

Remark 122. *According to Remark 120, if $g \in W^{2,1}(\mathbb{R}^+)$, then*

$$\widehat{V}(\rho) + \frac{2}{\pi}g(0) = o\left(\frac{1}{\rho}\right), \quad \rho \to \infty.$$

If $g(0) = -1$, then $\widehat{V}(\rho) - \frac{2}{\pi} = o\left(\frac{1}{\rho}\right)$, $\rho \to \infty$, and the integral $\Omega(x,z)$ converges uniformly and absolutely for all $x, z \in \mathbb{R}^+$. For the absolute convergence of $\Omega_x(x,z)$ we require that $g \in W^{3,1}(\mathbb{R}^+)$ and $g''(0) = 0$ in order to obtain $\widehat{V}(\rho) - \frac{2}{\pi} = o\left(\frac{1}{\rho^2}\right)$, $\rho \to \infty$.

Again, we use the Fourier-Legendre series (3.57) $G(x,z) = \sum_{n=0}^\infty \frac{\beta_n(x)}{x}P_{2n+1}\left(\frac{z}{x}\right)$ to transform the Gelfand-Levitan equation (6.11) into a system of linear algebraic equations where the unknowns are the coefficients $\{\beta_n(x)\}_{n=0}^\infty$. By formula (4.17), only the first coefficient β_0 is required to recover the potential q.

Theorem 123. *Suppose that $g \in W_1^3(\mathbb{R}^+)$ satisfy the conditions $g(0) = -1$ and $g''(0) = 0$. For every $x \in \mathbb{R}^+$ the Fourier-Legendre coefficients $\{\beta_n(x)\}_{n=0}^\infty$ satisfy the infinite system* *of linear algebraic equations*

$$\frac{\beta_m(x)}{4m+3}+\sum_{n=0}^{\infty}A_{n,m}(x)\beta_n(x)=\mathcal{T}_m(x)\quad \text{for } m\in\mathbb{N}_0, \tag{6.13}$$

where

$$A_{n,m}(x)=(-1)^{n+m}x\int_0^{\infty}j_{2n+1}(\rho x)j_{2m+1}(\rho x)d\sigma(\rho), \tag{6.14}$$

$$\mathcal{T}_m(x)=(-1)^{m+1}x\int_0^{\infty}\frac{\cos(\rho x)}{\rho}j_{2m+1}(\rho x)d\sigma(\rho), \tag{6.15}$$

and the series (6.13) converges pointwise.

Proof. By Remark 122, the conditions in g implies that $\widehat{V}(\rho)-\frac{2}{\pi}=o\left(\frac{1}{\rho^2}\right)$, $\rho\to\infty$, and hence the integrals in (6.12) converges absolutely for all $x,z\in\mathbb{R}^+$. Thus, $\Omega(x,z)$ and $\Omega_x(x,z)$ are uniformly bounded for $x,z\in\mathbb{R}$, and for every $x\in\mathbb{R}$, Eq. (6.11) is a Fredholm integral equation of the second kind with kernel $\Omega_x\in L_2\left((0,x)\times(0,x)\right)$. Then we only have to repeat the procedures of Theorems 36 and 86 and calculate the integrals $A_{m,n}(x)=\int_0^x\int_0^x P_{2m+1}\left(\frac{z}{x}\right)P_{2n+1}\left(\frac{\eta}{x}\right)\Omega_\eta(\eta,z)\frac{d\eta dz}{x}$ and $\mathcal{T}_m(x)=\int_0^x P_{2m+1}\left(\frac{z}{x}\right)\Omega(x,z)dz.$ By the absolutely convergence of the integrals, we can use the Fubini theorem and formula (2.27) to obtain

$$\begin{aligned}A_{m,n}(x)&=-\int_0^{\infty}\left[\int_0^x P_{2m+1}\left(\frac{z}{x}\right)\sin(\rho z)\frac{dz}{x}\right]\left[\int_0^x P_{2m+1}\left(\frac{\eta}{x}\right)\sin(\rho\eta)d\eta\right]d\sigma(\rho)\\&=(-1)^{n+m}x\int_0^{\infty}j_{2n+1}(\rho x)j_{2m+1}(\rho x)d\sigma(\rho).\end{aligned}$$

The proof of (6.15) is similar. Since for $\rho\in\mathbb{R}$

$$j_{2m+1}(\rho x)=\frac{\cos\left(\rho x-\frac{2m+1}{\pi}-\frac{\pi}{4}\right)}{|\rho x|}+O\left(\frac{1}{|\rho x|^2}\right),\ |\rho|\to\infty,$$

(see [1], formula (9.2.1)) the integrals in (6.14) and (6.15) converges absolutely. **Q.E.D.**

The proofs of the convergence and stability of the solutions of the corresponding approximate system are analogous to those of Theorems 37 and 87.

6.4 Numerical algorithm

Theorem 123 together with equality (4.17) lead to a direct algorithm for solving the inverse spectral problem, and with this the inverse problem for (6.2).

Algorithm 124. *Given $g \in W_1^3(\mathbb{R}^+)$ satisfying $g(0) = -1$ and $g''(0) = 0$. Fix $M \in \mathbb{N}$ and $\Lambda > 0$.*

1. *Compute the spectral function $\widehat{V}(\rho)$ using formula (6.9).*

2. *Compute $A_{n,m}^{\Lambda}(x)$ and $\mathcal{T}_m^{\Lambda}(x)$ by the formulas*

$$A_{n,m}^{\Lambda}(x) = (-1)^{n+m} x \int_0^{\Lambda} j_{2n+1}(\rho x) j_{2m+1}(\rho x) d\sigma(\rho), \tag{6.16}$$

$$\mathcal{T}_m^{\Lambda}(x) = (-1)^{m+1} x \int_0^{\Lambda} \frac{\cos(\rho x)}{\rho} j_{2m+1}(\rho x) d\sigma(\rho), \tag{6.17}$$

   for $n, m = \overline{0, M}$.

3. *For a set of points $\{x_l\}_{l=0}^{L} \subset (0, b]$, $b > 0$, solve the system*

$$\frac{\beta_m(x)}{4m+3} + \sum_{n=0}^{M} A_{n,m}^{\Lambda}(x) \beta_n(x) = \mathcal{T}_m^{\Lambda}(x) \quad \text{for } m = \overline{0, N}. \tag{6.18}$$

4. *Compute $q(x)$, $x \in (0, b]$, using (4.17), and $p(x)$ using (1.48).*

Chapter 7

Runge property and approximation by complete systems of solutions for strongly elliptic equations

In this chapter give an overview of some concepts related to the approximation of solutions of a strongly elliptic operator. A definition of a complete system of classical solutions is given, and we show how using the Runge property, it is possible to extend the completeness of these systems onto strictly internal domains with respect to the L_2, H^1 and H^2-norms. This result is applied to Schrödinger equations with potentials possessing some symmetries.

7.1 Relations between different types of solutions

Throughout the chapter Ω denotes a bounded domain in $\mathbb{R}^d$, $d \geqslant 2$. Let us consider the elliptic operator of the form

$$\mathbf{L}u(x) := -\operatorname{div}(A(x)\nabla u(x)) + q(x)u(x), \tag{7.1}$$

acting in a first instance on functions $u \in C^2(\Omega)$. The coefficient $q \in L_\infty(\Omega)$ is called the *potential* of $\mathbf{L}$, and in general it is supposed to be complex valued. The function $A : \Omega \to \mathbb{R}^{d\times d}$ is called the *principal coefficient* and supposed to be a symmetric matrix function, $A \in W^{1,\infty}\left(\Omega, \mathbb{R}^{d\times d}\right)$. We assume that there exists a constant $\mu_0 > 0$ such that

$\langle A(x)\xi,\xi\rangle_{\mathbb{C}^d} \geqslant \mu_0|\xi|^2$ for all $\xi\in\mathbb{C}^d$ and for almost all $x\in\Omega$. Under these conditions the operator $\mathbf{L}$ is called *strongly elliptic*. Following [120] we suppose additionally that there exists a constant $K\geqslant 1$ such that

1. $\frac{1}{K}|\xi|^2 \leqslant \langle A(x)\xi,\xi\rangle_{\mathbb{C}^d} \leqslant K|\xi|^2$, for all $\xi\in\mathbb{C}^d$ and for almost all $x\in\Omega$.

2. $\|q\|_{L_\infty(\Omega)}\leqslant K$.

3. If $d\geqslant 3$ then also $\|\nabla A_{i,j}\|_{L_\infty(\Omega)}\leqslant K,\ i,j=\overline{1,d}$.

As our main example we consider the *Schrödinger operator* $\mathbf{S}u := -\triangle_d\, u + q(x)u$. In this case the conditions 1,2 and 3 hold for $K=\max\{1,\|q\|_{L_\infty(\Omega)}\}$.

In the case when the coefficients satisfy the additional conditions: $A\in C^1(\Omega,\mathbb{R}^{d\times d})$ and $q\in C(\Omega)$ we introduce two spaces of classical solutions of the equation

$$\mathbf{L}u(x)=0 \quad \text{for } x\in\Omega. \tag{7.2}$$

1. The space of *classical solutions* is defined by

$$\mathrm{Sol}^{\mathbf{L}}(\Omega) := \left\{u\in C^2(\Omega)\,|\,u \text{ satisfies } (7.2)\right\}. \tag{7.3}$$

2. The *Bergman space* of solutions of (7.2) is defined by

$$\mathrm{Sol}_2^{\mathbf{L}}(\Omega) := \left\{u\in \mathrm{Sol}^{\mathbf{L}}(\Omega)\,\middle|\, \int_\Omega |u(x)|^2dx<\infty\right\}. \tag{7.4}$$

Note that $\mathrm{Sol}^{\mathbf{L}}(\Omega)$ is a subspace of $C(\Omega)$. $C(\Omega)$ can be endowed with a standard topology as follows. Let $\{K_n\}_{n=1}^\infty$ be a sequence of compact subsets of Ω such that

(i) $\forall n\in\mathbb{N}\quad K_n\subset \mathrm{Int}\,(K_{n+1})$ (where $\mathrm{Int}\,(K_{n+1})$ denotes the interior of the set K_{n+1}),

(ii) $\Omega=\bigcup_{n=1}^\infty K_n$.

One can choose, for example,

$$K_n := \overline{B_n^d(0)}\cap\left\{x\in\Omega\,\middle|\,\mathrm{dist}\,(x,\partial\Omega)\geqslant\frac{1}{n}\right\}.$$

By (i) and (ii), the sequence $\{K_n\}_{n=1}^\infty$ has the property that for any compact $K\subset\Omega$ there exists $N\in\mathbb{N}$ such that $K\subset K_N$. Then the topology of $C(\Omega)$ is generated by the family of semi-norms $\left\{ \|f\|_n := \max_{x \in K_n} |f(x)| \right\}_{n=1}^{\infty}$. With this topology $C(\Omega)$ is a Fréchet space, and the convergence induced is given by the uniform convergence on compact subsets of Ω. The topology is independent of the choice of $\{K_n\}_{n=1}^{\infty}$ (for the proof of these facts see [34, Ch. VII, Sec. 1] or [51, Prop. 4. 39]).

For some operators $\mathbf{L}$, $\mathrm{Sol}^{\mathbf{L}}(\Omega)$ is closed in $C(\Omega)$. For example, in the case $A \equiv I_d$, $q \equiv 0$, $\mathrm{Sol}^{\triangle_d}(\Omega) = Har(\Omega)$ is closed in $C(\Omega)$ (see [8, Th. 1.23]).

For the Bergman space of solutions the following result is known as Friedrich's property.

Proposition 125 ([53])**.** *Let $K \subset \Omega$ be compact. Then there exists a constant $C_K > 0$ such that for all $u \in \mathrm{Sol}_2^{\mathbf{L}}(\Omega)$ the inequality holds*

$$\max_{x \in K} |u(x)| \leqslant C_K \|u\|_{L_2(\Omega)}. \tag{7.5}$$

As a corollary, the following completeness property holds.

Proposition 126. *If $\mathrm{Sol}^{\mathbf{L}}(\Omega)$ is closed in $C(\Omega)$, then $\mathrm{Sol}_2^{\mathbf{L}}(\Omega)$ endowed with the L_2-norm is a Hilbert space. Furthermore, for each $x_0 \in \Omega$, the evaluation functional $\mathrm{Sol}_2^{\mathbf{L}}(\Omega) \ni u \mapsto u(x_0) \in \mathbb{C}$ is bounded in the L_2-norm, and hence the space has a reproducing kernel.*

Proof. Using (7.5), the proof is similar to the case of the analytic Bergman spaces (see, e.g. [44, Ch. I]). **Q.E.D.**

Again, $\mathrm{Sol}_2^{\triangle_d}(\Omega) = b_2(\Omega)$, the harmonic Bergman space.

Let us introduce the concept of *weak solutions* for (7.2). For this we suppose that q is *real valued* (this assumption is only for this class of solutions) and consider the sesquilinear form $\Phi : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ defined by

$$\Phi(u, v) := \int_{\Omega} \left\{ \langle \nabla v(x), A(x) \nabla u(x) \rangle_{\mathbb{C}^d} + q(x) v(x) \overline{u(x)} \right\} dx \quad \text{for } u, v \in H^1(\Omega).$$

The sesquilinear form is Hermitian, bounded and satisfies the inequality

$$|\Phi(u, v)| \leqslant \left(\|A\|_{L_\infty(\Omega, \mathbb{R}^{d \times d})} + \|q\|_{L_\infty(\Omega)} \right) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall u, v \in H^1(\Omega). \tag{7.6}$$

In this case, the space $H^1(\Omega)$ has a natural conjugation given by the standard conjugation of functions. By simplicity, we denote the associated operator $\mathbf{A} = \mathbf{A}_\Phi$. Hence $\mathbf{A} \in \mathcal{B}(H^1(\Omega), \widetilde{H}^{-1}(\Omega))$ and $(\mathbf{A}u, v)_{H^1(\Omega)} = \Phi(u, v)$ for all $u, v \in H^1(\Omega)$.

Now we introduce some operators derived from $\mathbf{A}$. We define $\mathbf{A}_1 : H^1(\Omega) \to H^{-1}(\Omega)$ given by $\mathbf{A}_1 u := \mathbf{A}u\big|_{H_0^1(\Omega)}$ for $u \in H^1(\Omega)$. Thus $\mathbf{A}_1 \in \mathcal{B}\left(H^1(\Omega), H^{-1}(\Omega)\right)$ and $(\mathbf{A}_1 u, v)_{H_0^1(\Omega)} = \Phi(u, v)$ for all $u \in H^1(\Omega)$, $v \in H_0^1(\Omega)$. In a similar way, we define $\mathbf{A}_0 : H_0^1(\Omega) \to H^{-1}(\Omega)$ by $\mathbf{A}_0 := \mathbf{A}_1|_{H_0^1(\Omega)}$. Note that $\mathbf{A}_0 \in \mathcal{B}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$ is the associated operator for $\Phi\big|_{H_0^1(\Omega)\times H_0^1(\Omega)}$.

A function $u \in H^1(\Omega)$ is called a **weak solution** of (7.2) if it satisfies the equation

$$\Phi(u, v) = 0 \quad \forall v \in H_0^1(\Omega), \tag{7.7}$$

or equivalently, if $\mathbf{A}_1 u = 0$. The space of all weak solutions is denoted by $\mathrm{Sol}_w^{\mathbf{L}}(\Omega)$, and due to (7.6) it is a closed subspace of $H^1(\Omega)$.

There are known several relations between the spaces of weak and classical solutions. Note that for any $V \Subset \Omega$: $u \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega)$ implies $u|_V \in \mathrm{Sol}_w^{\mathbf{L}}(V)$. Other relations are summarized in the following proposition.

Proposition 127. *Let $A \in C^1(\Omega, \mathbb{R}^{d\times d})$. Then the following properties are valid.*

1. $\mathrm{Sol}_w^{\mathbf{L}}(\Omega) \subset H_{loc}^2(\Omega)$ *and*

$$\forall u \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega) \quad \mathbf{L}u(x) = 0 \quad a.e.\ in\, \Omega.$$

2. *Given $V \Subset \Omega$, there exists a constant $C_1 = C_1(A, q, \Omega, V) > 0$ such that*

$$\forall u \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega) \quad \|u|_V\|_{H^2(V)} \leqslant C_1 \|u\|_{L_2(\Omega)}. \tag{7.8}$$

3. *If $q \in C(\Omega)$, then $\mathrm{Sol}^{\mathbf{L}}(\Omega) \subset H_{loc}^2(\Omega)$, and for $\omega \Subset V \Subset \Omega$ there exists a constant $C_2 = C_2(A, q, V, \omega)$ such that*

$$\forall u \in \mathrm{Sol}^{\mathbf{L}}(\Omega) \quad \|u|_\omega\|_{H^2(\omega)} \leqslant C_2 \|u|_V\|_{L_2(V)}. \tag{7.9}$$

4. *If Ω is a C^2-domain and $A \in C^1(\overline{\Omega}, \mathbb{R}^{d\times d})$, $q \in C(\overline{\Omega})$, then $\mathrm{Sol}_w^{\mathbf{L}}(\Omega) \subset H^2(\Omega)$.*

Proof. The proof of 1 and 2 can be found in [101, pp. 309-314].

For the part 3, it is clear that $\mathrm{Sol}^{\mathbf{L}}(\Omega) \subset H^2_{loc}(\Omega)$. Thus, if $\omega \Subset V \Subset \Omega$, then $u|_V \in \mathrm{Sol}^{\mathbf{L}}_w(V)$ for all $u \in \mathrm{Sol}^{\mathbf{L}}(\Omega)$, and applying the part 2 we obtain the constant $C_2(A, q, V, \omega)$ satisfying (7.9).

Finally, the proof of 4 can be found in [113, pp.232]. **Q.E.D.**

7.2 The Dirichlet and Neumann problems

In this section all the domains Ω are supposed to be bounded and of Lipschitz type, and we denote $\Gamma = \partial\Omega$.

The Dirichlet problem admits the following weak formulation. Given $f \in H^{-1}(\Omega)$ and $\phi \in H^{\frac{1}{2}}(\Gamma)$, find $u \in H^1(\Omega)$ satisfying the conditions

$$\mathfrak{D}_{(f,\phi)} = \begin{cases} \mathbf{A}_1 u = f, \\ \mathrm{tr}_\Gamma(u) = \phi. \end{cases} \tag{7.10}$$

The pair $(f, \phi) \in H^{-1}(\Omega) \times H^{\frac{1}{2}}(\Omega)$ is called the *Dirichlet data.* Of course, when $f \in L_2(\Omega)$, the corresponding functional is $(f, v)_{H^1(\Omega)} = \langle v, f\rangle_{L_2(\Omega)}$ and the first condition in (7.10) can be written as $\Phi(u, v) = \langle v, f\rangle_{L_2(\Omega)}$, that is the usual way for L_2-functions (see [113, Ch. IV]). Actually, a functional $f \in H^{-1}(\Omega)$ can be written in the form $(f, v)_{H^1(\Omega)} = \int_\Omega \overline{f_0(x)} v(x)dx + \sum_{j=1}^{d} \int_\Omega \overline{f_j(x)} \frac{\partial v(x)}{\partial x_j}$, for some functions $\{f_j\}_{j=0}^d \subset L_2(\Omega)$ (see [19], Prop. 9.20 and Remark 21 from pp. 220).

Remark 128. *Since $H^1(\Omega) \subset L_2(\Omega)$ is dense, $L_2(\Omega)$ is a pivot space for $H^1(\Omega)$ and we have the triple $H^1(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow \widetilde{H}^{-1}(\Omega)$. By the condition 1 of Section 7.1, if $u \in H^1(\Omega)$, then*

$$\begin{aligned} \operatorname{Re}\Phi(u, u) &= \operatorname{Re}\left(\int_\Omega \langle \nabla u(x), A(x)\nabla u(x)\rangle_{\mathbb{C}^d} dx\right) + \int_\Omega \operatorname{Re} q(x)|u(x)|^2 dx \\ &\geqslant \operatorname{Re}\left(\int_\Omega \langle \nabla A(x)u(x), \nabla u(x)\rangle_{\mathbb{C}^d} dx\right) - \|q\|_{L_\infty(\Omega)}\|u\|^2_{L_2(\Omega)} \\ &\geqslant \frac{1}{K}\int_\Omega |\nabla u(x)|^2 dx - \|q\|_{L_\infty}\|u\|^2_{L_2(\Omega)} \\ &= \frac{1}{K}\|u\|^2_{H^1(\Omega)} - \left(\|q\|_{L_\infty(\Omega)} + \frac{1}{K}\right)\|u\|^2_{L_2(\Omega)}. \end{aligned}$$

Thus, Φ is coercive on $H^1(\Omega)$ with respect to the pivot space $L_2(\Omega)$. In a similar way, $L_2(\Omega)$ is a pivot space for $H_0^1(\Omega)$, $H_0^1(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow H^{-1}(\Omega)$ and Φ is coercive on $H_0^1(\Omega)$ with respect to $L_2(\Omega)$.

Among the results on the existence of solutions of the Dirichlet problem $\mathfrak{D}_{(f,\phi)}$ we need one in particular that relates the Dirichlet data with the normal derivatives of the solutions of the homogeneous problem $\mathfrak{D}_{(0,0)}$. For a Lipschitz domain Ω a normal vector ν is defined a.e. on Γ. Then for $u, v \in C^\infty(\overline{\Omega})$ application of the Gauss-Ostrogradsky theorem gives us the Green identity

$$\int_\Omega \mathbf{L}u(x)\overline{v(x)}dx = \Phi(u,v) - \int_\Gamma \overline{v(x)}\langle A(x)\nabla u(x), \nu(x)\rangle dS. \tag{7.11}$$

Using the density of $C^\infty(\overline{\Omega})$ in $H^k(\Omega)$ for Lipschitz domains (Proposition 13 (2)), we can rewrite (7.11) as

$$\langle \mathbf{L}u, v\rangle_{L_2(\Omega)} = \Phi[u,v] - \langle \partial_\nu^A u, v\rangle_{L_2(\Gamma)}, \tag{7.12}$$

where $\partial_\nu^A u := \sum_{j=1}^d \sum_{k=1}^d A_{j,k} \mathrm{tr}_\Gamma \left(\frac{\partial u}{\partial x_k}\right) \nu_j$. Note that $\partial_\nu^A u$ defines a functional in $H^{-\frac{1}{2}}(\Gamma)$, given by $(\partial_\nu^A u, \phi)_{H^{\frac{1}{2}}(\Gamma)} := \langle \partial_\nu^A u, \phi\rangle_{L_2(\Gamma)}$ for $\phi \in H^{\frac{1}{2}}(\Gamma)$, and is related with $\mathbf{L}$ and Φ by (7.12). In this case $\mathbf{L}u$ defines a functional in $\tilde{H}^{-1}(\Omega)$. This indicates that the normal derivative can be defined for those functions that satisfy $\mathbf{L}u = f$ with $f \in \tilde{H}^{-1}(\Omega)$. This can be reformulated in terms of operator $\mathbf{A}$. The following result is a modification of [107, Lemma 4.3].

Proposition 129. *Let $u \in H^1(\Omega)$ and $f \in \widetilde{H}^{-1}(\Omega)$ be such that $\mathbf{A}_1 u = f\big|_{H_0^1(\Omega)}$. Then there exists $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ such that*

$$(\mathbf{A}u, v)_{H^1(\Omega)} = (f, v)_{H^1(\Omega)} + (\varphi, \mathrm{tr}_\Gamma(v))_{H^{\frac{1}{2}}(\Gamma)} \quad \forall v \in H^1(\Omega). \tag{7.13}$$

Furthermore, φ is uniquely determined by f and u and there exists a constant $C > 0$ for which

$$\|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)} \leqslant C\left(\|u\|_{H^1(\Omega)} + \|f\|_{\widetilde{H}^{-1}(\Omega)}\right). \tag{7.14}$$

Proof. Define $\varphi : H^{\frac{1}{2}}(\Gamma) \to \mathbb{C}$ given by

$$(\varphi, \phi)_{H^{\frac{1}{2}}(\Gamma)} := (\mathbf{A}u, \mathcal{E}\phi)_{H^1(\Omega)} - (f, \mathcal{E}\phi)_{H^1(\Omega)} \quad \text{for } \phi \in H^{\frac{1}{2}}(\Gamma),$$

where $\mathcal{E} \in \mathcal{B}\left(H^{\frac{1}{2}}(\Gamma), H^1(\Omega)\right)$ is a extension operator. It is clear that $\varphi \in H^{-\frac{1}{2}}(\Gamma)$. Take $v \in H^1(\Omega)$ and set $v_0 = v - \mathcal{E}\,\mathrm{tr}_\Gamma\, v$, in such a way that $v_0 \in H_0^1(\Omega)$. By hypothesis

$$
\begin{aligned}
0 &= (\mathbf{A}_1 u, v_0)_{H_0^1(\Omega)} - (f|_{H_0^1(\Omega)}, v_0)_{H_0^1(\Omega)} = (\mathbf{A}u, v_0)_{H^1(\Omega)} - (f, v_0)_{H^1(\Omega)} \\
&= (\mathbf{A}u, v)_{H^1(\Omega)} - (f, v)_{H^1(\Omega)} - \big((\mathbf{A}u, \mathcal{E}\,\mathrm{tr}_\Gamma\, v)_{H^1(\Omega)} - (f, \mathcal{E}\,\mathrm{tr}_\Gamma\, v)_{H^1(\Omega)}\big) \\
&= (\mathbf{A}u, v)_{H^1(\Omega)} - (f, v)_{H^1(\Omega)} - (\varphi, \mathrm{tr}_\Gamma(v))_{H^{\frac{1}{2}}(\Gamma)},
\end{aligned}
$$

from where we obtain (7.13). The bound (7.14) follows from definition of φ and the boundedness of $\mathbf{A}$ and f. For the uniqueness, suppose that $\psi \in H^{-\frac{1}{2}}(\Gamma)$ satisfies (7.13). Taking $\phi \in H^{\frac{1}{2}}(\Gamma)$ and $v \in H^1(\Omega)$ with $\mathrm{tr}_\Gamma(v) = \phi$, we obtain

$$
(\varphi - \psi, \phi)_{H^{\frac{1}{2}}(\Gamma)} = (\mathbf{A}u, v)_{H^1(\Omega)} - (f, v)_{H^1(\Omega)} - \big((\mathbf{A}u, v)_{H^1(\Omega)} - (f, v)_{H^1(\Omega)}\big) = 0.
$$

Hence $\phi = \psi$. **Q.E.D.**

Motivated by this proposition and following [107, 120], we introduce the definition of the *co-normal* derivative of a function $u \in H^1(\Omega)$.

Definition 130. *Let $u \in H^1(\Omega)$ and $f \in \tilde{H}^{-1}(\Omega)$. Suppose that $\mathbf{A}_1 u = f\big|_{H_0^1(\Omega)}$. The **co-normal** derivative of the function u with respect to f and $\mathbf{A}$ is the functional $\partial_\nu u \in H^{-\frac{1}{2}}(\Gamma)$ defined by*

$$
(\partial_\nu u, \phi)_{H^{\frac{1}{2}}(\Gamma)} := (\mathbf{A}u, \mathcal{E}\phi) - (f, \mathcal{E}\phi)_{H^1(\Omega)} \quad \text{for } \phi \in H^{\frac{1}{2}}(\Gamma), \tag{7.15}
$$

where $\mathcal{E} \in \mathcal{B}\left(H^{\frac{1}{2}}(\Gamma), H^1(\Omega)\right)$ is an extension operator.

Remark 131.

1. *If $u \in H^2(\Omega)$, using the Green identity (7.12) we obtain that $\mathbf{A}u = \mathbf{L}u$. Then taking $f = \mathbf{L}u$, by Green identity and Theorem 129, one obtains that the co-normal derivative coincides with $\partial_\nu^A u$. In particular, for the Schrödinger operator it coincides with the normal derivative $\partial_\nu u$.*

2. *From Theorem 129, the co-normal derivative $\partial_\nu u$ of a function $u \in H^1(\Omega)$ does not depend on the extension operator chosen.*

3. *The co-normal derivative is well defined for $u \in Sol_w^{\mathbf{L}}(\Omega)$ and is given by $(\partial_\nu u, \phi)_{H^{\frac{1}{2}}(\Gamma)} = (\mathbf{A}u, \mathcal{E}\phi)_{H^1(\Omega)}$.*

Based on this results, we formulate the Neumann problem in the following way (that can be found in [113, Ch. IV] and [107, Ch. IV]):

Neumann problem: Given $f \in \widetilde{H}^{-1}(\Omega)$ and $\varphi \in H^{-\frac{1}{2}}(\Gamma)$, find $u \in H^1(\Omega)$ satisfying the condition:

$$\mathfrak{N}_{(f,\varphi)} : \ (\mathbf{A}u, v)_{H^1(\Omega)} = (f, v)_{H^1(\Omega)} + (\varphi, \mathrm{tr}_\Gamma(v))_{H^{\frac{1}{2}}(\Gamma)} \quad \forall v \in H^1(\Omega). \tag{7.16}$$

The pair $(f, \varphi) \in \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$ is called the *Neumann data.*

About the existence of solutions, there are several results that relate the solutions of a non-homogeneous problem with the solutions of the homogeneous (i.e., with data $(0,0)$).

Theorem 132 (Existence of solutions for the Dirichlet problem)**.** *Denote by W the set of solutions of $\mathfrak{D}_{(0,0)}$. Then W has a finite dimension and there exist only two possibilities.*

1. *If $W = \{0\}$, then for any $(f, \phi) \in H^{-1}(\Omega) \times H^{\frac{1}{2}}(\Gamma)$ the Dirichlet problem $\mathfrak{D}_{(f,\phi)}$ has a unique solution $u \in H^1(\Omega)$, and there exists a constant $C > 0$ that does not depend on (f, ϕ) such that*

$$\|u\|_{H^1(\Omega)} \leqslant C \left(\|f\|_{H^{-1}(\Omega)} + \|\phi\|_{H^{\frac{1}{2}}(\Gamma)} \right). \tag{7.17}$$

2. *If $p = \dim W > 0$, $W = Span\{w_j\}_{j=1}^p$, the problem $\mathfrak{D}_{(f,\phi)}$ has a solution $u \in H^1(\Omega)$ iff the Dirichlet data satisfies the condition*

$$(f, w_j)_{H_0^1(\Omega)} = (\phi, \partial_\nu w_j)_{H^{-\frac{1}{2}}(\Gamma)}, \quad j = \overline{1, p}, \tag{7.18}$$

in which case the problem has exactly p linearly independent solutions. In this case if u is a solution then there exists $C > 0$ such that

$$\|u + W\|_{H^1(\Omega)/W} \leqslant C \left(\|f\|_{H^{-1}(\Omega)} + \|\phi\|_{H^{\frac{1}{2}}(\Gamma)} \right). \tag{7.19}$$

Proof. Consider the operator $\mathbf{A}_0$ associated to the sesquilinear form $\Phi\big|_{H_0^1(\Omega)\times H_0^1(\Omega)}$. By Remark 128, $L_2(\Omega)$ is a pivot space for $H_0^1(\Omega)$ and $\Phi\big|_{H_0^1(\Omega)\times H_0^1(\Omega)}$ is coercive with respect to this pivot space. By Proposition 13 the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, then it follows from Theorem 3 that $\mathbf{A}_0$ is a Fredholm operator with $\mathrm{Ind}(\mathbf{A}_0)=0$. Note that $W=\ker\mathbf{A}_0$. Set $(f,\phi)\in H^1(\Omega)\times H^{\frac{1}{2}}(\Gamma)$. Given $u\in H^1(\Omega)$, take $u_0=\mathcal{E}\phi$, where $\mathcal{E}\in\mathcal{B}\left(H^{\frac{1}{2}}(\Gamma),H^1(\Omega)\right)$ is an extension operator and set $y=u-u_0$. Hence $y\in H_0^1(\Omega)$ iff $\mathrm{tr}_\Gamma(u)=\phi$ and

$$\mathbf{A}_0 y=\mathbf{A}_1 y=\mathbf{A}_1 u-\mathbf{A}_1 u_0.$$

Writing $g=f-\mathbf{A}_1 u_0$, we have $g\in H^{-1}(\Omega)$ and u is a solution of $\mathfrak{D}_{(f,\phi)}$ iff y is a solution of $\mathfrak{D}_{(g,0)}$ iff $\mathbf{A}_0 y=g$. By Theorem 2, if $W=\{0\}$ then the equation $\mathbf{A}_0 y=g$ has a unique solution $y\in H_0^1(\Omega)$. Thus $u=y+u_0$ is the unique solution of $\mathfrak{D}_{(f,\phi)}$. In this case $\mathbf{A}_0^{-1}\in\mathcal{B}(H^{-1}(\Omega),H_0^1(\Omega))$, hence $\|y\|_{H_0^1(\Omega)}\leqslant\|\mathbf{A}_0^{-1}\|_{\mathcal{B}(H^{-1}(\Omega),H_0^1(\Omega))}\|g\|_{H^{-1}(\Omega)}$, from where we obtain (7.17) with

$$C=\max\left\{\|\mathbf{A}_0^{-1}\|_{\mathcal{B}(H^{-1}(\Omega),H_0^1(\Omega))},\|\mathbf{A}_1\|_{\mathcal{B}(H^1(\Omega),H^{-1}(\Omega))}\|\mathcal{E}\|_{\mathcal{B}\left(H^{\frac{1}{2}}(\Gamma),H^1(\Omega)\right)},\|\mathcal{E}\|_{\mathcal{B}\left(H^{\frac{1}{2}}(\Gamma),H^1(\Omega)\right)}\right\}.$$

Now suppose that $p=\dim W>0$ and write $W=\mathrm{Span}\{w_j\}_{j=0}^p$. Since $\mathbf{A}_0$ is self-adjoint, by the second part of Theorem 2, $\mathbf{A}_0 y=g$ iff $(g,w_j)_{H_0^1(\Omega)}=0$, for $j=\overline{1,p}$, it is

$$0=(g,w_j)_{H_0^1(\Omega)}=(f,w_j)_{H_0^1(\Omega)}-(\mathbf{A}_1 u_0,w_j)_{H_0^1(\Omega)},$$

hence

$$\begin{aligned}(f,w_j)_{H_0^1(\Omega)}&=(\mathbf{A}_1 u_0,w_j)_{H_0^1(\Omega)}\\&=(\mathbf{A}u_0,w_j)_{H^1(\Omega)}\\&=(u_0,\mathbf{A}w_j)_{\widetilde{H}^{-1}(\Omega)}\\&=\overline{(\mathbf{A}w_j,\mathcal{E}\phi)_{H^1(\Omega)}}\\&=\overline{(\partial_\nu w_j,\phi)_{H^{\frac{1}{2}}(\Gamma)}}\\&=(\phi,\partial_\nu w_j)_{H^{-\frac{1}{2}}(\Gamma)},\end{aligned}$$

in which in the third equality we use that $\mathbf{A}$ is self-adjoint and in the fifth the Remark 131(3). Finally, to obtain (7.17), since $\mathscr{R}(\mathbf{A}_0)$ is closed, by the first isomorphism theorem the operator $\widetilde{\mathbf{A}_0}:H_0^1(\Omega)/W\to\mathscr{R}(\mathbf{A}_0)$ given by $\widetilde{\mathbf{A}_0}(u+W)=\mathbf{A}_0 u$, has a bounded inverse [107, Cor. 2.2]. Hence there exists a constant $C_1>0$ such that $\|y+W\|_{H_0^1(\Omega)/W}\leqslant C\|g\|_{H^{-1}(\Omega)}$. Note that $\|y+W\|_{H^1(\Omega)/W}\leqslant\|y+W\|_{H_0^1(\Omega)/W}$ (because $\{w\in H_0^1(\Omega)\,|\,w-y\in W\}\subset\{w\in H^1(\Omega)\,|\,w-y\in W\}$). Applying the same procedure of the case 1 we obtain (7.19). **Q.E.D.**

Corollary 133. *Set* $V = \{\mathrm{tr}_\Gamma(u) \,|\, u \in Sol^{\mathbf{L}}_w(\Omega)\}$. *If* $W = \{0\}$, *then* $V = H^{\frac{1}{2}}(\Gamma)$. *On the other case, if* $W = \mathrm{Span}\{w_j\}_{j=1}^p$, *then*

$$\left\{\varphi \in H^{-\frac{1}{2}}(\Gamma) \,|\, (\varphi, \phi)_{H^{\frac{1}{2}}(\Gamma)} = 0 \;\forall \phi \in V\right\} = \mathrm{Span}\{\partial_\nu w_j\}_{j=1}^p. \tag{7.20}$$

Proof. First suppose that $W = \{0\}$. By Theorem 132(1), given $\phi \in H^{\frac{1}{2}}(\Gamma)$, there exists $u \in H^1(\Omega)$ that is a solution of $\mathfrak{D}_{(0,\phi)}$, but then $u \in \mathrm{Sol}^{\mathbf{L}}_w(\Omega)$ and $\phi = \mathrm{tr}_\Gamma(u) \in V$. Hence $V = H^{\frac{1}{2}}(\Gamma)$.

On the other hand, set $\widehat{V} = \left\{\varphi \in H^{-\frac{1}{2}}(\Gamma) \,|\, (\varphi, \phi)_{H^{\frac{1}{2}}(\Gamma)} = 0 \;\forall \phi \in V\right\}$. Given $\phi \in H^{\frac{1}{2}}(\Gamma)$, note that $\phi \in V$ iff there exists a solution $u \in H^1(\Omega)$ of $\mathfrak{D}_{(0,\phi)}$, and by condition (7.18), this happens iff $(\phi, \partial_\nu w_j)_{H^{-\frac{1}{2}}(\Gamma)} = 0$ for $j = \overline{1,p}$. Hence

$$V = \left\{\phi \in H^{\frac{1}{2}}(\Gamma) \,|\, (\partial_\nu w, \phi)_{H^{\frac{1}{2}}(\Gamma)} = 0, \;\forall w \in W\right\} = \left\{\phi \in H^{\frac{1}{2}}(\Gamma) \,|\, (\partial_\nu w)^*(\phi) = 0, \;\forall w \in W\right\}$$
$$= \left(\mathrm{Span}\{(\partial_\nu w_j)^*\}_{j=1}^p\right)_a.$$

Since $\mathrm{Span}\{(\partial_\nu w_j)^*\}_{j=1}^p$ is finite dimensional, by Lemma 1

$$V^a = \left(\left(\mathrm{Span}\{(\partial_\nu w_j)^*\}_{j=1}^p\right)_a\right)^a = \mathrm{Span}\{(\partial_\nu w_j)^*\}_{j=1}^p.$$

Finally, note that $\widehat{V} = (V^*)^a = (V^a)^*$ (by (1.1)). Thus,

$$\widehat{V} = \left(\mathrm{Span}\{(\partial_\nu w_j)^*\}_{j=1}^p\right)^* = \mathrm{Span}\{(\partial_\nu w_j)^*\}_{j=1}^p.$$

Q.E.D.

A similar result can be found for the Neumann problem.

Theorem 134 (Existence of solutions for a Neumann problem)**.** *Let us denote by* Z *the set of solutions of* $\mathfrak{N}_{(0,0)}$. *Then* Z *has finite dimension and we have two possibilities.*

1. *If* $Z = \{0\}$, *then for any* $(f, \varphi) \in \widetilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$, *the Neumann problem* $\mathfrak{D}_{(f,\varphi)}$ *has a unique solution* $u \in H^1(\Omega)$ *and there exists a constant* $C > 0$ *that does not depend on* (f, φ) *such that for all solution* u

$$\|u\|_{H^1(\Omega)} \leqslant C\left(\|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\varphi\|_{H^{\frac{1}{2}}(\Gamma)}\right). \tag{7.21}$$

2. *If* $q = \dim Z > 0$, $Z = Span\{z_j\}_{j=1}^q$, *the problem* $\mathfrak{N}_{(f,\varphi)}$ *has a solution* $u \in H^1(\Omega)$ *iff the Neumann data satisfies the condition*

$$(f, z_j)_{H^1(\Omega)} = -(\varphi, \mathrm{tr}_\Gamma(z_j))_{H^{\frac{1}{2}}(\Gamma)} = 0, \quad j = \overline{1, q}. \tag{7.22}$$

In such case, the problem has exactly q *linearly independent solutions and there exists a constant* $C > 0$ *such that for all solution* u

$$\|u + Z\|_{H^1(\Omega)/Z} \leqslant C \left(\|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\varphi\|_{H^{\frac{1}{2}}(\Gamma)} \right). \tag{7.23}$$

Proof. By Remark 128, Φ is coercive on $H^1(\Omega)$ with respect to the pivot space $L_2(\Omega)$. Since the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact (Proposition 13), the operator $\mathbf{A}$ is Fredholm with $\mathrm{Ind}(\mathbf{A}) = 0$. Note that $Z = \ker \mathbf{A}$ and given $(f, \varphi) \in \widetilde{H}(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$, the Neumann problem (7.16) is equivalent to the equation

$$\mathbf{A}u = g \quad \text{where } g \in \widetilde{H}^{-1}(\Omega) \text{ is given by } (g, v)_{H^1(\Omega)} := (f, v)_{H^1(\Omega)} + (\varphi, \mathrm{tr}_\Gamma(v))_{H^{-\frac{1}{2}}(\Gamma)}.$$

If $Z = \{0\}$, then $\mathbf{A}u = g$ has a unique solution. On the other case, the solution exists iff $(g, z_j)_{H^1(\Omega)} = 0$ for $j = \overline{1, q}$, from where we obtain (7.22). The proof of (7.21) and (7.23) is similar to that of Theorem 132. **Q.E.D.**

Let us introduce the Cauchy problem on a segment of Γ. Here we suppose that all the functions involved are real valued. By a Lipschitz portion of Γ we understand an open subset $\Sigma \subset \Gamma$. Following [4] we denote by $H_c^1(\Omega \cup \Sigma)$ the class of functions $u \in H^1(\Omega)$ having a compact support in $\Omega \cup \Sigma$, by $H_c^{\frac{1}{2}}(\Sigma) = \left\{ \mathrm{tr}_\Gamma(u)|_\Sigma \,\middle|\, u \in H_c^1(\Omega \cup \Sigma) \right\}$, and $H^{-\frac{1}{2}}(\Sigma) := \left(H_c^{\frac{1}{2}}(\Sigma) \right)^\star$. Given the Cauchy data $(\phi, \varphi, f) \in H_c^{\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\Sigma) \times \tilde{H}^{-1}(\Omega)$, the Cauchy problem

$$\begin{cases} \mathbf{L}u = f & \text{in } \Omega, \\ u = \phi & \text{on } \Sigma, \\ \partial_\nu u = \varphi & \text{on } \Sigma \end{cases} \tag{7.24}$$

consists in finding a function $u \in H^1(\Omega)$ satisfying $\mathrm{tr}_\Gamma(u)|_\Sigma = \phi$ and

$$(\mathbf{A}u, v)_{H^1(\Omega)} = (\varphi, \mathrm{tr}_\Gamma(v)|_\Sigma)_{H_c^{\frac{1}{2}}(\Sigma)} + (f, v)_{H^1(\Omega)} \quad \forall v \in H_c^1(\Omega \cup \Sigma). \tag{7.25}$$

In particular, we can consider a Cauchy data in $H_c^{\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\Gamma) \times \tilde{H}^{-1}(\Omega)$. For the Cauchy problem the following uniqueness result is known.

Theorem 135 ([4]). *If $\phi = 0$, $\varphi = 0$, $f = 0$, and the principal coefficient $A \in W^{1,\infty}\left(\Omega, \mathbb{R}^{d\times d}\right)$ satisfies conditions 1,2 and 3 of Section 7.1, then the unique solution of the Cauchy problem (7.24) is zero.*

7.3 A Runge property for Lipschitz domains

As in [102], the uniqueness of the Cauchy problem (UCP) will be key for proving the Runge property (RP). The following lemma was used in [102] and [32] without proof.

Lemma 136. *Let $\mathcal{H}$ be a Hilbert space, and $S_2 \subset S_1$ subspaces. Then $S_1 \subset \overline{S_2}^{\mathcal{H}}$ iff $S_2^{\perp} \subset S_1^{\perp}$.*

Proof. [⇒] Take $x \in S_2^{\perp}$. If $y \in S_1$, by hypothesis there exists a sequence $\{x_n\} \subset S_2$ such that $x_n \to y$, $n \to \infty$. The continuity of the inner product implies $\langle x, y\rangle = \lim_{n\to\infty}\langle x, x_n\rangle = 0$ and hence $x \in S_1^{\perp}$. Therefore $S_2^{\perp} \subset S_1^{\perp}$.

[⇐] Suppose that there exists some $y \in S_1 \setminus \overline{S_2}^{\mathcal{H}}$. It follows that there exists a functional $\varphi \in \mathcal{H}^{\star}$ such that $\varphi(S_2) = \{0\}$ and $\varphi(y) = 1$. By the Riesz representation theorem $\varphi(x) = \langle x, z\rangle$ for a unique $z \in \mathcal{H}$. Then $z \in S_2^{\perp}$ but $z \notin S_1^{\perp}$ that contradicts the hypothesis. Hence $S_1 \subset \overline{S_2}^{\mathcal{H}}$. **Q.E.D.**

Theorem 137 (Runge Property). *If $\Omega_1 \Subset \Omega_2$ are both bounded Lipschitz domains, and $\Omega_2 \setminus \overline{\Omega_1}$ is connected, then for all $u \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega_1)$ and $\epsilon > 0$, there exists $v \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega_2)$ such that*

$$\|u - v|_{\Omega_1}\|_{H^1(\Omega_1)} < \epsilon.$$

Figure 7.3 depicts a schematic representation of the domains.

Proof. Without loss of generality we suppose that all the solutions under consideration are real valued.

Set $S_1 = \mathrm{Sol}_w^{\mathbf{L}}(\Omega_1)$ and $S_2 = \left\{ v|_{\Omega_1} \,\middle|\, v \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega_2)\right\}$. Note that $S_j \subset H^1(\Omega_1)$, $j = 1, 2$. Then by Lemma 136, the conclusion of the RP is equivalent to show $S_2^{\perp_{H^1(\Omega_1)}} \subset S_1^{\perp_{H^1(\Omega_1)}}$. We use the notation $\Gamma_j = \partial\Omega_j$, $j = 1, 2$.

Let $f \in S_2^{\perp_{H^1(\Omega_1)}}$. In this case f defines a functional in $\tilde{H}^{-1}(\Omega_1)$ by the inner product $(f, v)_{H^1(\Omega_1)} := \int_{\Omega_1} f(x)v(x)dx + \sum_{j=1}^{d}\int_{\Omega_1} f_j(x)\frac{\partial v(x)}{\partial x_j}dx$, with $f_j = \frac{\partial f}{\partial x_j}$. This functional

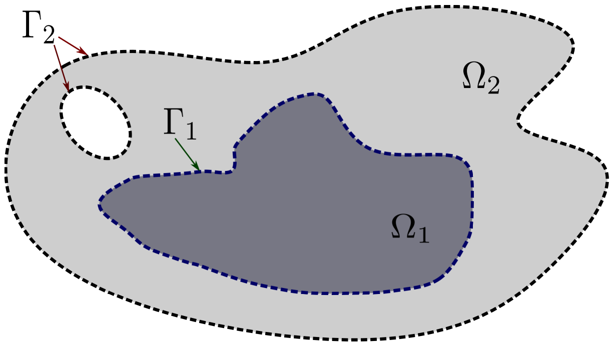

Figure 7.1: A schematic representation of the domains from the Runge property theorem.

can be extended onto $H^1(\Omega_2)$ as follows. Take $\tilde{f}_0 = f\chi_{\Omega_1}$ and $\tilde{f}_j = f_j\chi_{\Omega_1}$, for $j = \overline{1,d}$, (χ_{Ω_1} denote the characteristic function of Ω_1) and define $(F, v)_{H^1(\Omega_2)} := \int_{\Omega_2} \tilde{f}_0(x)v(x)dx + \sum_{j=1}^{d} \int_{\Omega_2} \tilde{f}_j(x)\frac{\partial v(x)}{\partial x_j}dx$. It is clear that $F \in \tilde{H}^{-1}(\Omega_2)$ and $(F, v)_{H^1(\Omega_1)} = (f, v|_{\Omega_1})_{H^1(\Omega_1)}$, for all $v \in H^1(\Omega_2)$.

Consider the Dirichlet problem $\mathfrak{D}_{(F,0)}$ in Ω_2. Set $W = \{w \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega_2) \,|\, \mathrm{tr}_{\Gamma_2}(w) = 0\}$. For any $w \in W$, by hypotheses we have $(F, w)_{H^1(\Omega_2)} = \langle f, w|_{\Omega_1}\rangle_{H^1(\Omega_1)} = 0$. Then the compatibility condition $(F, w)_{H^1(\Omega_2)} = (0, \partial_\nu w)_{H^{-\frac{1}{2}}(\Gamma_2)}$ is fulfilled. Due to the existence Theorem 132, the problem $\mathfrak{D}_{(F,0)}$ has a solution $u_0 \in H^1(\Omega_2)$.

If $v \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega_2)$, applying the co-normal derivative of u_0 to $\mathrm{tr}_{\Gamma_2}(v)$ we obtain

$$\begin{aligned}(\partial_\nu u_0, \mathrm{tr}_{\Gamma_2}(v))_{H^{\frac{1}{2}}(\Gamma_2)} &= (\mathbf{A}u_0, v)_{H^1(\Omega_2)} - (F, v)_{H^1(\Omega_2)} = (u_0, \mathbf{A}v)_{\widetilde{H}^{-1}(\Omega_2)} - \langle f, v|_{\Omega_1}\rangle_{H^1(\Omega_1)} \\ &= \overline{(\mathbf{A}v, u_0)_{H^1(\Omega_2)}} = 0,\end{aligned}$$

by definition of a weak solution. Hence $(\partial_\nu u_0, \mathrm{tr}_{\Gamma_2}(v))_{H^{\frac{1}{2}}(\Gamma_2)} = 0$ for all $v \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega_2)$. It follows from Corollary 133 that $\partial_\nu u_0 = \partial_\nu w_0$ for some $w_0 \in W$. Take $w_1 = u_0 - w_0$. Then $\mathrm{tr}_{\Gamma_2}(w_1) = 0$, $\partial_\nu w_1 = 0$.

Denote $D = \Omega_2 \setminus \overline{\Omega_1}$. Note that ∂D is the chain $\Gamma_2 - \Gamma_1$, then we can take the Lipschitz portion $\Sigma = \Gamma_2$. If $v \in H_c^1(D \cup \Sigma)$, consider the trivial extension $\tilde{v} = v\chi_{D\cup\Sigma}$.

Take $\varphi \in C_0^\infty(\Omega_2)$, and for $j = \overline{1,d}$ define $\tilde{v}_j = \frac{\partial v}{\partial x_j}\chi_{D\cup\Sigma}$. Then

$$\begin{aligned}\int_{\Omega_2}\left(\tilde{v}_j(x)\varphi(x) + \tilde{v}(x)\frac{\partial\varphi(x)}{\partial x_j}\right) &= \int_D\left(\frac{\partial v(x)}{\partial x_j}\varphi(x) + v(x)\frac{\partial\varphi(x)}{\partial x_j}\right)\\ &= \int_{\partial D}\mathrm{tr}_{\partial D}(v)(x)\mathrm{tr}_{\partial D}(\varphi)(x)\nu_j(x)dS\\ &= \int_{\Gamma_2}\mathrm{tr}_{\partial D}(v)(x)\mathrm{tr}_{\Gamma_2}(\varphi)(x)\nu_j dS\\ &\quad - \int_{\Gamma_1}\mathrm{tr}_{\partial D}(v)(x)\varphi|_{\Gamma_1}(x)\nu_j(x)dS = 0,\end{aligned}$$

because $\mathrm{tr}_{\Gamma_2}(\varphi) = 0$ and $\Gamma_1 \cap \mathrm{Supp}(v) = \emptyset$. Thus, $\frac{\partial\tilde{v}}{\partial x_j} = \tilde{v}_j \in L_2(\Omega_2)$ for $j = \overline{1,d}$, and then $\tilde{v} \in H^1(\Omega_2)$. Applying the definition of the co-normal derivative to w_1 we obtain

$$\begin{aligned}(\mathbf{A}w_1, v)_{H^1(D)} &= \Phi(w_1, v) = \Phi(w_1, \tilde{v}) = (\mathbf{A}w_1, \tilde{v})_{H^1(\Omega_2)}\\ &= (\partial_\nu w_1, \tilde{v})_{H^{\frac{1}{2}}(\Gamma_2)} + (F, \tilde{v})_{H^1(\Omega_2)} = (f, 0)_{H^1(\Omega_1)} = 0\end{aligned}$$

Comparing with (7.25), we conclude that w_1 is a solution to the Cauchy problem (7.24) in D with zero data. Due to Theorem 135, $w_1 = 0$ a.e. in D. It follows that w_1 satisfies

$$\begin{cases}\mathbf{L}w_1 = f & \text{in } \Omega_1,\\ \mathrm{tr}_{\Gamma_1}(w_1) = 0, & \\ \partial_\nu w_1\big|_{\Gamma_1} = 0. & \end{cases}$$

Now, if $u \in S_1$, we have

$$\begin{aligned}\langle u, f\rangle_{H^1(\Omega_1)} &= (f, u)_{H^1(\Omega_1)} = (\mathbf{A}w_1, u) - (\partial_\nu w_1, \mathrm{tr}_{\Gamma_1}(u))_{H^{\frac{1}{2}}(\Gamma_1)} = (w_1, \mathbf{A}u)_{\widetilde{H}^{-1}(\Omega_1)}\\ &= \overline{(\mathbf{A}u, w_1)_{H^1(\Omega)}} = 0,\end{aligned}$$

because $w_1 \in H_0^1(\Omega_1)$ and $u \in S_1$. $\therefore$ $f \in S_1^{\perp_{H^1(\Omega_1)}}$.

Q.E.D.

Since the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is continuous, we obtain as a corollary the RP for the L_2-norm.

Corollary 138. *Under the conditions of Theorem 137 for all $u \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega_1)$ and $\epsilon > 0$ there exists $v \in \mathrm{Sol}_w^{\mathbf{L}}(\Omega_2)$ such that*

$$\|u - v|_{\Omega_2}\|_{L_2(\Omega_1)} < \epsilon.$$

Remark 139. *The standard statement of the RP property corresponds to the situation when* $\Omega_1 \Subset \Omega_2$ *are both simply connected bounded domains. Lax in [102] and Colton in [32] in their proof used the fact that the simple connectivity implies that* $\Omega_2 \setminus \overline{\Omega_1}$ *is connected. In both cases the simple connectivity is considered in a stronger sense assuming that* $\mathbb{R}^d \setminus \overline{\Omega_i}$, $i = 1, 2$ *are connected.*

Under this condition, we obtain that the conclusions of Theorem 137 and Corollary 138 hold if $\Omega_1 \Subset \Omega_2$ *are simply connected.*

Remark 140. *The need to consider real coefficients appears only in relation with the UCP [4]. If the UCP were valid for complex valued potentials, the proof of Theorem 137 can be easily modified for this kind of potentials, but in this case, the equation to which UCP will apply involves the adjoint operator* $\mathbf{A}^\star$.

7.4 Approximation by complete systems of solutions

Next step is to apply Theorem 137 for studying approximation of weak solutions by complete systems of classical solutions. The model example of such system is the system of harmonic polynomials. If $K \subset \Omega$ is compact and the complement $\mathbb{R}^d \setminus K$ is connected, then any function which is harmonic in Ω can be approximated uniformly on K by harmonic polynomials (see [117]). If $d = 2$ and Ω is simply connected, the property is valid on any compact subset [121, Ch. 10]. Systems of transmuted harmonic polynomials were considered for the radial Schrödinger operator (which will be seen in the next chapter) and for the Schrödinger operator with a separable potential [24] with similar results obtained.

Now we propose two different definitions of complete systems of solutions. The first generalizes the completeness of harmonic polynomials. In order to consider classical solutions, from now on we suppose that $A \in C^1(\Omega, \mathbb{R}^{d\times d})$ and $q \in C(\Omega)$.

Definition 141 (Complete system of solutions). *A family of solutions* $\{U_n\}_{n=0}^{\infty} \subset \mathrm{Sol}^{\mathbf{L}}(\Omega)$ *is said to be a* ***complete system of solutions*** *for* $\mathrm{Sol}^{\mathbf{L}}(\Omega)$, *if for any* $u \in \mathrm{Sol}^{\mathbf{L}}(\Omega)$, $\epsilon > 0$ *and* $K \subset \Omega$ *compact with* $\mathbb{R}^d \setminus K$ *being connected, there exists* $S \in \mathit{Span}\,\{U_n\}_{n=0}^{\infty}$ *satisfying*

$$\max_{x\in K} |u(x) - S(x)| < \epsilon. \tag{7.26}$$

The second definition is proposed in order to generalize results for the Schrödinger equation on the plane obtained in [24].

Definition 142 (Strongly complete system of solutions)**.** *Let Ω be a bounded simply connected domain. A family of solutions $\{U_n\}_{n=0}^{\infty} \subset \mathrm{Sol}^{\mathbf{L}}(\Omega)$ is said to be a* ***strongly complete system of solutions*** *for $\mathrm{Sol}^{\mathbf{L}}(\Omega)$, if for any $u \in \mathrm{Sol}^{\mathbf{L}}(\Omega)$, $\epsilon > 0$ and $K \subset \Omega$ compact, there exists $S \in \mathrm{Span}\,\{U_n\}_{n=0}^{\infty}$ satisfying*

$$\max_{x\in K} |u(x) - S(x)| < \epsilon. \tag{7.27}$$

Assuming some additional smoothness of the coefficients, we obtain the following result concerning approximation of weak solutions in the L_2-norm.

Theorem 143. *Suppose that Ω is a Lipschitz domain , and $A \in C^{p+1}(\Omega, \mathbb{R}^{d\times d})$, $q \in C^{p+1}(\Omega)$, with $p = 1 + \left[\frac{d}{2}\right]$. Let $\{U_n\}_{n=0}^{\infty}$ be a complete system of solutions. If $\omega \Subset \Omega$ is a Lipschitz domain such that $\mathbb{R}^d \setminus \overline{\omega}$ and $\Omega \setminus \overline{\omega}$ are connected, then for any $u \in \mathrm{Sol}^{\mathbf{L}}_w(\omega)$ and $\epsilon > 0$, there exists $S \in \mathrm{Span}\,\{U_n\}_{n=0}^{\infty}$ such that*

$$\|u - S|_{\omega}\|_{L_2(\omega)} < \epsilon$$

Proof. Let $u \in \mathrm{Sol}^{\mathbf{L}}_w(\omega)$ and $\epsilon > 0$. By Corollary 138 there exists a solution $v \in \mathrm{Sol}^{\mathbf{L}}_w(\Omega)$ satisfying the inequality $\|u - v|_{\omega}\|_{L_2(\omega)} < \frac{\epsilon}{2}$.

On the other hand, since $A \in C^{p+1}(\Omega, \mathbb{R}^{d\times d})$ and $q \in C^{p+1}(\Omega)$, we have $v \in H^{p+2}_{loc}(\Omega)$ (see [101], pp. 314, Th. 2). Since $p = 1 + \left[\frac{d}{2}\right]$, and $H^{3+\left[\frac{d}{2}\right]}_{loc}(\Omega) \subset C^2(\Omega)$ (Theorem 15(1)), we have $v \in \mathrm{Sol}^{\mathbf{L}}(\Omega)$.

Since $\overline{\omega}$ is a compact subset with a connected complement, by Definition 141 there exists a linear combination $S \in \mathrm{Span}\,\{U_n\}_{n=0}^{\infty}$ such that $\max_{x\in\overline{\omega}} |v(x) - S(x)| < \dfrac{\epsilon}{2\sqrt{\mathrm{Vol}(\omega)}}$. Hence

$$\begin{aligned}\|u - S|_{\omega}\|_{L_2(\omega)} &\leqslant \|u - v|_{\omega}\|_{L_2(\omega)} + \|(v - S)|_{\omega}\|_{L_2(\omega)} < \frac{\epsilon}{2} + \left(\int_{\omega} |v(x) - S(x)|^2 dx\right)^{\frac{1}{2}} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2\sqrt{\mathrm{Vol}(\omega)}} \left(\int_{\omega} dx\right)^{\frac{1}{2}} = \epsilon\end{aligned}$$

Q.E.D.

Remark 144. *The statement of Theorem 143 is valid when $\omega \Subset \Omega$ are both simply connected and $\{U_n\}_{n=0}^{\infty}$ is a strongly complete system.*

In order to obtain the completeness in the H^1-norm we need an additional assumption on Ω. Given $\varepsilon > 0$, the ε-neighbourhood of Ω is the set $\Omega_\varepsilon := \left\{x \in \mathbb{R}^d \,\middle|\, \operatorname{dist}(x, \Omega) < \varepsilon\right\}$. A Lipschitz simply connected domain Ω is said to be *extendable* if there exists $\rho > 0$ sufficiently small such that Ω_ε is a Lipschitz simply connected domain, for all $0 < \varepsilon < \rho$. Examples of this kind of domains are balls, rectangles, and in general C^k domains with a normal pointing outside (one can obtain this applying the ε-neighbourhood theorem to the boundary $\Gamma = \partial\Omega$, for more details see [63, pp. 71] and [130, pp. 112]).

Now we prove the approximation theorem with respect to the H^1-norm.

Theorem 145. *Let Ω be a bounded Lipschitz simply connected domain. Suppose that $A \in C^{p+1}(\Omega, \mathbb{R}^{d\times d})$, $q \in C^{p+1}(\Omega)$, with $p = 1 + \left[\frac{d}{2}\right]$. Let $\{U_n\}_{n=0}^\infty \subset \operatorname{Sol}^{\mathbf{L}}(\Omega)$ be a strongly complete system of solutions. If $\omega \Subset \Omega$ is an extendable Lipschitz simply connected domain, then for any $u \in \operatorname{Sol}_w^{\mathbf{S}}(\omega)$ and $\epsilon > 0$ there exists $S \in \operatorname{Span}\{U_n\}_{n=0}^\infty$ such that*

$$\|u - S|_\omega\|_{H^1(\omega)} < \epsilon.$$

Proof. Since ω is extendable, we can take some $0 < \rho < \operatorname{dist}(\omega, \partial\Omega)$, such that ω_ρ be a simply connected Lipschitz domain. We have $\omega \Subset \omega_\rho \Subset \Omega$.

First we apply Theorem 137 to the pair of the domains $\omega \Subset \omega_\rho$ and obtain a solution $u_1 \in \operatorname{Sol}^{\mathbf{L}}(\omega_\rho)$ such that $\|u - u_1|_\omega\|_{H^1(\omega)} < \frac{\epsilon}{2}$.

Additionally, by Proposition 127(2), there exists a constant $C_1 = C_1(A, q, \omega_\rho, \omega)$ such that $\|v|_\omega\|_{H^2(\omega)} \leqslant C_1\|v\|_{L_2(\omega_\rho)}$ for all $v \in \operatorname{Sol}_w^{\mathbf{L}}(\omega_\rho)$.

Now we apply Theorem 143 to the pair of domains $\omega_\rho \Subset \Omega$, and we have a solution $S \in \operatorname{Span}\{U_n\}_{n=0}^\infty$ satisfying $\|u_1 - S|_{\omega_\rho}\|_{L_2(\omega_\rho)} < \frac{\epsilon}{2C_1}$ (a schematical representation of the domains and solutions is depicted on Figure 7.4).

Set $f = u_1 - S|_{\omega_\rho}$ and note that $f \in \operatorname{Sol}_w^{\mathbf{L}}(\omega_\rho)$, so then $\|f|_\omega\|_{H^2(\omega)} \leqslant C_1\|f\|_{L_2(\omega_\rho)}$. Thus,

$$\|u - S|_\omega\|_{H^1(\omega)} \leqslant \|u - u_0|_\omega\|_{H^1(\omega)} + \|f|_\omega\|_{H^1(\omega)} \leqslant \frac{\epsilon}{2} + C_1\|f\|_{L_2(\omega_\rho)} < \epsilon.$$

Q.E.D.

For operators with smooth coefficients there exists a result that establishes the completeness of classical solutions in the H^2-norm.

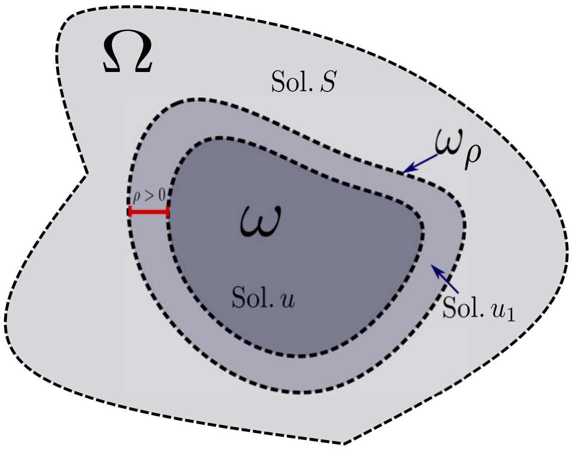

Figure 7.2: Schematical representation of the domains and solutions from the proof of Theorem 145.

Theorem 146. *Let D be a bounded domain, and suppose that there exists a neighborhood U of D, $D \Subset U$, such that $A \in C^\infty(U, \mathbb{R}^{d\times d})$, $q \in C^\infty(U)$. Then for all $u \in \mathrm{Sol}_w^{\mathbf{L}}(D) \cap H^2(D)$ and $\epsilon > 0$, there exists a neighborhood V of D, $D \Subset V \Subset U$, and $u_0 \in \mathrm{Sol}^{\mathbf{L}}(V)$ such that*

$$\|u - u_0|_D\|_{H^2(D)} < \epsilon.$$

This result is a modification given in [70] of a general theorem for elliptic systems due to Tarkhanov [134, Th. 8.1.3]. The following theorem establishes the completeness in the H^2-norm.

Theorem 147. *Under the hypotheses of Theorem 145, if ω is of the class C^2, and there exists a ρ-neighborhood of ω such that $A \in C^\infty(\omega_\rho, \mathbb{R}^{d\times d})$, $q \in C^\infty(\omega_\rho)$, then for all $u \in \mathrm{Sol}_w^{\mathbf{L}}(\omega)$ and $\epsilon > 0$, there exists $S \in \mathrm{Span}\{U_n\}_{n=0}^\infty$ such that*

$$\|u - S|_\omega\|_{H^2(\omega)} < \epsilon.$$

Proof. Since $\partial\omega$ is of the class C^2, by Proposition 127 (4) $u \in H^2(\omega)$, and by Theorem 146 we can take $\varepsilon < \rho$ and $u_0 \in \mathrm{Sol}^{\mathbf{S}}(\omega_\varepsilon)$ such that $\|u - u_0|_\omega\|_{H^2(\omega)} < \frac{\epsilon}{2}$.

Since ω is simply connected with a C^2-boundary, so is its neighborhood $\omega_{\frac{\varepsilon}{2}}$. We have $\omega \Subset \omega_{\frac{\varepsilon}{2}} \Subset \Omega$.

By Proposition 4 (3), there exists a constant $C_2 = C_2(A, q, \omega_{\frac{\varepsilon}{2}}, \omega)$ such that $\|v|_\omega\|_{H^2(\omega)} \leqslant C_2\|v\|_{L_2(\omega_{\frac{\varepsilon}{2}})}$ for all $v \in \mathrm{Sol}^{\mathbf{L}}(\omega_{\frac{\varepsilon}{2}})$.

Since $u_0|_{\omega_{\frac{\varepsilon}{2}}} \in \mathrm{Sol}_w^{\mathbf{L}}(\omega_{\frac{\varepsilon}{2}})$, due to Theorem 143 there exits $S \in \mathrm{Span}\{U_n\}_{n=0}^{\infty}$ satisfying $\left\| (u_0 - S)\,|_{\omega_{\frac{\varepsilon}{2}}} \right\|_{L_2(\omega_{\frac{\varepsilon}{2}})} < \frac{\epsilon}{2C_2}$ (a schematical representation of the domains and solutions is depicted on Figure 7.4).

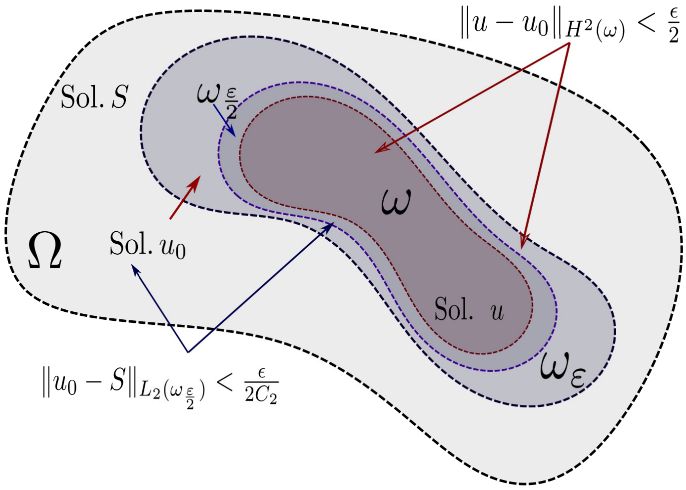

Figure 7.3: Schematical representation of the domains and solutions from the proof of Theorem 147.

Setting $f = (u_0 - S)\,|_{\omega_{\frac{\varepsilon}{2}}}$ we have $f \in \mathrm{Sol}^{\mathbf{L}}(\omega_{\frac{\varepsilon}{2}})$ and $\|f|_\omega\|_{H^2(\omega)} \leqslant C_2 \|f\|_{L_2(\omega_{\frac{\varepsilon}{2}})}$. Thus,

$$\|u - S|_\omega\|_{H^2(\omega)} \leqslant \|u - u_0|_\omega\|_{H^2(\omega)} + \|f|_\omega\|_{H^2(\omega)} \leqslant \frac{\epsilon}{2} + C_2 \|f\|_{L_2(\omega_{\frac{\varepsilon}{2}})} < \epsilon.$$

Q.E.D.

In the case of a normed space X, a system $\{U_n\}_{n=0}^{\infty}$ is called complete if $\overline{\mathrm{Span}\{U_n\}_{n=0}^{\infty}}^{X} = X$. Then Theorems 143, 145 and 147 can be summed up to establish the fact that every strongly complete system of solutions is also complete in $\mathrm{Sol}_w^{\mathbf{L}}(\omega)$ with respect to the L_2, H^1 and H^2-norms. As a corollary we obtain the completeness of boundary values of the complete system in the space of traces.

Corollary 148. *Under the conditions of Theorem 145 let* $\Gamma = \partial\omega$ *and zero be not an eigenvalue of the Dirichlet problem in* ω*. Then the set*

$$\widetilde{\mathcal{S}} = \left\{ U_n\big|_\Gamma \right\}_{n=0}^{\infty}$$

is a complete system in $H^{\frac{1}{2}}(\Gamma)$*.*

Proof. Let $\phi \in H^{\frac{1}{2}}(\Gamma)$. Then there exists a unique solution of the Dirichlet problem $u \in \mathrm{Sol}_w^{\mathbf{S}}(\omega)$ such that $\mathrm{tr}_\Gamma(u) = \phi$ (Theorem 132(1))

Due to Theorem 145, given $\epsilon_1 > 0$ there exists $S \in \mathrm{Span}\{U_n\}_{n=0}^{\infty}$ such that $\|u - S|_{\overline{\omega}}\|_{H^1(\omega)} < \epsilon_1$. Since $S|_{\overline{\omega}} \in H^1(\omega) \cap C(\overline{\omega})$ we have $\mathrm{tr}_\Gamma\left(S|_{\overline{\omega}}\right) = S\big|_\Gamma \in \mathrm{Span}(\widetilde{\mathcal{S}})$. Due to the boundedness of the trace operator we have

$$\left\|\phi - S\big|_\Gamma\right\|_{H^{\frac{1}{2}}(\Gamma)} = \left\|\mathrm{tr}_\Gamma\left(u - S|_{\overline{\omega}}\right)\right\|_{H^{\frac{1}{2}}(\Gamma)} \leqslant \|\mathrm{tr}_\Gamma\| \, \|u - S|_{\overline{\omega}}\|_{H^1(\omega)} < \|\mathrm{tr}_\Gamma\|\epsilon_1.$$

Choosing $\epsilon_1 = \frac{\epsilon}{\|\mathrm{tr}_\Gamma\|}$ finishes the proof. **Q.E.D.**

Remark 149. *(i) If zero is not an eigenvalue of the Dirichlet problem, by (7.17) we have that for any data* $\phi \in H^{\frac{1}{2}}(\Gamma)$ *there exists a solution* $u \in \mathrm{Sol}_w^{\mathbf{L}}(\omega)$ *and a constant* $C_3 > 0$ *such that* $\|u\|_{H^1(\omega)} \leqslant C_3\|\phi\|_{H^{\frac{1}{2}}(\Gamma)}$ *(see Theorem 132(2)). Due to this property, in order to approximate a solution of the Dirichlet problem it is sufficient to take a linear combination* S *of* $\widetilde{\mathcal{S}}$ *such that* $\phi \approx S\big|_\Gamma$*. Then* $u \approx S_N$ *in* $H^1(\omega)$*.*

(ii) Corollary 148 can be applied to Bergman spaces of solutions on the boundary Γ *(see [14]). Moreover, for such spaces there are procedures allowing one to obtain an orthonormal basis and the Bergman kernel derived from* $\widetilde{\mathcal{S}}$ *(see [21]).*

Remark 150. *Consider a complete system of solutions* $\{U_n\}_{n=0}^{\infty}$ *in the Bergman space* $\mathrm{Sol}_2^{\mathbf{L}}(\Omega)$ *(assuming that the Bergman space is complete). Modifying the proof of Theorem 143, since* $v \in H^1(\Omega) \cap C^2(\Omega)$*, then* $v \in \mathrm{Sol}_2^{\mathbf{L}}(\Omega)$*, and we can choose* $S \in \mathrm{Span}\{U_n\}_{n=0}^{\infty}$ *such that* $\|u - S\|_{L_2(\Omega)} < \frac{\epsilon}{2}$*, and the proof continues in the same way. Then a complete system in the Bergman space is complete as well in* $\mathrm{Sol}_2^{\mathbf{L}}(\omega)$ *in the* L_2*-norm. The completeness in the* H^k*-norm,* $k = 1, 2$ *is obtained directly from the completeness in the* L_2*-norm.*

7.5 Applications to the Schrödinger equation on the plane

Consider the case $d = 2$ and let $\Omega = (-a_1, a_1) \times (-a_2, a_2)$, with $a_j > 0$, $j = 1, 2$. Suppose that the potential $q \in C(\overline{\Omega})$ has the form $q(x, y) = q_1(x) + q_2(y)$, with $q_j \in C[-a_j, a_j]$

for $j = 1, 2$. We are interested in finding a transmutation operator that relates the Schrödinger operator $\mathbf{S} = -\triangle_2 + q_1(x) + q_2(y)$ with the Laplacian $\triangle_2$ in $C^2(\Omega)$.

Fix $j \in \{1, 2\}$ and let $f_j \in C^1[-a_j, a_j] \cap C^2(-a_j, a_j)$ be a solution of $-f_j'' + q_j(x)f = 0$, $x \in [-a_j, a_j]$, that does not vanish in $[-a_j, a_j]$ and satisfying the normalizing condition $f_j(0) = 1$. Consider the canonical transmutation operator

$$\mathbf{T}_j u(x) = u(x) + \int_{-x}^{x} K_j(x,t)u(t)dt \tag{7.28}$$

satisfying the relation

$$\left(\frac{\partial^2}{\partial x^2} - q_j(x)\right)\mathbf{T}_j u(x) = \mathbf{T}_j\left(\frac{\partial^2}{\partial t^2}u(t)\right), \qquad \text{for } u \in C^2[-a_j, a_j]. \tag{7.29}$$

Take $u \in C(\Omega)$. The operators $\mathbf{T}_j$ act on u as follows

$$\mathbf{T}_1 u(x,y) = u(x,y) + \int_{-x}^{x} K_1(x,t)u(t,y)dt,\ \mathbf{T}_2 u(x,y) = u(x,y) + \int_{-y}^{y} K_1(y,\tau)u(x,\tau)d\tau.$$

Is not difficult to see that $\mathbf{T}_1$ commutes with $\mathbf{T}_2$ in $C(\Omega)$. Let us define $\mathcal{T} := \mathbf{T}_1\mathbf{T}_2$. The operator $\mathcal{T}$ is well defined in Ω, and from relation (7.29) it follows that

$$\mathbf{S}\mathcal{T}u = \mathcal{T}\triangle_2 u, \qquad \text{for } u \in C^2(\Omega).$$

Since $\mathcal{T} : C(\overline{\Omega}) \to C(\overline{\Omega})$ is bounded (by boundedness of $\mathbf{T}_j$), $\mathcal{T}$ is continuous in the Fréchet space $C(\Omega)$. Furthermore, $\mathcal{T}$ is invertible with the inverse $\mathcal{T}^{-1} = \mathbf{T}_2^{-1}\mathbf{T}_1^{-1}$ being continuous as well. Thus, $\mathcal{T}$ is a transmutation operator for $\mathbf{S} = \triangle_2 - q_1(x) - q_2(y)$ and $\triangle_2$. By Remark 23, $\mathcal{T}(\mathrm{Har}(\Omega)) \subset \mathrm{Sol}^{\mathbf{S}}(\Omega)$, but in this case $\mathbf{T}_j\left(C^2[-a_j, a_j]\right) = C^2[-a_j, a_j]$, $j = 1, 2$ (Theorem 25) hence we have $\mathcal{T}\left(C^2(\Omega)\right) = C^2(\Omega)$. Thus, by Remark 24, $\mathcal{T}(\mathrm{Har}(\Omega)) = \mathrm{Sol}^{\mathbf{S}}(\Omega)$. Since $\mathbf{T}$ is an homeomorphism, $\mathrm{Sol}^{\mathbf{S}}(\Omega)$ is closed in $C(\Omega)$, and the corresponding Bergman space $\mathrm{Sol}_2^{\mathbf{S}}(\Omega)$ is complete.

Let us denote the associated formal powers to f_j by $\left\{\varphi_j^{(k)}\right\}_{k=0}^{\infty}$. With the aid of the formal powers the following family of functions is defined, $U_0(x,y) = f_1(x)f_2(y)$ and for $m > 0$

$$U_m(x,y) = \begin{cases} \displaystyle\sum_{\text{even } k=0}^{\frac{m+1}{2}} (-1)^{\frac{k}{2}} \binom{\frac{m+1}{2}}{k} \varphi_1^{\left(\frac{m+1}{2}-k\right)}(x)\varphi_2^{(k)}(y), & \text{if } m \text{ is odd,} \\ \displaystyle\sum_{\text{odd } k=0}^{\frac{m}{2}} (-1)^{\frac{k+1}{2}} \binom{\frac{m}{2}}{k} \varphi_1^{\left(\frac{m}{2}-k\right)}(x)\varphi_2^{(k)}(y), & \text{if } m \text{ is even.} \end{cases} \tag{7.30}$$

When $q_j \equiv 0$ and $f_j \equiv 1$, (7.30) reduces to the family of classical harmonic polynomials in 2 dimensions: $p_m(z) = \mathrm{Re}(z^m)$ if m is odd, $p_m(z) = \mathrm{Re}(iz^m)$ if m is even.

Theorem 151 ([24])**.** *For all* $m \in \mathbb{N}_0$

$$U_m(x, y) = \mathcal{T}[p_m(x, y)]. \tag{7.31}$$

Since the family of the harmonic polynomials represents a strongly complete system of solutions in $\mathrm{Sol}^{\triangle_2}(\Omega)$ (Theorem 18), the continuity of $\mathcal{T}$ and of its inverse implies the completeness of the formal powers $\{U_m\}_{m=0}^{\infty}$ [88, Th. 26]. Thus, in terms of Definition 142 we have the following statement.

Theorem 152 ([24])**.** *The family of the formal powers* $\{U_m\}_{m=0}^{\infty}$ *represents a strongly complete system of solutions in* $\mathrm{Sol}^{\mathbf{S}}(\Omega)$*.*

As an immediate corollary we obtain the completeness of the family of the formal powers in the space of weak solutions $\mathrm{Sol}_w^{\mathbf{S}}(\omega)$ considered in any simply connected Lipschitz domain $\omega \Subset \Omega$.

Theorem 153. *If* $q_j \in C^2(-a_j, a_j)$*,* $j = 1, 2$*, then in any simply connected Lipschitz domain* $\omega \Subset \Omega$ *the family of the formal powers* $\{U_n\}_{n=0}^{\infty}$ *represents a complete system in* $\mathrm{Sol}_w^{\mathbf{S}}(\omega)$ *with respect to the* L_2*-norm. If* ω *is extendable then the completeness holds with respect to the* H^1*-norm.*

In the case when ω *is of class* C^2*, and there exists a* ρ*-neighborhood of* ω *where* $q \in C^\infty(\omega_\rho)$*, the formal powers are complete with respect to the* H^2*-norm as well.*

Remark 154. *The construction of the formal powers (7.30) can be generalized to a general Schrödinger operator* $\mathbf{S} = -\triangle_2 + q(x, y)$ *in a bounded simply connected domain, if it admits a solution* $f \in C^2(\Omega)$ *that does not vanish in the whole* Ω*. Such construction is based on the theory of pseudoanalytic functions [75] and bicomplex-valued pseudoanalytic functions [22, 24]. Similar constructions can be applied to the operator* $\mathbf{L} = -div\,(p(x)\nabla) + q(x)$ *(see [75, Chs. III and IV]).*

Remark 155. *For higher dimensions, if the potential has the form* $q(x) = \sum_{j=1}^{d} q_j(x_j)$*, where* $q_j \in C[-a_j, a_j]$*, for* $j = \overline{1, d}$*, then using the same procedure one can construct* *a transmutation operator* $\mathcal{T} : C(\Omega) \to C(\Omega)$ *in the rectangle* $\Omega = \prod_{j=1}^{d}(-a_j, a_j)$, *as the product* $\mathcal{T} = \prod_{j=1}^{d} \mathbf{T}_j$ *of the corresponding transmutation operators of* $-\frac{\partial^2}{\partial x_j^2} + q_j(x_j)$. *Then* $\mathrm{Sol}^{\mathbf{S}}(\Omega)$ *is closed in* $C(\Omega)$ *and the Bergman space* $\mathrm{Sol}_2^{\mathbf{S}}(\Omega)$ *is complete. The construction of a complete system is reduced to finding an explicit form for the harmonic polynomials in* Ω.

Chapter 8

Transmutation operators and complete system of solutions for the radial Schrödinger equation

This chapter is dedicated to the construction of a transmutation operator for the radial Schrödinger equation in a star-shaped domain Ω, as well as its application to construct a complete system of solutions. Several new properties of the transmutation operator are established, including their continuity on the Fréchet space $C(\Omega)$ and its boundedness on the Bergman space. A Fourier-Jacobi series expansion of the integral transmutation kernel is derived and with its aid an infinite system of solutions of the radial Schrödinger equation is obtained which is shown to be complete with respect to the uniform norm. Explicit construction of the system is derived. In the case of Ω being an open ball centered at the origin the system of solutions represents an orthogonal basis of the corresponding Bergman space.

8.1 Some facts concerning to the radial Schrödinger equation

In this chapter, we study the radial Schrödinger equation

$$(\triangle_d - q(|x|))\, u(x) = 0, \tag{8.1}$$

where q is a C^1-function that depends on the radial component of x. The equation is considered in a bounded domain $\Omega \subset \mathbb{R}^d$, star-shaped with respect to the origin. In [14], S. Bergman showed for the case $d = 2$ and the potential q being an analytic function of the radial component $r = |x|$, that any solution u of (8.1) can be written in the form

$$u(x) = H(x) + \int_0^1 \sigma G(r, 1-\sigma^2) H(\sigma^2 x) d\sigma$$

where $H(x) = \int_{-1}^{1} h\left(\frac{x}{2}[1-t^2]\right) \frac{dt}{(1-t^2)^{\frac{1}{2}}}$ is the *Bergman transform* of a harmonic function h, and G is an analytic function of r. In [56], R. Gilbert showed that for any solution u there exists a unique harmonic function h such that

$$u(x) = \mathbf{T}h(x) = h(x) + \int_o^1 \sigma^{d-1} G(r, 1-\sigma^2) h(\sigma^2 x) d\sigma. \tag{8.2}$$

The representation (8.2) can be generalized for $d \geqslant 3$ and C^1-potentials (see [57, 58, 59] and [11, Ch. V]). The kernel G satisfies some initial value problem for a hyperbolic PDE (see [57]). When the potential q is analytic, the kernel G is an analytic function on the radial component (see [14, 56, 57]). In [138], I. Vekua constructed the operator (8.2) explicitly for the Helmholtz equation and showed its invertibility. In a general context, the invertibility of the operator (8.2) was shown in [57, 59]. The operator (8.2) is usually called a *transformation* or *transmutation operator* ([11]). It was applied, for example, to solving the Dirichlet problem on an admissible domain (see [32, 58, 59]), as well as to studying properties of generalized sub-harmonic functions [59].

8.2 Transmutation operators

For Eq. (8.1), it is required to specify a certain type of geometry in the domains.

Definition 156. *A bounded domain $\Omega \subset \mathbb{R}^d$ is said to be* ***star-shaped with respect to the origin*** *if $0 \in \Omega$ and for any $x \in \Omega$ the segment $[0, x] := \{tx \,|\, 0 \leqslant t \leqslant 1\}$ belongs to Ω. In this case denote $b := \sup_{x \in \Omega} |x|$.*

An important special case is an open ball $B_R^d(0)$. Note that Ω is contained in $B_b^d(0)$, which is the minimum ball centered at the origin containing Ω (see figure 8.2).

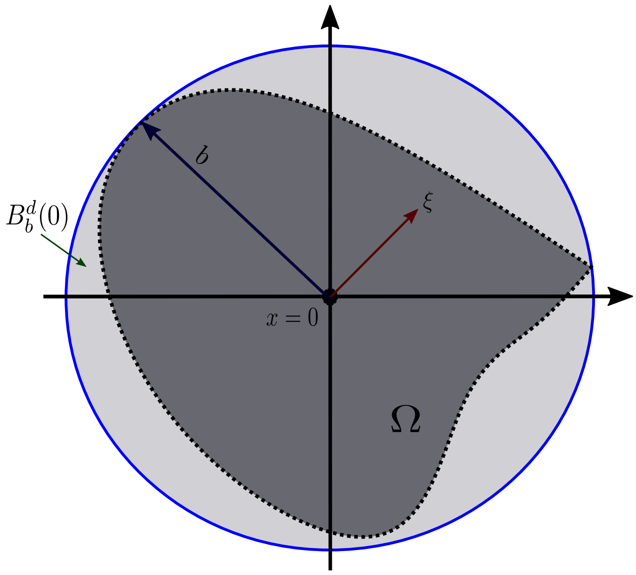

Figure 8.1: A star-shaped domain and the minimum ball containing it.

We assume that $q \in C^1[0, b]$ and consider the radial Schrödinger equation (8.1) in Ω (RS, for short). In spherical coordinates (8.1) can be written in the form

$$(\triangle_d - q(r))\, u(r, x') = 0 \quad \text{for} \quad (r, x') \in (0, b) \times \mathbb{S}^{d-1}, \tag{8.3}$$

where $r := |x|$ and $x' := \frac{x}{|x|}$. By (1.17), the Laplacian $\triangle_d$ can be written as

$$\triangle_d = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left(r^{d-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \triangle_{\mathbb{S}^{d-1}} = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \triangle_{\mathbb{S}^{d-1}},$$

where $\triangle_{\mathbb{S}^{d-1}}$ is the *spherical Laplacian*. The action of this operator does not affect the radial component of functions that can be written in separate variables.

We look for solutions $u \in C^2(\Omega)$. It is well known (see, for example, [11, 56, 57, 59]) that all solution u of (8.1) is an image of some harmonic function h under the action of the integral operator of the form

$$u(x) = h(x) + \int_0^1 \sigma^{d-1} G(r, 1 - \sigma^2) h(\sigma^2 x) d\sigma. \tag{8.4}$$

Here the kernel $G \in C^2([0, b] \times [0, 1])$ is the unique solution of the equation

$$r\left(G_{rr}(r,t) - q(r)G(r,t)\right) - G_r(r,t) + 2(1-t)G_{rt}(r,t) = 0 \quad \text{for} \quad (r,t) \in [0,b] \times [0,1], \tag{8.5}$$

satisfying the initial conditions

$$G(r, 0) = \int_0^r \tau q(\tau) d\tau \ \text{ for all } \ r \in [0, b]; \quad G(0, t) = 0 \ \text{ for all } \ t \in [0, 1]. \tag{8.6}$$

Note that the kernel G does not depend on the dimension d. The existence of the kernel for an analytic potential q is established in [14], and for a C^1-potential in [59]. In summary,

Theorem 157 ([59])**.** *Let $q \in C^1[0,b]$ (in general, complex-valued), and $G \in C^2([0,b]\times[0,1])$ be a solution of (8.5), (8.6). Then for any $h \in \mathrm{Har}(\Omega)$, (8.4) is a solution of (8.1). Reciprocally, if $u \in C^2(\Omega)$ is a solution of (8.1) then there exists such $h \in \mathrm{Har}(\Omega)$ that u has the form (8.4).*

Example 158. *Let $d = 2$ and $\kappa \in \mathbb{R}$. Consider the Helmholtz equation in $\mathbb{B}^2$,*

$$\left(\triangle_2 + \kappa^2\right) u(z) = 0 \quad \text{for } z \in \mathbb{B}^2. \tag{8.7}$$

In this case it is known (see [11, 138]) that every solution u has the form

$$u(z) = h(z) - \kappa r \int_0^1 \sigma \frac{J_1(\kappa r\sqrt{1-\sigma^2})}{\sqrt{1-\sigma^2}} h(\sigma^2 z) d\sigma,$$

where $h \in \mathrm{Har}(\mathbb{B}^2)$ and $J_1(\zeta)$ is a Bessel function of the first kind and first order. The kernel thus has the form

$$G(r,t) = -\kappa r \frac{J_1(\kappa r\sqrt{t})}{\sqrt{t}}.$$

As in chapters 1 and 2, denote by $\mathbf{S} := \triangle_d - q(r)$ the *radial Schrödinger operator*. Based on the integral representation (8.4), we introduce the operator: $\mathbf{T} : C(\Omega) \to C(\Omega)$ defined by $\mathbf{T} = \mathbf{I} + \mathbf{G}$, where $\mathbf{G}$ is the Fredholm integral operator

$$\mathbf{G}h(x) := \int_0^1 \sigma^{d-1} G(r, 1-\sigma^2) h(\sigma^2 x) d\sigma = \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1} G(r,t) h((1-t)x) dt. \tag{8.8}$$

By Theorem 157, the operator $\mathbf{T}$ has the property $\mathbf{T}(\mathrm{Har}(\Omega)) = \ker(\mathbf{S})$. We want to show that $\mathbf{T}$ is a transmutation operator for $\mathbf{S}$ and $\triangle_d$. For this, take $E_1 := C^2(\Omega)$ and $\mathbf{S}, \triangle_d : E_1 \to E$, and note that E_1 is $\mathbf{T}$-invariant. Note that in $\mathrm{Har}(\Omega)$, the equality holds, but we are interested in a more general common domain for the operators $\mathbf{S}, \triangle_d$, specifically $C^2(\Omega)$. However, as the following counterexample shows, the transmutation property is not valid on all $C^2(\Omega)$.

Example 159. *Let $d = 2$ and $\Omega = \mathbb{B}^2$. Consider $q \equiv -1$, and the Helmholtz operator $(\triangle_2 + 1)$ in $C^2(\mathbb{B}^2)$. Then from Example 158 we have*

$$\mathbf{T}h(z) = h(z) - r\int_0^1 \sigma \frac{J_1(r\sqrt{1-\sigma^2})}{\sqrt{1-\sigma^2}} h(\sigma^2 z) d\sigma = h(z) - \int_0^1 \frac{\partial}{\partial \sigma}\left(J\left(r\sqrt{1-\sigma^2}\right)\right) h(\sigma^2 z) d\sigma.$$

Take $f \in C^2(\mathbb{B}^2)$ given by $f(z) = |z|^2$. Then $\triangle_2 f(z) = 4$, and application of $\mathbf{T}$ leads to the equality

$$\begin{aligned}\mathbf{T}\left[\triangle_2 f(z)\right] &= 4\left[1 - \int_0^1 \frac{\partial}{\partial\sigma}\left(J_0\left(r\sqrt{1-\sigma^2}\right)\right) d\sigma\right] \\ &= 4\left[1 - J_0\left(r\sqrt{1-\sigma^2}\right)\Big|_{\sigma=0}^{1}\right] = 4J_0(r).\end{aligned}$$

On the other hand

$$\begin{aligned}\mathbf{T}[f(z)] &= r^2\left[-\int_0^1 \frac{\partial}{\partial\sigma}\left(J_0\left(r\sqrt{1-\sigma^2}\right)\right)\sigma^4 d\sigma\right] \\ &= r^2\left[1 - \sigma^4 J_0\left(r\sqrt{1-\sigma^2}\right)\Big|_{\sigma=0}^{1} + 4\int_0^1 \sigma^3 J_0\left(r\sqrt{1-\sigma^2}\right) d\sigma\right] \\ &= 4\int_0^r \left(1 - \frac{u^2}{r^2}\right) uJ_0(u)du\end{aligned}$$

where $u = r\sqrt{1-\sigma^2}$. Direct computation shows that $\int_0^r \left(1 - \frac{u^2}{r^2}\right) uJ_0(u)du = 2J_2(r)$, and hence $\mathbf{T}[f(z)] = 8J_2(r)$. Finally,

$$\begin{aligned}\left(\triangle_2 + 1\right)\mathbf{T}[f(z)] &= 8\left[J_2(r) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}J_2(r)\right)\right] \\ &= 8\left[J_2(r) + \frac{1}{r}\left(\frac{4}{r} - r\right)J_2(r)\right] = \frac{32}{r^2}J_2(r).\end{aligned}$$

Hence $\left(\triangle_2 + 1\right)\mathbf{T}[f(z)] \neq \mathbf{T}\left[\triangle_2 f(z)\right]$.

This example shows that in general $\mathbf{ST} \neq \mathbf{T}\triangle_d$, and hence $\mathbf{T}$ is not a transmutation operator for $\mathbf{S}$ and $\triangle_2$. However, from the same example it is easy to see that $r^2\left(\triangle_2 + 1\right)\mathbf{T}[f(z)] = \mathbf{T}[r^2\,\triangle_2\, f(z)]$. The following theorem extends this equality onto a general situation.

Theorem 160. *Let $G \in C^2\left([0,b] \times [0,1]\right)$ be the unique solution of (8.5) satisfying the conditions (8.6), and $\widehat{\mathbf{S}} := r^2\left(\triangle_d - q(r)\right)$ and $\widehat{\mathbf{L}} := r^2\triangle_d$. Then the operator $\mathbf{T} = \mathbf{I} + \mathbf{G}$ where $\mathbf{G}$ is by (8.8), satisfies the relation*

$$\widehat{\mathbf{S}}\mathbf{T}f = \mathbf{T}\widehat{\mathbf{L}}f \quad \forall f \in C^2(\Omega). \tag{8.9}$$

Proof. Let $f \in C^2(\Omega)$. In spherical coordinates the operator $\mathbf{T}$ has the form

$$u(r,x') = \mathbf{T}f(r,x') = f(r,x') + \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}G(r,t)f((1-t)r,x')dt.$$

The integration here affects only the radial component of f. Applying the operator $\mathbf{S}$ we have

$$\begin{aligned}
\mathbf{S}u(r,x') &= \triangle_d f(r,x') - q(r)f(r,x') + \frac{1}{2}\triangle_d\left(\int_0^1 (1-t)^{\frac{d}{2}-1}G(r,t)f((1-t)r,x')dt\right) + \\
&- \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}q(r)G(r,t)f((1-t)r,x')dt \\
&= \triangle_d f(r,x') - q(r)f(r,x') - \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}q(r)G(r,t)f((1-t)r,x')dt \\
&+ \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}G(r,t)\left\{\frac{1}{r^2}\triangle_{\mathbb{S}^{d-1}}\left[f((1-t)r,x')\right]\right\}dt \\
&+ \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}\frac{\partial^2}{\partial r^2}\left(G(r,t)f((1-t)r,x')\right)dt \\
&+ \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}\frac{d-1}{r}\frac{\partial}{\partial r}\left(G(r,t)f((1-t)r,x')\right)dt.
\end{aligned}$$

Since the spherical Laplacian does not affect the radial component, we have $\triangle_{\mathbb{S}^{d-1}}\left[f((1-t)r,x')\right] = \left(\triangle_{\mathbb{S}^{d-1}}f\right)((1-t)r,x')$. Expanding the partial derivative with respect to r and taking into account that $\frac{\partial}{\partial r}f((1-t)r,x') = (1-t)f_\rho((1-t)r,x')$, where $\rho = (1-t)r$, we obtain

$$\begin{aligned}
\mathbf{S}u(r,x') &= \triangle_d f - q(r)f - \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}q(r)G(r,t)f dt \\
&+ \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}G(r,t)\left[\frac{1}{r^2}\triangle_{\mathbb{S}^{d-1}}f\right]dt \\
&+ \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}(d-1)\left\{\frac{G_r(r,t)}{r}f + \frac{(1-t)G(r,t)}{r}f_\rho\right\}dt \\
&+ \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}\left\{G_{rr}(r,t)f + 2(1-t)G_r(r,t)f_\rho + (1-t)^2G(r,t)f_{\rho\rho}\right\}dt \\
&= \triangle_d f + \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}G(r,t)\left[(1-t)^2\left(\triangle_d f\right)((1-t)r,x')\right]dt - q(r)f \\
&+ \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}\left\{G_{rr}(r,t) - q(r)G(r,t) + \frac{d-1}{r}G_r(r,t)\right\}f dt \\
&+ \int_0^1 (1-t)^{\frac{d}{2}}G_r(r,t)f_\rho dt.
\end{aligned}$$

Note that $\frac{\partial}{\partial t}f((1-t)r,x') = -rf_\rho((1-t)r,x')$. Hence the integral $\int_0^1 (1-t)^{\frac{d}{2}}G_r(r,t)f_\rho dt$ can be written in terms of the partial derivative with respect to t. Integrating by parts we obtain

$$\begin{aligned}
\int_0^1 (1-t)^{\frac{d}{2}} G_r(r,t) f_\rho dt &= -\int_0^1 \frac{(1-t)^{\frac{d}{2}} G_r(r,t)}{r} \frac{\partial}{\partial t} f dt \\
&= -\frac{(1-t)^{\frac{d}{2}} G_r(r,t)}{r} f((1-t)r, x')\Big|_{t=0}^1 + \int_0^1 \frac{\partial}{\partial t}\left[\frac{(1-t)^{\frac{d}{2}} G_r(r,t)}{r}\right] f dt \\
&= \frac{G_r(r,0)}{r} f(r,x') + \int_0^1 \left[\frac{(1-t)^{\frac{d}{2}} G_{rt}(r,t)}{r} - \frac{d}{2}(1-t)^{\frac{d}{2}-1}\frac{G_r(r,t)}{r}\right] f dt.
\end{aligned}$$

Substituting the last expression into the expression of $\mathbf{S}u(r,x')$ we arrive at the equality

$$\begin{aligned}
\mathbf{S}u(r,x') &= \triangle_d f + \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1} G(r,t)\left[(1-t)^2 \left(\triangle_d f\right)((1-t)r,x')\right] dt \\
&+ \left(\frac{G_r(r,0)}{r} - q(r)\right) f(r,x') + \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}\left\{G_{rr}(r,t)\right\} f((1-t)r,x') dt \\
&+ \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1}\left\{2\frac{(1-t)G_{rt}(r,t)}{r} - q(r)G(r,t) - \frac{G_r(r,t)}{r}\right\} f((1-t)r,x') dt.
\end{aligned}$$

By hypothesis, G satisfies Eq. (8.5) and the condition $G_r(r,0) = rq(r)$, then

$$\mathbf{S}u(r,x') = \triangle_d f + \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1} G(r,t)\left[(1-t)^2 \left(\triangle_d f\right)((1-t)r,x')\right] dt.$$

Finally, multiplying the last expression by r^2 we obtain

$$\begin{aligned}
\widehat{\mathbf{S}}\mathbf{T} f(r,x') &= \widehat{\mathbf{L}} f(r,x') + \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1} G(r,t)\left[(1-t)^2 r^2 \left(\triangle h\right)((1-t)r,x')\right] dt \\
&= \widehat{\mathbf{L}} f(r,x') + \frac{1}{2}\int_0^1 (1-t)^{\frac{d}{2}-1} G(r,t)\left(\widehat{\mathbf{L}} f\right)((1-t)r,x') dt = \mathbf{T}\widehat{\mathbf{L}} f(r,x').
\end{aligned}$$

Q.E.D.

Let us study some properties of the operator $\mathbf{T}$. Consider the operator $\mathbf{G} : C(\Omega) \to C(\Omega)$, and write $\mathcal{R}_b := [0,b] \times [0,1]$. The main observation to establish the continuity of $\mathbf{G}$ is the fact that $G \in C(\mathcal{R}_b)$ (see [59]). In order to obtain some estimates for the approximation of operators, we need certain properties of the star-shaped domains.

Definition 161. *Given $A \subset \mathbb{R}^d$, we define the* ***star hull*** *(with respect to $x = 0$) of A as*

$$\mathrm{Star}(A) := \bigcup_{x \in A} [0, x].$$

Thus, $\mathrm{Star}(A)$ is the smallest star-shaped domain (with respect to $x = 0$) that contains A (In fact, the intersection of star-shaped domains with respect to $x = 0$ is a star-shaped domain, and $\mathrm{Star}(A)$ is the intersection of all that contain A).

Lemma 162. *Let $K \subset \mathbb{R}^d$ be compact. Then, the star hull* $\mathrm{Star}(K)$ *is compact.*

Proof. Consider the function $\phi : K \times [0,1] \ni (x,t) \mapsto tx \in \mathbb{R}^d$. The function ϕ is continuous and $\phi(K \times [0,1]) = \mathrm{Star}(K)$. By hypothesis, $K \times [0,1]$ is compact, hence also $\mathrm{Star}(K)$. **Q.E.D.**

Proposition 163. *The operator $\mathbf{G} : C(\Omega) \to C(\Omega)$ is continuous, and for all $f \in C(\Omega)$ and $K \subset \Omega$ compact, the inequality holds*

$$\max_{x \in K} |\mathbf{G}f(x)| \leqslant \frac{1}{d}\|G\|_{C(\mathcal{R}_b)} \cdot \max_{x \in \mathrm{Star}(K)} |f(x)|. \tag{8.10}$$

Proof. Let $f \in C(\Omega)$ and $K \subset \Omega$ be compact. Since Ω is star shaped, $\mathrm{Star}(K) \subset \Omega$, and by Lemma 162, $\mathrm{Star}(K)$ is compact. Take $x \in K$ and note that

$$\begin{aligned}
|\mathbf{G}f(x)| &\leqslant \int_0^1 \sigma^{d-1} |G(r, 1-\sigma^2) f(\sigma^2 x)| d\sigma \\
&= \int_0^1 \left|\sigma^{\frac{d-1}{2}} G(r, 1-\sigma^2)\right| \left|\sigma^{\frac{d-1}{2}} f(\sigma^2 x)\right| d\sigma \\
&\leqslant \left(\int_0^1 \sigma^{d-1} |G(r, 1-\sigma^2)|^2 d\sigma\right)^{\frac{1}{2}} \left(\int_0^1 \sigma^{d-1} |f(\sigma^2 x)|^2 d\sigma\right)^{\frac{1}{2}}
\end{aligned}$$

where the Cauchy–Bunyakovsky–Schwarz inequality was applied. Using the change of variables $t = 1 - \sigma^2$ in the first integral, and $t = \sigma^2$ in the second, we obtain

$$|\mathbf{G}f(x)| \leqslant \frac{1}{2} \left(\int_0^1 (1-t)^{\frac{d}{2}-1} |G(r,t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^1 t^{\frac{d}{2}-1} |f(tx)|^2 dt\right)^{\frac{1}{2}}.$$

The first integral can be estimated as follows

$$\int_0^1 (1-t)^{\frac{d}{2}-1} |G(r,t)|^2 dt \leqslant \|G\|^2_{C(\mathcal{R}_b)} \int_0^1 (1-t)^{\frac{d}{2}-1} dt = \frac{2}{d} \|G\|^2_{C(\mathcal{R}_b)}.$$

In the second integral, the function f is defined on the compact segment $[0, x] \subset \mathrm{Star}(K)$ and hence

$$\int_0^1 t^{\frac{d}{2}-1} |f(tx)|^2 dt \leqslant \left(\max_{x \in \mathrm{Star}(K)} |f(x)|\right)^2 \int_0^1 t^{\frac{d}{2}-1} dt = \frac{2}{d} \left(\max_{x \in \mathrm{Star}(K)} |f(x)|\right)^2.$$

Thus, we obtain the estimate

$$|\mathbf{G}f(x)| \leqslant \frac{1}{d}\|G\|_{C(\mathcal{R}_b)} \cdot \max_{x \in \mathrm{Star}(K)} |f(x)|.$$

In particular, for $n \in \mathbb{N}$, $\mathrm{Star}(K_n)$ is a compact subset of Ω, and there exists $N \in \mathbb{N}$ such that $\mathrm{Star}(K_n) \subset K_N$, then we have

$$\|\mathbf{G}f\|_n \leqslant \frac{1}{d}\|G\|_{C(\mathcal{R}_b)} \cdot \max_{x\in\mathrm{Star}(K_n)} |f(x)| \leqslant \frac{1}{d}\|G\|_{C(\mathcal{R}_b)} \cdot \|f\|_N,$$

which implies that $\mathbf{G}$ is continuous in the topology of $C(\Omega)$ (see [51, Prop. 5.15]). **Q.E.D.**

Corollary 164. *The transformation operator* $\mathbf{T} : C(\Omega) \to C(\Omega)$ *is continuous.*

Remark 165. *If* $f \in C(\overline{\Omega})$*, we have that* $|\mathbf{G}f(x)| \leqslant \frac{1}{d}\|G\|_{C(\mathcal{R}_b)}\|f\|_{C(\overline{\Omega})}$ *for all* $x \in \overline{\Omega}$. *Then* $\mathbf{G}$ *is bounded in the Banach space* $C(\overline{\Omega})$ *with* $\|\mathbf{G}\| \leqslant \frac{1}{d}\|G\|_{C(\mathcal{R}_b)}$ *(and consequently, also* $\mathbf{T}$*).*

To prove the invertibility of the operator $\mathbf{T}$ one can rewrite it as a Volterra integral operator. For $f \in C(\Omega)$ we write the integral $\mathbf{G}f(x)$ in spherical coordinates and changing the variable $\sigma^2 = \frac{\rho}{r}$ obtain

$$\begin{aligned}\mathbf{G}f(r,x') &= \int_0^1 \sigma^{d-1} G(r, 1-\sigma^2) f(\sigma^2 r, x') d\sigma = \int_0^r \left(\frac{\rho}{r}\right)^{\frac{d-1}{2}} G\left(r, 1-\frac{\rho}{r}\right) f(\rho, x') \frac{d\rho}{2\sqrt{\rho r}} \\ &= \int_0^r \left[\left(\frac{\rho}{r}\right)^{\frac{d}{2}-1} \frac{1}{2r} G\left(r, 1-\frac{\rho}{r}\right)\right] f(\rho, x') d\rho.\end{aligned}$$

Denote $K(r,\rho) := \left(\frac{\rho}{r}\right)^{\frac{d}{2}-1} \frac{1}{2r} G\left(r, 1-\frac{\rho}{r}\right)$. Then the operator $\mathbf{T}$ takes the form

$$\mathbf{T}h(r,x') = f(r,x') + \int_0^r K(r,\rho) f(\rho,x') d\rho, \tag{8.11}$$

(see [11, pp. 235]) of a Volterra integral operator of second kind. The invertibility of $\mathbf{T}$ depends on the behaviour of the kernel $K(r,\rho)$ in the triangle $\Pi := \{(r,\rho) \,|\, 0 \leqslant \rho \leqslant r \leqslant b\}$.

Note that $\frac{G(r,t)}{r}$ is continuous in $\mathcal{R}_b$. Indeed, the only singular point is $r = 0$, but taking into account the condition $G(0,t) = 0$ for all $t \in [0,1]$ (8.6) by the L'Hospital rule we have

$$\lim_{r\to 0^+} \frac{G(r,t)}{r} = \lim_{r\to 0^+} G_r(r,t) = G_r(0,t)$$

(since G is of class C^2 in $\mathcal{R}_b$). Denote $M_0 := \sup_{(r,t)\in\mathcal{R}_b} \left|\frac{G(r,t)}{r}\right|$. Note that from (8.6) $K(r,r) = \frac{1}{r}\int_0^r \tau q(\tau) d\tau$. By the L'Hospital rule, $\lim\limits_{r\to 0^+} \dfrac{G(r,0)}{r} = q(0)$, and hence $K(r,r)$ is well defined on the main diagonal of Π.

For the rest of Π, given $(r,\rho) \in \Pi$ we have $0 < \frac{\rho}{r} \leqslant 1$ and $t = 1 - \frac{\rho}{r} \in (0,1)$. Since $\frac{d}{2} - 1 \geqslant 1$, we obtain

$$|K(r,\rho)| = \left| \left(\frac{\rho}{r}\right)^{\frac{d}{2}-1} \frac{G(r,t)}{2r} \right| \leqslant \frac{M_0}{2}.$$

Due to the arbitrariness of (r,ρ), $K(r,\rho)$ is bounded on Π. Since $K \in L_2(\Pi)$ the Volterra integral operator (8.11) is invertible [72, Ch. X].

Furthermore, in [57] it was shown that the inverse operator $\mathbf{T}^{-1}$ has the form

$$\mathbf{T}^{-1}u(x) = u(x) + \int_0^1 \sigma^{d-1} g(r, 1-\sigma^2) u(\sigma^2 x) d\sigma \tag{8.12}$$

where g is related with the Riemann function of the equation $(\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{4}q(r))u(z) = 0$. Moreover, $h = \mathbf{T}^{-1}u \in Har(\Omega)$ for all solution u of (8.1). Then any solution u of (8.1) is of the form $u = \mathbf{T}h$. Since the inverse operator has the form of a Fredholm integral operator with a continuous kernel, we conclude that $\mathbf{T}^{-1} : C(\Omega) \to C(\Omega)$ is continuous. Consequently, we have the following statement.

Theorem 166. *The operator $\mathbf{T}$ is a transmutation operator for $\widehat{\mathbf{S}}$ and $\widehat{\mathbf{L}}$.*

Example 167 ([138])**.** *For the Helmholtz operator in $\mathbb{B}^2$ the transmutation operator can be written as a Volterra integral operator in polar coordinates as*

$$\mathbf{T}h(r,\theta) = h(r,\theta) - \int_0^r \frac{\partial}{\partial \rho} J_0\left(\kappa\sqrt{r(r-\rho)}\right) u(\rho,\theta) d\rho,$$

and the inverse is given by

$$\begin{aligned}\mathbf{T}^{-1}u(r,\theta) &= u(r,\theta) + \int_0^r \frac{r}{\rho}\frac{\partial}{\partial r} J_0\left(\kappa\sqrt{\rho(\rho-r)}\right) u(\rho,\theta) d\rho \\ &= u(\rho,\theta) + \kappa r \int_0^1 \frac{I_1\left(\kappa r \sigma\sqrt{1-\sigma^2}\right)}{\sqrt{1-\sigma^2}} u(\sigma^2 r, \theta) d\sigma.\end{aligned}$$

Hence the kernel of the inverse operator has the form $g(r,t) = \frac{I_1\left(\kappa r\sqrt{t(1-t)}\right)}{\sqrt{t(1-t)}}$, where $I_1(\zeta)$ is a modified Bessel function of first kind.

Remark 168. *Since $\mathbf{T}(\mathrm{Har}(\Omega)) = \mathrm{Sol}^{\mathbf{S}}(\Omega)$, by Remark 24, $\mathrm{Sol}^{\mathbf{S}}(\Omega)$ is a closed subspace of $C(\Omega)$.*

8.3 Boundedness and invertibility of the transmutation operator in the Bergman space

Now we establish the boundedness of the operator $\mathbf{G} : b_2(\Omega) \to L_2(\Omega)$ in the L_2-norm.

Theorem 169. *The operator* $\mathbf{G} : b_2(\Omega) \to L_2(\Omega)$ *is bounded. In consequence, the transmutation operator* $\mathbf{T} = \mathbf{I} + \mathbf{G}$ *is bounded as well.*

Proof. Given $h \in b_2(\Omega)$ we have

$$|\mathbf{G}h(x)|^2 \leqslant \frac{1}{2d}\|G\|^2_{C(\mathcal{R}_b)} \cdot \int_0^1 |h(tx)|^2 dt, \quad \text{for} \quad x \in \Omega.$$

Thus,

$$\begin{aligned}\int_\Omega |\mathbf{G}h(x)|^2 dx &\leqslant \frac{1}{2d}\|G\|^2_{C(\mathcal{R}_b)} \cdot \int_\Omega \left[\int_0^1 |h(tx)|^2 dt\right] dx \\ \text{(Fubini)} &= \frac{1}{2d}\|G\|^2_{C(\mathcal{R}_b)} \cdot \int_0^1 \left[\int_\Omega |h(tx)|^2 dx\right] dt.\end{aligned}$$

To study the integrals $\int_\Omega |h(tx)|^2 dx$, we consider the family of operators $\mathbf{B}_t : b_2(\Omega) \to b_2(\Omega)$ defined by the equality $\mathbf{B}_t h(x) = h(tx)$, for $0 \leqslant t \leqslant 1$. These can be seen as the composition $\mathbf{B}_t h = h \circ \varphi_t$, where $\varphi_t : \Omega \to t\Omega \subset \Omega$[1] is the homothetic transformation $\varphi_t(x) = \varphi(tx)$, so that $\mathbf{B}_t h$ is harmonic. For $t = 0$, $\mathbf{B}_0 h(x) = h(0)$ is just a functional evaluation, and in the Bergman space $b_2(\Omega)$ it is bounded. Take $\rho := \operatorname{dist}(0, \partial\Omega)$, then $B^d_\rho(0) \subset \Omega$. Hence the evaluation functional satisfies the inequality

$$|h(0)|^2 \leqslant \frac{1}{\rho^d V(\mathbb{B}^d)}\|h\|^2_{L_2(\Omega)}$$

(see [8, Prop. 8.1]). Then $\|\mathbf{B}_0 h\|^2_{L_2(\Omega)} \leqslant \dfrac{V(\Omega)}{\rho^d V(\mathbb{B}^d)}\|h\|^2_{L_2(\Omega)}$. For $t > 0$ with $\zeta = tx$, we have $\displaystyle\int_{t\Omega} |h(\zeta)|^2 dV_\zeta = \int_\Omega |h(tx)|^2 t^d dV_x$, so

$$\|\mathbf{B}_t h\|^2_{L_2(\Omega)} = \frac{1}{t^d}\int_{t\Omega} |h(\zeta)|^2 d\zeta \leqslant \frac{1}{t^d}\|h\|^2_{L_2(\Omega)}.$$

Thus, $\{\mathbf{B}_t\}_{0\leqslant t \leqslant 1} \subset \mathcal{B}(b_2(\Omega))$. Fix $h \in b_2(\Omega)$ and take $t \in [0,1]$. If $t = 0$, we know that $\|\mathbf{B}_0 h\|^2_{L_2(\Omega)} \leqslant \frac{V(\Omega)}{\rho^d V(\mathbb{B}^d)}\|h\|^2_{L_2(\Omega)}$. If $t > 0$, we have

$$\|\mathbf{B}_t h\|^2_{L_2(\Omega)} = \frac{1}{t^d}\int_{t\Omega} |h(\zeta)|^2 d\zeta = V(\Omega)\left\{\frac{1}{V(t\Omega)}\int_{t\Omega} |h(\zeta)|^2 d\zeta\right\}.$$

[1] If Ω is a star-shaped domain, then for any $t \in [0,1]$, $t\Omega$ is also a star-shaped domain and $t\Omega \subset \Omega$.

Note that $\rho\mathbb{B}^d \subset \Omega \subset b\mathbb{B}^d$. Let $0 < t < \frac{\rho}{b}$, and note that $t\Omega \subset (tb)\mathbb{B}^d \subset \Omega$, and

$$V(t\Omega) \geqslant V((t\rho)\mathbb{B}^d) = t^d\rho^d V(\mathbb{B}^d) = \left(\frac{\rho}{b}\right)^d (bt)^d V(\mathbb{B}^d) = \left(\frac{\rho}{b}\right)^d V((tb)\mathbb{B}^d).$$

Hence

$$\begin{aligned}\frac{1}{V(t\Omega)}\int_{t\Omega}|h(\zeta)|^2 d\zeta &\leqslant \frac{1}{V(t\Omega)}\int_{(tb)\mathbb{B}^d}|h(\zeta)|^2 d\zeta \\ &\leqslant \left(\frac{b}{\rho}\right)^d \cdot \frac{1}{V((tb)\mathbb{B}^d)}\int_{(tb)\mathbb{B}^d}|h(\zeta)|^2 d\zeta \\ &\leqslant \left(\frac{b}{\rho}\right)^d \cdot \sup_{0\leqslant t<\rho}\left\{\frac{1}{V(t\mathbb{B}^d)}\int_{t\mathbb{B}^d}|h(\zeta)|^2 d\zeta\right\}\end{aligned}$$

The supremum $H_{|h|^2}(0) := \sup_{0<t<\rho}\left\{\frac{1}{V(t\mathbb{B}^d)}\int_{t\mathbb{B}^d}|h(\zeta)|^2 d\zeta\right\}$ is the *Hardy-Littlewood maximal function* of $|h|^2$ evaluated in $x = 0$. It is known that for a continuous function in $\rho\mathbb{B}^d$, $x = 0$ is a Lebesgue point and this quantity is finite (furthermore, $\lim_{t\to 0^+}\frac{1}{V(t\mathbb{B}^d)}\int_{t\mathbb{B}^d}|h(\zeta)|^2 d\zeta = |h(0)|^2$, see [51, Ch. 3] or [121, Ch. 8]). Therefore, for $0 < t < \frac{\rho}{b}$, $\|\mathbf{B}_t h\|^2_{L_2(\Omega)} \leqslant \left(\frac{b}{\rho}\right)^d \cdot H_{|h|^2}(0)$. Finally, for $\frac{\rho}{b} \leqslant t \leqslant 1$ it is easily seen that $\|\mathbf{B}_t h\|^2_{L_2(\Omega)} \leqslant \left(\frac{b}{\rho}\right)^d \|h\|^2_{L_2(\Omega)}$. Then

$$\sup_{0\leqslant t\leqslant 1}\|\mathbf{B}_t h\|_{L_2(\Omega)} \leqslant \max\left\{\left(\frac{V(\Omega)}{\rho^d V(\mathbb{B}^d)}\right)^{\frac{1}{2}}\|h\|_{L_2(\Omega)}, \left(\frac{b}{\rho}\right)^{\frac{d}{2}}\cdot\left(H_{|h|^2}(0)\right)^{\frac{1}{2}}, \left(\frac{b}{\rho}\right)^{\frac{d}{2}}\|h\|_{L_2(\Omega)}\right\} < \infty$$

for all $h \in b_2(\Omega)$. It follows from the Uniform Boundedness Principle [51, Ch. 5], that $M := \sup_{0\leqslant t\leqslant 1}\|\mathbf{B}_t\| < \infty$. Thus,

$$\int_0^1\left[\int_\Omega |h(tx)|^2 dx\right]dt = \int_0^1 \|\mathbf{B}_t h\|^2_{L_2(\Omega)}dt \leqslant \int_0^1 \|\mathbf{B}_t\|^2\|h\|^2_{L_2(\Omega)}dt = M^2\|h\|^2_{L_2(\Omega)}.$$

From this we obtain $\|\mathbf{G}h\|^2_{L_2(\Omega)} \leqslant \frac{M^2}{2d}\|G\|^2_{C(\mathcal{R}_b)}\|h\|^2_{L_2(\Omega)}$. **Q.E.D.**

Remark 170. *(i) Since* $\mathrm{Sol}^{\mathbf{S}}(\Omega)$ *is closed in* $C(\Omega)$*, Proposition 126 implies that the Bergman space* $\mathrm{Sol}^{\mathbf{S}}_2(\Omega)$ *is complete.*

(ii) The previous result shows that $\mathbf{T}(b_2(\Omega)) \subset \mathrm{Sol}^{\mathbf{S}}_2(\Omega)$*. Actually, if* $u \in \mathrm{Sol}^{\mathbf{S}}_2(\Omega)$*, then* $h = \mathbf{T}^{-1}u \in \mathrm{Har}(\Omega)$*, and by the formula (8.12) one can show (in a similar way of the proof of Theorem 169) that* $h \in b_2(\Omega)$*, and hence* $\mathbf{T}(b_2(\Omega)) = \mathrm{Sol}^{\mathbf{S}}_2(\Omega)$*. Since* $\mathbf{T} \in \mathcal{B}\left(b_2(\Omega), \mathrm{Sol}^{\mathbf{S}}_2(\Omega)\right)$ *is a bijection, it follows from the open map theorem (see [51, Cor. 5. 11]) that* $\mathbf{T}^{-1} \in \mathcal{B}\left(\mathrm{Sol}^{\mathbf{S}}_2(\Omega), b_2(\Omega)\right)$*.*

8.4 Fourier-Jacobi series expansion of the integral transmutation kernel

In this section we propose a Fourier-Jacobi series representation for the kernel G and study some of its properties, and in the next section we derive convenient formulas for its coefficients. The interest in constructing G goes back to [57] where the possibility of its computation via the method of successive approximations was explored.

Following [81], we employ the fact that for each $r \in [0,1]$ fixed, the function $G(r,\cdot) \in L_2((0,1); w_d(t)dt)$, with $w_d(t) = t^{\frac{d}{2}-1}$. In Subsection 1.3.1 we saw that shifted Jacobi polynomials $\left\{\widehat{P}_n^{(d)}(t)\right\}_{n=0}^{\infty}$ given by (1.13) are an orthogonal basis for $L_2((0,1); w_d(t)dt)$ whose norm is given by $\|\widehat{P}_n^{(d)}\|_{L_2((0,1);w_d(t)dt)} = \sqrt{\frac{2}{4n+d}}$ (see (1.14)).

Proposition 171. *The integral kernel G admits the following Fourier-Jacobi series expansion, for each $r \in [0,1]$ fixed*

$$G(r,t) = \sum_{n=0}^{\infty} \alpha_n(r) \widehat{P}_n^{(d)}(t), \tag{8.13}$$

where

$$\alpha_n(r) = \left(2n + \frac{d}{2}\right) \int_0^1 t^{\frac{d}{2}-1} G(r,t) \widehat{P}_n^{(d)}(t) dt \quad \forall n \in \mathbb{N}. \tag{8.14}$$

The series (8.13) converges with respect to t in $L_2((0,1); w_d(t)dt)$.

Proof. Fix $r \in [0,1]$. Since $G(r,\cdot) \in L_2((0,1); w_d(t)dt)$, we can expand it in terms of the basis $\{\widehat{P}_n^{(d)}\}_{n=0}^{\infty}$ in the form (8.13), and the series converges in $L_2((0,1); w_d(t)dt)$ with respect to t. Also, for $m \in \mathbb{N}_0$ the coefficient α_m can be obtained multiplying (8.13) by $\widehat{P}_m^{(d)}$ and integrating,

$$\begin{aligned}\int_0^1 t^{\frac{d}{2}-1} G(r,t) \widehat{P}_m^{(d)}(t) dt &= \int_0^1 t^{\frac{d}{2}-1} \left(\sum_{n=0}^{\infty} \alpha_n(r) \widehat{P}_n^{(d)}(t)\right) \widehat{P}_m^{(d)}(t) dt \\ &= \sum_{n=0}^{\infty} \alpha_n(r) \int_0^1 t^{\frac{d}{2}-1} \widehat{P}_n^{(d)}(t) \widehat{P}_m^{(d)}(t) dt = \alpha_m(r) \|\widehat{P}_m^{(d)}\|^2_{L_2((0,1);w_d(t)dt)}.\end{aligned}$$

(The change of the order of the integral and the series is justified due to the fact that the integral is a bounded functional on $L_2((0,1); w_d(t)dt)$). Since $\|\widehat{P}_m^{(d)}\|^2_{L_2((0,1);w_d(t)dt)} = \frac{2}{4n+d}$, we obtain (8.14). **Q.E.D.**

Remark 172. *From (8.14) we can see that* $\{\alpha_n\}_{n=0}^{\infty} \subset C^2[0,1]$ *(because* $G(\cdot,t) \in C^2[0,b]$*).*

Now, we study the pointwise convergence of the series (8.13). In fact, due to the smoothness of G, one can obtain that the convergence is uniform.

Proposition 173. *Let* $G_N(r,t) := \sum_{n=0}^{N} \alpha_n(r)\widehat{P}_n^{(d)}(t)$ *denote the* N*-th partial sum of (8.13). Suppose that* $q \in C^p[0,b]$*, for some* $p > \frac{d}{2}-1$*, and define* $M_p := \max_{(r,t)\in\mathcal{R}_b} \left| \frac{\partial^{p+1}G(r,t)}{\partial t^{p+1}} \right|$*. Then there exists a constant* $c_p > 0$*, independent of* N *or* r*, such that for all* $N > p$ *the inequality holds*

$$\max_{(r,t)\in\mathcal{R}_b} |G(r,t) - G_N(r,t)| \leqslant \frac{c_p \cdot M_p}{N^{p-\frac{d}{2}+1}}. \tag{8.15}$$

Consequently, the Fourier-Jacobi series (8.13) converges uniformly to G *in* $\mathcal{R}_b$*.*

Proof. The following estimate for the remainder of the Fourier-Jacobi series of a smooth function is known (see the proof of Theorem 7.6 from [131], with Theorem 5.15 of [135]). *Suppose that* $q = \max\{\alpha,\beta\} > -\frac{1}{2}$*. Given* $p \in \mathbb{N}$ *with* $p > q$*, there exists a constant* $c_p > 0$ *such that, for all* $f \in C^{p+1}[-1,1]$ *and* $N > p$ *satisfies*

$$\max_{x\in[-1,1]} |f(x) - f_N(x)| \leqslant \frac{c_p}{N^{p-q}} \cdot \max_{t\in[-1,1]} |f^{(p+1)}(t)|, \tag{8.16}$$

where $f_N(x) := \sum_{n=0}^{N} a_n P_n^{(\alpha,\beta)}(x)$ *is an* N*-th partial sum of the Fourier-Jacobi series of* f. Since the change of variable $x = 2t-1$ is an isometry in the uniform norm between the spaces $C[-1,1]$ and $C[0,1]$, the same estimate is valid for the shifted Jacobi polynomials $\{\widehat{P}_n^{(d)}\}_{n=0}^{\infty}$. For this case, $q = \frac{d}{2} - 1$, and if $p > \frac{d}{2} - 1$, then (8.16) is fulfilled for all $f \in C^{p+1}[0,1]$. Since $q \in C^p[0,b]$, $G \in C^{p+1}(\mathcal{R}_b)$. In particular, for $r \in [0,b]$ fixed, $G(r,\cdot) \in C^{p+1}[0,1]$ and

$$|G(r,t) - G_N(r,t)| \leqslant \frac{c_p}{N^{p-q}} \cdot \max_{t\in[0,1]} \left| \frac{\partial^{p+1}G(r,t)}{\partial t^{p+1}} \right| \leqslant \frac{c_p M_p}{N^{p-\frac{d}{2}+1}} \quad \forall t \in [0,1].$$

The constant c_p does not depend on r and the last inequality is valid for all $r \in [0,b]$, from which we obtain (8.15), and thus the uniform convergence. **Q.E.D.**

For the dimensions $d = 2,3$ it is sufficient that $q \in C^1[0,b]$. Proposition 173 provides an estimate for the approximation of solutions of (8.1).

Theorem 174. *Suppose that $q \in C^p[0,b]$, for some $p > \frac{d}{2} - 1$. Let $u \in Sol^{\mathbf{S}}(\Omega)$ and $h \in \mathrm{Har}(\Omega)$ such that $u = \mathbf{T}h$. Then u can be represented as the series*

$$u(x) = h(x) + \sum_{n=0}^{\infty} \frac{(-1)^n \alpha_n(r)}{2} \int_0^1 t^{\frac{d}{2}-1} \widehat{P}_n^{(d)}(t) h(tx) dt, \tag{8.17}$$

which converges uniformly on each compact subset of Ω. More precisely, if $N > p$ and

$$u_N(x) := h(x) + \sum_{n=0}^{N} \frac{(-1)^n \alpha_n(r)}{2} \int_0^1 t^{\frac{d}{2}-1} \widehat{P}_n^{(d)}(t) h(tx) dt, \tag{8.18}$$

then for each compact $K \subset \Omega$ the estimate is valid

$$\max_{x \in K} |u(x) - u_N(x)| \leqslant \frac{c_p M_p}{d N^{p - \frac{d}{2} + 1}} \cdot \max_{x \in \partial(\mathrm{Star}(K))} |h(x)|, \tag{8.19}$$

where c_p and M_p are defined as in Proposition 173.

Proof. Let $u \in \mathrm{Sol}^{\mathbf{S}}(\Omega)$ and $h \in \mathrm{Har}(\Omega)$ such that $u = \mathbf{T}h$. Substituting the Fourier-Jacobi expansion (8.13) of the kernel G in the operator $\mathbf{T}$ and taking into account the uniform convergence we obtain

$$\begin{aligned} u(x) &= h(x) + \int_0^1 \sigma^{d-1} \left(\sum_{n=0}^{\infty} \alpha_n(r) \widehat{P}_n^{(d)}(1 - \sigma^2) \right) h(\sigma^2 x) d\sigma \\ &= h(x) + \sum_{n=0}^{\infty} \alpha_n(r) \int_0^1 \sigma^{d-1} P_n^{(0, \frac{d}{2}-1)}(-2\sigma^2 + 1) h(\sigma^2 x) d\sigma. \end{aligned}$$

The Jacobi polynomials satisfy the equality $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\alpha,\beta)}(x)$ [131, pp. 244]. Hence $P_n^{(0,\frac{d}{2}-1)}(-2\sigma^2+1) = (-1)^n P_n^{(0,\frac{d}{2}-1)}(2\sigma^2 - 1) = (-1)^n \widehat{P}_n^{(d)}(\sigma^2)$, and

$$\begin{aligned} u(x) &= h(x) + \sum_{n=0}^{\infty} (-1)^n \alpha_n(r) \int_0^1 \sigma^{d-1} \widehat{P}_n^{(d)}(\sigma^2) h(\sigma^2 x) d\sigma \\ &= h(x) + \sum_{n=0}^{\infty} \frac{(-1)^n \alpha_n(r)}{2} \int_0^1 t^{\frac{d}{2}-1} \widehat{P}_n^{(d)}(t) h(tx) dt. \end{aligned}$$

For the estimate, define the operator

$$\mathbf{G}_N h(x) = \int_0^1 \sigma^{d-1} G_N(r, 1-\sigma^2) h(\sigma^2 x) d\sigma = \sum_{n=0}^{N} \frac{(-1)^n \alpha_n(r)}{2} \int_0^1 t^{\frac{d}{2}-1} \widehat{P}_n^{(d)}(t) h(tx) dt,$$

where $G_N(r,t)$ is the N-th partial sum of the Fourier-Jacobi series of G. Then $u_N(x) =$

$\left(\mathbf{1}_{C(\Omega)} + \mathbf{G}_N\right) u(x)$. Now, given a compact $K \subset \Omega$ and $x \in K$, we have

$$\begin{aligned} |u(x) - u_N(x)| &= |\left(\mathbf{G} - \mathbf{G}_N\right) u(x)| \\ (\text{ By (8.10)}) &\leqslant \frac{1}{d} \cdot \max_{(r,t)\in\mathcal{R}_b} |G(r,t) - G_N(r,t)| \cdot \max_{x\in\mathrm{Star}(K)} |h(x)| \\ (\text{ By (8.15)}) &\leqslant \frac{1}{d} \cdot \frac{c_p M_p}{N^{p-\frac{d}{2}+1}} \cdot \max_{x\in\mathrm{Star}(K)} |h(x)|. \end{aligned}$$

This is fulfilled for all $x \in K$, so we have that

$$\max_{x\in K} |u(x) - u_N(x)| \leqslant \frac{c_p M_p}{dN^{p-\frac{d}{2}+1}} \cdot \max_{x\in\mathrm{Star}(K)} |h(x)|.$$

Since h is harmonic, by the maximum principle, $\max_{x\in\mathrm{Star}(K)} |h(x)| = \max_{x\in\partial(\mathrm{Star}(K))} |h(x)|$, we obtain (8.19). **Q.E.D.**

Remark 175. *The series (8.17) converges pointwise for all $r \in [0,b]$ (by the continuity of the inner product), even if $q \in C^1[0,b]$.*

8.5 A complete system of solutions for the Schrödinger equation

The next step is the construction of the coefficients $\{\alpha_n(r)\}_{n=0}^{\infty}$. We deduce a corresponding system of equations considering images of certain type of harmonic functions. First, observe from (8.17) that the action of the integral is focused on the radial variable, therefore, in case the integration variable is separated from h the expression can be simplified. Hence, let us consider the homogeneous harmonic functions.

8.5.1 The transmuted homogeneous harmonic polynomials

Taking $p \in \mathcal{H}_m(\mathbb{R}^d)$ and substituting it into the series (8.17) we have

$$\begin{aligned} \mathbf{T}p(x) &= p(x) + \sum_{n=0}^{\infty} \frac{(-1)^n \alpha_n(r)}{2} \int_0^1 t^{\frac{d}{2}-1} \widehat{P}_n^{(d)}(t) p(tx) dt \\ &= p(x) \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n \alpha_n(r)}{2} \int_0^1 t^{\frac{d}{2}-1} \widehat{P}_n^{(d)}(t) t^m dt\right]. \end{aligned}$$

The last integral is the inner product $\langle \widehat{P}_n^{(d)}, t^m \rangle_{L_2([0,1],w_d(t)dt)}$. Note that if $n > m$, then $\widehat{P}_n^{(d)} \perp t^m$, hence the last series is a finite sum for $n = \overline{0,m}$. Denoting

$$\gamma_{n,m} := \langle \widehat{P}_n^{(d)}, t^m \rangle_{L_2((0,1);w_d(t)dt)} \quad \text{for } m \in \mathbb{N}_0,\ 0 \leqslant n \leqslant m, \tag{8.20}$$

we obtain the representation

$$\mathbf{T}h(x) = \left[1 + \sum_{n=0}^{m} \frac{(-1)^n \alpha_n(r) \gamma_{n,m}}{2}\right] p(x), \tag{8.21}$$

valid for all $p \in \mathcal{H}_m(\mathbb{R}^d)$. Note that the representation does not depend on the choice of p, only on the degree m.

Now we derive an expression for the coefficients $\{\alpha_n(r)\}_{n=0}^{\infty}$. For this we establish the conditions fulfilled by them in the origin. For $r = 0$ we have the expansion $G(0,t) = \sum_{n=0}^{\infty} \alpha_n(0) \widehat{P}_n^{(d)}(t)$, however, by conditions (8.6), $G(0,t) = 0$, for all $t \in [0,1]$, so $\sum_{n=0}^{\infty} \alpha_n(0) \widehat{P}_n^{(d)}(t) = 0$. Due to the orthogonality of the shifted Jacobi polynomials we obtain that:

$$\alpha_n(0) = 0 \quad \forall n \in \mathbb{N}_0. \tag{8.22}$$

Theorem 176. *Let $\{\phi_m(r)\}_{m=0}^{\infty}$ be the system of functions defined by the equalities*

$$\phi_m(r) = \frac{y_m(r)}{r^{m+\frac{d-1}{2}}} \quad \text{for} \quad 0 < r \leqslant b,\ m \in \mathbb{N}_0 \tag{8.23}$$

where y_m is the unique solution of the ***perturbed Bessel equation***

$$\mathbf{P}_m y_m(r) := -\frac{d^2 y_m(r)}{dr^2} + \frac{\ell_m(\ell_m+1)}{r^2} y_m(r) + q(r) y_m(r) = 0 \ \text{ with } \ \ell_m := m + \frac{d-3}{2}, \tag{8.24}$$

satisfying the asymptotic conditions

$$y_m(r) \sim r^{\ell_m+1},\ y_m'(r) \sim (\ell_m+1) r^{\ell_m},\ r \to 0^+. \tag{8.25}$$

Then the coefficients $\{\alpha_n(r)\}_{n=0}^{\infty}$ can be obtained as follows. For $M \in \mathbb{N}_0$ the vectorial functions $\Phi_M := (\phi_m(r) - 1)_{m=0}^{M}$ and $A_M := (\alpha_m(r))_{m=0}^{M}$ are related by

$$\Gamma_M A_m = \Phi_M \tag{8.26}$$

where Γ_M is the lower triangular matrix

$$(\Gamma_M)_{i,j} := \begin{cases} \frac{(-1)^j \gamma_{j,i}}{2}, & \text{if } 0 \leqslant i \leqslant j, \\ 0, & \text{if } j > i. \end{cases} \tag{8.27}$$

Proof. Let $m \in \mathbb{N}_0$ and take $p \in \mathcal{H}_m(\mathbb{R}^d)$. Write $p(x) = r^m \hat{p}(x')$ with $r = |x|$ and $x' = \frac{x}{r}$ for $x \in \Omega \setminus \{0\}$, and $\hat{p}_m = p_m\big|_{\mathbb{S}^{d-1}}$. Denote

$$\phi_m(r) := 1 + \sum_{n=0}^{m} \frac{(-1)^n \alpha_n(r) \gamma_{n,m}}{2}. \tag{8.28}$$

Notice that by condition 8.22, $\phi_m(0) = 1$. By (8.21)

$$u_m(x) := \mathbf{T} p_m(x) = \phi_m(r) r^m \hat{p}_m(x').$$

Since $u_m \in \mathfrak{S}(\Omega)$, substituting it in (8.3) we have

$$\mathbf{S} u_m(x) = \left(\frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} - q(r) \right) \phi_m(r) r^m \hat{p}_m(x') + \phi_m(r) r^m \frac{\triangle_{\mathbb{S}^{d-1}}}{r^2} \hat{p}_m(x') = 0.$$

Using part 3 of Proposition 20 and expanding the left part of the equation we have

$$\begin{aligned}
\mathbf{S} u_m(x) &= \left(\phi_m'' r^m + 2m r^{m-1} \phi_m' + m(m-1) r^{m-2} \phi_m \right) \hat{p}_m + \frac{d-1}{r} \left(\phi_m' r^m + m r^{m-1} \phi_m \right) \hat{p}_m + \\
&\quad - q(r) \phi_m r^m \hat{p}_m - m(m+d-2) \phi_m r^{m-2} \hat{p}_m \\
&= \left[r^m \left(\phi_m'' - q(r) \phi_m \right) + r^{m-1} (2m+d-1) \phi_m' \right] \hat{p}_m \\
&\quad + \left[r^{m-2} \phi_m \left(m(m-1) + m - m(m+d-2) + m(d-1) \right) \right] \hat{p}_m \\
&= \left(\phi_m'' - q(r) \phi_m + \frac{2m+d-1}{r} \phi_m' \right) r^m \hat{p}_m = 0.
\end{aligned}$$

This is valid for any $\hat{p}_m \in \mathcal{H}_m(\mathbb{S}^{d-1})$, so we have that every function $\phi_m(r)$ satisfies the problem

$$L_m[\phi_m(r)] = \phi_m'' - q(r) \phi_m + \frac{2m+d-1}{r} \phi_m' = 0, \quad \phi_m(0) = 1.$$

We can rewrite the last equation as a perturbed Bessel equation. Consider $\phi_m(r) = \frac{y_m(r)}{r^{\ell_m+1}}$ with $\ell_m := m + \frac{d-3}{2} \geqslant -\frac{1}{2}$. Then y_m satisfies the perturbed Bessel equation

$$\mathbf{P}_m y_m(r) = -\frac{d^2 y_m(r)}{dr^2} + \frac{\ell_m(\ell_m+1)}{r^2} y_m(r) + q(r) y_m(r) = 0$$

which possesses a unique solution $y_m(r)$, satisfying the asymptotic condition

$$y_m(r) \sim r^{\ell_m+1} = r^{m+\frac{d-1}{2}}, \; y_m'(r) \sim (\ell_m+1) r^{\ell_m}, \; r \to 0^+$$

(see [73, 127]). Chosing this solution we obtain $\phi_m(0) = \lim\limits_{r \to 0^+} \frac{y_m(r)}{r^{m+\frac{d-1}{2}}} = 1$. Finally, we have the equality $\phi_m(r) = 1 + \sum_{n=0}^{m} \frac{(-1)^n \gamma_{n,m}}{2} \alpha_n(r)$ or, written in another form,

$$\begin{pmatrix} \frac{\gamma_{0,0}}{2} & 0 & \cdots & 0 \\ \frac{\gamma_{0,1}}{2} & -\frac{\gamma_{1,1}}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma_{0,m}}{2} & -\frac{\gamma_{1,m}}{2} & \cdots & \frac{(-1)^m \gamma_{m,m}}{2} \end{pmatrix} \begin{pmatrix} \alpha_0(r) \\ \alpha_1(r) \\ \vdots \\ \alpha_m(r) \end{pmatrix} = \begin{pmatrix} \phi_0(r) - 1 \\ \phi_1(r) - 1 \\ \vdots \\ \phi_m(r) - 1 \end{pmatrix} \tag{8.29}$$

which is precisely the representation (8.26). The determinant of the system (8.29) is $\prod_{j=1}^{m} \frac{(-1)^m \gamma_{m,m}}{2} \neq 0$, and thus it is uniquely solvable. **Q.E.D.**

Theorem 176 allows one to construct the kernel $G(r,t) = \sum_{n=0}^{\infty} \alpha_n(r) \widehat{P}_n(t)$ by solving system (8.26). However, as we show below, for practical purposes the functions $\{\phi_m(r)\}_{m=0}^{\infty}$ are even more useful than the coefficients α_n.

Definition 177. *Let $\{\phi_m(r)\}_{m=0}^{\infty}$ be the system of functions defined by $\phi_m(r) = r^{-m-\frac{d}{2}+\frac{1}{2}} y_m(r)$, where y_m is the unique solution of the perturbed Bessel equation (8.24) that satisfying the asymptotic condition $y_m(r) \sim r^{m+\frac{d-1}{2}}$, $r \to 0^+$. A **formal power of degree** m is a function of the form*

$$\mathcal{U}_m(x) := \mathbf{T} p_m(x) = \phi_m(r) r^m \hat{p}_m(x'), \tag{8.30}$$

where $p_m \in \mathcal{H}_m(\mathbb{R}^d)$. The set of all formal powers of degree m will be denoted by $\mathcal{S}_m(\Omega)$.

Note that a formal power of degree m is well defined in the ball $B_b^d(0)$.

Example 178. *In the case $d = 2$ the formal powers of degree m are generated by the functions*

$$\mathcal{V}_m(z) := \begin{cases} \phi_m(|z|) z^m = \phi_m(r) r^m e^{im\theta}, & \text{if } m \geqslant 0 \\ \phi_{|m|}(|z|) \bar{z}^{|m|} = \phi_{|m|}(r) r^{|m|} e^{im\theta}, & \text{if } m < 0. \end{cases}$$

where $\phi_m(r) = r^{-m-\frac{1}{2}} y_n(r)$, and y_m satisfies (8.24) with $\ell_m = m - \frac{1}{2}$ and the asymptotic condition $y_m(r) \sim r^{m+\frac{1}{2}}, r \to 0^+$.

Theorem 179. *If $K \subset \Omega$ is compact with $\mathbb{R}^d \setminus \mathrm{Star}(K)$ being connected, then for any $u \in Sol^{\mathbf{S}}(\Omega)$ there exists a sequence $\{S_n\}_{n=0}^{\infty}$ with $S_n \in \mathrm{Span}\left(\bigcup_{m=0}^{n} \mathcal{S}_m(\Omega)\right)$ such that S_n converges uniformly to u in K.*

Moreover, if $\Omega = B_R^d(0)$ *for some* $R > 0$*, then for any* $u \in \mathrm{Sol}^{\mathbf{S}}(B_R^d(0))$ *there exists a sequence of formal powers* $\{\mathcal{U}_m\}_{m=0}^{\infty}$*, with* $\mathcal{U}_m \in \mathcal{S}_m(B_R^d(0))$ *such that*

$$u(x) = \sum_{m=0}^{\infty} \mathcal{U}_m(x) \quad \textit{for } x \in B_R^d(0), \tag{8.31}$$

and the series converges uniformly on compact subsets of $B_R^d(0)$*.*

Proof. Let $u \in \mathrm{Sol}^{\mathbf{S}}(\Omega)$. Consider $h \in Har(\Omega)$ such that $u = \mathbf{T}h$. Since $\mathrm{Star}(K)$ is a compact subset of Ω with a connected complement, by Theorem 18, there exists a sequence $\{p_n\}_{n=0}^{\infty}$ with $p_n \in \mathrm{Span}\left(\bigcup_{m=0}^{n} \mathcal{H}_m(\mathbb{R}^d)\right)$, such that p_n converges uniformly to h in $\mathrm{Star}(K)$. Consider $S_n = \mathbf{T}p_n$. By definition, $S_n \in \mathrm{Span}\left(\bigcup_{m=0}^{n} \mathcal{S}_m(\Omega)\right)$ and by (8.10),

$$\max_{x \in K} |u(x) - S_n(x)| \leqslant \frac{1}{d} \|G\|_{C(\mathcal{R}_b)} \cdot \max_{x \in \mathrm{Star}(K)} |h(x) - p_n(x)|.$$

Since $\max\limits_{x \in \mathrm{Star}(K)} |h(x) - p_n(x)| \to 0$, $n \to \infty$, we have that S_n converges uniformly to u in K. For the second statement, if $\Omega = B_R^d(0)$, Theorem 17 establishes the existence of a sequence $\{p_m\}_{m=0}^{\infty}$ with $p_m \in \mathcal{H}_m(\mathbb{R}^d)$ such that $h(x) = \sum_{m=0}^{\infty} p_m(x)$ and the series converges in the topology of $C(B_R^d(0))$. Again, by the continuity of $\mathbf{T}$ we obtain

$$u(x) = \mathbf{T}h(x) = \mathbf{T}\left[\sum_{m=0}^{\infty} p_m(x)\right] = \sum_{m=0}^{\infty} \mathbf{T}p_m(x) = \sum_{m=0}^{\infty} \mathcal{U}_m(x),$$

and the series converges in $C(B_R^d(0))$, i.e., uniformly on compact subsets. **Q.E.D.**

Example 180. *For the case* $d = 2$*, every solution* $u \in \mathrm{Sol}_2^{\mathbf{S}}(B_R^2(0))$ *can be written as a series*

$$u(z) = \sum_{m=-\infty}^{+\infty} \mathcal{V}_m(z) = \sum_{m=-\infty}^{+\infty} r^{|m|} \phi_{|m|}(r) e^{im\theta},$$

and the series converges uniformly on compact subsets of $B_R^2(0)$*.*

Below, in Corollary 196 the absolute convergence of the series (8.31) is established. Actually, we can prove that the system is complete in the strong sense of Definition 142. For this purpose the following lemma will be useful.

Lemma 181. *Let* $D \subset \mathbb{R}^d$ *be bounded and star-shaped with respect to the origin. Then* $\mathbb{R}^d \setminus D$ *is connected.*

Proof. Let $x, y \in \mathbb{R}^d \setminus D$. We see that there a path joining x and y in $\mathbb{R}^d \setminus D$. Let us denote $[x, \infty) := \{tx \,|\, t \geqslant 1\}$. We claim that $[x, \infty) \cap D = \emptyset$. In fact, if there exists $\zeta \in [x, \infty) \cap D$, then $[x, \zeta] \subset [0, \zeta] \subset D$, thus $x \in D$, which contradicts our hypothesis. Hence $[x, \infty) \cap D = \emptyset$, and similarly $[y, \infty) \cap D = \emptyset$. By hypothesis D is bounded, then we take $R > 0$ with $\{x, y\} \cup D \subset B_R^d(0)$, and ζ_1, ζ_2, such that $\zeta_1 \in [x, \infty) \cap \left(\mathbb{R}^d \setminus \overline{B}_R^d(0)\right)$ and $\zeta_2 \in [y, \infty) \cap \left(\mathbb{R}^d \setminus \overline{B}_R^d(0)\right)$. Since $\mathbb{R}^d \setminus \overline{B}_R^d(0)$ is a domain, there exists a path γ joining ζ_1 with ζ_2 in $\mathbb{R}^d \setminus \overline{B}_R^d(0)$. Hence $[x, \zeta_1] \cup \gamma \cup [\zeta_2, y]$ join x with y in $\mathbb{R}^d \setminus D$. **Q.E.D.**

Theorem 182. *The formal powers* $\bigcup_{m=0}^{\infty} \mathcal{S}_m(\Omega)$ *are a strongly complete system of solutions for* $\mathrm{Sol}^{\mathbf{S}}(\Omega)$.

Proof. If $K \subset \Omega$ is compact, then $\mathrm{Star}(K)$ is a bounded star-shaped domain w.r.t. the origin, so then $\mathbb{R}^d \setminus \mathrm{Star}(K)$ is connected by Lemma 181. It follows from Theorem 179 that for any solution $u \in \mathrm{Sol}^{\mathbf{S}}(\Omega)$ and $\epsilon > 0$ there exists $S \in \mathrm{Span}\left(\bigcup_{m=0}^{n} \mathcal{S}_m(\Omega)\right)$ such that $\max_{x \in K} |u(x) - S(x)| < \epsilon$ **Q.E.D.**

An orthogonal basis for $\mathrm{Sol}_2^{\mathbf{S}}(B_R^d(0))$ is proposed in the following theorem.

Theorem 183. *For any* $m \in \mathbb{N}_0$, *fix an orthonormal basis* $\{Y_1^{(m)}, \cdots, Y_{d_m}^{(m)}\}$ *for* $\mathcal{H}_m(\mathbb{S}^{d-1})$ *and define*

$$\mathcal{V}_k^{(m)}(x) = r^m \phi_m(r) Y_k^{(m)}(x') \quad k = \overline{1, d_m}. \tag{8.32}$$

Then $\mathcal{S} = \{\{\mathcal{V}_k^{(m)}\}_{k=0}^{d_m}\}_{m=0}^{\infty}$ *is an orthogonal basis for the Bergman space* $\mathrm{Sol}_2^{\mathbf{S}}(B_R^d(0))$. *In particular* $\mathcal{S}_m(B_R^d(0)) \perp \mathcal{S}_n(B_R^d(0))$ *if* $n \neq m$.

Proof. First, note that $\mathcal{S}_m(B_R^d(0)) \subset L_2(B_R^d(0))$. Let $m, n \in \mathbb{N}_0$, and $k \in \{1, \cdots, d_m\}$ and $j \in \{1, \cdots, d_n\}$, with $(m, k) \neq (n, k)$. Then

$$\begin{aligned}
\langle \mathcal{V}_k^{(m)}, \mathcal{V}_j^{(n)} \rangle_{L_2(B_R^d(0))} &= \int_{B_R^d(0)} \mathcal{V}_k^{(m)}(x) \overline{\mathcal{V}_j^{(n)}(x)} dV_x \\
&= \int_0^R r^{d-1} \int_{\mathbb{S}^{d-1}} r^{n+m} \phi_m(r) \overline{\phi_m(r)} Y_k^{(m)}(x') \overline{Y_j^{(m)}(x')} d\sigma_{x'} dr \\
&= \left(\int_0^R r^{m+n+d-1} \phi_m(r) \overline{\phi_m(r)} dr \right) \left(\int_{\mathbb{S}^{d-1}} Y_k^{(m)}(x') \overline{Y_j^{(m)}(x')} d\sigma_{x'} \right).
\end{aligned}$$

Thus, $\langle \mathcal{V}_k^{(m)}, \mathcal{V}_j^{(n)} \rangle_{L_2(B_R^d(0))} = 0$ if $(m, k) \neq (n, j)$. Then $\{\mathcal{V}_k^{(m)}\}_{k=0}^{d_m}$ is an orthogonal basis for $\mathcal{S}_m(B_R^d(0))$, and $\mathcal{S}_m(B_R^d(0)) \perp \mathcal{S}_n(B_R^d(0))$ if $n \neq m$. Finally, if $u \in \mathrm{Sol}_2^{\mathbf{S}}(B_R^d(0))$, take $h \in$

$B_2(B_R^d(0))$ such that $u = \mathbf{T}h$. By Theorem 21 we can write $h(x) = \sum_{m=0}^{\infty} r^m \left(\sum_{k=0}^{d_m} \hat{h}_k^{(m)} Y_k^{(m)}(x') \right)$, where $\hat{h}_k^{(m)} = \langle h, r^m Y_k^{(m)} \rangle_{L_2(B_R^d(0))}$ and the series converges in the L_2 norm. Thus, due to the boundedness of $\mathbf{T}$,

$$u(x) = \mathbf{T}h(x) = \mathbf{T}\left[\sum_{m=0}^{\infty} r^m \left(\sum_{k=0}^{d_m} \hat{h}_k^{(m)} Y_k^{(m)}(x') \right) \right] = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{d_m} \hat{h}_k^{(m)} \mathbf{T}\left[r^m Y_k^{(m)}(x') \right] \right)$$

$$= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{d_m} \hat{h}_k^{(m)} \phi_m(r) r^m Y_k^{(m)}(x') \right) = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{d_m} \hat{h}_k^{(m)} \mathcal{V}_k^{(m)}(x) \right).$$

Hence u can be expanded into a Fourier series of formal powers (8.32). **Q.E.D.**

Example 184. *For $d = 2$ the formal powers $\mathcal{V}_n(r,\theta) := r^{|n|}\phi_{|n|}(r)e^{in\theta}$, $n \in \mathbb{Z}$ represent an orthogonal basis for* $\mathrm{Sol}_2^{\mathbf{S}}(B_R^2(0))$.

Example 185. *For $d = 3$ the functions $\phi_m(r) = \frac{y_m(r)}{r^{m+1}}$ are determined by the regular solutions $y_m(r)$ of the perturbed Bessel equation*

$$-y_m''(r) + \frac{m(m+1)}{r^2} y_m(r) + q(r) y_m(r) = 0.$$

Consider the orthonormal basis for the spherical harmonics $\mathcal{H}_m(\mathbb{S}^2)$ given by

$$Y_k^{(m)}(\theta,\phi) := \sqrt{\frac{(2m+1)(m-k)!}{4\pi(m+k)!}} P_k^{(m)}(cos(\theta)) e^{ik\phi} \quad \textit{for } \ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi < 2\pi;\ k = \overline{-m,m}$$

(the classical 3-dimensional spherical harmonics), where $P_k^{(m)}$ is the associated Legendre polynomial $P_k^{(m)}(x) := \frac{(-1)^k}{2^m m!}(1-x^2)^{\frac{k}{2}} \frac{d^{k+m}}{dx^{k+m}}(x^2-1)^m$.

Then $\mathcal{V}_k^{(m)}(r,\theta,\phi) := \sqrt{\frac{(2m+1)(m-k)!}{4\pi(m+k)!}} r^m \phi_m(r) P_k^{(m)}(cos(\theta)) e^{ik\phi}$, $m \in \mathbb{N}_0$, $k = \overline{-m,m}$ represent an orthogonal basis for $Sol_2^{\mathbf{S}}(B_b^3(0))$.

Remark 186. *For higher dimensions considering spherical coordinates on $\mathbb{S}^{d-1}$ for $x' = (x_1, \cdots, x_d) \in \mathbb{S}^{d-1}$, we write*

$$\begin{cases} x_1 = \sin(\theta_{d-1})\sin(\theta_{d-2})\cdots\sin(\theta_2)\sin(\theta_1), \\ x_2 = \sin(\theta_{d-1})\sin(\theta_{d-2})\cdots\sin(\theta_2)\cos(\theta_1) \\ x_3 = \sin(\theta_{d-1})\sin(\theta_{d-2})\cdots\sin(\theta_3)\cos(\theta_2) \\ \vdots \\ x_{d-1} = \sin(\theta_{d-1})\cos(\theta_{d-2}), \\ x_d = \cos(\theta_{d-1}), \end{cases}$$

where $0 \leqslant \theta_1 \leqslant 2\pi$, $0 \leqslant \theta_i \leqslant \pi$, *for* $i = \overline{2, d-1}$. *Then the surface measure on* $\mathbb{S}^{d-1}$ *is given by* $d\sigma = \left(\prod_{j=1}^{d-2} (\sin(\theta_{d-j})^{d-j-1}\right) d\theta_{d-1}\cdots d\theta_1$ *(see [37, Ch. 1]). For* $\alpha \in \mathbb{N}_0^d$ *with* $\alpha_d \in \{0,1\}$, *define*

$$Y_\alpha(x) := [h_\alpha]^{-1} g_\alpha(\theta_1) \prod_{j=1}^{d-2} (\sin(\theta_{d-j}))^{|\alpha^{j+1}|} C_{\alpha_j}^{(\lambda_j)}(\cos(\theta_{d-j})),$$

where $|\alpha^j| := \alpha_j + \cdots + \alpha_{d-1}$; $\lambda_j := |\alpha^{j+1}| + \frac{d-j-1}{2}$,

$$g_\alpha(\theta_1) := \begin{cases} \cos(\alpha_{d-1}\theta), & \text{if } \alpha_d = 0, \\ \sin(\alpha_{d-1}\theta), & \text{if } \alpha_d = 1. \end{cases}$$

and $C_n^{(\lambda)}(x) := \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(x)$ are the Gegenbauer polynomials and

$$[h_\alpha]^2 := b_\alpha \prod_{j=1}^{d-2} \frac{(\alpha_j!)\left(\frac{d-j+1}{2}\right)|\alpha^{j+1}|(\alpha_j+\lambda_j)}{(2\lambda_j)_{\alpha_j}\left(\frac{d-j}{2}\right)_{|\alpha^{j+1}|}\lambda_j},$$

in which

$$b_\alpha := \begin{cases} 2, & \text{if } \alpha_{d-1}+\alpha_d > 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then $\{Y_\alpha \,|\, |\alpha| = m,\ \alpha_d \in \{0,1\}\}$ is an orthonormal basis for $\mathcal{H}_m(\mathbb{S}^d - 1)$ [37, Th. 1.5.1]. Consequently, $\{\{r^m\phi_m(r)Y_\alpha \,|\, |\alpha| = m,\ \alpha_d \in \{0,1\}\}\}_{m=0}^\infty$ is an orthogonal basis for $\mathrm{Sol}_2^{\mathbf{S}}(B_b^d(0))$.

8.5.2 Approximation of weak solutions

Once the strong completeness of the formal powers has been obtained, we can apply the results of section 2.4 to approximate the weak solutions. Since the formal powers are well defined in the smallest ball $B_b(0)$ containing the star shaped domain Ω, the completeness of the formal powers can be formulated as follows. As a direct consequence of Theorems 143, 145 and 147 we have the completeness of the formal powers in the space of weak solutions.

Theorem 187. *Suppose that* $q \in C^p(B_R^d(0))$, *with* $p = 1 + \left[\frac{d}{2}\right]$. *Then in any simply connected Lipschitz domain* $\omega \Subset B_R^d(0)$ *the radial formal powers* $\bigcup_{m=0}^\infty \mathcal{S}_m(B_R^d(0))$ *are a* *complete system of solutions in* $\mathrm{Sol}_w^{\mathbf{S}}(\omega)$ *with respect to the* L_2*-norm. If* ω *is extendable, the completeness holds with respect to the* H^1*-norm.*

Additionally, in the case when ω *is of class* C^2*, and there exists a* ρ*-neighborhood of* ω *where* $q \in C^\infty(\omega_\rho)$*, the radial formal powers are complete with respect to the* H^2*-norm as well.*

8.6 Construction of the formal powers

As was shown above the construction of the formal powers reduces to the construction of the functions $\phi_m(r) = \frac{y_m(r)}{r^{m+\frac{d-3}{2}}}$ which implies solving the perturbed Bessel equation

$$\mathbf{P}_m y_m(r) := -y_m''(r) + q(r)y_m(r) + \frac{\ell_m(\ell_m+1)}{r^2}y_m(r) = 0 \quad \text{for } 0 < r \leqslant b, \qquad (8.33)$$

where $\ell_m = m + \frac{d-3}{2}$, for $m \in \mathbb{N}$. A special case is precisely when $d = 2$ and $m = 0$, and hence $\ell_0 = -\frac{1}{2}$. However, first we consider the case $\ell_m = m + \frac{d-3}{2} > 0$, for $d \neq 2$ or $d = 2$ and $m > 0$. It is well known that for $q \in C[0, b]$ there exists a unique regular solution y_m of (8.33) that satisfies the asymptotic conditions $y_m(r) \sim r^{\ell_m+1}$, $y_m'(r) \sim (\ell_m+1)r^{\ell_m}$, $r \to 0^+$ (see [73, 127]). First, consider the Bessel equation

$$\mathbf{B}_m\psi(r) = -\psi''(r) + \frac{\ell_m(\ell_m+1)}{r^2}\psi(r) = 0.$$

A fundamental set of solutions is given by $\{r^{\ell_m+1}, r^{-\ell_m}\}$, with the Wronskian $W(r^{\ell_m+1}, r^{-\ell_m}) = -(2\ell_m+1) = -2(m-1) - d$. Following [127], we are looking for a solution as a functional series

$$y_m(r) = \sum_{k=0}^{\infty} \psi_k^m(r). \qquad (8.34)$$

Substituting formally the series into (8.33) and establishing that $\psi_0^m(r) := r^{\ell_m+1}$, one can get a sufficient condition for (8.34) to be a solution, and it consists in the requirement that the system of functions $\{\psi_k^m(r)\}_{k=0}^{\infty}$ should satisfy the recursive relations

$$\mathbf{B}_m\psi_k^m(r) = -q(r)\psi_{k-1}^m(r) \quad \text{for } m \geqslant 1. \qquad (8.35)$$

Eq. (8.35) can be solved applying the method of variation of parameters. For this, we define the kernel function

$$\mathcal{L}_m(r,s) := \frac{1}{2\ell_m+1}\left(\frac{r^{\ell_m+1}}{s^{\ell_m}} - \frac{s^{\ell_m+1}}{r^{\ell_m}}\right) \quad \text{for } (r,s) \in (0,b] \times (0,b]. \qquad (8.36)$$

Then

$$\psi_k^m(r) := \begin{cases} r^{m+\frac{d-1}{2}}, & \text{for } \ k=0, \\ \displaystyle\int_0^r \mathcal{L}_m(r,s)q(s)\psi_{k-1}^m(s)ds, & \text{for } \ k \geqslant 1. \end{cases} \tag{8.37}$$

Remark 188. *1.* $\mathcal{L}_m \in C^2\left((0,b] \times (0,b]\right)$ *and* $\mathcal{L}_m(r,r) = 0$.

2. $\frac{\partial}{\partial r}\mathcal{L}_m(r,s)\Big|_{r=s} = \frac{1}{2\ell_m+1}\left((\ell_m+1)\frac{r^{\ell_m}}{s^{\ell_m}} + \ell_m\frac{s^{\ell_m+1}}{r^{\ell_m+1}}\right)\Big|_{r=s} = 1.$

3. $\frac{\partial^2}{\partial r^2}\mathcal{L}_m(r,s) = \frac{\ell_m(\ell_m+1)}{r^2}\mathcal{L}_m(r,s).$

The following lemma provides some bounds for the function $\mathcal{L}_m$ which allow one to establish the convergence of the integral in (8.37).

Lemma 189. *For all* $(r,s) \in (0,b] \times (0,b]$ *such that* $0 < s \leqslant r$*, the function* $\mathcal{L}_m$ *satisfies the inequalities*

(i) $|\mathcal{L}_m(r,s)| \leqslant \frac{2}{2\ell_m+1}\frac{r^{\ell_m+1}}{s^{\ell_m}}$.

(ii) $\left|\frac{\partial}{\partial r}\mathcal{L}_m(r,s)\right| \leqslant \left(\frac{r}{s}\right)^{\ell_m}$.

Proof. (i) First, note that $\frac{s}{r} \leqslant 1 \leqslant \frac{r}{s}$. Since $\ell_m > 0$ we have $\left(\frac{s}{r}\right)^{\ell_m} \leqslant \left(\frac{r}{s}\right)^{\ell_m}$. Then

$$|\mathcal{L}_m(r,s)| \leqslant \frac{2}{2\ell_m+1}\left[r\left(\frac{r}{s}\right)^{\ell_m} + s\left(\frac{s}{r}\right)^{\ell_m}\right] \leqslant \frac{2}{2\ell_m+1}r\left(\frac{r}{s}\right)^{\ell_m}.$$

(ii) Note that $\frac{s}{r} \leqslant 1 \Rightarrow \left(\frac{s}{r}\right)^{\ell_m+1} \leqslant \left(\frac{s}{r}\right)^{\ell_m}$, and so

$$\begin{aligned} \left|\frac{\partial}{\partial r}\mathcal{L}_m(r,s)\right| &\leqslant \frac{1}{2\ell_m+1}\left[(\ell_m+1)\left(\frac{r}{s}\right)^{\ell_m} + \ell_m\left(\frac{s}{r}\right)^{\ell_m+1}\right] \\ &\leqslant \frac{1}{2\ell_m+1}\left[(\ell_m+1)\left(\frac{r}{s}\right)^{\ell_m} + \ell_m\left(\frac{s}{r}\right)^{\ell_m}\right] \leqslant \left(\frac{r}{s}\right)^{\ell_m}. \end{aligned}$$

Q.E.D.

Using these bounds in the following statement we find majorants for the functions (8.37).

Proposition 190. *For all* $r \in [0,b]$ *the inequalities hold*

(i) $|\psi_k^m(r)| \leqslant \left(\dfrac{2}{2\ell_m+1}\right)^k \dfrac{r^{\ell_m+1}}{k!}\left(\displaystyle\int_0^r s|q(s)|ds\right)^k \quad \forall k \geqslant 0,$

(ii) $|(\psi_k^m)'(r)| \leqslant \left(\frac{2}{2\ell_m+1}\right)^{k-1} \frac{r^{\ell_m}}{k!} \left(\int_0^r s|q(s)|ds\right)^k \quad \forall k \geqslant 1.$

In consequence, $\{\psi_k^m\}_{k=0}^\infty \in C^1[0,b] \cap C^2(0,b]$*. Furthermore, for* $k \geqslant 1$*,* ψ_k^m *satisfies the recursive equation (8.35).*

Proof. For the first inequality, note that for $k=0$ it is fulfilled. For $k \geqslant 1$ we proceed by induction. For $k=1$ we have

$$\begin{aligned} |\psi_1^m(r)| &\leqslant \int_0^r |\mathcal{L}_m(r,s)||q(s)|s^{\ell_m+1}ds \\ \text{(Lemma 189(i))} &\leqslant \frac{2}{2\ell_m+1}\int_0^r \frac{r^{\ell_m+1}}{s^{\ell_m}}|q(s)|s^{\ell_m+1}ds \\ &= \frac{2}{2\ell_m+1} r^{\ell_m+1}\int_0^r s|q(s)|ds. \end{aligned}$$

This proves the case $k=1$. We assume the validity of inequality (i) for $k>1$, and prove it for $k+1$,

$$\begin{aligned} |\psi_{k+1}^m(r)| &\leqslant \int_0^r |\mathcal{L}_m(r,s)||q(s)||\psi_k^m(s)|ds \\ \text{(Lemma 189(i))} &\leqslant \frac{2}{2\ell_m+1}\int_0^r \frac{r^{\ell_m+1}}{s^{\ell_m}}|q(s)||\psi_k^m(s)|ds \\ \text{(Induction hypothesis)} &\leqslant \frac{2}{2\ell_m+1} r^{\ell_m+1}\int_0^r \frac{|q(s)|}{s^{\ell_m}}\left(\frac{2}{2\ell_m+1}\right)^k \frac{s^{\ell_m+1}}{k!}\left(\int_0^s \zeta|q(\zeta)|d\zeta\right)^k ds \\ &= \left(\frac{2}{2\ell_m+1}\right)^{k+1}\frac{r^{\ell_m+1}}{k!}\int_0^r s|q(s)|\left(\int_0^s \zeta|q(\zeta)|d\zeta\right)^k ds \\ &= \left(\frac{2}{2\ell_n+1}\right)^{k+1}\frac{r^{\ell_m+1}}{(k+1)!}\left(\int_0^r s|q(s)|ds\right)^{k+1}. \end{aligned}$$

This concludes the induction and proves (i) for all $k \geqslant 0$. For the derivative, first note that

$$\begin{aligned} (\psi_k^m)'(r) &= \mathcal{L}_m(r,r)q(r)\psi_{k-1}^m(r) + \int_0^r \frac{\partial}{\partial r}\mathcal{L}_m(r,s)q(s)\psi_{k-1}^m(s)ds \\ &= \int_0^r \frac{\partial}{\partial r}\mathcal{L}_m(r,s)q(s)\psi_{k-1}^m(s)ds, \end{aligned}$$

because $\mathcal{L}_m(r,r)=0$. Then, applying Lemma 189(i) and the inequality (i) we have

$$\begin{aligned} |(\psi_k^m)'(r)| &\leqslant \int_0^r \left|\frac{\partial}{\partial r}\mathcal{L}_m(r,s)q(s)\psi_{k-1}^m(s)\right| ds \\ &\leqslant \int_0^r \left(\frac{r}{s}\right)^{\ell_m}|q(s)|\left(\frac{2}{2\ell_m+1}\right)^{k-1}\frac{s^{\ell_m+1}}{(k-1)!}\left(\int_0^s \zeta|q(\zeta)|d\zeta\right)^{k-1} ds \\ &= \left(\frac{2}{2\ell_m+1}\right)^{k-1}\frac{r^{\ell_m}}{k!}\left(\int_0^r s|q(s)|ds\right)^k. \end{aligned}$$

Hence, $\{\psi_k^m\}_{k=0}^\infty$ and their derivatives are well defined with $\psi_k^m(0) = (\psi_k^m)'(0) = 0$. Thus, $\{\psi_k^m\}_{k=0}^\infty \subset C^1[0,b] \cap C^2(0,b]$. Finally, simple calculation shows that ψ_k^m satisfies the recursive relation (8.35). **Q.E.D.**

The convergence of the series (8.34) is established in the next theorem.

Theorem 191. *For $m \in \mathbb{N}$, the functional series (8.34), with the series of the first derivatives, converges absolutely and uniformly on the whole segment $[0,b]$, and the series of the second derivatives converges absolutely and uniformly on each compact interval $[\delta, b] \subset (0,b]$. The function y_m defined by the series (8.34) is a regular solution of the perturbed Bessel equation (8.33), that satisfies the asymptotic conditions $y_m(r) \sim r^{\ell_m+1}$, $y_m'(r) \sim (\ell_m+1)r^{\ell_m}$, $r \to 0^+$.*

Proof. We establish the convergence of the series (8.34). Let $x \in [0,b]$. By Proposition 190, we have

$$\begin{aligned}\sum_{k=0}^\infty |\psi_k^m(r)| &\leqslant \sum_{k=0}^\infty \left(\frac{2}{2\ell_m+1}\right)^k \frac{r^{\ell_m+1}}{k!}\left(\int_0^r s|q(s)|ds\right)^k \\ &\leqslant b^{\ell_m+1}\sum_{k=0}^\infty \frac{1}{k!}\left(\frac{2}{2\ell_m+1}\|sq(s)\|_{L_1[0,b]}\right)^k \\ &= b^{\ell_m+1}\exp\left(\frac{2}{2\ell_m+1}\|sq(s)\|_{L_1[0,b]}\right).\end{aligned}$$

Hence by the Weierstrass M-test, the series $y_m(r) = \sum_{k=0}^\infty \psi_k^m(r)$ converges absolutely and uniformly on $[0,b]$. In a similar way one can show that the series of the first derivatives converges on $[0,b]$ (using the bounds of Proposition 190 (ii)). For the second derivative, take $0 < \delta < b$ and $r \in [\delta, b]$. By the recursive relation $(\psi_k^m)'' = -q(r)\psi_{k-1}^m - \frac{\ell_m(\ell_n+1)}{r^2}\psi_k^m$, we have

$$\sum_{k=1}^\infty |(\psi_k^m)''(r)| \leqslant \max_{r\in[0,b]}|q(r)| \cdot \sum_{k=0}^\infty |\psi_k^m(r)| + \frac{\ell_m(\ell_m+1)}{\delta^2}\sum_{k=1}^\infty |\psi_k^m(r)|.$$

Since both series on the right hand side converge absolutely and uniformly, the same is true on the interval $[\delta, b]$ for the series of second derivatives as well. Then $y_m \in C^1[0,b] \cap C^2(0,b]$. Thus, substituting the series of y_m into (8.33) and using the recursive relation (8.35) we obtain that y_m is a regular solution. Finally, in order to establish the asymptotics of $y_m(r)$, we note that

$$|y_m(r) - r^{\ell_m+1}| \leqslant \sum_{k=1}^{\infty} \left(\frac{2}{2\ell_m+1}\right) \frac{r^{\ell_m+1}}{k!} \left(\int_0^r s|q(s)|ds\right)^k$$
$$= r^{\ell_m+1} \left\{\exp\left(\frac{2}{2\ell_m+1}\int_0^r s|q(s)|ds\right) - 1\right\}.$$

Then $\left|\frac{y_m(r)}{r^{\ell_m+1}} - 1\right| \leqslant \left\{\exp\left(\frac{2}{2\ell_m+1}\int_0^r s|q(s)|ds\right) - 1\right\}$. The right hand side tends to zero when $r \to 0^+$. Hence $\lim_{r\to 0^+} \frac{y_m(r)}{r^{\ell_n+1}} = 1$, which is the first asymptotic relation sought. The proof of the second is similar. **Q.E.D.**

Remark 192. *From the proof of Theorem 191 we obtain additionally two important facts.*

1. *For arbitrary $m \in \mathbb{N}$ we have the bound*

$$|\phi_m(r)| = \left|\frac{y_m(r)}{r^{\ell_m+1}}\right| \leqslant \exp\left(\frac{2}{2\ell_m+1}\|sq(s)\|_{L_1[0,b]}\right)$$
$$= \exp\left(\frac{\|sq(s)\|_{L_1[0,b]}}{2m+d-1}\right) \leqslant \exp\left(\|sq(s)\|_{L_1[0,b]}\right),$$

   so $\{\phi_m\}_{m=0}^{\infty} \subset C[0,b]$ is uniformly bounded.

2. *For the case $b = 1$ we have that*

$$|\phi_m(1) - 1| = |y_m(1) - 1| \leqslant \left\{\exp\left(\frac{\|sq(s)\|_{L_1[0,1]}}{2m+d-1}\right) - 1\right\}.$$

   The right hand side tends to zero when $m \to \infty$, hence we conclude that $\lim_{m\to\infty} \phi_m(1) = 1$.

In this way the existence of the functions $\phi_m(r)$ is established, $m \in \mathbb{N}$.

When $\ell_m = -\frac{1}{2}$, the idea is to implement a similar procedure for $\mathbf{B}_0\psi_0(r) = -\psi_0''(r) - \frac{1}{4r^2}\psi_0(r)$. In this case a fundamental set of solutions is given by $\{r^{\frac{1}{2}}, r^{\frac{1}{2}}\log(r)\}$, and the Wronskian is $W(r^{\frac{1}{2}}, r^{\frac{1}{2}}\log(r)) = 1$. The existence of a regular solution for $\mathbf{B}_0 y_0(r) = 0$ is established under the condition that the potential satisfies the condition $r(1-\log(r))q(r) \in L_1[0,b]$ (see [73]). This condition may be weakened of the type $r^{1-\varepsilon}q(r) \in L_1[0,b]$, for some $0 < \varepsilon < 1$ (see [93]). Either of these two conditions is satisfied by a continuous potential.

Following the same procedure we define

$$\mathcal{L}_0(r,s) = \sqrt{rs}\,(\log(r) - \log(s)) = \sqrt{rs}\log\left(\frac{r}{s}\right) \qquad \text{for } (r,s) \in (0,b] \times (0,b]$$

The function $\mathcal{L}_0$ satisfies the same conditions as those of $\mathcal{L}_0(r,r)$,

$$\frac{\partial}{\partial r}\mathcal{L}_0(r,s)\Big|_{r=s} = \sqrt{\frac{s}{r}}\left[\frac{1}{2}\log\left(\frac{r}{s}\right)+1\right]\Big|_{r=s} = 1,$$

and satisfies the equation $\frac{\partial^2}{\partial r^2}\mathcal{L}_0(r,s) = \frac{1}{4r^2}\mathcal{L}_0(r,s)$. Also the following bounds are valid.

Lemma 193. *The function $\mathcal{L}_0$ satisfies for $0 < s \leqslant r \leqslant b$ the following inequalities*

(i) $|\mathcal{L}_0(r,s)| \leqslant r\sqrt{\frac{r}{s}}$.

(ii) $\left|\frac{\partial}{\partial r}\mathcal{L}_0(r,s)\right| \leqslant \frac{3}{2}\sqrt{\frac{r}{s}}$.

Proof. Using the inequality

$$|\log(x)| \leqslant \frac{1}{2}\left|x-\frac{1}{x}\right| \qquad \text{for } \; x>0$$

and that $\frac{s}{r} \leqslant 1 \leqslant \frac{r}{s}$ we have

$$|\mathcal{L}_0(r,s)| \leqslant \frac{\sqrt{rs}}{2}\left|\frac{r}{s}-\frac{s}{r}\right| \leqslant \sqrt{rs}\cdot\frac{r}{s} = r\sqrt{\frac{r}{s}}.$$

For the derivative, we have

$$\begin{aligned}\left|\frac{\partial}{\partial r}\mathcal{L}_0(r,s)\right| &= \sqrt{\frac{s}{r}}\left|\frac{1}{2}\log\left(\frac{r}{s}\right)+1\right| \\ &\leqslant \sqrt{\frac{s}{r}}\left[\frac{1}{2}\left|\log\left(\frac{r}{s}\right)\right|+1\right] \\ &\leqslant \sqrt{\frac{s}{r}}\left[\frac{r}{2s}+1\right] \\ &= \frac{1}{2}\sqrt{\frac{r}{s}}+\sqrt{\frac{s}{r}} \leqslant \frac{3}{2}\sqrt{\frac{r}{s}}.\end{aligned}$$

Q.E.D.

Now, we define the system of functions

$$\psi_k^0(r) := \begin{cases}\sqrt{r}, & \text{for } \; k=0,\\ \displaystyle\int_0^r \mathcal{L}_0(r,s)q(r)\psi_{k-1}^0(s)ds, & \text{for } \; k\geqslant 1.\end{cases}$$

Repeating the same procedure as for the previous case, one can obtain the bounds for this system of functions.

Proposition 194. *For all $r \in [0, b]$ the functions $\{\psi_k^0\}_{k=0}^\infty$ satisfy the inequalities*

(i) $|\psi_k^0(r)| \leqslant \dfrac{\sqrt{r}b^k}{k!}\left(\int_0^r |q(t)|dt\right)^k \quad \forall k \geqslant 0.$

(ii) $|(\psi_k^0)'(r)| \leqslant \dfrac{\sqrt{r}b^{k-1}}{k!}\left(\int_0^r |q(t)|dt\right)^k \quad \forall k \geqslant 1.$

Similarly to Theorem 191 we obtain the following statement.

Theorem 195. *The functional series $y_0(r) = \sum_{k=0}^\infty \psi_k^0(r)$ and the series of the first derivatives converge absolutely and uniformly on $[0, b]$, and the series of the second derivatives converges absolutely and uniformly on each compact interval $[\delta, b] \subset (0, b]$. Furthermore, the function y_0 is a regular solution of the perturbed Bessel equation $\mathbf{P}_0 y_0(r) = 0$, that satisfies the asymptotic conditions $y_0(r) \sim \sqrt{r}$, $y_0'(r) \sim \frac{1}{2\sqrt{r}}$, $r \to 0^+$.*

Proof. The proof of the convergence of the series is similar to that from Theorem 191. Thus, $y_0 \in C^1[0, b] \cap C^2(0, b]$. Also the system of functions satisfies the recurrence relation $\mathbf{B}_0 \psi_k^0(r) = -q(r)\psi_{k-1}^0(r)$, for $k \geqslant 1$, hence y_0 is a regular solution of $\mathbf{B}_0 y_0(r) = 0$. Finally, we have the estimate

$$\left|\frac{y_0(r)}{\sqrt{r}} - 1\right| \leqslant \frac{1}{\sqrt{r}} \sum_{k=1}^\infty \frac{\sqrt{r}b^k}{k!}\left(\int_0^r |q(t)|dt\right)^k = \left\{\exp\left(b\int_0^r |q(s)|ds\right) - 1\right\}.$$

The right hand side tends to zero when $r \to 0^+$, from where we get that $\lim\limits_{r\to 0^+} \dfrac{y_0(r)}{\sqrt{r}} = 1$. The proof of the second asymptotic condition is similar. **Q.E.D.**

Hence, the function $\phi_0(r) = \frac{y_0(r)}{\sqrt{r}}$ is continuous, and the system of functions $\{\phi_m\}_{m=0}^\infty \subset C[0, b]$ is uniformly bounded. As a corollary of this observation, we obtain the absolute convergence of the expansion in terms of the formal powers.

Corollary 196. *Any function $u \in \mathrm{Sol}^{\mathbf{S}}(B_b^d(0))$ admits a series of the form*

$$u(x) = \sum_{m=0}^\infty \mathcal{U}_m(x),$$

where $\mathcal{U}_m \in \mathcal{S}_m(B_b^d(0))$, and the series converges ***absolutely and uniformly*** *on each compact subset of $B_b^d(0)$.*

Proof. Given $u \in \mathrm{Sol}^{\mathbf{S}}(B_b^d(0))$, take $h \in Har(B_b^d(0))$ with $u = \mathbf{T}h$. By Theorem 17, h admits an expansion $u(x) = \sum_{m=0}^{\infty} p_m(x)$, with $p_m \in \mathcal{H}_m(\mathbb{R}^d)$, and this series converges absolutely and uniformly on compact subsets of $B_b^d(0)$. Applying the transmutation operator we obtain the expansion $u(x) = \sum_{m=0}^{\infty} \mathcal{U}_m(x)$ and the uniform convergence. For the absolute convergence, note that

$$\sum_{m=0}^{\infty} |\mathcal{U}_m(x)| = \sum_{m=0}^{\infty} |\phi_m(r)||p_m(x)| \leqslant \left(\sup_{m\in\mathbb{N}_0} \|\phi_m\|_{C[0,b]}\right) \cdot \sum_{m=0}^{\infty} |p_m(x)|,$$

and the last series converges absolutely. Thus, the expansion of u converges absolutely.

Q.E.D.

As an application, we construct the solution of the Dirichlet $\mathfrak{D}_{(0,\varphi)}$ problem in the unit ball $\mathbb{B}^d$. We mention here another approach based on the transmutation operator $\mathbf{T}$ for solving the Dirichlet problem, developed in [57, 58, 59] where for the case of a positive potential q and $\varphi \in C(\mathbb{S}^{d-1})$ the unique solution u of $\mathfrak{D}_{(0.\varphi)}$ is constructed as $u = \mathbf{T}h$ with h being given as a double layer potential of some solution of an integral Fredholm equation.

Using the orthogonal basis of $\mathrm{Sol}_2^{\mathbf{S}}(\mathbb{B}^d)$ we obtain an explicit expression for the solution of $\mathfrak{D}_{(0,\varphi)}$, as a series of the formal powers.

Proposition 197. *Suppose that $\mathfrak{D}_{(0,0)}$ has only the trivial solution. Given $\varphi \in C^p(\mathbb{S}^{d-1})$ with $p = 2\left[\frac{d+3}{4}\right]$, the unique solution of the Dirichlet problem $\mathfrak{D}_{(0,\varphi)}$ can be written as the series*

$$u(x) = \sum_{m=0}^{\infty}\sum_{k=1}^{d_m} \hat{u}_k^{(m)} \mathcal{V}_k^{(m)}(x), \tag{8.38}$$

where

$$\hat{u}_k^{(m)} := \frac{\langle \varphi, Y_k^{(m)}\rangle_{L_2(\mathbb{S}^{d-1})}}{\phi_m(1)} \quad \text{for } m \in \mathbb{N}_0,\ k = \overline{1, d_m}. \tag{8.39}$$

The series (8.38) converges in $L_2(\mathbb{B}^d)$, and uniformly and absolutely in $\overline{\mathbb{B}^d}$.

Proof. We look for a solution in the form (8.38). On the boundary we have

$$\varphi(x') = u(1, x') = \sum_{m=0}^{\infty}\sum_{k=1}^{d_m} \hat{u}_k^{(m)} \mathcal{V}_k^{(m)}(1, x') = \sum_{m=0}^{\infty}\sum_{k=1}^{d_m} \hat{u}_k^{(m)} \phi_m(1) Y_k^{(m)}(x').$$

By Proposition 20 (5), $\{\{Y_k^{(m)}\}_{k=1}^{d_m}\}_{m=0}^{\infty}$ is an orthogonal basis for $L_2(\mathbb{S}^{d-1})$. Then the coefficients of the last expansion have the form

$$\hat{u}_k^{(m)}\phi_m(1) = \widehat{\varphi}_k^{(m)} := \langle \varphi, Y_k^{(m)} \rangle_{L_2(\mathbb{S}^{d-1})}, \quad m \in \mathbb{N}_0,\, k = \overline{1, d_m}.$$

If $\phi_m(1) = 0$ for some m, then $\mathcal{V}_k^{(m)}$ is a solution for the radial Schrödinger equation with $\mathcal{V}_k^{(m)}\big|_{\mathbb{S}^{d-1}} \equiv 0$, and is a non trivial solution for $\mathfrak{D}_{(0,0)}$, which contradicts the hypothesis. Thus, $\phi_m(1) \neq 0$ for all m and we obtain formula (8.39). By Remark 192 (2), $\lim_{n\to\infty} \phi_m(1) = 1$, and $\phi_m(1) \neq 1$ for all m, hence the sequence $\{\frac{1}{\phi_m(1)}\}_{m=0}^{\infty}$ is bounded, so we have

$$\sum_{m=0}^{\infty}\sum_{k=1}^{\infty} |\hat{u}_k^{(m)}|^2 \leqslant \left(\sup_{m\in\mathbb{N}_0} \frac{1}{|\phi_m(1)|}\right)^2 \cdot \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} |\widehat{\varphi}_k^{(m)}|^2 < +\infty.$$

Thus, the series (8.38) converges in $L_2(\mathbb{B}^d)$. Moreover, since $\varphi \in C^p(\mathbb{S}^{d-1})$ with $p = 2\left[\frac{d+3}{4}\right]$, its Fourier expansion in spherical harmonics converges absolutely and uniformly on $\mathbb{S}^{d-1}$ (see [67]), $\sum_{m=0}^{\infty}\sum_{k=1}^{d_m} |\widehat{\varphi}_k^{(m)}| < \infty$, and hence

$$\sum_{m=0}^{\infty}\sum_{k=1}^{d_m} |r^m \phi_m(r) \hat{u}_k^{(m)} Y_k^{(m)}(x')| \leqslant \left(\sup_{m\in\mathbb{N}_0} \|\phi_m\|_{C[0,b]}\right) \cdot \left(\sup_{m\in\mathbb{N}_0} \frac{1}{|\phi_m(1)|}\right) \cdot \sum_{m=0}^{\infty}\sum_{k=1}^{d_m} |\widehat{\varphi}_k^{(m)}| < \infty.$$

With the aid of the Weierstrass M-test we obtain the absolute and uniform convergence in $\overline{\mathbb{B}^d}$. **Q.E.D.**

Remark 198. *For $d = 2$ it is sufficient that $\varphi \in \mathrm{Lip}(\mathbb{S}^1)$ in order to guarantee the absolute and uniform convergence of its Fourier series on $\mathbb{S}^1$ (see [10, Ch. II]).*

Example 199. *Consider the Helmholtz equation $\triangle u(x) + \kappa^2 u(x) = 0$. In this case, the equation for the coefficients $\{\phi_m\}_{m=0}^{\infty}$ is given by*

$$r\phi_m''(r) + (2m + d - 1)\phi_m' + r\kappa^2\phi_m(r) = 0.$$

For $d = 3$, the solutions of the equation $r\phi_m''(r) + 2(m+1)\phi_m' + r\kappa^2\phi_m(r) = 0$ are given by $\phi_m(r) = \sqrt{\frac{2}{\pi}}\frac{j_m(\kappa r)}{r^m}$, where $j_m(r) = \sqrt{\frac{\pi}{2r}}J_{m+\frac{1}{2}}(r)$ are the spherical Bessel functions. Then an orthogonal complete system in three dimensions is

$$\mathcal{V}_k^{(m)}(r,\theta,\phi) = \sqrt{\frac{(2m+1)(m-k)!}{2\pi^2(m+k)!}}\, j_m(\kappa r) P_k^{(m)}(cos(\theta)) e^{ik\phi} \quad m \in \mathbb{N}_0; k = \overline{-m, m},$$

with $0 < r \leqslant b; 0 \leqslant \theta \leqslant \pi; 0 < \phi < 2\pi$. This coincides with known results for the Helmholtz equation in three dimensions (see, e.g., [43, p. 44]).

Example 200. *Consider in* $\mathbb{R}^3$ *on some bounded star-shaped domain the radial equation*

$$\triangle_3 u(x) - |x|^2 u(x) = 0.$$

In this case, the associated perturbed Bessel equation is

$$-y''(r) + \left(\frac{m(m+1)}{r^2} + r^2\right) y(r) = 0 \quad \text{for } 0 < r \leqslant b. \tag{8.40}$$

Applying formulas (8.37) for the system of functions $\{\psi_k^m\}_{k=0}^{\infty}$*, a direct computation shows that*

$$\psi_0^m(r) := r^{m+1};\ \psi_k^m(r) = \frac{r^{m+1+4k}}{4\cdot 8\cdots(4k)(2m+1)(2m+5)\cdots(2m+1+4k)} \quad \text{for } k \in \mathbb{N}.$$

The last expressions can be written as

$$\psi_k^m(r) = \frac{\Gamma\left(\frac{2m+5}{4}\right) r^{m+1+4k}}{4^{2k} k! \Gamma\left(\frac{2m+1}{4} + k + 1\right)} \quad \text{for } k \in \mathbb{N}_0.$$

By Theorem 191, the regular solution is given by

$$y_m(r) = r^{m+1} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{2m+5}{4}\right) r^{4k}}{4^{2k} k! \Gamma\left(\frac{2m+1}{4} + k + 1\right)} = \Gamma\left(\frac{2m+5}{4}\right) \cdot 2^{m+\frac{1}{2}} \cdot \sqrt{r} \cdot I_{\frac{2m+1}{4}}\left(\frac{r^2}{2}\right).$$

Hence an orthogonal complete system of solutions for (8.40) has the form

$$\mathcal{V}_k^{(m)}(r,\theta,\phi) = \sqrt{\frac{(2m+1)(m-k)!}{4\pi(m+k)!}} \cdot \frac{\Gamma\left(\frac{2m+5}{4}\right) \cdot 2^{m+\frac{1}{2}}}{r^{m+\frac{1}{2}}} \cdot I_{\frac{2m+1}{4}}\left(\frac{r^2}{2}\right) P_k^{(m)}(cos(\theta)) e^{ik\phi},$$

for $m \in \mathbb{N}_0$*,* $k = \overline{-m, m}$*; and* $0 < r \leqslant b$*;* $0 \leqslant \theta \leqslant \pi; 0 \leqslant \phi < 2\pi$*.*

8.7 A relation with a transmutation operator for the perturbed Bessel equation

Consider the radial part of the Laplacian $\triangle_d$ and define the radial operators

$$\mathbf{L} := \frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr}, \quad \mathbf{L}_q := \mathbf{L} - q(r),$$

defined on $C^2[0,b]$. Note that the Laplacian in polar coordinates has the formal form $\mathbf{L} + \frac{c^2}{r^2}$ with $c = \frac{\partial}{\partial\theta}$. The operator $\mathbf{T}$ can be seen as an operator in $C[0,b]$ having a Volterra type form

$$\mathbf{T}y(r) = y(r) + \int_0^r K(r,s) y(s) ds \quad \text{for} \quad y \in C[0,b]$$

with $K(r,s) = \frac{\rho^{\frac{d}{2}-1}}{2r^{\frac{d}{2}}} G\left(r, 1-\frac{s}{r}\right)$. It was shown in Section 8.2 that $\mathbf{T}$ is a transmutation operator for the pair $r^2\mathbf{L}_q$ and $r^2\mathbf{L}$. A direct computation shows that K satisfies the equation

$$r^2 K_{rr} + (d-1)rK_r + [(d-2) - r^2 q(r)]K = \rho^2 K_{\rho\rho} + (5-d)\rho K_\rho \quad \text{for } (r,s) \in \Pi$$

where $\Pi := \{(r,s) \,|\, 0 < s \leqslant r \leqslant b\}$ (see [11, Ch. IV]). Aditionally, K satisfies the condition

$$K(r,r) = \frac{1}{2r}\int_0^r tq(t)dt.$$

If we consider $K(r,s) = \frac{\kappa(r,s)}{s^{3-d}}$, then the function κ satisfies the problem:

$$\begin{cases} r^2\left(\triangle_r - q(r)\right)\kappa(r,s) = s^2 \triangle_s \kappa(r,s) & \text{for } (r,s) \in \Pi \\ \kappa(r,r) = -\frac{1}{2r^{d-2}}\int_0^r tq(t). \end{cases}$$

Denote $\widehat{\mathbf{L}}_q := r^2\mathbf{L}_q$ and $\widehat{\mathbf{L}} = r^2\mathbf{L}$. Then $\widehat{\mathbf{L}}_q\mathbf{T} = \mathbf{T}\widehat{\mathbf{L}}$ on $C^2[0,b]$.

Let $y \in C^2[0,b]$ be such that $u = r^{\frac{1-d}{2}}y \in C[0,b]$. Applying $\mathbf{L}_q$ to u and proceeding as in the proof of Theorem 176, we obtain

$$\mathbf{L}_q u = r^{\frac{1-d}{2}}\left[y'' - q(r)y - \left(\frac{d-1}{2}\right)\left(\frac{d-3}{2}\right)\frac{y}{r^2}\right] = r^{\frac{1-d}{2}}\left[y'' - q(r)y - \frac{\ell_d(\ell_d+1)}{r^2}y\right]$$

where $\ell_d = \frac{d-3}{2}$. The operator $\mathbf{P} := \frac{d^2}{dr^2} - q(r) - \frac{\ell_d(\ell_d+1)}{r^2}$ is the *perturbed Bessel* operator. We have the relation $\mathbf{L}_q\left(r^{\frac{1-d}{2}}y\right) = r^{\frac{1-d}{2}}\mathbf{P}y$.

Definition 201. *Let $\alpha \in \mathbb{R}$. We define the* ***radial multiplication*** *as the operator $\mathbf{M}^\alpha : \mathcal{D}(\mathbf{M}^\alpha) \subset C[0,b] \to C[0,b]$ given by*

$$\mathbf{M}^\alpha y(r) := r^\alpha y(r) \quad \text{for } y \in C[0,b] \text{ with } r^\alpha \in C[0,b].$$

Note that $\mathbf{M}^\alpha$ is a densely defined operator, and for $\alpha \geqslant 0$, $\mathcal{D}(\mathbf{M}^\alpha) = C[0,b]$. The operator satisfies the commutativity relation $\mathbf{M}^\alpha\mathbf{M}^\beta = \mathbf{M}^\beta\mathbf{M}^\alpha = \mathbf{M}^{\alpha+\beta}$.

Let $\boldsymbol{B} := \frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}$ denote the *spherical Bessel operator*. Then the following (transmutation) relations are valid

$$\mathbf{L}_q\mathbf{M}^{-\frac{d-1}{2}} = \mathbf{M}^{-\frac{d-1}{2}}\mathbf{P}, \quad \mathbf{L}\mathbf{M}^{-\frac{d-1}{2}} = \mathbf{M}^{-\frac{d-1}{2}}\mathbf{B}. \tag{8.41}$$

Due to the commutativity of the operators $\mathbf{M}^\alpha$ similar relations are valid for the operators $\widehat{\mathbf{P}} := \mathbf{M}^2\mathbf{P}$ and $\widehat{\mathbf{B}} := \mathbf{M}^2\mathbf{B}$.

It is well known that there exists an operator

$$\tilde{\mathbf{T}}f(x) = f(x) + \int_0^x V(x,t)f(t)dt, \tag{8.42}$$

such that the kernel V is a continuous function satisfying

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} - q(x) - \frac{\ell(\ell+1)}{x^2}\right) V(x,t) = \left(\frac{\partial^2}{\partial t^2} - \frac{\ell(\ell+1)}{t^2}\right) V(x,t), \\ V(x,x) = \dfrac{1}{2}\displaystyle\int_0^x tq(t)dt, \quad \lim_{t\to 0^+} t^\ell V(x,t) = 0 \end{cases}$$

and the condition $\sup\limits_{0\leqslant x\leqslant b} \displaystyle\int_0^x |V(x,t)|^2 dt < \infty$ (see [36]) . Operator $\tilde{\mathbf{T}}$ satisfies the condition

$$\mathbf{P}\tilde{\mathbf{T}} = \tilde{\mathbf{T}}\mathbf{B} \quad \text{in } \mathcal{E}_\ell[0,b], \tag{8.43}$$

where $\mathcal{E}_\ell[0,b] := \left\{y \in C^2[0,b] \,|\, y(0) = 0 \text{ and } y'(r) = \mathcal{O}(r^\ell), r \to 0^+, \text{ if } \ell = -\frac{1}{2}\right\}$ (see [94]). The following theorem establishes a relation between $\mathbf{T}$ and $\tilde{\mathbf{T}}$.

Theorem 202. *The operators $\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$ satisfy the equalities*

$$\mathbf{L}_q\left(\mathbf{M}^{-\frac{d-1}{2}}\tilde{\mathbf{T}}\right) = \left(\mathbf{M}^{-\frac{d-1}{2}}\tilde{\mathbf{T}}\right)\mathbf{B}. \tag{8.44}$$

and

$$\widehat{\mathbf{P}}\left(\mathbf{M}^{\frac{d-1}{2}}\mathbf{T}\mathbf{M}^{-\frac{d-1}{2}}\right) = \left(\mathbf{M}^{\frac{d-1}{2}}\mathbf{T}\mathbf{M}^{-\frac{d-1}{2}}\right)\widehat{\mathbf{B}}. \tag{8.45}$$

Proof. Note that

$$\begin{aligned} \mathbf{L}_q\left(\mathbf{M}^{-\frac{d-1}{2}}\tilde{\mathbf{T}}\right) &= \left(\mathbf{L}_q\mathbf{M}^{-\frac{d-1}{2}}\right)\tilde{\mathbf{T}} \\ \text{By (8.41)} &= \mathbf{M}^{-\frac{d-1}{2}}\left(\mathbf{P}\tilde{\mathbf{T}}\right) \\ \text{By (8.43)} &= \mathbf{M}^{-\frac{d-1}{2}}\left(\tilde{\mathbf{T}}\mathbf{B}\right), \end{aligned}$$

and we obtain the relation (8.44). On the other hand, since $\widehat{\mathbf{L}}_q = \mathbf{M}^{-\frac{d-1}{2}}\widehat{\mathbf{P}}\mathbf{M}^{\frac{d-1}{2}}$ and $\widehat{\mathbf{L}} = \mathbf{M}^{-\frac{d-1}{2}}\widehat{\mathbf{B}}\mathbf{M}^{\frac{d-1}{2}}$, we have

$$\left(\mathbf{M}^{-\frac{d-1}{2}}\widehat{\mathbf{P}}\mathbf{M}^{\frac{d-1}{2}}\right)\mathbf{T} = \mathbf{T}\left(\mathbf{M}^{-\frac{d-1}{2}}\widehat{\mathbf{B}}\mathbf{M}^{\frac{d-1}{2}}\right).$$

Hence

$$\widehat{\mathbf{P}}\left(\mathbf{M}^{\frac{d-1}{2}}\mathbf{T}\mathbf{M}^{-\frac{d-1}{2}}\right) = \left(\mathbf{M}^{\frac{d-1}{2}}\mathbf{T}\mathbf{M}^{-\frac{d-1}{2}}\right)\widehat{\mathbf{B}}.$$

Q.E.D.

Chapter 9

Conclusions and future work

Two new applications of transmutation operator theory to direct and inverse spectral problems for the SLEIF and to the construction of complete system solutions for the radial Schrödinger equation, were presented. The following results were obtained.

1. A transmutation operator that transforms solutions of $v'' + \rho^2 v = 0$ into solutions of the SLEIF is constructed. Its main properties of boundedness and invertibility are proved. A Fourier-Legendre series representation for the integral transmutation kernel is obtained with a recursive integration procedure for the expansion coefficients, together with a representations of the solutions $C(\rho, x)$ and $S_h(\rho, x)$ as NSBF.

2. A direct method for solving the inverse spectral problem for the SLEIF is presented. It is based on an integral representation for the solution with a continuous kernel, for which a Gelfand-Levitan equation is derived. The method reduces the inverse problem to an infinite system of linear algebraic equations for the Fourier-Legendre coefficients of the series expansion of the integral kernel, and the impedance function is recovered from the first coefficient of the solution vector. The numerical realization of the method is simple and involves nothing but the built-in functions of a modern numerical computing environment, such as Matlab.

3. An integral representation for the Jost solution and its derivative is obtained, together with an expansion in the form of a power series defined in the unit disk of the variable $z = \frac{\frac{1}{2}+i\rho}{\frac{1}{2}-i\rho}$. The coefficients of the series expansion can be computed

recursively. As was shown in [38, 78] this kind of representations are useful for the numerical computation of the Jost solution. The spectral data of the Sturm-Liouville problem are characterized and explicit formulas are obtained for the calculation of the eigenvalues, the normalizing constants and the spectral function.

4. Results on the Runge property for strongly elliptic equations are reported which allow one to extend the results on the completeness of systems of classical solutions onto essentially arbitrary bounded Lipschitz domains and spaces of weak solutions. This is especially important in the context of the transmutation operator theory which provides the possibility of efficient construction of complete systems of solutions for equations with variable coefficients.

5. The continuity and invertibility of the transmutation operator for the radial Schödinger equation, and its inverse, are established together with the transmutation property, a Fourier-Jacobi series representation for the integral transmutation kernel and uniform estimates for the approximations. From the Fourier-Jacobi series a complete system of solutions for (8.1) (called formal powers) is obtained. The completeness is established in the sense of uniform convergence on compact subsets and in L_2 and H^2-norms. The complete system can be applied to approximate solutions of boundary value problems by means of linear combinations of formal powers. The formal powers can also be used in the construction of the Green function or the Bergman reproducing kernels related with boundary value problems.

Of course, there are still several interesting questions to be answered, for example:

1. The construction of transmutation operators for less regular impedance functions, for example, for impedance functions from $W^{1,1}(-b, b)$.

2. The invertibility of the transmutation operators $\mathbf{T}_h$, with $h \neq 1$, as well as a formula to relate operators corresponding to different parameters.

3. The solution of the inverse problem for an impedance function $a \in W^{1,p}(0, \pi)$, for $1 \leqslant p \leqslant \infty$, which includes the derivation of a Gelfand-Levitan equation and the asymptotic relations for the spectral data.

4. The derivation of a Gelfand-Levitan equation for the Sturm-Liouville problem (5.1), (5.2) for $h \neq 0$, as well as its solution.

5. Develop a procedure for solving the dispersion problem for Eq. (1.46) in the whole line, analogous to the obtained for the Schrödinger equation [69, 77].

6. The construction of a transmutation operator and a complete system of solutions for the Schrödinger equation possessing other symmetries, for example, in cylindrical coordinates.

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