CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS
AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL

UNIDAD ZACATENCO
DEPARTAMENTO DE MATEMÁTICAS

Complejidad Topológica Simétrica y Plan Motriz

T E S I S

QUE PRESENTA

ISAAC ORTIGOZA SUÁREZ

PARA OBTENER EL GRADO DE
DOCTOR EN CIENCIAS

EN LA ESPECIALIDAD DE
MATEMÁTICAS

DIRECTOR DE TESIS:
DR. JESÚS GONZÁLEZ ESPINO BARROS

CIUDAD DE MÉXICO. NOVIEMBRE, 2022

Cinvestav

CENTER FOR RESEARCH AND ADVANCED
STUDIES OF THE NATIONAL POLYTECHNIC INSTITUTE

ZACATENCO CAMPUS
DEPARTMENT OF MATHEMATICS

Symmetrical Topological Complexity and Motion Planning

T H E S I S

SUBMITTED BY

ISAAC ORTIGOZA SUÁREZ

TO OBTAIN THE DEGREE OF
DOCTOR OF SCIENCE

IN THE SPECIALITY OF
MATHEMATICS

THESIS ADVISOR:
DR. JESÚS GONZÁLEZ ESPINO BARROS

MEXICO CITY. NOVEMBER, 2022

Contents

A mis padres

José Luis Ortigoza

y

Lucia Teresa Suárez

A mi esposa Lupita Reyes

A mis hijas Regina e Isabella Ortigoza

Agradecimientos

En esta ocasión quiero dividir los agradecimientos en cuatro grandes partes, que han sido pilares, que me sostienen y me alientan a seguir superándome: Mi familia, las instituciones que me han formado y a las que espero contribuir, los maravillosos amigos que he conocido en esta etapa de mi vida, y mi razón de ser quienes me alientan con su presencia a seguir superándome, así pues, en ese orden, antes que nada quiero agradecer a mi familia:

> A José Luis Ortigoza te extraño mucho papá, en gran medida, este logro es por ti. Siempre extrañaré esos momentos en los que platicábamos de matemáticas. El amor que le tengo a esta ciencia es gracias a ti.
>
> A Lucía Teresa Suárez, porque me haz dado amor, consejos, apoyo, comprensión y por haber sido un ejemplo en mi vida. Mamá ni en tres vidas me alcanzarían para devolverte todo lo que haz hecho por mí. Te amo mucho MAMÁ y te agradezco por todo.

Ambos me han formado como un ser de valores y principios, lo que considero, es la parte más difícil en la educación.

> A mi hermana: Ingrid Ortigoza a quien admiro por sus logros, quien siempre me ha dado su apoyo de manera incondicional.
>
> A mi tío Mauricio Ortigosa quien, en cierto modo, es mi hermano mayor, quien ha sido un modelo de éxito y desarrollo integral.
>
> A Adriana Paredes una gran amiga y ahora parte muy importante de mi familia, quien siempre tiene esa actitud positiva y llena de energía que nos contagia a todos.
>
> A mis primos Toño y Mary, por su ayuda en diversos aspectos de mi vida, por sus comentarios constructivos; por haber sido siempre un ejemplo de esfuerzo y determinación.

Al resto de mi familia por sus contribuciones a mi formación brindadas a lo largo de los años.

Mi agradecimiento a los académicos, las instituciones y a las personas que las integran, quienes contribuyeron para formar profesionistas de gran calidad que puedan ayudar a resolver algunas de las problemáticas del país, gracias por haber depositado su confianza en mí:

A Jesús González, quien fue un excelente maestro que supo motivarme y entusiasmarme por la topología algebráica; cada día contigo fue un aprendizaje, me hiciste reflexionar y creo que soy una mejor persona gracias a ti. También quiero darte las gracias por tu apoyo y paciencia, para la elaboración de esta tesis.

Quiero agradecer a los profesores del Cinvestav, por sus consejos, su tiempo y por los conocimientos complementarios a este trabajo, ya que sin ellos no habría notado ni disfrutado muchas de las posibles conexiones y ramificaciones del mismo.

Además no puedo dejar de mencionar a Miguel A. Xicoténcatl, Ruy Fabila Monroy, José Martínez Bernal, y a Carlos G. Pacheco por ser mis maestros, y quienes me ayudaron a descubrir, jugar y gozar las matemáticas en esta etapa.

Al Cinvestav por sus recursos y espacios, así como al apoyo administrativo, de secretaría (Norma Acosta, Anabel Lagos, Roxana Martínez, Adriana Aranda, Laura Valencia y Omar Hernández) de vigilancia, mantenimiento y de intendencia porque su labor es muy valiosa.

A Conacyt por el apoyo económico otorgado sin el cual este trabajo posiblemente no se habría logrado, y al fomento a la ciencia que realizan con diversas actividades y dinámicas.

A esos hermanos que la vida ha puesto en mi camino, y cuya influencia ha sido muy importante.

A mis grandes amigos que se convirtieron en hermanos Rex (Daniel Vázquez), Roberto Fierros, Duck (Francisco Leandro), Yannick de Icaza, Anaid Rosas, con quienes crecí y con quienes me gustaría tratar el resto de mi vida.

A mis amigos del Cinvestav: Nestor Colin, Alejandro González, Max E. Mitre, Yulieth K. Prieto, Fabiola Rodríguez, Rodolfo Salinas, Raul Alvarez, Miguel E. Uribe, Wincy A. Guerra, Bárbara M. Gutiérrez, Jonathan J. Gutiérrez, Christopher J. Roque, Carlos E. Vivares, Carlos Castro, Alejandra Fonseca, Hugo Corrales, Alejandro Avilés con ellos he aprendido, platicado y reído matemáticas, de no ser por ellos la vida matemática no tendría el sentido que tiene.

Al Acapulco (Jesus Angel Lara Rivera), Eduardo López y a Isidro Morales con quienes he compartido, no sólo pláticas interesantes de matemáticas, también cine, teatro, vacaciones, etcétera; ellos son, quienes para mí, le dan vida al departamento.

A Natalia Cadavid Aguilar muchas gracias por todo. Fuiste dos veces mi hermana academica. Les deseo mucha suerte a ti y a Mario en esta nueva etapa de su vida.

A Oscar Adrián Méndez Lara, con quien he compartido muchas platicas, café y se ha vuelto un gran amigo.

Quiero agradecer de manera muy especial a Nadia Huerta Sánchez por todo, no sólo ha sido una amiga, si no que se ha convertido en la compañera de muchas aventuras, siempre ha estado ayudándome en muchos aspectos, sin mencionar los momentos tan agradables que hemos compartido cafeteando matemáticas.

También quiero agradecer de manera muy especial a Álvaro Martínez Ramírez por ser un gran amigo y compañero en la vida, con él aprendí que a los matemáticos les gustan los polinomios, pero sólo hasta cierto grado.

Por último quiero agradecer a las personas más importantes en mi vida, a quienes me motivan, e inspiran para seguir superándome en todos los aspectos.

Quiero expresarle mi más sincera gratitud a mi esposa Lupita Reyes, por ser un breve cielo en la tierra, por ser el amor de mi vida y con quien espero compartir el resto de nuestros años en el mundo. Te doy gracias mi amor porque cuando sonríes serenas mi cansancio, cuando me abrazas me das consuelo cuando creo que las cosas estan saliendo mal. Tú eres la mejor esposa para mí y si hubiera que elegir otra vez te eligiría un millón de veces.

A mi hija Regina Ortigoza Reyes el día que escuche tu corazón, supe que mi vida había cambiado para siempre. Mi amor amo jugar contigo y gracias por cuidarme.

A mi pequeña Isabella Ortigoza Reyes, amo ser tu papá. Me encantan tus chinos y tu sonrisa.

Mis hijas son la luz de mis ojos y el motivo de éste y todos mis esfuerzos. Son el más grande tesoro de mi vida y tengo la fortuna de compartartirlo con su madre.

Resumen

La motivación de la tesis se debe a que M. Farber y M. Grant introducen el concepto de complejidad topológica simétrica $\mathrm{TC}^S(X)$ aunado a que I. Basabe, J. González, Y. B. Rudyak, y D. Tamaki introducen el concepto de complejidad topológica simetrizada $\mathrm{TC}^\Sigma(X)$, esto es por la necesidad de crear una planificación de movimiento que satisfaga la condición

$$\alpha(B,A)(t) = \alpha(A,B)(1-t) \quad \text{cuando} \quad A \neq B.$$

Estos conceptos se presentan en el primer capítulo. Definimos fórmulas análogas a las que dieron Armindo Costa and Michael Farber, para describir la acción de la monodromía, en la fibracion $\mathrm{NL}(X)/\mathbb{Z}_2 \longrightarrow B(X,2)$.

En el segundo capítulo explicamos las presentaciones que mostró J. Birman para los grupos de trenzas sobre el toro $P_2(T)$ y $B_2(T)$. Esta es la herramienta esencial para los resultados más importantes de la tesis.

El objetivo principal de la tesis es estudiar la categoría seccional de las fibraciones $\mathrm{NL}(T) \longrightarrow F(T,2)$ y $\mathrm{NL}(T)/\mathbb{Z}_2 \longrightarrow B(T,2)$. Para esto en el tercer capítulo construimos una resolución libre de $\mathbb{Z}$ sobre $\mathbb{Z}_{P_2(T)}$ con una cantidad mínima de generadores, y un levantamiento sobre la resolución barra de $\mathbb{Z}_{P_2(T)}$ que permite calcular los productos en la cohomología y mostramos un plan motriz para la primer fibración.

El resultado más importante de la tesis se encuentra en el capítulo cuatro. Construimos una resolución libre minimal de $\mathbb{Z}$ sobre $\mathbb{Z}_{B_2(T)}$ y se dieron formulas explícitas para una homotopía de contracción. Usando la técnica descrita por David Handel construimos una aproximación diagonal, con la cual obtenemos fórmulas explícitas que permiten calcular el producto cup.

En el capítulo cinco usamos la herramienta construida para calcular la complejidad topológica simétrica $\mathrm{TC}^S(T)$ y simetrizada sobre el toro $\mathrm{TC}^\Sigma(T)$, usando los resultados de Jesús González y Schwarz. Con este resultado se completan los cálculos para todas las superficies cerradas.

Abstract

The motivation of the thesis is due to the fact that M. Farber and M. Grant introduce the concept of symmetric topological complexity $\mathrm{TC}^S(X)$ and I. Basabe, J. González, Y. B. Rudyak and D. Tamaki introduce the concept of symmetrized topological complexity $\mathrm{TC}^\Sigma(X)$, this is due to the need to create a movement planning that satisfies the condition

$$\alpha(B,A)(t) = \alpha(A,B)(1-t) \quad \text{when} \quad A \neq B.$$

These concepts are presented in the first chapter. We define analogous formulas to those given by Armindo Costa and Michael Farber, to describe the action of monodromy, in the fibration $\mathrm{NL}(X)/\mathbb{Z}_2 \longrightarrow B(X,2)$.

In the second chapter we explain the presentations that J. Birman showed for the groups of braids on the torus $P_2(T)$ and $B_2(T)$. This is the essential tool for the most important results of the thesis.

The main goal of the thesis is to study the sectional category of the fibrations $\mathrm{NL}(T) \longrightarrow F(T,2)$ and $\mathrm{NL}(T)/\mathbb{Z}_2 \longrightarrow B(T,2)$. For this in the third chapter we build a free resolution of $\mathbb{Z}$ over $\mathbb{Z}_{P_2(T)}$ with a minimal number of generators, and a lifting over the bar resolution of $\mathbb{Z}_{P_2(T)}$ that allows us to calculate the products in the cohomology and we show a motion planning for the first fibration.

The most important result of the thesis is found in chapter 4. We construct a minimal free resolution of $\mathbb{Z}$ over $\mathbb{Z}_{B_2(T)}$ and explicit formulas for a contraction homotopy were given. Using the technique described by David Handel we construct a diagonal approximation. We obtain explicit formulas that allow us to calculate the cup products.

In chapter five we use these tools to compute the symmetric topological complexity $\mathrm{TC}^S(T)$ and symmetrized on the torus $\mathrm{TC}^\Sigma(T)$, using the results of Jesús González and Schwarz. With this result the calculations for all closed surfaces are completed.

Introduction

M. Farber and M. Grant introduce the concept of symmetric topological complexity $\mathrm{TC}^S(X)$ in [1] and I. Basabe, J. González, Y. B. Rudyak and D. Tamaki introduce the concept of symmetrized topological complexity $\mathrm{TC}^\Sigma(X)$ in [2], this is due to the need to create a movement planning that satisfies the condition

$$\alpha(B,A)(t) = \alpha(A,B)(1-t) \quad \text{when} \quad A \neq B.$$

The symmetrized topological complexity $\mathrm{TC}^\Sigma(X)$ is a homotopy invariant of X, whereas it is not known whether $\mathrm{TC}^S(X)$ satisfies the homotopy invariance. Much of the interest in TC^Σ and TC^S comes from the relation

$$\max\{\mathrm{TC}^S(X) - 1, \mathrm{TC}(X)\} \leqslant \mathrm{TC}^\Sigma(X) \leqslant \mathrm{TC}^S(X),$$

which holds for reasonably well-behaved spaces (see [2]). Together with Farber-Grant's upper estimate $\mathrm{TC}^S(X) \leqslant 2\dim(X)$ in [1] holding when X is a closed manifold, and Cohen-Vandembroucq's result that $\mathrm{TC}(S)$ is maximal possible for all closed surfaces $S_{\geq 2}$ of genus at least two (orientable of not), we see that $\mathrm{TC}^\Sigma(S_{\geq 2}) = \mathrm{TC}^S(S_{\geq 2}) = 4$. The situation for the projective plane $\mathbb{R}P^2$ is completely similar as the equality $\mathrm{TC}^\Sigma(\mathbb{R}P^2) = \mathrm{TC}^S(\mathbb{R}P^2) = 4$ is known from [3, Theorem 1.7]. However, the situation for the 2-sphere S^2 is somehow special as $\mathrm{TC}^\Sigma(S^2) = \mathrm{TC}^S(S^2) = 2$, in view of [2, Example 4.5]. As a consequence of the calculations in this thesis, we are able to settle (Theorem 5.2) the situation in the case of the torus T, the only remaining closed surface: $\mathrm{TC}^\Sigma(T) = \mathrm{TC}^S(T) = 3$.

One consequence that we have is that the Torus is an example where the general inequality

$$\mathrm{TC}^\Sigma(X \times Y) \leqslant \mathrm{TC}^\Sigma(X) + \mathrm{TC}^\Sigma(Y)$$

proven in [3, Lemma 4.9] is strict.

Chapter 1

Symmetric and Symmetrized TC

1.1 Configuration Spaces and Braid Groups

Given a space X, the configuration space of ordered pairs of distinct points in X is

$$F(X,2) = \{(x,y) \in X \times X \mid x \neq y\}.$$

The group $\mathbb{Z}_2$ acts on $F(X,2)$ by permutation of coordinates, and we write

$$B(X,2) = F(X,2)/\mathbb{Z}_2$$

for the configuration space of unordered pairs of distinct points in X. We use the notation

$$\pi_1(F(X,2)) = P_2(X) \quad \text{and} \quad \pi_1(B(X,2)) = B_2(X),$$

when the spaces are pathwise connected.

Using the Fadell-Newirth fibration and the associated homotopy exact sequence it can be shown (see for instance F. Cohen and J. Pakianathan [4]) that $F(X,2)$ is a $K(P_2(X),1)$-space, if X is $\mathbb{R}^2$ or a closed 2-manifold other than the sphere or the projective plane. In particular, from the 2-fold covering

$$\mathbb{Z}_2 \overset{j}{\hookrightarrow} F(X,2) \twoheadrightarrow B(X,2)$$

we get $\pi_n(F(X,2)) = 0 = \pi_n(B(X,2))$ for all $n \geqslant 2$, together with the short exact sequence

$$1 \longrightarrow P_2(X) \longrightarrow B_2(X) \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{1.1}$$

Then $B(X,2)$ is also a $K(B_2(X),1)$ and we can interpret $B_2(X)$ as the braid group of X on 2 strings, understanding $P_2(X)$ as the pure braid subgroup. Thus $P_2(X)$ is a normal subgroup of $B_2(X)$ of index 2.

G.P. Scott gives presentations of the braid groups $B_2(X)$ and $P_2(X)$ if X is a closed 2-manifold different of S^2 or $\mathbb{R}P^2$ [5]. In this work, we use the presentation given by D.L. Gonçalves, J. Guaschi, and M. Maldonado for $P_2(T)$ [6] and by K. Murasugi and B.I. Kurpita for $B_2(T)$ [7], where $T = S^1 \times S^1$ stands for the 2-torus.

1.2 Symmetric Motion Planning

The *sectional category* of a fibration $p : E \longrightarrow B$, denoted by $\mathrm{secat}(p)$, is the least integer n such that the base space B can be covered by $n+1$ open subspaces on each of which p admits a section. If no such n exists one sets $\mathrm{secat}(p) = \infty$.

The *topological complexity* $\mathrm{TC}(X)$ of a space X is defined as the sectional category of the evaluation map $e_{0,1} : P(X) \longrightarrow X \times X$ which takes a free path γ in X to its end points: $e_{0,1}(\gamma) = (\gamma(0), \gamma(1))$. This concept is intended to give a homotopical framework for studying the motion planning problem in robotics.

Farber-Grant introduced the concept of *symmetric topological complexity* [1], this is motivated by the need to create a motion planning that satisfies the condition

$$\alpha(B,A)(t) = \alpha(A,B)(1-t) \quad \text{for} \quad A \neq B.$$

We write $\mathrm{NL}(X)$ for the complement of the free loop space $\mathrm{L}(X)$ in the free path space on X. The group $\mathbb{Z}_2$ acts on a nonloop α by traveling it in the opposite direction, that is $\alpha \mapsto \overline{\alpha}$ where $\overline{\alpha}(t) = \alpha(1-t)$. The inclusion map $\mathrm{NL}(X) \longrightarrow PX$ gives us a restricted fibration $e_{0,1} : \mathrm{NL}(X) \longrightarrow F(X,2)$ which passes to the quotient giving the commutative diagram

$$\begin{array}{ccccc}
\widetilde{M} & \twoheadleftarrow & \widetilde{M} & = & \widetilde{M} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{NL}(X)/\mathbb{Z}_2 & \twoheadleftarrow & \mathrm{NL}(X) & \hookrightarrow & P(X) \\
\downarrow \varepsilon_{0,1} & & \downarrow e_{0,1} & & \downarrow e_{0,1} \\
B(X,2) & \twoheadleftarrow & F(X,2) & \hookrightarrow & X \times X.
\end{array} \tag{1.2}$$

Here $\widetilde{M}$ denotes the path space $\mathrm{Maps}([0,1],0,1;X,x_0,x_1)$ for a point (x_0,x_1) in $F(X,2)$. Alternatively, when X is path connected, we have the commutative diagram with fiber columns

$$\begin{array}{ccccc} \Omega X & = & \Omega X & = & \Omega X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{NL}(X)/\mathbb{Z}_2 & \twoheadleftarrow & \mathrm{NL}(X) & \hookrightarrow & P(X) \\ \downarrow{\scriptstyle \varepsilon_{0,1}} & & \downarrow{\scriptstyle e_{0,1}} & & \downarrow{\scriptstyle e_{0,1}} \\ B(X,2) & \twoheadleftarrow & F(X,2) & \hookrightarrow & X \times X. \end{array} \tag{1.3}$$

Remark 1.1
Observe that the diagram

$$\begin{array}{ccc} \mathrm{NL}(X) & \twoheadrightarrow & \mathrm{NL}(X)/\mathbb{Z}_2 \\ \downarrow{\scriptstyle e_{0,1}} & & \downarrow{\scriptstyle \varepsilon_{0,1}} \\ F(X,2) & \twoheadrightarrow & B(X,2). \end{array} \tag{1.4}$$

is a strict pullback, and that the fiber at $\{x_0, x_1\}$[1] is homeomorphic to $\mathrm{Maps}([0,1], 0, 1; X, x_0, x_1)$.

Definition 1.2 The ***symmetric topological complexity*** $\mathrm{TC}^S(X)$ of a space X is defined as one more than the sectional category of the evaluation map $\varepsilon_{0,1} : \mathrm{NL}(X)/\mathbb{Z}_2 \longrightarrow B(X,2)$ which takes the class of a unoriented nonloop α in X to the set of its end points on the configuration space of unordered pairs of points.

Definition 1.3 The ***symmetrized topological complexity*** of a space X, $\mathrm{TC}^\Sigma(X)$, is the smallest positive integer n for which $X \times X$ can be covered by $n+1$ open sets U each of which is closed under the switching involution κ on $X \times X$, and admits a continuous $\mathbb{Z}_2$-equivariant section $U \longrightarrow P(X)$ of the ($\mathbb{Z}_2$-equivariant) double evaluation map $e_{0,1} : P(X) \longrightarrow X \times X$.

Much of the interest in TC^Σ and TC^S comes from the relation

$$\max\{\mathrm{TC}^S(X) - 1, \mathrm{TC}(X)\} \leqslant \mathrm{TC}^\Sigma(X) \leqslant \mathrm{TC}^S(X), \tag{1.5}$$

which holds for reasonably well-behaved spaces (see [2]). Together with Farber-Grant's upper estimate $\mathrm{TC}^S(X) \leqslant 2\dim(X)$ in [1] holding when X is a closed manifold, and Cohen-Vandembroucq's result that $\mathrm{TC}(S)$ is maximal possible for all closed surfaces $S_{\geq 2}$ of genus at least two (orientable of not), we see that $\mathrm{TC}^\Sigma(S_{\geq 2}) = \mathrm{TC}^S(S_{\geq 2}) = 4$. The situation for the projective plane $\mathbb{R}P^2$

[1]The fiber at $\{x_0, x_1\}$ is $\mathrm{Maps}([0,1], 0, 1; X, x_0, x_1) \cup \mathrm{Maps}([0,1], 0, 1; X, x_1, x_0) / \sim$ where $\alpha \sim \overline{\alpha}$

is completely similar as the equality $\mathrm{TC}^{\Sigma}(\mathbb{R}P^2) = \mathrm{TC}^{S}(\mathbb{R}P^2) = 4$ is known from [3, Theorem 1.7]. However, the situation for the 2-sphere S^2 is somehow special as $\mathrm{TC}^{\Sigma}(S^2) = \mathrm{TC}^{S}(S^2) = 2$, in view of [2, Example 4.5]. As a consequence of the calculations in this thesis, we are able to settle (Theorem 5.2) the situation in the case of the torus T, the only remaining closed surface: $\mathrm{TC}^{\Sigma}(T) = \mathrm{TC}^{S}(T) = 3$.

1.3 Monodromy

There is a classical definition of sectional category in terms of fiberwise joins which is more appropriate for our purposes than the original one. We denote by $*^n E$ the n-fold fiberwise self-join of the fibration $p : E \longrightarrow B$ and by $j^n(p) : *^n\, E \longrightarrow B$ the n-join map. Then $\mathrm{secat}(p) \leqslant n$ if and only if $j^{n+1}(p)$ has a cross section [8].

We begin by giving a description of the action of $\pi_1(B(X, 2)) = B_2(X)$ on ΩX, which is the homotopic fiber of the map $\varepsilon_{0,1} : \mathrm{NL}(X)/\mathbb{Z}_2 \longrightarrow B(X, 2)$.

Given an element γ in $B_2(X)$ at the based point $\{x_0, x_1\}$ in $B(X, 2)$, we can interpret γ as an element in the braid group of X on 2 strings, that is $\alpha,\ \beta : [0, 1] \longrightarrow X$ where $|\{\alpha(t), \beta(t)\}| = 2$ for all t in $[0, 1]$, $\{\alpha(0), \beta(0)\} = \{\alpha(1), \beta(1)\} = \{x_0, x_1\}$. We pick an arbitrary ordering (x_0, x_1). Let α be the path that ends in x_0 and β the one that ends in x_1.

Let M denote the space $\mathrm{Maps}([0, 1], 0, 1; X, x_0, x_1) \cup \mathrm{Maps}([0, 1], 0, 1; X, x_1, x_0)\,/\mathbb{Z}_2$. By definition (see for instance [9]), we need to describe a homotopy $H : M \times [0, 1] \longrightarrow \mathrm{NL}(X)/\mathbb{Z}_2$ fitting in a strictly commutative diagram

$$\begin{array}{ccc} M & \hookrightarrow & \mathrm{NL}(X)/\mathbb{Z}_2 \\ \downarrow{\scriptstyle j_1} & \overset{H}{\nearrow} & \downarrow{\scriptstyle \varepsilon_{0,1}} \\ M \times [0,1] & \xrightarrow[\gamma\circ p_2]{} & B(X,2), \end{array} \tag{1.6}$$

where $j_1 : M \longrightarrow M \times [0, 1]$ is defined as $j_1(x) = (x, 1)$.

We start by describing the action of $B_2(X)$ on $\widetilde{M} = \mathrm{Maps}([0, 1], 0, 1; X, x_0, x_1)$ which is the fiber of the map $e_{0,1} : \mathrm{NL}(X) \longrightarrow F(X, 2)$, and then we pass it to the quotient.

Let $\widetilde{H} : \widetilde{M} \times [0, 1] \longrightarrow \mathrm{NL}(X)$ be defined by

$$\widetilde{H}(\omega, t)(s) = \begin{cases} \alpha(3s + t), & \text{if } \quad 0 \leqslant s \leqslant \frac{1-t}{3}; \\ \omega\left(\frac{3s+t-1}{1+2t}\right), & \text{if } \quad \frac{1-t}{3} \leqslant s \leqslant \frac{2+t}{3}; \\ \beta\left(t + 3 - 3s\right), & \text{if } \quad \frac{2+t}{3} \leqslant s \leqslant 1. \end{cases} \tag{1.7}$$

These formulas are analogous to those described by Armindo Costa and Michael Farber in [10].

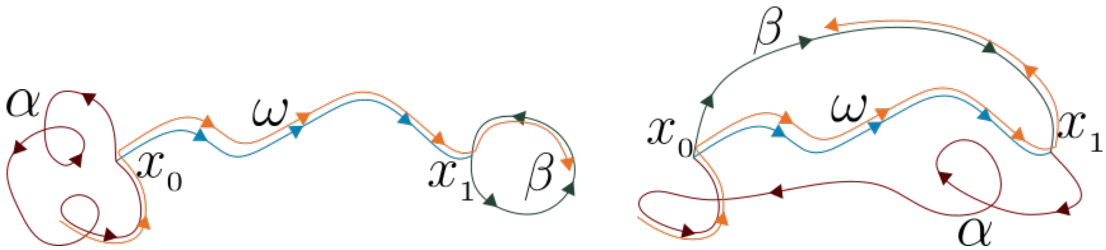


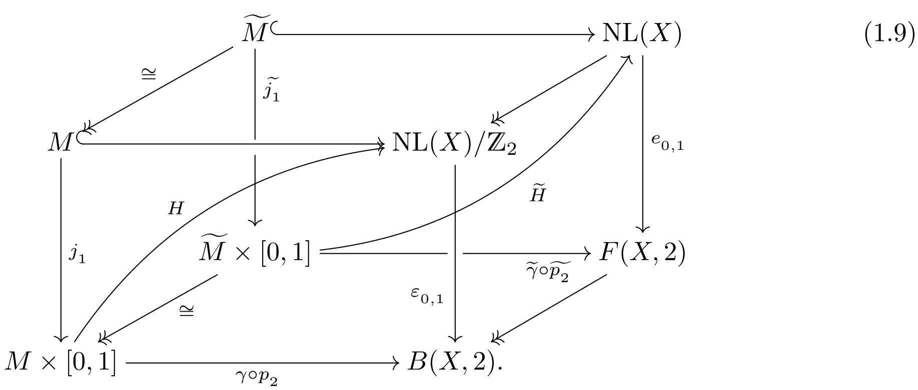
Figure 1.1: Monodromy Action

The map (1.7) renders the commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \hookrightarrow & \mathrm{NL}(X) \\ \downarrow{\scriptstyle \widetilde{j}_1} & \overset{\widetilde{H}}{\nearrow} & \downarrow{\scriptstyle e_{0,1}} \\ \widetilde{M}\times[0,1] & \xrightarrow[\widetilde{\gamma}\circ p_2]{} & F(X,2), \end{array} \tag{1.8}$$

were $\widetilde{\gamma}(t) = (\alpha(t), \beta(t))$.

We now take H to be the composition of the homeomorphism $M \times [0,1]$ to $\widetilde{M} \times [0,1]$, $\widetilde{H}$ and the projection from $\mathrm{NL}(X)$ onto $\mathrm{NL}(X)/\mathbb{Z}_2$. Then we get the following commutative diagram



Notice that the front face of (1.9) is exactly (1.6) and the triangles in the front commute in view of Lema (1.5) below.

Theorem 1.4
The monodromy action of $\gamma \in B_2(X)$ on M, denoted by $\cdot : B_2(X) \times M \longrightarrow M$, is given by

$[\gamma] \cdot [\omega] = [\{\alpha, \beta\}][\omega] = [\alpha\omega\overline{\beta}]$. Here ω is a path with endpoints x_0 and x_1, α is the path that ends in x_0 and β the one that ends in x_1.

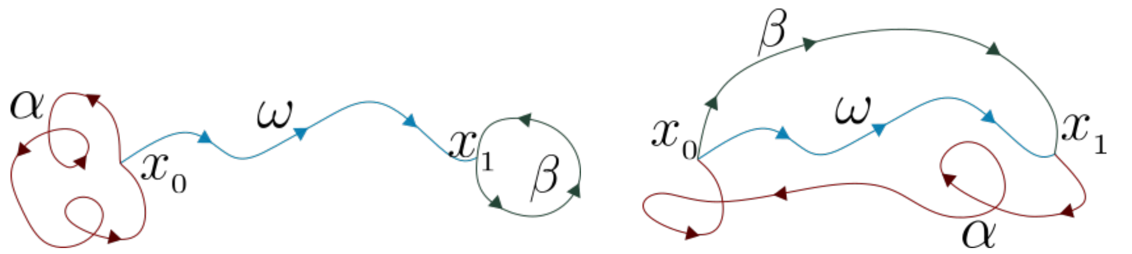
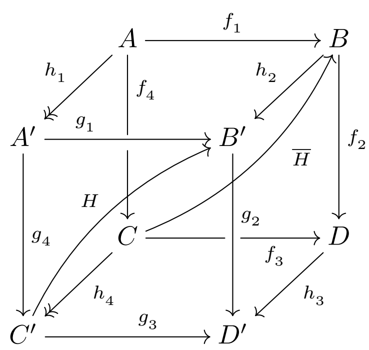


Figure 1.2: The Action of $B_2(X)$ into M.

The monodromy of the middle fiber in (1.2) is recovered as the restriction of the one on the right.

Lemma 1.5

Suppose that in the following diagram all the squares and back triangles commute and h_1, h_4 are epimorphism in some category. Then the triangles in front commute too.



Proof. The proof is routine. ■

Chapter 2

The 2-String Braid and Pure Braid Groups on the Torus

2.1 The Fundamental Groups of F(T,2) and B(T,2)

Let $T = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \cong S^1 \times S^1$ the oriented surface of genus one. We view a point on T in terms of its rectangular coordinates (u, v), where $u,\ v$ are real numbers module 1. The context will make it clear whether $(u, v) \in \mathbb{R} \times \mathbb{R}$ or $(u, v) \in T$. The configuration space of two distinct points in T is the space

$$F(T, 2) = \{(x, y) \in T \times T \mid x \neq y\}.$$

The group $\mathbb{Z}_2$ acts by permutation of coordinates. Write

$$B(T, 2) = F(T, 2)/\mathbb{Z}_2$$

for the configuration space of unordered two distinct points in X. The notation we have already used is

$$\pi_1(F(T, 2), (x_0, x_1)) = P_2(T) \quad \text{and} \quad \pi_1(B(T, 2), [x_0, x_1]) = B_2(T).^1$$

The group $P_2(T)$ is the 2-string pure braid group on T, and $B_2(T)$ is the 2-string full braid group on T. By the exact sequence (1.1) we can consider $P_2(T)$ as a subgroup of $B_2(T)$.

In [11], Joan S. Birman gives a presentation of the braids groups. We will give explicit formulas for generators of $B_2(T)$ and $P_2(T)$.[2] Consider the points

[1]For simplicity in notation, we omit writting base points whenever it is unnecessary to stress them explicity.

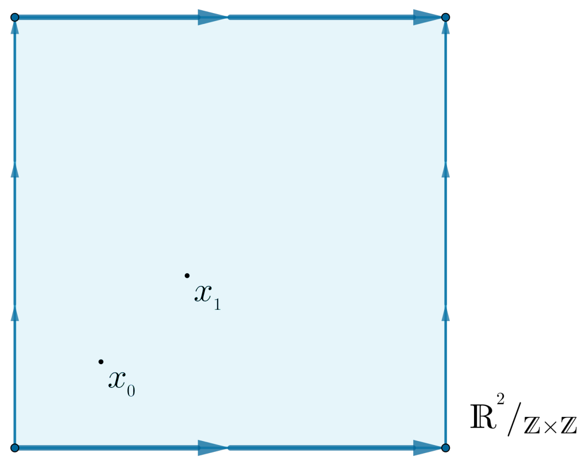
[2]In this work we will consider $P_2(T)$ and $B_2(T)$ as the groups based on (x_0, x_1) and $\{x_0, x_1\}$ respectively defined in (2.1).

$$x_{0} = \left(\frac{1}{5}, \frac{1}{5}\right), \quad x_{1} = \left(\frac{2}{5}, \frac{2}{5}\right). \tag{2.1}$$

Let $f_0,\ f_1,\ g_0,\ g_1 : [0,1] \longrightarrow T \times T$ be given by

$$\begin{aligned} f_0 =&(x_{0} + (0,t), x_{1}), \\ f_1 =&(x_{0}, x_{1} + (0,t)), \\ g_0 =&(x_{0} + (t,0), x_{1}), \\ g_1 =&(x_{0}, x_{1} + (t,0)). \end{aligned} \tag{2.2}$$

We define the elements $\tau_0,\ \tau_1, \in P_{2}(T) \subset B_{2}(T)$ as the classes represented by the elements f_0 and f_1 respectively. Consider also the elements $\rho_0,\ \rho_1$ represented by g_0 and g_1.



We now define $\gamma_{01} : [0,1] \longrightarrow T \times T$ to be the curve given by

$$\gamma_{01}(t) = \begin{cases} \left(x_0 + \left(\frac{2t}{5}, 0\right), x_1 - \left(\frac{2t}{5}, 0\right)\right) & if \quad t \in \left[0, \frac{1}{2}\right], \\ \left(\left(\frac{1}{5}, \frac{2}{5}\right) + \left(0, \frac{2t}{5}\right), \left(\frac{2}{5}, \frac{1}{5}\right) - \left(0, \frac{2t}{5}\right)\right) & if \quad t \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{2.3}$$

Then the element $\sigma \in B_{2}(T)$ is represented by γ_{01}. This element is illustrated by the representative loop in Figure(2.1).

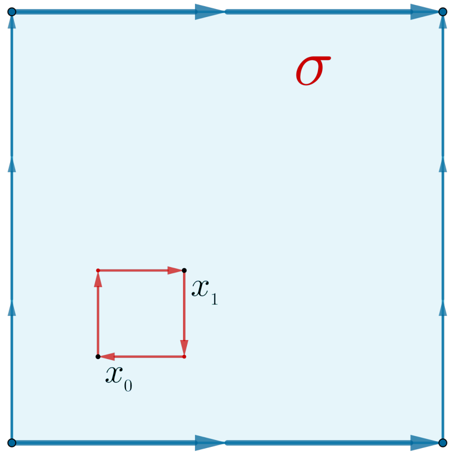


Figure 2.1: The loop $\sigma \in B_2(T)$.

Note that even though σ is not a pure braid, σ^2 is a pure braid. Then we define B as σ^2. The braid B is of the homotopy type of two loops, the first one begins at x_0 and surrounds x_1, and the second is the constant loop at x_1, as can be seen in Figure (2.2).

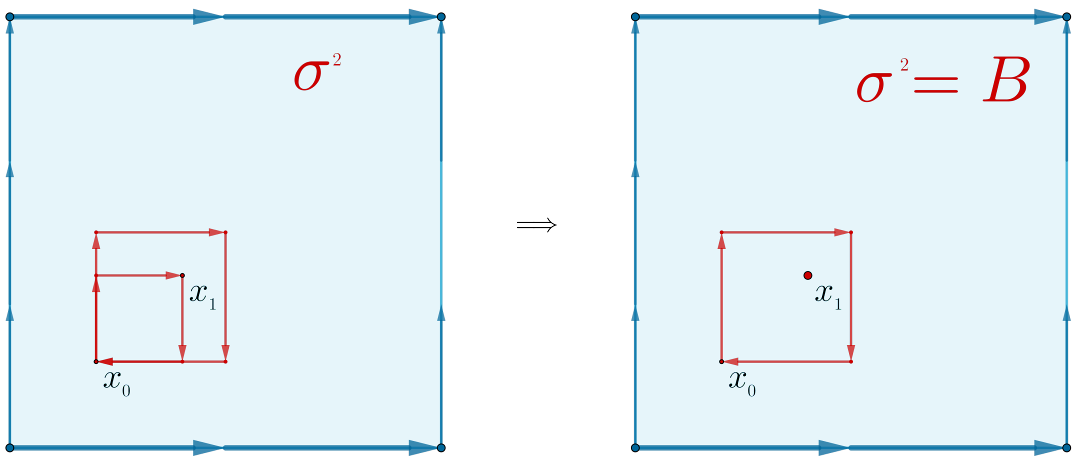


Figure 2.2: The homotopy type of B

To describe braids graphically on T, first consider $T \times [0,1]$, or equivalently consider the smaller torus T_1 inside the bigger torus T_0. The space between these two tori is denoted by $R \times S^1$, where R is an annulus. The smaller torus will also be referred to as the core of $R \times S^1$.

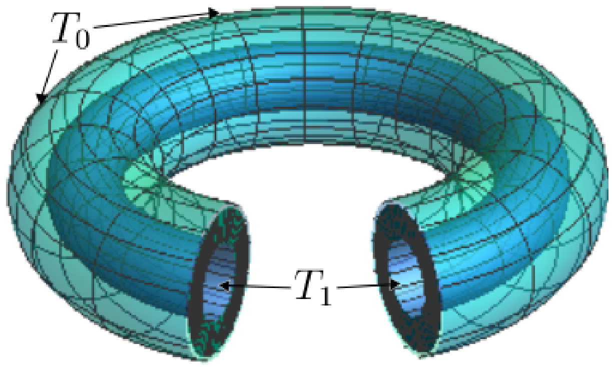


Figure 2.3: $R \times S^1$

Then, braids are strings in $R \times S^1$ that connect the base points x_0, x_1 on T_1 to the same points on T_0.

As can be seen from Figure(2.3), if we cut $R \times S^1$ open, then we can think of $R \times S^1$ as a cylinder whose ends we identify. See for example Figure (2.4).

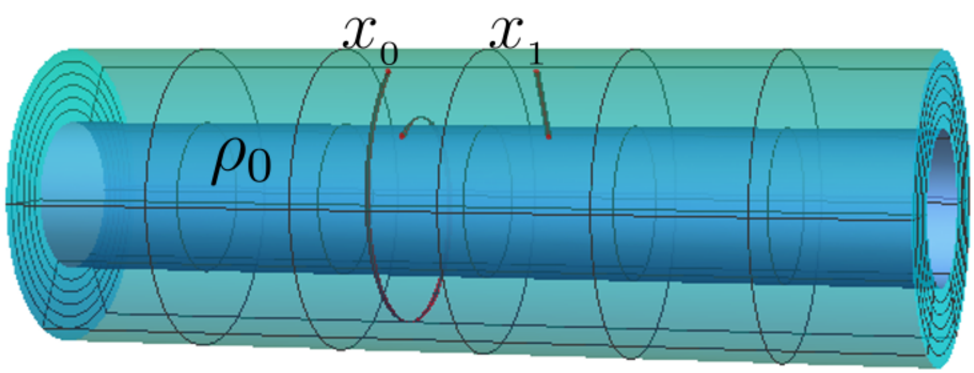


Figure 2.4: The braid ρ_0 in the Cylinder.

To further simplify the diagrams we need, we project the cylinder and its contents onto a vertical plane, this is just the analogue of a braid projection. In exactly the same way as for the a braid projection, we may add over "-" and undercrossing information to the projection, see for example Figure (2.5)

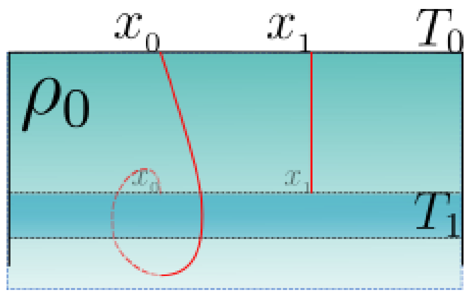


Figure 2.5: The braid ρ_0.

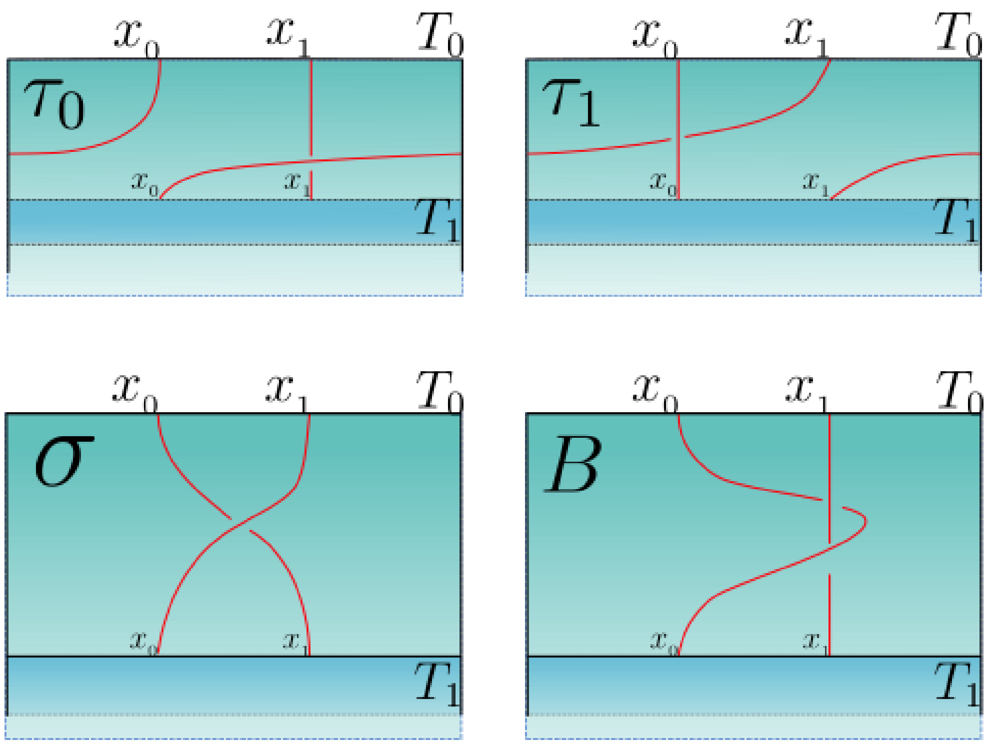


Figure 2.6: The braids τ_0, τ_1, σ and B.

As mentioned previusly, a presentation of the braids groups of the torus has been found and shown to be complete by J. Birman.

Theorem 2.1
For a torus T, the pure braid group, $P_2(T)$ has the following generators and relations, which in turn constitute a complete presentation for $P_2(T)$.

Generators:

$$\tau_0, \quad \tau_1, \quad \rho_0, \quad \rho_1, \quad B.$$

Relations:

$$\begin{array}{llllll}
1. & [\tau_0, \tau_1] & = & [\rho_0, \rho_1] & = & 1, \\
2. & \rho_0^{-1}\tau_1\rho_0\tau_1^{-1} & = & \rho_1\tau_0^{-1}\rho_1^{-1}\tau_0 & = & B, \\
3. & \tau_0^{-1}\tau_1^{-1}B\tau_1\tau_0 & = & \rho_0^{-1}\rho_1^{-1}B\rho_1\rho_0 & = & B, \\
4. & \rho_0^{-1}\tau_0^{-1}\rho_0\tau_0 & = & \rho_1\tau_1\rho_1^{-1}\tau_1^{-1} & = & B.
\end{array}$$

A presentation for the full braid group on T, $B_2(T)$, can then be obtained by adding σ to the set of generators for $P_2(T)$, and adding the following relations to the set of relations for $P_2(T)$:

$$\begin{array}{llllllll}
5. & \rho_1 & = & \sigma\rho_0\sigma, & & \tau_1 & = & \sigma\tau_0\sigma, \\
6. & \sigma^2 & = & B. & & & &
\end{array}$$

The proof can be found in [11].

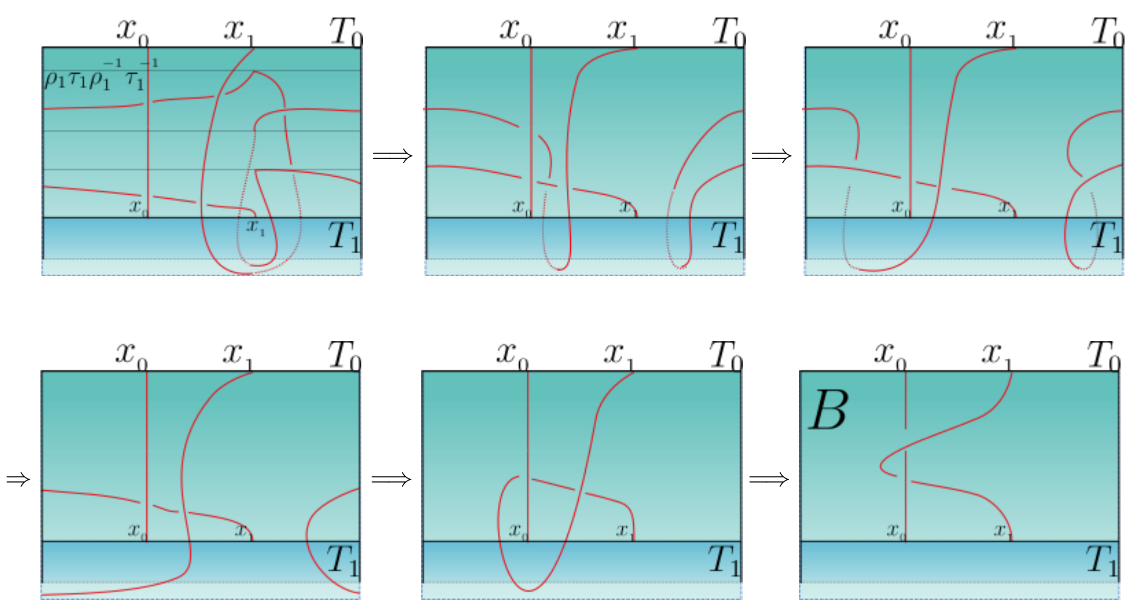


Figure 2.7: The relation $\rho_1\tau_1\rho_1^{-1}\tau_1^{-1} = B$.

In order to solve the word problem, we use a second presentation for $P_2(T)$ and $B_2(T)$.

Theorem 2.2
Define

$$a = \tau_0\tau_1, \quad b = \rho_0\rho_1, \quad x = \rho_1, \quad y = \tau_1.$$

Then the pure braid group $P_2(T)$ is generated by $a,\ b,\ x,\ y$ and B subject to the relations

$$[a,b]=[a,x]=[a,y]=[b,x]=[b,y]=1, \quad [x,y]=B. \tag{2.4}$$

That is; the group $P_2(T)$ is the direct sum of copies of $\mathbb{Z}$ generated by $a,\ b$ and a free group generated by x and y:

$$P_2(T)=\mathbb{Z}a\oplus\mathbb{Z}b\oplus\langle x,\ y\rangle. \tag{2.5}$$

Proof. The proof is routine, using the table of relations

$$\begin{array}{llllllll} a= & \tau_0\tau_1, & a^{-1}= & \tau_1^{-1}\tau_0^{-1}, & \tau_0= & ay^{-1}, & \tau_0^{-1}= & ya^{-1}, \\ b= & \rho_0\rho_1, & b^{-1}= & \rho_1^{-1}\rho_0^{-1}, & \rho_0= & bx^{-1}, & \rho_0^{-1}= & xb^{-1}, \\ x= & \rho_1, & x^{-1}= & \rho_1^{-1}, & \rho_1= & x, & \rho_1^{-1}= & x^{-1}, \\ y= & \tau_1, & y^{-1}= & \tau_1^{-1}, & \tau_1= & y, & \tau_1^{-1}= & y^{-1}. \end{array}$$

■

The assertion in (2.5) can also be observed in [11]. Actually, the above result is well known given that T is a topological group, for then $F(T,2)\simeq T\times(T\setminus\{x_1\})$. Using the presentation given by [12] and [5], D.L. Gonçalves, J. Guaschi and M. Maldonado give in [6] a different presentation of $P_2(T)$ leading to an isomorphism $P_2(T)\simeq\mathbb{Z}\oplus\mathbb{Z}\oplus F_2$[3] as in (2.5)

The next theorem can be found in [7].

Theorem 2.3
Define

$$\widetilde{a}=\rho_0\sigma, \quad \widetilde{b}=\sigma^{-1}\tau_0, \quad c=\widetilde{a}\widetilde{b}\sigma^{-1}=\rho_0\tau_0\sigma^{-1}.$$

The braid group $B_2(T)$ admits a presentation with generator $\widetilde{a},\ \widetilde{b},\ c$ subject to the relations

$$[\widetilde{a}^2,\widetilde{b}]=[\widetilde{a}^2,c]=[\widetilde{b}^2,\widetilde{a}]=[\widetilde{b}^2,c]=1, \quad \widetilde{a}^2\widetilde{b}^2=c^2. \tag{2.6}$$

Furthermore, the center of $B_2(T)$ is $\langle\widetilde{a}^2,\ \widetilde{b}^2 \mid \widetilde{a}^2\widetilde{b}^2=\widetilde{b}^2\widetilde{a}^2\rangle$, and

$$B_2(T)\Big/Z(B_2(T))=\langle\widetilde{a},\ \widetilde{b},\ c\mid\widetilde{a}^2=\widetilde{b}^2=c^2=1\rangle. \tag{2.7}$$

Proof. In accordance with [11], the full braid group $B_2(T)$ is generated by $\tau_0,\ \tau_1,\ \rho_0,\ \rho_1,\ \sigma$ and

[3] F_n is the free group generated by n elements.

B with relations given as in Theorem 2.1. The rest of the proof is routine. At any rate, alternative arguments are given at the end of the chapter implying (2.7). ■

Remark 2.4
In terms of the presentation in Theorem 2.3, the inclusion $P_2(T) \hookrightarrow B_2(T)$ is determined by

$$\begin{array}{rclrcl} a & \mapsto & \widetilde{b}^2, & b & \mapsto & \widetilde{a}^2, \\ x & \mapsto & c^{-1}\widetilde{a}\widetilde{b}\widetilde{a}, & y & \mapsto & \widetilde{a}^{-1}c. \end{array}$$

Proof. By Theorems 2.2 and 2.1, we have $a = \tau_0\tau_1$ so

$$a = \tau_0\tau_1 = (\sigma\widetilde{b})(\widetilde{b}\sigma^{-1}) = c^{-1}\widetilde{a}\widetilde{b}^2\widetilde{a}^{-1}c = \widetilde{b}^2.$$

Likewise, Theorems 2.2 and 2.1 yield $b = \rho_0\rho_1$ so

$$b = \rho_0\rho_1 = (\widetilde{a}\sigma^{-1})(\sigma\rho_0\sigma) = \widetilde{a}\sigma^{-1}\sigma\widetilde{a} = \widetilde{a}^2.$$

The rest of the proof is done in the same way. ■

2.2 The Word Problem of $B_2(T)$

The word problem for $P_2(T)$ is easy to solve due in part to the fact that we have a homotopic model for $F(T, 2)$. In this section we will solve the word problem for the full braid group $B_2(T)$.
Note that a word W in the group of braids can be rewritten of the form $\widetilde{a}^n\widetilde{b}^m w'$, where w' is a word in $\widetilde{a}$, $\widetilde{b}$, and c with exponents 1. For instance, the letter $\widetilde{a}^{-1}$ can be exchanged by $\widetilde{a}^{-2}\widetilde{a}$, where the central element $\widetilde{a}^{-2}$ can be moved to the beginning of the word. Likewise $\widetilde{b}^{-1} = \widetilde{b}^{-2}\widetilde{b}$ and $c^{-1} = c^{-2}c = \widetilde{a}^{-2}\widetilde{b}^{-2}c$.

The free abelian subgroup generated by braids a and b is central in the group of braids, in view of (2.5). In fact this subgroup is the center of B_2T.

We thus have the exact sequence

$$0 \longrightarrow \mathbb{Z}\widetilde{a}^2 \times \mathbb{Z}\widetilde{b}^2 \xrightarrow{\ i\ } B_2(T) \xrightarrow{\ \pi\ } \langle \widetilde{a},\ \widetilde{b},\ c \mid \widetilde{a}^2 = \widetilde{b}^2 = 1,\ c^2 = \widetilde{a}^2\widetilde{b}^2 \rangle \longrightarrow 1. \tag{2.8}$$

Note that the i-image of an element $(n, m) \in \mathbb{Z}\widetilde{a}^2 \times \mathbb{Z}\widetilde{b}^2$ has to be understood as $\widetilde{a}^{2n}\widetilde{b}^{2m}$ with n and m integers. The map π is the projection to the quotient.

The extension (2.8) gives rise to an action of $\mathbb{Z}_2\widetilde{a}*\mathbb{Z}_2\widetilde{b}*\mathbb{Z}_2c$ [4]$= \langle \widetilde{a},\ \widetilde{b},\ c \mid \widetilde{a}^2 = \widetilde{b}^2 = 1,\ c^2 = \widetilde{a}^2\widetilde{b}^2 \rangle$

[4]The free group generated by groups G_1 and G_2 is denoted by $G_1 * G_2$. The elements of this group are words in the free group generated by the elements of G_1 and G_2 subject solely to the relations in the groups G_i.

on $\mathbb{Z}\widetilde{a}^2 \times \mathbb{Z}\widetilde{b}^2$ given by conjugation on $B_2(T)$. This action is trivial since the elements in $\mathbb{Z}\widetilde{a}^2 \times \mathbb{Z}\widetilde{b}^2$ commute with any braid in $B_2(T)$.

Let $s : \mathbb{Z}_2\widetilde{a} * \mathbb{Z}_2\widetilde{b} * \mathbb{Z}_2c \longrightarrow B_2(T)$ be the set function defined as $s(w) = w$, that is, given a word w in $\langle \widetilde{a},\ \widetilde{b},\ c \mid \widetilde{a}^2 = \widetilde{b}^2 = 1,\ c^2 = \widetilde{a}^2\widetilde{b}^2 \rangle$ this function interprets it as a braid in $B_2(T)$. Then s turns out to be a set-theory cross-section of π, i.e. $\pi s = Id_{\mathbb{Z}_2\widetilde{a}*\mathbb{Z}_2\widetilde{b}*\mathbb{Z}_2c}$, moreover s is a normalized section in the sense that $s(1) = 1$.

For $w_1, w_2 \in \mathbb{Z}_2\widetilde{a} * \mathbb{Z}_2\widetilde{b} * \mathbb{Z}_2c$, both $s(w_1)s(w_2)$ and $s(w_1w_2)$ map to w_1w_2 in $\mathbb{Z}_2\widetilde{a} * \mathbb{Z}_2\widetilde{b} * \mathbb{Z}_2c$. Then we have that there exists a unique element $palin(w_1, w_2) \in \mathbb{Z}\widetilde{a}^2 \times \mathbb{Z}\widetilde{b}^2$ such that

$$s(w_1)s(w_2) = palin(w_1, w_2)s(w_1w_2). \tag{2.9}$$

So $palin : \mathbb{Z}_2\widetilde{a} * \mathbb{Z}_2\widetilde{b} * \mathbb{Z}_2c \times \mathbb{Z}_2\widetilde{a} * \mathbb{Z}_2\widetilde{b} * \mathbb{Z}_2c \longrightarrow \mathbb{Z}\widetilde{a}^2 \times \mathbb{Z}\widetilde{b}^2$ is a well-defined function that satisfies (2.9). Note that $palin(w_1, w_2)$ measures the deviation of w_1w_2 from being a palindrome. That is, reading w_1 from right to left, and w_2 from left to right, count the amount of letters $\widetilde{a}$, $\widetilde{b}$ and c appearing in the exact same order until a difference arises and, in the case that a c appears, add one to the counter both for $\widetilde{a}$ and for $\widetilde{b}$. For example, consider the words $w_1 = \widetilde{a}\widetilde{b}c\widetilde{b}\widetilde{a}c\widetilde{b}$ and $w_2 = \widetilde{b}c\widetilde{a}\widetilde{b}\widetilde{a} \in \mathbb{Z}_2\widetilde{a} * \mathbb{Z}_2\widetilde{b} * \mathbb{Z}_2c$ then $w_1w_2 = \widetilde{a}\widetilde{b}c\widetilde{a}$ then $palin(w_1, w_2) = \widetilde{a}^{2(2)}\widetilde{b}^{2(3)}$ because

$$(\widetilde{a}\widetilde{b}c\widetilde{b}\widetilde{a}c\widetilde{b})(\widetilde{b}c\widetilde{a}\widetilde{b}\widetilde{a}) = \widetilde{a}^{2(2)}\widetilde{b}^{2(3)}\widetilde{a}\widetilde{b}c\widetilde{a} \in B_2(T).$$

So, in this case, the function $palin(w_1, w_2)$ counts one $\widetilde{a}$, two $\widetilde{b}$'s and one c in the correct order and c increases the counter of $\widetilde{a}$ and $\widetilde{b}$ by one.

Consider the group law on $\left(\mathbb{Z}\widetilde{a}^2 \times \mathbb{Z}\widetilde{b}^2\right) \times \left(\mathbb{Z}_2\widetilde{a} * \mathbb{Z}_2\widetilde{b} * \mathbb{Z}_2c\right)$ defined by

$$(\widetilde{a}^{2(n_1)}\widetilde{b}^{2(m_1)}, w_1)(\widetilde{a}^{2(n_2)}\widetilde{b}^{2(m_2)}, w_2) = \left(\widetilde{a}^{2(n_1+n_2)}\widetilde{b}^{2(m_1+m_2)}palin(w_1, w_2), w_1w_2\right) \tag{2.10}$$

We denote this group by $\left(\mathbb{Z}\widetilde{a}^2 \times \mathbb{Z}\widetilde{b}^2\right) \rtimes_{palin} \left(\mathbb{Z}_2\widetilde{a} * \mathbb{Z}_2\widetilde{b} * \mathbb{Z}_2c\right)$ perturbed by $palin$.

Theorem 2.5
The braid group on the torus with two strings $B_2(T)$ is isomorphic to the group $\left(\mathbb{Z}\widetilde{a}^2 \times \mathbb{Z}\widetilde{b}^2\right) \rtimes_{palin} \left(\mathbb{Z}_2\widetilde{a} * \mathbb{Z}_2\widetilde{b} * \mathbb{Z}_2c\right)$ whith the group law (2.10).

Proof. The bijection $\left(\mathbb{Z}\widetilde{a}^2 \times \mathbb{Z}\widetilde{b}^2\right) \rtimes_{palin} \left(\mathbb{Z}_2\widetilde{a} * \mathbb{Z}_2\widetilde{b} * \mathbb{Z}_2c\right) \longrightarrow B_2(T)$ defined by $(\widetilde{a}^{2n}\widetilde{b}^{2m}, w) \mapsto \widetilde{a}^{2n}\widetilde{b}^{2m}s(w)$ is a group map. Furthermore

$$\left(\widetilde{a}^{2(n_1+n_2)}\widetilde{b}^{2(m_1+m_2)}palin(w_1,w_2), w_1w_2\right) \longmapsto \widetilde{a}^{2(n_1+n_2)}\widetilde{b}^{2(m_1+m_2)}palin(w_1,w_2)s(w_1w_2)$$

$$= a^{2(n_1+n_2)}\widetilde{b}^{2(m_1+m_2)}s(w_1)s(w_2)$$

$$= (\widetilde{a}^{2(n_1)}\widetilde{b}^{2(m_1)}s(w_1))(\widetilde{a}^{2(n_2)}\widetilde{b}^{2(m_2)}s(w_2)).$$

■

This theorem solves the word problem for the group of braids in the torus with two strings.

In the rest of this work, we will not make a distinction between the notation of $s(w)$ and w, that is, we will simply interpret the word w as the braid defined by the letters of w. So a word W in $B_2(T)$ has a canonical expression of the form $\widetilde{a}^{2n}\widetilde{b}^{2m}w$.

Chapter 3

The Cohomology of $F_2(T)$ and Obstruction Theory

3.1 Obstruction Theory

Using the work of A.S. Schwarz, for a fibration $p : E \longrightarrow B$, we denote by $*^n E$ the n-fold fiber join of the fibration and by $j^n(p) : *^n E \longrightarrow B$ the n-join map. Then $\mathrm{secat}(p) \leqslant n$ if and only if $j^{n+1}(p)$ has a cross section [8]. If F is the homotopic fiber of p, then $*^{n+1}F$ is the homotopic fiber of $j^{n+1}(p)$ in the fibration

$$*^{n+1}F \hookrightarrow *^{n+1}E \xrightarrow{j^{n+1}(p)} B.$$

It is well known that $*^{n+1}F \simeq \Sigma^n F^{\wedge n+1}$ so that $*^{n+1}F$ is $(n-1)$-connected. The primary obstruction to create a cross section in $j^{n+1}(p)$ is a cohomology class θ^{n+1} in $H^{n+1}(B; \pi_n(*^{n+1}F))$[1] [9]. Note that, if $n \geq 2$,

$$\begin{aligned} \pi_n(*^{n+1}F) &\simeq \pi_n(\Sigma^n F^{\wedge n+1}) \\ &\simeq \overline{H}_n(\Sigma^n F^{\wedge n+1}) \\ &\simeq \overline{H}_0(F^{\wedge n+1}) \\ &\simeq \bigotimes_{n+1} \overline{H}_0(F). \end{aligned} \tag{3.1}$$

Recall that the evaluation map $e_{0,1} : P(X) \longrightarrow X \times X$ takes a free path γ in X to its end points: $e_{0,1}(\gamma) = (\gamma(0), \gamma(1))$, and that $\mathrm{secat}(e_{0,1})$=TC(X). Recall also that NL(X) stands for the

[1] Here $H^*(X; G)$ stands for the cohomology of X with a local coefficient system G.

complement of the free loop space $\mathrm{L}(X)$ in the free path space on X. The group $\mathbb{Z}_2$ acts by taking a nonloop α to the reversed path $\overline{\alpha}$, i.e., $\alpha(1-t) = \overline{\alpha}(t)$. The inclusion map $\mathrm{NL}(X) \longrightarrow PX$ renders a restricted fibration $e_{0,1} : \mathrm{NL}(X) \longrightarrow F(X,2)$. This map passes to the $\mathbb{Z}_2$-quotient, and we get the commutative diagram with fiber columns

$$\begin{array}{ccccc}
\widetilde{M} & \twoheadleftarrow & \widetilde{M} & = & \widetilde{M} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{NL}(X)/\mathbb{Z}_2 & \twoheadleftarrow & \mathrm{NL}(X) & \hookrightarrow & P(X) \\
\downarrow{\scriptstyle \varepsilon_{0,1}} & & \downarrow{\scriptstyle e_{0,1}} & & \downarrow{\scriptstyle e_{0,1}} \\
B(X,2) & \twoheadleftarrow & F(X,2) & \hookrightarrow & X \times X.
\end{array} \tag{3.2}$$

Here $\widetilde{M}$ denotes the path space $\mathrm{Maps}([0,1], 0, 1; X, x_0, x_1)$ for a point (x_0, x_1) in $F(X,2)$. When X is path connected, we choose a fixed path $\delta : [0,1] \longrightarrow X$, such that $\delta(0) = x_0$ and $\delta(1) = x_1$. The map $\psi_\delta : \widetilde{M} \longrightarrow \Omega X$ defined as $\psi_\delta(\omega) = \omega\overline{\delta}$ is a homotopy equivalence and its homotopy inverse is given by $\psi^\delta(\sigma) = \sigma\delta$, where ΩX $=\mathrm{Maps}([0,1], 0, 1; X, x_0, x_0)$. We thus have the commutative diagram with fiber columns

$$\begin{array}{ccccc}
\Omega X & = & \Omega X & = & \Omega X \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{NL}(X)/\mathbb{Z}_2 & \twoheadleftarrow & \mathrm{NL}(X) & \hookrightarrow & P(X) \\
\downarrow{\scriptstyle \varepsilon_{0,1}} & & \downarrow{\scriptstyle e_{0,1}} & & \downarrow{\scriptstyle e_{0,1}} \\
B(X,2) & \twoheadleftarrow & F(X,2) & \hookrightarrow & X \times X
\end{array} \tag{3.3}$$

with common homotopy fiber ΩX.

Set $G = \pi_1(X, x_0)$ and let $I_0 = ker(\epsilon) \subset \mathbb{Z}_G$ be the kernel of the augmentation homomorphism $\epsilon : \mathbb{Z}_G \longrightarrow \mathbb{Z}$. An element of I_0 is a finite sum of the form $\sum n_i g_i$ where $n_i \in \mathbb{Z}$, $g_i \in G$ and $\sum n_i = 0$. In the above terms, the local coefficient system (3.1) takes the form

$$\pi_n(*^{n+1}\Omega X) \simeq \bigotimes_{n+1} \overline{H}_0(\Omega X) \simeq \bigotimes_{n+1} I_0 \tag{3.4}$$

and the primary obstructions θ^{n+1} associated to $\varepsilon_{0,1}$ and $e_{0,1}$ lie in

$$H^{n+1}(B(X,2); \bigotimes_{n+1} I_0)), \quad H^{n+1}(F(X,2); \bigotimes_{n+1} I_0)) \tag{3.5}$$

respectively. It will always be clear from the context whether we consider the obstruction as a cocycle or as a cohomology class.

3.2 Deleted Product, Mimick for $P_2(X)$

We define an action of $P_2(X)$ on $\mathbb{Z}_G$ and I_0 via

$$[\gamma]\sum n_i g_i = [(\alpha,\beta)]\sum n_i g_i = \sum n_i \alpha g_i \delta\overline{\beta}\,\overline{\delta}. \tag{3.6}$$

Here $\gamma = (\alpha,\beta)$ represents an element in the pure braid group of X on 2 strings, that is

$$\alpha,\ \beta : [0,1] \longrightarrow X,$$

$\alpha(t) \neq \beta(t)$ for all t in $[0,1]$ and $\alpha(0) = \alpha(1) = x_0$, $\beta(0) = \beta(1) = x_1$. The action is well defined, given that α and $\delta\overline{\beta}\,\overline{\delta}$ are loops based on x_0, and they are understood as elements of $\pi_1(X,(x_0))$ by taking their respective homotopy classes. These element are illustrated by their representative loops in Figure 3.1.

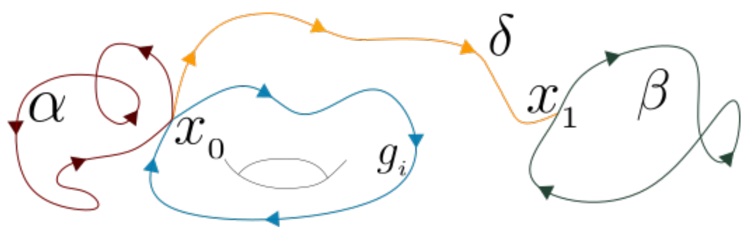


Figure 3.1: The braid γ, the loop g_i and the pad δ.

In Figure 3.2, which describes the action, $[\gamma]g_i$ is represented by the homotopy class of the loop $\alpha g_i \delta\overline{\beta}\,\overline{\delta}$. The path $g_i\delta$ describes the image of g_i, as an element in the loop space ΩX, of its corresponding path in the honest fiber at $\widetilde{M}$.

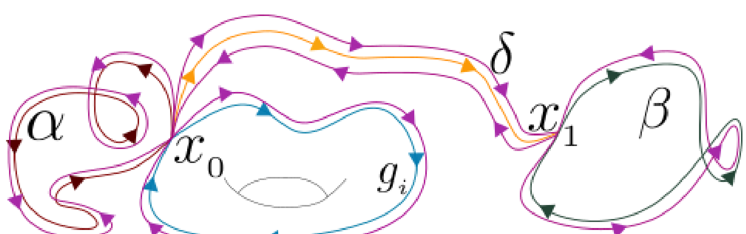


Figure 3.2: The Action of $P_2(X)$ on I_0

Remark 3.1
The concatenation $\psi^{\delta}(g_i) = g_i\delta$ lies in $\widetilde{M}$, and $[\gamma][\psi^{\delta}(g_i)] = [\alpha\psi^{\delta}(g_i)\beta]$, as defined in Theorem 1.4, is the analogous action defined in (3.6). That is $\psi_{\delta}([\gamma][\psi^{\delta}(g_i)]) = \alpha g_i \delta\overline{\beta}\,\overline{\delta}$.

Proposition 3.2
Consider the map $f : P_2(X) \longrightarrow I_0$ given by $f([\gamma]) = [\alpha\delta\overline{\beta}\,\overline{\delta}] - 1$, where $\gamma = (\alpha, \beta)$. Then f determines a one dimensional cohomology class $[f] = \mathfrak{b} \in H^1(F(X,2); I_0)$.

Proof. The function f is a crossed homomorphism, so we only have to check the equality $f([\gamma_1][\gamma_2]) = f([\gamma_1]) + [\gamma_1]f([\gamma_2])$. But

$$f([\gamma_1][\gamma_2]) = f([\alpha_1, \beta_1][\alpha_2, \beta_2]) = f([\alpha_1\alpha_2, \beta_1\beta_2]) = [\alpha_1\alpha_2\delta\overline{\beta_1\beta_2}\,\overline{\delta}] - 1,$$

whereas

$$f([\gamma_1]) + [\gamma_1]f([\gamma_2]) = [\alpha_1\delta\overline{\beta_1}\,\overline{\delta}] - 1 + (\alpha_1, \beta_1)[\alpha_2\delta\,\overline{\beta_2}\,\overline{\delta}] - 1 = s[\alpha_1\alpha_2\delta\overline{\beta_2}\,\overline{\beta_1}\,\overline{\delta}] - 1.$$

The result then follows from Whitehead's book [9]. ■

Remark 3.3
If we need to work with the actual fiber instead of the homotopic fiber, the action in the fiber is defined just as in Theorem 1.4 via (1.7). That is,

$$[\gamma] \cdot [\omega] = [\alpha\omega\overline{\beta}].$$

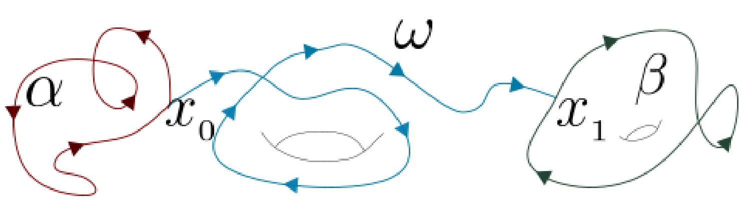


Figure 3.3: The action of $P_2(X)$ on the true fiber $\widetilde{M}$.

3.3 The Sectional Category for the Fibration $e_{0,1}$ over $F(T, 2)$, Torus Case

According to Theorem 2.2, the 2-string pure braid group on T is

$$P_2(T) = \mathbb{Z}a \oplus \mathbb{Z}b \oplus F_2 = \langle a,\ b,\ x,\ y \mid [a, b] = [a, x] = [a, y] = [b, x] = [b, y] = 1\rangle,$$

where $\{a,\ b\}$ is the basis of the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$, and $F_2 = \langle x, y\rangle$ is the rank-2 free group generated by x and y. $F(T, 2)$ has the homotopy type of $T \times \left(S^1 \vee S^1\right)$ because T is a topological group, so $F(T, 2)$ is a $K(P_2(T), 1)$ space. In their book [13], Milnor and Stasheff describe the cohomology of a deleted product. In this work we follow Brown's construction in [14] in order to calculate a free resolution of $\mathbb{Z}$ over the direct sum of two groups in terms of explicit free resolution of $\mathbb{Z}$ over each group. Thus, for the infinite cyclic group $\langle t\rangle$ generated by t, it is easy to see that the sequence

$$0 \longrightarrow \mathbb{Z}_{\langle t\rangle}\sigma_1 \xrightarrow{\quad t-1 \quad} \mathbb{Z}_{\langle t\rangle}\sigma_0 \xrightarrow{\quad \varepsilon \quad} \mathbb{Z} \longrightarrow 0 \tag{3.7}$$

is a free resolution. Then

$$0 \longrightarrow \mathbb{Z}_{F_2}k_1^x \oplus \mathbb{Z}_{F_2}k_1^y \xrightarrow{\quad \partial \quad} \mathbb{Z}_{F_2}k_0 \xrightarrow{\quad \varepsilon \quad} \mathbb{Z} \longrightarrow 0 \tag{3.8}$$

is $\mathbb{Z}_{F_2}$-free resolution, where $\partial k_1^s = (s-1)k_0$, for $s = x,\ y$.

If G and G' are groups and M, M' are G-module and G'-module respectively, then $M \bigotimes M'$ is a $G \times G'$-module via $(g, g') \cdot (m \otimes m') = gm \otimes g'm'$. If M is projective (e.g. free) over $\mathbb{Z}_G$ and M' is projective (e.g. free) over $\mathbb{Z}_{G'}$, then $M \bigotimes M'$ is projective over $\mathbb{Z}_{G\times G'}$, moreover the obvious homomorphism $\mathbb{Z}_G \bigotimes \mathbb{Z}_{G'} \longrightarrow \mathbb{Z}_{[G\times G']}$ is isomorphism.

Let $\epsilon : F \longrightarrow \mathbb{Z}$ and $\epsilon : F' \longrightarrow \mathbb{Z}$ be projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}_G$ and $\mathbb{Z}_{G'}$ respectively, and consider the complex $F \bigotimes F'$. This is a complex of projective $\mathbb{Z}_{[G\times G']}$-modules, and it is augmented over $\mathbb{Z}$ by $\epsilon \otimes \epsilon' : F \bigotimes F' \longrightarrow \mathbb{Z} \bigotimes \mathbb{Z} = \mathbb{Z}$.

Proposition 3.4
If $\epsilon : F \longrightarrow \mathbb{Z}$ and $\epsilon : F' \longrightarrow \mathbb{Z}$ are projetive resolutions of $\mathbb{Z}$ over $\mathbb{Z}_G$ and $\mathbb{Z}_{G'}$ respectively, then $\epsilon \otimes \epsilon' : F \bigotimes F' \longrightarrow \mathbb{Z}$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}_{[G\times G']}$. Moreover, if $\epsilon : F \longrightarrow \mathbb{Z}$ and $\epsilon : F' \longrightarrow \mathbb{Z}$ are projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}_G$, then so is $\epsilon \otimes \epsilon' : F \bigotimes F' \longrightarrow \mathbb{Z}$, where G acts diagonally on $F \bigotimes F'$.

Proof. This is a standard result, see for instance [14]. ■

Using the free resolution for $\pi_1(S^1)$ in (3.7), we obtain the free resolution over $\pi_1(T) \cong \pi_1(S^1 \times S^1) = \pi_1(S^1) \oplus \pi_1(S^1)$

$$0 \longrightarrow \mathbb{Z}_{\pi_1(T)}(g_1 h_1) \xrightarrow{\delta_2} \mathbb{Z}_{\pi_1(T)}(g_1 h_0) \bigoplus \mathbb{Z}_{\pi_1(T)}(g_0 h_1) \xrightarrow{\delta_1} \mathbb{Z}_{\pi_1(T)}(g_0 h_0) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0. \tag{3.9}$$

In turn, from the free resolution of $\mathbb{Z}$ over $\mathbb{Z}_{F_2}$ (3.8) and the free resolution of $\mathbb{Z}$ over $\mathbb{Z}_{\pi_1(T)}$ (3.9) we obtain a resolution $F \xrightarrow{\varepsilon} \mathbb{Z}$. Explicitly

$$\cdots \to \mathbb{Z}_{P_2(T)}e_x \bigoplus \mathbb{Z}_{P_2(T)}e_y \xrightarrow{D_3} \bigoplus_{i=1}^{5} \mathbb{Z}_{P_2(T)}\widetilde{e_i} \xrightarrow{D_2} \bigoplus_{i=1}^{4} \mathbb{Z}_{P_2(T)}e_i \xrightarrow{D_1} \mathbb{Z}_{P_2(T)}e \xrightarrow{\varepsilon} \mathbb{Z} \to 0, \tag{3.10}$$

which is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}_{P_2(T)}$, and whose differentials are defined in the following way.

If $a^i b^j w \in P_2(T)$[2], then D_1 is defined by

$$\begin{aligned} D_1(a^i b^j w)e_1 =& (a^i b^j (wx - w))e = a^i b^j w(x-1)e, \\ D_1(a^i b^j w)e_2 =& (a^i b^j (wy - w))e = a^i b^j w(y-1)e, \\ D_1(a^i b^j w)e_3 =& (a^i (b^{j+1} - b^j) w)e = a^i b^j w(b-1)e, \\ D_1(a^i b^j w)e_4 =& ((a^{i+1} - a^i) b^j w)e = a^i b^j w(a-1)e. \end{aligned} \tag{3.11}$$

D_2 is defined by

$$\begin{aligned} D_2(a^i b^j w)\widetilde{e_1} =& a^i b^j w(a-1)e_1 - a^i b^j w(x-1)e_4, \\ D_2(a^i b^j w)\widetilde{e_2} =& a^i b^j w(a-1)e_2 - a^i b^j w(y-1)e_4, \\ D_2(a^i b^j w)\widetilde{e_3} =& a^i b^j w(b-1)e_1 - a^i b^j w(x-1)e_3, \\ D_2(a^i b^j w)\widetilde{e_4} =& a^i b^j w(b-1)e_2 - a^i b^j w(y-1)e_3, \\ D_2(a^i b^j w)\widetilde{e_5} =& a^i b^j w(a-1)e_3 - a^i b^j w(b-1)e_4. \end{aligned} \tag{3.12}$$

Finally D_3 is defined by

$$\begin{aligned} D_3(a^i b^j w)e_x =& a^i b^j w(a-1)\widetilde{e_3} - a^i b^j w(b-1)\widetilde{e_1} + a^i b^j w(x-1)\widetilde{e_5}, \\ D_3(a^i b^j w)e_y =& a^i b^j w(a-1)\widetilde{e_4} - a^i b^j w(b-1)\widetilde{e_2} + a^i b^j w(y-1)\widetilde{e_5}. \end{aligned} \tag{3.13}$$

The generators are obtained in the following order: For $\mathbb{Z}_{P_2(T)}e$,

$$e = g_0 h_0 k_0 = g_0 \otimes h_0 \otimes k_0.$$

For the generators of $\bigoplus_{i=1}^{4} \mathbb{Z}_{P_2(T)} e_i$[3],

$$e_1 = g_0 h_0 k_1^x, \quad e_2 = g_0 h_0 k_1^y, \quad e_3 = g_0 h_1 k_0, \quad e_4 = g_1 h_0 k_0.$$

For the generators of $\bigoplus_{i=1}^{5} \mathbb{Z}_{P_2(T)} \widetilde{e_i}$,

$$\widetilde{e_1} = g_1 h_0 k_1^x, \quad \widetilde{e_2} = g_1 h_0 k_1^y, \quad \widetilde{e_3} = g_0 h_1 k_1^x, \quad \widetilde{e_4} = g_0 h_1 k_1^y, \quad \widetilde{e_5} = g_1 h_1 k_0.$$

Lastly, for $e_x,\ e_y$ we have

$$e_x = g_1 h_1 k_1^x, \quad e_y = g_1 h_1 k_1^y.$$

[2] Where w is a word in $F_2 = \langle x,\ y \rangle$.

[3] Throughout the rest of the chapter, we will omit the $\otimes$ symbol for simplicity for the generators.

As in (2.1), without loss of generality, we take the base point of $F_2(T)$ by choosing the points x_0, $x_1 \in T = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ as

$$x_0 = \left(\frac{1}{5}, \frac{1}{5}\right), \quad x_1 = \left(\frac{2}{5}, \frac{2}{5}\right).$$

Consider the path $d_{0,1} : [0,1] \longrightarrow T$ given by

$$d_{0,1}(t) = x_0 + t(x_1 - x_0) = \left(\frac{t+1}{5}, \frac{t+1}{5}\right),$$

and let δ be the corresponding homotopy class. By Proposition 3.2, the map $f : P_2(T) \longrightarrow I_0$ given by $f([\gamma]) = [\alpha\delta\overline{\beta}\,\overline{\delta}] - 1$ where $\gamma = (\alpha, \beta)$ is a crossed homomorphism and $[f] = \mathfrak{b} \in H^1(F(T,2), I_0)$.

Lemma 3.5

The crossed morphism f restricted to the subgroup generated by a and b in $P_2(T)$ is trivial. Thus the evaluation of the class $\mathfrak{b}$ at $a^i b^j w$ agrees with the evaluation at w, where w is any word in x and y.

Proof. Notice that classes $a = \tau_0\tau_1$ and $b = \rho_0\rho_1$ are represented by $k_a,\ k_b : [0,1] \longrightarrow T$ given by $k_a = (x_0 + (0,t), x_1 + (0,t))$ and respectively. Then the evaluation of f in a is given in the following way

$$\begin{aligned} f(a) =& f(\tau_0\tau_1) \\ =& [x_0 + (0,t)]\delta\overline{[x_1 + (0,t)]}\,\overline{\delta} - 1 \\ =& [x_0 + (0,t)][d_{0,1}][x_1 - (0,t)]\,\overline{d_{0,1}} - 1 \\ =& [x_0 + (0,t)][x_0 + t(x_1 - x_0)][x_1 - (0,t)]\,[x_1 + t(x_0 - x_1)] - 1. \end{aligned} \tag{3.14}$$

Consider $K : [0,1] \times [0,1] \longrightarrow T$ given by

$$K(t,s) = [x_0 + st(x_1 - x_0)][sx_1 + (1-s)x_0 - (0,t)]\,[sx_1 + (1-s)x_0 + st(x_0 - x_1)].$$

Note that $K(0,s) = K(1,s) = x_0$ for all $s \in [0,1]$, and

$$\begin{aligned} K(t,1) =& [x_0 + t(x_1 - x_0)][x_1 - (0,t)]\,[x_1 + t(x_0 - x_1)] \\ K(t,0) =& [x_0][x_0 - (0,t)][x_0]. \end{aligned}$$

It is clear that $K(t,0) = [x_0 - (0,t)] = \overline{[x_0 + (0,t)]}$, so $f(a) = [x_0 + (0,t)]\overline{[x_0 + (0,t)]} - 1 = 0$.

Note that a^i can be represented by the braid $(x_0 + (0,it), x_1 + (0,it))$. Using the same idea, we get $\delta\overline{[x_1 + (0,it)]}\,\overline{\delta} = \overline{[x_0 + (0,it)]}$. In a completely analogous way we see that $f(b^j) = 0$, so $f(a^i b^j) = f(a^i) + a^i f(b^j) = 0$. Thus f vanishes on the subgroup generated by a and b in $P_2(T)$.

Finally, if w is any word in $\langle x,\ y\rangle$, then $f(a^i b^j w) = f(wa^i b^j) = f(w) + wf(a^i b^j) = f(w) + w0 = f(w)$, which completes the proof. ■

Recall that $[x_0 + (0,t)]$ and $[x_0 + (t,0)]$ are the generators of $\pi_1(T, x_0) \cong \mathbb{Z} \bigoplus \mathbb{Z}$. The free non-abelian part of $P_2(T)$ is generated by $x,\ y$ and are represented by

$$g_1 = (x_0, x_1 + (t,0)) \quad \text{and} \quad f_1 = (x_0, x_1 + (0,t))$$

respectively. Note that one of the braids is constant and the other is homotopic to one of the fundamental cycles, that is,

$$\delta[x_1 + (t,0)]\overline{\delta} = [x_0 + (t,0)] \quad \text{and} \quad \delta[x_1 + (0,t)]\overline{\delta} = [x_0 + (0,t)].$$

Without this representing any confusion, we will consider x and y as the classes of homotopy based loops on x_1 when it is convenient[4]. For example,

$$\delta x\overline{\delta} = [x_0 + (t,0)] \text{ and } \delta y\overline{\delta} = [x_0 + (0,t)].$$

Corollary 3.6
Let w be a word in the free group generated by x and y, let n_x be the degree of $[w] \in \langle x,\ y\rangle / y$ and n_y be the degree of $[w] \in \langle x,\ y\rangle / x$. Then

$$f(w) = \delta\overline{x}^{n_x}\,\overline{y}^{n_y}\,\overline{\delta} - 1.$$

Proof. The result follows from the fact that $\pi_1(T, x_0)$ is abelian group. ■

Corollary 3.7
Then the subgroup spanned by $\{a,\ b\}$ acts trivially on $\mathbb{Z}_{\pi_1(T)}$. In detail, if w is a word in the free part of $P_2(T)$, and we let n_x and n_y be defined as in Corollary 3.6, then

$$w \sum n_k g_k = \sum n_k g_k \delta\overline{x}^{n_x}\,\overline{y}^{n_x}\,\overline{\delta}, \tag{3.15}$$

moreover, for any $a^i b^j w \in P_2(T)$,

$$a^i b^j w \sum n_k g_k = \sum n_k g_k \delta\overline{x}^{n_x}\,\overline{y}^{n_x}\,\overline{\delta}. \tag{3.16}$$

Proof. Using the same formulas as in the proof of Lemma (3.5), we have

[4]For this we consider only the non-constant braid given by the representatives g_1 and f_1 respectively.

$$\begin{aligned}
a^i b^j \sum n_k g_k &= \sum n_k [x_0 + (0,it)][x_0 + (jt,0)] g_k \delta [x_1 - (0,it)][x_1 - (jt,0)]\,\overline{\delta}\\
&= \sum n_k g_k \Big([x_0 + (0,it)]\,[x_0 + (jt,0)] \Big) \Big(\delta [x_1 - (0,it)]\,\overline{\delta}\,\delta [x_1 - (jt,0)]\,\overline{\delta} \Big)\\
&= \sum n_k g_k \Big([x_0 + (0,it)] \delta [x_1 - (0,it)]\,\overline{\delta} \Big)\,\Big([x_0 + (jt,0)] \delta [x_1 - (jt,0)]\,\overline{\delta} \Big)\\
&= \sum n_k g_k (1)(1) = \sum n_k g_k .
\end{aligned}$$

Finally, for $a^i b^j w \in P_2(T)$,

$$a^i b^j w \sum n_k g_k = \Big(a^i b^j\Big) w \sum n_k g_k = w \sum n_k g_k = \sum n_k g_k \delta \overline{x}^{n_x}\,\overline{y}^{n_x}\,\overline{\delta},$$

which completes the proof. ■

Corollary 3.8
The local system of coefficients associated to the fibration $e_{0,1} : NL(T) \longrightarrow F(T,2)$, as described in Remark 3.3, depends only on the action restricted to $\pi_1(T, x_0)$.

Proof. Note that the morphism defined on fundamental groups by the homeomorphism

$$(T, x_1) \xrightarrow{(\_ -1/5,\_ -1/5)} (T, x_0)$$

agrees with the homomorphism $\delta \_\_\, \overline{\delta} \colon \pi_1(T, x_1) \to \pi_1(T, x_0)$. ■

The monodromy action we found, is the one described by Costa-Farber in [10] restricted to $(1, \_\_)$.

The Alexander-Whitney diagonal approximation $\Delta : B \longrightarrow B \otimes B$ for the bar resolution[5]

$$B\colon \cdots \longrightarrow B_3 \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 = \mathbb{Z}_{P_2(T)} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}$ over $\mathbb{Z}_{P_2 T}$, is given by

$$\Delta \Big[\gamma_1 \mid \gamma_2 \mid \ldots \mid \gamma_n \Big] = \sum_{p=0}^{n} \Big[\gamma_1 \mid \ldots \mid \gamma_p \Big] \otimes \gamma_1 \cdots \gamma_p \Big[\gamma_{p+1} \mid \ldots \mid \gamma_q \Big].$$

Then the class $\mathfrak{b}^2 \in H^2(F(T,2), I_0 \bigotimes I_0)$ is represented by $f \smile f = (f \times f) \circ \Delta : B_2 \longrightarrow I_0 \bigotimes I_0$, which is defined as

$$f \smile f \Big[\gamma_1 \mid \gamma_2 \Big] = -f(\gamma_1) \otimes \gamma_1 f(\gamma_2). \tag{3.17}$$

[5]The bar resolution is also called the standard resolution, [14].

Then, if $\gamma_1 = a^{i_1} b^{j_1} w_1$, $\gamma_2 = a^{i_2} b^{j_2} w_2$, we have

$$
\begin{aligned}
f \smile f\Big[\gamma_1 \mid \gamma_2\Big] &= - f(w_1) \otimes w_1 f(w_2) \\
&= - \left(\delta \overline{w_1}\,\overline{\delta} - 1\right) \otimes w_1 \left(\delta \overline{w_2}\,\overline{\delta} - 1\right) \\
&= - \left(\delta \overline{w_1}\,\overline{\delta} - 1\right) \otimes \left(\delta \overline{w_2}\,\overline{w_1}\,\overline{\delta} - \delta \overline{w_1}\,\overline{\delta}\right). \qquad (3.18)
\end{aligned}
$$

We have an augmentation-preserving chain map $\widetilde{f} : F \longrightarrow B$

$$
\begin{array}{ccccccccc}
\cdots 0 \longrightarrow & \mathbb{Z}_{P_2(T)} e_x \bigoplus \mathbb{Z}_{P_2(T)} e_y & \xrightarrow{D_3} & \bigoplus_{i=1}^{5} \mathbb{Z}_{P_2(T)} \widetilde{e_i} & \xrightarrow{D_2} & \bigoplus_{i=1}^{4} \mathbb{Z}_{P_2(T)} e_i & \xrightarrow{D_1} & \mathbb{Z}_{P_2(T)} e \\
\downarrow f_4 & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \Updownarrow f_0 = Id_{\mathbb{Z}_{P_2(T)}} \\
\cdots B_4 \longrightarrow & B_3 & \longrightarrow & B_2 & \longrightarrow & B_1 & \longrightarrow & B_0 = \mathbb{Z}_{P_2(T)}
\end{array} \qquad (3.19)
$$

which is a homotopy equivalence, and is defined as $f_0 = Id_{\mathbb{Z}_{P_2}} : \mathbb{Z}_{P_2(T)} e \longrightarrow B_0 = \mathbb{Z}_{P_2(T)}$, $f_k : 0 \longrightarrow B_k$ for $k \geqslant 4$ in the only possible way, and

$$
\begin{aligned}
f_1\left((a^i b^j w) e_1\right) &= \left(a^i b^j w,\ a^i b^j w x\right) = a^i b^j w\,(1, x) = a^i b^j w\Big[x\Big], \\
f_1\left((a^i b^j w) e_2\right) &= \left(a^i b^j w,\ a^i b^j w y\right) = a^i b^j w\,(1, y) = a^i b^j w\Big[y\Big], \\
f_1\left((a^i b^j w) e_3\right) &= \left(a^i b^j w,\ a^i b^{j+1} w\right) = a^i b^j w\,(1, b) = a^i b^j w\Big[b\Big], \\
f_1\left((a^i b^j w) e_4\right) &= \left(a^i b^j w,\ a^{i+1} b^j w\right) = a^i b^j w\,(1, a) = a^i b^j w\Big[a\Big],
\end{aligned}
$$

$$
\begin{aligned}
f_2\left((a^i b^j w)\widetilde{e_1}\right) &= \left(a^i b^j w,\ a^{i+1} b^j w,\ a^i b^j w x\right) - \left(a^{i+1} b^j w,\ a^i b^j w x,\ a^{i+1} b^j w x\right) \\
&= a^i b^j w\Big[a \mid a^{-1} x\Big] - a^{i+1} b^j w\Big[a^{-1} x \mid a\Big], \\
f_2\left((a^i b^j w)\widetilde{e_2}\right) &= \left(a^i b^j w,\ a^{i+1} b^j w,\ a^i b^j w y\right) - \left(a^{i+1} b^j w,\ a^i b^j w y,\ a^{i+1} b^j w y\right) \\
&= a^i b^j w\Big[a \mid a^{-1} y\Big] - a^{i+1} b^j w\Big[a^{-1} y \mid a\Big], \\
f_2\left((a^i b^j w)\widetilde{e_3}\right) &= \left(a^i b^j w,\ a^i b^{j+1} w,\ a^i b^j w x\right) - \left(a^i b^{j+1} w,\ a^i b^j w x,\ a^i b^{j+1} w x\right) \\
&= a^i b^j w\Big[b \mid b^{-1} x\Big] - a^i b^{j+1} w\Big[b^{-1} x \mid b\Big], \\
f_2\left((a^i b^j w)\widetilde{e_4}\right) &= \left(a^i b^j w,\ a^i b^{j+1} w,\ a^i b^j w y\right) - \left(a^i b^{j+1} w,\ a^i b^j w y, a^i b^{j+1} w y\right) \\
&= a^i b^j w\Big[b \mid b^{-1} y\Big] - a^i b^{j+1} w\Big[b^{-1} y \mid b\Big],
\end{aligned}
$$

$$f_2\left((a^i b^j w)\widetilde{e_5}\right) = \left(a^i b^j w,\ a^{i+1} b^j w,\ a^{i+1} b^{j+1} w\right) - \left(a^i b^j w,\ a^i b^{j+1} w,\ a^{i+1} b^{j+1} w\right)$$
$$= a^i b^j w\left[a \mid b\right] - a^i b^j w\left[b \mid a\right] = a^i b^j w\left(\left[a \mid b\right] - \left[b \mid a\right]\right),$$

$$f_3\left((a^i b^j w)e_x\right) = a^i b^j w\Big((1,b,x,ab) - (1,a,x,ab) + (b,x,ab,bx) - (a,x,ab,ax)$$
$$+ (x,ab,bx,abx) - (x,ab,ax,abx)\Big)$$
$$= \left(a^i b^j w\right)\left(\left[b \mid b^{-1}x \mid abx^{-1}\right] - \left[a \mid a^{-1}x \mid abx^{-1}\right] + b\left[b^{-1}x \mid abx^{-1} \mid a^{-1}\right]\right.$$
$$\left. - a\left[a^{-1}x \mid abx^{-1} \mid b^{-1}\right] + x\left[abx^{-1} \mid a^{-1}x \mid a\right] - x\left[abx^{-1} \mid b^{-1}x \mid b\right]\right),$$
$$f_3\left((a^i b^j w)e_y\right) = a^i b^j w\Big((1,b,y,ab) - (1,a,y,ab) + (b,y,ab,by) - (a,y,ab,ay)$$
$$+ (y,ab,by,aby) - (y,ab,ay,aby)\Big)$$
$$= \left(a^i b^j w\right)\left(\left[b \mid b^{-1}y \mid aby^{-1}\right] - \left[a \mid a^{-1}y \mid aby^{-1}\right] + b\left[b^{-1}y \mid aby^{-1} \mid a^{-1}\right]\right.$$
$$\left. - a\left[a^{-1}y \mid aby^{-1} \mid b^{-1}\right] + y\left[aby^{-1} \mid a^{-1}y \mid a\right] - y\left[aby^{-1} \mid b^{-1}y \mid b\right]\right).$$

Theorem 3.9
The primary obstruction $\mathfrak{b}^2 \in H^2(F(T,2); I_0 \bigotimes I_0)$ to cross section $j^2(e_{0,1}) : *^2\mathrm{NL}(T) \longrightarrow F(T,2)$ vanishes.

Proof. The chain map $\widetilde{f} : F \longrightarrow B$ between the free resolution (3.10) and the bar resolution of $\mathbb{Z}$ over $\mathbb{Z}_{P_2(T)}$ induces an isomorphism $\widetilde{f}^* : H^2(F(T,2); I_0 \bigotimes I_0) \longrightarrow H^2(F(T,2); I_0 \bigotimes I_0)$. Then $\mathfrak{b}^2 = 0$ if and only if $\widetilde{f}^*(\mathfrak{b}^2) = 0$. The class of $\mathfrak{b}^2$ is represented by $f \smile f$, so it is enough to prove the vanishing of $f \smile f \circ f_2 : \bigoplus_{i=1}^{5} \mathbb{Z}_{P_2(T)}\widetilde{e_i} \longrightarrow I_0 \bigotimes I_0$:

$$f \smile f \circ f_2\left((a^i b^j w)\widetilde{e_1}\right) = a^i b^j w f \smile f\left[a \mid a^{-1}x\right] - a^{i+1} b^j w f \smile f\left[a^{-1}x \mid a\right]$$
$$= a^i b^j w(f(a) \otimes f(a^{-1}x)) - a^{i+1} b^j w(f(a^{-1}x)) \otimes x f(a)) = a^i b^j w(\ 0 \otimes f(x)) - a^{i+1} b^j w(f(x) \otimes\ 0) = 0.$$

In a similar way, we have $f \smile f \circ f_2\left((a^i b^j w)\widetilde{e_i}\right) = 0$ for $i = 2,3,4,5$:

$$f \smile f \circ f_2\left((a^i b^j w)\widetilde{e_2}\right) = a^i b^j w f \smile f\left[a \mid a^{-1}y\right] - a^{i+1} b^j w f \smile f\left[a^{-1}y \mid a\right] = 0,$$
$$f \smile f \circ f_2\left((a^i b^j w)\widetilde{e_3}\right) = a^i b^j w f \smile f\left[b \mid b^{-1}x\right] - a^i b^{j+1} w f \smile f\left[b^{-1}x \mid b\right] = 0,$$
$$f \smile f \circ f_2\left((a^i b^j w)\widetilde{e_4}\right) = a^i b^j w f \smile f\left[b \mid b^{-1}y\right] - a^i b^{j+1} w f \smile f\left[b^{-1}y \mid b\right] = 0,$$
$$f \smile f \circ f_2\left((a^i b^j w)\widetilde{e_5}\right) = a^i b^j w\left(f \smile f\left[a \mid b\right] - f \smile f\left[b \mid a\right]\right) = 0,$$

which completes the proof. ■

Since $\sum(\Omega T \wedge \Omega T) \simeq \bigvee_{\mathbb{Z}\times\mathbb{Z}} S^1$, Theorem 3.9 implies in fact that there are no homological obstructions to cross section $j^2(e_{0,1})$. However, we are prevented from deducing that $\mathrm{secat}(e_{0,1}) \leq 1$ because the Hurewicz homomorphism that we use when describing the coefficient system in (3.1), from the n-th homotopy group to the n-th homology group

$$\pi_n(\Sigma^n F^{\wedge n+1}) \longrightarrow \overline{H}_n(\Sigma^n F^{\wedge n+1})$$

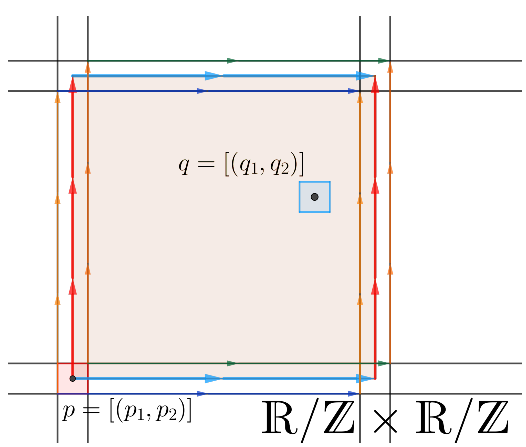
is an isomorphism for $n > 1$, but is just abelianization for $n = 1$ (which is the situation in Theorem 3.9).

3.4 Motion Planning

As discussed above, Theorem 3.9 does not give us a conclusive secat-argument. So, at this point in the work, the approach arises to explicitly solve the motion planning problem in $F(T, 2)$.

For $p \in T = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ there are representatives $(p_1, p_2) \in \mathbb{R}^2$ such that $[(p_1, p_2)] = p$. We define the set U as the collection of points $(p, q) \in F(T, 2)$, such that, given a element (p_1, p_2) in the class p there exists a representative (q_1, q_2) of q such that $(q_1, q_2) \in (p_1, p_1 + 1) \times (p_2, p_2 + 1)$. That is

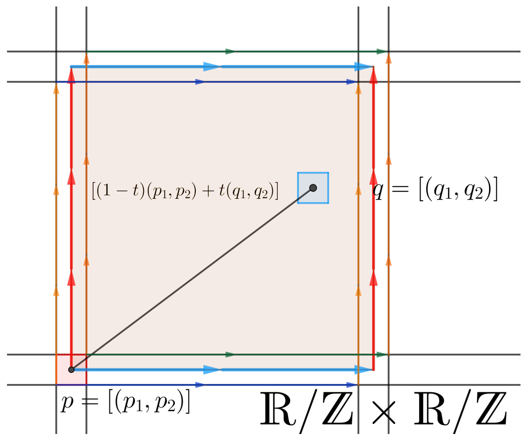
$$U = \{(p, q) \in F(T, 2) \mid p = [(p_1, p_2)] \wedge \exists (q_1, q_2) \in q \wedge (q_1, q_2) \in (p_1, p_1 + 1) \times (p_2, p_2 + 1)\}.$$



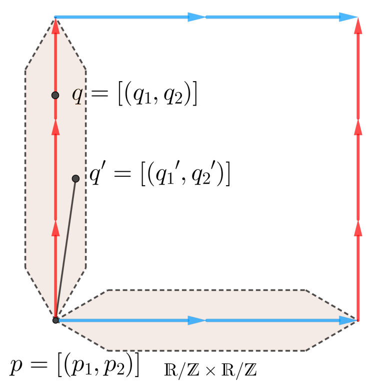
It is easy to see that U is an open subset of $F(T,2)$: just consider a neighborhood

$$((p_1-\varepsilon, p_1+\varepsilon)\times(p_2-\varepsilon, p_2+\varepsilon))\times((q_1-\varepsilon, q_1+\varepsilon)\times(q_2-\varepsilon, q_2+\varepsilon))$$

with a small enough epsilon. Next define $S : U \longrightarrow \mathrm{NL}(T)$ as $S(p,q) = [(1-t)(p_1,p_2)+t(q_1,q_2)]$. This is a continuous section of $e_{0,1} : \mathrm{NL}(T) \to F(T,2)$.



The complement of U is not open, but it is a deformation retract of one. Explicitly, define $\widetilde{S} : F(T,2)\setminus U \longrightarrow \mathrm{NL}(T)$ by $\widetilde{S}(p,q) = [(1-t)(p_1,p_2)+t(q_1,q_2)]$, and extend $\widetilde{S}$ in the obvious way over the open set described in the following figure.



So, we have covered $F(T,2)$ with two open sets over each of which there is a section for the map $e_{0,1} : \mathrm{NL}(T) \to F(T,2)$. Consequently $\mathrm{secat}\Big(e_{0,1} : \mathrm{NL}(T) \to F(T,2)\Big) \leq 1$. It should be noted that the latter assertion does not take into account the $\mathbb{Z}_2$ action used in the definition of TC^S and TC^Σ. The latter invariants will be addressed in the following chapters.

Chapter 4

Minimal $B_2(T)$-Free Resolution of $\mathbb{Z}$

From this chapter on we will rename the generators of $B_2(T)$ to simplify the notation. According to the notation used in Theorem 2.1, we define

$$a = \rho_0\sigma, \quad b = \sigma^{-1}\tau_0, \quad c = \rho_0\tau_0\sigma^{-1}$$

Then by Theorem 2.3,

$$B_2(T) = \langle a,\ b,\ c \mid [a^2, b] = [a^2, c] = [b^2, a] = [b^2, c] = 1, \quad a^2 b^2 = c^2 \rangle.$$

Also the center of $B_2(T)$ is $\langle a^2, b^2 \rangle$ and

$$B_2(T)\Big/ Z(B_2(T)) = \langle a,\ b,\ c \mid a^2 = b^2 = c^2 = 1 \rangle = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$

In section §2.2 we studied the exact sequence

$$0 \longrightarrow \mathbb{Z}a^2 \times \mathbb{Z}b^2 \longrightarrow B_2(T) \longrightarrow \mathbb{Z}/2\mathbb{Z}a * \mathbb{Z}/2\mathbb{Z}b * \mathbb{Z}/2\mathbb{Z}c \longrightarrow 1,$$

showing that $B_2(T)$ is a semi-direct product perturbed by *palin*[1]:

$$\left(\mathbb{Z}a^2 \times \mathbb{Z}b^2\right) \rtimes_{palin} \left(\mathbb{Z}_2 a * \mathbb{Z}_2 b * \mathbb{Z}_2 c\right).$$

See Theorem (2.5). In particular we have a canonical word form for each element in $B_2(T)$, that is, if $W \in B_2(T)$, then

$$W = a^{2n} b^{2m} w, \quad \text{where } w \in \mathbb{Z}_2 a * \mathbb{Z}_2 b * \mathbb{Z}_2 c,$$

[1]*palin* is the function defined by the equation (2.9).

and we will simply interpret the word w as the braid defined by the letters of w. Summarizing, an element W in $B_2(T)$ has a canonical word expression of the form $a^{2n}b^{2m}w$.

The objective of this chapter is to develop the algebraic tools that will allow us to do the fiber analysis of $\varepsilon_{0,1} : \mathrm{NL}(T)/\mathbb{Z}_2 \longrightarrow B(T,2)$. Such results will allow us calculate the symmetric topological complexity of the torus. At any rate, the results in this chapter are highly non-trivial and interesting in themselves.

Haefliger describes in [15] the cohomology ring $H^*(B(X,2))$ for certain manifolds X. Using Haefliger's calculations, Farber and Grant study $\mathrm{TC}^S X$ in [1] when X is a closed smooth manifold.

A well known fact is that if (C,d) is a chain complex and $s_n : C_n \longrightarrow C_{n+1}$, $s_{n-1} : C_{n-1} \longrightarrow C_n$ are additive homomorphisms such that $d_{n+1}s_n + s_{n-1}d_n = Id_{C_n}$, then C is exact in C_n. This is the case when $s_*\colon C \to C$ is in fact a contracting homotopy, i.e., when the latter equality holds for every n. The main objective of this chapter is to construct a free resolution of $\mathbb{Z}$ over $B_2(T)$ together with a contracting homotopy.

We define the sequence of modules $C_3 = \bigoplus_{i=1}^{3} \mathbb{Z}_{B_2(T)}P_i$, $C_2 = \bigoplus_{i=1}^{5} \mathbb{Z}_{B_2(T)}R_i$, $C_1 = \bigoplus_{i=a}^{c} \mathbb{Z}_{B_2(T)}[i]$, $C_0 = \mathbb{Z}_{B_2(T)}$, $C_{-1} = \mathbb{Z}$ and $C_n = 0$ for all n other than $3,\ 2,\ 1,\ 0,\ -1$. Then we define the graded $\mathbb{Z}_{B_2(T)}$-module $C = \bigoplus_{n\in\mathbb{Z}} C_n$ and we define a $\mathbb{Z}_{B_2(T)}$-endomorphism $d : C \longrightarrow C$ of degree -1. The map components are defined on generators and then extend by linearly.

Let $d_3 : \bigoplus_{i=1}^{3} \mathbb{Z}_{B_2(T)}P_i \longrightarrow \bigoplus_{i=1}^{5} \mathbb{Z}_{B_2(T)}R_i$ be defined by

$$\begin{aligned} d_3(P_1) &= (1+b)R_1 + (1+a)R_3, \\ d_3(P_2) &= (1+b)R_1 + (1+c)R_4 - (1-b^2)R_5, \\ d_3(P_3) &= R_2 + a^2 R_4 - (1-c)R_5. \end{aligned} \tag{4.1}$$

Let $d_2 : \bigoplus_{i=1}^{5} \mathbb{Z}_{B_2(T)} R_i \longrightarrow \bigoplus_{i=a}^{c} \mathbb{Z}_{B_2(T)}[i]$ be defined by

$$
\begin{aligned}
d_2(R_1) =& (1+a-b-ba)[a] - (1-a^2)[b] = (1-b)(1+a)[a] - (1-a^2)[b], \\
d_2(R_2) =& (1+a-c-ca)[a] - (1-a^2)[c] = (1-c)(1+a)[a] - (1-a^2)[c], \\
d_2(R_3) =& -(1-b^2)[a] + (1-a+b-ab)[b] = -(1-b^2)[a] + (1-a)(1+b)[b], \\
d_2(R_4) =& (1+b-c-cb)[b] - (1-b^2)[c] = (1-c)(1+b)[b] - (1-b^2)[c], \\
d_2(R_5) =& (1+a)[a] + a^2(1+b)[b] - (1+c)[c].
\end{aligned} \tag{4.2}
$$

Let $d_1 : \bigoplus_{i=a}^{c} \mathbb{Z}_{B_2(T)}[i] \longrightarrow \bigoplus_{i=a}^{c} \mathbb{Z}_{B_2(T)}$ be defined by $d_1([i]) = i - 1$.

Lastly, $d_0 = \varepsilon : \mathbb{Z}_{B_2(T)} \longrightarrow \mathbb{Z}$ is the augmentation map.

The rest of the boundary operators are forced to be zero: $d_n : C_n = \{0\} \longrightarrow C_{n-1}$ for all n diferent than $\{3,\ 2,\ 1,\ 0,\ -1\}$ and $d_{-1} : \mathbb{Z} \longrightarrow 0$.

Theorem 4.1

The sequence (C, d) below is a $\mathbb{Z}_{B_2(T)}$-chain complex.

$$
\cdots 0 \longrightarrow \bigoplus_{i=1}^{3} \mathbb{Z}_{B_2(T)} P_i \xrightarrow{d_3} \bigoplus_{i=1}^{5} \mathbb{Z}_{B_2(T)} R_i \xrightarrow{d_2} \cdots \xrightarrow{d_2} \bigoplus_{i=a}^{c} \mathbb{Z}_{B_2(T)}[i] \xrightarrow{d_1} \mathbb{Z}_{B_2(T)} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0. \tag{4.3}
$$

Proof. The proof is a straightforward calculation generator by generator, degree by degree.

$$
\begin{aligned}
d_2 d_3(P_1) =& d_2\left((1+b)R_1 + (1+a)R_3\right) \\
=& (1+b)d_2(R_1) + (1+a)d_2(R_3) \\
=& (1+b)\Big((1-b)(1+a)[a] - (1-a^2)[b]\Big) + (1+a)\Big(-(1-b^2)[a] + (1-a)(1+b)[b]\Big).
\end{aligned}
$$

The elements $(1-a^2)$ and $(1-b^2)$ lie in the center of $\mathbb{Z}_{B_2}(T)$, so $d_2 d_3(P_1) = 0$.

$$\begin{aligned} d_2 d_3(P_2) =& d_2 \Big((1+b)R_1 + (1+c)R_4 - (1-b^2)R_5\Big) \\ =& (1+b)d_2(R_1) + (1+c)d_2(R_4) - (1-b^2)d_2(R_5) \\ =& (1+b)\Big((1-b)(1+a)[a] - (1-a^2)[b]\Big) + (1+c)\Big((1-c)(1+b)[b] - (1-b^2)[c]\Big) \\ & - (1-b^2)\Big((1+a)[a] + a^2(1+b)[b] - (1+c)[c]\Big). \end{aligned}$$

The element $(1-b^2)$ lies in the center of $\mathbb{Z}_{B_2}(T)$ and $c^2 = a^2b^2$, so

$$d_2 d_3(P_2) = -(1-a^2)(1+b)[b] + (1-a^2b^2)(1+b)[b] - a^2(1-b^2)(1+b)[b] = 0.$$

Likewise,

$$\begin{aligned} d_2 d_3(P_3) =& d_2\Big(R_2 + a^2R_4 - (1-c)R_5\Big) = d_2(R_2) + a^2 d_2(R_4) - (1-c)d_2(R_5) \\ =& \Big((1-c)(1+a)[a] - (1-a^2)[c]\Big) + a^2\Big((1-c)(1+b)[b] - (1-b^2)[c]\Big) \\ & - (1-c)\Big((1+a)[a] + a^2(1+b)[b] - (1+c)[c]\Big) \end{aligned}$$

and the element a^2 lies in the center of $\mathbb{Z}_{B_2}(T)$, thereupon

$$= -(1-a^2)[c] - (a^2 - a^2b^2)[c] + (1-c^2)[c] = 0.$$

For the degree-two generators $R_1,\ R_2,\ R_3,\ R_4$ and R_5:

$$\begin{aligned} d_1 d_2(R_1) =& d_1\Big((1-b)(1+a)[a] - (1-a^2)[b]\Big) = (1-b)(1+a)d_1([a]) - (1-a^2)d_1([b]) \\ =& (1-b)(1+a)(a-1) - (1-a^2)(b-1) = 0. \\ d_1 d_2(R_2) =& d_1\Big((1-c)(1+a)[a] - (1-a^2)[c]\Big) = (1-c)(1+a)(a-1) - (1-a^2)(c-1) = 0. \\ d_1 d_2(R_3) =& d_1\Big(-(1-b^2)[a] + (1-a)(1+b)[b]\Big) = (b^2-1)(a-1) + (1-a)(1+b)(b-1) = 0. \\ d_1 d_2(R_4) =& d_1\Big((1-c)(1+b)[b] - (1-b^2)[c]\Big) = (1-c)(1+b)(b-1) - (1-b^2)(c-1) = 0. \\ d_1 d_2(R_5) =& d_1\Big((1+a)[a] + a^2(1+b)[b] - (1+c)[c]\Big) \\ =& (1+a)(a-1) + a^2(1+b)(b-1) - (1+c)(c-1) = a^2 - 1 + a^2b^2 - a^2 - c^2 + 1 = 0. \end{aligned}$$

Lastly, for $i \in \{a,\ b,\ c\}$ we have $\varepsilon\ d_1([i]) = \varepsilon(i-1) = 1 - 1 = 0$, which completes the proof. ■

4.1 A Contracting Homotopy of (C, d)

Let us now consider C as a graded $\mathbb{Z}$-module. Thus, the group C_n is zero, if n is diferent than $\{3,\ 2,\ 1,\ 0,\ -1\}$, and

$$C_{-1} = \mathbb{Z} \quad C_0 = \mathbb{Z}_{B_2(T)}, \quad C_1 = \bigoplus_{i=a}^{c} \mathbb{Z}_{B_2(T)}[i], \quad C_2 = \bigoplus_{i=1}^{5} \mathbb{Z}_{B_2(T)} R_i, \quad C_3 = \bigoplus_{i=1}^{3} \mathbb{Z}_{B_2(T)} P_i.$$

In each dimension, C is $\mathbb{Z}$ generated by $a^{2n} b^{2m} w\ E^i$, where E^i stands for generic generator of C_i as $\mathbb{Z}_{B_2(T)}$-module and $a^{2n} b^{2m} w$ is an element of $\mathbb{Z}_{B_2(T)}$ in its canonical form. Just for emphasis, let us note that the generator of $C_{-1} = \mathbb{Z}$ is the number 1.

We now define a degree-1 $\mathbb{Z}$-endomorphism $s : C \longrightarrow C$ via its value on $\mathbb{Z}$-generators and then extend it in a linear way. Explicitly, $s_n = 0$ for all $n \leqslant -2$ and $n \geqslant 4$, whereas:

- $s_{-1} : \mathbb{Z} \longrightarrow \mathbb{Z}_{B_2(T)}$ is given by $s_{-1}(n) = n$.
- $s_0 : \mathbb{Z}_{B_2(T)} \longrightarrow C_1$ is given by

$$s_0(g_1 g_2 \cdots g_u) = [g_1] + g_1[g_2] + \cdots + g_1 g_2 \cdots g_{u-1}[g_u], \tag{4.4}$$

where $g_1 g_2 \cdots g_u$ is in its canonical form, and with the consideration that, if $g^{-1} = g_i$, then $[g_i] = [g^{-1}] = -g^{-1}[g]$, where $g = a$ or b and $s_0(1) = 0$. In particualr, the elements of the sum in (4.4) may fail to appear in their canonical form.

A property that satisfies s_0 that will be very useful to us later is the following

$$s_0(g_1 g_2 \cdots g_k g_{k+1} \cdots g_u) = s_0(g_1 g_2 \cdots g_k) + g_1 g_2 \cdots g_k s_0(g_{k+1} \cdots g_u), \tag{4.5}$$

when $g_1 g_2 \ldots g_k g_{k+1} \ldots g_u$ and $g_{k+1} \ldots g_u$ are words in their canonical form.

In preparation for the definition of $s_1 : C_1 \longrightarrow C_2$, we need the following preliminary considerations. Let $a^{2n} b^{2m} w$ be an element in $B_2(T)$ written in canonical form, so that $w = g_1 g_2 \cdots g_u g_{u+1}$ is a word in $\langle a,\ b,\ c \mid a^2 = b^2 = c^2 = 1\rangle$. Set in addition $w_k = g_1 g_2 \cdots g_k$ and $w_0 = 1$. Then:

$$s_1(a^{2n} b^{2m} w[a]) = \begin{cases} -a^{2n} \displaystyle\sum_{r=0}^{2m-1} b^r R_1 - a^{2n} b^{2m} \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})}, & \text{if} \quad g_{u+1} = a, \quad n \in \mathbb{Z}, \quad m \geqslant 0; \\ a^{2n} \displaystyle\sum_{r=1}^{-2m} b^{-r} R_1 - a^{2n} b^{2m} \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})}, & \text{if} \quad g_{u+1} = a, \quad n \in \mathbb{Z}, \quad m < 0; \\ 0, & \text{if} \quad g_{u+1} \neq a, \end{cases}$$

where

$$R_{\varphi(g_k)} = \begin{cases} 0, & \text{if} \quad g_k = a; \\ R_1, & \text{if} \quad g_k = b; \\ R_2, & \text{if} \quad g_k = c. \end{cases}$$

$$s_1(a^{2n} b^{2m} w_u a[a]) = -a^{2n} \sum_{r=0}^{2m-1} b^r R_1 - a^{2n} b^{2m} \left(R_{\varphi(g_1)} + g_1 R_{\varphi(g_2)} + \cdots + g_1 g_2 \cdots g_{u-1} R_{\varphi(g_u)} \right).$$

$$s_1(a^{2n} b^{2m} w[b]) = \begin{cases} -a^{2n} b^{2m} \sum_{k=0}^{k=u-1} w_k R_{\psi(g_{k+1})}, & \text{if} \quad g_{u+1} = b, \quad n, m \in \mathbb{Z}; \\ 0, & \text{if} \quad g_{u+1} \neq b, \end{cases}$$

where

$$R_{\psi(g_k)} = \begin{cases} R_3, & \text{if} \quad g_k = a; \\ 0, & \text{if} \quad g_k = b; \\ R_4, & \text{if} \quad g_k = c. \end{cases}$$

$$s_1(a^{2n} b^{2m} w_u b[b]) = -a^{2n} b^{2m} \left(R_{\psi(g_1)} + g_1 R_{\psi(g_2)} + \cdots + g_1 g_2 \cdots g_{u-1} R_{\psi(g_u)} \right).$$

$$s_1(a^{2n} b^{2m} w_u c[c]) = -a^{2n} \sum_{r=0}^{2m-1} b^r R_1 - a^{2n} b^{2m} R_5 - a^{2n} b^{2m} \big(\eta(g_1) + g_1 \eta\varphi(g_2) + \cdots + g_1 g_2 \cdots g_{u-1} \eta(g_u) \big).$$

$$s_1(a^{2n} b^{2m} w[c]) = \begin{cases} -a^{2n} \sum_{r=0}^{2m-1} b^r R_1 - a^{2n} b^{2m} R_5 \\ \quad - a^{2n} b^{2m} \sum_{k=0}^{u-1} w_k \eta(g_{k+1}), & \text{if} \quad g_{u+1} = c, \quad n \in \mathbb{Z},\ m \geqslant 0; \\ a^{2n} \sum_{r=1}^{-2m} b^{-r} R_1 - a^{2n} b^{2m} R_5 \\ \quad - a^{2n} b^{2m} \sum_{k=0}^{u-1} w_k \eta(g_{k+1}), & \text{if} \quad g_{u+1} = c, \quad n \in \mathbb{Z},\ m < 0; \\ 0, & \text{if} \quad g_{u+1} \neq c, \end{cases}$$

where

$$\eta(g_k) = \begin{cases} a^2 R_3 - (1-a) R_5, & \text{if} \quad g_k = a; \\ R_1 - (1-b) R_5, & \text{if} \quad g_k = b; \\ 0, & \text{if} \quad g_k = c. \end{cases}$$

The map $s_2\colon C_2 \longrightarrow C_3$ is defined to be $(\mathbb{Z}a^2 \times \mathbb{Z}b^2)$-linear. Thus, it suffices to define it under the following conditions: For a word W in $\langle a,\ b,\ c \mid a^2 = b^2 = c^2 = 1\rangle$, considered as an element of $B_2(T)$, say $W = g_1 g_2 \cdots g_u g_{u+1}$, set $w_k = g_1 g_2 \cdots g_k$ and

$$s_2(WR_1) = \begin{cases} \sum w_k \xi_1(g_{k+1}), & \text{if} \quad W = wb; \\ 0, & \text{if} \quad g_{u+1} \neq b, \end{cases} \quad \text{where} \quad \xi_1(x) = \begin{cases} (a-1)P_1, & \text{if} \quad x = a; \\ 0, & \text{if} \quad x = b; \\ (c-1)P_2 + (1-b^2)P_3, & \text{if} \quad x = c. \end{cases}$$

$$s_2(WR_2) = \begin{cases} w(a^2(a-1)P_1 - a^2 aP_2 + a(1+c)P_3) + \sum w_k \xi_2(g_{k+1}), & \text{if} \quad W = wac; \\ wb(-a^2 P_2 + (1+c)P_3) + \sum w_k \xi_2(g_{k+1}), & \text{if} \quad W = wbc; \\ -a^2 P_2 + (1+c)P_3, & \text{if} \quad W = c; \\ 0, & \text{if} \quad g_{u+1} \neq c, \end{cases}$$

where

$$\xi_2(x) = \begin{cases} a^2(a-1)P_1, & \text{if} \quad x = a; \\ 0, & \text{if} \quad x = b; \\ (1-c)(-a^2 P_2 + (1+c)P_3), & \text{if} \quad x = c. \end{cases}$$

$$s_2(WR_3) = \begin{cases} wbP_1 + \sum w_k \xi_3(g_{k+1}), & \text{if} \quad W = wba; \\ w(cP_1 + (1-c)P_2 + (b^2-1)P_3) + \sum w_k \xi_3(g_{k+1}), & \text{if} \quad W = wca; \\ P_1, & \text{if} \quad W = a; \\ 0, & \text{if} \quad g_{u+1} \neq a, \end{cases}$$

where

$$\xi_3(x) = \begin{cases} (1-a)P_1, & \text{if} \quad x = a; \\ 0, & \text{if} \quad x = b; \\ (1-c)P_2 + (b^2-1)P_3, & \text{if} \quad x = c. \end{cases}$$

$$s_2(WR_4) = \begin{cases} w\left((1-a)P_1 + aP_2\right) + \sum w_k \xi_4(g_{k+1}), & \text{if} \quad W = wac; \\ wbP_2 + \sum w_k \xi_4(g_{k+1}), & \text{if} \quad W = wbc; \\ P_2, & \text{if} \quad W = c; \\ 0, & \text{if} \quad g_{u+1} \neq c, \end{cases}$$

where

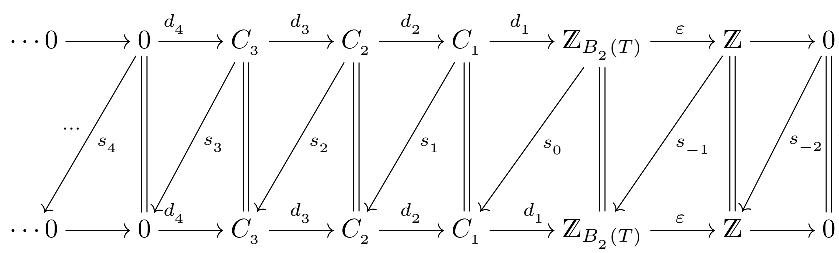
$$\xi_4(x) = \begin{cases} (1-a)P_1, & \text{if} \quad x = a; \\ 0, & \text{if} \quad x = b; \\ (1-c)P_2, & \text{if} \quad x = c. \end{cases}$$

Lastly, $s_2(WR_5) = \sum \chi_c(g_{k+1}) w_k P_3$, where χ_c is the characteristic function of $\{c\}$.

The main result of this section is:

Theorem 4.2
The map s is a contracting homotopy of (C, d). Thus (C, d) is a free resolution of $\mathbb{Z}$ over $B_2(T)$.

Proof. The proof will be complete once we check the homotopy relation $d_{\ell+1} s_\ell + s_{\ell-1} d_\ell = Id_{C_\ell}$ for all ℓ in $\mathbb{Z}$ —so that the chain complex (C, d) is exact in C_ℓ.



Case $\ell = -1$**.** Observe that $\varepsilon s_{-1}(n) = \varepsilon(n) = n$. So is (C, d) is exact in C_{-1}.

Case $\ell = 0$**.** Start by observing that $d_1([g_1]+g_1[g_2]+\cdots+g_1 g_2 \cdots g_{u-1}[g_u])$ is a telescopic sum equal to $g_1 g_2 \cdots g_{u-1} g_u - 1$. Using the convention we pointed out in (4.4) and the fact that $d_1(w[g^{-1}]) = d_1(-wg^{-1}[g])$, we get

$$d_1(w[g^{-1}]) = wg^{-1} - w.$$

Then

$$d_1 s_0 + s_{-1}\varepsilon\left(g_1 g_2 \cdots g_{u-1} g_u\right) = g_1 g_2 \cdots g_{u-1} g_u,$$

where $g_1 g_2 \cdots g_{u-1} g_u \in B_2(T)$ is written in its canonical form. The chain complex (C, d) is thus exact in C_0.

Case $\ell = 1$**.** Let $a^{2n} b^{2m} W$ be an element of $B_2(T)$ expressed in its canonical form, that is,

$$W = g_1 g_2 \cdots g_{u-1} g_u g_{u+1} \in \langle a,\ b,\ c \mid a^2 = b^2 = c^2 = 1\rangle \subset B_2(T).$$

If g_{u+1} is different from a, then $d_2 s_1(a^{2n} b^{2m} w_u g_{u+1}[a]) = d_2(0) = 0$, whereas using the identity (4.5)

$$\begin{aligned} s_0 d_1(a^{2n} b^{2m} w_u g_{u+1}[a]) =& s_0\left(a^{2n} b^{2m} w_u g_{u+1} a - a^{2n} b^{2m} w_u g_{u+1}\right) \\ =& s_0\left(a^{2n} b^{2m} w_u g_{u+1}\right) + a^{2n} b^{2m} w_u g_{u+1} s_0(a) - s_0\left(a^{2n} b^{2m} w_u g_{u+1}\right) \\ =& s_0\left(a^{2n} b^{2m} w_u g_{u+1}\right) + a^{2n} b^{2m} w_u g_{u+1}[a] - s_0\left(a^{2n} b^{2m} w_u g_{u+1}\right) \\ =& a^{2n} b^{2m} w_u g_{u+1}[a] \end{aligned} \quad (4.6)$$

To illustrate this, consider n, $m \geqslant 0$ the latter expression is

$$\begin{aligned} s_0 d_1(a^{2n} b^{2m} w_u g_{u+1}[a]) =& \left(\sum_{r=0}^{2n-1} a^r\right)[a] + a^{2n}\left(\sum_{r=0}^{2m-1} b^r\right)[b] + a^{2n} b^{2m} s_0(w_{u+1}) + a^{2n} b^{2m} w_u g_{u+1}[a] \\ & - \left(\sum_{r=0}^{2n-1} a^r\right)[a] - a^{2n}\left(\sum_{r=0}^{2m-1} b^r\right)[b] - a^{2n} b^{2m} s_0(w_{u+1}) \\ =& a^{2n} b^{2m} w_u g_{u+1}[a]. \end{aligned}$$

The other cases are analogous and differ by a sign and an exponent depending on the sign of n and m. Then from (4.6), for all n, $m \in \mathbb{Z}$, $\left(d_2 s_1 + s_0 d_1\right)\left(a^{2n} b^{2m} w_u g_{u+1}[a]\right) = Id_{C_1}\left(a^{2n} b^{2m} w_u g_{u+1}[a]\right)$ if $g_{u+1} \neq a$. In a parallel way we have $\left(d_2 s_1 + s_0 d_1\right)\left(a^{2n} b^{2m} w_u g_{u+1}[b]\right) = a^{2n} b^{2m} w_u g_{u+1}[b]$ if $g_{u+1} \neq b$, and $\left(d_2 s_1 + s_0 d_1\right)\left(a^{2n} b^{2m} w_u g_{u+1}[c]\right) = a^{2n} b^{2m} w_u g_{u+1}[c]$ if $g_{u+1} \neq c$. Thus, the bulk of the argument in the current case ($\ell = 1$) focuses on the situations for the elements of $a^{2n} b^{2m} w_u a[a]$, $a^{2n} b^{2m} w_u b[b]$ and $a^{2n} b^{2m} w_u c[c]$, which we now consider.

If $g_{u+1} = a$, the addends of $d_1\left(a^{2n} b^{2m} w_u a[a]\right)$ are not written in canonical form. So, for $n, m \geqslant 0$,

$$\begin{aligned} & s_0 d_1(a^{2n} b^{2m} w_u a[a]) \\ & \quad = s_0\left(a^{2n} b^{2m} w_u a^2 - a^{2n} b^{2m} w_u a\right) \\ & \quad = \left(\sum_{r=0}^{2n+1} a^r\right)[a] + a^{2(n+1)}\left(\sum_{r=0}^{2m-1} b^r\right)[b] + a^{2(n+1)} b^{2m} s_0(w_u) \\ & \qquad - \left(\sum_{r=0}^{2n-1} a^r\right)[a] - a^{2n}\left(\sum_{r=0}^{2m-1} b^r\right)[b] - a^{2n} b^{2m} s_0(w_u) - a^{2n} b^{2m} w_u[a] \\ & \quad = a^{2n}(1 + a - b^{2m} w_u)[a] + a^{2n}(a^2 - 1)\left(\sum_{r=0}^{2m-1} b^r\right)[b] + a^{2n} b^{2m}\left((a^2-1)s_0(w_u)\right). \end{aligned}$$

Then $a^{2n} b^{2m} w_u a[a] - s_0 d_1(a^{2n} b^{2m} w_u a[a])$ is equal to

$$\begin{aligned} & -a^{2n}\left((1 + a - b^{2m} w_u - b^{2m} w_u a)\right)[a] - a^{2n}\left((a^2 - 1)\left(\sum_{r=0}^{2m-1} b^r\right)[b] + b^{2m}\left((a^2-1)s_0(w_u)\right)\right) \\ & = -a^{2n}\left((1 + a - b^{2m} w_u(1+a))\right)[a] - a^{2n}\left(\left(\sum_{r=0}^{2m-1} b^r\right)(a^2 - 1)[b] + b^{2m}\left((a^2-1)s_0(w_u)\right)\right) \\ & = -a^{2n}\left((1 - b^{2m} w_u)(1+a)\right)[a] - a^{2n}\left(\left(\sum_{r=0}^{2m-1} b^r\right)(a^2 - 1)[b] + b^{2m}(a^2-1)s_0(w_u)\right). \end{aligned} \tag{4.7}$$

On the other hand, in $\mathbb{Z}_{B_2(T)}$ we have the identity

$$\begin{aligned} 1 - b^{2m} w_u &= 1 - b^{2m} - b^{2n} w_u + b^{2m} \\ &= \left(\sum_{r=0}^{2m-1} b^r \right) (1-b) + b^{2m} (1 - w_u). \end{aligned} \tag{4.8}$$

Then, using (4.8) in (4.7) we have

$$\begin{aligned} (Id_{c_1} - s_0 d_1)(a^{2n} b^{2m} w_u a[a]) = &-a^{2n} \left(\left(\left(\sum_{r=0}^{2m-1} b^r \right) (1-b) + b^{2m} (1 - w_u) \right) (1+a) \right) [a] \\ &- a^{2n} \left(\left(\sum_{r=0}^{2m-1} b^r \right) (a^2 - 1)[b] + b^{2m} (a^2 - 1) s_0(w_u) \right) \\ = &- a^{2n} \left(\sum_{r=0}^{2m-1} b^r \right) \Big((1-b)(1+a)[a] + (a^2-1)[b] \Big) - a^{2n} b^{2m} \big((1 - w_u)(1+a)[a] + (a^2 - 1) s_0(w_u) \big) \end{aligned}$$

and

$$(Id_{C_1} - s_0 d_1)(a^{2n} b^{2m} w_u a[a]) = -a^{2n} \left(\sum_{r=0}^{2m-1} b^r \right) d_2(R_1) - a^{2n} b^{2m} \big((1-w_u)(1+a)[a] + (a^2-1) s_0(w_u) \big). \tag{4.9}$$

Note that $w_{u-1} g_u = w_u$ since we have set $w_k = g_1 g_2 \cdots g_k$. Then

$$\begin{aligned} (1 - w_u)(1+a)[a] + (a^2-1) s_0(w_u) = &(1 - w_{u-1} + w_{u-1} - w_u)(1+a)[a] \\ &+ (a^2 - 1) \left(s_0(w_{u-1}) + w_{u-1}[g_u] \right) \\ = &(1 - w_{u-1})(1+a)[a] + (w_{u-1} - w_{u-1} g_u)(1+a)[a] + w_{u-1}(a^2 - 1)[g_u] + (a^2 - 1)(s_0(w_{u-1})) \\ = &w_{u-1} \Big((1 - g_u)(1+a)[a] + (a^2 - 1)[g_u] \Big) + (1 - w_{u-1})(1+a)[a] + (a^2 - 1)(s_0(w_{u-1})) \\ = &d_2 \left(w_{u-1} R_{\varphi(g_u)} \right) + (1 - w_{u-1})(1+a)[a] + (a^2 - 1)(s_0(w_{u-1})). \end{aligned}$$

Proceeding inductively we have

$$\begin{aligned} (1 - w_u)(1+a)[a] + (a^2-1) s_0(w_u) = &d_2 \left(w_{u-1} R_{\varphi(g_u)} \right) + d_2 \left(w_{u-2} R_{\varphi(g_{u-1})} \right) + \cdots + d_2 \left(R_{\varphi(g_1)} \right) \\ = &d_2 \left(w_{u-1} R_{\varphi(g_u)} + w_{u-2} R_{\varphi(g_{u-1})} + \cdots + R_{\varphi(g_1)} \right) = d_2 \left(\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} \right). \end{aligned}$$

Then, using (4.9), we finally have the required

$$(Id_{C_1} - s_0 d_1)(a^{2n} b^{2m} w_u a[a]) = -a^{2n} \left(\sum_{r=0}^{2m-1} b^r \right) d_2(R_1) - a^{2n} b^{2m} \left(d_2 \left(\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} \right) \right)$$
$$= d_2 \left(-a^{2n} \left(\sum_{r=0}^{2m-1} b^r \right) R_1 - a^{2n} b^{2m} \left(\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} \right) \right) = d_2 \left(s_1 \left(a^{2n} b^{2m} w_u a[a] \right) \right). \quad (4.10)$$

Before dealing with the situations where one (or both) of n and m are negative, note that, if $g = a$ or b, then

$$s_0(g^{-2k}) = [g^{-1}] + g^{-1}[g^{-1}] + \cdots + g^{-2k+1}[g^{-1}] = \sum_{r=0}^{2k-1} g^{-r}[g^{-1}] = -\sum_{r=0}^{2k-1} g^{-r} g^{-1}[g] = -\sum_{r=1}^{2k} g^{-r}[g].$$

Thus, when $n > 0$, $m \geqslant 0$ (so that $-n < 0$), we have

$$\begin{aligned} s_0 d_1(a^{-2n} b^{2m} w_u a[a]) &= s_0 \left(a^{-2n} b^{2m} w_u a^2 - a^{-2n} b^{2m} w_u a \right) \\ &= -\left(\sum_{r=1}^{2n-2} a^{-r} \right)[a] + a^{-2n+2} \left(\sum_{r=0}^{2m-1} b^r \right)[b] + a^{-2n+2} b^{2m} s_0(w_u) \\ &\quad + \left(\sum_{r=1}^{2n} a^{-r} \right)[a] - a^{-2n} \left(\sum_{r=0}^{2m-1} b^r \right)[b] - a^{-2n} b^{2m} s_0(w_u) - a^{-2n} b^{2m} w_u[a] \\ &= (a^{-2n+1} + a^{-2n} - a^{-2n} b^{2m} w_u)[a] \\ &\quad + a^{-2n} \left(\sum_{r=0}^{2m-1} b^r \right)(a^2 - 1)[b] + a^{-2n} b^{2m} \left((a^2 - 1) s_0(w_u) \right) \\ &= a^{-2n}(1 + a - b^{2m} w_u)[a] + a^{-2n}(a^2 - 1) \left(\sum_{r=0}^{2m-1} b^r \right)[b] + a^{-2n} b^{2m} \left((a^2 - 1) s_0(w_u) \right). \end{aligned}$$

Then $a^{-2n} b^{2m} w_u a[a] - s_0 d_1(a^{-2n} b^{2m} w_u a[a])$ is equal to (4.7) except for the sign in the exponent and a^{-2n} can be factored to the final conclusion in (4.10). We thus get the required

$$(Id_{C_1} - s_0 d_1)(a^{-2n} b^{2m} w_u a[a]) = d_2 \left(s_1 \left(a^{2n} b^{2m} w_u a[a] \right) \right). \quad (4.11)$$

Likewise, for the case $-m < 0$ and $n > 0$ we have

$$\begin{aligned}
s_0 d_1(a^{2n}b^{-2m}w_u a[a]) &= s_0\left(a^{2n}b^{-2m}w_u a^2 - a^{2n}b^{-2m}w_u a\right) \\
&= \left(\sum_{r=0}^{2n+1} a^r\right)[a] - a^{2(n+1)}\left(\sum_{r=1}^{2m} b^{-r}\right)[b] + a^{2(n+1)}b^{-2m}s_0(w_u) \\
&\quad - \left(\sum_{r=0}^{2n-1} a^r\right)[a] + a^{2n}\left(\sum_{r=1}^{2m} b^{-r}\right)[b] - a^{2n}b^{-2m}s_0(w_u) - a^{2n}b^{-2m}w_u[a] \\
&= a^{2n}(1+a-b^{-2m}w_u)[a] - a^{2n}(a^2-1)\left(\sum_{r=1}^{2m} b^{-r}\right)[b] + a^{2n}b^{-2m}\left((a^2-1)s_0(w_u)\right).
\end{aligned}$$

Then $a^{2n}b^{-2m}w_u a[a] - s_0 d_1(a^{2n}b^{-2m}w_u a[a])$ is equal to

$$\begin{aligned}
&-a^{2n}\left((1+a-b^{-2m}w_u - b^{-2m}w_u a)\right)[a] - a^{2n}\left((1-a^2)\left(\sum_{r=1}^{2m} b^{-r}\right)[b] + b^{-2m}\left((a^2-1)s_0(w_u)\right)\right) \\
&= -a^{2n}\left((1-b^{-2m}w_u)(1+a)\right)[a] - a^{2n}\left(\left(\sum_{r=1}^{2m} b^{-r}\right)(1-a^2)[b] + b^{-2m}(a^2-1)s_0(w_u)\right). \quad (4.12)
\end{aligned}$$

On the other hand, the following identity, analogous to that in (4.8), holds in $\mathbb{Z}_{B_2}(T)$:

$$1 - b^{-2m}w_u = -\left(\sum_{r=1}^{2m} b^{-r}\right)(1-b) + b^{-2m}(1-w_u). \quad (4.13)$$

Then, using (4.13) in (4.12), we get

$$\begin{aligned}
a^{2n}b^{-2m}w_u a[a] - s_0 d_1(a^{2n}b^{-2m}w_u a[a]) &= a^{2n}\left(\sum_{r=1}^{2m} b^{-r}\right)\left((1-b)(1+a)[a] + (a^2-1)[b]\right) \\
&\quad - a^{2n}b^{-2m}\left((1-w_u))(1+a)[a] + (a^2-1)s_0(w_u)\right)
\end{aligned}$$

which, in view of the definition of d_2 and (4.10), yields the required

$$\begin{aligned}
&= a^{2n}\left(\sum_{r=1}^{2m} b^{-r}\right)d_2(R_1) - a^{2n}b^{-2m}d_2\left(\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})}\right) \\
&= d_2\left(a^{2n}\left(\sum_{r=1}^{2m} b^{-r}\right)R_1 - a^{2n}b^{-2m}\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})}\right) = d_2\left(s_1\left(a^{2n}b^{-2m}w_u a[a]\right)\right). \quad (4.14)
\end{aligned}$$

Lastly, for the case $-n$ and $-m$ with n, $m > 0$:

$$\begin{aligned}
s_0 d_1(a^{-2n} b^{-2m} w_u a[a]) &= s_0 \left(a^{-2n} b^{-2m} w_u a^2 - a^{-2n} b^{-2m} w_u a \right) \\
&= - \left(\sum_{r=1}^{2n-2} a^{-r} \right) [a] - a^{-2n+2} \left(\sum_{r=1}^{2m} b^{-r} \right) [b] + a^{-2n+2} b^{-2m} s_0(w_u) \\
&\quad + \left(\sum_{r=1}^{2n} a^{-r} \right) [a] + a^{-2n} \left(\sum_{r=1}^{2m} b^{-r} \right) [b] - a^{-2n} b^{-2m} s_0(w_u) - a^{-2n} b^{-2m} w_u [a] \\
&= a^{-2n}(1 + a - b^{-2m} w_u)[a] - a^{-2n}(a^2 - 1) \left(\sum_{r=1}^{-2m} b^{-r} \right) [b] + a^{2n} b^{-2m} \left((a^2 - 1) s_0(w_u) \right).
\end{aligned}$$

Then $a^{-2n} b^{-2m} w_u a[a] - s_0 d_1(a^{-2n} b^{-2m} w_u a[a])$ is equal to

$$\begin{aligned}
&-a^{-2n} \left((1 + a - b^{-2m} w_u - b^{-2m} w_u a) \right) [a] - a^{-2n} \left((1 - a^2) \left(\sum_{r=1}^{2m} b^{-r} \right) [b] + b^{-2m} \left((a^2 - 1) s_0(w_u) \right) \right) \\
&\quad = - a^{-2n} \left((1 - b^{-2m} w_u)(1 + a) \right) [a] - a^{-2n} \left(\left(\sum_{r=1}^{2m} b^{-r} \right) (1 - a^2)[b] + b^{-2m} (a^2 - 1) s_0(w_u) \right).
\end{aligned} \tag{4.15}$$

Note that $a^{-2n} b^{-2m} w_u a[a] - s_0 d_1(a^{-2n} b^{-2m} w_u a[a])$ is equal to (4.12) except for the sign in the exponent and a^{-2n} can be factored to the final conclusion in (4.14). This yields the required

$$(Id_{C_1} - s_0 d_1)(a^{-2n} b^{2m} w_u a[a]) = d_2 \left(s_1 \left(a^{2n} b^{2m} w_u a[a] \right) \right). \tag{4.16}$$

We now deal with the situation for $a^{2n} b^{2m} w_u b[b]$. As above, we divide the analysis into cases according to the signs of n and m in the exponents of a and b. For $n, m > 0$, we have

$$\begin{aligned}
s_0 d_1(a^{2n} b^{2m} w_u b[b]) =& s_0 \left(a^{2n} b^{2m} w_u b^2 - a^{2n} b^{2m} w_u b \right) \\
=& \left(\sum_{r=0}^{2n-1} a^r \right) [a] + a^{2n} \left(\sum_{r=0}^{2m+1} b^r \right) [b] + a^{2n} b^{2(m+1)} s_0(w_u) \\
& - \left(\sum_{r=0}^{2n-1} a^r \right) [a] - a^{2n} \left(\sum_{r=0}^{2m-1} b^r \right) [b] - a^{2n} b^{2m} s_0(w_u) - a^{2n} b^{2m} w_u [b] \\
=& a^{2n}(b^{2m} + b^{2m+1})[b] + a^{2n} b^{2m} \left((b^2 - 1) s_0(w_u) - w_u [b] \right).
\end{aligned} \tag{4.17}$$

Then $a^{2n}b^{2m}w_u b[b] - s_0 d_1(a^{2n}b^{2m}w_u b[b])$ is equal to

$$a^{2n}b^{2m}\left((-1-b+w_u+w_u b)[b]-(b^2-1)s_0(w_u)\right)$$
$$=a^{2n}b^{2m}\Big((-(1{+}b)+w_u(1{+}b))[b]-(b^2{-}1)s_0(w_u)\Big)=a^{2n}b^{2m}\Big((w_u{-}1)(1{+}b)[b]-(b^2{-}1)s_0(w_u)\Big). \tag{4.18}$$

Recalling that $w_{u-1}g_u = w_u$, we have

$$\begin{aligned}(w_u-1)(1+b)[b]&-(b^2-1)s_0(w_u)\\ &=(w_u-w_{u-1}+w_{u-1}-1)(1+b)[b]-(b^2-1)\left(s_0(w_{u-1})+w_{u-1}[g_u]\right)\\ &=w_{u-1}\Big((g_u-1)(1+b)[b]-(b^2-1)[g_u]\Big)+(w_{u-1}-1)(1+b)[b]-(b^2-1)s_0(w_{u-1})\\ &=d_2\left(w_{u-1}R_{\psi(g_u)}\right)+\Big((w_{u-1}-1)(1+b)[b]-(b^2-1)s_0(w_{u-1})\Big).\end{aligned}$$

Proceeding inductively we have

$$\begin{aligned}(w_u-1)(1+b)[b]-(b^2-1)s_0(w_u)&=d_2\left(w_{u-1}R_{\psi(g_u)}\right)+d_2\Big(w_{u-2}R_{\psi(g_{u-1})}\Big)+\cdots+d_2\Big(R_{\psi(g_1)}\Big)\\ &=d_2\Big(w_{u-1}R_{\psi(g_u)}+w_{u-2}R_{\psi(g_{u-1})}+\cdots+R_{\psi(g_1)}\Big)=d_2\left(\sum_{k=0}^{u-1}w_k R_{\psi(g_{k+1})}\right).\end{aligned}$$

So, substituting the previous identity in (4.18), we are led to the required

$$\begin{aligned}(Id_{C_1}-s_0d_1)(a^{2n}b^{2m}w_u b[b])&=a^{2n}b^{2m}\left(d_2\left(\sum_{k=0}^{u-1}w_k R_{\psi(g_{k+1})}\right)\right)\\ &=d_2\left(a^{2n}b^{2m}\sum_{k=0}^{u-1}w_k R_{\psi(g_{k+1})}\right)=d_2\left(s_1\left(a^{2n}b^{2m}w_u b[b]\right)\right).\end{aligned} \tag{4.19}$$

For the case $-n$, with n, $m>0$:

$$\begin{aligned}s_0d_1(a^{-2n}b^{2m}w_u b[b])&=s_0\left(a^{-2n}b^{2m}w_u b^2-a^{-2n}b^{2m}w_u b\right)\\ &=-\left(\sum_{r=1}^{2n}a^{-r}\right)[a]+a^{-2n}\left(\sum_{r=0}^{2m+1}b^r\right)[b]+a^{-2n}b^{2(m+1)}s_0(w_u)\\ &\quad+\left(\sum_{r=1}^{2n}a^{-r}\right)[a]-a^{-2n}\left(\sum_{r=0}^{2m-1}b^r\right)[b]-a^{-2n}b^{2m}s_0(w_u)-a^{-2n}b^{2m}w_u[b]\\ &=a^{-2n}b^{2m}(1+b)[b]+a^{-2n}b^{2m}\Big((b^2-1)s_0(w_u)-w_u[b]\Big).\end{aligned} \tag{4.20}$$

Equality (4.20) is analogous to (4.18) except for the sign in the exponent $-2n$ and the term a^{-2n}, which can be factored until reaching a conclusion analogous to (4.19):

$$(Id_{C_1} - s_0 d_1)(a^{-2n} b^{2m} w_u b[b]) = d_2 \left(s_1 \left(a^{-2n} b^{2m} w_u b[b] \right) \right). \tag{4.21}$$

For the case $-m$, with n, $m > 0$:

$$\begin{aligned}
s_0 d_1 (a^{2n} b^{-2m} w_u b[b]) &= s_0 \left(a^{2n} b^{-2m} w_u b^2 - a^{2n} b^{-2m} w_u b \right) \\
&= \left(\sum_{r=0}^{2n-1} a^r \right) [a] - a^{2n} \left(\sum_{r=1}^{2m-2} b^{-r} \right) [b] + a^{2n} b^{-2m+2} s_0(w_u) \\
&\quad - \left(\sum_{r=0}^{2n-1} a^r \right) [a] + a^{2n} \left(\sum_{r=1}^{2m} b^{-r} \right) [b] - a^{2n} b^{-2m} s_0(w_u) - a^{2n} b^{-2m} w_u [b] \\
&= a^{2n} (b^{-2m+1} + b^{-2m})[b] + a^{2n} b^{-2m} \left((b^2 - 1) s_0(w_u) - w_u [b] \right).
\end{aligned} \tag{4.22}$$

Equality (4.22) is analogous to (4.18) except for the sign in the exponent $-2m$ and the term b^{-2m}, which can be factored until reaching a conclusion analogous to (4.19):

$$(Id_{C_1} - s_0 d_1)(a^{2n} b^{-2m} w_u b[b]) = d_2 \left(s_1 \left(a^{2n} b^{-2m} w_u b[b] \right) \right). \tag{4.23}$$

For the case $-n$, $-m$, with n, $m > 0$:

$$\begin{aligned}
s_0 d_1 (a^{-2n} b^{-2m} w_u b[b]) &= s_0 \left(a^{-2n} b^{-2m} w_u b^2 - a^{-2n} b^{-2m} w_u b \right) \\
&= - \left(\sum_{r=1}^{2n} a^{-r} \right) [a] - a^{-2n} \left(\sum_{r=1}^{2m-2} b^{-r} \right) [b] + a^{-2n} b^{-2m+2} s_0(w_u) \\
&\quad + \left(\sum_{r=1}^{2n} a^{-r} \right) [a] + a^{-2n} \left(\sum_{r=1}^{2m} b^{-r} \right) [b] - a^{-2n} b^{-2m} s_0(w_u) - a^{-2n} b^{-2m} w_u [b] \\
&= a^{-2n} (b^{-2m+1} + b^{-2m})[b] + a^{-2n} b^{-2m} \left((b^2 - 1) s_0(w_u) - w_u [b] \right).
\end{aligned} \tag{4.24}$$

Equality (4.24) is analogous to (4.18) except for the sign in the exponents $-2n$, $-2m$ and the term $a^{-2n} b^{-2m}$, which can be factored until reaching a conclusion analogous to (4.19):

$$(Id_{C_1} - s_0 d_1)(a^{-2n} b^{-2m} w_u b[b]) = d_2 \left(s_1 \left(a^{-2n} b^{-2m} w_u b[b] \right) \right). \tag{4.25}$$

Lastly, we deal with the case of $a^{2n} b^{2m} w_u c[c]$, starting with the case where both n and m are positive. We have

$$
\begin{aligned}
&s_0 d_1(a^{2n} b^{2m} w_u c[c]) \\
&\quad = s_0 \left(a^{2n} b^{2m} w_u c^2 - a^{2n} b^{2m} w_u c \right) \\
&\quad = s_0 \left(a^{2n+2} b^{2m+2} w_u - a^{2n} b^{2m} w_u c \right) \\
&\quad = \left(\sum_{r=0}^{2n+1} a^r \right) [a] + a^{2n+2} \left(\sum_{r=0}^{2m+1} b^r \right) [b] + a^{2n+2} b^{2m+2} s_0(w_u) \\
&\qquad - \left(\sum_{r=0}^{2n-1} a^r \right) [a] - a^{2n} \left(\sum_{r=0}^{2m-1} b^r \right) [b] - a^{2n} b^{2m} s_0(w_u) - a^{2n} b^{2m} w_u [c] \\
&\quad = a^{2n}(1+a)[a] + a^{2n}(a^2-1) \left(\sum_{r=0}^{2m-1} b^r \right) [b] + a^{2n} b^{2m} \left((a^2 b^2 - 1) s_0(w_u) \right) \\
&\qquad + a^{2n} b^{2m} \left(a^2 (1+b)[b] - w_u [c] \right).
\end{aligned} \tag{4.26}
$$

Then $a^{2n} b^{2m} w_u c[c] - s_0 d_1(a^{2n} b^{2m} w_u c[c])$ is equal to

$$
\begin{aligned}
&-a^{2n} \left((1+a)[a] + \left(\sum_{r=0}^{2m-1} b^r \right) (a^2-1)[b] \right) - a^{2n} b^{2m} \left(a^2(1+b)[b] + (a^2 b^2 - 1) s_0(w_u) - w_u (1+c)[c] \right) \\
&\quad = -a^{2n} \left((1+a)[a] + \left(\sum_{r=0}^{2m-1} b^r \right) (a^2-1)[b] \right) + a^{2n} b^{2m} w_u (1+a)[a] \\
&\qquad - a^{2n} b^{2m} \left(a^2 (1+b)[b] + (a^2 b^2 - 1) s_0(w_u) \right) - a^{2n} b^{2m} w_u (1+a)[a] \\
&\qquad - a^{2n} b^{2m} \left(-w_u + 1 - 1 \right) (1+c)[c].
\end{aligned}
$$

Using ([4.8](#))

$$
\begin{aligned}
&= -a^{2n} \left(\sum_{r=0}^{2m-1} b^r \right) \left((1-b)(1+a)[a] + (a^2-1)[b] \right) \\
&\quad - a^{2n} b^{2m} \left((1+a)[a] + a^2 (1-b)[b] - (1+c)[c] \right) - a^{2n} b^{2m} \left((a^2 b^2 - 1) s_0(w_u) + (1-w)(1+c)[c] \right),
\end{aligned}
$$

and using the definition of d_2 ([4.2](#)),

$$
= -a^{2n} \left(\sum_{r=0}^{2m-1} b^r \right) d_2(R_1) - a^{2n} b^{2m} d_2(R_5) - a^{2n} b^{2m} \left((a^2 b^2 - 1) s_0(w_u) + (1-w)(1+c)[c] \right). \tag{4.27}
$$

On the other hand, note that elements of the form $(a^2b^2-1)[g]+(1-g)(1+c)[c]$, with $g=a$, b or c, lie in the image of d_2. The above assertion is elementary for $g=c$ in view of the relation $c^2=a^2b^2$, whereas for the other two options we have in fact

$$\begin{aligned} d_2\eta(a) =& d_2\left(a^2R_3-(1-a)R_5\right)\\ =& a^2\left((b^2-1)[a]+(1-a)(1+b)[b]\right)-(1-a)\left((1+a)[a]+a^2(1+b)[b]-(1+c)[c]\right)\\ =& \left(a^2(b^2-1)-(1-a)(1+a)\right)[a]+(1-a)(1+c)[c]=(a^2b^2-1)[a]+(1-a)(1+c)[c] \end{aligned}$$

and

$$\begin{aligned} d_2\eta(b) =& d_2\left(R_1-(1-b)R_5\right)\\ =& (a^2-1)[b]+(1-b)(1+a)[a]-(1-b)\left((1+a)[a]+a^2(1+b)[b]-(1+c)[c]\right)\\ =& \left(a^2(b^2-1)+a^2-1\right)[b]+(1-b)(1+c)[c]=(a^2b^2-1)[b]+(1-b)(1+c)[c]. \end{aligned}$$

Then

$$\begin{aligned} &(1-w_u)(1+c)[c]+(a^2b^2-1)s_0(w_u)\\ &\quad=(1-w_{u-1}+w_{u-1}-w_{u-1}g_u)(1+c)[c]+(a^2b^2-1)\left(s_0(w_{u-1})+w_{u-1}[g_u]\right)\\ &\quad=w_{u-1}\left((a^2b^2-1)[g_u]+(1-g_u)(1+c)[c]\right)+(a^2b^2-1)s_0(w_{u-1})+(1-w_{u-1})(1+c)[c]\\ &\quad=w_{u-1}d_2\left(\eta(g_u)\right)+(a^2b^2-1)s_0(w_{u-1})+(1-w_{u-1})(1+c)[c]. \end{aligned}$$

Proceeding inductively we have

$$(a^2b^2-1)s_0(w_u)+(1-w_u)(1+c)[c]=\sum_{k=0}^{u-1}w_kd_2\left(\eta(g_{k+1})\right). \tag{4.28}$$

Finally, using (4.28) in (4.27), we get the required

$$\begin{aligned} &a^{2n}b^{2m}w_uc[c]-s_0d_1(a^{2n}b^{2m}w_uc[c])\\ &\quad=d_2\left(-a^{2n}\left(\sum_{r=0}^{2m-1}b^r\right)R_1-a^2b^2R_5\right)-d_2\left(a^{2n}b^{2m}\sum_{k=0}^{u-1}w_k\eta(g_{k+1})\right)\\ &\quad=d_2\left(-a^{2n}\left(\sum_{r=0}^{2m-1}b^r\right)R_1-a^2b^2R_5-a^{2n}b^{2m}\sum_{k=0}^{u-1}w_k\eta(g_{k+1})\right)=d_2\left(s_1\left(a^{2n}b^{2m}w_uc[c]\right)\right). \end{aligned}$$

For negative n:

$$\begin{aligned}
s_0 d_1(a^{-2n} b^{2m} w_u c[c]) = s_0 \left(a^{-2n} b^{2m} w_u c^2 - a^{-2n} b^{2m} w_u c\right) = s_0 \left(a^{-2n+2} b^{2m+2} w_u - a^{-2n} b^{2m} w_u c\right) \\
= - \left(\sum_{r=1}^{2n-2} a^{-r}\right)[a] + a^{-2n+2} \left(\sum_{r=0}^{2m+1} b^{r}\right)[b] + a^{-2n+2} b^{2m+2} s_0(w_u) \\
+ \left(\sum_{r=1}^{2n} a^{-r}\right)[a] - a^{-2n} \left(\sum_{r=0}^{2m-1} b^{r}\right)[b] - a^{-2n} b^{2m} s_0(w_u) - a^{-2n} b^{2m} w_u[c] \\
= a^{-2n}(1+a)[a] + a^{-2n}(a^2-1) \left(\sum_{r=0}^{2m-1} b^{r}\right)[b] + a^{-2n} b^{2m} \left((a^2 b^2 - 1) s_0(w_u)\right) \\
+ a^{-2n} b^{2m} \left(a^2(1+b)[b] - w_u[c]\right).
\end{aligned}$$

The latter expression is identical to (4.26) except for the sign in the exponent at $-2n$. The rest of the process is parallel and we get the required

$$a^{-2n} b^{2m} w_u c[c] - s_0 d_1(a^{-2n} b^{2m} w_u c[c]) = d_2\left(s_1(a^{-2n} b^{2m} w_u c[c])\right). \tag{4.29}$$

For negative m:

$$\begin{aligned}
s_0 d_1(a^{2n} b^{-2m} w_u c[c]) \\
= s_0 \left(a^{2n} b^{-2m} w_u c^2 - a^{2n} b^{-2m} w_u c\right) \\
= s_0 \left(a^{2n+2} b^{-2m+2} w_u - a^{2n} b^{-2m} w_u c\right) \\
= \left(\sum_{r=0}^{2n+1} a^{r}\right)[a] - a^{2n+2} \left(\sum_{r=1}^{2m-2} b^{-r}\right)[b] + a^{2n+2} b^{-2m+2} s_0(w_u) \\
- \left(\sum_{r=0}^{2n-1} a^{r}\right)[a] + a^{2n} \left(\sum_{r=1}^{2m} b^{-r}\right)[b] - a^{2n} b^{-2m} s_0(w_u) - a^{2n} b^{-2m} w_u[c] \\
= a^{2n}(1+a)[a] - a^{2n}(a^2-1) \left(\sum_{r=1}^{2m-2} b^{-r}\right)[b] + a^{2n} b^{-2m} \left((a^2 b^2 - 1) s_0(w_u)\right) \\
+ a^{2n} b^{-2m} \left((1+b)[b] - w_u[c]\right).
\end{aligned} \tag{4.30}$$

Then $a^{2n}b^{-2m}w_u c[c] - s_0 d_1(a^{2n}b^{-2m}w_u c[c])$ is equal to

$$
\begin{aligned}
-a^{2n}&\left((1+a)[a] - \left(\sum_{r=1}^{2m-2} b^{-r}\right)(a^2-1)[b]\right) - a^{2n}b^{-2m}\Big((1+b)[b] + (a^2b^2-1)s_0(w_u) - w_u(1+c)[c]\Big) \\
&= -a^{2n}\left((1+a)[a] - \left(\sum_{r=1}^{2m-2} b^{-r}\right)(a^2-1)[b]\right) + a^{2n}b^{-2m}w_u(1+a)[a] \\
&\quad - a^{2n}b^{-2m}\Big((1+b)[b] + (a^2b^2-1)s_0(w_u)\Big) - a^{2n}b^{-2m}w_u(1+a)[a] \\
&\quad - a^{2n}b^{-2m}\Big((-w_u+1-1)(1+c)[c] + a^2(1+b)[b] - a^2(1+b)[b]\Big).
\end{aligned}
$$

Using (4.13) together with the relation $\sum_{r=1}^{2m} b^{-r} = \sum_{r=1}^{2m-2} b^{-r} + b^{-2m}(b+1)$

$$
\begin{aligned}
=&a^{2n}\left(\sum_{r=1}^{2m} b^{-r}\right)\Big((1-b)(1+a)[a] + (a^2-1)[b]\Big) - a^{2n}b^{-2m}\Big((1+a)[a] + a^2(1-b)[b] - (1+c)[c]\Big) \\
&- a^{2n}b^{-2m}\Big((a^2b^2-1)s_0(w_u) + (1-w)(1+c)[c]\Big).
\end{aligned}
$$

Using the definition of d_2 (4.2) and (4.28)

$$
=a^{2n}\left(\sum_{r=1}^{2m} b^{-r}\right)d_2(R_1) - a^{2n}b^{-2m}d_2(R_5) - a^{2n}b^{-2m}\left(\sum_{k=0}^{u-1} w_k d_2\left(\eta(g_{k+1})\right)\right).
$$

The last expression readily yields the required

$$
a^{2n}b^{-2m}w_u c[c] - s_0 d_1(a^{2n}b^{-2m}w_u c[c]) = d_2\left(s_1(a^{2n}b^{-2m}w_u c[c])\right).
$$

Lastly, for both m and n negative:

$$\begin{aligned}
s_0 d_1(a^{-2n} b^{-2m} w_u c[c]) &= s_0\Big(a^{-2n} b^{-2m} w_u c^2 - a^{-2n} b^{-2m} w_u c\Big) = s_0\Big(a^{-2n+2} b^{-2m+2} w_u - a^{-2n} b^{-2m} w_u c\Big) \\
&= -\left(\sum_{r=1}^{2n-2} a^{-r}\right)[a] - a^{-2n+2}\left(\sum_{r=1}^{2m-2} b^{-r}\right)[b] + a^{-2n+2} b^{-2m+2} s_0(w_u) \\
&\quad + \left(\sum_{r=1}^{2n} a^{-r}\right)[a] + a^{-2n}\left(\sum_{r=1}^{2m} b^{-r}\right)[b] - a^{-2n} b^{-2m} s_0(w_u) - a^{-2n} b^{-2m} w_u[c] \\
&= a^{-2n}(1+a)[a] - a^{-2n}(a^2-1)\left(\sum_{r=1}^{2m-2} b^{-r}\right)[b] + a^{-2n} b^{-2m}\Big((a^2b^2-1)s_0(w_u)\Big) \\
&\quad + a^{-2n} b^{-2m}\left((1+b)[b] - w_u[c]\right). \qquad (4.31)
\end{aligned}$$

Equation (4.31) is identical to (4.30) except for the sign in the exponent at $-2n$, the rest of the process is analogous and we get the required

$$a^{-2n} b^{-2m} w_u c[c] - s_0 d_1(a^{-2n} b^{-2m} w_u c[c]) = d_2\Big(s_1(a^{-2n} b^{-2m} w_u c[c])\Big). \qquad (4.32)$$

This completes the case $\ell = 1$, thus showing that (C, d) is exact in C_1.

Case $\ell = 2$**.** As before, we verify the relation $s_1 d_2 + d_3 s_2 = Id_{C_2}$ on each $\mathbb{Z}$ generator $a^{2n} b^{2m} W R_i$ of C_2. We set $W = a^{2n} b^{2m} w_u z \in B_2(T)$ written in canonical form, say $w_u = g_1 g_2 \cdots g_u$ and with z a word (to be specified below, but not starting with g_u) on the letters $\{a, b, c\}$[2]. We start by considering the generators involving R_1.

For $n, m \geq 0$, we have

$$\begin{aligned}
&a^{2n} b^{2m} w_u a R_1 - s_1 d_2\Big(a^{2n} b^{2m} w_u a R_1\Big) \\
&\qquad = a^{2n} b^{2m} w_u a R_1 - s_1\Big(a^{2n} b^{2m} w_u a(1 + a - b - ba)[a] + a^{2n} b^{2m}(a^2-1) w_u a[b]\Big).
\end{aligned}$$

Since s_1 vanishes on the three elements

$$a^2 a^{2n} b^{2m} w_u[a], \quad -a^{2n} b^{2m} w_u ab[a] \quad \text{and} \quad a^{2n} b^{2m}(a^2-1) w_u a[b],$$

[2]Recall that, for canonical-form purposes, we think of $\langle a,\ b,\ c \mid a^2 = b^2 = c^2 = 1\rangle$ as contained in $B_2(T)$.

we get

$$a^{2n}b^{2m}w_u aR_1 - s_1 d_2\left(a^{2n}b^{2m}w_u aR_1\right) = a^{2n}b^{2m}w_u aR_1 - s_1\left(a^{2n}b^{2m}w_u a[a] - a^{2n}b^{2m}w_u aba[a]\right)$$
$$= a^{2n}b^{2m}w_u aR_1 + a^{2n}\sum_{r=0}^{2m-1} b^r R_1 + a^{2n}b^{2m}\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} - a^{2n}\sum_{r=0}^{2m-1} b^r R_1$$
$$- a^{2n}b^{2m}\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} - a^{2n}b^{2m}w_u R_{\varphi(a)} - a^{2n}b^{2m}w_u aR_{\varphi(b)} - a^{2n}b^{2m}w_u abR_{\varphi(a)}.$$

But $R_{\varphi(a)} = 0$ and $R_{\varphi(b)} = R_1$, so that $a^{2n}b^{2m}w_u aR_1 - s_1 d_2\left(a^{2n}b^{2m}w_u aR_1\right) = 0$. The latter equality holds in fact for all integers n and m. Indeed, note that

$$s_1\left(a^{2n}b^{2m}w_u a[a]\right) - a^{2n}b^{2m}s_1\left(w_u a[a]\right) = s_1\left(a^{2n}b^{2m}w_u aba[a]\right) - a^{2n}b^{2m}s_1\left(w_u aba[a]\right).$$

So, if we set $\Upsilon\left(a^{2n}b^{2m}W'[g]\right) = s_1\left(a^{2n}b^{2m}W'[g]\right) - a^{2n}b^{2m}s_1(W'[g])$ for W' as above, then the difference

$$\Upsilon\left(a^{2n}b^{2m}w_u a[a]\right) - \Upsilon\left(a^{2n}b^{2m}w_u aba[a]\right)$$

can be used to take care of sign adjustments in the exponents n and m. In detail, for all n and m in $\mathbb{Z}$, we have

$$a^{2n}b^{2m}w_u aR_1 - s_1 d_2\left(a^{2n}b^{2m}w_u aR_1\right)$$
$$= a^{2n}b^{2m}w_u aR_1 - s_1\left(a^{2n}b^{2m}w_u a(1+a-b-ba)[a] + a^{2n}b^{2m}(a^2-1)w_u a[b]\right)$$
$$= a^{2n}b^{2m}w_u aR_1 - \Upsilon(a^{2n}b^{2m}w_u a[a]) - a^{2n}b^{2m}s_1(w_u a[a]) + \Upsilon(a^{2n}b^{2m}w_u aba[a])$$
$$+ a^{2n}b^{2m}s_1(w_u aba[a])$$
$$= a^{2n}b^{2m}w_u aR_1 + a^{2n}b^{2m}\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} - a^{2n}b^{2m}\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} - a^{2n}b^{2m}w_u aR_1 = 0,$$

which yields the required relation because the equality $s_2\left(a^{2n}b^{2m}w_u aR_1\right) = 0$ holds by definition. Likewise

$$a^{2n}b^{2m}w_u cR_1 - s_1 d_2\left(a^{2n}b^{2m}w_u cR_1\right)$$
$$= a^{2n}b^{2m}w_u cR_1 - s_1\left(a^{2n}b^{2m}w_u c(1+a-b-ba)[a] + a^{2n}b^{2m}(a^2-1)w_u c[b]\right)$$
$$= a^{2n}b^{2m}w_u cR_1 - \Upsilon(a^{2n}b^{2m}w_u ca[a]) - a^{2n}b^{2m}s_1(w_u ca[a]) + \Upsilon(a^{2n}b^{2m}w_u cba[a])$$
$$+ a^{2n}b^{2m}s_1(w_u cba[a]).$$

But $\Upsilon(a^{2n}b^{2m}w_u ca[a]) = \Upsilon(a^{2n}b^{2m}w_u cba[a])$, so that

$$\begin{aligned}
&a^{2n}b^{2m}w_u cR_1 - s_1 d_2\left(a^{2n}b^{2m}w_u cR_1\right)\\
&\quad = a^{2n}b^{2m}w_u cR_1 + a^{2n}b^{2m}\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + a^{2n}b^{2m}w_u R_{\varphi(c)} + a^{2n}b^{2m}w_u cR_{\varphi(a)}\\
&\qquad - a^{2n}b^{2m}\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} - a^{2n}b^{2m}w_u R_{\varphi(c)} - a^{2n}b^{2m}w_u cR_{\varphi(b)} - a^{2n}b^{2m}w_u cbR_{\varphi(a)}.
\end{aligned}$$

Since $R_{\varphi(a)} = 0$ and $R_{\varphi(b)} = R_1$, we then get $a^{2n}b^{2m}w_u cR_1 - s_1 d_2\left(a^{2n}b^{2m}w_u cR_1\right) = 0$, which yields the required relation because the equality $s_2(a^{2n}b^{2m}w_u cR_1) = 0$ holds by definition.

In order to analyze the situation for $a^{2n}b^{2m}w_u bR_1$, we make a slight detour and establish the following two identities:

$$(w_u - 1) = \sum_{k=0}^{u-1} w_k (g_{k+1} - 1). \tag{4.33}$$

$$d_3(\xi_1(x)) = (x-1)(1+b)R_1 + (1-b^2)R_{\varphi(x)} + (a^2-1)R_{\psi(x)},\ \text{for } x = a,\, b \text{ or } c. \tag{4.34}$$

The former one is obvious, while in the case of the latter one we have

$$(a-1)(1+b)R_1 + (a^2-1)R_3 = (a-1)\left((1+b)R_1 + (1+a)R_3\right) = (a-1)d_3(P_1) = sd_3(\xi_1(a)),$$

$$(b^2-1)R_1 + (1-b^2)R_1 = 0 = d_3(\xi_1(b)) \quad \text{and}$$

$$(c-1)(1+b)R_1 + (1-b^2)R_2 + (a^2-1)R_4 = d_3((c-1)P_2 + (b^2-1)P_3) = d_3(\xi_1(c)),$$

which are (4.34) for $x = a, b, c$, respectively.

Then, for all $n, m \in \mathbb{Z}$, we have

$$\begin{aligned}
&a^{2n}b^{2m}w_u cbR_1 - s_1 d_2\left(a^{2n}b^{2m}w_u cbR_1\right)\\
&\quad = a^{2n}b^{2m}w_u cbR_1 - s_1\left(a^{2n}b^{2m}w_u cb(1+a-b-ba)[a] + a^{2n}b^{2m}(a^2-1)w_u cb[b]\right)\\
&\quad = a^{2n}b^{2m}w_u cbR_1 - s_1\left(a^{2n}b^{2m}\left(w_u cba[a] - b^2 w_u ca[a] + (a^2-1)w_u cb[b]\right)\right)\\
&\quad = a^{2n}b^{2m}\Bigg(w_u cbR_1 + \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + w_u R_2 + w_u cR_1 - (1+b)R_1\\
&\qquad - b^2\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} - b^2 w_u R_2 + (a^2-1)\sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + (a^2-1)w_u R_4\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= a^{2n} b^{2m} \Bigg((w_u c - w_u + w_u - 1)(1+b)R_1 + w_u(1-b^2)R_2 \\
&\qquad + w_u(a^2-1)R_4 + \sum_{k=0}^{u-1} w_k \Big((1-b^2)R_{\varphi(g_{k+1})} + (a^2-1)R_{\psi(g_{k+1})} \Big) \Bigg) \\
&= a^{2n} b^{2m} \Bigg(w_u \Big((c-1)(1+b)R_1 + (1-b^2)R_2 + (a^2-1)R_4 \Big) \\
&\qquad + (w_u - 1)(1+b)R_1 + \sum_{k=0}^{u-1} w_{k+1} \Big((1-b^2)R_{\varphi(g_{k+1})} + (a^2-1)R_{\psi(g_{k+1})} \Big) \Bigg).
\end{aligned}
$$

Using (4.33) we then get

$$
\begin{aligned}
a^{2n} b^{2m} w_u cbR_1 - s_1 d_2 \left(a^{2n} b^{2m} w_u cbR_1 \right) &= a^{2n} b^{2m} \Bigg(w_u d_3((c-1)P_2 + (1-b^2)P_3) \\
&\quad + \sum_{k=0}^{u-1} w_k \Big((g_{k+1}-1)(1+b)R_1 + (1-b^2)R_{\varphi(g_{k+1})} + (a^2-1)R_{\psi(g_{k+1})} \Big) \Bigg)
\end{aligned}
$$

and, by (4.34), we obtain the required

$$
\begin{aligned}
&a^{2n} b^{2m} w_u cbR_1 - s_1 d_2 \left(a^{2n} b^{2m} w_u cbR_1 \right) \\
&= a^{2n} b^{2m} \left(d_3(w_u((c-1)P_2 + (1-b^2)P_3)) + \sum_{k=0}^{u-1} w_k d_3(\xi_1(g_{k+1})) \right) \\
&= d_3 \left(a^{2n} b^{2m} \left(w_u((c-1)P_2 + (1-b^2)P_3) + \sum_{k=0}^{u-1} w_k \xi_1(g_{k+1}) \right) \right) = d_3 \left(s_2 \left(a^{2n} b^{2m} w_u cbR_1 \right) \right).
\end{aligned}
$$

We proceed in a similar way for $a^{2n} b^{2m} w_u abR_1$:

$$
\begin{aligned}
&a^{2n} b^{2m} w_u abR_1 - s_1 d_2 \left(a^{2n} b^{2m} w_u abR_1 \right) \\
&\qquad = a^{2n} b^{2m} w_u abR_1 - s_1 \left(a^{2n} b^{2m} w_u ab(1+a-b-ba)[a] + a^{2n} b^{2m} (a^2-1) w_u ab[b] \right) \\
&\qquad = a^{2n} b^{2m} w_u abR_1 - s_1 \left(a^{2n} b^{2m} \left(w_u aba[a] - b^2 w_u a[a] + (a^2-1) w_u ab[b] \right) \right) \\
&\qquad = a^{2n} b^{2m} \Bigg(w_u abR_1 + \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + w_u aR_1 - (1+b)R_1 - b^2 \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} \\
&\qquad\quad + (a^2-1) \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + (a^2-1) w_u R_3 \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= a^{2n} b^{2m} \Bigg((w_u a - 1)(1+b)R_1 + w_u(a-1)(a+1)R_3 \\
&\quad + \sum_{k=0}^{u-1} w_k \Big((1-b^2)R_{\varphi(g_{k+1})} + (a^2-1)R_{\psi(g_{k+1})} \Big) + (-w_u + w_u)(1+b)R_1 \Bigg) \\
&= a^{2n} b^{2m} \Bigg(w_u(a-1)\Big((1+b)R_1 + (a+1)R_3\Big) + (w_u - 1)(1+b)R_1 \\
&\quad + \sum_{k=0}^{u-1} w_k \Big((1-b^2)R_{\varphi(g_{k+1})} + (a^2-1)R_{\psi(g_{k+1})} \Big) \Bigg).
\end{aligned}
$$

Using (4.33) we then have

$$
\begin{aligned}
&a^{2n} b^{2m} w_u abR_1 - s_1 d_2 \left(a^{2n} b^{2m} w_u abR_1 \right) \\
&\quad = a^{2n} b^{2m} \left(w_u d_3(P_1) + \sum_{k=0}^{u-1} w_k \Big((g_{k+1} - 1)(1+b)R_1 + (1-b^2)R_{\varphi(g_{k+1})} + (a^2-1)R_{\psi(g_{k+1})} \Big) \right)
\end{aligned}
$$

and, by (4.34), we get the required

$$
\begin{aligned}
a^{2n} b^{2m} w_u abR_1 - s_1 d_2 \left(a^{2n} b^{2m} w_u abR_1 \right) &= a^{2n} b^{2m} \left(d_3(w_u P_1) + \sum_{k=0}^{u-1} w_k d_3(\xi_1(g_{k+1})) \right) \\
&= d_3 \left(a^{2n} b^{2m} \left(w_u P_1 + \sum_{k=0}^{u-1} w_k \xi_1(g_{k+1}) \right) \right) = d_3 \left(s_2 \left(a^{2n} b^{2m} w_u abR_1 \right) \right). \qquad (4.35)
\end{aligned}
$$

This completes the analysis of the $\mathbb{Z}$ generators involving R_1 and we consider next those involving R_2. For any $n, m \in \mathbb{Z}$ we have

$$
\begin{aligned}
&a^{2n} b^{2m} w_u aR_2 - s_1 d_2 \left(a^{2n} b^{2m} w_u aR_2 \right) \\
&\quad = a^{2n} b^{2m} w_u aR_2 - s_1 \left(a^{2n} b^{2m} w_u a \Big((1 + a - c - ca)[a] + (a^2 - 1)[c] \Big) \right) \\
&\quad = a^{2n} b^{2m} w_u aR_2 - \Upsilon(a^{2n} b^{2m} w_u a[a]) - a^{2n} b^{2m} s_1(w_u a[a]) \\
&\qquad + \Upsilon(a^{2n} b^{2m} w_u aca[a]) + a^{2n} b^{2m} s_1(w_u aca[a]).
\end{aligned}
$$

But $\Upsilon(a^{2n} b^{2m} w_u a[a]) = \Upsilon(a^{2n} b^{2m} w_u aca[a])$, so

$$
= a^{2n} b^{2m} w_u aR_2 + a^{2n} b^{2m} \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} - a^{2n} b^{2m} \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} - a^{2n} b^{2m} w_u aR_2 = 0,
$$

which yields the required relation because the equality $s_2(a^{2n} b^{2m} w_u aR_2) = 0$ holds by definition.

Likewise,

$$\begin{aligned}
a^{2n}b^{2m}w_u bR_2 - s_1 d_2 \left(a^{2n}b^{2m}w_u bR_2\right) \\
&= a^{2n}b^{2m}w_u bR_2 - s_1 \left(a^{2n}b^{2m}w_u \left(b(1+a-c-ca)[a] + b(a^2-1)[c]\right)\right) \\
&= a^{2n}b^{2m}w_u bR_2 - \Upsilon(a^{2n}b^{2m}w_u ba[a]) - a^{2n}b^{2m}s_1(w_u ba[a]) \\
&\quad + \Upsilon(a^{2n}b^{2m}w_u bca[a]) + a^{2n}b^{2m}s_1(w_u bca[a]).
\end{aligned}$$

But $\Upsilon(a^{2n}b^{2m}w_u ba[a]) = \Upsilon(a^{2n}b^{2m}w_u bca[a])$, so

$$\begin{aligned}
&= a^{2n}b^{2m}w_u aR_2 + a^{2n}b^{2m}\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + a^{2n}b^{2m}w_u R_1 - a^{2n}b^{2m}\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} \\
&\quad - a^{2n}b^{2m}w_u R_1 - a^{2n}b^{2m}w_u aR_2 = 0,
\end{aligned}$$

which yields the required relation because $s_2(a^{2n}b^{2m}w_u bR_2) = 0$ by definition.

The analysis of the situation for $a^{2n}b^{2m}w_u cR_2$ requires establishing first the following identity analogous to (4.34). For $x = a$, b or c:

$$d_3(\xi_2(x)) = a^2(x-1)(1+b)R_1 + (1-a^2)(x-1)R_5 + (1-a^2b^2)R_{\varphi(x)} + (a^2-1)\eta(x). \tag{4.36}$$

The details of the straighforward verification follow:

For $x = a$,

$$\begin{aligned}
a^2(a-1)(1+b)R_1 + &(1-a^2)(a-1)R_5 + (a^2-1)a^2R_3 - (a^2-1)(1-a)R_5 \\
&= a^2(a-1)\left((1+b)R_1 + (1+a)R_3\right) = a^2(a-1)d_3(P_1) = d_3(\xi_2(a)).
\end{aligned}$$

For $x = b$,

$$\begin{aligned}
a^2(b^2-1)R_1 + &(1-a^2)(b-1)R_5 + (1-a^2b^2)R_1 + (a^2-1)R_1 + (a^2-1)(b-1)R_5 \\
&= \left(a^2b^2 - a^2 + 1 - a^2b^2 + a^2 - 1\right) R_1 = 0 = d_3(\xi_1(b)).
\end{aligned}$$

For $x = c$,

$$\begin{aligned}
a^2(c-1)(1+b)R_1 &+ (1-a^2b^2)R_2 + (1-a^2)(c-1)R_5 \\
&= (1-c)\left(-a^2(1+b)R_1 + (1+c)R_2 + (a^2-1)R_5\right) \\
&= (1-c)d_3((1+c)P_3 - a^2P_2) = d_3\left((1-c)((1+c)P_3 - a^2P_2)\right) = d_3(\xi_2(c)).
\end{aligned}$$

Then, for all $n, m \in \mathbb{Z}$, we have

$$\begin{aligned}
&a^{2n} b^{2m} w_u acR_2 - s_1 d_2 \left(a^{2n} b^{2m} w_u acR_2\right) \\
&\quad = a^{2n} b^{2m} w_u acR_2 - s_1 \left(a^{2n} b^{2m} w_u ac(1 + a - c - ca)[a] + a^{2n} b^{2m} (a^2 - 1) w_u ac[c]\right) \\
&\quad = a^{2n} b^{2m} w_u acR_2 - s_1 \left(a^{2n} b^{2m} \left(w_u aca[a] - a^2 b^2 w_u a[a] + (a^2 - 1) w_u ac[c]\right)\right) \\
&\quad = a^{2n} b^{2m} \Bigg(w_u acR_2 + \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + w_u aR_2 - a^2 b^2 \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} \\
&\qquad + (a^2 - 1)\Big(R_5 + \sum_{k=0}^{u-1} w_k \eta(g_{k+1}) + w_u (a^2 R_3 - (1 - a) R_5)\Big)\Bigg) \\
&\qquad + \Upsilon(a^{2n} b^{2m} w_u aca[a]) - \Upsilon(a^{2n+2} b^{2m+2} w_u a[a]) + \Upsilon(a^{2n} b^{2m} (a^2 - 1) w_u ac[c]).
\end{aligned}$$

But

$$\Upsilon(a^{2n} b^{2m} w_u aca[a]) - \Upsilon(a^{2n+2} b^{2m+2} w_u a[a]) + \Upsilon(a^{2n} b^{2m} (a^2 - 1) w_u ac[c]) = a^{2n} b^{2m} (-a^2 (1 + b)) R_1,$$

so that

$$\begin{aligned}
&a^{2n} b^{2m} w_u acR_2 - s_1 d_2 \left(a^{2n} b^{2m} w_u acR_2\right) \\
&\quad = a^{2n} b^{2m} \Bigg(- a^2 (1 + b) R_1 + w_u a(1 + c) R_2 + (a^2 - 1) a^2 w_u R_3 + (a^2 - 1)(1 - w_u + w_u a) R_5 \\
&\qquad + \sum_{k=0}^{u-1} w_k \Big((1 - a^2 b^2) R_{\varphi(g_{k+1})} + (a^2 - 1)\eta(g_{k+1})\Big) + (-w_u + w_u) a^2 (1 + b) R_1 \Bigg) \\
&= a^{2n} b^{2m} \Bigg(w_u \Big(- a^2 (1 + b) R_1 + a(1 + c) R_2 + a^2 (a^2 - 1) w_u R_3 + (a^2) aR_5 \Big) + (w_u - 1) a^2 (1 + b) R_1 \\
&\qquad + (w_u - 1)(1 - a^2) R_5 + \sum_{k=0}^{u-1} w_k \Big((1 - a^2 b^2) R_{\varphi(g_{k+1})} + (a^2 - 1)\eta(g_{k+1})\Big)\Bigg).
\end{aligned}$$

Using (4.33) we then have

$$\begin{aligned}
&a^{2n} b^{2m} w_u acR_2 - s_1 d_2 \left(a^{2n} b^{2m} w_u acR_2\right) \\
&\quad = a^{2n} b^{2m} \Bigg(w_u d_3 (a^2 (a - 1) P_1 - a^2 aP_2 + a(1 + c) P_3) + \sum_{k=0}^{u-1} w_k \Big(a^2 (g_{k+1} - 1)(1 + b) R_1 \\
&\qquad + (1 - a^2)(g_{k+1} - 1) R_5 + (1 - a^2 b^2) R_{\varphi(g_{k+1})} + (a^2 - 1)\eta(g_{k+1})\Big)\Bigg)
\end{aligned}$$

and, by (4.36), we get the required

$$
\begin{aligned}
&a^{2n}b^{2m}w_u acR_2 - s_1 d_2\left(a^{2n}b^{2m}w_u acR_2\right)\\
&\quad= a^{2n}b^{2m}\left(d_3\left(w_u(a^2(a-1)P_1 - a^2aP_2 + a(1+c)P_3)\right) + \sum_{k=0}^{u-1} w_k d_3(\xi_2(g_{k+1}))\right)\\
&\quad= d_3\left(a^{2n}b^{2m}\left(w_u(a^2(a-1)P_1 - a^2aP_2 + a(1+c)P_3) + \sum_{k=0}^{u-1} w_k \xi_2(g_{k+1})\right)\right)\\
&\quad= d_3\left(s_2\left(a^{2n}b^{2m}w_u acR_2\right)\right).
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&a^{2n}b^{2m}w_u bcR_2 - s_1 d_2\left(a^{2n}b^{2m}w_u bcR_2\right)\\
&\quad= a^{2n}b^{2m}w_u bcR_2 - s_1\left(a^{2n}b^{2m}w_u bc(1+a-c-ca)[a] + a^{2n}b^{2m}(a^2-1)w_u bc[c]\right)\\
&\quad= a^{2n}b^{2m}w_u bcR_2 - s_1\left(a^{2n}b^{2m}\left(w_u bca[a] - a^2b^2w_u ba[a] + (a^2-1)w_u bc[c]\right)\right)\\
&\quad= a^{2n}b^{2m}\Bigg(w_u bcR_2 + \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + w_u aR_1 + w_u bR_2 - a^2b^2\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})}\\
&\qquad - a^2b^2w_u R_1 + (a^2-1)\Big(R_5 + \sum_{k=0}^{u-1} w_k \eta(g_{k+1}) + w_u(R_1 - (1-b)R_5)\Big)\Bigg)\\
&\qquad + \Upsilon(a^{2n}b^{2m}w_u bca[a]) - \Upsilon(a^{2n+2}b^{2m+2}w_u ba[a]) + \Upsilon(a^{2n}b^{2m}(a^2-1)w_u bc[c]).
\end{aligned}
$$

But $\Upsilon(a^{2n}b^{2m}w_u bca[a]) - \Upsilon(a^{2n+2}b^{2m+2}w_u ba[a]) + \Upsilon(a^{2n}b^{2m}(a^2-1)w_u bc[c]) = a^{2n}b^{2m}(-a^2(1+b))R_1$, so that

$$
\begin{aligned}
&a^{2n}b^{2m}w_u bcR_2 - s_1 d_2\left(a^{2n}b^{2m}w_u bcR_2\right)\\
&\quad= a^{2n}b^{2m}\Bigg(-a^2(1+b)R_1 + w_u b(1+c)R_2 + (a^2-1)w_u bR_5 + (1-a^2)(w_u-1)R_5\\
&\qquad + a^2(1-b^2)w_u R_1 + \sum_{k=0}^{u-1} w_k\Big((1-a^2b^2)R_{\varphi(g_{k+1})} + (a^2-1)\eta(g_{k+1})\Big) + a^2(w_u b - w_u b)R_1\Bigg)\\
&\quad= a^{2n}b^{2m}\Bigg(w_u b\Big(-a^2(1+b)R_1 + (1+c)R_2 + (a^2-1)w_u R_5\Big) + (w_u-1)a^2(1+b)R_1\\
&\qquad + (w_u-1)(1-a^2)R_5 + \sum_{k=0}^{u-1} w_k\Big((1-a^2b^2)R_{\varphi(g_{k+1})} + (a^2-1)\eta(g_{k+1})\Big)\Bigg).
\end{aligned}
$$

Using (4.33) we then have

$$
\begin{aligned}
&a^{2n}b^{2m}w_u bcR_2 - s_1 d_2\left(a^{2n}b^{2m}w_u bcR_2\right)\\
&\quad = a^{2n}b^{2m}\Bigg(w_u b\ d_3(-a^2P_2 + (1+c)P_3) + \sum_{k=0}^{u-1} w_k\Big(a^2(g_{k+1}-1)(1+b)R_1\\
&\qquad + (1-a^2)(g_{k+1}-1)R_5 + (1-a^2b^2)R_{\varphi(g_{k+1})} + (a^2-1)\eta(g_{k+1})\Big)\Bigg)
\end{aligned}
$$

and, by (4.36), we get the required

$$
\begin{aligned}
&a^{2n}b^{2m}w_u bcR_2 - s_1 d_2\left(a^{2n}b^{2m}w_u bcR_2\right) = a^{2n}b^{2m}\left(d_3\left(w_u b(-a^2P_2 + (1+c)P_3)\right) + \sum_{k=0}^{u-1} w_k d_3(\xi_2(g_{k+1}))\right)\\
&=d_3\left(a^{2n}b^{2m}\left(w_u b(-a^2P_2 + (1+c)P_3) + \sum_{k=0}^{u-1} w_k \xi_2(g_{k+1})\right)\right)\\
&=d_3\left(s_2\left(a^{2n}b^{2m}w_u bcR_2\right)\right). \qquad (4.37)
\end{aligned}
$$

Next we consider the generators involving R_3. For $n, m \in \mathbb{Z}$,

$$
\begin{aligned}
&a^{2n}b^{2m}w_u bR_3 - s_1 d_2\left(a^{2n}b^{2m}w_u bR_3\right)\\
&\quad = a^{2n}b^{2m}w_u bR_3 - s_1\left(a^{2n}b^{2m}w_u b\left((b^2-1)[a] + (1-a+b-ab)[b]\right)\right)\\
&\quad = a^{2n}b^{2m}w_u bR_3 - \Upsilon(a^{2n}b^{2m}w_u b[b]) - a^{2n}b^{2m}s_1(w_u b[b])\\
&\qquad + \Upsilon(a^{2n}b^{2m}w_u bab[b]) + a^{2n}b^{2m}s_1(w_u bab[b]).
\end{aligned}
$$

But $\Upsilon(a^{2n}b^{2m}w_u b[b]) = \Upsilon(a^{2n}b^{2m}w_u bab[b])$, so

$$
= a^{2n}b^{2m}\left(w_u bR_3 + \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} - \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} - w_u bR_3\right) = 0,
$$

which yields the required relation because $s_2(a^{2n}b^{2m}w_u bR_3) = 0$ by definition. Likewise,

$$
\begin{aligned}
&a^{2n}b^{2m}w_u cR_3 - s_1 d_2\left(a^{2n}b^{2m}w_u cR_3\right)\\
&\quad = a^{2n}b^{2m}w_u cR_3 - s_1\left(a^{2n}b^{2m}w_u c\left((b^2-1)[a] + (1-a+b-ab)[c]\right)\right)\\
&\quad = a^{2n}b^{2m}w_u bR_2 - \Upsilon(a^{2n}b^{2m}w_u cb[b]) - a^{2n}b^{2m}s_1(w_u cb[b])\\
&\qquad + \Upsilon(a^{2n}b^{2m}w_u cab[b]) + a^{2n}b^{2m}s_1(w_u cab[b]).
\end{aligned}
$$

But $\Upsilon(a^{2n}b^{2m}w_u cb[b]) = \Upsilon(a^{2n}b^{2m}w_u cab[b])$, so

$$=a^{2n}b^{2m}\left(w_u cR_3 + \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + w_u R_4 - \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} - w_u R_4 - w_u cR_3\right) = 0,$$

which yield the required relation because $s_2(a^{2n}b^{2m}w_u cR_3) = 0$ by definition.

The analysis of the situation for $a^{2n}b^{2m}w_u aR_3$ requires establishing first the following identity analogous to (4.34). For $x = a$, b or c:

$$d_3(\xi_3(x)) = (x-1)(1+b)R_1 + (b^2-1)R_{\varphi(x)} + (1-a^2)R_{\psi(x)}. \tag{4.38}$$

This time the details of the straighforward verification are left to the reader. Then, for generators of the form $a^{2n}b^{2m}w_u baR_3$, we have

$$\begin{aligned}
&a^{2n}b^{2m}w_u baR_3 - s_1 d_2\left(a^{2n}b^{2m}w_u baR_3\right)\\
&\quad= a^{2n}b^{2m}w_u baR_3 - s_1\left(a^{2n}b^{2m}w_u ba(b^2-1)[a] + a^{2n}b^{2m}w_u ba(1-a+b-ab)[b]\right)\\
&\quad= a^{2n}b^{2m}w_u baR_3 - s_1\left(a^{2n}b^{2m}\left(w_u ba(b^2-1)[a] + w_u ba(-a+b)[b]\right)\right)\\
&\quad= a^{2n}b^{2m}\Bigg(w_u baR_3 + b^2\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + b^2 w_u R_1 - \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} - w_u R_1\\
&\qquad - a^2\sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + w_u bR_3\Bigg) + \Upsilon(a^{2n}b^{2m}b^2 w_u ba[a]) - \Upsilon(a^{2n}b^{2m}w_u ba[a]).
\end{aligned}$$

But $\Upsilon(a^{2n}b^{2m+2}w_u ba[a]) - \Upsilon(a^{2n}b^{2m}w_u ba[a]) = a^{2n}b^{2m}(1+b)R_1$, so

$$\begin{aligned}
&a^{2n}b^{2m}w_u baR_3 - s_1 d_2\left(a^{2n}b^{2m}w_u baR_3\right)\\
&\quad= a^{2n}b^{2m}\Bigg((1+b-w_u+w_u b^2)R_1 + w_u b(1+a)R_3\\
&\qquad + \sum_{k=0}^{u-1} w_k\left((b^2-1)R_{\varphi(g_{k+1})} + (1-a^2)R_{\psi(g_k)}\right) + (w_u b - w_u b)R_1\Bigg)\\
&\quad= a^{2n}b^{2m}\Bigg(w_u b\Big((1+b)R_1 + (1+a)R_3\Big)\\
&\qquad + (1-w_u)(1+b)R_1 + \sum_{k=0}^{u-1} w_k\left((1-a^2b^2)R_{\varphi(g_{k+1})} + (a^2-1)\eta(g_{k+1})\right)\Bigg).
\end{aligned}$$

Using (4.33) we have

$$a^{2n}b^{2m}w_u baR_3 - s_1 d_2\left(a^{2n}b^{2m}w_u baR_3\right)$$
$$= a^{2n}b^{2m}\left(w_u bd_3(P_1) + \sum_{k=0}^{u-1} w_k\left((1-g_{k+1})(1+b)R_1 + (b^2-1)R_{\varphi(g_{k+1})} + (1-a^2)R_{\psi(g_{k+1})}\right)\right)$$

and, by (4.38), we get the required

$$a^{2n}b^{2m}w_u baR_3 - s_1 d_2\left(a^{2n}b^{2m}w_u baR_3\right)$$
$$= a^{2n}b^{2m}\left(d_3\left(w_u bP_1\right) + \sum_{k=0}^{u-1} w_k d_3(\xi_3(g_{k+1}))\right) = d_3\left(s_2\left(a^{2n}b^{2m}w_u baR_3\right)\right). \tag{4.39}$$

Likewise, for generators of the form $a^{2n}b^{2m}w_u caR_3$, we have

$$a^{2n}b^{2m}w_u caR_3 - s_1 d_2\left(a^{2n}b^{2m}w_u caR_3\right)$$
$$= a^{2n}b^{2m}w_u caR_3 - s_1\left(a^{2n}b^{2m}w_u ca(b^2-1)[a] + a^{2n}b^{2m}w_u ca(1-a+b-ab)[b]\right)$$
$$= a^{2n}b^{2m}w_u caR_3 - s_1\left(a^{2n}b^{2m}\left(w_u ca(b^2-1)[a] + w_u ca(b-ab)[b]\right)\right)$$
$$= a^{2n}b^{2m}\Bigg(w_u caR_3 + b^2\sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + b^2 w_u R_2 - \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})}$$
$$- w_u R_2 + \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + w_u R_4 + w_u cR_3 - a^2\sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} - a^2 w_u bR_4\Bigg)$$
$$+ \Upsilon(a^{2n}b^{2m}b^2 w_u ca[a]) - \Upsilon(a^{2n+2}b^{2m+2}w_u ca[a]) + a^{2n}b^{2m}(w_u - w_u)(1+b)R_1.$$

But $\Upsilon(a^{2n}b^{2m+2}w_u ca[a]) - \Upsilon(a^{2n}b^{2m}w_u ca[a]) = a^{2n}b^{2m}(1+b)R_1$, so

$$a^{2n}b^{2m}w_u caR_3 - s_1 d_2\left(a^{2n}b^{2m}w_u caR_3\right)$$
$$= a^{2n}b^{2m}\Bigg(w_u\left((1+b)R_1 + (b^2-1)R_2 + c(a+1)R_3 + (1-a^2)R_4\right)$$
$$+ (1-w_u)(1+b)R_1 + \sum_{k=0}^{u-1} w_k\left((b^2-1)R_{\varphi(g_{k+1})} + (1-a^2)R_{\psi(g_{k+1})}\right)\Bigg).$$

Using (4.33) we then have

$$
\begin{aligned}
a^{2n}b^{2m}w_u caR_3 &- s_1 d_2\left(a^{2n}b^{2m}w_u caR_3\right) \\
&= a^{2n}b^{2m}\Bigg(w_u b\ d_3\left(cP_1+(1-c)P_2+(b^2-1)P_3\right) \\
&\quad + \sum_{k=0}^{u-1} w_k\left((1-g_{k+1})(1+b)R_1+(b^2-1)R_{\varphi(g_{k+1})}+(1-a^2)R_{\psi(g_{k+1})}\right)\Bigg),
\end{aligned}
$$

and, by (4.38), we get the required

$$
\begin{aligned}
a^{2n}b^{2m}w_u caR_3 &- s_1 d_2\left(a^{2n}b^{2m}w_u caR_3\right) \\
&= a^{2n}b^{2m}\left(d_3\left(w_u(cP_1+(1-c)P_2+(b^2-1)P_3)\right)+\sum_{k=0}^{u-1} w_k d_3(\xi_3(g_{k+1}))\right) \\
&= d_3\left(s_2\left(a^{2n}b^{2m}w_u caR_3\right)\right).
\end{aligned}
\tag{4.40}
$$

It is now the turn of $\mathbb{Z}$-generators involving R_4. For $n, m \in \mathbb{Z}$, we have

$$
\begin{aligned}
a^{2n}b^{2m}w_u aR_4 &- s_1 d_2\left(a^{2n}b^{2m}w_u aR_4\right) \\
&= a^{2n}b^{2m}w_u aR_4 - s_1\left(a^{2n}b^{2m}w_u a\left((1+b-c-cb)[b]+(b^2-1)[c]\right)\right) \\
&= a^{2n}b^{2m}w_u aR_4 - a^{2n}b^{2m}s_1(w_u ab[b]) + a^{2n}b^{2m}s_1(w_u acb[b]) \\
&= a^{2n}b^{2m}\left(w_u aR_4 + \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + w_u R_3 - \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} - w_u R_3 - w_u aR_4\right) \\
&= 0,
\end{aligned}
$$

which yields the required relation because $s_2(a^{2n}b^{2m}w_u aR_4) = 0$ by definition. Likewise,

$$
\begin{aligned}
a^{2n}b^{2m}w_u bR_4 &- s_1 d_2\left(a^{2n}b^{2m}w_u bR_4\right) \\
&= a^{2n}b^{2m}w_u bR_4 - s_1\left(a^{2n}b^{2m}w_u b\left((1+b-c-cb)[b]+(b^2-1)[c]\right)\right) \\
&= a^{2n}b^{2m}w_u bR_2 - a^{2n}b^{2m}s_1(w_u b[b]) + a^{2n}b^{2m}s_1(w_u bcb[b]) \\
&= a^{2n}b^{2m}\left(w_u bR_4 + \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} - \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} - w_u bR_4\right) = 0,
\end{aligned}
$$

As in previous instances, the analysis of the generators of the form $a^{2n}b^{2m}w_u cR_4$ requires the following identity, whose straightforward verification is once again left as an exercise for the reader.

For $x = a$, b or c:

$$d_3(\xi_4(x)) = (1-x)(1+b)R_1 + (1-a^2b^2)R_{\psi(x)} + (b^2-1)\eta(x) + (1-x)(b^2-1)R_5. \qquad (4.41)$$

Then, for generators of the form $a^{2n}b^{2m}w_u acR_4$, we have

$$\begin{aligned}
a^{2n}b^{2m}w_u acR_4 - s_1 d_2 &\left(a^{2n}b^{2m}w_u acR_4\right)\\
&= a^{2n}b^{2m}w_u acR_4 - s_1\left(a^{2n}b^{2m}w_u ac(1+b-c-cb)[b] + a^{2n}b^{2m}(b^2-1)w_u ac[c]\right)\\
&= a^{2n}b^{2m}w_u bcR_4 - s_1\left(a^{2n}b^{2m}\left(w_u ac(b-cb)[b] + (b^2-1)w_u ac[c]\right)\right)\\
&= a^{2n}b^{2m}\Bigg(w_u acR_4 + \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + w_u R_3 + w_u aR_4 - a^2b^2\sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})}\\
&\quad - a^2b^2w_u R_3 + b^2R_5 - b^2\sum_{k=0}^{u-1} w_k\eta(g_{k+1}) + a^2b^2w_u R_3 - b^2w_u(1-a)R_5\\
&\quad - R_5 - \sum_{k=0}^{u-1} w_k\eta(g_{k+1}) - w_u a^2R_3 + w_u(1-a)R_5 + (w_u - w_u)(1+b)R_1\Bigg)\\
&\quad + \Upsilon(a^{2n}b^{2m}b^2w_u ac[c]) - \Upsilon(a^{2n}b^{2m}w_u ac[c]).
\end{aligned}$$

But $\Upsilon(a^{2n}b^{2m}b^2w_u ac[c]) - \Upsilon(a^{2n}b^{2m}w_u ac[c]) = a^{2n}b^{2m}(1+b)R_1$, so

$$\begin{aligned}
a^{2n}b^{2m}w_u acR_4 - s_1 d_2 &\left(a^{2n}b^{2m}w_u acR_4\right)\\
&= a^{2n}b^{2m}\Bigg(w_u\left((1+b)R_1 + (1-a^2)R_3 + a(1+c)R_4 + a(b^2-1)R_5\right)\\
&\quad + (1-w_u)\left((1+b)R_1 + (b^2-1)R_5\right) + \sum_{k=0}^{u-1} w_k\Big((1-a^2b^2)R_{\psi(g_{k+1})}\\
&\quad + (b^2-1)\eta(g_{k+1})\Big)\Bigg).
\end{aligned}$$

Using (4.33) and (4.41) we then get the required

$$\begin{aligned}
&a^{2n}b^{2m}w_u acR_4 - s_1 d_2\left(a^{2n}b^{2m}w_u acR_4\right)\\
&\quad = a^{2n}b^{2m}\left(w_u d_3\left((1-a)P_1 + aP_2\right) + \sum_{k=0}^{u-1} w_k d_3\left(\xi_4(g_{k+1})\right)\right)\\
&\quad = d_3\left(s_2\left(a^{2n}b^{2m}w_u acR_4\right)\right).
\end{aligned} \tag{4.42}$$

Likewise, for generators of the form $a^{2n}b^{2m}w_u bcR_4$, we have

$$\begin{aligned}
&a^{2n}b^{2m}w_u bcR_4 - s_1 d_2\left(a^{2n}b^{2m}w_u bcR_4\right)\\
&\quad = a^{2n}b^{2m}w_u bcR_4 - s_1\left(a^{2n}b^{2m}w_u bc(1+b-c-cb)[b] + a^{2n}b^{2m}(b^2-1)w_u bc[c]\right)\\
&\quad = a^{2n}b^{2m}w_u bcR_4 - s_1\left(a^{2n}b^{2m}\left(w_u bc(b-c)[b] + (b^2-1)w_u bc[c]\right)\right)\\
&\quad = a^{2n}b^{2m}\Bigg(w_u bcR_4 + \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + w_u bR_4 - a^2b^2\sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})}\\
&\qquad + b^2R_5 + b^2\sum_{k=0}^{u-1} w_k \eta(g_{k+1}) + b^2 w_u\left(R_1 - (1-b)R_5\right) - R_5 - \sum_{k=0}^{u-1} w_k\eta(g_{k+1})\\
&\qquad - w_u\left(R_1 - (1-b)R_5\right)\Bigg) - \Upsilon(a^{2n}b^{2m}b^2 w_u bc[c]) + \Upsilon(a^{2n}b^{2m}w_u bc[c]).
\end{aligned}$$

But $-\Upsilon(a^{2n}b^{2m}b^2w_u bc[c]) + \Upsilon(a^{2n}b^{2m}w_u bc[c]) = a^{2n}b^{2m}(1+b)R_1$, so

$$\begin{aligned}
&a^{2n}b^{2m}w_u bcR_4 - s_1 d_2\left(a^{2n}b^{2m}w_u bcR_4\right)\\
&\quad = a^{2n}b^{2m}\Bigg(w_u b\left((1+b)R_1 + (1+c)R_4 + +(b^2-1)R_5\right)\\
&\qquad + (1-w_u)\left((1+b)R_1 + (b^2-1)R_5\right) + \sum_{k=0}^{u-1} w_k\left((1-c^2)R_{\psi(g_{k+1})} + (b^2-1)\eta(g_{k+1})\right)\Bigg).
\end{aligned}$$

Using (4.33) and (4.41) we then get the required

$$\begin{aligned}
&a^{2n}b^{2m}w_u bcR_4 - s_1 d_2\left(a^{2n}b^{2m}w_u bcR_4\right)\\
&\quad = a^{2n}b^{2m}\left(w_u b\ d_3\left(P_2\right) + \sum_{k=0}^{u-1} w_k d_3\left(\xi_4(g_{k+1})\right)\right) = d_3\left(s_2\left(a^{2n}b^{2m}w_u bcR_4\right)\right).
\end{aligned} \tag{4.43}$$

Lastly, for $\mathbb{Z}$-generators involving R_5 we use the following identity, whose straightforward verification is again left as an exercise for the reader. For $x = a$, b or c:

$$d_3(\chi_c(x)P_3) = R_{\varphi(x)} + a^2 R_{\psi(x)} - \eta(x) + (1-x)R_5, \tag{4.44}$$

where

$$\chi_c(x) = \begin{cases} 0, & \text{if} \quad x = a,\ b; \\ 1, & \text{if} \quad x = c. \end{cases}$$

Then, for generators of the form $a^{2n}b^{2m}w_u aR_5$, we have

$$\begin{aligned} a^{2n}b^{2m}w_u aR_5 - s_1 d_2\left(a^{2n}b^{2m}w_u aR_5\right) \\ &= a^{2n}b^{2m}w_u aR_5 - s_1\left(a^{2n}b^{2m}\left(w_u a(1+a)[a] + a^2 w_u a(1+b)[b] - w_u a(1+c)[c]\right)\right) \\ &= a^{2n}b^{2m}w_u aR_5 - s_1\left(a^2b^2(w_u a[a] + a^2 w_u ab[b] - w_u ac[c])\right) \\ &= a^{2n}b^{2m}\Bigg(w_u aR_5 + \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + a^2\sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + a^2 w_u R_3 - R_5 \\ &\quad - \sum_{k=0}^{u-1} w_k \eta(g_{k+1}) - w_u\left(a^2R_3 - (1-a)R_5\right)\Bigg) - \Upsilon(a^2b^2w_u a[a]) + \Upsilon(a^2b^2w_u ac[c]). \end{aligned}$$

But $\Upsilon(a^2b^2w_u a[a]) = \Upsilon(a^2b^2w_u ac[c])$ so, by (4.44), we get the required

$$= a^{2n}b^{2m}d_3s_2\left(w_u aR_5\right) = d_3s_2\left(a^{2n}b^{2m}w_u aR_5\right).$$

Likewise, for $\mathbb{Z}$ generators $a^{2n}b^{2m}w_u bR_5$, we have

$$\begin{aligned} a^{2n}b^{2m}w_u bR_5 - s_1 d_2\left(a^{2n}b^{2m}w_u bR_5\right) \\ &= a^{2n}b^{2m}w_u bR_5 - s_1\left(a^{2n}b^{2m}\left(w_u b(1+a)[a] + a^2 w_u b(1+b)[b] - w_u b(1+c)[c]\right)\right) \\ &= a^{2n}b^{2m}w_u bR_5 - s_1\left(a^2b^2(w_u a[a] + a^2 w_u b[b] - w_u bc[c])\right) \\ &= a^{2n}b^{2m}\Bigg(w_u bR_5 + \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + w_u R_1 + a^2\sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} - R_5 \\ &\quad - \sum_{k=0}^{u-1} w_k \eta(g_{k+1}) - w_u R_1 - w_u(1-b)R_5\Bigg) - \Upsilon(a^2b^2w_u ba[a]) + \Upsilon(a^2b^2w_u bc[c]). \end{aligned}$$

But $\Upsilon(a^2 b^2 w_u ba[a]) = \Upsilon(a^2 b^2 w_u bc[c])$ so that, by (4.44), we get the required

$$= a^{2n} b^{2m} d_3 s_2 \left(w_u bR_5\right) = d_3 s_2 \left(a^{2n} b^{2m} w_u bR_5\right).$$

Lastly, for $\mathbb{Z}$ generators of the form $a^{2n} b^{2m} w_u cR_5$, we have

$$\begin{aligned} a^{2n} b^{2m} w_u cR_5 &- s_1 d_2 \left(a^{2n} b^{2m} w_u cR_5\right) \\ &= a^{2n} b^{2m} w_u cR_5 - s_1 \left(a^{2n} b^{2m} \left(w_u c(1+a)[a] + a^2 w_u c(1+b)[b] - w_u b(1+c)[c]\right)\right) \end{aligned}$$

$$\begin{aligned} &= a^{2n} b^{2m} w_u cR_5 - s_1 \left(a^2 b^2 (w_u ca[a] + a^2 w_u cb[b] - w_u c[c])\right) \\ &= a^{2n} b^{2m} \Bigg(w_u cR_5 + \sum_{k=0}^{u-1} w_k R_{\varphi(g_{k+1})} + w_u R_2 + a^2 \sum_{k=0}^{u-1} w_k R_{\psi(g_{k+1})} + a^2 w_u R_4 - R_5 \\ &\quad - \sum_{k=0}^{u-1} w_k \eta(g_{k+1}) \Bigg) - \Upsilon(a^2 b^2 w_u ca[a]) + \Upsilon(a^2 b^2 w_u c[c]). \end{aligned}$$

But $\Upsilon(a^2 b^2 w_u ca[a]) = \Upsilon(a^2 b^2 w_u c[c]$ so that, by (4.44), we get the required

$$= a^{2n} b^{2m} d_3 s_2 \left(w_u cR_5\right) = d_3 s_2 \left(a^{2n} b^{2m} w_u cR_5\right).$$

This completes the proof in the case $\ell = 2$.

Case $\ell = 3$**.** Since s_2 and d_3 are $(\mathbb{Z}a^2 \times \mathbb{Z}b^2)$-linear, we only have to check that $Id_{C_3} - s_2 d_3$ vanishes in WP_i for all $W \in \mathbb{Z}_2 a * \mathbb{Z}_2 b * \mathbb{Z}_2 c$ and $i \in \{1, 2, 3\}$. We start by considering the case of generators of the form $w_u xaP_1$ with $x \neq a$, for which

$$\left(Id_{C_3} - s_2 d_3\right)\left(w_u xaP_1\right) = w_u xaP_1 - s_2 \left(w_u xabR_1 + w_u xaR_3\right).$$

When $x = b$, we have

$$\begin{aligned} &\left(Id_{C_3} - s_2 d_3\right)\left(w_u baP_1\right) \\ &\quad = w_u baP_1 - \sum_{k=0}^{u-1} w_k \xi_1(g_{k+1}) - w_u \xi_1(b) - w_u b(a-1)P_1 - wbP_1 - \sum_{k=0}^{u-1} w_k \xi_3(g_{k+1}). \end{aligned}$$

But $\xi_1(b) = 0$ and $\xi_1(g_{k+1}) = -\xi_3(g_{k+1})$, so we get the required

$$\left(Id_{C_3} - s_2 d_3\right)\left(w_u baP_1\right) = 0.$$

When $x = c$, we have

$$\Big(Id_{C_3} - s_2 d_3\Big)\Big(w_u caP_1\Big) = w_u caP_1 - \sum_{k=0}^{u-1} w_k \xi_1(g_{k+1}) - w_u \xi_1(c) - w_u c(a-1)P_1$$
$$- w(cP_1 + (1-c)P_2 + (b^2 - 1)P_3) - \sum_{k=0}^{u-1} w_k \xi_3(g_{k+1}).$$

Since $\xi_1(c) = (c-1)P_2 + (1-b^2)P_3$, we get the required $\Big(Id_{C_3} - s_2 d_3\Big)\Big(w_u caP_1\Big) = 0$.
When $w_u x = 1$, we have

$$\Big(Id_{C_3} - s_2 d_3\Big)\Big(aP_1\Big) = aP_1 - (a-1)P_1 - P_1 = 0.$$

Now, for generators of the form $w_u xbP_1$ with $x \neq b$, we have

$$\Big(Id_{C_3} - s_2 d_3\Big)\Big(w_u xbP_1\Big) = w_u xbP_1 - s_2\Big(w_u xbR_1 + w_u xbaR_3\Big)$$
$$= w_u xbP_1 - \sum_{k=0}^{u-1} w_k \xi_1(g_{k+1}) - w_u \xi_1(x) - w_u xbP_1 - \sum_{k=0}^{u-1} w_k \xi_3(g_{k+1}) - w_u \xi_3(x) = 0,$$

because $\xi_1 = -\xi_3$, and for the generators of the form $w_u xcP_1$ with $x \neq c$, we have

$$\Big(Id_{C_3} - s_2 d_3\Big)\Big(w_u xcP_1\Big) = w_u xcP_1 - s_2\Big(w_u xcbR_1 + w_u xcaR_3\Big)$$
$$= w_u xcP_1 - \sum_{k=0}^{u-1} w_k \xi_1(g_{k+1}) - w_u \xi_1(x) - w_u x\Big((c-1)P_2 + (b^2 - 1)P_3\Big)$$
$$- w_u x\Big(cP_1 + (1-c)P_2 + (1-b^2)P_3\Big) - \sum_{k=0}^{u-1} w_k \xi_3(g_{k+1}) - w_u \xi_3(x) = 0.$$

Next, in order to deal with $\mathbb{Z}$ generators of the form WP_2, let us note that, for any x, we have

$$\xi_1(x) + \xi_4(x) + (b^2 - 1)\chi_c(x)P_3 = 0. \tag{4.45}$$

Suppose $W = w_u xa$, with $x \neq a$, then

$$\Big(Id_{C_3} - s_2 d_3\Big)\Big(w_u xaP_2\Big) = w_u xaP_2 - s_2\Big(w_u xabR_1 + w_u xacR_4 + (b^2 - 1)w_u xaR_5\Big)$$

$$= w_u xaP_2 - \sum_{k=0}^{u-1} w_k \xi_1(g_{k+1}) - w_u \xi_1(x) - w_u x(a-1)P_1 - \sum_{k=0}^{u-1} w_k \xi_4(g_{k+1})$$
$$- w_u \xi_4(x) - w_u x\Big((1-a)P_1 + aP_2\Big) + (1-b^2)\sum_{k=0}^{u-1} \chi_c(g_{k+1}) w_k P_3 + (1-b^2) w_u \chi_c(x) P_3$$

$$= -\sum_{k=0}^{u-1} w_k \Big(\xi_1(g_{k+1}) + \xi_4(g_{k+1}) + (b^2 - 1)\chi_c(g_{k+1})P_3\Big)$$
$$- w_u \Big(\xi_1(x) + \xi_4(x) + (b^2 - 1)\chi_c(x)P_3\Big) = 0,$$

in view of (4.45).

For $W = w_u xb$, with $x \neq b$,

$$\Big(Id_{C_3} - s_2 d_3\Big)\Big(w_u xbP_2\Big) = w_u xbP_2 - s_2\Big(w_u xbR_1 + w_u xbcR_4 + (b^2 - 1)w_u xbR_5\Big)$$

$$= w_u xbP_2 - \sum_{k=0}^{u-1} w_k \xi_1(g_{k+1}) - w_u \xi_1(x) - \sum_{k=0}^{u-1} w_k \xi_4(g_{k+1}) - w_u \xi_4(x)$$

$$- w_u xbP_2 + \sum_{k=0}^{u-1} (1-b^2) w_k \chi_c(g_{k+1}) P_3 + (1-b^2) w_u \chi_c(x) P_3 = 0.$$

For $W = w_u xc$, with $x \neq c$, we define first

$$\theta(w_u, x) = \begin{cases} w_u\left((1-a)P_1 + aP_2\right), & \text{if } x = a; \\ w_u bP_2, & \text{if } x = b. \end{cases} \tag{4.46}$$

Then

$$
\begin{aligned}
\Big(Id_{C_3} - s_2 d_3\Big)\Big(w_u xcP_2\Big) &= w_u xcP_2 - s_2\Big(w_u xcbR_1 + w_u xcR_4 + (b^2-1)w_u xcR_5\Big) \\
&= w_u xcP_2 - \sum_{k=0}^{u-1} w_k \xi_1(g_{k+1}) - w_u \xi_1(x) - w_u xP_4 - \sum_{k=0}^{u-1} w_k \xi_4(g_{k+1}) - \theta(w_u, x) \\
&\quad + \sum_{k=0}^{u-1}(1-b^2)w_k \chi_c(g_{k+1})P_3 + (1-b^2)w_u \chi_c(x)P_3 \\
&= w_u\Big(xcP_2 - \xi_1(x) + x(1-c)P_2 + x(b^2-1)P_3 - \theta(1,x) + (1-b^2)xP_3\Big) \\
&= w_u(xP_2 - \xi_1(x) - \theta(1,x)) = 0.
\end{aligned}
$$

Finally for when $W = c$, we have

$$
\begin{aligned}
\Big(Id_{C_3} - s_2 d_3\Big)\Big(cP_2\Big) &= cP_2 - s_2(cbR_1 + cR_4 + (b^2-1)cR_5) \\
&= cP_2 - (c-1)P_2 + (b^2-1)P_3 - P_2 + (1-b^2)P_5 = 0.
\end{aligned}
$$

Lastly, we consider the $\mathbb{Z}$ generators involving P_3. Suppose $y = a$ or b, then for generators of the form $w_u xyP_3$, with $x \neq y$ we have

$$
\begin{aligned}
\Big(Id_{C_3} - s_2 d_3\Big)\Big(w_u xyP_3\Big) &= w_u xyP_3 - s_2(w_u xyR_2 + a^2 w_u xyR_4 + w_u xycR_5 - w_u xyR_5) \\
&= w_u xyP_3 - \sum_{k=0}^{u-1} \chi_c(g_{k+1})w_k P_3 - w_u xyP_3 + \sum_{k=0}^{u-1} \chi_c(g_{k+1})w_k P_3 = 0.
\end{aligned}
$$

For the case $y = c$, it is convenient to define

$$
\Phi(w_u, x) = \begin{cases} w_u(a^2(a-1)P_1 - a^2 aP_2 + a(1+c)P_3), & \text{if } x = a; \\ w_u b(-a^2 P_2 + (1+c)P_3), & \text{if } x = b. \end{cases} \tag{4.47}
$$

Furthermore, for $x = a$ or b, we have the identity $x(c+1)P_3 - \Phi(1,x) - a^2\theta(1,x) = 0$ and, for $x = a$, b or c, we have the identity $\xi_2(x) + a^2\xi_4(x) + \chi_c(x)R_5 = 0$, both verified by simple substitution using the definitions of the corresponding functions.

Then, to finish the proof, it only remains to consider the case of $\Big(Id_{C_3} - s_2 d_3\Big)\Big(w_u xcP_3\Big)$ with

$x \neq c$. Direct substitution gives

$$\Big(Id_{C_3} - s_2 d_3\Big)\Big(w_u xcP_3\Big) = w_u xcP_3 - s_2(w_u xcR_2 + a^2 w_u xcR_4 + a^2 b^2 w_u xR_5 - w_u xcR_5)$$

$$= w_u xcP_5 - \sum_{k=0}^{u-1} w_k \xi_2(g_{k+1}) - w_u \Phi(1,x) - \sum_{k=0}^{u-1} w_k \xi_4(g_{k+1})$$

$$- a^2 w_u \theta(1,x) - \sum_{k=0}^{u-1} \chi_c(g_{k+1}) w_k P_3 + w_u xP_3$$

$$= w_u \Big(x(c+1)P_3 - \Phi(1,x) - a^2\theta(1,x)\Big) - \sum_{k=0}^{u-1} w_{k+1}\Big(\xi_2(x) + a^2\xi_4(x) + \chi_c(x)R_5\Big) = 0.$$

This completes the proof of the case $\ell = 3$. We have thus proved that (C, d) is a free resolution of $\mathbb{Z}$ over $Z_{B_2(T)}$. ■

4.2 Diagonal Approximation

In order to calculate the cup product of two cohomology classes it is necessary to construct a diagonal approximation. In Chapter 3 we used the Alexander diagonal approximation to the bar resolution, which helped us compute the cup product of the obstruction cohomology class. We are going to use the technique described by David Handel in his article [16].

Proposition 4.3
Let G be a group, X a free resolution of $\mathbb{Z}$ over $\mathbb{Z}_G$, and T a contracting homotopy for X. Extend $T_{-1}\epsilon : X_0 \longrightarrow X_0$ to a chain map $T_{-1}\epsilon : X \longrightarrow X$ over $\mathbb{Z}$ by defining $(T_{-1}\epsilon)_i = 0$ if $i \neq 0$. Let $U_q : (X \bigotimes X)_q \longrightarrow (X \bigotimes X)_{q+1}$ for $q \geqslant -1$ be the $\mathbb{Z}$-homomorphisms given by $U_{-1} = T_{-1} \otimes T_{-1} : \mathbb{Z} \otimes \mathbb{Z} \longrightarrow X_0 \bigotimes X_0$, and $U_q(u \otimes v) = T_i(u) \otimes v + (T_{-1}\epsilon)_i(u) \otimes T_{q-i}(v)$ for $u \in X_i$, $v \in X_{q-i}$, $0 \leqslant i \leqslant q$. Then the U_q constitute a contracting homotopy for $X \bigotimes X$.

Proposition 4.4
Let G be a group, X a free resolution of $\mathbb{Z}$ over $\mathbb{Z}_G$, and U a contracting homotopy for $X \bigotimes X$. Suppose that for each $q \geqslant 0$, B_q is a $\mathbb{Z}_G$-basis for X_q such that $\epsilon(b) = 1$ for each $b \in B_0$. Let $\Psi_0 : X_0 \longrightarrow X_0 \bigotimes X_0$ be the left $\mathbb{Z}_G$-module homomorphism determined by $\Psi_0(b) = b \otimes b$ for $b \in B_0$. For $q > 0$ let $\Psi_q : X_q \longrightarrow (X \bigotimes X)_q$ be the left $\mathbb{Z}_G$-module homomorphism determined inductively by $\Psi_q(b) = U_{q-1}\Psi_{q-1}d_q(b)$ for $b \in B_q$. Then Ψ is a diagonal approximation for X.

The previous propositions state how, given a finitely generated free resolution F of the trivial $\mathbb{Z}_G$-module $\mathbb{Z}$ over $\mathbb{Z}_G$, we can construct a diagonal approximation $\Psi : F \to F \bigotimes F$. Proofs of these propositions are routine and details can be found in Handel's article [16].

In this section we are going to follow Handel's construction in proposition 4.4 inductively, degree by degree and generator by generator, for the case of the chain complex (C, d) and the contration $s : C \longrightarrow C$ defined in the section §4.1. Let $U : C \bigotimes C \longrightarrow C \bigotimes C$ be the contraction defined in proposition 4.3.

We define $\Delta : C \longrightarrow C \bigotimes C$, at each degree, using Handel's construction, which yields commutative squares of the diagram

$$\begin{array}{ccccccccccc}
\cdots 0 & \longrightarrow & 0 & \longrightarrow & C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & \mathbb{Z}_{B_2(T)} \\
\cdots \downarrow \Delta_5 & & \downarrow \Delta_4 & & \downarrow \Delta_3 & & \downarrow \Delta_2 & & \downarrow \Delta_1 & & \downarrow \Delta_0 \\
\cdots (C \bigotimes C)_5 & \longrightarrow & (C \bigotimes C)_4 & \longrightarrow & (C \bigotimes C)_3 & \longrightarrow & (C \bigotimes C)_2 & \longrightarrow & (C \bigotimes C)_1 & \longrightarrow & \mathbb{Z}_{B_2(T)} \bigotimes \mathbb{Z}_{B_2(T)}
\end{array}$$

For degree 0 we define $\Delta_0 : \mathbb{Z}_{B_2(T)} \longrightarrow \mathbb{Z}_{B_2(T)} \bigotimes \mathbb{Z}_{B_2(T)}$ by

$$\Delta_0 \left(\sum_{i=1}^{p} k_i a^{2n_i} b^{2m_i} w_i \right) = \sum_{i=1}^{p} k_i \left(a^{2n_i} b^{2m_i} w_i \otimes a^{2n_i} b^{2m_i} w_i \right). \tag{4.48}$$

The situation for degree 1 is also immediate: we define $\Delta_1 : \bigoplus_{i=a}^{c} \mathbb{Z}_{B_2(T)}[i] \longrightarrow \left(C \bigotimes C \right)_1$ by

$$\Delta_1 ([i]) = [i] \otimes i + 1 \otimes [i] \quad \text{for } i = a,\ b,\ c. \tag{4.49}$$

For degree 2 we next define $\Delta_2 : \bigoplus_{i=1}^{5} \mathbb{Z}_{B_2(T)} R_i \longrightarrow \left(C \bigotimes C \right)_2$ on each generator.
Recall from (4.2) that $d_2(R_1) = (1 + a - b - ba)[a] - (1 - a^2)[b]$, then

$$\Delta_1 d_2(R_1) = (1 + a - b - ba)([a] \otimes a + 1 \otimes [a]) - (1 - a^2)([b] \otimes b + 1 \otimes [b]).$$

The last expression expands to

$$\Delta_1 d_2(R_1) = \begin{array}{rcrc}
 & [a] \otimes a & + & 1 \otimes [a] \\
+ & a[a] \otimes a^2 & + & a \otimes a[a] \\
- & b[a] \otimes ba & - & b \otimes b[a] \\
- & ba[a] \otimes a^2 b & - & ba \otimes ba[a] \\
+ & a^2[b] \otimes a^2 b & + & a^2 \otimes a^2[b] \\
- & [b] \otimes b & - & 1 \otimes [b]
\end{array}.$$

Then

$$U_1\Delta_1 d_2(R_1) = \begin{array}{rrlrrlrll} & s_1([a]) & \otimes a & + & s_0(1) & \otimes[a] & + & 1 & \otimes s_1([a]) \\ + & s_1(a[a]) & \otimes a^2 & + & s_0(a) & \otimes a[a] & + & 1 & \otimes s_1(a[a]) \\ - & s_1(b[a]) & \otimes ba & - & s_0(b) & \otimes b[a] & - & 1 & \otimes s_1(b[a]) \\ - & s_1(ba[a]) & \otimes a^2 b & - & s_0(ba) & \otimes ba[a] & - & 1 & \otimes s_1(ba[a]) \\ + & s_1(a^2[b]) & \otimes a^2 b & + & s_0(a^2) & \otimes a^2[b] & + & 1 & \otimes s_1(a^2[b]) \\ - & s_1([b]) & \otimes b & - & s_0(1) & \otimes[b] & - & 1 & \otimes s_1([b]). \end{array}$$

Using the definition of s_1 we then get

$$\begin{aligned} U_1\Delta_1 d_2(R_1) = {} & R_1 \otimes a^2 b + [a] \otimes a[a] - [b] \otimes b[a] - [b] \otimes ba[a] \\ & -b[a] \otimes ba[a] + [a] \otimes a^2[b] + a[a] \otimes a^2[b] + 1 \otimes R_1 \\ = {} & R_1 \otimes a^2 b + 1 \otimes R_1 + (1-b)\,([a] \otimes a[a]) \\ & + ((1+a)\,[a]) \otimes a^2[b] - [b] \otimes (b\,(1+a)\,[a])\,. \end{aligned}$$

Likewise, we have $d_2(R_2) = (1 + a - c - ca)[a] - (1 - a^2)[c]$, so

$$\Delta_1 d_2(R_2) = (1 + a - c - ca)([a] \otimes a + 1 \otimes [a]) - (1 - a^2)([c] \otimes c + 1 \otimes [c]),$$

which expands to

$$\Delta_1 d_2(R_2) = \begin{array}{rcrc} & [a] \otimes a & + & 1 \otimes [a] \\ + & a[a] \otimes a^2 & + & a \otimes a[a] \\ - & c[a] \otimes ca & - & c \otimes c[a] \\ - & ca[a] \otimes a^2 c & - & ca \otimes ca[a] \\ + & a^2[c] \otimes a^2 c & + & a^2 \otimes a^2[c] \\ - & [c] \otimes c & - & 1 \otimes [c]. \end{array}$$

Therefore $U_1\Delta_1 d_2(R_2)$ becomes

$$U_1\Delta_1 d_2(R_2) = \begin{array}{rrlrrlrll} & s_1([a]) & \otimes a & + & s_0(1) & \otimes[a] & + & 1 & \otimes s_1([a]) \\ + & s_1(a[a]) & \otimes a^2 & + & s_0(a) & \otimes a[a] & + & 1 & \otimes s_1(a[a]) \\ - & s_1(c[a]) & \otimes ca & - & s_0(c) & \otimes c[a] & - & 1 & \otimes s_1(c[a]) \\ - & s_1(ca[a]) & \otimes a^2 c & - & s_0(ca) & \otimes ca[a] & - & 1 & \otimes s_1(ca[a]) \\ + & s_1(a^2[c]) & \otimes a^2 c & + & s_0(a^2) & \otimes a^2[c] & + & 1 & \otimes s_1(a^2[c]) \\ - & s_1([c]) & \otimes c & - & s_0(1) & \otimes[c] & - & 1 & \otimes s_1([c]) \end{array}$$

Simplification yields

$$\begin{aligned} U_1\Delta_1 d_2(R_2) = \;& R_2\otimes a^2c+[a]\otimes a[a]-[c]\otimes c[a]-[c]\otimes ca[a]\\ & -c[a]\otimes ca[a]+[a]\otimes a^2[c]+a[a]\otimes a^2[c]+1\otimes R_2\\ = \;& R_2\otimes a^2c+1\otimes R_2+(1-c)\left([a]\otimes a[a]\right)\\ & +\left((1+a)\,[a]\right)\otimes a^2[c]-[c]\otimes\left(c\,(1+a)\,[a]\right). \end{aligned}$$

For R_3, we have $d_2(R_3)=(1+b-a-ab)[b]-(1-b^2)[a]$, so

$$\Delta_1 d_2(R_3)=(1+b-a-ab)([b]\otimes b+1\otimes[b])-(1-b^2)([a]\otimes a+1\otimes[a]),$$

which expands to

$$\Delta_1 d_2(R_3)=\begin{array}{rcrc} & [b]\otimes b & + & 1\otimes[b]\\ + & b[b]\otimes b^2 & + & b\otimes b[b]\\ - & a[b]\otimes ab & - & a\otimes a[b]\\ - & ab[b]\otimes b^2a & - & ab\otimes ab[b]\\ + & b^2[a]\otimes b^2a & + & b^2\otimes b^2[a]\\ - & [a]\otimes a & - & 1\otimes[a] \end{array}$$

Then

$$U_1\Delta_1 d_2(R_3)=\begin{array}{rllrllrll} & s_1([b]) & \otimes b & + & s_0(1) & \otimes[b] & + & 1 & \otimes s_1([b])\\ + & s_1(b[b]) & \otimes b^2 & + & s_0(b) & \otimes b[b] & + & 1 & \otimes s_1(b[b])\\ - & s_1(a[b]) & \otimes ab & - & s_0(a) & \otimes a[b] & - & 1 & \otimes s_1(a[b])\\ - & s_1(ab[b]) & \otimes b^2a & - & s_0(ab) & \otimes ab[b] & - & 1 & \otimes s_1(ab[b])\\ + & s_1(b^2[a]) & \otimes b^2a & + & s_0(b^2) & \otimes b^2[a] & + & 1 & \otimes s_1(b^2[a])\\ - & s_1([a]) & \otimes a & - & s_0(1) & \otimes[a] & - & 1 & \otimes s_1([a]), \end{array}$$

or

$$\begin{aligned} U_1\Delta_1 d_2(R_3) = \;& R_3\otimes b^2a+[b]\otimes b[b]-[a]\otimes a[b]-[a]\otimes ab[b]\\ & -a[b]\otimes ab[b]+[b]\otimes b^2[a]+b[b]\otimes b^2[a]+1\otimes R_3\\ = \;& R_3\otimes b^2a+1\otimes R_3+(1-a)\left([b]\otimes b[b]\right)\\ & +\left((1+b)\,[b]\right)\otimes b^2[a]-[a]\otimes\left(a\,(1+b)\,[b]\right). \end{aligned}$$

For R_4, $d_2(R_4)=(1+b-c-cb)[b]-(1-b^2)[c]$, so

$$\Delta_1 d_2(R_4)=(1+b-c-cb)([b]\otimes b+1\otimes[b])-(1-b^2)([c]\otimes c+1\otimes[c]),$$

which expands to

$$\Delta_1 d_2(R_4) = \begin{array}{rcrcr} & [b]\otimes b & + & 1\otimes [b] \\ + & b[b]\otimes b^2 & + & b\otimes b[b] \\ - & c[b]\otimes cb & - & c\otimes c[b] \\ - & cb[b]\otimes b^2 c & - & cb\otimes cb[b] \\ + & b^2[c]\otimes b^2 c & + & b^2\otimes b^2[c] \\ - & [c]\otimes c & - & 1\otimes [c]. \end{array}$$

Therefore

$$U_1\Delta_1 d_2(R_4) = \begin{array}{rllrllrll} & s_1([b]) & \otimes b & + & s_0(1) & \otimes [b] & + & 1 & \otimes s_1([b]) \\ + & s_1(b[b]) & \otimes b^2 & + & s_0(b) & \otimes b[b] & + & 1 & \otimes s_1(b[b]) \\ - & s_1(c[b]) & \otimes cb & - & s_0(c) & \otimes c[b] & - & 1 & \otimes s_1(c[b]) \\ - & s_1(cb[b]) & \otimes b^2 c & - & s_0(cb) & \otimes cb[b] & - & 1 & \otimes s_1(cb[b]) \\ + & s_1(b^2[c]) & \otimes b^2 c & + & s_0(b^2) & \otimes b^2[c] & + & 1 & \otimes s_1(b^2[c]) \\ - & s_1([c]) & \otimes c & - & s_0(1) & \otimes [c] & - & 1 & \otimes s_1([c]), \end{array}$$

or

$$\begin{aligned} U_1\Delta_1 d_2(R_4) = {} & R_4\otimes b^2 c + [b]\otimes b[b] - [c]\otimes c[b] - [c]\otimes cb[b] \\ & -c[b]\otimes cb[b] + [b]\otimes b^2[c] + b[b]\otimes b^2[c] + 1\otimes R_4 \\ = {} & R_4\otimes b^2 c + 1\otimes R_4 + (1-c)\left([b]\otimes b[b]\right) \\ & + \left((1+b)\,[b]\right)\otimes b^2[c] - [c]\otimes\left(c\,(1+b)\,[b]\right). \end{aligned}$$

Finally, for the last generator in C_2, we have $d_2(R_5) = (1+a)[a] + a^2(1+b)[b] - (1+c)[c]$, so

$$\Delta_1 d_2(R_5) = (1+a)([a]\otimes a + 1\otimes [a]) + a^2(1+b)([b]\otimes b + 1\otimes [b]) - (1+c)([c]\otimes c + 1\otimes [c]),$$

which expands to

$$\Delta_1 d_2(R_5) = \begin{array}{rcrcr} & [a]\otimes a & + & 1\otimes [a] \\ + & a[a]\otimes a^2 & + & a\otimes a[a] \\ + & a^2[b]\otimes a^2 b & + & a^2\otimes a^2[b] \\ + & a^2 b[b]\otimes a^2 b^2 & + & a^2 b\otimes a^2 b[b] \\ - & [c]\otimes c & - & 1\otimes [c] \\ - & c[c]\otimes a^2 b^2 & - & c\otimes c[c]. \end{array}$$

Then

$$U_1\Delta_1 d_2(R_5) = \begin{array}{rrlrrlrll} & s_1([a]) & \otimes a & + & s_0(1) & \otimes[a] & + & 1 & \otimes s_1([a]) \\ + & s_1(a[a]) & \otimes a^2 & + & s_0(a) & \otimes a[a] & + & 1 & \otimes s_1(a[a]) \\ + & s_1(a^2[b]) & \otimes a^2 b & + & s_0(a^2) & \otimes a^2[b] & + & 1 & \otimes s_1(a^2[b]) \\ + & s_1(a^2b[b]) & \otimes a^2b^2 & + & s_0(a^2b) & \otimes a^2b[b] & + & 1 & \otimes s_1(a^2b[b]) \\ - & s_1([c]) & \otimes c & - & s_0(1) & \otimes[c] & - & 1 & \otimes s_1([c]) \\ - & s_1(c[c]) & \otimes a^2b^2 & - & s_0(c) & \otimes c[c] & - & 1 & \otimes s_1(c[c]), \end{array}$$

or

$$\begin{aligned} U_1\Delta_1 d_2(R_5) = \quad & R_5 \otimes a^2b^2 + [a]\otimes a[a] + [a]\otimes a^2[b] + a[a]\otimes a^2[b] + [a]\otimes a^2b[b] \\ + \quad & a[a]\otimes a^2b[b] + a^2[b]\otimes a^2b[b] - [c]\otimes c[c] + 1\otimes R_5. \end{aligned}$$

Then, by Handel's proposition (4.4), we can define $\Delta_2 : \bigoplus_{i=1}^{5}\mathbb{Z}_{B_2(T)}R_i \longrightarrow \left(C\bigotimes C\right)_2$ by

$$\Delta_2(R_i) = \begin{cases} R_1\otimes a^2b + [a]\otimes a[a] - [b]\otimes b[a] - [b]\otimes ba[a] & \text{if} \quad i=1; \\ -b[a]\otimes ba[a] + [a]\otimes a^2[b] + a[a]\otimes a^2[b] + 1\otimes R_1, & \\ R_2\otimes a^2c + [a]\otimes a[a] - [c]\otimes c[a] - [c]\otimes ca[a] & \text{if} \quad i=2; \\ -c[a]\otimes ca[a] + [a]\otimes a^2[c] + a[a]\otimes a^2[c] + 1\otimes R_2, & \\ R_3\otimes b^2a + [b]\otimes b[b] - [a]\otimes a[b] - [a]\otimes ab[b] & \text{if} \quad i=3; \\ -a[b]\otimes ab[b] + [b]\otimes b^2[a] + b[b]\otimes b^2[a] + 1\otimes R_3, & \\ R_4\otimes b^2c + [b]\otimes b[b] - [c]\otimes c[b] - [c]\otimes cb[b] & \text{if} \quad i=4; \\ -c[b]\otimes cb[b] + [b]\otimes b^2[c] + b[b]\otimes b^2[c] + 1\otimes R_4, & \\ R_5\otimes a^2b^2 + [a]\otimes a[a] + [a]\otimes a^2[b] + a[a]\otimes a^2[b] & \text{if} \quad i=5. \\ +[a]\otimes a^2b[b] + a[a]\otimes a^2b[b] + a^2[b]\otimes a^2b[b] - [c]\otimes c[c] + 1\otimes R_5, & \end{cases} \tag{4.50}$$

The map $\Delta_3 : \bigoplus_{i=1}^{3}\mathbb{Z}_{B_2(T)}P_i \longrightarrow \left(C\bigotimes C\right)_3$ is defined analogously. In detail, the expression $\Delta_2 d_3(P_1) = (1+b)\Delta_2 R_1 + (1+a)\Delta_2 R_3$ expands to

$$\begin{aligned} \Delta_2 d_3(P_1) = \quad & (1+b)\Big[R_1\otimes a^2b + [a]\otimes a[a] - [b]\otimes b[a] - [b]\otimes ba[a] \\ & - b[a]\otimes ba[a] + [a]\otimes a^2[b] + a[a]\otimes a^2[b] + 1\otimes R_1\Big] \\ & +(1+a)\Big[R_3\otimes b^2a + [b]\otimes b[b] - [a]\otimes a[b] - [a]\otimes ab[b] \\ & - a[b]\otimes ab[b] + [b]\otimes b^2[a] + b[b]\otimes b^2[a] + 1\otimes R_3\Big] \end{aligned}$$

$$
\begin{aligned}
= \quad & R_1 \otimes a^2 b + [a] \otimes a[a] - [b] \otimes b[a] - [b] \otimes ba[a] \\
& -b[a] \otimes ba[a] + [a] \otimes a^2[b] + a[a] \otimes a^2[b] + 1 \otimes R_1 \\
& +bR_1 \otimes a^2 b^2 + b[a] \otimes ba[a] - b[b] \otimes b^2[a] - b[b] \otimes b^2 a[a] \\
& -b^2[a] \otimes b^2 a[a] + b[a] \otimes a^2 b[b] + ba[a] \otimes a^2 b[b] + b \otimes bR_1 \\
& +R_3 \otimes b^2 a + [b] \otimes b[b] - [a] \otimes a[b] - [a] \otimes ab[b] \\
& -a[b] \otimes ab[b] + [b] \otimes b^2[a] + b[b] \otimes b^2[a] + 1 \otimes R_3 \\
& +aR_3 \otimes a^2 b^2 + a[b] \otimes ab[b] - a[a] \otimes a^2[b] - a[a] \otimes a^2 b[b] \\
& -a^2[b] \otimes a^2 b[b] + a[b] \otimes b^2 a[a] + ab[b] \otimes b^2 a[a] + a \otimes aR_3.
\end{aligned}
$$

Then, according to proposition (4.4), $\Delta_3(P_1) = U_2 \Delta_2 d_3(P_1)$, which expands to

$$
\Delta_3(P_1) = \quad P_1 \otimes a^2 b^2 - R_1 \otimes a^2 b[b] - R_3 \otimes b^2 a[a] + [b] \otimes bR_1 + [a] \otimes aR_3 + 1 \otimes P_1.
$$

For the second $\mathbb{Z}_{B_2(T)}$ generator, P_2 in C_2, we have

$$
\begin{aligned}
\Delta_2 d_3(P_2) = \quad & (1+b)\Delta_2 R_1 + (1+c)\Delta_2 R_4 + (b^2 - 1)\Delta_2 R_5 \\
= \quad & (1+b)\Big[R_1 \otimes a^2 b + [a] \otimes a[a] - [b] \otimes b[a] - [b] \otimes ba[a] \\
& - b[a] \otimes ba[a] + [a] \otimes a^2[b] + a[a] \otimes a^2[b] + 1 \otimes R_1\Big] \\
& +(1+c)\Big[R_4 \otimes b^2 c + [b] \otimes b[b] - [c] \otimes c[b] - [c] \otimes cb[b] \\
& - c[b] \otimes cb[b] + [b] \otimes b^2[c] + b[b] \otimes b^2[c] + 1 \otimes R_4\Big] \\
& +(b^2 - 1)\Big[R_5 \otimes a^2 b^2 + [a] \otimes a[a] + [a] \otimes a^2[b] + a[a] \otimes a^2[b] \\
& + [a] \otimes a^2 b[b] + a[a] \otimes a^2 b[b] + a^2[b] \otimes a^2 b[b] - [c] \otimes c[c] + 1 \otimes R_5\Big] \\
= \quad & R_1 \otimes a^2 b + [a] \otimes a[a] - [b] \otimes b[a] - [b] \otimes ba[a] \\
& -b[a] \otimes ba[a] + [a] \otimes a^2[b] + a[a] \otimes a^2[b] + 1 \otimes R_1 \\
& +bR_1 \otimes a^2 b^2 + b[a] \otimes ba[a] - b[b] \otimes b^2[a] - b[b] \otimes b^2 a[a] \\
& -b^2[a] \otimes b^2 a[a] + b[a] \otimes a^2 b[b] + ba[a] \otimes a^2 b[b] + b \otimes bR_1 \\
& +R_4 \otimes b^2 c + [b] \otimes b[b] - [c] \otimes c[b] - [c] \otimes cb[b] \\
& -c[b] \otimes cb[b] + [b] \otimes b^2[c] + b[b] \otimes b^2[c] + 1 \otimes R_4 \\
& +cR_4 \otimes a^2 b^4 + c[b] \otimes cb[b] - c[c] \otimes a^2 b^2[b] - c[c] \otimes a^2 b^2 b[b] \\
& -a^2 b^2[b] \otimes a^2 b^2 b[b] + c[b] \otimes b^2 c[c] + cb[b] \otimes b^2 c[c] + c \otimes cR_4 \\
& +b^2 R_5 \otimes a^2 b^4 + b^2[a] \otimes b^2 a[a] + b^2[a] \otimes b^2 a^2[b] + b^2 a[a] \otimes a^2 b^2[b] \\
& +b^2[a] \otimes a^2 b^2 b[b] + b^2 a[a] \otimes a^2 b^2 b[b] + a^2 b^2[b] \otimes a^2 b^2 b[b] - b^2[c] \otimes b^2 c[c] + b^2 \otimes b^2 R_5 \\
& -R_5 \otimes a^2 b^2 - [a] \otimes a[a] - [a] \otimes a^2[b] - a[a] \otimes a^2[b] \\
& -[a] \otimes a^2 b[b] - a[a] \otimes a^2 b[b] - a^2[b] \otimes a^2 b[b] + [c] \otimes c[c] - 1 \otimes R_5.
\end{aligned}
$$

Then the diagonal approximation in P_2 is defined as

$$\begin{aligned}\Delta_3(P_2) = \;& P_2 \otimes a^2b^4 - R_1 \otimes a^2b[b] + R_5 \otimes a^2b^2[b] + R_5 \otimes a^2b^2b[b] \\ & -R_4 \otimes b^2c[c] - R_1 \otimes a^2b^2[b] - bR_1 \otimes a^2b^2[b] - R_1 \otimes a^2b^2b[b] - bR_1 \otimes a^2b^2b[b] \\ & +[b] \otimes bR_1 + [c] \otimes cR_4 + [b] \otimes b^2R_5 + b[b] \otimes b^2R_5 + 1 \otimes P_2.\end{aligned}$$

For the last generator of C_3, the expression $\Delta_2 d_3(P_3) = \Delta_2 R_2 + a^2\Delta_2 R_4 + (c-1)\Delta_2 R_5$ expands to

$$\begin{aligned}\Delta_2 d_3(P_3) = \;& R_2 \otimes a^2c + [a] \otimes a[a] - [c] \otimes c[a] - [c] \otimes ca[a] \\ & -c[a] \otimes ca[a] + [a] \otimes a^2[c] + a[a] \otimes a^2[c] + 1 \otimes R_2 \\ & +(a^2)\Big[R_4 \otimes b^2c + [b] \otimes b[b] - [c] \otimes c[b] - [c] \otimes cb[b] \\ & - c[b] \otimes cb[b] + [b] \otimes b^2[c] + b[b] \otimes b^2[c] + 1 \otimes R_4\Big] \\ & +(c-1)\Big[R_5 \otimes a^2b^2 + [a] \otimes a[a] + [a] \otimes a^2[b] + a[a] \otimes a^2[b] \\ & + [a] \otimes a^2b[b] + a[a] \otimes a^2b[b] + a^2[b] \otimes a^2b[b] - [c] \otimes c[c] + 1 \otimes R_5\Big] \\ = \;& R_2 \otimes a^2c + [a] \otimes a[a] - [c] \otimes c[a] - [c] \otimes ca[a] \\ & -c[a] \otimes ca[a] + [a] \otimes a^2[c] + a[a] \otimes a^2[c] + 1 \otimes R_2 \\ & +a^2R_4 \otimes a^2b^2c + a^2[b] \otimes a^2b[b] - a^2[c] \otimes a^2c[b] - a^2[c] \otimes a^2cb[b] \\ & -a^2c[b] \otimes a^2cb[b] + a^2[b] \otimes a^2b^2[c] + a^2b[b] \otimes a^2b^2[c] + a^2 \otimes a^2R_4 \\ & +cR_5 \otimes a^2b^2c + c[a] \otimes ca[a] + c[a] \otimes a^2c[b] + ca[a] \otimes a^2c[b] \\ & +c[a] \otimes a^2cb[b] + ca[a] \otimes a^2cb[b] + a^2c[b] \otimes a^2cb[b] - c[c] \otimes a^2b^2[c] + c \otimes cR_5 \\ & -R_5 \otimes a^2b^2 - [a] \otimes a[a] - [a] \otimes a^2[b] - a[a] \otimes a^2[b] \\ & -[a] \otimes a^2b[b] - a[a] \otimes a^2b[b] - a^2[b] \otimes a^2b[b] + [c] \otimes c[c] - 1 \otimes R_5.\end{aligned}$$

Then the diagonal approximation in the last generator is defined as

$$\begin{aligned}\Delta_3(P_3) = \;& P_3 \otimes a^2b^2c - R_2 \otimes a^2c[b] - R_2 \otimes a^2cb[b] + R_5 \otimes a^2b^2[c] \\ & +[a] \otimes a^2R_4 + a[a] \otimes a^2R_4 + [c] \otimes cR_5 + 1 \otimes P_3.\end{aligned}$$

Summarizing,

$$\Delta_3(P_i) = \begin{cases} P_1 \otimes a^2b^2 - R_1 \otimes a^2b[b] - R_3 \otimes b^2a[a] + [b] \otimes bR_1 & \text{if} \quad i=1; \\ \quad +[a] \otimes aR_3 + 1 \otimes P_1, & \\ P_2 \otimes a^2b^4 - R_1 \otimes a^2b[b] + R_5 \otimes a^2b^2[b] + R_5 \otimes a^2b^2b[b] & \\ -R_4 \otimes b^2c[c] - R_1 \otimes a^2b^2[b] - bR_1 \otimes a^2b^2[b] - R_1 \otimes a^2b^2b[b] & \text{if} \quad i=2; \\ -bR_1 \otimes a^2b^2b[b] + [b] \otimes bR_1 + [c] \otimes cR_4 + [b] \otimes b^2R_5 & \\ \quad +b[b] \otimes b^2R_5 + 1 \otimes P_2, & \\ P_3 \otimes a^2b^2c - R_2 \otimes a^2c[b] - R_2 \otimes a^2cb[b] + R_5 \otimes a^2b^2[c] & \text{if} \quad i=3. \\ \quad +[a] \otimes a^2R_4 + a[a] \otimes a^2R_4 + [c] \otimes cR_5 + 1 \otimes P_3, & \end{cases} \tag{4.51}$$

Chapter 5

The Symmetric and Symmetrized Topological Complexity of the Torus

In this chapter we compute the *symmetric topological complexity* of the torus $\mathrm{TC}^S(T)$, i.e., the sectional category of the fibration $\varepsilon_{0,1} : \mathrm{NL}(T)/\mathbb{Z}_2 \longrightarrow B(T,2)$. We will use the alternative definition given by Schwarz in [8], which says that the sectional category of $\varepsilon_{0,1}$ is less than or equal to n whenever the fiberwise join $j^{n+1}(\varepsilon_{0,1}) : *^{n+1}\mathrm{NL}(T)/\mathbb{Z}_2 \longrightarrow B(T,2)$ has a global cross section.

Consider a pair of points x_0 and x_1 in the torus. As in subsection 1.3, let M denote the space $\mathrm{Maps}([0,1],0,1;T,x_0,x_1) \cup \mathrm{Maps}([0,1],0,1;T,x_1,x_0)\,/\mathbb{Z}_2$, so we have a fiber sequence

$$*^{n+1}M \hookrightarrow *^{n+1}\mathrm{NL}(T)/\mathbb{Z}_2 \xrightarrow{j^{n+1}(\varepsilon_{0,1})} B(T,2).$$

It is clear that the fiber M has the homotopy type of ΩT. By the connectivity of the fiber, the primary obstruction for having $\mathrm{TC}^\Sigma(T) \leq n$ is a twisted cohomology class

$$\theta^{n+1} \in H^{n+1}\left(B(T,2); \pi_n(*^{n+1}\Omega T)\right).$$

As in (3.4) and (3.5), the additive structure of the local system of coefficients is $\pi_n(*^{n+1}\Omega T) \simeq \bigotimes_{n+1} I_0$, where I_0 is the kernel of the the augmentation homomorphism $\epsilon : \mathbb{Z}_{\pi_1(T,x_0)} \longrightarrow \mathbb{Z}$.

5.1 Mimick for $B_2(X)$

In this section I_0 stands for the kernel of the augmentation homomorphism $\epsilon : \mathbb{Z}_{\pi_1(X)} \longrightarrow \mathbb{Z}$, i.e. $I_0 = ker(\epsilon)$. In later sections we will specialize back to the case $X = T$.

An element γ in $B_2(X)$ at the based point $\{x_0, x_1\}$ in $B(X,2)$, we can interpret γ as an element in the braid group of X on 2 strings, that is

$$\alpha,\ \beta : [0,1] \longrightarrow X \quad \text{where} \quad |\{\alpha(t), \beta(t)\}| = 2 \text{ for all } t \in [0,1],$$

and $\{\alpha(0), \beta(0)\} = \{\alpha(1), \beta(1)\} = \{x_0, x_1\}$. We pick an arbitrary ordering (x_0, x_1). Let α be the path that ends in x_0 and β the one that ends in x_1. We define an action of $B_2(X)$ on $\mathbb{Z}_{\pi_1(X)}$ and I_0 as an extension to the action defined in (3.6) via

$$[\gamma]\sum n_i g_i = \begin{cases} \sum n_i \alpha g_i \delta\overline{\beta}\,\overline{\delta}, & \text{if} \quad [\gamma] \in P_2(X); \\ \sum n_i \beta\overline{\delta}\,\overline{g_i}\,\overline{\alpha}\,\overline{\delta}, & \text{if} \quad [\gamma] \notin P_2(X), \end{cases} \tag{5.1}$$

and for all i, the element g_i is an element in $\pi(X, x_0)$ and δ is a fixed path from x_0 to x_1. The elements $\alpha g_i \delta\overline{\beta}\,\overline{\delta}$ and $\beta\overline{\delta}\,\overline{g_i}\,\overline{\alpha}\,\overline{\delta}$ are understood as the corresponding homotopy classes, given by the concatenation of their respective paths taking a representative of the class g_i.

This action is analogous to the one described in (3.6), which is how monodromy is described for the homotopy fiber in Theorem (1.4). The equivalence between the actual fiber and the homotopy fiber was defined in Chapter 3, via the map $\psi_\delta : \widetilde{M} \longrightarrow \Omega X$ defined as $\psi_\delta(\omega) = \omega\overline{\delta}$. To visualize the action, the reader should go back to images (1.2) and (3.2).

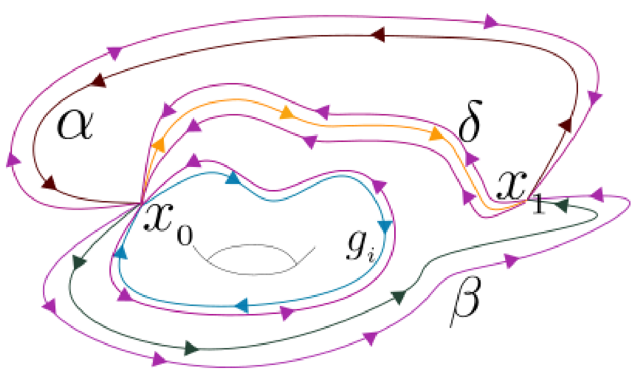


Figure 5.1: The Action of $B_2(X) \setminus P_2(X)$ on I_0

As discussed in the preliminary parts of this thesis, the primary obstruction for the inequality $\mathrm{TC}^S(T) \leq 3$ is the third power of the cohomology class spelled out next:

Proposition 5.1

Consider the map $f : B_2(X) \longrightarrow I_0$ given by

$$f([\gamma]) = \begin{cases} [\alpha\delta\overline{\beta}\,\overline{\delta}] - 1, & \text{if} \quad [\gamma] \in P_2(X); \\ [\beta\overline{\delta}\,\overline{\alpha}\,\overline{\delta}] - 1, & \text{if} \quad [\gamma] \notin P_2(X), \end{cases} \tag{5.2}$$

where $\gamma = (\alpha, \beta)$. Then f determines a one dimensional cohomology class $[f] \in H^1(B(X,2); I_0)$.

Proof. Consider a product $\varrho = [\gamma_1\gamma_2]$ in $B_2(X)$ with $\gamma_1 = [\alpha_1, \beta_1]$ and $\gamma_2 = [\alpha_2, \beta_2]$, where we are using the notation described above. If γ_1 and γ_2 are pure braids, the proof can be seen in Proposition 3.2. Suppose $\gamma_1 \notin P_2(X)$ and $\gamma_2 \in P_2(X)$ so that $\varrho \notin P_2(X)$. Then

$$\begin{aligned} f(\varrho) &= f([\gamma_1][\gamma_2]) = f([\alpha_1,\beta_1][\alpha_2,\beta_2]) \\ &= f([\alpha_1\alpha_2, \beta_1\beta_2]) = [\beta_1\beta_2\overline{\delta}\,\overline{\alpha_1\alpha_2}\,\overline{\delta}] - 1 = [\beta_1\beta_2\overline{\delta}\,\overline{\alpha}_2\,\overline{\alpha}_1\,\overline{\delta}] - 1, \end{aligned}$$

whereas

$$\begin{aligned} f([\gamma_1]) + [\gamma_1]f([\gamma_2]) &= [\beta_1\overline{\delta}\,\overline{\alpha_1}\,\overline{\delta}] - 1 + [\alpha_1,\beta_1]\left([\alpha_2\delta\,\overline{\beta_2}\,\overline{\delta}] - 1\right) \\ &= [\beta_1\overline{\delta}\delta\beta_2\overline{\delta}\,\overline{\alpha}_2\,\overline{\alpha}_1\,\overline{\delta}] - 1 = [\beta_1\beta_2\overline{\delta}\,\overline{\alpha}_2\,\overline{\alpha}_1\,\overline{\delta}] - 1. \end{aligned}$$

Consider now the case where $\gamma_1 \notin P_2(X)$ and $\gamma_2 \in P_2(X)$ so that $\varrho \notin P_2(X)$. Then

$$\begin{aligned} f(\varrho) &= f([\gamma_1][\gamma_2]) = f([\alpha_1,\beta_1][\alpha_2,\beta_2]) \\ &= f([\beta_1\alpha_2, \alpha_1\beta_2]) = [\alpha_1\beta_2\overline{\delta}\,\overline{\beta_1\alpha_2}\,\overline{\delta}] - 1 = [\alpha_1\beta_2\overline{\delta}\,\overline{\alpha}_2\overline{\beta}_1\,\overline{\delta}] - 1, \end{aligned}$$

whereas

$$\begin{aligned} f([\gamma_1]) + [\gamma_1]f([\gamma_2]) =& [\alpha_1\delta\overline{\beta}_1\,\overline{\delta}] - 1 + [\alpha_1,\beta_1]\left([\beta_2\overline{\delta}\,\overline{\alpha}_2\,\overline{\delta}] - 1\right) \\ =& [\alpha_1\beta_2\overline{\delta}\,\overline{\alpha}_2\overline{\delta}\,\delta\overline{\beta}_1\,\overline{\delta}] - 1 = [\alpha_1\beta_2\overline{\delta}\,\overline{\alpha}_2\overline{\beta}_1\,\overline{\delta}] - 1. \end{aligned}$$

Finally, in the last case, we have that γ_1 and γ_2 are not elements in $P_2(X)$ but $\varrho \in P_2(X)$. Then

$$f(\varrho) = f([\gamma_1][\gamma_2]) = f([\alpha_1,\beta_1][\alpha_2,\beta_2]) = f([\beta_1\alpha_2,\alpha_1\beta_2]) = [\beta_1\alpha_2\delta\overline{\alpha_1\beta_2}\,\overline{\delta}] - 1 = [\beta_1\alpha_2\delta\overline{\beta}_2\overline{\alpha}_1\,\overline{\delta}] - 1,$$

whereas

$$\begin{aligned} f([\gamma_1]) + [\gamma_1]f([\gamma_2]) =& [\beta_1\overline{\delta}\,\overline{\alpha}_1\,\overline{\delta}] - 1 + [\alpha_1,\beta_1]\left([\beta_2\overline{\delta}\,\overline{\alpha}_2\,\overline{\delta}] - 1\right) \\ =& [\beta_1\overline{\delta}\,\overline{\beta_2\overline{\delta}\,\overline{\alpha}_2\overline{\delta}}\,\overline{\alpha}_1\,\overline{\delta}] - 1 = [\beta_1\alpha_2\delta\overline{\beta}_2\overline{\alpha}_1\,\overline{\delta}] - 1. \end{aligned}$$

Then, for all $\gamma_1, \gamma_2 \in B_2(X)$, we have $f([\gamma_1][\gamma_2]) = f([\gamma_1]) + [\gamma_1]f([\gamma_2])$. In accordance with Whitehead's book [9], f determines a one dimensional cohomology class $[f] = \mathfrak{b} \in H^1(B(X,2); I_0)$. ∎

5.2 The Third Power

For the particular case of the torus we have that the action defined in (5.1) and the crossed morphism defined in (5.1) can be further simplified.

Recall that $B_2(T)$ has a presentation of the form

$$B_2(T) = \langle a,\ b,\ c \mid [a^2, b], [a^2, c], [b^2, a], [b^2, c],\ a^2 b^2 = c^2 \rangle \simeq \left(\mathbb{Z}a^2 \times \mathbb{Z}b^2\right) \rtimes_{palin} (\mathbb{Z}_2 a * \mathbb{Z}_2 b * \mathbb{Z}_2 c).$$

On the other hand $\pi_1(T, x_0) = \mathbb{Z}\rho \times \mathbb{Z}\tau$, where τ and ρ are the homotopy class given by the loops $\widetilde{f}, \widetilde{g} : [0,1] \longrightarrow T = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ defined as

$$\widetilde{f}(t) = (x_0 + (0,t)) \quad \text{and} \quad \widetilde{g}(t) = (x_0 + (t,0))$$

respectively. These loops are clearly related to the braids, $\tau_0,\ \tau_1,\ \rho_0,\ \rho_1,\ \sigma$ in chapter 2. Here we give explicit formulas and representatives for these homotopy classes —see (2.2):

$$a = \rho_0 \sigma, \quad b = \sigma^{-1} \tau_0, \quad c = \rho_0 \tau_0 \sigma^{-1}.$$

Let δ be the homotopy class of the projection to the first coordinate of some representative of the class of σ. For instance, in (2.3) we gave γ_{01} as a representative of the σ class. Then the action on $\mathbb{Z}_{\pi_1(X)}$ and on I_0 of the generators of $B_2(T)$ is given by

$$a g_i = \rho \overline{g}_i, \quad b g_i = \overline{\tau}\, \overline{g}_i \quad c g_i = \rho \tau \overline{g}_i.$$

By Corollary 3.6, $a^2 g_i = g_i$ and $b^2 g_i = g_i$. The latter fact is also deduced from the previous formulas. For instance, $a^2 g_i = a(a g_i) = a(\rho \overline{g}_i) = \rho(\overline{\rho \overline{g}_i}) = \rho g_i \overline{\rho} = g_i$, by the commutativity of $\pi_1(T)$. Then the action on the generators of $B_2(T)$ of the crossed morphism f in (5.1) is

$$f[a] = \rho - 1, \quad f[b] = \overline{\tau} - 1, \quad f[c] = \rho\tau - 1. \tag{5.3}$$

Theorem 5.2
The symmetric and the symmetrized topological complexity of the torus are both equal to three, $\mathrm{TC}^S(T) = \mathrm{TC}^\Sigma(T) = 3$.

Proof. It is known from [3, Example 4.7] that $3 \leq \mathrm{TC}^\Sigma(T)$. In view of (1.5), it then suffices to prove that $\mathrm{TC}^S(T) \leqslant 3$, i.e., that the fibration $j^3(\varepsilon_{0,1}) : *^3 \mathrm{NL}(T)/\mathbb{Z}_2 \longrightarrow B(T,2)$ admits a cross section. Now, according to Schwarz [8], the primary obstruction to the existence of such a section is given by $\mathfrak{b}^3$. Further, by dimensionality reasons, the primary obstruction is in fact the only one. So, the proof will be complete once we show the triviality of $\mathfrak{b}^3$.

In order to do the calculation we will use the free resolution (4.3), the diagonal approximation given in §4.2, and the representative f of the class $\mathfrak{b}$. To understand the behavior of $[f]^3$, it is enough to understand its behavior in each of the generators. This is done via the relation $[f]^3(P_i) = [f]^2 \times [f] \circ \Delta(P_i)$, where $[f]^2 \times [f]$ is the *cross-product*, of the classes $[f]^2$ and $[f]$.

For the generator P_1, we have

$$\begin{aligned} [f]^3(P_1) = \ & [f]^2 \times [f] \circ \Delta(P_1) \\ = \ & (-1)^{2\cdot 1}\Big[[f]^2 P_1 \otimes a^2 b^2 [f]1 - [f]^2 R_1 \otimes a^2 b[f][b] - [f]^2 R_3 \otimes b^2 a[f][a] + [f]^2[b] \otimes b[f]R_1 \\ & + [f]^2[a] \otimes a[f]R_3 + [f]^2 1 \otimes [f]P_1\Big]. \end{aligned}$$

The latter expression can be simplified as follows, since the evaluation is only different from zero in the corresponding dimensions:

$$[f]^3(P_1) = -[f]^2 R_1 \otimes a^2 b[f][b] - [f]^2 R_3 \otimes b^2 a[f][a] = -[f]^2 R_1 \otimes b[f][b] - [f]^2 R_3 \otimes a[f][a].$$

Similarly, using that $[f]^2 = [f] \times [f] \circ \Delta$, we have

$$\begin{aligned} [f]^3(P_1) = (-1)^{1\cdot 1}\Big[& -[f]R_1 \otimes a^2 b[f] \otimes b[f][b] - [f][a] \otimes a[f][a] \otimes b[f][b] + [f][b] \otimes b[f][a] \otimes b[f][b] \\ & + [f][b] \otimes ba[f][a] \otimes b[f][b] + b[f][a] \otimes ba[f][a] \otimes b[f][b] - [f][a] \otimes a^2[f][b] \otimes b[f][b] \\ & - a[f][a] \otimes a^2[f][b] \otimes b[f][b] - [f]1 \otimes [f]R_1 \otimes b[f][b] \\ & - [f]R_3 \otimes b^2 a[f]1 \otimes a[f][a] - [f][b] \otimes b[f][b] \otimes a[f][a] + [f][a] \otimes a[f][b] \otimes a[f][a] \\ & + [f][a] \otimes ab[f][b] \otimes a[f][a] + a[f][b] \otimes ab[f][b] \otimes a[f][a] - [f][b] \otimes b^2[f][a] \otimes a[f][a] \\ & - b[f][b] \otimes b^2[f][a] \otimes a[f][a] - [f]1 \otimes [f]R_3 \otimes a[f][a]\Big]. \end{aligned}$$

Simplifying and using that the action of the squares is trivial, we have

$$\begin{aligned} [f]^3(P_1) = \ & [[f][a] \otimes a[f][a] \otimes b[f][b] - [f][b] \otimes b[f][a] \otimes b[f][b] - [f][b] \otimes ba[f][a] \otimes b[f][b] \\ & -b[f][a] \otimes ba[f][a] \otimes b[f][b] + [f][a] \otimes a^2[f][b] \otimes b[f][b] + a[f][a] \otimes a^2[f][b] \otimes b[f][b] \\ & +[f][b] \otimes b[f][b] \otimes a[f][a] - [f][a] \otimes a[f][b] \otimes a[f][a] - [f][a] \otimes ab[f][b] \otimes a[f][a] \\ & -a[f][b] \otimes ab[f][b] \otimes a[f][a] + [f][b] \otimes b^2[f][a] \otimes a[f][a] + b[f][b] \otimes b^2[f][a] \otimes a[f][a]\Big] \\ = \ & [[f][a] \otimes [f][a] \otimes [f][b] - [f][b] \otimes b[f][a] \otimes b[f][b] + [f][b] \otimes b[f][a] \otimes b[f][b] \\ & +b[f][a] \otimes b[f][a] \otimes b[f][b] + [f][a] \otimes [f][b] \otimes b[f][b] - [f][a] \otimes [f][b] \otimes b[f][b] \\ & +[f][b] \otimes [f][b] \otimes [f][a] - [f][a] \otimes a[f][b] \otimes a[f][a] + [f][a] \otimes a[f][b] \otimes a[f][a] \\ & +a[f][b] \otimes a[f][b] \otimes a[f][a] + [f][b] \otimes [f][a] \otimes a[f][a] - [f][b] \otimes [f][a] \otimes a[f][a]] \\ = \ & [f][a] \otimes [f][a] \otimes [f][b] + b\,[[f][a] \otimes [f][a] \otimes [f][b]] \\ & +[f][b] \otimes [f][b] \otimes [f][a] + a\,[[f][b] \otimes [f][b] \otimes [f][a]] \\ = \ & (b+1)\,([f][a] \otimes [f][a] \otimes [f][b]) + (a+1)\,([f][b] \otimes [f][b] \otimes [f][a]). \end{aligned}$$

Evaluating the class of $[f]$ in the respective generators (5.3), we have

$$\begin{aligned}[f]^3(P_1) &= (b+1)\left([f][a]\otimes[f][a]\otimes[f][b]\right) + (a+1)\left([f][b]\otimes[f][b]\otimes[f][a]\right)\\ &= (b+1)\left((\rho-1)\otimes(\rho-1)\otimes(\overline{\tau}-1)\right) + (a+1)\left((\overline{\tau}-1)\otimes(\overline{\tau}-1)\otimes(\rho-1)\right).\end{aligned}$$

Now for the generator P_2 we have

$$\begin{aligned}[f]^3(P_2) =&[f]^2\times[f]\circ\Delta(P_3)\\ = (-1)^2 &\Big[[f]^2P_2\otimes a^2b^4[f] - [f]^2R_1\otimes a^2b[f][b] + [f]^2R_5\otimes a^2b^2[f][b] + [f]^2R_5\otimes a^2b^2b[f][b]\\ &-[f]^2R_4\otimes b^2c[f][c] - [f]^2R_1\otimes a^2b^2[f][b] - b[f]^2R_1\otimes a^2b^2[f][b] - [f]^2R_1\otimes a^2b^2b[f][b]\\ &-b[f]^2R_1\otimes a^2b^2b[f][b] + [f]^2[b]\otimes b[f]R_1 + [f]^2[c]\otimes c[f]R_4 + [f]^2[b]\otimes b^2[f][f]R_5\\ &+ [f]^2b[b]\otimes b^2[f]R_5 + [f]^2 1\otimes[f]P_2\Big].\end{aligned}$$

Simplifying and using de definition $[f]\smile[f] = [f]^2$ we have,

$$\begin{aligned}= (b+1)&\left([f][a]\otimes[f][a]\otimes[f][b]\right)\\ &-[f][b]\otimes b[f][a]\otimes[f][b] + [f][b]\otimes b[f][a]\otimes[f][b]\\ &+[f][a]\otimes[f][b]\otimes[f][b] - [f][a]\otimes[f][b]\otimes[f][b]\\ + (c+1)&\left([f][b]\otimes[f][b]\otimes[f][c]\right)\\ &-[f][c]\otimes c[f][b]\otimes[f][c] + [f][c]\otimes c[f][b]\otimes[f][c]\\ &+[f][b]\otimes[f][c]\otimes[f][c] - [f][b]\otimes[f][c]\otimes[f][c].\end{aligned}$$

Evaluation of $[f]$ at $[a]$, $[b]$, $[c]$ according to (5.3), yields

$$\begin{aligned}[f]^3(P_2) &= (b+1)\left([f][a]\otimes[f][a]\otimes[f][b]\right) + (c+1)\left([f][b]\otimes[f][b]\otimes[f][c]\right)\\ &= (b+1)\left((\rho-1)\otimes(\rho-1)\otimes(\overline{\tau}-1)\right) + (c+1)\left((\overline{\tau}-1)\otimes(\overline{\tau}-1)\otimes(\rho\,\tau-1)\right).\end{aligned} \tag{5.4}$$

Finally for the last generator, we have

$$\begin{aligned}[f]^3(P_3) &= [f]^2\times[f]\circ\Delta(P_3)\\ &=[f][a]\otimes[f][a]\otimes[f][c] - [f][c]\otimes[f][c]\otimes[f][c] + [f][b]\otimes[f][b]\otimes[f][c]\\ &=(\rho-1)\otimes(\rho-1)\otimes(\rho\tau-1) - (\rho\,\tau-1)\otimes(\rho\,\tau-1)\otimes(\rho\,\tau-1) + (\overline{\tau}-1)\otimes(\overline{\tau}-1)\otimes(\rho\,\tau-1).\end{aligned}$$

Hence $[f]^3 : \bigoplus_{i=1}^{5}\mathbb{Z}_{B_2(T)}P_i \longrightarrow \bigotimes_{i=1}^{3} I_0$ is defined in generators as

$$[f]^3(P_i) = \begin{cases} (b+1)\left((\rho-1)\otimes(\rho-1)\otimes(\overline{\tau}-1)\right) & \text{if } i=1; \\ +(a+1)\left((\overline{\tau}-1)\otimes(\overline{\tau}-1)\otimes(\rho-1)\right), & \\ (b+1)\left((\rho-1)\otimes(\rho-1)\otimes(\overline{\tau}-1)\right) & \text{if } i=2; \\ +(c+1)\left((\overline{\tau}-1)\otimes(\overline{\tau}-1)\otimes(\rho\,\tau-1)\right), & \\ (\rho-1)\otimes(\rho-1)\otimes(\rho\,\tau-1) & \text{if } i=3. \\ -(\rho\,\tau-1)\otimes(\rho\,\tau-1)\otimes(\rho\,\tau-1) & \\ +(\overline{\tau}-1)\otimes(\overline{\tau}-1)\otimes(\rho\,\tau-1), & \end{cases}$$

We define $\Psi : \bigoplus_{i=1}^{5} \mathbb{Z}_{B_2(T)} R_i \longrightarrow \bigotimes_{i=1}^{3} I_0$ as the $\mathbb{Z}_{B_2(T)}$ morphism given by

$$\begin{aligned} \Psi(R_1) =& (\rho-1)\otimes(\rho-1)\otimes(\overline{\tau}-1), \\ \Psi(R_2) =& (\rho-1)\otimes(\rho-1)\otimes(\rho\,\tau-1)-(\rho\,\tau-1)\otimes(\rho\,\tau-1)\otimes(\rho\,\tau-1), \\ \Psi(R_3) =& (\overline{\tau}-1)\otimes(\overline{\tau}-1)\otimes(\rho-1), \\ \Psi(R_4) =& (\overline{\tau}-1)\otimes(\overline{\tau}-1)\otimes(\rho\,\tau-1), \\ \Psi(R_5) =& 0 = 0\otimes 0\otimes 0. \end{aligned} \tag{5.5}$$

The proof is complete by noticing that $d_3^*\Psi = \Psi d_3 = [f]^3$, so the class $\mathfrak{b}^3$ vanishes. ∎

As noted in the introduction, we now have for a closed surface X:

$$\mathrm{TC}^S(X) = \mathrm{TC}^\Sigma(X) = \begin{cases} 4, & \text{if } X \text{ has genus at least 2;} \\ 2, & \text{if } X = S^2; \\ 3, & \text{if } X = T; \\ 4, & \text{if } X = \mathbb{R}P^2. \end{cases} \tag{5.6}$$

Chapter 6

Conclusions and Future Work

From the practical point of view, the symmetric forms of Topological Complexity TC^S and TC^Σ are important given their naturalness within the planning problem in robotics. One possibility to continue with this work is to build a motor plan associated with

$$\Omega(T) \longrightarrow \mathrm{NL}(T)/\mathbb{Z}_2 \longrightarrow B(T,2).$$

From a theoretical point of view, although symmetric topological complexity is a potentially different invariant from symmetrized topological complexity, no example is known so far in which the two invariants are different. Until before this thesis it was known that both invariants coincide in the case of all closed surfaces, with the possible exception of the torus. But now with the Theorem (5.2) the coincidence has been established also in that case.

A direct consequence is that the Torus is an example where the general inequality

$$\mathrm{TC}^\Sigma(X \times Y) \leqslant \mathrm{TC}^\Sigma(X) + \mathrm{TC}^\Sigma(Y)$$

proven in [3] is strict, since $\mathrm{TC}^\Sigma(S^1) = 2$.

In addition, the $B_2(T)$-free resolution of $\mathbb{Z}$ and the diagonal approximation shown in chapter 4, gives us explicit formulas to calculate the comology ring $H^*(B_2(T), G)$, when G is a $\mathbb{Z}_{B_2(T)}$-module.

Bibliography

[1] M. Farber and M. Grant, *Symmetric motion planning*, in *Topology and robotics*, vol. 438 of *Contemp. Math.*, pp. 85–104. Amer. Math. Soc., Providence, RI, 2007.

[2] I. Basabe, J. González, Y. B. Rudyak, and D. Tamaki, *Higher topological complexity and its symmetrization*, Algebr. Geom. Topol. **14** (2014), no. 4 2103–2124.

[3] J. González, *Symmetric bi-skew maps and symmetrized motion planning in projective spaces*, Proc. Edinb. Math. Soc. (2) **61** (2018), no. 4 1087–1100.

[4] F. Cohen and J. Pakianathan, *Configuration spaces and braid groups*, . Available at *http://web.math.rochester.edu/people/faculty/jonpak/newbraid.pdf* .

[5] G. P. Scott, *Braid groups and the group of homeomorphisms of a surface*, Proc. Cambridge Philos. Soc. **68** (1970) 605–617.

[6] D. Lima Gonçalves, J. Guaschi, and M. Maldonado, *Embeddings and the (virtual) cohomological dimension of the braid and mapping class groups of surfaces*, Confluentes Math. **10** (2018), no. 1 41–61. Available at https://arxiv.org/pdf/1610.03288.pdf and https://doi.org/10.5802/cml.45.

[7] K. Murasugi and B. I. Kurpita, *A study of braids*, vol. 484 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1999.

[8] A. S. Schwarz, *The genus of a fiber space*, Amer. Math. Soc. Transl. (2) **55** (1966) 49–140.

[9] G. W. Whitehead, *Elements of homotopy theory*, vol. 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978.

[10] A. Costa and M. Farber, *Motion planning in spaces with small fundamental groups*, Commun. Contemp. Math. **12** (2010), no. 1 107–119.

[11] J. S. Birman, *On braid groups*, Comm. Pure Appl. Math. **22** (1969) 41–72. Available at https://www.math.columbia.edu/~jb/papers.html *and* http://www.math.columbia.edu/%7Ejb/On_bd_gps.djvu.

[12] E. Fadell and S. Husseini, *The Nielsen number on surfaces*, in *Topological methods in nonlinear functional analysis (Toronto, Ont., 1982)*, vol. 21 of *Contemp. Math.*, pp. 59–98. Amer. Math. Soc., Providence, RI, 1983. Available at https://doi.org/10.1090/conm/021/729505.

[13] J. Milnor, J. Stasheff, and P. University, *Characteristic Classes*. Annals of mathematics studies. Princeton University Press, 1974.

[14] K. S. Brown, *Cohomology of groups*, vol. 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2012. Corrected reprint of the 1982 original.

[15] A. Haefliger, *Points multiples d'une application et produit cyclique rédut*, America Journal Math **83** (1961) 57–70. Available at *DOI:*10.1007/s40879-014-0031-3 *and* https://doi.org/10.1007/s40879-014-0031-3.

[16] D. Handel, *On products in the cohomology of the dihedral groups*, Tohoku Math. J. (2) **45.1** (1993) 13–42. Available at *DOI:*10.2748/tmj/1178225952 *and* https: //doi.org/10.2748/tmj/1178225952.

[17] D. C. Cohen and L. Vandembroucq, *Topological complexity of the klein bottle*, Journal of Applied and Computational Topology (May, 2017). Available at https://arxiv.org/abs/1612.03133 or https://doi.org/10.1007/s41468-017-0002-0.

[18] N. Cadavid-Aguilar and J. González, *Effective topological complexity of orientable-surface groups*, Topology Appl. **290** (2021) Paper No. 107575, 25.

[19] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[20] D. Lima G. and S. Tadao M., *Diagonal approximation and the cohomology ring of the fundamental groups of surfaces*, European Journal of Mathematics **1.1** (Mar. 2015) 122–137. Available at https://doi.org/10.1007/s40879-014-0031-3.