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Resoluciones libres minimales de ideales monomiales

Tesis que presenta

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To my parents and my sister, who have always been my support and I hope I can always be there for them.

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Resumen

Las resoluciones libres minimales han sido objeto de estudio debido a que albergan gran cantidad de invariantes algebraicos del modulo para el que son calculadas. Hilbert demostró que para un módulo finitamente generado, su resolución libre minimal siempre es finita, sin embargo, el proceso para encontrar una resolución libre minimal es de forma recursiva y compleja.

Desde los años 60's Irving Kaplansky propuso el problema de encontrar algoritmos que permitan encontrar resoluciones libres minimales de módulos finitamente generados de forma no recursiva, es decir, lo más explícita posible. Desde entonces se han hecho muchos esfuerzos para encontrar dicho algoritmo sin que haya sido posible. Algunos de estos esfuerzos han resultado en algoritmos como la resolución de Taylor, las resoluciones celulares, o en otros problemas supuestamente más simple y equivalente, aunque la realidad es que son igual o más complejo que encontrar una resolución libre minimal por definición.

Hoy en día es posible encontrar una resolución libre minimal para cualquier ideal monomial de un anillo de polinomios sobre un campo usando *softwares* como Macaulay2 ([12]) de forma bastante rápida sin que esto signifique que el problema de Kaplansky esté resuelto, pues Macaulay2 usa bases de Gröbner, lo que sigue siendo un algoritmo recursivo.

En el problema de Kaplansky se busca un algoritmo no recursivo que permita encontrar resoluciones libres minimales de forma explícita, sin que sea claro qué significa "de forma explícita". Para nosotros esto significa que el algoritmo tenga una complejidad algorítmica polinomial. Para algunos casos muy especiales se ha logrado encontrar dichos algoritmos, por ejemplo para algunos ideales de aristas de grafos (vea [28, 32]).

Nuestro estudio empezó preguntándonos sobre las propiedades combinatorias que tiene una resolución libre minimal. El primer inconveniente que identificamos fue que una resolución libre minimal graduada puede tener más de un generador

en un módulo libre con el mismo grado y sin embargo que estos generadores se comporten de forma distinta sin que estos grados permitan hacer notorio este comportamiento distinto. Para entender mejor esto empezamos a estudiar los ideales de aristas del grafo completo con n vértices. A este grafo pudimos asociarle subgrafos que representan los generadores de cada módulo libre de forma que simbolizan cada grado de cada generador minimal del módulo libre y además logran distinguir el comportamiento diferenciado de dos generadores del mismo grado en los diferenciales de una resolución libre minimal de este ideal. Estos subgrafos los llamamos grafos base (vea Capítulo 3).

Sin embargo el problema no terminaba. Hasta aquí habiamos logrado encontrar un complejo libre para el ideal de aristas del grafo completo descrito mediante el uso de grafos base pero no habiamos logrado demostrar que este complejo era exacto y minimal aunque el complejo tuviera toda la apariencia de serlo. En la literatura se encuentrar algunos criterios que permiten demostrar que un complejo es exacto sin tener que irse a la definición pero estos criterios no son fáciles de usar en la mayoría de casos, incluyendo el nuestro (vea [3, 26]). A partir de esto decidimos encontrar un criterio que fuera más fácil de aplicar y que nos permitiera demostrar que un complejo libre, cuyos números de Betti son los correctos y cuyos diferenciales son irredundantes, sea una resolución libre minimal graduada de un módulo. Este resultado es el criterio dado en el capítulo 3.

Posterios a esto, abordamos una operación utilizada en grafos llamada la duplicación de un vértice, pero esta vez aplicada a una variable en un ideal monomial. En el Capítulo 4 definimos la duplicación de un ideal monomial y definimos una resolución libre minimal para este nuevo ideal. Esta resolución se obtiene de nuevo de forma explícita aplicando algunas operaciones a los módulos libres y diferenciales de la resolución libre minimal del ideal original. Para demostrar que esta resolución de la duplicación de un ideal monomial es exacta y minimal, usamos de nuevo el criterio que dimos en el Capítulo 3.

Como aplicación de la duplicación de un ideal monomial, en el Capítulo 5 damos una resolución libre minimal para el ideal de aristas de un grafo multipartito completo. Previamente ya se había dado una resolución para el ideal de aristas de un grafo bipartito completo pero no para un grafo multipartito completo (vea [32]). Esta resolución también se da en términos de grafos base de un grafo multipartito completo.

El concepto de grafo base se puede generalizar a conjuntos base y las resoluciones basadas en grafos base las definimos en el Capítulo 4. En este capítulo introducimos dos nuevas formas de representar una resolución libre minimal: la resolución poset y la resolución combinatoria. Cada una de estas representaciones tienen sus ventajas y desventajas que serán estudiadas en trabajos futuros. En esta tesis se dan algunas pocas de estas características de cada una. Por ejemplo, se sabe que todas las resoluciones libres minimales de un ideal monomial son únicas salvo isomorfismo, pero con estas nuevas dos representaciones es fácil ver que, a pesar de ser isomorfas como sucesión de módulos libres, como resolución poset o resolución combinatoria pueden ser muy distinta, aportando nuevas formas de abordar este estudio.

En el Capítulo 5 estudiamos lo números de Betti para la unión de grafos. En particular damos los número de Betti para todos los cografos. También incluimos una revisión de los ideales monomiales en n variables cuya dimensión proyectiva es n. Estos ideales monomiales con dimensión proyectiva máxima no pueden ser ideales de aristas, lo que nos permite concluir que la dimesión proyective del ideal de aristas de un grafo con n vértices es a lo más $n-1$. Esta caracterización fue previamente hecha en [1].

En el Capítulo 6 hacemos un resumen de la dimensión proyectiva de los grafos hasta con 10 vértices, incluyendo el dibujo y la dimensión proyective de los grafos hasta con 7 vértices. Estos grafos serán importantes en el estudio de las resoluciones combinatorias que se hará más adelante.

Abstract

Minimal free resolutions have been studied because they have a lot of algebraic invariants of the module that are calculated. Hilbert proved that for a finitely generated module, its minimal free resolution is always finite, however, the process to find a minimal free resolution is in a recursive way and very convoluted.

Since the earlies 1960's, Irvin Kaplansky proposed the problem to find an algorithm that allows to find a minimal free resolution of a finitely generated module in a non-recursive way and as explicit as possible. Since then a lot of effort has been made to find the algorithm but without success. Some of these efforts have resulted in algorithms like the Taylor's resolution, the cellular resolutions, or in other problems which are supposedly easier and equivalent but they are really equal or more complex than to find a minimal free resolution by definition.

Nowadays is possible to find a minimal free resolution of any monomial ideal of a polynomial ring over a field using softwares as Macaulay2 ([12]) quite quickly without this meaning that the problem is resolved, because Macaulay2 uses Gröbner basis, which is a recursive algorithm.

The Kaplansky's problem looks for a non-recursive algorithm that allows to find minimal free resolutions in an explicit way, but it is not clear what means "in an explicit way". For us this means that the algorithm has a polynomial algorithmic complexity. In some very special cases it has been found those algorithms, for instance for the edge ideal of some graphs (see [28, 32]).

Our study began by asking us about the combinatoric properties that a minimal free resolution has. The first drawback was that we identify that a graded minimal free resolution can have more than one generator in a free module with the same degree and however these generators behave differently and that the degrees do not make noticeable this different behaviour. To a better understanding we started to study the edge ideals of the complete graph with n vertices. We associated to this graph some subgraphs that symbolize each one of the generators of the free modules

and distinguish the different behaviour of two generators with the same degree in the differentials of a minimal free resolution of the edge ideal. These subgraphs were called basis graphs (see Chapter 3).

However, the problem was not resolved. Until here we have accomplished to find a free complex for the edge ideal of the complete graph described using basis graphs but we could not prove that this free complex was exact and minimal even when this complex seemed to be. In literature we found some criterions that prove that a free complex is exact and minimal without using the definition but these criterions are not easy to use in most of the cases, including ours (see [3, 26]). From this we decided to find a criterion that was easy to apply and that allow us to prove that a free complex, which Betti numbers were correct and which differentials were irredundant, be a graded minimal free resolution of a module. This result is the criterion given in Chapter 3.

After that, we start the study of an operation used in graphs called the duplication of a vertex, but applied to a variable in a monomial ideal. In Chapter 4 we define the duplication of a monomial ideal and we define a minimal free resolution for this monomial ideal. This resolutions is obtained explicitly by applying some operations on the free modules and the differentials of a minimal free resolution of the original ideal. To show that the resolution of a monomial ideal is exact and minimal, we use the criterion given in Chapter 3.

As an application of the duplication of a monomial ideal, in Chapter 5 we give a minimal free resolution for the edge ideal of a complete multitpartite graph. Previously in [32], was given a minimal free resolution for the edge ideal of a bipartite graph but not for a complete mulitpartite graph. These resolutions are given in terms of basis graphs of the complete multipartite graph.

The concept of basis graph can be generalized to basis set and the resolutions based on basis graphs are defined in Chapter 4. In this chapter we introduce two new representations of a minimal free resolutions: the poset resolution and the combinatorial resolution. Each one of these representations give some advantages and disadvantages that will be studied in future work. In this thesis we give some the properties of each one. For instance, it is known that a minimal free resolution is unique up to isomorphism, but with these new representations is easy to see that, even when they are isomorphic as sequence of free modules, as poset resolution or combinatorial resolution they are very different, giving us new way to study this.

In Chapter 5 we study the Betti numbers for the join of graphs. In particular we give the Betti numbers for all the cographs. Moreover, we make a review of the monomial ideals with n variables whose projective dimension is n. These monomial ideals with maximum projective dimension can not be edge ideals which allow us to conclude that the projective dimension of the edge ideal of a graph with n vertices is at most $n-1$. This characterization was done previously in [1].

In Chapter 6 we make a summary of the projective dimension of the edge ideal of graphs up to 10 vertices, including a draw and the projective dimension of the graphs up to 7 vertices. These graphs will be useful in the study of combinatorial resolutions that will be done in a future work.

Contents

Chapter 1

Introduction

Minimal free resolutions are a central topic in algebra because they provide algebraic invariants of modules. Nowadays there are softwares, as Macaulay2 ([12]), that allow to compute almost (not to say every) minimal free resolution of any module and obtain all the algebraic invariants with few lines of code. However, these softwares use recursive algorithms as Gröbner Basis making these calculations computationally expensive and, in case that we want to make these calculations by hand, these algorithms are tedious and in most of the cases they are very difficult. Therefore, today the main problem the study of minimal free resolution is focused in obtain formulas or algorithms that are not recursive or that its computational complexity be polynomial.

In previous attempts, it has been possible to find some formulas to obtain free resolutions without the minimality of these resolutions or complexes of free modules without the exactness, see for instance the Taylor Resolution and the Scarft Complex in Chapter 2. Although these resolutions allow to obtain some invariants as the Hilbert Series, they do not give us all of them. There are also some attempts to obtain explicit minimal free resolutions of some modules as for example the edge ideal of a bipartite graph or the complete graph, see for instance [20, 28, 32]. However, sometimes these resolutions are based on some other objects as a cellular complex, a lattice or a poset that in some cases the calculations on these new objects result with the same difficult as to compute a minimal free resolution by definition, see for instance [5, 13].

One of the most difficult part to obtain a minimal free resolution is to prove that a free complex is exact. In this thesis we give a very simple but powerful criterion to make this proof easier. As we will see in almost all the thesis, the criterion will be used in every proof where we prove that a free complex is exact and minimal.

We start chapter two by giving a short overview of some classical theory about minimal free resolutions and some very known resolutions of monomial ideals. Here we give the Hochster's Formula (see Theorem 2.3.3) which is a standard method to find the Betti numbers of a minimal free resolution. Moreover, we give some free resolutions and free complexes that are minimal free resolution but only for some specific monomial ideals.

In Chapter 3 we start by making a study about good gradings with the aim to obtain the most general grading for the criterion given in this chapter. The Criterion is the following:

Lemma 3.2.2. *Let N be a positively graded finitely generated S-module. If Γ is a homogeneous minimal generating set of N and Λ is an irredundant homogeneous subset of N with $|\Gamma_{\mathbf{c}}| = |\Lambda_{\mathbf{c}}|$ for all $\mathbf{c} \in \mathbb{M}$, then there exists an automorphism φ of N such that*

$$\varphi(\Lambda_{\mathbf{c}}) = \Gamma_{\mathbf{c}}$$

and whose restriction on $\Lambda_{\mathbf{c}}$ is a k-linear map for all $\mathbf{c} \in \mathbb{M}$. Moreover, if M is a matrix representation of φ where Λ and Γ are ordered by its multidegree on a nondecreasing way, then it is an upper triangular block matrix.

The criterion will be used in this chapter to give an explicit minimal free resolution of the edge ideal of the complete graph but in Chapter 4 and 5 we also use this criterion to prove that the free complexes for the duplication of a monomial ideal and the disjoint union are exact and minimal. In this sense we consider that this criterion is powerful and in the simplicity of the proofs we see that this criterion makes easier the proofs.

In Chapter 4 we define the duplication of a monomial ideal and we give a minimal free resolution for this new ideal. The duplication of a variable in a monomial ideal I is a new monomial ideal with the copy of some special minimal generators of I. This duplication is motivated by the duplication of a vertex of a graph. The main theorem of this chapter is:

Theorem 4.2.15. *Let $I_{G^\diamond}$ be the duplication of a monomial ideal I_G. Then the free resolution $\mathbf{F}_\bullet^\diamond$ is a minimal free resolution of $I_{G^\diamond}$.*

Moreover, we introduce two new representations of a minimal free resolutions and we give the duplication of them.

As an application of the minimal free resolution of the duplication of a monomial ideal, in Chapter 5 we give a minimal free resolution of the edge ideal of a complete multipartite graph. It is not difficult to see that a complete multipartite graphs is the duplication of some vertices of a complete graph. Moreover, we give a minimal free resolution of the monomial ideal generated by the disjoint union of two minimal set of generators. This minimal free resolution will be defined from the minimal free resolutions of each one of the monomial ideals in the disjoint union. We also calculate the Betti numbers of the edge ideal of the join graphs and we make a review about the monomial ideals with maximum projective dimension.

As we will see, in this thesis we stablish a method to give a graded minimal free resolution of a graded module M. We first need a sequence of free modules that will be our candidate to be the minimal free resolution. Second, we have to show that this sequence is a free complex. Then, to show the exactness of the free complex, we use the criterion. That is, we have to compute the Betti numbers of a minimal free resolution of M and then we have to show that the columns of the differentials are irredundant. Finally, by the Criterion given in Chapter 3, we prove that the free complex is a minimal free resolution.

In the last chapter we give a summary of projective dimension of graphs with the draw of all the graphs from 2 to 7 vertices sorted by the projective dimension and the number of vertices.

Chapter 2

Preliminars

In this chapter we give some basic definitions about minimal free resolution and we make a short overview of the classical theory in the study of minimal free resolutions of monomial ideals.

First we give the definitions of a grading and a minimal free resolution. We study how to obtain a minimal free resolution by definition and how to compute the Betti numbers of a minimal free resolution. This last method is called the Hochster's Formula, and it will be very important in all this thesis.

Once we have a minimal free resolution of a monomial ideal, we will study some important algebraic invariants that come from it. Moreover, we give some complexes and resolutions for a monomial ideal and, in each case, we give the definition and the conditions to that each complex or resolution be exact and minimal. We will see that, although these resolutions are non recursive, they usually are not minimal or exact. At the end we give an explanation of why we always want a free resolution to be minimal.

2.1 Gradings

Let R be a commutative ring with identity. R is **graded** if there exist some additive subgroups R_i of R, for $i \in \mathbb{M}$, such that $R = \bigoplus_{i\in\mathbb{M}} R_i$ and $R_i \cdot R_j \subseteq R_{i+j}$. Notice that these definitions implies that the set of indices $\mathbb{M}$ should have an operation. We denote this operation with $+$ because we will only work with commutative monoids in this thesis. Then $\mathbb{M}$ is, at least, a monoid $\mathbb{M} = (M, +)$. In this case we say that the ring R is $\mathbb{M}$-graded to make emphasis in the set of indices. As we will see in Chapter 3, the monoid will play an important role when we compute some minimal free resolutions. Unless stated otherwise, we use as monoid the set $\mathbb{N}^n$, for some $n \geqslant 1$, and when $n \geqslant 2$, we can say that the ring R is **multigraded** or that R is $\mathbb{N}^n$-graded.

In the same way, an ideal I of a $\mathbb{M}$-graded ring R is $\mathbb{M}$-graded if we can write I as $I = \bigoplus_{i \in \mathbb{M}} R_i \cap I$.

Now, if N is an R-module and $\mathbb{M}$ is a monoid, we say that the module N is graded if R is $\mathbb{M}$-graded and N can be written as a direct sum of some additive subgroups N_i of N, with $i \in \mathbb{M}$, and the structure of module is respected, that is, $N = \bigoplus_{i \in \mathbb{M}} N_i$ and $R_i \cdot N_j \subseteq N_{i+j}$. Here we also say that the module N is $\mathbb{M}$-graded, and if the monoid is $\mathbb{N}^n$ with $n \geqslant 2$, we will say that the module N is multigraded.

A submodule F of a $\mathbb{M}$-graded S-module N is graded if we can write F as $F = \bigoplus_{i \in \mathbb{M}} F \cap N_i$.

Remark 2.1.1. *From the condition that $R_i N_j \subseteq N_{i+j}$ we have that every additive group N_i is in fact an R_0-module. Indeed, when R_0 is a field, the additive groups N_i will be R_0-vector spaces.*

Definition 2.1.2. *Let R be a $\mathbb{M}$-graded ring and N be a $\mathbb{M}$-graded R-module. The additive subgroups R_i and N_i are called the* **homogeneous components** *of degree $i \in \mathbb{M}$ of R and N, respectively. If $r \in R_i$, for some $i \in \mathbb{M}$, we say that r is a homogeneous element of degree i; if $n \in N_i$, for some $i \in \mathbb{M}$, we say that n is a homogeneous element of degree i.*

Instead of say that an element $r \in R$ is homogeneous and it is in the homogeneous component R_a of R, we just say that the degree of r is a and we denote it as $\deg(r) = a$. The same will be said for elements of a graded module.

Remark 2.1.3. *Since every homogeneous component is an additive group, then the zero element 0 will be in every homogeneous component. Thus we say that 0 is a homogeneous element of indeterminate degree.*

In Chapter 3 we make a deeper study about gradings and what would be a good grading for the study of minimal free resolutions.

2.2 Minimal Free Resolutions

From now on, let $S = k[\mathbf{x}]$ be the polynomial ring over a field k with set of variables $\mathbf{x} = \{x_1, \dots, x_n\}$. Given a sequence of modules and homomorphisms d_i

$$\mathbf{F}_\bullet : 0 \leftarrow F_{-1} \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \leftarrow \cdots \leftarrow F_{i-1} \xleftarrow{d_i} F_i \leftarrow \dots \xleftarrow{d_p} F_p \leftarrow 0$$

we say that $\mathbf{F}_\bullet$ is a **free complex** if each F_i is a free S-module and the image of d_{i+1} is contained in the kernel of d_i, that is, $\mathrm{img}(d_{i+1}) \subseteq \ker(d_i)$ for all i. Moreover, if the last containment is an equality, this means, if $\mathrm{img}(d_{i+1}) = \ker(d_i)$ for all i, then we will say that $\mathbf{F}_\bullet$ is an **exact complex**.

Definition 2.2.1. *Let I be a monomial ideal of S and $\mathbf{F}_\bullet$ be a sequence of free modules such that $F_{-1} = S/I$, $F_0 = S$ and d_0 is the projection map π of S onto S/I, that is,*

$$\mathbf{F}_\bullet : 0 \leftarrow S/I \xleftarrow{\pi} S \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \leftarrow \cdots \leftarrow F_{i-1} \xleftarrow{d_i} F_i \leftarrow \dots \xleftarrow{d_p} F_p \leftarrow 0.$$

If $\mathbf{F}_\bullet$ is an exact complex then we say that $\mathbf{F}_\bullet$ is a **free resolution** *of I.*

Notice that from the definition of exact complex, if $\mathbf{F}_\bullet$ is a free resolution of I, then the image of d_1 should be the kernel of d_0, which implies that the image of d_1 should be the set of generators of I because $\ker(d_0) = I$. If we apply the definition of been an exact complex but now to the map d_1, then we have that the image of d_2 should be the kernel of d_1, that is, the image of d_2 should be the set of relations between the generators of I. Continuing with this idea we have that each map d_{i+1} is defined from the kernel of the previous map d_i. Here arises a natural question: is this process finite? To answer this question we have the Hilbert's Syzygy Theorem, but before to enunciate the theorem we need to know what a syzygy is.

Definition 2.2.2. *Let $\mathbf{F}_\bullet$ be a free resolution of a monomial ideal I. The set of generators of $\ker(d_i)$ for some map d_i of $\mathbf{F}_\bullet$ are called the* **i-th syzygies** *and the S-module generated by these elements is the* **i-th module of syzygies**.

Moreover, if for each set of generators of $\ker(d_i)$ we consider a minimal set of generators to define the next map d_{i+1}, then we say that the resolution $\mathbf{F}_\bullet$ obtained is a **minimal free resolution** *of I.*

Now we can enunciate the theorem that answers the question about how long are the minimal free resolutions of monomial ideals.

Theorem 2.2.3 (Hilbert's Syzygies Theorem). *Let $S = k[\mathbf{x}]$ be a polynomial ring over a field k and a set of n variables $\mathbf{x}$. If I is a monomial ideal of S then any minimal free resolution of I has length at most n.*

For instance, if I is the monomial ideal of $S = \mathbb{R}[x,y,z]$ minimally generated by $\{x,y,z\}$, then its minimal free resolutions has length 3. On the other hand, if I is generated by $\{xy,z\}$, its minimal free resolutions has lenght 2.

2.2.1 Graded Minimal Free Resolutions

In this section we use the gradings of the ring S and the S-modules to define a grading on the minimal free resolutions.

Let $S = k[\mathbf{x}]$ be the polynomial ring over a field k and set of variables $\mathbf{x}$ with the $\mathbb{N}^n$-grading given by the homogeneous components $S_{\mathbf{a}} = \langle\{\mathbf{x}^{\mathbf{a}}\}\rangle_k$ for all $\mathbf{a} \in \mathbb{N}^n$. This grading will be called the **standard multigrading** of the polynomial ring S. When the monoid $\mathbb{N}^n$ is $\mathbb{N}$, that is, when $n = 1$, we call the grading the **standard grading** of S and it will have homogenenous components $S_a = \langle\{\mathbf{x}^{\mathbf{b}} \in S : b_1 + b_2 + \cdots + b_n = a\}\rangle_k$.

Given a map $d : F \to G$ between two graded S-modules F and G, we say that d is a **homogeneous map** if $d(f)$ is an homogeneous element of G for every homogeneous element f of F. If we have that $\deg(d(f)) = \deg(f) + c$ for some constant c, we say that d is homogeneous of degree c. We will focus on homogeneous maps of degree zero.

With the aim to define homogeneous maps of degree zero we need to define the shifting of free modules. Consider S as S-module with the standard multigrading and let $\mathbf{b}$ be a multidegree, that is, $\mathbf{b} \in \mathbb{N}^n$. We define the **shifted module** by $\mathbf{b}$, denoted as $S(-\mathbf{b})$ or $S(-\mathbf{x}^{\mathbf{b}})$, as the $\mathbb{N}^n$-graded S-module generated by $\mathbf{x}^{\mathbf{b}}$ and with homogeneous components give as $S(-\mathbf{b})_{\mathbf{a}} = S_{\mathbf{a}-\mathbf{b}}$. Here notice that $S_{\mathbf{a}} = 0$ for all $\mathbf{a} \notin \mathbb{N}^n$. In other words, the shifted module $S(-\mathbf{b})$ is the same module S but with its homogeneous components shifted by $\mathbf{b}$ because the homogeneous component of degree zero of S is now in the homogeneous component $\mathbf{b}$ in $S(-\mathbf{b})$.

Definition 2.2.4. *Let I be a monomial ideal of the polynomial ring $S = k[\mathbf{x}]$ and*

$$\mathbf{F}_\bullet : 0 \leftarrow S/I \xleftarrow{\pi} S \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \leftarrow \cdots \leftarrow F_{i-1} \xleftarrow{d_i} F_i \leftarrow \ldots \xleftarrow{d_p} F_p \leftarrow 0$$

a minimal free resolution of I. We say that $\mathbf{F}_\bullet$ is a **homogeneous graded minimal free resolution** *of I if each free module F_i is graded and each map d_i is a homogeneous map.*

We will focus on homogeneous $\mathbb{N}$-graded minimal free resolutions of monomial ideals with their free modules with the standard multigrading and mappings of degree zero. In most of the times we will only say that $\mathbf{F}_\bullet$ is a minimal free resolution which means that it is homogeneous and graded.

Next lemma says that given a finitely generated monomial ideal in the polynomial ring $S = k[\mathbf{x}]$, there is always a homogeneous multigraded minimal free resolution of that monomial ideal.

Lemma 2.2.5. *[26, Proposition 2.1] Let $S = k[\mathbf{x}]$ be a polynomial ring over a field k and a finitely generated monomial ideal I. Then every submodule of a graded free S-module has a minimal generating set of homogeneous elements.*

Since the kernel of a mapping of free S-modules is a submodule of a graded free S-module, then we always can choose a homogeneous minimal generating set for the syzygy module.

Once we have defined a graded minimal free resolution of a monomial ideal I, one natural question that arises is how many minimal free resolutions are there? Next proposition answers this question saying that there are only one up to isomorphism.

Theorem 2.2.6. *[26, Theorem 7.5] Let I be a monomial ideal of a polynomial ring $S = k[\mathbf{x}]$ and let $\mathbf{F}_\bullet$ and $\mathbf{G}_\bullet$ be two minimal free resolutions of I. Then there exist $\phi_i : F_i \to G_i$ isomorphisms such that the next diagram commutes for all i:*

$$\begin{tikzcd} F_i \arrow[d, "\phi_i"'] & F_{i+1} \arrow[l, "d_{i+1}"'] \arrow[d, "\phi_{i+1}"] \\ G_i & G_{i+1} \arrow[l, "\delta_{i+1}"] \end{tikzcd} .$$

We have described how is a minimal free resolution of a monomial ideal I but until now, we do not know how to construct a minimal free resolution. Next subsection will give a method to construct a minimal free resolution in a recursive way.

2.2.2 Minimal Free Resolutions By Hand

We want to emphasize that is always possible to give a minimal free resolution of a monomial ideal using the following construction but what is really an open problem is to give a non-recursive way to compute minimal free resolutions. From

Section 2.4 to Section 2.7 we give some other ways to give minimal free resolution and in Chapters 3, 4 and 5 we give non-recursive ways to compute minimal free resolutions for some special monomial ideals.

Let $S = k[\mathbf{x}]$ be a polynomial ring and I a monomial ideal of S. On S and S/I we will consider the standard multigrading. The process can be done also with any other grading.

The recursive process is: given a homogeneous map f, we will calculate the kernel of f and we will consider the free module generate by the minimal generators of $\ker(f)$. Then we define a new function whose image will be the $\ker(f)$ and we repeat the process with this new function.

More precisely, recall that a minimal free resolution starts with the function $\pi : S \to S/I$ from S to S/I, where I is the monomial ideal and π is usually called the projection function. We start calculating the kernel of this function π. It is easy to see that $\ker(\pi) = \langle \mathbf{g}_1, \ldots, \mathbf{g}_t \rangle$ where $\mathbf{G} = \{\mathbf{g}_1, \ldots, \mathbf{g}_t\}$ is a minimal generator set of I. Now we define a free module $F_1 = S(-\mathbf{g}_1) \oplus \cdots \oplus S(-\mathbf{g}_t)$ and the homogeneous map in the matricial form $d_1 = \begin{pmatrix} \mathbf{g}_1 & \cdots & \mathbf{g}_t \end{pmatrix}$.

Now we repeat the previous process to the new map d_1. That is, we compute the minimal generator of kernel of d_1, lets say $\ker(d_1)$ is minimally generated by $\langle \mathbf{c}_1, \ldots, \mathbf{c}_r \rangle$. Again, we define the free module $F_2 = S(-\mathbf{b}_1) \oplus \cdots S(-\mathbf{b}_r)$ and a homogeneous map $d_2 = \begin{pmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_r \end{pmatrix}$ with the vectors $\mathbf{c}_i$ written as columns and $\deg(\mathbf{c}_i) = \mathbf{b}_i$ for $1 \leqslant i \leqslant r$. By Hilbert's Syzygy Theorem 2.2.3 this process is finite. Then, after some finite steps we obtain a minimal free resolution $\mathbf{F}_\bullet$ of I,

$$\mathbf{F}_\bullet : 0 \leftarrow S/I \xleftarrow{\pi} S \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \leftarrow \cdots \leftarrow F_{i-1} \xleftarrow{d_i} F_i \leftarrow \ldots \xleftarrow{d_p} F_p \leftarrow 0.$$

The resolution $\mathbf{F}_\bullet$ is minimal because we are getting a minimal and homogeneous generator set of the kernel of each map d_i, and it is homogeneous because the free modules F_i are shifted by the degree of the generators of $\ker(d_{i-1})$.

Next example illustrates the process.

Example 2.2.7. *Consider the monomial ideal* $I = \langle x_1x_2, x_1x_3, x_2x_3 \rangle$ *and the projection function* $\pi : S \to S/I$*. As we saw above, the first function* d_1 *is given by the minimal generators of* I*, that is* $d_1 = \begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 \end{pmatrix}$*, and the first free module is* $F_1 = S(-x_1x_2) \oplus S(-x_1x_3) \oplus S(-x_2x_3)$*.*

Next step is calculate a minimal and homogeneous generator set of $\ker(d_1)$*. After some small calculations we find that* $\ker(d_1)$ *is minimally generated by the column* *vectors* $(-x_3, x_2, 0)$ *and* $(-x_3, 0, x_1)$. *These two vectors are homogeneous of multidegree* $x_1x_2x_3$, *then we define the free module* F_2 *as* $S(-x_1x_2x_3)^2$ *and the second map as*

$$d_2 = \begin{pmatrix} -x_3 & -x_3 \\ x_2 & 0 \\ 0 & x_1 \end{pmatrix}.$$

If we calculate the kernel of d_2 *we get that it is zero, then we have finished. Thus, a minimal free resolution of* I *is*

$$0 \longleftarrow S/I \xleftarrow{\pi} S \xleftarrow{\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 \end{pmatrix}} \begin{matrix} S(-x_1x_2) \\ \oplus \\ S(-x_1x_3) \\ \oplus \\ S(-x_2x_3) \end{matrix} \xleftarrow{\begin{pmatrix} -x_3 & -x_3 \\ x_2 & 0 \\ 0 & x_1 \end{pmatrix}} S(-x_1x_2x_3)^2 \longleftarrow 0.$$

Figure 2.1: A minimal free resolution by hand.

Since every minimal free resolution of a monomial ideal is isomorphic, next subsection treats about some important invariants.

2.3 Invariants Of Minimal Free Resolutions

Most of the times, the hard part of giving a minimal free resolution is to obtain the maps between the free S-modules, also called differentials. However, if we can compute all the free modules of a minimal free resolution of I, even without knowing the differentials between them, we obtain some invariants.

2.3.1 Betti Numbers

From Theorem 2.2.6 we have that the rank of the free S-modules is an invariant. This is our first definition of a numerical invariant of minimal free resolution.

Definition 2.3.1. *Let* $\mathbf{F}_\bullet$ *be a graded minimal free resolution of a monomial ideal* I. *The* i*-th* **total Betti number** *of* $\mathbf{F}_\bullet$ *is the rank of the* i*-th free module of* $\mathbf{F}$, *that is,*

$$\beta_i(S/I) = \text{rank} F_i.$$

The i-th **graded Betti number** *of degree* $\mathbf{b}$ *of* $\mathbf{F}_\bullet$ *is the number of summands of the form* $S(-\mathbf{b})$ *in* F_i*, and it is denoted as* $\beta_{i,\mathbf{b}}(S/I)$*.*

Remark 2.3.2. *Notice that the total Betti number can be obtained from the graded Betti numbers just by summing all the i-th graded Betti numbers for all degree* $\mathbf{b}$*, that is,* $\beta_i(S/I) = \sum_{\mathbf{b}\in\mathbb{N}^n} \beta_{i,\mathbf{b}}(S/I)$*.*

There is a formula to calculate the Betti numbers of a minimal free resolution of a monomial ideal known as the Hochster's formula. Before we give this important formula we need to define some special simplicial complexes that will be used in the definition.

Given a monomial ideal I and a degree $\mathbf{a} \in \mathbb{N}^n$, the **upper Koszul simplicial complex** of I is $K^{\mathbf{a}}(I) = \{\tau \in \{0,1\}^n : \mathbf{x}^{\mathbf{a}-\tau} \in I\}$. In the same way, the **lower Koszul simplicial complex** of I is $K_{\mathbf{a}}(I) = \{\tau \in \{0,1\}^n : \mathbf{x}^{\mathbf{a}-\mathbf{1}+\tau} \notin I\}$. In this last simplicial complex, the vector $\mathbf{a}-\mathbf{1}$ is obtained by subtracting one in each non-zero entry from $\mathbf{a}$. These two simplicial complexes are Alexander dual, see [23] for more details.

Theorem 2.3.3 (Hochster's formula)**.** *The i-th Betti number of I in multidegree* $\mathbf{a}$ *can be expressed as*

$$\begin{aligned}\beta_{i+1,\mathbf{a}}(S/I) &= \dim_k \tilde{H}_{i-1}(K^{\mathbf{a}}(I);k)\\ &= \dim_k \tilde{H}^{n-i-3}(K_{\mathbf{a}}(I);k).\end{aligned}$$

Here $\tilde{H}_i$ is the homology of the chains complex of the simplicial complex $K^{\mathbf{a}}(I)$ in homological degree i and $\tilde{H}^i$ is the cohomology of the same chains complex.

If we apply the Universal Coefficient Theorem, which says that $\tilde{H}_i(K^{\mathbf{a}}(I);k) = \tilde{H}_{n-i-3}(K_{\mathbf{a}}(I);k)$, to the Hochster's formula, we obtain a formula to compute the Betti number using the homology of the lower Koszul simplicial complex.

Corollary 2.3.4. *The i-th Betti number of I in degree* $\mathbf{a}$ *can be expressed as*

$$\beta_{i,\mathbf{a}}(S/I) = \dim_k \tilde{H}_{n-i-1}(K_{\mathbf{a}}(I);k).$$

This last formula is easier than compute the cohomology of the lower Koszul simplicial complex and it will be useful to compute the Betti numbers of some edge ideals.

There are other simplicial complexes that can be used to calculate the Hochster's Formula, for instance, in [10] is defined a simplicial complex in terms of the low common multiple lattice.

Now we continue with more invariants of a minimal free resolution.

2.3.2 Projective Dimension

Once we have calculated the Betti numbers of the minimal free resolution of a monomial ideal we can give some other invariants. One of them is the projective dimension of the monomial ideal I.

Definition 2.3.5. *Let I be a monomial ideal and $\beta_{i,\mathbf{a}}(S/I)$ the Betti numbers of a minimal free resolution of I. The* **projective dimension** *of I, denoted as $\mathrm{pd}(I)$, is the greatest i for which $\beta_{i,\mathbf{a}}(S/I)$ is not zero, that is,*

$$\mathrm{pd}(I) = \max\{i : \beta_{i,\mathbf{a}}(S/I) \neq 0\}.$$

The projective dimension of I is counting how many free modules are needed for describe I in terms of free modules. It is easy to see that if I is free, then its minimal generators do not have relations and thus we only need one free module to describe I. That free module is I itself. In other words, the projective dimension of I is measuring how far is I to be a free module.

2.3.3 The regularity

Let $S = k[\mathbf{x}]$ be a polynomial ring and I a monomial ideal of S. The **regularity** of a graded monomial ideal I is defined as

$$\mathrm{reg}(I) = max\{|\mathbf{b}| - i : \beta_{i,\mathbf{b}}(S/I) \neq 0\}$$

where $|\mathbf{b}| = \mathbf{b}_1 + \ldots + \mathbf{b}_{n+1}$ is the sum of its entries.

We give an example of previous two invariants.

Example 2.3.6. *Let $\mathbf{F}_\bullet$ be a minimal free resolution of the monomial ideal $I = \langle\{x_1x_2^3, x_1x_2x_3^2, x_1^2x_3^3\}\rangle$ of $S = k[x_1, x_2, x_3]$.*

$$\mathbf{F}_\bullet : 0 \longleftarrow S/I_G \xleftarrow{\pi} S \xleftarrow{\begin{pmatrix} x_1x_2^3 & x_1x_2x_3^2 & x_1^2x_3^3 \end{pmatrix}} \begin{matrix} S(-x_1x_2^3) \\ \oplus \\ S(-x_1x_2x_3^2) \\ \oplus \\ S(-x_1^2x_3^3) \end{matrix} \xleftarrow{\begin{pmatrix} -x_3^2 & 0 \\ x_2^2 & -x_1x_3 \\ 0 & x_2 \end{pmatrix}} \begin{matrix} S(-x_1x_2^3x_3^2) \\ \oplus \\ S(-x_1^2x_2x_3^3) \end{matrix} \longleftarrow 0$$

Figure 2.2: A minimal free resolution $\mathbf{F}_\bullet$.

The projective dimension of I is 2 and the regularity is 4.

2.3.4 Hilbert Series

In this part we present one important difference between consider an $\mathbb{N}$-grading on the polynomial ring and consider an $\mathbb{N}^n$-grading on the polynomial ring. Recall that the standard grading of the polynomial ring is given by the homogeneous components $S_a = \{\mathbf{x}^{\mathbf{b}} : b_1 + \cdots + b_n = a\}$, and the standard multigrading of the polynomial ring S is given by homogeneous components $S_{\mathbf{a}} = \{\mathbf{x}^{\mathbf{a}}\}$.

The first difference between these two gradings is the dimension of each homogeneous component. Using the method of *stars and bars* we can calculate how many monomials of degree d there are in a polynomial ring with n variables, that is, $\dim_k(S_d) = \binom{d+n-1}{n-1}$ with $d \in \mathbb{N}$. On the other hand, the dimension of any homogeneous component of degree $\mathbf{b}$ in the standard multigrading of S is 1, that is, $\dim_k(S_{\mathbf{b}}) = 1$ for any $\mathbf{b} \in \mathbb{N}^n$. Thus, the standard multigrading of S is finer than the standard grading of S because this last grading does not distinguish between two monomials of the same degree while the standard multigrading does.

Now we give the definition of the Hilbert Series, which is closely related with the dimension of the homogeneous components.

Definition 2.3.7. *Let $\mathbb{M}$ be a monoid and $\Omega = (\mathbb{M}, \{N_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}})$ an $\mathbb{M}$-grading of an S-module N. The $\mathbb{M}$-**Hilbert Series** of N is*

$$\mathrm{HS}(N;\mathbf{x}) = \sum_{\mathbf{a}\in\mathbb{M}} \dim_k(N_{\mathbf{a}})\mathbf{x}^{\mathbf{a}}.$$

When Ω is the standard grading of M and we set $x_i = t$ for all $x_i \in \mathbf{x}$, we obtain the $\mathbb{N}$-Hilbert Series of N, denoted as $\mathrm{HS}(N;t)$.

The ring of formal series in which Hilbert series lives is $\mathbb{Z}[[\mathbf{x}]]$. In this ring, each element $1-x_i$ is invertible and the series $\frac{1}{1-x_i} = 1 + x_i + x_i^2 + \dots$ is its inverse.

Example 2.3.8. *Consider $S = k[\mathbf{x}]$ with the standard multigrading. The $\mathbb{N}^n$-Hilbert series of S is the rational function*

$$\mathrm{HS}(S;\mathbf{x}) = \sum_{\mathbf{a}\in\mathbb{N}^n} \mathbf{x}^{\mathbf{a}} = \prod_{i=1}^{n} \frac{1}{1-x_i}$$

and the $\mathbb{N}$-Hilbert series of S is $\mathrm{HS}(S;t) = 1/(1-t)^n$. Recall that $S(-\mathbf{a})$ is the free S-module generated by $\mathbf{x}^a$. The $\mathbb{N}^n$-Hilbert series of $S(-\mathbf{a})$ is

$$\mathrm{HS}(S(-\mathbf{a});\mathbf{x}) = \frac{\mathbf{x}^a}{\prod_{i=1}^{n}(1-x_i)} = \mathbf{x}^{\mathbf{a}}\mathrm{HS}(S;\mathbf{x}).$$

The $\mathbb{N}^n$*-Hilbert series of the quotient module* S/I *is*

$$\mathrm{HS}(S/I;\mathbf{x}) = \mathrm{HS}(S;\mathbf{x}) - \mathrm{HS}(I;\mathbf{x}).$$

The $\mathbb{N}^n$-Hilbert series of a monomial ideal I can be obtained from its minimal free resolution using the following lemma.

Lemma 2.3.9. *Let* W*,* U *and* V *be three* S*-modules with the standard grading and a short exact sequence*

$$0 \leftarrow W \leftarrow U \leftarrow V \leftarrow 0.$$

Then,

$$\mathrm{HS}(U;\mathbf{x}) = \mathrm{HS}(W;\mathbf{x}) + \mathrm{HS}(V;\mathbf{x}).$$

Recall from previous subsection that we can construct any minimal free resolution just by calculating the kernel of each one of the differentials. The first differential is $\pi : S \to S/I$ and its kernel is I. With this kernel we define a free module F_1 and the mapping d_1 where d_1 is the composition of ι_1 inclusion map and ρ_1 the projection map. By construction, each piece of the sequence is exact, for example, in next diagram the sequence in red color is exact

$$\begin{tikzcd}
 & & & & & & & 0 & & & & & \\
 & & & \ker(\pi) \arrow[dl, "\iota_1"'] & & & & & \ker(d_{t-2}) \arrow[ul] \arrow[dl, "\iota_{t-1}"'] & & & & \\
0 & S/I \arrow[l] & S \arrow[l, "\pi"'] & & F_1 \arrow[ll, dashed, "d_1"'] \arrow[ul, "\rho_1"'] & & \cdots & F_{t-2} & & F_{t-1} \arrow[ll, dashed, "d_{t-1}"'] \arrow[ul, "\rho_{t-1}"'] & & F_t \arrow[ll, dashed, "d_t"'] \arrow[dl, "\rho_t"] & 0=\ker(d_t) \arrow[l] \\
 & & & & & \ker(d_1) \arrow[ul, "\iota_2"] & & & & & \ker(d_{t-1}) \arrow[ul, "\iota_t"] & & \\
 & & & & & & & & & & & 0 \arrow[ul] &
\end{tikzcd}$$

then, applying the lemma in the first part of the resolution, we get that:

$$\mathrm{HS}(S;\mathbf{x}) = \mathrm{HS}(S/I;\mathbf{x}) + \mathrm{HS}(\ker(\pi);\mathbf{x}).$$

If we continue applying the lemma in each exact part of the resolution we obtain that $\mathrm{HS}(F_1;\mathbf{x}) = \mathrm{HS}(\ker(\pi);\mathbf{x}) + \mathrm{HS}(\ker(d_1);\mathbf{x})$ and in general we have that

$$\mathrm{HS}(F_i;\mathbf{x}) = \mathrm{HS}(\ker(d_{i-1});\mathbf{x}) + \mathrm{HS}(\ker(d_i);\mathbf{x}).$$

Replacing the equations in the first one, we have the following formula for the Hilbert series of S/I in terms of the Hilbert series of the free modules of a minimal free resolution of I:

$$\mathrm{HS}(S/I;\mathbf{x}) = -\mathrm{HS}(S;\mathbf{x}) + \mathrm{HS}(F_1;\mathbf{x}) + \ldots + (-1)^{i+1}\mathrm{HS}(F_i;\mathbf{x}) + \ldots + (-1)^{t+1}\mathrm{HS}(F_t;\mathbf{x}).$$

Example 2.3.10. *A multigraded minimal free resolution of the edge ideal I_{K_3} of the complete graph with three vertices is:*

$$0 \longleftarrow S/I_3 \xleftarrow{\pi} S \xleftarrow{\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 \end{pmatrix}} \begin{array}{c} S(-x_1x_2) \\ \oplus \\ S(-x_1x_3) \\ \oplus \\ S(-x_2x_3) \end{array} \xleftarrow{\begin{pmatrix} -x_3 & -x_3 \\ x_2 & 0 \\ 0 & x_1 \end{pmatrix}} S(-x_1x_2x_3)^2 \longleftarrow 0$$

The $\mathbb{N}^n$-Hilbert series of S/I_3 is

$$\begin{aligned} \mathrm{HS}(S/I;\mathbf{x}) &= -\mathrm{HS}(S) + \mathrm{HS}(-x_1x_2) + \mathrm{HS}(-x_1x_3) + \mathrm{HS}(-x_2x_3) - 2\mathrm{HS}(-x_1x_2x_3) \\ &= \frac{-1 + x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3}{(1-x_1)(1-x_2)(1-x_3)}. \end{aligned}$$

2.4 Koszul Complex

We have given a recursive way to calculate a minimal free resolution of a monomial ideal I. This tedious process was called "by hand" because it implies to compute many kernels and their minimal generating set by definition. Although this process can be done it is very difficult when we have some special types of monomial ideals.

In this section we introduce the **Koszul Complex** of a monomial ideal and we show when this complex is a minimal free resolution of I. This complex is very well known and it was introduced by Jean-Louis Koszul to define a cohomology theory for Lie Algebras.

Let S^r be the free S-module of finite rank r. We write $\bigwedge^i S^r$ for the i-th exterior power of S^r. That is, $\bigwedge^i S^r$ is the k-vector space with basis $\{e_{j_1} \wedge \ldots \wedge e_{j_i} : j_1 < \ldots < j_i \text{ and } e_{j_t} \in S^r\}$. Then, given a S-linear map $s : S^r \to S$, the Koszul complex associated to s is the chain complex of S-modules

$$\mathbf{K}_\bullet(s) : 0 \leftarrow S \xleftarrow{d_1} \bigwedge^{1} S^r \leftarrow \cdots \leftarrow \bigwedge^{r-1} S^r \xleftarrow{d_r} \bigwedge^{r} S^r \leftarrow 0,$$

where the differential d_t is given by

$$d_t(e_{j_1} \wedge \ldots \wedge e_{j_t}) = \sum_{i=1}^{t} (-1)^{i+1} s(e_{j_i}) e_{j_1} \wedge \ldots \wedge \widehat{e_{j_i}} \wedge \ldots \wedge e_{j_t},$$

for any $e_i \in S^r$. Here $\widehat{e_i}$ means that the term e_i is omitted.

It is not difficult to see that $d_t \circ d_{t+1} = 0$, which means that $\mathbf{K}_\bullet(s)$ is a complex.

Note that $\bigwedge^1 S^r = S^r$ and $d_1 = s$. To emphasize in the finite sequence $s_1, \ldots, s_r$ defined by $s : S^r \to S$ we some times denote this Koszul complex as $\mathbf{K}_\bullet(s_1, \ldots, s_r)$.

Example 2.4.1. *Let $s : S^3 \to S$ given by the row matrix $\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 \end{pmatrix}$. The Koszul complex of this sequence is*

$$\mathbf{K}_\bullet(s) : 0 \leftarrow S \xleftarrow{\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 \end{pmatrix}} \bigwedge^1 S^3 \xleftarrow{\begin{pmatrix} -x_1x_3 & -x_2x_3 & 0 \\ x_1x_2 & 0 & -x_2x_3 \\ 0 & x_1x_2 & x_1x_3 \end{pmatrix}} \bigwedge^2 S^3 \xleftarrow{\begin{pmatrix} x_2x_3 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}} \bigwedge^3 S^3 \leftarrow 0.$$

Notice that $d_2(e_1 \wedge e_3) = -s(e_3)e_1 + s(e_1)e_3 = -x_2x_3e_1 + x_1x_2e_3$ which is the second column of the second differential in $\mathbf{K}_\bullet(s)$. In the same way

$$d_3(e_1 \wedge e_2 \wedge e_3) = x_2x_3e_1 \wedge e_2 - x_1x_3e_1 \wedge e_3 + x_1x_2e_2 \wedge e_3$$

is the third matrix in the chain complex. It is not difficult to see that the product of each two consecutive matrices is zero, that is, this sequence of modules is a complex.

The Koszul complex of a sequence of elements $s_1, \ldots, s_t$ is not exact nor minimal in general. This is the motivation for the next definition.

Definition 2.4.2. *Let M be an S-module and $s_1, \ldots, s_r$ be a finite sequence of elements in S. We say that $s_1, \ldots, s_r$ is a regular sequence on M if $\langle s_1, \ldots, s_r \rangle M \neq M$ and for each i, the element s_i is not a zero divisor on $M/\langle s_1, \ldots s_{i-1} \rangle M$.*

In previous definition $\langle s_1, \ldots, s_r \rangle$ is the ideal of S generated by the sequence $s_1, \ldots, s_r$.

Notice that on the polynomial ring $S = k[x_1, x_2, x_3]$ the sequence x_1, x_2, x_3 is a regular sequence on S because $\langle x_1, x_2, x_3 \rangle S$ is not S and x_1 is not a zero divisor in S, x_2 is not a zero divisor in $S/\langle x_1 \rangle S$ and x_3 is not a zero divisor in $S/\langle x_1, x_2 \rangle S$.

On the other hand, the sequence x_1x_2, x_1x_3, x_2x_3 is not a regular sequence on S because, although $S \neq \langle x_1x_2, x_1x_3, x_2x_3\rangle S$, x_1x_3 is a zero divisor in $S/\langle x_1x_2\rangle S$, since $x_1x_3 \cdot x_2 = x_1x_2 \cdot x_3$.

In case that a sequence s is regular, then its Koszul complex will be exact and minimal. Next theorem says that.

Theorem 2.4.3. *Let s be a regular sequence on S. Then the Koszul complex $\mathbf{K}_\bullet(s)$ is a minimal free resolution of s.*

Next we give an example of a minimal free resolution of a regular sequence.

Example 2.4.4. *Let $I = \langle x, y, z\rangle$ be the monomial ideal generated by the variables of S. It is not difficult to see that these variables form a regular sequence and thus $\mathbf{K}_\bullet(x, y, z)$ is a minimal free resolution of them.*

$$\mathbf{K}_\bullet(x,y,z) : 0 \leftarrow S \xleftarrow{\begin{pmatrix} x & y & z \end{pmatrix}} \bigwedge^1 S^3 \xleftarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} \bigwedge^2 S^3 \xleftarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} \bigwedge^3 S^3 \leftarrow 0.$$

There is a simple case in which we can give a characterization of a sequence of elements in S is regular, it is when these elements are monomials. Next proposition shows that.

Proposition 2.4.5. *Let $m_1, \ldots, m_t$ be a sequence of monomials in $S = k[\mathbf{x}]$ such that $m_i \notin k$ for all $1 \leqslant i \leqslant t$. If $\gcd(m_i, m_j) = 1$ for all $1 \leqslant i < j \leqslant t$, then these sequence of monomials form a regular sequence.*

Proof. Since none of the monomials are in k, then $S \neq \langle m_1, \ldots, m_t\rangle S$. Now, lets suppose that m_i is a zero divisor in $S/\langle m_1, \ldots m_{i-1}\rangle S$. That is, there exists a monomial n such that $m_i n$ is divided by some m_j, with $1 \leqslant j \leqslant i-1$. Since $\gcd(m_j, m_i) = 1$, then m_j divides only to n, but that implies that $n = 0$ in $S/\langle m_1, \ldots, m_{i-1}\rangle S$, which is a contradiction to that m_i is a zero divisor. Therefore, $m_1, \ldots, m_t$ is a regular sequence. □

The Koszul complex works good with regular sequence but in general it only give us a complex for the module. The main problem is that this complex is not exact. As we will see in following subsections, there are some other ways to compute complexes which will be exact but not minimal.

2.5 Taylor Resolution

We now introduce a resolution which is very simple to construct but highly no minimal. This sequence of free modules for a monomial ideal I is always a resolution, that is, is always an exact complex. Its name is due to Diana Taylor, a student of Kaplansky, who was the first to describe it in her thesis. This resolution is non recursive and give us upper bounds for the Betti numbers of a minimal free resolution.

For a monomial ideal with minimal generating set $\{m_1,\ldots,m_r\}$ and any subset A of $[r]=\{1,2,\ldots,r\}$ we define the monomial m_A as the low common multiple of the elements indexed by A, that is, $m_A=\operatorname{lcm}(m_i : m_i \in A)$.

The **Taylor's Resolution** $\mathbf{T}_\bullet=\{F_i,d_i\}$ of a monomial ideal with minimal generating set $\{m_1,\ldots,m_t\}$ has free modules $F_i=\bigoplus_{A\subseteq[r]} S(-m_A)$, where all the subset $A\subseteq[r]$ have cardinality i. And for each basis element e_A of F_i we define the differential as

$$d(e_A)=\sum_{j\in A}(-1)^{t+1}\frac{m_A}{m_{A\smallsetminus\{j_t\}}}\cdot e_{A\smallsetminus\{j_t\}}$$

where j_t is the t-th element in A.

Next we give an example.

Example 2.5.1. *Let I be a monomial ideal with minimal generating set $\{x_1x_2,x_1x_3,x_2x_3\}=\{m_1,m_2,m_3\}$. The Taylor's resolution of I is*

$$\mathbf{T}_\bullet : 0\leftarrow S \xleftarrow{\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 \end{pmatrix}} F_1 \xleftarrow{\begin{pmatrix} -x_3 & -x_3 & 0 \\ x_2 & 0 & -x_2 \\ 0 & x_1 & x_1 \end{pmatrix}} F_2 \xleftarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} F_3 \leftarrow 0$$

where $F_1=S(-x_1x_2)\oplus S(-x_1x_3)\oplus S(-x_2x_3)$, $F_2=S(-x_1x_2x_3)^3$ and $F_3=S(-x_1x_2x_3)$. Notice that $\operatorname{lcm}(m_i,m_j)=x_1x_2x_3$ for all $1\leqslant i<j\leqslant 3$, which implies that all three summands in F_2 have the same multidegree.

Unlike the Koszul complex, we have called this construction a resolution, that is because it is always an exact complex, as next theorem says.

Theorem 2.5.2. *Let I be a monomial ideal. The Taylor's resolution $\mathbf{T}_\bullet$ of I is always an exact complex.*

Taylor's resolutions is far from be minimal in most of cases, but there is a case when it is a minimal free resolution of a monomial ideal.

Proposition 2.5.3. *Let $\{m_1,\dots,m_r\}$ be a minimal generating set of a monomial ideal I such that $m_A \neq m_B$ for all non empty sets $A \neq B \subseteq [r]$. Then the Taylor's resolution of I is a minimal free resolution.*

Proof. Since $m_A \neq m_B$ for all $A \neq B$ subsets of $[r]$, then $\frac{m_A}{m_{A\smallsetminus j_t}} \neq 1$ for all A and all $j_t \in A$. This implies that $T_\bullet$ is minimal. □

Another difference between the Taylor's resolution and the Koszul complex is their grading and their differentials. Previous example shows that even for the same monomial ideal both complexes are different. There are some special cases where they coincide.

We have that the Koszul complex and the Taylor complex are isomorphic for regular sequences.

Proposition 2.5.4. *If the minimal generators of I form a regular sequence, the Taylor's resolution and the Koszul complex are isomorphic.*

Proof. Let $\{m_1,\dots m_r\}$ be a minimal generating set of a monomial ideal I and $s : S^r \to S$ such that $s(e_i) = m_i$. Since $\gcd(m_i,m_j) = 1$ for all $1 \leqslant i < j \leqslant r$, we have that $m_A = \operatorname{lcm}(m_i : i \in A) = \prod_{i\in A} m_i$ and thus

$$\frac{m_A}{m_{A\smallsetminus\{j_t\}}} = m_{j_t} = s(e_{j_t}).$$

That is, the differentials of $\mathbf{K}_\bullet(s)$ and $\mathbf{T}_\bullet$ coincide and thus, after an isomorphism between the free modules defined in each complex, we get the desired isomorphism. □

Given a minimal generating set $\mathbf{G} = \{m_1,\dots,m_r\}$, the set of all its low common multiple is denoted as LCM and is defined as $LCM = \{\operatorname{lcm}(m_i : i \in A \subseteq [r])\}$.

Taylor's resolution give us an upper bound for the Betti numbers in a minimal free resolution since it contains all the possible low common multiple of its minimal generating set and any other multidegree should have Betti number zero, as next theorem shows.

Theorem 2.5.5. *Let I be a monomial ideal with minimal generating set $\{m_1,\dots,m_r\}$. Then if $\mathbf{a}$ is a multidegree such that $\mathbf{x^a} \notin LCM$, then $\beta_{i,\mathbf{a}}(S/I) = 0$, for all $i \in \mathbb{N}$.*

Proof. If $\mathbf{x}^{\mathbf{a}} \in LCM$, by Taylor's resolution, $\beta_{i,\mathbf{a}}(S/I)$ can be non zero. But if $\mathbf{x}^{\mathbf{a}} \notin LCM$, by Hochster's formula (see Theorem 2.3.3) $\beta_{i,\mathbf{a}}(S/I) = 0$ for all $i \in \mathbb{N}$. □

In [10] is given a construction to calculate the Betti numbers of a monomial ideal. This construction is given by the lattice generated by the elements in *LCM* ordered by divisibility. On this lattice is defined a simplicial complex that give us the Betti numbers of the monomial ideal using Hochster Formula.

2.6 Scarf Complex

Taylor's resolution is an exact complex which is highly non minimal. In some cases as a regular sequence, it is minimal but these cases are not common. We can see this as the Taylor's resolution is above of a minimal free resolution. In this subsection we introduce the Scarf Complex of a monomial ideal and, as will see, this sequence of free modules will be always a complex but it is not exact and this complex is always contained in a minimal free resolution. In short words, the Scarf Complex is under a minimal free resolution.

Let I be a monomial ideal with minimal generating set $\mathbf{G} = \{m_1, \ldots, m_r\}$ and let *LCM* be the set of all the low common multiples for all the non empty subset of $\mathbf{G}$. The **Scarf simplicial complex** Δ is the collection of all subset of $\mathbf{G}$ whose least common multiple is unique:

$$\Delta = \{A \subseteq [r] : m_A = m_B \Rightarrow A = B\}.$$

For example, for the monomial ideal $I = \langle x_1x_2, x_1x_3, x_2x_3 \rangle$, the Scarf simplicial complex, where $m_1 = x_1x_2$, $m_2 = x_1x_3$ and $m_3 = x_2x_3$, is $\Delta = \{\{1\}, \{2\}, \{3\}\}$.

With the Scarf simplicial complex we define a sequence of free modules which is a complex as follows: the free modules are $F_t = \bigoplus_{A \in \Delta_t} S(m_A)$, where Δ_t are the elements in Δ with cardinality t. The differentials are defined as in the Taylor's Resolution, that is,

$$d(e_A) = \sum_{j \in A} (-1)^{t+1} \frac{m_A}{m_{A \setminus \{j_t\}}} \cdot e_{A \setminus \{j_t\}}.$$

Remark 2.6.1. *It is not difficult to see that Scarf simplicial complex is in fact a simplicial complex. Then the differentials are well defined.*

This sequence of free modules is known as the **Scarf Complex** or the Taylor Complex supported on Δ, and we will denote this complex as $\mathbf{Sc}_\bullet$.

Next we give an example:

Example 2.6.2. *Let I be a monomial ideal with minimal generating set $\{x_1x_2, x_1x_3, x_2x_3\} = \{m_1, m_2, m_3\}$. The Scarf Complex of I is*

$$\mathbf{Sc}_\bullet : 0 \leftarrow S \xleftarrow{\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 \end{pmatrix}} F_1 \leftarrow 0$$

where $F_1 = S(-x_1x_2) \oplus S(-x_1x_3) \oplus S(-x_2x_3)$. Notice that this complex is not exact because the kernel of the first differential is different from zero which is the second differential.

As we said, $\mathbf{Sc}_\bullet$ is a complex. We put this in next proposition.

Proposition 2.6.3. *The sequence of free modules $\mathbf{Sc}_\bullet$ is a complex.*

As in Taylor's Resolution or Koszul Complex, for some special monomial ideals these complexes are minimal free resolutions.

Definition 2.6.4. *[23, Definition 6.5] A monomial m'* **strictly divides** *another monomial m if m' divides m/x_i for all variables x_i dividing m. A monomial ideal $\langle m_1, \ldots, m_r\rangle$ is* **generic** *if whenever two distinct minimal generators m_i and m_j have the same positive (nonzero) degree in some variable, a third generator m_k strictly divides their least common multiple $\operatorname{lcm}(m_i, m_j)$.*

For instance the ideal $\langle x_1^2x_3, x_1x_2, x_2^2x_3, x_3^2\rangle$ is generic but the ideal $\langle x_1x_2, x_1x_3, x_2x_3\rangle$ is not generic.

There is another definition similar to be generic that is a monomial ideal is **strongly generic**. A monomial ideal that is strongly generic is generic but not the reciprocal. For the precise definition see [23, Chapter 3].

The generic ideals are interesting because they are good with the Scarf complex. Next theorem say what means to be "good".

Theorem 2.6.5. *[23, Theorem 6.13] If I is a generic ideal then the Scarf complex $\mathbf{Sc}_\bullet$ is a minimal free resolution of I.*

Since the Scarf complex is built from the Scarf simplicial complex Δ, its Betti numbers are related with the maximal faces of Δ. Next corollary says what are the Betti numbers of a generic monomial ideal and the resolution of these ideals are independent from the characteristic of the field.

Corollary 2.6.6. *[23, Corollary 6.15] The minimal free resolution of a generic monomial ideal I is independent of the characteristic of the field k.*

The total Betti number $\beta_i(S/I) = \sum_{\mathbf{a}\in\mathbb{N}^n} \beta_{i,\mathbf{a}}(S/I)$ equals the number of i-dimensional faces of its Scarf simplicial complex Δ.

Next we give an example of this resolution.

Example 2.6.7. *Let I be the monomial ideal minimally generated by $\{x_1^2x_3, x_1x_2, x_2^2x_3, x_3^2\}$. The Scarf Simplicial Complex of I is*

$$\Delta = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,4\},\{2,3,4\}\}.$$

Then the Scarf Complex $\mathbf{Sc}_\bullet$ is

$$0 \longleftarrow S/I \xleftarrow{\pi} S \xleftarrow{\begin{pmatrix} x_1^2x_3 & x_1x_2 & x_2^2x_3 & x_3^2 \end{pmatrix}} \begin{matrix} S(-x_1^2x_3) \\ \oplus \\ S(-x_1x_2) \\ \oplus \\ S(-x_2^2x_3) \\ \oplus \\ S(-x_3^2) \end{matrix} \xleftarrow{\begin{pmatrix} -x_2 & -x_3 & 0 & 0 & 0 \\ x_1x_3 & 0 & -x_2x_3 & -x_3^2 & 0 \\ 0 & 0 & x_1 & 0 & x_3 \\ 0 & x_1^2 & 0 & x_1x_2 & x_2^2 \end{pmatrix}} \begin{matrix} S(-x_1^2x_2x_3) \\ \oplus \\ S(-x_1^2x_3^2) \\ \oplus \\ S(-x_1x_2^2x_3) \\ \oplus \\ S(-x_1x_2x_3^2) \\ \oplus \\ S(-x_2^2x_3^2) \end{matrix} \xleftarrow{\begin{pmatrix} x_3 & 0 \\ -x_2 & 0 \\ 0 & x_3 \\ x_1 & -x_2 \\ 0 & x_1 \end{pmatrix}} \begin{matrix} S(-x_1^2x_2x_3^2) \\ \oplus \\ S(-x_1x_2^2x_3^2). \end{matrix}$$

We recall that, since I is a monomial generic ideal, then its Scarf Complex $\mathbf{Sc}_\bullet$ is a minimal free resolution of I. Moreover, notice that for example, $m_{\{1,2\}} = x_1^2x_2x_3$ and $m_{\{2,3,4\}} = x_1x_2^2x_3^2$ and the differentials of this resolution are obtained as in the Taylor resolution.

2.7 Cellular Resolutions

Cellular resolutions were introduced by Dave Bayer and Bernd Sturmfels. These resolutions are supported in a cellular complex and in some more general cases are supported on a CW-complex. Here we define this resolution following the book [23]. As we will see, in Chapter 4 we give a minimal free resolution for the complete bipartite graph, but using before was given a cellular resolution for the edge ideal of a bipartite graph, see [32].

The main idea of these resolutions is to find a labeled cellular complex that keeps all the information of a minimal free resolution. First we define a labeled cellular complex and then we give the resolution obtained from it.

Definition 2.7.1. *[23, Definition 4.2] Suppose X is a* **labeled cell complex***, by which we mean that its r verices have labels that are vectors $\mathbf{a}_1,\ldots,\mathbf{a}_r \in \mathbb{N}^n$. The label on an arbitrary face F of X is the exponent $\mathbf{a}_F$ on the least common multiple $\mathrm{lcm}(\mathbf{x}^{\mathbf{a}_i} : i \in F)$ of the monomial labels $\mathbf{x}^{\mathbf{a}_i}$ on vertices in F.*

The labels of all the faces will be crucial to define the free modules and the differentials of the minimal free resolution that is supported in a cellular complex X.

By convention, the label on the empty face $\emptyset \in X$ is $\mathbf{0} \in \mathbb{N}^n$, which is the exponent of $1 \in S$, the least common multiple of no monomials.

A free resolution $\mathbf{X}_\bullet$ supported in a labeled cell complex X is given by the free modules $F_i = \bigoplus_{F \in X_{i-1}} S(-\mathbf{a}_F)$ where X_{i-1} are the faces of X of dimension $i-1$ and the differentials are given by:

$$d(F) = \sum_{\text{facets } G \text{ of } F} \mathrm{sign}(G,F)\mathbf{x}^{\mathbf{a}_F - \mathbf{a}_G} G.$$

The symbols F and G are faces of X and are thought as basis vectors in degrees $\mathbf{a}_F$ and $\mathbf{a}_G$. The sign for (G,F) equals ± 1 and is part of the data in the boundary map of the chain complex of X.

This complex is a resolution of $I = \langle \mathbf{x}^{\mathbf{a}_v} : v \in X \text{ is a vertex}\rangle$ if the cellular complex is acyclic, as next proposition says.

Proposition 2.7.2. *[23, Proposition 4.5] The cellular free complex $\mathbf{X}_\bullet$ supported on X is a cellular resolution if and only if $X_{\leqslant \mathbf{b}}$ is acyclic over k for all $\mathbf{b} \in \mathbb{N}^n$. When $\mathbf{X}_\bullet$ is acyclic, it is a free resolution of S/I, where $I = \langle \mathbf{x}^{\mathbf{a}_v} : v \in X \text{ is a vertex}\rangle$ is generated by the monomial labels of vertices.*

Although cellular resolutions are a good approximation to a non recursive calculation for a minimal free resolution, not always it is clear how to obtain the cellular complex where it is supported. Moreover, in [31] there is an example of a minimal free resolution that is not supported in a CW-complex, that is, not every minimal free resolution is a cellular resolution.

2.8 Why minimality is important

In this chapter we gave some basics definitions and some ways to calculate a minimal free resolution, an exact complex or a complex of a module. It depends on how we make these calculations. Some of them are recursive and some other are simpler, in any case we always want to know when these calculations yield a minimal free resolution. But why are we always interested in free resolution that be minimal?

The answer are the algebraic invariants of the module, in this case, the algebraic invariants of the monomial ideal.

The importance of minimal free resolutions is that they are the key to find important invariants of the module. We have shown some ways to calculate a resolution, a complex of the module but in most of the cases this constructions are not minimal, which means that we do not obtain all the invariants of the module.

Once we have a minimal free resolution of a module, we are sure that any other minimal free resolution of the module will have the same algebraic properties, because all of them are isomorphic. For instance, with a minimal free resolution we can calculate the projective dimension, that is the length of the minimal resolution, and the regularity that is the width of the resolution. However, there are some invariants that can be calculated without a minimal free resolution. For instance, if we have a free resolution of a module that is not minimal, we also can calculate the Hilbert series that is also an invariant

In the following chapters we develop some theory to calculate minimal free resolutions of some edge ideals in a non recursive way. The goal is to obtain all these invariants without have an explicit minimal free resolution.

Chapter 3

When a graded free complex is exact?

In the early 1960's Irving Kaplansky raised the problem of construct a minimal free resolution of a monomial ideal I in a polynomial ring $S = k[\mathbf{x}]$ over a field k in a nonrecursive way. Give an explicit description of minimal free resolutions of monomial ideals has been a central problem of combinatorial commutative algebra since then. See for instance [8, 13, 23, 25, 26] and the references contained there. Since the 1970's, Hochster's formula [16] has given us a way to calculate the ranks of the free modules in a minimal free resolution of S/I, but it is rare to find a good description of its differentials.

In contrast, it is not strange to guess how a resolution of a monomial ideal looks like in which case is not so difficult to check that it is a complex. However, in general to prove that a complex is exact and minimal is the difficult part. There are various tools which can be used to establish exactness, but in general are not easy to apply. For instance, in [26, Theorem 6.4] is given an homotopic criterion for that a graded complex be exact and in [3] is given a criterion for exactness in a more general setting.

On the other hand, it is common to consider that modules and its free resolutions are graded, which gives some advantages. There are many possible graded structures in S and its modules. For instance, the standard grading on S given by

$$\deg(c\mathbf{x}^{\mathbf{g}}) = \mathbf{g}_1 + \cdots + \mathbf{g}_n \text{ for all } \mathbf{g} \in \mathbb{N}^n \text{ and } c \in k$$

is one of the most used. A little bit less common it is consider the polynomial ring S with the so called standard multigrading induced by $\operatorname{mdeg}(c\mathbf{x}^{\mathbf{g}}) = \mathbf{g}$ for all $\mathbf{g} \in \mathbb{N}^n$ and $c \in k$.

The main purpose in this chapter is to give a more manageable (at least in the monomial case) criterion to check when a free complex of a graded S-module is exact and minimal. The criterion is given in terms of the ranks of the free modules in a free resolution (which can be obtained by Hochster's formula) in each multidegree and the set of columns of the differential matrices.

In the first section we review how can be graded a ring and their modules. We discuss some of the properties that must satisfies in order to get a good grading for our purposes. Briefly speaking, we need that the base monoid of the grading be non cancellative, reduced or torsion free. Moreover, by Grillet Theorem's see [29, Theorem 3.11]) the monoid should be a positive affine monoid. We put emphasis on the properties of the natural order induced over the monoid. We finish by presenting the concepts of non-negative and positive gradings.

The criterion is given in second section. We begin with the following lemma which is a little bit more general.

Lemma 3.2.2. *Let N be a positively graded finitely generated S-module. If Γ is a homogeneous minimal generating set of N and Λ is an irredundant homogeneous subset of N with $|\Gamma_{\mathbf{c}}| = |\Lambda_{\mathbf{c}}|$ for all $\mathbf{c} \in \mathbb{M}$, then there exists an automorphism φ of N such that*

$$\varphi(\Lambda_{\mathbf{c}}) = \Gamma_{\mathbf{c}}$$

and whose restriction on $\Lambda_{\mathbf{c}}$ is a k-linear map for all $\mathbf{c} \in \mathbb{M}$. Moreover, if M is a matrix representation of φ where Λ and Γ are ordered by its multidegree on a nondecreasing way, then it is an upper triangular block matrix.

Here a set of vectors $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ in an S-module N is called irredundant whenever $\gamma_i \notin \langle \gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_s \rangle$ for all $1 \leq i \leq s$. After that we have the following criterion to check when a complex is indeed exact.

Theorem 3.2.6. *If M is a finitely generated positively graded S-module,*

$$\mathbf{F}_\bullet : 0 \leftarrow M \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \leftarrow \ldots \xleftarrow{d_p} F_p \leftarrow 0$$

is a graded minimal free resolution of M and

$$\mathbf{C}_\bullet : 0 \leftarrow M \xleftarrow{\delta_0} C_0 \xleftarrow{\delta_1} C_1 \leftarrow \ldots \xleftarrow{\delta_p} C_p \leftarrow 0$$

is a graded free complex of M such that

$$F_i = \bigoplus_{\mathbf{a} \in A_i \subset \mathbb{M}} S(-\mathbf{a}) = C_i$$

as free graded S-modules and the column sets $C(D_i)$ of the matrix representations D_i of the differentials δ_i are irredundant for all $0 \leqslant i \leqslant p$, then $\mathbf{C}_\bullet$ is isomorphic to $\mathbf{F}_\bullet$.

In the third section we construct a complex for the edge ideal of the complete graph in terms of some of its induced subgraphs as those given in [24] which is equivalent to the given in [9]. We use the criterion to prove that this complex is indeed exact.

3.1 Graded rings and modules

Before talking about graded complexes, we must first define what it means for a ring and module to be graded. Briefly, a grading of a ring or module consists of a decomposition of its additive structure indexed by a monoid. In the first subsection we define, in the most general setting, a grading over a ring and a module.

On the other hand, any monoid is naturally endowed with a preorder, which becomes an order whenever the monoid is commutative, cancellative and reduced. Furthermore, this order induces an order on its homogeneous components and, therefore, also on the elements of the ring or module which we are grading. In the second subsection we establish the conditions that must be satisfies the base monoid in order that this natural order will be a partial well order. This order plays an important role on the study of grading rings or modules.

In the third subsection we concentrate on gradings over the polynomial ring $S = k[\mathbf{x}]$ and their free modules. We finish this section by introducing shifted gradings and homogeneous homomorphisms between grading modules.

3.1.1 Graded rings and modules

A ***grading*** over a ring R is a pair $\Omega = (\mathbb{M}, \{R_\mathbf{a}\}_{\mathbf{a}\in\mathbb{M}})$, which consists of a monoid $\mathbb{M} = (M, \cdot)$ and a sequence $\{R_\mathbf{a}\}_{\mathbf{a}\in\mathbb{M}}$ of subgroups of the additive group of R such that

$$R = \bigoplus_{\mathbf{a}\in\mathbb{M}} R_\mathbf{a} \text{ as additive groups and } R_\mathbf{a}R_\mathbf{b} \subseteq R_{\mathbf{a}\cdot\mathbf{b}} \text{ for all } \mathbf{a}, \mathbf{b} \in \mathbb{M}.$$

That is, a ring is endowed with a grading whenever it can be decomposed into a direct sum of some of its additive subgroups in such a way that the multiplicative structure of the ring is compatible with the monoid operation. We say that $\mathbb{M}$ is the base monoid of the grading. If the ring is commutative, then the monoid which we graded it with must also be commutative. Therefore, since we only deal with commutative rings, from here on out all the base monoids will be commutative and the monoid operation will be denoted by $+$. Although two different gradings can have the same base monoid (see Subsection 3.1.3 for an example), we simply say that a ring R is $\mathbb{M}$-graded.

In a similar way, a module N over an $\mathbb{M}$-graded ring R is $\mathbb{M}$-***graded*** whenever we have a sequence $\{N_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}}$ of subgroups of the additive group of N such that

$$N = \bigoplus_{\mathbf{a}\in\mathbb{M}} N_{\mathbf{a}} \text{ as additive groups and } R_{\mathbf{a}}N_{\mathbf{b}} \subseteq N_{\mathbf{a}+\mathbf{b}} \text{ for all } \mathbf{a},\mathbf{b} \in \mathbb{M}.$$

That is, in a similar way that with a ring, a module is endowed with a grading whenever its additive group can be decomposed as a direct sum of some of its subgroups in such a way that the multiplicative structure of the components of the decomposition of the module and the base ring is compatible with the monoid operation. We recall that when we say that an R-module is $\mathbb{M}$-graded, we are necessarily assuming that the base ring R is also $\mathbb{M}$-graded.

Caution 3.1.1. *The multiplicative condition $R_{\mathbf{a}}N_{\mathbf{b}} \subseteq N_{\mathbf{a}+\mathbf{b}}$ for graded modules corresponds to the multiplicative condition for rings when it is considered as a module over itself.*

Definition 3.1.2. *The additive subgroups $N_{\mathbf{a}}$ in the decomposition of a grading are their* ***homogeneous components*** *and their elements are called* ***homogeneous*** *of degree* $\mathbf{a}$. *We write* $\operatorname{mdeg}_{\Omega}(m) = \mathbf{a}$ *for* $m \in N$ *when* $m \in N_{\mathbf{a}}$.

In a similar way, a subset A is homogeneous whenever its elements are homogeneous. A grading allows decomposing each element of the ring or module on its homogeneous parts, which in many cases makes it more manageable. Several ring and module concepts can be specialized to take advantage of the fact that they are endowed with a grading. For instance, it is not difficult to check that any graded R-module has a homogeneous minimal set of generators, see Proposition 2.1 [26]. Thus, homogeneity is a key concept in graded rings and modules.

Caution 3.1.3. *We recall that the zero (additive identity) of the ring or module belongs to all the homogeneous components of a grading. Thus, the zero is considered of undetermined degree.*

Any ring can be graded in a trivial way over the zero monoid by taking $R_0 = R$. Thus not just any graduation contributes with an interesting additional structure over a ring or a module. In general, it is not required that the homogeneous components of a grading be non zero.

To avoid the uncorrespondence between the base monoid and the grading we introduce the concept of a faithful grading. A grading is called ***faithful*** whenever all its homogeneous components are not equal to zero. We would like to say that every grading $\Omega = (\mathbb{M}, \{R_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}})$ over a ring is equivalent to the faithful grading $\Omega' = (\mathbb{M}', \{R_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}'})$ where $\mathbb{M}'$ is the submonoid of $\mathbb{M}$ given by $\mathbb{M}' = \{\mathbf{a} \in \mathbb{M} : R_{\mathbf{a}} \neq 0\}$. Unfortunately, $\mathbb{M}'$ as defined is not a monoid. Still, we can find a monoid that does that job.

Definition 3.1.4. *A grading $\Omega' = (\mathbb{M}', \{R'_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}'})$ is said to be a* **corefinement** *of $\Omega = (\mathbb{M}, \{R_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}})$ whenever there exists a monoid homomorphism $\psi : \mathbb{M} \to \mathbb{M}'$ such that*

$$R_{\mathbf{a}} \simeq R'_{\psi(\mathbf{a})} \text{ as additive groups for all } \mathbf{a} \in \mathbb{M} \text{ such that } R_{\mathbf{a}} \neq 0$$

and $\psi|_{\{\mathbf{a}\in M : R_{\mathbf{a}}\neq 0\}}$ is a bijection onto $\{\mathbf{b} \in M' : R_{\mathbf{b}} \neq 0\}$.

Example 3.1.5. *The induced $\mathbb{Z}-$grading Ω on $R = k[x]/\langle x^2\rangle$ is given by $R_0 = k$, $R_1 = \langle x\rangle_k$ and $R_n = 0$ for $n \in \mathbb{Z} \smallsetminus \{0,1\}$. We can consider a $\mathbb{Z}_2-$grading Ω' on R given by $R_{[0]} = k, R_{[1]} = \langle x\rangle_k$. Then the canonical projection $\pi : \mathbb{Z} \to \mathbb{Z}_2$ satisfies the conditions for Ω' to be a corefinement of Ω.*

Example 3.1.6. *We can also define a $\mathbb{Z}_4$-grading on $R = k[x]/\langle x^2\rangle$ by $R_{[0]} = k$, $R_{[1]} = \langle x\rangle_k$,$R_{[2]} = R_{[3]} = 0$. This one is also a corefinement of Ω, although it still has null components.*

Proposition 3.1.7. *Every $\mathbb{M}$-grading of a ring has a faithful corefinement.*

Proof. Let Ω be an $\mathbb{M}$-grading of a ring R. Define $\mathbf{a} \sim \mathbf{b}$ if $R_{\mathbf{a}} = R_{\mathbf{b}}$. Then $\sim$ is a congruence over M. Indeed, if $\mathbf{a} \sim \mathbf{b}$ and $\mathbf{c} \sim \mathbf{d}$, then

$$R_{\mathbf{a}+\mathbf{c}} = R_{\mathbf{a}}R_{\mathbf{c}} = R_{\mathbf{b}}R_{\mathbf{d}} = R_{\mathbf{b}+\mathbf{d}}$$

which means $\mathbf{a}+\mathbf{c} \sim \mathbf{b}+\mathbf{d}$. This means that we can define a quotient monoid $\mathbb{M}' = (\mathbb{M}/\sim) \smallsetminus [\mathbf{p}]$, where $R_{\mathbf{p}} = \{0\}$, and the induced grading given by $R_{[\mathbf{a}]} = R_{\mathbf{a}}$ is faithful. $\square$

From here on out all the gradings are assumed to be faithful.

Remark 3.1.8. *Not any ring can be graded in a non-trivial way. For instance, the ring of the integers $\mathbb{Z}$ can not be graded in a non-trivial way because its proper subgroups are of the form $k\mathbb{Z}$ for some $2 \leq k \in \mathbb{N}_+$ and therefore can't be the direct sum of some of these subgroups.*

When either the ring or module that we are grading is finitely generated, then the base monoid that we can use to grade it must also be finitely generated. Thus, since we deal with finitely generated modules, it is desirable that the base monoid will be finitely generated.

Grading imposes some structural restrictions on rings and their modules. For instance, if N is a graded R-module, then $R_{\mathbf{0}}N_{\mathbf{a}} \subseteq N_{\mathbf{a}}$ and therefore $N_{\mathbf{a}}$ is not only an additive group but an $R_{\mathbf{0}}$-module for all $\mathbf{a} \in \mathbb{M}$. In particular, when $R_{\mathbf{0}}$ is a field k, we get that homogeneous components are actually k-vector spaces. Moreover, if additionally N is finitely generated R-module, then their homogeneous components are finitely dimensional k-vector spaces. Thus we can briefly think a finitely generated graded R-module with $R_{\mathbf{0}}$ a field, as a kind of a sheaf of finitely dimensional space vectors over a monoid.

At first sight there is no big difference between the structure imposed by different gradings. For instance, there is not an apparent difference when a ring or module is either graded or multigraded. However, as we show after, depending of the base ring and the module some gradings are more convenient than others.

Here, we are mostly interested in modules with base ring a polynomial ring over a field. In a very particular way, we are interested in the kernel of a homogeneous homomorphism between free S-modules.

To finish this subsection we define when two gradings are equivalent.

Definition 3.1.9. *Two gradings $\Omega = (\mathbb{M}, \{R_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}})$ and $\Omega' = (\mathbb{M}', \{R'_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}'})$ over a ring R are **equivalent**, denoted by $\Omega \sim_\psi \Omega'$, whenever there exists a monoid isomorphism $\psi : \mathbb{M} \to \mathbb{M}'$ such that*

$$R_{\mathbf{a}} \simeq R'_{\psi(\mathbf{a})} \text{ as additive groups for all } \mathbf{a} \in \mathbb{M}.$$

*In a similar way, two gradings $\Pi = (\mathbb{M}, \{N_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}})$ and $\Pi' = (\mathbb{M}', \{N'_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}'})$ over an R-module N with gradings Ω and Ω' over the base ring R are **equivalent**, denoted by $\Pi \sim_\psi \Pi'$, whenever there exists a monoid isomorphism $\psi : \mathbb{M} \to \mathbb{M}'$ such that $\Omega \sim_\psi \Omega'$ and $N_{\mathbf{a}} \simeq N'_{\psi(\mathbf{a})}$ as additive groups for all $\mathbf{a} \in \mathbb{M}$.*

The next very simple example illustrates the concept of equivalence between graded rings.

Example 3.1.10. *Consider the grading over the polynomial ring $k[x]$ in one variable given by*

$$k[x]_t = \begin{cases} \langle x^{t/2} \rangle_k & \text{if } t \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we are considering the variable x with degree two instead of degree one as in the classical standard grading. It is not difficult to check that it is an $\mathbb{N}$-grading which is equivalent to the standard grading (see next subsection for the formal definition) over $k[x]$.

If the base monoid contains an ***idempotent*** $\mathbf{a}$ (that is, an element such that $\mathbf{a}+\mathbf{a}=\mathbf{a}$) and $p \in R_{\mathbf{a}}$, then $p^n \in R_{\mathbf{a}}$ for all $n \in \mathbb{N}$, which is not a desirable property because the grading can not distinguish the elements on the set $\{p^n : n \in \mathbb{N}\}$. In the next subsection, we do a deeper analysis in order to establish which properties of the monoid implies desirable property on the grading using the natural order induced on the base monoid as a guide.

3.1.2 Positive monotone partial well orders on the base monoid

In this subsection we study the possible orders over a monoid that are compatible with its operation, we put a particular emphasis on the natural order induced by the monoid operation. We are mainly interested when these orders are positive, monotone, and partial well orders.

First, any monoid is naturally endowed with a preorder structure over it. More precisely, let $\leq_{\mathbb{M}}$ be the binary relation given by

$$\mathbf{a} \leq_{\mathbb{M}} \mathbf{b} \text{ whenever } \mathbf{a}+\mathbf{c}=\mathbf{b} \text{ for some } \mathbf{c} \in \mathbb{M}.$$

It is not difficult to check that this binary relation is indeed a preorder, that is,

- For all $\mathbf{a} \in \mathbb{M}$, $\mathbf{a} \leq_{\mathbb{M}} \mathbf{a}$ (reflexive) and
- For all $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{M}$, if $\mathbf{a} \leq_{\mathbb{M}} \mathbf{b}$ and $\mathbf{b} \leq_{\mathbb{M}} \mathbf{c}$, then $\mathbf{a} \leq_{\mathbb{M}} \mathbf{c}$ (transitive).

Remark 3.1.11. *Reflexivity is due to the fact that $\mathbf{a}+\mathbf{0}=\mathbf{a}$. In a similar way, transitive is due because if $\mathbf{a}+\mathbf{d}=\mathbf{b}$ for some $\mathbf{d} \in \mathbb{M}$ and $\mathbf{b}+\mathbf{e}=\mathbf{c}$ for some $\mathbf{c} \in \mathbb{M}$, then $\mathbf{a}+\mathbf{d}+\mathbf{e}=\mathbf{b}+\mathbf{e}=\mathbf{c}$.*

As we will see next, several properties of the preorder $\leq_{\mathbb{M}}$ are directly related with properties of the monoid. For instance, an order $\leq$ on $\mathbb{M}$ is called positive whenever $0_{\mathbb{M}} \leq \mathbf{a}$ for all $\mathbf{a} \in M$ (that is, the zero of the monoid is a minimum element under $\leq$ and a monoid is called reduced whenever $\mathbf{a}+\mathbf{b}=0$ if and only if $\mathbf{a}=0$ (that is, a monoid is reduced whenever it has no inverses). The next result shows that these two concepts are equivalent.

Proposition 3.1.12. *A monoid is reduced if and only if the natural order $\leq_{\mathbb{M}}$ is positive.*

Proof. It follows directly from the definitions of reduced monoid and positive order. □

Now, in order that a preorder $\leq$ be an order we need that additionally to be antisymmetric. That is, if $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{b} \leq \mathbf{a}$, then $\mathbf{a}=\mathbf{b}$. On the other hand, a monoid is called cancellative whenever $\mathbf{a}+\mathbf{c}=\mathbf{b}+\mathbf{c}$ implies that $\mathbf{a}=\mathbf{b}$. The next result gives us conditions in such a way that $\leq_{\mathbb{M}}$ be indeed an order.

Proposition 3.1.13. *If a monoid $\mathbb{M}$ is cancellative and reduced, then $\leq_{\mathbb{M}}$ is antisymmetric.*

Proof. Let $\mathbf{a},\mathbf{b} \in M$ such that $\mathbf{a} \leq_{\mathbb{M}} \mathbf{b}$ and $\mathbf{b} \leq_{\mathbb{M}} \mathbf{a}$. Then there exists $\mathbf{c},\mathbf{d} \in M$ such that $\mathbf{a}+\mathbf{c}=\mathbf{b}$ and $\mathbf{b}+\mathbf{d}=\mathbf{a}$. Thus $\mathbf{a}+\mathbf{c}+\mathbf{d}=\mathbf{b}+\mathbf{d}=\mathbf{a}$. Since $\mathbb{M}$ is cancellative, then $\mathbf{c}+\mathbf{d}=0$, which means, since $\mathbb{M}$ is reduced, that $\mathbf{c}=0$, therefore $\mathbf{a}=\mathbf{b}$. □

We say that $\leq_{\mathbb{M}}$ is the ***natural order*** in $\mathbb{M}$. We have a partial converse of previous result.

Proposition 3.1.14. *Let $\mathbb{M}$ be a cancelative monoid. If $\leq_{\mathbb{M}}$ is antisymmetric, then $\mathbb{M}$ is reduced.*

Proof. We will proceed by contradiction. Assume that $\mathbb{M}$ is not reduced, that is, there exists $0 \neq \mathbf{b},\mathbf{c} \in M$ such that $\mathbf{b}+\mathbf{c}=0$. Now, let $0 \neq \mathbf{a} \in M$, by the definition of $\leq_{\mathbb{M}}$,

$$\mathbf{a} \leq_{\mathbb{M}} \mathbf{a}+\mathbf{b} \text{ and } (\mathbf{a}+\mathbf{b}) \leq_{\mathbb{M}} (\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+0=\mathbf{a}.$$

On the other hand, since M is cancellative and $\mathbf{b} \neq 0$, then $\mathbf{a} \neq \mathbf{a}+\mathbf{b}$; a contradiction to the fact that $\leq_{\mathbb{M}}$ is antisymmetric. □

It is not difficult to check that a cancellative monoid does not have idempotents, therefore for our purposes it's desirable for the base monoid to be cancellative and reduced.

On the other hand, we say that an order relation $\leq$ on a monoid $\mathbb{M}$ is ***monotone*** (with respect to the monoid operation) whenever $\mathbf{a} \leq \mathbf{b}$ implies that $\mathbf{a}+\mathbf{c} \leq \mathbf{b}+\mathbf{c}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{M}$. By definition the natural order on $\mathbb{M}$ is monotone.

Corollary 3.1.15. *If $\mathbb{M}$ is cancellative and reduced, then $\leq_{\mathbb{M}}$ is a positive monotone partial order.*

Proof. It follows by Propositions 3.1.13 and 3.1.14. □

Remark 3.1.16. *Given a monotone order $\leq$ on the base monoid $\mathbb{M}$ of a grading of a ring R the binary relation $\leq_R$ given by*

$$r_1 \leq_R r_2 \text{ whenever } r_1 \in R_{\mathbf{a}_1}, r_2 \in R_{\mathbf{a}_2} \text{ and } \mathbf{a}_1 \leq_{\mathbb{M}} \mathbf{a}_2$$

is a monotone order on $(R, \cdot)$.

On the other hand, we say that an order $\leq_2$ is a ***refinement*** of an another order $\leq_1$ whenever $\mathbf{a} \leq_1 \mathbf{b}$ implies that $\mathbf{a} \leq_2 \mathbf{b}$. In other words, if $(\leq)$ is the subset of $M \times M$ that defines the binary relation $\leq$, then $\leq_2$ is a refinement of $\leq_1$ if and only if $(\leq_1) \subseteq (\leq_2)$.

Proposition 3.1.17. *If M is a cancellative reduced monoid, then any positive monotone order $\leq$ is a refinement of the natural order $\leq_{\mathbb{M}}$.*

Proof. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in M$ such that $\mathbf{a}+\mathbf{b} = \mathbf{c}$, that is, $\mathbf{a} \leq_{\mathbb{M}} \mathbf{c}$. Since $\leq$ is positive, then $0 \leq \mathbf{b}$. Moreover, since $\leq$ is monotone, then $\mathbf{a} = \mathbf{a}+0 \leq \mathbf{a}+\mathbf{b} = \mathbf{c}$ and therefore $\leq$ is a refinement of $\leq_{\mathbb{M}}$. □

Remark 3.1.18. *In other words, the natural order $\leq_{\mathbb{M}}$ of a reduced cancellative monoid $\mathbb{M}$ is the minimum element in the set of all positive monotone orders over $\mathbb{M}$ and therefore some of its properties are inherited to any positive monotone order $\leq$ in $\mathbb{M}$.*

Now, we turn our attention to a central concept in order theory: antichains. Elements $\mathbf{a}, \mathbf{b}$ in M such that either $\mathbf{a} \leq \mathbf{b}$ or $\mathbf{b} \leq \mathbf{a}$ are called ***comparable***. Otherwise, they are called ***incomparable***, denoted by $\mathbf{a} \perp \mathbf{b}$. A set of incomparable elements in

M is an ***antichain***. It is not difficult to check that if $\leq_{\mathbb{M}}$ has no infinite antichains, then neither any positive monotone order $\leq$ in $\mathbb{M}$ does have.

On the other hand, a finite set $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_q\} \subseteq M$ generates $\mathbb{M}$ whenever for all $\mathbf{a} \in M$ there exists $\mathbf{r} \in \mathbb{N}^q$ such that $\mathbf{a} = \sum_{i=1}^q r_i \mathbf{g}_i$. In this case, we say that $\mathbb{M}$ is finitely generated. The next result shows that if $\mathbb{M}$ is a reduced cancellative monoid, then concepts of no infinite antichain and finitely generated are equivalent.

Proposition 3.1.19. *If $\mathbb{M}$ is a cancellative reduced monoid, then it is finitely generated if and only if $\leq_{\mathbb{M}}$ does not contain infinite antichains.*

Before we proceed with the proof of Proposition 3.1.19 we will introduce the concept of a representation of an element of the monoid. Given a finite generating set $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_q\}$ of $\mathbb{M}$ a ***G-representation*** of $\mathbf{a} \in M$ is a vector $\mathbf{r} \in \mathbb{N}^q$ such that $\mathbf{a} = \sum_{i=1}^q r_i \mathbf{g}_i$. On the other hand, let $\leq_{\mathbb{N}^q}$ be the natural partial order in the monoid $\mathbb{N}^q$, that is, $\mathbf{r} \leq \mathbf{s}$ if and only if $r_i \leq s_i$ for all $1 \leq i \leq q$.

Proof. ($\Rightarrow$) Let $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_q\}$ be a minimal generating set of $\mathbb{M}$. It is not difficult to check that $\mathbf{a}$ and $\mathbf{b}$ in M are incomparable under $\leq_{\mathbb{M}}$ if and only if any G-representations of $\mathbf{a}$ are incomparable under $\leq_{\mathbb{N}^q}$ with any G-representations of $\mathbf{b}$. Thus, if A is an antichain in $\mathbb{M}$, then

$$\mathbb{A} = \{\mathbf{r}_i : \mathbf{r}_i \text{ is a } G\text{-representation of } \mathbf{a}_i \in A\},$$

is also an antichain of $\mathbb{N}^q$ and therefore by [6, Lemma A] $|A| = |\mathbb{A}|$ is finite.

($\Leftarrow$) We will prove that if G is a minimal generating set for $\mathbb{M}$, then it is an antichain. If $\mathbf{g}_i, \mathbf{g}_j \in G$ are such that $\mathbf{g}_i \leq_{\mathbb{M}} \mathbf{g}_j$, then $\mathbf{g}_i + \mathbf{a} = \mathbf{g}_j$. Since G is a minimal generating of $\mathbb{M}$, then $\mathbf{a} = \sum_{\mathbf{g} \in G} r_{\mathbf{g}} \mathbf{g}$ for some $r_{\mathbf{g}} \in \mathbb{N}$ with $r_{\mathbf{g}} = 0$ for all but a finite number of $\mathbf{g}$.

First, $r_{\mathbf{g}_j} \neq 0$, otherwise G will not be a minimal generating set. In a similar way $\mathbf{a} \neq \mathbf{g}_j$, otherwise $\mathbf{g}_i + \mathbf{g}_j = \mathbf{g}_j$ and since $\mathbb{M}$ is cancellative, then $\mathbf{g}_i = 0$; a contradiction to the fact that G is a minimal generating set. On the other hand, since $\mathbb{M}$ is cancellative, then $\mathbf{g}_i + \sum_{\mathbf{g}_j \neq \mathbf{g} \in G} r_{\mathbf{g}} \mathbf{g} + (r_{\mathbf{g}_j} - 1)\mathbf{g}_j = 0$ with $\sum_{\mathbf{g}_j \neq \mathbf{g} \in G} r_{\mathbf{g}} \mathbf{g} + (r_{\mathbf{g}_j} - 1)\mathbf{g}_j \neq 0$; which is a contradiction to the fact that $\mathbb{M}$ is reduced. Thus all the elements of G are incomparable for $\leq_{\mathbb{M}}$ and therefore it is an antichain. Since $\mathbb{M}$ has no infinite antichains, it means G is finite too, which means G is finitely generated. $\square$

Caution 3.1.20. *In general, the G-representation of an element in $\mathbb{M}$ is not necessarily unique. For instance, a set $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_q\}$ of a monoid $\mathbb{M}$ is a minimal generating set if and only if G generates $\mathbb{M}$ and the G-representation of each $\mathbf{g}_i \in G$ is unique.*

At next, we will show that under some assumptions many properties of the natural induced order of a monoid are inherited from natural order $\leq_{\mathbb{N}^q}$ of $\mathbb{N}^q$.

Lemma 3.1.21. *Let $\mathbb{M}$ be a cancellative reduced finitely generated monoid and $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_q\}$ be a subset of $\mathbb{M}$ such that $\mathbf{g}_i \neq 0$ for all $1 \leq i \leq q$. If $\mathbf{r}$ and $\mathbf{s}$ are two different G-representations of $\mathbf{a} \in M$, then they are incomparable in $(\mathbb{N}^q, \leq_{\mathbb{N}^q})$.*

Proof. We will proceed by contradiction. Assume that $\mathbf{r} \leq_{\mathbb{N}^q} \mathbf{s}$. Thus since $\mathbb{M}$ is cancellative and $\sum_{i=1}^q r_i\mathbf{g}_i = \sum_{i=1}^q s_i\mathbf{g}_i$ we get that $\sum_{i=1}^q (s_i - r_i)\mathbf{g}_i = 0$. Moreover, since $\mathbf{r} \neq \mathbf{s}$, then $s_j - r_j \neq 0$ for at least some $1 \leq j \leq q$; which is a contradiction to the fact that $\mathbb{M}$ is reduced. □

Now, we turn our attention to descending sequences. A ***descending chain*** of $\leq$ is a sequence $\{\mathbf{a}_i\}_{i\in\mathbb{N}}$ of elements such that $\mathbf{a}_{i+1} \leq \mathbf{a}_i$. An order is called a well order whenever has no infinite descending sequences and infinite antichains.

Proposition 3.1.22. *If $\mathbb{M}$ is a finitely generated monoid, then the natural order $\leq_{\mathbb{M}}$ does not contain infinite descending sequences.*

Proof. Let $\{\mathbf{a}_i\}_{i\in\mathbb{N}}$ be a descending sequence for $\leq_{\mathbb{M}}$ and $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_q\}$ be a minimal generating set of $\mathbb{M}$. Thus $\mathbf{a}_0 = \sum_{j=1}^q r_i\mathbf{g}_i$ for some $r_i \in \mathbb{N}$ for all $1 \leq j \leq q$.

On the other hand, if $\mathbf{a} \leq_{\mathbb{M}} \mathbf{b}$, then there exists a G-representation $\mathbf{r_a}$ of $\mathbf{a}$ and a G-representation $\mathbf{r_b}$ of $\mathbf{b}$ such that $\mathbf{r_a} \leq_{\mathbb{N}^q} \mathbf{r_b}$. Thus, since any two representations of $\mathbf{a}_0$ are incomparable and $\mathbb{N}^q$ has no infinite antichains, then only there exists a finite number of elements in $\mathbb{M}$ such that are less or equal to $\mathbf{a}_0$ under $\leq_{\mathbb{M}}$ and we get the result. □

Using previous results we get that the natural order of a cancellative reduced finitely generated monoid is a partial well order.

Corollary 3.1.23. *Let $\mathbb{M}$ be a cancellative reduced monoid. If $\mathbb{M}$ is finitely generated, then $\leq_{\mathbb{M}}$ is a partial well order.*

Proof. It follows by Propositions 3.1.19 and 3.1.22. □

Moreover, we have that any positive monotone order over a cancellative reduced finitely generated monoid is a partial well order.

Proposition 3.1.24. *Let $\mathbb{M}$ be a cancellative reduced monoid. If $\mathbb{M}$ is finitely generated, then any positive monotone order $\leq$ over $\mathbb{M}$ is a partial well order.*

Proof. By Proposition 3.1.17 $\leq$ is a refinement of $\leq_{\mathbb{M}}$. Thus, if $A = \{\mathbf{a}_i\}_{i \in I}$ is an antichain of $\leq$, then it is also an antichain of $\leq_{\mathbb{M}}$ and therefore A must be finite.

Now, let $A = \{\mathbf{a}_i\}_{i \in I \subseteq \mathbb{N}}$ be a descending sequence in $\mathbb{M}$ with respect to $\leq$. Only remains to prove that A must be finite. Let $B_0 = \{i \in I : \mathbf{a}_i \leq_{\mathbb{M}} \mathbf{a}_0\}$ and $C_1 = I - B_0$ and in general

$$B_j = \{i \in C_j : \mathbf{a}_i \leq_{\mathbb{M}} \mathbf{a}_{s_j}\} \text{ where } s_j = \min\{i : i \in C_j\} \text{ and } C_{j+1} = C_j - B_j.$$

Also, let $J = \{k_j : k_j = \min\{i : i \in B_j\}\}$ and $A' = \{\mathbf{a}_j : j \in J\}$ be a subsequence of A.

By construction, the subsequence A' of A is an antichain with respect to $\leq_{\mathbb{M}}$ and therefore finite. Using similar arguments to those given in Proposition 3.1.22 we have that all the sets B_j's are finite. Finally, since $I = \sqcup_{j \in J} B_j$, then I is finite and therefore also A. □

Thus, from here on out we will assume that the base monoid which we use to grade as well as commutative is cancellative, reduced and finitely generated.

Now, we discuss the effect of torsion on gradings. Torsion on monoids generalizes the classical notion of torsion on groups.

Definition 3.1.25. *We say that a monoid $\mathbb{M}$ is **torsion-free** if $k\mathbf{a} = k\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in M$ and $k \in \mathbb{N}_+$, then $\mathbf{a} = \mathbf{b}$. Otherwise, we say that $\mathbb{M}$ has torsion.*

Remark 3.1.26. *We say a monoid is **cyclic torsion-free** whenever $k\mathbf{a} = 0$ for some $k \in \mathbb{N}_+$ with $\mathbf{a} \in M$ implies that $\mathbf{a} = 0$. It is not difficult to check that reduced implies cyclic torsion-free. We recall that a group is torsion free when it is cyclic torsion-free.*

As we mentioned before a desirable property of a grading is that its zero homogeneous component will be a field. Gradings with a base monoid with torsion have the disadvantage that we can not guarantee that the zero homogeneous component is a field. For instance, consider the $\mathbb{Z}_2$-grading over S given by

$$S_0 = \langle\{\mathbf{x}^{\mathbf{b}} : \mathbf{b}_1 + \cdots + \mathbf{b}_n \equiv 0 \pmod 2\}\rangle_k \text{ and } S_1 = \langle\{\mathbf{x}^{\mathbf{b}} : \mathbf{b}_1 + \cdots + \mathbf{b}_n \equiv 1 \pmod 2\}\rangle_k.$$

Even more, in this case, the zero homogeneous component is a vector space of infinite dimension. In some sense, this example results to be a little bit pathological in part because the binary relation $\leq_{\mathbb{Z}_2}$ is not even an order. In general, the torsion in the base monoid does not imply this behaviour, but it's still not enough good for our purposes.

The most studied gradings are ones in which their base monoids are ***positive affine monoids***, that is, finitely generated submonoids of $\mathbb{N}^q$ for some $q \in \mathbb{N}$. The next result say us that any positive affine monoid is isomorphic to a commutative, cancellative, reduced, finitely generated and torsion-free monoid. If we drop the condition of being reduced, we get affine monoids which are finitely generated submonoids of $\mathbb{Z}^q$ for some $q \in \mathbb{N}$.

Theorem 3.1.27 (Grillet Theorem's, see [29] Theorem 3.11). *Let $\mathbb{M}$ be a finitely generated monoid. Then $\mathbb{M}$ is commutative, cancellative, reduced and torsion-free if and only if it is isomorphic to a positive affine monoid.*

Remark 3.1.28. *Any monomial order corresponds to an order induced by gradings of the polynomial ring S with the natural numbers as base monoid and k-vector space $\langle \mathbf{x}^{\mathbf{a}} \rangle_k$ for all $\mathbf{a} \in \mathbb{N}^n$ as homogeneous components.*

We finish this subsection by presenting the main concept of this section. First, an $\mathbb{M}$-grading over a module N is called ***non-negative*** whenever $\mathbb{M}$ can be endowed with a monotone positive partial well order. At next we show an example of a non-negative grading. Let $S = k[\mathbf{x}]$ polynomial ring over a field k and consider the $\mathbb{N}$-grading defined by the decomposition $S = \bigoplus_{d \in \mathbb{N}} T_d$ where $T_d = \langle \{\mathbf{x}^{\mathbf{a}} : \mathbf{a}_n = d\} \rangle_k$. It is not difficult to check that it is faithful non-negative grading. However, it still has the disadvantage that can not distinguish between polynomials in the first $n-1$ variables.

Definition 3.1.29. *A non-negative grading over a polynomial ring over a field is called* ***positive*** *whenever the zero homogeneous component is equal to the field.*

When S is graded by a positive grading we say that it is ***positively graded***. In the next we present an example of a positive grading where the base monoid has torsion. Let $\mathbb{M}$ be the commutative monoid generated by a and b subject to $2a = 2b$. It is not difficult to check that it can be described as the set $M = \{sa : s \in \mathbb{N}\} \sqcup \{sa + b : s \in \mathbb{N}\}$ with an operation given by

$$(s_1 a + t_1 b) + (s_2 a + t_2 b) = (s_1 + s_2 + w)a + (t_1 + t_2)(\text{mod } 2)b \text{ where } w = \left\lfloor \frac{t_1 + t_2}{2} \right\rfloor.$$

Now, if $S_{sa+tb} = \langle \{x^u y^v : u + v = s + t \text{ and } u, v \in \mathbb{N}\} \rangle_k$, then $\Omega = (\mathbb{M}, \{S_{\mathbf{m}}\}_{\mathbf{m} \in \mathbb{M}})$ is an $\mathbb{M}$-grading of the polynomial ring $S = k[x, y]$ over a field k.

In [23, Chapter 8] can be found a similar discussion about which gradings have some desirable properties. Our approach is different of these in the sense that we use the natural order on the base monoid as a guide to deduce which properties must satisfy the base monoid in order to get a partial well order, which is good for our purposes.

Once we have discussed what it means to be graded and their positive monotone partial well orders, we turn our attention to the particular case of how to grade the polynomial ring S.

3.1.3 Grading the polynomial ring S and their free modules.

Now, we will focus on gradings over the polynomial ring $S = k[\mathbf{x}]$ and their free modules.

The most common grading over the polynomial ring S is the $\mathbb{N}$-grading defined by the decomposition

$$S = \bigoplus_{d \in \mathbb{N}} S_d \text{ where } S_d = \langle \{\mathbf{x}^{\mathbf{a}} : \mathbf{a}_1 + \cdots + \mathbf{a}_n = d\} \rangle_k,$$

which is called the ***standard grading***. We recall that, given a subset $\mathbf{A}$ of $\mathbb{N}^n$, $\langle \{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathbf{A}\} \rangle_k$ denotes the additive subgroup of $S = k[\mathbf{x}]$ generated by $\{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathbf{A}\}$. Since $S_{\mathbf{0}} = k$, then in a natural way $\langle \{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathbf{A}\} \rangle_k$ is also endowed with the structure of k-vector space.

Another grading over S is the $\mathbb{N}^n$-grading defined by the decomposition

$$S = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S_{\mathbf{a}} \text{ where } S_{\mathbf{a}} = \langle \mathbf{x}^{\mathbf{a}} \rangle_k,$$

which is called the ***standard multigrading*** over S. It is not difficult to check that when $n = 1$ these two gradings are equivalent. By contrast, when $n \geq 2$ it can be checked that they are not equivalent.

Moreover, the dimension of the k-vector spaces S_d and T_d from the grading defined in the previous subsection are different and therefore they can not be equivalent. Thus a module can have non equivalent gradings with the same base monoid. In a more general setting, S has the following different gradings.

On the other hand, given a multiset $\mathbf{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_t\}$ of $\mathbb{Z}^m$, let $\mathbb{M}_{\mathbf{D}}$ be the affine monoid of $\mathbb{Z}^m$ generated by $\mathbf{D}$ and $T_{\mathbf{m}}^{\mathbf{D}} = \langle \{\mathbf{x}^{\mathbf{a}} : D\mathbf{a} = \mathbf{m}\} \rangle_k$, where D is the matrix whose columns are the vectors in $\mathbf{D}$. It is not difficult to check that $\Gamma_{\mathbf{D}} = (\mathbb{M}_{\mathbf{D}}, \{T_{\mathbf{m}}^{\mathbf{D}}\}_{\mathbf{m} \in \mathbb{M}_{\mathbf{D}}})$ is a grading of S.

Proposition 3.1.30. *Two gradings $\Gamma_{\mathbf{D}}$ and $\Gamma_{\mathbf{D}'}$ are equivalent if and only if the base monoids $\mathbb{M}_{\mathbf{D}}$ and $\mathbb{M}_{\mathbf{D}'}$ are isomorphic.*

Proof. If $\mathbb{M}_{\mathbf{D}}$ and $\mathbb{M}_{\mathbf{D}'}$ are isormophic, then there exists an isomorphism $\psi : \mathbb{M}_{\mathbf{D}} \to \mathbb{M}_{\mathbf{D}'}$ such that $\psi(\mathbf{d_i}) = \mathbf{d}'_{\mathbf{i}}$. Take $\mathbf{x^a}$ in $T^{\mathbf{D}}_m$, it means, $D\mathbf{a} = m$, which is the same that $a_1\mathbf{d}_1 + \dots a_t\mathbf{d}_t = m$. Applying ψ on both sides we have that $a_1\mathbf{d'}_1 + \dots a_t\mathbf{d'}_t = \psi(m)$, that is, $D'\mathbf{a} = \psi(m)$, and thus $\mathbf{x^a}$ is in $T^{\mathbf{D}'}_{\psi(m)}$. Therefore $T^{\mathbf{D}}_m \simeq T^{\mathbf{D}'}_{\psi(m)}$ and $\Gamma_{\mathbf{D}}$ and $\Gamma_{\mathbf{D}'}$ are equivalent. The converse is clear from the definition. $\square$

Remark 3.1.31. *The standard degree is the grading induced by the row matrix $\mathbf{D} = (1 \cdots 1)$ and the standard multigrading is the grading induced by the identity matrix I_n.*

In a more general setting, as the next two results show any grading of S comes from a monoid homomorphism.

Proposition 3.1.32. *If $\Gamma = (\mathbb{M}, \{S_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{M}})$ is a faithful grading of S, then $\phi_\Gamma : \mathbb{N}^n \to \mathbb{M}$ given by $\phi_\Gamma(\mathbf{a}) = \mathbf{m}$ whenever $\mathbf{x^a} \in S_{\mathbf{m}}$ and $\phi_\Gamma(\mathbf{0}) = 0_{\mathbb{M}}$, is a surjective monoid homomorphism.*

Proof. First, ϕ_Γ is well defined because $S_{\mathbf{m}} \cap S_{\mathbf{m}'} = 0$ for all $\mathbf{m}, \mathbf{m}' \in \mathbb{M}$. Now, let $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ and $\mathbf{m}, \mathbf{n} \in \mathbb{M}$ such that $\phi_\Gamma(\mathbf{a}) = \mathbf{m}$ and $\phi_\Gamma(\mathbf{b}) = \mathbf{n}$. That is, $\mathbf{x^a} \in S_{\mathbf{m}}$ and $\mathbf{x^b} \in S_{\mathbf{n}}$. Thus $\mathbf{x^{a+b}} = \mathbf{x^a x^b} \in S_{\mathbf{m}}S_{\mathbf{n}} \subseteq S_{\mathbf{m+n}}$ and therefore $\phi_\Gamma(\mathbf{a}+\mathbf{b}) = \mathbf{m}+\mathbf{n} = \phi_\Gamma(\mathbf{a}) + \phi_\Gamma(\mathbf{b})$. Finally, it is clear that ϕ_Γ is surjective if and only if Γ is faithful. $\square$

Next result sort of a converse of the previous one.

Proposition 3.1.33. *If $\phi : \mathbb{N}^n \to \mathbb{M}$ is a surjective monoid homomorphism and*

$$S_{\mathbf{a}} = \langle \{\mathbf{x^b} : \mathbf{b} \in \phi^{-1}(\mathbf{a})\} \rangle_k \textit{ for all } \mathbf{a} \in \mathbb{M},$$

then the pair $\Phi = (\mathbb{M}, \{S_{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{M}})$ is a faithful $\mathbb{M}$-grading over S.

Proof. Since ϕ is a function, it is easy to see that $S_a \cap S_b = 0$. Then, from the definition of ϕ, we have that $S = \bigoplus_{\mathbf{a}\in\mathbb{M}} S_{\mathbf{a}}$. On the other hand, since ϕ is a monoid homomorphism, then $\phi(\mathbf{c}+\mathbf{d}) = \mathbf{a}+\mathbf{b}$ for all $\mathbf{c} \in \phi^{-1}(\mathbf{a})$ and $\mathbf{d} \in \phi^{-1}(\mathbf{b})$ and therefore

$$\phi^{-1}(\mathbf{a}) + \phi^{-1}(\mathbf{b}) = \{\mathbf{c}+\mathbf{d} : \mathbf{c} \in \phi^{-1}(\mathbf{a}) \text{ and } \mathbf{d} \in \phi^{-1}(\mathbf{b})\} \subseteq \phi^{-1}(\mathbf{a}+\mathbf{b}).$$

Thus $S_{\mathbf{a}}S_{\mathbf{b}} \subseteq S_{\mathbf{a+b}}$ and therefore Φ is an $\mathbb{M}$-grading over S. $\square$

Remark 3.1.34. *The standard grading is induced by the map $\phi : \mathbb{N}^n \to \mathbb{N}$ given by $\phi(\mathbf{a}) = \mathbf{a}_1 + \cdots + \mathbf{a}_n$ and the standard multigrading is induced by the identity map.*

Now, we turn our attention to the gradings over free S-modules. First, we define the classical standard multigrading of S^r.

Definition 3.1.35. *The **standard multigrading** over S^r is the $\mathbb{N}^n$-grading defined by the decomposition*

$$S^r = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} (S_\mathbf{a})^r = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} (S_\mathbf{a} \oplus \cdots \oplus S_\mathbf{a})$$

where $S_\mathbf{a}$ is the homogeneous component in multidegree $\mathbf{a}$ in the standard multigrading over S.

In other words, the standard multigrading over S^r decomposes it into the k-vector spaces $(S_\mathbf{a})^r$ of dimension r over the field k. Its homogeneous elements are vectors with a term of the form $c\mathbf{x}^\mathbf{a}$ in all its entries. For instance, consider $S = k[x,y]$ and S^2 be the free S-module of rank two. In this case the vector $v_1 = (2xy, x) \in S^2$ is not homogeneous because $v_1 = (2xy,0) + (0,x)$ and $(2xy,0) \in S_{xy} \oplus S_{xy}$ while $(0,x) \in S_x \oplus S_x$. For simplicity, sometimes $S_{(a,b)}$ will be denoted by $S_{x^a y^b}$.

The standard multigrading over S^r can be easily generalized by replacing the standard grading on each copy of S.

Definition 3.1.36. *Given a sequence $\Phi = \{(\mathbb{M}, \{S_{\mathbf{a},i}\}_{\mathbf{a}\in\mathbb{M}})\}_{i=1}^r$ of $\mathbb{M}$-gradings over the polynomial ring S, let Γ_Φ be the $\mathbb{M}$-grading over S^r defined by the decomposition*

$$S^r = \bigoplus_{\mathbf{a}\in\mathbb{M}} (S_{\mathbf{a},1} \oplus \cdots \oplus S_{\mathbf{a},r}).$$

*Moreover, the $\mathbb{M}$-grading Γ_Φ is a **positive** $\mathbb{M}$-**grading** over S^r whenever all the M-gradings in Φ over S are positive.*

To finish we introduce shifted gradings and homogeneous homomorphisms between them.

3.1.4 Homogeneous homomorphisms and shifted gradings.

We begin by introducing the shifted grading of a module.

Definition 3.1.37. *Given an $\mathbb{M}$-graded R-module N the R-module N* ***shifted*** *by $\mathbf{a} \in \mathbb{M}$, denoted by $N(-\mathbf{a})$, is the R-module N but generated in degree $\mathbf{a}$. In other words, $N(-\mathbf{a})_{\mathbf{a}+\mathbf{b}} = N_{\mathbf{b}}$ for all $\mathbf{b} \in \mathbb{M}$.*

For simplicity, sometimes $S(-\mathbf{a})$ will be denoted by $S(-\mathbf{x}^{\mathbf{a}})$. For instance, if $S = k[x,y]$ is the S-module with the standard multidegree shifted by $(1,2)$, then $1 \in S(-xy^2)_{xy^2}$ and $xy \in S(-xy^2)_{x^2y^3}$.

In a similar way, given a finite multiset $\mathbf{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_r\}$ in $\mathbb{M}$, the free R-module R^r shifted by $\mathbf{A}$, denoted by $R(-\mathbf{A})$, is the direct sum $\bigoplus_{\mathbf{a}_i \in \mathbf{A}} R(-\mathbf{a}_i)$ of R-modules shifted by each element in $\mathbf{A}$. That is, $R(-\mathbf{A})$ is the free R-module minimally generated by elements of degrees $\mathbf{a}_1, \dots, \mathbf{a}_r$ and its grading is given by

$$R(-\mathbf{A}) = \bigoplus_{\mathbf{b} \in \mathbb{M}} \left(\bigoplus_{1 \le i \le r} R(-\mathbf{a}_i)_{\mathbf{b}} \right) = \bigoplus_{\mathbf{b} \in \mathbb{M}} (R(-\mathbf{a}_1)_{\mathbf{b}} \oplus \cdots \oplus R(-\mathbf{a}_r)_{\mathbf{b}}).$$

Now, we are ready to define homogeneous homomorphisms between graded free S-modules.

Definition 3.1.38. *A homomorphism $\phi : M \to N$ of $\mathbb{M}$-graded R-modules is called* ***graded*** *or* ***homogeneous*** *whenever there exists $\mathbf{c} \in \mathbb{M}$ such that for all $\mathbf{a} \in \mathbb{M}$,*

$$\phi(M_{\mathbf{a}}) \subseteq N_{\mathbf{a}+\mathbf{c}}.$$

For instance, if $\mathbf{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_s\}$ and $\mathbf{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_t\}$ are finite multisets in $\mathbb{M}$, then a homomorphism of R-modules

$$d : R(-\mathbf{A}) \to R(-\mathbf{B})$$

is homogeneous if and only if the columns of its matrix representation Δ are homogeneous in the standard shifted $\mathbb{M}$-grading of $R(-\mathbf{B})$. For instance, if S is graded with the standard multigrading, then the entries of the matrix representation of a homogeneous homomorphism $d : S(-\mathbf{A}) \to S(-\mathbf{B})$ are terms. That is, if $\delta = (\delta_1, \dots, \delta_r)$ is a column of Δ, then each δ_i is a term $e\mathbf{x}^{\mathbf{c}_i}$ with $e \in k$ and $\mathbf{c}_i + \mathbf{b}_i = \mathbf{c}_j + \mathbf{b}_j$ for all $1 \le i, j \le r$. By contrast, this is not necessarily true if we use the standard degree to grade S. Which is a slight but important difference between these two gradings.

3.2 The criterion.

Once we have defined what it means for a free S-module to be graded, we are almost ready to establish a criterion to check when a set of elements of a finitely generated graded free S-module is indeed a minimal generating set. But before we need to introduce the concept of irredundancy, which plays a central role in the criterion.

Definition 3.2.1. *A set of vectors $\Gamma = \{\gamma_1, \dots, \gamma_s\}$ in an S-module is called* **irredundant** *whenever*

$$\gamma_i \notin \langle \gamma_1, \dots, \hat{\gamma}_i, \dots, \gamma_s \rangle \text{ for all } 1 \leq i \leq s.$$

We recall that if Γ is a generating set, then to being irredundant is equivalent to be minimal. Also, irredundancy shares some of the spirit of the condition of being linearly independent in linear algebra. For instance, if Γ is irredundant, then

$$\sum_{j \in J} r_j \gamma_j \neq 0 \text{ for all } r_j \in k \setminus 0 \text{ and } J \subseteq [s] = \{1, \dots, s\}.$$

Checking irredundancy is more complicated than checking linear independence. However, it is simpler than checking that it is a minimal generating set of an S-module. Especially, when the entries of the vectors in Γ are monomials, checking irredundancy is a manageable problem, see for instance Theorem 3.3.15.

From here on out we assume that any S-module is endowed with a non-negative $\mathbb{M}$-grading Ω and $\leq_\Omega$ is the corresponding monotone positive partial well order in $\mathbb{M}$. Now, given any set Γ of a graded S-module N, let

$$\mathbb{M}_\Gamma := \{\mathbf{c} : \Gamma_{\mathbf{c}} \neq \emptyset\} \subseteq \mathbb{M},$$

$\mathrm{Min}_{\leq_\Omega}(\mathbb{M}_\Gamma)$ be its minimal set of elements under $\leq_\Omega$ and $\mathrm{Min}_{\leq_\Omega}(\Gamma) := \bigsqcup_{\mathbf{c} \in \mathrm{Min}_{\leq_\Omega}(\mathbb{M}_\Gamma)} \Gamma_{\mathbf{c}}$, see next commutative diagram

$$\begin{tikzcd} \Gamma \subseteq N \arrow[r, "\Omega"] \arrow[d, dashed] & \mathbb{M}_\Gamma \subseteq \mathbb{M} \arrow[d, "\leq_\Omega"] \\ \mathrm{Min}_{\leq_\Omega}(\Gamma) \arrow[r, "\Omega"] & \mathrm{Min}_{\leq_\Omega}(\mathbb{M}_\Gamma) \end{tikzcd} .$$

We recall that $\mathrm{Min}_{\leq_\Omega}(\mathbb{M}_\Gamma)$ it is well defined and finite because $\leq_\Omega$ has neither infinite descending chains or infinite antichains. Thus, let

$$\Gamma^i = \begin{cases} \mathrm{Min}_{\leq_\Omega}(\Gamma) & \text{if } i = 1, \\ \mathrm{Min}_{\leq_\Omega}(\Gamma \setminus \Gamma^1 \cup \cdots \cup \Gamma^{i-1}) & \text{if } i \geq 2. \end{cases}$$

Since $\mathrm{Min}_{\leq_\Omega}(\Gamma) \neq \emptyset$ for all $\Gamma \neq \emptyset$, then if Γ is finite, then there exists a natural number $c(\Gamma) < \infty$ such that

$$\Gamma = \bigcup_{1 \leq i \leq c(\Gamma)} \Gamma^i \text{ with } \Gamma^i \neq \emptyset \text{ for all } 1 \leq i \leq c(\Gamma).$$

We call the number $c(\Gamma)$ as the complexity number of Γ with respect to the grading Ω. Finally, we are ready to present the main result of this section. From here on out we assume that all the free S-modules are ***positively*** graded by an $\mathbb{M}$-grading Ω.

Lemma 3.2.2. *Let N be a positively graded finitely generated S-module. If Γ is a homogeneous minimal generating set of N and Λ is an irredundant homogeneous subset of N with $|\Gamma_{\mathbf{c}}| = |\Lambda_{\mathbf{c}}|$ for all $\mathbf{c} \in \mathbb{M}$, then there exists an automorphism φ of N such that*

$$\varphi(\Lambda_{\mathbf{c}}) = \Gamma_{\mathbf{c}}$$

and whose restriction on $\Lambda_{\mathbf{c}}$ is a k-linear map for all $\mathbf{c} \in \mathbb{M}$. Moreover, if M is a matrix representation of φ where Λ and Γ are ordered by their multidegrees on a nondecreasing way, then it is an upper triangular block matrix.

Proof. Firstly, given $\lambda \in \Lambda$, let $\Gamma_{<\lambda} = \{\gamma \in \Gamma : \mathrm{mdeg}_\Omega(\gamma) < \mathrm{mdeg}_\Omega(\lambda)\}$, $\Gamma_{>\lambda} = \{\gamma \in \Gamma : \mathrm{mdeg}_\Omega(\gamma) > \mathrm{mdeg}_\Omega(\lambda)\}$, $\Gamma_{=\lambda} = \{\gamma \in \Gamma : \mathrm{mdeg}_\Omega(\gamma) = \mathrm{mdeg}_\Omega(\lambda)\}$ and $\Gamma_{\perp\lambda} = \Gamma \setminus (\Gamma_{<\lambda} \cup \Gamma_{>\lambda} \cup \Gamma_{=\lambda})$. That is, $\Gamma_{\perp\lambda}$ are the elements in Γ that are not comparable with λ.

Since Γ is a generating set of N, then for all $\lambda \in \Lambda \subset N$ there exists r_γ's in S such that

$$\lambda = \sum_{\gamma \in \Gamma} r_\gamma \gamma = \sum_{\gamma \in \Gamma_{<\lambda}} r_\gamma \gamma + \sum_{\gamma \in \Gamma_{=\lambda}} r_\gamma \gamma + \sum_{\gamma \in \Gamma_{\perp\lambda}} r_\gamma \gamma + \sum_{\gamma \in \Gamma_{>\lambda}} r_\gamma \gamma.$$

Note that the r_γ's are not necessarily different from zero and the r_γ's are not necessarily unique. Now, let $h_1, \ldots, h_r$ be the homogeneous components of $\sum_{\gamma \in \Gamma_{>\lambda}} r_\gamma \gamma$. That is, $\sum_{\gamma \in \Gamma_{>\lambda}} r_\gamma \gamma = \sum_{i=1}^r h_i$ where the h_i's are homogeneous and different from zero. Since the γ's are homogeneous, then

$$\mathrm{mdeg}_\Omega(h_i) > \mathrm{mdeg}_\Omega(\lambda) \text{ for all } 1 \leq i \leq r.$$

Thus h_i must be equal to zero for all $1 \le i \le r$ and therefore $\sum_{\gamma \in \Gamma_{>\lambda}} r_\gamma \gamma$ is equal to zero. We remark that if we do not assume that the γ's are homogeneous, then this is not necessarily true.

Using similar arguments we also get that $\sum_{\gamma \in \Gamma_{\perp\lambda}} r_\gamma \gamma = 0$ and since $S^r(-\mathbf{A})$ is positively graded, $r_\gamma \in k$ for all $\gamma \in \Gamma_{=\lambda}$. That is, for all $\lambda \in \Lambda$ there exist $\gamma_1, \ldots \gamma_{s+t} \in \Gamma$ with $\mathrm{mdeg}_\Omega(\gamma_i) < \mathrm{mdeg}_\Omega(\lambda)$ for all $1 \le i \le s$ and $\mathrm{mdeg}_\Omega(\gamma_{s+i}) = \mathrm{mdeg}_\Omega(\lambda)$ for all $1 \le i \le t$, $r_i \in S$ for all $1 \le i \le s$ and $r_{s+i} \in k$ for all $1 \le i \le t$ such that

$$\lambda = \sum_{i=1}^{s} r_i \gamma_i + \sum_{i=1}^{t} r_{s+i} \gamma_{s+i} \text{ with } \sum_{i \in I} r_i \gamma_i \neq 0 \text{ for all } I \subseteq [s+t].$$

We recall that this representation is not necessarily unique. Given one of these representations of $\lambda \in \Lambda$, let $r_{\Gamma,\lambda} \in S^\Gamma$ given by

$$(r_{\Gamma,\lambda})_\gamma = \begin{cases} r_i & \text{if } \gamma = \gamma_i \text{ for some } 1 \le i \le s+t, \\ 0 & \text{otherwise.} \end{cases}$$

Also, let M_φ be the matrix whose columns are indexed by the elements of Λ, whose rows are indexed by the elements of Γ and whose columns are the vectors $r_{\Gamma,\lambda}$. It is not difficult to check that if Λ and Γ are ordered by their multidegrees on a nondecreasing way by $\le_\Omega$, then M_φ is a square upper triangular block matrix with diagonal blocks for each $\mathbf{c} \in \mathbb{M}$ such that $\Gamma_\mathbf{c} \neq \emptyset$. The matrix M_φ can also be seen as an upper triangular block matrix with diagonal blocks for each pair (Γ^i, Λ^i) and this diagonal block with entries in the field k.

Now, let φ be the endomorphism of N given by $\varphi(\gamma) = M_\varphi \mathbf{e}_\gamma$ for all $\gamma \in \Gamma$ where $\mathbf{e}_\gamma \in S^\Gamma$ is the canonical vector given by

$$(\mathbf{e}_\gamma)_{\gamma'} = \begin{cases} 1 & \text{if } \gamma' = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $\varphi(\Lambda_\mathbf{c}) = \Gamma_\mathbf{c}$ and its restriction on $\Lambda_\mathbf{c}$ is a k-linear map for all $\mathbf{c} \in \mathbb{M}$.

When the diagonal blocks of an upper triangular block matrix $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are invertible its inverse is equal to $\begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}$. Thus using induction on the number of diagonal blocks we have that an upper triangular block matrix is invertible if and only if each of its diagonal blocks are invertible. Thus to prove that φ is an automorphism only remains to prove that the diagonal blocks of M_φ are invertible. In order to do that we will use induction on the complexity of Γ. If $c(\Gamma)=1$, then the entries of M_φ are in the field k. Thus, if M_φ is not invertible, then there exist $0\neq \mathbf{r}\in k^\Lambda$ such that $M_\varphi \mathbf{r}=\mathbf{0}$. That is, $\sum_{\lambda\in\Lambda}\mathbf{r}_\lambda\lambda=0$ and therefore Λ is not irredundant which is a contradiction. Now, assume that M_φ is invertible for all the finitely generated submodules N of a shifted free S-module $S^r(-\mathbf{A})$ with $c(\Gamma)\leq i-1$.

Now, we will prove the result when $c(\Gamma)=i$. For all $\lambda\in\Lambda^i$, let

$$\mathbf{r}'_{\Gamma,\lambda}=\begin{cases}\mathbf{r}_{\Gamma,\lambda} & \text{if } \lambda\notin\Lambda^i,\\ 0 & \text{if } \lambda\in\Lambda^i,\end{cases}$$

where $\mathbf{r}_{\Gamma,\lambda}$ is the column of M_φ corresponding to λ and let $\lambda'=\sum_{\gamma\in\Gamma}(\mathbf{r}_{\Gamma,\lambda}-\mathbf{r}'_{\Gamma,\lambda})_\gamma\gamma=\lambda-\sum_{\gamma\in\Gamma}(\mathbf{r}'_{\Gamma,\lambda})_\gamma\gamma$. Let $\Lambda^{<i}=\cup_{j=1}^{i-1}\Lambda^i$, $\Gamma^{<i}=\cup_{j=1}^{i-1}\Gamma^i$ and $M_\varphi^{<i}$ be the submatrix of M_φ obtained by deleting the columns not indexed by the elements in $\Lambda^{<i}$ and the rows not indexed by the elements in $\Gamma^{<i}$. By induction hypothesis $M_\varphi^{<i}$ is invertible. Thus

$$\sum_{\gamma\in\Gamma}(\mathbf{r}'_{\Gamma,\lambda})_\gamma\gamma=\sum_{\lambda\in\Lambda^{<i}}s_\lambda\lambda \text{ for some } s_\lambda\text{'s in } S.$$

Now, let M_φ^i be the diagonal block of M_φ whose columns are indexed by Λ^i and whose rows are indexed by Γ^i. If M_φ^i is not invertible, then there exists $0\neq\mathbf{r}\in k^{\Lambda^i}$ such that $M_\varphi^i\mathbf{r}=\mathbf{0}$, that is, $\sum_{\lambda\in\Lambda^i}\mathbf{r}_\lambda\lambda'=0$. Thus

$$0=\sum_{\lambda\in\Lambda^i}\mathbf{r}_\lambda\lambda'=\sum_{\lambda\in\Lambda^i}\mathbf{r}_\lambda(\lambda-\sum_{\gamma\in\Gamma}(\mathbf{r}'_{\Gamma,\lambda})_\gamma\gamma)=\sum_{\lambda\in\Lambda^i}\mathbf{r}_\lambda(\lambda-\sum_{\lambda\in\Lambda^{<i}}s_\lambda\lambda)=\sum_{\lambda\in\Lambda^i}\mathbf{r}_\lambda\lambda-\sum_{\lambda\in\Lambda^i}\mathbf{r}_\lambda\sum_{\lambda\in\Lambda^{<i}}s_\lambda\lambda.$$

That is, Λ is not irredundant which is a contradiction and therefore we get that M_φ is invertible and φ an automorphism of N. □

Remark 3.2.3. *Lemma 3.2.2 is similar to the Foundational Theorem given in [26, Theorem 2.12]. However, there exists a crucial difference between them, Lemma 3.2.2 does not assume that Γ and Λ are both minimal homogeneous generators of N as in [26, foundational Theorem 2.12]. Actually we deduce that an irredundant homogeneous subset of N is a minimal homogeneous generator of N by comparing the ranks at each degree with a minimal homogeneous generator of N. The first part of the proof of Lemma 3.2.2 uses similar ideas to the used in the graded Nakayama's Lemma.*

We are mostly interested when the S-submodule N is the kernel of a homogeneous homomorphism between graded free S-modules. In this case applying Lemma 3.2.2 we get a criterion to check when a set of elements in the kernel is indeed a minimal generating set.

Corollary 3.2.4. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be multisets in* $\mathbb{M}$ *and* $d : S^r(-\mathbf{A}) \to S^t(-\mathbf{B})$ *be a homogeneous homomorphism of* S*-modules. If* Γ *is a homogeneous minimal generating set of* $\ker(d)$ *and* Λ *is an irredundant homogeneous subset of* $\ker(d)$ *such that*

$$|\Gamma_{\mathbf{c}}| = |\Lambda_{\mathbf{c}}| \text{ for all } \mathbf{c} \in \mathbb{M},$$

then there exists an automorphism φ *of* $\ker(d)$ *such that* $\varphi(\Lambda_{\mathbf{c}}) = \Gamma_{\mathbf{c}}$ *for all* $\mathbf{c} \in \mathbb{M}$ *and whose restriction on each* $\Lambda_{\mathbf{c}}$ *is a* k*-linear map. Moreover, if* M *is the matrix representation of* φ *with respect to* Λ *and* Γ *ordered by their multidegrees on a nondecreasing way, then it is an upper triangular block matrix.*

Proof. It follows directly from Lemma 3.2.2 because $\ker(d)$ is a finitely generated S-submodule of $S^r(-\mathbf{A})$. □

The next example illustrates how it works the previous result.

Example 3.2.5. *Let* $d : S^9(-\mathbf{B}) \to S^6(-\mathbf{A})$ *be the homogeneous (under the standard multidegree) homomorphism whose matrix representation is the matrix* D *given in Figure 3.1.*

$$
\begin{array}{c}
S^6(-\mathbf{A})\\
\shortparallel\\
S(-x_1x_2)\oplus S(-x_3x_4)\\
\oplus\\
S(-x_1x_5)\oplus S(-x_2x_5)\\
\oplus\\
S(-x_3x_5)\oplus S(-x_4x_5)
\end{array}
\xleftarrow{\;D=\begin{pmatrix}
-x_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3x_4\\
0 & 0 & 0 & 0 & 0 & 0 & -x_5 & 0 & x_1x_2\\
x_2 & -x_2 & -x_3 & 0 & -x_4 & 0 & 0 & 0 & 0\\
0 & x_1 & 0 & -x_3 & 0 & -x_4 & 0 & 0 & 0\\
0 & 0 & x_1 & x_2 & 0 & 0 & x_4 & -x_4 & 0\\
0 & 0 & 0 & 0 & x_1 & x_2 & 0 & x_3 & 0
\end{pmatrix}\;}
\begin{array}{c}
S^9(-\mathbf{B})\\
\shortparallel\\
S(-x_1x_2x_5)^2\\
\oplus\\
S(-x_1x_3x_5)\\
\oplus\\
S(-x_2x_3x_5)\\
\oplus\\
S(-x_1x_4x_5)\\
\oplus\\
S(-x_2x_4x_5)\\
\oplus\\
S(-x_3x_4x_5)^2\\
\oplus\\
S(-x_1x_2x_3x_4)
\end{array}
$$

$$
G=\begin{pmatrix}
0 & 0 & 0 & 0 & -x_3x_4\\
x_3 & x_4 & 0 & 0 & 0\\
-x_2 & 0 & x_4 & 0 & -x_2x_4\\
x_1 & 0 & 0 & x_4 & 0\\
0 & -x_2 & -x_3 & 0 & 0\\
0 & x_1 & 0 & -x_3 & 0\\
0 & 0 & 0 & 0 & x_1x_2\\
0 & 0 & x_1 & x_2 & 0\\
0 & 0 & 0 & 0 & x_5
\end{pmatrix}\;\Gamma
\qquad
\Gamma \xleftarrow{\;F=\left(\begin{array}{cccc|c}
1 & 0 & 0 & 0 & x_4\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & -x_1\\
\hline
0 & 0 & 0 & 0 & -1
\end{array}\right)\;} \Lambda
\qquad
L=\begin{pmatrix}
0 & 0 & 0 & 0 & x_3x_4\\
x_3 & x_4 & 0 & 0 & x_3x_4\\
-x_2 & 0 & 0 & x_4 & 0\\
x_1 & 0 & x_4 & 0 & 0\\
0 & -x_2 & 0 & -x_3 & 0\\
0 & x_1 & -x_3 & 0 & x_1x_3\\
0 & 0 & 0 & 0 & -x_1x_2\\
0 & 0 & x_2 & x_1 & -x_1x_2\\
0 & 0 & 0 & 0 & -x_5
\end{pmatrix}\;\Lambda
$$

Figure 3.1: The matrix representation of the first differential $d : S^9(-\mathbf{B}) \to S^6(-\mathbf{A})$ of a minimal free resolution of the edge ideal of the bowtie graph $I_G = \langle x_1x_2, x_2x_5, x_5x_1, x_5x_3, x_3x_4, x_4x_5\rangle$ and two possible minimal generating set for its kernel.

Let Γ *and* Λ *be the columns of the matrices* G *and* L *respectively. It is not difficult to check, using for instance Macaulay2 [12], that* Γ *and* Λ *are homogeneous minimal generator sets of* $\ker(d)$ *with* $|G_{\mathbf{a}}| = |L_{\mathbf{a}}|$ *for all* $\mathbf{a} \in \mathbb{N}^n$*. We recall that the multidegrees of the columns of* G *are* $x_1x_2x_3x_5$, $x_1x_2x_4x_5$, $x_1x_3x_4x_5$, $x_2x_3x_4x_5$ *and* $x_1x_2x_3x_4x_5$ *respectively. And the multidegrees of the columns of* L *are* $x_1x_2x_3x_5$, $x_1x_2x_4x_5$, $x_2x_3x_4x_5$, $x_1x_3x_4x_5$ *and* $x_1x_2x_3x_4x_5$ *respectively.*

It is not difficult to check that F *is the matrix representation of an automorphism as in Corollary 3.2.4. The first diagonal block of* F *is clearly invertible because is a permutation. The second diagonal block is equal to the matrix* (-1)*. Also*

$$
\lambda_5=\begin{pmatrix} x_3x_4\\ x_3x_4\\ 0\\ 0\\ 0\\ x_1x_3\\ -x_1x_2\\ -x_1x_2\\ -x_5\end{pmatrix} = x_4\begin{pmatrix} 0\\ x_3\\ -x_2\\ x_1\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix} - x_1\begin{pmatrix} 0\\ 0\\ 0\\ x_4\\ 0\\ -x_3\\ 0\\ x_2\\ 0\end{pmatrix} - \begin{pmatrix} -x_3x_4\\ 0\\ -x_2x_4\\ 0\\ 0\\ 0\\ x_1x_2\\ 0\\ x_5\end{pmatrix} = x_4\gamma_1 - x_1\gamma_4 - \gamma_5.
$$

and $\lambda_5' = \lambda_5 - (x_4\lambda_1 - x_1\lambda_3) = -\gamma_5$*.*

Now, we apply Corollary 3.2.4 to get a criterion for that a graded free complex be exact and minimal. Before doing this we introduce the concept of complex.

A ***free complex*** of F is a sequence of homomorphisms $\mathbf{F}_\bullet = \{F_i, d_i\}_{i=-1}^{p}$ between free S-modules, which are called differentials,

$$\mathbf{F}_\bullet : 0 \leftarrow F \xleftarrow{\pi = d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} \dots \xleftarrow{d_p} F_p \leftarrow 0$$

such that $d_{i-1}d_i = 0$ for all $1 \leqslant i \leqslant p$. We say that it is graded whenever the modules F_i are graded and the d_i's are homogeneous. Moreover, it is ***exact*** whenever $\mathrm{im}(d_i) = \ker(d_{i-1})$ for all $1 \leqslant i \leqslant p$ in which case it is a ***free resolution*** of F_{-1}. We say that two complexes $\mathbf{F}_\bullet$ and $\mathbf{C}_\bullet$ are isomorphic whenever there exists a series of homogeneous isomomorphisms $T_i : F_i \to C_i$ for all $-1 \leq i \leq p$ such that the following diagram commutes.

$$\begin{tikzcd}
0 & C \arrow[l] \arrow[d, leftrightarrow, "T_{-1}"] & C_0 \arrow[l, "\delta_0"'] \arrow[d, leftrightarrow, "T_0"] & \cdots \arrow[l, "\delta_1"'] & C_{p-1} \arrow[l, "\delta_{p-1}"'] \arrow[d, leftrightarrow, "T_{p-1}"] & C_p \arrow[l, "\delta_p"'] \arrow[d, leftrightarrow, "T_p"] \\
0 & F \arrow[l] & F_0 \arrow[l, "d_0"'] & \cdots \arrow[l, "d_1"'] & F_{p-1} \arrow[l, "d_{p-1}"'] & F_p \arrow[l, "d_p"']
\end{tikzcd}$$

Theorem 3.2.6. *If M is a finitely generated positively graded S-module,*

$$\mathbf{F}_\bullet : 0 \leftarrow M \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \leftarrow \dots \xleftarrow{d_p} F_p \leftarrow 0$$

is a graded minimal free resolution of M and

$$\mathbf{C}_\bullet : 0 \leftarrow M \xleftarrow{\delta_0} C_0 \xleftarrow{\delta_1} C_1 \leftarrow \dots \xleftarrow{\delta_p} C_p \leftarrow 0$$

is a graded free complex of M such that

$$F_i = \bigoplus_{\mathbf{a} \in A_i \subset \mathbb{M}} S(-\mathbf{a}) = C_i$$

as free graded S-modules and the column sets $C(D_i)$ of the matrix representations D_i of the differentials δ_i are irredundant for all $0 \leqslant i \leqslant p$, then $\mathbf{C}_\bullet$ is isomorphic to $\mathbf{F}_\bullet$.

Proof. We will use induction on the homological degree of $\mathbf{F}_\bullet$. Note that T_{-1} is the identity map on M. We begin by proving that T_0 is an isomorphism. Let q be the rank of the free modules F_0 and C_0 and $\{\mathbf{e}_j\}_{1 \leqslant j \leqslant q}$ its canonical basis. Let $\mathbf{G} = \{d_0(\mathbf{e}_j)\}_{1 \leqslant j \leqslant q} := \{g_j\}_{1 \leqslant j \leqslant q}$ and $\mathbf{H} = \{\delta_0(\mathbf{e}_j)\}_{1 \leqslant j \leqslant q} := \{h_j\}_{1 \leqslant j \leqslant q}$. That is, $\mathbf{G}$ and

$\mathbf{H}$ are the columns of the matrix representation of d_0 and δ_0 respectively. Since d_0 and δ_0 are homogeneous maps, $\mathbf{H}$ and $\mathbf{G}$ are homogeneous of the same multidegrees. Thus by Lemma 3.2.2 there exists an isomorphism φ between $\mathbf{G}$ and $\mathbf{H}$ such that $\{\varphi(g_j)\}_{1\leqslant j\leqslant q} = \{h_j\}_{1\leqslant j\leqslant q}$ and T_0 given by

$$T_0(\mathbf{e}_j) = \sum_{l=1}^{q} r_l \mathbf{e}_l \text{ where } \varphi(g_j) = \sum_{l=1}^{q} r_l g_l \text{ with } r_l \in S \text{ for all } 1 \leq j \leq q$$

is an isomorphism between C_0 and F_0.

Now, let's assume that there exist homogeneous isomomorphisms T_j for all $0 \leq j \leq i$ such that previous diagram commutes up to that point. Thus we need to prove that there exists a homogeneous isomomorphism T_{i+1} such that the diagram commutes

$$\begin{tikzcd}
C_{i-1} & C_i \arrow[l, "\delta_i"'] \arrow[d, leftrightarrow, "T_i"] & C_{i+1} \arrow[l, "\delta_{i+1}"'] \arrow[d, leftrightarrow, dashed, "T_{i+1}"] \\
F_{i-1} & F_i \arrow[l, "d_i"'] & F_{i+1} \arrow[l, "d_{i+1}"']
\end{tikzcd}.$$

Since F_{i+1} and C_{i+1} are equal as free graded S-modules, they have the same rank q. Let $\{\mathbf{e}_j\}_{1\leq j\leq q}$ be their canonical basis. Now, let

$$\mathbf{G} = \{d_{i+1}(\mathbf{e}_j)\}_{1\leq j\leq q} := \{\mathbf{g}_j\}_{1\leq j\leq q} \text{ and } \mathbf{H} = \{\delta_{i+1}(\mathbf{e}_j)\}_{1\leq j\leq q} := \{\mathbf{h}_j\}_{1\leq j\leq q}.$$

That is, $\mathbf{G}$ and $\mathbf{H}$ are the columns of the matrix representations of d_{i+1} and δ_{i+1} respectively which are homogeneous. Since $\mathbf{F}_\bullet$ and $\mathbf{C}_\bullet$ are complexes, then $\mathbf{g}_j \in \ker(d_i)$ and $\mathbf{h}_j \in \ker(\delta_i)$ for all $1 \leq j \leq q$. Moreover, since $\mathbf{F}_\bullet$ is exact, then $\mathbf{G}$ is a minimal generator of $\ker(d_i)$.

On the other hand, since $d_iT_i = T_{i-1}\delta_i$, then $d_iT_i(\mathbf{h}_j) = T_{i-1}\delta_i(\mathbf{h}_j) = T_{i-1}(0) = 0$ and therefore $T_i(\mathbf{h}_j) \in \ker(d_i)$ for all $1 \leq j \leq q$. Moreover, since T_i is an automorphism and $\mathbf{H}$ is irredundant, then $\{T_i(\mathbf{h}_j)\}_{1\leq i\leq q}$ is irredundant and homogeneous. Thus, by Corollary 3.2.4 there exists homogeneous isomorphism φ such that $\{T_i(\mathbf{h}_j)\}_{1\leq j\leq q} = \{\varphi(\mathbf{g}_j)\}_{1\leq j\leq q}$. Finally, φ induces an isomorphism T_{i+1} between C_{i+1} and F_{i+1} given by

$$T_{i+1}(\mathbf{e}_j) = \sum_{l=1}^{q} r_l \mathbf{e}_l \text{ where } \varphi(\mathbf{g}_j) = \sum_{l=1}^{q} r_l \mathbf{g}_l \text{ with } r_l \in S \text{ for all } 1 \leq j \leq q.$$

$\square$

This criterion simplify the highly nontrivial part of showing that a free complex is exact and minimal, that is, a minimal free resolution of a module. Now, instead of showing that the equality $\ker(d_i) = \operatorname{im}(d_{i+1})$ holds, we only have to show that a free complex has the correct Betti numbers and each column set of any differential is an irredundant set.

We finish this section with an example of how Theorem 3.2.6 works for a non-monomial ideal.

Example 3.2.7. *Let $I = \langle x_1+x_2, x_2^2+x_1x_3, x_4^3\rangle$ be a homogeneous non-monomial ideal of the polynomial ring $S = k[x_1,x_2,x_3,x_4]$ with the standard grading. Using Macaulay2 [12] we get the minimal free resolution of I in the top line of Figure 3.2. In the bottom line of Figure 3.2 we show a free complex $\mathbf{C}_\bullet$ of I with the columns of its differentials irredundant.*

By Theorem 3.2.6, $\mathbf{C}_\bullet$ is also a minimal free resolution of I as shown in the isomorphisms between $\mathbf{F}_\bullet$ and $\mathbf{C}_\bullet$ given in Figure 3.2.

$$
\begin{tikzcd}[column sep=large]
S/I & S \arrow[l] \arrow[d, "\mathrm{Id}"'] & S(-1)\oplus S(-2)\oplus S(-3) \arrow[l, "\begin{pmatrix} x_1+x_2 & x_2^2+x_1x_3 & x_4^3 \end{pmatrix}"'] \arrow[d, "\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}"] & S(-3)\oplus S(-4)\oplus S(-5) \arrow[l, "\begin{pmatrix} -x_2^2-x_1x_3 & -x_4^3 & x_3x_4^3 \\ x_1+x_2 & 0 & -x_4^3 \\ 0 & x_1+x_2 & x_2^2-x_2x_3 \end{pmatrix}"'] \arrow[d, "\begin{pmatrix} -1&0&0\\0&-1&x_3\\0&0&-1 \end{pmatrix}"] & S(-6) \arrow[l, "\begin{pmatrix} x_4^3 \\ -x_2^2+x_2x_3 \\ x_1+x_2 \end{pmatrix}"'] \arrow[d, "\mathrm{Id}"] & 0 \arrow[l] \\
S/I & S \arrow[l] & S(-1)\oplus S(-2)\oplus S(-3) \arrow[l, "\begin{pmatrix} x_1+x_2 & x_2^2+x_1x_3 & x_4^3 \end{pmatrix}"] & S(-3)\oplus S(-4)\oplus S(-5) \arrow[l, "\begin{pmatrix} x_2^2+x_1x_3 & x_4^3 & 0 \\ -x_1-x_2 & 0 & x_4^3 \\ 0 & -x_1-x_2 & -x_2^2-x_1x_3 \end{pmatrix}"] & S(-6) \arrow[l, "\begin{pmatrix} -x_4^3 \\ x_2^2+x_1x_3 \\ -x_1-x_2 \end{pmatrix}"] & 0 \arrow[l]
\end{tikzcd}
$$

Figure 3.2: Two minimal free resolutions of the ideal $I = \langle x_1+x_2, x_2^2+x_1x_3, x_4^3\rangle$ and isomorphism between them.

3.3 Multigraded minimal free resolution of the complete graph.

One way to prove that a sequence of free S-modules and homomorphism between them is actually a minimal free resolution is to break it down into two steps: first prove that it is a complex and then prove that it is exact. Usually, the second step is the more complicated of these two. In this section, we present the case of the edge ideal of the complete graph to show how Theorem 3.2.6 can be used to accomplish this second step. Finding a minimal free resolution of the edge ideal of the complete graph is one better-understood case. However, in almost all cases only are given their graded Betti numbers. Here we give in an explicit way its differentials.

To the authors knowledge, an explicit minimal free resolution of the edge ideal of the complete graph has been given at least twice before. The first one by Reiner in Welker in 2001. More precisely, in [28] was given a description of a graded minimal free resolution of a matroidal ideal. However, the procedure given is very convoluted. The second one, was given in 2020 by Galetto in [9] using standard Young tableaux with hook shape, this resolution is exactly the same that is given here. However, at difference of these two previous approaches, our method is of general purpose, that is, it is applicable to any monomial ideal for which we have a hunch about a minimal free resolution. For instance in Chapter 4 is used the criterion given in Theorem 3.2.6 to prove that a given complex is indeed a minimal free resolution of the duplication of a monomial ideal.

Briefly, our approach consists of introducing some subset of subgraphs of the complete graph, which we called basis graphs. And then we use them to construct a sequence of free S-modules and homomorphism between them. After that we prove, using the combinatorics of these basis graphs, that it is indeed a complex. Finally, we use Theorem 3.2.6 to prove that this complex is exact and therefore a minimal free resolution. The minimal free resolution presented is as those given in [24].

The **complete graph**, denoted by K_n, is the graph with vertex set $V(K_n) = \{v_1, \dots, v_n\}$ and edge set $E(K_n) = \{v_i v_j : 1 \leq i \neq j \leq n\}$. We recall that its **edge ideal** is the monomial ideal

$$I_{K_n} = \langle \{x_i x_j : v_i v_j \in E(K_n)\} \rangle \subset S.$$

Also, recall that we are considering that the variables in S inherit the ordering of their indices. More precisely, $x_i < x_j$ if and only if $i < j$. Now, let's define basis graphs of the complete graph.

Definition 3.3.1. *Given $A = (i_1, \dots, i_a) \subseteq [n]$ with $i_1 < i_2 < \cdots < i_a$ and $i_1 \neq i \in A$, the* **basis graph** B_A^i *of K_n with* **support** *A and order a is the subgraph of K_n with edge set*

$$E(\mathrm{B}_A^i) = \{v_i v_a : a \in A\} \cup \{v_a v_{a'} : i < a, a' \in A\}.$$

In other words, if $A_{\leq j} = \{a \in A : a \leq j\}$ and $A_{\geq j} = \{a \in A : a \geq j\}$ for all $j \in A$, then B_A^i is such that its induced subgraphs in $A_{\leq i}$ and $A_{\geq i}$ are a star with center in v_i and a complete graph respectively. Thus, we say that B_A^i is **rooted** in v_i. In the next example, we illustrate this concept by presenting basis graphs of K_4 of order four.

Example 3.3.2. *The complete graph with four vertices has three basis graphs with support $A = \{1,2,3,4\}$, see figure 3.3 (b)-(d).*

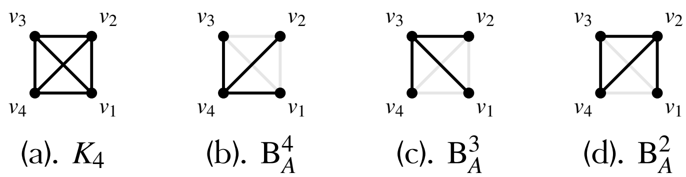


Figure 3.3: The complete graph K_4 and its three possible basis graphs with support $A = \{1,2,3,4\}$.

Remark 3.3.3. *It is not difficult to check that there are $|A|-1$ basis graphs with support $A \subseteq [n]$ and there are $\binom{n}{j}(j-1)$ basis graphs of the complete graph with n vertices of order j.*

The poset of basis graphs of the complete graph under the subgraph relation will play the role of a type of skeleton of a minimal free resolution for its edge ideal. Thus, we turn our attention to characterizing when a basis graph is a subgraph of another one.

Lemma 3.3.4. *If $i \in A \subseteq [n]$ and $j \in C \subseteq [n]$, then*

$$\mathrm{B}_A^i \subseteq \mathrm{B}_C^j \text{ if and only if either } \begin{cases} A \subseteq C & \text{when } i = j, \text{ or} \\ A \subseteq C_{\geq j} & \text{when } i \neq j. \end{cases}$$

Proof. When $i = j$ the result it follows directly from the definition of the basis graphs of K_n. On the other hand when $i \neq j$ we have the following: $(\Rightarrow)$ If there exists $k \in A$ such that $k < j$, then $v_i v_k \in E(\mathrm{B}_A^i)$ and $v_i v_k \notin E(\mathrm{B}_C^j)$ which is a contradiction. $(\Leftarrow)$ It follows because $\mathrm{B}_C^j[C_{\geq j}]$ is a complete graph. □

Now, let $\mathscr{B}_j$ be the set of basis graphs of K_n of order j, $\mathbf{x}^A = \prod_{a \in A} \mathbf{x}^a$ and

$$F_i = \begin{cases} S/I_n & \text{if } i = -1, \\ S & \text{if } i = 0, \\ F_i = \bigoplus_{\mathrm{B}_A^k \in \mathscr{B}_{i+1}} S(-\mathbf{x}^A) & \text{if } 1 \leq i \leq n-1, \end{cases}$$

be a sequence of free S-modules. That is, we have a shifted copy of S in F_i for each basis graph of K_n of order i.

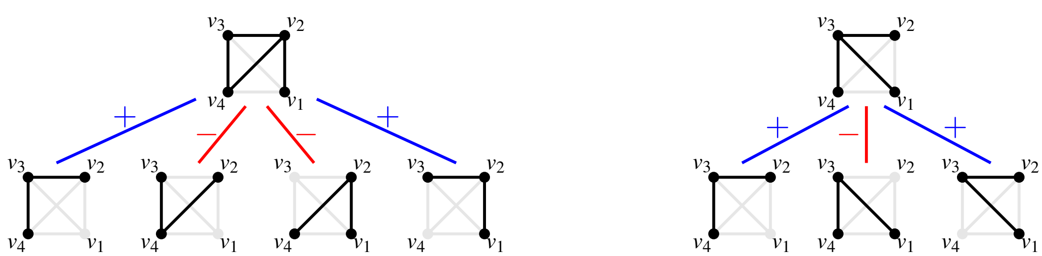
The next ingredient that we need to define the homogeneous homomorphism between the free S-modules F_i and F_{i-1} is a scalar function between the basis graphs of K_n.

Definition 3.3.5. *If* B_A^i *and* B_C^j *are basis graphs of* K_n *with* $C = A \cup \{l\}$*, then the scalar function between them is given by*

$$\sigma(\mathrm{B}_A^i, \mathrm{B}_C^j) = \begin{cases} (-1)^{|A_{\leq l} \setminus i|} & \text{if } i = j, \\ (-1)^{|A_{\leq i}|} & \text{if } l < j < i. \end{cases}$$

Note that the scalar function it is only defined whenever B_A^i is a proper subgraph of B_C^j of order one plus. But, it is convenient to think that scalar function is equal to zero in the other cases. In this case, it only takes the values either of zero, one or minus one, but in general, takes any value in the field k. Moreover, it is not difficult to check that the basis graph B_A^i has $a-1$ basis graph as subgraphs whenever $i \neq i_2$ and $2(a-2)$ whenever $i = i_2$. In the next example, we illustrate this property of basis graphs of K_n.

Example 3.3.6. *Let* $A = (1,2,3,4)$ *and consider the basis graphs* B_A^2 *and* B_A^3*. It is not difficult to check that* B_A^2 *has* $4 = 2(|A|-2)$ *basis graphs and* B_A^3 *has only* $3 = |A|-1$ *basis subgraphs.*



(a). B_A^2 (b). B_A^3

Figure 3.4: Basis subgraphs B_A^2 and B_A^3 and its basis subgraphs. Arrows code scalar function between them.

Now, let $d_k : F_k \to F_{k-1}$ whose matrix representation is given by

$$(d_k)_{\mathrm{B}_A^i, \mathrm{B}_C^j} = \sigma(\mathrm{B}_A^i, \mathrm{B}_C^j)\mathbf{x}^{C \setminus A}.$$

That is, the columns and rows of d_k correspond to elements in $\mathscr{B}_{k+1}$ and $\mathscr{B}_k$ respectively. For instance, the first column of the matrix d_3 given in Example 3.3.7 correspond to the basis graph $K^3_{(1,2,3,4)}$ whose entries different from zero correspond to its basis subgraphs $K^3_{(2,3,4)}$, $K^4_{(2,3,4)}$, $K^2_{(1,2,4)}$ and $K^2_{(1,2,3)}$ as in Example 3.3.6. For simplicity, we say that the column (row) associated to the basis graph B^i_A is the B^i_A column (row). Finally, taking $d_0 = \pi$ as the projection of F_0 over the quotient module F_{-1} we get the sequence

$$\mathrm{K}_\bullet(n) : 0 \leftarrow S/I \xleftarrow{\pi=d_0} S \xleftarrow{d_1} F_1 \xleftarrow{d_2} \cdots \xleftarrow{d_{n-1}} F_{n-1} \leftarrow 0$$

of free S-modules and graded homomorphism between them.

The next example illustrates the construction of $\mathrm{K}_\bullet(4)$.

Example 3.3.7. *For $n = 4$, the sequence of free modules $\mathrm{K}_\bullet(n)$ is given by:*

$$\mathrm{K}_\bullet(4) : 0 \leftarrow S/I \xleftarrow{\pi=d_0} S \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} F_3 \leftarrow 0,$$

where $F_1 = S(-x_1x_3) \oplus S(-x_2x_3) \oplus S(-x_1x_2) \oplus S(-x_1x_4) \oplus S(-x_2x_4) \oplus S(-x_3x_4)$, $F_2 = S(-x_1x_2x_3)^2 \oplus S(-x_1x_2x_4)^2 \oplus S(-x_1x_3x_4)^2 \oplus S(-x_2x_3x_4)^2$, $F_3 = S(-x_1x_2x_3x_4)^3$ and the differentials are given by:

$$d_1 = \emptyset \; \begin{pmatrix} x_1x_3 & x_2x_3 & x_1x_2 & x_1x_4 & x_2x_4 & x_3x_4 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} -x_2 & 0 & 0 & 0 & 0 & -x_4 & 0 & 0 \\ x_1 & x_1 & 0 & 0 & 0 & 0 & 0 & -x_4 \\ 0 & -x_3 & 0 & -x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & x_1 & x_1 & 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_1 & x_2 & x_2 \end{pmatrix} \text{ and } d_3 = \begin{pmatrix} 0 & x_4 & 0 \\ 0 & 0 & x_4 \\ x_3 & 0 & 0 \\ 0 & 0 & -x_3 \\ -x_2 & 0 & 0 \\ 0 & -x_2 & 0 \\ x_1 & 0 & -x_1 \\ 0 & x_1 & x_1 \end{pmatrix}.$$

Once we have a candidate to a minimal free resolution the next step is to prove that it is indeed a complex, that is, the products d_kd_{k+1} are equal to zero. In general, this part it is not that difficult to check. When, as in our case, the sequence of free S-modules and differentials is given in terms of the combinatorics of the monomial ideal, the fact of being a complex relies significantly on this.

Next, we present some basic properties of basis graphs of K_n in which rely on the fact that the sequence free S-modules and differentials is a complex.

Next, lemma tells us that between two basis graphs $B^i_A \subsetneq B^j_C$ of K_n whose respective orders differ by two there are exactly two basis subgraphs.

Lemma 3.3.8. *Let $i \in A \subseteq [n]$, $r \in F \subseteq [n]$ and $j \in C = (j_1, j_2, \dots, j_c) \subseteq [n]$ with $j_1 < j_2 < \cdots < j_c$. If $C = A \cup \{g,h\}$ with $g < h$ and $\mathrm{B}^i_A \subsetneq \mathrm{B}^r_F \subsetneq \mathrm{B}^j_C$, then*

$$\mathrm{B}^r_F \text{ equals one of } \begin{cases} \mathrm{B}^i_{A\cup\{g\}} \text{ or } \mathrm{B}^i_{A\cup\{h\}} & \text{if } i = j, \\ \mathrm{B}^i_{A\cup\{j_2\}} \text{ or } \mathrm{B}^{j_3}_{A\cup\{j_2\}} & \text{if } i > j = j_2 \text{ and } A = \{j_3, \dots, j_c\}, \\ \mathrm{B}^j_{A\cup\{j_1\}} \text{ or } \mathrm{B}^j_{A\cup\{j_2\}} & \text{if } i > j = j_3 \text{ and } A = \{j_3, \dots, j_c\}, \\ \mathrm{B}^j_{A\cup\{j_1\}} \text{ or } \mathrm{B}^i_{A\cup\{h\}} & \text{if } i > j = j_2 \in A. \end{cases}$$

Proof. First, by Lemma 3.3.4, $A \subsetneq F \subsetneq C$ and $j \leq r \leq i$. Thus, since $C = A \cup \{g,h\}$ we get that F equals $A \cup \{g\}$ or $A \cup \{h\}$. Now, if $i = j$, then $r = i$ and by Lemma 3.3.4 we get that B^r_F equals $\mathrm{B}^i_{A\cup\{g\}}$ or $\mathrm{B}^i_{A\cup\{h\}}$. Thus, from here we assume that $i > j$. We divide the proof in two cases: when $j \in A$ and when $j \notin A$.

First, if $j \notin A$, we have that $g = j_1$, $h = j_2 = j$ and $i \geq j_4$. Now, if $F = A \cup \{j_1\}$, then $j_1 r \in E(\mathrm{B}^r_F)$ and $j_1 r \notin E(\mathrm{B}^j_C)$, a contradiction to the fact that $\mathrm{B}^r_F \subsetneq \mathrm{B}^j_C$. Thus $F = A \cup \{j_2\}$ and therefore $r \neq j_2$. In a similar way, if $j_3 < r < i$, then $j_3 i \in E(\mathrm{B}^i_A)$ and $j_3 i \notin E(\mathrm{B}^r_F)$, a contradiction to the fact that $\mathrm{B}^i_A \subsetneq \mathrm{B}^r_F$. Thus r equals i or j_3 and by Lemma 3.3.4 we get that B^r_F equals $\mathrm{B}^i_{A\cup\{j_2\}}$ or $\mathrm{B}^{j_3}_{A\cup\{j_2\}}$.

If $j \in A$ we need to consider two additional cases: when either $j = j_3$ or $j = j_2$. In the first case, it is not difficult to check that $g = j_1$ and $h = j_2$. Moreover, if $r \neq j$ and $j_1 \in F$, then $j_1 r \in \mathrm{B}^r_F$ and $j_1 r \notin \mathrm{B}^j_C$ a contradiction to the fact that $\mathrm{B}^r_F \subsetneq \mathrm{B}^j_C$. A similar argument can be used when $r \neq j$ and $j_2 \in F$. Since $\{j_1, j_2\} \cap F \neq \emptyset$, $j = r$ and by Lemma 3.3.4 we get that B^r_F equals $\mathrm{B}^j_{A\cup\{j_1\}}$ or $\mathrm{B}^j_{A\cup\{j_2\}}$.

Finally, if $j = j_2$, then $g = j_1$. Moreover, if $r \neq j$, then $j_1 \notin F$ otherwise $j_1 r \in \mathrm{B}^r_F$ and $j_1 r \notin \mathrm{B}^j_C$ a contradiction to the fact that $\mathrm{B}^r_F \subsetneq \mathrm{B}^j_C$. Moreover, $r = i$ otherwise $ij \in \mathrm{B}^i_A$ and $ij \notin \mathrm{B}^r_F$ a contradiction to the fact that $\mathrm{B}^i_A \subsetneq \mathrm{B}^r_F$. Thus r can only be either i or j and by Lemma 3.3.4 we get that B^r_F equals $\mathrm{B}^j_{A\cup\{j_1\}}$ or $\mathrm{B}^i_{A\cup\{h\}}$. □

As next result proves the fact that $\mathrm{K}_\bullet(n)$ is indeed a complex relies on the previous property of the basis graphs of K_n.

Proposition 3.3.9. *The sequence* $\mathrm{K}_\bullet(n)$ *of free* S*-modules and differentials is a complex.*

Proof. To prove that $\mathrm{K}_\bullet(n)$ is a complex, we need to prove that the product of two consecutive differentials $d_k d_{k+1}$ is always equal to zero. Indeed, the product of the matrices d_k and d_{k+1} is equal to zero if and only if the dot product of each row of d_k with each column of d_{k+1} is equal to zero.

We recall that the entries in d_k are determined by pairs of basis graphs. More precisely, the entries of the column (row) B_A^i are determined by the basis subgraph of B_A^i and the scalar function between them. Thus, the dot product of rows and columns is also determined by the relation between basis graphs.

For instance, let B_A^i be the basis graph associated to a row of the differential d_k and B_C^j be the basis graph associated to a column of the differential d_{k+1}. An entry of the column B_C^j of d_{k+1} is different from zero if and only if there exists a basis subgraph B_F^r such that $\mathrm{B}_F^r \subsetneq \mathrm{B}_C^j$ and an entry of the row B_A^i of d_k is different from zero if and only if there exists basis a subgraph B_F^r such that $\mathrm{B}_A^i \subsetneq \mathrm{B}_F^r$. Thus, if $\mathrm{B}_A^i \not\subset \mathrm{B}_C^j$, then its dot product is zero because the intersection between the support of the column B_C^j and the support of the row B_A^i is empty. That is, not there exists B_F^r such that $\mathrm{B}_A^i \subsetneq \mathrm{B}_F^r \subsetneq \mathrm{B}_C^j$.

Now, we calculate the dot product of the column B_C^j with row B_A^i with $\mathrm{B}_A^i \subsetneq \mathrm{B}_C^j$ and $|C \setminus A| = 2$. Lemma 3.3.8 establish that there exist four possible cases all of them with only two product in the dot product which are different form zero. Following the notation, in Lemma 3.3.8 next diagrams describe the four possible cases and the associated basis graphs to the entries which give products different from zero.

$$\begin{tikzcd}[column sep=small] & \mathrm{B}^i_C \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{B}^i_{A\cup\{g\}} \arrow[dr, dash] & & \mathrm{B}^i_{A\cup\{h\}} \arrow[dl, dash] \\ & \mathrm{B}^i_A & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathrm{B}^{j_2}_C \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{B}^i_{A\cup\{j_2\}} \arrow[dr, dash] & & \mathrm{B}^{j_3}_{A\cup\{j_2\}} \arrow[dl, dash] \\ & \mathrm{B}^i_A & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathrm{B}^{j_3}_C \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{B}^{j_3}_{A\cup\{j_1\}} \arrow[dr, dash] & & \mathrm{B}^{j_3}_{A\cup\{j_2\}} \arrow[dl, dash] \\ & \mathrm{B}^i_A & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathrm{B}^{j}_C \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{B}^{j}_{A\cup\{j_1\}} \arrow[dr, dash] & & \mathrm{B}^{i}_{A\cup\{h\}} \arrow[dl, dash] \\ & \mathrm{B}^i_A & \end{tikzcd}$$

(a) (b) (c) (d)

Figure 3.5: The four possible cases of products different from zero in the dot product of a row of d_k with a column of d_{k+1}.

Thus, if $\mathrm{B}^{r_1}_{F_1}$ and $\mathrm{B}^{r_2}_{F_2}$ are the unique basis graphs such that $\mathrm{B}^i_A \subsetneq \mathrm{B}^{r_1}_{F_1}, \mathrm{B}^{r_2}_{F_2} \subsetneq \mathrm{B}^j_C$, then the dot product of the row B^i_A and column B^j_C is zero if and only if $r(\mathrm{B}^i_A,\mathrm{B}^{r_1}_{F_1})c(\mathrm{B}^{r_1}_{F_1},\mathrm{B}^j_C) + r(\mathrm{B}^i_A,\mathrm{B}^{r_2}_{F_2})c(\mathrm{B}^{r_2}_{F_2},\mathrm{B}^j_C) = 0$, where $c(\mathrm{B}^i_A,\mathrm{B}^j_C)$ is the entry of the column B^j_C corresponding to B^i_A and $r(\mathrm{B}^i_A,\mathrm{B}^j_C)$ is the entry of the row B^i_A corresponding to B^j_C. For instance, for the first case

$$\sigma(\mathrm{B}^i_{A\cup\{g\}},\mathrm{B}^i_C)\sigma(\mathrm{B}^i_A,\mathrm{B}^i_{A\cup\{g\}})x_g x_h + \sigma(\mathrm{B}^i_{A\cup\{h\}},\mathrm{B}^i_C)\sigma(\mathrm{B}^i_A,\mathrm{B}^i_{A\cup\{h\}})x_g x_h = 0$$

if and only if $\sigma(\mathrm{B}^i_{A\cup\{g\}},\mathrm{B}^i_C)\sigma(\mathrm{B}^i_A,\mathrm{B}^i_{A\cup\{g\}}) + \sigma(\mathrm{B}^i_{A\cup\{h\}},\mathrm{B}^i_C)\sigma(\mathrm{B}^i_A,\mathrm{B}^i_{A\cup\{h\}}) = 0$ and therefore we only need to check that the scalar function on the edges of each square in Figure 3.5 has an odd number of minus signs. Using a similar argument it is not difficult to see that in the other cases it is also only necessary to verify the same condition on the scalar function. This condition is what is called unbalanced scalar function in [24]. Now, by the definition of the scalar function, we have that

$$\sigma(\mathrm{B}^i_A,\mathrm{B}^i_{A\cup\{h\}})\sigma(\mathrm{B}^i_{A\cup\{g\}},\mathrm{B}^i_C) = (-1)^{|A_{\le h}\setminus i|}(-1)^{|(A\cup\{g\})_{\le h}\setminus i|} = -1$$

and

$$\sigma(\mathrm{B}^i_A,\mathrm{B}^i_{A\cup\{g\}})\sigma(\mathrm{B}^i_{A\cup\{h\}},\mathrm{B}^i_C) = (-1)^{|A_{\le g}\setminus i|}(-1)^{|(A\cup\{h\})_{\le g}\setminus i|} = 1$$

because $g < h$. For the second case we have that

$$\sigma(\mathrm{B}^i_A,\mathrm{B}^i_{A\cup\{j_2\}})\sigma(\mathrm{B}^{j_3}_{A\cup\{j_2\}},\mathrm{B}^{j_2}_C) = (-1)^{|A_{\le j_2}\setminus i|}(-1)^{|(A\cup\{j_2\})_{\le j_3}|} = (-1)^0(-1)^2 = 1$$

and

$$\sigma(\mathrm{B}^i_A,\mathrm{B}^{j_3}_{A\cup\{j_2\}})\sigma(\mathrm{B}^i_{A\cup\{j_2\}},\mathrm{B}^{j_2}_C) = (-1)^{|A_{\le i}|}(-1)^{|(A\cup\{j_2\})_{\le i}|} = -1$$

because $j_2 < j_3 < i$ and $A = \{j_3, \dots, j_c\}$. For the third case we have that

$$\sigma(\mathrm{B}^i_A, \mathrm{B}^{j_3}_{A\cup\{j_2\}})\sigma(\mathrm{B}^i_A, \mathrm{B}^{j_3}_{A\cup\{j_1\}}) = (-1)^{|A_{\leq i}|}(-1)^{|A_{\leq i}|} = 1$$

and

$$\sigma(\mathrm{B}^{j_3}_{A\cup\{j_1\}}, \mathrm{B}^{j_3}_C)\sigma(\mathrm{B}^{j_3}_{A\cup\{j_2\}}, \mathrm{B}^{j_3}_C) = (-1)^{|(A\cup\{j_1\})_{\leq j_2}\setminus j_3|}(-1)^{|(A\cup\{j_2\})_{\leq j_1}\setminus j_3|} = (-1)^1(-1)^0 = -1$$

because $A = \{j_3, \dots, j_c\}$. Finally, for the fourth case we have that

$$\sigma(\mathrm{B}^i_A, \mathrm{B}^j_{A\cup\{j_1\}})\sigma(\mathrm{B}^i_{A\cup\{h\}}, \mathrm{B}^j_C) = (-1)^{|A_{\leq i}|}(-1)^{|(A\cup\{h\})_{\leq i}|} = \begin{cases} -1 & \text{if } h < i, \\ 1 & \text{if } h > i, \end{cases}$$

and

$$\sigma(\mathrm{B}^i_A, \mathrm{B}^i_{A\cup\{h\}})\sigma(\mathrm{B}^j_{A\cup\{j_1\}}, \mathrm{B}^j_C) = (-1)^{|A_{\leq h}\setminus i|}(-1)^{|(A\cup\{j_1\})_{\leq h}\setminus j|} = \begin{cases} 1 & \text{if } h < i, \\ -1 & \text{if } h > i. \end{cases}$$

□

The next step is to prove that the complex $\mathrm{K}_\bullet(n)$ is exact. In order to apply Theorem 3.2.6 we first need to calculate the Betti numbers of the edge ideal of the complete graph. We recall the definition of Betti numbers of an ideal.

Definition 3.3.10. *The i-th* **Betti number** *in multidegree* $\mathbf{b}$ *of an ideal I, denoted as $\beta_{i,\mathbf{a}}(I)$, is the number of summands equal to $S(-\mathbf{a})$ in the i-th free module F_i of a minimal free resolution $\mathrm{F}_\bullet = \{F_i, \delta_i\}_{i=-1}^p$ of I.*

We will calculate the Betti numbers by using Hochster's formula, that is, by computing the reduced homology of the lower Koszul simplicial complex.

Definition 3.3.11. *Given a monomial ideal I and $\mathbf{a} \in \mathbb{N}^n$, the* **lower and upper Koszul simplicial complex** *are given by*

$K_{\mathbf{a}}(I) = \{$*squarefree vectors* $\tau \leqslant \mathbf{a} : \mathbf{x}^{\mathbf{a}-\mathbf{1}+\tau} \notin I\}$ *and* $K^{\mathbf{a}}(I) = \{$*squarefree vectors* $\tau : \mathbf{x}^{\mathbf{a}-\tau} \in I\}$.

Theorem 3.3.12 (Hochster's formula)**.** *If $\beta_{i,\mathbf{g}}(I)$ is the i-th Betti number of a monomial ideal I in multidegree* $\mathbf{a}$, *then*

$$\beta_{i,\mathbf{a}}(I) = \begin{cases} \dim_k \tilde{H}_{i-1}(K^{\mathbf{a}}(I); k), \\ \dim_k \tilde{H}^{n-i-2}(K_{\mathbf{a}}(I); k), \\ \dim_k \tilde{H}^{i-1}(K^{\mathbf{a}}(I); k), \\ \dim_k \tilde{H}_{n-i-2}(K_{\mathbf{a}}(I); k). \end{cases}$$

Proof. A version of this classical formula appears by first time in [16]. Several of these versions can be found in the literature, for instance, the first two can be found as [23, Theorems 1.34 and 5.11] respectively. Last two versions it follows by applying the Universal Coefficient Theorem for cohomology to the first two. □

Before we calculate the Betti numbers, we will state some notation. Given a vector $\mathbf{a} \in \mathbb{N}^n$, we set $\operatorname{supp}(\mathbf{a}) = \{i \in [n] : \mathbf{a}_i \neq 0\}$ and given $A \subseteq [n]$ and a monomial ideal I, we set $I(A) = \langle \mathbf{x}^{\mathbf{a}} \in I : \operatorname{supp}(\mathbf{a}) \subseteq A \rangle$. Finally, let e_i be the i-th vector in the canonical basis of $\mathbb{R}^n$, that is, the vector with a 1 in position i and 0 in the other positions.

Proposition 3.3.13. *If $\mathbf{a} \in \{0,1\}^n$ and $\mathscr{B}_{\mathbf{a}}$ is the set of basis graphs of K_n with base $A = \operatorname{supp}(\mathbf{a})$, then*

$$\beta_{|A|-1,\mathbf{a}}(I_{K_n}) = |A| - 1 = |\mathscr{B}_{\mathbf{a}}|.$$

Proof. It is not difficult to see that $K_{\mathbf{a}}(I_{K_n}) = K_{\mathbf{a}}(I_{K_n}(A))$ and

$$K_{\mathbf{b}}(I_{K_n}(A)) = \begin{cases} \{e_i : i \in A\} & \text{if } \mathbf{b} = \mathbf{a}, \\ \{\mathbf{0}\} & \text{if } \mathbf{b} \neq \mathbf{a}. \end{cases}$$

Thus, $\tilde{H}_i(K_{\mathbf{a}}(I);k)$ is equal to zero with exception for $i = 0$ where its dimension is equal to the number of connected components of $K_{\mathbf{a}}$ minus one. Therefore, by Hochster's formula, we conclude the result. □

Remark 3.3.14. *The Betti numbers of the edge ideal of a complete graph are very easy to calculate and it has been done several times before.*

Now, we prove that the set of columns of the differentials of $\mathrm{K}_\bullet(n)$ are irredundant.

Theorem 3.3.15. *The columns of the differentials of the complex $\mathrm{K}_\bullet(n)$ are irredundant.*

Proof. We will proceed by contradiction, that is, we will assume that the columns $\{c_1, \ldots, c_r\}$ of a differential d_i in $\mathrm{K}_\bullet(n)$ are redundant. Without loss of generality, we can assume that

$$c_1 = s_2c_2 + \cdots + s_rc_r \text{ with } s_i \in S \text{ for all } 2 \leqslant j \leqslant r.$$

Let $h_1, \ldots, h_t$ homogeneous such that $\sum_{i=2}^{r} s_ic_i = \sum_{i=1}^{l} h_i$. Since the c_i's are homogeneous of multidegree $\mathbf{x}^A$ with $|A| = i+1$ for some $A \subseteq [n]$, then $h_i = 0$ whenever

$\text{mdeg}(h_i) \neq \text{mdeg}(c_1)$ and if $\text{mdeg}(h_i) = \text{mdeg}(c_1)$, then $\text{mdeg}(h_i) = \sum_{j=1}^{t} s_{i_j} c_{i_j}$ with $\text{mdeg}(c_{i_j}) = \text{mdeg}(c_1)$ and $s_{i_j} \in k$ for all $1 \leq j \leq t$.

Thus without loss of generality we can assume that

$$c_1 = s_2 c_2 + \dots s_t c_t \text{ where } s_j \in k \text{ and } \text{mdeg}(c_j) = \text{mdeg}(c_1) \text{ for all } 2 \leqslant j \leqslant t.$$

Now, let $i_u \in A$ and $i_u \neq \min(\mathrm{A})$ and $\mathrm{B}_A^{i_1}, \dots, \mathrm{B}_A^{i_t}$ be the basis graphs associated to the columns $c_1, \dots, c_t$, respectively. By Lemma 3.3.4, $\mathrm{B}_{A \smallsetminus i_u}^{i_1}$ is a subgraph of $\mathrm{B}_A^{i_1}$ and not a subgraph of $\mathrm{B}_{A \smallsetminus i_u}^{i_j}$ for $2 \leqslant j \leqslant t$. Therefore $(c_1)_{\mathrm{B}_{A \smallsetminus i_u}^{i_1}} \neq 0$ and $(c_i)_{\mathrm{B}_{A \smallsetminus i_u}^{i_1}} = 0$ for all $2 \leqslant j \leqslant t$ which is a contradiction to the fact that $c_1 = s_2 c_2 + \dots s_t c_t$. □

Finally, putting all together we can conclude that the complex $\mathrm{K}_\bullet(n)$ is exact.

Corollary 3.3.16. *The complex* $\mathrm{K}_\bullet(n)$ *is a minimal free resolution of the edge ideal of the complete graph with n vertices.*

Proof. It follows from Theorems 3.2.6, 3.3.15 and Proposition 3.3.13. □

Chapter 4

A minimal free resolution of the duplication

Current studies in minimal free resolutions are focus in calculate the Betti numbers and the differentials in a minimal free resolutions with simple computations or in a non recursive way. In some cases this calculations are translate to make other calculations that can be the same or more difficult than the original problem, or it can be still a recursive calculation.

In this chapter we aboard the problem in other form: we will construct a minimal free resolution of certain monomial ideal from another minimal free resolution that is known. The new minimal free resolution will have as sub-resolution the minimal free resolution that is known. This process can be done with few calculations and in a non recursive way.

In the first section we introduce the basic definitions and the concept of duplication for monomial ideals and the copy of monomials, matrices, sets and vectors.

In the second section we make the duplication of a minimal free resolution to obtain a minimal free resolution of the duplication ideal. The main result of this chapter is the following:

Theorem 4.2.15. *Let $I_{G^\diamond}$ be the duplication of a monomial ideal I_G. Then the free resolution $\mathbf{F}_\bullet^\diamond$ is a minimal free resolution of $I_{G^\diamond}$.*

In third section we give two new representations of a minimal free resolution: a combinatorial resolution and a poset resolution. For these two representations we also make the duplication and we show the advantages and disadvantages of each representaion.

In final section, we give a minimal free resolution in the three representations for the complete multipartite graph. This resolution is very explicit and it is calculated using the duplication of some subgraphs.

4.1 The duplication of a monomial ideal

Given $n,m \in \mathbb{N}_+$, let $\mathbf{x} := \{x_1, x_2, \ldots, x_n\}$, $\mathbf{y} := \{y_1, y_2, \ldots, y_m\}$ and $\mathbf{z} := \mathbf{x} \sqcup \mathbf{y}$ be set of variables.

Given a set of variables $\mathbf{w}$ of cardinality $c = |\mathbf{w}|$ there exists a bijection between $[c]$ and $\mathbf{w}$ given by $i \leftrightarrow w_i$. For simplicity sometimes we identify x_i with i and vice-versa. In a similar way, there exists a bijection between $\mathbb{N}^c$ and $\mathbb{N}^{\mathbf{w}}$ given by

$$\mathbf{g}_i \leftrightarrow \mathbf{g}_{w_i} \text{ for all } \mathbf{g} \in \mathbb{N}^c \text{ and } i \in [c].$$

Given a set of variable $\mathbf{w}$, let $k[\mathbf{w}]$ be the polynomial ring over the field k with variables $\mathbf{w}$. We denote as $\mathrm{Mon}(S)$ the set of monomials in the polynomial ring S. Given $\mathbf{g} \in \mathbb{N}^c$, let $\mathbf{w}^{\mathbf{g}} := \prod_{i=1}^{c} w_i^{\mathbf{g}_i} \in \mathrm{Mon}(k[\mathbf{w}])$. This map is a bijection between $\mathbb{N}^c$ and $\mathrm{Mon}(k[\mathbf{w}])$. For simplicity sometimes we identify $\mathbf{x}^{\mathbf{g}}$ with $\mathbf{g}$. Abusing the notation we write $\mathbf{w}^R = \prod_{w_i \in R} w_i$ for $R \subseteq \mathbf{w}$.

Next definition is fundamental in this chapter. In particular, we will give the definition for the case of monomials but later we also give the definition in terms of graphs.

Definition 4.1.1. *Let $S = k[\mathbf{x}]$ and $T = k[\mathbf{y}]$. Given $\mathbf{m} \in \mathrm{Mon}(S[y])$ and $\mathbf{g} \in \mathrm{Mon}(T)$, the* **g-copy** *of $\mathbf{m}$ is the monomial $\mathbf{m}(y \to \mathbf{g}) \in S[\mathbf{y}]$ obtained from $\mathbf{m}$ by replacing the variable y in $\mathbf{m}$ by the monomial $\mathbf{g}$. When $\mathbf{g} = \mathbf{y}^R$, with $R \subseteq \mathbf{y}$, for simplicity we write $\mathbf{m}^R$ instead of $\mathbf{m}(y \to \mathbf{y}^R)$.*

Remark 4.1.2. *The variable y of $S[y]$ is a distinguished variable that it is not in $\mathbf{x}$ nor $\mathbf{y}$. Moreover, if $y \nmid \mathbf{m}$, then $\mathbf{m}(y \to \mathbf{g}) = \mathbf{m}$.*

Next example illustrates previous definition.

Example 4.1.3. *Let $\mathbf{m} = x_1 x_2^3 y^2$ be a monomial in $S[y]$ and $\mathbf{g} = y_1 y_2^2$ be a monomial in T.*

The $\mathbf{g}$-copy of $\mathbf{m}$ is

$$\mathbf{m}(y \to y_1 y_2^2) = x_1 x_2^3 (y_1 y_2^2)^2.$$

In case that $\mathbf{g} = y_1y_3$*, the* $\mathbf{g}$*-copy of* $\mathbf{m}$ *is*

$$\mathbf{m}^{y_1y_3} = x_1x_2^3(y_1y_3)^2.$$

This copy operation can be easily extended to sets and multisets in $S[y]$. For instance, if $\mathbf{M}$ is a multiset in $\mathrm{Mon}(S[y])$ and $R \subseteq \mathbf{y}$, let

$$\mathbf{M}^R = \mathbf{M}(y \to \mathbf{y}^R) := \{\mathbf{m}^R : \mathbf{m} \in \mathbf{M}\}$$

be a multiset of $\mathrm{Mon}(S[\mathbf{y}])$. The multiset $\mathbf{M}^R$ is called the R-**copy** of $\mathbf{M}$. It also can be extended to matrices with entries in $S[y]$, vectors in $\mathbb{N}^{\mathbf{w}}$, etc. For instance, if $\mathbf{v} \in \mathbb{N}^{n+1}$ and $R \subseteq [m]$, let $\mathbf{v}^R \in \mathbb{N}^{n+m}$ given by

$$(\mathbf{v}^R)_j = \begin{cases} \mathbf{v}_j & \text{if } 1 \leqslant j \leqslant n, \\ \mathbf{v}_{n+1} & \text{if } j - n \in R, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.1.4. *Let* $\mathbf{M} = \{x_1x_2, x_1x_2^2y^2, x_1x_2^2y^2\}$ *be a multiset in* $S[y]$ *and* $R = \{y_1, y_3\}$ *a subset of* $\mathbf{y}$*. Then* $\mathbf{M}^R = \{x_1x_2, x_1x_2^2(y_1y_3)^2, x_1x_2^2(y_1y_3)^2\}$ *is the* $\mathbf{g}$*-copy of* $\mathbf{M}$ *with* $\mathbf{g} = \mathbf{y}^R$*.*

Now let $\mathbf{v} = (2,1,3)$ *be a vector in* $\mathbb{N}^3$ *and* $R = \{y_2, y_3\} \subset \mathbf{y}$*, then* $\mathbf{v}^R = (2,1,0,3,3)$ *is the copy of* $\mathbf{v}$*.*

Given a matrix D *with entries in* $S[y]$ *and* $R = \{y_1, y_2\}$*, the* R*-copy of* D *is:*

$$\begin{pmatrix} -y^2 & 0 \\ x_2^2 & -x_1y \\ 0 & x_2 \end{pmatrix} \qquad \begin{pmatrix} -(y_1y_2)^2 & 0 \\ x_2^2 & -x_1y_1y_2 \\ 0 & x_2 \end{pmatrix}$$

Matrix D. The R-copy D^R .

Figure 4.1: The copy of a matrix.

The $\mathbf{g}$-copy is, as its name says, a copy of the original object. This definition is motivated by the duplication of a vertex in graphs: in this operation, we make a copy of the graph with a new vertex and we join (as set of edges) the original graph with the copied graph to obtain the duplication. Using this as motivation, we give the copy of a monomial ideal as follows.

Definition 4.1.5. *Given a set of monomials G generating an ideal I of $S[y]$ and a subset $R \subseteq \mathbf{y}$ of cardinality m, let*

$$G^{\diamond(R)} = \bigcup_{y_i \in R} G^{y_i} \subset \mathrm{Mon}(S[\mathbf{y}])$$

be the **duplication of** *G and $I_{G^\diamond}$ is the* **duplication of the monomial ideal** *generated by G. In other words, the duplication of a monomial ideal is the union of the y_i-copies for every $y_i \in R$.*

As we see later, the duplication is closely related to the duplication of graphs. In fact, this is a generalization of this operation.

For simplicity, we write $G^\diamond$ instead of $G^{\diamond(\mathbf{y})}$. Moreover, let $G[\emptyset] = \{\mathbf{g} \in G : y \not| \mathbf{g}\}$ and

$$G[R] = G^{\diamond(R)} \smallsetminus G[\emptyset] \text{ for all } \emptyset \neq R \subseteq \mathbf{y}.$$

In other words, $G[R] = \{\mathbf{g} \in G^{\diamond(R)} : y_r | \mathbf{g} \text{ for some } r \in R\}$.

Remark 4.1.6. *Note that $G[R] \neq G^R$. For instance, $G[\emptyset] \subseteq G^\emptyset$ with $G[\emptyset] = G^\emptyset$ if and only if $G \subset S$. In general there not exists this type of relation between $G[R]$ and G^R. In particular, $G[y_i] = G^{y_i} \smallsetminus G[\emptyset]$.*

Next example illustrates previous definitions.

Example 4.1.7. *Let $G = \{x_1x_2, x_1^3y, x_2x_3y^2\}$ be a set of monomials in $S[y]$ and $R = \{y_1, y_3\} \subseteq \mathbf{y}$.*

The duplication of G is:

$$\begin{aligned} G^{\diamond(R)} &= G^{y_1} \cup G^{y_3} \\ &= \{x_1x_2, x_1^3y_1, x_2x_3y_1^2\} \cup \{x_1x_2, x_1^3y_3, x_2x_3y_3^2\} \\ &= \{x_1x_2, x_1^3y_1, x_2x_3y_1^2, x_1^3y_3, x_2x_3y_3^2\}. \end{aligned}$$

Moreover, $G[\emptyset] = \{x_1x_2\}$ and $G[R] = \{x_1^3y_1, x_2x_3y_1^2, x_1^3y_3, x_2x_3y_3^2\}$.

Also notice that $G^R = \{x_1x_2, x_1^3y_1y_3, x_2x_3(y_1y_3)^2\} \neq G[R]$.

Remark 4.1.8. *Notice that any duplication $G^{\diamond(R)}$ with $R = \{y_j\}$ of any set of monomials G of $S[y]$ are isomorphic because $G^{\diamond(R)}$ is a relabeling of the monomials in G.*

We finish this section with the following description of a matrix that will be useful in next section.

Given a matrix $D \in \mathrm{M}_{n,m}(S[y])$ whose rows and columns are indexed by multisets $\mathbf{N}$ and $\mathbf{M}$ in $\mathrm{Mon}(S[y])$ respectively, let A, B and C be matrices with entries in $S[y]$ such that

$$D = \begin{array}{c c} & \begin{array}{cc} \mathbf{M}[\emptyset] & \mathbf{M}[y] \end{array} \\ \begin{array}{c} \mathbf{N}[\emptyset] \\ \mathbf{N}[y] \end{array} & \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \end{array}$$

where $\mathbf{K}[\emptyset] = \{\mathbf{m} \in \mathbf{K} : y \not| \mathbf{m}\}$ and $\mathbf{K}[y] = \{\mathbf{m} \in \mathbf{K} : y | \mathbf{m}\}$ for any multiset $\mathbf{K}$ of monomials in $S[y]$.

In next section we introduce the main topic of this chapter.

4.2 The duplication of a minimal free resolution

We start this section by making a review of the concept of a minimal free resolution. We recall that the purpose of this chapter is to give a minimal free resolution of the duplication of a monomial ideal.

An $\mathbb{N}^n$-graded, or simply multigraded, free resolution of a monomial ideal I is a sequence of $\mathbb{N}^n$-graded free modules which is an exact complex and its first non trivial module is S/I. More in detail, an $\mathbb{N}^n$-complex is a sequence $\mathbf{F}_\bullet = \{F_i, d_i\}_{i\in\mathbb{Z}}$ of $\mathbb{N}^n$-graded free modules

$$F_i = \bigoplus_{\mathbf{a}\in\mathbf{A}_i} S(-\mathbf{a}),$$

where $\mathbf{A}_i$ is a multiset of $\mathbb{N}^n$, $S(-\mathbf{a})$ is the free S-module obtained by shifting S by the multidegree $\mathbf{a}$, a sequence of homogeneous $\mathbb{N}^n$-graded maps $d_i : F_i \to F_{i-1}$ such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. Last condition implies that $\mathrm{im}(d_{i+1}) \subseteq \ker(d_i)$. Moreover, when the complex $\mathbf{F}_\bullet$ is exact, or more precisely $\mathrm{im}(d_{i+1}) = \ker(d_i)$, and

$$\mathbf{F}_\bullet : 0 \leftarrow S/I \xleftarrow{\pi} S \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \leftarrow \cdots \leftarrow F_{i-1} \xleftarrow{d_i} F_i \leftarrow \ldots \xleftarrow{d_p} F_p \leftarrow 0,$$

where $d_0 = \pi$ is the projection or quotient map from S to S/I, $F_0 = S$ and $F_{-1} = S/I$, then $\mathbf{F}_\bullet$ is a free resolution of either the ideal I or the module S/I. Furthermore, a free resolution $\mathbf{F}_\bullet$ is minimal if the entries of the maps d_i are in the maximal ideal $\langle x_1, \ldots, x_n\rangle$.

We will denote as I_G the monomial ideal generated by G. Moreover, we assume that G is a minimal generating set of I_G.

The goal of this section is to give a minimal free resolution of the monomial ideal $I_{G^\diamond}$. To achive this, we start by giving a sequence of free modules $\mathbf{F}_\bullet^\diamond$ that will be obtained from a minimal free resolution $\mathbf{F}_\bullet$ of I_G. Then, using the properties of $\mathbf{F}_\bullet$ we will give a proof that this new sequence $\mathbf{F}_\bullet^\diamond$ is a complex and later a minimal free resolution.

In next, let

$$\mathbf{F}_\bullet : 0 \leftarrow S[y]/I_G \xleftarrow{\pi} S[y] \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \leftarrow \cdots \leftarrow F_{i-1} \xleftarrow{d_i} F_i \leftarrow \ldots \xleftarrow{d_p} F_p \leftarrow 0$$

be a minimal free resolution of I_G where D_i is the matrix representation of the differentials d_i.

Definition 4.2.1. *Let Y_i be the matrix given by*

$$(Y_i)_{r,s} = \begin{cases} y^{a_r} & \text{if } r = s, \text{ where } y^{a_r}m \in \mathbf{M}_i[y] \text{ with } y \not| m, \\ 0 & \text{otherwise,} \end{cases}$$

$\mathbf{M}_i^\diamond = \mathbf{M}_i[\emptyset] \cup \bigcup_{\emptyset \neq R \subseteq \mathbf{y}} \mathbf{M}_{i-|R|+1}[y]^R$, $F_i^\diamond = \bigoplus_{\mathbf{m} \in \mathbf{M}_i^\diamond} S[\mathbf{y}](-\mathbf{m})$ *and* $D_i^\diamond$ *be the block matrix given by*

$$D_i^\diamond[R', R] = \begin{cases} A_i & \text{if } R', R = \emptyset, \\ B_i^{y_j} & \text{if } R' = \emptyset \text{ and } R = \{y_j\}, \\ (-1)^{|R|-1} C_{i-|R|+1}^R & \text{if } \emptyset \neq R' = R, \\ (-1)^{t+1} Y_{i-|R|+1}^{y_{j_t}} & \text{if } R \setminus R' = \{y_{j_t}\} \text{ and } R = \{y_{j_s}, \ldots, y_{j_1}\} \text{ with } j_s < \cdots < j_1, \\ 0 & \text{otherwise,} \end{cases}$$

with one block for each subset of $\mathbf{y}$ *and whose rows and columns are index by* $\mathbf{M}_{i-1}^\diamond$ *and* $\mathbf{M}_i^\diamond$*, respectively. Putting all this together we get the following sequence of free* S*-modules and homomorphisms between them*

$$\mathbf{F}_\bullet^\diamond : 0 \leftarrow S[\mathbf{y}]/I_{G^\diamond} \xleftarrow{\pi} S[\mathbf{y}] \xleftarrow{d_1^\diamond} \bigoplus_{\mathbf{m} \in \mathbf{M}_1^\diamond} S[\mathbf{y}](-\mathbf{m}) \xleftarrow{d_2^\diamond} \ldots \xleftarrow{d_i^\diamond} \bigoplus_{\mathbf{m} \in \mathbf{M}_i^\diamond} S[\mathbf{y}](-\mathbf{m}) \xleftarrow{d_{i+1}^\diamond} \ldots \xleftarrow{d_{p+1}^\diamond} \bigoplus_{\mathbf{m} \in \mathbf{M}_{p+m-1}^\diamond} S[\mathbf{y}](-\mathbf{m}) \leftarrow 0$$

where $d_i^\diamond$ is the differential induced by the matrix $D_i^\diamond$.

Remark 4.2.2. *Note that we are assuming that the multisets $\mathbf{M}_i[y]$ are ordered.*

In what follows we illustrate with some example how it works this construction. For instance, if $m=3$, then the matrix $D_i^{\diamond}$ looks like:

$$D_i^{\diamond} = \begin{array}{c c} & \begin{array}{cccccccc} \mathbf{M}_i[\emptyset] & \mathbf{M}_i[y]^{y_1} & \mathbf{M}_i[y]^{y_2} & \mathbf{M}_i[y]^{y_3} & \mathbf{M}_{i-1}[y]^{y_1y_2} & \mathbf{M}_{i-1}[y]^{y_1y_3} & \mathbf{M}_{i-1}[y]^{y_2y_3} & \mathbf{M}_{i-2}[y]^{y_1y_2y_3} \end{array} \\ \begin{array}{c} \mathbf{M}_{i-1}[\emptyset] \\ \mathbf{M}_{i-1}[y]^{y_1} \\ \mathbf{M}_{i-1}[y]^{y_2} \\ \mathbf{M}_{i-1}[y]^{y_3} \\ \mathbf{M}_{i-2}[y]^{y_1y_2} \\ \mathbf{M}_{i-2}[y]^{y_1y_3} \\ \mathbf{M}_{i-2}[y]^{y_2y_3} \\ \mathbf{M}_{i-3}[y]^{y_1y_2y_3} \end{array} & \left(\begin{array}{cccccccc} A_i & B_i^{y_1} & B_i^{y_2} & B_i^{y_3} & 0 & 0 & 0 & 0 \\ 0 & C_i^{y_1} & 0 & 0 & -Y_{i-1}^{y_2} & -Y_{i-1}^{y_3} & 0 & 0 \\ 0 & 0 & C_i^{y_2} & 0 & Y_{i-1}^{y_1} & 0 & -Y_{i-1}^{y_3} & 0 \\ 0 & 0 & 0 & C_i^{y_3} & 0 & Y_{i-1}^{y_1} & Y_{i-1}^{y_2} & 0 \\ 0 & 0 & 0 & 0 & -C_{i-1}^{y_1y_2} & 0 & 0 & Y_{i-2}^{y_3} \\ 0 & 0 & 0 & 0 & 0 & -C_{i-1}^{y_1y_3} & 0 & -Y_{i-2}^{y_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -C_{i-1}^{y_2y_3} & Y_{i-2}^{y_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_{i-2}^{y_1y_2y_3} \end{array}\right) \end{array}.$$

The following example illustrates how to construct $\mathbf{F}_\bullet^{\diamond}$ from a given $\mathbf{F}_\bullet$.

Example 4.2.3. *Let*

$$\mathbf{F}_\bullet : 0 \longleftarrow S/I_G \overset{\pi}{\longleftarrow} S \xleftarrow{\begin{pmatrix} x_1x_2^3 & x_1x_2y^2 & x_1^2y^3 \end{pmatrix}} \begin{array}{c} S(-x_1x_2^3) \\ \oplus \\ S(-x_1x_2y^2) \\ \oplus \\ S(-x_1^2y^3) \end{array} \xleftarrow{\begin{pmatrix} -y^2 & 0 \\ x_2^2 & -x_1y \\ 0 & x_2 \end{pmatrix}} \begin{array}{c} S(-x_1x_2^3y^2) \\ \oplus \\ S(-x_1^2x_2y^3) \end{array} \longleftarrow 0$$

be a minimal free resolution of the monomial ideal $I_G = \langle\{x_1x_2^3, x_1x_2y^2, x_1^2y^3\}\rangle$. Its free modules has multidegrees $\mathbf{M}_1 = \{x_1x_2^3, x_1x_2y^2, x_1^2y^3\}$ and $\mathbf{M}_2 = \{x_1x_2^3y^2, x_1^2x_2y^3\}$. The matrix representations of its differentials have blocks $A_1 = \begin{pmatrix} x_1x_2^3 \end{pmatrix}$, $B_1 = \begin{pmatrix} x_1x_2y^2 & x_1^2y^3 \end{pmatrix}$, $B_2 = \begin{pmatrix} y^2 & 0 \end{pmatrix}$ and $C_2 = \begin{pmatrix} x_2^2 & -x_1y \\ 0 & x_2 \end{pmatrix}$. The blocks C_1 and A_1 do not appear in the matrix representation of the differentials because the multidegrees that indexes their columns and rows are empty.

For $R=\{y_1,y_2\}$, we have that $I_{G^{\diamond(R)}} = \langle x_1x_2^3, x_1x_2y_1^2, x_1^2y_1^3, x_1x_2y_2^2, x_1^2y_2^3\rangle$. Moreover, $\mathbf{M}_1^{\diamond(R)} = G^{\diamond(R)}$,

$\mathbf{M}_2^{\diamond(R)} = \{x_1x_2^3y_1^2, x_1^2x_2y_1^3, x_1x_2^3y_2^2, x_1^2x_2y_2^3, x_1x_2(y_1y_2)^2, x_1^2(y_1y_2)^3\}, \mathbf{M}_3^{\diamond(R)} = \{x_1x_2^3(y_1y_2)^2, x_1^2x_2(y_1y_2)^3\}$,

and

$$\mathbf{F}_\bullet^{\diamond(R)} : 0 \longleftarrow T/I_{G^{\diamond}} \overset{\pi}{\longleftarrow} T \xleftarrow{\begin{pmatrix} x_1x_2^3 & x_1x_2y_1^2 & x_1^2y_1^3 & x_1x_2y_2^2 & x_1^2y_2^3 \end{pmatrix}} F_1^{\diamond} \xleftarrow{\begin{pmatrix} -y_1^2 & 0 & -y_2^2 & 0 & 0 & 0 \\ x_2^2 & -x_1y_1 & 0 & 0 & -y_2^2 & 0 \\ 0 & x_2 & 0 & 0 & 0 & -y_2^3 \\ 0 & 0 & x_2^2 & -x_1y_2 & y_1^2 & 0 \\ 0 & 0 & 0 & x_2 & 0 & y_1^3 \end{pmatrix}} F_2^{\diamond} \xleftarrow{\begin{pmatrix} -y_2^2 & 0 \\ 0 & -y_2^3 \\ y_1^2 & 0 \\ 0 & y_1^3 \\ -x_2^2 & x_1y_1y_2 \\ 0 & -x_2 \end{pmatrix}} F_3^{\diamond} \longleftarrow 0,$$

where $F_i^{\diamond(R)}$ be the free modules with multidegrees $\mathbf{M}_i^{\diamond(R)}$. In what follows we will prove that the sequence $\mathbf{F}_\bullet^{\diamond}$ is indeed a minimal free resolution of $I_{G^{\diamond}}$.

4.2.1 The sequence $\mathbf{F}^\diamond_\bullet$ is a complex

We start by proving that $\mathbf{F}^\diamond_\bullet$ is a complex, which is simpler than to prove its exactness. We recall that a free sequence $\mathbf{F}_\bullet = \{F_i, d_i\}_{i\in\mathbb{Z}}$ is a free complex whenever $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$. And clearly, $d_{i-1} \circ d_i = 0$ if and only if its matrix representations satisfies that $D_{i-1} \cdot D_i = 0$.

Proposition 4.2.4. *If $\mathbf{F}_\bullet$ is a minimal free resolution of the monomial ideal I_G minimally generated by G, then the free sequence $\mathbf{F}^{\diamond(R)}_\bullet$ is a free complex for $I_{G^{\diamond(R)}}$.*

Proof. We will use induction on $|R| = m$. For $m = 1$ there is nothing to prove because $I_{G^{\diamond(y_1)}} = I_G(y \to y_1)$ and therefore $\mathbf{F}_\bullet$ is a minimal free resolution of $I_{G^{\diamond(y_1)}}$, just by changing y by y_1. For $m = 2$, we need to prove that

$$D^\diamond_{i-1}\cdot D^\diamond_i = \begin{array}{rc} & \begin{array}{cccc} \mathbf{M}_{i-1}[\emptyset] & \mathbf{M}_{i-1}[y]^{y_1} & \mathbf{M}_{i-1}[y]^{y_2} & \mathbf{M}_{i-2}[y]^{y_1y_2} \end{array} \\ \begin{array}{c} \mathbf{M}_{i-2}[\emptyset] \\ \mathbf{M}_{i-2}[y]^{y_1} \\ \mathbf{M}_{i-2}[y]^{y_2} \\ \mathbf{M}_{i-3}[y]^{y_1y_2} \end{array} & \left(\begin{array}{cccc} A_{i-1} & B^{y_1}_{i-1} & B^{y_2}_{i-1} & 0 \\ 0 & C^{y_1}_{i-1} & 0 & -Y^{y_2}_{i-2} \\ 0 & 0 & C^{y_2}_{i-1} & Y^{y_1}_{i-2} \\ 0 & 0 & 0 & -C^{y_1y_2}_{i-2} \end{array}\right) \end{array} \begin{array}{cl} \begin{array}{cccc} \mathbf{M}_{i}[\emptyset] & \mathbf{M}_{i}[y]^{y_1} & \mathbf{M}_{i}[y]^{y_2} & \mathbf{M}_{i-1}[y]^{y_1y_2} \end{array} & \\ \left(\begin{array}{cccc} A_{i} & B^{y_1}_{i} & B^{y_2}_{i} & 0 \\ 0 & C^{y_1}_{i} & 0 & -Y^{y_2}_{i-1} \\ 0 & 0 & C^{y_2}_{i} & Y^{y_1}_{i-1} \\ 0 & 0 & 0 & -C^{y_1y_2}_{i-1} \end{array}\right) & \begin{array}{l} \mathbf{M}_{i-1}[\emptyset] \\ \mathbf{M}_{i-1}[y]^{y_1} \\ \mathbf{M}_{i-1}[y]^{y_2} \\ \mathbf{M}_{i-2}[y]^{y_1y_2} \end{array} \end{array} = 0,$$

which follows by the following identities:

a) $A_{i-1}A_i = 0$.

b) $A_{i-1}B^{y_1}_i + B^{y_1}_{i-1}C^{y_1}_i = 0$

c) $A_{i-1}B^{y_2}_i + B^{y_2}_{i-1}C^{y_2}_i = 0$.

d) $-B^{y_1}_{i-1}Y^{y_2}_{i-1} + B^{y_2}_{i-1}Y^{y_1}_{i-1} = 0$.

e) $C^{y_1}_{i-1}C^{y_1}_i = 0$.

f) $-C^{y_1}_{i-1}Y^{y_2}_{i-1} + Y^{y_2}_{i-2}C^{y_1y_2}_{i-1} = 0$.

g) $C^{y_2}_{i-1}C^{y_2}_i = 0$.

h) $C^{y_2}_{i-1}Y^{y_1}_{i-1} - Y^{y_1}_{i-2}C^{y_1y_2}_{i-1} = 0$.

i) $C^{y_1y_2}_{i-2}C^{y_1y_2}_{i-1} = 0$.

The identities $a), b), c), d), e), g)$ and $i)$ are follow because $\mathbf{F}_\bullet$ is a complex. Thus, it only remains to prove the identities $f)$ and $h)$. For these, we will consider the matrices Y_i's as matrices in the ring of fractions of $S[\mathbf{y}]$. Thus, it is not difficult to check that $Y^{1/y_2}_{i-2}M(y \to y_1)Y^{y_2}_{i-1} = M(y \to y_1y_2)$ for any matrix M with monomial entries in $S[y]$. Applying this to $C^{y_1}_{i-1}$ we get that $Y^{1/y_2}_{i-2}C^{y_1}_{i-1}Y^{y_2}_{i-1} = C^{y_1y_2}_{i-1}$ and multiplying both sides by $Y^{y_2}_{i-2}$ we get $f)$. In a similar way we get that $C^{y_2}_{i-1}Y^{y_1}_{i-1} = Y^{y_1}_{i-2}C^{y_1y_2}_{i-1}$ which implies identity $h)$.

Now assume that the result is true for $m-1$. Since the matrix representation of the differentials can be decomposed as

$$D_i^\diamond = \begin{array}{c} \\ \mathbf{M}_{i-1}^\diamond[\emptyset] \\ \mathbf{M}_i^\diamond[y_{m-1}] \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{M}_i^\diamond[\emptyset] & \mathbf{M}_i^\diamond[y_{m-1}] \end{array} \\ \begin{pmatrix} A_i' & B_i' \\ 0 & C_i' \end{pmatrix} \end{array}$$

then the result follows because $I_G^{\diamond(R)}$ is equal to $(I_G^{\diamond(R\setminus y_m)})^{\diamond(y_{m-1},y_m)}$. □

Now, we turn our attention to prove the exactness of $\mathbf{F}_\bullet^\diamond$.

4.2.2 The complex $\mathbf{F}_\bullet^\diamond$ is exact

By the criterion given in Theorem 3.2.6 the exactness of $\mathbf{F}_\bullet^\diamond$ can be reduced to the following next two conditions:

1. The columns of the matrices $D_i^\diamond$ are irredundant, and
2. The free S-modules $F_i^\diamond = \bigoplus_{\mathbf{m}\in\mathbf{M}_i^\diamond} S[\mathbf{y}](-\mathbf{m})$ have the correct multidegrees, that is, the multidegrees of the non zero Betti numbers of $I_{G^\diamond}$ correspond to the multidegrees in $\mathbf{M}_i^\diamond$.

First, we recall the definition of irredundant.

Definition 4.2.5. *A set of vectors* $\Gamma = \{\gamma_1,\ldots,\gamma_s\}$ *in a* S*-module is called irredundant whenever*

$$\gamma_i \notin \langle \gamma_1,\ldots,\hat{\gamma}_i,\ldots,\gamma_s\rangle \text{ for all } 1\le i\le s.$$

The irredundancy of the columns of the matrices $D_i^\diamond$ is not difficult. Actually, it follows by the definition of the matrices $D_i^\diamond$.

Proposition 4.2.6. *If* $\mathbf{F}_\bullet$ *is a minimal free resolution of the monomial ideal* I_G *minimally generated by* G*, then the columns of the diferentials* $D_i^\diamond$ *in* $\mathbf{F}_\bullet^\diamond$ *are irredundant.*

Proof. It follows from the fact that the matrices $D_i^\diamond$ are block upper triangle form and the blocks of $D_i^\diamond$ are irredundant. □

The correct Betti numbers

The Betti number in multidegree $\mathbf{a}$ and homological degree i of a monomial ideal I is the number $\beta_{i,\mathbf{a}}(S/I)$ of summands of the form $S(-\mathbf{a})$ in the free module F_i of any minimal free resolution $\mathbf{F}_\bullet$ of I. Thus, we need to calculate the Betti numbers of $I_{G^\diamond}$ and then to show that these are equal to the number of summands in the free modules of $\mathbf{F}^\diamond_\bullet$. In order to do that we will use Hochster's formula.

Theorem 4.2.7 (Hochster's formula, see for instance Theorem 1.34 in [23])**.** *The Betti numbers of a monomial ideal I can be expressed as*

$$\beta_{i+1,\mathbf{a}}(S/I) = \dim_k \tilde{H}_{i-1}(K^{\mathbf{a}}(I);k),$$

where $\tilde{H}_{i-1}$ is the reduced homology and $K^{\mathbf{a}}(I)$ is the upper Koszul simplicial complex.

We recall what means the upper Koszul simplicial complex $K^{\mathbf{a}}(I)$.

Definition 4.2.8. *Given a monomial ideal I and $\mathbf{a}$ in $\mathbb{N}^n$, the* **upper Koszul simplicial complex** *of I in multidegree $\mathbf{a}$ is given by*

$$K^{\mathbf{a}}(I) = \{\tau \in \{0,1\}^n : \mathbf{x}^{\mathbf{a}-\tau} \in I\}.$$

Remark 4.2.9. *Formally a simplicial complex is a family of sets which are closed under taking subsets. Here, as usual in the literature, using the bijection between sets and zero-one vectors, we consider a simplicial complex as a set of zero-one vectors that are closed under the partial order $\leq$ on $\mathbb{N}^n$.*

On the other hand, given a vector $\mathbf{c} \in \mathbb{N}^n$ its support is defined by

$$s(\mathbf{c})_i = \begin{cases} 1 & \text{if } \mathbf{c}_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now, if G is a minimal generator set of a monomial ideal I and $\mathbf{a} \in \mathbb{N}^n$, then $s(\mathbf{a}-\mathbf{g}) \in K^{\mathbf{a}}(I_G)$ for all $\mathbf{g} \in G$ with $\mathbf{g} \leqslant \mathbf{a}$ because $s(\mathbf{a}-\mathbf{g}) \leq \mathbf{a}-\mathbf{g}$ implies that $\mathbf{g} \leq \mathbf{a} - s(\mathbf{a}-\mathbf{g})$ and therefore $\mathbf{a} - s(\mathbf{a}-\mathbf{g}) \in I$. Moreover, we get the following alternative description of the upper Koszul simplicial complex.

Lemma 4.2.10. *If I_G is a monomial ideal minimally generated by G and $\mathbf{a} \in \mathbb{N}^{n+1}$, then*

$$F(K^{\mathbf{a}}(I_G)) \subseteq \{s(\mathbf{a}-\mathbf{g}) : \mathbf{g} \in G \text{ with } \mathbf{g} \leqslant \mathbf{a}\},$$

where $F(\Delta)$ is the set of facets of a simplicial complex Δ.

Proof. First, by the definition of the upper Koszul simlicial complex of I it is clear that if τ is a face of $K^{\mathbf{a}}(I_G)$, then there exists $\mathbf{g} \in G$ such that $\mathbf{g} \leq \mathbf{a} - \tau$ or equivalently $\tau \leq \mathbf{a} - \mathbf{g}$. Thus $\tau = s(\tau) \leq s(\mathbf{a} - \mathbf{g}) \in \{0,1\}^n$ for all $\mathbf{g} \in G$ with $\mathbf{g} \leq \mathbf{a} - \tau$. Therefore, if $\tau_i = 1$ for some $1 \leq i \leq n$, then clearly $s(\mathbf{a} - \mathbf{g})_i = 1$.

Additionally, if τ is a facet, then $s(\mathbf{a} - \mathbf{g}) = \tau$ for any $\mathbf{g} \in G$ with $\mathbf{g} \leq \mathbf{a} - \tau$. Otherwise, if $\tau_i = 0$ and $s(\mathbf{a} - \mathbf{g})_i = 1$ for some $1 \leq i \leq n$, then $\mathbf{g}_i \leq \mathbf{a}_i - 1$ and therefore $\mathbf{g} \leq \mathbf{a} - (\tau + \mathbf{e}_i)$. That is, $\tau' = \tau + \mathbf{e}_i \neq \tau$ is a face of $K^{\mathbf{a}}(I_G)$, which is a contradiction to the fact that τ is a facet. □

In other words, $K^{\mathbf{a}}(I_G) = \langle \{s(\mathbf{a} - \mathbf{g}) : \mathbf{g} \in G \text{ with } \mathbf{g} \leqslant \mathbf{a}\}\rangle$, where $\langle T \rangle$ denotes the minimal simplicial complex containing T.

Remark 4.2.11. *Note that not for all $\mathbf{g} \in G$ with $\mathbf{g} \leqslant \mathbf{a}$, $s(\mathbf{a} - \mathbf{g})$ is a facet of $K^{\mathbf{a}}(I_G)$. For instance, if $G = \{x_1x_3^2, x_1^2x_2x_3\}$ and $\mathbf{a} = (3,1,3)$, then $s(\mathbf{a} - \mathbf{g}_1) = (1,1,1)$ and $s(\mathbf{a} - \mathbf{g}_2) = (1,0,1)$.*

On the other hand, the join of two simplicial complexes $\Delta \subseteq \{0,1\}^n$ and $\Delta' \subseteq \{0,1\}^m$ is the simplicial complex given by

$$j(\Delta, \Delta') = \{(\tau, \tau') : \tau \in \Delta \text{ and } \tau' \in \Delta'\} \subseteq \{0,1\}^{n+m}.$$

When Δ' is a point v (that is, a 0-simplex), $j(\Delta, v)$ is called the cone of Δ with v. It is well know that the cone of any simplicial complex is contractible and therefore has reduced homology 0 at any dimension, see for instance [19, Corollary 4.23].

Also, it is well known that $K^{\mathbf{a}}(I_G)$ is the cone of some simplicial complex whenever $\mathbf{a}$ is not the least common multiple of a subset of G. Therefore, if $\mathbf{a}$ is not in the lcm-lattice, then the Betti numbers $\beta_{*,\mathbf{a}}$ of I_G are equal to zero. In what follows we will prove that $K^{\mathbf{b}}(I_{G^{\diamond(\{y_1,y_2\})}})$ is contractible for all $\mathbf{b}$ with $\mathbf{b}_{n+1} \neq \mathbf{b}_{n+2}$ both different from zero.

Lemma 4.2.12. *If $\mathbf{b} \in \mathbb{N}^{n+2}$ with $\mathbf{b}_{n+1} > \mathbf{b}_{n+2} \geq 0$, then $K^{\mathbf{b}}(I_{G^{\diamond(\{y_1,y_2\})}})$ is contractible.*

Proof. To simplify the notation $K^{\mathbf{b}}(I_{G^{\diamond(\{y_1,y_2\})}})$ will be denoted by $K^{\mathbf{b}}(I_{G^\diamond})$. Since $G^\diamond = G^{y_1} \cup G^{y_2}$, by Lemma 4.2.10 $K^{\mathbf{b}}(I_{G^\diamond}) = \langle \{s(\mathbf{b} - \mathbf{g}) : \mathbf{g} \in \mathbf{G}^{y_1} \text{ with } \mathbf{g} \leqslant \mathbf{b}\} \cup \{s(\mathbf{b} - \mathbf{g}) : \mathbf{g} \in \mathbf{G}^{y_2} \text{ with } \mathbf{g} \leqslant \mathbf{b}\}\rangle$.

First, we will prove that $\{s(\mathbf{b} - \mathbf{g}) : \mathbf{g} \in \mathbf{G}^{y_2} \text{ with } \mathbf{g} \leqslant \mathbf{b}\} \subseteq \langle \{s(\mathbf{b} - \mathbf{g}) : \mathbf{g} \in \mathbf{G}^{y_1} \text{ with } \mathbf{g} \leqslant \mathbf{b}\}\rangle$. For any $\mathbf{g} \in \mathbf{G}^{y_2}$ there exists $\tilde{\mathbf{g}} \in \mathbf{G}$ such that $\tilde{\mathbf{g}}^{y_2} = \mathbf{g}$. Moreover, since $\mathbf{b}_{n+1} > \mathbf{b}_{n+2}$, $\mathbf{g} \leqslant \mathbf{b}$ implies that $\tilde{\mathbf{g}} \leqslant \tilde{\mathbf{b}}$ where $\tilde{\mathbf{b}}$ is the vector obtained from $\mathbf{b}$ by erasing its last entry. On the other hand, $s(\mathbf{b}-\mathbf{g})_i = s(\mathbf{b}-\tilde{\mathbf{g}})_i$ for all $1 \leqslant i \leqslant n$, $s(\mathbf{b}-\tilde{\mathbf{g}})_{n+2} = 1$ because $\tilde{\mathbf{g}}_{n+2} = 0$ and $\mathbf{b}_{n+2} > 0$ and $s(\mathbf{b}-\tilde{\mathbf{g}})_{n+1} = 1$ because $\tilde{\mathbf{g}}_{n+1} = \mathbf{g}_{n+2} \leqslant \mathbf{b}_{n+2} < \mathbf{b}_{n+1}$. Thus $s(\mathbf{b}-\mathbf{g}) \leqslant s(\mathbf{b}-\tilde{\mathbf{g}})$ and therefore $\{s(\mathbf{b}-\mathbf{g}) : \mathbf{g} \in \mathbf{G}^{y_2}\} \subseteq \langle\{s(\mathbf{b}-\mathbf{g}) : \mathbf{g} \in \mathbf{G}^{y_1}\}\rangle$. That is, $K^{\mathbf{b}}(I_{G^\diamond}) = \langle\{s(\mathbf{b}-\mathbf{g}) : \mathbf{g} \in \mathbf{G}^{y_1}\}\rangle$.

Finally, since $\{s(\mathbf{b}-\mathbf{g}) : \mathbf{g} \in \mathbf{G}^{y_1}\} = \{(s(\tilde{\mathbf{b}}-\tilde{\mathbf{g}}), 1) : \tilde{\mathbf{g}} \in \mathbf{G}[y]\}$, then $K^{\mathbf{b}}(I_{G^\diamond})$ is $j(K^{\tilde{\mathbf{b}}}(I_G), e_{n+2})$ and therefore contractible. $\square$

The case when $0 \leq \mathbf{b}_{n+1} < \mathbf{b}_{n+2}$ follows by using similar arguments. Therefore we can imply that $K^{\mathbf{b}}(I_{G^\diamond(\{y_1,y_2\})})$ is possibly non contractible whenever

$$\mathbf{b} = \begin{cases} (\mathbf{a}, a, 0), \\ (\mathbf{a}, 0, a), \\ (\mathbf{a}, a, a) \end{cases}$$

for some $\mathbf{a} \in \mathbb{N}^n$ and $a \in \mathbb{N}$. It is not difficult to check that $K^{(\mathbf{a},a,0)}(I_{G^\diamond(\{y_1,y_2\})})$ and $K^{(\mathbf{a},0,a)}(I_{G^\diamond(\{y_1,y_2\})})$ are isomorphic to $K^{(\mathbf{a},a)}(I_G)$. Therefore only remains to consider the case when $K^{(\mathbf{a},a,a)}(I_{G^\diamond(\{y_1,y_2\})})$, which we will prove that it has almost the same reduced homology than $K^{(\mathbf{a},a)}(I_G)$. To do this, let

$$\Delta_1 = j(K^{\mathbf{a}}(I_G), \mathbf{e}_{n+2}) \subset \{0,1\}^{n+2} \text{ and } \Delta_2 = \{(\tau, 0) : \tau \in j(K^{\mathbf{a}}(I_G)', \mathbf{e}_{n+1})\} \subset \{0,1\}^{n+2},$$

where $K^{\mathbf{a}}(I_G)' = \{\tau : (\tau, 0) \in K^{\mathbf{a}}(I_G)\}$. It is not difficult to check that $K^{\mathbf{a}}(I_G) = \Delta_1 \cap \Delta_2$. Moreover, as next result shows $K^{\mathbf{b}}(I_{G^\diamond(\{y_1,y_2\})}) = \Delta_1 \cup \Delta_2$.

Lemma 4.2.13. *If* $\mathbf{a} \in \mathbb{N}^{n+1}$ *with* $\mathbf{a}_{n+1} \neq 0$ *and* $\mathbf{b} = (\mathbf{a}, \mathbf{a}_{n+1})$, *then*

$$K^{\mathbf{b}}(I_{G^\diamond(\{y_1,y_2\})}) = \Delta_1 \cup \Delta_2.$$

Proof. Again, to simplify the notation $K^{\mathbf{b}}(I_{G^\diamond(\{y_1,y_2\})})$ will be denoted by $K^{\mathbf{b}}(I_{G^\diamond})$. By Lemma 4.2.10 any facet $\tilde{\tau}$ of $F(K^{\mathbf{b}}(I_{G^\diamond}))$ is equal to $s(\mathbf{b}-\tilde{\mathbf{g}})$ for some $\tilde{\mathbf{g}} \in \mathbf{G}^\diamond$ with $\tilde{\mathbf{g}} \leq \mathbf{b}$. Since $\mathbf{G}^\diamond = \mathbf{G}^{y_1} \cup \mathbf{G}^{y_2}$ and $\mathbf{g} \leq \mathbf{a}$ if and only if $\mathbf{g}^{y_i} \leq \mathbf{b}$ for all $\mathbf{g} \in G$ and $i = 1, 2$, then it is not difficult to check that

$$s(\mathbf{b}-\tilde{\mathbf{g}}) = \begin{cases} (s(\mathbf{a}-\mathbf{g}), 1) \text{ for some } \mathbf{g} \in \mathbf{G}[y] & \text{if } \tilde{\mathbf{g}} \in \mathbf{G}^{y_1}, \\ (s(\mathbf{a}-\mathbf{g}), 1) \text{ for some } \mathbf{g} \in \mathbf{G}[y] & \text{if } \tilde{\mathbf{g}} \in \mathbf{G}^{y_2} \text{ and } s(\mathbf{a}-\mathbf{g})_{n+1} = 1, \\ (s(\mathbf{a}-\mathbf{g}) + \mathbf{e}_{n+1}, 0) \text{ for some } \mathbf{g} \in \mathbf{G}[y] & \text{if } \tilde{\mathbf{g}} \in \mathbf{G}^{y_2} \text{ and } s(\mathbf{a}-\mathbf{g})_{n+1} = 0 \end{cases}$$

for all $\tilde{\mathbf{g}} \in \mathbf{G}^\diamond$ with $\tilde{\mathbf{g}} \leq \mathbf{b}$. Finally, we get the result because $(s(\mathbf{a}-\mathbf{g}), 1) \in \Delta_1$ and $(s(\mathbf{a}-\mathbf{g}) + \mathbf{e}_{n+1}, 0) \in \Delta_2$ whenever $s(\mathbf{a}-\mathbf{g})_{n+1} = 0$. $\square$

Using Mayer-Vietoris sequence to calculate the reduced homology of $K^{\mathbf{b}}(I_{G^\diamond})$ we get that $\beta_{i,\mathbf{b}}(T/I_{G^\diamond}) = \beta_{i-1,\mathbf{a}}(S/I_G)$.

Proposition 4.2.14. *If* $\mathbf{a} \in \mathbb{N}^{n+1}$ *with* $\mathbf{a}_{n+1} \neq 0$ *and* $\mathbf{b} = (\mathbf{a}, \mathbf{a}_{n+1})$*, then*

$$\beta_{i,\mathbf{b}}(T/I_{G^\diamond}) = \beta_{i-1,\mathbf{a}}(S/I_G) \text{ for all } i.$$

Proof. By the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_i(\Delta_1 \cap \Delta_2) \to \tilde{H}_i(\Delta_1) \oplus \tilde{H}_i(\Delta_2) \to \tilde{H}_i(\Delta_1 \cup \Delta_2) \to \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2) \to \cdots$$

and since Δ_1 and Δ_2 are contractible, $K^{\mathbf{b}}(I_{G^\diamond}) = \Delta_1 \cup \Delta_2$ and $K^{\mathbf{a}}(I_G) = \Delta_1 \cap \Delta_2$, then $\tilde{H}_i(K^{\mathbf{b}}(I_{G^\diamond})) = \tilde{H}_{i-1}(K^{\mathbf{a}}(I_G))$ and therefore $\beta_{i,\mathbf{b}}(T/I_{G^\diamond}) = \beta_{i-1,\mathbf{a}}(S/I_G)$. □

Finally, putting all together we get that $\mathbf{F}_\bullet^\diamond$ is a minimal free resolution of $I_{G^\diamond}$.

Theorem 4.2.15. *Let I_G be a monomial ideal in $S[y]$ minimally generated by G. If $\mathbf{F}_\bullet$ is a multigraded minimal free resolution of I_G, then $\mathbf{F}_\bullet^\diamond$ is a minimal free resolution of $I_{G^\diamond}$.*

Proof. We will use induction on m. For $m = 1$, the result it is clear. For $m = 2$, it follows by Theorem 3.2.6 and Propositions 4.2.6 and 4.2.14. The rest follows because $I_G^{\diamond(R)}$ is equal to $(I_G^{\diamond(R\setminus y_m)})^{\diamond(y_{m-1},y_m)}$. □

Corollary 4.2.16. *Let I_G be a monomial ideal in the polynomial ring $S = k[\mathbf{x}, y]$ minimally generated by G, and $I_{G^\diamond}$ its duplication. If I_G has projective dimension* $\mathrm{pd}(I_G) = p$*, then*

$$\mathrm{pd}(I_{G^\diamond}) = \begin{cases} p+1 & \text{if } \mathbf{M}_p[y] \neq \emptyset, \\ p & \text{if } \mathbf{M}_p[y] = \emptyset, \end{cases}$$

where $\mathbf{M}_p$ are the multidegrees that generate the p-th free module of a minimal free resolution of I_G.

Proof. It follows from the definition of the multigraded free modules of $\mathbf{F}_\bullet^\diamond$. □

Remark 4.2.17. *If we make the duplication of I_G with an $R \subset \mathbf{y}$ such that $|\mathbf{y}| = 3$, then the projective dimension increases if $\mathbf{M}_p[y] \neq \emptyset$ or $\mathbf{M}_{p-1}[y] \neq \emptyset$. In general for a duplication big enough we have that the dimension should increase because at least $\mathbf{M}_1[y]$ is not empty. Once the projective dimension has increased, if we make another duplication, the projective dimension will keep growing.*

Next corollary says how the depth of the duplication changes.

Corollary 4.2.18. *Let I_G a monomial ideal minimally generated by G in a polynomial ring $S = k[\mathbf{x}, y]$ and $I_{G^\diamond}$ its duplication in $T = k[\mathbf{x}, \mathbf{y}]$. If S/I_G has depth e in S, then the duplication $T/I_{G^\diamond}$ has depth:*

$$\operatorname{depth}(T/I_{G^\diamond}) = \begin{cases} e & \text{if } \mathbf{M}_p[y] \neq \emptyset, \\ e+1 & \text{if } \mathbf{M}_p[y] = \emptyset, \end{cases}$$

where $\mathbf{M}_p$ are the multidegrees that generate the p-th free module of a minimal free resolution of I_G.

Proof. Use the Auslander-Buchsbaum formula: $\operatorname{pd}(S/I_G) = n+1-\operatorname{depth}_S(S/I_G)$ where $n+1$ is the number of variables of the polynomial ring S. Then, by the previous corollary we have the result. $\square$

Next example shows a monomial ideal such that the projective dimension of its duplication does not change but the depth does increase.

Example 4.2.19. *Let I_G be a monomial ideal with a minimal free resolution $\mathbf{F}_\bullet$ as the following:*

$$\mathbf{F}_\bullet : 0 \longleftarrow S/I_G \xleftarrow{\pi} S \xleftarrow{\begin{pmatrix} x_1x_3 & x_2x_3 & x_3x_4 & x_4y \end{pmatrix}} \bigoplus_{\mathbf{m}\in\mathbf{M}_1} S(-\mathbf{m}) \xleftarrow{\begin{pmatrix} -x_2 & -x_4 & 0 & 0 \\ x_1 & 0 & -x_4 & 0 \\ 0 & x_1 & x_2 & -y \\ 0 & 0 & 0 & x_3 \end{pmatrix}} \bigoplus_{\mathbf{m}\in\mathbf{M}_2} S(-\mathbf{m}) \xleftarrow{\begin{pmatrix} x_4 \\ -x_2 \\ x_1 \\ 0 \end{pmatrix}} \bigoplus_{\mathbf{m}\in\mathbf{M}_3} S(-\mathbf{m}) \longleftarrow 0$$

Figure 4.2: A minimal free resolution of I_G.

In particular the multidegrees of the free modules are:

$$\begin{aligned} \mathbf{M}_1 &= \{x_1x_3, x_2x_3, x_3x_4, x_4y\}, \\ \mathbf{M}_2 &= \{x_1x_2x_3, x_1x_3x_4, x_2x_3x_4, x_3x_4y\}, \\ \mathbf{M}_3 &= \{x_1x_2x_3x_4\}. \end{aligned}$$

The duplication of I_G with $R = \{y_1, y_2\}$ has minimal free resolutions $\mathbf{F}^\diamond_\bullet$ as next figure shows:

$$\mathbf{F}_\bullet^\diamond : \ 0 \longleftarrow T/I_{G^\diamond} \xleftarrow{\pi} T \xleftarrow{\begin{pmatrix} x_1x_3 & x_2x_3 & x_3x_4 & x_4y_1 & x_4y_2 \end{pmatrix}} \bigoplus_{\mathbf{m}\in\mathbf{M}_1^\diamond} T(-\mathbf{m}) \xleftarrow{\begin{pmatrix} -x_2 & -x_4 & 0 & 0 & 0 & 0 \\ x_1 & 0 & -x_4 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & -y_1 & -y_2 & 0 \\ 0 & 0 & 0 & x_3 & 0 & -y_2 \\ 0 & 0 & 0 & 0 & x_3 & y_1 \end{pmatrix}} \bigoplus_{\mathbf{m}\in\mathbf{M}_2^\diamond} T(-\mathbf{m}) \xleftarrow{\begin{pmatrix} x_4 & 0 \\ -x_2 & 0 \\ x_1 & 0 \\ 0 & -y_2 \\ 0 & y_1 \\ 0 & -x_3 \end{pmatrix}} \bigoplus_{\mathbf{m}\in\mathbf{M}_3^\diamond} T(-\mathbf{m}) \longleftarrow 0$$

Figure 4.3: A minimal free resolution of $I_{G^\diamond}$.

The multidegrees of the free modules in $\mathbf{F}_\bullet^\diamond$ *are:*

$$\begin{aligned} \mathbf{M}_1^\diamond &= \{x_1x_3, x_2x_3, x_3x_4, x_4y_1, x_4y_2\}, \\ \mathbf{M}_2^\diamond &= \{x_1x_2x_3, x_1x_3x_4, x_2x_3x_4, x_3x_4y_1, x_3x_4y_1, x_4y_1y_2\}, \\ \mathbf{M}_3^\diamond &= \{x_1x_2x_3x_4, x_3x_4y_1y_2\}. \end{aligned}$$

Notice that there are not elements in $\mathbf{M}_4^\diamond$ *because* $\mathbf{M}_3[y]$ *is empty.*

A T-module M is Cohen-Macaulay when $\mathrm{depth}_{\mathrm{T}}(M) = \dim(M)$, where $\dim(M)$ is the Krull dimension of M. We have that Cohen-Macaulay property is lost under duplication.

Example 4.2.20. *Let* $I_G = \langle x_1x_2, x_1y, x_2y\rangle$ *be the edge ideal of the complete graph with three vertices. This edge ideal* I_G *is Cohen-Macaulay.*

On the other hand, the duplication ideal $I_{G^\diamond} = \langle\{x_1x_2, x_1y_1, x_2y_1, x_1y_2, x_2y_2\}\rangle$ *is not Cohen-Macaulay because the depth of* $I_{G^\diamond}$ *stays equal as the depth of* I_G *but the Krull dimension increases in one.*

Remark 4.2.21. *In case of edge ideals, the Krull dimension equals to the stability number of the graph G. Then if we duplicated a vertex of the graph G a large number of times, we have that the vertex and its duplications form a large stable set, which means that its Krull dimension is big. On the other hand, after all these duplication we will have that the projective dimension has increased and it implies that the depth has not increased. In short terms, if we make the duplication of a vertex of a graph a large number of times, the duplicated graph is not Cohen-Macaulay.*

We recall that the regularity of a multigraded ideal I_G is defined as

$$\mathrm{reg}(I_G) = max\{|\mathbf{b}| - i : \beta_{i,\mathbf{b}}(I_G) \neq 0\}$$

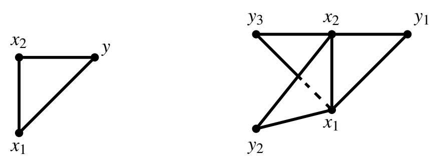
where $|\mathbf{b}| = \mathbf{b}_1 + \ldots + \mathbf{b}_{n+1}$ is the sum of its entries.

In general the regularity changes under duplication.

Example 4.2.22. *Let* $I_G = \langle x_1x_2, x_1y^3, x_2y^3\rangle$ *and its duplication* $I_{G^\diamond} = \langle x_1x_2, x_1y_1^3, x_2y_1^3, x_1y_2^3, x_2y_2^3\rangle$. *Using Macaulay2 [12] we can see that the regularity of* I_G *is* 4 *while the regularity of* $I_{G^\diamond}$ *is* 6.

We finish this section with an example of a duplication of a monomial ideal.

Example 4.2.23. *Let* $I_G = \langle\{x_1x_2, x_1x_3, x_2x_3\}\rangle$ *be the edge ideal of the complete graph with three vertices, see figure 4.4 (a).*



(a) Graph G. (b) Graph $G^\diamond$.

Figure 4.4: Duplication of the graph G with $R = \{y_1, y_2, y_3\}$.

A minimal free resolution of I_G *is:*

$$\mathbf{F}_\bullet:\ 0 \longleftarrow S/I_G \xleftarrow{\ \pi\ } S \xleftarrow{\ \emptyset\ \begin{pmatrix} x_1x_2 & x_1y & x_2y \end{pmatrix}} \begin{matrix} S(-x_1x_2) \\ \oplus \\ S(-x_1y) \\ \oplus \\ S(-x_2y) \end{matrix} \xleftarrow{\ \begin{pmatrix} -y & 0 \\ 0 & -x_2 \\ x_1 & x_1 \end{pmatrix}\ } S(-x_1x_2y)^2 \longleftarrow 0$$

Figure 4.5: A minimal free resolution of I_G.

This is not a coincidence that we can label the rows and columns of the differentials of $\mathbf{F}_\bullet$ *using some special subgraphs of the graph. In next section we will describe these subgraphs.*

Now we will give a minimal free resolution of the duplication of the graph. This is a new graph, see figure 4.4 (b).

$$\mathbf{F}^\diamond_\bullet:\ 0 \longleftarrow T/I_{G^\diamond} \xleftarrow{\ \pi\ } T \xleftarrow{\ D_1^\diamond\ } \bigoplus_{\mathbf{m}\in\mathbf{M}_1^\diamond} T(-\mathbf{m}) \xleftarrow{\ D_2^\diamond\ } \bigoplus_{\mathbf{m}\in\mathbf{M}_2^\diamond} T(-\mathbf{m}) \xleftarrow{\ D_3^\diamond\ } \bigoplus_{\mathbf{m}\in\mathbf{M}_3^\diamond} T(-\mathbf{m}) \xleftarrow{\ D_4^\diamond\ } \bigoplus_{\mathbf{m}\in\mathbf{M}_4^\diamond} T(-\mathbf{m}) \longleftarrow 0$$

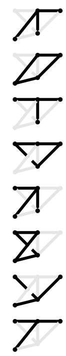
Figure 4.6: A minimal free resolution of $I_{G^\diamond}$.

The differentials of this minimal free resolution $\mathbf{F}_\bullet^\diamond$ *are:*

$$D_1^\diamond = \emptyset \begin{pmatrix} x_1x_2 & x_1y_1 & x_2y_1 & x_1y_2 & x_2y_2 & x_1y_3 & x_2y_3 \end{pmatrix}$$

$$D_2^\diamond = \begin{pmatrix}
-y_1 & 0 & -y_2 & 0 & -y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_2 & 0 & 0 & 0 & 0 & -y_2 & 0 & -y_3 & 0 & 0 & 0 \\
x_1 & x_1 & 0 & 0 & 0 & 0 & 0 & -y_2 & 0 & -y_3 & 0 & 0 \\
0 & 0 & 0 & -x_2 & 0 & 0 & y_1 & 0 & 0 & 0 & -y_3 & 0 \\
0 & 0 & x_1 & x_1 & 0 & 0 & 0 & y_1 & 0 & 0 & 0 & -y_3 \\
0 & 0 & 0 & 0 & 0 & -x_2 & 0 & 0 & y_1 & 0 & y_2 & 0 \\
0 & 0 & 0 & 0 & x_1 & x_1 & 0 & 0 & 0 & y_1 & 0 & y_2
\end{pmatrix}$$

$$D_3^\diamond = \begin{pmatrix}
-y_2 & 0 & -y_3 & 0 & 0 & 0 & 0 & 0 \\
0 & -y_2 & 0 & -y_3 & 0 & 0 & 0 & 0 \\
y_1 & 0 & 0 & 0 & -y_3 & 0 & 0 & 0 \\
0 & y_1 & 0 & 0 & 0 & -y_3 & 0 & 0 \\
0 & 0 & y_1 & 0 & y_2 & 0 & 0 & 0 \\
0 & 0 & 0 & y_1 & 0 & y_2 & 0 & 0 \\
0 & x_2 & 0 & 0 & 0 & 0 & y_3 & 0 \\
-x_1 & -x_1 & 0 & 0 & 0 & 0 & 0 & y_3 \\
0 & 0 & 0 & x_2 & 0 & 0 & -y_2 & 0 \\
0 & 0 & -x_1 & -x_1 & 0 & 0 & 0 & -y_2 \\
0 & 0 & 0 & 0 & 0 & x_2 & y_1 & 0 \\
0 & 0 & 0 & 0 & -x_1 & -x_1 & 0 & y_1
\end{pmatrix}$$



$$D_4^{\diamond} = \begin{pmatrix} y_3 & 0 \\ 0 & y_3 \\ -y_2 & 0 \\ 0 & -y_2 \\ y_1 & 0 \\ 0 & y_1 \\ 0 & -x_2 \\ x_1 & x_1 \end{pmatrix}$$

We will see in next sections that these graphs that are labeling the columns and rows of the differentials $D_i^{\diamond}$ are very important. In particular this kind of resolution will be called a Combinatorial Resolution.

4.3 The poset resolution and its duplication

Until now we have defined a minimal free resolution of a monomial ideal as a sequence of free modules which is exact and minimal.

From now on, if $\mathbf{F}_\bullet = \{F_i, d_i\}_{i\in\mathbb{Z}}$ is a sequence of free modules that is exact and minimal, we will say that $\mathbf{F}_\bullet$ is an **algebraic resolution**, that is, we will say that the minimal free resolutions as always were presented will be called algebraic resolutions.

In this section we give another representation of a minimal free resolution of a monomial ideal. This new representation will be called a **poset resolution**. First we make a review of some definitions of algebraic resolutions.

Given an algebraic resolution $\mathbf{F}_\bullet = \{F_i, d_i\}_{i\in\mathbb{Z}}$ the first free module is $F_0 = S$ which is generated by the multidegree 1. Any other free module F_i, with $i \geqslant 1$, is generated by the multidegrees in the multiset $\mathbf{M}_i$. Moreover, a differential $d_i : F_i \to F_{i-1}$ has a matrix representation D_i where each column and each row are labeled by a multidegree in $\mathbf{M}_i$ and $\mathbf{M}_{i-1}$, respectively. Thus, we can identify an entry (u, v) of D_i with the pair $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a}$ is the multidegree labeling the u-th row and $\mathbf{b}$ is the multidegree labeling the v-th column of D_i.

Now we are ready to introduce the poset resolution of a monomial ideal I_G.

Definition 4.3.1. *Let $\mathbf{F}_\bullet = \{F_i, d_i\}_{i\in\mathbb{Z}}$ be an algebraic resolution of I_G and D_i are the matrix representation of the differentials d_i. A* **poset resolution** *of I_G is a pair $(V,E)_\bullet$, where V is a set of ranked and labeled vertices, defined as*

$$V = \{v_{i,\mathbf{m}} : \mathbf{m} \in \mathbf{M}_i\}.$$

And a set of labeled edges E defined as

$$E = \{(v_{i,\mathbf{a}}, v_{i+1,\mathbf{b}})_c : \text{ the entry } (\mathbf{a},\mathbf{b}) \text{ of } D_{i+1} \text{ is } c\mathbf{x}^{\mathbf{b}-\mathbf{a}} \text{ with } c \in k\}.$$

A vertex $v_{i,\mathbf{m}}$ has rank i and is labeled by $\mathbf{m}$; an edge $(v_{i,\mathbf{a}}, v_{i+1,\mathbf{b}})_c$ is labeled by c.

Next example shows previous definition.

Example 4.3.2. *Let*

$$(\Lambda,\mu)_\bullet : 0 \longleftarrow S/I_G \xleftarrow{\pi} S \xleftarrow{\emptyset\ \begin{pmatrix} x_1x_4 & x_1x_2 & x_1x_3 \end{pmatrix}} \begin{matrix} S(-x_1x_4) \\ \oplus \\ S(-x_1x_2) \\ \oplus \\ S(-x_1x_3) \end{matrix} \xleftarrow{\begin{pmatrix} -x_3 & 0 & -x_2 \\ 0 & x_3 & x_4 \\ x_4 & -x_2 & 0 \end{pmatrix}} \begin{matrix} S(-x_1x_3x_4) \\ \oplus \\ S(-x_1x_2x_3) \\ \oplus \\ S(-x_1x_2x_4) \end{matrix} \xleftarrow{\begin{pmatrix} x_2 \\ x_4 \\ -x_3 \end{pmatrix}} S(-x_1x_2x_3x_4) \longleftarrow 0$$

Figure 4.7: An algebraic resolution of I_G.

be an algebraic resolution of I_G. Notice that

$$\begin{aligned} \mathbf{M}_0 &= \{1\}, \\ \mathbf{M}_1 &= \{x_1x_4, x_1x_2, x_1x_3\}, \\ \mathbf{M}_2 &= \{x_1x_3x_4, x_1x_2x_3, x_1x_2x_4\}, \\ \mathbf{M}_3 &= \{x_1x_2x_3x_4\}. \end{aligned}$$

Then the poset resolution $(V,E)_\bullet$ of I_G is

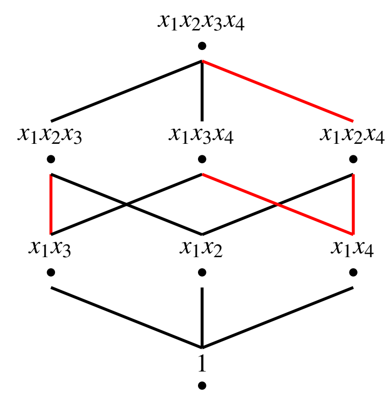


Figure 4.8: A poset resolution of I_G.

The red lines in previous poset are the edges labeled with -1 *and black lines are labeled with* 1*. For instance, the red line between the vertices* x_1x_4 *and* $x_1x_3x_4$ *represents the same as the entry* $(x_1x_4, x_1x_3x_4)$ *in the matrix* D_2 *of* $\mathbf{F}_\bullet$*.*

Once we have a poset resolution of a monomial ideal I_G we can recover the algebraic resolution of I_G using the information contained in the vertices and the edges. For instance, to recover the differential D_i of $\mathbf{F}_\bullet$ we focus in the vertices of rank i and $i-1$ and the edges between them: the entry $(\mathbf{a}, \mathbf{b})$ of D_i is $c\mathbf{x}^{\mathbf{b}-\mathbf{a}}$ which is given by the edge $(v_{i-1,\mathbf{a}}, v_{i,\mathbf{b}})_c$ of the poset.

We call this new representation of a minimal free resolution as poset because the vertices of this graph are ordered by an order relation given by the edges of the graph.

In [10] is defined a lattice which vertices are labeled by the minimum common multiple of the minimal generators of I and this lattice is ordered by divisibility. This lattice is called the LCM-lattice. With this LCM-lattice is possible to calculate the Betti numbers of I but not the differentials between the free modules of a minimal free resolution of I.

In [5] they use a poset to generate a sequence of k-vector spaces but this sequence in general is not a complex nor it is exact. For us, our poset will be represent a resolution, that is, it will be generate a sequence of free modules that is an exact complex and minimal.

Remark 4.3.3. *Is important to notice that in general a poset resolution is not a lattice because not any pair of elements has a least upper bound.*

4.3.1 The duplication of a poset resolution

In next we define the duplication of a poset resolution. First we need some definitions.

Definition 4.3.4. *Let $(V,E)_\bullet$ a poset resolution of a monomial ideal I_G. The subposet $(V[\mathbf{g}],E[\mathbf{g}])$ is the poset with set of vertices $V[\mathbf{g}] = \{v_{i,\mathbf{a}} \in V : \mathbf{g}|\mathbf{a}\}$ and set of edges $E[\mathbf{g}] = \{(v_{i,\mathbf{a}}, v_{i+1,\mathbf{b}})_c \in E : \mathbf{g}|\mathbf{a}$ and $\mathbf{g}|\mathbf{b}\}$.*

For example, the subposet $(V[x_4],E[x_4])$ of the poset given in Figure 4.8 is

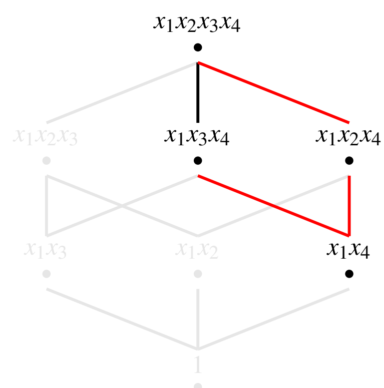


Figure 4.9: Subposet of a poset resolution.

Now we are ready to define the duplication of a poset resolution.

Definition 4.3.5. *Let $(V,E)_\bullet$ be a poset resolution of a monomial ideal $I_G \subset S[y]$. The poset resolution of the duplication $I_{G^\diamond}$ is the pair $(V^\diamond,E^\diamond)_\bullet$ defined as follows:*

$$V_i^\diamond = V[\emptyset] \cup \bigcup_{\emptyset \neq R \subseteq \mathbf{y}} V_{i-|R|+1}[y]^R$$

where $V_i[y]^R = \{v_{i,\mathbf{a}^R} : y|\mathbf{a}\}$ and $\mathbf{a}^R$ is the R-copy of $\mathbf{a}$. And the edges are labeled as

$$\begin{aligned} E^\diamond = E \quad & \cup \quad \{(v_{i,\mathbf{a}}, v_{i+1,\mathbf{b}^{y_t}})_c : (v_{i,\mathbf{a}}, v_{i+1,\mathbf{b}})_c \in E,\ y \nmid \mathbf{a} \text{ and } y_t \in \mathbf{y}\} \\ & \cup \quad \{(v_{i,\mathbf{a}^{R'}}, v_{i+1,\mathbf{b}^R})_{(-1)^{|R|-1}c} : (v_{i,\mathbf{a}}, v_{i+1,\mathbf{b}})_c \in E \text{ and } R' = R\} \\ & \cup \quad \{(v_{i,\mathbf{a}^{R'}}, v_{i+1,\mathbf{b}^R})_{(-1)^{t+1}c} : (v_{i,\mathbf{a}}, v_{i+1,\mathbf{b}})_c \in E,\ R \setminus R' = \{y_{j_t}\} \text{ with} \\ & \qquad R = \{y_{j_s}, \ldots, y_{j_1}\} \text{ and } y_{j_s} < \ldots y_{j_1}\}. \end{aligned}$$

Graphically is easy to see previous definition.

Example 4.3.6. *The duplication of the poset given in Figure 4.8 is the poset* $(V^\diamond, E^\diamond)$*:*

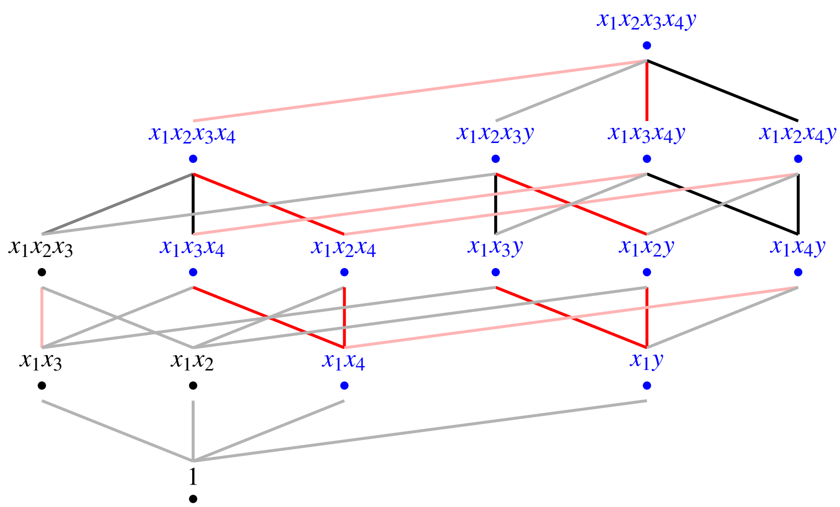


Figure 4.10: The poset resolution of $I_{G^\diamond}$.

Notice that the duplication $(V^\diamond, E^\diamond)$ *is the same poset* (V, E) *with the subposet* $(V[x_4], E[x_4])$ *pasted twice and with some other edges. We draw with blue color the vertices of* $(V[x_4], E[x_4])$ *and the edges a little more transparent to make noticeable this last affirmation.*

As we said before, from the poset resolution we can recover the algebraic resolution of a monomial ideal. With this idea we can give a proof of the next theorem.

Theorem 4.3.7. *Give a poset resolution* $(V, E)_\bullet$ *of a monomial ideal* I_G*, the poset* $(V^\diamond, E^\diamond)_\bullet$ *is a minimal free resolution of* $I_{G^\diamond}$*.*

Proof. Notice that the labeled vertices of $V^\diamond$ coincides with the multidegrees $\mathbf{M}^\diamond$ of $\mathbf{F}^\diamond_\bullet$. Moreover, the edges $E^\diamond$ defines the same differential matrix $D^\diamond_i$ as in $\mathbf{F}^\diamond_\bullet$. Thus, $(V^\diamond, E^\diamond)_\bullet$ is a poset resolution of $I_{G^\diamond}$. $\square$

Although the poset resolution contains the same information as an algebraic resolution and it is shorter to write than an algebraic resolution, it has some advantages and disadvantages respect to an algebraic resolution. For instance, an advantage is that the poset will be usefull when we calculate the signature (see [17]) and it has the disadvantage that if we have two algebraic resolution of a monomial ideal I (this two resolutions are isomorphic), the poset resolution can be different. Next example shows that.

Example 4.3.8. *Next two posets are poset resolutions of the edge ideal of the complete graph with four vertices K_4. The poset of the left side in Figure 4.11 is a combinatorial resolution (see next section), and the poset of the right side is the minimal free resolution obtained from Macaulay2 ([12]).*

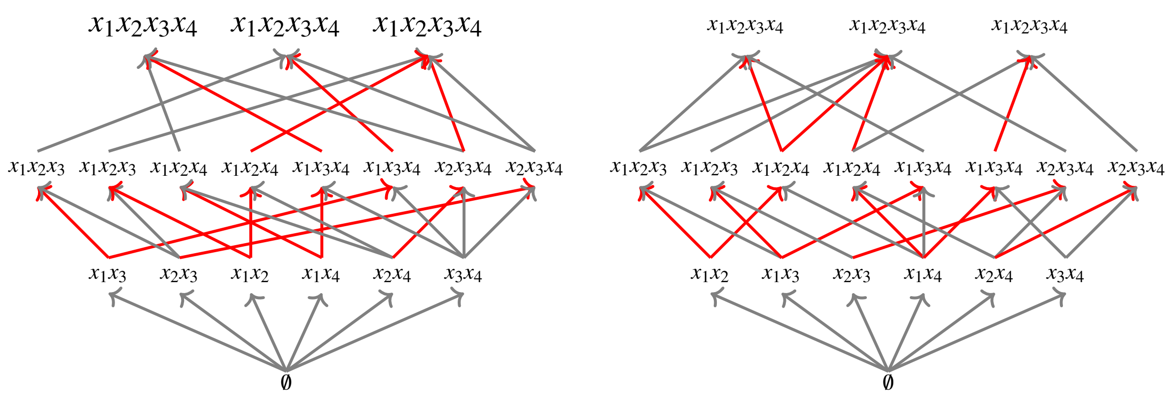


Figure 4.11: Two poset resolutions of K_4.

These two poset are different, for instance, the number of edges between the vertices of rank 3 and 4 is greater in the right poset than in the left poset. In some way that shows that the poset on the left side is minimal respect to the poset of the right side. As we will see later, the poset on the left side is the poset of a combinatorial resolution. However, these two poset resolutions are isomorphic as algebraic resolutions, as next commutative diagram shows.

$$
\begin{tikzcd}[column sep=large, row sep=huge]
S \arrow[d, "\mathrm{Id}"'] & \begin{array}{c} S(-x_1x_3)\oplus S(-x_2x_3) \\ \oplus \\ S(-x_1x_2)\oplus S(-x_1x_4) \\ \oplus \\ S(-x_2x_4)\oplus S(-x_3x_4) \end{array} \arrow[l, "{\begin{pmatrix} x_1x_3 & x_2x_3 & x_1x_2 & x_1x_4 & x_2x_4 & x_3x_4 \end{pmatrix}}"'] \arrow[d, "{\begin{pmatrix} 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}}"] & \begin{array}{c} S(-x_1x_2x_3)^2 \\ \oplus \\ S(-x_1x_2x_4)^2 \\ \oplus \\ S(-x_1x_3x_4)^2 \\ \oplus \\ S(-x_2x_3x_4)^2 \end{array} \arrow[l, "{\begin{pmatrix} -x_2&0&0&0&0&-x_4&0&0\\ x_1&x_1&0&0&0&0&0&-x_4\\ 0&-x_3&0&-x_4&0&0&0&0\\ 0&0&-x_2&0&-x_3&0&0&0\\ 0&0&x_1&x_1&0&0&-x_3&0\\ 0&0&0&0&x_1&x_1&x_2&x_2 \end{pmatrix}}"'] \arrow[d, "{\begin{pmatrix} 0&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1 \end{pmatrix}}"] & S(-x_1x_2x_3x_4)^3 \arrow[l, "{\begin{pmatrix} 0&x_4&0\\ 0&0&x_4\\ x_3&0&0\\ 0&0&-x_3\\ -x_2&0&0\\ 0&-x_2&0\\ x_1&0&-x_1\\ 0&x_1&x_1 \end{pmatrix}}"'] \arrow[d, "{\begin{pmatrix} 0&-1&0\\ 0&1&1\\ 1&1&0 \end{pmatrix}}"] \\
S & \begin{array}{c} S(-x_1x_2)\oplus S(-x_1x_3) \\ \oplus \\ S(-x_2x_3)\oplus S(-x_1x_4) \\ \oplus \\ S(-x_2x_4)\oplus S(-x_3x_4) \end{array} \arrow[l, "{\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 & x_1x_4 & x_2x_4 & x_3x_4 \end{pmatrix}}"'] & \begin{array}{c} S(-x_1x_2x_3)^2 \\ \oplus \\ S(-x_1x_2x_4)^2 \\ \oplus \\ S(-x_1x_3x_4)^2 \\ \oplus \\ S(-x_2x_3x_4)^2 \end{array} \arrow[l, "{\begin{pmatrix} -x_3&0&-x_4&0&0&0&0&0\\ x_2&-x_2&0&0&-x_4&0&0&0\\ 0&x_1&0&0&0&0&-x_4&0\\ 0&0&x_2&-x_2&x_3&-x_3&0&0\\ 0&0&0&x_1&0&0&x_3&-x_3\\ 0&0&0&0&0&x_1&0&x_2 \end{pmatrix}}"'] & S(-x_1x_2x_3x_4)^3 \arrow[l, "{\begin{pmatrix} x_4&x_4&0\\ 0&x_4&0\\ -x_3&-x_3&0\\ 0&-x_3&x_3\\ x_2&0&0\\ 0&0&-x_2\\ 0&x_1&0\\ 0&0&x_1 \end{pmatrix}}"']
\end{tikzcd}
$$

Figure 4.12: Two isomorphic minimal free resolutions of K_4.

In the top line is the algebraic resolution of the poset in the left side, that is, is the resolution of the combinatorial poset. In the bottom line is the algebraic resolution obtained from Macaulay2 ([12]) which is not combinatorial. And between these two resolution we put isomorphism that makes that the diagram commutes.

Next we give the third representation of a minimal free resolution.

4.4 The combinatorial resolution and its duplication

In this section we introduce a third new form to represent a minimal free resolution of a monomial ideal. As we will see, these new representations is also shorter than the algebraic resolution, however, each representation has some advantages and some disadvantages.

The study of the minimal free resolution of the edge ideal of the complete graph (see Chapter 3) was the starting point for the study of this kind of resolutions.

We start by giving the basic concept of a combinatorial resolution.

4.4.1 Basis sets

A basis set will be a set of some subgraphs of a graph G that will be labeling the rows and columns of the differentials of a minimal free resolution, and will be generators of the free modules. To complete a minimal free resolution, we introduce a scalar function that will give the coefficients in the monomial differentials of the resolution.

First we give some basic definitions and notations.

Given a graph G with set of vertices V we denote as $\mathbf{x}^V = \prod_{i\in V} x_i$ the monomial generated by the set of vertices. If $H = \{G_1, \ldots, G_t\}$ is a set of graphs, we denote as $\mathbf{x}^H$ the monomial defined by $\operatorname{lcm}(\mathbf{x}^{G_i} : G_i \in G)$. For instance, if G is an edge with vertices $V = \{1,2\}$, the monomial $\mathbf{x}^V$ is x_1x_2.

In next we assume that all graphs have no isolated vertices.

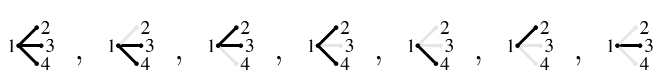
Moreover, if G is a graph with set of edges E, the **edge ideal** of G is the monomial ideal I_G minimally generated by $\{\mathbf{x}^f : f \in E\}$. That is, the edge ideal of G is the monomial ideal minimally generated by the monomials induced by the edges of G.

Definition 4.4.1. *A* **basis set** *of a graph G is a set $\Lambda(G)$, or simply Λ when is clear the context, that contains some subsets of subgraphs of G and the empty set such that they are pairwise different.*

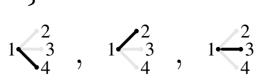
We will denote as $\Lambda_i(G)$, or simply Λ_i when is clear the context, a subset of Λ with $\beta_i(S/I_G)$ elements. Here $\beta_i(S/I_G)$ is the total Betti number of I_G. The set Λ_i is called a **basis set of dimension** *i.*

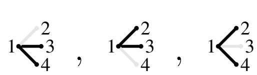
When an element of $\Lambda(G)$ has only one element, we do not use the braces of the notation of set.

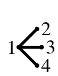
Example 4.4.2. *Let G be the star graph with four vertices. A basis set of G and the basis sets of dimension $i \in \{0,1,2,3\}$ are:*

$\Lambda = \{$  $, \emptyset \}$

$\Lambda_0 = \{ \emptyset \}$

$\Lambda_1 = \{$  $\}$

$\Lambda_2 = \{$  $\}$

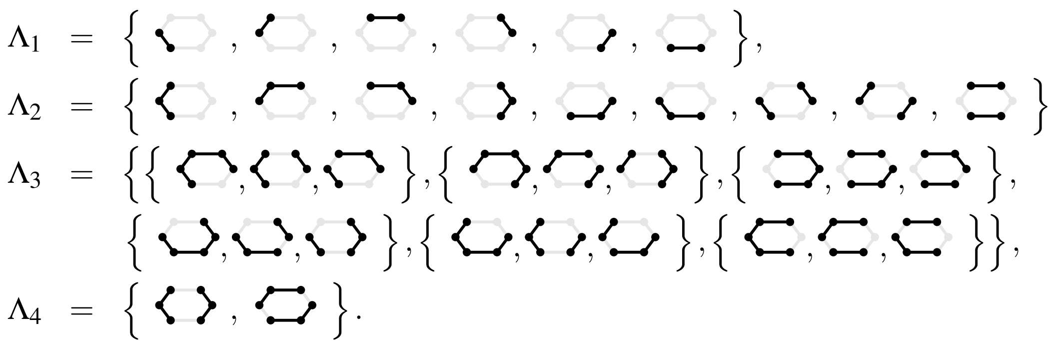
$\Lambda_3 = \{$  $\}$

In this case all the subsets of subgraphs of G have one element and notice that the graph G can be an element of $\Lambda(G)$ and the empty see too.

Next example shows a basis set for the cycle with six vertices C_6. This basis set is more interesting because some subsets of subgraphs have more than one element.

Example 4.4.3. *Let C_6 be the cycle with six vertices. The basis set of dimension $i \in \{0,1,2,3,4\}$ are:*

$$\Lambda_0 = \{\emptyset\},$$



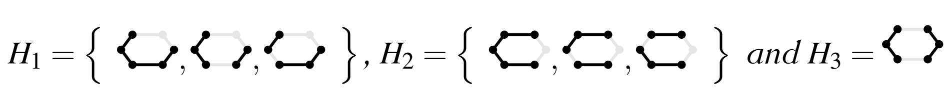
As we saw in this example, the basis set of dimension 3 has all its subsets with three elements and all of them are different. The complete Betti number $\beta_3(S/I_{C_6})$ is six, which coincides with the cardinality of Λ_3.

Once we have the set of basis sets, we need an scalar function to relate these sets. We recall that this scalar function will be the scalar that appears as coefficient in the entries of an algebraic resolution.

Definition 4.4.4. *Let Λ_{i-1} and Λ_i basis sets of G of dimension $i-1$ and i, respectively. A* **scalar function** *μ is a function $\mu : \Lambda_{i-1} \times \Lambda_i \to k$ such that $\mu(H_{i-1}, H_i) = 0$ if and only if no element of H_{i-1} is contained in H_i.*

To make the notation easier, we sometimes denote as $\mu_{H,G}$ the image of $\mu(H,G)$. Next example shows a scalar function.

Example 4.4.5. *Let*



where $H_1, H_2 \in \Lambda_3(C_6)$ *and* $H_3 \in \Lambda_4(C_6)$. *Then* $\mu_{H_1,H_3} = -1$ *and* $\mu_{H_2,H_3} = 0$ *is an scalar function because the second element of* H_1 *is contained in* H_3 *and no element of* H_2 *is contained in* H_3.

Another scalar function is $\mu'_{H_1,H_3} = 2$ *and* $\mu'_{H_2,H_3} = 0$. *If we define* μ_{H_2,H_3} *as a non-zero integer, then this function is not an scalar function.*

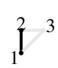
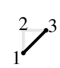
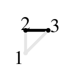
We have defined the poset resolution from an algebraic resolution but also we can get a poset resolution from a basis set Λ and an scalar function μ as follows: put a vertex 1 in rank 0. Then for each element in Λ_i, put a vertex in rank i labeled with the respective element of Λ_i. The edges of this poset are defined by containment between the vertices of rank $i-1$ and i and the label is given by the scalar function μ. In short terms, given Λ and μ we can define a poset resolution $(V,E)_\bullet$ as

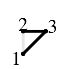
$$V = \{v_{i,\lambda} : \lambda \in \Lambda_i\}$$

and the set of edges is $E = \{(v_{i,\lambda}, v_{i+1,\tau})_{\mu(\lambda,\mu)}\}$.

Example 4.4.6. *Let* Λ *and* μ *be a basis set and a scalar function of the edge ideal* $I_G = \langle x_1x_2, x_1x_3, x_2x_3\rangle$ *defined as follows:*

$\Lambda_0 = \{\emptyset\}$

$\Lambda_1 = \Big\{ \Lambda_{1,1} =$  $, \Lambda_{1,2} =$  $, \Lambda_{1,3} =$  $\Big\}$

$\Lambda_2 = \Big\{ \Lambda_{2,1} =$  $, \Lambda_{2,2} =$  $\Big\}$

and the μ *function is:*

$\mu(\Lambda_0, \Lambda_{1,j}) = 1$ *for* $j \in \{1,2,3\}$.
$\mu(\Lambda_{1,1}, \Lambda_{2,1}) = -1$
$\mu(\Lambda_{1,2}, \Lambda_{2,2}) = -1$
$\mu(\Lambda_{1,3}, \Lambda_{2,1}) = 1$
$\mu(\Lambda_{1,3}, \Lambda_{2,2}) = 1$

The poset resolution obtained from this is:

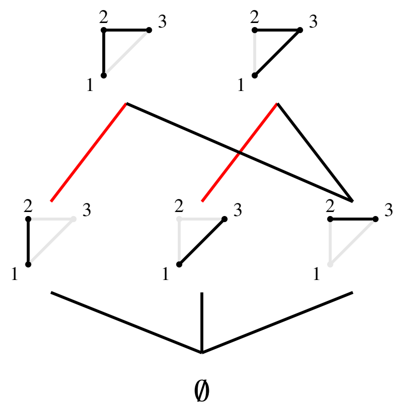


Figure 4.13: A poset resolution from a Λ and μ.

Examples 4.3.2 and 4.3.6 are also examples of posets that can be obtained from basis set and an scalar function.

Moreover, if we have a basis set and an scalar function we can define an algebraic resolutions based on them. Given a basis set Λ for a graph G and an scalar function, we can define a sequence of free modules as follows: between each free module F_i and F_{i-1} we define differentials $d_i : F_i \to F_{i-1}$ such that for each $H \in \Lambda_i$ we have

$$d_i(H) = \sum_{L \in \Lambda_{i-1}} \mu_{L,H} \mathbf{x}^{H \setminus L} \cdot L$$

where $H \setminus L$ are the vertices in H and not in L. Notice that if L is not contained in H then $\mu_{L,H}\mathbf{x}^{H \setminus L} = 0$. Between S and S/I_G we use the projection function π.

Remark 4.4.7. *Notice that the sets $L \in \Lambda_{i-1}$ are labeling the image $d_i(H)$, that is, we can use the sets $H \in \Lambda_i$ and $L \in \Lambda_{i-1}$ to label the rows and columns of the matrix representation of d_i.*

Thus we have the following sequence of free modules denoted $(\Lambda, \mu)_\bullet$:

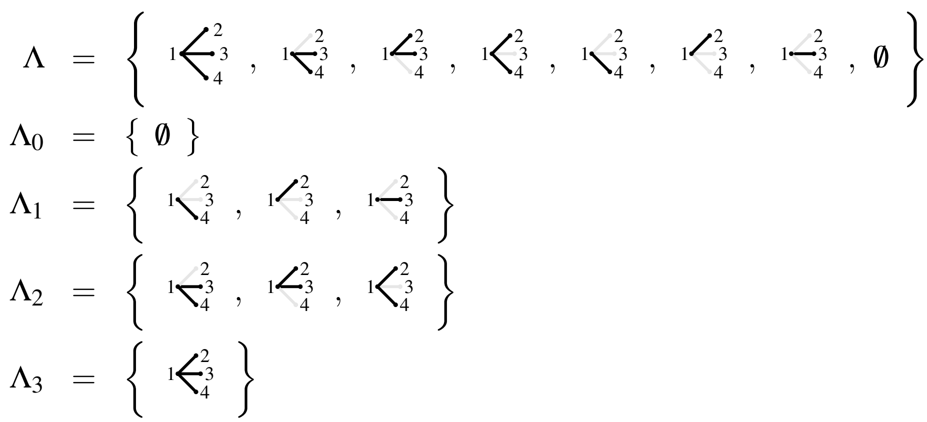
$$(\Lambda,\mu)_\bullet : 0 \leftarrow S/I_G \xleftarrow{\pi} S \xleftarrow{d_1} \bigoplus_{H\in\Lambda_1} S(H) \xleftarrow{d_2} \bigoplus_{H\in\Lambda_2} S(H) \leftarrow \cdots \leftarrow \bigoplus_{H\in\Lambda_{i-1}} S(H) \xleftarrow{d_i} \bigoplus_{H\in\Lambda_i} S(H) \leftarrow \ldots \xleftarrow{d_p} \bigoplus_{H\in\Lambda_p} S(H) \leftarrow 0.$$

This sequence of free modules always exists for the edge ideal of the complete graph (see Chapter 3), the start graph, and we will show that for the edge ideal of the complete multipartite graphs also exist.

In next example we show a free sequence $(\Lambda, \mu)_\bullet$ for the star graph with four vertices.

Example 4.4.8. *Let G be the star graph with four vertices. The following is the free sequence of free modules using a basis set of G.*

First, note that the basis set of G and the basis sets in each dimension are:



The free sequence defined before for the star graph is:

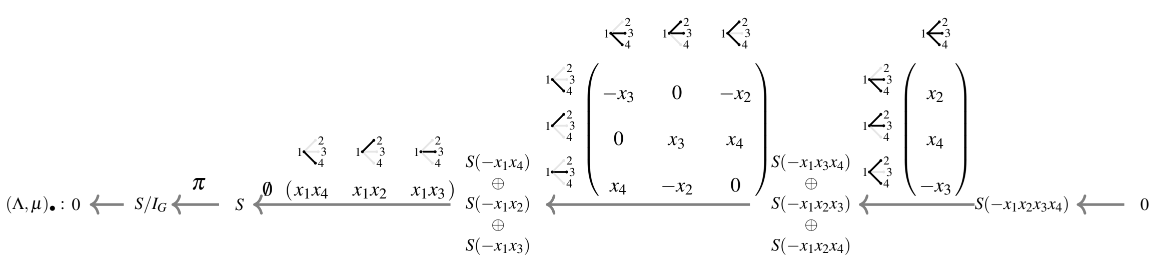


Figure 4.14: A free sequence for the star graph.

Remark 4.4.9. *Notice that this free sequence for the edge ideal of the star graph is an algebraic resolution of the edge ideal of the star graph.*

This motivates the following definition.

Definition 4.4.10. *Let G be a graph, Λ a basis set of G and μ a scalar function. If the sequence of free modules $(\Lambda, \mu)_\bullet$ is an algebraic resolution, then it is called a* **combinatorial resolution**.

As examples of combinatorial resolutions we have the examples 4.2.23 and the resolutions given for the edge ideal of the complete grah in Chapter 3.

Combinatorial resolutions have many advantages, for instance, we can save all the information of the minimal free resolution in the set Λ and the scalar function μ. However, not every minimal free resolution of a monomial ideal is a combinatorial resolution. Next example shows a non combinatorial resolution.

Example 4.4.11. *Let I_G be the edge ideal of the complete graph with four vertices. The following minimal free resolution was calculated using Macaulay2 ([12]).*

$$0 \longleftarrow S/I_G \longleftarrow S \xleftarrow{\begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 & x_1x_4 & x_2x_4 & x_3x_4 \end{pmatrix}} \begin{matrix} S(-x_1x_2)\oplus S(-x_1x_3) \\ \oplus \\ S(-x_2x_3)\oplus S(-x_1x_4) \\ \oplus \\ S(-x_2x_4)\oplus S(-x_3x_4) \end{matrix} \xleftarrow{\begin{pmatrix} -x_3 & 0 & -x_4 & 0 & 0 & 0 & 0 & 0 \\ x_2 & -x_2 & 0 & 0 & -x_4 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & -x_4 & 0 \\ 0 & 0 & x_2 & -x_2 & x_3 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & x_3 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & x_2 \end{pmatrix}} \begin{matrix} S(-x_1x_2x_3)^2 \\ \oplus \\ S(-x_1x_2x_4)^2 \\ \oplus \\ S(-x_1x_3x_4)^2 \\ \oplus \\ S(-x_2x_3x_4)^2 \end{matrix} \xleftarrow{\begin{pmatrix} x_4 & x_4 & 0 \\ 0 & x_4 & 0 \\ -x_3 & -x_3 & 0 \\ 0 & -x_3 & x_3 \\ x_2 & 0 & 0 \\ 0 & 0 & -x_2 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 \end{pmatrix}} S(-x_1x_2x_3x_4)^3 \longleftarrow 0$$

Figure 4.15: A no combinatorial resolution of K_4.

This minimal free resolution is not combinatorial because there is not a basis set Λ and an scalar function such that the third differential of this minimal free resolution coincides with the third differential of a combinatorial resolution.

4.4.2 The duplication of a combinatorial resolution

In this subsection we translate the duplication of a minimal free resolution to the duplication of a combinatorial resolution. To do this, we need to apply the R-copy to a basis set.

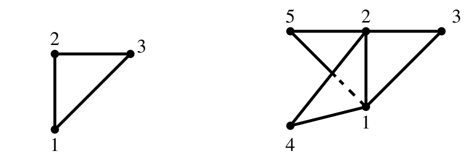
First we introduce the duplication of graphs.

Definition 4.4.12. *Let G be a graph and $V = [n+1]$ the set of vertices of G. The R-copy of G, where $R = \{n+1,\dots,n+m\} \subset \mathbb{N}$ is a set of vertices, is the graph G^R with set of $V = [n+m]$ and edges*

$$E(G^R) = E(G) \cup \{(i,j) : (n+1,j) \in E(G) \text{ and } i \in R\}.$$

We give an example of previous definition.

Example 4.4.13. *Let G be the graph in Figure 4.16 (a), with set of vertices $\{1,2,3\}$ and let R be the new set of vertices $\{3,4,5\}$. The R-copy of G is the graph G^R, see Figure 4.16 (b).*



(a) Graph G. (b) Graph G^R.

Figure 4.16: R-copy of the graph G.

Remark 4.4.14. *Notice that we only need to talk about the R-copy because for edge ideals the monomials are always square-free.*

Moreover, notice that the edge ideal of G^R is the same as the duplication of the edge ideal I_G.

As before, if we apply the R-copy on a graph G without the vertex $n+1$, then the R-copy of G is the same graph G, that is, $G^R = G$.

Given a set of graphs Λ and a vertex v, we denote as $\Lambda[v]$ the subset of Λ whose elements are the graphs of Λ that contains the vertex v, that is,

$$\Lambda[v] = \{G \in \Lambda : v \in V(G)\}.$$

If we have a set of graphs Λ, we denote as Λ^R the set of R-copies of the graphs in Λ. That is $\Lambda^R = \{G^R : G \in \Lambda\}$.

Now we are ready to give the duplication of a combinatorial resolution $(\Lambda,\mu)_\bullet$ of a graph G.

Definition 4.4.15. *Let $(\Lambda,\mu)_\bullet$ be a combinatorial resolution of a graph G with $n+1$ vertices and $Y = \{n+1,\ldots,n+m\}$ a set of vertices.*

Let

$$\Lambda_i^\diamond = \Lambda_i \cup \bigcup_{\emptyset \neq R \subseteq Y} \Lambda_{i-|R|+1}[n+1]^R$$

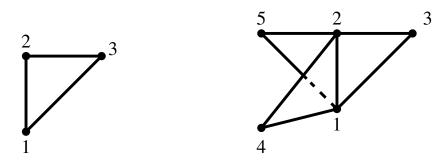
and $\mu^\diamond : \Lambda^\diamond_{i-1} \times \Lambda^\diamond_i \to \mathbb{Z}$ *defined as*

$$\mu^\diamond(H^{R'}, H^R) = \begin{cases} \mu(H', H) & \text{if } R = \{j\} = R' \text{ with } j \in Y, \\ (-1)^{|R|-1}\mu(H', H) & \text{if } \emptyset \neq R' = R, \\ (-1)^{t+1}\mu(H', H) & \text{if } R \smallsetminus R' = y_t \text{ and } R = \{y_1, \ldots, y_s\} \text{ with } y_1 < \cdots < y_s, \\ 0 & \text{otherwise,} \end{cases}$$

where $R, R' \subseteq Y$.

The sequence of free modules $(\Lambda^\diamond, \mu^\diamond)_\bullet$ *is the duplication of* $(\Lambda, \mu)_\bullet$.

Example 4.4.16. *In Example 4.2.23 we give a graph* G *and a duplication of* G *with* $R = \{3,4,5\}$ *denoted* $G^\diamond$.



(a) Graph G. (b) Graph $G^\diamond$.

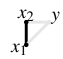
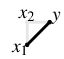
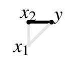
Figure 4.17: Duplication of the graph G.

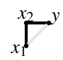
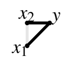
A combinatorial resolution of I_G *is:*

$$\mathbf{F}_\bullet : 0 \longleftarrow S/I_G \xleftarrow{\pi} S \xleftarrow{\emptyset\ \begin{pmatrix} x_1x_2 & x_1x_3 & x_2x_3 \end{pmatrix}} \begin{matrix} S(-x_1x_2) \\ \oplus \\ S(-x_1y) \\ \oplus \\ S(-x_2y) \end{matrix} \xleftarrow{\begin{pmatrix} -y & 0 \\ 0 & -x_2 \\ x_1 & x_1 \end{pmatrix}} S(-x_1x_2y)^2 \longleftarrow 0$$

Figure 4.18: A combinatorial resolution of I_G.

Where

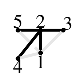
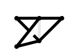
$$\begin{aligned}
\Lambda_0 &= \{\emptyset\} \\
\Lambda_1 &= \left\{ \Lambda_{1,1} = \right.
\end{aligned}$$  $, \Lambda_{1,2} =$  $, \Lambda_{1,3} =$  $\left. \right\}$

$$\Lambda_2 = \left\{ \Lambda_{2,1} = \right.$$  $, \Lambda_{2,2} =$  $\left. \right\}$

and the μ function is:

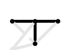
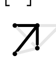
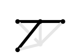
$\mu(\Lambda_0, \Lambda_{1,j}) = 1$ *for* $j \in \{1,2,3\}$.
$\mu(\Lambda_{1,1}, \Lambda_{2,1}) = -1$
$\mu(\Lambda_{1,2}, \Lambda_{2,2}) = -1$
$\mu(\Lambda_{1,3}, \Lambda_{2,1}) = 1$
$\mu(\Lambda_{1,3}, \Lambda_{2,2}) = 1$

With this basis set Λ and the scalar function μ we can define the duplication of $(\Lambda, \mu)_\bullet$. For instance, if $Y = \{3,4,5\}$, then

$$\begin{aligned}
\Lambda_4^\diamond &= \Lambda_4 \cup \bigcup_{\emptyset \neq R \subseteq Y} \Lambda_{4-|R|+1}[3]^R = \Lambda_2[3]^R \text{ with } R = \{3,4,5\} \\
&= \{\Lambda_{2,1}^R \cup \Lambda_{2,2}^R\} = \left\{ \Lambda_{2,1}^R = \right.
\end{aligned}$$  $, \Lambda_{2,2}^R =$  $\left. \right\}$

Also we have for $\Lambda_2^\diamond$:

$$\begin{aligned}
\Lambda_3^\diamond &= \Lambda_3 \cup \bigcup_{\emptyset \neq R \subseteq Y} \Lambda_{3-|R|+1}[3]^R \\
&= \Lambda_2[3]^{\{3,4\}} \cup \Lambda_2[3]^{\{3,5\}} \cup \Lambda_2[3]^{\{4,5\}} \cup \Lambda_1[3]^Y
\end{aligned}$$

$= \{$  ,  $\} \cup \{$  ,  $\} \cup \{$  ,  $\} \cup \{$  ,  $\}$

And the fourth differential is

$$D_4^{\diamond} = \begin{pmatrix} y_3 & 0 \\ 0 & y_3 \\ -y_2 & 0 \\ 0 & -y_2 \\ y_1 & 0 \\ 0 & y_1 \\ 0 & -x_2 \\ x_1 & x_1 \end{pmatrix}$$

The complete resolution $(\Lambda^{\diamond}, \mu^{\diamond})$ *is given in Example 4.2.23.*

We finish this section with the next theorem.

Theorem 4.4.17. *Let* $(\Lambda, \mu)_{\bullet}$ *be a combinatorial resolution of a graph* G*. Then* $(\Lambda^{\diamond}, \mu^{\diamond})_{\bullet}$ *is also a combinatorial resolution of* $G^{\diamond}$*.*

Proof. First notice that if H is a basis set, then H^R also is: it is not difficult to see that the R-copy of different graphs are also different graphs. Thus $\Lambda^{\diamond}$ is a basis set of $G^{\diamond}$ and the rest of the proof follows from Theorem 4.2.15. $\square$

Chapter 5

Other results

In this chapter we group some results that can be seen as the application of previous two chapters.

First, we give a minimal free resolution of the edge ideal of the complete multipartite graph. This minimal free resolution can be defined from a combinatorial resolution of a complete graph contained in the complete multipartite graph, but this can also be done from a poset or an algebraic resolution of some complete graph contained in the multiparte graph.

Secondly, we give the Betti numbers of the edge ideal of join graphs and cographs. All the description is based on the presence of a complete multipartite graphs as spanning graph of a graph. Moreover, if a graph is a subgraph of a complete multipartite graph, then its projective dimension is bounded by the projective dimension of the complete multipartite graph.

From the Betti numbers of join graphs we saw that the projective dimension of these edge ideals are bounded by $n-1$, where n is the number of vertices of the graph. Then, a natural next step is to wonder about the monomial ideals whose projective dimension is maximum. In the second section of this chapter we develop this study which was done first in [1], however here we make different proofs.

5.1 A minimal free resolution of the complete multipartite graph

In this subsection we give a minimal free resolution for a complete multipartite graph. Indeed, we give a combinatorial resolution, a poset resolution and an algebraic resolution. We use the basis set and the scalar function defined in Chapter 3 for the edge ideal of the complete graph.

In [32] is given a minimal free resolution only for the edge ideal of complete bipartite graph. This resolution is defined from a cellular complex using a cellular resolution.

First we define what is a complete multipartite graph.

Definition 5.1.1. *A graph $K_{k_1,\dots,k_t} = (V,E)$ is a* **complete multipartite graph** *if $V = \bigsqcup_{i=1}^{t} V_{k_i}$ with $|V_{k_i}| = k_i$ and*

$$E = \{(v_i, v_j) : v_i \in V_{k_i}, v_j \in V_{k_j} \text{ and } i \neq j\}.$$

In this case we say that $K_{k_1,\dots,k_t}$ is a complete t-partite graph.

Notice that if G is a complete t-partite graph, then G contains as induced subgraph a complete graph with t vertices. This induced complete graph will be the begining to define the combinatorial resolution of a t-partite graph.

On the set of vertices of a t-partite graph we can define an order as follows: $v_i < v_j$ if and only if $v_i \in V_{k_i}$, $v_j \in V_{k_j}$ and $i < j$. In this way we have that the complete graph with t vertices that is contained in the t-partite graph, has an order over its vertices.

With all this, we are ready to give a combinatorial resolution for a complete t-partite graph.

Let K_t the complete graph with set of vertices $V' = \{v_i : v_i \in V_{k_i}\}$ and let $\Lambda = \{B_A^i : A \subseteq V'\}$ be a basis set where B_A^i as in Definition 3.3.1. On Λ consider the scalar function σ defined in Definition 3.3.5. Finally let $(\Lambda, \sigma)_\bullet$ be the combinatorial resolution of K_t.

Theorem 5.1.2. *Let Λ_1 be a basis set for $K_{k_1,1,\dots,1}$ obtained as $\Lambda_1 = \Lambda^R$ with $R = V_{k_1}$ and σ_1 the scalar function as in definition 4.4.15. Then the combinatorial resolution of $K_{k_1,\dots,k_t}$ is $\Lambda^\diamond = \Lambda_{t-1}^R$ with $R = V_{k_t}$, and scalar function $\sigma^\diamond$ obtained from σ_{t-1}.*

With this process we obtain a combinatorial resolution of the complete multipartite graph $K_{k_1,\dots,k_t}$ after at most t steps.

If we have an algebraic resolution of the edge ideal of a complete subgraph of G, then applying the duplication of algebraic resolutions, we obtain an algebraic resolution of the edge ideal of G. The same happens if we start with a poset resolution of the edge ideal of a complete subgraph of G.

In Example 4.2.23 G is the complete induced subgraph of $K_{3,1,1} = G^{\diamond}$ and we make the duplication of both algebraic and combinatorial resolutions because we are labeling the columns of rows of the algebraic resolution with subgraphs of G.

Next we give another example.

Example 5.1.3. *Here We give a poset resolution of the edge ideal of $K_{3,2}$. We start with the poset resolution of the complete graph with two vertices, that is, the poset resolution of the graph with vertices labeled with v_1 and v_2 which is in the bottom left.*

From the poset resolution of K_2 we obtain the poset resolution of $K_{3,1}$ applying duplication on the vertex v_1. The poset resolution is the poset in the gray block in Figure 5.1.

Again, applying duplication on the vertex v_2 to the poset resolution of $K_{3,1}$ we get the poset resolution of $K_{3,2}$.

In this poset the red edges are edges labeled with -1, and gray edges are labeled with 1. The complete poset resolution is given next:

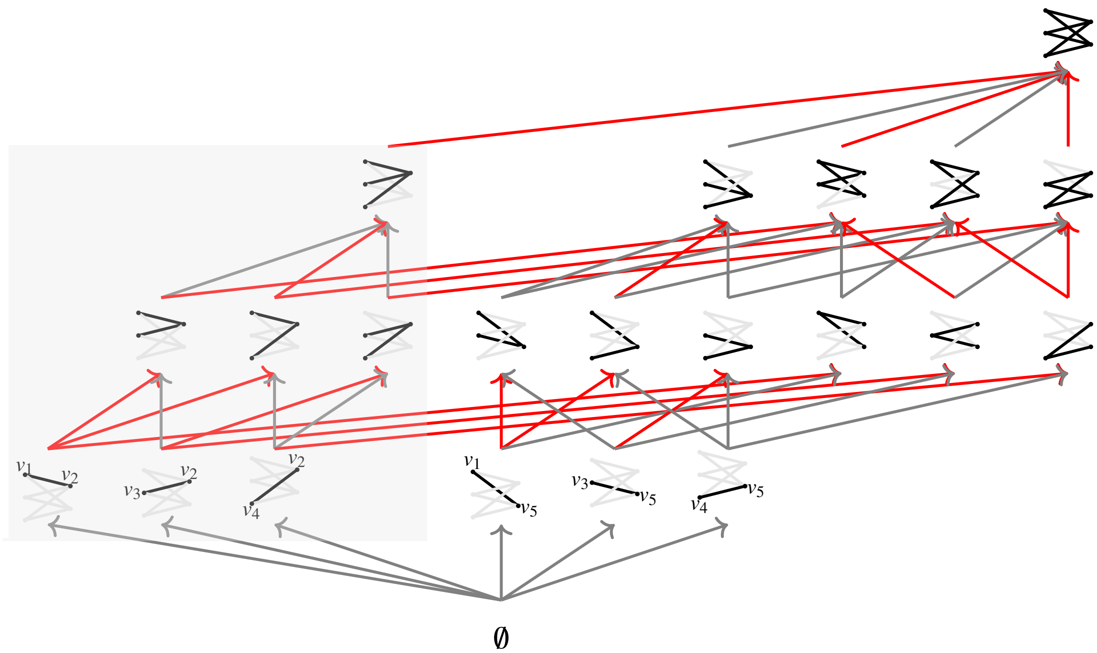


Figure 5.1: Poset Resolution of $K_{3,2}$.

5.2 A minimal free resolution of the disjoint union of monomial ideals

In this section we give a minimal free resolution for the disjoint union of two monomial ideals. This minimal free resolution will be given in terms of the minimal free resolution of each monomial ideal.

We start by recalling the definition of a minimal free resolution of a monomial ideal.

Let $S = k[\mathbf{x}]$ be a polynomial ring over a field k and set of variables $\mathbf{x} = \{x_1, \ldots, x_n\}$. A graded free resolution of a monomial ideal I is a graded exact sequence of free modules which first non trivial module is S/I. More in detail, an $\mathbb{N}^n$-graded complex is a sequence $\mathbf{F}_\bullet = \{F_i, d_i\}_{i\in\mathbb{Z}}$ of $\mathbb{N}^n$-graded free modules $F_i = \bigoplus_{\mathbf{a}\in\mathbf{A}_i} S(-\mathbf{a})$ where $\mathbf{A}_i$ is a multiset of $\mathbb{N}^n$, $S(-\mathbf{a})$ is the free S-module obtained by shifting S by the degree $\mathbf{a}$, and a sequence of homogeneous $\mathbb{N}^n$-graded maps $d_i : F_i \to F_{i-1}$, such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. Moreover, when the complex $\mathbf{F}_\bullet$ is exact, or more precisely

$\mathrm{Im}(d_{i+1}) = \ker(d_i)$, and

$$\mathrm{F}_\bullet : 0 \leftarrow S/I \xleftarrow{\pi} S \xleftarrow{d_1} F_1 \xleftarrow{d_2} \cdots \xleftarrow{d_{i-1}} F_{i-1} \xleftarrow{d_i} F_i \xleftarrow{d_{i+1}} \cdots \xleftarrow{d_p} F_p \leftarrow 0$$

where $d_0 = \pi$ is the projection map from S to S/I, $F_0 = S$ and $F_{-1} = S/I$, then $\mathbf{F}_\bullet$ is a free resolution of the ideal I or of the module S/I. And we say that $\mathbf{F}_\bullet$ is minimal if the image of each map d_i is in the maximal ideal $\langle x_1, \ldots, x_n \rangle$.

In following, we will consider two monomial ideals I and J in two different polynomial rings in such a way that the intersection of these two monomial ideals is empty. From some minimal free resolutions of each monomial ideal I and J, we will define explicitly a minimal free resolution for the monomial ideal obtained from the disjoint union of I and J, that is, a minimal free resolution of $I \sqcup J$.

Let $I = \langle \mathbf{g}_1, \ldots, \mathbf{g}_t \rangle$ be a monomial ideal of a polynomial ring $S = k[\mathbf{x}]$ and $J = \langle \mathbf{m}_1, \ldots, \mathbf{m}_u \rangle$ be a monomial ideal of a polynomial ring $R = k[\mathbf{y}]$, where $\mathbf{x} \cap \mathbf{y} = \emptyset$, that is, both I and J are disjoint. Moreover, let $\mathbf{F}_\bullet$ and $\mathbf{H}_\bullet$ be two graded minimal free resolutions of I and J, respectively. That is,

$$\mathrm{F}_\bullet : 0 \leftarrow S/I \xleftarrow{\pi} S \xleftarrow{d_1} F_1 \xleftarrow{d_2} \cdots \xleftarrow{d_{i-1}} F_{i-1} \xleftarrow{d_i} F_i \xleftarrow{d_{i+1}} \cdots \xleftarrow{d_p} F_p \leftarrow 0$$

and

$$\mathrm{H}_\bullet : 0 \leftarrow R/J \xleftarrow{\pi} R \xleftarrow{h_1} H_1 \xleftarrow{h_2} \cdots \xleftarrow{h_{i-1}} H_{i-1} \xleftarrow{h_i} H_i \xleftarrow{h_{i+1}} \cdots \xleftarrow{h_q} H_q \leftarrow 0.$$

Caution 5.2.1. *Here we need that both minimal free resolutions are graded with the same monoid.*

Thus, we assume that $\mathbf{F}_\bullet$ is $\mathbb{N}^n$-graded and $\mathbf{H}_\bullet$ is $\mathbb{N}^m$-graded.

Let $I \sqcup J$ be the monomial ideal minimally generated by $\{\mathbf{g}_1, \ldots, \mathbf{g}_t\} \sqcup \{\mathbf{m}_1, \ldots, \mathbf{m}_u\}$. From both resolution $\mathbf{F}_\bullet$ and $\mathbf{H}_\bullet$ we define the following sequence of free modules: $L_{-1} = T/I \sqcup J$, where $T = k[\mathbf{x} \sqcup \mathbf{y}]$ is the polynomial ring over a field k and set of variables $\mathbf{x} \sqcup \mathbf{y}$, $L_0 = T$ and for all $i \geqslant 1$ we have

$$L_i = \bigoplus_{l=0}^{i} \left(\bigoplus_{(\mathbf{a},\mathbf{b}) \in \mathbf{A}_{i-l} \times \mathbf{B}_l} T(-\mathbf{ab}) \right),$$

where $\mathbf{A}_i$ and $\mathbf{B}_i$ are the degrees of the free modules F_i and H_i, respectively, and $\mathbf{ab}$ is the vector in $\mathbb{N}^{n+m}$ defined as:

$$\mathbf{ab}_i = \begin{cases} \mathbf{a}_i & \text{if } 1 \leqslant i \leqslant n, \\ \mathbf{b}_i & \text{if } n+1 \leqslant i \leqslant m. \end{cases}$$

With the aim to make short the notation, we will say that a degree $\mathbf{ab}$ is in $\mathbf{A}_i \times \mathbf{B}_j$ whenever $(\mathbf{a}, \mathbf{b}) \in \mathbf{A}_i \times \mathbf{B}_j$.

Given a degree $\mathbf{ab} \in L_i$, with $\mathbf{a} \in \mathbf{A}_{i-l}$ and $\mathbf{b} \in \mathbf{B}_l$, we define the differential δ_i defined as:

$$\delta_i(\mathbf{x^a y^b}) = d_{i-l}(\mathbf{x^a}) \cdot \mathbf{y^b} + (-1)^{i-l-1} h_l(\mathbf{y^b}) \cdot \mathbf{x^a},$$

for all $i \geqslant 1$. Here $\mathbf{x^a}$ and $\mathbf{y^b}$ denote basis elements in degrees $\mathbf{a}$ and $\mathbf{b}$, respectively.

Then we have the following graded sequence of free modules of $I \sqcup J$:

$$\mathrm{L}_\bullet : 0 \leftarrow T/I \sqcup J \xleftarrow{\pi} T \xleftarrow{\delta_1} L_1 \xleftarrow{\delta_2} \cdots \xleftarrow{\delta_{i-1}} L_{i-1} \xleftarrow{\delta_i} L_i \xleftarrow{\delta_{i+1}} \cdots \xleftarrow{\delta_{p+q}} L_{p+q} \leftarrow 0.$$

Next we have to show that this sequence $\mathbf{L}_\bullet$ is a minimal free resolution of $I \sqcup J$, but first we give an example to illustrate previous construction.

Example 5.2.2. *Let $\mathbf{F}_\bullet$ be a minimal free resolution of a monomial ideal $I = \langle x_1^2, x_1x_2^2 \rangle$ of $S = k[\mathbf{x}]$ and $\mathbf{H}_\bullet$ be a minimal free resolution of a monomial ideal $J = \langle y_1y_3, y_2y_3 \rangle$ of $R = k[\mathbf{y}]$:*

$$\mathbf{F}_\bullet : 0 \longleftarrow S/I \xleftarrow{\pi} S \xleftarrow{\begin{array}{cc} & \begin{array}{cc} \mathbf{x}_1^2 & \mathbf{x}_1\mathbf{x}_2^2 \end{array} \\ \mathbf{1} & \begin{pmatrix} x_1^2 & x_1x_2^2 \end{pmatrix} \end{array}} F_1 \xleftarrow{\begin{array}{cc} & \mathbf{x}_1^2\mathbf{x}_2^2 \\ \begin{array}{c} \mathbf{x}_1^2 \\ \mathbf{x}_1\mathbf{x}_2^2 \end{array} & \begin{pmatrix} -x_2^2 \\ x_1 \end{pmatrix} \end{array}} F_2 \longleftarrow 0,$$

$$\mathbf{H}_\bullet : 0 \longleftarrow R/J \xleftarrow{\pi} R \xleftarrow{\begin{array}{cc} & \begin{array}{cc} \mathbf{y}_1\mathbf{y}_3 & \mathbf{y}_2\mathbf{y}_3 \end{array} \\ \mathbf{1} & \begin{pmatrix} y_1y_3 & y_2y_3 \end{pmatrix} \end{array}} H_1 \xleftarrow{\begin{array}{cc} & \mathbf{y}_1\mathbf{y}_2\mathbf{y}_3 \\ \begin{array}{c} \mathbf{y}_1\mathbf{y}_3 \\ \mathbf{y}_2\mathbf{y}_3 \end{array} & \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \end{array}} H_2 \longleftarrow 0.$$

The sequence of free modules $\mathbf{L}_\bullet$ defined for the monomial ideal $I \sqcup J = \langle x_1^2, x_1x_2^2, y_1y_3, y_2y_3 \rangle$ in the polynomial ring $T = k[x_1, x_2, y_1, y_2, y_3]$ is:

$$\mathbf{L}_\bullet : 0 \leftarrow T/I \sqcup J \xleftarrow{\pi} T \xleftarrow{\begin{array}{c|cccc} & \mathbf{x_1^2} & \mathbf{x_1x_2^2} & \mathbf{y_1y_3} & \mathbf{y_2y_3} \\ \hline \mathbf{1} & \multicolumn{4}{c}{\begin{pmatrix} x_1^2 & x_1x_2^2 & y_1y_3 & y_2y_3 \end{pmatrix}} \end{array}} L_1 \xleftarrow{\begin{array}{c|cccccc} & \mathbf{x_1^2x_2^2} & \mathbf{x_1^2y_1y_3} & \mathbf{x_1^2y_2y_3} & \mathbf{x_1x_2^2y_1y_3} & \mathbf{x_1x_2^2y_2y_3} & \mathbf{y_1y_2y_3} \\ \hline \mathbf{x_1^2} & \multicolumn{6}{c}{\multirow{4}{*}{}} \end{array}\;\begin{array}{l}\mathbf{x_1^2}\\ \mathbf{x_1x_2^2}\\ \mathbf{y_1y_3}\\ \mathbf{y_2y_3}\end{array}\begin{pmatrix} -x_2^2 & -y_1y_3 & -y_2y_3 & 0 & 0 & 0 \\ x_1 & 0 & 0 & -y_1y_3 & -y_2y_3 & 0 \\ 0 & x_1^2 & 0 & x_1x_2^2 & 0 & -y_2 \\ 0 & 0 & x_1^2 & 0 & x_1x_2^2 & y_1 \end{pmatrix}} L_2$$

$$L_2 \xleftarrow{\begin{array}{c} \begin{array}{cccc} \mathbf{x_1^2x_2^2y_1y_3} & \mathbf{x_1^2x_2^2y_2y_3} & \mathbf{x_1^2y_1y_2y_3} & \mathbf{x_1x_2^2y_1y_2y_3} \end{array} \\ \begin{array}{l}\mathbf{x_1^2x_2^2}\\ \mathbf{x_1^2y_1y_3}\\ \mathbf{x_1^2y_2y_3}\\ \mathbf{x_1x_2^2y_1y_3}\\ \mathbf{x_1x_2^2y_2y_3}\\ \mathbf{y_1y_2y_3}\end{array}\begin{pmatrix} y_1y_3 & y_2y_3 & 0 & 0 \\ -x_2^2 & 0 & y_2 & 0 \\ 0 & -x_2^2 & -y_1 & 0 \\ x_1 & 0 & 0 & y_2 \\ 0 & x_1 & 0 & -y_1 \\ 0 & 0 & x_1^2 & x_1x_2^2 \end{pmatrix} \end{array}} L_3 \xleftarrow{\begin{array}{c} \mathbf{x_1^2x_2^2y_1y_2y_3} \\ \begin{array}{l}\mathbf{x_1^2x_2^2y_1y_3}\\ \mathbf{x_1^2x_2^2y_2y_3}\\ \mathbf{x_1^2y_1y_2y_3}\\ \mathbf{x_1x_2^2y_1y_2y_3}\end{array}\begin{pmatrix} -y_2 \\ y_1 \\ -x_2^2 \\ x_1 \end{pmatrix} \end{array}} L_4 \longleftarrow 0$$

In these sequences we are labeling the rows and columns of the differentials with their correspondent degree. We can do this because all these sequences are graded.

Now we should see that the free sequence $\mathbf{L}_\bullet$ is a minimal free resolution of $I \sqcup J$. Lets see first that it is a complex.

Proposition 5.2.3. *Let I and J be two monomial ideals in the polynomial rings $S = k[\mathbf{x}]$ and $R = k[\mathbf{y}]$, respectively, with $\mathbf{x} \cap \mathbf{y} = \emptyset$. Then the sequence of free modules $\mathbf{L}_\bullet$ defined as before, is a free complex of the monomial ideal $I \sqcup J$.*

Proof. We need to show that $\mathrm{Im}(\delta_i) \subseteq \ker(\delta_{i-1})$, or equivalently, we need to show that $\delta_{i-1}(\delta_i(\mathbf{x^a y^b})) = 0$ for any i and any $\mathbf{ab}$ in L_i.

Let $\mathbf{ab} \in \mathbf{A}_{i-l} \times \mathbf{B}_l$ a degree of L_i and notice that:

$$\begin{aligned} \delta_{i-1}(\delta_i(\mathbf{x^a y^y})) &= \delta_{i-1}(d_{i-l}(\mathbf{x^a})\mathbf{y^b} + (-1)^{i-l-1}h_l(\mathbf{y^b})\mathbf{x^a}) \\ &= \delta_{i-1}(d_{i-l}(\mathbf{x^a})\mathbf{y^b}) + (-1)^{i-l-1}\delta_{i-1}(\mathbf{x^a}h_l(\mathbf{y^b})) \\ &= d_{i-l-1}(d_{i-l}(\mathbf{x^a}))\mathbf{y^b} + (-1)^{i-l-2}d_{i-l}(\mathbf{x^a})h_l(\mathbf{y^b}) \\ &\quad +(-1)^{i-l-1}\left(d_{i-l}(\mathbf{x^a})h_l(\mathbf{y^b}) + (-1)^{i-l-2}h_{l-1}(h_l(\mathbf{y^b}))\mathbf{x^a}\right). \end{aligned}$$

At this point we should recall that $\mathbf{F}_\bullet$ and $\mathbf{H}_\bullet$ are complexes of I and J, respectively. That is, $d_{i-1}(d_i(\mathbf{x^a})) = 0$ for all i and $h_{l-1}(h_l(\mathbf{y^b})) = 0$ for all l. Therefore,

$$\delta_{i-1}(\delta_i(\mathbf{x^a y^b})) = (-1)^{i-l-2}d_{i-l}(\mathbf{x^a})h_l(\mathbf{y^b}) + (-1)^{i-l-1}d_{i-l}(\mathbf{x^a})h_l(\mathbf{y^b}) = 0,$$

which means that $\mathbf{L}_\bullet$ is a complex. $\square$

It only remains to show that the free complex $\mathbf{L}_\bullet$ is exact. To see this we will apply the Criterion given in this thesis in Theorem 3.2.6. That is, we need to compute the Betti numbers of a minimal free resolution of $I \sqcup J$ and show that these Betti numbers coincide with the ranks of the free modules of $\mathbf{L}_\bullet$, and we will show that the columns of the differentials δ_i are irredundant.

To calculate the Betti numbers of a minimal free resolution we use the Hochster's formula (see Theorem 2.3.3). Before continue, we need some previous results.

Definition 5.2.4. *Let Δ and Δ' be two simplicial complexes. The* **join** *of $\Delta \subset \{0,1\}^n$ and $\Delta' \subset \{0,1\}^m$ is the simplicial complex given by*

$$j(\Delta,\Delta') = \{(\tau,\tau') : \tau \in \Delta \text{ and } \tau' \in \Delta'\} \subseteq \{0,1\}^{n+m}.$$

Observe that $j(\Delta,\emptyset) = \emptyset$ and $j(\Delta,\{\emptyset\}) = \Delta$.

The following Theorem says how is the homology of the join of two simplicial complexes.

Theorem 5.2.5. *Let Δ be a simplicial complex such that all homology groups of Δ are free k-modules. Let Δ' be any simplicial complex. Then*

$$\tilde{H}_n(j(\Delta,\Delta');k) \simeq \bigoplus_{i+j=n-1} \tilde{H}_i(\Delta;k) \otimes_k \tilde{H}_j(\Delta';k).$$

Proof. See [19, Corollary 4.23]. □

A very well known tool in algebraic topology is the Mayer-Vietoris sequence. Next we recall it. Let X be a topological space and A, B be two subspaces whose interiors covers X. The **Mayer-Vietoris Sequence** for the triad (X,A,B) is the long exact sequence:

$$\ldots \to \tilde{H}_{n+1}(X) \xrightarrow{\partial_*} \tilde{H}_n(A\cap B) \xrightarrow{\binom{i_*}{j_*}} \tilde{H}_n(A) \oplus \tilde{H}_n(B) \xrightarrow{k_*-l_*} \tilde{H}_n(X) \xrightarrow{\partial_*} \tilde{H}_{n-1}(A\cap B) \to \ldots$$

Here $i : A\cap B \hookrightarrow A$, $j : A\cap B \hookrightarrow B$, $k : A \hookrightarrow X$ and $l : B \hookrightarrow X$ are inclusion maps. Since this sequence is exact, if we show that $\tilde{H}_n(A) \oplus \tilde{H}(B) = 0$ for all n, then we will have that $\tilde{H}_n(X) \simeq \tilde{H}_{n-1}(A\cap B)$. This idea will be useful in next lemma.

Lemma 5.2.6. *Let I and J be two disjoint monomial ideals and* **a**, **b** *be two degrees. Then*

$$\tilde{H}_n(K^{\mathbf{ab}}(I\sqcup J);k) \simeq \bigoplus_{i+j=n-2} \tilde{H}_i(K^{\mathbf{a}}(I);K) \otimes_k \tilde{H}_j(K^{\mathbf{b}}(J);k).$$

Proof. From the definition of the upper Koszul simplicial complex, it is easy to see that $K^{\mathbf{ab}}(I \sqcup J) = j(K^{\mathbf{a}}(I), \mathbf{1}') \cup j(\mathbf{1}, K^{\mathbf{b}}(J))$, where $\mathbf{1} \subseteq \{0,1\}^n$ is the simplicial complex generated by the simplex $\mathbf{1} = (1,\ldots,1)$. Moreover, it is easy to see that $j(K^{\mathbf{a}}(I), \mathbf{1}') \cap j(\mathbf{1}, K^{\mathbf{b}}(J)) = j(K^{\mathbf{a}}(I), K^{\mathbf{b}}(J))$. Then, applying the Mayer-Vietoris sequence we have that

$$\tilde{H}_n(K^{\mathbf{ab}}(I \sqcup J); k) \simeq \tilde{H}_{n-1}(j(K^{\mathbf{a}}(I), K^{\mathbf{b}}(J)); k).$$

Then, the result follows from previous isomorphism and Theorem 5.2.5. □

As a corollary we have formulas for the Betti numbers of the disjoint union of two monomial ideals.

Corollary 5.2.7. *Let I and J two disjoint monomial ideals and $\mathbf{a}$ and $\mathbf{b}$ be two degrees. Then the Betti number of $I \sqcup J$ in degree $\mathbf{ab}$ is:*

$$\begin{aligned}
\beta_{n+1,\mathbf{ab}}(T/I \sqcup J) &= \dim_k \tilde{H}_{n-1}(K^{\mathbf{ab}}(I \sqcup J); k) \\
&= \dim_k \bigoplus_{i+j=n-3} \tilde{H}_i(K^{\mathbf{a}}(I); k) \otimes_k \tilde{H}_j(K^{\mathbf{b}}(J); k) \\
&= \sum_{i+j=n-3} \beta_{i+2,\mathbf{a}}(S/I)\beta_{j+2,\mathbf{b}}(R/J) \\
&= \sum_{i+j=n+1} \beta_{i,\mathbf{a}}(S/I)\beta_{j,\mathbf{b}}(R/J).
\end{aligned}$$

Since we have a formula for the Betti numbers of $I \sqcup J$ in terms of each Betti number of I and J, now we can show that the free complex $\mathbf{L}_\bullet$ is exact and minimal.

Theorem 5.2.8. *Let I and J be two disjoint monomial ideals. The free complex $\mathbf{L}_\bullet$ is a minimal free resolution of the monomial ideal $I \sqcup J$.*

Proof. We will apply the Criterion given in Theorem 3.2.6, that is, first we will show that for each degree $\mathbf{ab}$ we have that $\beta_{i,\mathbf{ab}}(T/I \sqcup J) = |\mathbf{M}_{i,\mathbf{ab}}|$ where $|\mathbf{M}_{i,\mathbf{ab}}|$ is the number of free modules shifted by $\mathbf{ab}$ in the i-th free module of $\mathbf{L}_\bullet$. Secondly, we will show that the columns of each differential δ_i of $\mathbf{L}_\bullet$ form an irredundant set.

Let $\mathbf{W}_\bullet = \{W_i, f_i\}_{i\in\mathbb{Z}}$ a minimal free resolution of $I \sqcup J$ where $W_i = \bigoplus_{\mathbf{c}\in\mathbf{M}_i} T(-\mathbf{c})$ and let $\mathbf{L}_\bullet = \{L_i, \delta_i\}$ be the free complex of $I \sqcup J$ defined before. From the definition of Betti number we have that $\beta_{t,\mathbf{ab}}(T/I \sqcup J) = |\mathbf{M}_{t,\mathbf{ab}}|$ and by Corollary 5.2.7 we also have that $\beta_{t,\mathbf{ab}}(T/I \sqcup J) = \sum_{i+j=t} |\mathbf{A}_{i,\mathbf{a}}||\mathbf{B}_{j,\mathbf{b}}|$. Since $L_t = \bigoplus_{i+j=t} \left(\bigoplus_{(\mathbf{a},\mathbf{b})\in\mathbf{A}_i\times\mathbf{B}_j} T(-\mathbf{ab})\right)$, then we have that $\sum_{i+j=t} |\mathbf{A}_{i,\mathbf{a}}||\mathbf{B}_{j,\mathbf{b}}|$ is the number of summands of the form $T(-\mathbf{ab})$

in the free module L_i. Therefore, $|\mathbf{M}_{t,\mathbf{ab}}| = \sum_{i+j=t} |\mathbf{A}_{i,\mathbf{a}}||\mathbf{B}_{j,\mathbf{b}}|$ and $W_t \simeq L_t$ as graded free modules.

It only remains to show that the set of columns $\{\mathbf{v}_i\}_{i=1}^{u}$ of a differential δ_j of $\mathbf{L}_\bullet$ is an irredundant set. Let's suppose opposite, that is, we can suppose that there exists $t_i \in T$ such that

$$\mathbf{v}_u = \sum_{i=1}^{u-1} t_i \mathbf{v}_i. \tag{5.1}$$

If $\deg(\mathbf{v}_u) \in \mathbf{A}_r \times \mathbf{B}_s$, then all the entries of $\mathbf{v}_u$ are zero except possibly at the rows in degree $\mathbf{cd} \in \mathbf{A}_{r-1} \times \mathbf{B}_s$ or $\mathbf{cd} \in \mathbf{A}_r \times \mathbf{B}_{s-1}$. Thus, we can rewrite the equation 5.1 as

$$\mathbf{v}_u = \sum_{i \in J} t_i \mathbf{v}_i \tag{5.2}$$

where $J = \{i \in [u-1] : \deg(\mathbf{v}_i) \in \mathbf{A}_p \times \mathbf{B}_q \text{ and } (p,q) \in \{(r,s),(r-1,s+1),(r+1,s-1)\}\}$ because any $\mathbf{v}_i$ with $i \notin J$ has its entries equal to zero in the rows of degree $\mathbf{cd}$. Now let m be the entry in a row of degree $\mathbf{cd} \in \mathbf{A}_{r-1} \times \mathbf{B}_s$. Then we have the following possibilities:

$$m \in \begin{cases} S & \text{if } \deg(\mathbf{v}_i) \in \mathbf{A}_r \times \mathbf{B}_s \\ R & \text{if } \deg(\mathbf{v}_i) \in \mathbf{A}_{r-1} \times \mathbf{B}_{s+1} \\ \{0\} & \text{if } \deg(\mathbf{v}_i) \in \mathbf{A}_{r+1} \times \mathbf{B}_{s-1}. \end{cases}$$

On the other hand, if m is in a row of degree $\mathbf{cd} \in \mathbf{A}_r \times \mathbf{B}_{s-1}$ we have analogous possibilities. Since $\deg(\mathbf{v}_u) \in \mathbf{A}_r \times \mathbf{B}_s$, we can rewrite the equation 5.2 as

$$\mathbf{v}_u = \sum_{i \in J'} t_i \mathbf{v}_i$$

where $J' = \{i \in J : \deg(\mathbf{v}_i) \in \mathbf{A}_r \times \mathbf{B}_s\}$. But this last equation implies that the columns of d_i and h_i are not irredundant, which is a contradiction. $\square$

5.3 Betti numbers of the join graph

In this section we show that if a graph G contains as spanning graph the complete multipartite graph $K_{n_1,\dots,n_t}$ then the projective dimension of G is the same as the projective dimension of $K_{n_1,\dots,n_t}$. As a particular case, if G is the join of two graphs G_1 and G_2, then we can calculate all the Betti numbers of G whenever we know the Betti numbers of G_1 and G_2.

Some times we will refer to the projective dimension or the Betti number of a graph G but it should be interpreted as the projective dimension of the Betti number of the edge ideal defined by G.

We start by giving some definitions of graphs and then we recall the definition of the lower Koszul simplicial complex and the Hochster's formula.

An **spanning graph** H of a graph G is a subgraph such that the vertices of H are the same as G, that is, H is a subgraph of G and $V(H) = V(G)$.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the **join graph**, denoted $G_1 + G_2$, is the graph with set of vertices $V_1 \cup V_2$ and set of edges

$$E(G_1 + G_2) = E_1 \cup E_2 \cup \{\{v_1, v_2\} : v_1 \in V_1, v_2 \in V_2\}.$$

We can extend the definition of the join graph recursively to make the join graph of n graphs. For instance, the join graph of three graphs G_1, G_2 and G_3 is the join graph of $G_1 + G_2$ and G_3.

An example of a join graph is the complete bipartite graph $K_{n,m}$: it is the join graph of two trivial graphs with n and m vertices.

Remark 5.3.1. *Notice that if we take the subgraph H of $G_1 + G_2$ defined as $V(H) = V(G_1 + G_2)$ and $E(H) = \{\{v_1, v_2\} : v_1 \in V_1, v_2 \in V_2\}$, then this subgraph is an spanning graph of $G_1 + G_2$. Indeed, H is the complete bipartite graph $K_{n,m}$ where $|V_1| = n$ and $|V_2| = m$. Therefore, the join graph always contains a complete multipartite graph as spanning graph.*

For a monomial ideal I and a degree $\mathbf{a} \in \mathbb{N}^n$, the lower Koszul simplicial complex of I is $K_{\mathbf{a}}(I) = \{\text{squarefree vector } \tau : \mathbf{x}^{\mathbf{a}-\mathbf{1}+\tau} \notin I\}$. Notice that if we are working only with edge ideals, then the degree $\mathbf{a}$ is a vector with entries zero or ones. Most of the times in this section we will work with the degree $\mathbf{a} = \mathbf{1}$, that is, the vector of ones in $\mathbb{N}^n$. In this case the lower Koszul simplicial complex is the simplicial complex generated by the monomials that are not in I, that is,

$$K_{\mathbf{1}}(I) = \{\text{squarefree vector } \tau : \mathbf{x}^{\tau} \notin I\}.$$

Sometimes we will use the lower Koszul simplicial complex of a subgraph H in degree $\chi(V(H))$. Here this degree is the vector in $\mathbb{N}^n$ with entries given by the characteristic vector of $V(H)$. That is,

$$\chi(V(H))_i = \begin{cases} 0 & \text{if } i \notin V(H), \\ 1 & \text{if } i \in V(H), \end{cases}$$

where the vertices of H are labeled by some subset of $\mathbb{N}$.

Next formula is a corollary of the Hochster's formula.

Corollary 5.3.2. *Let I be a monomial ideal, $\mathbf{a} \in \mathbb{N}^n$ a degree and $K_{\mathbf{a}}(S/I)$ the lower Koszul simplicial complex of I. The i-th Betti number of I in degree $\mathbf{a}$ is*

$$\beta_{i,\mathbf{a}}(S/I) = \dim_k \tilde{H}_{n-i-1}(K_{\mathbf{a}}(I);k).$$

For more details about Hochster's formula and Betti numbers see Section 2.3.1.

Next theorem shows that if a graph G contains a complete multipartite graph as spanning graph then the Betti number of I_G in dimension $n-1$ is non-zero. Moreover, any other subgraph H with less than n vertices will have Betti number in dimension $n-1$ equals to zero.

Theorem 5.3.3. *Let G and H be two graphs such that $H \subsetneq K_{n_1,\ldots,n_t} \subseteq G$, where $K_{n_1,\ldots,n_t}$ is the complete multipartite graph with $V(K_{n_1,\ldots,n_t}) = V(G)$. Then*

1) $\beta_{n-1,\mathbf{1}}(S/I_H) = \dim_k \tilde{H}_0(K_{\mathbf{1}}(I_H;k)) \leqslant t-2.$

2) $\beta_{n-1,\mathbf{1}}(S/I_G) = \dim_k \tilde{H}_0(K_{\mathbf{1}}(I_G;k)) \neq 0.$

3) $\beta_{n,\mathbf{1}}(S/I_G) = \dim_k \tilde{H}_{-1}(K_{\mathbf{1}}(I_G;k)) = 0.$

Here $n = n_1 + n_2 + \cdots + n_t$ is the number of vertices of G.

Proof. (1). Suppose that $\mathbf{x_i} = \{x_{i,1}, \ldots, x_{i,n_i}\}$ are the sets of vertices of $K_{n_1,\ldots,n_t}$ with $i \in \{1,\ldots,t\}$. Since $H \subsetneq K_{n_1,\ldots,n_t}$ then any pair of vertices $x_{i,u}, x_{i,v} \in \mathbf{x_i}$ of H are connected in $K_{\mathbf{1}}(I_H)$ because $x_{i,u}x_{i,v} \notin I_H$, for all $i \in \{1,\ldots,t\}$. If $\{x_{i,u}, x_{j,v}\} \notin E(H)$, then $\{x_{i,u}, x_{j,v}\}$ is an edge of $K_{\mathbf{1}}(I_H)$ and thus every vertex that is connected to $x_{i,u}$ is also connected to any vertex that is connected to $x_{j,v}$. This is, $K_{\mathbf{1}}(I_H)$ has at most $t-1$ connected components. Therefore, $\beta_{n-1,\mathbf{1}}(S/I_H) = \dim_k \tilde{H}_0(K_{\mathbf{1}}(I_H);k) \leqslant t-2$.
(2). If $K_{n_1,\ldots,n_t}$ is an spanning graph of G, then it is enough to prove that $K_{\mathbf{1}}(I_G)$ has homology at dimension zero. But this is more general: G is the join graph of t graphs $G_1, G_2, \ldots, G_t$ and $K_{\mathbf{1}}(I_G) = K_{\chi(V(G_1))}(I_{G_1}) \sqcup \ldots \sqcup K_{\chi(V(G_t))}(I_{G_t})$. Then $K_{\mathbf{1}}(I_G)$ has homology at dimension zero because it has more than one connected component.
(3). Recall that if $\Delta \neq \{\emptyset\}$ then $\tilde{H}_{-1}(\Delta;k) = 0$. Since $K_{\mathbf{1}}(I_G)$ has at least the vertices $\mathbf{x_i}$, then $\tilde{H}_{-1}(K_{\mathbf{1}}(I_G);k) = 0$. □

Previous theorem can be interpreted in terms of the projective dimension: the projective dimension of the edge ideal of a graph G that contains a complete multipartite graph as spanning graph, depends on the projective dimension of the complete multipartite graphs, and the projective dimension of any subgraph of $K_{n_1,\dots,n_t}$ is bounded by the projective dimension of the edge ideal of $K_{n_1,\dots,n_t}$. Next corollary shows that.

Corollary 5.3.4. *Let G and H be two graphs such that $H \subsetneq K_{n_1,\dots,n_t} \subseteq G$, where $K_{n_1,\dots,n_t}$ is an spanning graph of G. Then $\mathrm{pd}(I_G) = n_1 + \cdots + n_t - 1$ and $\mathrm{pd}(I_H) \leqslant n_1 + \cdots + n_t - 1$.*

Proof. By previous theorem we have that $\beta_{n-1,\mathbf{1}}(S/I_G) \neq 0$ and $\beta_{n,\mathbf{1}}(S/I_G) = 0$, where $n = n_1 + \cdots + n_t$. Thus, $\mathrm{pd}(I_G) = n - 1$

If H is a subgraph of $K_{n_1,\dots,n_t}$ then $\mathrm{pd}(I_H) \leqslant n - 1 = \mathrm{pd}(I_{K_{n_1,\dots,n_t}})$. In particular notice that if H is a subgraph of a complete bipartite graph K_{n_1,n_2}, then $\mathrm{pd}(I_H) \leqslant n_1 + n_2 - 2$. $\square$

Next corollary shows that if G is the join graph of some graphs, then the projective dimension of G is determined by the number of vertices of G.

Corollary 5.3.5. *Let $\{G_i = (V_i, E_i)\}_{i=1}^t$ be a set of graphs where $|V_i| = n_i$ for every $i \in \{1, \dots t\}$. The join graph $G = G_1 + \cdots + G_t$ has projective dimension $n - 1 = n_1 + \cdots + n_t - 1$.*

Proof. The projective dimension follows from previous corollary taking $G = G_1 + \cdots + G_t$. $\square$

For instance, the cone $c(G)$ of a graph G is the join graph of a graph G with a trivial graph with one vertex. Then the projective dimension of the cone $c(G)$ is the number of vertices of G.

Theorem 5.3.3 give us more than the projective dimension of the join graph. It give us the Betti numbers of the join graph $G = G_1 + \cdots + G_t$ in terms of the Betti numbers of G_i with $i \in \{1, \dots, t\}$. Next theorem shows that.

Theorem 5.3.6. *Let $\{G_i = (V_i, E_i)\}_{i=1}^t$ be a set of graphs with $|V_i| = n_i$ for all i. The join graph $G = G_1 + \cdots + G_t$ with n vertices has Betti numbers:*

$$\beta_{n-i,\mathbf{1}}(S/I_G) = \begin{cases} \sum_{j=1}^t \beta_{n_j - i, \chi(V_j)}(S/I_{G_j}) + t - 1 & \text{if } i = 1, \\ \sum_{j=1}^t \beta_{n_j - i, \chi(V_j)}(S/I_{G_j}) & \text{if } i \neq 1. \end{cases}$$

Proof. Since G is the join graph of the graph $G_1,\ldots,G_t$, then we have that $K_{\mathbf{1}}(I_G) = K_{\chi(V_1)}(I_{G_1}) \sqcup \ldots \sqcup K_{\chi(V_t)}(I_{G_t})$ which implies that $\tilde{H}_{i-1}(K_{\mathbf{1}}(I_G);k) = \tilde{H}_{i-1}(K_{\chi(V_1)}(I_{G_1});k) \oplus \ldots \oplus \tilde{H}_{i-1}(K_{\chi(V_t)}(I_{G_t});k)$ for every $i \neq 1$ and $\tilde{H}_{i-1}(K_{\mathbf{1}}(I_G);k) = \tilde{H}_{i-1}(K_{\chi(V_1)}(I_{G_1});k) \oplus \ldots \oplus \tilde{H}_{i-1}(K_{\chi(V_t)}(I_{G_t});k) \oplus k^{t-1}$ for $i=1$. Then, taking the dimension on the homologies we obtain the result. □

Next example illustrates previous theorems.

Example 5.3.7. *Let B be the bow tie graph and K_3 be the complete graph with three vertices and their edge ideals $I_B = \langle x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_3x_5, x_4x_5\rangle$ and $I_{K_3} = \langle y_1y_2, y_1y_3, y_2y_3\rangle$. Since $K_{3,5}$ is an spanning graph of $G = B + K_3$, applying theorem 5.3.3 we conclude that* $\mathrm{pd}(I_G) = 7$.

Moreover, its Betti numbers in multidegree $\mathbf{1}$ *are*

$$\begin{aligned}\beta_{7,\mathbf{1}}(S/I_G) &= \beta_4(S/I_B) + \beta_2(S/I_{K_3}) + 1 = 1+2+1 = 4,\\ \beta_{6,\mathbf{1}}(S/I_G) &= \beta_3(S/I_B) + \beta_1(S/I_{K_3}) = 1+0 = 1,\end{aligned}$$

and $\beta_{i,\mathbf{1}}(S/I_G) = 0$ for $i \notin \{6,7\}$. Doing the same for all induced subgraph of G, we obtain the Betti numbers of the minimal free resolution of S/I_G:

$$\mathrm{F}_\bullet : 0 \leftarrow S/I_G \leftarrow S \leftarrow S^{24} \leftarrow S^{89} \leftarrow S^{155} \leftarrow S^{154} \leftarrow S^{190} \leftarrow S^{29} \leftarrow S^4 \leftarrow 0.$$

To finish this section we give the following conjectures.

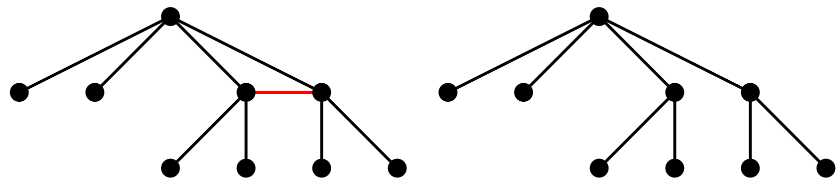
Conjecture 5.3.8. *Let G be a graph with m vertices that contains a complete multipartite graph $K_{n_1,\ldots,n_t}$ with n vertices, where $m \geqslant n$. Then* $\mathrm{pd}(I_G) \geqslant n-1$.

Next conjecture can be seen as a reciprocal result of previous theorems.

Conjecture 5.3.9. *Let G be a connected graph such that* $\mathrm{pd}(I_G) = n-1$, *where n is the number of vertices of G. Then G contains a complete multipartite graph of n vertices.*

In general it is not true that if a graph G contains a subgraph H such that $\mathrm{pd}(H) = t$ then $\mathrm{pd}(G) \geqslant t$. Next example illustrates an example of this.

Example 5.3.10. *Given a graph G (see Figure 5.2 (a)) and H a subgraph of G (see Figure 5.2 (b)), it is not true that* $\mathrm{pd}(G) \geqslant \mathrm{pd}(H)$.



(a) Projective dimension 4. (b) Projective dimension 5.

Figure 5.2: A subgraph H with projective dimension greater than the graph G.

Notice that this example also shows that is not true that if G is a graph with projective dimension t, then $G \setminus e$, with e an edge of G, implies that $\mathrm{pd}(G \setminus e) \leqslant t$.

Next conjecture is an analogous result.

Conjecture 5.3.11. *Let H, K and G be graphs such that $H \subseteq K \subseteq G$. If* $\mathrm{pd}(I_H) = \mathrm{pd}(I_G)$ *then* $\mathrm{pd}(K) = \mathrm{pd}(G)$.

Next conjecture is related with the combinatorial resolutions.

Conjecture 5.3.12. *If two graphs G and H have combinatorial resolutions, then the join graph $G+H$ has a combinatorial resolution.*

5.3.1 Betti numbers of cographs

In this section we apply previous theorems to cographs which can be seen as a join graph or a disjoint union of join graphs. As a consequence, the projective dimension of a threshold graph will be calculated.

We give a recursive definition of cographs and next lemma shows that alternatively we can use the join operation.

Definition 5.3.13. *Any* ***cograph*** *may be constructed using following rules:*

1. *any single vertex is a cograph.*
2. *If G is a cograph, so is its complement graph G^c.*
3. *If G and H are cographs, so is their disjoint union $G \sqcup H$.*

With the aim to see a cograph as a join of some cographs we give the following lemma.

Lemma 5.3.14. *$G+H$ is a cograph if and only if G and H are cographs.*

Proof. Note that $G+H$ is the same as $(G^c \sqcup H^c)^c$. From the definition we know that if $(G^c \sqcup H^c)^c$ is a cograph, then $G^c \sqcup H^c$,G^c and H^c are cographs. Thus G and H are cographs too.

Since G and H are cographs, so G^c and H^c are cographs. Also $G^c \sqcup H^c$ is a cograph and is not difficult to check that $G^c \sqcup H^c$ is the same graph as the join graph $G+H$. □

Then a cograph can be obtained following these equivalent rules:

1. any single vertex is a cograph.

2. If G and H are cographs, so is their join graph $G+H$.

3. If G and H are cographs, so is their disjoint union $G \sqcup H$.

With this new equivalent definition, a cograph is indeed either a join graph of two cographs or the disjoint union of two cographs. Next theorem give us the projective dimension of a connected cograph.

Theorem 5.3.15. *If G is a connected cograph with n vertices, then $\mathrm{pd}(G) = n-1$.*

Actually, whenever a graph G is connected, it is the join graph of some two cographs G_1 and G_2 and the theorem 5.3.6 give us the Betti numbers of G from the Betti numbers of G_1 and G_2.

In case that a cograph G is disconnected, the projective dimension can be also computed. Indeed, a minimal free resolution of any disjoint union $G = G_1 \sqcup G_2$ can be obtained from the minimal free resolutions of G_1 and G_2, as is shown in Section 5.2. Next theorem give us the projective dimension of a disjoint union of two cographs.

Theorem 5.3.16. *Let G be a cograph such that G is the disjoint union of cographs G_1 and G_2. If $\mathrm{pd}(G_1) = n$ and $\mathrm{pd}(G_2) = m$ then $\mathrm{pd}(G) = n+m$.*

Remark 5.3.17. *Every **threshold graph** is also a cograph, see [2] for more details. A threshold graph may be formed by repeatedly adding one vertex, either connected to all previous vertices or to none of them; each such operation is one of the disjoint union or join operations by which a cotree may be formed.*

Corollary 5.3.18. *If G is a connected threshold graph with n vertices, then $\mathrm{pd}(G) = n-1$.*

5.4 Monomial ideals with maximum projective dimension

In this section we study the monomial ideals with projective dimension n in a polynomial ring with n variables. These ideas are contained in [1]. Here we give different proofs and we apply this characterization for the case of edge ideals.

From Hilbert Syzygies's Theorem (see Theorem 2.2.3) we know that a monomial ideal I in a polynomial ring $S = k[\mathbf{x}]$ has projective dimension less or equal than n, where n is the number of variables of S, that is, $n = |\mathbf{x}|$. Then we are interested in characterize monomial ideals for which $\beta_{n,\mathbf{a}}(S/I) \neq 0$.

We start by giving two lemmas that will be important to characterize the minimum number of minimal generators that can have a monomial ideal with maximum projective dimension.

Lemma 5.4.1. *Let I be a monomial ideal with minimal set of generators $\{\mathbf{g}_1, \dots, \mathbf{g}_r\}$ in the polynomial ring $S = k[\mathbf{x}]$ with n variables. If the projective dimension of I is n, then $r \geqslant n$.*

Proof. If $\mathrm{pd}(I) = n$, then there is a degree $\mathbf{b}$ with $x_i | \mathbf{x}^{\mathbf{b}}$ for all $i \in \{1, \dots n\}$, corresponding to the least common multiple γ of some of the generators, such that $\tilde{H}_{-1}(K_{\mathbf{b}}(I); k) \neq 0$. This occurs if and only if $K_{\mathbf{b}}(I) \neq \emptyset$, which by Alexander duality (see Section 2.3.1 for more details) is the same as saying that $K^{\mathbf{b}}(I)$ is the $n-2$-sphere with facets $\sigma_i = \{1, \dots, n\} \setminus \{i\}$. There is at least one generator for each facet, so $r \geqslant n$. □

Next proposition give a first characterization of the monomial ideal with maximum projective dimension and with n generators.

Proposition 5.4.2. *Let I be a monomial ideal of the polynomial ring $S = k[\mathbf{x}]$ with minimal set of generators $\{\mathbf{g}_1, \dots, \mathbf{g}_n\}$. Then $\mathrm{pd}(I) = n$ if and only if $I = \langle x_1^{a_1} m_1, \dots, x_n^{a_n} m_n \rangle$ such that for each $i \in \{1, \dots, n\}$, m_i are monomials such that $x_i \nmid m_i$ and $x_i^{a_i} \nmid \mathbf{g}_j$ for $j \neq i$.*

Proof. If $I = \langle x_1^{a_1} m_1, \dots, x_n^{a_n} m_n \rangle$, then $\gamma = \mathrm{lcm}(x_1^{a_1} m_1, \dots, x_n^{a_n} m_n) = x_1^{a_1} \dots x_n^{a_n} = \mathbf{x}^{\mathbf{a}}$, and

$$\frac{\gamma}{\mathbf{g}_i} = x_1^{b_1} \dots \hat{x}_i \dots x_n^{b_n}$$

where $\hat{x}_i$ means that the variable x_i does not appear in the monomial. Notice that this means that the facet associated to $\mathbf{g}_i$ is just $\sigma_i = \{1, \dots, n\} \setminus \{i\}$. That is, $K^{\mathbf{a}}(I)$ is the simplicial $n-2$-sphere. The result follows from the proof of the Lemma 5.4.1.

Conversely suppose that $\mathrm{pd}(I) = n$. Then, by the proof of the Lemma 5.4.1 we have that $K^{\mathbf{a}}(I)$ is the simplicial $n-2$-sphere, with facet $\sigma_i = \{1, \ldots, n\} \setminus \{i\}$. This means that, potentially after a rearrangement of the generators:

$$\frac{\gamma}{\mathbf{g}_i} = x_1^{b_1} \ldots \hat{x}_i \ldots x_n^{b_n}$$

for some $b_1, \ldots b_{i-1}, b_{i+1}, \ldots, b_n \geqslant 1$. Then $\mathbf{g}_i$ has the highest power of a_i of x_i between all the generators and does not have the highest power of x_j for $i \neq j$, since all the remaining variables appear in the fraction. Also there can not be two generators both having the same highest power of some x_i because the association, for each generator, of a facet, is a surjection between two sets with the same number of elements. So we can just write each generator as

$$\mathbf{g}_i = x_i^{a_i} m_i$$

where $x_i \nmid m_i$.

Now, $x_i^{a_i} \nmid \mathbf{g}_j$ for $j \neq i$, otherwise $\gamma / \mathbf{g}_j$ would not be divisible by x_i, which is impossible. □

Before we continue we need to recall the following definition.

Definition 5.4.3. *A monomial m' strictly divides another monomial m if m' divides m/x_i for all variables x_i dividing m.*

Next we give the characterization of the monomial ideals with maximum projective dimension.

Theorem 5.4.4. *Let I be a monomial ideal of the polynomial ring $S = k[\mathbf{x}]$ minimally generated by a set G. Then $\mathrm{pd}(I) = n$ if and only if there is a subset $H = \{\mathbf{g}_1, \ldots, \mathbf{g}_n\} \subseteq G$ such that for each $i \in \{1, \ldots n\}$, $\mathbf{g}_i = x_i^{a_i} m_i$, $x_i \nmid m_i$, $x_i^{a_i} \nmid \mathbf{g}_j$ and no element of $G \setminus H$ strictly divides to $\gamma = \mathrm{lcm}(\mathbf{g}_1, \ldots, \mathbf{g}_n)$.*

Proof. Suppose first that $\mathrm{I} = n$. From the proof of Lemma 5.4.1 it follows that the ideal I satisfies $\mathrm{pd}(I) = n$ if and only if there is some degree $\mathbf{b}$ such that $\Delta = K^{\mathbf{b}}(I)$ is the simplicial $n-2$-sphere. Let $H = \{\mathbf{g}_1, \ldots, \mathbf{g}_n\}$ be a set defined by choosing a monomial $\mathbf{g}_i$ corresponding to each facet $\sigma_i = \{1, \ldots, n\} \setminus \{i\}$ of Δ. Then from the proof of Proposition 5.4.2 it follows that for each i, there exists $a_i \in \mathbb{N}$ and a monomial $m_i \in S$ such that $\mathbf{g}_i = x_i^{a_i} m_i$, $x_i \nmid m_i$ and $x_i^{a_i} \nmid \mathbf{g}_j$ for $j \neq i$.

Now take $\delta \in G$ and suppose that δ strictly divides γ. Then the face of Δ associated to δ is the full face $\{1,\ldots,n\}$ since γ/δ is divisible by all the variables. This is impossible. Therefore, the remaining elements of G can not strictly divide γ.

Conversely, suppose $I = \langle H \cup (G-H)\rangle$ where $H = \{\mathbf{g}_1,\ldots,\mathbf{g}_n\rangle$ such that for each $i,j \in \{1,\ldots,n\}$, $\mathbf{g}_i = x_i^{a_i} m_i$, $x_i \not| m_i$, $x_i^{a_i} \not| \mathbf{g}_j$ if $j \neq i$, and no elements of $G \setminus H$ strictly divides γ. Then, if $\mathbf{b}$ is the exponent of $\gamma = \operatorname{lcm}(\mathbf{g}_1,\ldots,\mathbf{g}_n)$, $K^{\mathbf{b}}(I)$ consists of the facets $\sigma_i = \{1,\ldots,n\} \setminus \{i\}$ associated to each $\mathbf{g}_i$ and other faces τ associated to each $\delta \in G \setminus H$ such that $\delta|\gamma$. Since δ does not strictly divide γ then there is some variable x_i such that $x_i \not| \frac{\gamma}{\delta}$, which means that $\tau \subseteq \sigma_i$. Therefore, $K^{\mathbf{b}}(I)$ is the simplicial $n-2$-sphere and $\operatorname{pd}(I) = n$. □

Notice that previous theorem ensures the existence of a subset H of a set of generators whenever the ideal I generated by G has maximum projective dimension but the theorem does not say how to obtain this subset. In case that the set G has many elements, finding the subset H can be tedious.

Next we give an example of previous theorem.

Example 5.4.5. *Let I be the monomial ideal minimally generated by $G = \{x_1^3x_3, x_1x_2^3, x_2x_3^2, x_3^3\}$. In this case the subset H of G is $H = \{x_1^3x_3, x_1x_2^3, x_2x_3^2\}$ and notice that*

$$x_3^3 \not| x_1^3x_2^3x_3^2 = \operatorname{lcm}(x_1^3x_3, x_1x_2^3, x_2x_3^2).$$

Then by Theorem 5.4.4 the monomial ideal I has projective dimension 3*.*

Moreover, the existence of the subset H is not unique, for example, if we take $H_1 = \{x_1^3x_3, x_2x_3^2, x_3^3\}$ we obtain another subset that satisfies the conditions of Theorem 5.4.4.

As a simple consequence of this characterization we have the following corollary.

Corollary 5.4.6. *Let G be a graph and I_G its edge ideal. Then the projective dimension of I_G is always less than n.*

Proof. It follows directly from the Theorem 5.4.4. □

Chapter 6

The projective dimension of some edge ideals

In this chapter we present some computations of the projective dimension of the edge ideal of some graphs, the computations were done using Macaulay2 ([12]). We begin by summarizing the projective dimension of the edge ideal of graphs with less or equal than 10 vertices and then we draw these graphs grouped by their number of vertices and projective dimension.

These computations were very helpful in the understanding of the minimal free resolutions for some graphs and also to generate some conjectures. For instance, it was very useful to find the Betti numbers of the join graph (see Chapter 5) and figure out the basis graphs on its combinatorial resolution, see [24].

6.1 Table with the projective dimension of some edge ideals

In this section we give a table with the projective dimension of the edge ideal of some graphs and the number of graphs with that projective dimension.

For a given number of vertices we have the number of connected graphs with this number of vertices and the different projective dimensions of the edge ideals of these graphs. Moreover, we give the number of graphs that have this projective dimension for each number of vertices.

# Vertices	# Graphs	Projective dimension	# Graphs
2	1	1	1
3	2	2	2
4	6	2	1
		3	5
5	7	3	8
		4	13
6	112	3	4
		4	64
		5	44
7	853	4	58
		5	604
		6	191
8	11,117	4	19
		5	1,230
		6	8,639
		7	1,229
9	261,080	4	1
		5	707
		6	37,949
		7	208,835
		8	13,588
10	11'716,571	5	155
		6	38641
		7	2'025,587
		8	9'363,591
		9	288,597

Table 6.1: Number of graphs by vertices and projective dimension.

6.2 Graphs and their projective dimension

In this section we draw all the graphs from 2 to 7 vertices and we group them by number of vertices and projective dimension.

Table 6.2: Graphs with 2 vertices and projective dimension 1.

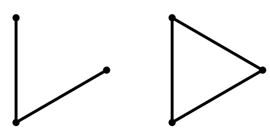


Table 6.3: Graphs with 3 vertices and projective dimension 2.

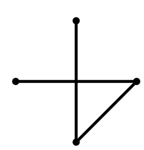


Table 6.4: Graphs with 4 vertices and projective dimension 2.

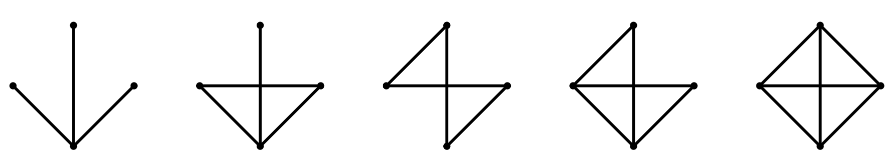


Table 6.5: Graphs with 4 vertices and projective dimension 3.

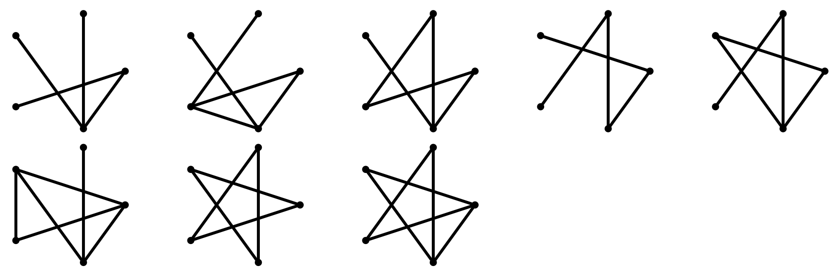


Table 6.6: Graphs with 5 vertices and projective dimension 3.

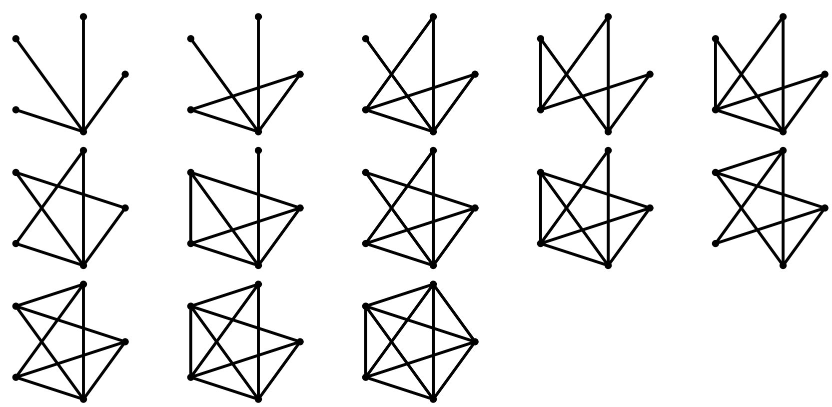


Table 6.7: Graphs with 5 vertices and projective dimension 4.

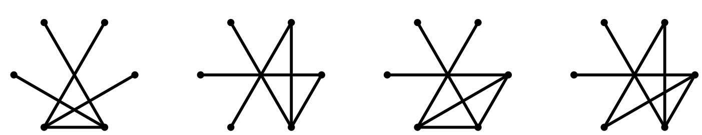


Table 6.8: Graphs with 6 vertices and projective dimension 3.

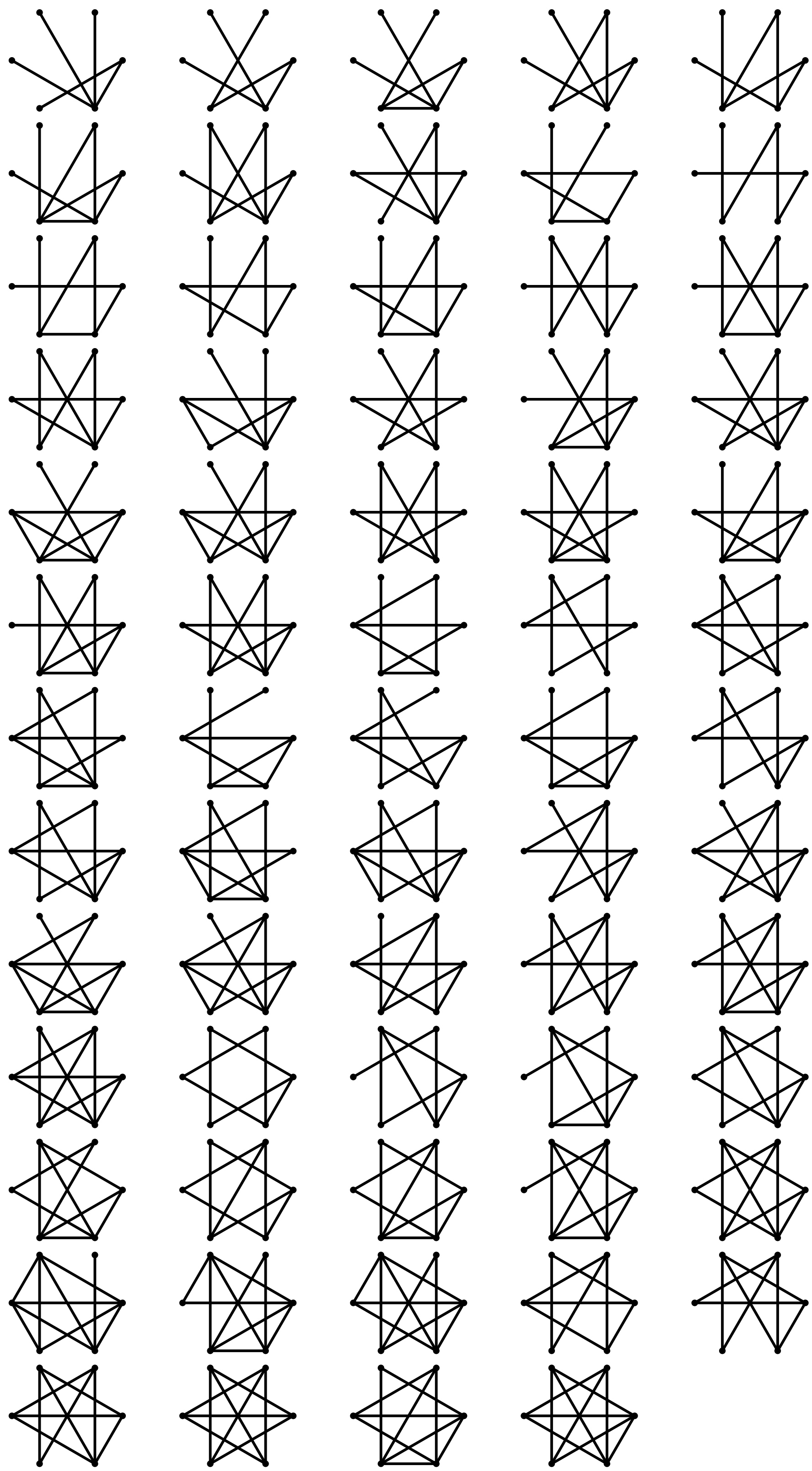


Table 6.9: Graphs with 6 vertices and projective dimension 4.

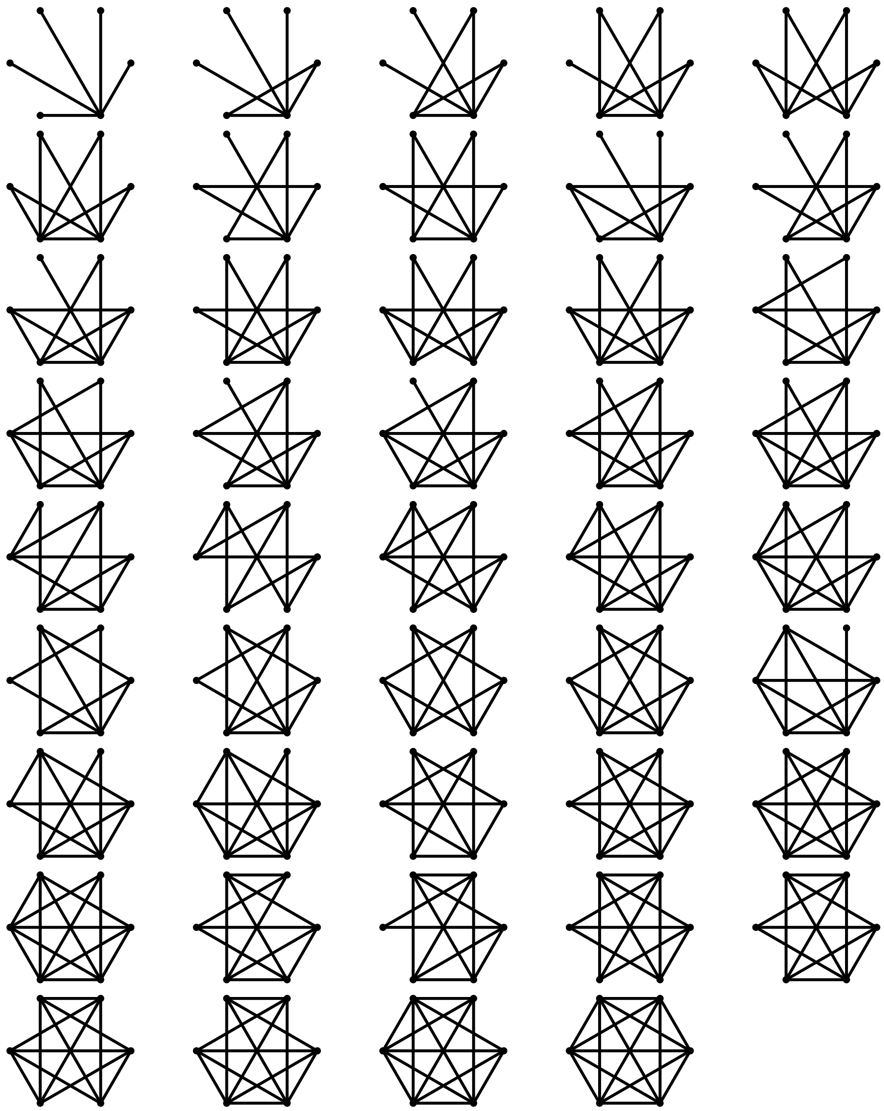
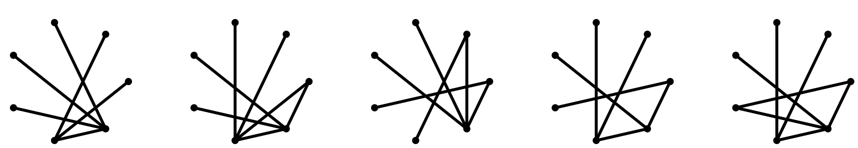
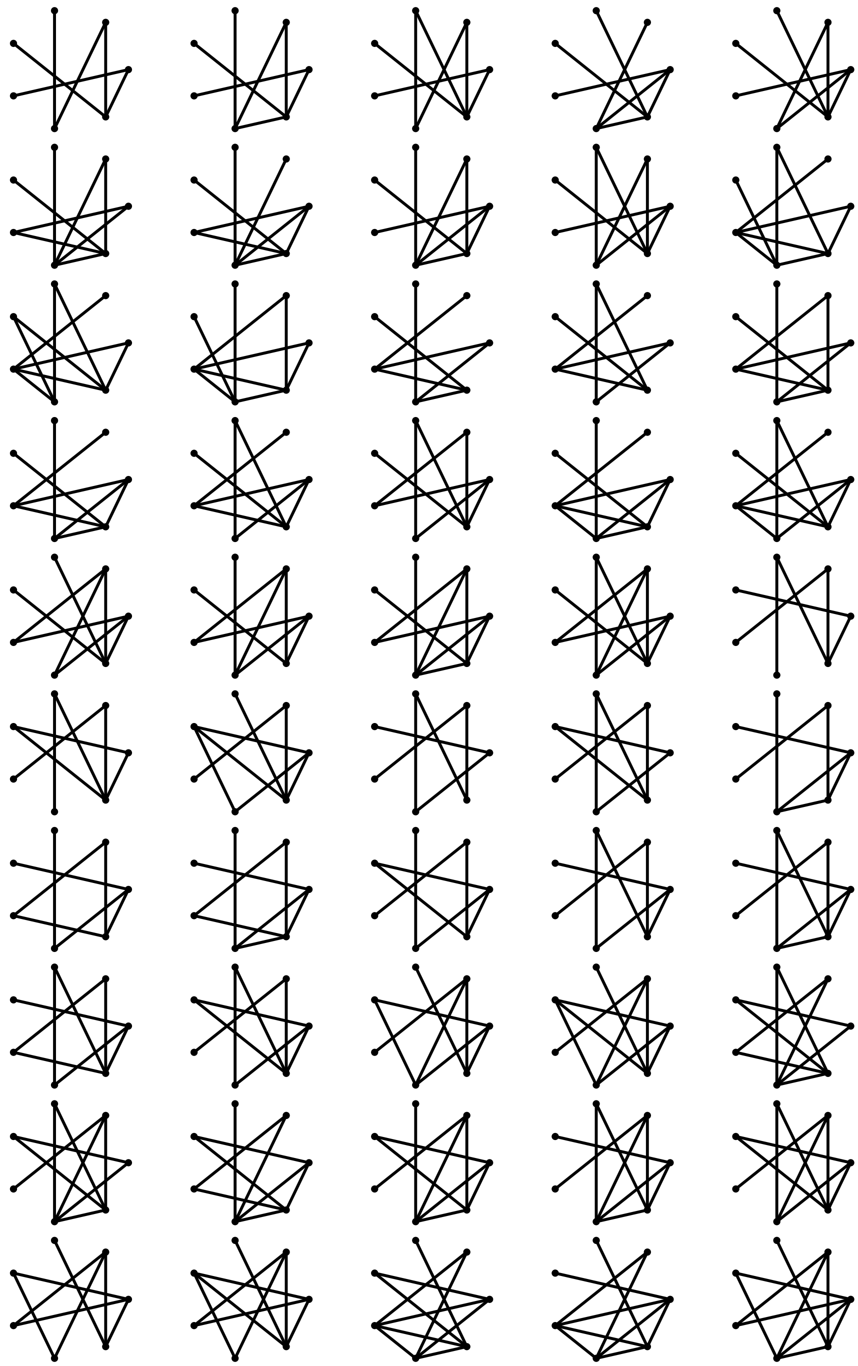


Table 6.10: Graphs with 6 vertices and projective dimension 5.





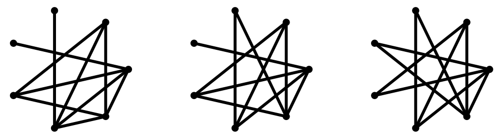
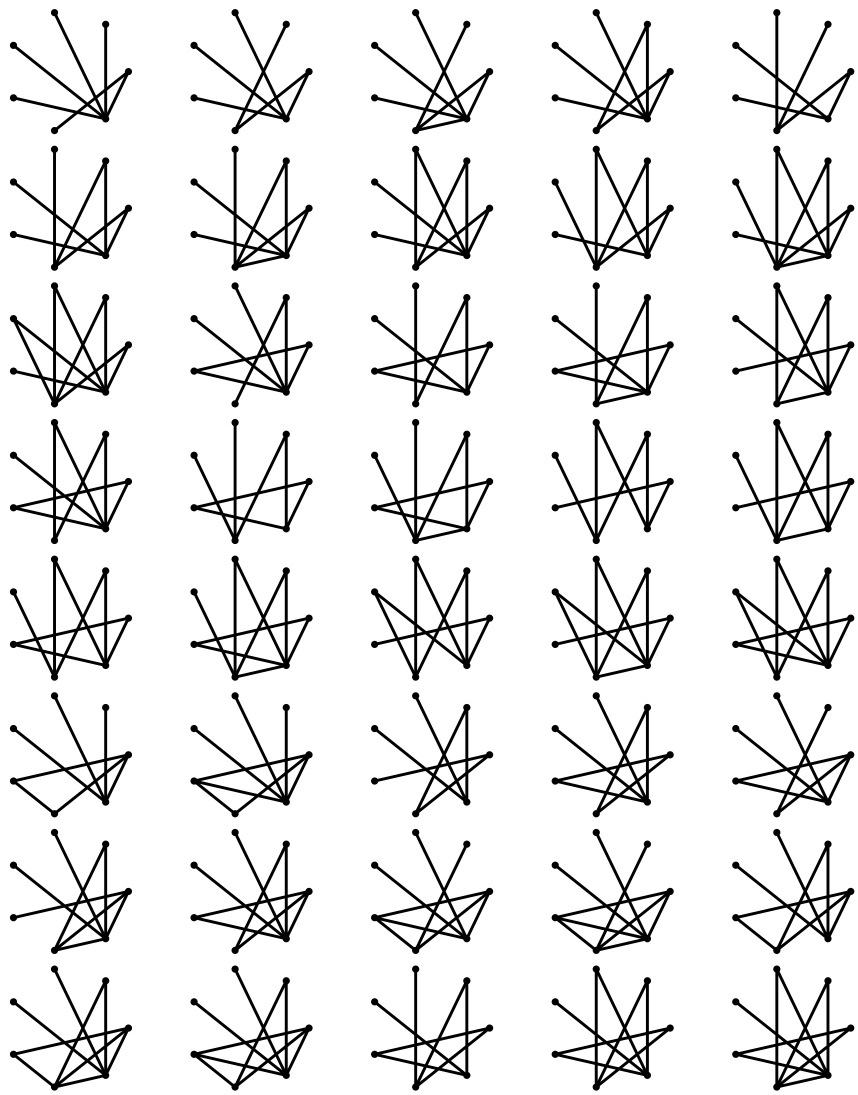
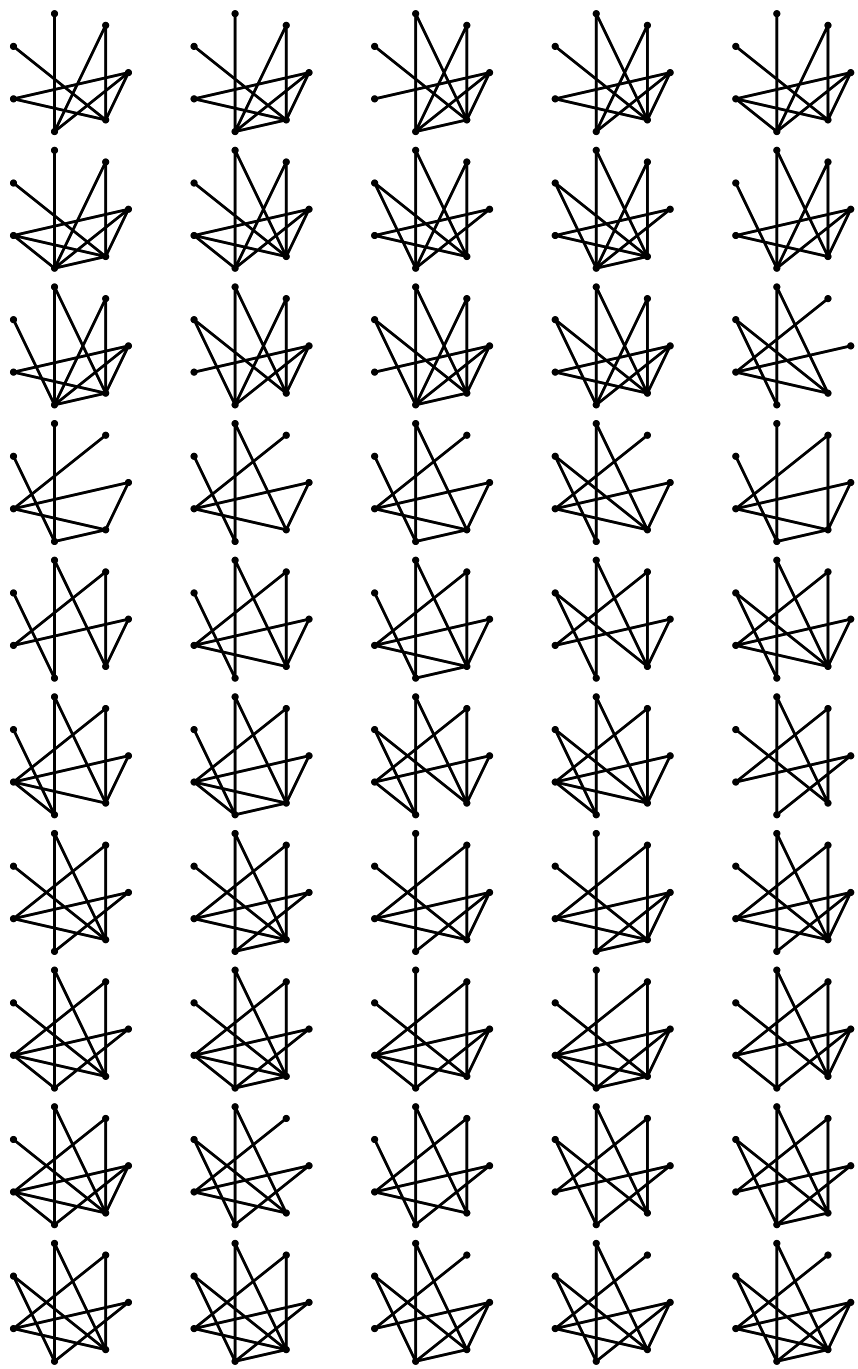
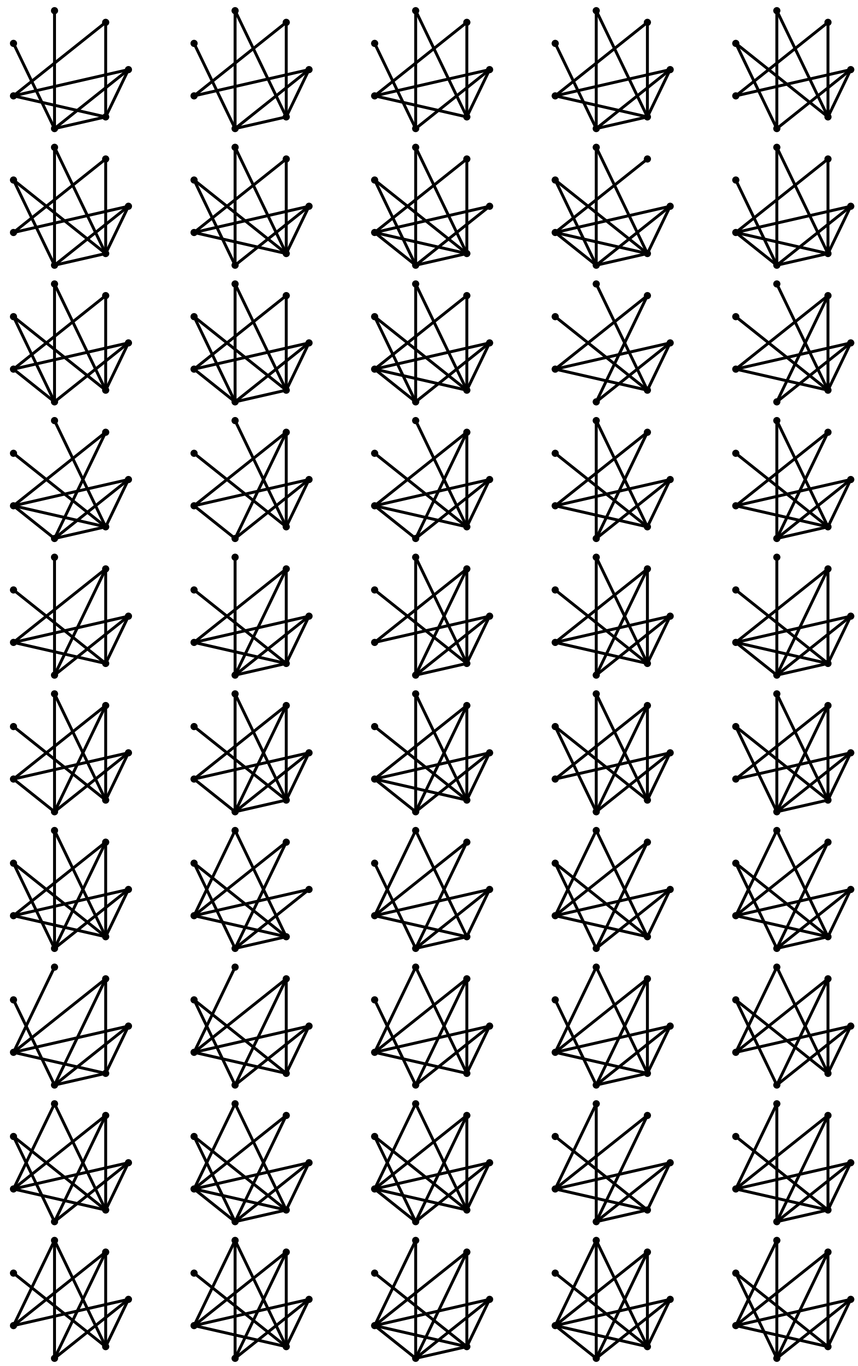
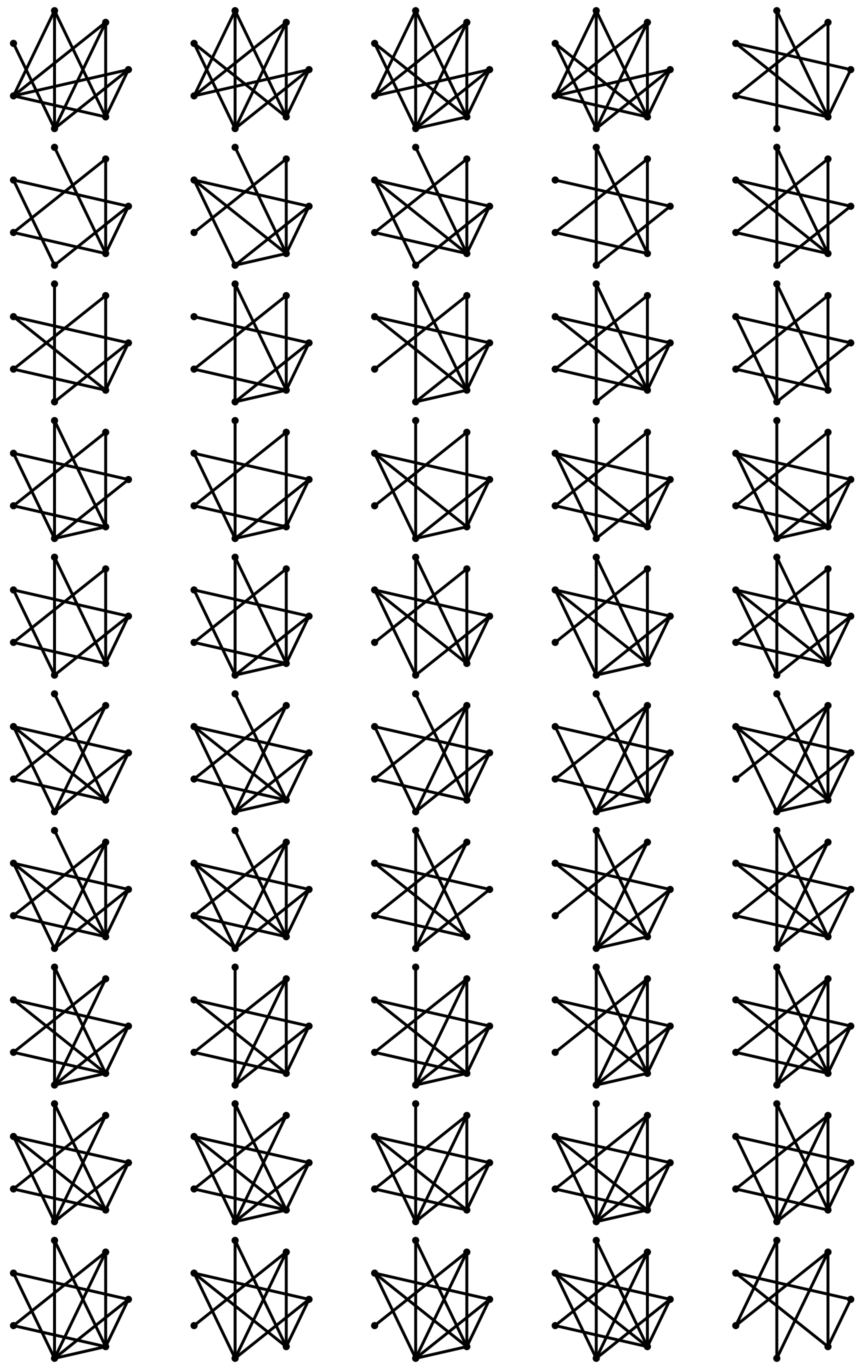


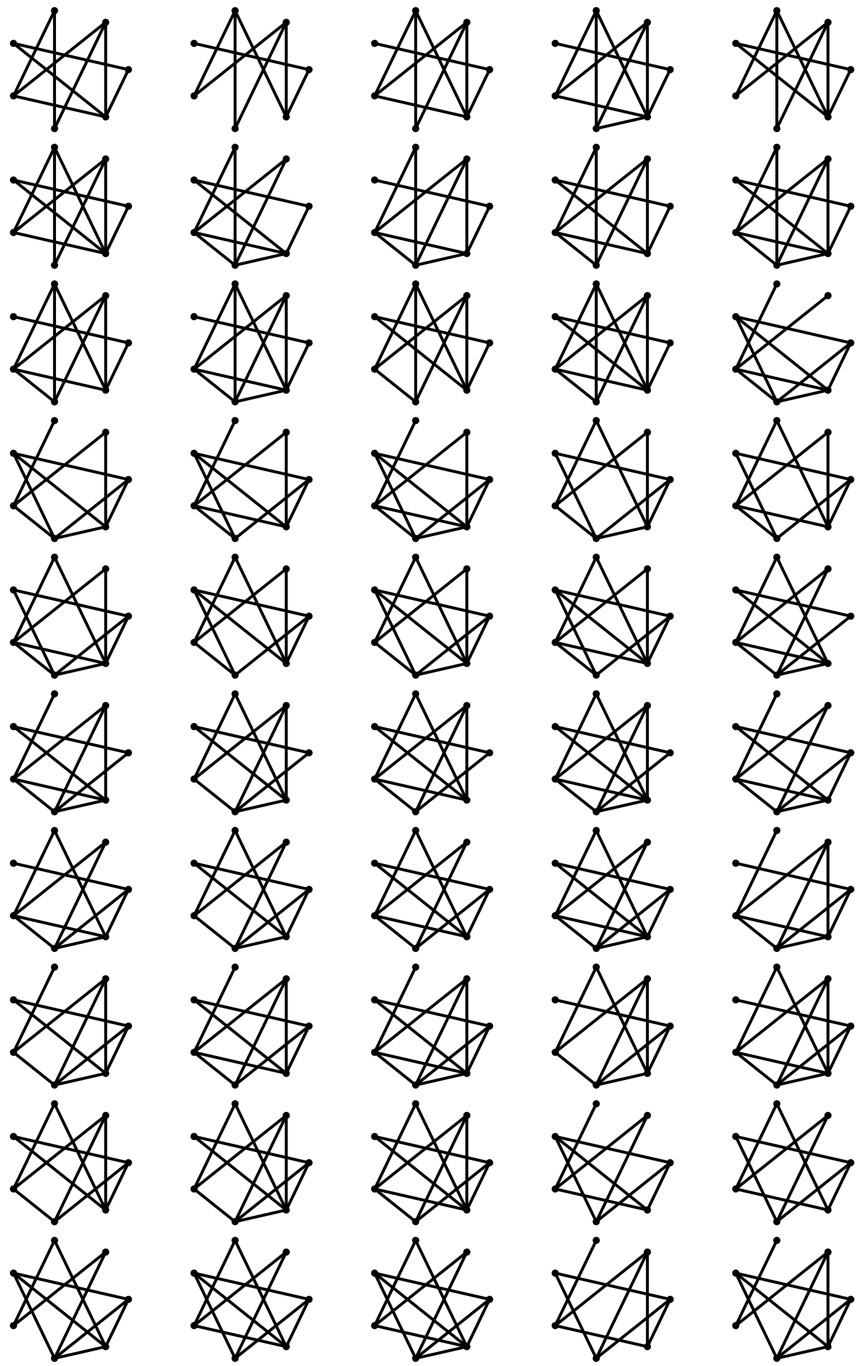
Table 6.11: Graphs with 7 vertices and projective dimension 4.

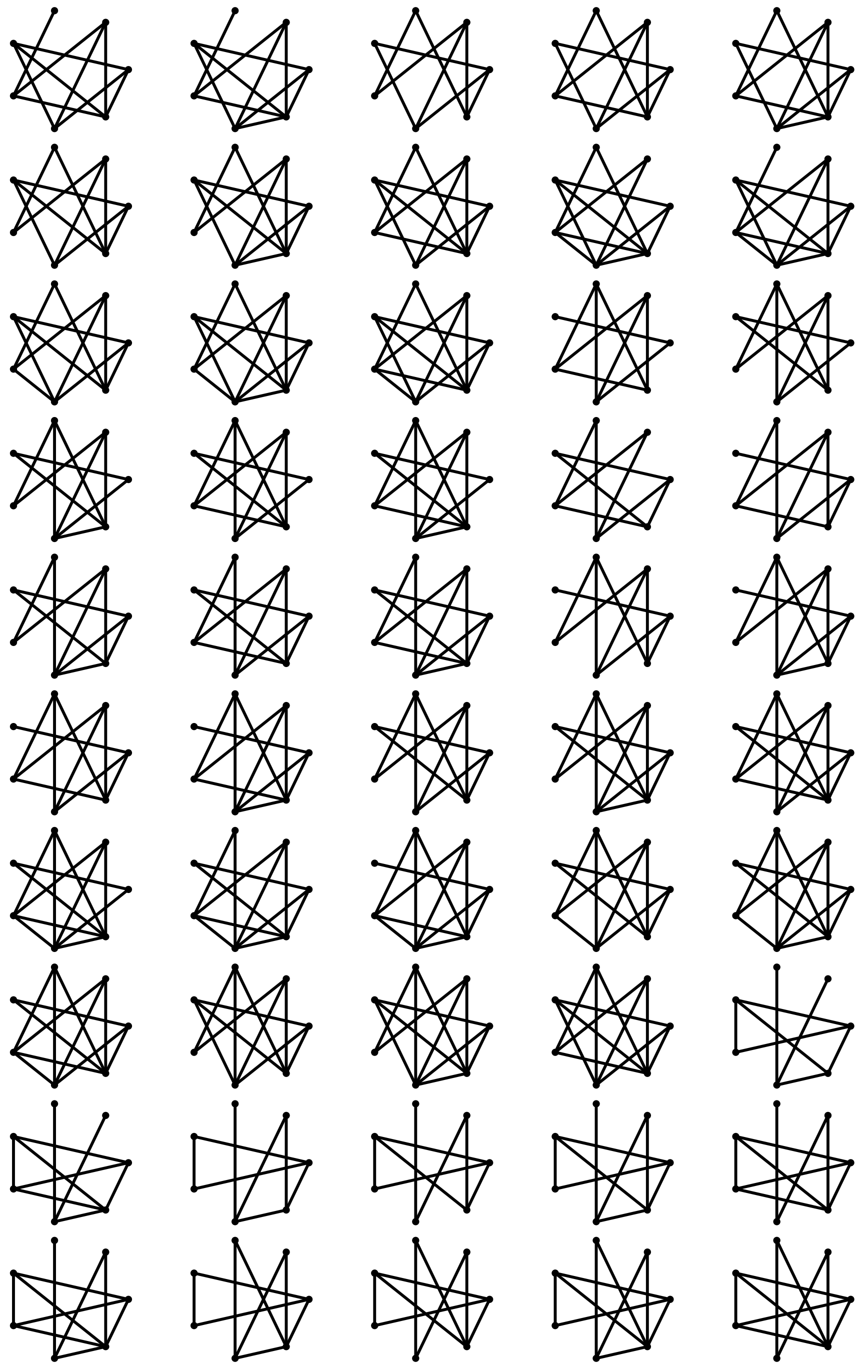


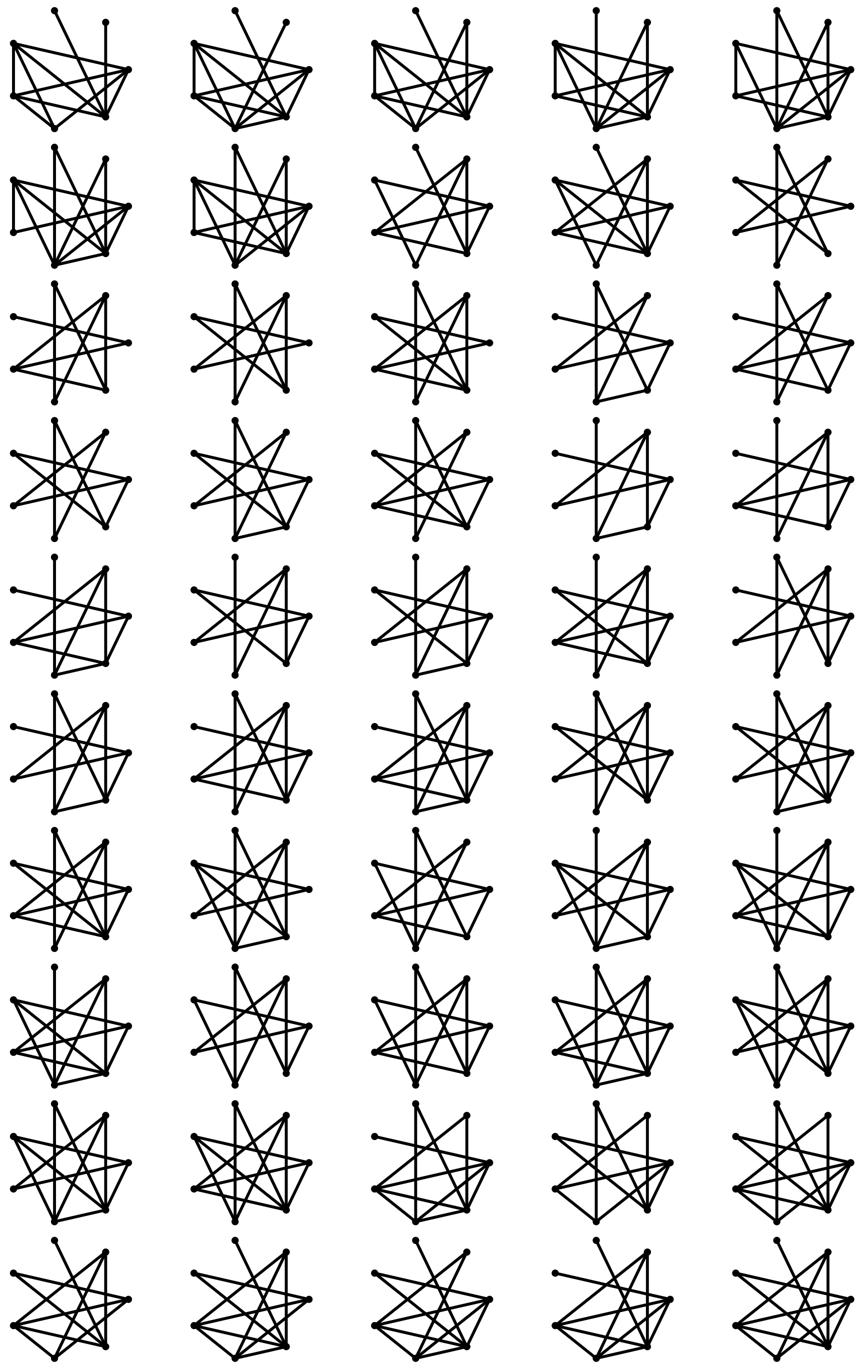


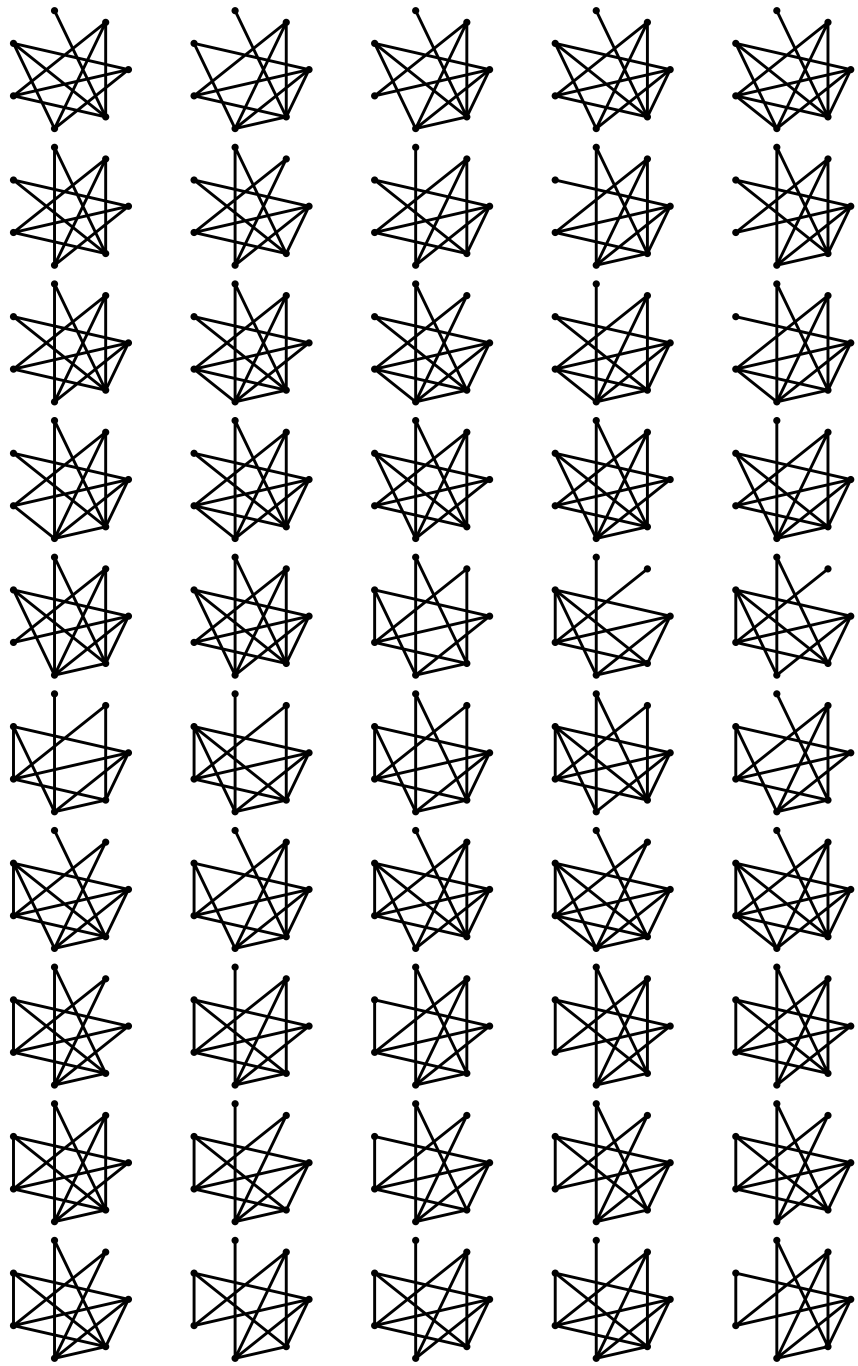


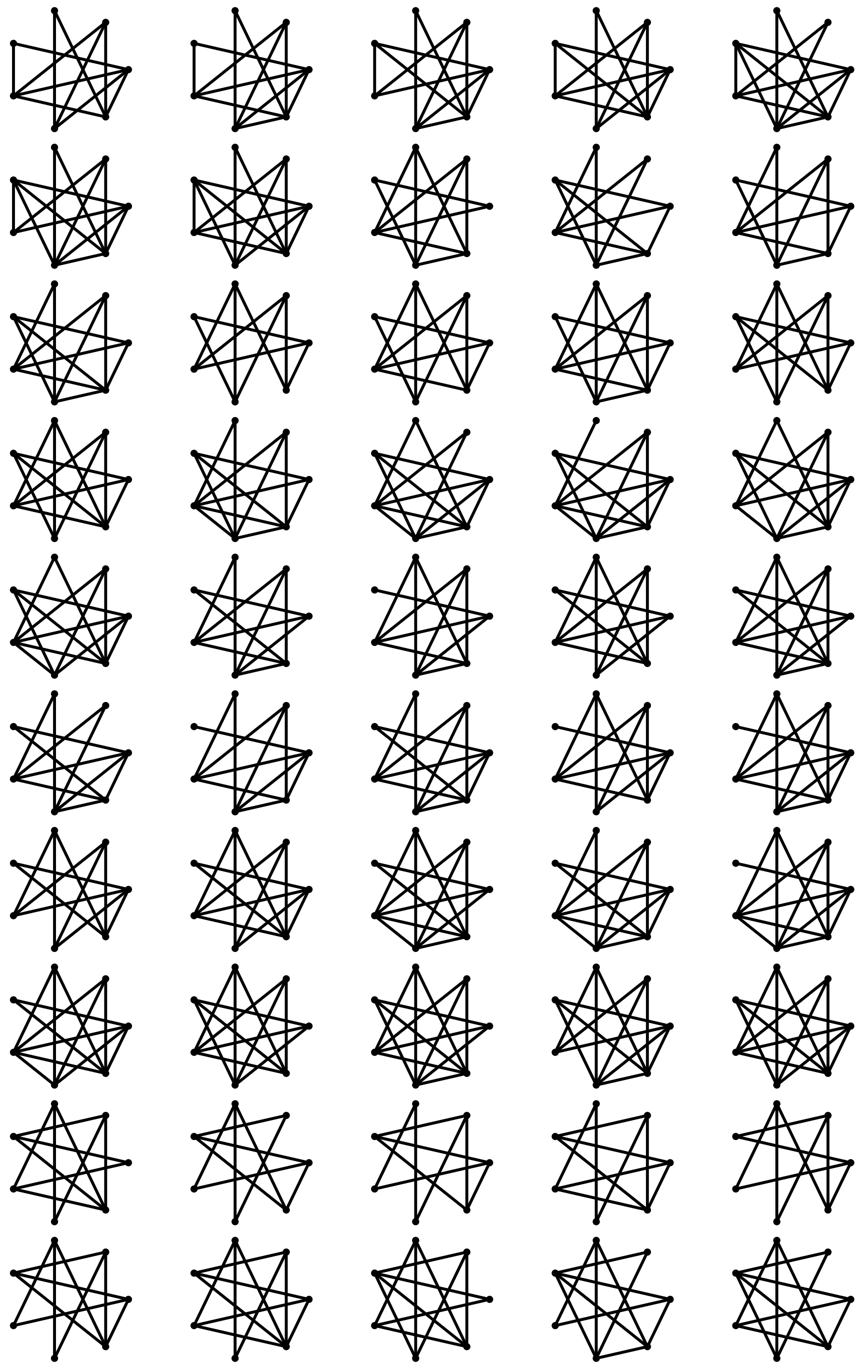


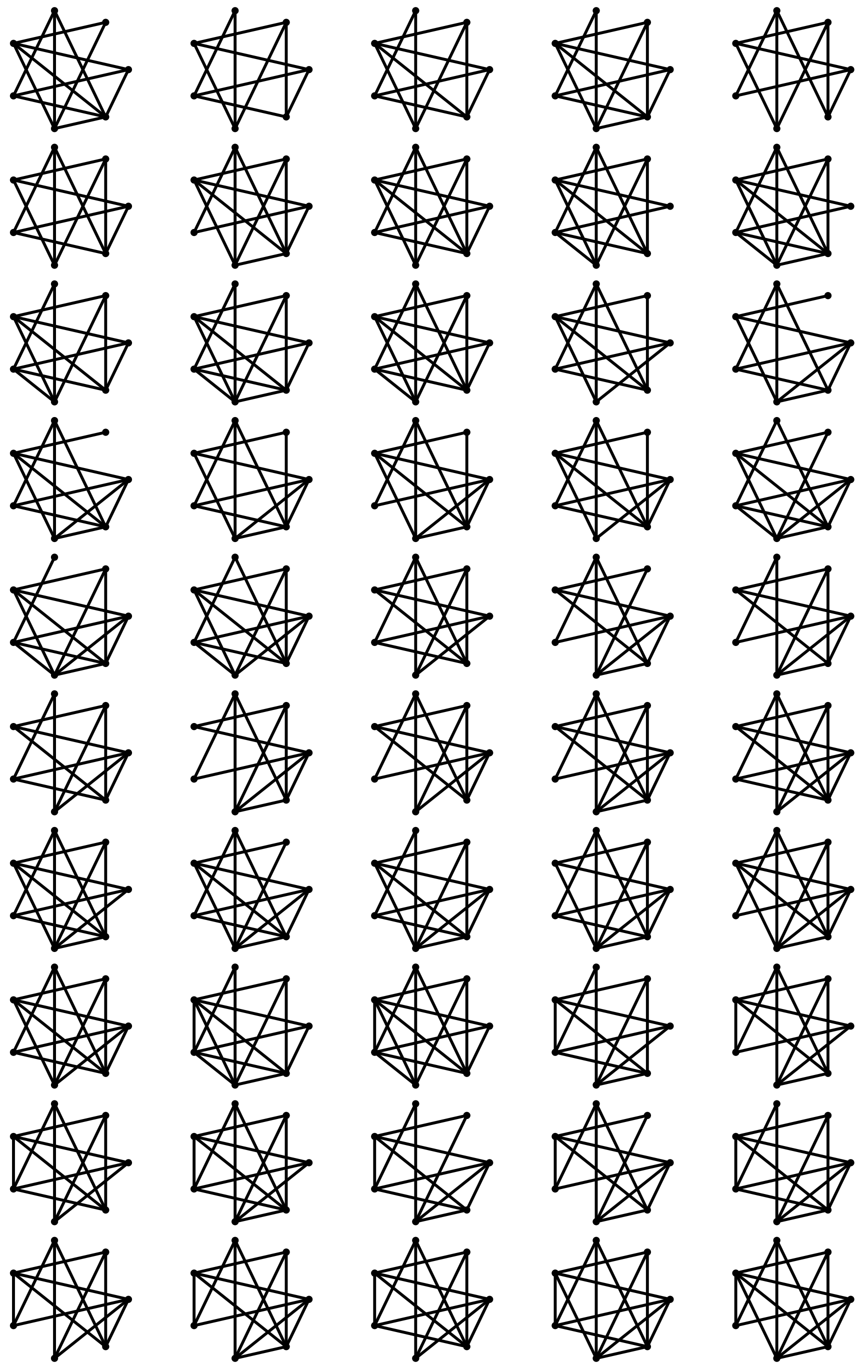


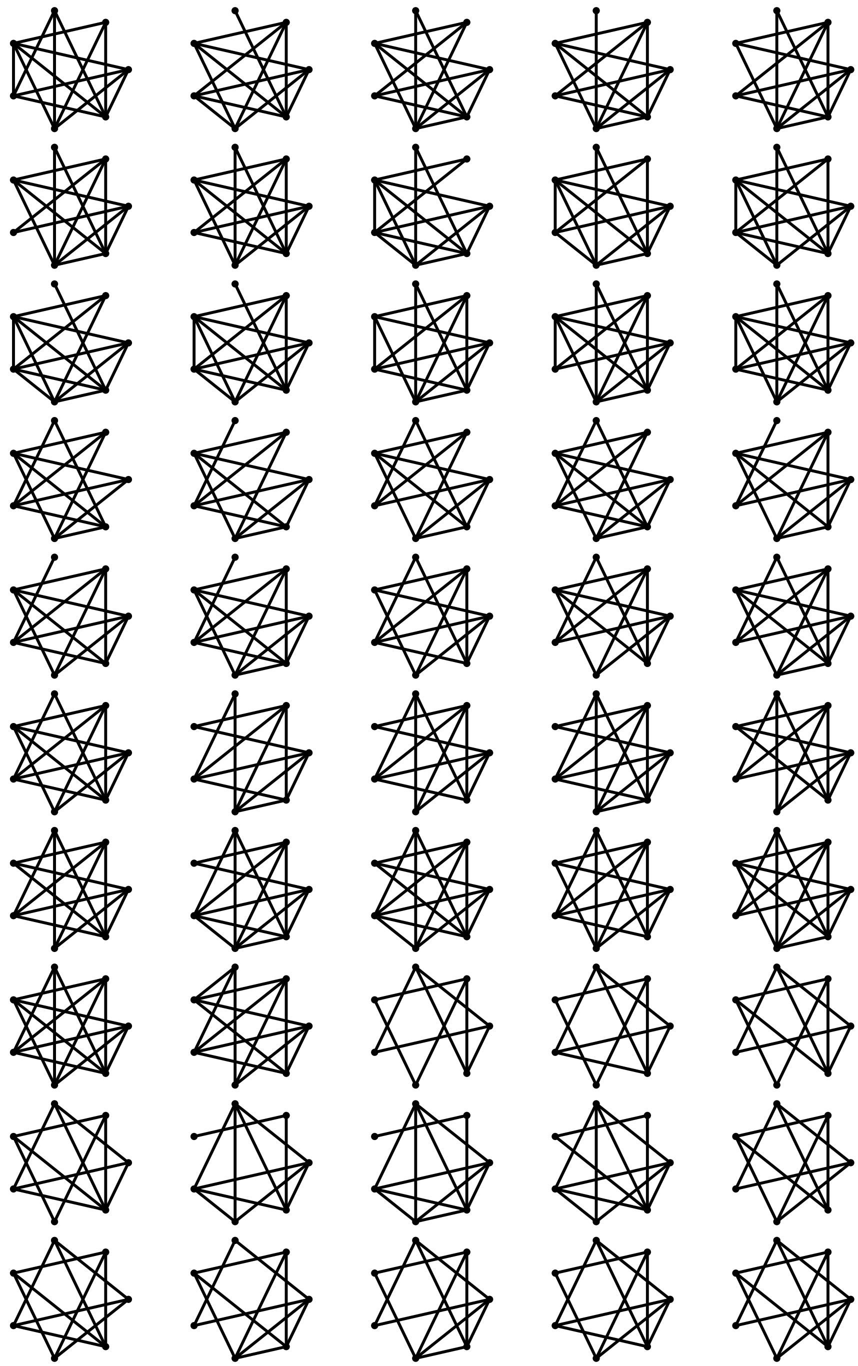


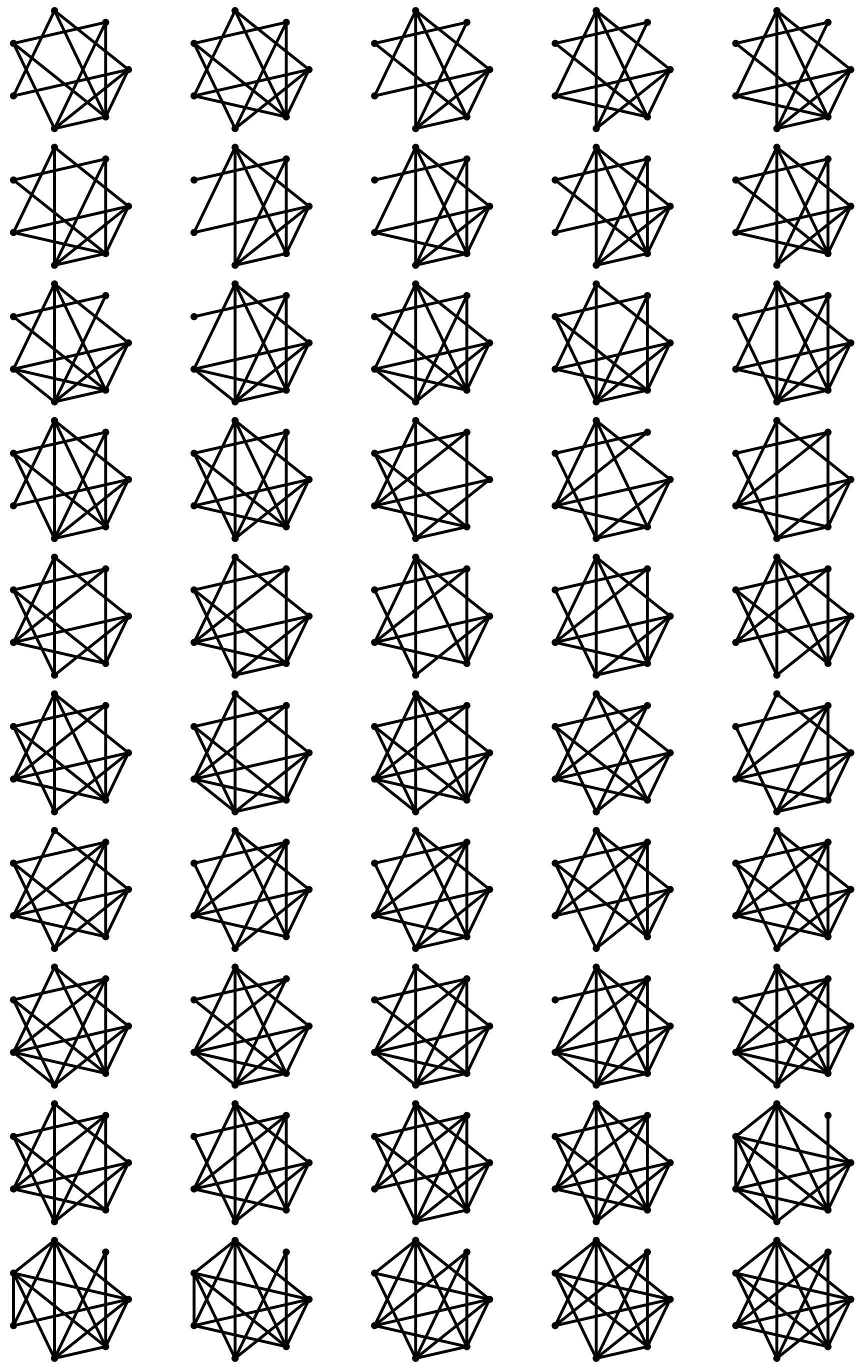












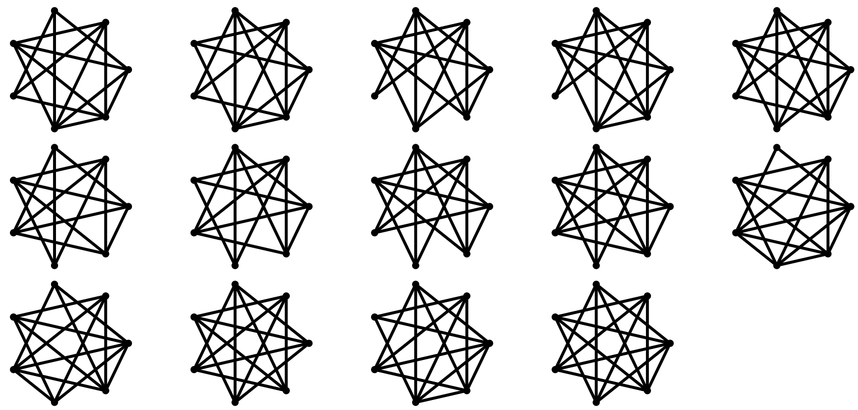
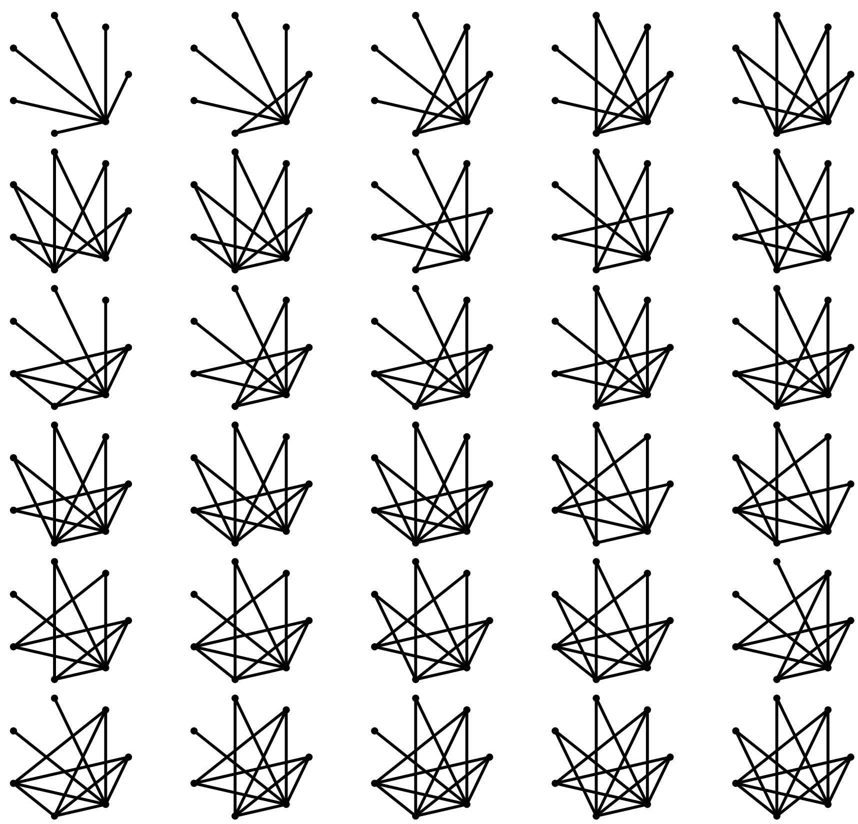
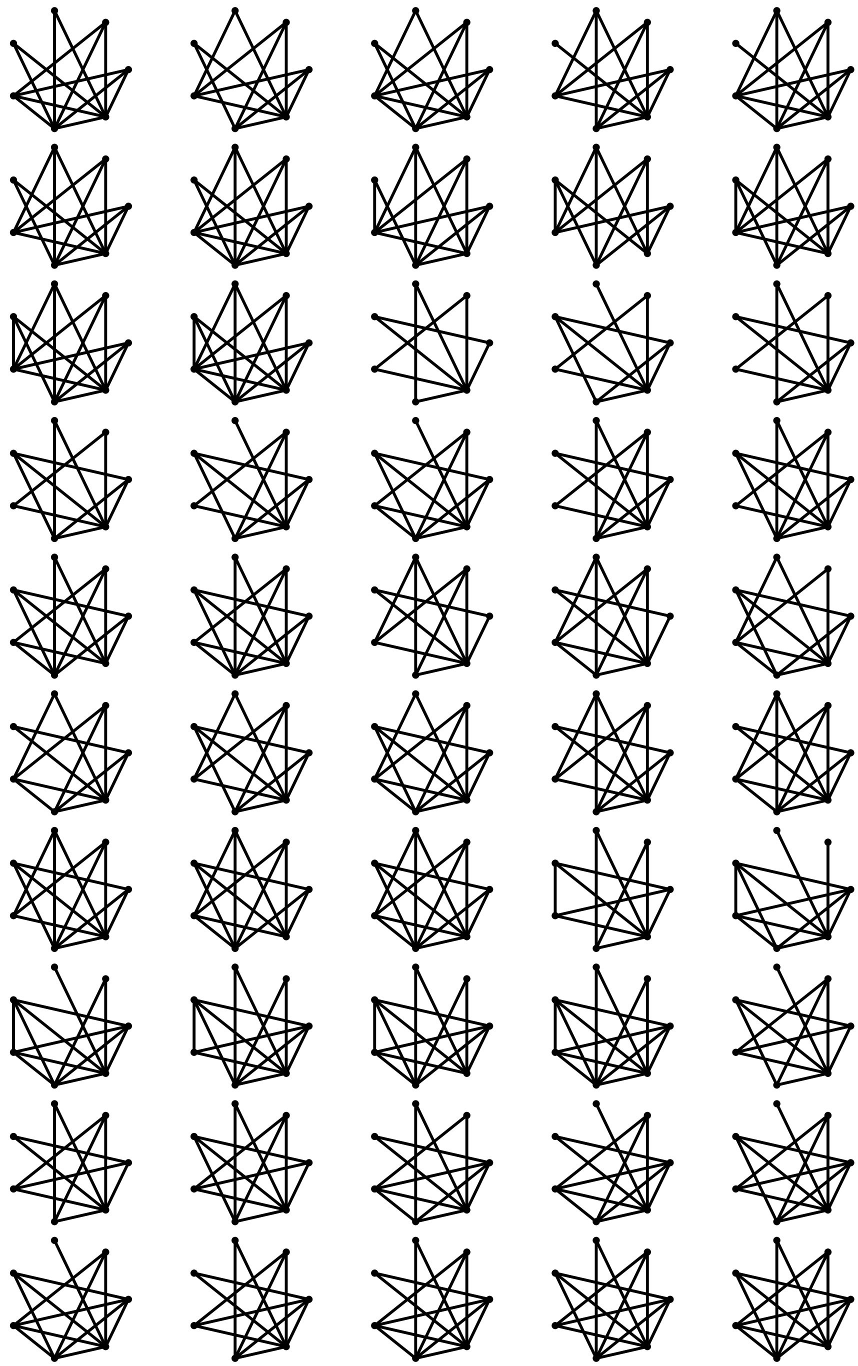
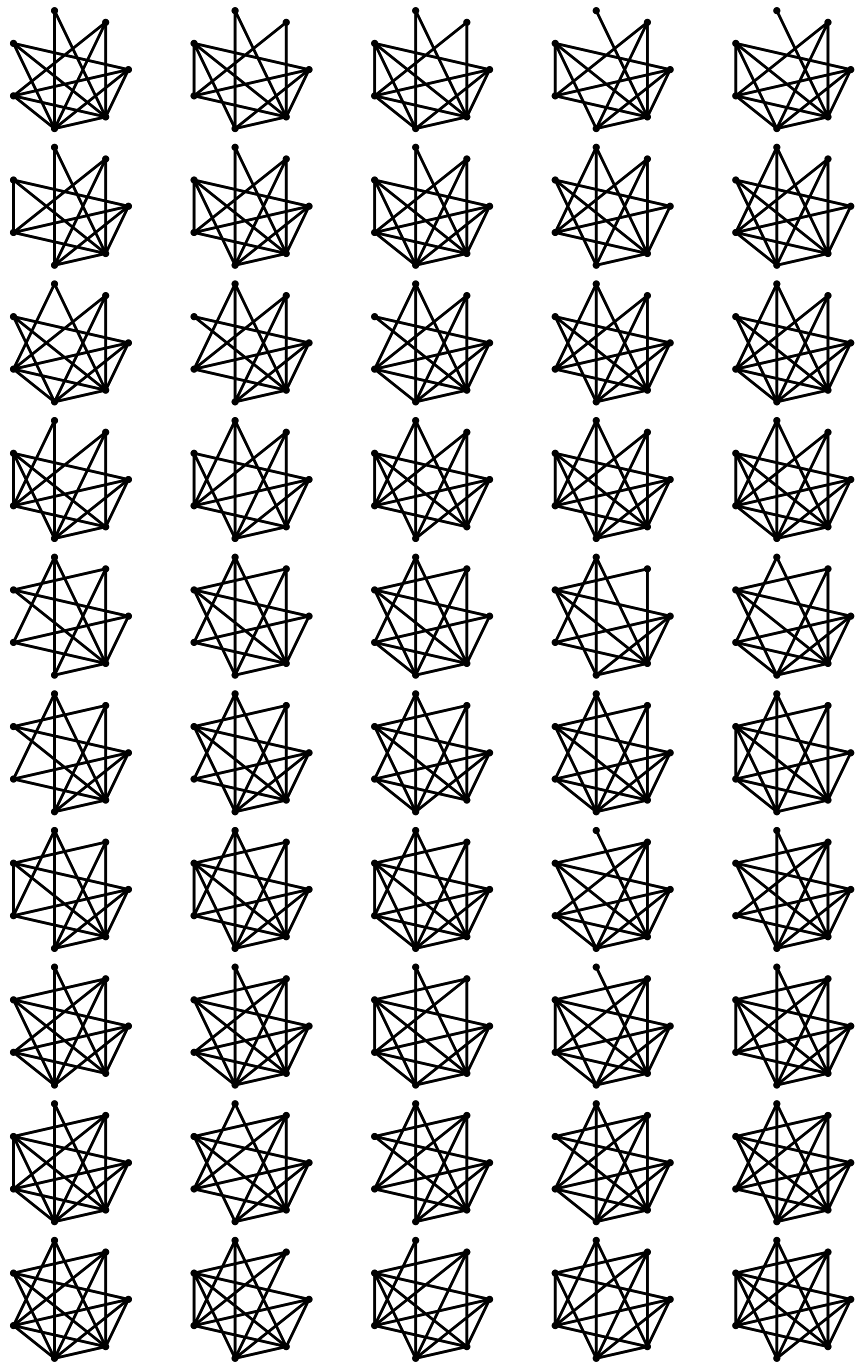
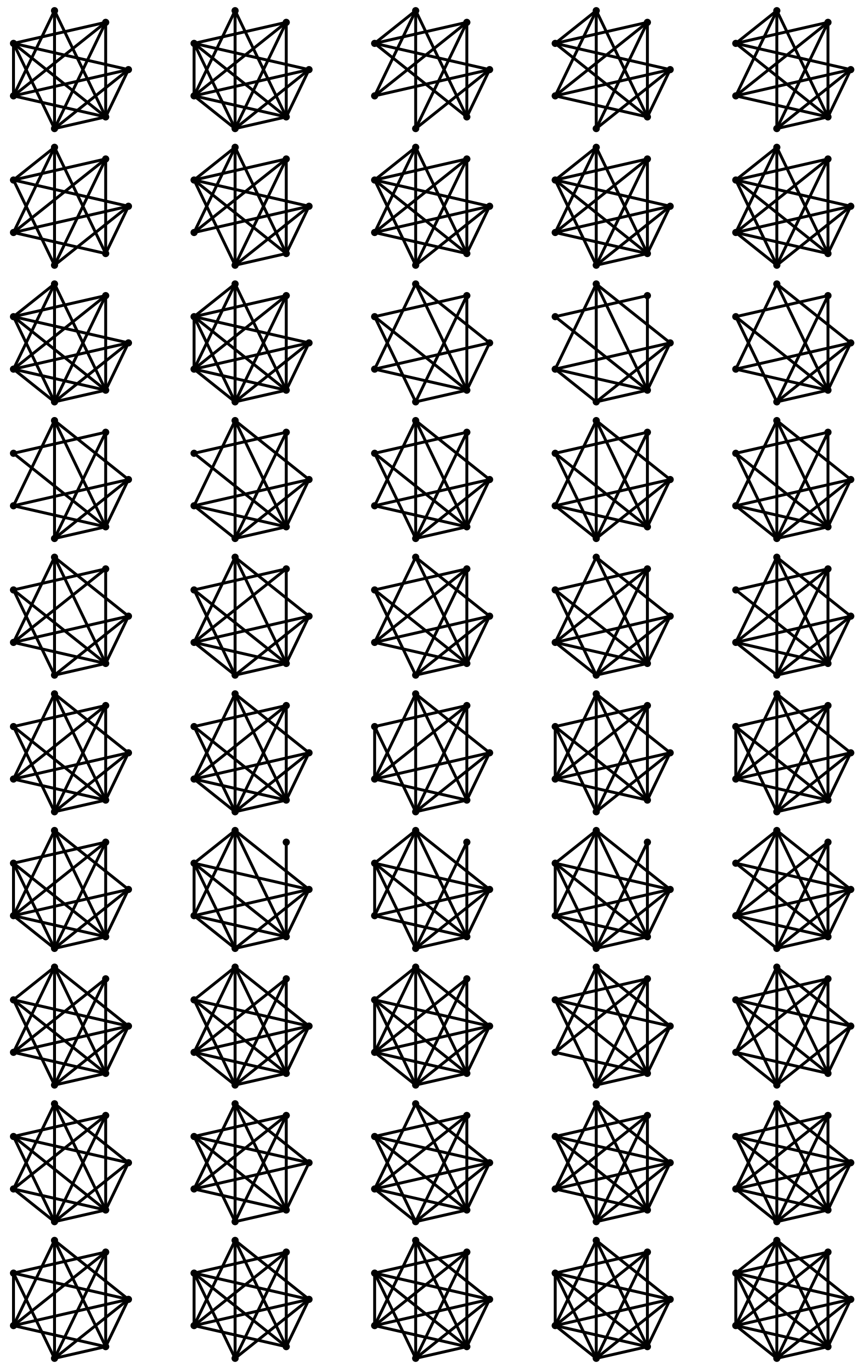


Table 6.12: Graphs with 7 vertices and projective dimension 5.









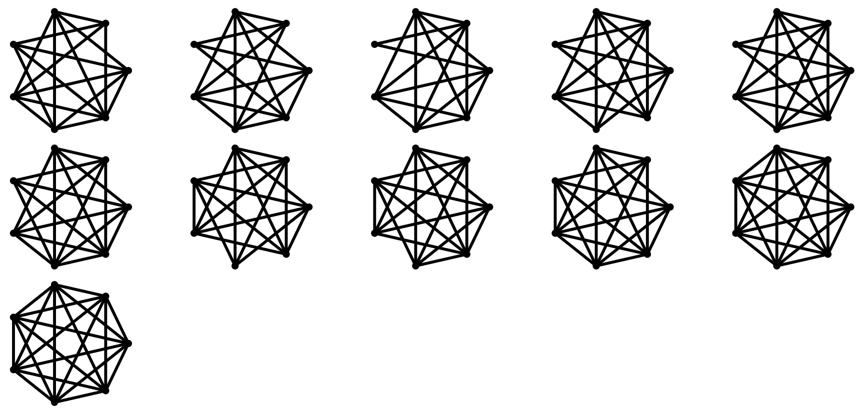


Table 6.13: Graphs with 7 vertices and projective dimension 6.

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