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Monomial subrings and ideals associated to graphs

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Subanillos e ideales monomiales asociados a gráficas

TESIS

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ABSTRACT

Let G be a simple graph with vertex set and edge set: $V(G) = \{x_1, \ldots, x_n\}$ and $E(G)$, respectively. We take an edge orientation $\mathcal{O}$ and a vertex-weight w of G. In this thesis, we study the following algebraic objects associated to G: the homogeneous monomial subring of G, the toric ideal of $(G, \mathcal{O})$ and the edge ideal of $(G, \mathcal{O}, w)$.

An ideal is unmixed if each one of its associated primes has the same height. The edge ideal of $D = (G, \mathcal{O}, w)$ is $I(D) := \left(x_i x_j^{w(x_j)} \mid (x_i, x_j) \in E(D)\right) \subseteq R$ where $R := \mathbb{K}[x_1, \ldots, x_n]$ and $\mathbb{K}$ is a field. In Chapter 2, we characterize when $I(D)$ is unmixed, if G is in one of the following families: perfect, König, SCQ, simplicial, chordal, without 3- and 5-cycles, without 4- and 5-cycles, $girth(G) \geqslant 5$ or $girth(G) \geqslant 6$.

A monomial algebra is Gorenstein if it is Cohen–Macaulay and its canonical module is a principal ideal. The homogeneous monomial subring of G is $S := \mathbb{K}[x_1 t, \ldots, x_n t]\ [\{x_i x_j t \mid \{x_i, x_j\} \in E(G)\}] \subseteq R[t]$ where t is a new variable. Assume S is normal. In Chapter 3, we study when S is Gorenstein. We prove that if S is Gorenstein, then G is unmixed with a strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction. Also, we give sufficient conditions for S to be Gorenstein when G is unmixed with a $\lceil \frac{n}{2} \rceil$-$\tau$-reduction.

An ideal I is a binomial complete intersection if it can be generated with $ht(I)$ binomials. The toric ideal P_D of $D = (G, \mathcal{O})$ is the kernel of the morphism of $\mathbb{K}$-algebras $\varphi : \mathbb{K}[\{y \mid y \in E(D)\}] \to \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by $\varphi(y) = x_i x_j^{-1}$ where $y = (x_j, x_i) \in E(D)$. In Chapter 4, we define the $\mathcal{Q}$-sums, the $\mathcal{Q}$-ring graphs and we prove the following results: the binomial complete intersection property is closed under $\mathcal{Q}$-sums; the toric ideal of a $\mathcal{Q}$-ring graph is a binomial complete intersection; and a theta-ring graph is a $\mathcal{Q}$-ring graph. Also, we characterize when P_D is a binomial complete intersection if G is a Truemper configuration.

Resumen

Sea G una gráfica simple cuyos conjuntos de vértices y de aristas son: $V(G) = \{x_1, \ldots, x_n\}$ y $E(G)$, respectivamente. Tomamos una orientación de aristas $\mathcal{O}$ y una función de pesos w de G. En esta tesis, estudiamos los siguientes objetos algebraicos asociados a G: el subanillo monomial homogéneo de G, el ideal tórico de $(G, \mathcal{O})$ y el ideal de aristas de $(G, \mathcal{O}, w)$.

Un ideal es no mezclado si cada uno de sus primos asociados tiene la misma altura. El ideal de aristas de $D = (G, \mathcal{O}, w)$ es $I(D) := (x_i x_j^{w(x_j)} \mid (x_i, x_j) \in E(D)) \subseteq R$ donde $R := \mathbb{K}[x_1, \ldots, x_n]$ con $\mathbb{K}$ un campo. En el Capítulo 2, caracterizamos cuando $I(D)$ es no mezclado, si G está en alguna de las siguientes familias: perfectas, König, SCQ, simpliciales, cordadas, sin 3- ni 5-ciclos, sin 4- ni 5-ciclos, $girth(G) \geqslant 5$ o $girth(G) \geqslant 6$.

Un álgebra monomial es Gorenstein si es Cohen–Macaulay y su módulo canónico es un ideal principal. El subanillo monomial homogéneo de G es $S := \mathbb{K}[x_1 t, \ldots, x_n t]\ [\{x_i x_j t \mid \{x_i, x_j\} \in E(G)\}] \subseteq R[t]$ donde t es una nueva variable. Asumimos que S es normal. En el Capítulo 3, estudiamos cuando S es Gorenstein. Probamos que si S es Gorenstein, entonces G es no mezclada con una $\lceil \frac{n}{2} \rceil$-$\tau$-reducción fuerte. Además, damos condiciones suficientes para que S sea Gorenstein cuando G es no mezclada con una $\lceil \frac{n}{2} \rceil$-$\tau$-reducción.

Un ideal I es una intersección binomial completa si se puede generar con $ht(I)$ binomios. El ideal tórico P_D de $D = (G, \mathcal{O})$ es el núcleo del morfismo de $\mathbb{K}$-álgebras $\varphi : \mathbb{K}[\{y \mid y \in E(D)\}] \to \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ dado por $\varphi(y) = x_i x_j^{-1}$ donde $y = (x_j, x_i) \in E(D)$. En el Capítulo 4, definimos las $\mathcal{Q}$-sumas, las gráficas $\mathcal{Q}$-anilladas y demostramos los siguientes resultados: la propiedad de intersección binomial completa es cerrada bajo $\mathcal{Q}$-sumas; el ideal tórico de una gráfica $\mathcal{Q}$-anillada es una intersección binomial completa y una gráfica theta-anillada es una gráfica $\mathcal{Q}$-anillada. Además, caracterizamos cuando P_D es una intersección binomial completa si G es una configuración de Truemper.

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INTRODUCTION

In this thesis, we study three algebraic objects associated to graphs: 1) unmixed edge ideals of weighted oriented graphs; 2) Gorenstein homogeneous monomials subrings of graphs; and 3) complete intersection toric ideals of oriented graphs. In Chapter 1, we give the algebraic and combinatorial definitions, properties and known results that we will use in the following chapters. In Chapters 2, 3 and 4 we give the original results of this thesis. The results of these chapters are published in [9], [10] and [11].

We consider G a simple graph with vertex set $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$. The edge ideal of G is the ideal $I(G) = (\{x_i x_j \mid \{x_i, x_j\} \in E(G)\})$ in the polynomial ring $R := \mathbb{K}[x_1, \dots, x_n]$ over a field $\mathbb{K}$. An edge orientation is a function $\mathcal{O} : E(G) \to V(G) \times V(G)$ which gives an orientation to each edge of G as follows: $\mathcal{O}(\{x_i, x_j\}) = (x_i, x_j)$ or $\mathcal{O}(\{x_i, x_j\}) = (x_j, x_i)$. Furthermore, a vertex-weight is a function $w : V(G) \to \mathbb{N}$, in this case $w(x_i)$ is the weight of a vertex $x_i \in V(G)$. The triplet $D = (G, \mathcal{O}, w)$ is a weighted oriented graph (Definition 1.60) whose edge ideal is $I(D) = (\{x_i x_j^{w(x_j)} \mid (x_i, x_j) \in E(D)\}) \subseteq R$. The edge ideal $I(D)$ (introduced in [33] and [23]) generalizes the usual definition of edge ideal of a graph $I(G)$, since $I(D) = I(G)$ if $w(x) = 1$ for each $x \in V(D)$. Some algebraic properties of $I(D)$ are studied in [40], [23], [26], [33], [34] and [48]. In Chapter 2, we study when $I(D)$ is unmixed (Definition 1.72 and Theorem 1.85) if G is in one of the following families of graphs: perfect (Definition 1.13); SCQ (Definition 1.39); simplicial (Definition 1.36); chordal (Definition 1.9); without 3- and 5-cycles; without 4- and 5-cycles; and $girth(G) \geqslant 5$ (Definition 1.8). In Section 2.1, we introduce the generating $\star$-semi-forests of D (Definitions 2.4 and 2.10). Furthermore, in Theorem 2.13 we prove that if G is a perfect graph, then G has a τ-reduction (Definition 1.28) in complete graphs (Definition 1.4) and we characterize when $I(D)$ is unmixed.

Theorem 2.13 Let $D = (G, \mathcal{O}, w)$ be a weighted oriented graph where G is a perfect graph, then G has a τ-reduction $H_1, \dots, H_s$ in complete subgraphs. Furthermore, $I(D)$ is unmixed if and only if each H_i has no generating $\star$-semi-forests.

The SCQ-graphs are special well-covered graphs (Definition 1.17 and Proposition 1.40). These graphs have a τ-reduction in: simplexes (Definition 1.36), basic 5-cycles (Definition 1.37) and edges. In Section 2.2, we introduce the $\star$-property of a

5-cycle (Definition 2.17). Using this property and $V^+ = \{x \in V(D) \mid w(x) > 1\}$ (Definition 1.60), in Theorem 2.23, we characterize when $I(D)$ is unmixed if G is an SCQ-graph.

Theorem 2.23 Let D be a weighted oriented graph where G is an SCQ graph. Hence, $I(D)$ is unmixed if and only if D satisfies the following conditions:

(a) Each basic 5-cycle of G has the $\star$-property.
(b) Each simplex of D has no generating $\star$-semi-forests.
(c) $N_D(b) \subseteq N_D^+(a)$ when $a \in V^+$, $\{b, b'\} \in Q_G$ and $b' \in N_D^+(a)$.

Recall G is well-covered if and only if G is unmixed (Remark 1.22). Furthermore, unmixed chordal graphs and unmixed simplicial graphs are SCQ-graphs (Theorem 1.38). Hence, using Theorem 2.23, in Corollary 2.24 we characterize when $I(D)$ is unmixed if G is chordal or simplicial.

Corollary 2.24 Let D be a weighted oriented graph where G is chordal or simplicial. Hence, $I(D)$ is unmixed if and only if D satisfies the following conditions:

(a) Each vertex is in exactly one simplex of D.
(b) Each simplex of D has not a generating $\star$-semi-forest.

In Section 2.3, we study $I(D)$ when G has at most one of the following families of cycles: 3-cycles, 4-cycles and 5-cycles. In particular in Theorem 2.28, we characterize when $I(D)$ is unmixed if G has no 3- and 5-cycles.

Theorem 2.28 Let D be a weighted oriented graph such that G has no 3- and 5-cycles. Hence, $I(D)$ is unmixed if and only if D satisfies the following conditions:

(a) G is well-covered.
(b) If $(z, x) \in E(D)$ and $z \in V^+$, then $N_D(x') \subseteq N_D^+(z)$ for some $x' \in N_D(x) \setminus z$.

Furthermore, we introduce the $\star$-property for 1-simplexes and 2-simplexes (Definition 2.37). The k-cycle and n-complete graph are denoted by C_k and K_n, respectively. A vertex $x \in V(D)$ is a sink if $N_D^+(x) = \emptyset$ (Definition 1.57). Also, the graph T_{10} (Figure 1.1) is a well-covered graph (Theorem 1.42). With these properties and graphs; in Theorem 2.38, we characterize when $I(D)$ is unmixed if G is a graph without 4- and 5-cycles.

Theorem 2.38 Let D be a connected weighted oriented graph without 4- and 5-cycles. Hence, $I(D)$ is unmixed if and only if D satisfies one of the following conditions:

(a) $G \in \{K_1, C_7, T_{10}\}$ and the vertices of V^+ are sinks.
(b) 1-simplexes and 2-simplexes have the $\star$-property and $\{V(H) \mid H \text{ is a 1-simplex or a 2-simplex}\}$ is a partition of $V(G)$.

In Corollary 2.40, using Theorem 2.38, we characterize when $I(D)$ is unmixed if $girth(G) \geqslant 6$.

Corollary 2.40 Let D be a connected weighted oriented graph with $girth(G) \geqslant 6$. Hence, $I(D)$ is unmixed if and only if D satisfies one of following properties:

(a) $G \in \{K_1, C_7\}$ and the vertices of V^+ are sinks.
(b) G has a perfect matching $y_1 = \{x_1, x_1'\}, \ldots, y_r = \{x_r, x_r'\}$ with $deg_D(x_1) = \cdots = deg_D(x_r) = 1$, furthermore, $(x_j, x_j') \in E(D)$ when $x_j' \in V^+$.

Corollary 2.40 permits to prove that if $girth(G) \geqslant 6$, then $I(D)$ is unmixed if and only if $I(D)$ is Cohen-Macaulay (see Corollary 2.41). Now, the graphs P_{10}, P_{13}, P_{14} and Q_{13} (Figure 1.1) are special well-covered graphs (Theorem 1.46). With these graphs in Theorem 2.43, we characterize when $I(D)$ is unmixed if $girth(G) \geqslant 5$.

Theorem 2.43 Let D be a connected weighted oriented graph with $\text{girth}(G) \geqslant 5$. Hence, $I(D)$ is unmixed if and only if D satisfies one of the following properties:

(a) $G \in \{K_1, C_7, Q_{13}, P_{13}, P_{14}\}$ and the vertices of V^+ are sinks.
(b) $G = P_{10}$, furthermore if x is not a sink in V^+, then $x = d_1$ with $N_D^+(x) = \{g_1, b_2\}$ or $x = d_2$ with $N_D^+(x) = \{g_2, b_1\}$.
(c) $\{V(H) \mid H \text{ is a 1-simplex or a basic 5-cycle}\}$ is a partition of $V(G)$, furthermore the 1-simplexes and the basic 5-cycles of G have the $\star$-property.

On the other hand, the homogeneous monomial subring of G is the ring $S := \mathbb{K}[x_1t, \ldots, x_nt, x^{v_1}t, \ldots, x^{v_m}t, t] \subseteq R[t]$ with $R = \mathbb{K}[x_1, \ldots, x_n]$ and t a new variable. A monomial algebra $\mathcal{A}$ is Gorenstein if $\mathcal{A}$ is Cohen-Macaulay and its canonical module $\omega_{\mathcal{A}}$ is a principal ideal. In [14], Dupont, Rentería and Villarreal prove that if G is bipartite, then S is Gorenstein if and only if G is unmixed (Proposition 1.98). Recall that S is normal if G is bipartite (Corollary 1.97). In Chapter 3, we study when S is Gorenstein if S is normal. Danilov characterizes the canonical module ω_S when S is normal (Proposition 1.101) in terms of $\mathbb{N}B \cap (\mathbb{R}_+B)^\circ$ where $\mathbb{R}_+B$ is a cone (Definition 1.99). In Section 3.1, we study this cone, in particular in Proposition 3.2 and Lemma 3.5, we characterize the elements of $(\mathbb{R}_+B)^\circ$. We set the characteristic vectors $v_1, \ldots, v_m$ of G (Definition 1.91) and by Proposition 3.2, we have that $\mathbb{R}_+B = H^+_{(e_1,0)} \cap \cdots \cap H^+_{(e_n,0)} \cap H^+_{(-\ell_1,1)} \cap \cdots \cap H^+_{(-\ell_q,1)}$. If G is not bipartite, in Lemma 3.8, we give a proof of $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \in \{\ell_1, \ldots, \ell_q\}$. In Section 3.2, we study when S is Gorenstein if S is normal.

Using $S \simeq \mathbb{K}[x_1z, \ldots, x_nz, x^{v_1}, \ldots, x^{v_m}, z]$ and some results of [3] and [32], we can show that if S is normal and n is even, then S is Gorenstein if and only if G is a unmixed bipartite graph. In Corollary 3.11, we give a new proof of this result. In Definition 3.12, we introduce the strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction. Using the strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction, in Theorem 3.14, we given necessary conditions for S to be Gorenstein if S is normal.

Theorem 3.14 If S is normal and Gorenstein, then G is unmixed, $\tau(G) = \lceil \frac{n}{2} \rceil$ and G has a strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction.

In Definition 3.16, we define the principal representation of a vector in $\mathbb{N}B$. Furthermore, using this definition, in Theorem 3.17, we give sufficient conditions for S to be Gorenstein

Theorem 3.17 If S is normal, G is unmixed with a $\lceil \frac{n}{2} \rceil$-$\tau$-reduction and each $w \in (\mathbb{R}_+B)^\circ \cap \mathbb{N}B$ has a principal representation, then S is Gorenstein.

Finally, we conjecture (Conjecture 3.19) that the necessary conditions of Theorem 3.14 are sufficient if n is odd.

Conjecture 3.19 Assume S is normal and n is odd. Then, S is Gorenstein if and only if G is unmixed with a strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction.

On the other hand, the toric ideal P_D of an oriented graph $D = (G, \mathcal{O})$ is the kernel of the epimorphism of $\mathbb{K}$-algebras $\varphi : \mathbb{K}[y_1, \ldots, y_m] \to \mathbb{K}[D]$ given by $y_i \to x^{v_i}$ where $\mathbb{K}[D] := \mathbb{K}[x^{v_1}, \ldots, x^{v_m}]$ and $v_1, \ldots, v_m$ are the edge characteristic vectors of D (Definition 1.104). The toric ideal P_D is a binomial prime ideal of height $ht(P_D) = m - n + r$ where r is the number of connected components of D. Furthermore, P_D is a binomial complete intersection if it can be generated with $ht(P_D)$ binomials. The complete intersection has been studied for some families of toric ideals in [1], [2], [22], [25], [29] and [42]. Others algebraic properties of these toric ideals are studied in [5] and [27]. In [30], Morris studies when P_D is a binomial complete intersection if D has not oriented cycles. In Chapter 4, we study the general case, i.e. when D can have oriented cycles. In Definition 4.2, we introduce the $\mathcal{Q}$-sums and in Theorem 4.6, we prove that binomial complete intersection property is closed under $\mathcal{Q}$-sums.

Theorem 4.6 Let D the $\mathcal{Q}$-sum of D_1 and D_2. If P_{D_1} and P_{D_2} are binomial complete intersections, then P_D is a binomial complete intersection.

Furthermore, in Definition 4.8, we introduce the $\mathcal{Q}$-ring graphs. In Corollary 4.9, we prove this family of oriented graphs has the binomial complete intersection property.

Corollary 4.9 If D is a $\mathcal{Q}$-ring graph, then P_D is a binomial complete intersection.

Give a graph G, there is an edge orientation $\mathcal{O}$ of G such that P_D is a binomial complete intersection where $D = (G, \mathcal{O})$ ([25] and [38]). A graph is a CI$\mathcal{O}$-graph if the toric ideals associated to each edge orientation of this graph is a binomial complete intersection (Definition 1.116). In [24], Gitler, Reyes and Vega prove that the theta-ring graphs (Definition 1.51) are the CI$\mathcal{O}$-graphs (Theorem 1.120). In Theorem 4.14, we prove the $\mathcal{Q}$-ring graphs generalize the theta ring graphs.

Theorem 4.14 If G is a connected theta-ring graph, then D is a $\mathcal{Q}$-ring graph.

Thetas, pyramids, prisms (Definition 1.47) and θ-partial wheels (Definition 1.48) are the Truemper configurations. These graphs appear in the study of β-balanceable graphs and the excluded minor of ternary matroids (see [44]). Also, the minimal forbidden induced subgraphs (obstructions) of the theta-rings graphs are the Truemper configurations (Theorem 1.53). In Section 4.2 (Propositions 4.16, 4.22, 4.24 and 4.25), we study the edge orientations of the Truemper configurations whose toric ideals are binomial complete intersections. In the following results we use the description and notation of the Truemper configuration given in Figure 1.2.

Proposition 4.16 If G is a theta graph, then P_D is a binomial complete intersection if and only if at least one principal path of D is oriented.

Proposition 4.22 Let G be a pyramid. Then, P_D is a binomial complete intersection if and only if $(x_i, x_j, \mathcal{L}_j, z)$ is an oriented path for some $i, j \in \{1, 2, 3\}$ with $i \neq j$.

Proposition 4.24 Let G be a prism. Then, P_D is a binomial complete intersection if and only if at least one path $(x_i, x_j, \mathcal{L}_j, z_j, z_k)$ is oriented with $i, j, k \in \{1, 2, 3\}$ and $j \notin \{i, k\}$.

Proposition 4.25 Let D be an oriented partial wheel with center z and rim C. If z is neither a source nor a sink; or $C = (x, \mathcal{L}, x', \mathcal{L}', x'', \mathcal{L}'', x)$ where $\mathcal{L}, \mathcal{L}''$ are oriented, $x, x', x'' \in N_D(z)$, $N_D(z) \cap V\big((\mathcal{L}')^{\circ}\big) = \emptyset$ and $(z, x, \mathcal{L}, x')$ or $(z, x, \mathcal{L}'', x'')$ is an oriented path, then P_D is a binomial complete intersection.

In Example 4.35, D' is an induced oriented subgraph of D and P_D is a binomial complete intersection, but $P_{D'}$ is not a binomial complete intersection. Hence, the binomial complete intersection property is not closed under induced subgraphs. Nevertheless, in Proposition 4.26, we prove the binomial complete intersection property is closed under some special induced subgraphs.

Proposition 4.26 Let $D = (G, \mathcal{O})$ be an oriented graph with a non-oriented path $\mathcal{P}$ such that $deg_G(x) = 2$ for each $x \in V(\mathcal{P}^{\circ})$ and $G' = G \setminus V(\mathcal{P}^{\circ})$ is connected. If P_D is a binomial complete intersection, then $P_{D'}$ is a binomial complete intersection with $D' = G'_{\mathcal{O}}$.

Although binomial complete intersection is not closed under induced oriented subgraphs, in Proposition 4.33, we prove that special oriented theta is an obstruction of the binomial complete intersection property.

Proposition 4.33 Let θ be a theta of G with end vertices x and z such that $deg_G(a) = 2$ for each $a \in V(\theta) \setminus \{x, z\}$. If D is connected and D has not oriented paths between x and z, then P_D is not a binomial complete intersection.

TABLE OF CONTENTS

CHAPTER 1

PRELIMINARIES

1.1 GRAPHS AND ORIENTED GRAPHS

In this section we give the combinatorial definitions and properties and some known results that we will use in chapters 2, 3 and 4.

1.1.1 BASIC DEFINITIONS OF GRAPHS

Definition 1.1 A **finite simple graph** G is an ordered pair of finite sets $(V(G), E(G))$, where the elements of $E(G)$ are subsets of cardinality two of $V(G)$. $V(G)$ is called the **vertex set** of G and $E(G)$ is called the **edge set** of G.

In this thesis all graphs are finite and simple. We assume G is a graph.

Definition 1.2 A vertex x is **incident** with an edge $y \in E(G)$ if $x \in y$. Furthermore, if $y = \{x, x'\} \in E(G)$, then we say x and x' are **adjacent vertices** or **neighbours**. The set of all neighbours in $V(G)$ of a vertex x is the **neighbourhood** of x in G and it is denoted by $N_G(x)$ and the **closed neighbourhood** of x in G is the set $N_G[x] := N_G(x) \cup \{x\}$.

Definition 1.3 The **degree** of a vertex x of G is $deg_G(x) = |N_G(x)|$. Furthermore, a vertex is **isolated** if its degree is zero.

Definition 1.4 A graph G is **complete** if $N_G[x] = V(G)$ for each $x \in V(G)$. In this case, G is denoted by K_n where $n = |V(G)|$.

Definition 1.5 A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given a subset A of $V(G)$, the **subgraph induced** by A in G, denoted by $G[A]$, is the subgraph G_1 of G with $V(G_1) = A$ and $E(G_1) = \{y \in E(G) \mid y \subseteq A\}$. A subgraph H of G is **induced** if there is $B \subseteq V(G)$ such that $H = G[B]$.

Definition 1.6 If H_1 and H_2 are subgraphs of G; we define the subgraphs $H_1 \cup H_2$ and $H_1 \cap H_2$ where $V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$, $E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$; $V(H_1 \cap H_2) = V(H_1) \cap V(H_2)$ and $E(H_1 \cap H_2) = E(H_1) \cap E(H_2)$.

Definition 1.7 A **walk** of G is a sequence of vertices $\mathcal{P} = (x_1, \dots, x_k)$ such that $\{x_i, x_{i+1}\} \in E(G)$ for $i = 1, \dots, k-1$. In this case, $\mathcal{P}$ is a subgraph of G, where $V(\mathcal{P}) = \{x_1, \dots, x_k\}$ and $E(\mathcal{P}) = \{\{x_i, x_{i+1}\} \mid i \in \{1, \dots, k-1\}\}$. It is possible that $x_i = x_j$ for some $i \neq j$. If $x_1, \dots, x_k$ are different, then $\mathcal{P}$ is a **path** and its **length** is the number of its edges.

Definition 1.8 A walk $C = (z_1, z_2, \dots, z_k, z_1)$ is an k**-cycle** if $(z_1, \dots, z_k)$ is a path. A k-cycle C is **even** (resp. **odd**) if k is even (resp. odd), in this case C is denoted by C_k. A cycle C of G is **induced** if C is an induced subgraph of G. The minimum length of a cycle (contained) in a graph G, is called the **girth of** G.

Definition 1.9 If G has no odd-cycles, then G is called **bipartite**. Furthermore, G is a **chordal graph** if the induced cycles are 3-cycles.

Definition 1.10 [21] A graph G satisfies the **odd cycle condition**, if for any two odd cycles either have a common vertex, or there exists a pair of vertices, one from each cycle, which are joined by an edge.

Definition 1.11 A k**-colouring of** G is a function $c : V(G) \to \{1, 2, \dots, k\}$ such that $c(x) \neq c(x')$ if $\{x, x'\} \in E(G)$. The smallest integer k such that G has a k-colouring is called the **chromatic number of** G and it is denoted by $\chi(G)$.

Proposition 1.12 [13, Proposition 1.6.1] A graph G is 2-colouring if and only if G is bipartite.

Definition 1.13 The **clique number**, denoted by $\omega(G)$ is the size of the largest complete subgraph of G. Furthermore, if $\chi(H) = \omega(H)$ for every induced subgraph H of G, then G is a **perfect** graph.

Definition 1.14 The **complement** of G, denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ such that for each pair $x, x' \in V(G)$, we have that $\{x, x'\} \in E(\overline{G})$ if and only if $\{x, x'\} \notin E(G)$.

Theorem 1.15 [13, Theorem 5.5.3] G is perfect if and only if $\overline{G}$ is perfect.

The following Theorem (called the strong perfect graph Theorem) characterizes the forbidden induced subgraphs for perfect graphs.

Theorem 1.16 [7] A graph G is perfect if and only if neither G nor $\bar{G}$ contains and odd cycle of length at least 5 as an induced subgraph.

1.1.2 WELL-COVERED GRAPHS

In this subsection, we give some definitions and known results related with the well-covered property of graphs.

Definition 1.17 A **stable set** of G is a subset of $V(G)$ containing no edge of G. The **stable number of** G, denoted by $\alpha(G)$, is $\alpha(G) := \max\{|S| \mid S \text{ is a stable set of } G\}$. A graph G is **well–covered** if $|S| = \alpha(G)$ for each maximal stable set S of G.

Remark 1.18 Let A be a subset of $V(G)$, then A is a stable set of G if and only if $\overline{G}[A]$ is a complete subgraph of $\overline{G}$. Hence, $\alpha(G) = \omega(\overline{G})$.

Definition 1.19 A **vertex cover** $\mathcal{C}$ of G is a subset of $V(G)$, such that if $\{x, x'\} \in E(G)$, then $x \in \mathcal{C}$ or $x' \in \mathcal{C}$. A vertex cover $\mathcal{C}$ of G is **minimal** if each proper subset of $\mathcal{C}$ is not a vertex cover of G. The **cover number** of G is $\tau(G) := \min\{|\mathcal{C}| \mid \mathcal{C} \text{ is a vertex cover of } G\}$. A graph G is **unmixed** if every minimal vertex cover has $\tau(G)$ elements.

Remark 1.20 Let $\mathcal{C}$ be a vertex cover of G and $x \in V(G)$. We take $y \in E(G)$, then $\mathcal{C} \cap y \neq \emptyset$. Hence, $y \cap (\mathcal{C} \setminus x) \neq \emptyset$ if $x \notin y$, furthermore $x' \in N_D(x)$ if $y = \{x, x'\}$. Therefore, $(\mathcal{C} \setminus x) \cup N_D(x)$ is a vertex cover of G.

Remark 1.21 Let $\mathcal{C}$ be a vertex cover of G with $x \in V(G) \setminus \mathcal{C}$. If $x' \in N_G(x)$, then $y := \{x, x'\} \in E(G)$ and $y \cap \mathcal{C} \neq \emptyset$. So, $x' \in \mathcal{C}$, since $x \notin \mathcal{C}$. Hence, $N_G(x) \subseteq \mathcal{C}$.

Remark 1.22 If S is a stable set of G and $y \in E(G)$, then $y \nsubseteq S$. Thus, $y \cap (V(G) \setminus S) \neq \emptyset$. Hence, a subset F of $V(G)$ is a (maximal) stable set if and only if $V(G) \setminus F$ is a (minimal) vertex cover. Therefore, $\tau(G) + \alpha(G) = |V(G)|$ and G is unmixed if and only if G is well-covered.

Definition 1.23 A collection of pairwise disjoint edges of G is called a **matching**. A **perfect matching** is a matching whose union is $V(G)$. The maximum cardinality of a matching of G is denoted by $\nu(G)$. Furthermore, G is a **König graph** if $\tau(G) = \nu(G)$.

Definition 1.24 Let $y = \{b, b'\}$ be an edge of G. We say that y has the **property (P)** if for each pair of edges $\{a, b\}$, $\{a', b'\} \in E(G)$, we have that $\{a, a'\} \in E(G)$. On the other hand, we say that a matching P of G has the **property (P)** if each edge of P has the property **(P)**.

Lemma 1.25 [34, Lemma 3.1] Let P be a matching of G with the property **(P)**. If $b_1, b_2 \in N_G(a)$ and $\{b_1, b_1'\}$, $\{b_2, b_2'\} \in P$, then $\{b_1', b_2'\} \notin E(G)$.

Proof. By contradiction, suppose $\{b_1', b_2'\} \in E(G)$. Since $\{a, b_1\} \in E(G)$, $\{b_1, b_1'\} \in P$ and P has the property **(P)**, we have $\{a, b_2'\} \in E(G)$. Similarly, $\{a, a\} \in E(G)$, since $\{a, b_2\}, \{a, b_2'\} \in E(G)$, $\{b_2, b_2'\} \in P$ and P has the property **(P)**. A contradiction, since G is a simple graph. Therefore, $\{b_1', b_2'\} \notin E(G)$. □

Proposition 1.26 [47, Theorem 1.1] Let G be a bipartite graph without isolated vertices. Then, G is unmixed if and only if G has a perfect matching with the property **(P)**.

Theorem 1.27 [6, Proposition 15] Let G be a König graph without isolated vertices. Then, G is well–covered if and only if G has a perfect matching with property **(P)**.

Definition 1.28 A **τ-reduction** of G is a collection of pairwise disjoint induced subgraphs $G_1, \ldots, G_s$ of G such that $V(G) = \cup_{i=1}^{s} V(G_i)$ and $\tau(G) = \sum_{i=1}^{s} \tau(G_i)$.

Remark 1.29 If $G_1, \ldots, G_s$ is a τ-reduction of G, then $|V(G)| = \sum_{i=1}^{s} |V(G_i)|$ and $\tau(G) = \sum_{i=1}^{s} \tau(G_i)$. Furthermore, by Remark 1.22, $\alpha(G) = |V(G)| - \tau(G)$. Hence, $\alpha(G) = \sum_{i=1}^{s} |V(G_i)| - \sum_{i=1}^{s} \tau(G_i) = \sum_{i=1}^{s} \Big(|V(G_i)| - \tau(G_i)\Big) = \sum_{i=1}^{s} \alpha(G_i)$.

Lemma 1.30 If G is unmixed with a τ-reduction $G_1, \ldots, G_s$, then for each F maximal stable $\alpha(G_i) = |F \cap V(G_i)|$ for $i = 1, \ldots, s$.

Proof. Let F be a maximal stable set. Then, $F \cap V(G_i)$ is a stable set of G_i. Thus, $|F \cap V(G_i)| \leq \alpha(G_i)$. Hence, by Remark 1.29, $\alpha(G) = \sum_{i=1}^{s} \alpha(G_i) \geq \sum_{i=1}^{s}$

$|F \cap V(G_i)| = |F|$. But G is well-covered, then $|F| = \alpha(G)$. Therefore $\alpha(G_i) = |F \cap V(G_i)|$. □

Definition 1.31 G is **very well-covered** if G is well-covered without isolated vertices and $|V(G)| = 2\alpha(G)$ (equivalently, $|V(G)| = 2\tau(G)$).

Proposition 1.32 [15, Theorem 1.2] G is very well-covered if and only if there is a perfect matching $\{y_1, \ldots, y_s\}$ with the property **(P)**.

Remark 1.33 In the previous proposition $y_1, \ldots, y_s$ is a τ-reduction.

Proof. Since $\{y_1, \ldots, y_s\}$ is a perfect matching $V(G) = \cup_{i=1}^{s} y_i$ implies $|V(G)| = 2s$. Also, G is very well-covered, then $\tau(G) = s$. Furthermore, $\alpha(y_i) = 1$, then $\tau(G) = s = \sum_{i=1}^{s} \tau(y_i)$. Therefore, $y_1, \ldots, y_s$ is a τ-reduction of G. □

Proposition 1.34 [36, Lemma 14] If G is unmixed, with $\tau(G) = \frac{n+1}{2}$, then there exists a τ-reduction $G_1, \ldots, G_s$ of G such that $G_i \in E(G)$ for $1 \leq i \leq s-1$ and G_s is a j-cycle where $j \in \{3, 5, 7\}$.

Remark 1.35 In the previous proposition, $\{G_1, \ldots, G_{s-1}\}$ has the property **(P)**.

Proof. By contradiction, suppose that $G_j = \{x, x'\}$ has no the property **(P)**. Thus, there are $\{x, z\}, \{x', z'\} \in E(G)$ such that $\{z, z'\} \notin E(G)$. Then, there is a maximal stable set F such that $\{z, z'\} \subseteq F$. Hence, $|F \cap V(G_j)| = 0$, since F is a stable set. A contradiction, by Lemma 1.30, since $\alpha(G_j) = 1$. Therefore, $\{G_1, \ldots, G_{s-1}\}$ has the property **(P)**. □

Definition 1.36 A vertex x of G is a **simplicial vertex** if the induced subgraph $H = G[N_G[x]]$ is a complete graph. In this case, H is called k-**simplex** (or **simplex**) where $k = |V(H)| - 1$. The set of simplexes of G is denoted by S_G. Furthermore, G is a **simplicial graph** if every vertex of G is a simplicial vertex or is adjacent to a simplicial vertex.

Definition 1.37 An induced 5-cycle C of G is called **basic** if C does not contain two adjacent vertices of degree three or more in G.

Theorem 1.38 [35, Theorems 1 and 2] If G is a chordal or simplicial graph, then G is well-covered if and only if every vertex of G belongs to exactly one simplex of G.

Now, we define SCQ-graphs. These graphs generalize the graphs defined in [37].

Definition 1.39 G is an **SCQ graph** (or $G \in SCQ$) if $\{V(H) \mid H \in S_G \cup C_G \cup Q_G\}$ is a partition of $V(G)$, where C_G is the set of basic 5-cycles and $Q_G = \emptyset$ or Q_G is a matching with the property **(P)**.

Proposition 1.40 If G is an SCQ-graph, then G is well-covered and $\alpha(G) = |S_G| + 2|C_G| + |Q_G|$.

Proof. We take a maximal stable set S of G and $H \in S_G \cup C_G \cup Q_G$.
First assume $H \in S_G$, then $H = G[N_G[v]]$ is a complete graph for some $v \in V(H)$. So, $|S \cap V(H)| \leqslant 1$. But if $S \cap V(H) = \emptyset$, then $S \cup \{v\}$ is a stable set. A contradiction, since S is maximal. Hence $|S \cap V(H)| = 1$.
Now, suppose $H \in C_G$. So, $H = (z_1, z_2, z_3, z_4, z_5, z_1)$ is a 5-cycle with $deg_G(z_1) = deg_G(z_3) = deg_G(z_4) = 2$. Thus, $|V(H) \cap S| \leqslant 2$. If $\{z_3, z_4\} \cap S = \emptyset$, then $z_2, z_5 \in S$, since $deg_G(z_3) = deg_G(z_4) = 2$ and S is a maximal stable set. Then, $|V(H) \cap S| = 2$. Now, if $\{z_3, z_4\} \cap S \neq \emptyset$, then we can assume $z_3 \in S$ implies $z_2 \notin S$. Consequently, $z_1 \in S$ or $z_5 \in S$, since $deg_G(z_1) = 2$ and $z_2 \notin S$. Hence $|V(H) \cap S| = 2$, since $|V(H) \cap S| \leqslant 2$.
Finally, assume $H \in Q_G$, then $H = \{x_1, x_2\} \in E(G)$ and H has the property **(P)**. Thus, $|V(H) \cap S| \leqslant 1$. But if $V(H) \cap S = \emptyset$, then there are $x_1', x_2' \in S$ such that $\{x_1, x_1'\}, \{x_2, x_2'\} \in E(G)$, since S is maximal . So, $\{x_1', x_2'\} \in E(G)$ since H has the property **(P)**. A contradiction, since $x_1', x_2' \in S$. Hence, $|V(H) \cap S| = 1$.
Then $|S| = |S_G| + 2|C_G| + |Q_G|$, since $\{V(H) \mid H \in S_G \cup C_G \cup Q_G\}$ is a partition of $V(G)$. Therefore G is well-covered and $\alpha(G) = |S_G| + 2|C_G| + |Q_G|$. □

Remark 1.41 If G is König well-covered graph, then G is an SCQ graph.

In the following results, we use the graphs of Figure 1.1.

Theorem 1.42 [17, Theorem 1.1] Let G be a connected graph without 4- and 5-cycles. Then, G is well-covered if and only if $G \in \{C_7, T_{10}\}$ or $\{V(H) \mid H \in S_G\}$ is a partition of $V(G)$.

Remark 1.43 Suppose G is a well-covered graph such that $G \notin \{C_7, T_{10}\}$. If G is simplicial, or G is chordal or G is a graph without 4- and 5-cycles, then by Theorems 1.38 and 1.42, $\{V(H) \mid H \in S_G\}$ is a partition of $V(G)$. Therefore, G is an SCQ graph with $C_G = Q_G = \emptyset$.

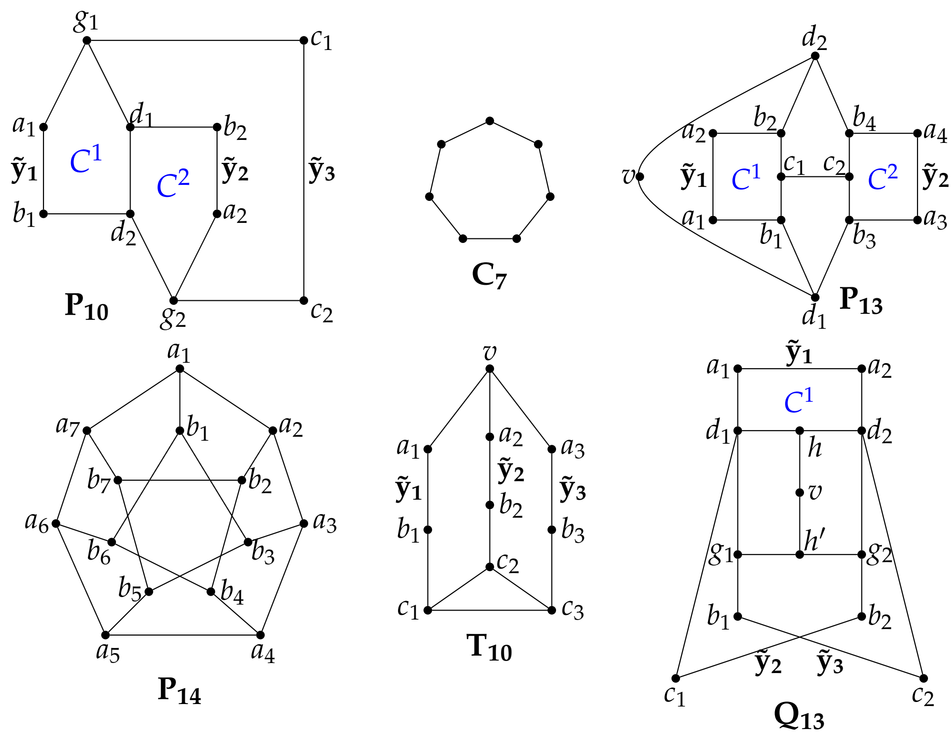


Figure 1.1 Special well–covered graphs

Definition 1.44 An edge y is **pendant** if y has a vertex of degree 1 in G.

Corollary 1.45 If G is connected and unmixed with $girth(G) \geqslant 6$ such that $G \notin \{K_1, C_7\}$, then its pendant edges form a perfect matching of G.

Proof. Since $girth(G) \geqslant 6$, we have G has no 3-, 4- and 5-cycles. Thus, by Theorem 1.42, $\mathcal{A} = \{V(H) \mid H \in S_G\}$ is a partition of $V(G)$, since $G \neq C_7$ and T_{10} has a 3-cycle. Now, if $H \in S_G$, then $|V(H)| = 2$, since G has no 3-cycles and $G \neq K_1$. Hence, $H \in S_G$ if and only if H is a pendant edge. Therefore, the pendant edge of G is a perfect matching of G, since $\mathcal{A}$ is a partition of $V(G)$. □

Theorem 1.46 [16, Theorems 2 and 3] If G is connected without 3- and 4-cycles, then G is well-covered if and only if $G \in \{K_1, C_7, P_{10}, P_{13}, P_{14}, Q_{13}\}$ or $\{V(H) \mid H \in S_G \cup C_G\}$ is a partition of $V(G)$.

1.1.3 TRUEMPER CONFIGURATIONS AND THETA-RING GRAPHS

Truemper configurations (see Definitions 1.47, 1.48 and 1.49) appear in the study of β-balanceable graphs and the excluded minor for ternary matroids. In [44], Truemper showed that G is β-balanceable if and only if for every induced subgraph H such that H is K_4 or a Truemper configuration, we have that H is β^H-balanceable. After, in [8], Conforti, Cornuéjols, Kapoor and Vušković proved that G is a universally signable graph if and only if G has no Truemper configurations as induced subgraphs. On the other hand, theta-ring graphs (see Definition 1.51) are introduced in [24], in that paper Gitler, Reyes and Vega proved that G is a theta-ring graph if and only if the toric ideal $P_{G_{\mathcal{O}}}$ (see Definition 1.105) is a binomial complete intersection for each edge orientation $\mathcal{O}$ of G. Also, they proved in [24] that the minimal forbidden induced subgraphs for theta-ring graphs are the Truemper configurations. Hence, theta-ring graphs and universal signable graphs are equivalent. In this subsection, we give the definitions and some known results about Truemper configurations and theta-ring graphs.

Definition 1.47 A **theta** is a graph consisting of two non adjacent vertices x and z, and three paths $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ with ends x and z, such that the union of every two of $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ is an induced cycle.
A **pyramid** is a graph consisting of a vertex z, a triangle $C = (x_1, x_2, x_3, x_1)$, and three paths $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, such that: $\mathcal{L}_i$ is a path between x_i and z; $V(\mathcal{L}_i) \cap V(\mathcal{L}_j) = \{z\}$ for different $i, j \in \{1,2,3\}$ and at most one of the $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ has only one edge.
A **prism** is a graph consisting of two vertex-disjoint triangles $C = (x_1, x_2, x_3, x_1)$ and $C' = (z_1, z_2, z_3, z_1)$, and three vertex-disjoint paths $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ such that $\mathcal{L}_i$ is a path between x_i and z_i for $i = 1, 2, 3$.
A **partial wheel** W is a graph where $V(W) = \{z, x_1, \ldots, x_s\}$, such that $C = (x_1, \ldots, x_s, x_1)$ is a cycle in W and the edges of W are the edges of C and some edges between z and vertices of C. In this case, C is the **rim** of W and z is the **center** of W.

If G is a theta or a pyramid or a prism and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are as in the last definition, then $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are called **principal paths of** G.

Definition 1.48 A partial wheel W with rim C and center z is a θ**-partial wheel** if there exist two non adjacent vertices in $V(C) \cap N_W(z)$.

Definition 1.49 Thetas, pyramids, prisms and θ-partial wheels are the **Truemper configurations**.

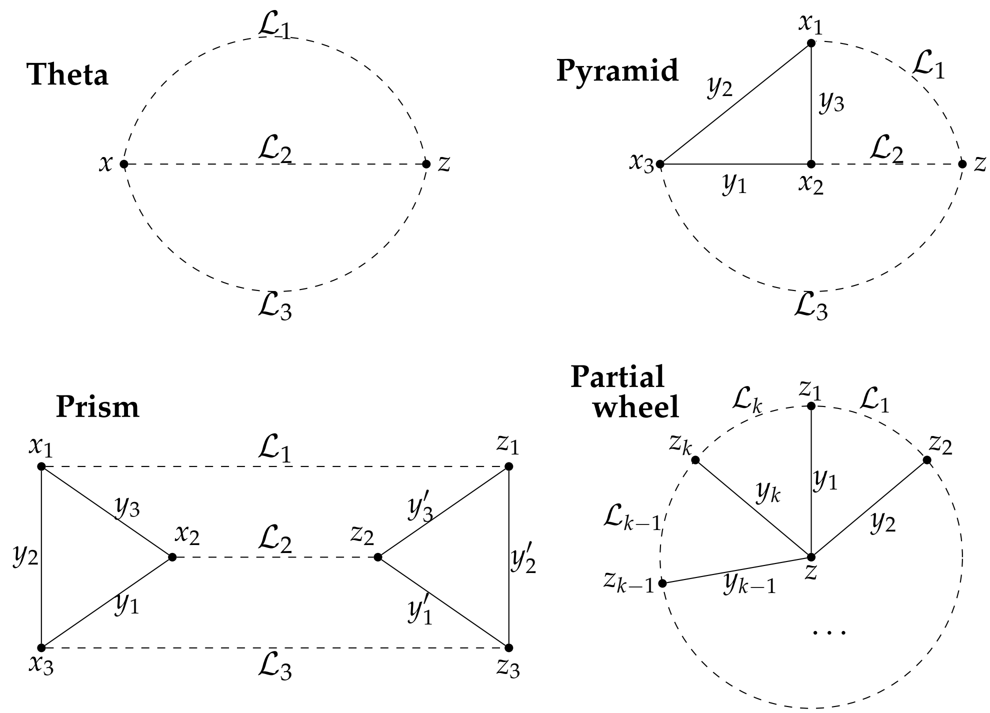


Figure 1.2 Truemper configurations

Definition 1.50 A **chorded-theta subgraph** T of G is a subgraph induced by three paths $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ each between two different non adjacent vertices x, x' such that $V(\mathcal{L}_i) \cap V(\mathcal{L}_j) = \{x, x'\}$ for $1 \leqslant i < j \leqslant 3$. A **transversal triangle** C of T is a 3-cycle such that $V(C) = \{z_1, z_2, z_3\}$ with $z_i \in V(\mathcal{L}_i) \setminus \{x, x'\}$ for $i = 1, 2, 3$.

Definition 1.51 A graph G is called a **theta-ring graph** if each chorded-theta of G has a transversal triangle.

Definition 1.52 If G_1 and G_2 are subgraphs of G such that $H = G_1 \cup G_2$ and $K = G_1 \cap G_2$ is a complete subgraph, then H is the **clique-sum** of G_1 and G_2.

Theorem 1.53 [24, Theorem 4] The following conditions are equivalent:

(i) G is a theta-ring graph.
(ii) G can be constructed by $0, 1, 2$-clique-sums of chordal graphs and/or cycles.
(iii) G can be constructed by clique-sums of complete graphs and/or cycles.
(iv) G has no Truemper configurations as induced subgraphs.

1.1.4 ORIENTED GRAPHS

Definition 1.54 An **oriented graph** D is an ordered pair $(G, \mathcal{O})$ where G is a graph and $\mathcal{O}$ is a function $\mathcal{O} : E(G) \to V(G) \times V(G)$ such that $\mathcal{O}(\{x, x'\}) = (x, x')$ or $\mathcal{O}(\{x, x'\}) = (x', x)$. In this case, G is the **underlying graph** of D, $\mathcal{O}$ is an **edge orientation** of G. Furthermore, the vertex set of D is $V(D) := V(G)$ and the edge set of D is $E(D) := \{\mathcal{O}(y) \mid y \in E(G)\} \subseteq V(G) \times V(G)$.

In this subsection, we assume that $\mathcal{O}$ is an edge orientation of G and $D = (G, \mathcal{O})$.

Definition 1.55 An oriented graph H is an **oriented subgraph** of D if $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$.

Definition 1.56 If H is a subgraph of G and $D = (G, \mathcal{O})$, then $\mathbf{H}_{\mathcal{O}}$ is the oriented subgraph of D where $V(H_{\mathcal{O}}) = V(H)$ and $E(H_{\mathcal{O}}) = \{\mathcal{O}(y) \mid y \in E(H)\}$.

Definition 1.57 Let x be a vertex of D, the sets $N_D^+(x) = \{z \mid (x, z) \in E(D)\}$ and $N_D^-(x) = \{z \mid (z, x) \in E(D)\}$ are called the **out-neighbourhood** and the **in-neighbourhood** of x, respectively. If $N_D^+(x) = \emptyset$ or $N_D^-(x) = \emptyset$, then x is called **sink** or **source**, respectively. Furthermore, the **neighbourhood** of x is the set $N_D(x) = N_D^+(x) \cup N_D^-(x)$. If $A \subseteq V(D)$, then $N_D^+(A) = \bigcup_{a \in A} N_D^+(a)$.

Remark 1.58 If $x \in V(D)$, then $N_D(x) = N_G(x)$.

Definition 1.59 An **oriented walk** in D is a walk $\mathcal{P} = (x_1, \dots, x_k)$ of G such that $(x_i, x_{i+1}) \in E(D)$ for $i = 1, \dots, k-1$. If furthermore $\mathcal{P}$ is a path, we say that $\mathcal{P}$ is an **oriented path**. A cycle $C = (z_1, z_2, \dots, z_k, z_1)$ of G is an **oriented k-cycle** of D if $(z_1, \dots, z_k)$ is an oriented path.

Definition 1.60 A **vertex-weight** of a graph G is a function $w : V(G) \to \mathbb{N}$ and if $x \in V(G)$, then $w(x)$ is called the **weight** of x. An oriented graph $D = (G, \mathcal{O})$ is called a **weighted oriented graph** if G has a vertex-weight w and we denote it by $D = (G, \mathcal{O}, w)$. In this case, we denote the set $\{x \in V(D) \mid w(x) > 1\}$ by V^+.

1.2 COMBINATORIAL COMMUTATIVE ALGEBRA

1.2.1 BASIC DEFINITIONS

In this subsection, we consider R is a commutative ring with unit 1.

Definition 1.61 An **ideal** I of R is a subset of R such that I is an additive subgroup and if $r \in R$ and $a \in I$, then $ra \in I$. A subset X of I is called a **generator set** of the ideal I, if $I = \{\sum_{i=1}^{s} r_i a_i \mid r_i \in R, a_i \in X \text{ and } s \in \mathbb{N}\}$ and we denoted by $I = (X)$. In this case, if X is a finite set, then we say that I is **finitely generated**. Furthermore, X is called a **minimal generator set** of I, if no one proper subset of X is a generator set of I.

Definition 1.62 An ideal I is **principal** if I is generated by one element of R.

Definition 1.63 An ideal P of R is **prime** if $P \neq R$ and $ab \in P$ implies $a \in P$ or $b \in P$. The **height of a prime ideal** P is

$$ht(P) := \max\{s \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_s = P \text{ is a chain of prime ideals}\}.$$

Definition 1.64 An ideal J of R is **primary** if $J \neq R$ and $ab \in J$ implies $a \in J$ or $b^k \in J$ for some $k \geqslant 0$.

Definition 1.65 The **radical** of an ideal I is the ideal $r(I) := \{r \in R \mid r^k \in I$ for some $k \in \mathbb{Z}^+ \setminus \{0\}\}$.

Proposition 1.66 [39, Proposition 6.110] If J is a primary ideal, then $r(J)$ is a prime ideal.

Definition 1.67 We say I has a **primary decomposition** if $I = \bigcap_{i=1}^{s} q_i$ where q_i is a primary ideal for each $i = 1, \ldots, s$. In this case, the decomposition is **irredundant** if $q_1 \cap \cdots \cap \hat{q}_j \cap \cdots \cap q_s \neq I$ for each $1 \leqslant j \leqslant s$.

Proposition 1.68 gives the existence and Proposition 1.71 gives the uniqueness under radicals of the irredundant primary decompositions of each ideal in a Noetherian ring. This result is called Lasker-Noether Theorem.

Proposition 1.68 [39, Theorem 6.115] If R is Noetherian and I is an ideal of R with $I \neq R$, then I has a primary decomposition.

Definition 1.69 If I is an ideal of R and $x \in R$, we define the ideal $(I : x) := \{a \in R \mid ax \in I\}$.

Definition 1.70 Assume R is Noetherian and I is an ideal of R. A prime ideal P of R is an **associated prime** of I if $P = (I : x)$ for some $x \in R$. The set of associated primes of I is denoted by $\mathrm{Ass}(I)$.

Proposition 1.71 [39, Theorem 6.116] Assume R is Noetherian and I is an ideal of R. If $I = \bigcap_{i=1}^{s} q_i$ is an irredundant primary decomposition of I, then $\mathrm{Ass}(I) = \{r(q_1), \ldots, r(q_s)\}$.

Definition 1.72 An ideal I of R is **unmixed** if each one of its associated primes has the same height.

Definition 1.73 Let M be a finitely generated R-module where R is Noetherian. If I is an ideal of R such that $IM \neq M$, then the **grade of** M in I, denoted by $G(I, M)$ is the length of a maximal M-regular sequence in I. Furthermore, if R is Noetherian and local where the maximal ideal is m, then the **depth** of M is $depth(M) := G(m, M)$.

The depth and regular sequence of monomial ideals and their powers are studied in [18], [19] and [20].

Definition 1.74 A monomial algebra $\mathcal{A}$ is **Gorenstein** if $\mathcal{A}$ is Cohen-Macaulay and its canonical module $\omega_{\mathcal{A}}$ is a principal ideal.

Proposition 1.75 [28] If $\mathcal{A}$ is normal, then $\mathcal{A}$ is Cohen-Macaulay. Hence, if $\mathcal{A}$ is normal, then $\mathcal{A}$ is Gorenstein if and only if $\omega_{\mathcal{A}}$ is principal.

1.2.2 EDGE IDEALS

In this subsection, we assume $D = (G, \mathcal{O}, w)$ is a weighted oriented graph (see Definition 1.60) and $\mathbb{K}[x_1, \ldots, x_n]$ is a polynomial ring where $V(G) = \{x_1, \ldots, x_n\}$ and $\mathbb{K}$ is a field. We give some known results of edge ideals of G and D. In particular, we study the primary decomposition and the unmixed property of these ideals.

Definition 1.76 The **edge ideal** of the graph G is the ideal

$$I(G) = \big(\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}\big) \text{ of } \mathbb{K}[x_1, \ldots, x_n].$$

Proposition 1.77 [46, Corollary 6.1.18] If G is a graph, then $ht\big(I(G)\big) = \tau(G)$.

Remark 1.78 [34, Remark 2.12] $I(G)$ is unmixed if and only if G is well-covered.

Definition 1.79 The **edge ideal** of the weighted oriented graph D is the ideal $I(D) = \big(\{x_i x_j^{w(x_j)} \mid (x_i, x_j) \in E(D)\}\big)$ of $\mathbb{K}[x_1, \ldots, x_n]$.

Definition 1.80 $\mathcal{C}$ is a vertex cover of D if $\mathcal{C}$ is a vertex cover of G.

Remark 1.81 Consider the weighted oriented graph $\tilde{D} = (G, \mathcal{O}, \tilde{w})$ where $\tilde{w}(x) = 1$ if x is a source and $\tilde{w}(x) = w(x)$ if x is not a source. Hence, $I(\tilde{D}) = I(D)$. Therefore, in this thesis, we can assume that if x is a source, then $w(x) = 1$.

Definition 1.82 Let $\mathcal{C}$ be a vertex cover of D, we define the following three sets:

- $\mathbf{L_1}(\mathcal{C}) := \{x \in \mathcal{C} \mid N_D^+(x) \cap \mathcal{C}^c \neq \emptyset\}$ where $\mathcal{C}^c = V(D) \setminus \mathcal{C}$,
- $\mathbf{L_2}(\mathcal{C}) := \{x \in \mathcal{C} \mid x \notin L_1(\mathcal{C}) \text{ and } N_D^-(x) \cap \mathcal{C}^c \neq \emptyset\}$,
- $\mathbf{L_3}(\mathcal{C}) := \mathcal{C} \setminus (L_1(\mathcal{C}) \cup L_2(\mathcal{C}))$.

Remark 1.83 Let $\mathcal{C}$ be a vertex cover of D, then $x \in L_3(\mathcal{C})$ if and only if $N_D[x] \subseteq \mathcal{C}$. Hence, $L_3(\mathcal{C}) = \emptyset$ if and only if $\mathcal{C}$ is minimal.

Definition 1.84 A vertex cover $\mathcal{C}$ of D is **strong** if for each $x \in L_3(\mathcal{C})$ there is $(x', x) \in E(D)$ such that $x' \in L_2(\mathcal{C}) \cup L_3(\mathcal{C}) = \mathcal{C} \setminus L_1(\mathcal{C})$ with $x' \in V^+$ (i.e. $w(x') > 1$).

Theorem 1.85 [33, Theorem 31] The following conditions are equivalent:

(1) $I(D)$ is unmixed.
(2) Each strong vertex cover of D has the same cardinality.
(3) $I(G)$ is unmixed and $L_3(\mathcal{C}) = \emptyset$ for each strong vertex cover $\mathcal{C}$ of D.

Remark 1.86 We have $\tau(G) = |\mathcal{C}_1|$, for some vertex cover $\mathcal{C}_1$. So, $\mathcal{C}_1$ is minimal. Thus, by Remark 1.83, $L_3(\mathcal{C}_1) = \emptyset$. Hence, $\mathcal{C}_1$ is strong. Now, if $I(D)$ is unmixed, then by (2) in Theorem 1.85, $|\mathcal{C}| = |\mathcal{C}_1| = \tau(G)$ for each strong vertex cover $\mathcal{C}$ of D.

Theorem 1.87 [34, Theorem 3.4] If G is König, then $I(D)$ is unmixed if and only if D satisfies the following two conditions:

(1) G has a perfect matching P with property **(P)**.
(2) If $a \in V(D)$, $w(a) > 1$, $b' \in N_D^+(a)$ and $\{b, b'\} \in P$, then $N_D(b) \subseteq N_D^+(a)$.

Proposition 1.88 [33, Proposition 51] If $I(D)$ is Cohen-Macaulay, then $I(D)$ is unmixed.

Theorem 1.89 [34, Theorem 4.3 and Proposition 4.5] If G is König or G has neither 3- nor 5-cycles, then $I(D)$ is Cohen-Macaulay if and only if D satisfies the following two conditions:

(1) G has a perfect matching P with property **(P)** and G has no 4-cycles with two edges in P.
(2) If $a \in V(D)$, $w(a) > 1$, $b' \in N_D^+(a)$ and $\{b, b'\} \in P$, then $N_D(b) \subseteq N_D^+(a)$.

Corollary 1.90 [34, Corollary 4.4] Let D be a weighted oriented graph, where G is a König graph without 4-cycles. Hence, $I(D)$ is unmixed if and only if $I(D)$ is Cohen–Macaulay.

1.2.3 HOMOGENEOUS SUBRINGS OF GRAPHS

In this subsection, we give some properties of the homogeneous monomial subrings associated to graphs. In particular, we give some known results of Gorenstein and normal properties of these monomial subrings. We assume G is a connected graph with $V(G) = \{x_1, \dots, x_n\}$ and $|E(G)| = m \neq 0$.

Definition 1.91 If $y = \{x_i, x_j\} \in E(G)$, then the **characteristic vector** of y is the vector in $\{0,1\}^n \subseteq \mathbb{Z}^n$ such that its i-th entry is 1, its j-th entry is 1, and the remaining entries are zero.

We assume $v_1, \dots, v_m$ the characteristic vectors of the edges of G.

Definition 1.92 Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, the **homogeneous monomial subring** of G is the ring:

$$S = \mathbb{K}[x_1 t, \cdots, x_n t, x^{v_1} t, \cdots, x^{v_m} t, t] \subset R[t], \text{ where } t \text{ is a new variable.}$$

Remark 1.93 S is a standard $\mathbb{K}$-algebra, where a monomial $x^a t^b$ has degree b. We assume S has this grading.

Definition 1.94 Let G_z be the graph with loops where $V(G_z) = V(G) \cup \{z\}$ and $E(G_z) = E(G) \cup \{\{z, x_i\} \mid i = 1, \ldots, n\} \cup \{(z, z)\}$. The **edge ring** of G_z is $R(G_z) := \mathbb{K}[x_1 z, \ldots, x_n z, x^{v_1}, \ldots, x^{v_m}, z^2]$.

The following Lemma is the result of change of grading.

Lemma 1.95 [46, Lemma 8.4.15] If G is a graph, then $S \cong R(G_z)$.

Since we assume G is connected, we have:

Proposition 1.96 [14, Theorems 3.2 and 3.3] S is normal if and only if G satisfies the odd cycle condition.

Others proofs of Proposition 1.96 are given in [41, Proposition 2.1] and [31, Corollary 2.3].

Corollary 1.97 If G is bipartite, then S is normal.

Proof. G has no odd cycles, since G is bipartite. Then, G satisfies the odd cycle condition. Hence, by Proposition 1.96, S is normal. □

Proposition 1.98 [14, Corollary 4.3] If G is bipartite, then S is Gorenstein if and only if G is unmixed.

Definition 1.99 We consider the set $B := \{(e_1, 1), \ldots, (e_n, 1), (v_1, 1), \ldots, (v_m, 1), e_{n+1}\} \subseteq \mathbb{R}^n \times \mathbb{R}$ where $e_1, \ldots, e_n$ are the canonical vectors in $\mathbb{R}^n$ and $e_{n+1} = (\underbrace{0, \ldots, 0}_{n}, 1)$. Furthermore,

$$\mathbb{R}_+B = \left\{ \sum_{i=1}^{m} \alpha_i(v_i, 1) + \sum_{i=1}^{n} \beta_i(e_i, 1) + \lambda e_{n+1} \mid \alpha_i, \beta_i, \lambda \in \mathbb{R}_+ \right\} \subseteq \mathbb{R}^n \times \mathbb{R}.$$

Remark 1.100 We have $\operatorname{aff}(\mathbb{R}_+B) = \mathbb{R}^{n+1}$, since $(e_1, 1), \ldots, (e_n, 1), e_{n+1}$ are linearly independent.

The following is a result given by Danilov in [12]. This result permits to characterize the canonical module of S when S is normal.

Proposition 1.101 [4, Theorem 6.3.5] If S is normal, then the canonical module of S is given by

$$\omega_S = \left(\{x^a t^b \mid (a,b) \in \mathbb{N}B \cap (\mathbb{R}_+B)^\circ\}\right),$$

where $(\mathbb{R}_+B)^\circ$ is the interior of $\mathbb{R}_+B$ relative to $\operatorname{aff}(\mathbb{R}_+B)$.

The following results characterize the generator of ω_S if S is normal and ω_S is principal.

Proposition 1.102 [11, Proposition 3.7] If S is normal and ω_S is principal, then $\omega_S = (x^{\mathbf{1}} t^{\beta})$ where $\beta \leq \lfloor \frac{n}{2} \rfloor + 1$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$.

Proposition 1.103 [11, Proposition 3.12] If G is not bipartite, S is normal and Gorenstein, then G is unmixed, $\tau(G) = \lceil \frac{n}{2} \rceil$ and $\omega_S = (x^{\mathbf{1}} t^b)$ with $b = \lfloor \frac{n}{2} \rfloor + 1$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$.

1.2.4 TORIC IDEALS OF ORIENTED GRAPHS

In this subsection, we assume that $D = (G, \mathcal{O})$ is an oriented graph (see Definition 1.54) where $V(D) = \{x_1, \ldots, x_n\}$ and $E(D) = \{y_1, \ldots, y_m\}$.

Definition 1.104 For each oriented edge $y = (x_i, x_j) \in E(D)$, the **characteristic vector** (or **edge characteristic vector**) of y is $v_y = (v_y^1, \ldots, v_y^n) \in \{0, 1, -1\}^n \subseteq \mathbb{Z}^n$ such that $v_y^i = -1$, $v_y^j = 1$ and $v_y^l = 0$ for $l \notin \{i, j\}$.

In this subsection, we assume $v_1, \ldots, v_m$ are the characteristic vectors of $y_1, \ldots, y_m$, respectively.

Definition 1.105 The **toric ideal of** D, denoted by P_D, is the kernel of the epimorphism of $\mathbb{K}$-algebras:

$$\varphi : \mathbb{K}[y_1, \ldots, y_m] \to \mathbb{K}[D] \text{ where } y_i \to x^{v_i} \tag{1.1}$$

where $\mathbb{K}[y_1, \ldots, y_m]$, $\mathbb{K}[D] := \mathbb{K}[x^{v_1}, \ldots, x^{v_m}] \subseteq \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\mathbb{K}$ is a field.

Proposition 1.106 [24] P_D is a binomial prime ideal of height $ht(P_D) = m - n + r$ where r is the number of connected components of D.

Definition 1.107 P_D is a **binomial complete intersection** if P_D can be generated by $m - n + r$ binomials.

Definition 1.108 Let $\mathcal{L} = (x = x_1, \ldots, x_r = x')$ be a walk in D. Then, we define $\mathcal{L}^+ = \{(x_i, x_{i+1}) \mid (x_i, x_{i+1}) \in E(D)\}$ and $\mathcal{L}^- = \{(x_{i-1}, x_i) \mid (x_{i-1}, x_i) \in E(D)\}$. Thus, $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^-$ and we define the monomials

$$y_{\mathcal{L}^+} = \prod_{y_j \in \mathcal{L}^+} y_j \qquad \text{and} \qquad y_{\mathcal{L}^-} = \prod_{y_j \in \mathcal{L}^-} y_j.$$

In particular, if $\mathcal{L} = C$ is a cycle of D, then we define the binomial $y_C = y_{C^+} - y_{C^-}$.

Remark 1.109 If $\mathcal{L}$ is an oriented walk, then $\mathcal{L}^+ = \emptyset$ or $\mathcal{L}^- = \emptyset$, i.e., $y_{\mathcal{L}^+} = 1$ or $y_{\mathcal{L}^-} = 1$ respectively.

Proposition 1.110 [24] and [25] If C is a cycle of D, then $y_C \in P_D$.

Theorem 1.111 [24, Theorem 1] If $0 \neq y^\alpha - y^\beta \in P_D$ then there exist a cycle C of D such that $y_C = y^{\alpha'} - y^{\beta'}$, $y^{\alpha'} \mid y^\alpha$ and $y^{\beta'} \mid y^\beta$.

Corollary 1.112 [24, Corollary 2] If $0 \neq y^\alpha - y^\beta = f$ in P_D with $gcd(y^\alpha, y^\beta) = 1$, then there exist cycles $C^1, \ldots, C^s$ of D such that $y^\alpha = y^{\alpha_1} \cdots y^{\alpha_s}$ and $y^\beta = y^{\beta_1} \cdots y^{\beta_s}$ where $y_{C^i} = y^{\alpha_i} - y^{\beta_i}$ for $i = 1, \ldots, s$.

Proposition 1.113 [25, Corollary 4.5] P_D is generated by the set of binomials corresponding to cycles without chords.

Remark 1.114 Let $\mathcal{B}$ be a binomial generating set of P_D. If $u - u' \in \mathcal{B}$, then we can assume $gcd(u, u') = 1$.

Definition 1.115 Let $\mathcal{A}$ be a binomial subset of $R = \mathbb{K}[y_1, \ldots, y_m]$. Then, we define $Mon(R) = \{m \in R \mid m \text{ is a monomial}\}$ and

$$\mathbf{Mon}(\mathcal{A}) = \{m \in Mon(R) \mid m - m' \in \mathcal{A} \text{ or } m' - m \in \mathcal{A} \text{ with } m' \in Mon(R)\}.$$

Definition 1.116 G is a **CI**$\mathcal{O}$ graph if for each edge orientation $\mathcal{O}$ of G we have $P_{(G,\mathcal{O})}$ is a binomial complete intersection.

Proposition 1.117 [45, Proposition 2.51] Truemper configurations are not CI$\mathcal{O}$-graphs.

Lemma 1.118 [24, Lemma 2] Let C^1, C^2 be two cycles of D whose intersection is an oriented path $\mathcal{L}$. Then $C^3 = (C^1 \cup C^2) \setminus V(\mathcal{L}^\circ)$ is a cycle and $y_{C^3} \in (y_{C^1}, y_{C^2})$.

Proposition 1.119 [24, Lemma 7 and Proposition 11] Let G_1, G_2 be two connected graphs and let G be the k-clique-sum of G_1 and G_2 with $k \in \{0, 1, 2\}$. If G_1 and G_2 are $CI\mathcal{O}$ graphs, then G is a $CI\mathcal{O}$ graph.

Theorem 1.120 [24] G is a CI$\mathcal{O}$-graph if and only if G is a theta-ring graph.

Chapter 2
Unmixed weighted oriented graphs

In this chapter we assume $D = (G, \mathcal{O}, w)$ is a weighted oriented graph. By Remark 1.81, if x is a source of D, then $w(x) = 1$. In this chapter, we characterize the unmixed property of $I(D)$ when G is in one of the following families of graphs: SCQ, chordal, simplicial, perfect, graphs without 3- and 5-cycles, graphs without 4- and 5-cycles, or graphs with *girth* $\geqslant 5$.

2.1 Strong vertex cover and $\star$-semi-forest

In this section we introduce the $\star$-semi-forest (Definition 2.4). With this definition, we characterize when a subset of $V(G)$ is contained in a strong vertex cover (Theorem 2.11). Furthermore, we characterize when $I(D)$ is unmixed if G is perfect (Theorem 2.13).

Proposition 2.1 If $\mathcal{C}$ is a vertex cover of D and $A \subseteq V^+$ such that $N_D^+(A) \subseteq \mathcal{C}$, then there is a strong vertex cover $\mathcal{C}'$ of D, such that $N_D^+(A) \subseteq \mathcal{C}' \subseteq \mathcal{C}$.

Proof. First, we prove that there is a vertex cover $\mathcal{C}'$ such that $L_3(\mathcal{C}') \subseteq N_D^+(A) \subseteq \mathcal{C}' \subseteq \mathcal{C}$. We take $L := N_D^+(A)$. If $L_3(\mathcal{C}) \subseteq L$, then we take $\mathcal{C}' = \mathcal{C}$. Now, we suppose there is $a_1 \in L_3(\mathcal{C}) \setminus L$, then by Remark 1.83, $N_D[a_1] \subseteq \mathcal{C}$. Thus, $\mathcal{C}_1 = \mathcal{C} \setminus \{a_1\}$ is a vertex cover and $L \subseteq \mathcal{C}_1$, since $L \subseteq \mathcal{C}$ and $a_1 \notin L$. Now, we suppose that there are vertex covers $\mathcal{C}_0, \ldots, \mathcal{C}_k$, such that $L \subseteq \mathcal{C}_i = \mathcal{C}_{i-1} \setminus \{a_i\}$ and $a_i \in L_3(\mathcal{C}_{i-1}) \setminus L$ for $i = 1, \ldots, k$ where $\mathcal{C}_0 = \mathcal{C}$ and we give the following recursively process: If $L_3(\mathcal{C}_k) \subseteq L$, then we take $\mathcal{C}' = \mathcal{C}_k$. Now, if there is $a_{k+1} \in L_3(\mathcal{C}_k) \setminus L$, then by Remark 1.83, $N_D[a_{k+1}] \subseteq \mathcal{C}_k$. Consequently, $\mathcal{C}_{k+1} := \mathcal{C}_k \setminus \{a_{k+1}\}$ is a vertex cover. Also, $L \subseteq \mathcal{C}_{k+1}$, since $L \subseteq \mathcal{C}_k$ and $a_{k+1} \notin L$. This process is finite, since $|V(D)|$ is finite. Hence, there is m such that $\mathcal{C}' = \mathcal{C}_m$, i.e. $L_3(\mathcal{C}_m) \subseteq L \subseteq \mathcal{C}_m \subseteq \mathcal{C}$.

Now, we prove that $\mathcal{C}'$ is strong. We take $x \in L_3(\mathcal{C}')$, then $x \in L = N_D^+(A)$, since $L_3(\mathcal{C}') \subseteq L$. Thus, $(x', x) \in E(D)$ for some $x' \in A \subseteq V^+$. Hence, $x' \in \mathcal{C}'$, since $x \in L_3(\mathcal{C}')$. Also, $x' \notin L_1(\mathcal{C}')$, since $N_D^+(x') \subseteq N_D^+(A) \subseteq \mathcal{C}'$. Hence, $x' \in \big(\mathcal{C}' \setminus L_1(\mathcal{C}')\big) \cap V^+$. Therefore, $\mathcal{C}'$ is strong. □

Definition 2.2 If B is a weighted oriented subgraph of D with exactly one cycle C, then B is called **unicycle oriented graph** when B satisfies the following conditions:

(i) C is an oriented cycle in B and for each $x \in V(B) \setminus V(C)$, there is an oriented path (in B) from C to x.
(ii) If $x \in V(B)$ with $w(x) = 1$, then $deg_B(x) = 1$.

Definition 2.3 A weighted oriented subgraph T of D without cycles, is a **rooted oriented tree** (**ROT**) with **root** $v \in V(T)$ when T satisfies the following properties:

(i) If $x \in V(T) \setminus \{v\}$, there is an oriented path $\mathcal{P}$ in T from v to x.
(ii) If $x \in V(T)$ with $w(x) = 1$, then $deg_T(x) = 1$ when $x \neq v$ or $V(T) = \{v\}$ when $x = v$.

Definition 2.4 A weighted oriented subgraph H of D is a **$\star$-semi-forest** if there are rooted oriented trees $T_1, \ldots, T_r$ whose roots are $v_1, \ldots, v_r$ and unicycle oriented subgraphs $B_1, \ldots, B_s$ such that $H = \big(\cup_{i=1}^{r} T_i\big) \cup \big(\cup_{j=1}^{s} B_j\big)$ with the following conditions:

(i) $V(T_1), \ldots, V(T_r), V(B_1), \ldots, V(B_s)$ is a partition of $V(H)$.
(ii) There is $W = \{w_1, \ldots, w_r\} \subseteq V(D) \setminus V(H)$ such that $w_i \in N_D(v_i)$ for $i = 1, \ldots, r$ (it is possible that $w_i = w_j$ for some $1 \leqslant i < j \leqslant r$).
(iii) There is a partition W_1, W_2 of W such that W_1 is a stable set of D, $W_2 \subseteq V^+$ and $(w_i, v_i) \in E(D)$ if $w_i \in W_2$. Also, $N_D^+(W_2 \cup \tilde{H}) \cap W_1 = \emptyset$, where

$$\tilde{H} = \{x \in V(H) \mid deg_H(x) \geqslant 2\} \cup \{v_i \mid deg_H(v_i) = 1\}.$$

Remark 2.5 If v_i is a root vertex of T_i, with $deg_H(v_i) \geqslant 1$, then $v_i \in \tilde{H}$. Furthermore, by (ii) in Definition 2.2 and Definition 2.3, we have $\tilde{H} \subseteq V^+$.

Remark 2.6 Let H be a connected $\star$-semi-forest, then H is a ROT or H is an unicycle oriented graph. If H is a ROT with root v, then $W = W_1 \cup W_2 = \{w\}$ and $w \in N_D(v)$. Furthermore, (iii) in Definition 2.4 is equivalent to: $w \notin N_D^+(\tilde{H})$ if $w \in W_1$; or $w \in N_D^-(v) \cap V^+$ if $w \in W_2$.

Lemma 2.7 If H is a $\star$-semi-forest of D, then

$$V(H) \subseteq N_D(W_1) \cup N_D^+(W_2 \cup \tilde{H}).$$

Proof. We take $x \in V(H)$. Since $H = \big(\cup_{i=1}^{r} T_i \big) \cup \big(\cup_{j=1}^{s} B_j \big)$, we have two cases:

Case 1) $x \in V(B_j)$ for some $1 \leqslant j \leqslant s$. Let C be the oriented cycle of B_j. If $x \in V(C)$, then there is $z_1 \in V(C)$ such that $(z_1, x) \in E(C)$. Furthermore, $deg_H(z_1) \geqslant deg_C(z_1) = 2$, then $z_1 \in \tilde{H}$. Hence, $x \in N_D^+(z_1) \subseteq N_D^+(\tilde{H})$. Now, if $x \in V(B_j) \setminus V(C)$, then there is an oriented path $\mathcal{P}$ in B_j from C to x. Thus, there is $z_2 \in V(\mathcal{P})$ such that $(z_2, x) \in E(\mathcal{P})$. If $|V(\mathcal{P})| > 2$, then $deg_H(z_2) \geqslant deg_{\mathcal{P}}(z_2) = 2$. If $|V(\mathcal{P})| = 2$, then $z_2 \in V(C)$ and $deg_H(z_2) > deg_C(z_2) = 2$. Therefore, $z_2 \in \tilde{H}$ and $x \in N_D^+(\tilde{H})$.

Case 2) $x \in V(T_i)$ for some $1 \leqslant i \leqslant r$. First, assume $x = v_i$, then there is $w_i \in W = W_1 \cup W_2$ such that $x \in N_D(w_i)$. If $w_i \in W_1$, then $x \in N_D(W_1)$. Also, if $w_i \in W_2$, then by (iii) of Definition 2.4, $x \in N_D^+(w_i) \subseteq N_D^+(W_2)$. Now, we suppose $x \neq v_i$, then there is an oriented path $\mathcal{L}$, from v_i to x. Consequently, there is $z_3 \in V(\mathcal{L})$ such that $(z_3, x) \in E(D)$. If $z_3 \neq v_i$, then $deg_H(z_3) \geqslant deg_{\mathcal{L}}(z_3) = 2$. Thus, $z_3 \in \tilde{H}$ and $x \in N_D^+(\tilde{H})$. Finally, if $z_3 = v_i$, then $deg_H(z_3) \geqslant 1$. Hence, by Remark 2.5, $z_3 \in \tilde{H}$ and $x \in N_D^+(\tilde{H})$. □

Remark 2.8 Sometimes to identify the relation between W and H in Definition 2.4, W is denoted by W^H. Similarly, W_1^H and W_2^H. Furthermore, $\{v_1, \ldots, v_r\}$ is denoted by V^H. If $\{T_1, \ldots, T_r\} = \emptyset$, then $W^H = W_1^H = W_2^H = \emptyset$. Also, if $H = \cup_{i=1}^{s} B_i$, then $V^H = \emptyset$.

Lemma 2.9 Let K be a weighted oriented subgraph of D. If H is a maximal ROT in K with root v, or H is a maximal unicycle oriented subgraph in K whose cycle is C, then there is not $(x', x) \in E(K)$ with $x' \in V(H) \cap V^+$ and $x \in V(K) \setminus V(H)$.

Proof. By contradiction, suppose there is $(x', x) \in E(K)$ with $x' \in V(H) \cap V^+$ and $x \in V(K) \setminus V(H)$. Thus, $H \subsetneq H_1 := H \cup \{(x', x)\} \subseteq K$. If H is a unicycle oriented subgraph (resp. H is a ROT) with cycle C (resp. with root v), then there is an oriented path $\mathcal{P}$ from C (resp. from v) to x'. Consequently, $\mathcal{P} \cup \{(x', x)\}$ is an oriented path from C (resp. from v) to x in H_1. Furthermore, H_1 has exactly one cycle (resp. has no cycles), since $deg_{H_1}(x) = 1$ and $V(H_1) \setminus V(H) = \{x\}$.

Now, we take $z \in V(H_1)$ with $w(z) = 1$, then $z = x$ or $z \in V(H)$. We will prove $deg_{H_1}(z) = 1$. If $z = x$, then $deg_{H_1}(x) = 1$. Now, if $z \in V(H)$, then $z \neq x'$, since $x' \in V^+$. So, $deg_{H_1}(z) = deg_H(z)$, since $N_{H_1}(x) = \{x'\}$. If H is a ROT with $V(H) = \{v\}$, then $x' = z = v$. A contradiction, since $w(z) = 1$ and $x' \in V^+$. Consequently, by (ii) in Definitions 2.2 and 2.3, $deg_{H_1}(z) = deg_H(z) = 1$. Hence, H_1 is a unicycle oriented subgraph with cycle C (resp. is a ROT with root v) of K.

This is a contradiction, since $H \subsetneq H_1 \subseteq K$ and H is maximal. $\square$

Definition 2.10 Let K be a weighted oriented subgraph of D and H a $\star$-semi-forest of D. We say H is a **generating $\star$-semi-forest** of K if $V(K) = V(H)$.

Theorem 2.11 Let K be an induced weighted oriented subgraph of D. Hence, the following conditions are equivalent:

(1) There is a strong vertex cover $\mathcal{C}$ of D, such that $V(K) \subseteq \mathcal{C}$.
(2) K has a generating $\star$-semi-forest.

Proof. (2) $\Rightarrow$ (1) Let $\mathcal{C}_1$ be a minimal vertex cover of D. By (2), K has a generating $\star$-semi-forest H. Now, using the notations of Definition 2.4, we take $\mathcal{C}_2 = \big(\mathcal{C}_1 \setminus W_1\big) \cup N_D(W_1) \cup N_D^+(W_2 \cup \tilde{H})$. By Remark 1.20, $\mathcal{C}_2$ is a vertex cover of D. Since W_1 is a stable set, $N_D(W_1) \cap W_1 = \emptyset$. Then, $\mathcal{C}_2 \cap W_1 = \emptyset$, since $N_D^+(W_2 \cup \tilde{H}) \cap W_1 = \emptyset$. By Remark 2.5 and (iii) in Definition 2.4, $\tilde{H} \cup W_2 \subseteq V^+$. So, by Proposition 2.1, there is a strong vertex cover $\mathcal{C}$ of D such that $N_D^+(W_2 \cup \tilde{H}) \subseteq \mathcal{C} \subseteq \mathcal{C}_2$. Consequently, $\mathcal{C} \cap W_1 = \emptyset$, since $\mathcal{C}_2 \cap W_1 = \emptyset$. Thus, by Remark 1.21, $N_D(W_1) \subseteq \mathcal{C}$. Then, by Lemma 2.7, $V(H) \subseteq N_D(W_1) \cup N_D^+(W_2 \cup \tilde{H}) \subseteq \mathcal{C}$. Furthermore, $V(K) = V(H)$, since H is a generating $\star$-semi-forest of K. Therefore, $V(K) \subseteq \mathcal{C}$.

(1) $\Rightarrow$ (2) We have $\mathcal{C}$ is a strong vertex cover such that $V(K) \subseteq \mathcal{C}$. If $A := L_1(\mathcal{C}) \cap V(K) = \{v_1, \dots, v_s\}$, then there is $w_i \in V(D) \setminus \mathcal{C} \subseteq V(D) \setminus V(K)$ such that $(v_i, w_i) \in E(D)$. We take the ROTs $M_1 = \{v_1\}, \dots, M_s = \{v_s\}$ and sets $W_1^i = W_1^{M_i} = \{w_i\}$ and $W_2^i = W_2^{M_i} = \emptyset$ for $i = 1, \dots, s$.

Now, we will give a recursive process to obtain a generating $\star$-semi-forest of K. For this purpose, suppose we have connected $\star$-semi-forests $M_{s+1}, \dots, M_l$ of $K \setminus A$ with subsets $W_1^{s+1}, \dots, W_1^l, W_2^{s+1}, \dots, W_2^l \subseteq V(D) \setminus V(K)$ and $V^{s+1}, \dots, V^l \subseteq V(K)$ such that for each $s < j \leqslant l$, they satisfy the following conditions:

(a) $W_1^{M_j} = W_1^j$, $W_2^{M_j} = W_2^j$ and $V^j = \{v_j\}$ if M_j is a ROT with root v_j; or $W_1^j = W_2^j = \emptyset$ and V^j is the cycle of M_j if M_j is a unicycle oriented subgraph.
(b) M_j is a maximal ROT in $K^j := K \setminus \cup_{i=1}^{j-1} V(M_i)$ whose root is v_j or M_j is a maximal unicycle oriented subgraph in K^j whose cycle is V^j.
(c) $W_1^j \cap \mathcal{C} = \emptyset$ and $W_2^j \subseteq \big(\mathcal{C} \setminus (L_1(\mathcal{C}) \cup V(K))\big) \cap V^+$.

Continuing with the recursive process, we take $K^{l+1} := K \setminus \big(\cup_{i=1}^l V(M_i)\big)$. This process begins with $l = s$; in this case, $K^{s+1} := K \setminus \big(\cup_{i=1}^s V(M_i)\big) = K \setminus A$; furthermore, if $A = \emptyset$, then $K^1 = K$. Now, if $K^{l+1} = \emptyset$, then $V(K) = \cup_{i=1}^l V(M_i)$

and we end the process. On the other hand, if $K^{l+1} \neq \emptyset$, then we will construct a connected $\star$-semi-forest M_{l+1} of K^{l+1} in each one of the following cases:

Case (1) There is $z \in L_2(\mathcal{C}) \cap V(K^{l+1})$. Thus, there is $(z', z) \in E(D)$ with $z' \notin \mathcal{C}$. We take a maximal ROT M_{l+1} in K^{l+1}, whose root is z. Also, we take $V^{l+1} = \{v_{l+1}\} = \{z\}$, $W_1^{M_{l+1}} = W_1^{l+1} = \{z'\}$ and $W_2^{M_{l+1}} = W_2^{l+1} = \emptyset$. Furthermore, $V(M_{l+1}) \subseteq V(K^{l+1}) \subseteq V(K) \setminus A \subseteq \mathcal{C} \setminus L_1(\mathcal{C})$, then $z' \notin N_D^+(M_{l+1})$, since $z' \notin \mathcal{C}$. Hence, by Remark 2.6, M_{l+1} is a connected $\star$semi-forest. Furthermore, M_{l+1} satisfies (a), (b) and (c), since $z' \in N_D(z) \setminus \mathcal{C}$ and $W_2^{l+1} = \emptyset$.

Case (2) $L_2(\mathcal{C}) \cap V(K^{l+1}) = \emptyset$. So, $V(K^{l+1}) \subseteq L_3(\mathcal{C})$, since $V(K^{l+1}) \subseteq V(K) \setminus A \subseteq \mathcal{C} \setminus L_1(\mathcal{C})$. Consequently, if $x \in V(K^{l+1})$, then there is $x_1 \in \big(\mathcal{C} \setminus L_1(\mathcal{C})\big) \cap V^+$ such that $(x_1, x) \in E(D)$, since $\mathcal{C}$ is strong. If $x_1 \in V(K^{l+1})$, then there is $x_2 \in \big(\mathcal{C} \setminus L_1(\mathcal{C})\big) \cap V^+$ such that $(x_2, x_1) \in E(D)$, since $\mathcal{C}$ is strong. Continuing with this process we obtain a maximal path $\mathcal{P} = (x_r, x_{r-1}, \ldots, x_1, x)$ such that $x_{r-1}, \ldots, x_1, x$ are different in $V(K^{l+1})$ and $x_1, \ldots, x_r \in \big(\mathcal{C} \setminus L_1(\mathcal{C})\big) \cap V^+$. Then, $x_r \notin \cup_{j=1}^{s} V(M_j)$, since $x_r \notin L_1(\mathcal{C})$. Now, suppose $x_r \in V(M_j)$ for some $s < j \leqslant l$. Thus, $(x_r, x_{r-1}) \in E(K^j)$, $x_{r-1} \in V(K^j) \setminus V(M_j)$ and $x_r \in V^+ \cap V(M_j)$. A contradiction, by Lemma 2.9, since M_j is a maximal ROT in K^j with root v_j. Hence, $x_r \notin \cup_{j=1}^{l} V(M_j)$. This implies $x_r \notin V(K)$ or $x_r \in V(K^{l+1})$.

<u>Case (2.a)</u> $x_r \notin V(K)$. We take a maximal ROT M_{l+1} in K^{l+1} whose root is x_{r-1}. Also, we take $V^{l+1} = \{v_{l+1}\} = \{x_{r-1}\}$, $W_1^{l+1} = W_1^{M_{l+1}} = \emptyset$; and $W_2^{l+1} = W_2^{M_{l+1}} = \{x_r\}$. Thus, by Remark 2.6, M_{l+1} is a connected $\star$-semi-forest. Furthermore, M_{l+1} satisfies (a), (b) and (c), since $W_1^{l+1} = \emptyset$, $x_r \in \big(\mathcal{C} \setminus L_1(\mathcal{C})\big) \cap V^+$ and $x_r \notin V(K)$.

<u>Case (2.b)</u> $x_r \in V(K^{l+1})$. Then, $x_r \in L_3(\mathcal{C})$, since $V(K^{l+1}) \subseteq L_3(\mathcal{C})$. Consequently, there is $x_{r+1} \in \big(\mathcal{C} \setminus L_1(\mathcal{C})\big) \cap V^+$ such that $(x_{r+1}, x_r) \in E(D)$, Then $\tilde{\mathcal{P}} = (x_{r+1}, x_r, \ldots, x_1, x)$ is an oriented walk. By the maximality of $\mathcal{P}$, we have that $x_r \in \{x_{r-1}, \ldots, x_1, x\}$. So, $\mathcal{P} = (x_r, \ldots, x_1, x)$ contains an oriented cycle C. We take a maximal unicycle oriented subgraph M_{l+1} of K^{l+1} with cycle C, $V^{l+1} = C$ and $W_1^{l+1} = W_2^{l+1} = \emptyset$. Hence, M_{l+1} satisfies (a), (b) and (c).

Therefore, we obtain a connected $\star$-semi-forest M_{l+1} such that it satisfies (a), (b) and (c). We take $K^{l+2} = K \setminus \big(\cup_{j=1}^{l+1} V(M_j)\big)$ and we continue with the recursive process. Since K is finite, the recursive process stops and we obtain $M_1, \ldots, M_t \subseteq K$ such that $V(K) = \cup_{j=1}^{t} V(M_j)$, $W_1^j \cap \mathcal{C} = \emptyset$ and $W_2^j \subseteq \big(\mathcal{C} \setminus L_1(\mathcal{C})\big) \cap V^+$ for $j = 1, \ldots t$. Now, we take $H := \cup_{j=1}^{t} M_j$ and $W^i = \cup_{j=1}^{t} W_i^j$ for $i = 1, 2$. So, $V(H) = V(K)$. Also, $W^1 \cap \mathcal{C} = \emptyset$, then W^1 is a stable set, since $\mathcal{C}$ is a vertex cover.

Furthermore, $W^2 \subseteq V^+$ and $W^2 \subseteq \mathcal{C} \setminus L_1(\mathcal{C})$, then $N_D^+(W^2) \subseteq \mathcal{C}$. Thus, $N_D^+(W^2) \cap W^1 = \emptyset$, since $\mathcal{C} \cap W^1 = \emptyset$. On the other hand, if $x \in L_1(\mathcal{C}) \cap V(K) = A$, then there is $1 \leqslant i \leqslant s$ such that $x = v_i$ and $M_i = \{v_i\}$. So, $deg_H(x) = deg_{M_i}(v_i) = 0$. Consequently, $\tilde{H} \cap L_1(\mathcal{C}) = \emptyset$ implies $N_D^+(\tilde{H}) \subseteq \mathcal{C}$, since $V(H) \subseteq \mathcal{C}$. Hence, $N_D^+(\tilde{H}) \cap W^1 = \emptyset$, since $W^1 \cap \mathcal{C} = \emptyset$. Therefore, H is a generating $\star$-semi-forest of K, with $W_i^H = W^i$ for $i = 1, 2$. □

Corollary 2.12 Let K be a complete weighted oriented subgraph of D. Hence, the following conditions are equivalent:

(1) $V(K) \not\subseteq \mathcal{C}$ for each strong vertex cover $\mathcal{C}$ of D.
(2) K has no generating $\star$-semi-forests.
(3) $|\mathcal{C} \cap V(K)| = |V(K)| - 1$ for each strong vertex cover $\mathcal{C}$.

Proof. (1) ⇔ (2) By Theorem 2.11.

(3) ⇒ (1) If $\mathcal{C}$ is a strong vertex cover of D, then $|V(K) \cap \mathcal{C}| = |V(K)| - 1$. So, $V(K) \not\subseteq \mathcal{C}$.

(1) ⇒ (3) If $\mathcal{C}$ is a strong vertex cover of D, then $\mathcal{C}_K := \mathcal{C} \cap V(K)$ is a vertex cover of K. Hence, $|\mathcal{C}_K| \geqslant \tau(K) = |V(K)| - 1$. Therefore, $|\mathcal{C}_K| = |V(K)| - 1$, since $V(K) \not\subseteq \mathcal{C}$. □

Theorem 2.13 Let $D = (G, \mathcal{O}, w)$ be a weighted oriented graph where G is a perfect graph, then G has a τ-reduction $G_1, \ldots, G_s$ in complete subgraphs. Furthermore, $I(D)$ is unmixed if and only if each G_i has no generating $\star$-semi-forests.

Proof. First, we prove G has a τ-reduction in complete graphs. By Theorem 1.15, $\overline{G}$ is perfect. Thus, $s := \omega(\overline{G}) = \chi(\overline{G})$. So, there is a s-colouring $c : V(\overline{G}) \to \{1, \ldots, s\}$. We take $V_i := c^{-1}(i)$ for $i = 1, \ldots, s$. Then, V_i is a stable set in $\overline{G}$, since c is a s-colouring. Hence, by Remark 1.18, $G_i := G[V_i]$ is a complete graph in G and $s = \omega(\overline{G}) = \beta(G)$. Furthermore, $V_1, \ldots, V_s$ is a partition of $V(\overline{G}) = V(G)$, since c is a function. Consequently,

$$\sum_{i=1}^{s} \tau(G_i) = \sum_{i=1}^{s} \big(|V_i| - 1\big) = \Big(\sum_{i=1}^{s} |V_i|\Big) - s = |V(G)| - \beta(G).$$

By Remark 1.22, $|V(G)| - \beta(G) = \tau(G)$, then $G_1, \ldots, G_s$ is a τ-reduction of G.

Now, we prove that $I(D)$ is unmixed if and only if each G_j has no generating $\star$-semi-forests. Recall, $V_j = V(G_j)$.

⇒) By contradiction, assume G_j has a generating $\star$-semi-forest, then by Theorem 2.11 there is a strong vertex $\mathcal{C}$ such that $V_j \subseteq \mathcal{C}$. Furthermore, $\mathcal{C} \cap V_i$ is a vertex cover of G_i, then $|\mathcal{C} \cap V_i| \geqslant \tau(G_i) = |V_i| - 1$ for $i \neq j$. Thus, $|\mathcal{C}| = \sum_{i=1}^{s} |\mathcal{C} \cap V_i| \geqslant |V_j| + \sum_{\substack{i=1 \\ i \neq j}}^{s} (|V_i| - 1)$, since $V_1, \ldots, V_s$ is a partition of $V(G)$. Hence, by Remark 1.22, $|\mathcal{C}| > |V(G)| - s = \tau(G)$, since $s = \beta(G)$. A contradiction, by Remark 1.86, since $I(D)$ is unmixed.

$\Leftarrow$) Let $\mathcal{C}$ be a strong vertex cover, then $\mathcal{C} \cap V_i$ is a vertex cover of G_i. So, $|\mathcal{C} \cap V_i| \geqslant \tau(G_i) = |V_i| - 1$ for $i = 1, \ldots, s$. Furthermore, by Theorem 2.11, $V_i \nsubseteq \mathcal{C}$. Consequently, $|\mathcal{C} \cap V_i| = |V_i| - 1$. Thus, $|\mathcal{C}| = \sum_{i=1}^{s} \big(|V_i| - 1\big)$, since $V_1, \ldots, V_s$ is a partition of $V(G)$. Therefore, by (2) in Theorem 1.85, $I(D)$ is unmixed. $\square$

2.2 UNMIXEDNESS OF WEIGHTED ORIENTED SCQ GRAPHS

In this section (in Theorem 2.23) we characterize when $I(D)$ is unmixed if G is an SCQ-graph (see Definition 1.39). Using this result, we characterize (in Corollary 2.24) the unmixedness of $I(D)$ when G is chordal or simplicial (see Definitions 1.9 and 1.36).

Proposition 2.14 If $y \in E(G)$, then the following conditions are equivalent:

(1) $|\mathcal{C} \cap y| = 1$ for each strong vertex cover $\mathcal{C}$ of D.
(2) y has the property **(P)**; furthermore, $N_D(b') \subseteq N_D^+(a)$ if $(a, b) \in E(D)$ where $y = \{b, b'\}$ and $a \in V^+$.

Proof. (1) $\Rightarrow$ (2) First, we show $y = \{b, b'\}$ has the property **(P)**. By contradiction, suppose there are $\{a, b\}, \{a', b'\} \in E(G)$ such that $\{a, a'\} \notin E(G)$. This implies, there is a maximal stable set S such that $\{a, a'\} \subseteq S$. By Remark 1.22, $\tilde{\mathcal{C}} := V(G) \setminus S$ is a minimal vertex cover. So, by Remark 1.83, $\tilde{\mathcal{C}}$ is strong. Furthermore, $a, a' \notin \tilde{\mathcal{C}}$, then $b, b' \in \tilde{\mathcal{C}}$, since $\{a, b\}, \{a', b'\} \in E(G)$. A contradiction by (1). Now, assume $(a, b) \in E(D)$ with $a \in V^+$, then we will prove that $N_D(b') \subseteq N_D^+(a)$. By contradiction, suppose there is $c \in N_D(b') \setminus N_D^+(a)$. We take a vertex cover $\mathcal{C}_1$. By Remark 1.20, $\mathcal{C} = \big(\mathcal{C}_1 \setminus \{c\}\big) \cup N_D(c) \cup N_D^+(a)$ is a vertex cover. Furthermore, $c \notin \mathcal{C}$, since $c \notin N_D^+(a)$. By Proposition 2.1, there is a strong vertex cover $\mathcal{C}'$ such that $N_D^+(a) \subseteq \mathcal{C}' \subseteq \mathcal{C}$, since $a \in V^+$. Consequently, $b \in \mathcal{C}'$ and $c \notin \mathcal{C}'$, since $b \in N_D^+(a)$ and $c \notin \mathcal{C}$. Then, by Remark 1.21, $b' \in N_D(c) \subseteq \mathcal{C}'$. Hence, $\{b, b'\} \subseteq \mathcal{C}'$. A contradiction, by (1).

(2) $\Rightarrow$ (1) By contradiction, assume there is a strong vertex cover $\mathcal{C}$ of D such that $|\mathcal{C} \cap y| \neq 1$. So, $|\mathcal{C} \cap y| = 2$, since $\mathcal{C}$ is a vertex cover. Hence, by Theorem 2.11, there is a generating $\star$-semi-forest H of y. We set $y = \{z, z'\}$. First, assume H

is not connected. Then, using the Definition 2.4, we have $H = M_1 \cup M_2$ where $M_1 = \{v_1\}$, $M_2 = \{v_2\}$ and $w_1, w_2 \in W^H$ such that $w_i \in N_D(v_i)$ for $i = 1,2$. Thus, $\{z, z'\} = \{v_1, v_2\}$ and $\{w_1, w_2\} \in E(G)$, since y satisfies the property **(P)**. This implies $|W_1^H \cap \{w_1, w_2\}| \leqslant 1$, since W_1^H is a stable set. Hence, we can suppose $w_2 \in W_2^H$, then $w_2 \in V^+$ and $z = v_2$ imply $(w_2, z) \in E(D)$ and $z' = v_1$. Consequently, by (2) with $a = w_2$, $w_1 \in N_D(z') \subseteq N_D^+(w_2)$, then $(w_2, w_1) \in E(D)$. Furthermore, by (iii) in Definition 2.4, $N_D^+(W_2^H) \cap W_1^H = \emptyset$, then $w_1 \in W_2^H$. So, $w_1 \in V^+$ and $(w_1, z') \in E(D)$. Thus, by (2) with $a = w_1$, we have $(w_1, w_2) \in E(D)$. A contradiction, then H is connected. Hence, H is a ROT with $V(H) = \{z, z'\}$. We can suppose $v_1 = z$ and $W^H = \{w_1\}$, then $(z, z') \in E(D)$, $w_1 \in N_D(z)$ and $z = v_1 \in \tilde{H}$, since $deg_H(v_1) = 1$. If $w_1 \in N_D^+(z)$, then $w_1 \in W_1^H$, since $z = v_1$. A contradiction, since $N_D^+(\tilde{H}) \cap W_1^H = \emptyset$. Then, $w_1 \notin N_D^+(z)$. By Remark 2.5, $z = v_1 \in \tilde{H} \subseteq V^+$. Therefore, by (2) (taking $a = b' = z$ and $b = z'$), we have $N_D(z) \subseteq N_D^+(z)$, since $(z, z') \in E(D)$, $y = \{z, z'\}$ and $z \in V^+$. A contradiction, since $w_1 \in N_D(z) \setminus N_D^+(z)$. □

Corollary 2.15 [34, Theorem 3.4] Let D be a weighted oriented graph, where G is König without isolated vertices. Hence, $I(D)$ is unmixed if and only if D satisfies the following two conditions:

(a) G has a perfect matching P with the property **(P)**.
(b) $N_D(b') \subseteq N_D^+(a)$, when $a \in V^+$, $\{b, b'\} \in P$ and $b \in N_D^+(a)$.

Proof. $\Rightarrow$) By Theorem 1.85, $I(G)$ is unmixed. Thus, by Remark 1.78 and Theorem 1.27, G has a perfect matching P with the property **(P)**. Thus, $\nu(G) = |P|$. Also, $\tau(G) = \nu(G)$, since G is König. So, $\tau(G) = |P|$. Now, we take a strong vertex cover $\mathcal{C}$ of D and $y \in P$, then $|\mathcal{C} \cap y| \geqslant 1$. Furthermore, by Remark 1.86, $|\mathcal{C}| = \tau(G) = |P|$. Hence, $|\mathcal{C} \cap y| = 1$, since $\mathcal{C} = \cup_{\tilde{y} \in P} \mathcal{C} \cap \tilde{y}$. Therefore, by Proposition 2.14, D satisfies (b).

$\Leftarrow$) We take a strong vertex cover $\mathcal{C}$ of D. By Proposition 2.14, $|\mathcal{C} \cap y| = 1$ for each $y \in P$, since D satisfies (a) and (b). This implies $|\mathcal{C}| = |P|$, since P is a perfect matching. Therefore, by (2) in Theorem 1.85, $I(D)$ is unmixed. □

Lemma 2.16 If there is a basic 5-cycle $C = (z_1, z_2, z_3, z_4, z_5, z_1)$ with $(z_1, z_2), (z_2, z_3) \in E(D)$, $z_2 \in V^+$ and C satisfies one of the following conditions:

(a) $(z_3, z_4) \in E(D)$ with $z_3 \in V^+$
(b) $(z_1, z_5), (z_5, z_4) \in E(D)$ with $z_5 \in V^+$,

then there is a strong vertex cover $\mathcal{C}$ such that $|\mathcal{C} \cap V(C)| = 4$.

Proof. We take $\mathcal{C}_1 = \big(\mathcal{C}_0 \setminus V(C)\big) \cup N_D(z_1) \cup N_D^+(z_2, x)$ where $\mathcal{C}_0$ is a vertex cover such that $x = z_3$ if C satisfies (a), or $x = z_5$ if C satisfies (b). Thus, $x \in V^+$. Also, $z_4 \in N_D^+(z_3)$ if C satisfies (a) or $z_4 \in N_D^+(z_5)$ if C satisfies (b). Hence, $\{z_2, z_3, z_4, z_5\} \subseteq N_D(z_1) \cup N_D^+(z_2, x)$ implies $\{z_2, z_3, z_4, z_5\} \subseteq \mathcal{C}_1$. So, by Remark 1.20, $\mathcal{C}_1$ is a vertex cover. Furthermore, $z_1 \notin \mathcal{C}_1$, since $z_1 \notin N_D(z_1) \cup N_D^+(z_2, z_3)$ and $z_1 \notin N_D^+(z_5)$ if C satisfies (b). By Proposition 2.1, there is a strong vertex cover $\mathcal{C}'$ such that $N_D^+(z_2, x) \subseteq \mathcal{C}' \subseteq \mathcal{C}_1$, since $\{z_2, x\} \subseteq V^+$. Also, $z_1 \notin \mathcal{C}'$, since $z_1 \notin \mathcal{C}_1$. Then, by Remark 1.21, $N_D(z_1) \subseteq \mathcal{C}'$. Hence, $\{z_2, z_3, z_4, z_5\} \subseteq N_D(z_1) \cup N_D^+(z_2, x) \subseteq \mathcal{C}'$. Therefore, $|\mathcal{C}' \cap V(C)| = 4$, since $z_1 \notin \mathcal{C}'$. □

Definition 2.17 Let C be an induced 5-cycle, we say that C has the **$\star$-property** if for each $(a, b) \in E(C)$ such that $a \in V^+$, we have that $C = (a', a, b, b', c, a')$ with the following properties:

($\star$.1) $(a', a) \in E(D)$ and $w(a') = 1$.
($\star$.2) $N_D^-(a) \subseteq N_D(c)$ and $N_D^-(a) \cap V^+ \subseteq N_D^-(c)$.
($\star$.3) $N_D(b') \subseteq N_D(a') \cup N_D^+(a)$ and $N_D^-(b') \cap V^+ \subseteq N_D^-(a')$.

Remark 2.18 Let $C = (a', a, b, b', c, a')$ be a 5-cycle with the ($\star$.1) property of Definition 2.17, such that $(a, b) \in E(C)$ and $a \in V^+$. So, $(a', a) \in E(D)$ and $w(a') = 1$. Now, suppose $N_D^-(a) \subseteq V(C)$, then $N_D^-(a) = \{a'\} \subseteq N_D(c)$ and $N_D^-(a) \cap V^+ = \emptyset \subseteq N_D^-(c)$. Hence, C satisfies ($\star$.2). On the other hand, if $N_D^-(a) \not\subseteq V(C)$ with C a basic cycle, such that satisfies ($\star$.2) property of Definition 2.17, then there is $w \in N_D^-(a) \setminus V(C) \subseteq N_D(c)$. Thus, $deg_D(b') = 2$, since C is basic. Consequently, $N_D(b') = \{b, c\} \subseteq N_D(a') \cup N_D^+(a)$. Hence, C satisfies the first part of ($\star$.3) in Definition 2.17.

Lemma 2.19 Let $C = (a_1', a_1, b_1, b_1', c_1, a_1')$ be a basic 5-cycle of D such that $(a_1', a_1) \in E(D)$, $deg_D(a_1) \geqslant 3$, $deg_D(c_1) \geqslant 3$ and $w(b_1) = 1$. If there is a strong vertex cover $\mathcal{C}$ of D, such that $V(C) \subseteq \mathcal{C}$, then C has no the $\star$-property.

Proof. By contradiction, suppose C has the $\star$-property and there is a strong vertex cover $\mathcal{C}$, such that $V(C) \subseteq \mathcal{C}$. Then, $deg_D(a_1') = deg_D(b_1') = 2$, since C is a basic cycle and $deg_D(a_1), deg_D(c_1) \geqslant 3$. Hence, $a_1', b_1' \in L_3(\mathcal{C})$ and $N_D^-(a_1') \subseteq \{c_1\}$, since $V(C) \subseteq \mathcal{C}$ and $(a_1', a_1) \in E(D)$. Thus, $(c_1, a_1') \in E(D)$ and $w(c_1) \neq 1$, since $a_1' \in L_3(\mathcal{C})$ and $\mathcal{C}$ is strong. By ($\star$.1) with $(a, b) = (c_1, a_1')$, we have that $(b_1', c_1) \in E(D)$. Hence, $N_D^-(b_1') \subseteq \{b_1\}$, since $deg_D(b_1') = 2$. This is a contradiction, since $b_1' \in L_3(\mathcal{C})$, $w(b_1) = 1$ and $\mathcal{C}$ is strong. □

Lemma 2.20 Let $\mathcal{C}$ be a vertex cover of D and C a 5-cycle. If $A \subseteq V^+$ and $B \subseteq V(D)$ such that $N_D^+(A) \subseteq \mathcal{C}$, $B \cap \mathcal{C} = \emptyset$ and $|V(C) \cap (N_D^+(A) \cup N_D(B))| \geqslant 4$, then there is a strong vertex cover $\mathcal{C}'$ such that $|\mathcal{C}' \cap V(C)| \geqslant 4$.

Proof. By Proposition 2.1, there is a vertex cover $\mathcal{C}'$ such that $N_D^+(A) \subseteq \mathcal{C}' \subseteq \mathcal{C}$. So, $B \cap \mathcal{C}' = \emptyset$, since $B \cap \mathcal{C} = \emptyset$. Thus, by Remark 1.21, $N_D(B) \subseteq \mathcal{C}'$. Then, $N_D^+(A) \cup N_D(B) \subseteq \mathcal{C}'$. Therefore, $|V(C) \cap \mathcal{C}'| \geqslant 4$, since $|V(C) \cap (N_D^+(A) \cup N_D(B))| \geqslant 4$. □

Proposition 2.21 Let C be a basic 5-cycle, then C has the $\star$-property if and only if $|\mathcal{C} \cap V(C)| = 3$ for each strong vertex cover $\mathcal{C}$ of D.

Proof. $\Rightarrow$) By contradiction, we suppose there is a strong vertex cover $\mathcal{C}$ such that $|\mathcal{C} \cap V(C)| \geqslant 4$. Thus, there is a path $L = (d_1, d_2, d_3, d_4) \subseteq C$ such that $V(L) \subseteq \mathcal{C}$. Then, $deg_D(d_2) = 2$ or $deg_D(d_3) = 2$, since C is basic. We can suppose $deg_D(d_2) = 2$, then $N_D[d_2] \subseteq \mathcal{C}$. This implies $b_1 := d_2 \in L_3(\mathcal{C})$. So, there is $(a_1, b_1) \in E(D)$ with $a_1 \in (\mathcal{C} \setminus L_1(\mathcal{C})) \cap V^+$, since $\mathcal{C}$ is strong. Since, $N_D[b_1] \subseteq \mathcal{C}$, we can set $C = (a_1', a_1, b_1, b_1', c_1, a_1')$. Consequently, $\{a_1, b_1'\} = N_D(b_1) = N_D(d_2) = \{d_1, d_3\} \subseteq \mathcal{C}$. By ($\star$.1), $(a_1', a_1) \in E(D)$ and $w(a_1') = 1$. If $b_1 \in V^+$, then by Remark 1.81, b_1 is not a sink. This implies, $(b_1, b_1') \in E(D)$. Then, by ($\star$.1) with $(a, b) = (b_1, b_1')$, $w(a_1) = 1$. A contradiction, since $a_1 \in V^+$. Hence, $w(b_1) = 1$.

Now we prove $a_1' \in \mathcal{C}$. By contradiction assume $a_1' \notin \mathcal{C}$, then $\{a_1, b_1, b_1', c_1\} \subseteq \mathcal{C}$, since $|\mathcal{C} \cap V(C)| \geqslant 4$. Suppose $b_1' \in L_3(\mathcal{C})$, then there is $x \in (\mathcal{C} \cap V^+) \setminus L_1(\mathcal{C})$ such that $(x, b_1') \in E(D)$, since $\mathcal{C}$ is strong. Then, by ($\star$.3) with $(a, b) = (a_1, b_1)$, we have that $x \in N_D^-(a_1')$, i.e. $(x, a_1') \in E(D)$. Consequently, $x \in L_1(\mathcal{C})$, since $x \in \mathcal{C}$ and $a_1' \notin \mathcal{C}$. This is a contradiction. Hence, $b_1' \notin L_3(\mathcal{C})$, i.e. there is $x' \in N_D(b_1') \setminus \mathcal{C}$, since $b_1' \in \mathcal{C}$. By ($\star$.3), $x' \in N_D(a_1') \cup N_D^+(a_1)$. Furthermore, by Remark 1.21, $N_D(a_1') \subseteq \mathcal{C}$, then $x' \in N_D^+(a_1)$, since $x' \notin \mathcal{C}$. This implies $a_1 \in L_1(\mathcal{C})$, since $a_1 \in \mathcal{C}$ and $x' \notin \mathcal{C}$. A contradiction, since $a_1 \notin L_1(\mathcal{C})$. Therefore, $a_1' \in \mathcal{C}$.

Thus, $\{a_1, a_1', b_1, b_1'\} \subseteq \mathcal{C}$. Now, we prove $c_1 \in \mathcal{C}$, $deg_D(a_1) \geqslant 3$ and $deg_D(c_1) \geqslant 3$.

Case (1) $a_1 \in L_3(\mathcal{C})$. Thus, there is $z \in N_D^-(a_1) \cap V^+$ such that $z \in \mathcal{C} \setminus L_1(\mathcal{C})$. Then, $z \notin V(C)$, since $N_D^-(a_1) \cap V(C) = \{a_1'\}$ and $w(a_1') = 1$. Furthermore, by ($\star$.2), $z \in N_D^-(c_1)$ i.e., $(z, c_1) \in E(D)$. Consequently, $c_1 \in \mathcal{C}$ and $deg_D(a_1), deg_D(c_1) \geqslant 3$, since $z \in \mathcal{C} \setminus L_1(\mathcal{C})$ and $z \in N_D(a_1) \cap N_D(c_1)$.

Case (2) $a_1 \notin L_3(\mathcal{C})$. Then, there is $z' \in N_D(a_1)$ such that $z' \notin \mathcal{C}$. So, $z' \notin V(C)$, since $N_D(a_1) \cap V(C) = \{a_1', b_1\} \subseteq \mathcal{C}$. Thus, $z' \in N_D^-(a_1)$, since $a_1 \in \mathcal{C} \setminus L_1(\mathcal{C})$. Consequently, by ($\star$.2), $z' \in N_D(c_1)$. Hence, by Remark 1.21, $c_1 \in \mathcal{C}$, since $z' \notin \mathcal{C}$.

Furthermore, $deg_D(a_1), deg_D(c_1) \geqslant 3$, since $z' \in N_D(a_1) \cap N_D(c_1)$.

This implies, $V(C) \subseteq \mathcal{C}$. A contradiction, by Lemma 2.19, since C has the $\star$-property.

$\Leftarrow$) Assume $C = (a', a, b, b', c, a')$ with $(a, b) \in E(C)$ such that $w(a) \neq 1$. We take a minimal vertex cover $\mathcal{C}$ of D. We will prove ($\star$.1), ($\star$.2) and ($\star$.3).

($\star$.**1**) First we prove $(a', a) \in E(D)$. By contradiction, suppose $(a, a') \in E(D)$. By Remark 1.81, there is $x \in N_D^-(a)$, since $a \in V^+$. Thus, $x \notin V(C)$ and $deg_D(a) \geq 3$. Consequently, $deg_D(a') = deg_D(b) = 2$, since C is basic. Also, $deg_D(b') = 2$ or $deg_D(c) = 2$, since C is basic. We can assume $deg_D(c) = 2$, then $N_D(c) = \{a', b'\}$. So, by Remark 1.20, $\mathcal{C}_1 = \big(\mathcal{C} \setminus \{x, c\}\big) \cup N_D(x, b) \cup N_D^+(a)$ is a vertex cover, since $N_D(c) = \{a', b'\} \subseteq N_D^+(a) \cup N_D(b) \subseteq \mathcal{C}_1$. Furthermore, $c \notin N_D(x)$ and $c \notin N_D(b) \cup N_D^+(a)$, since $deg_D(c) = 2$ and C is induced. Then, $c \notin \mathcal{C}_1$. Also, $N_D(b) = \{b', a\}$, implies $x \notin \mathcal{C}_1$, since $x \notin N_D^+(a)$. Hence, $N_D^+(a) \subseteq \mathcal{C}_1$, $\{x, c\} \cap \mathcal{C}_1 = \emptyset$ and $\{a, a', b, b'\} \subseteq N_D^+(a) \cup N_D(x, c)$. A contradiction, by Lemma 2.20, since $a \in V^+$.

Now, we prove $w(a') = 1$. By contradiction, assume $w(a') \neq 1$. So, we have $(a', a) \in E(D)$ and $a' \in V^+$. Hence, by the last argument, $(c, a') \in E(D)$. A contradiction, by (a) in Lemma 2.16.

($\star$.**2**) We will prove $N_D^-(a) \subseteq N_D(c)$. By contradiction, suppose there is $x \in N_D^-(a) \setminus N_D(c)$. Also, $N_D^-(a) \cap V(C) \subseteq \{a'\} \subseteq N_D(c)$, since $(a, b) \in E(D)$. Thus, $x \notin V(C)$. By Remark 1.20, $\mathcal{C}_2 = \big(\mathcal{C} \setminus \{x, c\}\big) \cup N_D(x, c) \cup N_D^+(a)$ is a vertex cover. Furthermore, $\{x, c\} \cap \mathcal{C}_2 = \emptyset$, since $x \in N_D^-(a) \setminus N_D(c)$ and $c \notin N_D(a, x)$. Also, $\{a, a', b, b'\} \subseteq N_D^+(a) \cup N_D(x, c)$. A contradiction, by Lemma 2.20, since $a \in V^+$.

Now, we prove $N_D^-(a) \cap V^+ \subseteq N_D^-(c)$. By contradiction, suppose there is $x \in N_D^-(a) \cap V^+ \setminus N_D^-(c)$. By Remark 1.20, $\mathcal{C}_3 = (\mathcal{C} \setminus \{c\}) \cup N_D(c) \cup N_D^+(a, x)$ is a vertex cover. Furthermore, $c \notin N_D^+(a, x)$, since $x \notin N_D^-(c)$. Then, $c \notin \mathcal{C}_3$. Furthermore, $\{b, a, a', b'\} \subseteq N_D^+(a, x) \cup N_D(c)$. A contradiction by Lemma 2.20, since $\{a, x\} \subseteq V^+$.

($\star$.**3**) We prove $N_D(b') \subseteq N_D(a') \cup N_D^+(a)$. By contradiction, we suppose there is $x \in N_D(b') \setminus \big(N_D(a') \cup N_D^+(a)\big)$. Thus, $x \notin C$, since $N_D(b') \cap V(C) = \{c, b\} \subseteq N_D(a') \cup N_D^+(a)$. By Remark 1.20, $\mathcal{C}_4 = \big(\mathcal{C} \setminus \{x, a'\}\big) \cup N_D(x, a') \cup N_D^+(a)$ is a vertex cover. Also, $x \notin \mathcal{C}_4$, since $x \notin N_D(a') \cup N_D^+(a)$. By ($\star$.1), $(a', a) \in E(D)$, then $a' \notin \mathcal{C}_4$, since $a' \notin N_D(x) \cup N_D^+(a)$. So, $\{x, a'\} \cap \mathcal{C}_4 = \emptyset$ and $\{b, b'a, c\} \subseteq N_D^+(a) \cup N_D(x, a')$. A contradiction, by Lemma 2.20.

Finally, we prove $N_D^-(b') \cap V^+ \subseteq N_D^-(a')$. By contradiction, we suppose there is

$x \in \big(N_D^-(b') \cap V^+\big) \setminus N_D^-(a')$. By ($\star$.1), $(a',a) \in E(D)$. By Remark 1.20, $\mathcal{C}_5 = \big(\mathcal{C} \setminus \{a'\}\big) \cup N_D(a') \cup N_D^+(x,a)$ is a vertex cover. Furthermore, $a' \notin N_D^+(x,a)$, since $x \notin N_D^-(a')$ and $(a',a) \in E(D)$. Consequently, $a' \notin \mathcal{C}_5$. Also, $\{b,b',a,c\} \subseteq N_D^+(a,x) \cup N_D(a')$ and $\{a,x\} \subseteq V^+$. A contradiction, by Lemma 2.20. $\square$

Lemma 2.22 Let $\mathcal{C}$ be a vertex cover of D where G is an SCQ graph. Hence, $|\mathcal{C}| = \tau(G)$ if and only if $|\mathcal{C} \cap V(K)| = |V(K)| - 1$, $|\mathcal{C} \cap V(C)| = 3$ and $|\mathcal{C} \cap y| = 1$ for each $K \in S_G$, $C \in C_G$ and $y \in Q_G$, respectively.

Proof. We set $\mathcal{C}$ a vertex cover of D, $K \in S_G$, $C \in C_G$ and $y \in Q_G$. Then, there are $a, a' \in V(C)$ and $z \in V(G)$ such that $deg_G(a) = deg_G(a') = 2$, $\{a,a'\} \notin E(G)$ and $K = G[N_G[z]]$. We set $A_K := V(K) \setminus \{z\}$ and $B_C := V(C) \setminus \{a,a'\}$. Also, $\mathcal{C} \cap V(K)$ is a vertex cover of K, so $|\mathcal{C} \cap V(K)| \geqslant \tau(K) = |V(K)| - 1$. Similarly, $|\mathcal{C} \cap V(C)| \geqslant \tau(C) = 3$ and $|\mathcal{C} \cap y| \geqslant \tau(y) = 1$. Thus,

$$\begin{aligned} |\mathcal{C}| &= \sum_{K \in S_G} |\mathcal{C} \cap V(K)| + \sum_{C \in C_G} |\mathcal{C} \cap V(C)| + \sum_{y \in Q_G} |\mathcal{C} \cap y| \\ &\geqslant \sum_{K \in S_G} \big(|V(K)| - 1\big) + 3|C_G| + |Q_G|, \end{aligned} \tag{2.1}$$

since $\mathcal{H} = \{V(H) \mid H \in S_G \cup C_G \cup Q_G\}$ is a partition of $V(G)$. Now, we take a maximal stable set S contained in $V(Q_G) := \{x \in V(G) \mid x \in y \text{ and } y \in Q_G\}$. So, $|S \cap y| \leqslant 1$ for each $y \in Q_G$, since S is stable. If $S \cap y = \emptyset$ for some $y = \{x_1, x_2\} \in Q_G$, then there are $z_1, z_2 \in S$ such that $\{x_1, z_1\}$, $\{x_2, z_2\} \in E(G)$, since S is maximal. But Q_G satisfies the property **(P)**, then $\{z_1, z_2\} \in E(G)$. A contradiction, since S is stable. Hence, $|S \cap y| = 1$ for each $y \in Q_G$. Consequently, $|S| = |Q_G|$ and $|S'| = |Q_G|$, where $S' = V(Q_G) \setminus S$. Now, we take

$$\mathcal{C}(S') = \Big(\bigcup_{K \in S_G} A_K\Big) \cup \Big(\bigcup_{C \in C_G} B_C\Big) \cup S'.$$

We prove $\mathcal{C}(S')$ is a vertex cover of D. By contradiction, suppose there is $\hat{y} \in E(G)$ such that $\hat{y} \cap \mathcal{C}(S') = \emptyset$. We set $z' \in \hat{y}$, then $\hat{y} = \{z', z''\}$. If $z' \in V(\tilde{K})$ for some $\tilde{K} \in S_G$, then $z' = \tilde{z}$ where $\tilde{K} = G[N_G[\tilde{z}]]$, since $A_{\tilde{K}} \subseteq \mathcal{C}(S')$ and $z' \notin \mathcal{C}(S')$. So, $z'' \in N_G(z') \subseteq \tilde{K} \setminus \{z'\} = A_{\tilde{K}} \subseteq \mathcal{C}(S')$. A contradiction, since $\hat{y} \cap \mathcal{C}(S') = \emptyset$. Now, if $z' \in V(\tilde{C})$ for some $\tilde{C} \in C_G$, then $z' \notin B_{\tilde{C}}$. Thus, $deg_G(z') = 2$ implies $z'' \in B_{\tilde{C}} \subseteq \mathcal{C}(S')$, since $\{z', z''\} \in E(G)$. A contradiction. This implies, $\hat{y} \subseteq V(Q_G)$, since $\mathcal{H}$ is a partition of $V(G)$. Also, $\hat{y} \cap S' = \emptyset$, since $\hat{y} \notin \mathcal{C}(S')$. Consequently, $\hat{y} \subseteq V(Q_G) \setminus S' = S$. But S is a stable set, a contradiction. Hence, $\mathcal{C}(S')$ is a vertex cover of D. Furthermore,

$$|\mathcal{C}(S')| = \sum_{K \in S_G} |A_K| + \sum_{C \in C_G} |B_C| + |S'| = \sum_{K \in S_G} \big(|V(K)| - 1\big) + 3|C_G| + |Q_G|.$$

Therefore, by (2.1), $\tau(G) = \sum_{K \in S_G} \big(|V(K)| - 1\big) + 3|C_G| + |Q_G|$. Furthermore, $|\mathcal{C}| = \tau(G)$ if and only if $|\mathcal{C} \cap V(K)| = |K| - 1$, $|\mathcal{C} \cap V(C)| = 3$ and $|\mathcal{C} \cap y| = 1$ for each $K \in S_G$, $C \in C_G$ and $y \in Q_G$, respectively. □

Theorem 2.23 Let D be a weighted oriented graph where G is an SCQ graph. Hence, $I(D)$ is unmixed if and only if D satisfies the following conditions:

(a) Each basic 5-cycle of G has the $\star$-property.
(b) Each simplex of D has no generating $\star$-semi-forests.
(c) $N_D(b) \subseteq N_D^+(a)$ when $a \in V^+$, $\{b, b'\} \in Q_G$ and $b' \in N_D^+(a)$.

Proof. $\Rightarrow$) We take a strong vertex cover $\mathcal{C}$ of D, then by Remark 1.86, $|\mathcal{C}| = \tau(G)$. Consequently, by Lemma 2.22, $|\mathcal{C} \cap V(K)| = |V(K)| - 1$, $|\mathcal{C} \cap V(C)| = 3$ and $|\mathcal{C} \cap y| = 1$ for each $K \in S_G$, $C \in C_G$ and $y \in Q_G$. Thus, $V(K) \not\subseteq \mathcal{C}$. Consequently, by Theorem 2.11, D satisfies (b). Furthermore, by Propositions 2.21 and 2.14, D satisfies (a) and (c).

$\Leftarrow$) Let $\mathcal{C}$ be a strong vertex cover of D. By (a) and Proposition 2.21, we have $|\mathcal{C} \cap V(C)| = 3$ for each $C \in C_G$. Furthermore, by (b) and Corollary 2.12, $|\mathcal{C} \cap V(K)| = |V(K)| - 1$ for each $K \in S_G$. Now, if $y \in Q_G$, then by Definition 1.39, y has the property **(P)**. Thus, by (c) and Proposition 2.14, $|\mathcal{C} \cap y| = 1$. Hence, by Lemma 2.22, $|\mathcal{C}| = \tau(G)$. Therefore $I(D)$ is unmixed, by (2) in Theorem 1.85. □

Corollary 2.24 Let D be a weighted oriented graph where G is chordal or simplicial. Hence, $I(D)$ is unmixed if and only if D satisfies the following conditions:

(a) Each vertex is in exactly one simplex of D.
(b) Each simplex of D has not a generating $\star$-semi-forest.

Proof. $\Rightarrow$) By (3) in Theorem 1.85 and Remark 1.78, G is well-covered. Thus, by Theorem 1.38, G satisfies (a). Furthermore, by Remark 1.43, G is an SCQ graph with $C_G = Q_G = \emptyset$. Hence, by Theorem 2.23, D satisfies (b).

$\Leftarrow$) By (a), $\{V(H) \mid H \in S_G\}$ is a partition of $V(G)$. Hence, G is an SCQ graph with $C_G = \emptyset$ and $Q_G = \emptyset$. Therefore, by (b) and Theorem 2.23, $I(D)$ is unmixed. □

2.3 UNMIXEDNESS OF WEIGHTED ORIENTED GRAPHS WITHOUT SOME SMALL CYCLES

In this section we characterize the unmixedness of $I(D)$ when G is a graph with at most one of the following families of cycles: 3-cycles, 4-cycles and 5-cycles (Theorems 2.28, 2.38 and 2.43). In particular, we obtain an easy characterization of the unmixedness of $I(D)$ if $girth(G) \geqslant 6$ (see Corollary 2.40).

Proposition 2.25 If for each $(z, x) \in E(D)$ with $z \in V^+$, we have that $N_D(x') \subseteq N_D^+(z)$ for some $x' \in N_D(x) \setminus z$, then $L_3(\mathcal{C}) = \emptyset$ for each strong vertex cover $\mathcal{C}$ of D.

Proof. By contradiction, suppose there is a strong vertex cover $\mathcal{C}$ of D and $x \in L_3(\mathcal{C})$. Hence, there is $z \in \big(\mathcal{C} \setminus L_1(\mathcal{C})\big) \cap V^+$ with $(z, x) \in E(D)$. Then, $N_D(x) \subseteq \mathcal{C}$ and $N_D^+(z) \subseteq \mathcal{C}$, since $x \in L_3(\mathcal{C})$ and $z \in \mathcal{C} \setminus L_1(\mathcal{C})$. By hypothesis, there is a vertex $x' \in N_D(x) \setminus z \subseteq \mathcal{C}$ such that $N_D(x') \subseteq N_D^+(z) \subseteq \mathcal{C}$. Thus, $x' \in L_3(\mathcal{C})$. Since $\mathcal{C}$ is strong, there is $(z_1, x') \in E(D)$ with $z_1 \in V^+$. So, $z_1 \in N_D(x') \subseteq N_D^+(z)$. On the other hand, $(z_1, x_1) \in E(D)$ where $x_1 := x'$ and $z_1 \in V^+$, then by hypothesis, there is $x_2 \in N_D(x_1) \setminus z_1$ such that $N_D(x_2) \subseteq N_D^+(z_1)$. Hence, $x_2 \in N_D(x_1) = N_D(x') \subseteq N_D^+(z)$. Consequently, $z \in N_D(x_2) \subseteq N_D^+(z_1)$. A contradiction, since $z_1 \in N_D^+(z)$. □

Corollary 2.26 If G is unmixed and V^+ is a subset of sinks, then $I(D)$ is unmixed.

Proof. If $x' \in V^+$, then x' is a sink. Thus, $(x', x) \notin E(D)$ for each $x' \in V(D)$. Hence, by Proposition 2.25, $L_3(\mathcal{C}) = \emptyset$, for each strong vertex cover $\mathcal{C}$ of D. Furthermore, by Remark 1.78, $I(G)$ is unmixed. Therefore $I(D)$ is unmixed, by (3) in Theorem 1.85. □

Lemma 2.27 Let $(z, x'), (x', x)$ be edges of D with $x' \in V^+$ and $N_D(x) = \{x', x_1, \ldots, x_s\}$. If there are $z_i \in N_D(x_i) \setminus N_D^+(x')$ such that $\{z, x, z_1, \ldots, z_s\}$ is a stable set, then $I(D)$ is mixed.

Proof. We have $A \cup \{x\}$ is a stable set where $A := \{z, z_1, \ldots, z_s\}$. We can take a maximal stable set S of $V(G)$, such that $A \cup \{x\} \subseteq S$. So, $\tilde{\mathcal{C}} = V(G) \setminus S$ is a minimal vertex cover of D. Hence, $\mathcal{C} = \tilde{\mathcal{C}} \cup N_D^+(x')$ is a vertex cover of D. Also $A \cap \mathcal{C} = \emptyset$, since $A \subseteq S$, $z \in N_D^-(x')$ and $z_i \notin N_D^+(x')$. By Proposition 2.1, there is a strong vertex cover $\mathcal{C}'$ of D such that $N_D^+(x') \subseteq \mathcal{C}' \subseteq \mathcal{C}$, since $x' \in V^+$. Thus,

$A \cap \mathcal{C}' = \emptyset$, since $A \cap \mathcal{C} = \emptyset$. Then, by Remark 1.21, $N_D(A) \subseteq \mathcal{C}'$. Furthermore $N_D(x) = \{x', x_1, \ldots, x_s\} \subseteq N_D(A)$. Consequently, $N_D[x] \subseteq \mathcal{C}'$, since $x \in N_D^+(x') \subseteq \mathcal{C}'$. Hence, $x \in L_3(\mathcal{C}')$. Therefore, by (3) in Theorem 1.85, $I(D)$ is mixed. □

Theorem 2.28 Let D be a weighted oriented graph such that G has no 3- and 5-cycles. Hence, $I(D)$ is unmixed if and only if D satisfies the following conditions:

(a) G is well-covered.
(b) If $(z, x) \in E(D)$ and $z \in V^+$, then $N_D(x') \subseteq N_D^+(z)$ for some $x' \in N_D(x) \setminus z$.

Proof. $\Leftarrow$) By Proposition 2.25 and (b), we have that $L_3(\mathcal{C}) = \emptyset$ for each strong vertex cover $\mathcal{C}$ of D. Furthermore, by (a) and Remark 1.78, $I(G)$ is unmixed. Therefore, by (3) in Theorem 1.85, $I(D)$ is unmixed.

$\Rightarrow$) By (3) in Theorem 1.85 and Remark 1.78, D satisfies (a). Now, we take $(z, x) \in E(D)$ with $z \in V^+$. Then, by Remark 1.81, there is $z' \in N_D^-(z)$. Thus, $z' \neq x$ and $z' \notin N_D(x)$, since G has no 3-cycles. We set $N_D(x) \setminus \{z\} = \{x_1, \ldots, x_s\}$, then $z' \notin \{z, x_1, \ldots, x_s\}$. We will prove (b). By contradiction, suppose there is $z_i \in N_D(x_i) \setminus N_D^+(z)$ for each $i = 1, \ldots, s$. So, $x \notin \{z_1, \ldots, z_s\}$, since $x \in N_D^+(z)$. If $\{z_i, z_j\} \in E(G)$ for some $1 \leqslant i < j \leqslant s$, then $(x, x_i, z_i, z_j, x_j, x)$ is a 5-cycle. But G has no 5-cycles, then $\{z_1, \ldots, z_s\}$ is a stable set. If $z = z_i$, then (z, x, x_i, z) is a 3-cycle. A contradiction, this implies $z \notin \{z_1, \ldots, z_s\}$. Now, if $\{x, z_k\} \in E(G)$ or $\{z', z_k\} \in E(G)$ for some $k \in \{1, \ldots, s\}$, then (x, x_k, z_k, x) is a 3-cycle or (z', z, x, x_k, z_k, z') is a 5-cycle. Hence, $\{x, z', z_1, \ldots, z_s\}$ is a stable set. A contradiction, by Lemma 2.27, since $I(D)$ is unmixed. □

In the following results, we use the notation of Figure 1.1.

Remark 2.29 Let G be a graph in $\{C_7, T_{10}, P_{10}, P_{13}, P_{14}, Q_{13}\}$. Hence,

(a) G does not contain 4-cycles. Furthermore, if G has a 3-cycle, then $G = T_{10}$.
(b) If $deg_G(x) = 2$, then x is not in a 3-cycle of G.
(c) If $G \neq C_7$ and $\tilde{y} = \{v, u\} \in E(G)$ with $deg_D(v) = deg_D(u) = 2$, then $\tilde{y} \in \{\tilde{y}_1, \tilde{y}_2, \tilde{y}_3\}$. Also, if $\tilde{y}$ is in a 5-cycle C, then $G \in \{P_{10}, P_{13}\}$, $\tilde{y} \in \{\tilde{y}_1, \tilde{y}_2\}$ and $C \in \{C^1, C^2\}$ or $G = Q_{13}$, $\tilde{y} = \tilde{y}_1$ and $C = C^1$.
(d) If $P = (z_1, z_2, z_3)$ is a path in G with $deg_G(z_i) = 2$ for $i = 1, 2, 3$, then $G = C_7$.

Proof. (a) By Theorems 1.42 and 1.46, G has no 4-cycles. Now, if G has a 3-cycle then, by Theorem 1.46, $G = T_{10}$.

(b) By (a), the unique 3-cycle is (c_1, c_2, c_3, c_1) in T_{10} and $deg_{T_{10}}(c_i) = 3$ for $i = 1, 2, 3$.

(c) By Figure 1.1, $\tilde{y} \in \{\tilde{y}_1, \tilde{y}_2, \tilde{y}_3\}$ and $G \in \{T_{10}, P_{10}, P_{13}, Q_{13}\}$, since $G \neq C_7$. Now, assume $\tilde{y}$ is in a 5-cycle C. By Theorem 1.42, $G \neq T_{10}$. If $G = P_{10}$, then $\tilde{y}_3$ is not in a 5-cycle. Thus, $\tilde{y} \in \{\tilde{y}_1, \tilde{y}_2\}$ and $C \in \{C^1, C^2\}$. Now, if $G = P_{13}$, then $\tilde{y} \in \{\tilde{y}_1, \tilde{y}_2\}$ and $C \in \{C^1, C^2\}$. Finally, if $G = Q_{13}$, then $\tilde{y}_2, \tilde{y}_3$ are not in a 5-cycle. Hence, $\tilde{y} = \tilde{y}_1$ and $C = C^1$.

(d) By contradiction, suppose $G \neq C_7$. If $\tilde{y} \in E(P)$, then by (c), $\tilde{y} \in \{\tilde{y}_1, \tilde{y}_2, \tilde{y}_3\}$. But, $\tilde{y}_i \cap \tilde{y}_j = \emptyset$ for $i \neq j$. A contradiction, since P is a path. □

Lemma 2.30 Let G be a graph in $\{C_7, T_{10}, P_{10}, P_{13}, P_{14}, Q_{13}\}$ with $I(D)$ unmixed. If $(z, x'), (x', x) \in E(D)$ with $x' \in V^+$ and $N_D(x) \setminus \{x'\} = \{x_1\}$, then $deg_D(x_1) = 2$.

Proof. By contradiction, suppose $deg_D(x_1) \geqslant 3$. Hence, there are $z_1, z_1' \in N_D(x_1) \setminus \{x\}$. By hypothesis, $deg_D(x) = 2$. Then, by (b) in Remark 2.29, x is not in a 3-cycle. So, $x_1 \neq z$ and $x' \notin \{z_1, z_1'\}$. Furthermore, by (a) in Remark 2.29, G has no 4-cycles. Thus, $z \notin \{z_1', z_1\}$ and $z_1, z_1' \notin N_D(x')$. If $z_1', z_1 \in N_D(z)$, then (x_1, z_1', z, z_1, x_1) is a 4-cycle. A contradiction, then we can assume $z_1 \notin N_D(z)$. Consequently, $\{x, z, z_1\}$ is a stable set, since x is not in a 3-cycle. A contradiction, by Lemma 2.27, since $I(D)$ is unmixed. □

Lemma 2.31 If $I(D)$ is unmixed, $G \in \{C_7, T_{10}, P_{10}, P_{13}, P_{14}, Q_{13}\}$ and $\tilde{y} = (x', x) \in E(D)$ with $deg_D(x) = 2$ and $x' \in V^+$, then $G = P_{10}$ and $\tilde{y} = (d_i, b_j)$ with $\{i, j\} = \{1, 2\}$.

Proof. By Remark 1.81, there is $z \in N_D^-(x')$. We set $N_D(x) = \{x', x_1\}$. Thus, by Lemma 2.30, $deg_D(x_1) = 2$. Now, we set $N_D(x_1) = \{x, z_1\}$. By (b) in Remark 2.29, x is not in a 3-cycle. So, $z, z_1 \notin N_D(x)$ implies $z \neq x_1$ and $z_1 \neq x'$. Also, by (a) in Remark 2.29, G has no 4-cycles. Then, $z_1 \neq z$ and $z_1 \notin N_D(x')$. If $z \notin N_D(z_1)$, then $\{x, z, z_1\}$ is a stable set. A contradiction, by Lemma 2.27, since $I(D)$ is unmixed. Hence, $\{z_1, z\} \in E(G)$ and $C := (z, x', x, x_1, z_1, z)$ is a 5-cycle. Suppose $x'' \in N_D^-(x') \setminus \{z\}$, then $x'' \notin N_D(z_1)$, since (z_1, z, x', x'', z_1) is not in a 4-cycle in G. Consequently, $\{x'', x, z_1\}$ is a stable set, since $deg_D(x) = 2$. A contradiction, by Lemma 2.27, since $(x'', x'), (x', x) \in E(D)$. Hence, $N_D^-(x') = \{z\}$. Now, by (c) in Remark 2.29 and by symmetry of P_{10} and P_{13}, we can assume $\{x, x_1\} = \tilde{y}_1$, $C = C^1$ and $G \in \{P_{10}, P_{13}, Q_{13}\}$.

First, assume $G = P_{13}$. By symmetry and notations of Figure 1.1, we can suppose $x_1 = a_1$ and $x = a_2$, since $\tilde{y}_1 = \{x, x_1\}$. Then, $x' = b_2$, $z_1 = b_1$ and $z = c_1$. Thus, $(b_2, d_2) \in E(D)$, since $x' = b_2$ and $N_D^-(x') = \{z\} = \{c_1\}$. Furthermore, $(c_1, b_2) = (z, x') \in E(D)$, $(b_2, d_2) \in E(D)$, $b_2 = x' \in V^+$ and $N_D(d_2) = \{b_2, b_4, v\}$. Also, $a_4 \in N_D(b_4) \setminus N_D^+(b_2)$, $d_1 \in N_D(v) \setminus N_D^+(b_2)$ and $\{c_1, d_2, a_4, d_1\}$ is a stable set.

A contradiction, by Lemma 2.27, since $I(D)$ is unmixed.

Now, suppose $G = Q_{13}$. By symmetry of Q_{13}, we can suppose $x = a_2$ and $x_1 = a_1$, then $d_2 = x' \in V^+$, $z = h$ and $(h, d_2) = (z, x') \in E(D)$. So, $(d_2, c_2) \in E(D)$, since $N_D^-(d_2) = N_D^-(x') = \{z\} = \{h\}$. A contradiction, by Lemma 2.27, since $(h, d_2), (d_2, c_2) \in E(D)$, $d_2 \in V^+$, $N_D(c_2) = \{d_2, b_1\}$, $g_1 \in N_D(b_1) \setminus N_D^+(d_2)$ and $\{h, c_2, g_1\}$ is a stable set.

Hence, $G = P_{10}$ and $\{x, x_1\} = \tilde{y}_1 = \{a_1, b_1\}$. If $x = a_1$ and $x_1 = b_1$, then $g_1 = x' \in V^+$, $(d_1, g_1) = (z, x') \in E(D)$, since $C = C^1$. Furthermore, $(g_1, c_1) \in E(D)$, since $N_D^-(g_1) = N_D^-(x') = \{z\} = \{d_1\}$. A contradiction by Lemma 2.27, since $(d_1, g_1), (g_1, c_1) \in E(D)$, $g_1 \in V^+$, $N_D(c_1) = \{g_1, c_2\}$, $g_2 \in N_D(c_2) \setminus N_D^+(g_1)$ and $\{d_1, c_1, g_2\}$ is a stable set. Therefore, $x = b_1$ and $x_1 = a_1$, imply $x' = d_2$ and $(x', x) = (d_2, b_1)$, since $C = C^1$. □

Remark 2.32 Assume $I(D)$ is unmixed, $G \in \{C_7, T_{10}, Q_{13}, P_{13}, P_{14}\}$, $\mathcal{C}$ is a strong vertex cover of D and $x \in \mathcal{C} \cap V^+$ such that $N_G(x) \setminus \mathcal{C} \subseteq V_2 := \{a \in V(G) \mid deg_G(a) = 2\}$. We take $b \in N_G(x) \setminus \mathcal{C}$. By Lemma 2.31, $(x, b) \notin E(D)$, since $G \neq P_{10}$. Then, $(b, x) \in E(D)$. Consequently, $N_D(x) \setminus \mathcal{C} \subseteq N_D^-(x)$, i.e. $N_D^+(x) \subseteq \mathcal{C}$. Hence, $x \in \mathcal{C} \setminus L_1(\mathcal{C})$.

Proposition 2.33 If $I(D)$ is unmixed, with $G \in \{C_7, T_{10}, Q_{13}, P_{13}, P_{14}\}$, then the vertices of V^+ are sinks.

Proof. By contradiction, suppose there is $(x', x) \in E(D)$ with $x' \in V^+$. Then, by Lemma 2.31, $deg_D(x) \geqslant 3$, since $G \neq P_{10}$. Thus, $G \neq C_7$. By Remark 1.81, x' is not a source. So, there is $(z, x') \in E(D)$. We set $V_2 := \{a \in V(G) \mid deg_G(a) = 2\}$. By Theorem 1.85, $L_3(\tilde{\mathcal{C}}) = \emptyset$ for each strong vertex cover $\tilde{\mathcal{C}}$ of D, since $I(D)$ is unmixed. Hence, to obtain a contradiction, in each one of the following cases, we will give a vertex cover $\mathcal{C}$ of D such that $L_3(\mathcal{C}) = \{x\}$ and $x' \in \mathcal{C} \setminus L_1(\mathcal{C})$, since with these conditions $\mathcal{C}$ is strong and $L_3(\mathcal{C}) \neq \emptyset$. We will use the notations of Figure 1.1.

Case (1) If $D = T_{10}$, then $x \in \{v, c_1, c_2, c_3\}$, since $deg_G(x) \geqslant 3$.

Case (1.a) $x = v$. Then, $\mathcal{C}_1 = \{v, a_1, a_2, a_3, c_1, c_2, c_3\}$ is a vertex cover with $L_3(\mathcal{C}_1) = \{x\}$. By symmetry of P_{10}, we can suppose $x' = a_1$. Consequently, $z = b_1$ and $N_D^+(x') = \{v\} \subseteq \mathcal{C}_1$, since $a_1 \in V_2$. Hence, $x' \in \mathcal{C}_1 \setminus L_1(\mathcal{C}_1)$.

Case (1.b) $x \in \{c_1, c_2, c_3\}$. By symmetry of P_{10}, we can assume $x = c_1$ and $x' \in N_D(c_1) = \{b_1, c_2\}$. Thus, $\mathcal{C}_2 = \{v, a_2, a_3, b_1, c_1, c_2, c_3\}$ is a vertex cover with $L_3(\mathcal{C}_2) = \{x\}$ and $x' \in \mathcal{C}_2$. If $x' = b_1$, then $z = a_1$ and $N_D^+(x') = \{c_1\} \subseteq \mathcal{C}_2$, since $b_1 \in V_2$.

So, $x' \notin L_1(\mathcal{C}_2)$. Now, if $x' = c_2$, then by Lemma 2.31, $(c_2, b_2) \notin E(D)$, since $c_2 = x' \in V^+$, $b_2 \in V_2$ and $I(D)$ is unmixed. Consequently, $N_D^+(x') = N_D^+(c_2) \subseteq \{c_1, c_3\} \subseteq \mathcal{C}_2$. Hence, $x' \notin L_1(\mathcal{C}_2)$.

Case (2) If $D = P_{14}$, then, by symmetry, we can assume $x' = a_1$ and $x \in \{a_2, b_1\}$.

Case (2.a) $x = a_2$. Thus, $z \in \{a_7, b_1\}$. We take $\mathcal{C}_3 = \{a_1, a_2, a_3, a_4, a_6, b_1, b_2, b_5, b_6, b_7\}$ if $z = a_7$, or $\mathcal{C}_3 = \{a_1, a_2, a_3, a_5, a_7, b_2, b_3, b_4, b_5, b_6\}$ if $z = b_1$. So, $\mathcal{C}_3$ is a vertex cover of D, $x' \in \mathcal{C}_3$ and $L_3(\mathcal{C}_3) = \{x\}$. Furthermore, $N_D(x') \setminus \mathcal{C}_3 = \{z\}$ and $z \in N_D^-(x')$, then $x' \in \mathcal{C}_3 \setminus L_1(\mathcal{C}_3)$.

Case (2.b) $x = b_1$. By symmetry of P_{14}, we can suppose $z = a_2$. Then, $\mathcal{C}_4 = \{a_1, a_3, a_4, a_5, a_7, b_1, b_2, b_3, b_6, b_7\}$ is a vertex cover of D with $L_3(\mathcal{C}_4) = \{x\}$ and $x' = a_1 \in \mathcal{C}_4$. Also, $N_D^+(x') \subseteq \{b_1, a_7\} \subseteq \mathcal{C}_4$, since $z = a_2$. Hence, $x' \in \mathcal{C}_4 \setminus L_1(\mathcal{C}_4)$.

Case (3) If $D = P_{13}$, then we can assume $x \in \{b_1, c_2, d_1\}$, since $deg_G(x) \geqslant 3$.

Case (3.a) $x = c_2$. Then, $x' \in N_D(c_2) = \{b_3, b_4, c_1\}$. Without loss of generality, we can suppose $x' \in \{b_3, c_1\}$. We take $\mathcal{C}_5 = \{a_1, a_4, b_2, b_3, b_4, c_1, c_2, d_1, v\}$, then $\mathcal{C}_5$ is a vertex cover of D with $L_3(\mathcal{C}_5) = \{x\}$ and $x' \in \mathcal{C}_5$. If $x' = c_1$, then $z \in \{b_1, b_2\}$. By symmetry, we can assume $z = b_1$ so $N_D^+(x') = N_D^+(c_1) \subseteq \{b_2, c_2\} \subseteq \mathcal{C}_5$. Now, if $x' = b_3$, then $b_3 \in V^+ \cap \mathcal{C}_5$. Hence, by Lemma 2.31, $(a_3, b_3) \in E(D)$, since $a_3 \in V_2$ and $I(D)$ is unmixed. Consequently, $N_D^+(x') \subseteq \{c_2, d_1\} \subset \mathcal{C}_5$. Therefore, $x' \in \mathcal{C}_5 \setminus L_1(\mathcal{C}_5)$.

Case (3.b) $x = b_1$. Hence, $\mathcal{C}_6 = \{a_1, a_3, b_1, b_2, b_3, b_4, c_1, d_1, d_2\}$ is a vertex cover of D with $L_3(\mathcal{C}_6) = \{x\}$ and $x' \in N_D(b_1) = \{a_1, c_1, d_1\} \subseteq \mathcal{C}_6$. If $x' \in \{a_1, d_1\}$, then $N_D(x') \setminus \mathcal{C}_6 \subseteq \{a_2, v\} \subseteq V_2$. Consequently, by Lemma 2.31, $(b, x') \in E(D)$ for each $b \in N_D(x') \setminus \mathcal{C}_6$, since $x' \in V^+$. So, $N_D^+(x') \subseteq \mathcal{C}_6$ implies $x' \in \mathcal{C}_6 \setminus L_1(\mathcal{C}_6)$. Now, if $x' = c_1$, then we can assume $(c_2, c_1) \in E(D)$, since in another case we have the case (3.a) with $x = c_2$ and $x' = c_1$. Thus, $N_D^+(x') \subseteq \{b_1, b_2\} \subset \mathcal{C}_6$. Hence, $x' \in \mathcal{C}_6 \setminus L_1(\mathcal{C}_6)$.

Case (3.c) $x = d_1$. So, $x' \in N_G(d_1) = \{b_1, b_3, v\}$. By symmetry, we can assume $x' \in \{b_1, v\}$. Furthermore, $\mathcal{C}_7 = \{a_2, a_4, b_1, b_2, b_3, b_4, c_1, d_1, v\}$ is a vertex cover of D with $L_3(\mathcal{C}_7) = \{x\}$ and $x' \in \mathcal{C}_7$. If $x' = b_1$, then $N_D(x') \setminus \mathcal{C}_7 \subseteq \{a_1\}$. Also, by Lemma 2.31, $(a_1, b_1) \in E(D)$, since $a_1 \in V_2$ and $b_1 = x' \in V^+$. Thus, $N_D^+(x') \subseteq \{c_1, d_1\} \subseteq \mathcal{C}_7$. Now, if $x' = v$, then $z = d_2$, since $deg_D(v) = 2$. Then, $N_D^+(v) = \{d_1\} \subseteq \mathcal{C}_7$. Hence, $x' \in \mathcal{C}_7 \setminus L_1(\mathcal{C}_7)$.

Case (4) $D = Q_{13}$. Hence, $x \in \{d_1, d_2, g_1, g_2, h, h'\}$, since $deg_D(x) \geqslant 3$. By symmetry, we can suppose $x \in \{d_2, g_2, h, h'\}$.

Case (4.a) $x \in \{d_2, g_2\}$. We take $\mathcal{C}_8 = \{a_2, c_1, c_2, d_1, d_2, g_1, g_2, h, h'\}$ if $x = d_2$, or $\mathcal{C}_8 = \{a_1, b_2, c_2, d_1, d_2, g_1, g_2, h, h'\}$ if $x = g_2$. Thus, $\mathcal{C}_8$ is a vertex cover of D with $L_3(\mathcal{C}_8) = \{x\}$ and $V(G) \setminus \mathcal{C}_8 = \{a_1, b_1, b_2, v\} \cup \{a_2, b_1, c_1, v\} \subseteq V_2$. Consequently, by Lemma 2.31, $N_D(x') \setminus \mathcal{C}_8 \subseteq N_D^-(x')$, since $x' \in V^+$. This implies $N_D^+(x') \subseteq \mathcal{C}_8$. Therefore $x' \in \mathcal{C}_8 \setminus L_1(\mathcal{C}_8)$.

Case (4.b) $x \in \{h, h'\}$. We take $\mathcal{C}_9 = \{a_2, b_1, b_2, d_1, d_2, g_1, g_2, h, v\}$ if $x = h$, or $\mathcal{C}_9 = \{a_2, c_1, c_2, d_1, d_2, g_1, g_2, h', v\}$ if $x = h'$. Thus, $\mathcal{C}_9$ is a vertex cover of D with $L_3(\mathcal{C}_9) = \{x\}$. Also, $x' \in N_G(x) \subseteq \{v, d_1, d_2, g_1, g_2\}$. If $x' = v$, then $\{x, z\} \subseteq N_D(x') = \{h, h'\}$. Hence, $N_D^+(x') = \{x\} \subseteq \mathcal{C}_9$, then $x' \in \mathcal{C}_9 \setminus L_1(\mathcal{C}_9)$. Now, if $x' \neq v$, then $x' \in \{d_1, d_2\}$ when $x = h$ or $x' \in \{g_1, g_2\}$ when $x = h'$. Then, $N_D(x') \setminus \mathcal{C}_9 \subseteq \{a_1, c_1, c_2\} \subseteq V_2$ if $x = h$, or $N_D(x') \setminus \mathcal{C}_9 \subseteq \{b_1, b_2\} \subseteq V_2$ if $x = h'$. Consequently, by Lemma 2.31, $N_D(x') \setminus \mathcal{C}_9 \subseteq N_D^-(x')$, since $x' \in V^+$. Therefore, $N_D^+(x') \subseteq \mathcal{C}_9$ and $x' \in \mathcal{C}_9 \setminus L_1(\mathcal{C}_9)$. □

Corollary 2.34 If $G \in \{C_7, T_{10}, Q_{13}, P_{13}, P_{14}\}$. Then $I(D)$ is unmixed if and only if the vertices of V^+ are sinks.

Proof. $\Leftarrow$) By Theorems 1.42 and 1.46, G is well-covered. Thus, by Remark 1.78, $I(G)$ is unmixed. Therefore, by Corollary 2.26, $I(D)$ is unmixed.

$\Rightarrow$) By Proposition 2.33. □

Proposition 2.35 Let H be a 1-simplex of G with $V(H) = \{z, z'\}$ and $H = G[N_G[z]]$, then the following conditions are equivalent:

(1) $|V(H) \cap \mathcal{C}| = 1$ for each strong vertex cover $\mathcal{C}$ of D.
(2) If $z' \in V^+$, then $(z, z') \in E(D)$.

Proof. (2) $\Rightarrow$ (1) If $\{x, z\}, \{x', z'\} \in E(G)$, then $x = z'$, since $N_G(z) = \{z'\}$. So, $\{x, x'\} = \{z', x'\} \in E(G)$. Hence, H has the property **(P)**. Now, we take $(a, b) \in E(D)$ with $a \in V^+$ and $\{b, b'\} = \{z, z'\}$. If $b = z$, then $a = b' = z'$, since $N_D(z) = \{z'\}$. So, $(z', z) = (a, b) \in E(D)$. A contradiction by (2), since $z' = a \in V^+$. Thus, $b = z'$ and $b' = z$. Consequently, $N_D(b') = N_D(z) = \{z'\} = \{b\} \subseteq N_D^+(a)$, since $(a, b) \in E(D)$. Therefore, by Proposition 2.14, H satisfies (1).

(1) $\Rightarrow$ (2) By contradiction, suppose $z' \in V^+$ and $(z', z) \in E(D)$. Hence, by Proposition 2.14 (with $a = b' = z'$ and $b = z$), $N_D(z') \subseteq N_D^+(z')$. A contradiction, by Remark 1.81, since $z' \in V^+$. □

Proposition 2.36 Let G be a graph with a 2-simplex K such that $E(K) \cap E(C) = \emptyset$

for each 4-cycle C of G. Hence, the following conditions are equivalent:

(1) $|V(K) \cap \mathcal{C}| = 2$ for each strong vertex cover $\mathcal{C}$ of D.
(2) $V(K) \not\subseteq V^+$ and if $(a,c) \in E(K)$ with $a \in V^+$ and $N_D^-(a) \not\subseteq V(K)$, then $(b,a) \in E(K)$, $deg_D(b) = 2$ and we have that $w(c) = 1$ or $(b,c) \in E(K)$, where $V(K) = \{a,b,c\}$.

Proof. (1) $\Rightarrow$ (2) First, we suppose $(a,c) \in E(K)$ with $a \in V^+$ and $N_D^-(a) \not\subseteq V(K)$, then there is $w \in N_D^-(a) \setminus V(K)$.

If $(a,b) \in E(K)$, then H_1 is a generating $\star$-semi-forest of K where $E(H_1) = \{(a,b), (a,c)\}$, $V^{H_1} = \{a\}$ and $W^{H_1} = W_1^{H_1} = \{w\}$, since $\tilde{H}_1 = \{a\}$ and $N_D^+(a) \cap W_1^{H_1} = \emptyset$. A contradiction by (1) and Corollary 2.12. Hence, $(b,a) \in E(D)$.

Now, assume $deg_D(b) \neq 2$, then there is $w' \in N_D(b) \setminus V(K)$. If $\{w,w'\} \in E(G)$ or $w' \in N_D^+(a)$, then $C^1 = (w,w',b,a,w)$ or $C^1 = (a,w',b,c,a)$ is a 4-cycle. So, $\{a,b\} \in E(K) \cap E(C^1)$ or $\{b,c\} \in E(K) \cap E(C^1)$. A contradiction, by hypothesis, then $\{w,w'\} \notin E(G)$ and $w' \notin N_D^+(a)$. Thus, $H_2 = T_1 \cup T_2$ is a generating $\star$-semi-forest of K where $E(T_1) = \{(a,c)\}$, $V(T_2) = \{b\}$, $V^{H_2} = \{a,b\}$ and $W^{H_2} = W_1^{H_2} = \{w,w'\}$, since $W_1^{H_2}$ is a stable set, $\tilde{H}_2 = \{a\}$ and $N_D^+(\tilde{H}_2) \cap W_1^{H_2} = \emptyset$. A contradiction by (1) and Corollary 2.12. Hence, $deg_D(b) = 2$.

On the other hand, if $w \in N_D^+(c)$, then $C^2 = (a,w,c,b,a)$ is a 4-cycle and $\{a,b\} \in E(K) \cap E(C^2)$. A contradiction, then $w \notin N_D^+(c)$. Hence, if $c \in V^+$ and $(c,b) \in E(K)$, then H_3 is a generating $\star$-semi-forest of K where $E(H_3) = \{(a,c),(c,b)\}$, $V^{H_3} = \{a\}$ and $W^{H_3} = W_1^{H_3} = \{w\}$, since $\tilde{H}_3 = \{a,c\}$ and $N_D^+(\tilde{H}_3) \cap W_1^{H_3} = \emptyset$. A contradiction, then $w(c) = 1$ or $(b,c) \in E(K)$.

Now, assume $V(K) \subseteq V^+$. We can suppose $N_D[c] = V(K) = \{a,b,c\}$. By Remark 1.81, we can assume, $(a,c) \in E(D)$, since $c \in V(K) \subseteq V^+$. If $N_D^-(a) \not\subseteq V(K)$, then by the first argument $(b,a) \in E(D)$, $deg_D(b) = 2$ and $(b,c) \in E(D)$, since $c \in V^+$. A contradiction, by Remark 1.81, since $b \in V^+$. This implies, $N_D^-(a) \subseteq V(K)$. Hence, $(b,a) \in E(D)$, since $a \in V^+$. Consequently, by the last argument, $(c,b) \in E(D)$, since $(b,a) \in E(D)$ and $V(K) \subseteq V^+$. Therefore, $H_4 = K$ is a generating $\star$-semi-forest of K with $V^{H_4} = W^{H_4} = \emptyset$. A contradiction by Corollary 2.12.

(2) $\Rightarrow$ (1) By contradiction, using Corollary 2.12, we suppose there is a generating $\star$-semi-forest H of K. We set $V(K) = \{z_1,z_2,z_3\}$ such that $N_D[z_3] = V(K)$. Then, $z_3 \notin V^H$, since $N_D(z_3) \subseteq V(K)$. So, there is $z \in V^+$ such that $(z,z_3) \in E(H)$. We can assume $z = z_2$.

<u>Case (1)</u> There is $w \in N_D^-(z_2) \setminus V(K)$. Then, by (2), $(z_1,z_2) \in E(K)$, $deg_D(z_1) = 2$ and we have that $w(z_3) = 1$ or $(z_1,z_3) \in E(D)$. Thus, $z_1 \notin V^H$, since $N_D(z_1) \subseteq$

$V(K)$. Consequently, there is $z' \in N_D^-(z_1) \cap V^+ \cap V(K)$. Hence, $z' = z_3$, since $(z_1, z_2) \in E(D)$. But $w(z_3) = 1$ or $(z_1, z_3) \in E(D)$. A contradiction.

Case (2) $N_D^-(z_2) \subseteq V(K)$. This implies by Remark 1.81, $(z_1, z_2) \in E(D)$, since $z_2 \in V^+$. Thus, by Definition 2.4, $z_2 \notin V^H$, since $deg_H(z_2) \geqslant 1$ and $N_D^-(z_2) \subseteq V(K) = V(H)$. So, there is $z'' \in N_H^-(z_2) \cap V^+$. Then, $z'' = z_1$ and $w(z_3) = 1$, since by (2), $V(K) \not\subseteq V^+$. Hence, H is a ROT with $V^H = \{z_1\}$. Consequently, there is $w_1 \in N_D^-(z_1) \setminus V(K)$. Hence, by (2) (with $a = z_1$, $b = z_3$ and $c = z_2$), $w(z_2) = 1$ or $(z_3, z_2) \in E(D)$. A contradiction. □

Definition 2.37 If K is a 1-simplex (resp. 2-simplex) such that K satisfies (2) of Proposition 2.35 (resp. (2) of Proposition 2.36), then we say K has the **$\star$-property**.

Theorem 2.38 Let D be a connected weighted oriented graph without 4- and 5-cycles. Hence, $I(D)$ is unmixed if and only if D satisfies one of the following conditions:

(a) $G \in \{K_1, C_7, T_{10}\}$ and the vertices of V^+ are sinks.
(b) 1-simplexes and 2-simplexes have the $\star$-property and $\{V(H) \mid H \text{ is a 1-simplex or a 2-simplex}\}$ is a partition of $V(G)$.

Proof. $\Rightarrow$) By (3) in Theorem 1.85 and Remark 1.78, G is well-covered. Thus, by Theorem 1.42, $G \in \{C_7, T_{10}\}$ or $\{V(H) \mid H \in S_G\}$ is a partition of $V(G)$. If $G \in \{C_7, T_{10}\}$, then by Proposition 2.33, D satisfies (a). Now, if $\{V(H) \mid H \in S_G\}$ is a partition of $V(G)$, then G is an SCQ graph with $C_G = Q_G = \emptyset$. If S_G has a 0-simplex, then $G = K_1$, since G is connected. In this case, $V^+ = \emptyset$ and D satisfies (a). Hence, we can assume that if $H \in S_G$, then H is a 1-simplex or a 2-simplex, since G has no 4-cycles. Also, by Theorem 2.23 and Corollary 2.12, if $H \in S_G$, then $|\mathcal{C} \cap V(H)| = |V(H)| - 1$ for each strong vertex cover $\mathcal{C}$. Therefore, by Propositions 2.35 and 2.36, D satisfies (b), since G has no 4-cycles.

$\Leftarrow$) If D satisfies (a), then by Theorem 1.42, G is well-covered, since K_1 is well-covered. Consequently, by Corollary 2.26, $I(D)$ is unmixed. Now, we can suppose $G \neq K_1$, then G has no 0-simplexes, since G is connected. Hence, $S_G = \{V(H) \mid H \text{ is a 1-simplex or a 2-simplex of } D\}$, since G has no 4-cycles. Now, assume D satisfies (b), then G is an SCQ graph with $C_G = Q_G = \emptyset$. Furthermore, by Propositions 2.35 and 2.36 and Corollary 2.12, each simplex of D has no generating $\star$-semi-forests. Therefore, by Theorem 2.23, $I(D)$ is unmixed. □

Remark 2.39 Let H be a basic 5-cycle or a 1-simplex. By Propositions 2.21 and 2.35, H has the $\star$-property if and only if $|\mathcal{C} \cap V(H)| = \tau(H)$ for each strong vertex cover $\mathcal{C}$. By Proposition 2.36, we have the same result if H is a 2-simplex such that $E(H) \cap E(C) = \emptyset$ for each 4-cycle C of G.

Corollary 2.40 Let D be a connected weighted oriented graph with $girth(G) \geqslant 6$. Hence, $I(D)$ is unmixed if and only if D satisfies one of following properties:

(a) $G \in \{K_1, C_7\}$ and the vertices of V^+ are sinks.
(b) G has a perfect matching $y_1 = \{x_1, x_1'\}, \ldots, y_r = \{x_r, x_r'\}$ with $deg_D(x_1) = \cdots = deg_D(x_r) = 1$, furthermore, $(x_j, x_j') \in E(D)$ when $x_j' \in V^+$.

Proof. $\Leftarrow$) If D satisfies (a), then $I(D)$ is unmixed by (a) in Theorem 2.38. Now, assume D satisfies (b). G has no 2-simplexes, since G has no 3-cycles. We take $f = \{z, z'\}$ a 1-simplex of G. We can suppose $N_D[z] = \{z, z'\}$. Since, $y_1, \ldots, y_r$ is a perfect matching, there is some y_i such that $z \in y_i$. Then, $y_i = \{x_i, x_i'\} \subseteq N_D[z] = \{z, z'\} = f$. Thus, $y_1, \ldots, y_r$ are the 1-simplexes of D. Also, by (b), $y_1, \ldots, y_r$ has the $\star$-property. Hence, by Theorem 2.38, $I(D)$ is unmixed.

$\Rightarrow$) G has no 2-simplexes and $G \neq T_{10}$, since G has no 3-cycles. Hence, by Theorem 2.38, D satisfies (a) or $B := \{V(H) \mid H \text{ is a 1-simplex}\}$ is a partition of $V(G)$ and the 1-simplexes have the $\star$-property. We can suppose $B = \{y_1, \ldots, y_r\}$ where $y_i = \{x_i, x_i'\}$ and $deg_D(x_i) = 1$. Therefore, D satisfies (b), since each y_i has the $\star$-property. $\square$

Corollary 2.41 Let D be a weighted oriented graph with $girth(G) \geqslant 6$. Hence, $I(D)$ is unmixed if and only if $I(D)$ is Cohen-Macaulay.

Proof. $\Leftarrow$) By Proposition 1.88.

$\Rightarrow$) By (b) in Corollary 2.40, G is König. Furthermore, G has no 4-cycles, since $girth(G) \geqslant 6$. Hence, by Corollary 1.90, $I(D)$ is Cohen-Macaulay. $\square$

Proposition 2.42 If $G = P_{10}$, then the following properties are equivalent:

(1) $I(D)$ is unmixed.
(2) If $x' \in V^+$ and x' is not a sink, then $x' = d_1$ with $N_D^+(x') = \{g_1, b_2\}$ or $x' = d_2$ with $N_D^+(x') = \{g_2, b_1\}$.

Proof. (2) $\Rightarrow$ (1) Let $\mathcal{C}$ be a strong vertex cover of D. Suppose $x \in L_3(\mathcal{C})$. So, there is $x' \in \big(\mathcal{C} \setminus L_1(\mathcal{C})\big) \cap V^+$ such that $x \in N_D^+(x')$. Thus, by (2), $x' \in \{d_1, d_2\}$ and $x \in N_D^+(x') \subseteq \{b_1, b_2, g_1, g_2\}$. Hence, $L_3(\mathcal{C}) \subseteq \{b_1, b_2, g_1, g_2\}$. By symmetry of P_{10}, we can assume $x' = d_1$. Then by (2), $x \in N_D^+(x') = \{g_1, b_2\}$. Also, $\{g_1, d_1, b_2\} =$

$N_D^+[x'] \subseteq \mathcal{C}$, since $x' \in \mathcal{C} \setminus L_1(\mathcal{C})$. But $x' = d_1 \notin L_3(\mathcal{C}) \subseteq \{b_1, b_2, g_1, g_2\}$, then $N_D(x') \not\subseteq \mathcal{C}$. Thus, $d_2 \notin \mathcal{C}$. So, by Remark 1.21, $\{b_1, d_1, g_2\} = N_D(d_2) \subseteq \mathcal{C}$. Furthermore, $\{a_1, a_2\} \cap L_3(\mathcal{C}) = \emptyset$, since $L_3(\mathcal{C}) \subseteq \{b_1, b_2, g_1, g_2\}$. Consequently, $\{a_1, a_2\} \cap \mathcal{C} = \emptyset$, since $N_D(a_1, a_2) = \{g_1, b_1, g_2, b_2\} \subseteq \mathcal{C}$. But, $x \in \{g_1, b_2\} \cap L_3(\mathcal{C})$, then $a_1 \in N_D(g_1) \subseteq \mathcal{C}$ or $a_2 \in N_D(b_2) \subseteq \mathcal{C}$. A contradiction, then $L_3(\mathcal{C}) = \emptyset$. Also, by Theorem 1.46 and Remark 1.78, $I(G)$ is unmixed. Therefore, by (3) in Theorem 1.85, $I(D)$ is unmixed.

(1) $\Rightarrow$ (2) We take $V_2 := \{a \in V(G) \mid deg_G(a) = 2\}$ and $x' \in V^+$, such that x' is not a sink, then there is $(x', x) \in E(D)$. Also, by Remark 1.81, there is $z \in N_D^-(x')$.

We will prove $x' \in \{d_1, d_2\}$. If $deg_D(x) = 2$, then by Lemma 2.31, $x' \in \{d_1, d_2\}$. Now, we assume $deg_D(x) \geqslant 3$, then by symmetry of P_{10}, we can suppose $x \in \{g_1, d_1\}$.

First suppose $x = g_1$. Thus, $\mathcal{C}_1 = \{a_1, a_2, c_1, d_1, d_2, g_1, g_2\}$ is a vertex cover of D and $L_3(\mathcal{C}_1) = \{x\}$. Furthermore, $x' \in N_D(g_1) = \{a_1, c_1, d_1\}$. If $x' \in \{a_1, c_1\}$, then $x' \in \mathcal{C}_1 \cap V_2$. So, $N_D(x') = \{x, z\}$ implies $N_D^+(x') = \{x\} = \{g_1\} \subseteq \mathcal{C}_1$. Consequently, $x' \in \mathcal{C}_1 \setminus L_1(\mathcal{C}_1)$. Hence, $\mathcal{C}_1$ is strong, since $L_3(\mathcal{C}_1) = \{x\}$ and $x' \in V^+$. A contradiction, by Theorem 1.85, then $x' = d_1$.

Now, suppose $x = d_1$. Then, $\mathcal{C}_2 = \{a_1, b_2, c_2, d_1, d_2, g_1, g_2\}$ is a vertex cover of D with $L_3(\mathcal{C}_2) = \{x\}$. Also, $x' \in N_D(x) = \{g_1, b_2, d_2\}$. If $x' \in \{g_1, b_2\}$, then $N_D(x') \setminus \{d_1\} \subseteq N_D(g_1, b_2) \setminus \{d_1\} = \{a_1, a_2, c_1\} \subseteq V_2$. So, by Lemma 2.31, $N_D(x') \setminus \{d_1\} \subseteq N_D^-(x')$, since $x' \notin \{d_1, d_2\}$. Thus, $N_D^+(x') = \{d_1\} = \{x\} \subseteq \mathcal{C}_2$ implies $x' \in \mathcal{C}_2 \setminus L_1(\mathcal{C}_2)$. Consequently, $\mathcal{C}_2$ is strong with $L_3(\mathcal{C}_2) \neq \emptyset$. A contradiction, by Theorem 1.85, then $x' = d_2$.

Now, we will prove $N_D^+(x') = \{g_i, b_{i'}\}$ if $x' = d_i$ and $\{i, i'\} = \{1, 2\}$. By symmetry of P_{10}, we can assume $x' = d_1$. By contradiction, suppose $N_D^+(x') \neq \{g_1, b_2\}$. Hence, $g_1 \notin N_D^+(d_1)$ or $b_2 \notin N_D^+(d_1)$, since $x' = d_1$ and $z \in N_D^-(x')$.

Case (1) $g_1 \notin N_D^+(d_1)$ and $b_2 \in N_D^+(d_1)$. So, $N_D^+(d_1) \subseteq \{b_2, d_2\}$ and $\mathcal{C}_1' = \{a_1, a_2, b_2, c_1, c_2, d_1, d_2\}$ is a vertex cover of D with $L_3(\mathcal{C}_1') = \{b_2\}$. Also, $(d_1, b_2) \in E(D)$ and $x' = d_1 \in \big(\mathcal{C}_1' \setminus L_1(\mathcal{C}_1')\big) \cap V^+$, since $N_D^+[d_1] \subseteq \{d_1, b_2, d_2\} \subset \mathcal{C}_1'$. Hence, $\mathcal{C}_1'$ is a strong vertex cover of D with $L_3(\mathcal{C}_1') \neq \emptyset$. A contradiction, by Theorem 1.85.

Case (2) $b_2 \notin N_D^+(d_1)$ and $g_1 \in N_D^+(d_1)$. Then, $N_D^+(d_1) \subseteq \{g_1, d_2\}$ and $\mathcal{C}_2' = \{a_1, a_2, c_1, d_1, d_2, g_1, g_2\}$ is a vertex cover of D with $L_3(\mathcal{C}_2') = \{g_1\}$. Furthermore $(d_1, g_1) \in E(D)$ and $d_1 = x' \in \big(\mathcal{C}_2' \setminus L_1(\mathcal{C}_2')\big) \cap V^+$, since $N_D^+[d_1] \subseteq \{d_1, g_1, d_2\} \subset$

$\mathcal{C}_2'$. Consequently, $\mathcal{C}_2'$ is a strong vertex cover of D. A contradiction, since $L_3(\mathcal{C}_2') \neq \emptyset$.

Case (3) $b_2, g_1 \notin N_D^+(d_1)$. Thus, $x = d_2$, $N_D^+(d_1) = \{d_2\}$ and $\mathcal{C}_3' = \{a_2, b_1, c_1, d_1, d_2, g_1, g_2\}$ is a vertex cover of D with $L_3(\mathcal{C}_3') = \{d_2\}$. Also, $(d_1, d_2) \in E(D)$ and $d_1 = x' \in \big(\mathcal{C}_3' \setminus L_1(\mathcal{C}_3')\big) \cap V^+$, since $N_D^+[d_1] = \{d_1, d_2\} \subset \mathcal{C}_3'$. Hence, $\mathcal{C}_3'$ is a strong vertex cover of D. A contradiction. □

Theorem 2.43 Let D be a connected weighted oriented graph with girth$(G) \geqslant 5$. Hence, $I(D)$ is unmixed if and only if D satisfies one of the following properties:

(a) $G \in \{K_1, C_7, Q_{13}, P_{13}, P_{14}\}$ and the vertices of V^+ are sinks.
(b) $G = P_{10}$, furthermore if x is not a sink in V^+, then $x = d_1$ with $N_D^+(x) = \{g_1, b_2\}$ or $x = d_2$ with $N_D^+(x) = \{g_2, b_1\}$.
(c) $\{V(H) \mid H$ is a 1-simplex or a basic 5-cycle$\}$ is a partition of $V(G)$, furthermore the 1-simplexes and the basic 5-cycles of G have the $\star$-property.

Proof. If $G \neq K_1$, then G has no 0-simplexes, since G is connected. Hence, $S_G = S_G^1 := \{H \mid H$ is a 1-simplex$\}$, since G has no 3-cycles.

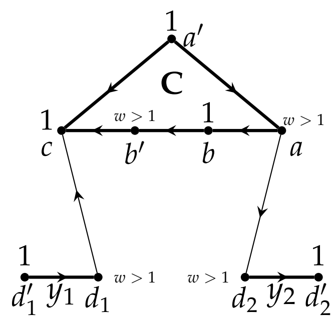
$\Rightarrow$) By (3) in Theorem 1.85 and Remark 1.78, G is well-covered. So, by Theorem 1.46, $G \in \{K_1, C_7, P_{10}, P_{13}, P_{14}, Q_{13}\}$ or $\{V(H) \mid H \in S_G \cup C_G\}$ is a partition of $V(G)$. If $G \in \{C_7, Q_{13}, P_{13}, P_{14}\}$ or $G = P_{10}$, then by Proposition 2.33 and Proposition 2.42, D satisfies (a) or (b). Now, if $G = K_1$, then by Remark 1.81, $V^+ = \emptyset$. Finally, if $G \neq K_1$ and $\{V(H) \mid H \in S_G \cup C_G\}$ is a partition of $V(G)$, then G is an SCQ graph with $Q_G = \emptyset$ and $S_G = S_G^1$. Thus, $\{V(H) \mid H \in S_G^1 \cup C_G\}$ is a partition of $V(G)$. By Theorem 2.23, Corollary 2.12 and Proposition 2.35, D satisfies (c).

$\Leftarrow$) If D satisfies (b), then by Proposition 2.42, $I(D)$ is unmixed. Now, if D satisfies (a), then by Theorem 1.46, G is well-covered. So, by Corollary 2.26, $I(D)$ is unmixed. Finally, if D satisfies (c), then G is an SCQ-graph, $Q_G = \emptyset$ and $G \neq K_1$. Thus, $S_G = S_G^1$. Hence, by Theorem 2.23, Corollary 2.12 and Proposition 2.35, $I(D)$ is unmixed. □

Remark 2.44 A graph is well-covered if and only if each connected component is well-covered. Hence, when D is no connected in Theorem 2.38 and Corollary 2.40 (resp. Theorem 2.43), $I(D)$ is unmixed if and only if each connected component of D satisfies (a) or (b) (resp. (a), (b) or (c)).

2.4 EXAMPLES

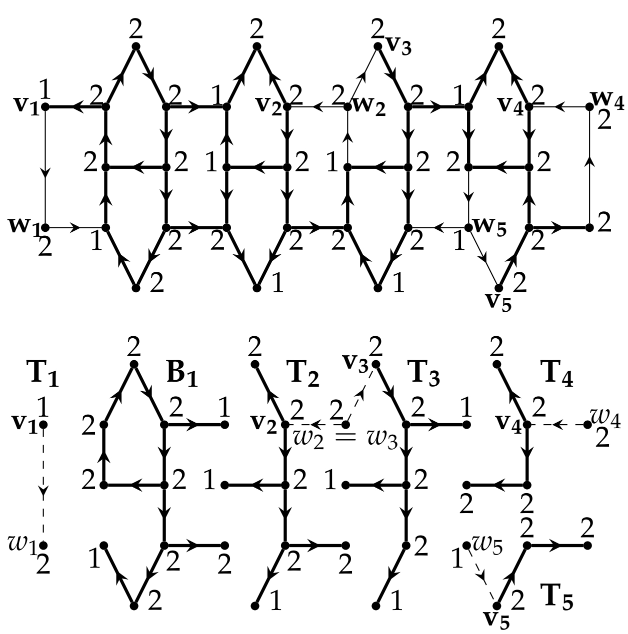
Example 2.45 Let $D = (G, \mathcal{O}, w)$ be the weighted oriented graph of Figure 2.1.



- G has no 3- and 4-cycles. Moreover, $girth(G) = 5$.
- G is an SCQ-graph with $S_G = \{y_1, y_2\}$, $C_G = \{C\}$ and $Q_G = \emptyset$, where $y_1 = \{d_1, d_1'\}$, $y_2 = \{d_2, d_2'\}$ and $C = (a', a, b, b', c, a')$.
- C and y_1 has the $\star$-property. But y_2 has no the $\star$-property. Hence, by Theorem 2.43, $I(D)$ is mixed.

Figure 2.1 Mixed weighted oriented graph without 3- and 4-cycles

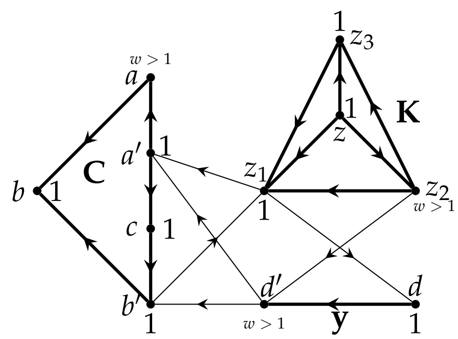
Example 2.46 Let D be the weighted oriented graph of Figure 2.2.



We take the weighted oriented subgraph K of D induced by $V(D) \setminus \{w_1, w_2, w_4, w_5\}$. In the figure: T_i is a ROT with root v_i for $i = 1, \ldots, 5$; and B_1 is an unicycle oriented subgraph. Furthermore, $H = \cup_{i=1}^{5} T_i \cup B_1$ is a $\star$-semi-forest with $W^H = \{w_1, w_2, w_4, w_5\}$ (where $w_3 = w_2$), $W_1^H = \{w_1, w_5\}$ and $W_2^H = \{w_2, w_4\}$. Finally, since $V(H) = V(K)$, H is a generating $\star$-semi-forest of K.

Figure 2.2. A generating $\star$-semi-forest of the graph $K = G[V(D) \setminus \{w_1, w_2, w_4, w_5\}]$

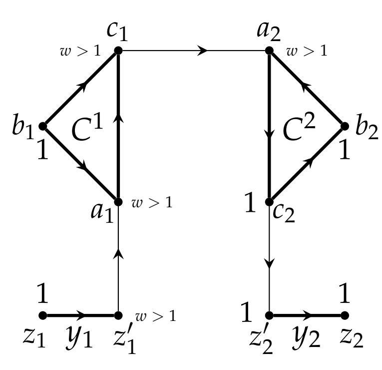
Example 2.47 Let $D = (G, \mathcal{O}, w)$ be the weighted oriented graph of Figure 2.3.



- G is an SCQ-graph, with $S_G = \{K\}$, $C_G = \{C\}$ and $Q_G = \{y\}$, where $K = D[N_D[z]]$, $C = (b, a, a', c, b', b)$ and $y = \{d, d'\}$.
- If H is a generating $\star$-semi-forest of K, then by Lemma 2.7, $z \in N_D(W_1^H)$, since z is a source and $V(H) = V(K)$. But $N_D(z) \subseteq V(K)$ and $W_1^H \cap V(H) = \emptyset$. Thus, K has not a generating $\star$-semi-forest.
- C has the $\star$-property.
- Furthermore, $V^+ \cap N_D^-(d, d') = \{z_2\}$, $N_D^+(z_2) \cap y = \{d'\}$ and $N_D(d) = \{d', z_1\} \subseteq N_D^+(z_2)$.
- Hence, by Theorem 2.23, $I(D)$ is unmixed.

Figure 2.3 Unmixed weighted oriented SCQ-graph

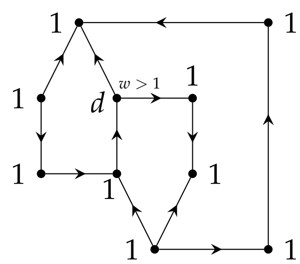
Example 2.48 Let $D = (G, \mathcal{O}, w)$ be the weighted oriented graph of Figure 2.4.



- G has no 4- and 5-cycles.
- G is an SCQ-graph with $S_G = \{C^1, C^2, y_1, y_2\}$ and $C_G = Q_G = \emptyset$, where $y_1 = \{z_1, z_1'\}$, $y_2 = \{z_2, z_2'\}$, $C^1 = (a_1, b_1, c_1, a_1)$ and $C^2 = (a_2, b_2, c_2, a_2)$.
- C^1, C^2, y_1 and y_2 have the $\star$-property.
- Hence, by Theorem 2.38, $I(D)$ is unmixed.

Figure 2.4 Unmixed weighted oriented graph without 4- and 5-cycles

Example 2.49 Let $D = (G, \mathcal{O}, w)$ be the weighted oriented graph of Figure 2.5.



- $G = P_{10}$, then by Theorem 1.46 and Remark 1.78, G is well-covered and $I(G)$ is unmixed.
- $V^+ = \{d\}$ and d is not a sink. But, by Proposition 2.42, $I(D)$ is unmixed.

Figure 2.5 Unmixed weighted oriented graph where $G = P_{10}$

Chapter 3

Gorenstein Homogeneous subrings of graphs

In this chapter, we study the homogeneous monomial subring S of a graph G (see Definition 1.92). In particular, we study when S is Gorenstein if S is normal.

3.1 Polyhedral cone of subrings of graphs

In this section we study the polyhedral cone $\mathbb{R}_+B$ (see Definition 1.99). In particular, we characterize: the elements of $(\mathbb{R}_+B)^\circ$ (Proposition 3.2 and Lemma 3.5) and some support hyperplanes (Lemma 3.8).

Remark 3.1 By Theorem 1.1.29 and Proposition 1.1.51 in [46], $\mathbb{R}_+B$ has the unique irreducible representation $\mathbb{R}_+B = H^+_{\lambda_1} \cap \cdots \cap H^+_{\lambda_{q_1}}$ where $H^+_{\lambda_i} = \{w \in \mathbb{R}^{n+1} \mid w \cdot \lambda_i \geq 0\}$. Also, by Theorem 1.1.44 in [46], if $F_i = H_{\lambda_i} \cap \mathbb{R}_+B$ and $H_{\lambda_i} = \{w \in \mathbb{R}^{n+1} \mid w \cdot \lambda_i = 0\}$ for $1 \leq i \leq q_1$, then $F_1, \ldots, F_{q_1}$ are the facets of $\mathbb{R}_+B$.

Proposition 3.2 There exist $\ell_1, \ldots, \ell_q \in \mathbb{R}^n$ such that $\mathbb{R}_+B = H^+_{(e_1,0)} \cap \cdots \cap H^+_{(e_n,0)} \cap H^+_{(-\ell_1,1)} \cap \cdots \cap H^+_{(-\ell_q,1)}$. Also, $w = (\tilde{w}, a) \in (\mathbb{R}_+B)^\circ$ if and only if $\tilde{w} \cdot e_i > 0$ for $1 \leq i \leq n$ and $w \cdot (-\ell_j, 1) > 0$ for $1 \leq j \leq q$.

Proof. We have $\mathbb{R}_+B = H^+_{\lambda_1} \cap \cdots \cap H^+_{\lambda_{q_1}}$. We will prove $\lambda_j = (\tilde{\lambda}_j, 0) \in \mathbb{R}^n \times \mathbb{R}$ if and only if $H_{\lambda_j} \in \{H_{(e_1,0)}, \ldots, H_{(e_n,0)}\}$. Assume $\lambda_j = (\tilde{\lambda}_j, 0)$, then $\tilde{\lambda}_j \cdot e_i \geq 0$, since $(e_i, 1) \in \mathbb{R}_+B \subseteq H^+_{\lambda_j}$ for $1 \leq i \leq n$. We take $I = \{i \mid \tilde{\lambda}_j \cdot e_i = 0\}$, then $(e_i, 1) \in H_{(\tilde{\lambda}_j,0)}$ if and only if $i \in I$. Furthermore, $(v_k, 1) \in H_{(\tilde{\lambda}_j,0)}$ if and only if $y_k = \{x_{i_1}, x_{i_2}\}$ with $i_1, i_2 \in I$, since $\tilde{\lambda}_j \cdot e_i \geq 0$. Thus, dim $F_j = |I| + 1$ where $F_j = H_{\lambda_j} \cap \mathbb{R}_+B$, since $e_{n+1} \in H_{\lambda_j}$. But F_j is a facet, then $|I| = n - 1$. Hence, $H_{(\tilde{\lambda}_j,0)} \in \{H_{(e_1,0)}, \ldots, H_{(e_n,0)}\}$. Now, we prove $H_{(e_k,0)} \in \{H_{\lambda_1}, \ldots, H_{\lambda_{q_1}}\}$. Since $(e_k, 0) \cdot e_{n+1} = 0$ and $(e_k, 0) \cdot (e_i, 1) = 0$ for $i \neq k$, we have $e_{n+1}, (e_i, 0) \in A_k :=$

$\mathbb{R}_+B \cap H_{(e_k,0)}$ for $i \neq k$. Also, $(e_k,0)\cdot(e_k,1) = 1$ and $(e_k,0)\cdot(v_i,1) \geq 0$, then $\mathbb{R}_+B \subseteq H^+_{(e_k,0)}$. Hence, A_k is a facet of $\mathbb{R}_+B$, so $H_{(e_k,0)} \in \{H_{\lambda_1},\dots,H_{\lambda_{q_1}}\}$.
Now, we take $\lambda_j = (\tilde{\lambda}_j, a_j) \in \mathbb{R}^n \times \mathbb{R}$ with $a_j \neq 0$. Since $e_{n+1} \in \mathbb{R}_+B \subseteq H^+_{\lambda_j}$, $a_j = e_{n+1}\cdot\lambda_j > 0$. Hence, $H_{(\tilde{\lambda}_j,a_j)} = H_{(\delta_j,1)}$ where $\delta_j = \frac{\tilde{\lambda}_j}{a_j}$.
Therefore, $\mathbb{R}_+B = H^+_{(e_1,0)} \cap \cdots \cap H^+_{(e_n,0)} \cap H^+_{(-\ell_1,1)} \cap \cdots \cap H_{(-\ell_q,1)}$.
Now, by Theorem 1.1.44 in [46], $w = (\tilde{w},a) \in (\mathbb{R}_+B)^\circ$ if and only if $w\cdot(-\ell_j,1) > 0$ for $1 \leq j \leq q$ and $\tilde{w}\cdot e_i = (\tilde{w},a)\cdot(e_i,0) > 0$ for $1 \leq i \leq n$. □

Notation 3.3 In this section we take $|a| = a\cdot\mathbf{1} = \sum_{i=1}^n a_i$, where $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^n$ and $a = (a_1,\dots,a_n) \in \mathbb{R}^n$. Furthermore, $\mathbf{0} = (0,\dots,0) \in \mathbb{R}^n$ and if C is a cycle, then $\mathbf{1}_C = \sum_{x_i \in V(C)} e_i$.

Lemma 3.4 Let $w = (\tilde{w},b)$ be a vector in $\mathbb{N}B$ with $\tilde{w} \in \mathbb{N}^n$ and $b \in \mathbb{N}$. Hence,

1) $|\tilde{w}| \leq 2b$.
2) If $|\tilde{w}| = 2b$, then $w \in \mathbb{N}\big((v_1,1),\dots,(v_m,1)\big)$.

Proof. Since $(\tilde{w},b) \in \mathbb{N}B$, $(\tilde{w},b) = \sum_{i=1}^m \alpha_i(v_i,1) + \sum_{i=1}^n \beta_i(e_i,1) + \lambda e_{n+1}$, where $\alpha_i,\beta_i,\lambda \in \mathbb{N}$. Thus, $b = \sum_{i=1}^m \alpha_i + \sum_{i=1}^n \beta_i + \lambda$. Also, $|\tilde{w}| = \tilde{w}\cdot\mathbf{1} = \sum_{i=1}^m \alpha_i(v_i\cdot\mathbf{1}) + \sum_{i=1}^n \beta_i(e_i\cdot\mathbf{1}) = 2\big(\sum_{i=1}^m \alpha_i\big) + \sum_{i=1}^n \beta_i$. Hence, $2b = |\tilde{w}| + \sum_{i=1}^n \beta_i + 2\lambda \geq |\tilde{w}|$. Furthermore, if $|\tilde{w}| = 2b$, then $\sum_{i=1}^n \beta_i = 0$ and $\lambda = 0$. Consequently, $\beta_i = 0$ for $1 \leq i \leq n$. Therefore, $w \in \mathbb{N}\big((v_1,1),\dots,(v_m,1)\big)$. □

Lemma 3.5 If $w = (\tilde{w},b) = \sum_{i=1}^m \alpha_i(v_i,1) + \sum_{i=1}^n \beta_i(e_i,1) + \lambda e_{n+1}$ with $\alpha_i,\beta_i \in \mathbb{R}_+$, $\lambda > 0$ and $\tilde{w}\cdot e_j > 0$ for each $1 \leq j \leq n$, then $w \in (\mathbb{R}_+B)^\circ$.

Proof. We have $(v_i,1),(e_j,1) \in \mathbb{R}_+B \subseteq H^+_{(-\ell_k,1)}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, then

$$\Big(\sum_{i=1}^m \alpha_i(v_i,1) + \sum_{i=1}^n \beta_i(e_i,1)\Big)\cdot(-\ell_k,1) \geq 0.$$

Furthermore, $\lambda e_{n+1}\cdot(-\ell_k,1) = \lambda > 0$. Hence, $w\cdot(-\ell_k,1) > 0$ for $1 \leq k \leq q$. Therefore, by Proposition 3.2, $w \in (\mathbb{R}_+B)^\circ$ since $\tilde{w}\cdot e_i > 0$ for $1 \leq i \leq n$. □

Remark 3.6 If τ is a spanning tree of G, then $|E(\tau)| = n-1$.

Lemma 3.7 If τ is a spanning tree of G, $e \in E(G)$ and $\tau \cup \{e\}$ has an odd cycle C, then the characteristic vectors of the edges of $E(\tau)\cup\{e\}$ are linearly independent.

Proof. We can assume $E(C) = \{y_1, \ldots, y_k\}$ with $y_i = \{x_i, x_{i+1}\}$ for $1 \leq i \leq k-1$ and $y_k = \{x_k, x_1\}$. Also, we can suppose $E(\tau) \cup \{e\} = \{y_1, \ldots, y_n\}$, since $|E(\tau)| = n-1$. We will do the proof by induction on $n-k$. If $n-k=0$, then $E(\tau) \cup \{e\} = E(C)$. Thus, $\sum_{i=1}^{k} (-1)^{i+1} v_i = (e_1+e_2) - (e_2+e_3) + \cdots + (e_k+e_1) = 2e_1$. So, $e_1 \in \mathbb{R}(v_1, \ldots, v_k)$. Similarly, $e_i \in \mathbb{R}(v_1, \ldots, v_k)$ for $1 \leq i \leq k$. Hence, $v_1, \ldots, v_k$ are linearly independent. Now, assume $n-k>0$. Then, there is $x \in V(G)$ such that $deg_{\tau \cup \{e\}}(x) = 1$. We can suppose $x = x_n \in y_n$ and $x_n \notin y_j$ for $1 \leq j \leq n-1$. Thus, $C \subseteq (\tau \setminus \{x_n\}) \cup \{e\} \subseteq G' := G \setminus x_n$ and $\tau \setminus \{x_n\}$ is a spanning tree of G'. Hence, by induction hypothesis, $v_1, \ldots, v_{n-1}$ are linearly independent, since $E(\tau \setminus \{x_n\}) = E(\tau) \setminus \{y_n\}$. Therefore, $v_1, \ldots, v_n$ are linearly independent, since $x_n \in y_n$ and $x_n \notin y_j$ for $1 \leq j \leq n-1$. □

In the following results $\ell_1, \ldots, \ell_q$ are as in Proposition 3.2.

Lemma 3.8 If G is not bipartite, then $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) = \frac{1}{2}(\mathbf{1}) \in \{\ell_1, \ldots, \ell_q\}$.

Proof. We take $\ell := \frac{1}{2}(\mathbf{1}) \in \mathbb{R}^n$, then $(e_i, 1) \cdot (-\ell, 1) = \frac{1}{2}$, $(v_j, 1) \cdot (-\ell, 1) = 0$ and $e_{n+1} \cdot (-\ell, 1) = 1$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Hence, $\mathbb{R}_+ B \subseteq H^+_{(-\ell,1)}$ and $\{(v_1, 1), \ldots, (v_m, 1)\} \subseteq H_{(-\ell,1)}$. Now, since G is not bipartite, there is an odd cycle C of G. We take $e \in E(C)$, then $C - e$ is a path and there is a spanning tree τ such that $C - e \subseteq \tau$. So, we can assume $E(\tau) \cup \{e\} = \{y_1, \ldots, y_n\}$, since $|E(\tau)| = n-1$. Now, if $(\mathbf{0}, 0) = \sum_{i=1}^{n} \alpha_i (v_i, 1)$, then $\sum_{i=1}^{n} \alpha_i v_i = \mathbf{0}$. But, by Lemma 3.7, $v_1, \ldots, v_n$ are linearly independent, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Hence, $(v_1, 1), \ldots, (v_n, 1)$ are linearly independent in $H_{(-\ell,1)} \cap \mathbb{R}_+ B$. Thus, $H_{(-\ell,1)} \cap \mathbb{R}_+ B$ is a facet of $\mathbb{R}_+ B$. Therefore, by Proposition 3.2, $\ell \in \{\ell_1, \ldots, \ell_q\}$. □

3.2 NORMAL GORENSTEIN SUBRINGS OF GRAPHS

In this section we study when the homogeneous subring S (of a graph G) is Gorenstein if S is normal. In particular, we prove that if S is normal and G is not bipartite, then G is unmixed, $\tau(G) = \lceil \frac{n}{2} \rceil$ and G has a strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction (Definition 3.12 and Theorem 3.14). Finally, we give necessary conditions for S to be Gorenstein if S is normal (see Theorem 3.17).

Lemma 3.9 Assume $\omega_S = (x^\alpha t^\beta)$, then $x^{\tilde{w}} t^a \in \omega_S$ if and only if $(\tilde{w}, a) - (\alpha, \beta) \in \mathbb{N}B$.

Proof. We have $x^{\tilde{w}} t^a \in \omega_S = (x^\alpha t^\beta)$ if and only if $x^{\tilde{w}} t^a = (x^u t^{a'})(x^\alpha t^\beta)$ with $x^u t^{a'} \in S$. Equivalently, $(\tilde{w}, a) - (\alpha, \beta) = (u, a') \in \mathbb{N}B$. □

Proposition 3.10 Assume S is normal and C is an odd cycle with $|V(C)| = k$.

1) If $w = (\tilde{w}, a) = w' + \left(\mathbf{1}_C, \frac{k+1}{2}\right)$ where $w' \in \mathbb{N}B$, $\tilde{w} \cdot e_i > 0$ for each $1 \leq i \leq n$, then $x^{\tilde{w}} t^a \in \omega_S$. Furthermore, if $\omega_S = (x^{\mathbf{1}} t^b)$, then $w - (\mathbf{1}, b) \in \mathbb{N}B$.

2) If $\omega_S = \left(x^{\mathbf{1}} t^{\frac{n+1}{2}}\right)$ and $\left(\mathbf{1} + e_j, \frac{n+1}{2}\right) \in \mathbb{N}B$, then there is $y \in E(G)$ such that $x_j \in y$ and $y \cap V(C) \neq \emptyset$.

Proof. 1) We can assume $C = (y_1, \ldots, y_k)$ where $\{x_1\} = y_1 \cap y_k$. Then, $\left(\mathbf{1}_C, \frac{k+1}{2}\right) = \sum_{\substack{1 \leq i \leq k \\ i \text{ even}}} (v_i, 1) + (e_1, 1) \in \mathbb{N}B$ (Recall $\mathbf{1}_C = \sum_{x_i \in V(C)} e_i$). Thus, $w \in \mathbb{N}B$ since $w' \in \mathbb{N}B$. Also, $w = w' + \frac{1}{2}\sum_{i=1}^{k}(v_i, 1) + \frac{1}{2}e_{n+1}$, since $\left(\mathbf{1}_C, \frac{k+1}{2}\right) = \frac{1}{2}\sum_{i=1}^{k}(v_i, 1) + \frac{1}{2}e_{n+1}$. Hence, by Lemma 3.5, $w \in (\mathbb{R}_+ B)^\circ$ since $\tilde{w} \cdot e_i > 0$ for $1 \leq i \leq n$. So, by Proposition 1.101, $x^{\tilde{w}} t^a \in \omega_S$. Now, if $\omega_S = (x^{\mathbf{1}} t^b)$, then by Lemma 3.9, $w - (\mathbf{1}, b) \in \mathbb{N}B$.
2) We take $w_1 = (\tilde{w}_1, a_1) = \left(\mathbf{1} + e_j, \frac{n+1}{2}\right) + \left(\mathbf{1}_C, \frac{k+1}{2}\right)$, then $\tilde{w}_1 \cdot e_i > 0$ for $1 \leq i \leq n$. Hence, by 1), $w_1 - \left(\mathbf{1}, \frac{n+1}{2}\right) \in \mathbb{N}B$, since $\left(\mathbf{1} + e_j, \frac{n+1}{2}\right) \in \mathbb{N}B$. But $w_1 - \left(\mathbf{1}, \frac{n+1}{2}\right) = \left(\mathbf{1}_C + e_j, \frac{k+1}{2}\right)$. Thus, by 2) in Lemma 3.4, $\left(\mathbf{1}_C + e_j, \frac{k+1}{2}\right) = \sum_{i=1}^{q} \alpha_i (v_i, 1)$ with $\alpha_i \in \mathbb{N}$, since $|\mathbf{1}_C + e_j| = 2\left(\frac{k+1}{2}\right)$. So, there is $i_1 \in \{1, \ldots, m\}$ such that $v_{i_1} = e_j + e_{j'}$ where $x_{j'} \in V(C)$. Therefore, $x_j \in y := y_{i_1}$ and $y \cap V(C) = \{x_{j'}\}$. □

Since $S \cong \mathbb{K}[x_1 z, \ldots, x_n z, x^{v_1}, \ldots, x^{v_m}, z]$, using [32, Theorem 2.1] and [3, Theorem 1.7], we obtain that if G is not bipartite and S is normal and Gorenstein, then n is odd. Now, in Corollary 3.11, we obtain another proof of this result using Proposition 3.10.

Corollary 3.11 If S is normal and n is even, then S is Gorenstein if and only if G is an unmixed bipartite graph.

Proof. $\Rightarrow$) By contradiction suppose G is not bipartite, then G has an odd k-cycle C. By Proposition 1.103, $\omega_S = (x^{\mathbf{1}} t^b)$ where $b = \frac{n}{2} + 1$, $\tau(G) = \frac{n}{2}$ and G is unmixed. Then, G is very well-covered and by Proposition 1.32, there is a τ-reduction $G_1, \ldots, G_s$ with $G_i \in E(G)$. We can assume $G_i = y_i$ for $1 \leq i \leq s$. We take $w = (\tilde{w}, a) := \sum_{i=1}^{s}(v_i, 1) + \left(\mathbf{1}_C, \frac{k+1}{2}\right)$. Since $y_1, \ldots, y_s$ is a partition of $V(G)$, $\sum_{i=1}^{s}(v_i, 1) = (\mathbf{1}, s)$ and $s = \frac{n}{2}$. Thus, $\tilde{w} \cdot e_i > 0$ for $1 \leq i \leq n$. Hence, by 1) in Proposition 3.10, $w - (\mathbf{1}, b) \in \mathbb{N}B$. But $w - (\mathbf{1}, b) = \left(\mathbf{1}_C, \frac{k-1}{2}\right)$, since $b = \frac{n}{2} + 1$. Then, by 1) in Lemma 3.4, $2\left(\frac{k-1}{2}\right) \geq |\mathbf{1}_C| = |V(C)| = k$. A contradiction, therefore G is bipartite. Also, by Proposition 1.98, G is unmixed.
$\Leftarrow$) By Proposition 1.98, S is Gorenstein. □

Definition 3.12 A τ-reduction $G_1, \ldots, G_s$ is a $\lceil \frac{\mathbf{n}}{\mathbf{2}} \rceil - \boldsymbol{\tau}$**-reduction** if $G_1, \ldots, G_{s-1} \in$

$E(G)$ and $G_s \in E(G)$ or $G_s \in \{C_3, C_5, C_7\}$ (i.e. G_s is a 3-, 5- or 7-cycle). A $\lceil \frac{n}{2} \rceil$-$\tau$-reduction $G_1, \ldots, G_s$ is **strong** when: $G_s \in E(G) \cup \{C_3, C_5, C_7\}$, then for each $x \in N_G[G_s]$ and each odd cycle C of G, there is $\{x, x'\} \in E(G)$ with $x' \in V(C)$.

Proposition 3.13 If S is normal, $\omega_S = (x^{\mathbf{1}} t^{\frac{n+1}{2}})$ and $G_1, \ldots, G_s$ is a $\lceil \frac{n}{2} \rceil$-$\tau$-reduction with $G_s \in \{C_3, C_5, C_7\}$, then $G_1, \ldots, G_s$ is strong.

Proof. Let C be an odd k'-cycle and $x \in N_G[G_s]$. We can suppose $G_1 = y_1, \ldots, G_{s-1} = y_{s-1}$ and $G_s = (y_{j_1}, \ldots, y_{j_k})$. First assume $x \in V(G_s)$, then we can assume $x = x_1 \in y_{j_1} \cap y_{j_k}$. Thus, $(\mathbf{1} + e_1, \frac{n+1}{2}) = \sum_{i=1}^{s-1}(v_i, 1) + \sum_{\substack{1 \le i \le k \\ i \text{ odd}}}(v_{j_i}, 1) \in \mathbb{N}B$, since $y_1, \ldots, y_{s-1}, V(G_s)$ is a partition of $V(G)$. Now, suppose $x \in N_G[G_s] \setminus V(G_s)$, then there is $y_{j'} = \{x, x'\} \in E(G)$ with $x' \in V(G_s)$. We can suppose $x' \in y_{j_1} \cap y_{j_k}$ and $x = x_1$, then $\left(\mathbf{1} + e_1, \frac{n+1}{2}\right) = \sum_{i=1}^{s-1}(v_i, 1) + \sum_{\substack{1 \le i \le k \\ i \text{ even}}}(v_{j_i}, 1) + (v_{j'}, 1) \in \mathbb{N}B$.
Hence, in both cases by 2) in Proposition 3.10, there is $y \in E(G)$ such that $x = x_1 \in y$ and $y \cap V(C) \neq \emptyset$. Therefore, $G_1, \ldots, G_s$ is a strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction. □

Theorem 3.14 If S is normal and Gorenstein, then G is unmixed, $\tau(G) = \lceil \frac{n}{2} \rceil$ and G has a strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction.

Proof. By Propositions 1.98 and 1.103, G is unmixed. Also, by Propositions 1.26 and 1.103, $\tau(G) = \lceil \frac{n}{2} \rceil$. Thus, by Propositions 1.32 (if n is even) and 1.34 (if n is odd), there is a $\lceil \frac{n}{2} \rceil$-$\tau$-reduction $G_1, \ldots, G_s$. If $G_s \in E(G)$, then $G_1, \ldots, G_s$ is strong. Now, assume $G_s \in \{C_3, C_5, C_7\}$. So, G is not bipartite and n is odd, since $V(G_1), \ldots, V(G_s)$ is a partition of $V(G)$. Then, by Proposition 1.103, $\omega_S = (x^{\mathbf{1}} t^{\frac{n+1}{2}})$. Hence, by Proposition 3.13, $G_1, \ldots, G_s$ is strong. □

Now, we will give sufficient conditions for S to be Gorenstein if S is normal and G is unmixed with a $\lceil \frac{n}{2} \rceil$-$\tau$-reduction (see Theorem 3.17).

Proposition 3.15 Let $y_1, \ldots, y_u$ be disjoint edges with the property **(P)**. If $w = (\tilde{w}, a) = \sum_{i=1}^{m} \alpha_i (v_i, 1) + \sum_{i=1}^{n} \beta_i (e_i, 1) + \lambda e_{n+1}$ with $\alpha_i, \beta_i, \lambda \in \mathbb{N}$ and $\sum_{i=1}^{u} \alpha_i$ maximal, then for each $k \in \{1, \ldots, u\}$, $\alpha_k = \tilde{w} \cdot e_{j_1}$ or $\alpha_k = \tilde{w} \cdot e_{j_2}$, where $y_k = \{x_{j_1}, x_{j_2}\}$.

Proof. Assume $y_k = \{x_{j_1}, x_{j_2}\}$ with $k \in \{1, \ldots, u\}$. First, we prove $\beta_{j_1} = 0$ or $\beta_{j_2} = 0$. By contradiction, suppose $\beta_{j_1} > 0$ and $\beta_{j_2} > 0$. Since $(e_{j_1}, 1) + (e_{j_2}, 1) = (v_k, 1) + e_{n+1}$, we have $w = \sum_{i=1}^{m} \tilde{\alpha}_i (v_i, 1) + \sum_{i=1}^{n} \tilde{\beta}_i (e_i, 1) + \tilde{\lambda} e_{n+1}$ where $\tilde{\alpha}_k = \alpha_k + 1$ and $\tilde{\alpha}_i = \alpha_i$ if $i \neq k$; $\tilde{\beta}_{j_1} = \beta_{j_1} - 1 \geq 0$, $\tilde{\beta}_{j_2} = \beta_{j_2} - 1 \geq 0$ and $\tilde{\beta}_j = \beta_j$ if $j \notin \{j_1, j_2\}$; and $\tilde{\lambda} = \lambda + 1$. But, $\sum_{i=1}^{u} \tilde{\alpha}_i = \left(\sum_{i=1}^{u} \alpha_i\right) + 1$. A contradiction, since $\sum_{i=1}^{u} \alpha_i$ is maximal. Hence, $\beta_{j_1} = 0$ or $\beta_{j_2} = 0$. We can assume $\beta_{j_2} = 0$ and we have two cases:

Case $\beta_{j_1} > 0$: We prove $\alpha_l = 0$ if $x_{j_2} \in y_l$ and $l \neq k$. By contradiction, suppose there is $y_l \in E(G)$ with $x_{j_2} \in y_l$, $\alpha_l \neq 0$ and $l \neq k$. Then, $l \notin \{1, \ldots, u\}$, since $y_1, \ldots, y_u$ are disjoint. We can assume $y_l = \{x_{j_2}, x_{j_3}\}$, then $(v_l, 1) + (e_{j_1}, 1) = (v_k, 1) + (e_{j_3}, 1)$. Thus, $w = \sum_{i=1}^{m} \alpha'_i(v_i, 1) + \sum_{i=1}^{n} \beta'_i(e_i, 1) + \lambda e_{n+1}$ where $\alpha'_k = \alpha_k + 1$, $\alpha'_l = \alpha_l - 1 \geq 0$ and $\alpha'_i = \alpha_i$ if $i \notin \{k, l\}$; $\beta'_{j_3} = \beta_{j_3} + 1$, $\beta'_{j_1} = \beta_{j_1} - 1 \geq 0$ and $\beta'_j = \beta_j$ if $j \notin \{j_1, j_3\}$. But, $\sum_{i=1}^{u} \alpha'_i = \sum_{i=1}^{u} \alpha_i + 1$ since $l \notin \{1, \ldots, u\}$. A contradiction, since $\sum_{i=1}^{u} \alpha_i$ is maximal. Hence, $\alpha_l = 0$ if $x_{j_2} \in y_l$ and $l \neq k$.

Case $\beta_{j_1} = 0$: We prove $\alpha_l = 0$ if $x_{j_1} \in y_l$ with $l \neq k$ or $\alpha_l = 0$ if $x_{j_2} \in y_l$ with $l \neq k$. By contradiction, suppose there are $l_1, l_2 \in \{1, \ldots, n\} \setminus \{k\}$ such that $x_{j_1} \in y_{l_1}, x_{j_2} \in y_{l_2}, \alpha_{l_1} \neq 0$ and $\alpha_{l_2} \neq 0$. Then, $l_1, l_2 \notin \{1, \ldots, u\}$, since $y_1, \ldots, y_u$ are disjoint. We assume $y_{l_1} = \{x_{j_1}, x_{j'_1}\}$ and $y_{l_2} = \{x_{j_2}, x_{j'_2}\}$. So, $\{x_{j'_1}, x_{j'_2}\} \in E(G)$, since y_k has the property **(P)**. We assume $y_{l'} = \{x_{j'_1}, x_{j'_2}\}$, then $(v_k, 1) + (v_{l'}, 1) = (v_{l_1}, 1) + (v_{l_2}, 1)$. Thus, $w = \sum_{i=1}^{m} \alpha''_i(v_i, 1) + \sum_{i=1}^{n} \beta_i(e_i, 1) + \lambda e_{n+1}$ where $\alpha''_k = \alpha_k + 1, \alpha''_{l'} = \alpha_{l'} + 1, \alpha''_{l_1} = \alpha_{l_1} - 1 \geq 0, \alpha''_{l_2} = \alpha_{l_2} - 1 \geq 0$ and $\alpha''_i = \alpha_i$ if $i \notin \{k, l', l_1, l_2\}$. But $\sum_{i=1}^{u} \alpha''_i \geq \sum_{i=1}^{u} \alpha_i + 1$, since $l_1, l_2 \notin \{1, \ldots, u\}$. A contradiction. Hence, we can suppose $\alpha_l = 0$ if $x_{j_2} \in y_l$ and $l \neq k$.
In both cases, $\tilde{w} \cdot e_{j_2} = \alpha_k$, since $\tilde{w} \cdot e_{j_2} = \sum_{x_{j_2} \in y_l} \alpha_l + \beta_{j_2}$ and $\beta_{j_2} = 0$. □

Definition 3.16 Let $G_1 = y_1, \ldots, G_{s-1} = y_{s-1}, G_s$ be a $\lceil \frac{n}{2} \rceil$-$\tau$-reduction. A representation $w = \sum_{i=1}^{m} \alpha_i(v_i, 1) + \sum_{i=1}^{n} \beta_i(e_i, 1) + \lambda e_{n+1} \in \mathbb{N}B$ (with $\alpha_i, \beta_j, \lambda \in \mathbb{N}$) is **principal** if it satisfies the following conditions:

1) $\sum_{i=1}^{u} \alpha_i$ is maximal where $u = s$ if $G_s \in E(G)$ or $u = s - 1$ if $G_s \notin E(G)$.
2) If $G_s \in E(G)$, then $\lambda > 0$.
3) If $G_s \in \{C_3, C_5, C_7\}$, then $G_s = (y_{j_1}, \ldots, y_{j_k})$ such that $\alpha_{j_i} > 0$ for each i even in $\{1, \ldots, k\}$ and $\beta_l > 0$ where $x_l \in y_{j_1} \cap y_{j_k}$.

Theorem 3.17 If S is normal, G is unmixed with a $\lceil \frac{n}{2} \rceil$-$\tau$-reduction and each $w \in (\mathbb{R}_+ B)^\circ \cap \mathbb{N}B$ has a principal representation, then S is Gorenstein.

Proof. Let $G_1, \ldots, G_s$ be a $\lceil \frac{n}{2} \rceil$-$\tau$-reduction. We can assume $G_1 = y_1, \ldots, G_{s-1} = y_{s-1}$; and $G_s = y_s$ if $G_s \in E(G)$. First we prove $x^{\mathbf{1}} t^b \in \omega_S$ with $b = \lfloor \frac{n}{2} \rfloor + 1$. If $G_s \in E(G)$, then $n = 2s$ and by Lemma 3.5, $(\mathbf{1}, b) = \sum_{i=1}^{s} (v_i, 1) + e_{n+1} \in \mathbb{N}B \cap (\mathbb{R}_+ B)^\circ$, since $y_1, \ldots, y_s$ is a partition of $V(G)$. Thus, by Proposition 1.101, $x^{\mathbf{1}} t^b \in \omega_S$. Now, if $G_s \in \{C_3, C_5, C_7\}$, then

$$(\mathbf{1}, b) = \sum_{i=1}^{s-1}(v_i, 1) + \left(\mathbf{1}_C, \frac{k+1}{2}\right) \tag{3.1}$$

where $C := G_s$ and $k := |V(C)|$, since $y_1, \ldots, y_{s-1}, G_s$ is a τ-reduction. Hence, by 1) in Proposition 3.10, $x^{\mathbf{1}}t^b \in \omega_S$.
Now, we take $x^{\tilde{w}}t^a \in \omega_S$ and we prove $x^{\tilde{w}}t^a \in (x^{\mathbf{1}}t^b)$. By Proposition 1.101, $w := (\tilde{w}, a) \in (\mathbb{R}_+B)^\circ \cap \mathbb{N}B$. Thus, w has a principal representation $w = \sum_{i=1}^m \alpha_i(v_i, 1) + \sum_{i=1}^n \beta_i(e_i, 1) + \lambda e_{n+1}$. We take $u = s$ if $G_s \in E(G)$ and $u = s-1$ if $G_s \notin E(G)$. So, by Proposition 3.15, for each $l \in \{1, \ldots, u\}$, $\tilde{w} \cdot e_{i_1} = \alpha_l$ or $\tilde{w} \cdot e_{i_2} = \alpha_l$ where $y_l = \{x_{i_1}, x_{i_2}\}$. Also, by Proposition 3.2, $\tilde{w} \cdot e_{i_1} > 0$ and $\tilde{w} \cdot e_{i_2} > 0$, since $w \in (\mathbb{R}_+B)^\circ$. Then, $\alpha_l > 0$ for $1 \leq l \leq u$. If $G_s \in E(G)$, then $u = s$, $\lambda > 0$ and $w' := w - (\mathbf{1}, b) = \sum_{i=1}^m \alpha'_i(v_i, 1) + \sum_{i=1}^n \beta_i(e_i, 1) + \lambda' e_{n+1}$, where $\alpha'_i = \alpha_i - 1 \geq 0$ if $i \in \{1, \ldots, u\}$, $\alpha'_i = \alpha_i$ in another case; and $\lambda' = \lambda - 1 \geq 0$. Hence, $w' \in \mathbb{N}B$ implies (by Lemma 3.9) $x^{\tilde{w}}t^a \in (x^{\mathbf{1}}t^b)$. Now, assume $G_s \in \{C_3, C_5, C_7\}$. Thus, $u = s-1$ and $G_s = C = (y_{j_1}, \ldots, y_{j_k})$ with $\alpha_{j_i} > 0$ for each i even in $\{1, \ldots, k\}$ and $\beta_1 > 0$ where $x_1 \in y_{j_1} \cap y_{j_k}$. Since $\left(\mathbf{1}_C, \frac{k+1}{2}\right) = \sum_{\substack{1 \leq i \leq k \\ i \text{ even}}}(v_{j_i}, 1) + (e_1, 1)$, by (3.1), $w'' := w - (\mathbf{1}, b) = \sum_{i=1}^m \alpha''_i(v_i, 1) + \sum_{i=1}^n \beta''_i(e_i, 1) + \lambda e_{n+1}$, where $\beta''_1 = \beta_1 - 1 \geq 0$, $\beta''_i = \beta_i$ if $i \neq 1$; $\alpha''_i = \alpha_i - 1 \geq 0$ if $i \in \{1, \ldots, u\} \cup \{j_2, j_4, \ldots, j_{k-1}\}$ and $\alpha''_i = \alpha_i$ in another case. So, $w'' \in \mathbb{N}B$. Hence, by Lemma 3.9, $x^{\tilde{w}}t^a \in (x^{\mathbf{1}}t^b)$.
This implies, $\omega_S = (x^{\mathbf{1}}t^b)$, i.e. ω_S is principal. Therefore, by Proposition 1.75, S is Gorenstein, since S is normal. □

Corollary 3.11 characterizes when S is Gorenstein if S is normal and n is even. Now, we give two conjectures when n is odd.

Conjecture 3.18 If S is normal, n is odd and G is unmixed with a strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction, then each $w \in (\mathbb{R}_+B)^\circ \cap \mathbb{N}B$ has a principal representation.

If Conjecture 3.18 is true, then by Theorems 3.14 and 3.17, the following Conjecture is also true.

Conjecture 3.19 Assume S is normal and n is odd. Then, S is Gorenstein if and only if G is unmixed with a strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction.

Conjecture 3.20 Let G be a graph whose pendant edges form a perfect matching. If S is Gorenstein, then G is bipartite and S is normal.

3.3 EXAMPLES

Example 3.21 Let G be the graph of Figure 3.1. By Proposition 1.96, S is normal. Furthermore, by Proposition 1.34, G is unmixed with $\tau(G) = \frac{n+1}{2}$ and y_1, y_2, $C = (y_3, y_4, y_5, y_6, y_7)$ is the unique $\lceil \frac{n}{2} \rceil$-$\tau$-reduction. But it is not strong, since $C' = (y_8, y_9, y_{10})$ is a 3-cycle and there are not edge between $x_3 \in N_G[C]$ and C'. Then, by Theorem 3.14, S is not Gorenstein, since S is normal. Hence, ω_S is not principal.

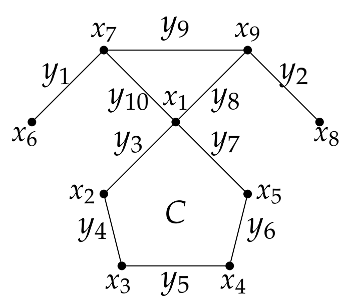


Figure 3.1 G is unmixed and S is not Gorenstein

Example 3.22 Let G be the graph of Figure 3.2. Then, $n = 6$ and G is not bipartite, since $C = (y_4, y_5, y_6)$ is a cycle. By Proposition 1.96, S is normal. Also, y_1, y_2, y_3 is a perfect matching with the property **(P)**. Thus, by Proposition 1.32, G is very-well-covered, i.e. G is unmixed and $\tau(G) = \frac{n}{2}$. Furthermore, y_1, y_2, y_3 is a strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction. But, by Corollary 3.11, S is not Gorenstein.

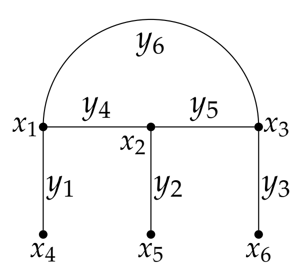


Figure 3.2 G is unmixed with a strong $\lceil \frac{n}{2} \rceil$-$\tau$-reduction. But S is not Gorenstein

CHAPTER 4

TORIC IDEALS OF ORIENTED GRAPHS

Let $D = (G, \mathcal{O})$ be an oriented graph whose toric ideal is P_D (see Definition 1.105). In this chapter, we study when P_D is a binomial complete intersection (see Definition 1.107).

4.1 $\mathcal{Q}$-SUMS AND $\mathcal{Q}$-RING GRAPHS

In this section we define the $\mathcal{Q}$-ring graphs (Definition 4.8) and we prove their toric ideals are binomial complete intersections (Corollary 4.9). Furthermore, we prove that if G is a theta-ring graph (see Definition 1.51), then $D = (G, \mathcal{O})$ is a $\mathcal{Q}$-ring graph for each edge orientation $\mathcal{O}$ of G (Theorem 4.14).

Definition 4.1 If D_1 and D_2 are oriented graphs, then $\mathbf{D} = \mathbf{D_1} \cap \mathbf{D_2}$ is the oriented graph, such that $V(D) = V(D_1) \cap V(D_2)$ and $E(D) = E(D_1) \cap E(D_2)$.

Definition 4.2 Let D_1 and D_2 be two oriented graphs such that $D_1 \cap D_2 = \mathcal{Q}$ is an oriented path. Then, the $\mathcal{Q}$**-sum** of D_1 and D_2 is the oriented graph $D = D_1 \cup D_2$.

Definition 4.3 Let $\mathcal{Q} = (z_1, z_2, \ldots, z_k)$ be a path of G. If C is a cycle of G, then a **subintersection** between $\mathcal{Q}$ and C is a subpath $I = (z_i, z_{i+1}, \ldots, z_j)$ of $\mathcal{Q}$ such that $E(I) \subseteq E(C)$ (i.e. $\{z_i, z_{i+1}\}, \ldots, \{z_{j-1}, z_j\} \in E(C)$), $\{z_{i-1}, z_i\} \notin E(C)$ and $\{z_j, z_{j+1}\} \notin E(C)$. It is possible that $i = j$, i.e. $I = (z_i)$. We denote $\mathbf{I}(\mathcal{Q}, \mathbf{C}) = \{I \mid I \text{ is a subintersection between } \mathcal{Q} \text{ and } C\}$.

Lemma 4.4 Let $D = (G, \mathcal{O})$ be the $\mathcal{Q}$-sum of $D_1 = (G_1, \mathcal{O}_1)$ and $D_2 = (G_2, \mathcal{O}_2)$. If C is a cycle of G such that $|I(\mathcal{Q}, C)| \leqslant 1$, then $C \subseteq G_1$ or $C \subseteq G_2$.

Proof. By contradiction suppose $C \nsubseteq G_1$ and $C \nsubseteq G_2$. Without loss of generality, we can assume $C = (a_1, \ldots, a_k, a_1)$ such that $\{a_1, a_2\} \notin E(G_2)$ and $s = \min \{i \mid \{a_i, a_{i+1}\} \notin E(G_1)\}$. Thus, $\{a_1, a_2\} \in E(G_1), s > 1$, and $\{a_{s-1}, a_s\} \in E(G_1)$, since $E(G) = E(G_1) \cup E(G_2)$ and s is minimal. So, $\{a_s, a_{s+1}\} \in E(G_2)$ and $a_s \in V(G_1) \cap V(G_2) = V(\mathcal{Q})$. Consequently, there is $I = (a_{i_1}, \ldots, a_s) \in I(\mathcal{Q}, C)$ for some $2 \leqslant i_1 \leqslant s$, since $\{a_1, a_2\} \notin E(G_2)$ and $\{a_s, a_{s+1}\} \notin E(G_1)$. We set $a_{k+1} := a_1$ and $a_{k+2} := a_2$ and we can take $s' = \min\{i \mid i > s \text{ such that } \{a_i, a_{i+1}\} \notin E(G_2)\}$. Hence, $s' \leqslant k+1$, since $\{a_{k+1}, a_{k+2}\} = \{a_1, a_2\} \notin E(G_2)$. So, $\{a_{s'}, a_{s'+1}\} \in E(G_1)$ and $\{a_{s'-1}, a_{s'}\} \in E(G_2)$. Then, $a_{s'} \in V(G_2) \cap V(G_1) = V(\mathcal{Q})$ and there is $s+1 \leqslant i_2 \leqslant s'$ such that $I' = (a_{i_2}, \ldots, a_{s'}) \in I(\mathcal{Q}, C)$, since $\{a_s, a_{s+1}\} \notin E(G_1)$ and $\{a_{s'}, a_{s'+1}\} \notin E(G_2)$. Hence, $I \neq I'$ and $|I(\mathcal{Q}, C)| \geqslant 2$. A contradiction, therefore $C \subseteq G_1$ or $C \subseteq G_2$. □

Proposition 4.5 Let D be the $\mathcal{Q}$-sum of D_1 and D_2, and C a cycle of D. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are generating sets of P_{D_1} and P_{D_2} respectively, then $y_C \in (\mathcal{B}_1 \cup \mathcal{B}_2)$.

Proof. By induction on $|I(\mathcal{Q}, C)|$. If $|I(\mathcal{Q}, C)| \leqslant 1$, then by Lemma 4.4, $C \subseteq D_1$ or $C \subseteq D_2$. Thus, $y_C \in (\mathcal{B}_1)$ or $y_C \in (\mathcal{B}_2)$ respectively. Now, assume $|I(\mathcal{Q}, C)| \geqslant 2$. So, we can suppose $\mathcal{Q} = (z_1, z_2, \ldots, z_k)$ and $I = (z_{i_1}, z_{i_1+1}, \ldots, z_{j_1}), I' = (z_{i_2}, z_{i_2+1}, \ldots, z_{j_2}) \in I(\mathcal{Q}, C)$ such that $i_1 = \min\{i \mid z_i \in V(C)\}$, $i_2 = \min\{i \mid i > j_1 \text{ such that } z_i \in V(C)\}$. Also, we can assume $C = (a_1, \ldots, a_s, a_1)$ such that $a_1 = z_{i_1}, a_2 = z_{i_1+1}, \ldots, a_{s_1} = z_{j_1}$ and $s_1 = j_1 - i_1 + 1$. Now, we take $s_2 = \min\{i \mid a_i \in V(I')\}$, then $s_2 > s_1$. Since $I' \in I(\mathcal{Q}, C)$, we have $a_{s_2} = z_{i_2}$ or $a_{s_2} = z_{j_2}$. If $a_{s_2} = z_{i_2}$, then $a_{s_2+1} = z_{i_2+1}, \ldots, a_{s_3} = z_{j_2}$ where $s_3 = s_2 + (j_2 - i_2)$; also, we take $C^1 = (z_{j_1} = a_{s_1}, a_{s_1+1}, \ldots, a_{s_2} = z_{i_2}, z_{i_2-1}, \ldots, z_{j_1})$ and $C^2 = (a_1 = z_{i_1}, \ldots, a_{s_1} = z_{j_1}, z_{j_1+1}, \ldots, z_{i_2}, \ldots, z_{j_2} = a_{s_2}, a_{s_2+1}, \ldots, a_s, a_1)$. Now, if $a_{s_2} = z_{j_2}$, then $a_{s_2+1} = z_{j_2-1}, \ldots, a_{s_3'} = z_{i_2}$ where $s_3' = s_2 + (i_2 - j_2)$ since $I' \in I(\mathcal{Q}, C)$; and we take $C^1 = (a_1 = z_{i_1}, a_2 = z_{i_1+1}, \ldots, a_{s_1} = z_{j_1}, z_{j_1+1}, \ldots, z_{i_2} = a_{s_3'}, a_{s_3'+1}, \ldots, a_s, a_1)$ and $C^2 = (z_{j_1} = a_{s_1}, a_{s_1+1}, \ldots, a_{s_2} = z_{j_2}, a_{s_2+1} = z_{j_2-1}, \ldots, a_{s_3'} = z_{i_2}, z_{i_2-1}, \ldots, z_{j_1})$. In both cases, C^1 and C^2 are cycles with $|I(\mathcal{Q}, C^1)| < |I(\mathcal{Q}, C)|$ and $|I(\mathcal{Q}, C^2)| < |I(\mathcal{Q}, C)|$. Then, by induction hypothesis $y_{C^1}, y_{C^2} \in (\mathcal{B}_1 \cup \mathcal{B}_2)$. Furthermore, $C = (C^1 \cup C^2) \setminus \mathcal{Q}'$, where $C^1 \cap C^2 = \mathcal{Q}' = (z_{j_1}, z_{j_1+1}, \ldots, z_{i_2})$. Therefore, by Lemma 1.118, $y_C \in (y_{C^1}, y_{C^2}) \subseteq (\mathcal{B}_1 \cup \mathcal{B}_2)$, since $\mathcal{Q}'$ is an oriented path. □

Theorem 4.6 Let D the $\mathcal{Q}$-sum of D_1 and D_2. If P_{D_1} and P_{D_2} are binomial complete intersections, then P_D is a binomial complete intersection.

Proof. Assume $|V(D_i)| = n_i$, $|E(D_i)| = m_i$, then $ht(P_{D_i}) = m_i - n_i + 1$ for $i = 1, 2$. Since D is the $\mathcal{Q}$-sum of D_1 and D_2, we have $|E(D)| = m_1 + m_2 - q$, $|V(D)| = n_1 + n_2 - (q+1)$ where $\mathcal{Q} = D_1 \cap D_2$ is an oriented path of D and $|E(\mathcal{Q})| = q$. Hence, $ht(P_D) = |E(D)| - |V(D)| + 1 = (m_1 - n_1 + 1) + (m_2 - n_2 + 1) = ht(P_{D_1}) + ht(P_{D_2})$. Since P_{D_i} is a binomial complete intersection, there is a binomial set of generators $\mathcal{B}_i$ of P_{D_i} such that $|\mathcal{B}_i| = ht(P_{D_i})$. We take $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Hence, by Propositions 1.113 and 4.5, $P_D = (y_C \mid C \text{ is a cycle of } D) \subseteq (\mathcal{B})$. Furthermore, $|\mathcal{B}| \leqslant |\mathcal{B}_1| + |\mathcal{B}_2| = ht(P_{D_1}) + ht(P_{D_2}) = ht(P_D)$. Therefore, P_D is a binomial complete intersection. □

Remark 4.7 In Example 4.35, P_D is a binomial complete intersection and D is a $\mathcal{Q}$-sum of a θ-partial wheel D' and the cycle without chords $C^5 = (x', \mathcal{L}_6, z_4, \mathcal{L}_3, z_3, \mathcal{L}_5, x')$, where $\mathcal{Q} = (z_4, \mathcal{L}_3, z_3)$. But $P_{D'}$ is not a binomial complete intersection.

In the following results, we use the notation of oriented subgraphs of Definition 1.56.

Definition 4.8 We define inductively the sets of oriented graphs $\mathcal{H}_i$ in the following form: $\mathcal{H}_1 := \{C_{\mathcal{O}} \mid C \text{ is a cycle or an edge and } \mathcal{O} \text{ is an orientation of } C\}$ and $\mathcal{H}_j := \{D \mid D \text{ is a } \mathcal{Q}\text{-sum of } D_1 \text{ and } D_2 \text{ such that } D_1 \in \mathcal{H}_{i_1}, D_2 \in \mathcal{H}_{i_2} \text{ and } i_1 + i_2 = j\}$. We say that an oriented graph D is a **$\mathcal{Q}$-ring graph** if $D \in \cup_{j=1}^{\infty} \mathcal{H}_j$.

Corollary 4.9 If D is a $\mathcal{Q}$-ring graph, then P_D is a binomial complete intersection.

Proof. If $D \in \mathcal{H}_1$, then by Theorem 1.120, P_D is a binomial complete intersection, since cycles and edges are theta-ring graphs. Now, if $D \in \mathcal{H}_j$ with $j \geqslant 2$, then D is a $\mathcal{Q}$-sum of D_1 and D_2 where $D_1 \in \mathcal{H}_{i_1}$, $D_2 \in \mathcal{H}_{i_2}$ and $i_1 + i_2 = j$. By induction hypothesis, P_{D_1} and P_{D_2} are binomial complete intersections. Therefore, by Theorem 4.6, P_D is a binomial complete intersection. □

In Lemmas 4.10, 4.12, 4.13 and Theorem 4.14, we assume $D = (G, \mathcal{O})$ is an oriented graph.

Lemma 4.10 If $G = K_n$, then D has an oriented path $\mathcal{L}$ such that $V(\mathcal{L}) = V(G)$.

Proof. By induction on n. If $n = 3$ is clear. We set $V(K_n) = \{x_1, \ldots, x_n\}$. By induction hypothesis, $D' := \big(K_n \setminus \{x_n\}\big)_{\mathcal{O}}$ has an oriented path $\mathcal{L}'$ with $V(D') = V(\mathcal{L}')$. We can assume $\mathcal{L}' = (x_1, \ldots, x_{n-1})$ with $(x_i, x_{i+1}) \in E(\mathcal{L}')$ for $1 \leqslant i \leqslant n-2$. Now, we prove $D = (K_n)_{\mathcal{O}}$ has an oriented path $\mathcal{L}$ with $V(D) = V(\mathcal{L})$. By contradiction assume D has no an oriented path $\mathcal{L}$ with $V(D) = V(\mathcal{L})$. Thus, $(x_1, x_n) \in E(D)$, because if $(x_n, x_1) \in E(D)$, then $(x_n, x_1, \ldots, x_{n-1})$ is an oriented path. Now, if $(x_n, x_2) \in E(D)$, then $(x_1, x_n, x_2, \ldots, x_{n-1})$ is an oriented path. So, $(x_2, x_n) \in E(D)$, implies $(x_3, x_n) \in E(D)$, because if $(x_n, x_3) \in E(D)$, then $(x_1, x_2, x_n, x_3, x_4, \ldots, x_{n-1})$ is an oriented path. Continuing with this process, we obtain $(x_{n-1}, x_n) \in E(D)$.

Hence, $(x_1,\dots,x_{n-1},x_n)$ is an oriented path. A contradiction. □

Definition 4.11 Let H be a graph and $x \notin V(H)$, then the **cone** of H on x, denoted by $C(H,x)$ is the graph such that $V\big(C(H,x)\big) = V(H) \cup \{x\}$ and $E\big(C(H,x)\big) = E(H) \cup \{\{x,a\} \mid a \in V(H)\}$.

Lemma 4.12 If $\mathcal{L}$ is a path of D such that $G = C(\mathcal{L},x)$, then D is a $\mathcal{Q}$-ring graph.

Proof. We assume $\mathcal{L} = (a_1,\dots,a_s)$. We prove the result by induction on s. If $s=1$ or $s=2$, then $G = K_2$ or $G = C_3$. Thus, D is a $\mathcal{Q}$-ring graph. Now, assume $s \geqslant 3$. By induction hypothesis, if $G' := C(\mathcal{L}',x)$ where $\mathcal{L}' = (a_1,\dots,a_{s-1})$, then $D' = G'_{\mathcal{O}}$ is a $\mathcal{Q}$-ring graph. Furthermore, D is the $\mathcal{Q}'$-sum of D' and the triangle $C = (x,a_{s-1},a_s,x)$, where $\mathcal{Q}' = (x,a_{s-1})$. Hence, G is a $\mathcal{Q}$-ring graph. □

Lemma 4.13 If G is a clique sum of G' and a complete graph K_n and $G'_{\mathcal{O}}$ is a $\mathcal{Q}$-ring graph, then D is a $\mathcal{Q}$-ring graph.

Proof. Since G is a clique-sum of G' and K_n, then $G = G' \cup K_n$ and $G' \cap K_n = K_s$ with $s \leqslant n$. By Lemma 4.10, there is an oriented path $\mathcal{L}_1 \subseteq (K_s)_{\mathcal{O}}$ such that $V(\mathcal{L}_1) = V(K_s)$. We set $V(K_n) \setminus V(K_s) = \{a_{s+1},\dots,a_n\}$. By Lemma 4.12, $H^1_{\mathcal{O}}$ is a $\mathcal{Q}$-ring graph where $H^1 = C(\mathcal{L}_1,a_{r+1})$. We take $D^2 = (G_2,\mathcal{O}_2)$ the $\mathcal{L}_1$-sum of $D^1 := G'_{\mathcal{O}}$ and $H^1_{\mathcal{O}}$, then D^2 is a $\mathcal{Q}$-ring graph. Since, $G_2[V(K_r) \cup \{a_{r+1}\}]$ is a complete graph, by Lemma 4.10, there is an oriented path $\mathcal{L}_2$ of D_2 such that $V(\mathcal{L}_2) = V(K_s) \cup \{a_{s+1}\}$. Also, by Lemma 4.12, $H^2_{\mathcal{O}}$ is a $\mathcal{Q}$-ring graph where $H^2 = C(\mathcal{L}_2,a_{r+2})$. We take $D^3 = (G_3,\mathcal{O}_3)$ the $\mathcal{L}_2$-sum of D^2 and $H^2_{\mathcal{O}}$, then D^3 is a $\mathcal{Q}$-ring graph. Continuing with this process, we obtain that $D^1, D^2, \dots, D^{n-s} = D$ are $\mathcal{Q}$-ring graphs. □

Theorem 4.14 If G is a connected theta-ring graph, then D is a $\mathcal{Q}$-ring graph.

Proof. By Theorem 1.53, G can be constructed recursively by clique sums of cycles and/or complete graphs, i.e. $G = \big(\dots((H_1 \oplus H_2) \oplus H_3)\dots\big) \oplus H_s$ where H_i is a cycle or H_i is a complete graph. We continue the proof by induction on s. If $s \geqslant 2$, then we have $G = G' \oplus H_s$ where $G' = \big(\dots(H_1 \oplus H_2) \oplus \dots\big) \oplus H_{s-1}$. By induction hypothesis $(G')_{\mathcal{O}}$ is a $\mathcal{Q}$-ring graph. Thus, by Lemma 4.13, D is a $\mathcal{Q}$-ring graph, if H_s is a complete graph. Now, if H_s is a cycle , then G is 1- or 2- clique sum of G' and H_s. Consequently, D is a $\mathcal{Q}$-sum of $(G')_{\mathcal{O}}$ and $(H_s)_{\mathcal{O}}$. Hence, D is a $\mathcal{Q}$-ring graph, since $(G')_{\mathcal{O}}$ is a $\mathcal{Q}$-ring graph. Finally, assume $s=1$, i.e. $G = C_n$ or $G = K_n$. If $G = C_n$, $G = K_2$ or $G = K_3$, then D is a $\mathcal{Q}$-ring graph. Now, assume $G = K_n$ with

$n \geqslant 4$. We prove D is a $\mathcal{Q}$-ring graph by induction on n. By induction hypothesis, we have $\tilde{D} = (K_{n-1})_{\mathcal{O}}$ is a $\mathcal{Q}$-ring graph. Furthermore, by Lemma 4.10, there is an oriented path $\mathcal{L}$ in $\tilde{D}$ such that $V(\mathcal{L}) = V(\tilde{D})$. We set $\{x\} = V(G) \setminus V(K_{n-1})$, then by Lemma 4.12, $H_{\mathcal{O}}$ is a $\mathcal{Q}$-ring graph where $H = C(\mathcal{L}, x)$. Therefore, D is a $\mathcal{Q}$-ring graph, since D is the $\mathcal{L}$-sum of $\tilde{D}$ and $H_{\mathcal{O}}$. □

4.2 ORIENTED TRUEMPER CONFIGURATIONS

In this section we study when the toric ideal P_D is a binomial complete intersection if G is a Truemper configuration: thetas (Proposition 4.16), pyramids (Proposition 4.22), prisms (Proposition 4.24) and partial wheels (Proposition 4.25). Furthermore, in this section we assume $D = (G, \mathcal{O})$ is an oriented graph and we use the notation of Figure 1.2.

Lemma 4.15 Let $\mathcal{L}$ be a non-oriented path of D such that $deg_D(x) = 2$ for each $x \in V(\mathcal{L}^\circ)$. Hence,

1) If C is a cycle of D such that $E(C) \cap E(\mathcal{L}) \neq \emptyset$, then $\mathcal{L} \subseteq C$ and C is not oriented.
2) If $u - u' \in P_D$ with $gcd(u, u') = 1$ and there is $y \in E(\mathcal{L})$ such that $y \mid u$, then $y_{\mathcal{L}^+} \mid u$ and $y_{\mathcal{L}^-} \mid u'$ when $y \in E(\mathcal{L}^+)$ or $y_{\mathcal{L}^-} \mid u$ and $y_{\mathcal{L}^+} \mid u'$ when $y \in E(\mathcal{L}^-)$.
3) If $\mathcal{B}$ is a binomial generating set of P_D, then $\mathcal{B}' = \{f \in \mathcal{B} \mid f \in P_{D'}\}$ is a binomial generating set of $P_{D'}$, where $D' = D \setminus V(\mathcal{L}^\circ)$.

Proof. Since $\mathcal{L}$ is not oriented, we have $\mathcal{L}_+ \neq \emptyset$ and $\mathcal{L}_- \neq \emptyset$.

1) We can assume $\mathcal{L} = (x_1, \ldots, x_s)$ and $(x_j, x_{j+1}) \in E(C) \cap E(\mathcal{L})$ or $(x_{j+1}, x_j) \in E(C) \cap E(\mathcal{L})$ for some $1 < j < s$. Then $\mathcal{L} \subseteq C$, since $deg_D(x_i) = 2$ for each $1 < i < s$. Hence, C is not oriented, since $\mathcal{L}$ is a non oriented path.

2) Without loss of generality, we can suppose $y \in E(\mathcal{L}^+)$, By Corollary 1.112, there are cycles $C^1, \ldots, C^s$ such that $y_{C^i} = u_i - u'_i$ for $1 \leqslant i \leqslant s$, and $u = u_1 \cdots u_s$ and $u' = u'_1 \cdots u'_s$. We can assume $y \mid u_1$, since $y \mid u$. Thus, $y \in E(C^1) \cap E(\mathcal{L})$. So, by 1), $\mathcal{L} \subseteq C$ implies $y_{\mathcal{L}^+} \mid u_1$ and $y_{\mathcal{L}^-} \mid u'_1$, since $y \mid u_1$ and $y \in E(\mathcal{L}^+)$. Hence, $y_{\mathcal{L}^+} \mid u$ and $y_{\mathcal{L}^-} \mid u'$.

3) Since $\mathcal{L}$ is not oriented, $\mathcal{L}^+ \neq \emptyset$ and $\mathcal{L}^- \neq \emptyset$. Then, $y_{\mathcal{L}^+} \neq 1$ and $y_{\mathcal{L}^-} \neq 1$. We set $E(\mathcal{L}) = \{y_1, \ldots, y_k\}$. We take $f \in P_{D'}$, then $f = \sum_{g_i \in \mathcal{B}'} f_i g_i + \sum_{g_i \in \mathcal{B} \setminus \mathcal{B}'} f_i g_i$ for some polynomials f_i, since $\mathcal{B}$ is a binomial generating set of P_D. If $g_i = u_i - u'_i \in \mathcal{B} \setminus \mathcal{B}'$, then there is $y \in E(\mathcal{L})$ such that $y \mid u_i u'_i$. By 2),

$y_{\mathcal{L}^+} \mid u_i$ and $y_{\mathcal{L}^-} \mid u_i'$ or $y_{\mathcal{L}^-} \mid u_i$ and $y_{\mathcal{L}^+} \mid u_i'$. In both cases, $g_i \mid_{y_1=\cdots=y_k=0} = 0-0=0$, since $y_{\mathcal{L}^+} \neq 1$ and $y_{\mathcal{L}^-} \neq 1$. Hence, $f = f \mid_{y_1=\cdots=y_k=0} = \sum_{g_i \in \mathcal{B}'} f_i' g_i$, where $f_i' = f_i \mid_{y_1=\cdots=y_k=0}$, since $f \in P_{D'}$ and $g_j \mid_{y_1=\cdots=y_k=0} = g_j$ for $g_j \in \mathcal{B}'$. Therefore, $\mathcal{B}'$ is a generating set of $P_{D'}$. □

Proposition 4.16 If G is a theta graph, then P_D is a binomial complete intersection if and only if at least one principal path of D is oriented.

Proof. We assume $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ are the principal paths of D between x and z, so the cycles of D are $C^1 = (x, \mathcal{L}_1, z, \mathcal{L}_2, x)$, $C^2 = (x, \mathcal{L}_2, z, \mathcal{L}_3, x)$ and $C^3 = (x, \mathcal{L}_1, z, \mathcal{L}_3, x)$.

$\Leftarrow$) Without loss of generality, we can suppose $\mathcal{L}_2$ is a principal oriented path of D. Thus, D is the $\mathcal{L}_2$-sum of the cycles C^1 and C^2 where $C^1 \cap C^2 = \mathcal{L}_2$. Hence, by Theorem 4.6, P_D is a binomial complete intersection.

$\Rightarrow$) By contradiction, suppose $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are not oriented. We have

$$\begin{aligned} y_{C^1} &= y_{\mathcal{L}_1^+} y_{\mathcal{L}_2^-} - y_{\mathcal{L}_1^-} y_{\mathcal{L}_2^+} \\ y_{C^2} &= y_{\mathcal{L}_2^+} y_{\mathcal{L}_3^-} - y_{\mathcal{L}_2^-} y_{\mathcal{L}_3^+} \\ y_{C^3} &= y_{\mathcal{L}_1^+} y_{\mathcal{L}_3^-} - y_{\mathcal{L}_1^-} y_{\mathcal{L}_3^+}. \end{aligned}$$

Let $\mathcal{B} = \{f_1, \ldots, f_s\}$ be a minimum binomial set of generators of P_D. Since $y_{C^1}, y_{C^2}, y_{C^3} \in P_D$, there is a monomial $y^\alpha \in Mon(\mathcal{B})$ such that $y^\alpha \mid m_1$, for each $m_1 \in \mathcal{A} := Mon(\{y_{C^1}, y_{C^2}, y_{C^3}\}) = \{y_{\mathcal{L}_1^+} y_{\mathcal{L}_2^-}, y_{\mathcal{L}_1^-} y_{\mathcal{L}_2^+}, y_{\mathcal{L}_2^+} y_{\mathcal{L}_3^-}, y_{\mathcal{L}_2^-} y_{\mathcal{L}_3^+}, y_{\mathcal{L}_1^+} y_{\mathcal{L}_3^-}, y_{\mathcal{L}_1^-} y_{\mathcal{L}_3^+}\}$. Also, $y_{\mathcal{L}_i^+} \neq 1$ and $y_{\mathcal{L}_i^-} \neq 1$, since $\mathcal{L}_i$ is not oriented. Then, $|\mathcal{A}| = 6$. By Theorem 1.111, there is a cycle C such that $y_C = m_2 - m_2'$ and $m_2 \mid y^\alpha$. Hence, $m_2 = y^\alpha = m_1$, since C^1, C^2 and C^3 are the cycles of D. Then, $\mathcal{A} \subseteq Mon(\mathcal{B})$ implies $|\mathcal{B}| \geqslant |\mathcal{A}|/2 = 3$. But $ht(P_D) = 2$. Therefore, P_D is not a binomial complete intersection. A contradiction. □

Definition 4.17 An oriented chordless cycle C of D is called **contractible** if $|N_D(x) \cap V(C)| \leqslant 1$ for each $x \in V(D) \setminus V(C)$. In this case, the **contraction** of C in D is the oriented graph $D_{\diagup C}$ with $V(D_{\diagup C}) = \big(V(D) \setminus V(C)\big) \cup \{v\}$ and $E(D_{\diagup C}) = \big(E(D) \setminus E(C)\big) \cup \{(v, x) \mid (a, x) \in E(D)$ with $a \in V(C)\} \cup \{(x, v) \mid (x, b) \in E(D)$ with $b \in V(C)\}$.

Definition 4.18 If u is a monomial of $R = \mathbb{K}[y_1, \ldots, y_m]$, then $supp(u) = \{y_i \mid y_i \mid u\}$. Furthermore, if $f = m_1 - m_2$ is a binomial of R, then $supp(f) = supp(m_1) \cup supp(m_2)$.

Proposition 4.19 Let C be a contractible cycle of D. If $\mathcal{B}$ is a binomial generating set of P_D with $|\mathcal{B}| = ht(P_D)$ and $\{f \in \mathcal{B} \mid supp(f) \subseteq E(C)\} \neq \emptyset$. Then, $P_{D'}$ is a binomial complete intersection, where $D' = D_{\diagup C}$.

Proof. We set $E\big(D \setminus V(C)\big) = \{y_1, \ldots, y_k\}$, $\{y \in E(D) \mid \ |y \cap V(C)| = 1\} = \{y_{k+1}, \ldots, y_{k+s}\}$, $E(C) = \{y_{k+s+1}, \ldots, y_m\}$. Furthermore, for each $k+1 \leqslant i \leqslant k+s$, we take $\tilde{y}_i = (v, x)$ or $\tilde{y}_i = (x, v)$ if $y_i = (z, x)$ or $y_i = (x, z)$ respectively, where $z \in V(C)$. Then, $E(D') = \{y_1, \ldots, y_k, \tilde{y}_{k+1}, \ldots, \tilde{y}_{k+s}\}$ and $\tilde{y}_i \neq \tilde{y}_j$, since C is contractible. We consider the morphism of k-algebras:

$$\phi : k[y_1, \ldots, y_m] \to k[y_1, \ldots, y_k, \tilde{y}_{k+1}, \ldots, \tilde{y}_{k+s}]$$

given by $\phi(y_i) = y_i$ for $1 \leqslant j \leqslant k$, $\phi(y_i) = \tilde{y}_i$ for $k < i \leqslant k+s$ and $\phi(y_i) = 1$ for $k+s < i \leqslant m$. We will prove $\phi(y_{C'}) \in P_{D'}$, where C' is a cycle of D. If $E(C') \subseteq E(C)$, then $C' = C$ and $supp(y_{C'}) \subseteq E(C)$. So, $\phi(y_{C'}) = 0 \in P_{D'}$. Now, assume $E(C') \nsubseteq E(C)$, then we can suppose $C' = (\mathcal{L}_1, \tilde{\mathcal{L}}_1, \ldots, \mathcal{L}_l, \tilde{\mathcal{L}}_l)$, where $\mathcal{L}_i, \tilde{\mathcal{L}}_i$ are paths such that $E(\mathcal{L}_i) \subseteq E(D) \setminus E(C)$ and $E(\tilde{\mathcal{L}}_i) \subseteq E(C)$ for $i = 1, \ldots, l$ (it is possible that $\tilde{\mathcal{L}}_i \subseteq V(C)$ or $l = 1$ with $\tilde{\mathcal{L}}_1 = \emptyset$). If $\tilde{\mathcal{L}}_1 = \emptyset$, then $C' \subseteq D \setminus V(C) \subseteq D'$ and $E(C') \subseteq \{y_1, \ldots, y_k\}$. Thus, C' is a cycle of D' and $\phi(y_{C'}) = y_{C'} \in P_{D'}$. Now, assume $\tilde{\mathcal{L}}_1 \neq \emptyset$, then we can assume $\mathcal{L}_i$ is a path between a_i and b_i such that $a_i, b_i \in V(C)$, i.e. $\mathcal{L}_i = (a_i, y_{j_i}, \ldots, y_{k_i}, b_i)$ where $y_{j_i}, y_{k_i} \in E(D)$, $y_{j_i} \cap V(C) = \{a_i\}$, $y_{k_i} \cap V(C) = \{b_i\}$ and $E(\mathcal{L}_i') \subseteq E\big(D \setminus V(C)\big)$, where $\mathcal{L}_i' = \mathcal{L}_i \setminus \{y_{j_i}, y_{k_i}\}$. Then, $C^i = (v, \tilde{y}_{j_i}, \mathcal{L}_i', \tilde{y}_{k_i}, v)$ is a cycle of D'. Thus, $\phi(y_{C'}) = y_{(C^1)^+} \cdots y_{(C^l)^+} - y_{(C^1)^-} \cdots y_{(C^l)^-}$, since $y_{C'} = y_{\mathcal{L}_1^+} y_{\tilde{\mathcal{L}}_1^+} \cdots y_{\mathcal{L}_l^+} y_{\tilde{\mathcal{L}}_l^+} - y_{\mathcal{L}_1^-} y_{\tilde{\mathcal{L}}_1^-} \cdots y_{\mathcal{L}_l^-} y_{\tilde{\mathcal{L}}_l^-}$, $\phi(y_{\tilde{\mathcal{L}}_i^+}) = \phi(y_{\tilde{\mathcal{L}}_i^-}) = 1$, $\phi(y_{\mathcal{L}_i^+}) = y_{(C^i)^+}$ and $\phi(y_{\mathcal{L}_i^-}) = y_{(C^i)^-}$. Hence, $\phi(y_{C'}) \in P_{D'}$, since $y_{C^i} = y_{(C^i)^+} - y_{(C^i)^-} \in P_{D'}$. Therefore, by Proposition 1.113, $\phi(P_D) \subseteq P_{D'}$.

We set $\mathcal{B} = \{f_1, \ldots, f_u\}$ where $u = ht(P_D) = m - n + 1$ and $n = |V(D)|$. We take a cycle C^1 of D' and we prove that there is a cycle C^2 of D such that $y_{C^1} = \phi(y_{C^2})$. If $v \notin V(C^1)$, then $V(C^1) \subseteq V(D) \setminus V(C)$ implies $C^1 \subseteq D$ and $E(C^1) \subseteq E\big(D \setminus V(C)\big) = \{y_1, \ldots, y_k\}$. Thus, $\phi(y_{C^1}) = y_{C^1}$, we can take $C^2 = C^1$. Now, assume $v \in V(C^1)$, then $C^1 = (v, \tilde{y}_{i_1}, x_{j_1}, \ldots, x_{j_{l-1}}, \tilde{y}_{i_l}, v)$ such that $\mathcal{L} = C^1 \setminus \{v\}$ is a path in $D \setminus V(C)$ and $y_{i_1}, y_{i_l} \in \{y_{k+1}, \ldots, y_{k+s}\}$. So, $\phi(y_{i_1}) = \tilde{y}_{i_1}$, $\phi(y_{i_l}) = \tilde{y}_{i_l}$ and there are $z, z' \in V(C)$ such that $z \in y_{i_1}$ and $z' \in y_{i_l}$. Furthermore, there is a path $\mathcal{L}'$ in C between z and z', (we take $\mathcal{L}' = (z)$ if $z = z'$). So, we take $C^2 = (z', \mathcal{L}', z, y_{i_1}, x_{j_1}, \mathcal{L}, x_{j_{l-1}}, y_{i_l}, z')$, then $\phi(y_{C^2}) = y_{C^1}$, since $\mathcal{L} \subseteq D \setminus V(C)$, $\mathcal{L}' \subseteq C$, $\phi(y_{i_1}) = \tilde{y}_{i_1}$ and $\phi(y_{i_2}) = \tilde{y}_{i_2}$. Furthermore, $y_{C^2} = \sum_{i=1}^{u} g_i f_i$ for some $g_1, \ldots, g_u \in k[y_1, \ldots, y_m]$, since $\mathcal{B}$ is a generating set of P_D. Thus, $y_{C^1} = \phi(y_{C^2}) = \sum_{i=1}^{u} \phi(g_i)\phi(f_i) \in \big(\phi(f_1), \ldots, \phi(f_u)\big) \subseteq k[y_1, \ldots, y_k, \tilde{y}_{k+1}, \ldots, \tilde{y}_{k+s}]$. Hence, by Proposition 1.113, $P_{D'} \subseteq (\mathcal{B}')$, where $\mathcal{B}' = \{\phi(f_1), \ldots, \phi(f_u)\} \setminus \{0\}$. Also, $\mathcal{B}' \subseteq P_{D'}$, since $\phi(P_D) \subseteq P_{D'}$ and $\mathcal{B} \subseteq P_D$. Then $\mathcal{B}'$ is a binomial generating set of P_D.

Now, if $f = m_1 - m_2$ with $m_1, m_2 \in Mon(k[y_1, \ldots, y_m])$ and $supp(f) \subseteq E(C)$, then $\phi(f) = 0$ since $\phi(m_1) = \phi(m_2) = 1$. Thus, $|\mathcal{B}'| < |\mathcal{B}| = m - n + 1$, since $\{f \in \mathcal{B} \mid supp(f) \subseteq E(C)\} \neq \emptyset$. Also, by Principal Ideal Theorem, $|\mathcal{B}'| \geqslant ht(P_{D'}) = m' - n' + 1$, where $m' = |E(D')| = m - q$, $n' = |V(D')| = n - q + 1$, where $q = |V(C)| = |E(C)|$. Hence, $m - n + 1 > |\mathcal{B}'| \geqslant ht(P_{D'}) = (m - q) - (n - q + 1) + 1 = m - n$. Therefore, $P_{D'}$ is a binomial complete intersection. □

Lemma 4.20 Let $\mathcal{C}$ be the set of oriented cycles in D. If $1 \leqslant |\mathcal{C}| \leqslant 2$, then there is a minimum binomial generating set $\mathcal{B}'$ of P_D where $\{f \in \mathcal{B}' \mid supp(f) \subseteq E(C)$ with $C \in \mathcal{C}\} \neq \emptyset$.

Proof. Let $\mathcal{B}$ be a minimum binomial generating set of P_D. Since $\mathcal{C} \neq \emptyset$, there is $C \in \mathcal{C}$ and $y_C = 1 - u'$. Then, there is $g = 1 - u \in \mathcal{B}$. If $|\mathcal{C}| = 1$, then $\mathcal{C} = \{C\}$ and by Corollary 1.112, $u = (u')^{\alpha}$. Thus, $supp(g) \subseteq E(C)$. Now, assume $\mathcal{C} = \{C^1, C^2\}$, then by Corollary 1.112, $u = u_1^{\alpha_1} u_2^{\alpha_2}$ where $y_{C^1} = 1 - u_1$ and $y_{C^2} = 1 - u_2$. By Remark 1.114, we can suppose that if $m_1 - m_2 \in \mathcal{B}$, then $gcd(m_1, m_2) = 1$. We take $\mathcal{A} = \{h \in \mathcal{B} \mid supp(h) \subseteq E(C^1) \cup E(C^2)\}$, then by Corollary 1.112, $g \in \mathcal{A}$, since $\mathcal{C} = \{C^1, C^2\}$. Furthermore, if $h \in \mathcal{A}$, then by Corollary 1.112, $h = 1 - u_1^{\beta_1} u_2^{\beta_2}$ or $h = u_1^{\beta_1} - u_2^{\beta_2}$, since C^1 and C^2 are all the cycles in $E(C^1) \cup E(C^2)$. Thus, $(\mathcal{A}) \subseteq (y_{C^1}, y_{C^2})$. If $|\mathcal{A}| \geqslant 2$, then $\mathcal{B}' = (\mathcal{B} \setminus \mathcal{A}) \cup (y_{C^1}, y_{C^2})$ is a minimum generating set of P_D, since $|\mathcal{B}'| \leqslant |\mathcal{B}|$. Also $supp(y_{C^1}) \subseteq E(C^1)$.

Now, assume $|\mathcal{A}| \leqslant 1$, then $\mathcal{A} = \{g\}$ and $g = 1 - u_1^{\gamma_1} u_2^{\gamma_2}$. We take $m_1' - m_2' \in \mathcal{B}$ such that $supp(m_1') \subseteq E(C^1) \cup E(C^2)$. We prove $m_1' - m_2' = g$. By contradiction, suppose there is $\tilde{y} \in E(D) \setminus \big(E(C^1) \cup E(C^2)\big)$ such that $\tilde{y} \mid m_2'$. Thus, by Corollary 1.112, there is a cycle C', such that $y_{C'} = n_1 - n_2$, $n_1 \mid m_1'$, $n_2 \mid m_2'$ and $\tilde{y} \mid n_2$. Then, $\tilde{y} \in E(C')$, implies C' is not oriented cycle and $n_1 \neq 1$, since $\tilde{y} \notin E(C^1) \cup E(C^2)$. We take a maximal path $\mathcal{L}$ such that $\tilde{y} \in E(\mathcal{L})$, $\mathcal{L} \subseteq C'$ and $E(\mathcal{L}) \cap \big(E(C^1) \cup E(C^2)\big) = \emptyset$. Then, $E(\mathcal{L}) \subseteq supp(n_2)$, since $supp(n_1) \subseteq supp(m_1') \subseteq E(C^1) \cup E(C^2)$. So, $\mathcal{L}$ is an oriented path. We set a, b the end-vertices of $\mathcal{L}$, then by the maximality of $\mathcal{L}$, $a \in V(C^i)$ and $b \in V(C^{i'})$ with $i, i' \in \{1, 2\}$. We prove $i \neq i'$. By contradiction, suppose $i = i'$, then $C^i = \mathcal{L}_1 \cup \mathcal{L}_2$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are oriented paths between a and b. Thus, $\mathcal{L} \cup \mathcal{L}_1$ or $\mathcal{L} \cup \mathcal{L}_2$ is an oriented cycle of D. A contradiction, since $\mathcal{C} = \{C^1, C^2\}$, $\tilde{y} \in E(\mathcal{L})$ and $E(\mathcal{L}) \cap \big(E(C^1) \cup E(C^2)\big) = \emptyset$. Hence, $i \neq i'$. Without loss of generality, we can assume $i = 1$, $i' = 2$ and $\mathcal{L}$ is an oriented path from a to b. Since C' is a cycle, there are paths $\tilde{\mathcal{L}}_2, \mathcal{L}'$ such that $\tilde{\mathcal{L}} := (a, \mathcal{L}, b, \mathcal{L}_2, b', \mathcal{L}', a') \subseteq C'$, $\mathcal{L}_2 \subseteq C^2$, $E(\mathcal{L}') \cap \big(E(C^1) \cup E(C^2)\big) = \emptyset$ and $a' \in V(C^1) \cup V(C^2)$. By last argument (on $\mathcal{L}$), we have $\mathcal{L}'$ is an oriented path and $a' \in V(C^1)$, since $b' \in V(\mathcal{L}_2) \subseteq V(C^2)$. Also, $E(\mathcal{L} \cup \mathcal{L}') \cap E(C^1 \cup C^2) = \emptyset$, then $y_{\mathcal{L}} y_{\mathcal{L}'} \mid n_2$ and $\mathcal{L}'$ is an oriented path from b' to a', since $\tilde{\mathcal{L}} \subseteq C'$. Since C^1 and C^2 are oriented cycles, there are oriented paths

$\mathcal{P}_1$ from a' to a in C^1 and $\mathcal{P}_2$ from b to b' in C^2. Hence, $(a, \mathcal{L}, b, \mathcal{P}_2, b', \mathcal{L}', a', \mathcal{P}_1, a)$ is an oriented cycle in D. A contradiction, since $\mathcal{C} = \{C^1, C^2\}$. Therefore, $m_1' - m_2' = g$.

Consequently, if $f = m_1 - m_2 \in \mathcal{B} \setminus \{g\}$, then there are $y_{i_1}, y_{i_2} \in E(D) \setminus E(C^1 \cup C^2)$ such that $y_{i_1} \mid m_1$ and $y_{i_2} \mid m_2$. So, $f \mid_{y_1 = \cdots = y_s = 0} = 0$ where $E(D) \setminus E(C^1 \cup C^2) = \{y_1, \ldots, y_s\}$. Now, $y_{C^1} = \sum_{g_i \in \mathcal{B}} f_i \cdot g_i$ with $f_i \in k[y_1, \ldots, y_m]$, since $\mathcal{B}$ is a generating set. If we evaluate $y_1 = \cdots = y_s = 0$ in the last equation, we obtain $1 - u_1 = y_{C^1} \mid_{y_1 = \cdots = y_s = 0} = h \cdot g$, where $g_1 = g$ and $f_1 \mid_{y_1 = \cdots = y_s = 0} = h$. So, $1 - u_1 = h \cdot (1 - u_1^{\alpha_1} u_2^{\alpha_2})$. If $y \in E(C^2)$, then $0 = deg_y(1 - u_1) \geqslant deg_y(1 - u_1^{\alpha_1} u_2^{\alpha_2}) \geqslant \alpha_2$. Hence, $\alpha_2 = 0$, $g = 1 - u_1^{\alpha_1}$ and $supp(g) \subseteq E(C^1)$. □

Lemma 4.21 If G is a pyramid and each path $(x_i, x_j, \mathcal{L}_j, z)$ is not oriented with $i \neq j$, then some $\mathcal{L}_k$ is not oriented with $k \in \{1, 2, 3\}$. Furthermore, if $C = (x_1, x_2, x_3, x_1)$ is an oriented cycle, then $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ are not oriented.

Proof. By contradiction, we can suppose $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are oriented and $E(\mathcal{L}_1) = \mathcal{L}_1^+$. Thus, $(x_1, x_i) \in E(D)$ for $i = 2, 3$, since $(x_i, x_1, \mathcal{L}_1, z)$ is not an oriented path. So, $E(\mathcal{L}_i) = \mathcal{L}_i^-$, since $(x_1, x_i, \mathcal{L}_i, z)$ is not oriented path. Hence, $(x_2, x_3, \mathcal{L}_3, z)$ or $(x_3, x_2, \mathcal{L}_2, z)$ is an oriented path, since $(x_3, x_2) \in E(D)$ or $(x_2, x_3) \in E(D)$. A contradiction.

Now, assume $C = (x_1, x_2, x_3, x_1)$ is an oriented cycle. If $\mathcal{L}_i$ is oriented, then $(x_j, x_i, \mathcal{L}_i, z)$ or $(x_k, x_i, \mathcal{L}_i, z)$ is oriented with $\{i, j, k\} = \{1, 2, 3\}$. A contradiction, then $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ are not oriented. □

Proposition 4.22 Let G be a pyramid. Then, P_D is a binomial complete intersection if and only if $(x_i, x_j, \mathcal{L}_j, z)$ is an oriented path for some $i, j \in \{1, 2, 3\}$ with $i \neq j$.

Proof. Since G is a pyramid, $ht(P_D) = |E(D)| - |V(D)| + 1 = 3$ and the chordless cycles in D are $C = (x_1, x_2, x_3, x_1)$, $C^1 = (x_2, x_1, \mathcal{L}_1, z, \mathcal{L}_2, x_2)$, $C^2 = (x_3, x_2, \mathcal{L}_2, z, \mathcal{L}_3, x_3)$ and $C^3 = (x_3, x_1, \mathcal{L}_1, z, \mathcal{L}_3, x_3)$.

$\Leftarrow$) Without loss of generality, we can suppose $\mathcal{L} = (x_1, x_2, \mathcal{L}_2, z)$ is an oriented path. Now, we take D_1 the $\mathcal{P}$-sum of the cycles C and C^2 where $\mathcal{P} = C \cap C^2$ is the path (x_2, x_3). Thus, D is the $\mathcal{L}$-sum of D_1 and the cycle C^1, since $D_1 \cap C^1 = \mathcal{L}$ is an oriented path. Therefore, by Theorem 4.6, P_D is a binomial complete intersection.

$\Rightarrow$) By contradiction, suppose $(x_i, x_j, \mathcal{L}_j, z)$ is not oriented for each $i, j \in \{1, 2, 3\}$ with $i \neq j$. We prove C is not oriented. By contradiction, assume C is oriented. By Lemma 4.21, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ are not oriented. Then, C is the unique oriented cycle in

D. Also, C is contractible, since $|N_D(x) \cap V(C)| \leqslant 1$ for each $x \in V(D) \setminus V(C)$. By Lemmas 4.20 and 4.19, $P_{D\diagup C}$ is a binomial complete intersection. But $D\diagup C$ is a theta graph whose principal paths are not oriented. A contradiction by Proposition 4.16. Hence, C is not oriented. So, without loss of generality, we can assume $y_1 = (x_3, x_2), y_2 = (x_3, x_1), y_3 = (x_1, x_2) \in E(C)$. Thus $\mathcal{L}_1^+ \neq \emptyset$, $\mathcal{L}_1^- \neq \emptyset$, $\mathcal{L}_2^- \neq \emptyset$ and $\mathcal{L}_3^+ \neq \emptyset$, since each $(x_i, x_j, \mathcal{L}_j, z)$ is not oriented. Furthermore,

$$\begin{aligned} y_C &= y_2\, y_3 - y_1 \\ y_{C^1} &= y_{\mathcal{L}_1^+}\, y_{\mathcal{L}_2^-} - y_3\, y_{\mathcal{L}_1^-}\, y_{\mathcal{L}_2^+} \\ y_{C^2} &= y_1\, y_{\mathcal{L}_2^+}\, y_{\mathcal{L}_3^-} - y_{\mathcal{L}_2^-}\, y_{\mathcal{L}_3^+} \\ y_{C^3} &= y_2\, y_{\mathcal{L}_1^+}\, y_{\mathcal{L}_3^-} - y_{\mathcal{L}_1^-}\, y_{\mathcal{L}_3^+}. \end{aligned}$$

Thus, $\mathcal{A}_1 = Mon(y_C, y_{C^1}, y_{C^2}, y_{C^3}) \setminus \{y_1\, y_{\mathcal{L}_2^+}\, y_{\mathcal{L}_3^-}\}$. Let $\mathcal{B}_1$ be a minimum binomial set of generators of P_D, then for each $m_1 \in \mathcal{A}_1$ there is $m' \in Mon(\mathcal{B}_1)$ such that $m' \mid m_1$. By Theorem 1.111, there is a cycle C' such that $y_{C'} = y^\alpha - y^\beta$ and $y^\alpha \mid m'$, then $y^\alpha \mid m_1$. We will prove $y^\alpha = m_1$. First assume $\mathcal{L}_1 \subseteq C'$. If $C' \neq C^1$ and $C' \neq C^3$, then $C' = (C^1)'$ or $C' = (C^3)'$ where $(C^1)' = (x_1, \mathcal{L}_1, z, \mathcal{L}_2, x_2, y_1, x_3, y_2, x_1)$ or $(C^3)' = (x_1, \mathcal{L}_1, z, \mathcal{L}_3, x_3, y_1, x_2, y_3, x_1)$. Thus, $y^\alpha \in \mathcal{A}_2 = Mon(y_{(C^1)'}, y_{(C^3)'}) = \{y_2 y_{\mathcal{L}_1^+} y_{\mathcal{L}_2^-}, y_1 y_{\mathcal{L}_1^-} y_{\mathcal{L}_2^+}, y_1 y_{\mathcal{L}_1^+} y_{\mathcal{L}_3^-}, y_3 y_{\mathcal{L}_1^-} y_{\mathcal{L}_3^+}\}$. A contradiction, since $y^\alpha \mid m_1$, $m_1 \in \mathcal{A}_1$, $\mathcal{L}_2^- \neq \emptyset$ and $\mathcal{L}_3^- \neq \emptyset$. Hence $C' = C^1$ or $C' = C^3$ implies $y^\alpha \in Mon(y_{C^1}, y_{C^3})$. Also, $y^\alpha \mid m_1$, then $y^\alpha = m_1$, since $m_1 \in \mathcal{A}_1$, $\mathcal{L}_2^- \neq \emptyset$ and $\mathcal{L}_3^+ \neq \emptyset$. Now, assume $\mathcal{L}_1 \nsubseteq C'$, then $C' \in \{C, C^2, (C^2)'\}$ where $(C^2)' = (x_2, \mathcal{L}_2, z, \mathcal{L}_3, x_3, y_2, x_1, y_3, x_2)$. But $y_{(C^2)'} = y_2 y_3 y_{\mathcal{L}_2^+} y_{\mathcal{L}_3^-} - y_{\mathcal{L}_2^-} y_{\mathcal{L}_3^+}$, $\mathcal{L}_2^- \neq \emptyset$, $\mathcal{L}_3^- \neq \emptyset$, $y^\alpha \mid m_1$ and $m_1 \in \mathcal{A}_1$. Hence, $y^\alpha = m_1$. So $m_1 = m'$, since $y^\alpha \mid m'$ and $m' \mid m_1$. Therefore, $\mathcal{A}_1 \subseteq Mon(\mathcal{B}_1)$. Then, $|\mathcal{B}_1| \geqslant |\mathcal{A}_1|/2 > 3$. A contradiction, since $|\mathcal{B}_1| = ht(P_D) = 3$. □

In the following results, we use the notation of Figure 1.2.

Lemma 4.23 If G is a prism such that each path $(x_i, x_j, \mathcal{L}_i, z_j, z_k)$ is not oriented with $j \notin \{i, k\}$, then the only possible oriented cycles are $C^1 = (x_1, x_2, x_3, x_1)$ and $C^2 = (z_1, z_2, z_3, z_1)$. Furthermore, if $y_{C^1} = y_1 - y_2 y_3$, $y_{C^2} = y'_{i_1} - y'_{i_2} y'_{i_3}$ where $\{i_1, i_2, i_3\} = \{1, 2, 3\}$ and $f = m_1 - m'_1$ is a binomial of P_D, then $\{m_1, m'_1\} \cap \mathcal{A} = \emptyset$ with $\mathcal{A} = \{y_2, y_3, y'_{i_2}, y'_{i_3}, y_{\mathcal{L}_1^+}, y_{\mathcal{L}_1^-}, y_{\mathcal{L}_2^+}, y_{\mathcal{L}_2^-}, y_{\mathcal{L}_3^+}, y_{\mathcal{L}_3^-}\}$.

Proof. By contradiction, suppose $\tilde{C}$ is an oriented cycle of D with $\tilde{C} \neq C^1$ and $\tilde{C} \neq C^2$. Consequently, there is $\mathcal{L}_j$ such that $\mathcal{L}_j \subseteq \tilde{C}$, implies there is $\mathcal{L} = (x_i, x_j, \mathcal{L}_i, z_j, z_k) \subseteq \tilde{C}$ with $j \notin \{i, k\}$. A contradiction, since $\mathcal{L}$ is not oriented.

Now, we take $f = m_1 - m'_1 \in P_D$ and by contradiction, suppose $\{m_1, m'_1\} \cap \mathcal{A} \neq \emptyset$.

Without loss of generality, we can assume $m_1 \in \mathcal{A}$. By Corollary 1.112, there is a cycle C such that $y_C = y^\alpha - y^\beta$ and $y^\alpha \mid m_1$. Since $y_{C^1} = y_1 - y_2y_3$ and $y_{C^2} = y_{i_1} - y_{i_2}y_{i_3}$, we have that C^1 and C^2 are not oriented cycles. Hence, D has not oriented cycles and C is not oriented. Thus, $y^\alpha \neq 1$ implies $m_1 \neq 1$. First, assume $m_1 = y_{\mathcal{L}_i^+}$ or $m_1 = y_{\mathcal{L}_i^-}$. We can suppose $m_1 = y_{\mathcal{L}_i^+}$, then $y^\alpha \mid y_{\mathcal{L}_i^+}$. Thus, by 1) in Lemma 4.15, $\mathcal{L}_i \subseteq C$ and $y^\alpha = y_{\mathcal{L}_i^+}$, since $y^\alpha \neq 1$. Consequently, $\mathcal{L}^1 = (x_i, x_j, \mathcal{L}_j, z_j, z_{i'}) \subseteq C$ or $\mathcal{L}^2 = (x_i, x_j, x_k, \mathcal{L}_k, z_k, z_{j'}) \subseteq C$ where $\{i, j, k\} = \{1, 2, 3\}$, $i' \neq j$ and $j' \neq k$. Then, $\mathcal{L}^1$ or $\mathcal{L}^2$ are oriented paths, since $y_{C^+} = y^\alpha = y_{\mathcal{L}_i^+}$. A contradiction. Now, assume $m_1 \in \{y_2, y_3, y'_{i_2}, y'_{i_3}\}$. Without loss of generality, we can assume $m_1 = y_2$, then $y^\alpha = y_2$, since $y^\alpha \mid m_1$ and $y^\alpha \neq 1$. So, $C^+ = \{y_2\}$ and $y_3 \notin V(C)$, since $(C^1)^- = \{y_2, y_3\}$. Thus, $\mathcal{L}' = (x_1, \mathcal{L}_1, z_1, z_i) \subseteq C$ and $E(\mathcal{L}') \subseteq C^-$ for some $i \in \{2, 3\}$. Hence, $\tilde{\mathcal{L}} = (x_2, x_1, \mathcal{L}_1, z_1, z_i)$ is an oriented path, since $E(\mathcal{L}') \subseteq C^-$, $C^+ = \{y_2\}$ and $(C^1)^- = \{y_2, y_3\}$. A contradiction. □

Proposition 4.24 Let G be a prism. Then, P_D is a binomial complete intersection if and only if at least one path $(x_i, x_j, \mathcal{L}_j, z_j, z_k)$ is oriented with $i, j, k \in \{1, 2, 3\}$ and $j \notin \{i, k\}$.

Proof. The chordless cycles in D are $C^1 = (x_1, x_2, x_3, x_1)$, $C^2 = (z_1, z_2, z_3, z_1)$, $C^3 = (x_2, x_1, \mathcal{L}_1, z_1, z_2, \mathcal{L}_2, x_2)$, $C^4 = (x_3, x_2, \mathcal{L}_2, z_2, z_3, \mathcal{L}_3, x_3)$ and $C^5 = (x_3, x_1, \mathcal{L}_1, z_1, z_3, \mathcal{L}_3, x_3)$.

$\Leftarrow$) Without loss of generality, we can suppose $\mathcal{L} = (x_1, x_2, \mathcal{L}_2, z_2, z_j)$ is an oriented path of D with $j \neq 2$. We take the following cases:

Case $j = 1$. We take D_1 the $\mathcal{P}_1$-sum of C^1 and C^4; and D_2 the $\mathcal{P}_2$-sum of D_1 and C^2 where $\mathcal{P}_1 = (x_2, x_3)$ and $\mathcal{P}_2 = (z_2, z_3)$. Hence, D is the $\mathcal{L}$-sum of D_2 and C^3, since $\mathcal{L}$ is an oriented path.

Case $j = 3$. We take D'_1 the $\mathcal{Q}_1$-sum of C^2 and C^3; and D'_2 the $\mathcal{Q}_2$-sum of C^1 and C^4 where $\mathcal{Q}_1 = (z_1, z_2)$ and $\mathcal{Q}_2 = (x_2, x_3)$. Hence, D is the $\mathcal{L}$-sum of D'_1 and D'_2, since $\mathcal{L}$ is an oriented path.

Therefore, in both cases (by Theorem 4.6), P_D is a binomial complete intersection.

$\Rightarrow$) By contradiction, assume each path $(x_i, x_j, \mathcal{L}_j, z_j, z_k)$ is not oriented.

Case C^1 or C^2 is oriented. By Lemma 4.23, $\mathcal{C} = \{C \subseteq D \mid C \text{ is an oriented cycle }\} \subseteq \{C^1, C^2\}$. Thus, by Lemma 4.20, there is a minimum binomial generating set $\mathcal{B}$ such that $y_{C^1} \in \mathcal{B}$ or $y_{C^2} \in \mathcal{B}$. Without loss of generality, we can assume $y_{C^2} \in \mathcal{B}$. Then, by Proposition 4.19, $P_{D_{/C^2}}$ is a binomial complete intersection. So, by Proposition

4.22, there is an oriented path $(x_i, x_j, \mathcal{L}_j, z)$ in $D_{\diagup C^2}$, since $D_{\diagup C^2}$ is a pyramid. Thus, there is $k \in \{1,2,3\} \setminus \{j\}$ such that $(x_i, x_j, \mathcal{L}_j, z_j, z_k)$ is an oriented path in D, since C^2 is an oriented cycle. A contradiction.

Case C^1 and C^2 are not oriented. Thus, by Lemma 4.23, D has not oriented cycles. We can assume $V(y_{j_1}) = \{x_{j_2}, x_{j_3}\}$ and $V(y'_{j_1}) = \{z_{j_2}, z_{j_3}\}$ for all $\{j_1, j_2, j_3\} = \{1,2,3\}$; furthermore $y_{C^1} = y_1 - y_2y_3$ and $y_{C^2} = y'_{r_1} - y'_{r_2}y'_{r_3}$. We take the cycles $C^{ij} = (x_i, \mathcal{L}_i, z_i, z_j, \mathcal{L}_j, x_j, x_i)$. Without loss of generality, we can assume $y_k = (x_i, x_j) \in E(D)$ where $\{i,j,k\} = \{1,2,3\}$, then $y_{C^{ij}} = y_k y_{\mathcal{L}_j^+} y_{\mathcal{L}_i^-} - y'_k y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+}$ if $y'_k = (z_i, z_j) \in E(D)$ or $y_{C^{ij}} = y_k y'_k y_{\mathcal{L}_j^+} y_{\mathcal{L}_i^-} - y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+}$ if $y'_k = (z_j, z_i) \in E(D)$. We take a minimum binomial generating set $\mathcal{B}$ of P_D. By Remark 1.114, we can assume $gcd(m'_1, m'_2) = 1$ for each $m'_1 - m'_2 \in \mathcal{B}$. Now, for each $m_1 \in \mathcal{A} := Mon(y_{C^1}, y_{C^2}, y_{C^{1,2}}, y_{C^{1,3}}, y_{C^{2,3}})$ there is $m_2 - m'_2 \in \mathcal{B}$ such that $m_2 \mid m_1$. By Theorem 1.111, there is a cycle C with $y_C = y^\alpha - y^\beta$ such that $y^\alpha \mid m_2$ and $y^\beta \mid m'_2$. Then, $y^\alpha \mid m_1$. Since D has no oriented cycles, $y^\alpha \neq 1$. If $m_1 \in \mathcal{A}_1 := \{y_1, y'_{r_1}, y_2y_3, y'_{r_2}y'_{r_3}\}$, then by Lemma 4.23, $y^\alpha = m_2 = m_1$. Now, if $m_1 \in \mathcal{A}'_2 := \{m \in \mathcal{A} \mid m = y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+}\}$, then $gcd(y^\alpha, y_{\mathcal{L}_j^-}) \neq 1$ or $gcd(y^\alpha, y_{\mathcal{L}_i^+}) \neq 1$, since $y^\alpha \neq 1$. Thus, by 2) in Lemma 4.15, $y_{\mathcal{L}_i^-} \mid y^\alpha$ or $y_{\mathcal{L}_i^+} \mid y^\alpha$. So, by Lemma 4.23 and 2) in Lemma 4.15, $y^\alpha = y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+} = m_2 = m_1$, since $y^\alpha \mid m_1$ and $y^\alpha \mid m_2$. Now, if $m_1 \in \mathcal{A}''_2 = \{m \in \mathcal{A} \mid m = y_k y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+}$ or $m = y'_k y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+}$ such that $y_k \neq y_1$ or $y'_k \neq y'_{r_1}\}$. We prove that $m_1 = y^\alpha$. By contradiction, suppose $m_1 \neq y^\alpha$. We can assume $m_1 = y'_k y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+}$. So, by Lemma 4.23 and 2) in Lemma 4.15, $y'_k y_{\mathcal{L}_j^-} \mid y^\alpha$ or $y'_k y_{\mathcal{L}_i^+} \mid y^\alpha$ or $y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+} \mid y^\alpha$, since $y^\alpha \mid m_1$ and $y'_k \neq y'_{r_1}$. But $y^\alpha \neq m_1$ implies $(z_i, z_k, \mathcal{L}_k, x_k, x'_i)$ or $(z_j, z_k, \mathcal{L}_k, x_k, x'_i)$ or (z_i, z_k, z_j) is an oriented path contained in C (where $i' \neq k$). A contradiction, by hypothesis and $y'_k \neq y'_{r_1}$. Hence, $y^\alpha = m_1$ implies $m_2 = m_1$.

If $m_2 \in \mathcal{A}_2 := \mathcal{A}'_2 \cup \mathcal{A}''_2$ and $m'_2 \in \mathcal{A}_1 \cup \mathcal{A}_2$, then we prove $\{m_2, m'_2\} = \{y_k y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+}, y'_k y_{\mathcal{L}_j^+} y_{\mathcal{L}_i^-}\}$. We have $m_2 = y^\alpha = y_{C^+}$ and $\mathcal{L}_j^- \cup \mathcal{L}_i^+ \subseteq C^+$, since $m_2 \in \mathcal{A}_2$. So, by 1) in Lemma 4.15, $\mathcal{L}_i \cup \mathcal{L}_j \subseteq C$. Furthermore, $\mathcal{L}_j^+ \cup \mathcal{L}_i^- \subseteq C^-$, since $m_2 = y_{C^+}$. Then, $y_{\mathcal{L}_j^+} y_{\mathcal{L}_i^-} \mid y^\beta$, since $y^\beta = y_{C^-}$. Now, if $y_k \notin E(C)$, then $y_i, y_j \in E(C)$, since $x_i \in V(\mathcal{L}_i) \subseteq V(C)$ and $x_j \in V(\mathcal{L}_j) \subseteq V(C)$. Hence, $y_k \in E(C)$ or $y_i, y_j \in E(C)$. Similarly, $y'_k \in E(C)$ or $y'_i y'_j \in E(C)$. Thus, if $m_2 \in \mathcal{A}'_2$, then $y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+} = m_2 = y^\alpha = y_{C^+}$ implies $y_k y'_i y'_j \mid y^\beta$ or $y_k y'_k \mid y^\beta$ or $y'_k y_i y_j \mid y^\beta$ or $y_i y_j y'_i y_{j'} \mid y^\beta$, since $y^\beta = y_{C^-}$. So, $m'_2 \notin \mathcal{A}_1 \cup \mathcal{A}_2$, since $y^\beta \mid m'_2$. A contradiction, then $m_2 \in \mathcal{A}''_2$. Thus, we can assume $m_2 = y_k y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+}$. Consequently, $y'_k y_{\mathcal{L}_j^+} y_{\mathcal{L}_i^-} \mid y^\beta$ or $y'_i y'_j y_{\mathcal{L}_j^+} y_{\mathcal{L}_i^-} \mid y^\beta$, since $y^\alpha = m_2$. But $m'_2 \in \mathcal{A}_1 \cup \mathcal{A}_2$, then $m'_2 = y'_k y_{\mathcal{L}_j^+} y_{\mathcal{L}_i^-}$, since $y^\beta \mid m'_2$.

Now, we study two cases:

Case $r_1 \neq 1$ or $y'_1 = (z_3, z_2)$. Then, for each $(i,j) \in \{(1,2),(2,3),(1,3)\}$ there is $f_{ij} = m_{ij} - m'_{ij} \in \mathcal{B}$ such that $m_{ij} = y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+}$ or $m_{ij} = y_k y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^+}$ or $m_{ij} = y'_k y_{\mathcal{L}_j^-} y_{\mathcal{L}_i^-}$; furthermore, if $m'_{ij} \in \mathcal{A}_1 \cup \mathcal{A}_2$, then $m_{ij}, m'_{ij} \in \mathcal{A}'_2$. So, $|\mathcal{B}'| = 3$, where $\mathcal{B}' = \{f_{1,2}, f_{2,3}, f_{1,3}\}$. Furthermore, $\mathcal{A}_1 \subseteq Mon(\mathcal{B} \setminus \mathcal{B}')$, then $|\mathcal{B}| \geqslant |\mathcal{B}'| + |\mathcal{A}_1|/2 = 3 + 2 = 5$. But $ht(P_D) = 4$. Therefore, P_D is not a binomial complete intersection.

Case $r_1 = 1$ and $y'_1 = (z_2, z_3)$. We have $y_1 = (x_2, x_3)$ and

$$\begin{aligned} y_{C^1} &= y_1 - y_2\, y_3; \quad y_{C_2} = y'_1 - y'_2\, y'_3 \\ y_{C^3} &= y_3\, y_{\mathcal{L}_1^+}\, y_{\mathcal{L}_2^-} - y'_3\, y_{\mathcal{L}_1^-}\, y_{\mathcal{L}_2^+} \\ y_{C^4} &= y_1\, y_{\mathcal{L}_3^+}\, y_{\mathcal{L}_2^-} - y'_1\, y_{\mathcal{L}_3^-}\, y_{\mathcal{L}_2^+} \\ y_{C^5} &= y_2\, y_{\mathcal{L}_1^-}\, y_{\mathcal{L}_3^+} - y'_2\, y_{\mathcal{L}_1^+}\, y_{\mathcal{L}_3^-}. \end{aligned}$$

In this case, $\mathcal{A}_2 = \mathcal{A}'_2 = Mon(y_{C^3}, y_{C^5})$ and $\mathcal{A}_1 = Mon(y_{C^1}, y_{C^2})$. Assume $|\mathcal{B}| = ht(P_D) = 4$, then $\mathcal{B} = \{y_{C^1}, y_{C^2}, y_{C^3}, y_{C^5}\}$, since $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq Mon(\mathcal{B})$. Hence,

$$y_{C^4} = g_1 y_{C^1} + g_2 y_{C^2} + g_3 y_{C^3} + g_5 y_{C^5}. \tag{4.1}$$

Furthermore, $y_{\mathcal{L}_1^-} \neq 1$ and $y_{\mathcal{L}_1^+} \neq 1$, since $(x_2, y_3, x_1, \mathcal{L}_1, z_1, y'_2, z_3)$ and $(x_3, y_2, x_1, \mathcal{L}_1, z_1, y'_3, z_2)$ are not oriented paths. If we evaluate $y_{\mathcal{L}_1^+} = y_{\mathcal{L}_1^-} = 0$, $y_{\mathcal{L}_3^+} = y_{\mathcal{L}_3^-} = y_{\mathcal{L}_2^+} = y_{\mathcal{L}_2^-} = 1$, $y_1 = y_2 = y_3 = x$ and $y'_1 = y'_2 = y'_3 = 0$, in the equation (4.1), then we obtain that $x = (x - x^2) f_1$ with $f_1 \in k[x]$. But $deg_x\big((x - x^2) f_1\big) \geqslant 2$. A contradiction, then $|\mathcal{B}| \geqslant ht(P_D)$. Therefore, P_D is not a binomial complete intersection.

□

Proposition 4.25 Let D be an oriented partial wheel with center z and rim C. If z is neither a source nor a sink; or $C = (x, \mathcal{L}, x', \mathcal{L}', x'', \mathcal{L}'', x)$ where $\mathcal{L}, \mathcal{L}''$ are oriented, $x, x', x'' \in N_D(z)$, $N_D(z) \cap V\big((\mathcal{L}')^\circ\big) = \emptyset$ and $(z, x, \mathcal{L}, x')$ or $(z, x, \mathcal{L}'', x'')$ is an oriented path, then P_D is a binomial complete intersection.

Proof. We set $C = (z_1, \mathcal{L}_1, z_2, \ldots, z_{k-1}, \mathcal{L}_{k-1}, z_k, \mathcal{L}_k, z_{k+1} = z_1)$ and $C^i = (z, z_i, \mathcal{L}_i, z_{i+1}, z)$ where $N_D(z) = \{z_1, \ldots, z_k\}$ and $\mathcal{L}_i$ is a path between z_i and z_{i+1}.

First, assume z is neither a source nor a sink, then without loss of generality, we can suppose $(z, z_1), (z_j, z) \in E(D)$. We take, H_2 the $\mathcal{P}_2$-sum of C^1 and C^2, where $\mathcal{P}_2 = (z, z_2)$; H_3 the $\mathcal{P}_3$-sum of H_2 and C^3 where $\mathcal{P}_3 = (z, z_3)$, continuing with this process, we obtain H_{j-1} the $\mathcal{P}_{j-1}$-sum of H_{j-2} and C^{j-1} where $\mathcal{P}_{j-1} = (z, z_{j-1})$. Now, we take H'_{j+1} the $\mathcal{P}_{j+1}$-sum of C^j and C^{j+1} where $\mathcal{P}_{j+1} = (z, z_{j+1})$; H'_{j+2} the

$\mathcal{P}_{j+2}$-sum of H'_{j+1} and C^{j+2} where $\mathcal{P}_{j+2} = (z, z_{j+2})$; continuing with this process we obtain H'_k the $\mathcal{P}_k$-sum of H'_{k-1} and C^k where $\mathcal{P}_k = (z, z_k)$. Then, D is the $\mathcal{P}$-sum of H_{j-1} and H'_k, where $\mathcal{P} = (z_1, z, z_j)$. Hence, by Theorem 4.6, P_D is a binomial complete intersection.

Now, assume $C = (x, \mathcal{L}, x', \mathcal{L}', x'', \mathcal{L}'', x)$ where $N_G(z) \cap V\big((\mathcal{L}')^\circ\big) = \emptyset$; $x, x', x'' \in N_G(z)$ and $(z, x, \mathcal{L}, x')$ or $(z, x, \mathcal{L}'', x'')$ is an oriented path. Then, without loss of generality, we can assume $x = z_1$, $x' = z_t$, $x'' = z_{t+1}$ and $\tilde{\mathcal{L}} := (z, x, \mathcal{L}, x') = (z, z_1, \mathcal{L}, z_t)$ is an oriented path, where $\mathcal{L} = (z_1, \mathcal{L}_1, z_2, \dots, \mathcal{L}_{t-1}, z_t)$. We take $\tilde{H}_2$ the $\tilde{\mathcal{P}}_2$-sum of C^1 and C^2, where $\tilde{\mathcal{P}}_2 = (z, z_2)$, $\tilde{H}_3$ the $\tilde{\mathcal{P}}_3$-sum of $\tilde{H}_2$ and C^3 where $\tilde{\mathcal{P}}_3 = (z, z_3)$; continuing with this process, we obtain $\tilde{H}_{t-1}$ the $\tilde{\mathcal{P}}_{t-1}$-sum of $\tilde{H}_{t-2}$ and C^{t-1} where $\tilde{\mathcal{P}}_{t-1} = (z, z_{t-1})$. Now, we take $\tilde{H}_{t+2}$ the $\tilde{\mathcal{P}}_{t+2}$-sum of C^{t+1} and C^{t+2} where $\tilde{\mathcal{P}}_{t+2} = (z, z_{t+2})$; $\tilde{H}_{t+3}$ the $\tilde{\mathcal{P}}_{t+3}$-sum of $\tilde{H}_{t+2}$ and C^{t+3} where $\tilde{P}_{t+3} = (z, z_{t+3})$. Continuing with this process, we obtain $\tilde{H}_k$ the $\tilde{\mathcal{P}}_k$-sum of $\tilde{H}_{k-1}$ and C^k where $\tilde{\mathcal{P}}_k = (z, z_k)$. Furthermore, we take D_1 the $\mathcal{L}''$-sum of $\tilde{H}_k$ and C, since $\tilde{H}_k \cap C = \mathcal{L}''$ is an oriented path. Hence, D is a $\tilde{\mathcal{L}}$-sum of D_1 and $\tilde{H}_{t-1}$, since $D_1 \cap \tilde{H}_{t-1} = \tilde{\mathcal{L}}$ is an oriented path. Therefore, by Theorem 4.6, P_D is a binomial complete intersection. □

4.3 SPECIAL ORIENTED SUBGRAPHS

In this section $D = (G, \mathcal{O})$ is an oriented graph. We prove that if P_D is a binomial complete intersection and we obtain D' from D deleting a special path, then $P_{D'}$ is also a binomial complete intersection (see Proposition 4.26). Furthermore, in Proposition 4.33, we prove that if G has a theta with some conditions, then P_D is not a binomial complete intersection. These results are interesting because the binomial complete intersection property is not closed under induced subgraphs (see Example 4.35).

Proposition 4.26 Let $\mathcal{P}$ be a non-oriented path of D such that $deg_G(x) = 2$ for each $x \in V(\mathcal{P}^\circ)$ and $G' = G \setminus V(\mathcal{P}^\circ)$ is connected. If P_D is a binomial complete intersection, then $P_{D'}$ is a binomial complete intersection with $D' = G'_{\mathcal{O}}$.

Proof. Since P_D is a binomial complete intersection, there is a minimal binomial generating set $\mathcal{G} = \{f_1, \dots, f_s\}$ of P_D, where $s = ht(P_D) = m - n + 1$ and $f_i = m_i - m'_i$ for $i = 1, \dots, s$. Also, $y_{\mathcal{P}^+} \neq 1$ and $y_{\mathcal{P}^-} \neq 1$, since $\mathcal{P}$ is not oriented. By Remark 1.114, we can assume $gdc(m_i, m'_i) = 1$. We take $\mathcal{G} = \{f_1, \dots, f_{s'}\} \cup \{f_{s'+1}, \dots, f_s\}$ where $\{f_1, \dots, f_{s'}\} = \{f_i \in \mathcal{G} \mid gcd(y_{\mathcal{P}}, m_i m'_i) \neq 1\}$. By 2) in Lemma 4.15, we can assume $f_i = y_{\mathcal{P}^+} n_i - y_{\mathcal{P}^-} n'_i$ for $i = 1, \dots, s'$ and $gcd(y_{\mathcal{P}}, m_i m'_i) = 1$ for $j =$

$s'+1,\ldots,s$. By 3) in Lemma 4.15, $\mathcal{B}' = \{f \in \mathcal{B} \mid f \in P_{D'}\} = \{f_{s'+1},\ldots,f_s\}$ is a binomial generating set of $P_{D'}$. Furthermore, $ht(P_{D'}) = (m-l)-(n-l+1)+1 = m-n = s-1$, since D' is connected. Then, $s-s' = |\mathcal{B}'| \geqslant ht(P_{D'}) = s-1$ implies $s' \leqslant 1$. Let x, x' be the end vertices of $\mathcal{P}$, then there is a path $\mathcal{Q}$ in D' between x and x', since D' is connected. Now, we take the cycle $C = (\mathcal{P}, \mathcal{Q})$, then $y_C \in P_D$. Consequently,

$$y_C = y_{\mathcal{P}^+}y_{\mathcal{Q}^-} - y_{\mathcal{P}^-}y_{\mathcal{Q}^+} = \sum_{i=1}^{s} h_i f_i \tag{4.2}$$

Evaluating $y_{l+1} = \cdots = y_m = 1$ in (4.2), we have $y_{\mathcal{P}^+} - y_{\mathcal{P}^-} = \sum_{i=1}^{s'} \tilde{h}_i f'_i$ where $\tilde{h}_i = h_i \mid_{y_{l+1}=\cdots=y_m=1}$ and $f'_i = f_i \mid_{y_{l+1}=\cdots=y_m=1}$ for $1 \leqslant i \leqslant s'$, since $f_j \mid_{y_{l+1}=\cdots=y_m=1} = 0$ for $s'+1 \leqslant j \leqslant s$. This implies $1 \leqslant s'$, since $y_{\mathcal{P}^+} \neq y_{\mathcal{P}^-}$. Therefore $s'=1$ and $P_{D'}$ is a binomial complete intersection. □

Definition 4.27 We define the following set of edges:

$$\mathcal{A}(\mathbf{D}) := \{y \in E(D) \mid y \text{ is contained in an oriented cycle of } D\}.$$

Lemma 4.28 Let $\mathcal{L}$ be a path between x and x'. If $E(\mathcal{L}^+) \subseteq \mathcal{A}(D)$ or $E(\mathcal{L}^-) \subseteq \mathcal{A}(D)$, then there is an oriented walk between x and x'.

Proof. Without loss of generality, we can assume $E(\mathcal{L}^-) \subseteq \mathcal{A}(D)$. So, if $y = (a, a') \in E(\mathcal{L}^-)$, then there is an oriented cycle C^y such that $y \in E(C^y)$. Thus, $\mathcal{L}_y := C^y - \{y\}$ is an oriented path from a' to a. We set $\mathcal{L} = (x_1, y_1, x_2, \ldots, x_s, y_s, x_{s+1})$ where $x = x_1$ and $x' = x_{s+1}$. We take $\mathcal{L}' = (x_1, \mathcal{L}_1, x_2, \ldots, x_s, \mathcal{L}_s, x_{s+1})$, where $\mathcal{L}_i = y_i$ if $y_i \in E(\mathcal{L}^+)$ or $\mathcal{L}_i = \mathcal{L}_{y_i}$ if $y_i \in E(\mathcal{L}^-)$. Hence, $\mathcal{L}'$ is an oriented walk between x and x'. □

Lemma 4.29 If $\mathcal{L}$ is an oriented walk from z to z', then there is an oriented path from z to z'.

Proof. For each walk $W = (x_{j_1}, y_{i_1}, x_{j_2}, \ldots, y_{i_r}, x_{j_{r+1}})$ we define $\text{int}(W) = |\{k \mid x_{j_k} = x_{j_{k'}} \text{ for some } k' \neq k\}|$. We take an oriented walk $\mathcal{L}' = (z = x_{j'_1}, y_{i'_1}, x_{j'_2}, \ldots, y_{i'_s}, x'_{j_s+1} = z')$ from z to z' such that $\text{int}(\mathcal{L}')$ is minimal. We prove $\text{int}(\mathcal{L}') = 0$. By contradiction, suppose $\text{int}(\mathcal{L}') > 0$, then there are k and k' such that $x_{j'_k} = x_{j'_{k'}}$. We can assume $k' > k$, then $\mathcal{L}'' = (x_{j'_1}, y_{i'_1}, \ldots, y_{i'_{k-1}}, x_{j'_k} = x_{j'_{k'}}, y_{i'_{k'}}, \ldots, y_{i'_s}, x_{j'_{s+1}})$ is a walk from z to z' and $\text{int}(\mathcal{L}'') < \text{int}(\mathcal{L}')$. A contradiction, since $\text{int}(\mathcal{L}')$ is minimal. Hence, int $(\mathcal{L}') = 0$ implies $\mathcal{L}'$ is a path. □

Remark 4.30 Assume $1 - u \in P_D$ and $y \mid u$, where $y \in E(D)$ and $u \in Mon(R)$. By Corollary 1.112, there are cycles $C^1, \ldots, C^s$ such that $1 = u_1 \cdots u_s$ and $u = u'_1 \cdots u'_s$ with $y_{C^i} = u_i - u'_i$ for $1 \leqslant i \leqslant s$. Since $y \mid u$, there is $j \in \{1, \ldots, s\}$ such that $y \mid u'_j$. Hence, $y \in V(C^j)$ and C^j is an oriented cycle, since $y_{C^j} = 1 - u'_j$. Therefore $y \in \mathcal{A}(D)$.

Definition 4.31 If $f \in R = \mathbb{K}[y_1, \ldots, y_m]$ and $A = \{y_{i_1}, \ldots, y_{i_s}\} \subseteq \{y_1, \ldots, y_m\}$, then we define $\mathbf{f} \mid_{\mathbf{A}} := f \mid_{y_{i_1} = \cdots = y_{i_s} = 1}$.

Lemma 4.32 Let $f = u - u'$ be a binomial of P_D. If $f \mid_{\mathcal{A}(D)} \neq 0$, then $u \mid_{\mathcal{A}(D)} \neq 1$ and $u' \mid_{\mathcal{A}(D)} \neq 1$.

Proof. By contradiction, suppose $u \mid_{\mathcal{A}(D)} = 1$ or $u' \mid_{\mathcal{A}(D)} = 1$. Without loss of generality, we can assume $u \mid_{\mathcal{A}(D)} = 1$. If $u = 1$, then by Remark 4.30, $u' \mid_{\mathcal{A}(D)} = 1$. Thus, $f \mid_{\mathcal{A}(D)} = 0$. A contradiction, then $u \neq 1$. Hence, $u = y_{i_1}^{\alpha_1} \cdots y_{i_s}^{\alpha_s}$ with $\{y_{i_1}, \ldots, y_{i_s}\} \subseteq \mathcal{A}(D)$. Thus, there are oriented cycles $C^1, \ldots, C^s$ such that $y_{i_j} \in E(C^j)$ for each $j = 1, \ldots, s$. Then, $\mathcal{L}_j = C^j - \{y_{i_j}\}$ is an oriented path and $y_{C^j} = 1 - y_{\mathcal{L}_j} y_{i_j} \in P_D$. Hence, $1 - (y_{i_1} y_{\mathcal{L}_1})^{\alpha_1} \cdots (y_{i_s} y_{\mathcal{L}_s})^{\alpha_s} \in P_D$ implies $1 - u'(y_{\mathcal{L}_1}^{\alpha_1} \cdots y_{\mathcal{L}_s}^{\alpha_s}) = \big(1 - (y_{i_1} y_{\mathcal{L}_1})^{\alpha_1} \cdots (y_{i_s} y_{\mathcal{L}_s})^{\alpha_s}\big) + (y_{\mathcal{L}_1}^{\alpha_1} \cdots y_{\mathcal{L}_s}^{\alpha_s})(u - u') \in P_D$. So, if $y \mid u'$, then by Remark 4.30, $y \in \mathcal{A}(D)$. Consequently, $u' \mid_{\mathcal{A}(D)} = 1$ implies $f \mid_{\mathcal{A}(D)} = u \mid_{\mathcal{A}(D)} - u' \mid_{\mathcal{A}(D)} = 1 - 1 = 0$. A contradiction, therefore, $u \mid_{\mathcal{A}(D)} \neq 1$ and $u' \mid_{\mathcal{A}(D)} \neq 1$. □

Proposition 4.33 Let θ be a theta of G with end vertices x and z such that $deg_G(a) = 2$ for each $a \in V(\theta) \setminus \{x, z\}$. If D is connected and D has no oriented paths between x and z, then P_D is not a binomial complete intersection.

Proof. By contradiction, suppose there is a binomial generating set $\mathcal{B}$ of P_D such that $|\mathcal{B}| = ht(P_D) = m - n + 1$, since D is connected. We can assume $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are the principal paths of θ. By hypothesis, $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are not oriented. We take $D' = D \setminus V(\mathcal{L}_1^{\circ})$. Then, D' is connected, since D is connected and $\mathcal{L}_2, \mathcal{L}_3 \subseteq D'$. Also, $ht(P_{D'}) = |E(D')| - |V(D')| + 1 = m - n = ht(P_D) - 1$, since $|E(\mathcal{L}_1^{\circ})| = |V(\mathcal{L}_1^{\circ})| + 1$. By 3) in Lemma 4.15, $\mathcal{B}' = \{f \in \mathcal{B} \mid f \in P_{D'}\}$ is a generating set of $P_{D'}$. So, $|\mathcal{B}'| \geqslant ht(P_{D'}) = ht(P_D) - 1 = |\mathcal{B}| - 1$ implies $|\mathcal{B} \setminus \mathcal{B}'| \leqslant 1$. We take the cycles $C^2 = \mathcal{L}_1 \cup \mathcal{L}_2$ and $C^3 = \mathcal{L}_1 \cup \mathcal{L}_3$, then $y_{C^j} = m_j - m'_j \in P_D$ where $m_j = y_{\mathcal{L}_1^+} y_{\mathcal{L}_j^-}$ and $m'_j = y_{\mathcal{L}_1^-} y_{\mathcal{L}_j^+}$ for $j = 2, 3$. Hence, $y_{C^j} = \sum_{g_i \in \mathcal{B}} f_i^j g_i$ where $f_i^j \in k[y_1, \ldots, y_m]$. We take $\mathcal{B}_1 = \{g_i \in \mathcal{B} \mid g_i \mid_{\mathcal{A}(D)} = 0\}$ and $\mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1$. By 1) in Lemma 4.15, $E(\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3) \cap \mathcal{A}(D) = \emptyset$. Thus, $m_j - m'_j = y_{C^j} = y_{C^j} \mid_{\mathcal{A}(D)} = \sum_{g_i \in \mathcal{B}_2} f_i^j \mid_{\mathcal{A}(D)}$

$g_i \mid_{\mathcal{A}(D)}$, since $g_i \mid_{\mathcal{A}(D)} = 0$ if $g_i \in \mathcal{B}_1$. So, for $j = 2, 3$, there is $g_{i_j} = m_{i_j} - m'_{i_j}$ such that $h_{i_j} \mid m_j$ or $h'_{i_j} \mid m'_j$ where $g_{i_j} \mid_{\mathcal{A}(D)} = h_{i_j} - h'_{i_j}$. By Lemma 4.32, $h_{i_j} \neq 1$ and $h'_{i_j} \neq 1$. Without loss of generality, we can suppose $h_{i_j} \mid m_j$. Since $h_{i_j} \neq 1$, there is $y \in E(D)$ such that $y \mid h_{i_j}$ implies $y \mid m_{i_j}$ and $y \mid m_j$. By Theorem 1.111, there is a cycle C such that $y \in E(C)$ and $y_{C^+} \mid m_{i_j}$ and $y_{C^-} \mid m'_{i_j}$. Furthermore, $y \mid m_j$, then $y \in E(\mathcal{L}_1) \cup E(\mathcal{L}_j)$, since $m_j = y_{\mathcal{L}_1^+} y_{\mathcal{L}_j^-}$. Without loss of generality, we can assume $y \in E(\mathcal{L}_1)$. Hence, by 1) in Lemma 4.15, $\mathcal{L}_1 \subseteq C$. Consequently, $C = \mathcal{L}_1 \cup \mathcal{L}$, where $\mathcal{L}$ is a path between x and z. Also, $C^+ = \mathcal{L}_1^+ \cup \mathcal{L}^-$ and $C^- = \mathcal{L}_1^- \cup \mathcal{L}^+$. By Lemmas 4.28 and 4.29, there is $y' \in E(D) \setminus \mathcal{A}(D)$ such that $y' \in E(\mathcal{L}^-) \subseteq E(C^+)$. Thus, $y' \mid y_{C^+}$ implies $y' \mid m_{i_j}$. Hence $y' \mid h_{i_j}$, since $y' \notin \mathcal{A}(D)$. Consequently, $y' \mid m_j$, since $h_{i_j} \mid m_j$. But $C = \mathcal{L} \cup \mathcal{L}_1$, then $y' \notin E(\mathcal{L}_1)$ implies $y' \nmid y_{\mathcal{L}_1^+}$. Thus, $y' \mid y_{\mathcal{L}_j^-}$, since $m_j = y_{\mathcal{L}_1^+} y_{\mathcal{L}_j^-}$. Then, by 2) in Lemma 4.15, $y_{\mathcal{L}_j^-} \mid y_{C^+}$ and $y_{\mathcal{L}_j^+} \mid y_{C^-}$, since $y' \in E(C^+) \cap E(\mathcal{L}_j^-)$. So, $y_{\mathcal{L}_j^-} \mid m_{i_j}$ and $y_{\mathcal{L}_1^+} \mid m_{i_j}$, since $y_{C^+} \mid m_{i_j}$, $y_{\mathcal{L}_j^-} \mid y_{C^+}$ and $\mathcal{L}_1^+ \subseteq C^+$. Thus, $m_j \mid m_{i_j}$, since $m_j = y_{\mathcal{L}_1^+} y_{\mathcal{L}_j^-}$ and $E(\mathcal{L}_1) \cap E(\mathcal{L}_j) = \emptyset$. Thus, by 2) in Lemma 4.15, $y_{\mathcal{L}_1^-} y_{\mathcal{L}_j^+} \mid m'_{i_j}$, i.e. $m'_j \mid m'_{i_j}$. Consequently, $g_{i_2} = m_2 \ell_2 - m'_2 \ell'_2$ and $g_{i_3} = m_3 \ell_3 - m'_3 \ell'_3$ where $supp(\ell_2) \cup supp(\ell_3) \subseteq \mathcal{A}(D)$, since $h_{i_j} \mid m_j$. Furthermore, by 1) in Lemma 4.15, $\big(E(\mathcal{L}_2) \cup E(\mathcal{L}_3)\big) \cap \mathcal{A}(D) = \emptyset$. Hence, $g_{i_2}, g_{i_3} \in \mathcal{B} \setminus \mathcal{B}'$ and $g_{i_2} \neq g_{i_3}$. This is a contradiction, since $|\mathcal{B} \setminus \mathcal{B}'| \leqslant 1$. □

Remark 4.34 Not oriented path condition is indispensable in Proposition 4.26. In Example 4.35, P_D is a binomial complete intersection, but $P_{D'}$ is not a binomial complete intersection, where $G' = G \setminus V(\mathcal{L}^\circ)$, $\mathcal{L} = \mathcal{L}_5 \cup \mathcal{L}_6$ and $deg_G(x) = 2$ for each $x \in V(\mathcal{L}^\circ)$.

4.4 EXAMPLES

Example 4.35 Let D' be the partial wheel of Figure 4.1 (b), whose rim is $C = (z_1, \mathcal{L}_1, z_2, \ldots, z_4, \mathcal{L}_4, z_5 = z_1)$ and center z such that $N_{G'}(z) = \{z_1, z_2, z_3, z_4\}$. In the Figure 4.1 (a), $D = D' \cup \mathcal{L}$ where $\mathcal{L}$ is the path $\mathcal{L} = (z_3, \mathcal{L}_5, x', \mathcal{L}_6, z_4)$. The cycles without chords of D are C, $C^i = (z, z_i, \mathcal{L}_i, z_{i+1}, z)$ for $1 \leqslant i \leqslant 4$, $C^5 = (x', \mathcal{L}_5, z_3, \mathcal{L}_3, z_4, \mathcal{L}_6, x')$ and $C' = (x', \mathcal{L}_6, z_4, \mathcal{L}_4, z_1, \mathcal{L}_1, z_2, \mathcal{L}_2, z_3, \mathcal{L}_5, x')$ whose binomials are:

$$y_{C^1} = y_1 - y_2\, y_{\mathcal{L}_1}$$

$$y_{C^2} = y_3 - y_2\, y_{\mathcal{L}_2}$$
$$y_{C^3} = y_3 - y_4\, y_{\mathcal{L}_3}$$
$$y_{C^4} = y_1 - y_4\, y_{\mathcal{L}_4}$$
$$y_{C^5} = 1 - y_{\mathcal{L}_3}\, y_{\mathcal{L}_5}\, y_{\mathcal{L}_6}$$
$$y_C = y_{\mathcal{L}_2}\, y_{\mathcal{L}_4} - y_{\mathcal{L}_1}\, y_{\mathcal{L}_3}$$
$$y_{C'} = y_{\mathcal{L}_1} - y_{\mathcal{L}_2}\, y_{\mathcal{L}_4}\, y_{\mathcal{L}_5}\, y_{\mathcal{L}_6}.$$

We have, $y_{C^4} \in (y_{C'}, y_{C^1}, y_{C^2}, y_{C^3}, y_{C^5})$ and $y_C \in (y_{C'}, y_{C^5})$. Hence, by Proposition 1.113, $\mathcal{G} = \{y_{C'}, y_{C^1}, y_{C^2}, y_{C^3}, y_{C^5}\}$ is a minimum binomial generating set of P_D, since $ht(P_D) = 5$. Therefore, P_D is a binomial complete intersection. Furthermore, $\mathcal{G}' = \{y_C, y_{C^1}, \ldots, y_{C^4}\}$ is a minimum binomial generating set of $P_{D'}$ but $ht(P_{D'}) = 4$. Therefore, $P_{D'}$ is not a binomial complete intersection.

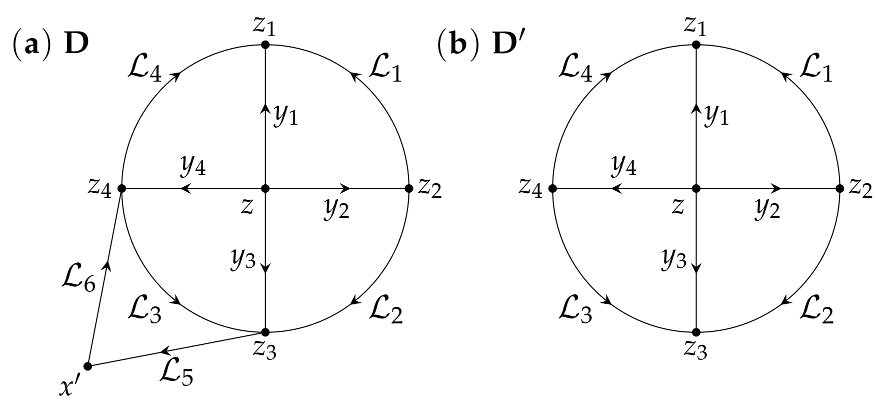


Figure 4.1. A graph D with the binomial complete intersection property and an induced subgraph of D' without this property

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NOTATION

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