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Resumen

En esta disertación presentamos una construcción matemática rigurosa de ciertas teorías cuánticas de campo p-ádicas, cuyas amplitudes de N puntos están dadas por los valores esperados de productos de operadores de vértice. Mostramos que este tipo de amplitudes admiten un desarrollo en serie donde cada término es una función zeta local de Igusa. El primer término de esta serie es una versión regularizada de la amplitud de cuerda abierta p-ádica de Koba-Nielsen. En la década de 1980 los físicos obtuvieron amplitudes de cuerdas abiertas p-ádicas de Koba-Nielsen mediante cálculos formales. El objetivo central de este trabajo es proporcionar un marco matemático para comprender dichos cálculos. Esta tesis está basada en la publicación: A.R. Fuquen-Tibatá, H. García-Compeán and W.A. Zúñiga-Galindo, *Euclidean quantum field formulation of p-adic open string amplitudes*, Nuclear Physics B. **975** (2022), 115684.

Abstract

In this dissertation we present a rigorous mathematical construction of certain p-adic quantum field theories, whose N-point amplitudes are the expectation of products of vertex operators. We show that this type of amplitudes admit a series expansion where each term is an Igusa local zeta function. The first term in this series is a regularized version of the p-adic Koba-Nielsen open string amplitude. In the 1980s physicists obtained p-adic Koba-Nielsen open string amplitudes by formal calculations. The central goal of this work is to provide a mathematical framework to understand such calculations. This dissertation is based on the publication: A.R. Fuquen-Tibatá, H. García-Compeán and W.A. Zúñiga-Galindo, *Euclidean quantum field formulation of p-adic open string amplitudes*, Nuclear Physics B. **975** (2022), 115684.

Contents

Introduction

In this thesis we provide a mathematical construction of a class of quantum field theories whose amplitudes are expectations of products of vertex operators. Using this approach, we carry out a rigorous mathematical derivation of the N-point Koba-Nielsen amplitudes.

String amplitudes were introduced by Veneziano in 1968 [53], further generalizations were obtained by Virasoro [54], Koba and Nielsen [43], among others. In the 80s, Freund, Witten and Volovich, studied string amplitudes at the tree-level over different number fields, and suggested the existence of connections between these amplitudes (see e.g. [14] and [56]). In this framework the connections with number theory, specifically with local zeta functions, occur naturally (see [4]-[6] and [9]-[12]).

p-Adic string theories have been studied over the time with some periodic fluctuations in their interest (see [14], [19], [39], [55]). Recently, a considerable amount of work has been developed on this topic in the context of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [20], [34], [33], [36]. String theory with a p-adic worldsheet was proposed and studied for the first time in [23]. Later, this theory was formally known as p-adic string theory. p-Adic strings are related to ordinary strings at least in two different ways. First, connections through the adelic relations [24], and second, through the limit when p tends to 1 in the tree-level p-adic string amplitudes [28], [29].

The tree-level string amplitudes (without loops) were explicitly computed in the case of p-adic string worldsheet in [14] and [22]. Since the 80s, it has been of interest the construction of field theories whose correlators are the p-adic tree-level string amplitudes (or p-adic Koba-Nielsen amplitudes). Spokoiny [51], Zhang [58], (see also [49]) constructed formally quantum field theories whose amplitudes are expectation values of products of vertex operators. In [57] Zabrodin established that the tree-level string amplitudes may be obtained starting with a discrete field theory on a Bruhat-Tits tree. These ideas were used by Ghoshal and Kawano in the study of p-adic strings in constant B-fields [30].

The naive Euclidean version of the p-adic N-point amplitudes is presented in [30], [51] and [58] by

$$\mathcal{A}^{(N)}(\boldsymbol{k}) = \left\langle \prod_{j=1}^{N} \int_{\mathbb{Q}_p} dx_j \ e^{\boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} \right\rangle_D = \frac{1}{Z_0} \int D\boldsymbol{\varphi} e^{-S(\boldsymbol{\varphi})} \left\{ \int_{\mathbb{Q}_p^N} d^N x \ e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} \right\} \tag{0.1}$$

where $\langle \cdot \rangle_D$ denotes the expected value with respect to the weighted measure $D\boldsymbol{\varphi} e^{-S(\boldsymbol{\varphi})}$, $\int_{\mathbb{Q}_p} dx_j \ e^{\boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)}$ is the tachyonic vertex operator of the j-th tachyon with momentum $\boldsymbol{k}_j = (k_{1,j}, \ldots, k_{D,j})$ and field $\boldsymbol{\varphi}(x_j) = (\varphi_1(x_j), \ldots, \varphi_D(x_j))$, the product $\boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)$ denotes the standard Euclidean scalar product, $Z_0 = \int D\boldsymbol{\varphi} e^{-S(\boldsymbol{\varphi})}$ is the normalization constant, and the action S is given by

$$S(\boldsymbol{\varphi}) = \frac{T_0}{2} \sum_{j=1}^{N} \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \left\{ \frac{\varphi_j(x_j) - \varphi_j(y_j)}{|x_j - y_j|_p} \right\}^2 dx_j dy_j. \tag{0.2}$$

However, the measure $D\boldsymbol{\varphi}$ appearing in (0.1) is given formally (there was no a mathematical construction for this). This yields in a non-rigorous formulation of the amplitudes $\mathcal{A}^{(N)}(\boldsymbol{k})$. For this reason a new construction of the amplitudes is given. Note that in (0.1) the fields $\boldsymbol{\varphi}$ are functions not distributions, thus we construct a Gaussian probability measure which is denoted by $\mathbb{P}_D$ in the suitable space of functions $\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)$ known as the D-dimensional Lizorkin space of second kind [2, Chapter 7]. In Archimedean and non-Archimedean cases free quantum fields correspond to Gaussian probability measures on suitable infinite dimensional spaces. The reader may consult [31, Section 6.2] for the classical case, and [7, Section 5.5], [59], [60] for the p-adic case. Thus, in this case the measure $\mathbb{P}_D$ corresponds to a free quantum field.

Since the functions in $\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)$ have compact support, the exponential becomes 1 outside the ball containing the support and the integral over $\mathbb{Q}_p^N$ in the right-hand side of (0.1) is divergent, then it is necessary to introduce a cut-off and define the amplitude by a limit process as follows.

With the aid of the measure $\mathbb{P}_D$ we define

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0}\int\limits_{B_R^N}\left\{\int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N}\boldsymbol{k}_j\cdot\boldsymbol{\varphi}(x_j)}\, d\mathbb{P}_D(\boldsymbol{\varphi})\right\}\prod_{\nu=1}^{N} dx_\nu, \tag{0.3}$$

where B_R^N denotes a N-dimensional ball of radius p^R and $Z_0 = \int_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} d\mathbb{P}_D(\boldsymbol{\varphi})$. The N-point amplitude is defined as $\mathcal{A}^{(N)}(\boldsymbol{k}) = \lim_{R\to\infty}\mathcal{A}_R^{(N)}(\boldsymbol{k})$.

Notice that the integral over $\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)$ can be rewritten as a product of integrals over $\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ (the one-dimensional Lizorkin space). Using the solution $\varphi_{L,m}$ of the motion equation $\boldsymbol{D}\varphi_{L,m} = J_{L,m}$ (see [2, Theorem 10.2.2]), where $J_{L,m}\in\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ is conveniently taken, the change of variable $\varphi = \varphi_{L,m} + \tilde{\varphi}$ in (0.3) is performed to obtain

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0}\int\limits_{B_R^N}\prod_{1\leq i<j\leq N}|x_j - x_i|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j}\left\{\int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N}\boldsymbol{k}_j\cdot\widetilde{\boldsymbol{\varphi}}(x_j)}\, d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}})\right\}\prod_{\nu=1}^{N} dx_\nu, \tag{0.4}$$

where $\widetilde{\mathbb{P}}_D$ is the measure associated to $\widetilde{\boldsymbol{\varphi}}$. Notice that in the classical quantum field theory (QFT) $\boldsymbol{k}$ is considered as a coupling constant. However, in our case we do not have this assumption and therefore there is no a classical perturbative expansion for (0.4). Recalling the classical normalization

$$x_1 = 0,\ x_{N-1} = 1,\ x_N = \infty$$

and the series expansion of the exponential function around zero, we show that (0.4) possesses a series expansion of the form

$$\begin{aligned} \mathcal{A}_R^{(N)}(\boldsymbol{k}) = & \frac{C_0}{Z_0} \sum_{r=0}^{\infty} \int_{B_R^{N-3}} \prod_{j=2}^{N-2} |x_j|_p^{\frac{2(p-1)}{p \ln p} \boldsymbol{k}_1 \cdot \boldsymbol{k}_j} |1 - x_j|_p^{\frac{2(p-1)}{p \ln p} \boldsymbol{k}_{N-1} \cdot \boldsymbol{k}_j} \times \\ & \prod_{2 \leq i < j \leq N-2} |x_j - x_i|_p^{\frac{2(p-1)}{p \ln p} \boldsymbol{k}_i \cdot \boldsymbol{k}_j} G_r(\boldsymbol{k}, \boldsymbol{x}) \prod_{\nu=2}^{N-2} dx_\nu, \end{aligned} \tag{0.5}$$

where C_0 and $G_0(\boldsymbol{k}, \boldsymbol{x})$ are constants. Since $G_r(\boldsymbol{k}, \boldsymbol{x})$ is a locally constant function in $\boldsymbol{x}$, the product $1_{B_R^{N-3}}(\boldsymbol{x})\, G_r(\boldsymbol{k}, \boldsymbol{x})$ is a test function in $\boldsymbol{x}$ depending on $\boldsymbol{k}$, for $r \geq 1$. Each term in the series (0.5)

$$\begin{aligned} Z_{G_r,R}^{(N)}(\boldsymbol{k}) := & \frac{C_0}{Z_0} \int_{\mathbb{Q}_p^{N-3}} \prod_{j=2}^{N-2} |x_j|_p^{\frac{2(p-1)}{p \ln p} \boldsymbol{k}_1 \cdot \boldsymbol{k}_j} |1 - x_j|_p^{\frac{2(p-1)}{p \ln p} \boldsymbol{k}_{N-1} \cdot \boldsymbol{k}_j} \times \\ & \prod_{2 \leq i < j \leq N-2} |x_j - x_i|_p^{\frac{2(p-1)}{p \ln p} \boldsymbol{k}_i \cdot \boldsymbol{k}_j} 1_{B_R^{N-3}}(\boldsymbol{x})\, G_r(\boldsymbol{k}, \boldsymbol{x}) \prod_{\nu=2}^{N-2} dx_\nu, \end{aligned}$$

is a particular case of a multivariate Igusa zeta function for each r. In [11]-[12] (see also [26]) it was established that all the integrals $Z_{G_r,R}^{(N)}(\boldsymbol{k})$ are holomorphic functions in a common domain and if $G_r(\boldsymbol{k}, \boldsymbol{x}) = 1$,

$$\begin{aligned} \lim_{R \to \infty} Z_R^{(N)}(\boldsymbol{k}) = & \frac{C_0}{Z_0} \int_{\mathbb{Q}_p^{N-3}} \prod_{j=2}^{N-2} |x_j|_p^{\frac{2(p-1)}{p \ln p} \boldsymbol{k}_1 \cdot \boldsymbol{k}_j} |1 - x_j|_p^{\frac{2(p-1)}{p \ln p} \boldsymbol{k}_{N-1} \cdot \boldsymbol{k}_j} \times \\ & \prod_{2 \leq i < j \leq N-2} |x_j - x_i|_p^{\frac{2(p-1)}{p \ln p} \boldsymbol{k}_i \cdot \boldsymbol{k}_j} \prod_{\nu=2}^{N-2} dx_\nu. \end{aligned} \tag{0.6}$$

Equation (0.6) coincides up to a multiplicative constant with $Z^{(N)}(\boldsymbol{k})$ introduced in [11] as a p-adic Koba-Nielsen string amplitude (see also [12]). Using this fact we show that the first term in the series (0.5) is a regularized version of the p-adic Koba-Nielsen amplitude for open strings.

This thesis is structured as follows. Chapter 1 summarizes some results on p-adic analysis, local zeta functions and white noise calculus. In Chapter 2 we present some facts in quantum field theory and string theory. In Chapter 3 it is constructed the probability measure $\mathbb{P}_D$. We propose a definition for the N-point amplitudes in Definition 3 from which we obtain the Koba-Nielsen amplitudes. For this purpose, a change of variables is given and using some results in white noise calculus and the series expansion of the exponential function we show that it is possible to obtain a series expansion for the amplitude whose first term converges to the p-adic Koba-Nielsen amplitude for open strings. This result is summarized in Theorem 4.

Chapter 1

Mathematical preliminaries

In this chapter we review some basic facts about p-adic analysis, local zeta functions and white noise calculus. For an in-depth discussion about p-adic analysis we refer the reader to consult [2], [52], [55] and references therein. From now on, we denote p as a fixed prime number.

1.1 p-Adic numbers

Given x a nonzero rational number, we can represent x as $p^{\gamma}\frac{a}{b}$ where a and b are coprime integers with p. We define $\gamma = \mathrm{ord}_p(x)$, to be the p-adic order of x. We also set $\mathrm{ord}_p(0) := \infty$.

The p-adic norm in $\mathbb{Q}$ is defined by

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ p^{-\mathrm{ord}_p(x)} & \text{if} \quad x \neq 0. \end{cases}$$

Note that for every x and y, rational numbers, $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ i.e. the norm $|\cdot|_p$ is non-Archimedean.

The metric space $(\mathbb{Q}_p, |\cdot|_p)$ is given by the completion of $(\mathbb{Q}, |\cdot|_p)$ and is called the field of p-adic numbers.

The p-adic numbers satisfying $|x|_p \leq 1$ are called *p-adic integers* and the set containing this numbers is denoted by $\mathbb{Z}_p$. The group of units (p-adic numbers x with $|x|_p = 1$) is denoted by $\mathbb{Z}_p^{\times}$.

Any nonzero p-adic number x has a representation as a power series in the form

$$x = p^{\mathrm{ord}_p(x)} \sum_{i=0}^{\infty} x_i p^i,$$

where $x_i \in \{0, \ldots, p-1\}$ and $x_0 \neq 0$. Using this expansion, we define the *fractional part of* $x \in \mathbb{Q}_p$, denoted by $\{x\}_p$, as the part of the series containig negative exponents. Note that if $x \in \mathbb{Z}_p$, there are no negative exponents in the expansion, consequently $\{x\}_p = 0$.

We extend the p-adic norm to $\mathbb{Q}_p^N$ by

$$\|\boldsymbol{x}\|_p = \max_{1 \leq i \leq N} |x_i|_p, \text{ for } \boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N.$$

We define $\mathrm{ord}(\boldsymbol{x}) = \min_{1 \leq i \leq N} \{\mathrm{ord}(x_i)\}$, then $\|\boldsymbol{x}\|_p = p^{-\mathrm{ord}(\boldsymbol{x})}$. With this norm, $\mathbb{Q}_p^N$ becomes an ultrametric space. In addition, every $\boldsymbol{x} \in \mathbb{Q}_p^N \setminus \{\boldsymbol{0}\}$ can be represented uniquely as $\boldsymbol{x} = p^{\mathrm{ord}(\boldsymbol{x})} v(\boldsymbol{x})$, where $\|v(\boldsymbol{x})\|_p = 1$.

For $r \in \mathbb{Z}$, $B_r^N(\boldsymbol{a}) = \left\{\boldsymbol{x} \in \mathbb{Q}_p^N ; \|\boldsymbol{x} - \boldsymbol{a}\|_p \leq p^r\right\}$ denotes *the ball of radius p^r with center* $\boldsymbol{a} = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N$, in the case when the center is $\boldsymbol{0}$ we use the notation $B_r^N := B_r^N(\boldsymbol{0})$. Note that $B_r^N(\boldsymbol{a}) = B_r(a_1) \times \cdots \times B_r(a_N)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p ; |x - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center $a_i \in \mathbb{Q}_p$, this implies that the product topology of $\mathbb{Q}_p^N$ is equal to the topology induced by the norm $\|\cdot\|_p$. We also denote by $S_r^N(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{Q}_p^N ; ||\boldsymbol{x} - \boldsymbol{a}||_p = p^r\}$ *the sphere of radius p^r with center* $\boldsymbol{a} \in \mathbb{Q}_p^N$ and S_r^N the sphere with center $\boldsymbol{0}$. Notice that $S_0^1 = \mathbb{Z}_p^{\times}$ but $\left(\mathbb{Z}_p^{\times}\right)^N \subsetneq S_0^N$.

As a result of the ultrametricity, balls and spheres are simultaneously open and closed subsets in $\mathbb{Q}_p^N$. In addition, two balls in $\mathbb{Q}_p^N$ are either disjoint or one is contained in the other. As a consequence, the topological space $\left(\mathbb{Q}_p^N, \|\cdot\|_p\right)$ is a totally disconnected space, i.e. the only connected subsets of $\mathbb{Q}_p^N$ are the empty set and the points. A subset of $\mathbb{Q}_p^N$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^N$ (see e.g. [2, Section 1.8] or [55, Section 1.3]). The balls and spheres are compact subsets. Therefore $\left(\mathbb{Q}_p^N, \|\cdot\|_p\right)$ is a locally compact topological space.

Since $(\mathbb{Q}_p^N, +)$ is a locally compact topological group, there exists a Haar measure $d^N\boldsymbol{x}$, which is invariant under translations, i.e. $d^N(\boldsymbol{x}+\boldsymbol{a}) = d^N\boldsymbol{x}$. If we normalize this measure by the condition $\int_{\mathbb{Z}_p^N} d\boldsymbol{x} = 1$, then $d^N\boldsymbol{x}$ is unique see e.g. [2, Section 4.3]. From now on, if we have an integral over $\mathbb{Q}_p^N$, $d^N\boldsymbol{x}$ stands for the normalized Haar measure on $(\mathbb{Q}_p^N, \|\cdot\|_p)$.

A complex-valuated function $\varphi : \mathbb{Q}_p^N \to \mathbb{C}$ is called *locally constant* if for any $\boldsymbol{x} \in \mathbb{Q}_p^N$, there exists $l(\boldsymbol{x}) \in \mathbb{Z}$ such that:

$$\varphi(\boldsymbol{y}) = \varphi(\boldsymbol{x});\quad \boldsymbol{y} \in B_{l(\boldsymbol{x})}^N(\boldsymbol{x}).$$

A *Bruhat-Schwartz* or *test function* $\varphi : \mathbb{Q}_p^N \to \mathbb{C}$ is a locally constant function with compact support. Since φ has compact support, there exists $l \in \mathbb{Z}$ such that for any $\boldsymbol{x} \in \mathbb{Q}_p^N$,

$$\varphi(\boldsymbol{x}+\boldsymbol{x}') = \varphi(\boldsymbol{x}) \text{ for any } \boldsymbol{x}' \in B_l^N(x). \tag{1.1}$$

Any test function can be represented by a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D} := \mathcal{D}(\mathbb{Q}_p^N)$. We denote by $\mathcal{D}_{\mathbb{R}} := \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ the $\mathbb{R}$-vector space of Bruhat-Schwartz functions.

For $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$, the largest number $l = l(\varphi)$ satisfying (1.1) is called *the exponent of local* *constancy (or the parameter of constancy) of* φ.

Notation 1. *We will use* $\Omega\left(p^{-r}\left\|\boldsymbol{x}-\boldsymbol{a}\right\|_p\right)$ *to denote the characteristic function of the ball* $B_r^N(\boldsymbol{a})$. *For more general sets, we will use the notation* 1_A *for the characteristic function of a set* A. *We denote*

$$\Delta_k\left(\boldsymbol{x}\right)=\Omega\left(p^{-k}\left\|\boldsymbol{x}\right\|_p\right),\ k\in\mathbb{Z}$$

and

$$\delta_k\left(\boldsymbol{x}\right)=p^{kN}\Omega\left(p^{k}\left\|\boldsymbol{x}\right\|_p\right),\ k\in\mathbb{Z}.$$

Note that the functions Δ_k and δ_k are Bruhat-Schwartz functions.

We denote by $\mathcal{D}_m^l(\mathbb{Q}_p^N)$ the finite-dimensional subspace of $\mathcal{D}(\mathbb{Q}_p^N)$ having supports in the ball B_m^N and with parameters of constancy greater than or equal to l. We now define a topology on $\mathcal{D}$ as follows. We say that a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ of functions in $\mathcal{D}$ converges to zero if the two following conditions hold:

(1) there are two fixed integers k_0 and m_0 such that $\varphi_j\in\mathcal{D}_{m_0}^{k_0}$, for each $j\in\mathbb{N}$.

(2) $\varphi_j\to 0$ uniformly.

$\mathcal{D}$ endowed with the above topology becomes a topological vector space.

Given $\rho\in[1,\infty)$, we denote by $L^\rho:=L^\rho\left(\mathbb{Q}_p^N\right):=L^\rho\left(\mathbb{Q}_p^N,d^N\boldsymbol{x}\right)$, the $\mathbb{C}$-vector space of all complex-valued and Borel measurable functions g satisfying

$$\int_{\mathbb{Q}_p^N}\left|g\left(\boldsymbol{x}\right)\right|^\rho d^N\boldsymbol{x}<\infty.$$

The corresponding $\mathbb{R}$-vector spaces are denoted as $L_{\mathbb{R}}^\rho:=L_{\mathbb{R}}^\rho\left(\mathbb{Q}_p^N\right)=L_{\mathbb{R}}^\rho\left(\mathbb{Q}_p^N,d^N\boldsymbol{x}\right)$. For

$g \in L^{\rho}$, we define

$$\|g\|_{\rho} = \left\{ \int_{\mathbb{Q}_p^N} \|g(\boldsymbol{x})\|^{\rho} \, d^N\boldsymbol{x} \right\}^{\frac{1}{\rho}},$$

as the norm of the function g. For $\rho = \infty$, $g \in L^{\infty}\left(\mathbb{Q}_p^N\right)$ if

$$\|g\|_{\infty} := \operatorname{ess} \sup_{\boldsymbol{x} \in \mathbb{Q}_p^N} |g(\boldsymbol{x})| < \infty,$$

where ess sup denotes the essential supremum of $|g(\boldsymbol{x})|$.

For $x \in \mathbb{Q}_p$ we set $\chi_p(x) = \exp(2\pi i\{x\}_p)$. The map $\chi_p(\cdot)$ is a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as a multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$, then $\chi_p(x)$ is an additive character on $\mathbb{Q}_p$, it is called the *standard additive character* of $(\mathbb{Q}_p, +)$.

Given $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N) \in \mathbb{Q}_p^N$, we define

$$\boldsymbol{x} \cdot \boldsymbol{\xi} = x_1\xi_1 + \cdots + x_N\xi_N.$$

The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$ is defined by

$$(\mathcal{F}\varphi)(\boldsymbol{\xi}) = \int_{\mathbb{Q}_p^N} \chi_p(\boldsymbol{\xi} \cdot \boldsymbol{x})\varphi(\boldsymbol{x}) d^N\boldsymbol{x} \quad \text{for } \boldsymbol{\xi} \in \mathbb{Q}_p^N.$$

The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^N)$ onto itself, satisfying

$$(\mathcal{F}(\mathcal{F}\varphi))(\boldsymbol{\xi}) = \varphi(-\boldsymbol{\xi}), \tag{1.2}$$

in addition if $\varphi \in \mathcal{D}_m^l$, $\widehat{\varphi}(\boldsymbol{\xi}) \in \mathcal{D}_{-l}^{-m}\left(\mathbb{Q}_p^N\right)$ (see e.g. [2, Lemma 4.8.3]). As an example, we see that $\mathcal{F}(\Delta_k)(\boldsymbol{x}) = \delta_k(\boldsymbol{x})$ for $k \in \mathbb{Z}$, $\boldsymbol{x} \in \mathbb{Q}_p^N$.

We will also use the notation $\mathcal{F}_{\boldsymbol{x} \to \boldsymbol{\kappa}}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

The Fourier transform has an extension to L^2 and it is unitary on L^2 i.e. $||f||_2 = ||\widehat{f}||_2$ for $f \in L^2$. Moreover (1.2) is also valid in L^2 (see e.g. [52, Chapter III, Section 2]).

The $\mathbb{C}$-vector space $\mathcal{D}' := \mathcal{D}'\left(\mathbb{Q}_p^N\right)$ of all continuous linear functionals on $\mathcal{D}(\mathbb{Q}_p^N)$ is called the *Bruhat-Schwartz space of distributions*. Every linear functional on $\mathcal{D}$ is continuous, i.e. $\mathcal{D}'$ agrees with the algebraic dual of $\mathcal{D}$ (see e.g. [55, Chapter 1, VI.3, Lemma]). We denote by $\mathcal{D}'_{\mathbb{R}} := \mathcal{D}'_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$ the dual space of $\mathcal{D}_{\mathbb{R}}$.

We endow $\mathcal{D}'$ with the weak topology, i.e. a sequence $\{T_j\}_{j\in\mathbb{N}}$ in $\mathcal{D}'$ converges to T if $\lim_{j\to\infty} T_j(\varphi) = T(\varphi)$ for any $\varphi \in \mathcal{D}$. The map

$$\begin{array}{ccc} \mathcal{D}' \times \mathcal{D} & \rightarrow & \mathbb{C} \\ (T, \varphi) & \mapsto & T(\varphi) \end{array}$$

is a bilinear form that is continuous in T and φ separately. We call this map the pairing between $\mathcal{D}'$ and $\mathcal{D}$. From now on it is used the notation (T, φ) instead of $T(\varphi)$.

Every f in L^1_{loc} defines a distribution $f \in \mathcal{D}'\left(\mathbb{Q}_p^N\right)$ by the formula

$$(f, \varphi) = \int_{\mathbb{Q}_p^N} f(\boldsymbol{x})\, \varphi(\boldsymbol{x})\, d^N\boldsymbol{x}.$$

Such distributions are called *regular distributions*.

Remark 1. *i. For $f \in L^2_{\mathbb{R}}$, $(f, \varphi) = \langle f, \varphi \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2_{\mathbb{R}}$.*

ii. The distributions generated by the functions δ_k converge to the distribution δ in $\mathcal{D}'\left(\mathbb{Q}_p^N\right)$. In fact, given $\varphi \in \mathcal{D}$, for all $k \geq -l(\varphi)$

$$\begin{aligned}(\delta_k, \varphi) &= \int_{\mathbb{Q}_p^N} p^{Nk} \Omega\left(p^k \|\boldsymbol{x}\|_p\right) \varphi(\boldsymbol{x})\, d^N \boldsymbol{x} \\ &= \varphi(\mathbf{0})\, p^{Nk} \int_{B_{-k}^N} d^N \boldsymbol{x} \\ &= \varphi(\mathbf{0}) = (\delta, \varphi)\,.\end{aligned}$$

The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'\left(\mathbb{Q}_p^N\right)$ is defined by

$$(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}(\mathbb{Q}_p^N).$$

$\mathcal{F}[T]$ is a linear and continuous isomorphism from $\mathcal{D}'\left(\mathbb{Q}_p^N\right)$ onto $\mathcal{D}'\left(\mathbb{Q}_p^N\right)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\boldsymbol{\xi})]$.

The Vladimirov operator $\boldsymbol{D} : \mathcal{D}(\mathbb{Q}_p) \to L^2(\mathbb{Q}_p)$ is defined by

$$\begin{aligned}\boldsymbol{D}\theta(x) &= \frac{p^2}{p+1} \int_{\mathbb{Q}_p} \frac{\theta(x) - \theta(y)}{|x-y|_p^2} dy = \frac{p^2}{p+1} \int_{\mathbb{Q}_p} \frac{\theta(x) - \theta(x-z)}{|z|_p^2} dz \\ &= \mathcal{F}_{\xi \to x}^{-1}\left[|\xi|_p \mathcal{F}_{x \to \xi} \theta\right].\end{aligned}$$

$\boldsymbol{D}$ is written as

$$\boldsymbol{D}\theta(x) = f_{-1}(x) * \varphi(x) = -\frac{p^2}{p+1} |x|_p^{-2} * \theta(x)\,, \text{ for } \theta \in \mathcal{D}(\mathbb{Q}_p)\,, \tag{1.3}$$

where the function $f_{-1}(x) = -\frac{p^2}{p+1} |x|_p^{-2}$ determines a distribution from $\mathcal{D}'$ (see [55, Chapter 2, Section IX.1]) this function is known as the *Riesz kernel*.

It is worthwhile to mention that Vladimirov operator does not leave invariant the space $\mathcal{D}(\mathbb{Q}_p)$. In order to introduce its inverse operator it is necessary to restrict its domain to an invariant subspace. One space holding this condition is the Lizorkin space

$$\mathcal{L} := \mathcal{L}(\mathbb{Q}_p) = \left\{ \varphi \in \mathcal{D}(\mathbb{Q}_p); \int_{\mathbb{Q}_p} \varphi(x)\, dx = 0 \right\},$$

which is a natural definition domain for this operator. This is a complete space with the topology inherited from $\mathcal{D}(\mathbb{Q}_p)$ and is dense in $L^\rho(\mathbb{Q}_p)$, $1 < \rho < \infty$ (see [2, Theorem 7.4.3] for more details).

Noting that the set $\mathcal{F}(\mathcal{L}(\mathbb{Q}_p))$ of Fourier transforms of functions in $\mathcal{L}(\mathbb{Q}_p)$ coincides with the set $\{\widehat{\varphi} \in \mathcal{D}(\mathbb{Q}_p); \widehat{\varphi}(0) = 0\}$ we obtain a $\mathbb{C}$-vector space isomorphism between $\mathcal{L}$ and $\mathcal{F}(\mathcal{L}(\mathbb{Q}_p))$.

We denote by $\mathcal{L}' = \mathcal{L}'(\mathbb{Q}_p)$ the topological dual of the space $\mathcal{L}(\mathbb{Q}_p)$, this space is called p-adic *Lizorkin space of distributions of second kind.*

If $\mathcal{L}^\perp(\mathbb{Q}_p)$ denotes the subspace of functionals in $\mathcal{D}'(\mathbb{Q}_p)$ orthogonal to $\mathcal{L}(\mathbb{Q}_p)$, we have the characterization

$$\mathcal{L}'(\mathbb{Q}_p) = \mathcal{D}'(\mathbb{Q}_p) / \mathcal{L}^\perp(\mathbb{Q}_p)$$

[2, Theorem 7.3.4].

We define the inverse of $\boldsymbol{D}$ as the operator

$$\begin{array}{rccc} \boldsymbol{D}^{-1}: & \mathcal{L}(\mathbb{Q}_p) & \to & \mathcal{L}(\mathbb{Q}_p) \\ & \theta & \mapsto & \boldsymbol{D}^{-1}\theta, \end{array}$$

where $\boldsymbol{D}^{-1}\theta(x) = \mathcal{F}^{-1}_{\xi \to x}\left[|\xi|_p^{-1} \mathcal{F}_{x \to \xi}\theta \right]$. Since $(\mathcal{F}_{x \to \xi}\theta)(0) = 0$, we have $\boldsymbol{D}^{-1}\theta(x) \in \mathcal{L}(\mathbb{Q}_p)$.

Consider the equation

$$\boldsymbol{D}\psi(x) = \theta(x) \text{ for } \theta \in \mathcal{L}(\mathbb{Q}_p), \tag{1.4}$$

this equation has a unique solution $\psi \in \mathcal{L}(\mathbb{Q}_p)$ ([55, Chapter 2, Section IX.2]). Setting

$$(f_1, \theta) = -\frac{(p-1)}{p \ln p} \int_{\mathbb{Q}_p} \theta(x) \ln |x|_p \, dx, \text{ for } \theta \in \mathcal{L}(\mathbb{Q}_p), \tag{1.5}$$

we have

$$\widehat{f_1}(\xi) = \frac{1}{|\xi|_p} \text{ in } \mathcal{L}'(\mathbb{Q}_p)$$

and

$$\psi(x) = \boldsymbol{D}^{-1}\theta(x) = f_1(x) * \theta(x),$$

is the solution of equation (1.4) see [2, Theorem 9.2.6].

1.2 Local zeta functions

We review some well-known results about local zeta functions. For an in-depth discussion of classical aspects see [18], [41], [46] or [47].

Definition 1. *Let $f(\boldsymbol{x})$ be a nonconstant polynomial in $\mathbb{Q}_p[x_1, \ldots, x_N]$ and let ϕ be a test function. The Igusa local zeta function attached to the pair (f, ϕ) is*

$$Z_\phi(s, f) = \int_{\mathbb{Q}_p^N \setminus f^{-1}(0)} \phi(\boldsymbol{x}) \, |f(\boldsymbol{x})|_p^s \, d^N\boldsymbol{x},$$

for $s \in \mathbb{C}$ and $Re(s) > 0$. In the case $\phi = \Delta_0(x)$, we use the notation $Z(s, f)$ instead of $Z_\phi(s, f)$.

Using the fact that for $\mathrm{Re}(s) > 0$ the function $\phi(\boldsymbol{x}) \, |f(\boldsymbol{x})|_p^s$ is continuous with compact support, and the Haar measure of any compact set is finite, we obtain that the integral $Z_\phi(s, f)$ converges for $\mathrm{Re}(s) > 0$. Moreover, for every nonconstant polynomial $f(\boldsymbol{x}) \in \mathbb{Q}_p[x_1, \ldots, x_N]$ and any test function $\phi(x_1, \ldots, x_N)$, the local zeta function $Z_\phi(s, f)$ attached to (f, ϕ) has a meromorphic extension to the whole complex plane as a rational function in p^{-s} (see [41, Theorem 5.4.1]).

The generalization of the Igusa zeta function was introduced by F. Loeser in [47] and is called multivariate local zeta function.

Definition 2. *Let $f_1, \ldots, f_l$ be nonconstant polynomials in $\mathbb{Q}_p[x_1, \ldots, x_N] \setminus \mathbb{Q}_p$ and $\phi(x_1, \ldots, x_N)$ a test function. The multivariate Igusa zeta function attached to $(f_1, \ldots, f_l, \phi)$ is defined as the integral*

$$Z_\phi(s_1, \ldots, s_l, f_1, \ldots, f_l) = \int_{\mathbb{Q}_p^N \setminus \bigcup_{i=1}^{l} f_i^{-1}(\mathbf{0})} \phi(\boldsymbol{x}) \prod_{i=1}^{l} |f_i(\boldsymbol{x})|_p^{s_i} \, d^N\boldsymbol{x},$$

for $(s_1, \ldots, s_l) \in \mathbb{C}^l$ and $Re(s_i) > 0$, for $i = 1, \ldots, l$.

Theorem 1. *(F. Loeser, [47, Theorem 1.1.4])*

The multivariate zeta function $Z_\phi(s_1, \ldots, s_l, f_1, \ldots, f_l)$ attached to $(f_1, \ldots, f_l, \phi)$ admits a meromorphic extension to $\mathbb{C}^l$ as a rational function in the p^{-s_i}, $i = 1, \ldots, l$, more precisely,

$$Z_\phi(s_1, \ldots, s_l, f_1, \ldots, f_l) = \frac{P_\phi(s_1, \ldots, s_l)}{\prod_{j \in T} \left(1 - p^{-N_0^j - \sum_{i=1}^{l} N_i^j s_i}\right)}, \tag{1.6}$$

where T is a finite set, the N_0, N_i are nonnegative integers, and $P_\phi(s_1, \ldots, s_l)$ is a polynomial in the variables p^{-s_i}.

1.3 Basic aspects of white noise calculus

In this section we summarize some important facts about white noise. For a deeper discussion we refer the reader to [37], [40], [44] or [48].

A seminorm $\|\cdot\|$ on a vector space V over $\mathbb{R}$ (resp. $\mathbb{C}$) is *Hilbertian* if it comes from some nonnegative, symmetric bilinear (resp. Hermitian sesquilinear) form $\langle \cdot, \cdot \rangle$ on $V \times V$.

A *countably Hilbert* space is a locally convex space V admitting a countable set of compatible Hilbertian norms $\{|\cdot|_n\}_{n\in\mathbb{N}}$ such that if $\{\xi_n,\ n\in\mathbb{N}\}$ is a sequence in V which converges to zero for a norm $|\cdot|_n$ and is a Cauchy sequence concerning to the norm $|\cdot|_m\,,\ n,m\in\mathbb{Z}$, then it goes to zero concerning to $|\cdot|_m$.

Suppose that these norms obey $|\cdot|_n \le |\cdot|_m$ for $n\le m$, denote by V_n the completion of V under $|\cdot|_n\,,\ n\in\mathbb{N}$, there is a continuous linear operator T_n^m, which maps the space V_m onto an everywhere dense set of V_n.

A countably Hilbert space V is called *nuclear*, if for any n there is an $m\ge n$ such that the operator T_n^m is nuclear, i.e. has the form

$$T_n^m v=\sum_{k=1}^{\infty}\lambda_k\langle v,v_k\rangle_m w_k,\ v\in V_m$$

where $\{v_k\}_{k\in\mathbb{N}}$ and $\{w_k\}_{k\in\mathbb{N}}$ are orthonormal systems of vectors in the spaces V_m and V_n respectively, $\lambda_k>0$ and $\sum\limits_{k=1}^{\infty}\lambda_k<\infty$.

Let $\mathcal{H}$ be a real separable (infinite dimensional) Hilbert space and let V be a real nuclear space densely and continuously embedded in $\mathcal{H}$. By using the representation theorem to identify $\mathcal{H}$ with its dual space $\mathcal{H}^*$, we get the triple

$$V\subset\mathcal{H}\subset V^*,$$

such a triple is called a Gel'fand triple [45, Chapter 2]. Note that $\mathcal{H}$ is dense in V^* with the weak topology of V^*. Let $(\cdot,\cdot)$ denote the pairing between V^* and V and let $|\cdot|$ be the norm of $\mathcal{H}$. Then $|\xi|^2=(\xi,\xi)$ for $\xi\in V$.

We choose any fixed elements $\varphi_1,\dots,\varphi_n\in V$. To each element $T\in V^*$ corresponds the point $((T,\varphi_1),\dots,(T,\varphi_n))$ in the n-dimensional space $\mathbb{R}^n$. The elements $\varphi_1,\dots,\varphi_n\in V$

define a mapping

$$T \mapsto ((T, \varphi_1), \ldots, (T, \varphi_n)),$$

from V^* into $\mathbb{R}^n$. Let A be a given set in $\mathbb{R}^n$, and consider the set Z of all linear functionals T such that

$$((T, \varphi_1), \ldots, (T, \varphi_n)) \in A.$$

We call Z the cylinder set defined by the elements $\varphi_1, \ldots, \varphi_n$ and the set $A \subset \mathbb{R}^n$.

We consider V^* equipped with the weak topology. $\mathcal{B}$ denotes the Borel σ-algebra on V^* i.e. the σ-algebra generated by the weak topology of V^*.

Theorem 2. *(Bochner-Minlos, [37, Theorem 1.1]) Let V be a nuclear space and C a function on V with the following properties:*

i. C is continuous on V.

ii. C is positive definite i.e.

$$\sum_{i,j=1}^{n} C(f_i - f_j) z_i \overline{z_j} \geq 0, \;\; f_1, \ldots, f_n \in V, \;\; z_1, \ldots, z_n \in \mathbb{C}.$$

iii. $C(0) = 1$.

Then, there exists a unique probability measure μ_C on $(V^*, \mathcal{B}(V^*))$ whose characteristic functional is equal to C, so, for all $f \in V$,

$$\int_{V^*} e^{i\langle W,f\rangle} \; d\mu_C(W) = C(f). \tag{1.7}$$

As an example, consider the function C on V given by:

$$C(\xi) := \exp\left(-\frac{1}{2}|\xi|^2\right), \tag{1.8}$$

then C is a characteristic functional on V and there exists a probability space $(V^*, \mathcal{B}, \mu)$ where μ is a Gaussian measure satisfying

$$\int_{V^*} e^{i\langle W,f\rangle}\, d\mu(W) = \exp\left(-\frac{1}{2}|f|^2\right)$$

for all $f \in V$. We call $(V^*, \mathcal{B}, \mu)$ the Gaussian space associated with $(V, |\cdot|)$.

For $1 \leq \rho < \infty$, we set $(L^\rho_{\mathbb{C}}) := (L^\rho(V^*), \mu; \mathbb{C})$ to denote the complex vector space of measurable functions $\Psi : V^* \to \mathbb{C}$ satisfying

$$\| \Psi\|^\rho_{(L^\rho_{\mathbb{C}})} = \int_{V^*} |\Psi(W)|^\rho\, d\mu(W) < \infty.$$

The space $(L^\rho_{\mathbb{R}}) := (L^\rho(V^*), \mu; \mathbb{R})$ is defined in a similar way.

Given a complex-valued polynomial P on $\mathbb{R}^n$, $n \in \mathbb{N}$ and $\xi_1, \ldots, \xi_n$ in V, we construct the V^*-functional P_ξ by

$$P_\xi\,(T) = P\left((T, \xi_1)\,, \ldots, (T, \xi_n)\right),\ T \in V^*.$$

This functional belongs to $(L^\rho_{\mathbb{C}})$ for all $\rho \in [1, \infty)$ [37, Proposition 1.6]. Denote the random variable $T \to (T, \xi)$ by T_ξ. The mapping $\xi \mapsto T_\xi$ is called the (canonical) coordinate process over V. For $\xi \in V$, $\alpha \in \mathbb{C}$ and using the random variable T_ξ it is possible to define the functionals $\exp(\alpha T_\xi)$ on V^* and with aid of (1.7) and (1.8) these exponentials belong to the space $(L^\rho_{\mathbb{C}})$ [37, Proposition 1.7].

Chapter 2

Quantum field theory and string theory: basic facts

In this chapter, we summarize some facts on quantum field theory (QFT) and string theory that we required in this work. For an in-depth presentation, we refer the reader to consult [16], [32], [42] and the references given there.

2.1 Quantum field theory

Historically, quantum field theory (QFT) has been developed as quantum mechanics for infinite degrees of freedom. It is obtained after quantizing the classical fields. A general theory of classical fields consists of a spacetime M, a space of fields $\mathcal{H}$ and a Lagrangian density L which is a density on M for each point of $\mathcal{H}$.

In the case of scalar fields, $\mathcal{H}$ is compose by fields $\varphi_i : M \to \mathbb{R}$ and the Lagrangian

$L = L(\varphi_1, \varphi_2, \ldots)$ with $\varphi_i \in \mathcal{H}$ is used to define the action

$$S = \int_M L(\varphi_1, \varphi_2, \ldots)$$

which is a functional acting on the fields. The equations of motion of the system can be derived from the action and the physically relevant fields are the critical points of this action. Looking for these critical points we find equations that are linear partial differential equations (PDEs) in the case of electromagnetic fields and nonlinear PDEs in the case of gravitational fields.

For example, consider a scalar field $\varphi : M \to \mathbb{R}$ defined over Minkowski's spacetime $M = \mathbb{R}^{1,3}$, with the convention $\nabla_\mu = \partial_\mu$, we take the Lagrangian

$$L(\varphi, \partial_\mu \varphi) = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2,$$

where m is a constant (mass), $\partial^\mu = \eta_{\mu\nu}\partial_\nu$ and $\eta_{\mu\nu}$ is the Minkowski metric tensor

$$[\eta_{\mu\nu}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then

$$\frac{\partial L}{\partial \varphi} = -m^2 \varphi, \quad \frac{\partial L}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi.$$

The Euler–Lagrange equations applied to this situation yield to the dynamical field equation

$$-m^2\varphi - \partial_\mu \left(\partial^\mu \varphi\right) = 0.$$

This equation can be written as

$$\left(\Box + m^2\right)\varphi = 0,$$

where $\Box = \partial_\mu \partial^\mu = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Lambertian operator. Thus, the extreme fields for the above Lagrangian satisfy the linear second-order hyperbolic PDE known as the Klein–Gordon equation.

The quantization of the fields could be obtained using canonical quantization or by path integral quantization. In the canonical quantization the observables (fields) are represented by operators in a Hilbert space $\mathcal{H}$, from which the quantum states of the system arise. In the case of classical free fields (Klein-Gordon, Dirac, Maxwell) the Lagrangians of these fields are quadratic and the corresponding Euler–Lagrange equations are linear PDEs. The Fourier transform can be used to diagonalize the quadratic form in the Lagrangian and thus decouple the Euler–Lagrange equations.

In the case of the Klein–Gordon equation corresponding to scalar bosons each Fourier mode is a harmonic oscillator whose quantization is known. Hence, we obtain a family of pairs of operators, one creates particles with a given momentum and the other destroys particles with that same momentum. From the so-called vacuum state (which is annihilated by all the operators that destroy particles) and the operators that create particles it is possible to construct states with a finite number of particles. Taking all possible linear combinations of these states we obtain a vector space with an inner product whose completion yields a Hilbert space. This is the space on which the operators that create and destroy particles act.

The path integral quantization was first devised by Feynman and developed by Dirac. In this method the fields are not considered operators but functions, like in classical theory. Feynman's path integral method for the quantization of fields starts from a functional integral of the form

$$Z = \int e^{iS(\varphi)} D\varphi. \tag{2.1}$$

It is called the *partition function* and is taken over the infinite-dimensional space of all fields, where

$$S(\varphi) = \int_M L(\varphi(x))\, d^N x$$

is the action, L is the Lagrangian density of the system and $D\varphi$ is a heuristic measure in the space of all fields. Here the quantization is given from the integral.

In general, physicists take the measure $D\varphi$ in a formal way which means that there is no mathematical construction for the measure. Starting from such a partition function one constructs the quantization of the system by writing the Wightman correlation functions. Given the correlation functions, we reconstruct the Hilbert space, the quantum fields, and the entire QFT satisfying the Wightman axioms (see Wightman's reconstruction Theorem [16, Chapter 6]).

The k-point Wightman correlation functions are defined by

$$\langle \varphi(x_1) \cdots \varphi(x_k) \rangle_D = \frac{1}{Z} \int \varphi(x_1) \cdots \varphi(x_k)\, e^{iS(\varphi)} D\varphi$$

where $\langle \cdot \rangle_D$ denotes the expected value with respect to the weighted measure $D\varphi e^{-S(\varphi)}$. Euclidean QFT can be obtained after a suitable rotation of the type $\varphi \mapsto \sqrt{-1}\varphi$ and the correlation functions are called the Schwinger k-point functions.

2.2 String theory

To understand the structure of matter, we need to know the dynamics of smaller components. The particles that do not show any substructure are known as elementary particles. In QFT, those are described as points in space. In string theory, particles are not described as points but as strings. We show some aspects of string theory, for more details reader can consult [9], [17], [32] and references therein.

The string is a 1-dimensional object. There exist open strings which have different endpoints and closed strings which do not have start or endpoints, they are like loops. As the strings propagate in spacetime it sweeps out a worldsheet, an underlying Riemannian or pseudo-Riemannian spacetime surface.

The theory is defined for a dynamical embedding map $X : \Sigma_{g,b,N} \to M$, where $\Sigma_{g,b,N}$ (the worldsheet) is a compact and oriented Riemann surface with a possible nonempty boundary characterized by the genus g, its number of boundaries b and N marked points. The worldsheet $\Sigma_{g,b,N}$ has local coordinates $\sigma = (\sigma_0, \sigma_1)$, where σ_0 is an evolution parameter and σ_1 the position along the string. Under a Wick rotation $\sigma_2 = \sqrt{-1}\sigma_0$, the metric written in terms of coordinates (σ_2, σ_1) becomes a metric with Euclidean signature. The intrinsic metric of the worldsheet is denoted by h_{ab}.
The string may oscillate in the target space M, it has an infinite number of quantum modes of oscillation characterized by a mass and a spin. In the bosonic string the spectrum consists of a tachyonic mode, this is the most simple mode and the one with less square mass. Here we consider M to be the flat Minkowskian space with metric $\eta_{\mu\nu}$.

In bosonic string theory the simplest invariant action is given by

$$S = \frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{h} h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \tag{2.2}$$

where $T = \dfrac{1}{2\pi\alpha'}$ is the string tension, $\alpha' = l_s^2$ with l_s the string length, $\sqrt{h}$ is the square root of the absolute value of the determinant of h_{ab} and h^{ab} is the inverse of h_{ab}. The action (2.2) is equivalent to the Nambu-Goto action (see [32, Chapter 2]). Classically (2.2) describes the propagation of a string in the Minkowski space. This action has worldsheet reparametrization invariance. A standard and convenient choice is a parametrization of the worldsheet such that $h_{ab} = \eta_{ab} e^{\phi}$ where η_{ab} is the metric of a flat worldsheet with Lorentzian signature $(-, +, \ldots, +)$ and e^{ϕ} is an unknown conformal factor. This parametrization is possible at least locally and (2.2) reduces to the free field action

$$S = -\frac{T}{2} \int d^2\sigma \eta_{\mu\nu} \eta^{ab} \partial_a X^{\mu} \partial_b X^{\nu}. \tag{2.3}$$

While the strings are extended they can overlap in space at the same time and interactions occur. The perturbative scattering amplitudes of N-particles are defined as the correlation function of N vertex operators $V_{\Lambda_i}(\boldsymbol{k}_i)$, $i = 1, \ldots, N$. The squared of these amplitudes gives the probability of having a configuration of certain outgoing states given a set of incoming states. The vertex operators are functionals of the embedding fields and their derivatives. They are given by

$$V_{\Lambda}(\boldsymbol{k}) = \int d^2\sigma \sqrt{h} W_{\Lambda}(\sigma) e^{i\boldsymbol{k}\cdot X(\sigma)}, \tag{2.4}$$

where $W_{\Lambda}(\sigma)$ represents a functional of X and its derivatives associated to the species of field in the string spectrum, $X(\sigma) = (x_1, \ldots, x_N)$ and $\boldsymbol{k} = (k_1, \ldots, k_N)$ are the position and momentum vectors in the target space M, and $\boldsymbol{k} \cdot X(\sigma)$ is the Minkowskian inner product. In the case in which the external particles are tachyons we have $\Lambda = t$ and $W_{\Lambda}(\sigma) = 1$. In the case of gauge fields $W_A(\sigma) = \epsilon_{\mu} \partial_t X^{\mu}$ here ϵ_{μ} is the polarization vector. For graviton $W_G(\sigma) = \epsilon_{\mu\nu} h^{ab} \partial_a X^{\mu} \partial_b X^{\nu}$ where $\epsilon_{\mu\nu}$ is the polarization tensor.

In the case of bosonic closed strings we consider $g = 0$ and for open strings $g = 0$ and $b = 1$ i.e. we work in the 2-sphere and the disk. In the open string case, the vertex operators are inserted on the boundary of the disk, using symmetry we can fix three points on the boundary which could be 0, 1 and ∞. The open string action (2.3) determines Neumann boundary conditions given by

$$\eta_{\mu\nu}\partial_\sigma X^\nu|_{\partial\Sigma} = 0,$$

where ∂_σ is the normal derivative of $\partial\Sigma$. These conditions are in general complex valued because of the Wick rotation. We are concerned in the case when the string worldsheet Σ is a disk, which corresponds to the open string. The disk is transformed into the upper half plane whose boundary is the real line, via a conformal transformation. In these variables the Neumann boundary conditions are [50]

$$\eta_{\mu\nu}(\partial - \overline{\partial})X^\nu|_{z=\overline{z}} = 0 \tag{2.5}$$

where z is the complex worldsheet coordinate with Im $z \geq 0$, $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial\overline{z}$. The expected value $\langle X^\mu(z)\, X^\nu(z')\rangle$ is the propagator which specifies the probability amplitude for a particle traveling from one place to another in a given period of time. This propagator restricted to the boundary conditions (2.5) gives

$$\langle X^\mu(\tau)\, X^\nu(\tau')\rangle = -\alpha'\eta^{\mu\nu}\log|\tau - \tau'|\,,\ \ \tau, \tau' \in \mathbb{R}. \tag{2.6}$$

The correlators in the path integral formalism can be computed as Gaussian integrals. Consider the integrals

$$\int DX \exp\left(\int d^2z\,(X\Delta X + iJX)\right) \sim \exp\left(\frac{1}{2}\int dzdz'\,J(z)\,G(z,z')\,J(z')\right), \tag{2.7}$$

here Δ is a differential operator, J is an arbitrary source and $G(z,z')$ is the inverse operator or Green's function, that satisfies $\Delta G(z,z') = \delta(z-z')$ (the symbol $\sim$ indicates that there is some proportional factor not relevant to the analysis).

We use this result to obtain the scattering amplitudes by choosing the appropriate source J. In the special case when $J(\tau) = \sum_{l=1}^{N} \delta(\tau - \tau_l)\boldsymbol{k}_l$, we obtain the scattering amplitudes integrating the expected values of the vertex operators (2.4) over the entire boundary of the worldsheet, which is just integrating (2.7) over the real variable τ. With the aid of (2.6) and (2.7)

$$\begin{aligned}
\int d^N\tau \langle V(\boldsymbol{k}_1,\tau_1)\cdots V(\boldsymbol{k}_N,\tau_N)\rangle &= \int d^N\tau \int DX \exp\left(-S + i\left(\boldsymbol{k}_1 X_1 + \cdots + \boldsymbol{k}_N X_N\right)\right) \\
&= \int d^N\tau \exp\left(\frac{1}{2}\int d\tau d\tau' J(\tau)\left(-\alpha'\eta^{\mu\nu}\log|\tau-\tau'|\right)J(\tau')\right) \\
&= \int d^N\tau \exp\left(\frac{1}{2}\int d\tau d\tau' \sum_{l=1}^{N}\delta(\tau-\tau_l)\boldsymbol{k}_l\left(-\alpha'\eta^{\mu\nu}\log|\tau-\tau'|\right)\right. \\
&\quad \left.\sum_{m=1}^{N}\delta(\tau-\tau_m)\boldsymbol{k}_m\right) \\
&= \int d^N\tau \prod_{l,m}|\tau_l-\tau_m|^{\alpha'\boldsymbol{k}^l_\mu\eta^{\mu\nu}\boldsymbol{k}^m_\nu},
\end{aligned}$$

where $d^N\tau = d\tau_1\cdots d\tau_N$. Fixing $\alpha'=1$, the order of the external momenta $\boldsymbol{k}_i$ and the three points $\tau_1 = 0, \tau_{N-1} = 1$ and $\tau_N = \infty$ we obtain the Koba-Nielsen amplitude

$$\tilde{A}^{(N)}_{\mathbb{R}}(\boldsymbol{k}) = \int\limits_{0<\tau_2<\cdots<\tau_{N-2}<1} d\tau_2\cdots d\tau_{N-2}\prod_{j=2}^{N-2}|\tau_j|^{\boldsymbol{k}_1\boldsymbol{k}_j}|1-\tau_j|^{\boldsymbol{k}_{N-1}\boldsymbol{k}_j}\prod_{2\le i<j\le N-2}|\tau_j-\tau_i|^{\boldsymbol{k}_i\boldsymbol{k}_j}.$$

The four-point amplitude is known as Veneziano's amplitude

$$\tilde{A}^{(4)}_{\mathbb{R}}(\boldsymbol{k}) = \int_0^1 dx|x|^{k_1k_2}|1-x|^{k_3k_2}.$$

A generalization of this amplitude [21] is given by

$$A_{\mathbb{R}}^{(4)}(\boldsymbol{k}) = \int_{\mathbb{R}} dx|x|^{k_1 k_2}|1-x|^{k_3 k_2}. \tag{2.8}$$

For N-points the generalization gives

$$A_{\mathbb{R}}^{(N)}(\boldsymbol{k}) \int_{\mathbb{R}^{N-3}} dx_2 \cdots dx_{N-2} \prod_{j=2}^{N-2} |x_j|^{\boldsymbol{k}_1 \boldsymbol{k}_j} |1-x_j|^{\boldsymbol{k}_{N-1} \boldsymbol{k}_j} \prod_{2\leq i<j\leq N-2} |x_j - x_i|^{\boldsymbol{k}_i \boldsymbol{k}_j}. \tag{2.9}$$

This is the Koba-Nielsen open string amplitude of N-points (see [11], [32] for more details).

2.2.1 p-Adic string amplitudes

The p-adic string theory started around 1987. In this year Volovich [56] considered modeling spacetime coordinates, string worldsheet coordinates and string amplitudes with p-adic values. Freund and Witten [24] proposed and studied string theory with p-adic worldsheet. Brekke, Freund, Olson and Witten [14] and Frampton and Okada [22] worked out N-point amplitudes in explicit form and investigated how these can be obtained from an effective Lagrangian p-adic string worldsheet, the tree-level string amplitudes (without loops).

We denote by $\mathbb{K}$ a local field of characteristic zero (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$). The Koba-Nielsen open string amplitudes for N-points over $\mathbb{K}$ are formally defined as

$$A_{\mathbb{K}}^{(N)}(\boldsymbol{k}) := \int_{\mathbb{K}^{N-3}} \prod_{j=2}^{N-2} |x_j|_{\mathbb{K}}^{\boldsymbol{k}_1 \boldsymbol{k}_j} |1-x_j|_{\mathbb{K}}^{\boldsymbol{k}_{N-1} \boldsymbol{k}_j} \prod_{2\leq i<j\leq N-2} |x_j - x_i|_{\mathbb{K}}^{\boldsymbol{k}_i \boldsymbol{k}_j} \prod_{\nu=2}^{N-2} dx_\nu, \tag{2.10}$$

where $\boldsymbol{k} = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)$ and $\boldsymbol{k}_i = (k_{1,i}, \ldots, k_{D,i}) \in \mathbb{R}^D$, is the momentum vector of the i-th tachyon for $i = 1, \ldots, N$ $(N \geq 4)$, $|\cdot|_{\mathbb{K}}$ is the norm in $\mathbb{K}$ and the product $\boldsymbol{k}_i \boldsymbol{k}_j$ is the Minkowski product. These vectors obey

$$\sum_{i=1}^{N} \boldsymbol{k}_i = 0,\ \ \boldsymbol{k}_i \boldsymbol{k}_i = 2 \text{ for } i = 1, \ldots, N.$$

The parameter D is an arbitrary positive integer. Typically, D is taken to be 26 for bosonic strings. In the case when $\mathbb{K} = \mathbb{Q}_p$ equation (2.10) is called the tree-level p-adic open string N-point amplitude. In [11] the authors studied these amplitudes and showed that it is possible regularize them by

$$A_{\mathbb{K}}^{(N)}(\boldsymbol{k}) = Z_{\mathbb{K}}^{(N)}(\boldsymbol{s})|_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j}$$

where $Z_{\mathbb{K}}^{(N)}(\boldsymbol{s})$ denotes the meromorphic continuation of the Koba-Nielsen local zeta function

$$Z_{\mathbb{K}}^{(N)}(\boldsymbol{s}) := \int_{\mathbb{K}^{N-3}} \prod_{j=2}^{N-2} |x_j|_{\mathbb{K}}^{s_{1j}} |1-x_j|_{\mathbb{K}}^{s_{(N-1)j}} \prod_{2\leq i<j\leq N-2} |x_j - x_i|_{\mathbb{K}}^{s_{ij}} \prod_{\nu=2}^{N-2} dx_\nu \tag{2.11}$$

in which s_{1j}, $s_{(N-1)j}$ for $2 \leq j \leq N-1$ and s_{ij} for $2 \leq i < j \leq N-2$ are complex numbers, $\boldsymbol{s} := (s_{ij}) \in \mathbb{C}^D$.

Chapter 3

Euclidean quantum field formulation of p-adic open string amplitudes

In this chapter we work in the real Lizorkin space $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ which is the space obtained by restricting the Lizorkin space to the functions taking real values i.e. the space $\mathcal{L}\left(\mathbb{Q}_p^N\right) \cap \mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$.

We set $\boldsymbol{k} := (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)$, where $\boldsymbol{k}_j = (k_{1,j}, \ldots, k_{D,j}) \in \mathbb{R}^D$ is the momentum of a tachyon, $j = 1, \ldots, N$. The dimension $D \geq 1$ is fixed along this chapter. We also set

$$\boldsymbol{\varphi}(\cdot) = (\varphi_1(\cdot), \ldots, \varphi_D(\cdot)) \in \left(\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p\right)\right)^D .$$

For $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^D$, $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the standard scalar product in $\mathbb{R}^D$. The space of fields is the D-dimensional space $\mathcal{L}_{\mathbb{R}}^D\left(\mathbb{Q}_p^N\right)$.

The naive Euclidean version of the p-adic N-point amplitudes is given by

$$\mathcal{A}^{(N)}\left(\boldsymbol{k}\right) = \frac{1}{Z_0} \int D\boldsymbol{\varphi} e^{-S(\boldsymbol{\varphi})} \int\limits_{\mathbb{Q}_p^N} d^N x \; e^{\sum\limits_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} \tag{3.1}$$

where $D\varphi$ is a measure in the space of fields which is constructed in the next section, $Z_0 = \int D\boldsymbol{\varphi} e^{-S(\boldsymbol{\varphi})}$ is constant, $d^N x = \prod_{\nu=1}^{N} dx_\nu$ and the action is $S(\boldsymbol{\varphi}) = \frac{T_0}{2} \sum_{j=1}^{D} S_j(\varphi_j)$, with

$$S_j(\varphi_j) = \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \left\{ \frac{\varphi_j(x_j) - \varphi_j(y_j)}{|x_j - y_j|_p} \right\}^2 dx_j dy_j.$$

Since there exists $l \in \mathbb{Z}$ such that $\varphi_j(x_j) = 0$ for $|x_j|_p > p^l$, it follows that the integral

$$\int_{\mathbb{Q}_p^N} d^N x \ e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)}$$

is divergent. To overcome this problem, it is necessary to introduce a cut-off and set

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0} \int D\boldsymbol{\varphi} e^{-S(\boldsymbol{\varphi})} \int_{B_R^N} d^N x e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)}$$

where R is a positive integer and $B_R^N = \left\{ x \in \mathbb{Q}_p^N; \|x\|_p \le p^R \right\}$.

We now construct the measure for the space of fields. For this purpose, we express the action in terms of an operator. For $\varphi_j \in \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p)$,

$$\begin{aligned} S_j(\varphi_j) =& \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \left\{ \frac{\varphi_j(x_j) - \varphi_j(y_j)}{|x_j - y_j|_p} \right\}^2 dx_j dy_j \\ =& 2 \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \frac{\varphi_j(x_j)(\varphi_j(x_j) - \varphi_j(y_j))}{|x_j - y_j|_p^2} dy_j dx_j \\ =& \frac{2(p+1)}{p^2} \int_{\mathbb{Q}_p} \varphi_j(x_j) \boldsymbol{D} \varphi_j(x_j) dx_j. \end{aligned}$$

Then,

$$S(\boldsymbol{\varphi}) = \frac{T_0(p+1)}{p^2} \sum_{j=1}^{D} \int_{\mathbb{Q}_p} \varphi_j(x_j) \boldsymbol{D} \varphi_j(x_j) dx_j.$$

This expression involving the Vladimirov operator gives us an idea about how to construct the measure.

Gaussian processes and free quantum fields

We use the Bochner-Minlos theorem to construct the measure over the fields. For this, we give a characteristic functional satisfying the conditions in Theorem 2. We define the bilinear form $\mathbb{B}$ by

$$\begin{array}{rccc}\mathbb{B}: & \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)\times\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right) & \to & \mathbb{R}\\ \\ & (\varphi,\theta) & \mapsto & \langle\varphi,\boldsymbol{D}^{-1}\theta\rangle\end{array}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2\left(\mathbb{Q}_p\right)$.

Lemma 1. *$\mathbb{B}$ is a positive, continuous bilinear form from $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)\times\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ into $\mathbb{R}$.*

Proof. Notice that for $\varphi\in\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$, we have

$$\mathbb{B}\left(\varphi,\varphi\right)=\left\langle\varphi,\boldsymbol{D}^{-1}\varphi\right\rangle=\left\langle\mathcal{F}^{-1}\varphi,\frac{\mathcal{F}\varphi}{|\xi|_p}\right\rangle=\int\limits_{\mathbb{Q}_p}\frac{|\widehat{\varphi}(\xi)|^2}{|\xi|_p}d\xi\geq 0.$$

Then $\mathbb{B}\left(\varphi,\varphi\right)=0$ implies that φ is zero almost everywhere. Since φ is continuous, $\varphi=0$. Let $(\varphi_n,\theta_n)\in\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)\times\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ be two sequences such that $\varphi_n\to 0$ and $\theta_n\to 0$ in $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$. We recall that the topology of $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ agrees with the topology of $\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$. Now,

$$\mathbb{B}\left(\theta_n,\varphi_n\right)=\int\limits_{\mathbb{Q}_p}\frac{\widehat{\theta}_n\left(\xi\right)\overline{\widehat{\varphi}}_n\left(\xi\right)}{|\xi|_p}d\xi=\int\limits_{\mathbb{Z}_p}\frac{\widehat{\theta}_n\left(\xi\right)\overline{\widehat{\varphi}}_n\left(\xi\right)}{|\xi|_p}d\xi+\int\limits_{\mathbb{Q}_p\smallsetminus\mathbb{Z}_p}\frac{\widehat{\theta}_n\left(\xi\right)\overline{\widehat{\varphi}}_n\left(\xi\right)}{|\xi|_p}d\xi. \tag{3.2}$$

We denote

$$I_1(\theta_n,\varphi_n)=\int\limits_{\mathbb{Z}_p}\frac{\widehat{\theta}_n(\xi)\,\overline{\widehat{\varphi}}_n(\xi)}{|\xi|_p}d\xi,$$

$$I_2(\theta_n,\varphi_n)=\int\limits_{\mathbb{Q}_p\smallsetminus\mathbb{Z}_p}\frac{\widehat{\theta}_n(\xi)\,\overline{\widehat{\varphi}}_n(\xi)}{|\xi|_p}d\xi.$$

Since $\theta_n\in\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p)$, there exist two integers m_0, l_0, independent of n, such that

$$\text{supp }\widehat{\theta}_n\subset p^{l_0}\mathbb{Z}_p \text{ and } \widehat{\theta}_n(\xi)\mid_{\xi_0+p^{m_0}\mathbb{Z}_p}=\widehat{\theta}_n(\xi_0) \tag{3.3}$$

for each $n\in\mathbb{N}$. Without loss of generality, we may assume that m_0 is a positive integer. From (3.3), $\widehat{\theta}_n(\xi)\mid_{p^{m_0}\mathbb{Z}_p}=\widehat{\theta}_n(0)=0$ for each $n\in\mathbb{N}$, and

$$\begin{aligned}|I_1(\varphi_n,\theta_n)| &\leq \|\widehat{\varphi}_n\|_\infty\int\limits_{p^{-m_0}<|\xi|_p\leq 1}\frac{\left|\widehat{\theta}_n(\xi)\right|}{|\xi|_p}d\xi\leq\|\varphi_n\|_1\|\widehat{\theta}_n\|_\infty\int\limits_{p^{-m_0}<|\xi|_p\leq 1}\frac{1}{|\xi|_p}d\xi\\ &\leq C_1\|\varphi_n\|_1\|\theta_n\|_1.\end{aligned}$$

For the second integral,

$$\begin{aligned}|I_2(\varphi_n,\theta_n)| &\leq \|\widehat{\varphi}_n\|_\infty\int\limits_{|\xi|_p>1}\frac{\left|\widehat{\theta}_n(\xi)\right|}{|\xi|_p}d\xi\leq\|\widehat{\varphi}_n\|_\infty\int\limits_{|\xi|_p>1}\left|\widehat{\theta}_n(\xi)\right|d\xi\\ &\leq \|\varphi_n\|_1\|\widehat{\theta}_n\|_1.\end{aligned}$$

Therefore, replacing the inequalities for $I_j(\varphi_n,\theta_n)$, $j=1,2$ in (3.2)

$$\mathbb{B}(\varphi_n,\theta_n)\leq C_1\|\varphi_n\|_1\|\theta_n\|_1+\|\varphi_n\|_1\|\widehat{\theta}_n\|_1.$$

The continuity of $\mathbb{B}$ follows from the fact that $\varphi_n\to 0$ and $\theta_n\to 0$ uniformly which imply that $\|\varphi_n\|_1 \to 0$, $\|\theta_n\|_1 \to 0$, and $\|\widehat{\theta}_n\|_1 \to 0$ as n tends to infinity. The convergence of $\widehat{\theta}_n$ follows from

$$\left\|\widehat{\theta}_n\right\|_1 = \int\limits_{p^{l_0}\mathbb{Z}_p} \left|\widehat{\theta}_n(\xi)\right| d\xi \leq p^{-l_0}\left\|\widehat{\theta}_n\right\|_\infty \leq p^{-l_0}\left\|\theta_n\right\|_1.$$

$\square$

We recall that $\mathcal{D}(\mathbb{Q}_p)$ is a nuclear space cf. [15, Section 4]. Since any subspace of a nuclear space is also nuclear, $\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ is a nuclear space that is dense and continuously embedded in $L^2_{\mathbb{R}}(\mathbb{Q}_p)$ cf. [2, Theorem 7.4.4]. Consequently, we have the following Gel'fand triple:

$$\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p) \hookrightarrow L^2_{\mathbb{R}}(\mathbb{Q}_p) \hookrightarrow \mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$$

(see [45, Chapter 2]). We denote by $\mathcal{B} := \mathcal{B}(\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p))$ the σ-algebra generated by the cylinder subsets of $\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$.

Consider the mapping

$$\begin{array}{cccc} \mathcal{C}: & \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p) & \to & \mathbb{C} \\ & & & \\ & f & \mapsto & e^{-\frac{1}{2}\mathbb{B}(f,f)}. \end{array}$$

This functional is a continuous, positive definite mapping cf. Lemma 1 and $\mathcal{C}(0) = 1$. Therefore $\mathcal{C}$ defines a characteristic functional in $\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$. By Bochner-Minlos theorem (see Theorem 2), there exists a unique probability measure $\mathbb{P}$ called the canonical Gaussian measure on $(\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p), \mathcal{B})$ given by its characteristic functional as

$$\int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sqrt{-1}(W,f)} d\mathbb{P}(W) = e^{-\frac{1}{2}\mathbb{B}(f,f)}, \quad f \in \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p), \tag{3.4}$$

where $(\cdot,\cdot)$ is the pairing between $\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$ and $\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$. The measure $\mathbb{P}$ corresponds to a free quantum field on $\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$. This identification is well-known in the Archimedean and non-Archimedean settings (see [31, Section 6.2] and [7, Section 5.5]). We have just constructed a measure in the space $\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$. However, our fields are in the D-dimensional space $\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)$, we define the measure in this space naturally by taking the product measure.

3.1 N-point amplitudes

We denote by

$$\mathbb{P}_D(\boldsymbol{\varphi}) := \bigotimes_{j=1}^{D} \mathbb{P}(\varphi_j),$$

the product probability measure on the product σ-algebra $\mathcal{B}^N$. We set

$$\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p) = \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p) \times \cdots \times \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p),\ D\text{-times}.$$

The probability measure

$$\frac{1_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)}(\boldsymbol{\varphi})\, d\mathbb{P}_D(\boldsymbol{\varphi})}{Z_0}, \tag{3.5}$$

where $Z_0 = \int_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} d\mathbb{P}_D(\boldsymbol{\varphi})$ represents a free quantum field in $\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)$.

Intuitively, the N-point amplitudes are the expectation values of products of vertex operators, with respect to the measure (3.5) this is written as

$$\left\langle \prod_{j=1}^{N} \int_{\mathbb{Q}_p} dx_j e^{\boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} \right\rangle_{\mathbb{P}_D} = \frac{1}{Z_0} \int_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} \int_{\mathbb{Q}_p^N} d^N x e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} d\mathbb{P}_D(\boldsymbol{\varphi}). \tag{3.6}$$

In the right hand side of (3.6) each $\boldsymbol{\varphi}(x_j)$ needs to be a function, for this reason, the factor $1_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)}$ is completely necessary in (3.5). Due to the divergence of the second integral in the right-hand side of (3.6), we define the N-point amplitudes as follows.

Definition 3. *For a positive integer R, we define*

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) := \frac{1}{Z_0} \int_{B_R^N} \left\{ \int_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} d\mathbb{P}_D(\boldsymbol{\varphi}) \right\} \prod_{\nu=1}^{N} dx_\nu.$$

The p-adic N-point amplitud is defined by $\mathcal{A}^{(N)}(\boldsymbol{k}) = \lim_{R\to\infty} \mathcal{A}_R^{(N)}(\boldsymbol{k})$.

We show that the ansatz proposed in the above definition allows us to obtain a regularized version of the p-adic Koba-Nielsen open string amplitude as the first term in the series expansion of $\mathcal{A}^{(N)}(\boldsymbol{k})$. This results are given in Theorems 3 and 4.

Using that

$$\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j) = \sum_{j=1}^{N}\sum_{l=1}^{D} k_{l,j}\varphi_l(x_j),$$

we have

$$\begin{aligned}\mathcal{A}_R^{(N)}(\boldsymbol{k}) &= \frac{1}{Z_0}\int_{B_R^N}\left\{\int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=1}^{N}\sum_{l=1}^{D} k_{l,j}\varphi_l(x_j)}\prod_{n=1}^{D} d\mathbb{P}(\varphi_n)\right\}\prod_{\nu=1}^{N} dx_\nu \\ &= \int_{B_R^N}\left\{\prod_{l=1}^{D}\frac{1}{Z_0^{1/D}}\int_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} k_{l,j}\varphi_l(x_j)} d\mathbb{P}(\varphi_l)\right\}\prod_{\nu=1}^{N} dx_\nu. \end{aligned} \tag{3.7}$$

Fixing l, we denote

$$\sum_{j=1}^{N} k_{l,j}\varphi_l(x_j) := \sum_{j=1}^{N} v_j\varphi(x_j). \tag{3.8}$$

Notice that in (3.8) $v_j \in \mathbb{R}$ and $\varphi \in \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$. Each term of the product in (3.7) is denoted by

$$\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x},\boldsymbol{v}) := \frac{1}{Z_0^{1/D}}\int_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j\varphi(x_j)} d\mathbb{P}(\varphi), \tag{3.9}$$

where $\boldsymbol{x} = (x_1,\ldots,x_N) \in \mathbb{Q}_p^N$, $\boldsymbol{v} = (v_1,\ldots,v_N) \in \mathbb{R}^N$. Using the Dirac distribution δ centered at x_j we write

$$\sum_{j=1}^{N} v_j\varphi(x_j) = \sum_{j=1}^{N} v_j\left(\delta(x-x_j),\varphi(x)\right).$$

Lemma 2. *With the above notation,* $\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x},\boldsymbol{v}) < \infty$ *for any* $R, N, \boldsymbol{x}, \boldsymbol{v}$. *Furthermore, for* $R, N, \boldsymbol{v}$ *fixed,* $\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x},\boldsymbol{v})$ *is a continuous function in* $\boldsymbol{x}$.

Proof. Recall that

$$\int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{(W,\theta)} d\mathbb{P}(W) < \infty, \tag{3.10}$$

for any $\theta \in \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$, where (W,θ) denotes the pairing between the space of distributions $\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$ and the Lizorkin space $\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ (cf. [37, Theorem 1.7]). Using that $\sum_{j=1}^{N} |v_j| |\varphi(x_j)| \delta(x - x_j) \in \mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$, for any $\varphi \in \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ and fixing $\theta \in \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ such that $\theta(x_j) > 1$, for $j = 1, \ldots, N$, we have

$$\begin{aligned}\sum_{j=1}^{N} v_j \varphi(x_j) &\leq \sum_{j=1}^{N} |v_j| |\varphi(x_j)| \leq \sum_{j=1}^{N} |v_j| |\varphi(x_j)| \theta(x_j) \\ &= \left(\sum_{j=1}^{N} |v_j| |\varphi(x_j)| \delta(x - x_j), \theta(x) \right),\end{aligned}$$

thus

$$\begin{aligned}\int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j \varphi(x_j)} d\mathbb{P}(\varphi) &\leq \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} |v_j||\varphi(x_j)|} d\mathbb{P}(\varphi) \\ &\leq \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\left(\sum_{j=1}^{N} |v_j||\varphi(x_j)|\delta(x-x_j), \theta(x) \right)} d\mathbb{P}(\varphi) \\ &\leq \int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{(W,\theta)} d\mathbb{P}(W) < \infty.\end{aligned}$$

Finally, the continuity in $\boldsymbol{x}$ follows from the dominated convergence theorem and the fact that

$$\int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} 1_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)}(\varphi) e^{\sum_{j=1}^{N} v_j \varphi(x_j)} d\mathbb{P}(\varphi) \leq \int\limits_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{(W,\theta)} d\mathbb{P}(W).$$

□

Corollary 1. *For R fixed, $\mathcal{A}_R^{(N)}(\boldsymbol{k}) < \infty$ for any $\boldsymbol{k}$. Furthermore,*

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0} \int_{B_R^N} \left\{ \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} d\mathbb{P}_D(\boldsymbol{\varphi}) \right\} \prod_{\nu=1}^{N} dx_\nu$$
$$= \frac{1}{Z_0} \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} \left\{ \int_{B_R^N} e^{\sum_{j=1}^{N} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j)} \prod_{\nu=1}^{N} dx_\nu \right\} d\mathbb{P}_D(\boldsymbol{\varphi}).$$

Proof. By Lemma 2, for $R, N, \boldsymbol{k}$ given,

$$\boldsymbol{x} \mapsto \prod_{l=1}^{D} \int_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} k_{l,j}\varphi_l(x_j)} d\mathbb{P}(\varphi_l) = \int_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=1}^{N}\sum_{l=1}^{D} k_{l,j}\varphi_l(x_j)} \prod_{n=1}^{D} d\mathbb{P}(\varphi_n) < \infty$$

is a well-defined and continuous function. Now, the announced formula is a consequence of Fubini's theorem. □

We set

$$\delta_n(x) = \begin{cases} p^n & |x|_p \le p^{-n} \\ 0 & |x|_p > p^{-n}, \end{cases}$$

for a positive integer n. In the space of distributions we have the convergence $\delta_n(x) \to \delta(x)$ when n goes to infinity (see Remark 1). Consider the approximation for $\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v})$ given by

$$\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v}; I) := \frac{1}{Z_0^{1/D}} \int_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j (\delta_I(x - x_j), \varphi(x))} d\mathbb{P}(\varphi), \tag{3.11}$$

where I is a positive integer.

Lemma 3. *With the above notation*

$$\lim_{I \to \infty} \widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v}; I) = \widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v}).$$

Proof. The proof is similar to the one given for Lemma 2. We show that

$$1_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)}\left(\varphi\right)e^{\sum\limits_{j=1}^{N}v_j(\delta_I(x-x_j),\varphi(x))}\leq 1_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)}\left(W\right)e^{(W,\theta)}, \tag{3.12}$$

where W is a distribution depending on x_j, v_j, for $j=1,\ldots,N$, but independent of I and $\theta\in\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ is a fixed positive function. Then the result follows by using (3.10) and the dominated convergence theorem. Notice that

$$\left|\sum_{j=1}^{N}v_j\left(\delta_I\left(x-x_j\right),\varphi\left(x\right)\right)\right|=\left|p^I\sum_{j=1}^{N}v_j\int\limits_{x_j+p^I\mathbb{Z}_p}\varphi\left(y\right)dy\right|\leq p^I\sum_{j=1}^{N}\left|v_j\right|\int\limits_{x_j+p^I\mathbb{Z}_p}\left|\varphi\left(y\right)\right|dy.$$

Let l_φ be the index of local constancy of φ and $I_\varphi=\max\left\{I,l_\varphi\right\}$, then $p^{I_\varphi}\mathbb{Z}_p$ is a subgroup of $p^I\mathbb{Z}_p$ and

$$G_j:=\left(x_j+p^I\mathbb{Z}_p\right)/p^{I_\varphi}\mathbb{Z}_p$$

is a finite set such that $x_j+p^I\mathbb{Z}_p=\bigsqcup_{\widetilde{x}\in G_j}\left(\widetilde{x}+p^{I_\varphi}\mathbb{Z}_p\right)$ (disjoint union). Considering $\theta\in\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ such that $\theta\left(\widetilde{x}\right)\geq 1$ for $\widetilde{x}\in G_j$, we have

$$\begin{aligned}
p^I\sum_{j=1}^{N}\left|v_j\right|\int\limits_{x_j+p^I\mathbb{Z}_p}\left|\varphi\left(y\right)\right|dy &=p^I\sum_{j=1}^{N}\sum_{\widetilde{x}\in G_j}\left|v_j\right|\int\limits_{\widetilde{x}+p^{I_\varphi}\mathbb{Z}_p}\left|\varphi\left(y\right)\right|dy\\
&=p^{I-I_\varphi}\sum_{j=1}^{N}\sum_{\widetilde{x}\in G_j}\left|v_j\right|\left|\varphi\left(\widetilde{x}\right)\right|\\
&\leq\sum_{j=1}^{N}\sum_{\widetilde{x}\in G_j}\left|v_j\right|\left|\varphi\left(\widetilde{x}\right)\right|\\
&\leq\sum_{j=1}^{N}\sum_{\widetilde{x}\in G_j}\left|v_j\right|\left|\varphi\left(\widetilde{x}\right)\right|\theta\left(\widetilde{x}\right)\\
&=\sum_{j=1}^{N}\sum_{\widetilde{x}\in G_j}\left|v_j\right|\left(\left|\varphi\left(\widetilde{x}\right)\right|\delta\left(x-\widetilde{x}\right),\theta\left(x\right)\right).
\end{aligned}$$

Thus (3.12) holds. □

For $L \geq 1$ and $m \in \mathbb{Q}_p^{\times}$ we define,

$$J_{L,m}(x) = \sum_{j=1}^{N} v_j \delta_L (x - x_j) - \sum_{j=1}^{N} v_j |m|_p^{-1} \Omega \left(\frac{|x|_p}{|m|_p} \right) * \delta_L (x - x_j).$$

Lemma 4. *With the above notation, the following sentences hold true:*

i. $J_{L,m}(x) \in \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ *for any* $L \geq 1$, $m \in \mathbb{Q}_p^{\times}$.

ii. $J_{L,m}(x) \to J_L(x) := \sum_{j=1}^{N} v_j \delta_L (x - x_j)$ *in* $L^{\rho}(\mathbb{Q}_p)$, $1 < \rho < \infty$ *when* $|m|_p \to \infty$.

iii. $J_{L,m}(x) \to J_L(x)$ *in* $\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$ *when* $|m|_p \to \infty$.

iv. *The equation* $\boldsymbol{D}\varphi_{L,m} = J_{L,m}$ *has a unique solution* $\varphi_{L,m} \in \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$ *given by* $\varphi_{L,m} = f_1 * J_{L,m}$, *where* f_1 *is defined in (1.5).*

v. $\varphi_{L,m} \to f_1 * J_L$ *in* $\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$ *as* $|m|_p \to \infty$.

vi. $f_1 * J_L = \frac{1-p}{p \ln p} \sum_{j=1}^{N} v_j \ln |x - x_j|_p$, *if* $|x - x_j|_p > p^{-L}$ *for* $j = 1, \ldots, N$.

Proof. *i.* Denote by $\Delta_m(\xi) = \Omega\left(|m\xi|_p\right)$, $m \in \mathbb{Q}_p^{\times}$, the characteristic function of the ball $B_{\log_p |m|_p^{-1}}$. Then

$$\widehat{J}_{L,m}(\xi) = \sum_{j=1}^{N} v_j \chi_p (\xi \cdot x_j) \Delta_L(\xi) (1 - \Delta_m(\xi)),$$

where $\Delta_L(\xi) = \Omega\left(p^{-L} |\xi|_p\right)$, which implies that $\widehat{J}_{L,m}$ is a test function satisfying $\widehat{J}_{L,m}(0) = 0$ for $|m|_p > 1$.

ii. Notice that

$$J_{L,m}(x) - \sum_{j=1}^{N} v_j \delta_L (x - x_j) = - \sum_{j=1}^{N} v_j |m|_p^{-1} \Omega \left(\frac{|x|_p}{|m|_p} \right) * \delta_L (x - x_j).$$

Since $|m|_p^{-1} \Omega\left(|m|_p^{-1} |x|_p\right) \in L^1(\mathbb{Q}_p)$ and $\delta_L(x) \in L^{\rho}$, $1 < \rho < \infty$ it follows that

$\Omega\left(\frac{|x|_p}{|m|_p}\right)*\delta_L(x-x_j)\to 0$ as $|m|_p\to\infty$ in L^ρ, $1<\rho<\infty$ [2, Lemma 7.4.2].

iii. Take $\theta\in\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)$, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\left|\int_{\mathbb{Q}_p}J_{L,m}(x)\,\theta(x)\,dx-\int_{\mathbb{Q}_p}J_L(x)\,\theta(x)\,dx\right| &= \left|\int_{\mathbb{Q}_p}\theta(x)\left(J_{L,m}(x)-J_L(x)\right)dx\right| \\ &\leq \|\theta\|_2\|J_{L,m}-J_L\|_2.\end{aligned}$$

From item *ii.*, $\|J_{L,m}-J_L\|_2\to 0$ as $|m|_p\to\infty$ and the convergence holds.

iv. This was proved in [2, Theorem 9.2.6], see also [55, Chapter 2, Section IX.2].

v. It follows from item *iii.* using the continuity of the convolution.

vi. If $|x-x_j|_p>p^{-L}$ for any $j=1,\ldots,N$,

$$\begin{aligned}f_1*J_L(x) &= \frac{1-p}{p\ln p}\sum_{j=1}^{N}v_j\ln|x|_p*\delta_L(x-x_j)\\ &= \frac{1-p}{p\ln p}\sum_{j=1}^{N}v_jp^L\int\limits_{x-x_j+p^L\mathbb{Z}_p}\ln|z|_p\,dz \qquad (3.13)\\ &= \frac{1-p}{p\ln p}\sum_{j=1}^{N}v_j\ln|x-x_j|_p\,. \qquad (3.14)\end{aligned}$$

$\square$

Let $\varphi_{L,m}$ be the function given in *iv.* Lemma 4. We use the measurable mapping

$$\begin{array}{ccc}\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p) & \to & \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)\\ \varphi & \mapsto & \widetilde{\varphi}-\varphi_{L,m},\end{array}$$

as a change of variable in (3.11). There exist a measure $\widetilde{\mathbb{P}}_{L,m}$ such that

$$\begin{aligned}\widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x},\boldsymbol{v};I) &= \frac{1}{Z_0^{1/D}} \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j\left(\delta_I(x-x_j),\widetilde{\varphi}-\varphi_{L,m}\right)} d\widetilde{\mathbb{P}}_{L,m}(\widetilde{\varphi}) \\ &= \frac{1}{Z_0^{1/D}} e^{\sum_{j=1}^{N} v_j\left(\delta_I(x-x_j),-\varphi_{L,m}\right)} \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j(\delta_I(x-x_j),\widetilde{\varphi})} d\widetilde{\mathbb{P}}_{L,m}(\widetilde{\varphi}). \qquad (3.15)\end{aligned}$$

We proceed to compute the limits $|m|_p \to \infty$, $L \to \infty$, $I \to \infty$ in (3.15) to obtain a formula for $\mathcal{A}_R^{(N)}(\boldsymbol{k})$.

Lemma 5. *The exponential term in (3.15) satisfies*

$$\lim_{I\to\infty} \lim_{L\to\infty} \lim_{|m|_p\to\infty} e^{\sum_{j=1}^{N} v_j\left(\delta_I(x-x_j),-\varphi_{L,m}\right)} = e^{\frac{p-1}{p\ln p}\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N} v_j v_i \ln|x_j-x_i|_p}.$$

Proof. With the aid of formula (3.14) and using the continuity of the pairing and the continuity of the convolution we obtain

$$\sum_{j=1}^{N} v_j\left(\delta_I(x-x_j), -\varphi_{L,m}\right) \to \sum_{j=1}^{N} v_j\left(\delta_I(x-x_j), \frac{p-1}{p\ln p}\sum_{i=1}^{N} v_i \ln|x-x_i|_p\right),$$

for any x_j with $|x-x_j|_p > p^{-L}$, this convergence is given in $\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)$ as $|m|_p \to \infty$, it follows that

$$e^{\sum_{j=1}^{N} v_j(\delta_I(x-x_j),-\varphi_{L,m})} \to e^{\sum_{j=1}^{N} v_j\left(\delta_I(x-x_j),\frac{p-1}{p\ln p}\sum_{i=1}^{N} v_i \ln|x-x_i|_p\right)}$$

as $|m|_p \to \infty$. Since $\ln|x|_p$ is locally constant in $\mathbb{Q}_p^{\times}$, and $\lim_{t\to-\infty} e^t = 0$, we have for I sufficiently large that

$$\left(\delta_I(x-x_j), \frac{p-1}{p\ln p}\sum_{i=1}^{N} v_i \ln|x-x_i|_p\right) = \frac{p-1}{p\ln p}\sum_{i=1}^{N} v_i \ln|x_j-x_i|_p$$

if $x_j \neq x_i$, and $-\infty$ otherwise. Therefore,

$$e^{\sum_{j=1}^{N} v_j \left(\delta_I(x-x_j), \frac{p-1}{p \ln p} \sum_{i=1}^{N} v_i \ln|x-x_i|_p \right)} = e^{\frac{p-1}{p \ln p} \sum_{j=1}^{N} \sum_{i=1, i\neq j}^{N} v_j v_i \ln|x_j - x_i|_p}$$

for I sufficiently large. $\square$

For the limit of the integral in (3.15), we proceed as follows. Using (3.4) and the change of variable $W = \widetilde{W} - \varphi_{L,m}$, we have

$$\int_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sqrt{-1}\left(\widetilde{W} - \varphi_{L,m}, g\right)} d\widetilde{\mathbb{P}}_{L,m}\left(\widetilde{W}\right) = e^{-\frac{1}{2}\mathbb{B}(g,g)},$$

i.e.

$$\int_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sqrt{-1}\left(\widetilde{W}, g\right)} d\widetilde{\mathbb{P}}_{L,m}\left(\widetilde{W}\right) = e^{\sqrt{-1}\left(\varphi_{L,m}, g\right) - \frac{1}{2}\mathbb{B}(g,g)} =: \mathcal{C}_{L,m}\left(g\right). \tag{3.16}$$

By Lemma 4 we have

$$\lim_{L\to\infty} \lim_{|m|_p\to\infty} \mathcal{C}_{L,m}\left(g\right) = e^{\sqrt{-1}\left(\frac{p-1}{p \ln p} \sum_{j=1}^{N} v_j \ln|x-x_j|_p, g \right) - \frac{1}{2}\mathbb{B}(g,g)} =: \mathcal{C}\left(g\right).$$

We denote by $\widetilde{\widetilde{\mathbb{P}}}$ the measure obtained from the Bochner-Minlos theorem applied to $\mathcal{C}\left(g\right)$. Recall that

$$\mathcal{C}(h) = \int_{\mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sqrt{-1}(W,h)} d\mathbb{P}\left(W\right) = \int_{\mathbb{R}} e^{\sqrt{-1}x} d\mathbb{P}_h\left(x\right)$$

where $\mathbb{P}_h\left(x\right)$ is the measure of the half-space $(W,h) \leq x$ in $\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p\right)$ (see e.g. [27, Chapter IV, Section 4.1]). If $\mathcal{C}(h_n) \to \mathcal{C}(\widetilde{h})$, and $\mathbb{P}_{h_n}\left(\mathbb{R}\right) \leq 1$ for all n, then $\mathbb{P}_{h_n} \Rightarrow \mathbb{P}_{\widetilde{h}}$, here $\mathcal{C}(\widetilde{h})$ is the characteristic function of $\mathbb{P}_{\widetilde{h}}$ (see e.g. [8, Theorem 7.8.11]). The arrow "$\Rightarrow$" means that

$$\int_{\mathbb{R}} l(x) d\mathbb{P}_{h_n}\left(x\right) \to \int_{\mathbb{R}} l(x) d\mathbb{P}_{\widetilde{h}}\left(x\right) \text{ for any bounded continuous function } l(x). \tag{3.17}$$

From this we conclude that

$$\widetilde{\mathbb{P}}_{L,m} \Rightarrow \widetilde{\mathbb{P}} \text{ when } |m|_p \to \infty, \text{ and } L \to \infty.$$

If $l(x) \in L^1(\mathbb{R}, \mathbb{P}_{h_n})$ for any n and $l(x) \in L^1\left(\mathbb{R}, \mathbb{P}_{\widetilde{h}}\right)$, using the fact that the bounded continuous functions are dense in both $L^1(\mathbb{R}, \mathbb{P}_{h_n})$ and $L^1\left(\mathbb{R}, \mathbb{P}_{\widetilde{h}}\right)$ (see [3, Proposition 1.3.22]), we assume in (3.17) that $l(x)$ is an integrable function.

Then, we have the following result.

Lemma 6. *The following equalities are satisfied*

$$\lim_{L\to\infty} \lim_{|m|_p\to\infty} \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j(\delta_I(x-x_j),\widetilde{\varphi})} d\widetilde{\mathbb{P}}_{L,m}(\widetilde{\varphi}) = \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j(\delta_I(x-x_j),\widetilde{\varphi})} d\widetilde{\mathbb{P}}(\widetilde{\varphi}),$$

and

$$\lim_{I\to\infty} \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j(\delta_I(x-x_j),\widetilde{\varphi})} d\widetilde{\mathbb{P}}(\widetilde{\varphi}) = \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j\widetilde{\varphi}(x_j)} d\widetilde{\mathbb{P}}(\widetilde{\varphi}).$$

Applying Lemmas 3, 5 and 6 to (3.15) we have

$$\lim_{I\to\infty} \lim_{L\to\infty} \lim_{|m|_p\to\infty,} \widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v}; I) = \widetilde{\mathcal{A}}_R^{(N)}(\boldsymbol{x}, \boldsymbol{v})$$
$$= \frac{1}{Z_0^{1/D}} e^{\frac{p-1}{p\ln p}\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N} v_j v_i \ln|x_j-x_i|_p} \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p)} e^{\sum_{j=1}^{N} v_j\widetilde{\varphi}(x_j)} d\widetilde{\mathbb{P}}(\widetilde{\varphi}).$$

Using this formula and the definition of $\mathcal{A}_R^{(N)}(\boldsymbol{k})$, we establish the following result.

Proposition 1. *The amplitude $\mathcal{A}_R^{(N)}(\boldsymbol{k})$ satisfies*

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = \frac{1}{Z_0} \int\limits_{B_R^N} \prod_{1\leq i<j\leq N} |x_j - x_i|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \int\limits_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=1}^{N}\boldsymbol{k}_j\cdot\widetilde{\boldsymbol{\varphi}}(x_j)} d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}) \prod_{\nu=1}^{N} dx_\nu. \tag{3.18}$$

We assume that the insertion points $x_1, \ldots, x_N$, with $N \geq 4$ belong to the p-adic projective line, and then by using the Möbius group, we choose the normalization

$$x_1 = 0, x_{N-1} = 1, x_N = \infty. \tag{3.19}$$

In our framework, the convention $x_N = \infty$ means that the N-point amplitudes do not depend on x_N. Replacing (3.19) in (3.18) $\mathcal{A}_R^{(N)}(\boldsymbol{k})$ takes the form

$$\begin{aligned}\mathcal{A}_R^{(N)}(\boldsymbol{k}) =& \frac{C_0}{Z_0} \int\limits_{B_R^{N-3}} \prod_{j=2}^{N-2} |x_j|_p^{\frac{2(p-1)}{p \ln p}\boldsymbol{k}_1 \cdot \boldsymbol{k}_j} |1 - x_j|_p^{\frac{2(p-1)}{p \ln p}\boldsymbol{k}_{N-1} \cdot \boldsymbol{k}_j} \times \\ & \prod_{2 \leq i < j \leq N-2} |x_j - x_i|_p^{\frac{2(p-1)}{p \ln p}\boldsymbol{k}_i \cdot \boldsymbol{k}_j} \int\limits_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=2}^{N-2} \boldsymbol{k}_j \cdot \widetilde{\boldsymbol{\varphi}}(x_j)} d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}) \prod_{\nu=2}^{N-2} dx_\nu,\end{aligned}$$

where the momenta vectors satisfy $\sum_{i=1}^{N} \boldsymbol{k}_i = \boldsymbol{0}$ and

$$C_0 = \int\limits_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\boldsymbol{k}_1 \cdot \widetilde{\boldsymbol{\varphi}}(0) + \boldsymbol{k}_{N-1} \cdot \widetilde{\boldsymbol{\varphi}}(1)} d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}).$$

Define the function

$$\Theta(\boldsymbol{k}, \boldsymbol{x}) := \Theta(\boldsymbol{k}, x_2, \ldots, x_{N-2}) = \int\limits_{\mathcal{L}_{\mathbb{R}}^D(\mathbb{Q}_p)} e^{\sum_{j=2}^{N-2} \boldsymbol{k}_j \cdot \widetilde{\boldsymbol{\varphi}}(x_j)} d\widetilde{\mathbb{P}}_D(\widetilde{\boldsymbol{\varphi}}).$$

We consider the expansion in series for the exponential function around zero

$$\begin{aligned} e^{\sum_{j=2}^{N-2} \boldsymbol{k}_j \cdot \widetilde{\boldsymbol{\varphi}}(x_j)} &= \lim_{M \to \infty} \sum_{r=0}^{M} \frac{\left(\sum_{j=2}^{N-2} \boldsymbol{k}_j \cdot \widetilde{\varphi}(x_j) \right)^r}{r!} \\ &= \lim_{M' \to \infty} \sum_{r=0}^{M'} F_r(\boldsymbol{k}, \widetilde{\boldsymbol{\varphi}}(x_2), \ldots, \widetilde{\boldsymbol{\varphi}}(x_{N-2})), \end{aligned}$$

where $F_r\left(\boldsymbol{k},\widetilde{\boldsymbol{\varphi}}\left(x_2\right),\ldots,\widetilde{\boldsymbol{\varphi}}\left(x_{N-2}\right)\right)$ is a homogeneous polynomial of degree r in the variables $k_{l,j}$, $l=1,\ldots,D$, $j=2,\ldots,N-2$, whose coefficients are polynomials in $\widetilde{\boldsymbol{\varphi}}\left(x_2\right),\ldots,\widetilde{\boldsymbol{\varphi}}\left(x_{N-2}\right)$. Since

$$\sum_{r=0}^{M'}\left|F_r\left(\boldsymbol{k},\widetilde{\boldsymbol{\varphi}}\left(x_2\right),\ldots,\widetilde{\boldsymbol{\varphi}}\left(x_{N-2}\right)\right)\right|\leq e^{\sum_{j=2}^{N-2}\sum_{l=1}^{D}\left|k_{l,j}\right|\left|\widetilde{\varphi}_l(x_j)\right|}\in L^1\left(\mathcal{L}_{\mathbb{R}}^{D}\left(\mathbb{Q}_p\right),\widetilde{\mathbb{P}}_D\right),$$

by the dominated convergence theorem and Corollary 1, we have

$$\begin{aligned}\Theta(\boldsymbol{k},\boldsymbol{x})&=\int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)}\left\{\lim_{M'\to\infty}\sum_{r=0}^{M'}F_r\left(\boldsymbol{k},\widetilde{\boldsymbol{\varphi}}\left(x_2\right),\ldots,\widetilde{\boldsymbol{\varphi}}\left(x_{N-2}\right)\right)\right\}d\widetilde{\mathbb{P}}_D\left(\boldsymbol{\varphi}\right)\\&=\lim_{M'\to\infty}\sum_{r=0}^{M'}\int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)}F_r\left(\boldsymbol{k},\widetilde{\boldsymbol{\varphi}}\left(x_2\right),\ldots,\widetilde{\boldsymbol{\varphi}}\left(x_{N-2}\right)\right)d\widetilde{\mathbb{P}}_D\left(\widetilde{\boldsymbol{\varphi}}\right)\\&=\int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)}d\widetilde{\mathbb{P}}_D\left(\widetilde{\boldsymbol{\varphi}}\right)+\sum_{r=1}^{\infty}\int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)}F_r\left(\boldsymbol{k},\widetilde{\boldsymbol{\varphi}}\left(x_2\right),\ldots,\widetilde{\boldsymbol{\varphi}}\left(x_{N-2}\right)\right)d\widetilde{\mathbb{P}}_D\left(\widetilde{\boldsymbol{\varphi}}\right).\end{aligned}$$

The functions $F_r\left(\boldsymbol{k},\widetilde{\boldsymbol{\varphi}}\left(x_2\right),\ldots,\widetilde{\boldsymbol{\varphi}}\left(x_{N-2}\right)\right)$ are integrable continuous functions in $\boldsymbol{x}$ for $\boldsymbol{k}$ fixed, we conclude that

$$G_r(\boldsymbol{k},\boldsymbol{x}):=\int\limits_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)}F_r\left(\boldsymbol{k},\widetilde{\boldsymbol{\varphi}}\left(x_2\right),\ldots,\widetilde{\boldsymbol{\varphi}}\left(x_{N-2}\right)\right)d\widetilde{\mathbb{P}}_D\left(\widetilde{\boldsymbol{\varphi}}\right)$$

is a continuous function in $\boldsymbol{x}$. Therefore $\Theta(\boldsymbol{k},\boldsymbol{x})$ is expanded as a series

$$\Theta(\boldsymbol{k},\boldsymbol{x})=C+\sum_{r=1}^{\infty}G_r(\boldsymbol{k},\boldsymbol{x}).$$

where each term is a continuous function in $\boldsymbol{x}$. Using the formula given in Proposition 1 and Fubini's theorem, we obtain the following result.

Theorem 3. *The amplitude* $\mathcal{A}_R^{(N)}(\boldsymbol{k})$ *admits the following expansion in the momenta:*

$$
\begin{aligned}
\mathcal{A}_R^{(N)}(\boldsymbol{k}) = & \frac{CC_0}{Z_0} \int\limits_{B_R^{N-3}} \prod_{j=2}^{N-2} |x_j|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_j} |1-x_j|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_j} \times \\
& \prod_{2\leq i<j\leq N-2} |x_j - x_i|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \prod_{\nu=2}^{N-2} dx_\nu + \\
& \frac{C_0}{Z_0} \sum_{r=1}^{\infty} \int\limits_{B_R^{N-3}} \prod_{j=2}^{N-2} |x_j|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_j} |1-x_j|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_j} \times \\
& \prod_{2\leq i<j\leq N-2} |x_j - x_i|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} G_r(\boldsymbol{k},\boldsymbol{x}) \prod_{\nu=2}^{N-2} dx_\nu .
\end{aligned}
$$

To continue the study of the amplitudes $\mathcal{A}_R^{(N)}(\boldsymbol{k})$, we introduce the following notation:

$$
\begin{aligned}
A_R^{(N)}(\boldsymbol{k}) = & \frac{CC_0}{Z_0} \int_{B_R^{N-3}} \prod_{j=2}^{N-2} |x_j|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_j} |1-x_j|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_j} \times \\
& \prod_{2\leq i<j\leq N-2} |x_j - x_i|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} \prod_{\nu=2}^{N-2} dx_\nu ,
\end{aligned}
$$

$$
\begin{aligned}
Z_{G_r,R}^{(N)}(\boldsymbol{k}) = & \frac{C_0}{Z_0} \int\limits_{\mathbb{Q}_p^{N-3}} \prod_{j=2}^{N-2} |x_j|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_j} |1-x_j|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_j} \times \\
& \prod_{2\leq i<j\leq N-2} |x_j - x_i|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} 1_{B_R^{N-3}}(\boldsymbol{x}) G_r(\boldsymbol{k},\boldsymbol{x}) \prod_{\nu=2}^{N-2} dx_\nu .
\end{aligned}
$$

Notice that $1_{B_R^{N-3}}(\boldsymbol{x})G_r(\boldsymbol{k},\boldsymbol{x})$ is a continuous function in $\boldsymbol{x}$ with support contained in B_R^{N-3}.

3.2 Regularization of p-adic open string amplitudes

3.2.1 The p-adic Koba-Nielsen local zeta functions

Take $N \geq 4$ and $s_{ij} \in \mathbb{C}$ satisfying $s_{ij} = s_{ji}$ for $1 \leq i < j \leq N-1$. The p-adic Koba-Nielsen local zeta function (or p-adic open string N-point zeta function) is defined as

$$Z^{(N)}(\mathbf{s}) = \int_{\mathbb{Q}_p^{N-3} \smallsetminus \Lambda} \prod_{j=2}^{N-2} |x_j|_p^{s_{1j}} |1-x_j|_p^{s_{(N-1)j}} \prod_{2 \leq i < j \leq N-2} |x_j - x_i|_p^{s_{ij}} \prod_{\nu=2}^{N-2} dx_\nu, \tag{3.20}$$

where $\boldsymbol{s} = (s_{ij}) \in \mathbb{C}^{D_0}$, D_0 denotes the total number of possible subsets $\{i, j\}$ and

$$\Lambda := \left\{ (x_2, \ldots, x_{N-2}) \in \mathbb{Q}_p^{N-3};\ \prod_{j=2}^{N-2} x_j (1-x_j) \prod_{2 \leq i < j \leq N-2} (x_j - x_i) = 0 \right\}.$$

These functions were introduced in [11] (see also [12]). The functions $Z^{(N)}(\mathbf{s})$ are holomorphic in a certain domain of $\mathbb{C}^{D_0}$ and admit analytic extension to $\mathbb{C}^{D_0}$ as rational functions in the

$$p^{-s_{ij}},\ i, j \in \{1, \ldots, N-1\},$$

which are denoted also as $Z^{(N)}(\mathbf{s})$ see [11, Theorem 1], [12, Theorem 6.1].

If $\phi(x_2, \ldots, x_{N-2})$ is a locally constant function with compact support, then

$$Z_\phi^{(N)}(\boldsymbol{s}) = \int_{\mathbb{Q}_p^{N-3} \smallsetminus \Lambda} \phi(x_2, \ldots, x_{N-2}) \prod_{j=2}^{N-2} |x_j|_p^{s_{1j}} |1-x_j|_p^{s_{(N-1)j}} \prod_{2 \leq i < j \leq N-2} |x_j - x_i|_p^{s_{ij}} \prod_{\nu=2}^{N-2} dx_\nu,$$

with $\mathrm{Re}(s_{ij}) > 0$, for any i, j, is a multivariate Igusa local zeta function. These functions admit an analytic extension as rational functions in the variables $p^{-s_{ij}}$, [47]. If we take ϕ to be the characteristic function of B_R^{N-3}, the ball centered at the origin with radius p^R, the dominated convergence theorem and [11, Theorem 1], imply that

$$\lim_{R\to\infty} Z_R^{(N)}(\boldsymbol{s}) := \lim_{R\to\infty} \int\limits_{B_R^{N-3}\smallsetminus\Lambda} \prod_{j=2}^{N-2} |x_j|_p^{s_{1j}} \, |1-x_j|_p^{s_{(N-1)j}} \prod_{2\leq i<j\leq N-2} |x_j - x_i|_p^{s_{ij}} \prod_{\nu=2}^{N-2} dx_\nu \quad (3.21)$$

$$= Z^{(N)}(\mathbf{s}),$$

for any $\boldsymbol{s}$ in the natural domain of $Z^{(N)}(\mathbf{s})$.

In [14], Brekke, Freund, Olson and Witten worked out the N-point amplitudes in explicit form and investigated how these are obtained from an effective Lagrangian. The tree-level p-adic open string amplitudes for N-points are defined as

$$A_{\mathcal{M}}^{(N)}(\boldsymbol{k}) = \quad (3.22)$$

$$\int\limits_{\mathbb{Q}_p^{N-3}} \prod_{j=2}^{N-2} |x_j|_p^{\boldsymbol{k}_1\boldsymbol{k}_j} \, |1-x_j|_p^{\boldsymbol{k}_{N-1}\boldsymbol{k}_j} \prod_{2\leq i<j\leq N-2} |x_j - x_i|_p^{\boldsymbol{k}_i\boldsymbol{k}_j} \prod_{\nu=2}^{N-2} dx_\nu,$$

where the momentum vectors obey

$$\sum_{i=1}^{N} \boldsymbol{k}_i = \boldsymbol{0}, \qquad \boldsymbol{k}_i\boldsymbol{k}_i = 2 \text{ for } i = 1, \ldots, N,$$

see Section 2.2.1, Chapter 2.

In [11], [12], the p-adic integrals $Z^{(N)}(\boldsymbol{s})$ are used as regularizations of the amplitudes $A_{\mathcal{M}}^{(N)}(\boldsymbol{k})$. More precisely, the amplitude $A_{\mathcal{M}}^{(N)}(\boldsymbol{k})$ can be redefined as

$$A_{\mathcal{M}}^{(N)}(\boldsymbol{k}) = Z^{(N)}(\boldsymbol{s}) \mid_{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j} \text{ with } i \in \{1, \ldots, N-1\}, \, j \in J \text{ or } i, j \in J,$$

where $J = \{2, \ldots, N-2\}$. Then the amplitudes $A_{\mathcal{M}}^{(N)}(\boldsymbol{k})$ are well-defined rational functions in the $p^{-\boldsymbol{k}_i\boldsymbol{k}_j}$, i, $j \in \{1, \ldots, N-1\}$, which agree with integrals (3.22) when they converge.

Remark 2. *In [11], [12], the local zeta functions $Z^{(N)}(\boldsymbol{s})$ were used to regularize Koba-*

Nielsen amplitudes $A_{\mathcal{M}}^{(N)}(\boldsymbol{k})$*, when the momenta* $\boldsymbol{k}$ *belongs to the Minkowski space. Here, we use the functions* $Z^{(N)}(\boldsymbol{s})$ *to regularize Koba-Nielsen amplitudes* $A^{(N)}(\boldsymbol{k})$ *when the momenta* $\boldsymbol{k}$ *belongs to the Euclidean space. This is possible because* $Z^{(N)}(\boldsymbol{s})$ *is a rational function in the* $p^{-s_{ij}}$*.*

Remark 3. *We denote by* $Z_{\cdot}^{(N)}(\boldsymbol{s})$ *the distribution* $\phi \mapsto Z_{\phi}^{(N)}(\boldsymbol{s})$*. Then the mapping*

$$\begin{array}{ccc} \mathbb{C}^{D_0} & \rightarrow & \mathcal{D}'(\mathbb{Q}_p^{N-3}) \\ & & \\ \boldsymbol{s} & \mapsto & Z_{\cdot}^{(N)}(\boldsymbol{s}) \end{array} \tag{3.23}$$

is a meromorphic function of $\boldsymbol{s}$*. By using the fact that* $\mathcal{D}(\mathbb{Q}_p^{N-3})$ *is dense in the space of continuous functions with compact support* $\mathcal{C}_c(\mathbb{Q}_p^{N-3})$*, the functional* $\phi \mapsto Z_{\phi}^{(N)}(\boldsymbol{s})$ *has a unique extension to* $\mathcal{C}_c(\mathbb{Q}_p^{N-3})$*. Furthermore, if* $\boldsymbol{s}_0$ *is a pole of* $Z_{\phi}^{(N)}(\boldsymbol{s})$*, by using Gel'fand-Shilov method of analytic continuation (see e.g. [41, pp. 65-67]), we have*

$$Z_{\phi}^{(N)}(\boldsymbol{s}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^{D_0}} c_{\boldsymbol{k}}(\phi)(\boldsymbol{s} - \boldsymbol{s}_0)^{\boldsymbol{k}},$$

where the $c_{\boldsymbol{k}}$*s are distributions from* $\mathcal{D}'(\mathbb{Q}_p^{N-3})$*. The density of* $\mathcal{D}(\mathbb{Q}_p^{N-3})$ *in* $\mathcal{C}_c(\mathbb{Q}_p^{N-3})$ *implies that* $c_{\boldsymbol{k}} \neq 0$ *in* $\mathcal{D}'(\mathbb{Q}_p^{N-3})$ *if and only if* $c_{\boldsymbol{k}} \neq 0$ *in* $\mathcal{C}_c'(\mathbb{Q}_p^{N-3})$*, the strong dual space of* $\mathcal{C}_c(\mathbb{Q}_p^{N-3})$ *and consequently the mapping*

$$\begin{array}{ccc} \mathbb{C}^{D_0} & \rightarrow & \mathcal{C}_c'(\mathbb{Q}_p^{N-3}) \\ & & \\ \boldsymbol{s} & \mapsto & Z_{\cdot}^{(N)}(\boldsymbol{s}) \end{array}$$

is a meromorphic function in $\boldsymbol{s}$ *having the same poles of the mapping (3.23).*

Notice that by (3.21) we have

$$\frac{Z_0}{CC_0} \lim_{R \to \infty} A_R^{(N)}(\boldsymbol{k}) = \lim_{R \to \infty} \left(Z_R^{(N)}(\boldsymbol{s}) \mid_{s_{ij} = \frac{2(p-1)}{p \ln p} \boldsymbol{k}_i \cdot \boldsymbol{k}_j} \right) = Z^{(N)}(\boldsymbol{s}) \mid_{s_{ij} = \frac{2(p-1)}{p \ln p} \boldsymbol{k}_i \cdot \boldsymbol{k}_j}.$$

Since $Z^{(N)}(\boldsymbol{s})$ is a holomorphic function in a certain domain of $\mathbb{C}^{D_0}$, we conclude that $\lim_{R\to\infty} A_R^{(N)}(\boldsymbol{k})$ exists for $\boldsymbol{k}$ belonging to a nonempty subset of $\mathbb{C}^{D_0}$.

By Remark 3, we may assume that $1_{B_R^{N-3}}(\boldsymbol{x})G_r(\boldsymbol{k},\boldsymbol{x}) =: \phi$ is a test function in $\boldsymbol{x}$, and then $Z^{(N)}_{G_r,R}(\boldsymbol{k}) = \frac{C_0}{Z_0} Z^{(N)}_{\phi}(\boldsymbol{s}) \mid_{s_{ij}=\frac{2(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j}$ is a multivariate local zeta function. Furthermore,

$$
\begin{aligned}
|Z^{(N)}_{G_r,R}(\boldsymbol{k})| \leq & \frac{C_0}{Z_0} \int_{\mathbb{Q}_p^{N-3}} \prod_{j=2}^{N-2} |x_j|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_1\cdot\boldsymbol{k}_j} |1-x_j|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_{N-1}\cdot\boldsymbol{k}_j} \times \\
& \prod_{2\leq i<j\leq N-2} |x_j - x_i|_p^{\frac{2(p-1)}{p\ln p}\boldsymbol{k}_i\cdot\boldsymbol{k}_j} |G_r(\boldsymbol{k},\boldsymbol{x})| \prod_{\nu=2}^{N-2} dx_\nu,
\end{aligned}
$$

which implies that $\left|Z^{(N)}_{G_r,R}(\boldsymbol{k})\right| \leq \frac{C_0\ C_r(\boldsymbol{k},R)}{Z_0} Z^{(N)}(\boldsymbol{k})$, where

$$
C_r(\boldsymbol{k}, R) = \sup_{\boldsymbol{x}\in B_R^{N-3}} |G_r(\boldsymbol{k},\boldsymbol{x})| .
$$

Since $Z^{(N)}(\boldsymbol{k})$ converges in a nonempty open set, we conclude that all the $Z^{(N)}_{G_r,R}(\boldsymbol{k})$s converge in the open set where $Z^{(N)}(\boldsymbol{k})$ converges.

In conclusion, we obtain the following result.

Theorem 4. *The functions $\mathcal{A}_R^{(N)}(\boldsymbol{k})$ possess the following representation:*

$$
\mathcal{A}_R^{(N)}(\boldsymbol{k}) = A_R^{(N)}(\boldsymbol{k}) + \sum_{r=1}^{\infty} Z^{(N)}_{G_r,R}(\boldsymbol{k}),
$$

for R fixed, where $A_R^{(N)}(\boldsymbol{k})$ and all the $Z^{(N)}_{G_r,R}(\boldsymbol{k})$s are multivariate Igusa's local zeta functions, all of them converging in a common nonempty open set. Furthermore,

$$
\lim_{R\to\infty} A_R^{(N)}(\boldsymbol{k}) = \frac{CC_0}{Z_0} Z^{(N)}(\boldsymbol{k}),
$$

which is the p-adic Koba-Nielsen open string amplitude.

3.2.2 $\boldsymbol{\varphi}^4$-Theories

Consider the family of $\boldsymbol{\varphi}^4$-interacting quantum field theories:

$$\frac{1_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)}(\boldsymbol{\varphi})\, e^{-\lambda E_{int}(\boldsymbol{\varphi})} d\mathbb{P}_D(\boldsymbol{\varphi})}{Z}, \text{ for } \lambda > 0,$$

where

$$E_{int}(\boldsymbol{\varphi}) = \sum_{j=1}^{D} \int_{\mathbb{Q}_p} \varphi_j^4(x)dx, \text{ and } \ Z = \int_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} e^{-\lambda E_{int}(\boldsymbol{\varphi})} d\mathbb{P}_D(\boldsymbol{\varphi}).$$

The amplitudes of such theories are defined as

$$\mathcal{A}_R^{(N)}(\boldsymbol{k}, \lambda) = \frac{1}{Z} \int_{B_R^{N-3}} \left\{ \int_{\mathcal{L}_{\mathbb{R}}^{D}(\mathbb{Q}_p)} e^{\sum_{j=2}^{N-2} \boldsymbol{k}_j \cdot \boldsymbol{\varphi}(x_j) - \lambda E_{int}(\boldsymbol{\varphi})} d\mathbb{P}_D(\boldsymbol{\varphi}) \right\} \prod_{\nu=2}^{N-2} dx_\nu.$$

These amplitudes admit expansions of the type given in Proposition 3, where the functions $G_r(\boldsymbol{k}, \boldsymbol{x})$ are replaced by continuous functions in $\boldsymbol{x}$ depending on $\boldsymbol{k}$ and λ. The behavior of these quantum field theories is completely different from the standard ones due to the fact that we are computing the correlation functions for a very particular class of observables, which are products of vertex operators.

Conclusions

In this thesis we provide a mathematical construction of a class of quantum field theories whose amplitudes are expectations of products of vertex operators. The following results were obtained.

1. A measure $\mathbb{P}_D(\boldsymbol{\varphi})$ is constructed in the Lizorkin space $\mathcal{L}^D(\mathbb{Q}_p)$ using the Bochner-Minlos theorem.

2. With the aid of the measure $\mathbb{P}_D(\boldsymbol{\varphi})$ the string amplitudes $\mathcal{A}^{(N)}(\boldsymbol{k})$ are defined by a limit process $\mathcal{A}^{(N)}(\boldsymbol{k}) = \lim\limits_{R\to\infty} \mathcal{A}_R^{(N)}(\boldsymbol{k})$.

3. The functions $\mathcal{A}_R^{(N)}(\boldsymbol{k})$ in the limit are finite for any $\boldsymbol{k}$ and admit a series representation in the form
$$\mathcal{A}_R^{(N)}(\boldsymbol{k}) = A_R^{(N)}(\boldsymbol{k}) + \sum_{r=1}^{\infty} Z_{G_r,R}^{(N)}(\boldsymbol{k}).$$
The functions $Z_{G_r,R}^{(N)}(\boldsymbol{k})$ are multivariate local Zeta functions bounded by the Koba-Nielsen open string amplitude $Z^{(N)}(\boldsymbol{k})$, and $Z_{G_r,R}^{(N)}(\boldsymbol{k})$ are well defined in a common subset of $\mathbb{C}^D$.

4. The sequence $\left\{A_R^{(N)}\right\}_{R\in\mathbb{N}}$ converges to the Koba-Nielsen amplitude when $R \to \infty$.

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