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Centro de Investigación y de Estudios Avanzados del I.P.N.
Unidad Guadalajara

El Problema de la Regulación no Lineal con Enfoque de Lógica Difusa

Tesis que presenta:

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Director de Tesis

Dr. Bernardino Castillo Toledo

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A dissertation submitted by:
Jesús Alberto Meda Campaña

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Para mi Papá †

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El Problema de la Regulación no Lineal con Enfoque de Lógica Difusa

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Resumen

La teoría de regulación clásica para sistemas no lineales está basada en la solución de un conjunto de ecuaciones diferenciales parciales conocidas como ecuaciones FIB (ecuaciones de Francis-Isidori-Byrnes). Estas ecuaciones son muy complicadas en general y en algunos casos resultan imposibles de resolver por lo que es necesario encontrar soluciones aproximadas.

Por otro lado, el modelado difuso tipo Takagi-Sugeno permite representar, al menos localmente, la dinámica de un sistema no lineal a través de sistemas lineales. Además, para los sistemas difusos tipo Takagi-Sugeno, el problema de estabilización puede resolverse empleando técnicas lineales lo que simplifica significativamente el proceso de diseño de controladores no lineales. Por está razón, el presente trabajo está dedicado al estudio de un método alternativo para resolver el problema de seguimiento de trayectorias basado en la teoría de regulación y en el modelado difuso tipo Takagi-Sugeno.

La primera opción que se presenta es la construcción del regulador difuso a partir de controladores lineales estáticos. Sin embargo, luego de analizar el controlador resultante, se observa que el diseño local basado en este tipo de controladores, en general, no es suficiente para resolver el problema de seguimiento exacto.

Este planteamiento, aún cuando sólo garantiza seguimiento aproximado de las señales de referencia, permite considerar la inclusión de reguladores robustos lineales los cuales mejoran claramente el funcionamiento del regulador difuso. Al igual que en caso anterior, se concluye que el seguimiento exacto no puede garantizarse mediante el diseño de controladores locales, ya que se debe de considerar el modelo difuso completo al momento de construir el regulador.

Por otro lado, la relativa simplicidad con que pueden generarse los controladores locales motiva la búsqueda de una metodología que permita reducir o incluso eliminar el error en estado estacionario producido por el regulador difuso. La propuesta llevada a cabo en este estudio consiste en agregar un término de modos deslizantes a la señal de control generada por el controlador difuso. Se demuestra que, bajo ciertas condiciones, esta técnica permite el seguimiento exacto de las referencias.

Es importante mencionar que las condiciones de existencia, de los controladores analizados a lo largo de esta tesis, son tratadas como problemas numéricos, lo que permite realizar el diseño en una forma práctica. Asímismo, en el caso del controlador difuso con modos deslizantes se propone un algoritmo que facilita el cálculo del término discontinuo necesario para preservar la propiedad de estabilidad en el sistema difuso a la vez que la condición de regulación es satisfecha.

The Nonlinear Regulation Problem from a Fuzzy Point of View

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Abstract

The classical regulation theory for nonlinear systems is based on the solution of a set of partial differential equations, named FIB equations (Francis-Isidori-Byrnes equations). These equations are in general hard or even impossible to solve and some approximated solutions have to be found.

On the other hand, the Takagi-Sugeno model formulation has allowed to describe, at least locally, the dynamics of a nonlinear system by means of linear subsystems. For these Takagi-Sugeno fuzzy systems, the problem of stabilization can be solved through linear techniques which greatly simplifies the process of designing nonlinear controllers. For that reason, this work is devoted to the study of an alternative method to solve the problem of tracking trajectories based on the theory of regulation and the Takagi-Sugeno fuzzy modelling.

A first attempt to build the fuzzy regulator is made from combining simple linear regulators. Nevertheless, after analyzing the resulting controller, it is observed that the local design based on static regulators is not sufficient to solve the problem when the exact tracking of the references is desired.

Even though this approach ensures approximated tracking, its study leads to consider the inclusion of linear robust design which clearly improves the behavior of the fuzzy controller. As in the previous case, it is concluded that the analysis of the overall fuzzy model becomes necessary when exact tracking is pursued.

However, the relative simplicity involved in the designing of fuzzy controllers from local techniques encourages to search for a method capable of reducing or even eliminating the tracking error. In that sense, the addition of a sliding mode term to the control signal generated by fuzzy controllers is suggested. It is also shown that this procedure allows the exact tracking of the references under the fulfilment of certain conditions.

It is worth mentioning that the existence conditions of the proposed controllers can be treated like numerical problems allowing the design process to be carried out in a practical way. An algorithm to compute the discontinuous term is developed to be applied when the fuzzy controller with sliding modes is considered. This procedure preserves the stability property of the fuzzy system while the regulation condition is satisfied.

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Chapter 1

Introduction

"Great is the art of beginning,
but greater is the art of ending."
— Lazurus Long

1.1 Preliminaries

Nowadays in the automatic control field, the tracking of desired trajectories is a very important problem that finds applications in several areas of science and technology such as telecommunications, robotics, aeronautics, biology, chemistry, etc.

Specifically, the problem of controlling the output of a system achieving asymptotical tracking of trajectories and/or asymptotical rejection of undesired disturbances has major importance in system theory. In this regard, the Linear Regulation Theory establishes the solution conditions when the problem includes linear, time-invariant and finite dimensional systems. The method takes into account that the exogenous inputs, namely references and disturbances, are defined by the trajectories of an autonomous linear system, the so-called *exosystem* [15, 22].

In the special situations where the reference is constant (i.e. set point control) or the exogenous disturbance is a sinusoidal signal of known frequency (i.e. the exosystem is a simple harmonic oscillator), the asymptotical convergence to zero of the tracking error for every possible external input (i.e. for every desired set point or every amplitude and phase of the disturbing sinusoid) is guaranteed by a controller including an internal model of the exosystem. The controller designed in this way is called robust because, under certain conditions, it achieves the control goals despite parameter uncertainties in a neighborhood of the nominal values.

This contrasts with alternative approaches, such as the method which addresses the tracking of a fixed trajectory, where instead of the assumption that the reference belongs to a class of signals generated by an exogenous system, one needs to assume complete knowledge of the past, present and future behavior of the trajectory to be tracked, as well as perfect knowledge of all system parameters.

In [14], [15] and [16] an exhaustive presentation of the theory of output regulation for multivariable linear, time-invariant and finite dimensional systems is given, while the extension to the analysis of the corresponding design problem for nonlinear systems is treated in [18] and [19].

Basically, the output regulation problem consists of designing either a state feedback or an error feedback controller such that, when the plant is not affected by external signal the equilibrium of the closed-loop system is asymptotically stable and the tracking error converges asymptotically to zero when the plant is under the influence of the *exosystem.* The control scheme for both the state feedback and error feedback regulator are depicted in Figure 1.1 and Figure 1.2, respectively.

In [15], both the linear regulator problem and the linear robust regulator are analyzed and the conditions allowing their solution are derived. The control law for the first case is obtained from the solution of a set of linear matrix equations, henceforth named *Francis equations*, while the robust problem is solved by means of a controller designed using the internal model principle. For this case, the internal model is the immersion of the exosystem into an observable dynamical system which

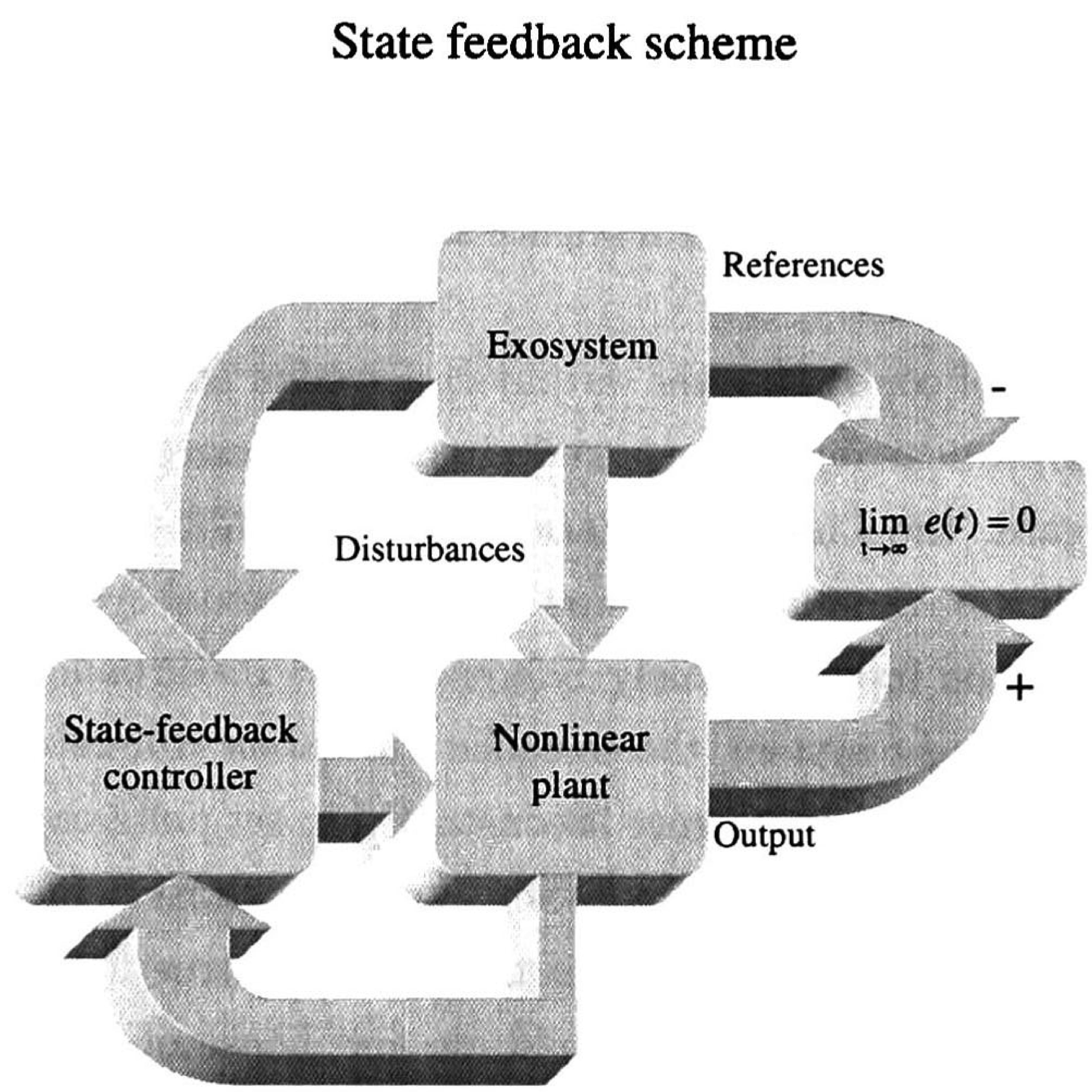

Figure 1.1: State feedback control scheme.

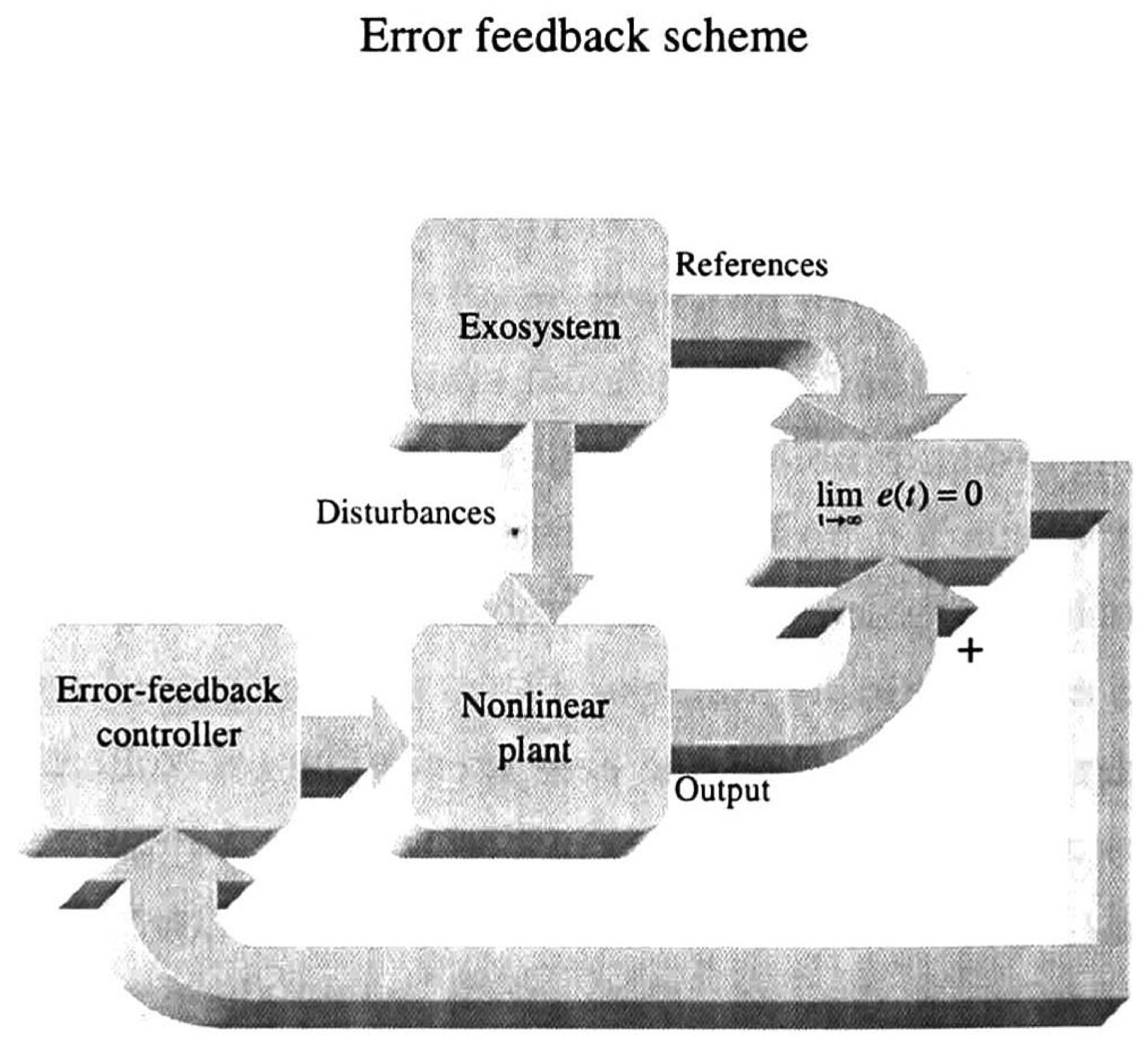

Figure 1.2: Error feedback control scheme.

generates every possible steady-state input for any allowed parameter value.

As mentioned before, these results have been extended to the nonlinear setting by Isidori and Byrnes [18, 19]. They have shown that the solution of the nonlinear output regulator problem depends on the solvability of a set of partial differential equations henceforth known as *Francis-Isidori-Byrnes equations* (FIB equations). Also in this case, the inclusion of an internal model is a necessary condition to ensure robustness under parameter variations.

In spite of all these studies, the nonlinear problem is still an interesting research topic because its typical solution includes some very complex aspects, which will be addressed in the following section.

On the other hand, Takagi-Sugeno (TS) fuzzy models have received great interest during the last years due to the relative simplicity for constructing them. For instance, the TS fuzzy dynamics can be inferred by linearizing a nonlinear plant at different operation points. Afterwards, the resulting linear systems are combined using some information of the behavior of the original system. Basically, linear controllers can be designed for each local subsystem and the final fuzzy controller is obtained by blending these local compensators using the same information of the TS fuzzy model.

Another important feature of the TS fuzzy models is that the overall controller can applied to the original nonlinear plant and the result will depend only on the degree of approximation given by the fuzzy system.

It is important to say that if the equation of the nonlinear system is available, in many cases an exact fuzzy representation of the original dynamics can be obtained [39]. This is sufficient to guarantee the correct performance of the fuzzy controller on the original nonlinear plant.

1.2 Motivation

It is worth noting that although the nonlinear regulator approach proposed by Isidori and Byrnes [18, 19] gives conditions to solve the nonlinear problem in general, in many cases its application is not trivially performed, mainly due to:

1) the solution to the set of nonlinear differential equations may be very difficult or even impossible to obtain,

2) the existence of an immersion of finite dimension for the exosystem can not be guaranteed in general [18].

On the other hand, in recent works some authors have tried to avoid the complexity of the nonlinear regulation problem by applying linear results to the Takagi-Sugeno models [6, 9, 38, 44].

Nevertheless, as discussed in [12] and [24], in general linear regulators do not ensure the zero output tracking error in a Takagi-Sugeno fuzzy model.

Therefore, the major encouragement for this dissertation is to develop an alternative method such that the fuzzy regulator design procedure becomes as simple as possible, while the tracking error is bounded.

1.3 Objectives

The main objective of this research work is the study of the tracking problem when both the plant and the exosystem are represented by Takagi-Sugeno fuzzy models. In that sense, the results presented in this dissertation satisfy both stability and regulation properties for the fuzzy model, and the behavior of the original nonlinear plant under the effects of the fuzzy regulator will depend on the approximation degree provided by the Takagi-Sugeno fuzzy model [43].

To this end, in a preliminary analysis, several partial goals are identified which must be fulfilled to complete the research work in a satisfactory way. In the following list the most important intermediate activities carried out are itemized.

- Integration of Regulation Theory into a Takagi-Sugeno fuzzy modelling scheme in order to find the mathematical conditions, which allow the tracking error to converge asymptotically to zero when the fuzzy regulator is applied to the fuzzy plant.

- Identification of the special cases, if any, which can be solved by fuzzy regulators designed by means of linear local controllers exclusively.

- Obtaining a practical approach to develop both the state feedback fuzzy regulator and the error feedback fuzzy regulator using numerical techniques and to compare their performance.

- To study the inclusion of a sliding mode term into the fuzzy controller and to find the conditions, which allow the regulator to fulfill the control goal.

- Validation of the fuzzy regulator approach by applying the controller to the synchronization of chaotic systems.

In the following chapters, these points are analyzed in detail as they all together give form to this thesis report.

1.4 Thesis structure

The document is organized as follows:

- Chapter 2 presents a brief review on regulation theory, Takagi-Sugeno fuzzy modelling and the Parallel Distributed Compensation approach. Finally, A result establishing that the fuzzy systems can be considered as universal approximators is included in this chapter, also.

- In Chapter 3, the fuzzy regulation problem when the controller is designed on the basis of linear local regulators is analyzed. This chapter includes a section devoted to identify the particular cases where this controller solves the tracking problem in an exact way. Finally, a numerical example showing how the approach can be applied is presented.

- In Chapter 4, an approach based upon linear robust regulators is studied. This technique allows improving the performance of the fuzzy aggregate controller. By the end of the chapter a numerical example is used to illustrate the viability of the method.

- In Chapter 5, an analysis of the fuzzy regulator which includes a discontinuous term is given. It is shown that, under certain conditions, this controller eliminates the steady-state error remaining when the overall controller is obtained from linear regulators. Besides, a numerical algorithm to obtain the sliding mode controller in a practical way is deduced. Finally, an example is used to show the performance of the algorithm.

- Chapter 6 is devoted to the application of the fuzzy controller in order to synchronize chaotic systems. In this chapter both Complete Synchronization and Generalized Synchronization are studied. Some examples are performed to show the validity of the fuzzy regulator.

- In Chapter 7, the conclusions and final comments are drawn, while some research lines which may complement the present work are suggested as well.

- In Appendix A, the fuzzy regulator designed on the basis of linear robust controllers is extended to the fuzzy time-delay systems.

- In Appendix B, a list of the publications generated during the research work is given.

Chapter 2

Mathematical background

> "Parents can only give good
> advice or put them on the right paths,
> but the final forming of a person's
> character lies in their own hands."
>
> — Anne Frank

2.1 Introduction

This chapter is intended to provide an overview of the most important tools used along the research work; namely, the regulation theory, the Takagi-Sugeno fuzzy modelling and the Parallel Distributed Compensation approach.

As mentioned before, the classical regulation theory for nonlinear systems is based on the solution of a set of partial differential equations better known as the FIB equations.

In Section 2.2, the main results on nonlinear regulation are briefly reviewed, the interested reader is referred to [18], [19] and [35] for a more detailed description of this framework.

Afterwards, Section 2.3 summarizes the Takagi-Sugeno modelling, which allows describing, at least locally, the dynamics of a nonlinear system by means of linear subsystems.

The stabilization problem for this kind of systems can be solved by means of a controller computed through linear techniques [38, 39]. This design method, known as the Parallel Distributed Compensation approach, is presented in Section 2.4.

Finally, in Section 2.5 a result giving conditions for the fuzzy system to be considered as an universal approximator is presented.

2.2 Nonlinear regulation theory

The nonlinear regulation problem, for a system defined by the following set of equations

$$\dot{x}(t) = f(x(t), w(t), u(t)) \tag{2.1}$$

$$\dot{w}(t) = s(w(t)) \tag{2.2}$$

$$e(t) = h(x(t), w(t)) \tag{2.3}$$

where $x(t) \in U \subset \mathbb{R}^n$ is the state vector of the plant, $w(t) \in W \subset \mathbb{R}^s$ is the state vector of the exosystem and $u(t) \in \mathbb{R}^m$ is the input signal; consists of finding a controller capable of driving the output of the plant towards the reference signal given by the external system.

Equation (2.3) describes the tracking error $e(t) \in \mathbb{R}^m$, which is usually defined as the difference between the reference signals $y_r(t)$ and the measurable outputs of the plant $y_i(t)$ with $i = 1, .., m$. It is also assumed that $f(\cdot, \cdot, \cdot)$, $s(\cdot)$ and $h(\cdot, \cdot)$ are analytical functions satisfying $s(0) = 0$, $f(0, 0, 0) = 0$ and $h(0, 0) = 0$.

Formally, the regulation problem, when full state information of both the plant and the exosystem is available, can be defined as the problem of designing a state feedback controller of the form

$$u(t) = \alpha(x(t), w(t)) \tag{2.4}$$

such that:

S$_{\text{fi}}$) the equilibrium $x(t) = 0$ of the closed-loop system

$$\dot{x}(t) = f(x(t), 0, \alpha(x(t), 0))$$

is asymptotically stable (stability condition),

R$_{\text{fi}}$) there exists a neighborhood $V \subset U \times W$ around $(0, 0)$ such that, for every initial condition $(x(0), w(0))$, the solution of the closed-loop system

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t), \alpha(x(t), w(t))), \\ \dot{w}(t) &= s(w(t)), \\ e(t) &= h(x(t), w(t)), \end{aligned}$$

satisfies

$$\lim_{t \to \infty} e(t) = 0,$$

(regulation condition).

Considering

$$\begin{aligned} A &= \tfrac{\partial f(x,w,u)}{\partial x} \Big|_{(0,0,0)}, & B &= \tfrac{\partial f(x,w,u)}{\partial u} \Big|_{(0,0,0)}, & P &= \tfrac{\partial f(x,w,u)}{\partial w} \Big|_{(0,0,0)}, \\ C &= \tfrac{\partial h(x,w)}{\partial x} \Big|_{(0,0)}, & S &= \tfrac{\partial s(w)}{\partial w} \Big|_{(0,0)}, & Q &= \tfrac{\partial h(x,w)}{\partial w} \Big|_{(0,0)}, \end{aligned}$$

the main result in nonlinear regulation with *full information* is summarized as follows [18]:

Theorem 1 *If*

R1$_{\text{fi}}$) *every trajectory $w(t)$ defined by $\dot{w}(t) = s(w(t))$ and $w(0)$ is bounded,*

R2$_{\text{fi}}$) *there exists a matrix K such that $A + BK$ is Hurwitz,*

R3$_{\text{fi}}$) *there exist steady-state mappings $x_{ss}(t) = \pi(w(t))$ and $u_{ss}(t) = c(w(t))$ with $\pi(0) = 0$ and $c(0) = 0$, defined in a neighborhood $W^\circ \subset W$ around the origin, satisfying*

$$\frac{\partial \pi(w(t))}{\partial w(t)} s(w(t)) = f(\pi(w(t)), w(t), \alpha(\pi(w(t)), w(t))), \tag{2.5}$$

$$0 = h(\pi(w(t)), w(t)), \tag{2.6}$$

for all $w(t) \in W^\circ$,

then, the regulation problem can be solved by a state feedback controller of the form $u(t) = \alpha(x(t), w(t)) = K\left[x(t) - \pi(w(t))\right] + c(w(t))$.

Proof. See [18]. ■

Nonetheless, in real situations full information about the states of the plant is not often available and the design of an error feedback controller becomes necessary. According to Isidori [19], the setting for this case includes a dynamical controller of the form

$$\dot{\xi}(t) = \eta(\xi(t), e(t)), \tag{2.7}$$

$$u(t) = \theta(\xi(t)), \tag{2.8}$$

with $\xi(t) \in \Xi \subset R^{\nu}$ and $\nu = n + s$, which solves the tracking problem if

$\mathbf{S_e}$) the equilibrium $(x(t), \xi(t)) = (0, 0)$ of the closed-loop system

$$\begin{aligned} \dot{x}(t) &= f(x(t), 0, \theta(\xi(t))), \\ \dot{\xi}(t) &= \eta(\xi(t), h(x(t), 0)), \end{aligned}$$

is asymptotically stable (stability condition),

$\mathbf{R_e}$) there exists a neighborhood $V \subset U \times \Xi \times W$ around $(0, 0, 0)$ such that, for every initial condition $(x(0), \xi(0), w(0))$, the solution of the closed-loop system

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t), \theta(\xi(t))), \\ \dot{w}(t) &= s(w(t)), \\ \dot{\xi}(t) &= \eta(\xi(t), e(t)), \\ e(t) &= h(x(t), w(t)), \end{aligned}$$

satisfies

$$\lim_{t \to \infty} e(t) = 0,$$

(regulation condition).

Thus, considering

$$F = \frac{\partial \eta(\xi, e)}{\partial \xi}\Big|_{(0,0)}, \quad G = \frac{\partial \eta(\xi, e)}{\partial e}\Big|_{(0,0)}, \text{ and } \quad H = \frac{\partial \theta(\xi)}{\partial \xi}\Big|_{(0)},$$

the result giving the existence conditions of the error feedback controller can be written as follows [18]:

Theorem 2 *If*

R1$_e$) *every trajectory $w(t)$ defined by $\dot{w}(t) = s(w(t))$ and $w(0)$ is bounded,*

R2$_e$) *there exists a matrix G such that $\begin{pmatrix} A & BH \\ GC & F \end{pmatrix}$ is Hurwitz,*

R3$_e$) *there exist mappings $x(t) = \pi(w(t))$ and $\xi(t) = \sigma(w(t))$ with $\pi(0) = 0$ and $\xi(0) = 0$, defined in a neighborhood $W^\circ \subset W$ around the origin, satisfying*

$$\begin{aligned} \frac{\partial \pi(w(t))}{\partial w(t)} s(w(t)) &= f(\pi(w(t)), w(t), \theta(\sigma(w(t)))), \\ \frac{\partial \sigma(w(t))}{\partial w(t)} s(w(t)) &= \eta(\sigma(w(t)), 0) \\ 0 &= h(\pi(w(t)), w(t)), \end{aligned}$$

for all $w(t) \in W^\circ$;

then, the regulation problem can be solved by an error feedback controller.

Proof. See [18]. ∎

Another interesting situation appears when the mathematical model of the plant includes parameters which are assumed fixed but unknown. This uncertainty may be caused by temperature variations, aged components, dust, etc.

On the other hand, in this work the exosystem is considered as an auxiliary system which generates the reference and perturbation signals. For that reason, it is assumed to be free of uncertainties.

The regulation problem under these considerations, i.e., ensuring the tracking of the reference signals in the presence of parameter uncertainties is named *the robust regulation problem.* The equations describing this kind of systems must explicitly include an unknown parameter vector $\mu \in \mathbb{R}^p$, such that the nominal parametric values are represented when $\mu = 0$, i.e.

$$\dot{x}(t) = f(x(t), w(t), u(t), \mu), \tag{2.9}$$
$$\dot{w}(t) = s(w(t)), \tag{2.10}$$
$$e(t) = h(x(t), w(t), \mu). \tag{2.11}$$

The design of the robust regulator requires the existence of a neighborhood $\mathcal{P}$ around $\mu = 0$, such that it is possible to solve the regulation problem for every value inside this vicinity (well-defined problem). Similarly to the error feedback controller, a dynamical regulator must be designed, taking care of ensuring:

S$_\mathbf{r}$) the equilibrium $(x(t), \xi(t)) = (0, 0)$ of the closed-loop system

$$\begin{aligned} \dot{x}(t) &= f(x(t), 0, \theta(\xi(t)), \mu) \\ \dot{\xi}(t) &= \eta(\xi(t), h(x(t), 0, \mu)) \end{aligned}$$

is asymptotically stable (stability condition),

R$_\mathbf{r}$) there exists a neighborhood $V \subset U \times \Xi \times W$ around $(0, 0, 0)$ such that, for every initial condition $(x(0), \xi(0), w(0))$ and for every μ, the solution of the closed-loop system

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t), \gamma(\xi(t), \mu), \\ \dot{w}(t) &= s(w(t)), \\ \dot{\xi}(t) &= \eta(\xi(t), h(x(t), w(t), \mu)), \\ e(t) &= h(x(t), w(t), \mu), \end{aligned}$$

satisfies

$$\lim_{t \to \infty} e(t) = 0,$$

when the plant is under the effects of the exosystem (regulation condition).

Defining

$$A(\mu) = \frac{\partial f(x,w,u,\mu)}{\partial x}\Big|_{(0,0,0,\mu)}, \quad B(\mu) = \frac{\partial f(x,w,u,\mu)}{\partial u}\Big|_{(0,0,0,\mu)}, \quad C(\mu) = \frac{\partial h(x,w,\mu)}{\partial x}\Big|_{(0,0,\mu)},$$

and considering the extended external system

$$\dot{w}_a(t) = s_a(w_a(t)) = \begin{bmatrix} s(w(t)) \\ 0 \end{bmatrix}$$

which generates the signals

$$w_a(t) = \begin{bmatrix} w(t) \\ \mu \end{bmatrix},$$

it is possible to deduce the result establishing the existence conditions for the nonlinear robust regulator [18]:

Theorem 3 *The nonlinear robust regulation problem is solvable if and only if there exist mappings $x_{ss}(t) = \pi_a(w(t), \mu)$ and $u_{ss}(t) = c_a(w(t), \mu)$, with $\pi_a(0, \mu) = 0$ and $c_a(0, \mu) = 0$, both defined in a* *neighborhood* $W^\circ \times \mathcal{P} \subset W \times \mathbb{R}^p$ *around the origin, satisfying*

$$\begin{aligned} \frac{\partial \pi_a(w(t),\mu)}{\partial w(t)} s(w(t)) &= f(\pi_a(w(t),\mu), w(t), c_a(w(t),\mu),\mu) \\ 0 &= h(\pi_a(w(t),\mu), w(t),\mu), \end{aligned}$$

for all $(w(t),\mu) \in W^\circ \times \mathcal{P}$, *and such that the autonomous system with output* $\{W^\circ \times \mathcal{P}, s_a, c_a\}$ *is immersed into a system of the form*

$$\begin{aligned} \dot{\xi}(t) &= \varphi(\xi(t)) \\ u &= \gamma(\xi(t)), \end{aligned}$$

defined in a neighborhood Ξ° *around the origin* $\mathbb{R}^\nu$, *where* $\varphi(0) = 0$, $\gamma(0) = 0$ *and the matrices*

$$\Phi = \frac{\partial \varphi(\xi)}{\partial \xi}\Big|_{(0)}, \quad \Gamma = \frac{\partial \gamma(\xi)}{\partial \xi}\Big|_{(0)},$$

guarantee the stability of

$$\begin{pmatrix} A(0) & 0 \\ NC(0) & \Phi \end{pmatrix}, \begin{pmatrix} B(0) \\ 0 \end{pmatrix}$$

for some matrix N, *and the detectability of the pair*

$$\begin{pmatrix} C(0) & 0 \end{pmatrix}, \begin{pmatrix} A(0) & B(0)\Gamma \\ 0 & \Phi \end{pmatrix}$$

Proof. See [18]. ■

2.3 Takagi-Sugeno fuzzy modelling

The Takagi-Sugeno (TS) fuzzy models, first introduced in [37] by Takagi and Sugeno and later by Sugeno and Kang in [36], are used to represent nonlinear dynamics by means of a set of IF-THEN rules. The most important and useful feature of this kind of mathematical models is that the consequent part of the rules are local linear systems computed from the original nonlinear plant, for example, by linearizing at different operation points. This is also the main difference between TS fuzzy models and pure fuzzy systems [43].

The aggregate TS fuzzy model is obtained when the local linear systems are "combined"; such that, a nonlinear interpolation is achieved among them, resulting in an approximation of the original nonlinear dynamics. Besides, it is worth mentioning that nonlinear systems can be arbitrarily approximated using TS fuzzy models (see Section 2.5); however, as it can be easily inferred, the degree of approximation depends on the complexity of the fuzzy model. On the other hand; in many cases, it is possible to obtain an exact representation for the original dynamics instead of a simple approximation [39].

The structure of a TS fuzzy model is given below, where $z(t) = [z_1(t), \dots, z_p(t)]$ are premise variables that can be functions of the state, perturbations and/or time; M_j^i is a fuzzy set and r is the number of rules in the fuzzy model; $x \in \mathbb{R}^n$ is the state vector; $u \in \mathbb{R}^m$ is the input vector; $y \in R^q$ is the output vector; with $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{q\times n}$

Continuous-time case

Rule i

IF $z_1(t)$ is M_1^i and ... and $z_n(t)$ is M_p^i

$$THEN \begin{cases} \dot{x}(t) = & A_i x(t) + B_i u(t) \\ y(t) = & C_i x(t) \end{cases} \quad i = 1, \dots, r. \tag{2.12}$$

The overall TS fuzzy system, for a pair $(x(t), u(t))$, is inferred using a *singleton fuzzier*[1], a *product inference engine* and a *center average defuzzier* [39, 43]. It will be assumed in the rest of this work that the premise variables do not depend on the input variable $u(t)$ in order to reduce the complexity during the defuzzification process of the controllers. Thus, the aggregate TS fuzzy model is:

Continuous-time

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} \varpi_i(z(t))\{A_i x(t) + B_i u(t)\}}{\sum_{i=1}^{r} \varpi_i(z(t))} \tag{2.13}$$

$$= \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) + B_i u(t)\} \tag{2.14}$$

$$y(t) = \frac{\sum_{i=1}^{r} \varpi_i(z(t)) C_i x(t)}{\sum_{i=1}^{r} \varpi_i(z(t))} \tag{2.15}$$

$$= \sum_{i=1}^{r} h_i(z(t)) C_i x(t), \tag{2.16}$$

[1]A singleton fuzzier maps the crisp point $x \in U$ into a fuzzy set M with support x_i where $\varpi(x_i) = 1$ for $x_i = x$ and $\varpi(x_i) = 0$ for $x_i \neq x$ [27].

where

$$\varpi_i(z(t)) = \prod_{j=1}^{p} M_j^i(z_j(t)), \tag{2.17}$$

$$h_i(z(t)) = \frac{\varpi_i(z(t))}{\sum_{i=1}^{r} \varpi_i(z(t))}, \tag{2.18}$$

for all t, and the term $M_j^i(z_j(t))$ is the membership value for $z_j(t)$ in M_j^i. Also, since

$$\begin{aligned} &\sum_{i=1}^{r} \varpi_i(z(t)) > 0, \\ &\varpi_i(z(t)) \geq 0 \qquad i = 1, \dots, r, \end{aligned} \tag{2.19}$$

one has

$$\begin{aligned} &\sum_{i=1}^{r} h_i(z(t)) = 1, \\ &h_i(z(t)) \geq 0 \qquad i = 1, \dots, r, \end{aligned} \tag{2.20}$$

for all t.

2.4 Parallel Distributed Compensation (PDC)

The PDC offers a procedure to design a fuzzy controller from a given Takagi-Sugeno Fuzzy model. In the PDC design, each control rule corresponds to a rule of the TS fuzzy model, i.e., the fuzzy controller shares the same fuzzy sets with the TS fuzzy model in the premise parts. Thus, for the fuzzy model (2.12) one has:

Rule i

IF $z_1(t)$ is M_1^i and $\cdots$ and $z_n(t)$ is M_p^i

$$THEN \quad u(t) = \quad K_i x(t) \quad i = 1, \dots, r. \tag{2.21}$$

The fuzzy control rules have a linear controller in the consequent parts. Therefore, the overall fuzzy controller is represented by

$$u(t) = \sum_{i=1}^{r} h_i(z(t)) k_i x(t). \tag{2.22}$$

In this case, state feedback laws are used to stabilize the local subsystems; however, dynamic controllers can be also considered. The control design consists in determining the local feedback gains K_i which results in a simple way to handle nonlinear systems.

It is important to notice that although the fuzzy controller (2.22) is constructed on the basis of local design, the feedback gains K_i should be determined using global design conditions. These conditions, which are needed to guarantee the global stability of the overall fuzzy system are established in the following theorem [39].

Theorem 4 *The equilibrium of the closed-loop (2.14)-(2.22) is globally asymptotically stable if there exist a common positive definite matrix* $\mathbf{P}$ *satisfying*

$$\begin{aligned} \tilde{A}_{ii}^T\mathbf{P} + \mathbf{P}\tilde{A}_{ii} &< 0 \text{ for all } i = 1,\ldots,r, \\ \left(\tilde{A}_{ij} + \tilde{A}_{ji}\right)^T \mathbf{P} + \mathbf{P}\left(\tilde{A}_{ij} + \tilde{A}_{ji}\right) &\leq 0 \text{ for all } i,j = 1,\ldots,r, \text{ such that } h_i(z(t)) \cdot h_j(z(t)) \neq 0, \end{aligned}$$

where $\tilde{A}_{ij} = A_i + B_i K_j$.

Proof. See [39]. ■

2.5 The fuzzy system as an universal approximator

As early mentioned, this work considers that the plant and the exosystem are described by means of Tagaki-Sugeno fuzzy models. For that reason the following questions raise naturally: What will happen when the fuzzy regulator is applied on the original nonlinear plant? The behavior observed on the original plant will satisfy the desired control goals?

Unfortunately, the approach proposed along this thesis cannot assure a proper performance when the fuzzy controller is applied over the nonlinear plant. This behavior will depend on the approximation degree provided by the Takagi-Sugeno fuzzy model.

However, nonlinear systems can be arbitrarily approximated by means of fuzzy models as stated in the following theorem:

Theorem 5 *Suppose that the universe of discourse* U *is a compact set in* $\mathbb{R}^n$. *Then, for any given real continuous function* $g(x)$ *on* U *and arbitrary* $\epsilon > 0$, *there exists a fuzzy system* $f(x)$ *in the form*

$$f(x) = \frac{\sum_{i=1}^{r} x_i \left(\prod_{j=1}^{p} a_j^i e^{-\left(\frac{z_j - \bar{z}_j^i}{\sigma_j^i}\right)^2} \right)}{\sum_{i=1}^{r} \left(\prod_{j=1}^{p} a_j^i e^{-\left(\frac{z_j - \bar{z}_j^i}{\sigma_j^i}\right)^2} \right)}, \tag{2.23}$$

with $a_j^i \in (0, 1]$, $\sigma_j^i \in (0, \infty)$ *and* $x_i, \bar{z}_j^i$ *are real-valued parameters, such that*

$$\sup_{x \in U} |f(x) - g(x)| < \epsilon.$$

That is, the fuzzy system with singleton fuzzier, product inference engine, center average defuzzier and Gaussian membership functions is an universal approximator.

Proof. See [43] ■

Clearly, the fuzzy system (2.23) can be rewritten as

$$\begin{aligned} f(x) &= \frac{\sum_{i=1}^{r} \varpi_i(z) x_i}{\sum_{i=1}^{r} \varpi_i(z)}, \\ &= \sum_{i=1}^{r} h_i(z) x_i, \end{aligned}$$

where

$$\begin{aligned} \varpi_i(z) &= \prod_{j=1}^{p} a_j^i e^{-\left(\frac{z_j - \bar{z}_j^i}{\sigma_j^i}\right)^2}, \\ h_i(z) &= \frac{\varpi_i(z)}{\sum_{i=1}^{r} \varpi_i(z)} \end{aligned}$$

Remark 6 *Under certain conditions, it is possible to obtain not only an approximation, but an exact representation of the nonlinear system through a fuzzy model [39].*

Chapter 3

Fuzzy regulation through linear controllers

" The greatest obstacle
to discovery is not ignorance
– it is the illusion of knowledge."
— Daniel J. Boorstin

3.1 Introduction

Consider a system with m (vector u) inputs and l (vector y) outputs which has an internal description of n (vector x) states. A natural way to represent many physical systems is by nonlinear state-space models of the form:

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x,u), \end{aligned} \tag{3.1}$$

where f and h are nonlinear functions.

Linear state-space models may then be derived from linearizing such models resulting:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3.2}$$

where A, B, C and D are real matrices. Thus, if (3.2) is obtained by linearizing (3.2) then $A = \dfrac{\partial f}{\partial x}$, $B = \dfrac{\partial f}{\partial u}$, $C = \dfrac{\partial h}{\partial x}$ and $D = \dfrac{\partial h}{\partial u}$.

It is important to have into account that system (3.2) does not describe the entire dynamics of (3.1) but only the behavior around the linearization point [13, 21, 25].

However, the design and analysis of nonlinear control systems requires more sophisticated mathematical tools than those typically used for linear systems.

For that reason, the advantages provided by the Takagi-Sugeno fuzzy modelling in the approximation of nonlinear systems, allowing the extension of the linear results to a larger region, become interesting.

It has to be noticed that the results developed in the following sections consider that the plant and the exosystem are described by means of Takagi-Sugeno fuzzy models. Therefore, stability and regulation conditions obtained along this work can be only guaranteed on this kind of systems.

In other words, the behavior of the original nonlinear plant influenced by the fuzzy controller will depend on the approximation degree of the Takagi-Sugeno fuzzy model (see Section 2.5).

This chapter is devoted to the analysis of the fuzzy regulation problem when static controllers are designed on the local linear subsystems.

In Section 3.2, the approach to solve the fuzzy regulation problem by means of linear controllers is addressed. It is shown that this procedure does not guarantee the exact tracking of the reference signals in general.

Carrying this analysis further, conditions allowing the output of the plant to tend asymptotically towards the desired reference are established in Section 3.3.

Finally; in Section 3.4, a simple example which illustrates the mathematical approach presented in this chapter is developed.

3.2 Fuzzy regulation using simple linear controllers

Considering a r-rule TS fuzzy model of the form:

Rule i

IF $z_1(t)$ is M_1^i and ... and $z_n(t)$ is M_p^i

$$THEN \begin{cases} \dot{x}(t) = & A_i x(t) + B_i u(t) + P_i w(t) \\ \dot{w}(t) = & S_i w(t) \\ e(t) = & C_i x(t) - Q_i w(t) \qquad\qquad i = 1, \dots, r, \end{cases} \tag{3.3}$$

then, according to Chapter 2 the resulting composite system is defined by

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) + B_i u(t) + P_i w(t)\} \tag{3.4}$$

$$\dot{w}(t) = \sum_{i=1}^{r} h_i(z(t)) S_i w(t) \tag{3.5}$$

$$e(t) = \sum_{i=1}^{r} h_i(z(t))\{C_i x(t) - Q_i w(t)\}. \tag{3.6}$$

Thus, in order to solve the tracking problem for system (3.4)–(3.6) the following fuzzy regulation equations need to be satisfied:

$$\frac{\partial \pi(w(t))}{\partial w(t)} \sum_{i=1}^{r} h_i(z(t)) S_i w(t) = \sum_{i=1}^{r} h_i(z(t))\{A_i \pi(w(t)) + B_i \gamma(w(t)) + P_i w(t)\}, \tag{3.7}$$

$$0 = \sum_{i=1}^{r} h_i(z(t))\{C_i \pi(w(t)) - Q_i w(t)\}. \tag{3.8}$$

As a result, if there exists a fuzzy stabilizer of the form $u(t) = \sum_{i=1}^{r} h_i(z(t))K_i x(t)$ [39], and if it is possible to solve equations (3.7)–(3.8), then the fuzzy regulation problem has a solution and the tracking error converges asymptotically to zero. However, as it can be easily observed, the solution for this set of equations may be difficult to be obtained.

A first attempt to develop a method to solve the fuzzy regulation problem was based on the design of local linear controllers [3, 44].

In this approach, it is assumed that the mappings $\pi(w(t))$ and $\gamma(w(t))$ can be approximated by

$$\tilde{\pi}(w(t)) = \sum_{i=1}^{r} h_i(z(t))\Pi_i w(t) \tag{3.9}$$

$$\tilde{\gamma}(w(t)) = \sum_{i=1}^{r} h_i(z(t))\Gamma_i w(t), \tag{3.10}$$

respectively, with Π_i and Γ_i resulting from the solution of the r linear local regulation problems [22]

$$\Pi_i S_i = A_i\Pi_i + B_i\Gamma_i + P_i, \tag{3.11}$$

$$0 = C_i\Pi_i - Q_i, \tag{3.12}$$

for $i = 1, \ldots, r$. As consequence, the following controller is obtained

$$u(t) = \sum_{i=1}^{r} h_i(z(t))K_i \left[x(t) - \sum_{i=1}^{r} h_i(z(t))\Pi_i w(t)\right] + \sum_{i=1}^{r} h_i(z(t))\Gamma_i w(t). \tag{3.13}$$

To determine whether or not these approximations fulfill the regulation condition is necessary to substitute (3.9)–(3.10) in equations (3.7)–(3.8). Simple algebraic manipulations yield:

$$\left[\begin{array}{c} \sum_{i=1}^{r} \dot{h}_i(z(t))\Pi_i + \\ \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))\Pi_i S_j \end{array}\right] = \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))A_i\Pi_j + \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))B_i\Gamma_j + \sum_{i=1}^{r} h_i(z(t))P_i, \tag{3.14}$$

$$0 = \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))C_i\Pi_j - \sum_{i=1}^{r} h_i(z(t))Q_i. \tag{3.15}$$

At this point, it can be readily observed that a condition to solve exactly the fuzzy regulation problem by means of linear controllers is:

$$\sum_{i=1}^{r} \dot{h}_i(z(t))\Pi_i = 0, \tag{3.16}$$

because this term is not eliminated by the local design in general. On the other hand, a deeper view to equations (3.14)–(3.15) allows to infer that although if condition (3.16) is satisfied the solution of the fuzzy regulation problem cannot be achieved by solving the local linear regulation problems, in general. This is due to the crossed terms, which result from the expansion of equations (3.14)–(3.15). Hence, the techniques proposed by Tanaka and Wang [39, 43] to stabilize TS fuzzy models may not be sufficient to fulfill the control goal when the exact tracking is pursued [12, 24].

In the next section, this study is extended and some special cases, which are solvable by designing a fuzzy controller using local regulators are identified.

3.3 Particular cases

Supposing that the steady-state manifold of the fuzzy regulation problem coincides with the local steady-state manifold of each linear subsystem, i.e., $x_{ss}(t) = \pi(w(t)) = \Pi_1 = \ldots = \Pi_r = \Pi$, then condition (3.16) is trivially guaranteed because $\sum_{i=1}^{r} h_i(z(t)) = 1$ and $\sum_{i=1}^{r} \dot{h}_i(z(t)) = 0$, i.e.,

$$\sum_{i=1}^{r} \dot{h}_i(z(t))\Pi_i = \sum_{i=1}^{r} \dot{h}_i(z(t))\Pi = 0.$$

Therefore, equations (3.14)–(3.15) can be rewritten as:

$$\begin{aligned} \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))\Pi S_j &= \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))A_i\Pi \\ &+ \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))B_i\Gamma_j + \sum_{i=1}^{r} h_i(z(t))P_i, \qquad (3.17) \\ 0 &= \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))C_i\Pi - \sum_{i=1}^{r} h_i(z(t))Q_i. \qquad (3.18) \end{aligned}$$

The set of linear matrix equations defined by Equation (3.17) and Equation (3.18) includes the r linear matrix equations (3.11) and (3.12). This means that the fuzzy regulator must have the ability of ensuring the tracking of the reference for every single linear subsystem of the TS fuzzy model, which is an important feature to be considered in the design of fuzzy controllers.

The remaining linear matrix equations obtained from Equation (3.17) and Equation (3.18) are

$$\Pi S_i + \Pi S_j = A_i\Pi + B_i\Gamma_j + P_i + A_j\Pi + B_j\Gamma_i + P_j, \tag{3.19}$$
$$0 = C_i\Pi - Q_i + C_j\Pi - Q_j, \tag{3.20}$$

for $i = 1, \ldots, r$ and $j = 1, \ldots, r$, such that $h_i(z(t)) \cdot h_j(z(t)) \neq 0$. As it can be seen, these equations include the crossed terms of equations (3.17)–(3.18). In other words, the exact tracking in the interpolation regions of the fuzzy models is carried out only when equations (3.19)–(3.20) have solution. Thus, equations (3.11)–(3.12) and (3.19)–(3.20) have to be solved simultaneously. Nevertheless, if there exists a solution for equations (3.11)–(3.12) then Π_i and Γ_i are unique [22].

But, even though the exact tracking of references can not be guaranteed in general when a fuzzy controller based on linear regulators is applied to system (3.4)–(3.6), it is possible to identify two particular cases where the design of local controllers proves to be sufficient for solving the fuzzy regulation problem.

Case 1.- $B_1 = \ldots = B_r = B$. Under these circumstances, equations (3.19)–(3.20) are transformed into

$$\Pi S_i + \Pi S_j = A_i\Pi + B\Gamma_j + P_i + A_j\Pi + B\Gamma_i + P_j, \tag{3.21}$$
$$0 = C_i\Pi - Q_i + C_j\Pi - Q_j, \tag{3.22}$$

which do not include any crossed term.

Case 2.- $\Gamma_1 = \ldots = \Gamma_r = \Gamma$. When this situation is reached, equations (3.19)–(3.20) can be rewritten as

$$\Pi S_i + \Pi S_j = A_i\Pi + B_i\Gamma + P_i + A_j\Pi + B_j\Gamma + P_j, \tag{3.23}$$
$$0 = C_i\Pi - Q_i + C_j\Pi - Q_j. \tag{3.24}$$

As in the previous case, the crossed terms are removed from the design process.

Remark 6 *In both cases, the solution of equations (3.17)–(3.18) implies the solution of equations (3.19)–(3.20). That is, the local design ensures the exact tracking of the references for system (3.4)–(3.6).*

Remark 7 *If the conditions of Case 2 are satisfied, then the solution of the fuzzy regulation problem described by equations (3.4)–(3.6) is completely linear and is given by* Π *and* Γ.

The following theorem summarizes the existence conditions of the fuzzy controller designed on linear regulators [9, 11, 39].

Theorem 8 *Suppose the following assumptions hold:*

H1$_{\text{lr}}$) *every trajectory $w(t)$ defined by (3.5) and $w(0)$ is bounded,*

H2$_{\text{lr}}$) *taking $\tilde{A}_{ij} = A_i + B_iK_j$, there exist matrices K_j and $\mathbf{P}$ satisfying*

$$0 > \tilde{A}_{ii}^T\mathbf{P} + \mathbf{P}\tilde{A}_{ii}$$

$$0 > (\tilde{A}_{ij} + \tilde{A}_{ji})^T\mathbf{P} + \mathbf{P}(\tilde{A}_{ij} + \tilde{A}_{ji})$$

$$\mathbf{P} > 0$$

for all $i = 1, \ldots, r$ and $j = 1, \ldots, r$ such that $h_i(z(t)) \cdot h_j(z(t) \neq 0$,

H3$_{\text{lr}}$) *there exist mappings $\tilde{\pi}(w(t)) = \sum_{i=1}^{r} h_i(z(t))\Pi_i w$ and $\tilde{\gamma}(w(t)) = \sum_{i=1}^{r} h_i(z(t))\Gamma_i w$ whose matrices Π_i and Γ_i are computed from*

$$\begin{aligned} \Pi_i S_i &= A_i\Pi_i + B_i\Gamma_i + P_i \\ 0 &= C_i\Pi_i - Q_i \end{aligned}$$

for $i = 1, \ldots, r$,

H4$_{\text{lr}}$) *$\Pi_1 = \ldots = \Pi_r = \Pi$ and conditions for Case 1 or Case 2 are fulfilled,*

then, the tracking error for the Fuzzy Regulation Problem by means of Local Linear converges asymptotically to zero.

Proof. The satisfaction of H2$_{\text{lr}}$ implies the existence of a stabilizer of the form

$$u(t) = \sum_{i=1}^{r} h_i(z(t))K_i x(t). \tag{3.25}$$

On the other hand, when H1$_{\text{lr}}$, and H3$_{\text{lr}}$ are satisfied the solution of the local regulator problems is ensured. And from the previous analysis it follows that the design based on local controllers solve the fuzzy regulation problem if H4$_{\text{lr}}$ is achieved. ■

If condition H2$_{\text{lr}}$ is transformed into a numerical problem, then the stabilizer can be computed in a practical way by taking advantage of LMI design [5, 39] and Theorem 8 can be rewritten as:

Theorem 9 *Suppose the following assumptions hold:*

H1$_{\mathbf{lr}}$) *every trajectory $w(t)$ defined by (3.5) and $w(0)$ is bounded,*

H2$^*_{\mathbf{lr}}$) *the LMIs*

$$XA_i^T + A_iX + B_iY_i + Y_i^TB_i^T < 0$$

for all $i = 1, \ldots, r$,

$$X(A_i + A_j)^T + (A_i + A_j)X + B_iY_j + Y_j^TB_i^T + B_jY_i + Y_i^TB_j^T < 0$$

are feasible, for all $i = 1, \ldots, r$ and $j = 1, \ldots, r$ such that $h_i(z(t)) \cdot h_j(z(t) \neq 0$, and

$$X > 0,$$

H3$_{\mathbf{lr}}$) *there exist mappings $\tilde{\pi}(w(t)) = \sum_{i=1}^{r} h_i(z(t))\Pi_i w$ and $\tilde{\gamma}(w(t)) = \sum_{i=1}^{r} h_i(z(t))\Gamma_i w$ obtained from*

$$\begin{aligned} \Pi_i S_i &= A_i\Pi_i + B_i\Gamma_i + P_i \\ 0 &= C_i\Pi_i - Q_i \end{aligned}$$

for $i = 1, \ldots, r$,

H4$_{\mathbf{lr}}$) *$\Pi_1 = \ldots = \Pi_r = \Pi$ and conditions for Case 1 or Case 2 are fulfilled,*

then, Fuzzy Regulation Problem is solved by means of linear local design. Moreover, $\mathbf{P} = X^{-1}$ and $K_i = Y_i\mathbf{P}$

Proof. Follows directly from Theorem 8. ■

Now that conditions for then exact solution of the fuzzy regulation problem are given, it will be analyzed the problem when assumption H4$_{\text{lr}}$ is not satisfied.

To this end, it is assumed the existence of the exact mappings $\pi(w(t))$ and $\gamma(w(t))$ solving the fuzzy regulation problem, i.e.,

$$\begin{aligned} \frac{\partial\pi(w(t))}{\partial w(t)} \sum_{i=1}^{r} h_i(z(t))S_i w(t) &= \sum_{i=1}^{r} h_i(z(t))\{A_i\pi(w(t)) + B_i\gamma(w(t)) + P_i w(t)\}, \\ 0 &= \sum_{i=1}^{r} h_i(z(t))\{C_i\pi(w(t)) - Q_i w(t)\}, \end{aligned}$$

where $\pi(w(t))$ is the center manifold rendered invariant by $\gamma(w(t))$.

Besides, if assumptions $H2_{lr}$ (or $H2^*_{lr}$) and $H3_{lr}$ hold, then the fuzzy stabilizer (3.25) and the local regulators can be computed while the bounded behavior of the external signals is also guaranteed by Condition $H1_{lr}$.

On the other hand, it has been explained that if (3.9) and (3.10) are considered to solve the fuzzy regulation problem then equations (3.14)–(3.15) are obtained. From these equations follows immediately

$$\begin{aligned}\mathcal{N}(\hat{\pi}(w(t))) &= \Bigg\{-\sum_{i=1}^{r}\dot{h}_i(z(t))\Pi_i - \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))\Pi_i S_j + \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))A_i\Pi_j \\ &+ \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))B_i\Gamma_j + \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))P_i\Bigg\} w(t) = \mathcal{O}(\|w(t)\|^p), \quad (3.26)\end{aligned}$$

where $p > 1$ because the premise variable $z(t)$ ultimately depends on $w(t)$.

Therefore, if $\sum_{i=1}^{r}\dot{h}_i(z(t))$ is bounded then

$$\begin{aligned}\|\mathcal{N}(\hat{\pi}(w(t)))\| &= \Bigg\|\Bigg\{-\sum_{i=1}^{r}\dot{h}_i(z(t))\Pi_i - \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))\Pi_i S_j + \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))A_i\Pi_j \\ &+ \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))B_i\Gamma_j + \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))P_i\Bigg\}\Bigg\| \|w(t)\| \le \beta,\end{aligned}$$

since $0 < h_i(z(t)) \le 1$ and Condition $H1_{lr}$.

In other words, the manifold $\hat{\pi}(w(t))$ defined by the fuzzy mappings (3.9) and (3.10) satisfies the conditions of the *Approximation of the Center Manifold Theorem* [8]:

Theorem 10 *If a continuously differentiable function $\tilde{\pi}(w))$ with $\pi(0) = 0$ and $\frac{\partial\tilde{\pi}}{\partial w}(0) = 0$ can be found such that $\mathcal{N}(\tilde{\pi}(w)) = \mathcal{O}(\|w\|p)$ for some $p > 1$, then*

$$\pi(w) - \tilde{\pi}(w) = \mathcal{O}(\|w\|^{p+1}).$$

Proof. See [8] ∎

Remark 11 *$\tilde{\pi}(w(t))$ and $\hat{\pi}(w(t))$ are not the same in general, because $\hat{\pi}(w(t))$ represents the real steady-state manifold for the fuzzy regulation problem and it depends on approximations (3.9)–(3.10).*

Remark 12 *If (3.9)–(3.10) do not solve the fuzzy regulation problem then at least a bounded error is ensured.*

These results are stated in the following corollary:

Corollary 13 *If*

H1$^{c}_{lr}$) *every trajectory $w(t)$ defined by (3.5) and $w(0)$ is bounded,*

H2$^{c}_{lr}$) *the LMIs*

$$XA_i^T + A_iX + B_iY_i + Y_i^TB_i^T < 0$$

for all $i = 1, \dots, r$,

$$X(A_i + A_j)^T + (A_i + A_j)X + B_iY_j + Y_j^TB_i^T + B_jY_i + Y_i^TB_j^T < 0$$

are feasible, for all $i = 1, \dots, r$ and $j = 1, \dots, r$ such that $h_i(z(t)) \cdot h_j(z(t) \neq 0$, and

$$X > 0,$$

H3$^{c}_{lr}$) *there exist mappings $\tilde{\pi}(w(t)) = \sum_{i=1}^{r} h_i(z(t))\Pi_i w$ and $\tilde{\gamma}(w(t)) = \sum_{i=1}^{r} h_i(z(t))\Gamma_i w$ obtained from*

$$\begin{aligned} \Pi_i S_i &= A_i\Pi_i + B_i\Gamma_i + P_i \\ 0 &= C_i\Pi_i - Q_i \end{aligned}$$

for $i = 1, \dots, r$,

H4$^{c}_{lr}$) *there exist $\pi(w(t))$ and $\gamma(w(t))$ solving exactly*

$$\begin{aligned} \frac{\partial\pi(w(t))}{\partial w(t)} \sum_{i=1}^{r} h_i(z(t))S_iw(t) &= \sum_{i=1}^{r} h_i(z(t))\{A_i\pi(w(t)) + B_i\gamma(w(t)) + P_iw(t)\}, \\ 0 &= \sum_{i=1}^{r} h_i(z(t))\{C_i\pi(w(t)) - Q_iw(t)\}, \end{aligned}$$

H5$^{c}_{lr}$) *the term $\sum_{i=1}^{r} \dot{h}_i(z(t))$ is bounded,*

then, the tracking error for the Fuzzy Regulation Problem solved through simple linear local controllers is bounded.

Proof. Follows directly from the previous analysis. ■

Remark 14 *It is important to notice that the tracking error is bounded even if assumption $H4_{lr}$ is not satisfied. For that reason, the fuzzy regulator built from simple linear controllers is an excellent choice when the exact tracking of the references is not necessary.*

3.4 Numerical example

In this section, a stiff link system is used to illustrate the performance of the fuzzy regulator analyzed previously. The mathematical model describing the nonlinear dynamics of the plant is [44]:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), &(3.27)\\ \dot{x}_2(t) &= a\sin(x_1(t)) + bu(t), &(3.28)\end{aligned}$$

with $a = -\frac{Mgl}{Ml^2+I}$, $b = \frac{1}{Ml^2+I}$, $g = 9.81m/s^2$ $M = 20kg$, $l = 0.5m$ and $I = 0.8kg \cdot m^2$; while the reference signal y_r is the state $w_1(t)$ of the exosystem

$$\begin{aligned}\dot{w}_1(t) &= w_2(t)\\ \dot{w}_2(t) &= -w_1(t).\end{aligned}$$

First of all, it is necessary to find a fuzzy model, such that an approximation of the nonlinear behavior of the equations is obtained. For this case, the following two-rule TS fuzzy model accomplishes such a task:

Rule 1

IF $x_1(t)$ is about 0

$$THEN \begin{cases} \dot{x}(t) = & A_1x(t) + Bu(t) \\ \dot{w}(t) = & Sw(t) \\ e_1(t) = & C_1x(t) - Q_1w(t) \end{cases}$$

Rule 2

IF $x_1(t)$ is about $\frac{\pi}{2}$

$$THEN \begin{cases} \dot{x}(t) = & A_2x(t) + Bu(t) \\ \dot{w}(t) = & Sw(t) \\ e_2(t) = & C_2x(t) - Q_2w(t), \end{cases}$$

with matrices:

$$A_1 = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ \frac{2}{\pi}a & 0 \end{pmatrix}, B_1 = B_2 = \begin{pmatrix} 0 \\ b \end{pmatrix}, C_1 = C_2 = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$S_1 = S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Q_1 = Q_2 = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

while the membership functions representing the nonlinearity of the original system are:

$$h_1(x_1(t)) = \left[1 - \frac{1}{1+e^{-7(x_1(t)-\pi/4)}}\right]\left[\frac{1}{1+e^{-7(x_1(t)+\pi/4)}}\right]; h_2(x_1(t)) = 1 - h_1(x_1(t)).$$

Because of the simplicity of the fuzzy model, it is not difficult to obtain the exact mappings x_{ss} and u_{ss} from equations (3.7)–(3.8):

$$\begin{aligned} x_{1ss} &= w_1, \\ x_{2ss} &= w_2, \\ u_{ss} &= -\left(\frac{1 + ah_1(w_1) + \frac{2}{\pi}ah_2(w_1)}{b}\right) w_1. \end{aligned} \tag{3.29}$$

Now, if the local design approach is used to find a solution for the fuzzy problem then equations (3.11)–(3.12) must to be solved, arising:

$$\begin{aligned} \Pi_1 = \Pi_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \Gamma_1 &= \begin{pmatrix} \frac{-1-a}{b} & 0 \end{pmatrix}, \\ \Gamma_2 &= \begin{pmatrix} \frac{-1-\frac{2a}{\pi}}{b} & 0 \end{pmatrix}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} x_{1ss} &= w_1 \\ x_{2ss} &= w_2 \\ u_{ss} &= \left(h_1(w_1)\frac{-1-a}{b} + h_2(w_1)\frac{-1-\frac{2a}{\pi}}{b}\right) w_1 = -\left(\frac{1 + ah_1(w_1) + \frac{2a}{\pi}h_2(w_1)}{b}\right) w_1. \end{aligned} \tag{3.30}$$

It is clearly seen that Π_1 equals Π_2 and B_1 equals B_2, consequently this example satisfies Case 1. This means that the exact steady-state input (3.29) and the steady-state input computed from linear controllers (3.30) are identical.

The fuzzy stabilizer for this example is computed from:

$$\begin{aligned} K_1 &= \begin{pmatrix} 91.027 & -4.124 \end{pmatrix} \text{ and} \\ K_2 &= \begin{pmatrix} 55.379 & -4.124 \end{pmatrix} \end{aligned}$$

which were obtained using the MATLAB LMI Toolbox.

In Figure 3.1, Figure 3.2 and Figure 3.3 appear the simulation outcomes showing how the fuzzy regulator designed from local controllers solves the fuzzy regulation problem under these particular conditions.

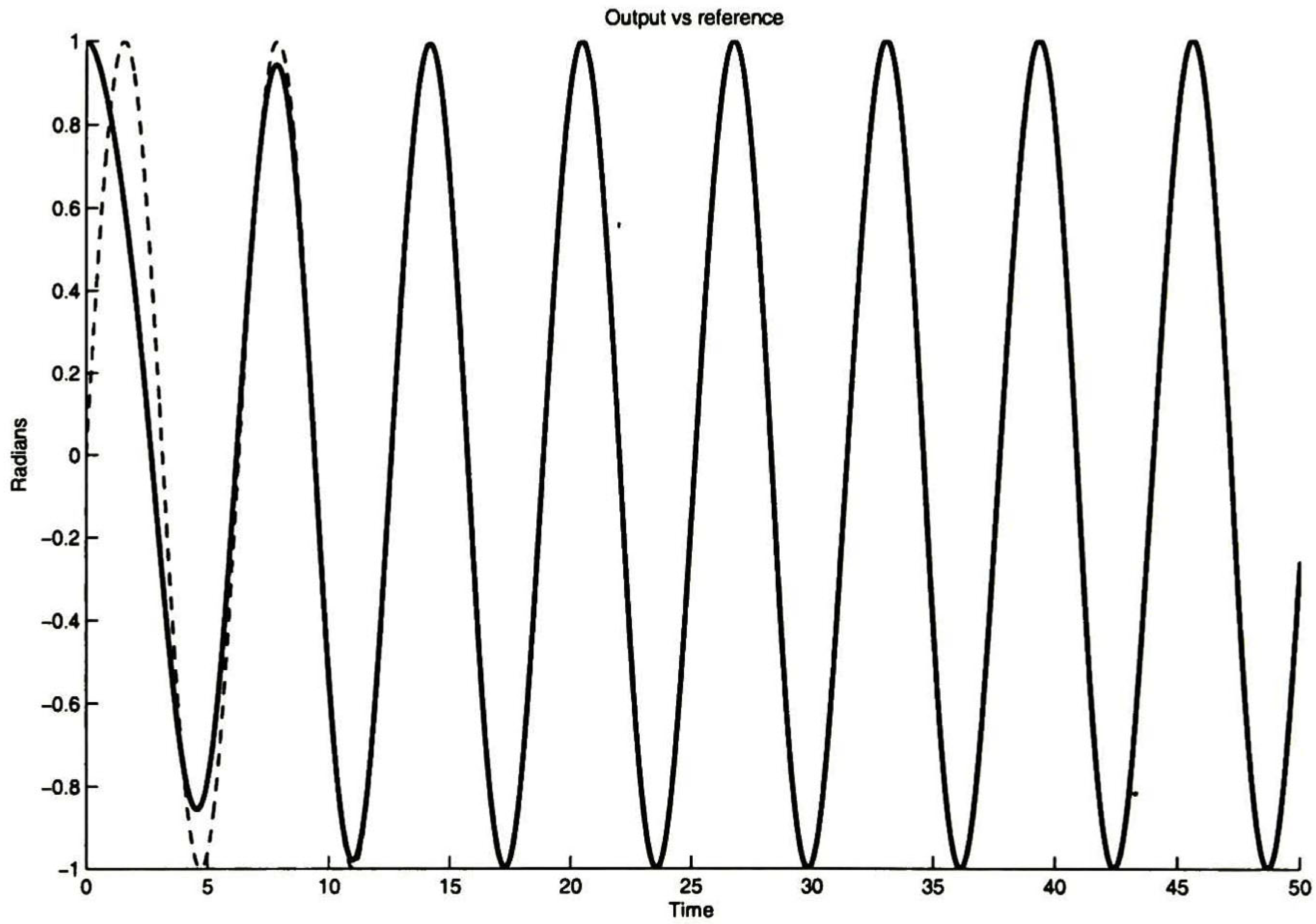

Figure 3.1: Output vs reference when $B_1 = B_2$ and the fuzzy regulator is designed from local linear controllers.

A little change in the scenario will be introduced to demonstrate the lack of effectiveness of the local design under much more general situations. To this end, it will be considered a similar TS fuzzy model involving small differences between matrix B_1 and matrix B_2, namely:

Rule 1

IF $x_1(t)$ is about 0.

$$THEN \begin{cases} \dot{x}(t) = & A_1 x(t) + B_1 u(t) \\ \dot{w}(t) = & Sw(t) \\ e(t) = & Cx(t) - Qw(t) \end{cases}$$

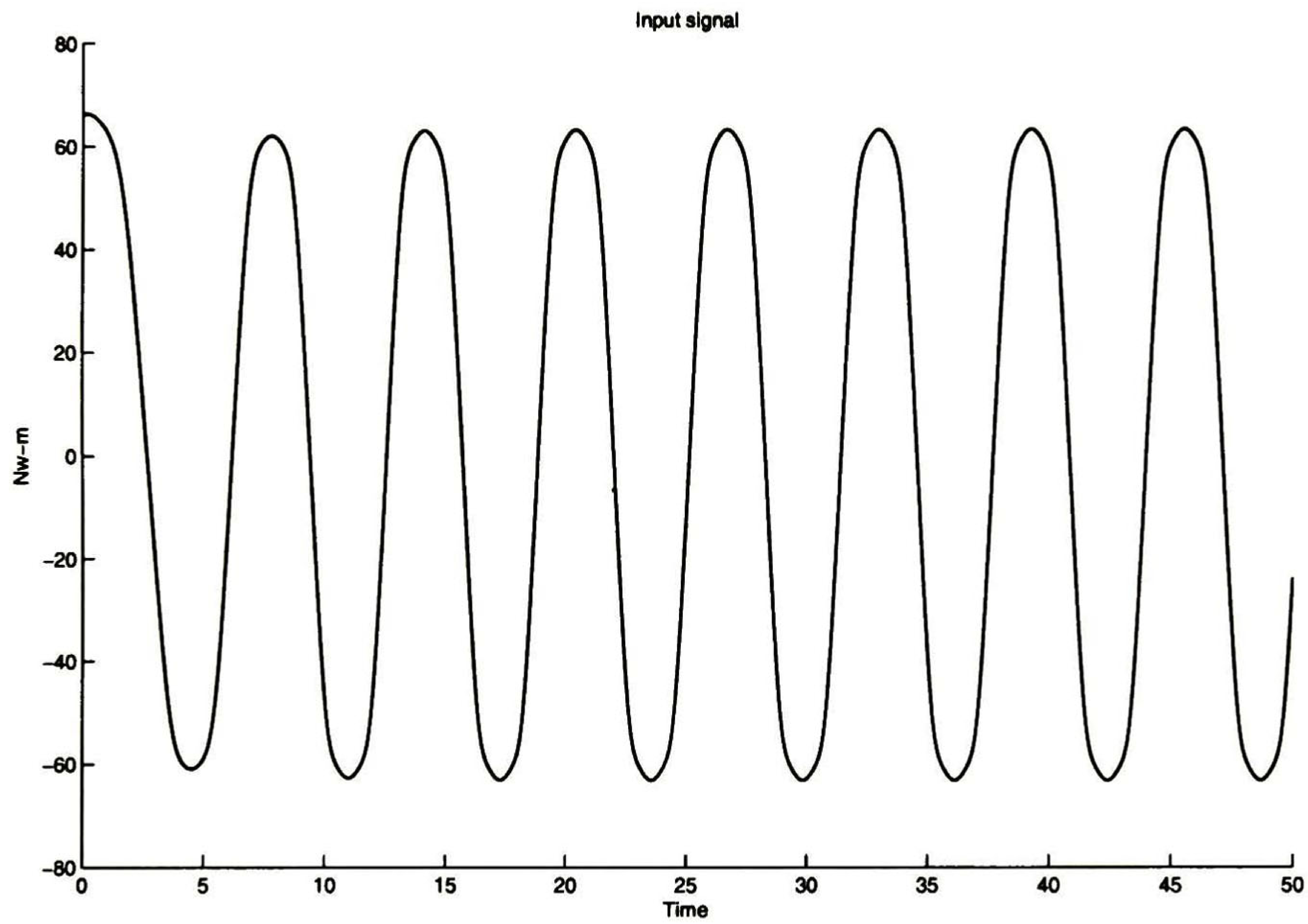

Figure 3.2: Input signal when $B_1 = B_2$ and the fuzzy regulator is designed from local linear controllers.

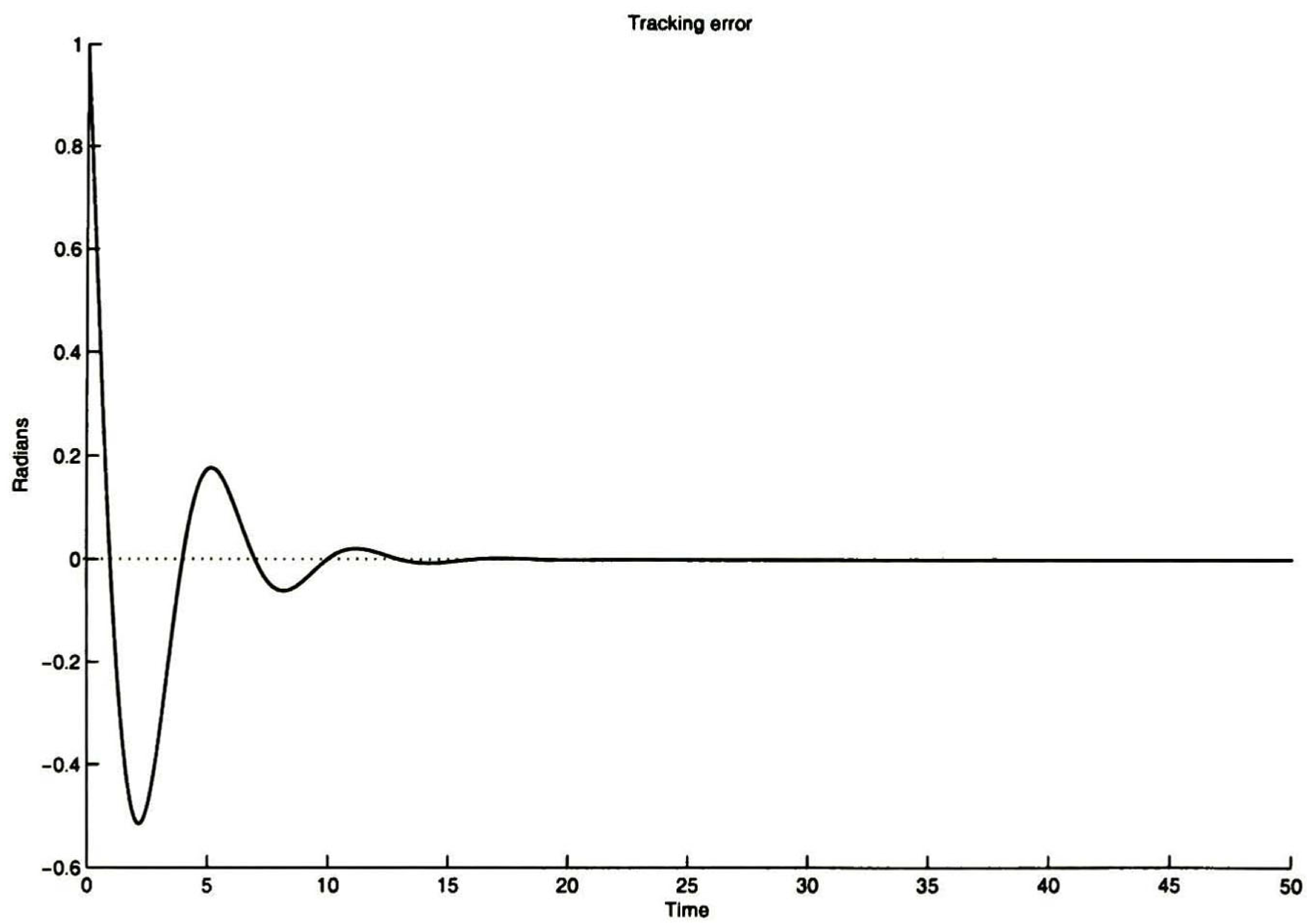

Figure 3.3: Tracking error when $B_1 = B_2$ and the fuzzy regulator is designed from local linear controllers.

Rule 2

IF $x_1(t)$ is about $\frac{\pi}{2}$

$$THEN \begin{cases} \dot{x}(t) = & A_2 x(t) + B_2 u(t) \\ \dot{w}(t) = & S w(t) \\ e(t) = & C x(t) - Q w(t), \end{cases}$$

and where the matrices defining the linear systems are:

$$A_1 = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ \frac{2}{\pi}a & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 \\ b \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ \alpha b \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$S_1 = S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

where $\alpha \in \mathbb{R}$ and with the same membership functions mentioned earlier.

For this case the exact mappings solving the regulation problem are

$$\begin{aligned} x_{1ss} &= w_1, \\ x_{2ss} &= w_2, \\ u_{ss} &= -\left(\frac{1 + a h_1(w_1) + \frac{2}{\pi} a h_2(w_1)}{b h_1(w_1) + \alpha b h_2(w_1)}\right) w_1. \end{aligned} \tag{3.31}$$

But, when the local design is considered to solve the fuzzy regulation problem, it results:

$$\begin{aligned} \Pi_1 = \Pi_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \Gamma_1 &= \begin{pmatrix} \frac{-1-a}{b} & 0 \end{pmatrix}, \\ \Gamma_2 &= \begin{pmatrix} \frac{-1-\frac{2a}{\pi}}{\alpha b} & 0 \end{pmatrix}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} x_{1ss} &= w_1 \\ x_{2ss} &= w_2 \\ u_{ss} &= \left(h_1(w_1)\frac{-1-a}{b} + h_2(w_1)\frac{-1-\frac{2a}{\pi}}{\alpha b}\right) w_1. \end{aligned} \tag{3.32}$$

Once more Π_1 equals Π_2 but B_1 is different from B_2 and Γ_1 is different from Γ_2 when $\alpha \neq 1$. For that reason, neither conditions of Case 1 nor conditions of Case 2 would be satisfied in this example. And of course, in accordance with the analysis carried out in this chapter, the steady-state input computed from linear controllers (3.32) would not coincide with the exact steady-state input (3.31) either. As consequence, the tracking error will not converge to zero.

The overall regulator for this case is completed by the fuzzy stabilizer computed from:

$$\begin{aligned} K_1 &= \begin{pmatrix} -25.795 & -11.585 \end{pmatrix} \text{ and} \\ K_2 &= \begin{pmatrix} -30.551 & -6.2132 \end{pmatrix}. \end{aligned}$$

Figure 3.4, Figure 3.5 and Figure 3.6 are the simulation results of taking $\alpha = 2$. These graphics confirm the inefficiency of the local design method when the problem does not fit neither in Case 1 nor in Case 2.

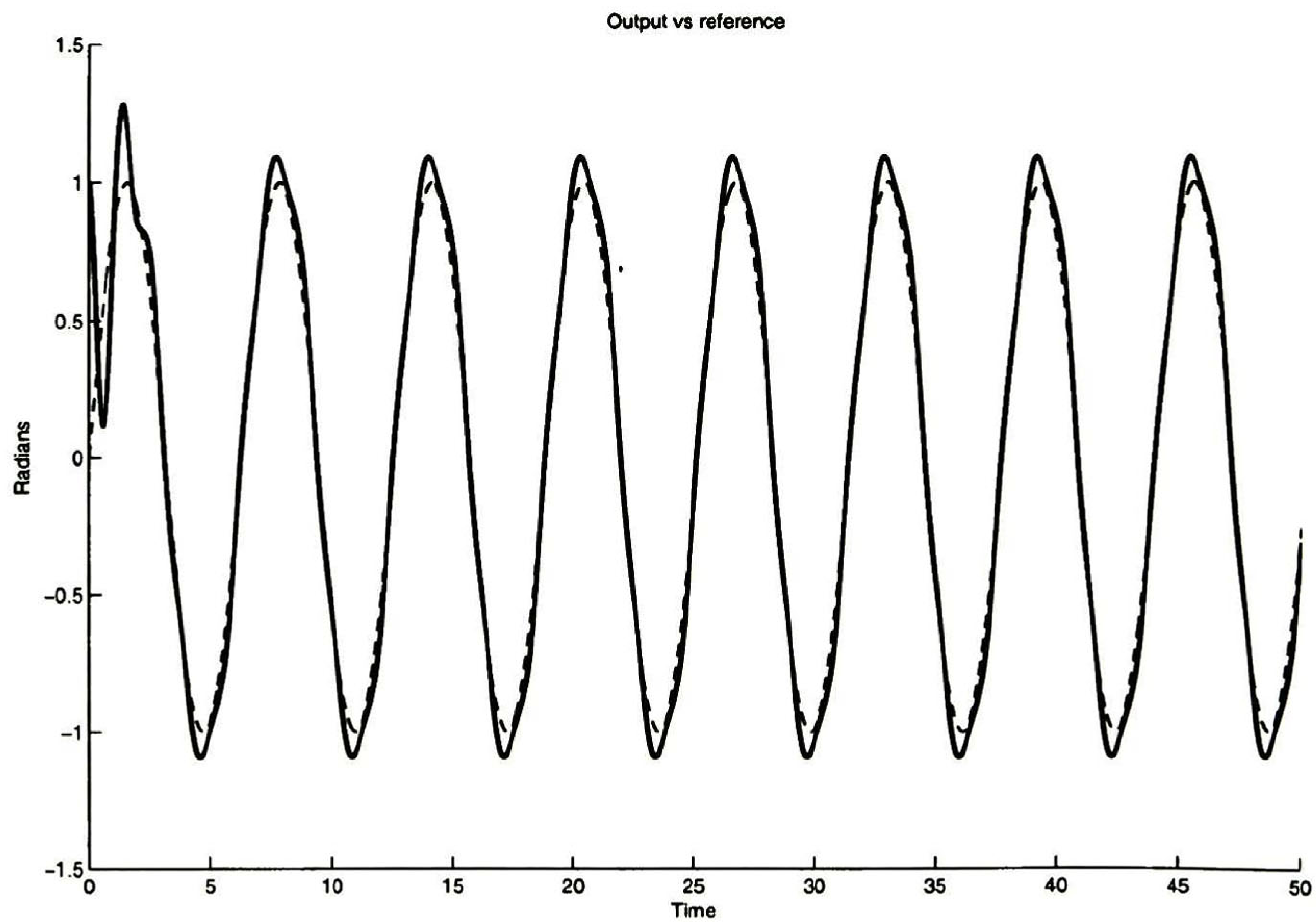

Figure 3.4: Output vs reference when $B_1 \neq B_2$ with $\alpha = 2$ and the fuzzy regulator is designed from local linear controllers.

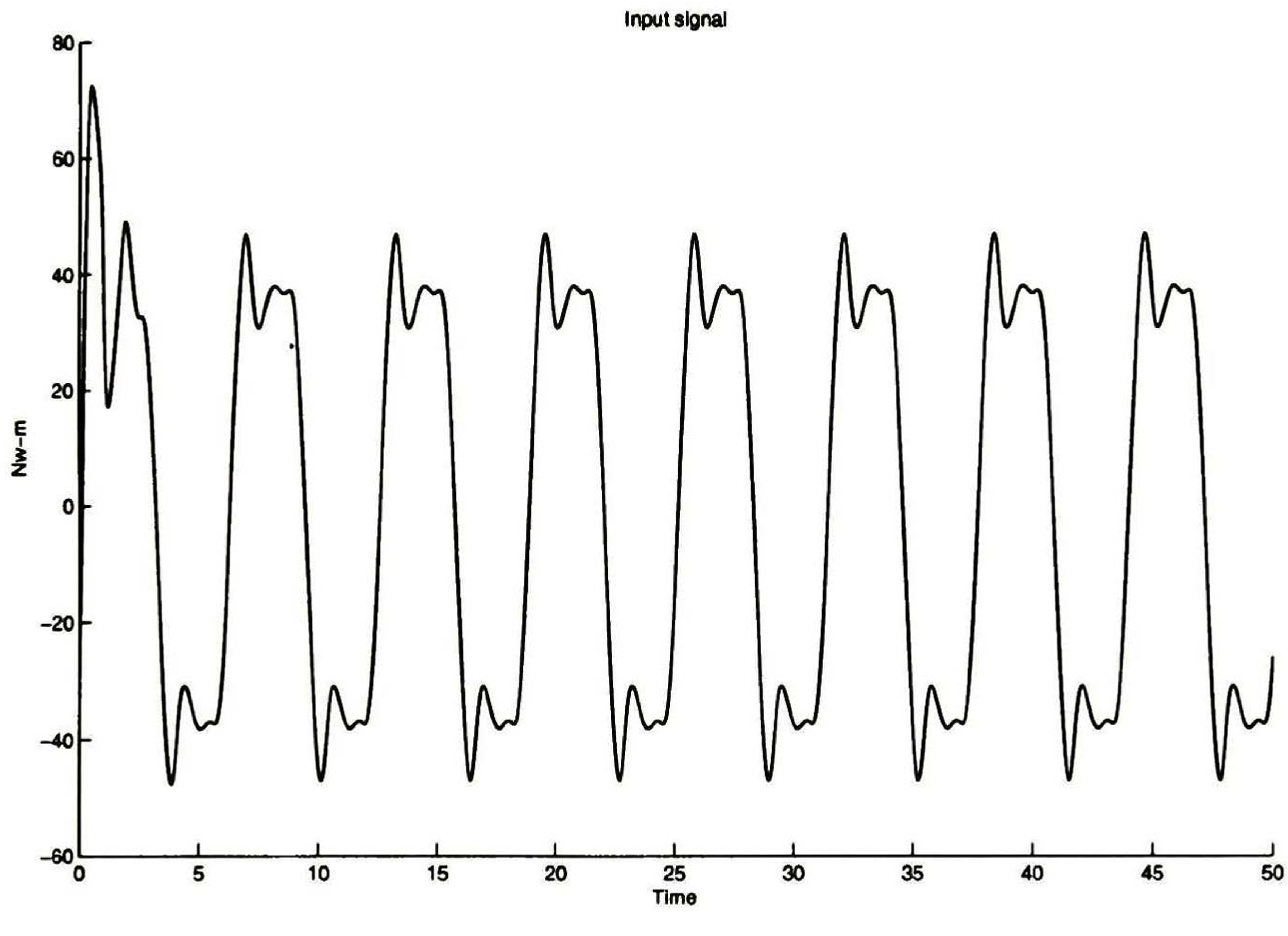

Figure 3.5: Input signal when $B_1 \neq B_2$ with $\alpha = 2$ and the fuzzy regulator is designed from local linear controllers.

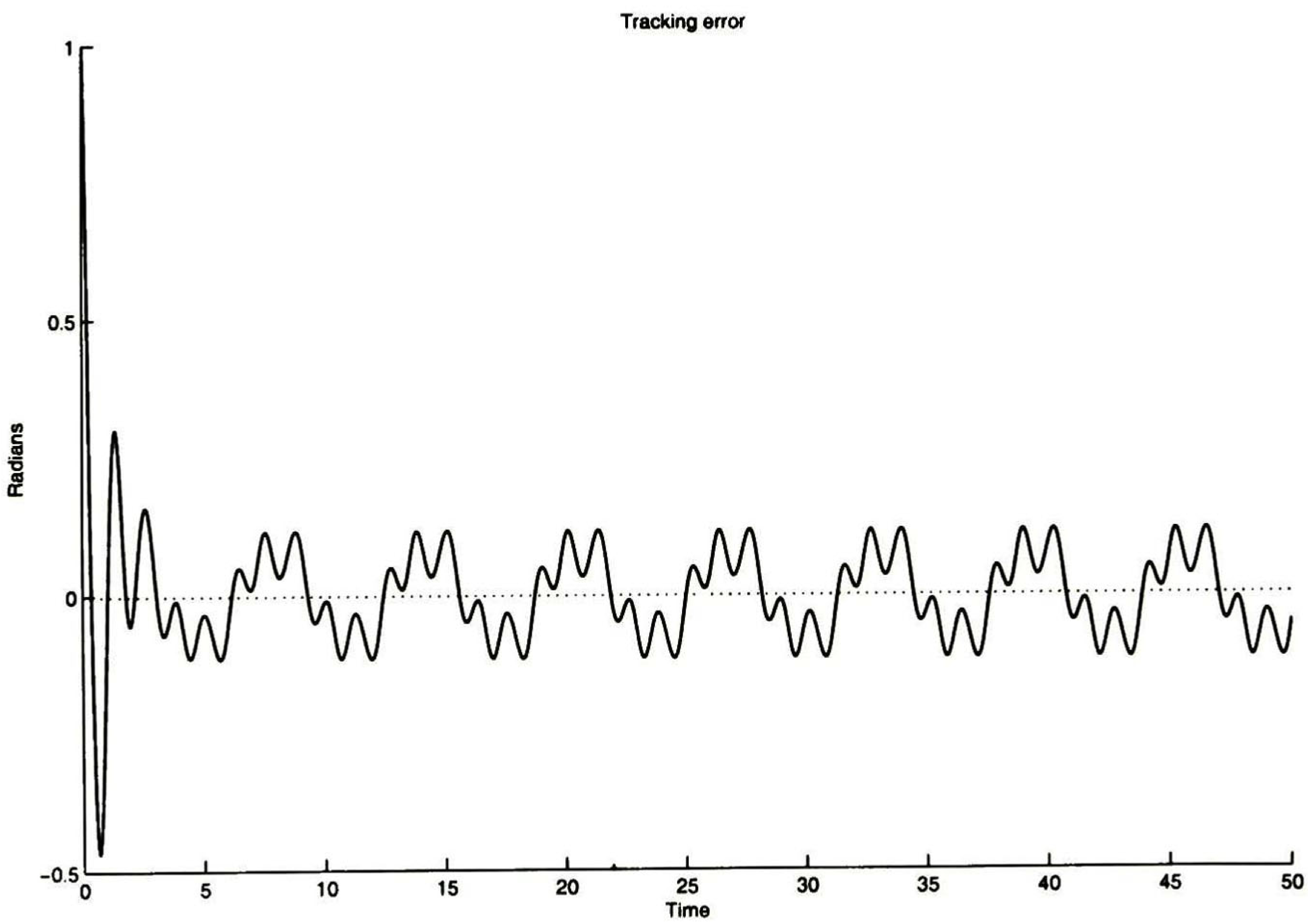

Figure 3.6: Tracking error when $B_1 \neq B_2$ with $\alpha = 2$ and the fuzzy regulator is designed from local linear controllers.

Chapter 4

Fuzzy regulation based on linear robust design

' It does not matter
how slowly you go so
long as you do not stop."
— Confucius

4.1 Introduction

Some authors had shown that the stabilization for TS fuzzy systems can be achieved by means of linear controllers designed on the local subsystems included in the fuzzy model [38, 39, 42]. Unfortunately, from the preceding chapter, it is observed that these results can not be easily extended to the fuzzy regulation problem because its exact solution implies the explicit inclusion of the membership functions in the analysis [12].

Nevertheless, the simplicity involved in the design of linear controllers motivates the inclusion of dynamic controllers instead of static ones during the local design. Before proceeding with this study, the immersion concept is briefly introduced [18].

Consider a pair of smooth autonomous systems

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned}$$

and

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}) \\ \bar{y} &= \bar{h}(\bar{x}) \end{aligned}$$

defined on two different space states, X and $\bar{X}$ respectively, but having the same output space $Y = \mathbb{R}^m$ Assume $f(0) = 0$, $h(0) = 0$, $\bar{f}(0) = 0$ and $\bar{h}(0) = 0$.

Then, system $\{X, f, h\}$ is immersed into system $\{\bar{X}, \bar{f}, \bar{h}\}$ if there exists a C^k mapping $\tau : X \to \bar{X}$, satisfying $\tau(0) = 0$ and

$$h(x) \neq h(z) \Rightarrow \bar{h}(\tau(x)) \neq \bar{h}(\tau(z))$$

such that

$$\begin{aligned} \frac{\partial \tau}{\partial x} f(x) &= \bar{f}(\tau(x)) \\ h(x) &= \bar{h}(\tau(x)) \end{aligned}$$

for all $x \in X$.

This means that any output response generated by $\{X, f, h\}$ is also an output response of $\{\bar{X}, \bar{f}, \bar{h}\}$. The relevance of immersed systems is because, sometimes, $\{\bar{X}, \bar{f}, \bar{h}\}$ may have special properties that $\{X, f, h\}$ does not have.

For instance, any linear systems can always be immersed into an observable linear system. This fact will be used in the present chapter in order to design linear dynamic controllers capable of observing the steady-state input which solves the local regulation problem.

As a result, in Section 4.2 linear robust regulators are considered to replace the static linear controllers proposed in Chapter 3.

The linear robust approach provides the ability of compensating certain parameter uncertainties [18]. As it will be explained later in this chapter, the performance of the fuzzy regulator is increased by taking into account this kind of local design.

An application to time-delay systems is considered in Appendix A to illustrate how this "minor" change in the local design is transformed in a significative improvement of the aggregate controller.

4.2 Fuzzy regulation using linear robust controllers

As previously mentioned, in this method the overall fuzzy regulator is obtained by *combining* linear robust regulators [22], such that the controller can be described by:

$$\dot{\xi}(t) = \sum_{i=1}^{r} h_i(z(t))[F_i\xi(t) + G_i e(t)] \tag{4.1}$$

$$u(t) = \sum_{i=1}^{r} h_i(z(t))\mathcal{H}_i\xi(t) \tag{4.2}$$

where

$$F_i = \begin{pmatrix} A_i + B_iK_i - G_{i0}C_i & 0 \\ -G_{i1}C_i & \Phi_i \end{pmatrix},\ G_i = \begin{pmatrix} G_{i0} \\ G_{i1} \end{pmatrix},\ \mathcal{H}_i = \begin{pmatrix} K_i & H \end{pmatrix},\ \Phi_i = diag\{\Phi_{i1}, \ldots, \Phi_{im}\},$$

$$\Phi_{ij} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a^i_{j,0} & -a^i_{j,1} & -a^i_{j,2} & \ldots & -a^i_{j,s_j-1} \end{pmatrix},\ H = \begin{pmatrix} H_1 & 0 & \ldots & 0 \\ 0 & H_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & H_m \end{pmatrix},$$
$$H_j = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}_{1\times s_j},$$

with $i = 1 \ldots r$, $j = 1 \ldots m$ and a^i_* as the coefficients of the characteristic polynomial of S_i.

Thus, matrices (K_i, G_{0i}, G_{1i}) must be computed such that

$$\bar{A}_{ii}^T\mathbf{P} + \mathbf{P}\bar{A}_{ii} < 0 \tag{4.3}$$

for all $i = 1 \ldots r$,

$$\left(\frac{\bar{A}_{ij} + \bar{A}_{ji}}{2}\right)^T \mathbf{P} + \mathbf{P}\left(\frac{\bar{A}_{ij} + \bar{A}_{ji}}{2}\right) < 0 \tag{4.4}$$

for all $i = 1, \ldots, r$, $j = 1, \ldots, r$ satisfying $h_i(z(t)) \cdot h_j(z(t)) \neq 0$. Where

$$\bar{A}_{ij} = \begin{pmatrix} A_i & B_i\mathcal{H}_j \\ G_iC_j & F_i \end{pmatrix} \tag{4.5}$$

and $\mathbf{P} > 0$.

The following theorem presents the conditions for the solution of the Fuzzy Regulation Problem by means of Linear Robust Regulators [9, 11, 39]

Theorem 15 *If*

H1$_{\mathbf{lrr}}$) *every trajectory $w(t)$ defined by (3.5) and $w(0)$ is bounded,*

H2$_{\mathbf{lrr}}$) *for $i = 1, \ldots, r$ there exist K_i such that $\dot{x}(t) = A_ix(t) + B_iK_ix(t)$ is stable,*

H3$_{\mathbf{lrr}}$) *for $i = 1, \ldots, r$ there exist G_i such that*

$$\dot{\xi}(t) = \begin{pmatrix} A_i & -B_iH \\ 0 & \Phi_i \end{pmatrix} \xi(t) - G_i \begin{pmatrix} C_i & 0 \end{pmatrix} \xi(t)$$

is stable,

H4$_{\mathbf{lrr}}$) *there exist mappings $\tilde{\pi}(w(t)) \sum_{i=1}^{r} h_i(z(t))\Pi_iw$ and $\tilde{\gamma}(w(t)) = \sum_{i=1}^{r} h_i(z(t))\Gamma_iw$, where matrices Π_i and Γ_i solve*

$$\begin{aligned} \Pi_iS_i &= A_i\Pi_i + B_i\Gamma_i + P_i \\ 0 &= C_i\Pi_i - Q_i \end{aligned}$$

for $i = 1, \ldots, r$,

H5$_{\mathbf{lrr}}$) *there exist triplets $(K_i, G_{0,i}, G_{1,i})$ and matrix P satisfying*

$$\begin{aligned} 0 &> \bar{A}_{ii}^TP + P\bar{A}_{ii} \\ 0 &> (\bar{A}_{ij} + \bar{A}_{ji})^TP + P(\bar{A}_{ij} + \bar{A}_{ji}) \\ P &> 0 \end{aligned}$$

for all $i = 1, \ldots, r$ and $j = 1, \ldots, r$ such that $h_i(z(t)) \cdot h_j(z(t) \neq 0$,

H6$_{\mathrm{lrr}}$) *there exist $\pi(w(t))$ and $\gamma(w(t))$ solving*

$$\begin{aligned}\frac{\partial\pi(w(t))}{\partial w(t)}\sum_{i=1}^{r}h_i(z(t))S_iw(t) &= \sum_{i=1}^{r}h_i(z(t))\{A_i\pi(w(t))+B_i\gamma(w(t))+P_iw(t)\},\\ 0 &= \sum_{i=1}^{r}h_i(z(t))\{C_i\pi(w(t))-Q_iw(t)\},\end{aligned}$$

H7$_{\mathrm{lrr}}$) *the term $\sum_{i=1}^{r}\dot{h}_i(z(t))$ is bounded,*

then, the tracking error for the Fuzzy Regulation Problem by means of Linear Robust Regulators is bounded.

Proof. If H2$_{\mathrm{lrr}}$, H3$_{\mathrm{lrr}}$ and H4$_{\mathrm{lrr}}$ are satisfied, then the existence of local robust regulators is ensured. In that sense, only the stability and regulation properties for the overall fuzzy system will be inspected.

Stability.- Considering the closed-loop system with $w(t)=0$

$$\begin{pmatrix}\dot{x}(t)\\ \dot{\xi}(t)\end{pmatrix} = \sum_{i=1}^{r}h_i^2(z(t))\bar{A}_{ii}\begin{pmatrix}x(t)\\ \xi(t)\end{pmatrix}+\sum_{i<j}^{r}h_i(z(t))h_j(z(t))\left[(\bar{A}_{ij}+\bar{A}_{ji})\right]\begin{pmatrix}x(t)\\ \xi(t)\end{pmatrix}$$

with $\bar{A}_{ij}$ defined as above. Thus, according to the Parallel Distributed Compensation approach proposed by Tanaka and Wang in [39] and [42], when H5$_{\mathrm{lrr}}$ is satisfied, the asymptotically stability of the equilibrium point $(x(t),\xi(t))=(0,0)$ is achieved.

Regulation.- As in the previous chapter, this property is analyzed directly from the fact that every local controller is a robust regulator for its respective subsystem.

In other words, if assumptions H2$_{\mathrm{lrr}}$, H3$_{\mathrm{lrr}}$, H4$_{\mathrm{lrr}}$ and H5$_{\mathrm{lrr}}$ are satisfied, then the existence of a fuzzy stabilizer and r local robust regulators is ensured while the bounded evolvement of the exosystem is guaranteed also by Condition H1$_{\mathrm{lrr}}$. According to this explanation and since $0<h_i(z(t))\leq 1$ and $\sum_{i=1}^{r}\dot{h}_i(z(t))$ is bounded (Assumption H7$_{\mathrm{lrr}}$), from the previous chapter it can be easily proven that $\hat{\pi}(w(t))$ is an approximation of $\pi(w(t))$, where $\hat{\pi}(w(t))$ is the real steady-state manifold defined by (3.9) and (3.10) while $\pi(w(t))$ is the exact solution for the fuzzy regulation problem.

Obviously, the existence of the exact mappings solving the fuzzy regulation problem is granted by Assumption H6$_{\mathrm{lrr}}$. ■

At this point, the solution of the fuzzy regulation problem on the basis of linear robust regulators depends on the a priori calculation of matrices K_i, $G_{0,i}$, $G_{1,i}$, and afterwards a search for matrix P ensuring the stability becomes necessary. If P is not found, new K_i, $G_{0,i}$, $G_{1,i}$ need to be computed and then again the search for matrix P must to be carried out. Thus, in the following analysis, numerical techniques are included in order to solve the fuzzy regulation problem through linear robust regulators in a more efficient way. Using this approach both the calculation of K_i, $G_{0,i}$, $G_{1,i}$ and the search for P are performed at the same time.

When matrix (4.5) is expanded, one gets:

$$\bar{A}_{ij} = \begin{pmatrix} A_i & B_iK_j & B_iH_j \\ G_{0,i}C_j & A_i + B_iK_j - G_{0,i}C_j & 0 \\ G_{1,i}C_j & -G_{1,i}C_j & \Phi_i \end{pmatrix}, \tag{4.6}$$

which by means of transformation $T = \begin{pmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{pmatrix}$, is similar to $T\bar{A}_{ij}T^{-1}$, i.e.,

$$\begin{pmatrix} A_i + B_iK_j & B_iK_j & B_iH_j \\ 0 & A_i - G_{0,i}C_j & -B_iH_j \\ 0 & -G_{1,i}C_j & \Phi_i \end{pmatrix} \tag{4.7}$$

Therefore, the problem is to stabilize

$$\dot{x}(t) = (A_i + B_iK_j)x(t) \tag{4.8}$$

and

$$\dot{\xi}(t) = \left\{ \begin{pmatrix} A_i & -B_iH_j \\ 0 & \Phi_i \end{pmatrix} - \begin{pmatrix} G_{0,i} \\ G_{1,i} \end{pmatrix} \begin{pmatrix} C_i & 0 \end{pmatrix} \right\} \xi(t). \tag{4.9}$$

To this end, suitable LMIs (Linear matrix Inequalities) are introduced to replace conditions $H2_{\text{lrr}}$, $H3_{\text{lrr}}$ and $H5_{\text{lrr}}$. In this sense, it can be proved that Equation (4.8) is stable if the following LMIs are feasible [5]:

$$X_1A_i^T + A_iX_1 + B_iY_i + Y_i^TB_i^T < 0 \tag{4.10}$$

for all $i = 1, \ldots, r$,

$$X_1\left(A_i^T + A_j^T\right) + \left(A_i + A_j\right)X_1 + B_iY_j + Y_j^TB_i^T + B_jY_i + Y_i^TB_j^T < 0 \tag{4.11}$$

for $i < j \le r$ such that $h_i(z(t)) \cdot h_j(z(t)) \neq 0$, and

$$X_1 > 0 \tag{4.12}$$

for $i = 1, \ldots, r$ with $\mathbf{P}_1 = X_1^{-1}$ and $K_i = Y_i\mathbf{P}_1$.

On the other hand, to analyze the stability of Equation (4.9), it is considered

$$H = H_1 = \ldots H_r,\ A_{e,i} = \begin{pmatrix} A_i & -B_iH \\ 0 & \Phi_i \end{pmatrix},\ G_i = \begin{pmatrix} G_{0,i} \\ G_{1,i} \end{pmatrix} \text{ and } C_{e,i} = \begin{pmatrix} C_i & 0 \end{pmatrix}.$$

The procedure follows in a similar way to that presented previously, that is, Equation (4.9) is stable if the following LMIs are feasible:

$$A_{e,i}^T \mathbf{P}_2 + \mathbf{P}_2 A_{e,i} - M_i C_{e,i} - C_{e,i}^T M_i^T \ < \ 0 \tag{4.13}$$

for all $i = 1, \ldots, r$,

$$\left(A_{e,i}^T + A_{e,j}^T\right) \mathbf{P}_2 + \mathbf{P}_2 \left(A_{e,i} + A_{e,j}\right) - M_i C_{e,j} - C_{e,j}^T M_i^T - M_j C_{e,i} - C_{e,i}^T M_j^T \ < \ 0 \tag{4.14}$$

for all i and j such that $h_i(z(t)) \cdot h_j(z(t)) \neq 0$ and

$$\mathbf{P}_2 > 0. \tag{4.15}$$

Finally, the gains for the observers are obtained from $G_i = \mathbf{P}_2^{-1} M_i$ while the common matrix guaranteeing the stability of the system is given by

$$\mathbf{P} = T^{-1} diag(\mathbf{P}_1, \mathbf{P}_2) T,$$

with T defined as above. This analysis is summarized as follows:

Theorem 16 *If*

$\mathbf{H1_{lrr}}$) *every trajectory $w(t)$ defined by (3.5) and $w(0)$ is bounded,*

$\mathbf{H2_{lrr}}$) *there exist mappings $\tilde{\pi}(w(t)) = \sum_{i=1}^{r} \Pi_i w$ and $\tilde{\gamma}(w(t)) = \sum_{i=1}^{r} \Gamma_i w$, whose matrices Π_i and Γ_i solve*

$$\begin{aligned} \Pi_i S_i &= A_i \Pi_i + B_i \Gamma_i + P_i \\ 0 &= C_i \Pi_i - Q_i, \end{aligned}$$

$\mathbf{H3^*_{lrr}}$) *LMIs (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15) are feasible,*

$\mathbf{H4_{lrr}}$) *there exist $\pi(w(t))$ and $\gamma(w(t))$ solving*

$$\begin{aligned} \frac{\partial \pi(w(t))}{\partial w(t)} \sum_{i=1}^{r} h_i(z(t)) S_i w(t) &= \sum_{i=1}^{r} h_i(z(t)) \left\{ A_i \pi(w(t)) + B_i \gamma(w(t)) + P_i w(t) \right\}. \\ 0 &= \sum_{i=1}^{r} h_i(z(t)) \{ C_i \pi(w(t)) - Q_i w(t) \}, \end{aligned}$$

H5$_{\text{lrr}}$) *the term* $\sum_{i=1}^{r} \dot{h}_i(z(t))$ *is bounded,*

then, the tracking error for the Fuzzy Regulation Problem by means of Linear Robust Regulators is bounded. Moreover, the solution is given by $K_i = Y_i\mathbf{P}_1$ *and* $G_i = \mathbf{P}_2^{-1}M_i$.

Proof. Follows directly from Theorem 15 and the previous discussion. ■

Remark 17 *The immersion included in the robust regulator generates the steady-state input signals needed to solve the regulation problem for all the values of the system parameters located in a neighborhood* $\mathcal{P}$*, around* $\mu = 0$*, where* $\mu \in \mathbb{R}^p$ *is the vector of unknown parameters [18, 22].*

This property improves the performance of the fuzzy regulator; since in this case, the fuzzy model (3.4)-(3.6) could be considered as a variation of the system solved by the aggregate controller (4.1)-(4.2) designed from linear robust regulators.

Remark 18 *Although this design technique shows a better behavior of the closed-loop system, it is not possible to ensure the asymptotical convergence of the tracking error because the membership functions are not included in the analysis yet.*

4.2.1 Example

In this section, the previous formulation is applied on the example of Chapter 3 with $\alpha = 2$. Then, local linear robust regulator problems are solved and the aggregate controller is obtained by its fuzzy combination. The gain matrices were computed through the LMI Toolbox of MATLAB getting:

$$\begin{aligned}
K_1 &= \begin{pmatrix} -11.515 & -10.057 \end{pmatrix}, \\
K_2 &= \begin{pmatrix} -23.2 & -5.4056 \end{pmatrix}, \\
G_1 &= \begin{pmatrix} 1.1576 & -14.807 & -0.71843 & 0.95171 \end{pmatrix}^T \text{ and} \\
G_2 &= \begin{pmatrix} 1.1326 & -8.7182 & -0.64677 & 0.93617 \end{pmatrix}^T
\end{aligned}$$

Clearly, the steady-state mappings are similar to those presented in Section 3.4, i.e.,

$$\Pi_1 = \Pi_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} \frac{-1-a}{b} & 0 \end{pmatrix},$$

$$\Gamma_2 = \begin{pmatrix} \frac{-1-\frac{2a}{\pi}}{\alpha b} & 0 \end{pmatrix},$$

which means that the tracking error will not converge asymptotically to zero. The simulation outcomes appear in Figure 4.1, Figure 4.2 and Figure 4.3.

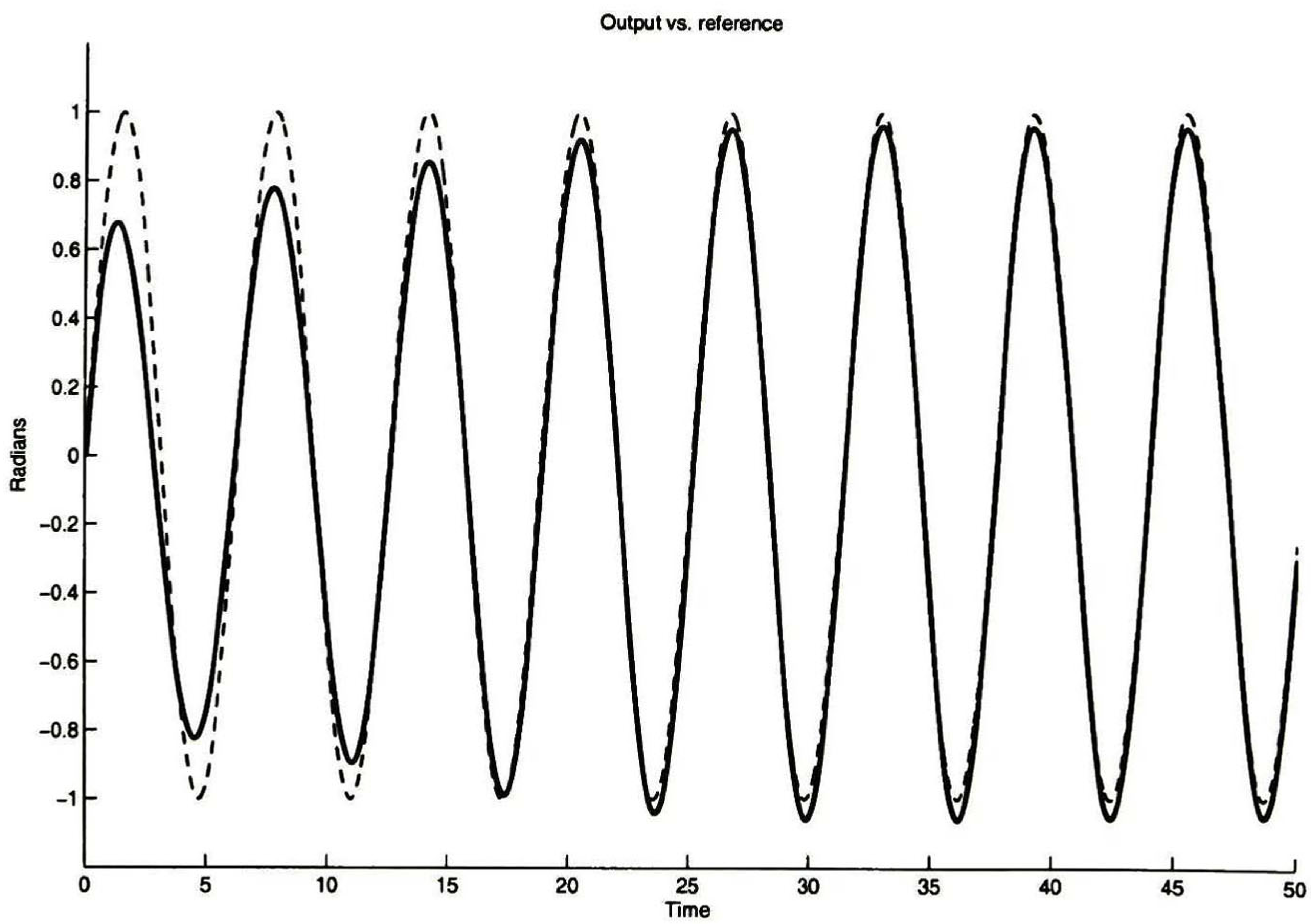

Figure 4.1: Output vs reference when $B_1 \neq B_2$ with $\alpha = 2$ and the fuzzy controller is designed from linear robust regulators.

A significative reduction of the magnitude of the tracking error and a smoother input signal are observed when these results are compared with those presented in Figure 3.4, Figure 3.5 and Figure 3.6 of the previous chapter.

As previously mentioned, although the tracking of reference signals is better with this approach, it is not yet possible to guarantee zero tracking error because the composite fuzzy model is not considered during the design process.

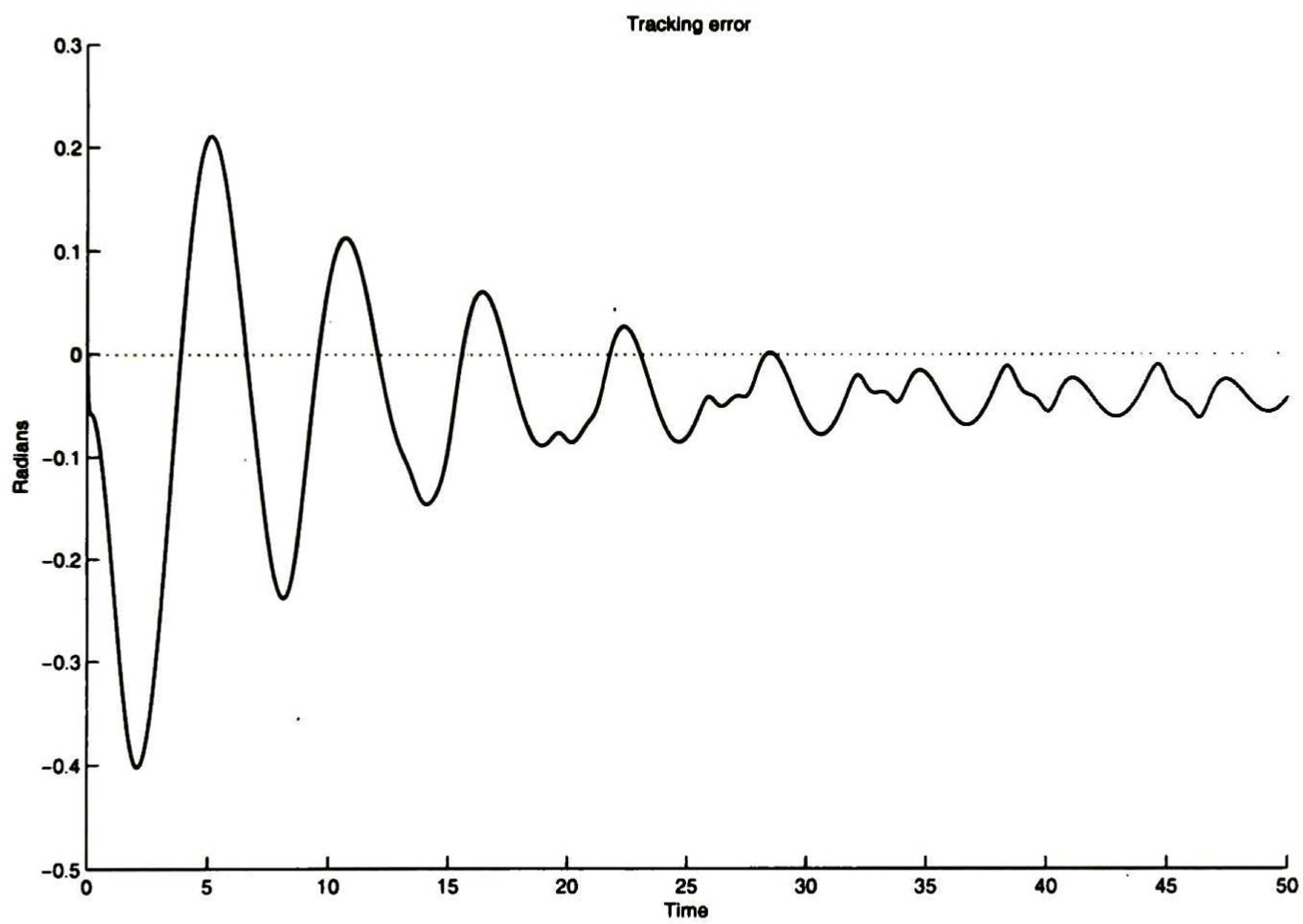

Figure 4.2: Tracking error when $B_1 \neq B_2$ with $\alpha = 2$ and the fuzzy controller is designed from linear robust regulators.

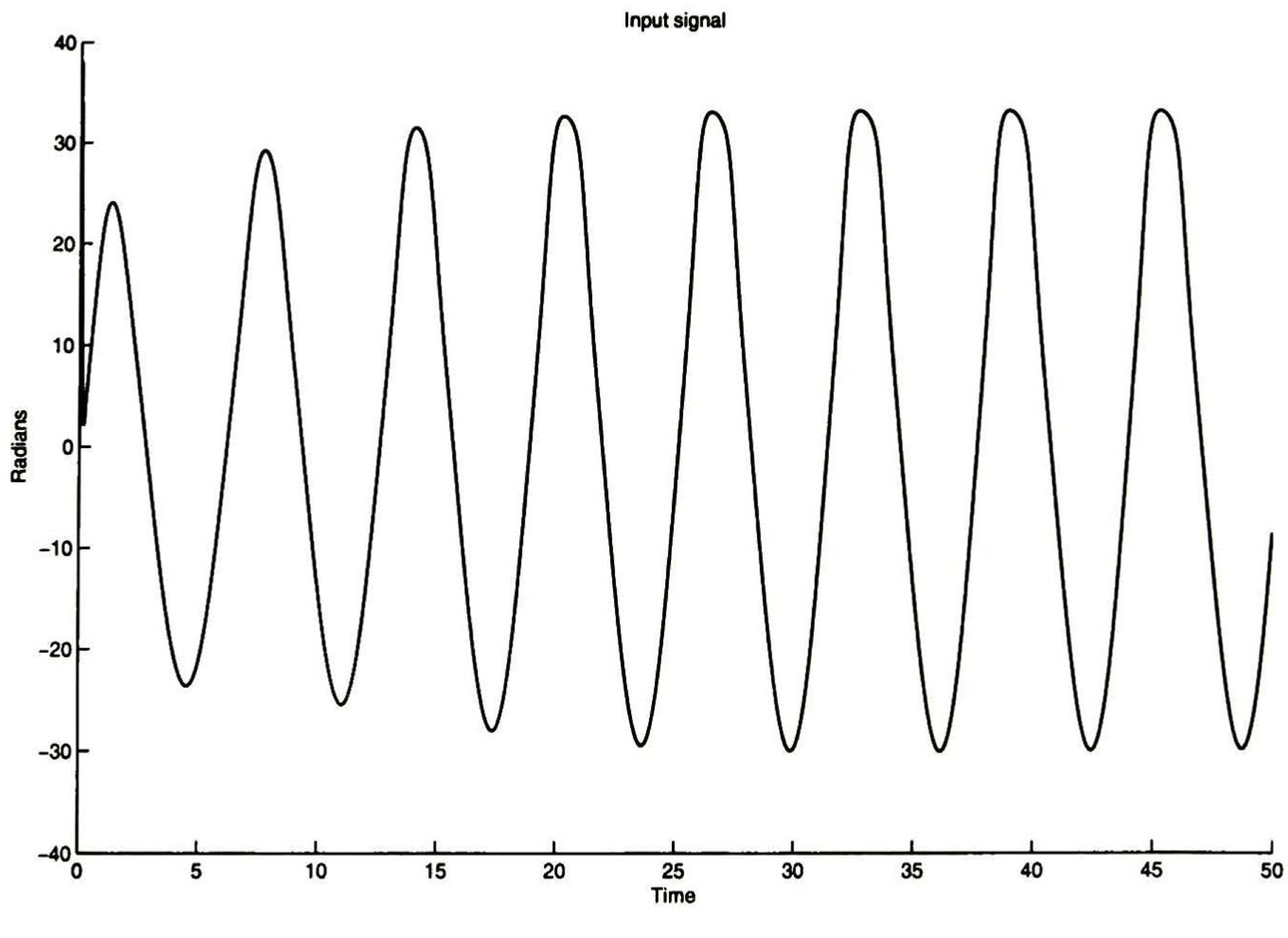

Figure 4.3: Input signal when $B_1 \neq B_2$ with $\alpha = 2$ and the fuzzy controller is designed from linear robust regulators.

Chapter 5

Fuzzy regulators with sliding modes

" If I have a thousand ideas
and only one turns out to be good,
I am satisfied.'
— Alfred Nobel

5.1 Introduction

Sliding mode control makes system motion robust with respect to system parameter variations, no-modelled dynamics and external disturbances. Roughly speaking, the sliding mode design approach consists of two steps. In the first step, a sliding surface is designed so that the controller satisfies some time-domain or frequency-domain requirements. The second step is to design a control law such that the system remains on (or close to) the sliding surface [40].

In addition, this technique provides efficient control laws for linear and nonlinear plants. Another distinguishing feature is its order reduction capability, which enables simplification of design and system decoupling [41].

For those reasons, it results desirable to explore if this methodology can be used to reduce the steady-state error into acceptable bounds when is combined with a fuzzy controller.

Then, taking into account the simplicity of the method presented in Chapter 3 and the features of the sliding mode control; in Section 5.2, the ability of sliding mode control of dealing with no-modelled dynamics is studied when a discontinuous term is added to the fuzzy regulator built through simple linear controllers.

In Section 5.3, a numerical algorithm which allows the fuzzy controller with sliding modes to be obtained in a practical manner is developed.

Finally, in Section 5.4, a simple numerical example is used to illustrate the effectiveness of the algorithm; showing that in some cases the tracking error tends asymptotically to zero, while in other cases the steady-state error is considerably reduced.

5.2 The fuzzy regulator with discontinuous term

In this section, an alternative method based on sliding modes is presented, showing that, under certain conditions, the tracking error tends to zero when the system is defined by equations (3.4)–(3.6).

Basically, the addition of a discontinuous term to the input signal generated by the fuzzy controller built from linear regulators is proposed, such that the aggregate controller turns out to be:

$$u = u_f + v(e) \tag{5.1}$$

where

$$\begin{aligned} u_f &= \left(\sum_{i=1}^{r} h_i(z(t))K_i\right)x + \left(\sum_{i=1}^{r} h_i(z(t))\Gamma_i - \sum_{i=1}^{r} h_i(z(t))K_i \sum_{j=1}^{r} h_j(z(t))\Pi_j\right)w \\ v(e) &= g \cdot sign(e). \end{aligned}$$

The problem consists of finding triplets (K_i, Π_i, Γ_i) for all $i = 1,..,r$ and g such that the following conditions are satisfied:

S$_{sm}$) the equilibrium $(x, w) = (0, 0)$ of system

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))A_i x(t) + \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))B_i K_j x(t)$$

is asymptotically stable (stability condition).

R$_{sm}$) the solution for the closed-loop system

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) + B_i u(t) + P_i w(t)\} \\ \dot{w}(t) &= \sum_{i=1}^{r} h_i(z(t))S_i w(t) \\ e(t) &= \sum_{i=1}^{r} h_i(z(t))\{C_i x(t) - Q_i w(t)\}, \end{aligned}$$

with $u(t)$ defined by Equation (5.1), satisfies

$$\lim_{t\to\infty} e(t) = 0,$$

(regulation condition).

First, the stability property is examined when controller (5.1) is used on the TS fuzzy model. To this end, a fuzzy plant not affected by exosystem is considered, i.e., with $w = 0$. As consequence the tracking error becomes:

$$e = \sum_{i=1}^{r} h_i(z(t))C_i x,$$

and the closed-loop system (3.4), (5.13) can be rewritten as

$$\dot{x}(t) = \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))N_{ij}x + \sum_{i=1}^{r} h_i(z(t))B_i g \cdot sign(e), \tag{5.2}$$

with

$$N_{ij} = (A_i + B_i K_j).$$

Now, considering the Lyapunov function

$$V = x^T \mathbf{P} x,$$

one has

$$\begin{aligned}\dot{V} &= x^T\mathbf{Q}x + 2x^TP\sum_{i=1}^{r}h_i(z(t))B_ig\cdot sign(e)\\ &\leq x^T\mathbf{Q}x + x^TP\sum_{i=1}^{r}h_i(z(t))B_i\sum_{i=1}^{r}h_i(z(t))B_i^TPx\\ &+ g^2 sign(e)^T sign(e),\end{aligned} \tag{5.3}$$

with

$$\mathbf{Q} = \left(\sum_{i,j=1}^{r}h_i(z(t))h_j(z(t))N_{ij}^T\mathbf{P} + \mathbf{P}\sum_{i,j=1}^{r}h_i(z(t))h_j(z(t))N_{ij}\right)$$

But, from the Parallel Distributed Compensation (PDC) analysis [39, 42], it is deduced that $\mathbf{Q} < 0$ for all $i, j = 1, \ldots, r$ if there exist matrices K_i and $\mathbf{P}$ such that

$$N_{ii}^T\mathbf{P} + \mathbf{P}N_{ii} < 0$$

for all $i = 1, ..., r$ and

$$\left(\frac{N_{ij}+N_{ji}}{2}\right)^T\mathbf{P} + \mathbf{P}\left(\frac{N_{ij}+N_{ji}}{2}\right) < 0$$

for all $i, j = 1, \ldots, r$ when $h_i(z(t))\cdot h_j(z(t)) \neq 0$.

On the other hand, it is clear that $\sum_{i=1}^{r}h_i(z(t))\|B_i\| \leq \sum_{i=1}^{r}\|B_i\|$ because $\sum_{i=1}^{r}h_i(z(t)) = 1$ and $0 \leq h_i(z(t)) \leq 1$ for all $i = 1\ldots r$ [20].

Thus, $\dot{V} < 0$ when

$$\left(\|\mathbf{Q}\| - \|\mathbf{P}\|^2\left(\sum_{i=1}^{r}\|B_i\|\right)^2\right)\|x\|^2 - g^2q > 0$$

where q is the error signal dimension.

In this manner, it can be directly inferred that the equilibrium $(x, w) = (0, 0)$ is asymptotically stable if

$$|g| < \alpha_1$$

with α_1 defined as

$$\alpha_1 \equiv \sqrt{\frac{\left(\|\mathbf{Q}\| - \|\mathbf{P}\|^2\left(\sum_{i=1}^{r}\|B_i\|\right)^2\right)\|x\|^2}{q}} > 0. \tag{5.4}$$

In situations where the previous square root has not solution, the recalculation of matrices K_i and $\mathbf{P}$ will be needed.

To complete the study of the fuzzy regulator with a discontinuous term, the inspection of the regulation property must be performed. To this end, the steady-state error

$$\tilde{x} = x - \pi(w)$$

have to be considered which allows the tracking error (3.6) to be represented as follows:

$$e = \sum_{i=1}^{r} h_i(z(t))C_i\tilde{x} + \sum_{i=1}^{r} h_i(z(t))C_i\pi(w) + \sum_{i=1}^{r} h_i(z(t))Q_iw.$$

From regulation theory [18, 19] and Chapter 3, it can be proved that the fuzzy regulation problem has a solution if and only if

$$\frac{\partial\pi}{\partial w}s(w) = \sum_{i=1}^{r} h_i(z(t))A_i\pi(w) + \sum_{i=1}^{r} h_i(z(t))B_i\gamma(w) + \sum_{i=1}^{r} h_i(z(t))P_iw \tag{5.5}$$

and

$$0 = \sum_{i=1}^{r} h_i(z(t))C_i\pi(w) + \sum_{i=1}^{r} h_i(z(t))Q_iw \tag{5.6}$$

are fulfilled.

In this way,

$$e = \sum_{i=1}^{r} h_i(z(t))C_i\tilde{x}$$

arises, whose derivative is

$$\begin{aligned}
\dot{e} &= \sum_{i=1}^{r} \dot{h}_i(z(t))C_i\tilde{x} + \sum_{i=1}^{r} h_i(z(t))C_i\dot{\tilde{x}} \\
&= \sum_{i=1}^{r} \dot{h}_i(z(t))C_i\tilde{x} \\
&+ \sum_{i=1}^{r} h_i(z(t))C_i\left(\sum_{j=1}^{r} h_j(z(t))A_jx + \sum_{j=1}^{r} h_j(z(t))B_ju + \sum_{j=1}^{r} h_j(z(t))P_jw - \frac{\partial\pi}{\partial w}s(w)\right)
\end{aligned}$$

By adding $\gamma(w)$ to and subtracting $\gamma(w)$ from Equation (5.1), one gets

$$\begin{aligned}
\dot{e} &= \sum_{i=1}^{r} \dot{h}_i(z(t))C_i\tilde{x} + \sum_{i=1}^{r} h_i(z(t))C_i\left\{\left(\sum_{j=1}^{r} h_j(z(t))A_j + \sum_{j=1}^{r} h_j(z(t))B_j\sum_{k=1}^{r} h_k(z(t))K_k\right)\tilde{x}\right. \\
&+ \sum_{j=1}^{r} h_j(z(t))A_j\pi(w) + \sum_{j=1}^{r} h_j(z(t))B_j\gamma(w) + \sum_{j=1}^{r} h_j(z(t))P_jw - \frac{\partial\pi}{\partial w}s(w) \\
&+ \sum_{j=1}^{r} h_j(z(t))B_j\left[\sum_{k=1}^{r} h_k(z(t))K_k\pi(w) - \sum_{k=1}^{r} h_k(z(t))K_k\left(\sum_{\ell=1}^{r} h_\ell(z(t))\Pi_\ell\right)w + \sum_{k=1}^{r} h_k(z(t))\Gamma_kw\right. \\
&- \left.\left.\gamma(w) + v(e)\right]\right\}.
\end{aligned}$$

which, by means of Equation (5.5), is transformed into

$$\begin{aligned}
\dot{e} &= \sum_{i=1}^{r} \dot{h}_i(z(t))C_i\tilde{x} + \sum_{i=1}^{r} h_i(z(t))C_i \left(\sum_{j=1}^{r} h_j(z(t))A_j + \sum_{j=1}^{r} h_j(z(t))B_j \sum_{k=1}^{r} h_k(z(t))K_k \right) \tilde{x} \\
&+ \sum_{i=1}^{r} h_i(z(t))C_i \left\{ \sum_{j=1}^{r} h_j(z(t))B_j \left[\sum_{k=1}^{r} h_k(z(t))K_k\pi(w) - \sum_{k=1}^{r} h_k(z(t))K_k \left(\sum_{\ell=1}^{r} h_\ell(z(t))\Pi_\ell \right) w \right.\right. \\
&+ \left.\left. \sum_{k=1}^{r} h_k(z(t))\Gamma_k w - \gamma(w) + v(e) \right] \right\}
\end{aligned}$$

Now, if

$$V = \frac{1}{2}e^T e$$

is considered as the Lyapunov function, then the derivative $\dot{V}$ is given by:

$$\begin{aligned}
\dot{V} &= e^T\dot{e} \\
&= e^T \sum_{i=1}^{r} \dot{h}_i(z(t))C_i\tilde{x} + e^T \sum_{i=1}^{r} h_i(z(t))C_i \left(\sum_{j=1}^{r} h_j(z(t))A_j \right. \\
&+ \left. \sum_{j=1}^{r} h_j(z(t))B_j \sum_{k=1}^{r} h_k(z(t))K_k \right) \tilde{x} + e^T \sum_{i=1}^{r} h_i(z(t))C_i \left\{ \sum_{j=1}^{r} h_j(z(t))B_j \right. \\
&\times \left[\sum_{k=1}^{r} h_k(z(t))K_k\pi(w) - \sum_{k=1}^{r} h_k(z(t))K_k \left(\sum_{\ell=1}^{r} h_\ell(z(t))\Pi_\ell \right) w \right. \\
&+ \left.\left. \sum_{k=1}^{r} h_k(z(t))\Gamma_k w - \gamma(w) + g \cdot sign(e) \right] \right\} \\
&\leq \|e\|\|M_1\| + \|e\|\|M_2\| + \|e\|\|M_3\| + \|e\|\|M_4\|g,
\end{aligned}$$

with

$$\begin{aligned}
M_1 &= \sum_{i=1}^{r} \dot{h}_i(z(t))C_i\tilde{x}, \\
M_2 &= \sum_{i=1}^{r} h_i(z(t))C_i \left(\sum_{j=1}^{r} h_j(z(t))A_j + \sum_{j=1}^{r} h_j(z(t))B_j \sum_{k=1}^{r} h_k(z(t))K_k \right) \tilde{x}, \\
M_3 &= \sum_{i=1}^{r} h_i(z(t))C_i \left\{ \sum_{j=1}^{r} h_j(z(t))B_j \left[\sum_{k=1}^{r} h_k(z(t))K_k\pi(w) \right.\right. \\
&- \left.\left. \sum_{k=1}^{r} h_k(z(t))K_k \left(\sum_{\ell=1}^{r} h_\ell(z(t))\Pi_\ell \right) w + \sum_{k=1}^{r} h_k(z(t))\Gamma_k w - \gamma(w) \right] \right\} \text{ and} \\
M_4 &= \sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))C_iB_j > 0,
\end{aligned}$$

where $\left(\sum_{i,j=1}^{r} h_i(z(t))h_j(z(t))C_iB_j\right)^{-1}$ must exist.

Then, it is readily deduced that the regulation condition is satisfied if

$$g < -\alpha_2,$$

where

$$\alpha_2 \equiv \frac{\|M_1\| + \|M_2\| + \|M_3\|}{\|M_4\|} > 0. \tag{5.7}$$

Finally, it is possible to conclude that the fuzzy regulation problem is solved by means of a discontinuous term if

$$-\alpha_1 \quad < g < \quad 0 \tag{5.8}$$

and

$$g < -\alpha_2. \tag{5.9}$$

At this point, the existence conditions for a fuzzy controller with a discontinuous term can be arranged in the following theorem.

Theorem 19 *If*

H1$_{\mathbf{sm}}$) *every trajectory $w(t)$ defined by (3.5) and $w(0)$ is bounded,*

H2$_{\mathbf{sm}}$) *the pairs (A_i, B_i) are stabilizable for all $i = 1, \ldots, r$,*

H3$_{\mathbf{sm}}$) *there exist matrices Π_i and Γ_i solving*

$$\Pi_i S_i = A_i\Pi_i + B_i\Gamma_i + P_i \tag{5.10}$$
$$0 = C_i\Pi_i + Q_i \tag{5.11}$$

for all $i = 1, \ldots, r$,

H4$_{\mathbf{sm}}$) *there exist matrices K_i and $\mathbf{P}$ such that*

$$N_{ii}^T\mathbf{P} + \mathbf{P}N_{ii} < 0$$

for all $i = 1, ..., r$ and

$$\left(\frac{N_{ij} + N_{ji}}{2}\right)^T \mathbf{P} + \mathbf{P}\left(\frac{N_{ij} + N_{ji}}{2}\right) < 0$$

for all $i, j = 1, \ldots, r$ when $h_i(z(t)) \cdot h_j(z(t)) \neq 0$ with

$$N_{ij} = (A_i + B_iK_j), \tag{5.12}$$

H5$_{\mathbf{sm}}$) *there exists a real number g such that $-\alpha_1 < g < 0$ and $g < -\alpha_2$ with α_1 and α_2 defined as above,*

H6$_{\mathbf{sm}}$) *there exist $\pi(w(t))$ and $\gamma(w(t))$ solving exactly*

$$\begin{aligned}\frac{\partial\pi(w(t))}{\partial w(t)}\sum_{i=1}^{r} h_i(z(t))S_i w(t) &= \sum_{i=1}^{r} h_i(z(t))\left\{A_i\pi(w(t)) + B_i\gamma(w(t)) + P_i w(t)\right\},\\ 0 &= \sum_{i=1}^{r} h_i(z(t))\{C_i\pi(w(t)) - Q_i w(t)\},\end{aligned}$$

H7$_{\mathbf{sm}}$) *the term $\sum_{i=1}^{r} \dot{h}_i(z(t))$ is bounded,*

then the fuzzy regulation with sliding modes problem has a solution. Moreover the controller has the form

$$\begin{aligned} u &= \left(\sum_{i=1}^{r} h_i(z(t))K_i\right)x + \left(\sum_{i=1}^{r} h_i(z(t))\Gamma_i - \sum_{i=1}^{r} h_i(z(t))K_i\sum_{j=1}^{r} h_j(z(t))\Pi_j\right)w \\ &+ g\cdot sign(e). \end{aligned} \tag{5.13}$$

Proof. Follows directly from the previous analysis. ■

Clearly, this approach includes the overall fuzzy system, which allows the fuzzy regulator to drive the output of the plant towards the reference signal when the design conditions are satisfied.

Nonetheless, although condition (5.8) could be verified in a relative simple way, inequality (5.9) is too complex to be satisfied because it depends on the exact mappings $\pi(w(t))$ and $\gamma(w(t))$ which are unknown.

Another scenario to solve this problem is obtained when it is supposed that

$$\pi(w) = \sum_{i=1}^{r} h_i(z(t))\Pi_i w + f_1(w)$$

and

$$\gamma(w) = \sum_{i=1}^{r} h_i(z(t))\Gamma_i w + f_2(w),$$

where $f_1(w)$ and $f_2(w)$ indicate the difference between the exact mappings, namely $\pi(w)$ and $\gamma(w)$, and their respective fuzzy approximations.

Obviously, when these assumptions are considered the stability condition is not affected. However; in this case, after deriving the Lyapunov function $V = e^T Pe$, one gets

$$
\begin{aligned}
\dot{V} &= e^T\dot{e} \\
&= e^T\sum_{i=1}^{r}\dot{h}_i(z(t))C_i\tilde{x} + e^T\sum_{i=1}^{r}h_i(z(t))C_i\left(\sum_{j=1}^{r}h_j(z(t))A_j\right. \\
&+ \left.\sum_{j=1}^{r}h_j(z(t))B_j\sum_{k=1}^{r}h_k(z(t))K_k\right)\tilde{x} + e^T\sum_{i=1}^{r}h_i(z(t))C_i\left\{\sum_{j=1}^{r}h_j(z(t))B_j\right. \\
&\times \left[\sum_{k=1}^{r}h_k(z(t))K_k\left(\sum_{\ell=1}^{r}h_\ell(z(t))\Pi_\ell\right)w + f_1(w) - \sum_{k=1}^{r}h_k(z(t))K_k\left(\sum_{\ell=1}^{r}h_\ell(z(t))\Pi_\ell\right)w\right. \\
&+ \left.\left.\sum_{k=1}^{r}h_k(z(t))\Gamma_k w - \sum_{k=1}^{r}h_k(z(t))\Gamma_k w - f_2(w) + g\cdot sign(e)\right]\right\} \\
&= e^T\sum_{i=1}^{r}\dot{h}_i(z(t))C_i\tilde{x} + e^T\sum_{i=1}^{r}h_i(z(t))C_i\left(\sum_{j=1}^{r}h_j(z(t))A_j + \sum_{j=1}^{r}h_j(z(t))B_j\sum_{k=1}^{r}h_k(z(t))K_k\right)\tilde{x} \\
&+ e^T\sum_{i=1}^{r}h_i(z(t))C_i\left\{\sum_{j=1}^{r}h_j(z(t))B_j\left[f_1(w) - f_2(w) + g\cdot sign(e)\right]\right\} \\
&\leq \|e\|\|M_1^*\| + \|e\|\|M_2^*\| + \|e\|\|M_3^*\| + \|e\|\|M_4^*\|g,
\end{aligned}
$$

getting

$$
\begin{aligned}
M_1^* &= \sum_{i=1}^{r}\dot{h}_i(z(t))C_i\tilde{x}, \\
M_2^* &= \sum_{i=1}^{r}h_i(z(t))C_i\left(\sum_{j=1}^{r}h_j(z(t))A_j + \sum_{j=1}^{r}h_j(z(t))B_j\sum_{k=1}^{r}h_k(z(t))K_k\right)\tilde{x}, \\
M_3^* &= \sum_{i=1}^{r}h_i(z(t))C_i\left\{\sum_{j=1}^{r}h_j(z(t))B_j\left[f_1(w) - f_2(w)\right]\right\} \text{ and} \\
M_4^* &= \sum_{i,j=1}^{r}h_i(z(t))h_j(z(t))C_iB_j > 0,
\end{aligned}
$$

where $\left(\sum_{i,j=1}^{r}h_i(z(t))h_j(z(t))C_iB_j\right)^{-1}$ must exist.

Hence, the steady-state error tends asymptotically to zero when

$$g < -\alpha_2^*,$$

where

$$\alpha_2^* \equiv \frac{(\|M_1^*\| + \|M_2^*\| + \|M_3^*\|)}{\|M_4^*\|} > 0, \tag{5.14}$$

then, it is possible to solve the regulation problem using sliding modes if

$$-\alpha_1 \quad < g < \quad 0 \tag{5.15}$$

and

$$g < -\alpha_2^*. \tag{5.16}$$

Unfortunately, Condition (5.16) cannot be verified directly because functions $f_1(w)$ and $f_2(w)$ are unknown. Therefore, the application of numerical techniques is proposed to obtain an approximation for the fuzzy regulator with sliding modes.

5.3 Numerical approach to design the fuzzy regulator with sliding modes

From the previous analysis can be concluded that the fuzzy regulation problem is solved by means of sliding modes when there exists an intersection between restrictions (5.8) and (5.9). In other words, the arbitrary increment of the magnitude of g may produce instability instead of reducing the tracking error.

That is why in this section the discontinuous term will be obtained from a predefined value, namely β. On this basis, a practical way to compute matrices K_i and $\mathbf{P}$ ensuring the stability of the overall fuzzy system and the reduction of the tracking error is presented.

To this end, Assumption H2$_{\text{sm}}$ is rewritten as an LMI [5], which includes the necessary terms to ensure the existence of $\alpha_1 > 0 \in \mathbb{R}$. Then, the resulting linear matrix inequality is

$$-\beta I > \mathcal{Q}_1 A_i^T + X_i^T B_i^T + A_i \mathcal{Q}_1 + B_i X_i + \lambda I \tag{5.17}$$

for all $i = 1 \ldots r$, with $\lambda = \left(\sum_{i=1}^{r} \|B_i\| \right)$. $\mathcal{Q}_1$ and X_i are the unknowns; with $X_i = K_i \mathcal{Q}_1$ and $\mathcal{Q}_1 > 0$. β is a design parameter satisfying $\beta > 0$ and allowing the method to obtain different values for α_1.

By proceeding in a similar way with Assumption H4$_{\text{sm}}$, the following LMI is obtained

$$-2\beta I \quad > \quad \mathcal{Q}_1 A_i^T + X_j^T B_i^T + \mathcal{Q}_1 A_j^T + X_i^T B_j^T + A_i \mathcal{Q}_1 + B_i X_j + A_j \mathcal{Q}_1 + B_j X_i + 2\lambda^2 I \tag{5.18}$$

for all $i, j = 1, \ldots, r$ when $h_i(z(t)) \cdot h_j(z(t)) \neq 0$. Finally, the common matrix $\mathbf{P}$ can be computed from $\mathcal{Q}_1^{-1}$ [39].

By equations (5.3), (5.4), (5.17) and (5.18), it is possible to infer that the stability property is satisfied when $g = -\beta$.

The following algorithm starts with $g = 0$ and produces a new value for the sliding modes gain in each iteration by incrementing β systematically, which can be considered as an approximation of α_1.

Algorithm 20 *Computing K_i, **P** and g.*

Step 1: *set $\beta = 0$ and take any $\Delta\beta$ as an increment.*

Step 2: *solve the LMIs (5.17) and (5.18). Construct the regulator (5.13) considering $g = -\beta$.*

Step 3: *reset $\beta = \beta + \Delta\beta$, goto Step 2.*

5.4 Example

In this section, Algorithm 20 is applied on the fuzzy model presented in Chapter 3 with $\alpha = 2.5$. If β and $\Delta\beta$ are initialized as $\beta = 0$ and $\Delta\beta = 2$, then it is possible to obtain the following controllers using the MATLAB LMI toolbox:

Controller 1 with $\beta = 0$:

$$\begin{aligned} K_1 &= \begin{pmatrix} -53.5904 & -20.9328 \end{pmatrix} \\ K_2 &= \begin{pmatrix} -36.4632 & -8.8450 \end{pmatrix} \\ g &= 0 \end{aligned}$$

Controller 2 with $\beta = 2$:

$$\begin{aligned} K_1 &= \begin{pmatrix} -271.1921 & -69.5392 \end{pmatrix} \\ K_2 &= \begin{pmatrix} -127.2090 & -29.0904 \end{pmatrix} \\ g &= -2 \end{aligned}$$

Controller 3 with $\beta = 4$:

$$\begin{aligned} K_1 &= \begin{pmatrix} -787.8985 & -184.5593 \end{pmatrix} \\ K_2 &= \begin{pmatrix} -342.1423 & -76.9143 \end{pmatrix} \\ g &= -4 \end{aligned}$$

The behavior of the plant under the action of each one of the previous controllers is simulated by means of SIMULINK. The tracking errors are presented in Figure 5.1. This graphic is useful to illustrate how the performance of the controllers is improved as the absolute value of g grows. Figure 5.2 compares the output of the plant and the reference signal for the three cases while Figure 5.3 shows the control inputs in the same conditions.

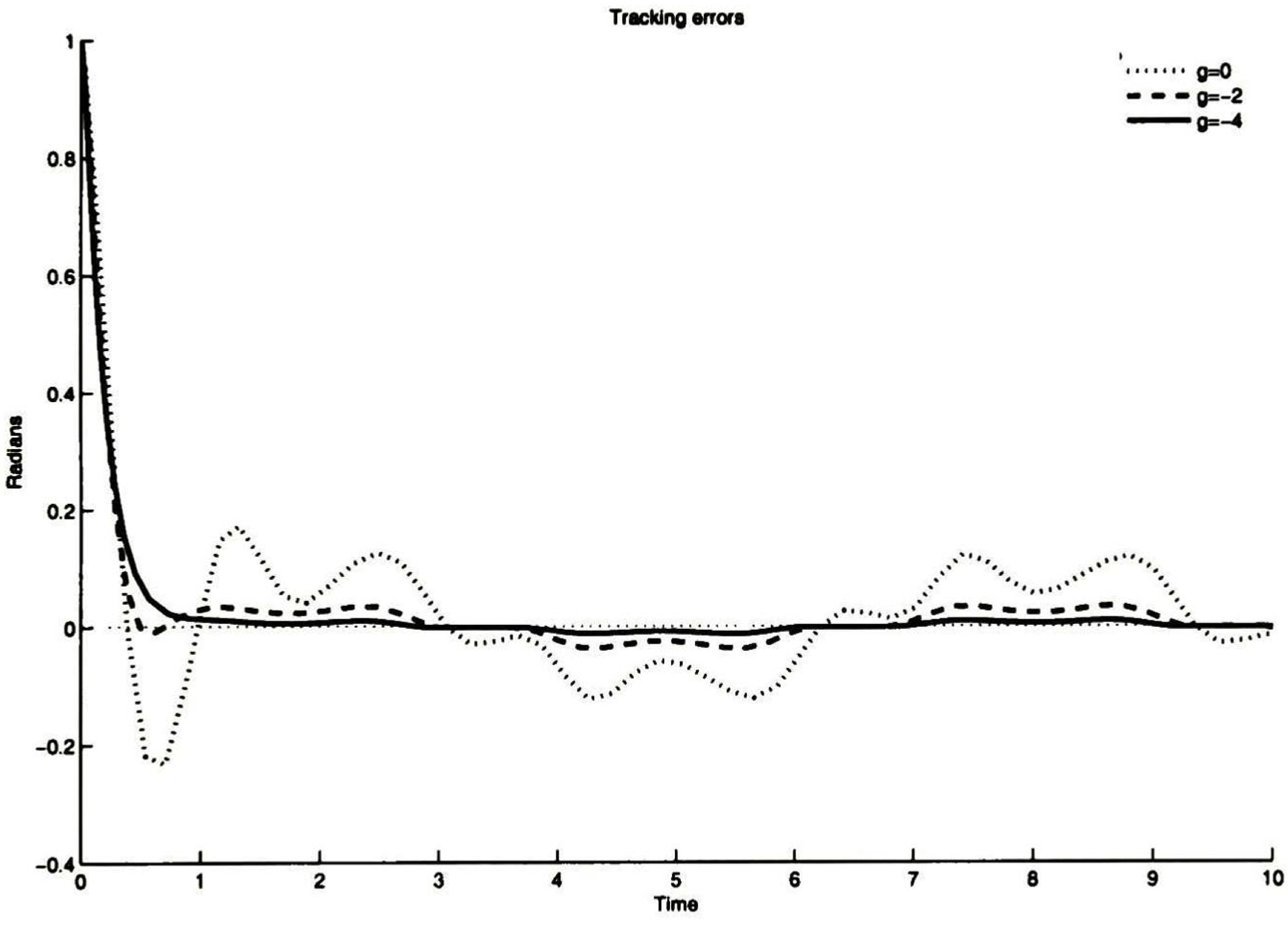

Figure 5.1: Tracking errors under the action of the fuzzy controllers with sliding modes.

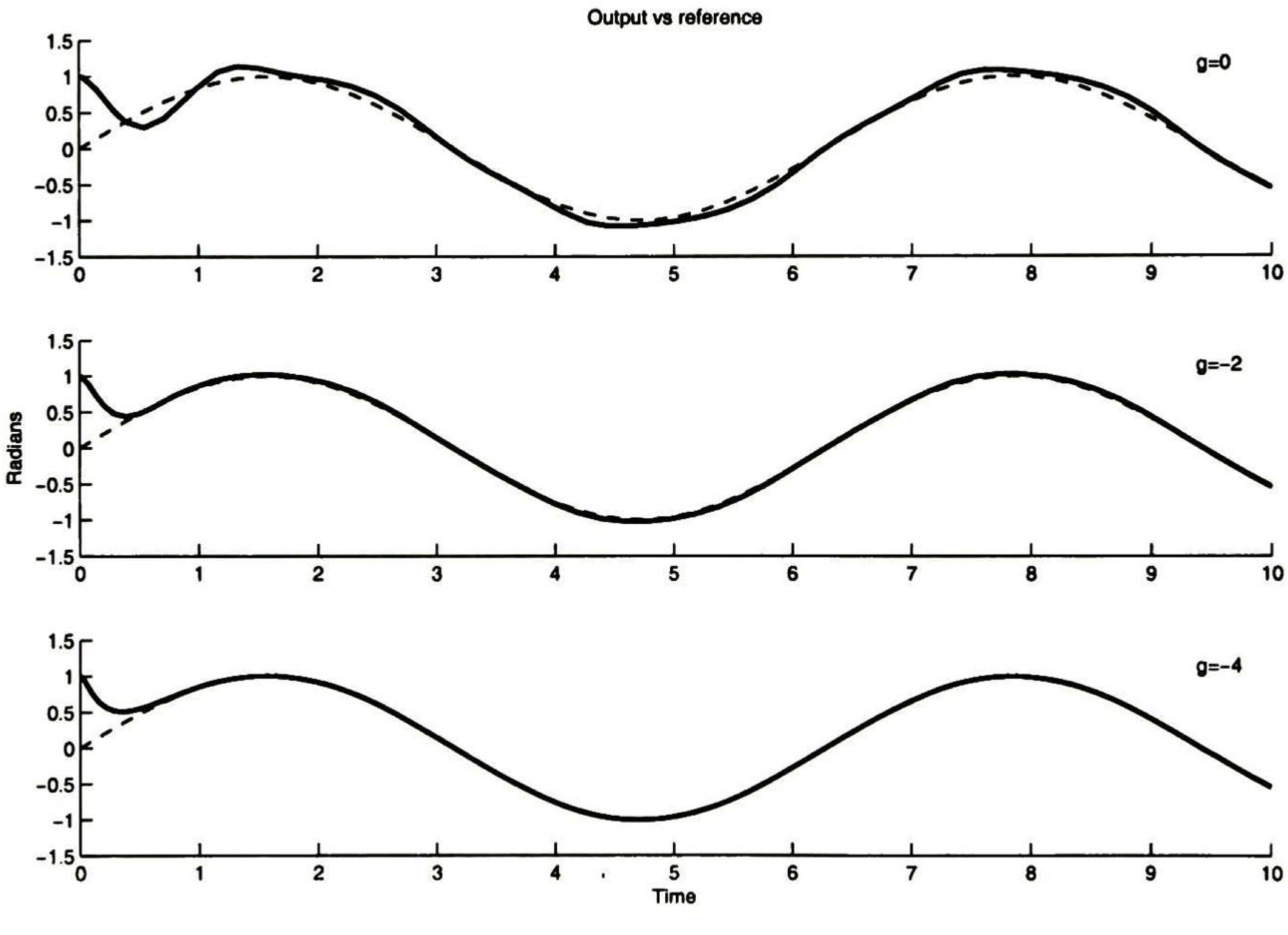

Figure 5.2: Output vs reference under the action of the fuzzy controllers with sliding modes.

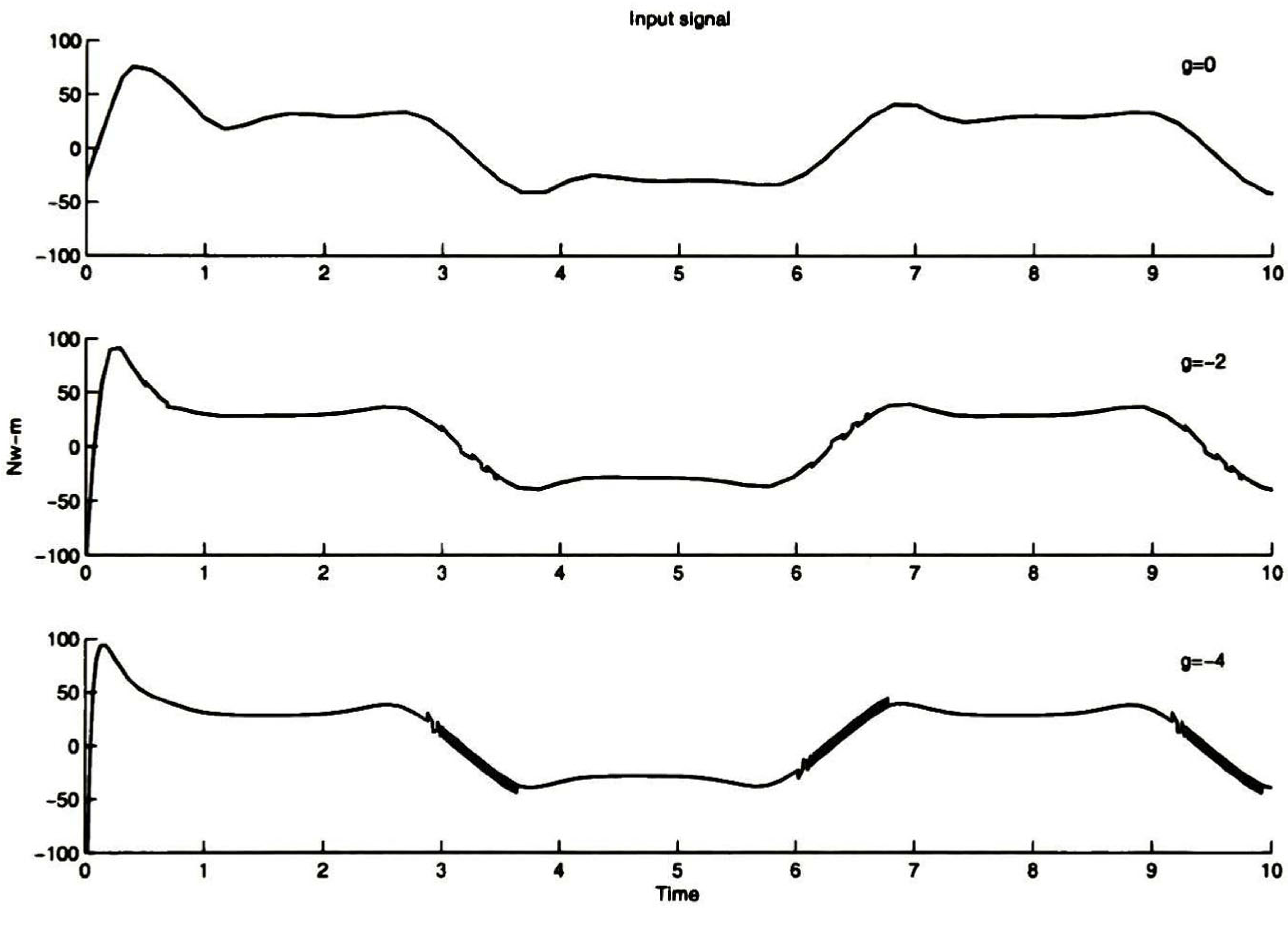

Figure 5.3: Control signals under the action of the fuzzy controllers with sliding modes.

Chapter 6

Application to chaotic systems

" A little knowledge that acts
is worth infinitely more than
much knowledge that is idle."

— Kahlil Gibran

6.1 Introduction

Chaotic systems can be found in very common real-life processes, for instance the weather, turbulence in liquids, human heart beating, brain activity, financial markets, population explosion, among many others. Therefore, the study of chaos has received great importance in recent years. It is worth mentioning that the control of chaos does not only implies suppressing the chaotic behavior of the systems, but to modify it or even generate it as well. One of the most interesting aspects in chaos control is *synchronization*.

The origin of the word synchronization is a Greek root (*syn*–together and *chronos*–time) which means "to share the common time". The original meaning of synchronization has been maintained up to now in the colloquial use of this word, as agreement or correlation in time of different processes [26]. Historically, the analysis of synchronization phenomena in the evolution of dynamical systems has been a subject of active investigation since the earlier days of physics. It started in the 17^{th} century with the finding of Huygens that two very weakly coupled pendulum clocks (hanging at the same beam) become synchronized in phase. Other examples are the synchronized lightning of fireflies, or the peculiarities of adjacent organ pipes which can almost reduce one another to silence or speak in absolute unison. For an exhaustive overview of the classic examples of synchronization of periodic systems the reader is referred to [4].

Recently, the search for synchronization has moved to chaotic systems. In this framework, the appearance of collective (synchronized) dynamics is, in general, not trivial. Indeed, a dynamical system is called chaotic whenever its evolution sensitively depends on the initial conditions. This implies that two trajectories emerging from two different initial conditions separate exponentially in the course of the time. As consequence, chaotic systems intrinsically defy synchronization, because even two identical systems starting from slightly different initial conditions would evolve in time in an unsynchronized manner (the differences in the states of the systems would grow exponentially). This is a relevant practical problem because initial conditions are never known perfectly in experimental situations. In that sense, the setting of some collective (synchronized) behavior in coupled chaotic systems has a great importance and interest.

On the other hand, most of the better-known chaotic systems can be easily represented by TS fuzzy models. Hence, in this chapter the fuzzy techniques based entirely on linear controllers are applied in order to obtain synchronization between chaotic systems.

6.2 Complete synchronization

As stated above, chaotic systems defy synchronization, due to their essential feature of displaying high sensitivity to initial conditions. As a result, two *identical* chaotic systems starting at nearly the same initial points in phase space develop onto trajectories which become uncorrelated in the course of the time. Nevertheless, it has been shown that it is possible to synchronize these kinds of systems, to make them evolving on the same chaotic trajectory [2, 28, 29, 31]. When one deals with coupled identical systems, synchronization appears as the equality of the state variables while evolving in time. This type of synchronization is referred as Complete Synchronization (CS). Other names were given in the literature, such as Conventional Synchronization [32] or Identical Synchronization.

Considering the following chaotic attractors

$$\begin{aligned} \dot{w} &= f(w) \text{ as the \textit{driver} system and} \\ \dot{x} &= f'(x, w, u) \text{ as \textit{response} system,} \end{aligned} \tag{6.1}$$

the existence of CS implies that $\lim_{t\to\infty} \|e(t)\| = 0$; where $e(t)$ is the synchronization error defined by $e(t) = \|x - w\|$. In other words, the response system forgets its initial conditions, though the chaotic behavior is still observed.

Naturally, this kind of synchronization can be seen as a regulation problem where $\pi(w)$ turns out to be the identity. In [10], an approach to synchronize two Chen's attractors using fuzzy robust regulators is presented.

In the remainder of this section the fuzzy regulator built on simple linear controllers is considered as the synchronization signal between chaotic attractors, showing that this approach guarantees Complete Synchronization if the conditions given in Chapter 3 are fulfilled.

First, the fuzzy approach is used to synchronize two Rössler attractors described by the following equations:

Driver system

$$\begin{aligned} \dot{w}_1 &= -(w_2 + w_3) \\ \dot{w}_2 &= w_1 + a w_2 \\ \dot{w}_3 &= b w_1 - (c - w_1) w_3 \end{aligned}$$

Response system

$$\begin{aligned}
\dot{x}_1 &= -(x_2 + x_3) \\
\dot{x}_2 &= x_1 + a x_2 \\
\dot{x}_3 &= b x_1 - (c - x_1) x_3 + u
\end{aligned}$$

with $a = 0.34$, $b = 0.4$ and $c = 4.5$.

These systems can be exactly represented by means of the following two-rule TS fuzzy model when $x_1 \in [c - d, c + d]$ [39]:

Rule 1

IF $x_1(t)$ is M_1

$$THEN \begin{cases} \dot{x}(t) = & A_1 x(t) + Bu(t) \\ \dot{w}(t) = & S_1 w(t) \\ e(t) = & Cx(t) - Qw(t) \end{cases}$$

Rule 2

IF $x_1(t)$ is M_2

$$THEN \begin{cases} \dot{x}(t) = & A_2 x(t) + Bu(t) \\ \dot{w}(t) = & S_2 w(t) \\ e(t) = & Cx(t) - Qw(t), \end{cases}$$

while the matrices defining the linear systems are

$$A_1 = S_1 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & -d \end{pmatrix}, \; A_2 = S_2 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & d \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \text{ and}$$

$$C = Q = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

For this case, the membership functions representing the nonlinearity of the original system are given by

$$\begin{aligned}
h_1(x_1) = M_1(x_1) &= \frac{1}{2}\left(1 + \frac{c - x_1}{d}\right) \text{ and} \\
h_2(x_1) = M_2(x_1) &= \frac{1}{2}\left(1 - \frac{c - x_1(t)}{d}\right),
\end{aligned}$$

where $d = 10$.

The controller capable of synchronize these chaotic systems is obtained from:

$$\begin{aligned}
\Pi_1 = \Pi_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\Gamma_1 = \Gamma_2 &= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \\
K_1 &= \begin{pmatrix} 2.4856 & 0.6387 & 8.2318 \end{pmatrix} \text{ and} \\
K_2 &= \begin{pmatrix} 2.4856 & 0.6387 & -11.7682 \end{pmatrix}
\end{aligned}$$

The simulation process is started with the following initial conditions $x_1 = 2$, $x_2 = 5$, $x_3 = 20$, $w_1 = 1$, $w_2 = 0$, $w_3 = 0$ allowing the systems to evolve during 50 seconds. The results are depicted in Figure 6.1, Figure 6.2 and Figure 6.3 showing an acceptable performance of the fuzzy regulator.

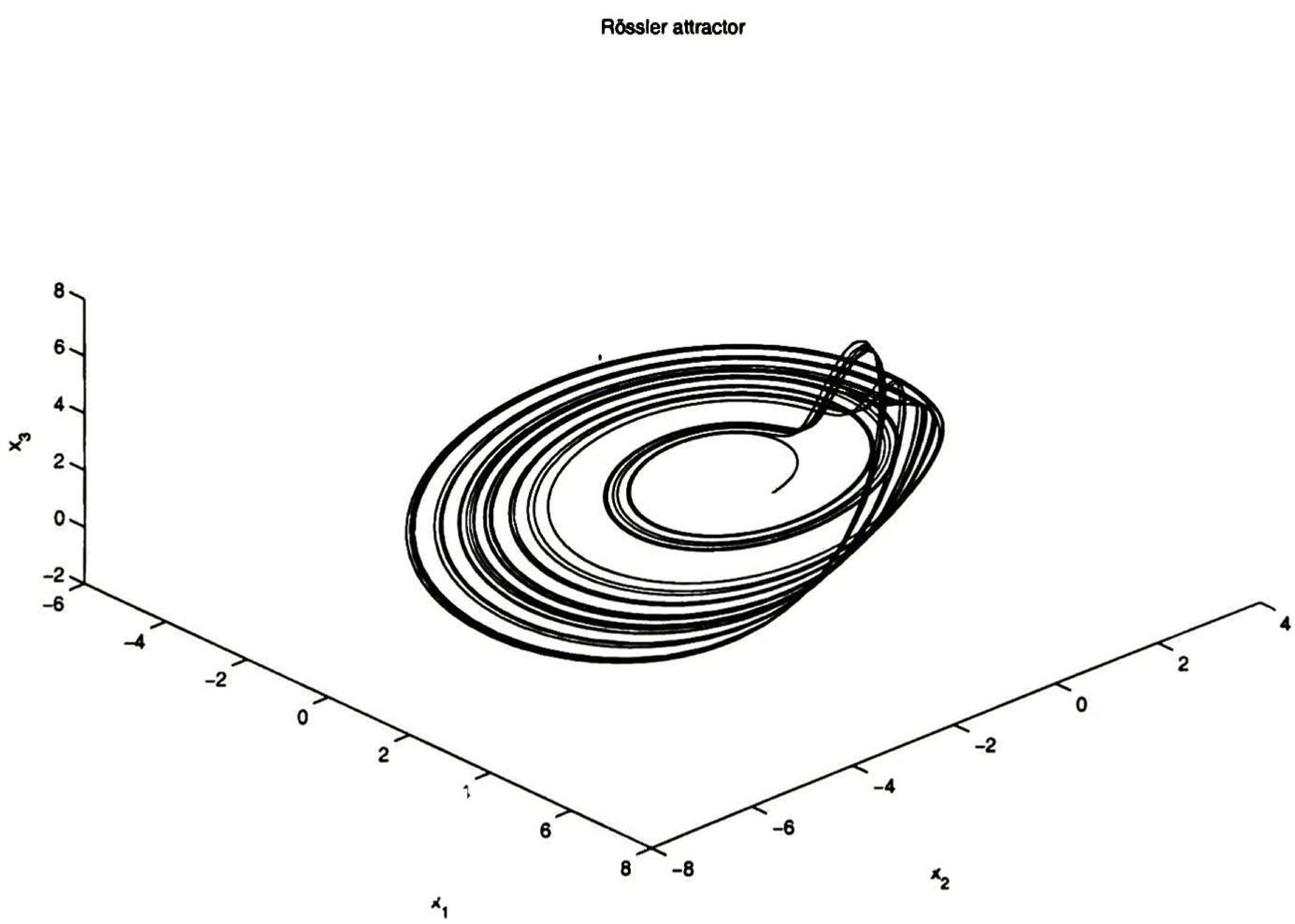

Figure 6.1: Rössler attractor.

Similar results can be obtained when the fuzzy regulator is used to synchronize two Lorenz attractors of the form:

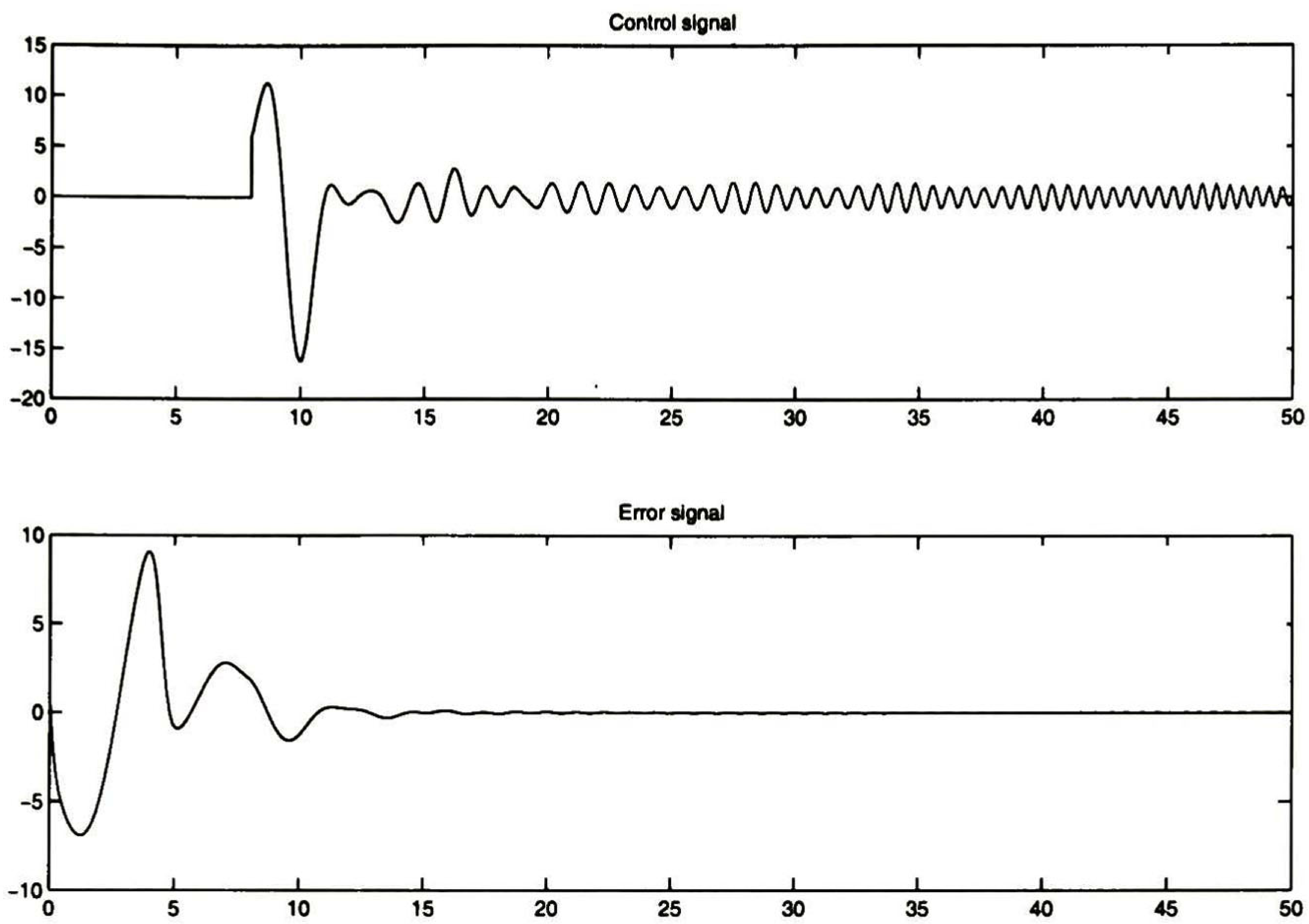

Figure 6.2: Control signal and tracking error for the Complete Synchronization of Rössler systems.

Driver system

$$\begin{aligned}\dot{w}_1 &= -aw_1 + aw_2\\ \dot{w}_2 &= cw_1 - w_2 - w_1w_3\\ \dot{w}_3 &= w_1w_2 - bw_3\end{aligned}$$

Response system

$$\begin{aligned}\dot{x}_1 &= -ax_1 + ax_2\\ \dot{x}_2 &= cx_1 - x_2 - x_1x_3 + u\\ \dot{x}_3 &= x_1x_2 - bx_3\end{aligned}$$

with $a = 10$, $b = \frac{8}{3}$ and $c = 28$.

Like the previous example, these systems can be locally exactly represented by means of a two-rule TS fuzzy model in the region $x_1 \in [-d, d]$ [39]. The matrices and membership function used in this case are

$$A_1 = S_1 = \begin{pmatrix} -a & a & 0\\ c & 1 & d\\ 0 & -d & b\end{pmatrix},\ A_2 = S_2 = \begin{pmatrix} -a & a & 0\\ c & 1 & -d\\ 0 & d & b\end{pmatrix}.\ B = \begin{pmatrix} 0 & 1 & 0\end{pmatrix},\ C = Q = \begin{pmatrix} 1 & 0 & 0\end{pmatrix},$$

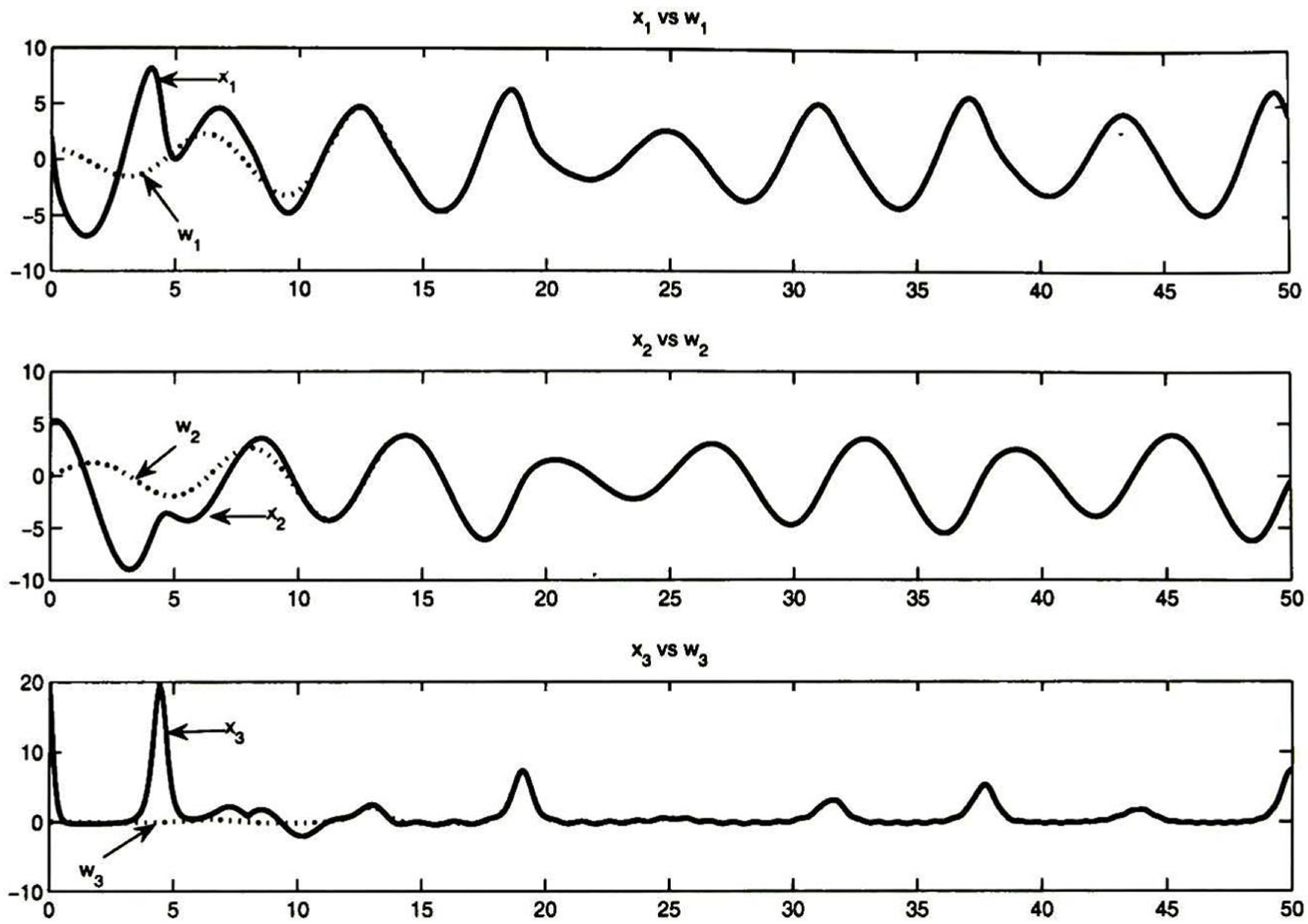

Figure 6.3: Driver states versus response states for the Complete Synchronization of Rössler systems.

$$h_1(x_1) = M_1(x_1) = \frac{1}{2}\left(\frac{-x_1+d}{d}\right) \text{ and } h_2(x_1) = M_2(x_1) = \frac{1}{2}\left(\frac{x_1+d}{d}\right),$$

where $d = 30$.

After applying the design approach, the controller is defined by the following matrices which can be obtained using MATLAB:

$$\begin{aligned}
\Pi_1 = \Pi_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\Gamma_1 = \Gamma_2 &= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \\
K_1 &= \begin{pmatrix} -154.4886 & 1.1762 & 76.4093 \end{pmatrix} \text{ and} \\
K_2 &= \begin{pmatrix} -154.4886 & 1.1762 & -76.4093 \end{pmatrix}
\end{aligned}$$

The initial conditions considered for this example are $x_1 = 2$, $x_2 = 5$, $x_3 = 20$, $w_1 = 1$, $w_2 = 0$, $w_3 = 0$. Figure 6.4, Figure 6.5 and Figure 6.6 show again an acceptable performance of the fuzzy regulator.

In the same way, the fuzzy regulator can be applied to synchronize two Chen's chaotic systems described by:

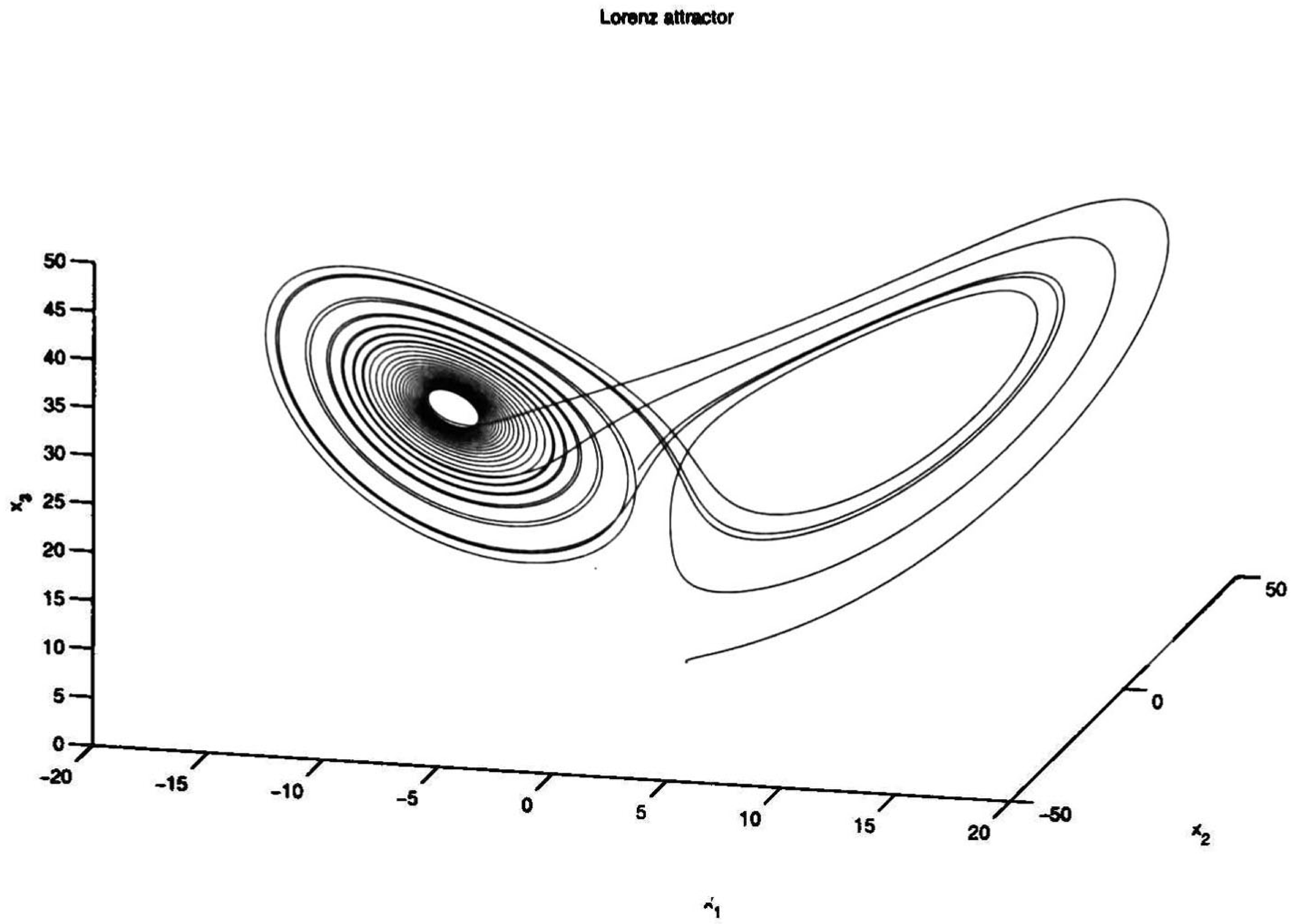

Figure 6.4: Lorenz attractor.

Driver system

$$\begin{aligned}\dot{w}_1 &= -aw_1 + aw_2 \\ \dot{w}_2 &= (c-a)w_1 + cw_2 - w_1w_3 \\ \dot{w}_3 &= w_1w_2 - bw_3\end{aligned}$$

Response system

$$\begin{aligned}\dot{x}_1 &= -ax_1 + ax_2 \\ \dot{x}_2 &= (c-a)x_1 + cx_2 - x_1x_3 + u \\ \dot{x}_3 &= x_1x_2 - bx_3\end{aligned}$$

with $a = 35$, $b = 3$ and $c = 28$.

The region where a TS fuzzy model of two rules represents exactly the dynamics of the Chen attractor is $x_1 \in [-d, d]$ with the same membership functions in the Lorenz case. The matrices defining the local subsystems for this example are

$$A_1 = S_1 = \begin{pmatrix} -a & a & 0 \\ c-a & c & d \\ 0 & -d & b \end{pmatrix},\ A_2 = S_2 = \begin{pmatrix} -a & a & 0 \\ c-a & c & -d \\ 0 & d & b \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \text{ and}$$

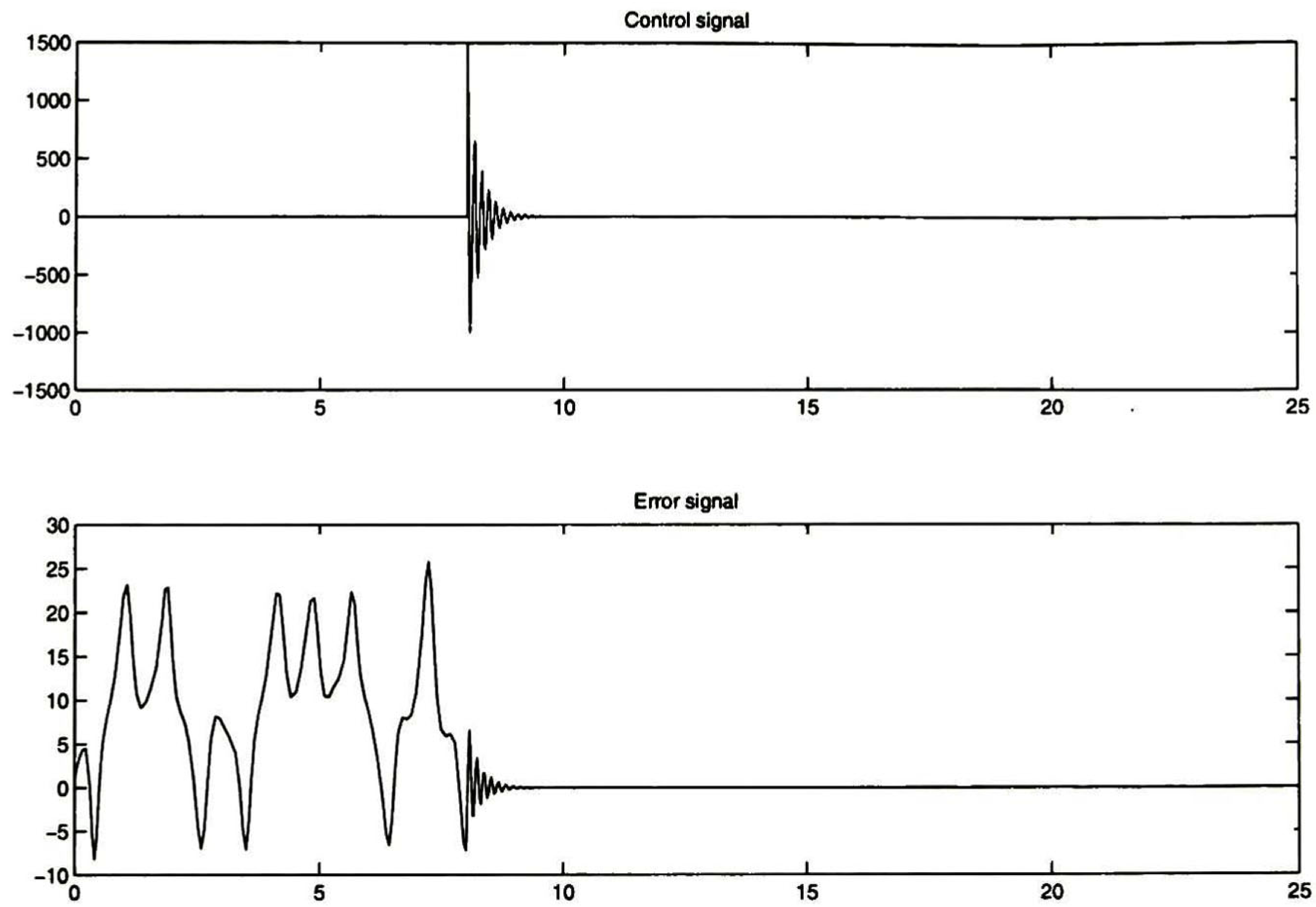

Figure 6.5: Control signal and tracking error for the Complete Synchronization of Lorenz systems.

$$C = Q = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

while the matrices solving the local regulator problem and therefore achieving the CS for Chen systems are:

$$\begin{aligned} \Pi_1 = \Pi_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \Gamma_1 = \Gamma_2 &= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \\ K_1 &= \begin{pmatrix} -1002.4 & -13.979 & 90.705 \end{pmatrix} \text{ and} \\ K_2 &= \begin{pmatrix} -1002.4 & -13.979 & -90.705 \end{pmatrix}. \end{aligned}$$

The simulation results under the same conditions considered for the Lorenz synchronization are given in Figure 6.7, Figure 6.8 and Figure 6.9.

So far, the synchronization problems presented satisfy conditions of Case 1 analyzed in Section 3.3. For that reason, the CS can be easily reached by means of fuzzy regulators designed on local controllers. However, an even more interesting situation arises when the synchronization of two

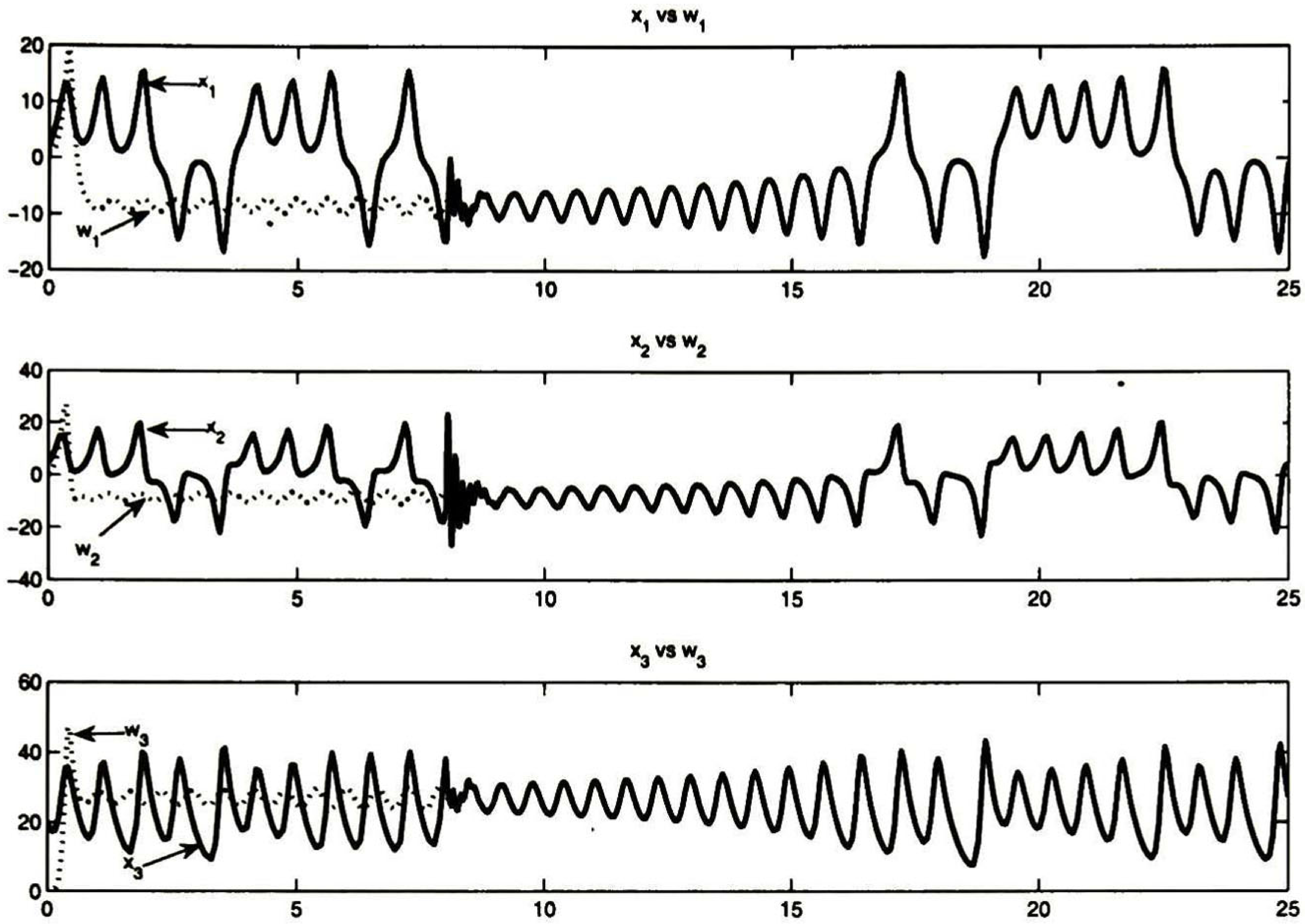

Figure 6.6: Driver states versus response states for the Complete Synchronization of Lorenz systems.

different chaotic systems is required. In the following section an analysis of this problem is carried out using fuzzy control.

6.3 Generalized synchronization

In the previous section, it has been reviewed the synchronization phenomena between identical chaotic oscillators. In general, when there exists an essential difference between the coupled systems, there is no hope to have a trivial manifold in the phase space attracting the system trajectories, and hence it is not clear at a first glance if nonidentical chaotic systems can synchronize. Two central issues are the most studied and interesting aspect of the subject. The first is that one should generalize the concept of synchronization to include non-identity manifold between the coupled systems. The second is that one should design some tests to detect it. Many works have shown that this type of chaotic synchronization can exist and have called it Generalized Synchronization (GS) [2, 33]. In most cases, evidence of it has been provided for unidirectional coupling schemes.

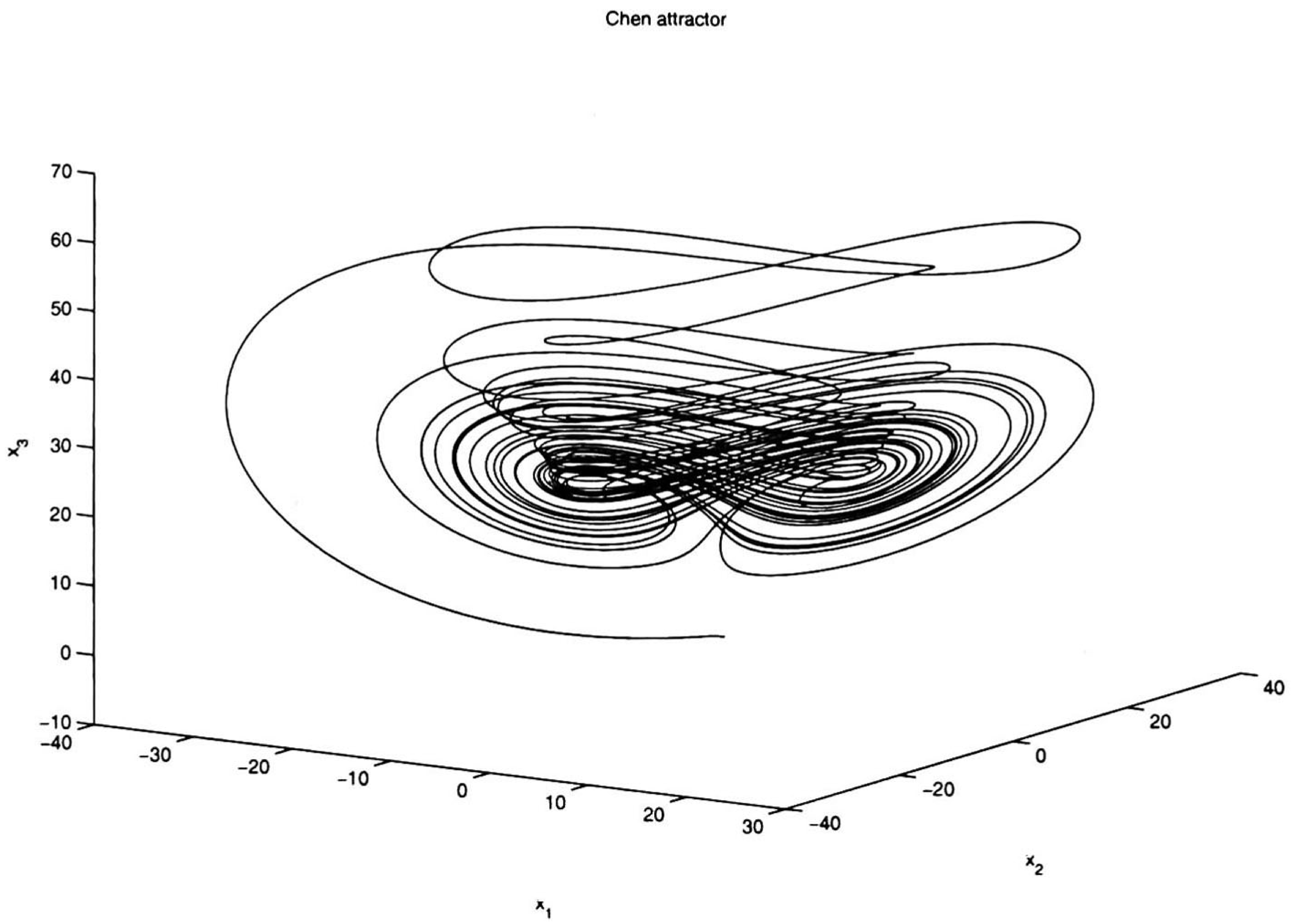

Figure 6.7: Chen attractor.

Taking the following chaotic systems

$$\begin{aligned} \dot{w} &= s(w) \text{ as the } \textit{driver} \text{ system and} \\ \dot{x} &= f(x, \phi(w), u) \text{ as } \textit{response} \text{ system,} \end{aligned} \tag{6.2}$$

where w is the r-dimensional state vector of the driver, x is the n-dimensional state vector of the response; and with s and f as vector fields satisfying $s : R^r \to R^r$ and $f : R^n \to R^n$, respectively; the chaotic trajectories of the two systems are said to be synchronized in a generalized sense if there exists a transformation $\pi : w \to x$ which is able to map asymptotically the trajectories of the driver attractor into the ones of the response attractor $x(t) = \pi(w(t))$, regardless on the initial conditions in the basin of the synchronization manifold $M = \{(w; x) : x = \pi(w)\}$.

Kocarev and Parlitz [23] formulated the necessary and sufficient conditions for the occurrence of GS. As in the case of CS, the notion of GS is equivalent to

$$\lim_{t\to\infty} \|x(t; x_1(0); w(0)) - x(t; x_2(0); w(0))\| = 0,$$

where $(x_1(0), w(0))$ and $(x_2(0), w(0))$ are two generic initial conditions of system (6.2) in the basin

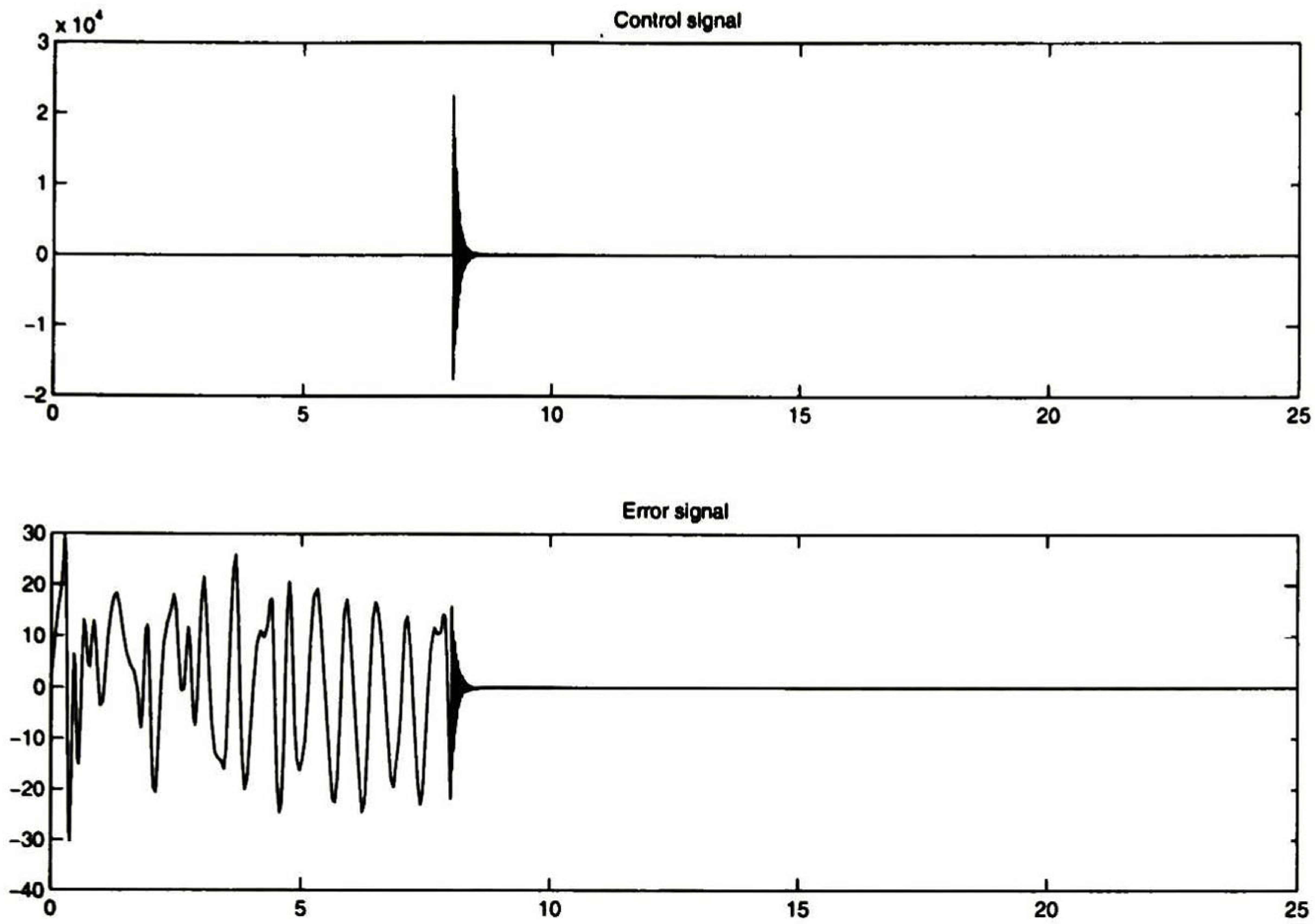

Figure 6.8: Control signal and tracking error for the Complete Synchronization of Chen systems.

of the synchronization attractor. In other words, a map $x = \pi(w)$ exists whenever the action of the driver is responsible for causing that the response system forgets its initial conditions [1, 32].

In literature, one can find many works oriented to analyze the relationship between the states of different systems. For instance, Pecora, Carroll and Heagy presented a complete discussion on how to characterize a functional relationship between two dynamical variables whose temporal behavior is obtained from time series in an experiment [30]. In this work, suitable statistical parameters are introduced and applied to test the mathematical properties of the map, i.e., to test whether or not the mapping is continuous, injective, differentiable and with a continuous inverse.

Another method to describe dynamical interdependence among nonlinear systems based on mutual nonlinear prediction is given in [34]. This method provides information on the directionality of the coupling, that is why it can be used to detect GS between dynamical variables. This technique was applied to detect GS in a neuronal ensemble.

On the other hand, an experimental approach to detect GS in an experiment is described in [1] and [32]. In these works, the authors propose to construct an auxiliary system (a system identical with the response), driven by the same driving system, namely

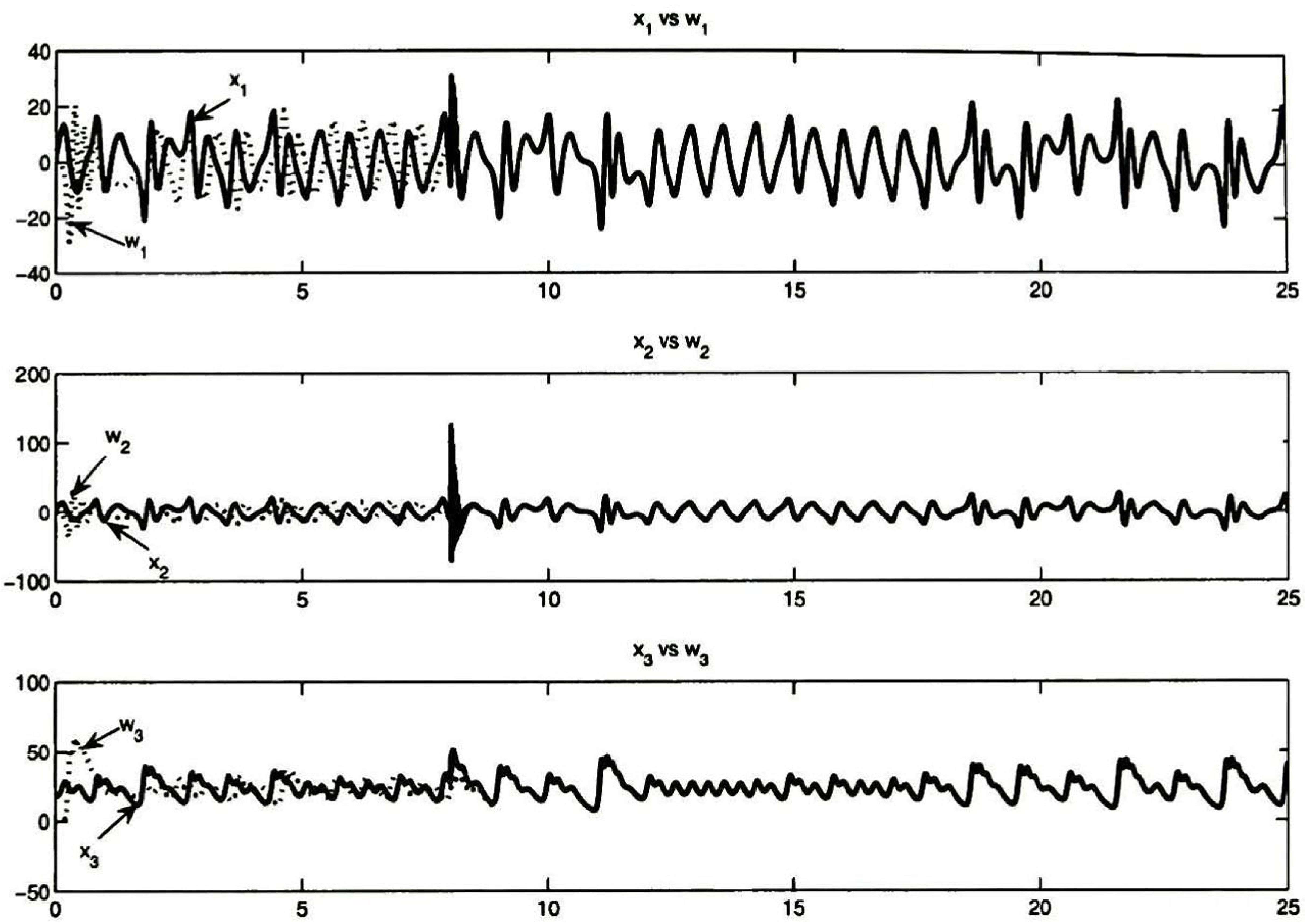

Figure 6.9: Driver states versus response states for the Complete Synchronization of Chen systems.

$$\dot{x}' = f(x', \phi(w), u'). \tag{6.3}$$

Then, $x(t)$ is said to be synchronized with $w(t)$ in a generalized way if $lim_{t\to\infty}\|x(t)-x'(t)\| = 0$. This equals to state that GS between $w(t)$ and $x(t)$ occurs if CS takes place between $x(t)$ and $x'(t)$. The main advantage of this criterion is the easy detection of CS between $x(t)$ and $x'(t)$. On the other hand, the possibility of building an identical copy of an experimental device is typically a very difficult task.

Based upon the concepts of latter approach, it will be shown how the fuzzy regulator designed on local controllers synchronizes chaotic systems in a general way.

In this example, it is considered that the Driver system is the Lorenz attractor and the Response system is the Chen attractor, both described by a TS fuzzy model of two rules:

IF $x_1(t)$ is M_1

$$THEN \begin{cases} \dot{x}(t) = & A_1x(t) + Bu(t) \\ \dot{w}(t) = & S_1w(t) \\ e(t) = & Cx(t) - Qw(t) \end{cases}$$

Rule 2

IF $x_1(t)$ is M_2

$$THEN \begin{cases} \dot{x}(t) = & A_2 x(t) + Bu(t) \\ \dot{w}(t) = & S_2 w(t) \\ e(t) = & Cx(t) - Qw(t), \end{cases}$$

where

$$A_1 = \begin{pmatrix} -a & a & 0 \\ c-a & c & d \\ 0 & -d & b \end{pmatrix}, A_2 = \begin{pmatrix} -a & a & 0 \\ c-a & c & -d \\ 0 & d & b \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} -a_w & a_w & 0 \\ c_w & 1 & d \\ 0 & -d & b_w \end{pmatrix},$$

$$S_2 = \begin{pmatrix} -a_w & a_w & 0 \\ c_w & 1 & -d \\ 0 & d & b_w \end{pmatrix} \text{ and } C = Q = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

with $a = 35$, $b = 3$, $c = 28$, $a_w = 10$, $b_w = \frac{8}{3}$, $c_w = 28$ and $d = 30$. The membership functions for this case are the same presented in the previous section.

The aggregate controller is constructed from the following matrices:

$$\begin{aligned} \Pi_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0.7143 & 0.2857 & 0 \\ 3.0041 & -0.0143 & 1.2861 \end{pmatrix}, \\ \Pi_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0.7143 & 0.2857 & 0 \\ -3.0041 & 0.0143 & 1.2861 \end{pmatrix} \\ \Gamma_1 &= \begin{pmatrix} -102.2648 & -0.7142 & -30.0121 \end{pmatrix}, \\ \Gamma_2 &= \begin{pmatrix} -102.2648 & -0.7142 & 30.0121 \end{pmatrix}, \\ K_1 &= \begin{pmatrix} -1055 & -14.694 & 99.768 \end{pmatrix} \text{ and} \\ K_2 &= \begin{pmatrix} -1055 & -14.694 & -99.768 \end{pmatrix} \end{aligned}$$

As it can be easily seen $\Pi_1 \neq \Pi_2$, as a result, and according to the analysis given in Chapter 3 the exact regulation cannot be achieved. This fact is depicted in Figure 6.10 and Figure 6.11.

Still, this kind of controller achieves GS because the fuzzy regulator always takes the states of the response system to the approximated manifold $\hat{\pi}(w)$ in spite of the initial conditions (See Chapter 3).

This fact will be clarified in Section 6.4 where a simple communication example is used to verify the GS through the fuzzy regulators.

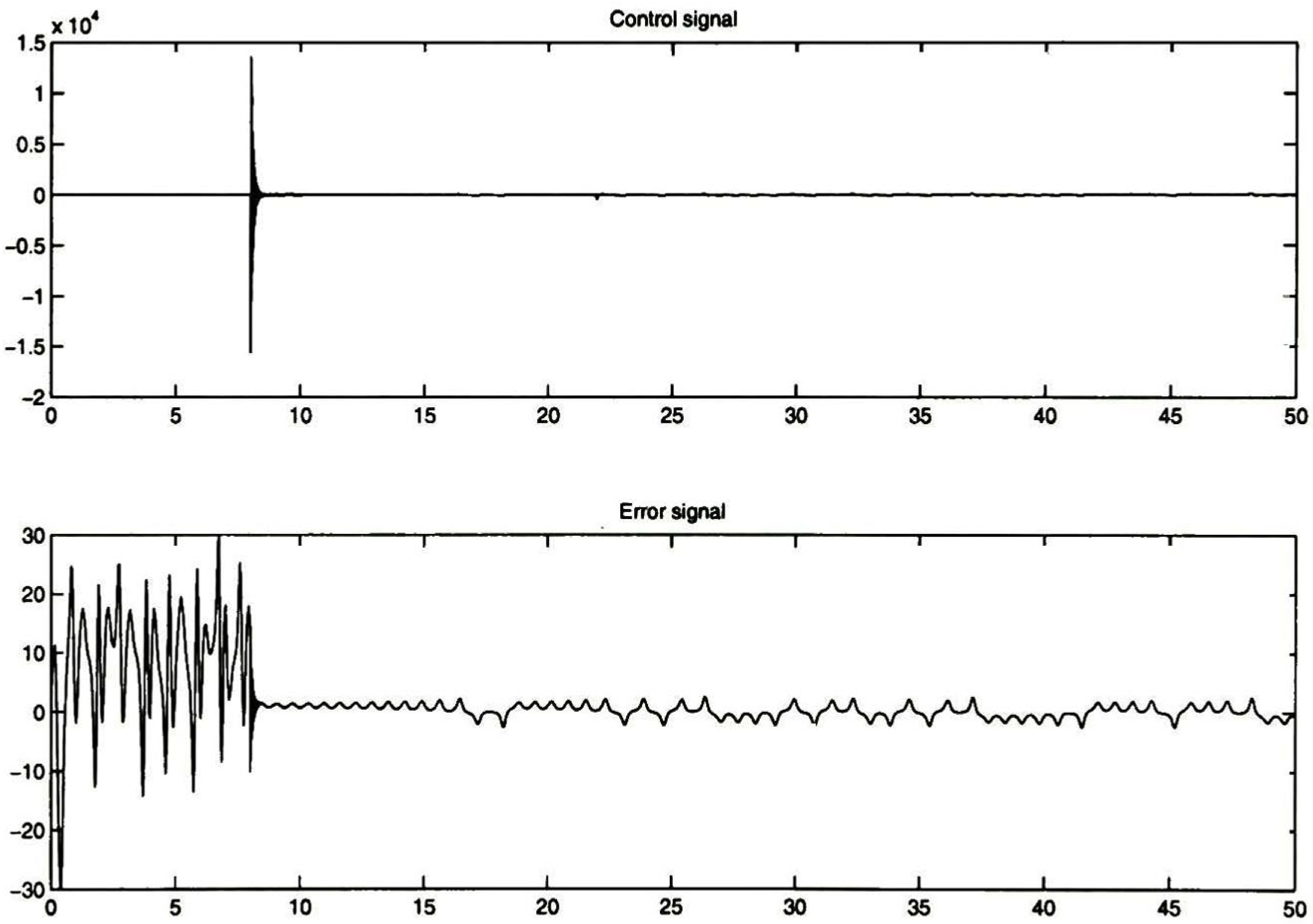

Figure 6.10: Control signal and tracking error for the Generalized Synchronization of Lorenz-Chen systems.

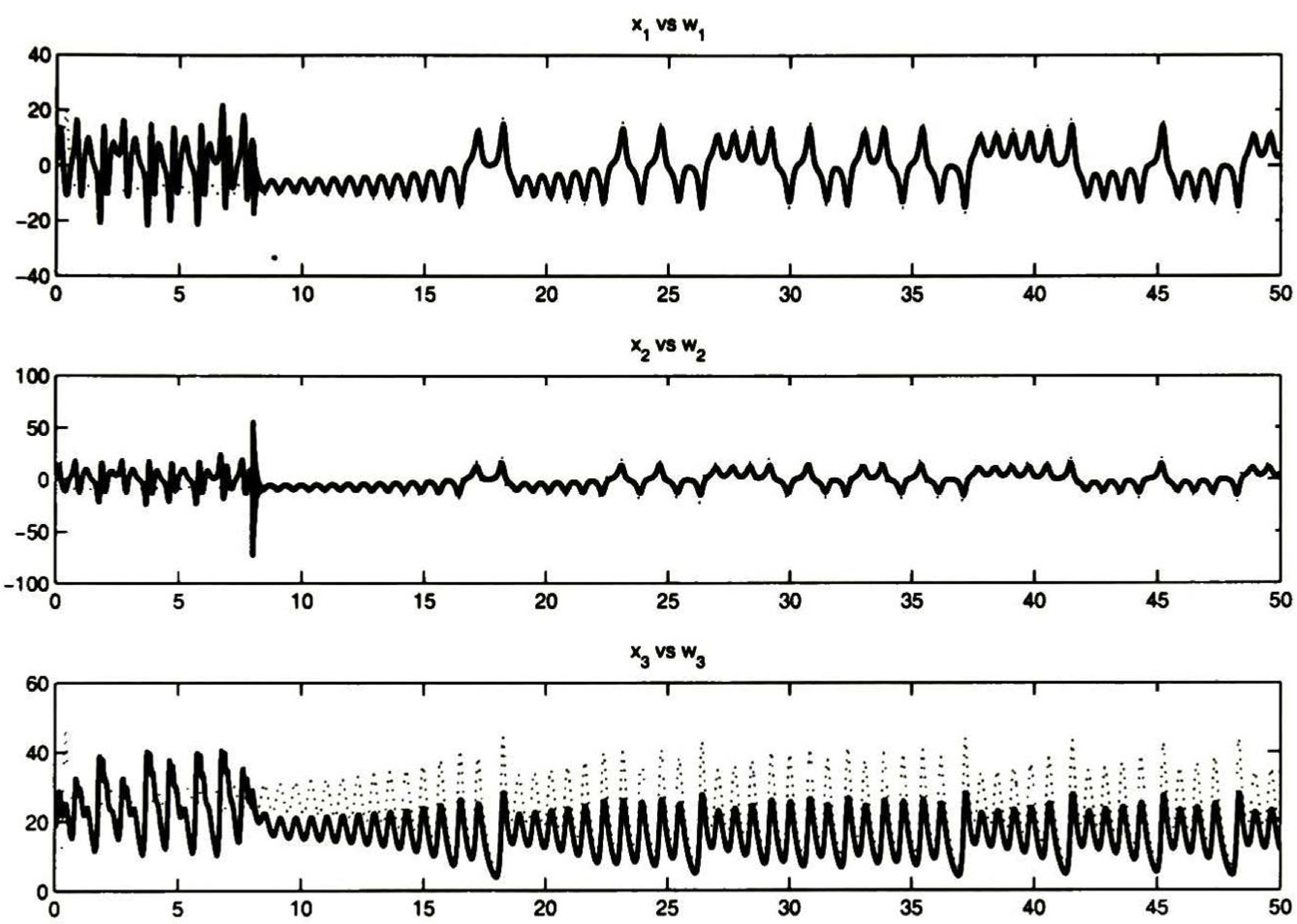

Figure 6.11: Driver states versus response states for the Generalized Synchronization of Lorenz-Chen systems.

6.4 Using fuzzy regulation in a transmission scheme

In this section, a simple way to apply the results obtained on fuzzy regulation in order to transmit signals ciphered by chaos is presented.

The source-receiver approach for this example is given in Figure 6.12 and Figure 6.13. In words, the fuzzy controller is used to synchronize the source's response system with the source's driver system. When the synchronization is achieved, the message signal is added to one state of the regulated system. Then, the resulting signal is transmitted to the receiver.

In the receiver there is a response system which is controlled by another fuzzy regulator. Both, the response system and the controller are exact replicas of those considered in the source. Thus, it is sufficient to activate the controller in the receiver and subtract the same state considered for mounting the message from the received signal in order to decode the ciphered signal.

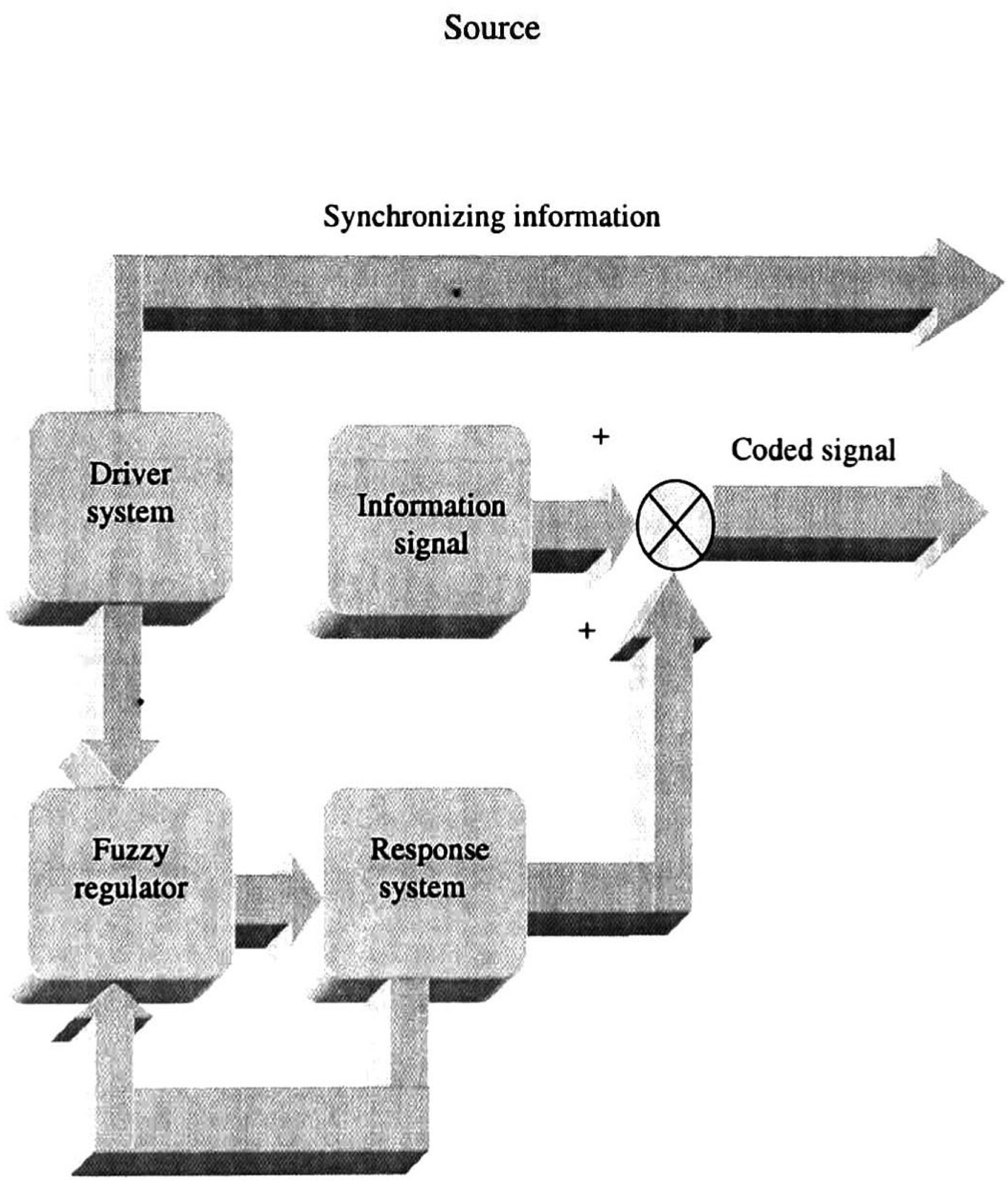

Figure 6.12: Source scheme.

For this application, it is considered that the driver system is a Lorenz attractor and the response system are Chen attractors. Therefore, the controller is designed based on the following TS

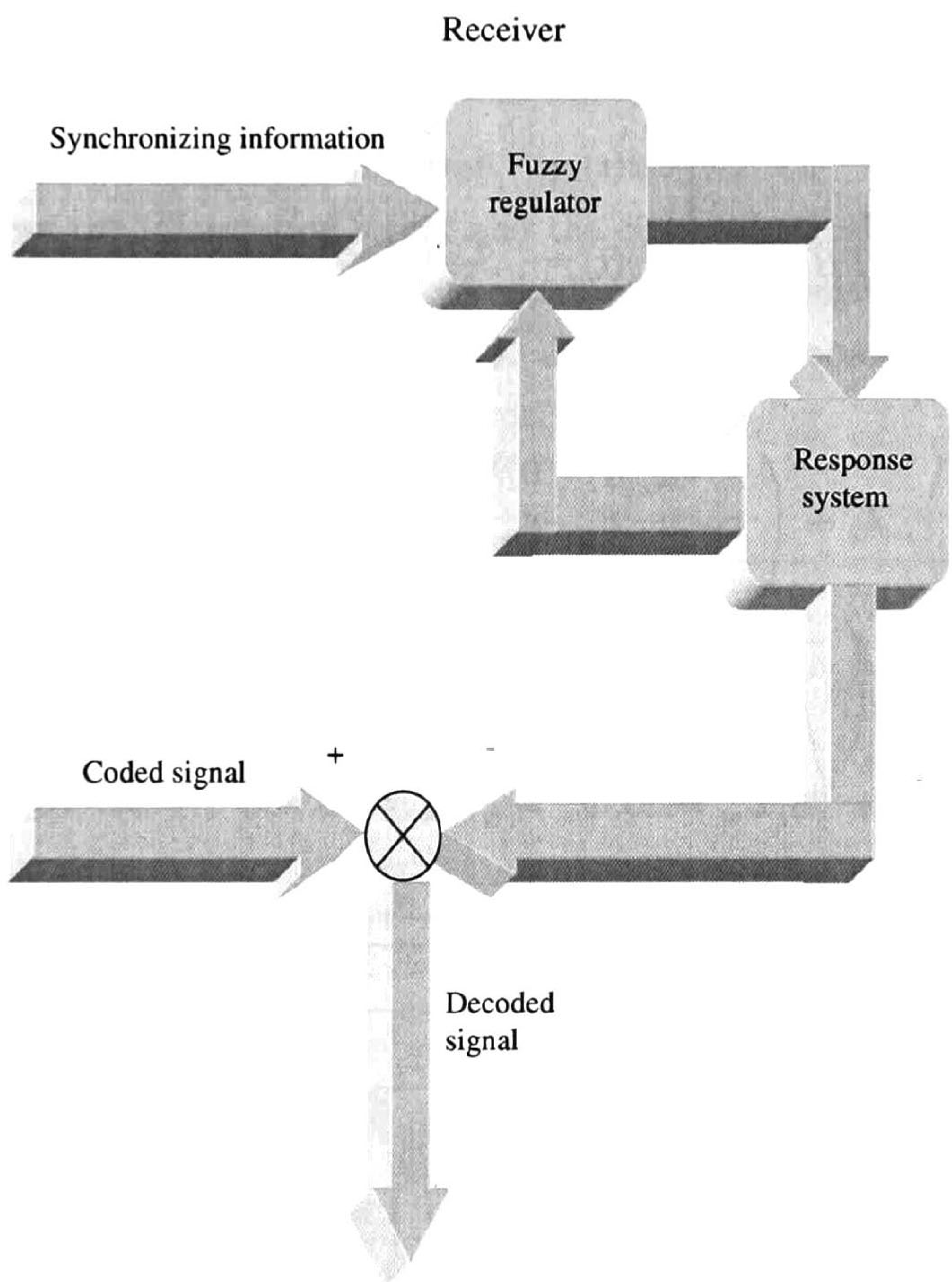

Figure 6.13: Receiver scheme.

fuzzy model which appears in the previous section:

IF $x_1(t)$ is M_1

$$THEN \begin{cases} \dot{x}(t) = & A_1x(t) + Bu(t) \\ \dot{w}(t) = & S_1w(t) \\ e(t) = & Cx(t) - Qw(t) \end{cases}$$

Rule 2

IF $x_1(t)$ is M_2

$$THEN \begin{cases} \dot{x}(t) = & A_2x(t) + Bu(t) \\ \dot{w}(t) = & S_2w(t) \\ e(t) = & Cx(t) - Qw(t), \end{cases}$$

where

$$A_1 = \begin{pmatrix} -a & a & 0 \\ c-a & c & d \\ 0 & -d & b \end{pmatrix}, A_2 = \begin{pmatrix} -a & a & 0 \\ c-a & c & -d \\ 0 & d & b \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} -a_w & a_w & 0 \\ c_w & 1 & d \\ 0 & -d & b_w \end{pmatrix},$$

$$S_2 = \begin{pmatrix} -a_w & a_w & 0 \\ c_w & 1 & -d \\ 0 & d & b_w \end{pmatrix} \text{ and } C = Q = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

With $a = 35$, $b = 3$, $c = 28$, $a_w = 10$, $b_w = \frac{8}{3}$, $c_w = 28$ and $d = 30$. Again, the membership functions for this case are

$$h_1(x_1) = M_1(x_1) := \frac{1}{2}\left(\frac{-x_1 + d}{d}\right) \text{ and}$$

$$h_2(x_1) = M_2(x_1) = \frac{1}{2}\left(\frac{x_1 + d}{d}\right),$$

and the controller is defined by

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.7143 & 0.2857 & 0 \\ 3.0041 & -0.0143 & 1.2861 \end{pmatrix},$$

$$\Pi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0.7143 & 0.2857 & 0 \\ -3.0041 & 0.0143 & 1.2861 \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} -102.2648 & -0.7142 & -30.0121 \end{pmatrix},$$

$$\Gamma_2 = \begin{pmatrix} -102.2648 & -0.7142 & 30.0121 \end{pmatrix},$$

$$K_1 = \begin{pmatrix} -1055 & -14.694 & 99.768 \end{pmatrix} \text{ and}$$

$$K_2 = \begin{pmatrix} -1055 & -14.694 & -99.768 \end{pmatrix}$$

The message signal used to test the performance of the transmission system is the state x_1 of a Rössler system defined by

$$\begin{aligned}
\dot{x}_1 &= -(x_2 + x_3) \\
\dot{x}_2 &= x_1 + a x_2 \\
\dot{x}_3 &= b x_1 - (c - x_1) x_3
\end{aligned}$$

with $a = 0.34$, $b = 0.4$ and $c = 4$.

Figure 6.14 shows the information signal and the ciphered signal which is transmitted from the source to the receiver. In Figure 6.15, the original information signal and the signal generated through the synchronization process are depicted. The decode procedure is started at $t = 10s$. Finally, in Figure 6.16 appears the deciphering error.

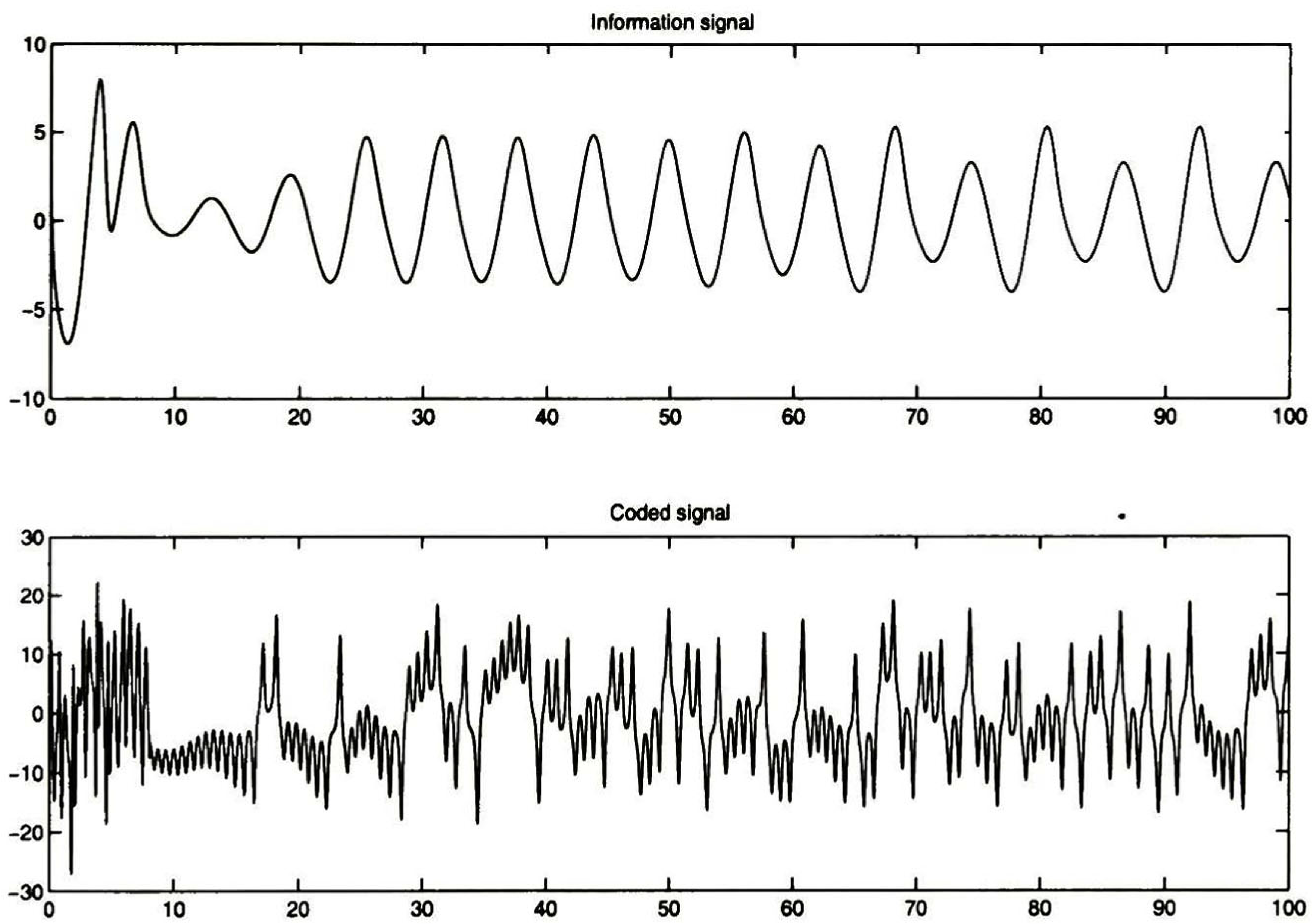

Figure 6.14: Information signal and transmitted signal.

As it can be seen, the response systems forget their initial conditions because of the control signal. For that reason, the proposed approach allows the original message to be reconstructed from the received chaotic signal. Although this communication scheme needs to be refined, it clearly

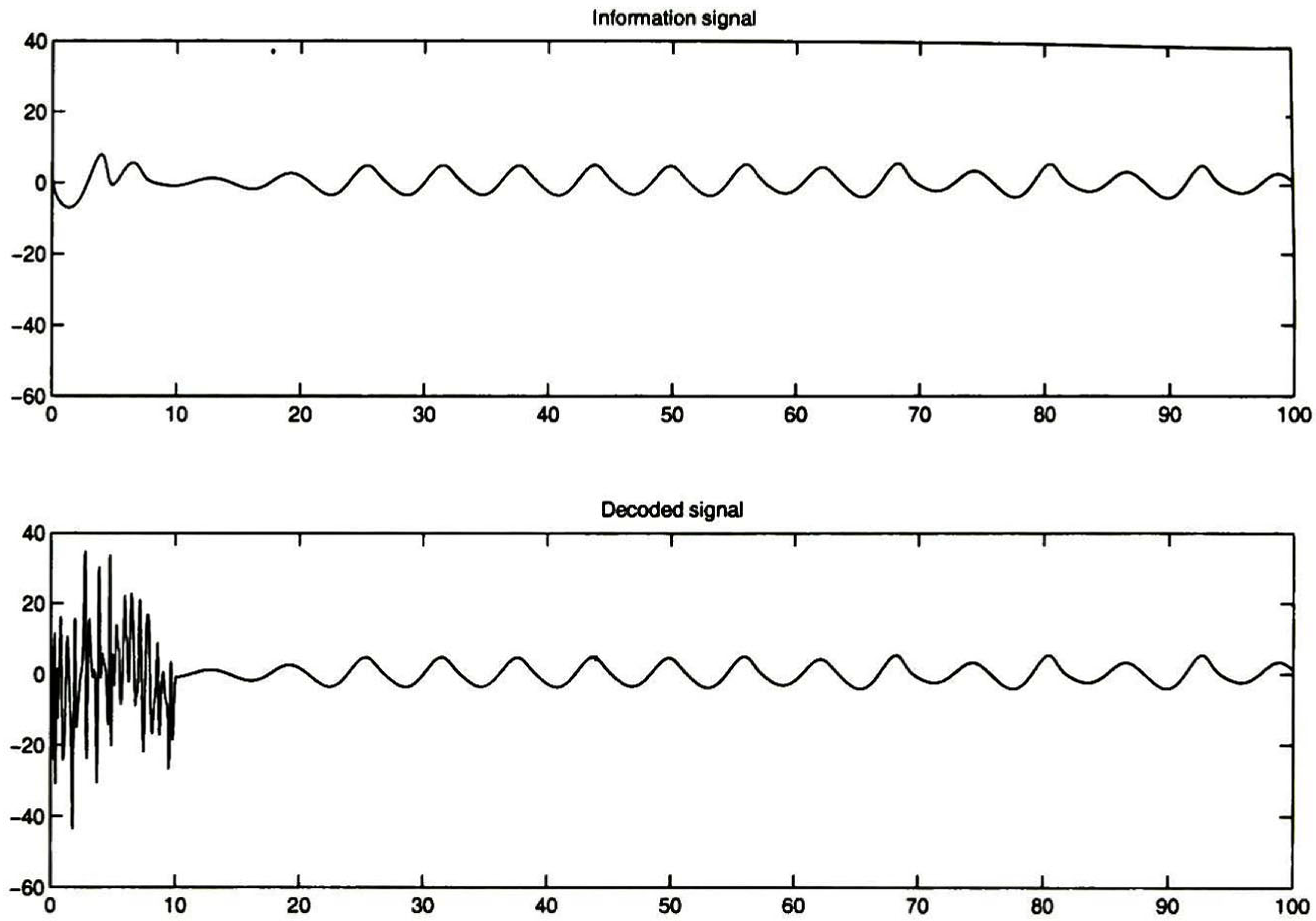

Figure 6.15: Information signal and decoded signal.

represents a possible real-life application for the fuzzy regulator. In this sense, some advantages and drawbacks of the method are given below.

Advantages

- The driver system and the response system can be different chaotic plants.
- Because the tracking error between the driver system and the response system does not need to be zeroed, the fuzzy controller can be computed entirely from linear controllers.

Drawbacks

- The complete state of the driver system must be sent to the receiver.

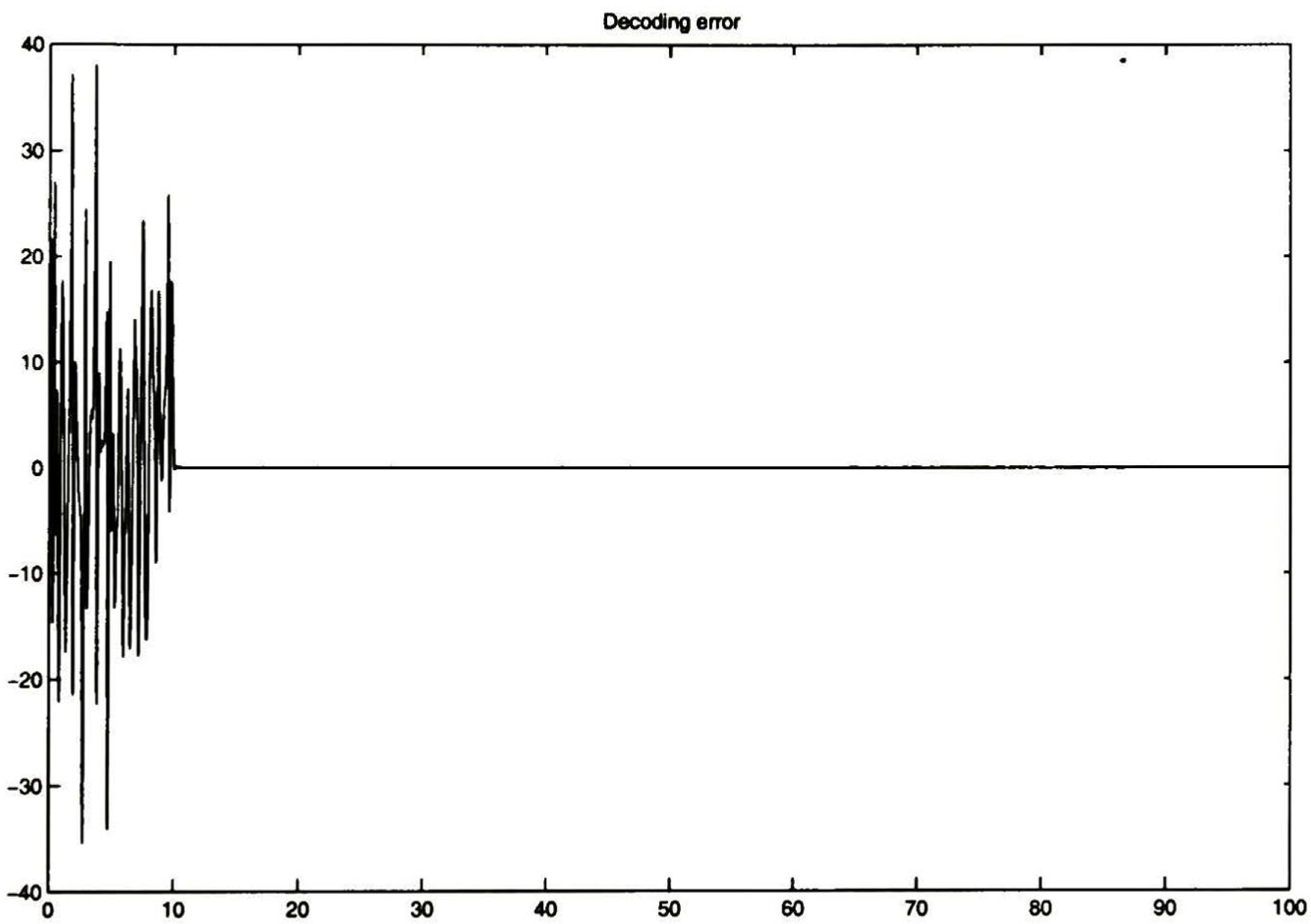

Figure 6.16: Decoding error.

Chapter 7

Conclusions and future work

"And in the end,
it's not the years
in your life that count.
It's the life in your years."
— Abraham Lincoln

7.1 Final remarks

Along this dissertation, the regulation problem when TS fuzzy models are available to represent both the plant and the exosystem has been reviewed.

After devoting Chapter 2 to recall the basic techniques which are the basis for this study, namely regulation theory, Takagi-Sugeno fuzzy modelling and the Parallel Distributed Compensation approach; in Chapter 3, the regulator designing based on simple linear controllers is analyzed, showing that this kind of control guarantees the exact tracking only if very specific conditions are satisfied. Nonetheless, it was also shown there that this scheme guarantees a bounded error when more relaxed conditions are fulfilled.

Then, in Chapter 4, the use of linear robust regulator proves to be a good alternative to reduce the steady-state error, although the exact tracking can not be guaranteed either.

However, even though these linear-based approaches do not solve the fuzzy regulation problem in general, the simplicity involved in their computing motivates a deeper analysis leading to the inclusion of a discontinuous term which is presented in Chapter 5. It was proven that under certain conditions this technique compensate the remaining tracking error when the fuzzy controller is obtained from simple linear regulators.

Another important consideration, directly affecting the application viability of this approach, is the use of numerical techniques in order to build the controller. Thus, by expressing the existence conditions as LMIs [5], computational tools can be employed to solve the problem in polynomial time, allowing the fuzzy regulator with sliding modes to be designed in a practical way.

The examples included in every chapter were intended to show how numerical tools, as MATLAB, facilitate the design process when the dynamics of the plant and the exosystem can be approximated by a Takagi-Sugeno fuzzy model.

It was also shown in Chapter 6 that the fuzzy regulator can be applied in order to synchronize chaotic systems. Particularly, Complete Synchronization can be achieved by means of the fuzzy regulator designed entirely on linear controllers when the driver and the response systems are described by the same dynamical equations.

Concerning to General Synchronization, the linear approach is unable to guarantee the exact tracking of the driver signal. . Nevertheless, this approach drives the output signal to an invariant manifold, not necessarily the desired one, causing that the response system forgets its initial conditions. This fact is of great importance in the communication scheme proposed also in Chapter 6.

On the other hand, it becomes obvious that the time and computational resources needed to solve the numerical problem depend on the number of linear matrix inequalities to be satisfied.

Hence, very complex TS fuzzy models could be less susceptible to be treated in this way.

But, considering the studies oriented to the development of new numerical techniques and the accelerated improvement in the capabilities of electronic processing, one supposes that in a near future this drawback could be less important, which leads to conclude that in some cases the approach discussed in this work can be taken into account as a valid choice to solve the regulation problem for TS fuzzy models

7.2 Future work

Of course, the study of the fuzzy regulation problem is very far from be exhausted, and it offers a wide open space for new research works which might include:

- Validation of the fuzzy regulator with sliding modes by applying it on real systems.
- Inclusion of the immersion of $\pi(w(t))$ in the fuzzy controller.
- Application of adaptive techniques in order to obtain the exact mappings $\pi(w(t))$ and $\gamma(w(t))$.
- Continue with the application to chaotic systems.
- Discretization of the continuous-time TS fuzzy model in order to design a discrete-time fuzzy controller capable to guarantee the regulation for the continuous-time plant.

Appendix A

An application to time-delay fuzzy systems

Nonlinear dynamics and time-delays are usually present in real-life processes which must be controlled and in many cases the outputs of these systems need to track reference signals. Hence, it is not surprising the existence of works addressing the nonlinear output regulation problem for nonlinear time-delay systems [17]. However, the solution proposed by the authors is difficult and in many cases impossible to obtain.

As referred earlier, in [44] an approach to construct the output regulator when the plant and the exosystem are described by a Takagi-Sugeno fuzzy model is presented. In that work the authors proposed a control signal, based in both state feedback and error feedback, designed on the local subsystems. However, as explained in Section 3.3, these techniques do not solve the regulation problem in general because the nonlinear interpolation among fuzzy rules is not considered [12, 24].

On the other hand, in section 4.2 it was proposed an approach to construct a fuzzy regulator on the basis of linear robust regulators in order to track references in a very efficient way. The relative advantages of this technique is that the design process involves linear techniques exclusively and its behavior is better than the performance shown by the fuzzy regulator formulated from simple linear controllers. Therefore, the fuzzy regulator for a nonlinear time-delay system will be obtained by applying such a method, i.e., combining both the linear robust regulation theory and the TS fuzzy modelling.

Considering the nonlinear time-delay system

$$\dot{x}(t) = f(x(t), x(t-\tau), w(t), u(t)) \tag{A.1}$$

$$\dot{w}(t) = s(w(t)) \tag{A.2}$$

$$e(t) = h(x(t), w(t)) \tag{A.3}$$

$$x(t_0 + d) = \varphi(d) \tag{A.4}$$

where $u(t) \in \mathbb{R}^m$ is the input signal, $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^s$ is the state vector of the exosystem and $\tau \in \mathbb{R}$ is a positive time delay. The tracking error $e(t) \in \mathbb{R}^m$, given by Equation (A.3), is the difference between the system output and the reference signal and again, it is assumed that $f(\cdot, \cdot, \mu)$, $s(\cdot)$ and $h(\cdot, \cdot)$ are analytical functions, where $s(0) = 0$, $f(0,0,0,0) = 0$ and $h(0,0) = 0$ with $0 = t_0 < \tau$ and the initial condition $\varphi(d)$ is a continuous function defined along the interval $[-\tau \;\; 0]$.

The TS fuzzy approximation for system (A.1)–(A.3) is given by the following set of conditional statements [38, 43]:

Rule i
IF $z_1(t)$ is M_1^i and $z_2(t)$ is M_2^i and ... and $z_p(t)$ is M_p^i
THEN

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + A_i^\tau x(t-\tau) + B_i u(t) + P_i w(t) \\ \dot{w}(t) &= S_i w(t) \\ e(t) &= C_i x(t) - Q_i(t) w(t), \end{aligned}$$

for all $i = 1, \ldots, r$, where r is the number of rules in the model, $z_i(t)$ are the measurable output signals, and M_j^i are fuzzy sets.

Consequently, the aggregate fuzzy model is:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + A_i^\tau x(t-\tau) + B_i u(t) + P_i w(t)] \tag{A.5}$$

$$\dot{w}(t) = \sum_{i=1}^{r} h_i(z(t)) S_i w(t) \tag{A.6}$$

$$e(t) = \sum_{i=1}^{r} h_i(z(t))[C_i x(t) - Q_i w(t)]. \tag{A.7}$$

As before, h_i is the normalized weight for each rule. These weights depends on the membership function for $z(t)$ in M_j^i, namely

$$h_i(z(t)) = \frac{\varpi_i(z(t))}{\sum_{J=1}^{r} \varpi_j(z(t))} \tag{A.8}$$

$$\varpi_i(z(t)) = \prod_{j=1}^{p} M_j^i(z(t)) \tag{A.9}$$

$$\sum_{i=1}^{r} h_i(z(t)) = 1 \tag{A.10}$$

$$h_i(z(t)) \geq 0 \tag{A.11}$$

with $z(t) = \left[\ z_1(t) \quad z_2(t) \quad \ldots \quad z_p(t)\ \right]$ and $i = 1, \ldots, r$.

Thus, for the TS fuzzy model (A.5)–(A.7), the Fuzzy Time-Delay Regulator Problem (FTDRP) is reformulated as the problem of finding, if possible, a dynamical controller of the form

$$\dot{\xi}(t) = F(\xi(t), \xi(t-\tau), z(t), e(t)) \tag{A.12}$$

$$u(t) = \mathcal{H}(\xi(t)) \tag{A.13}$$

such that:

$\mathbf{S_{td}}$) the equilibrium point $(x(t), x(t-\tau), \xi(t), \xi(t-\tau)) = (0, 0, 0, 0)$ of the closed-loop system with no external signals

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + A_i^{\tau} x(t-\tau) + B_i \mathcal{H}(\xi(t))] \\
\dot{\xi}(t) &= F(\xi(t), \xi(t-\tau), z(t), 0)
\end{aligned}$$

is asymptotically stable (stability condition),

$\mathbf{R_{td}}$) the solution of the closed-loop system

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))[A_i x(t) + A_i^{\tau} x(t-\tau) + B_i \mathcal{H}(\xi(t)) + P_i w(t)] \\
\dot{w}(t) &= \sum_{i=1}^{r} h_i(z(t)) S_i w(t) \\
e(t) &= \sum_{i=1}^{r} h_i(z(t))[C_i x(t) - Q_i w(t)] \\
\dot{\xi}(t) &= F(\xi(t), \xi(t-\tau), z(t), e(t))
\end{aligned}$$

satisfies

$$\lim_{t \to \infty} e(t) = 0,$$

(regulation condition).

The typical method for constructing a fuzzy controller consists of designing r local controllers, such that the fulfillment of the control goals for their corresponding subsystem is achieved and at the same time the aggregate regulator ensures the control goals for the composite system. Similarly, a solution for the FTDRP is proposed by solving linear local robust regulation problems.

To this end, the PDC results presented in [42] are applied, that is, the design of linear time-delay robust regulators for each subsystem of the TS fuzzy model at the nominal values of the parameters is suggested, such that the global stability of the fuzzy model is also guaranteed [9]. In other words, regulator problems for the r subsystems

$$\begin{aligned}\dot{x}(t) &= A_i x(t) + A_i^{\tau} x(t-\tau) + B_i u(t) + P_i w_i(t)\\ \dot{w}(t) &= S_i w(t)\\ e(t) &= C_i x(t) - Q_i w(t) \qquad i = 1, \dots, r\end{aligned}$$

have to be solved. Then, the fuzzy controller will be composed by rules of the form:

Rule i

IF $z_1(t)$ is M_1^i and $z_2(t)$ is M_2^i and ... and $z_p(t)$ is M_p^i

THEN

$$\dot{\xi}(t) = F_i \xi(t) + F_i^{\tau} \xi(t-\tau) + G_i e(t)$$

$$u(t) = \mathcal{H}_i \xi(t)$$

for all $i = 1 \dots r$, with the total controller defined by

$$\dot{\xi}(t) = \sum_{i=1}^{r} h(z(t))[F_i \xi(t) + F_i^{\tau} \xi(t-\tau) + G_i e(t)] \tag{A.14}$$

$$u(t) = \sum_{i=1}^{r} h(z(t)) \mathcal{H}_i \xi(t), \tag{A.15}$$

where

$$F_i = \begin{pmatrix} A_i + B_i K - G_{0,i} C_i & 0 \\ -G_{1,i} C_i & \Phi_i \end{pmatrix}, F_i^{\tau} = \begin{pmatrix} A_i^{\tau} & 0 \\ 0 & 0 \end{pmatrix}, G_i = \begin{pmatrix} G_{0,i} \\ G_{1,i} \end{pmatrix}, \mathcal{H}_i = \begin{pmatrix} K_i & H \end{pmatrix},$$

$$\Phi_i = diag\{\Phi_{i1}, \dots, \Phi_{im}\},\ \Phi_{ij} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_{j,0}^i & -a_{j,1}^i & -a_{j,2}^i & \dots & -a_{j,s_j-1}^i \end{pmatrix},$$

$$H = diag\{H_1, \ldots, H_m\},\ H_j = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}_{1\times s_j}$$

and $S_i = diag\{S_{i1}, \ldots, S_{im}\}$ with $i = 1 \ldots r$, $j = 1 \ldots m$ and a^i_{j*} as the coefficients of the characteristic polynomial of S_{ij} [9, 18].

The following theorem gives conditions for the solution of the FTDRP.

Theorem 21 *If*

H1$_{\mathbf{td}}$) *every trajectory $w(t)$ defined by (A.6) and $w(0)$ is bounded,*

H2$_{\mathbf{td}}$) *for $i = 1, \ldots, r$ there exist K_i such that $\dot{x}(t) = A_i x(t) + A_i^\tau (x - \tau) + B_i K_i x(t)$ is stable,*

H3$_{\mathbf{td}}$) *for $i = 1, \ldots, r$ there exist G_i such that*

$$\dot{\xi}(t) = \begin{pmatrix} A_i & -B_i H \\ 0 & \Phi_i \end{pmatrix} \xi(t) + \begin{pmatrix} A_i^\tau & 0 \\ 0 & 0 \end{pmatrix} \xi(t-\tau) - G_i \begin{pmatrix} C_i & 0 \end{pmatrix} \xi(t)$$

is stable,

H4$_{\mathbf{td}}$) *there exist mappings $\tilde{\pi}(w(t)) = \sum_{i=1}^{r} \Pi_i w$ and $\tilde{\gamma}(w(t)) = \sum_{i=1}^{r} \Gamma_i w$, whose matrices Π_i and Γ_i solve*

$$\begin{aligned} \Pi_i S_i &= A_i \Pi_i + A_i^\tau \Pi_i e^{-\tau S_i} + B_i \Gamma_i + P_i \\ 0 &= C_i \Pi_i - Q_i \end{aligned}$$

for $i = 1, \ldots, r$,

H5$_{\mathbf{td}}$) *there exist triplets $(K_i, G_{0,i}, G_{1,i})$ and matrices $\mathbf{P}$ and W_i [7, 39, 45] satisfying*

$$0 > \bar{A}_{ii}^T \mathbf{P} + \mathbf{P}\bar{A}_{ii} + \mathbf{P} + \mathbf{P}\bar{A}_{ii}^\tau W_i \bar{A}_{ii}^{\tau T} \mathbf{P}$$

$$\begin{aligned} 0 &> (\bar{A}_{ij} + \bar{A}_{ji})^T \mathbf{P} + \mathbf{P}(\bar{A}_{ij} + \bar{A}_{ji}) + \mathbf{P} \\ &+ \mathbf{P}(\bar{A}_i^\tau + \bar{A}_j^\tau) W_i (\bar{A}_i^\tau + \bar{A}_j^\tau)^T \mathbf{P} \end{aligned}$$

$$\mathbf{P} > 0$$

$$W_i > 0$$

$$\mathbf{P} \geq W_i^{-1}$$

for all i and j such that $h_i(z(t)) \cdot h_j(z(t) \neq 0$, with

$$\bar{A}_{ij} = \begin{pmatrix} A_i & B_i\mathcal{H}_j \\ G_iC_j & F_i \end{pmatrix} \tag{A.16}$$

and

$$\bar{A}_i^\tau = \begin{pmatrix} A_{0,i}^\tau & 0 \\ 0 & F_i^\tau \end{pmatrix} \tag{A.17}$$

H6$_{\mathbf{td}}$) *there exist $\pi(w(t))$ and $\gamma(w(t))$ solving exactly*

$$\begin{aligned}
\frac{\partial\pi(w(t))}{\partial w(t)}\sum_{i=1}^{r} h_i(z(t))S_iw(t) &= \sum_{i=1}^{r} h_i(z(t))\left\{A_i\pi(w(t)) + A_i^\tau\pi(w(t-\tau)) + B_i\gamma(w(t))\right. \\
&+ \left. P_iw(t)\right\}.
\end{aligned}$$

$$0 = \sum_{i=1}^{r} h_i(z(t))\{C_i\pi(w(t)) - Q_iw(t)\},$$

H7$_{\mathbf{td}}$) *the term $\sum_{i=1}^{r} \dot{h}_i(z(t))$ is bounded,*

then, the tracking error for the Fuzzy Time-Delay Regulation Problem is bounded.

Proof. As in Section 4.2, if H1$_{\mathrm{td}}$, H2$_{\mathrm{td}}$, H3$_{\mathrm{td}}$ and H4$_{\mathrm{td}}$ hold, then the existence of the local robust regulators is ensured, hence only the stability and the behavior of tracking error for the overall fuzzy system will be verified.

Stability.- Considering the closed-loop system with $w(t) = 0$

$$\begin{aligned}
\begin{pmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{pmatrix} &= \sum_{i=1}^{r} h_i^2(z(t))\left[\bar{A}_{ii}\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} + \bar{A}_i^\tau\begin{pmatrix} x(t-\tau) \\ \xi(t-\tau) \end{pmatrix}\right] \\
&+ \sum_{i<j}^{r} h_i(z(t))h_j(z(t))\left[(\bar{A}_{ij} + \bar{A}_{ji})\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} + (\bar{A}_i^\tau + \bar{A}_j^\tau)\begin{pmatrix} x(t-\tau) \\ \xi(t-\tau) \end{pmatrix}\right]
\end{aligned}$$

with $\bar{A}_{ij}$ and $\bar{A}_i^\tau$ defined as above. By proceeding such as in [7], it is concluded that when H5$_{\mathrm{td}}$ is satisfied, the equilibrium point $(x(t), x(t-\tau), \xi(t), \xi(t-\tau)) = (0, 0, 0, 0)$ is asymptotically stable.

Regulation.- Follows directly from Theorem 15. ■

As it can be easily seen, the solution of the FTDRP depends on the a priori calculation of matrices K_i, $G_{0,i}$, $G_{1,i}$, and afterwards a search for matrices $\mathbf{P}$ and W_i ensuring the stability becomes necessary. If $\mathbf{P}$ and W_i are not found, new K_i, $G_{0,i}$, $G_{1,i}$ have to be computed and then again the search for matrices $\mathbf{P}$ and W_i must to be carried out. Thus, in the following analysis, numerical techniques are included in order to solve the FTDRP in a more efficient way. Using this approach both the calculation of K_i, $G_{0,i}$, $G_{1,i}$ and the search for $\mathbf{P}$ and W_i are performed at the same time.

When matrix (A.16) is expanded, one gets:

$$\bar{A}_{ij} = \begin{pmatrix} A_i & B_iK_j & B_iH_j \\ G_{0,i}C_j & A_i + B_iK_j - G_{0,i}C_j & 0 \\ G_{1,i}C_j & -G_{1,i}C_j & \Phi_i \end{pmatrix}, \tag{A.18}$$

which by means of transformation $T = \begin{pmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{pmatrix}$, is similar to $T\bar{A}_{ij}T^{-1}$, i.e.,

$$\begin{pmatrix} A_i + B_iK_j & B_iK_j & B_iH_j \\ 0 & A_i - G_{0,i}C_j & -B_iH_j \\ 0 & -G_{1,i}C_j & \Phi_i \end{pmatrix} \tag{A.19}$$

If the same transformation is applied on matrix (A.17), then

$$T\bar{A}_i^{\tau}T^{-1} = \begin{pmatrix} A_i^{\tau} & 0 & 0 \\ 0 & A_i^{\tau} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.20}$$

is obtained.

Therefore, the problem is to stabilize

$$\dot{x}(t) = (A_i + B_iK_j)x(t) + A_i^{\tau}x(t-\tau) \tag{A.21}$$

and

$$\dot{\xi}(t) \;=\; \left\{ \begin{pmatrix} A_i & -B_iH_j \\ 0 & \Phi_i \end{pmatrix} - \begin{pmatrix} G_{0,i} \\ G_{1,i} \end{pmatrix} \begin{pmatrix} C_i & 0 \end{pmatrix} \right\} \xi(t) + \begin{pmatrix} A_i^{\tau} & 0 \\ 0 & 0 \end{pmatrix} \xi(t-\tau). \tag{A.22}$$

To this end, suitable LMIs (Linear matrix Inequalities) are introduced to replace conditions $H2_{td}$, $H3_{td}$ and $H5_{td}$ [5]. In this sense, it can be proved that Equation (A.21) is stable if the following LMIs are feasible [7]

$$X_1A_i^T + A_iX_1 + B_iY_i + Y_i^TB_i^T + X_1 + A_i^{\tau}W_iA_i^{\tau T} \;<\; 0 \tag{A.23}$$

for all $i = 1, \ldots, r$,

$$\begin{aligned} &X_1\left(A_i^T + A_j^T\right) + \left(A_i + A_j\right)X_1 + B_iY_j + Y_j^TB_i^T \;+ \\ &B_jY_i + Y_i^TB_j^T + X_1 + (A_i^{\tau} + A_j^{\tau})W_i(A_i^{\tau} + A_j^{\tau})^T \;<\; 0 \end{aligned} \tag{A.24}$$

for $i < j \leq r$ such that $h_i(z(t)) \cdot h_j(z(t)) \neq 0$,

$$X_1 > 0 \tag{A.25}$$

and

$$W_i \geq X_1, \tag{A.26}$$

for $i = 1, \ldots, r$ with $\mathbf{P}_1 = X_1^{-1}$ and $K_i = Y_i \mathbf{P}_1$. On the other hand, to analyze the stability of Equation (A.22) it is considered

$$H = H_1 = \ldots H_r,\ A_{e,i} = \begin{pmatrix} A_i & -B_i H \\ 0 & \Phi_i \end{pmatrix},\ G_i = \begin{pmatrix} G_{0,i} \\ G_{1,i} \end{pmatrix},\ C_{e,i} = \begin{pmatrix} C_i & 0 \end{pmatrix} \text{ and } A^{\tau}_{e,i} = \begin{pmatrix} A^{\tau}_i & 0 \\ 0 & 0 \end{pmatrix}.$$

The procedure follows in a similar way to that presented previously, that is, Equation (A.22) is stable if the following LMIs are feasible

$$\begin{pmatrix} A^T_{e,i}\mathbf{P}_2 + \mathbf{P}_2 A_{e,i} - M_i C_{e,i} - C^T_{e,i} M^T_i + \mathbf{P}_2 & * \\ A^{\tau T}_{e,i}\mathbf{P}_2 & -Z_i \end{pmatrix} < 0 \tag{A.27}$$

$$\begin{pmatrix} \begin{matrix} \left(A^T_{e,i} + A^T_{e,j}\right)\mathbf{P}_2 + \mathbf{P}_2\left(A_{e,i} + A_{e,j}\right) - M_i C_{e,j} \\ -C^T_{e,j} M^T_i - M_j C_{e,i} - C^T_{e,i} M^T_j + \mathbf{P}_2 \end{matrix} & * \\ \left(A^{\tau T}_{e,i} + A^{\tau T}_{e,j}\right)\mathbf{P}_2 & -Z_i \end{pmatrix} \dot{<} 0 \tag{A.28}$$

$$\mathbf{P}_2 > 0 \tag{A.29}$$

$$\mathbf{P}_2 \leq Z_i \tag{A.30}$$

for all i and j such that $h_i(z(t)) \cdot h_j(z(t)) \neq 0$, with $G_i = \mathbf{P}_2^{-1} M_i$ and where $*$ represents blocks that can be easily inferred by symmetry.

Finally, the common matrix guaranteeing the stability of the system is given by

$$\mathbf{P} = T^{-1} diag(\mathbf{P}_1, \mathbf{P}_2) T,$$

with T defined as above. This analysis is summarized as follows:

Theorem 22 *If*

$\mathbf{H1_{td}}$) *every trajectory $w(t)$ defined by (A.6) and $w(0)$ is bounded,*

H2$_{\mathbf{td}}$) *there exist mappings* $\tilde{\pi}(w(t)) = \sum_{i=1}^{r} \Pi_i w$ *and* $\tilde{\gamma}(w(t)) = \sum_{i=1}^{r} \Gamma_i w$, *whose matrices* Π_i *and* Γ_i *solve*

$$
\begin{aligned}
\Pi_i S_i &= A_i \Pi_i + A_i^{\tau} \Pi_i e^{-\tau S_i} + B_i \Gamma_i + P_i \\
0 &= C_i \Pi_i - Q_i,
\end{aligned}
$$

H3$^*_{\mathbf{td}}$) *LMIs (A.23), (A.24), (A.25), (A.26), (A.27), (A.28), (A.29) and (A.30) are feasible,*

H4$_{\mathbf{td}}$) *there exist* $\pi(w(t))$ *and* $\gamma(w(t))$ *solving exactly*

$$
\begin{aligned}
\frac{\partial \pi(w(t))}{\partial w(t)} \sum_{i=1}^{r} h_i(z(t)) S_i w(t) &= \sum_{i=1}^{r} h_i(z(t)) \left\{ A_i \pi(w(t)) + A_i^{\tau} \pi(w(t-\tau)) + B_i \gamma(w(t)) \right. \\
&+ \left. P_i w(t) \right\}. \\
0 &= \sum_{i=1}^{r} h_i(z(t)) \{ C_i \pi(w(t)) - Q_i w(t) \},
\end{aligned}
$$

H5$_{\mathbf{td}}$) *the term* $\sum_{i=1}^{r} \dot{h}_i(z(t))$ *is bounded,*

then, the tracking error for Fuzzy Time-Delay Regulator Problem is bounded. Moreover, the solution is given by $K_i = Y_i \mathbf{P}_1$ *and* $G_i = \mathbf{P}_2^{-1} M_i$.

Proof. Follows directly from Theorem 21 and the previous discussion. ∎

A.1 Numeric simulation

To illustrate the latter analysis, the numeric design approach is applied on the TS fuzzy model defined by

Rule 1: *IF* $x_2(t)$ is about M_1 (small) *THEN*

$$
\sum\nolimits_1 : \begin{cases}
\dot{x}(t) = & A_1 x(t) + A^{\tau} x(t-\tau) + P w(t) + B u(t) \\
\dot{w}(t) = & S w(t) \\
e(t) = & C x(t) - Q w(t)
\end{cases}
$$

Rule 2: *IF* $x_2(t)$ is about M_2 (big) *THEN*

$$\sum_2 : \begin{cases} \dot{x}(t) = & A_2x(t) + A^{\tau}x(t-\tau) + Pw(t) + Bu(t) \\ \dot{w}(t) = & Sw(t) \\ e(t) = & Cx(t) - Qw(t) \end{cases}$$

with

$$A_1 = \begin{pmatrix} 3 & -2 \\ 2 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 3 & -3 \\ 1 & 0 \end{pmatrix}, A^{\tau} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \end{pmatrix} \text{ and } \tau = 1.$$

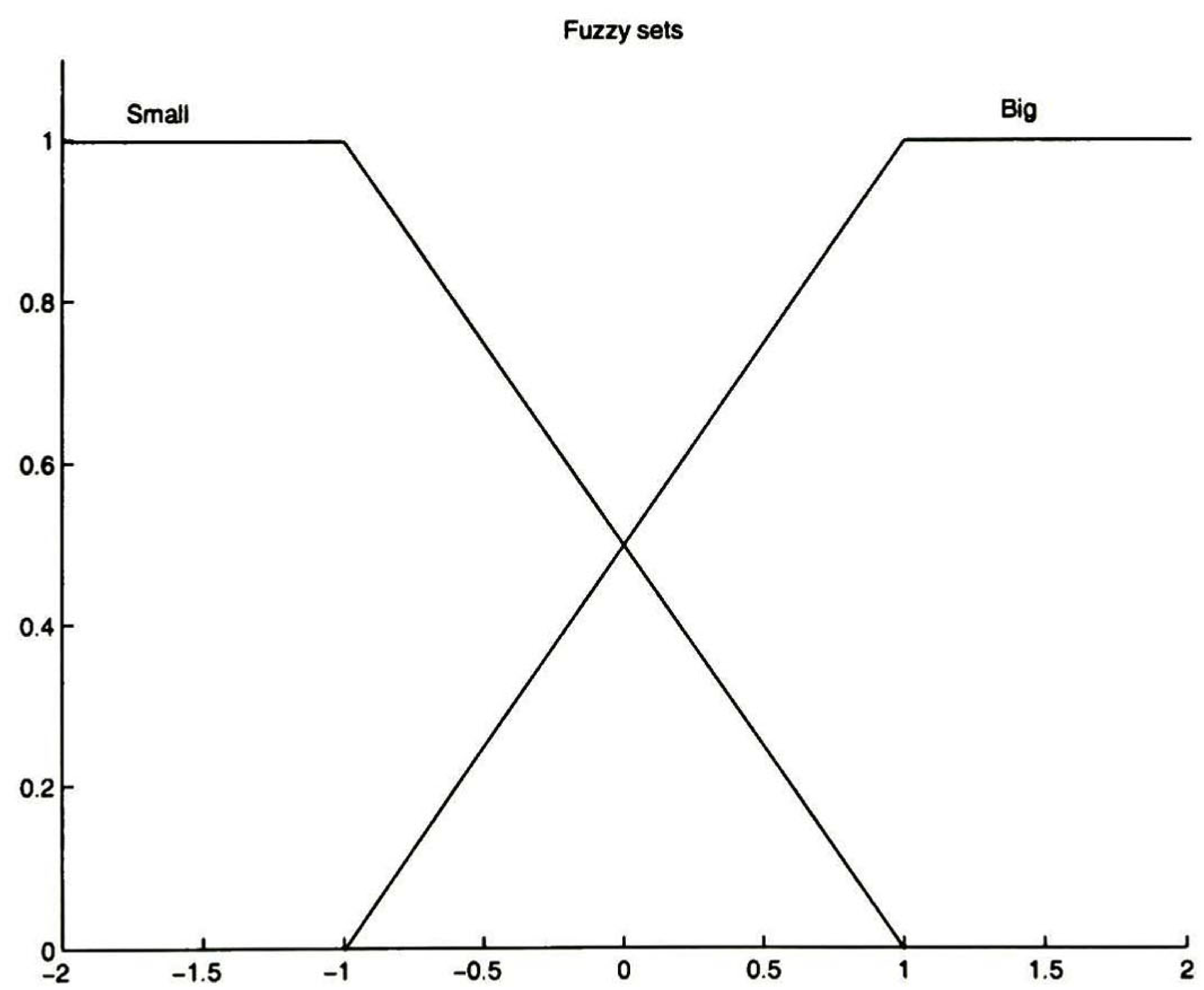

Figure A.1: Membership functions

The membership functions for this example are shown in Figure A.1. While, the solution obtained using the MATLAB LMI toolbox is given by

$$\begin{aligned} K_1 &= \begin{pmatrix} -9.3777 & -7.3230 \end{pmatrix}, \\ K_2 &= \begin{pmatrix} -6.7428 & -1.6044 \end{pmatrix}, \\ G_1 &= \begin{pmatrix} 8.4308 & -3.2404 & -7.4243 & -18.6196 \end{pmatrix}^T \text{ and} \\ G_2 &= \begin{pmatrix} 9.4903 & -11.3126 & 0.2990 & -37.5209 \end{pmatrix}^T \end{aligned}$$

The steady-state mappings for locals subsystems can also be computed with the MATLAB LMI toolbox. For this example, one gets

$$\Pi_1 = \begin{pmatrix} 1 & 0 \\ 0.5793 & -2.2702 \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} -2.0428 & -2.6833 \end{pmatrix}.$$

$$\Pi_2 = \begin{pmatrix} 1 & 0 \\ 0.5793 & -1.2702 \end{pmatrix} \text{ and}$$

$$\Gamma_2 = \begin{pmatrix} -1.8843 & -2.2236 \end{pmatrix}$$

As it can be seen, $\Pi_1 \neq \Pi_2$ which means that the steady-state error will not converge to zero. Figure A.2, Figure A.3 and Figure A.4 were obtained after constructing the overall controller through the fuzzy combination of linear robust regulators. These graphics show the output signal versus the reference, the tracking error and the input signal, respectively.

At $t = 25s$ parameter variations were introduced to test the robustness of the controller. Matrices A_1, and A_2 were changed to

$$A_1 = \begin{pmatrix} 3 & -3 \\ 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}, \text{ respectively.}$$

As expected, this approach no longer guarantees robust regulation. Nevertheless, the fuzzy regulator constructed in this way is more efficient than the fuzzy controller designed on simple linear controllers (see Figure A.5, Figure A.6 and Figure A.7).

As explained above, the rationale behind this is that local robust regulators "assume" some of the non-considered fuzzy behavior as parametric variations and the controller "try" to compensate them.

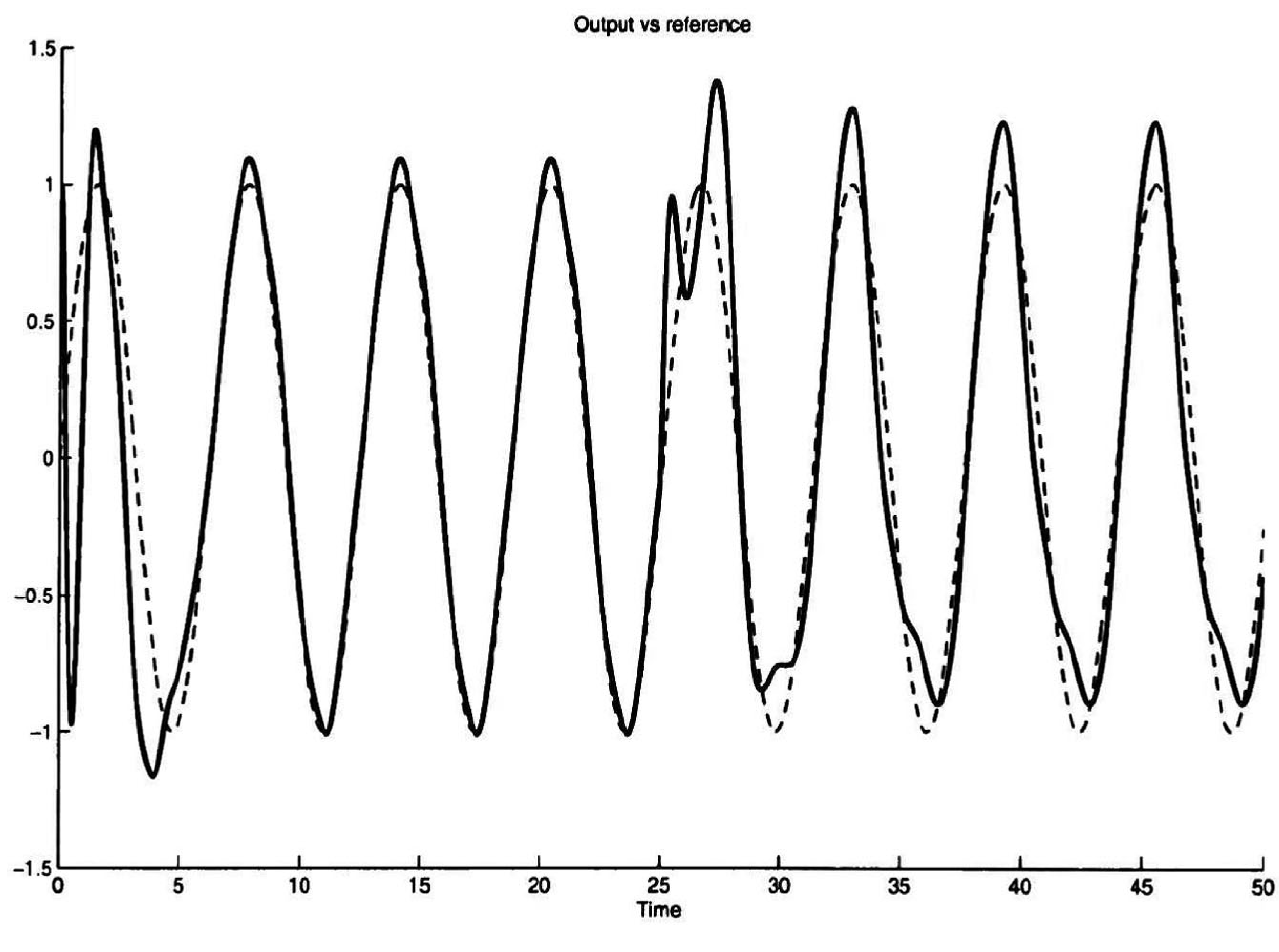

Figure A.2: Output versus reference for the time-delay system when the fuzzy controller is designed from linear robust regulators.

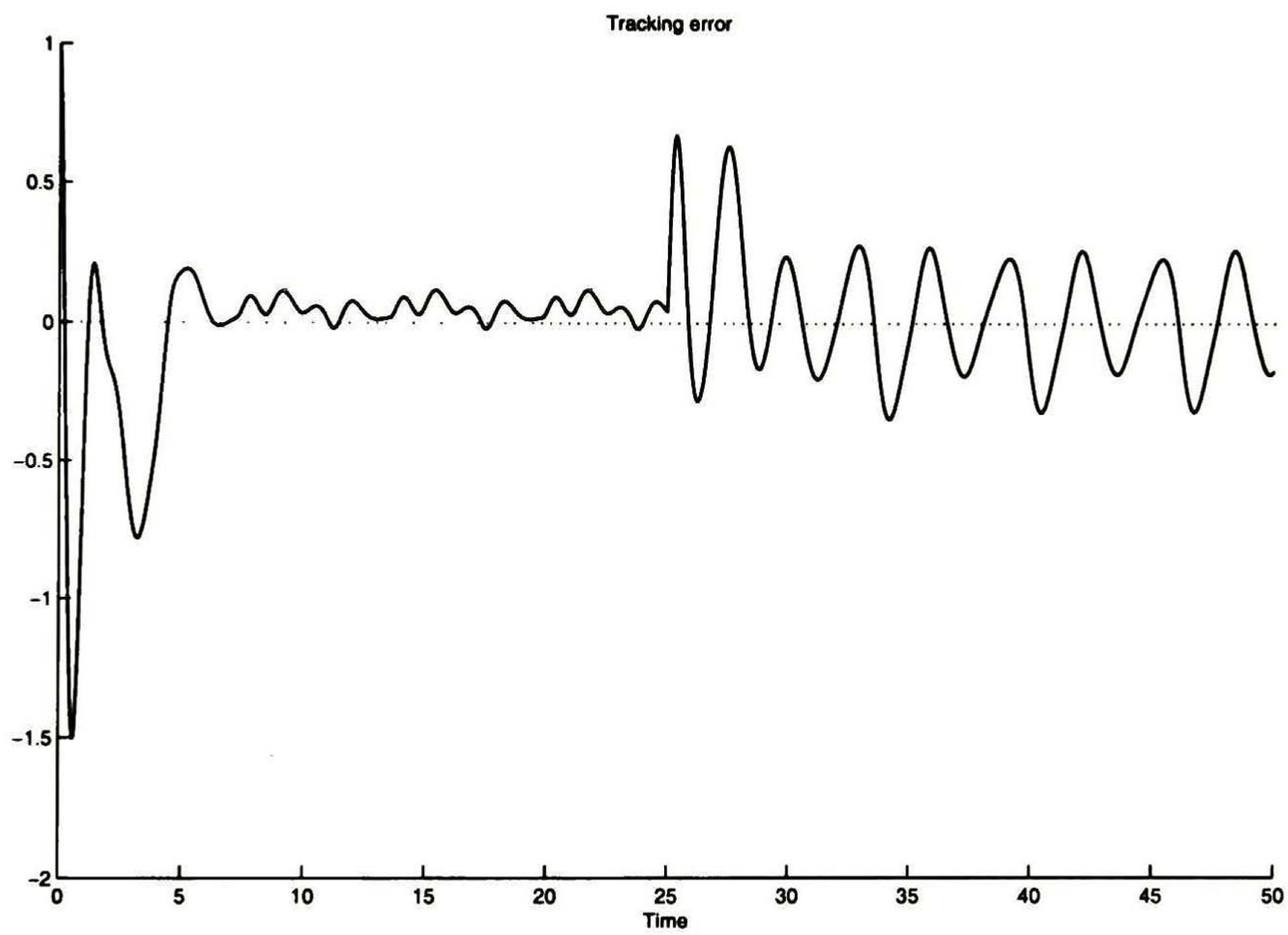

Figure A.3: Tracking error for the time-delay system when the fuzzy controller is designed from linear robust regulators.

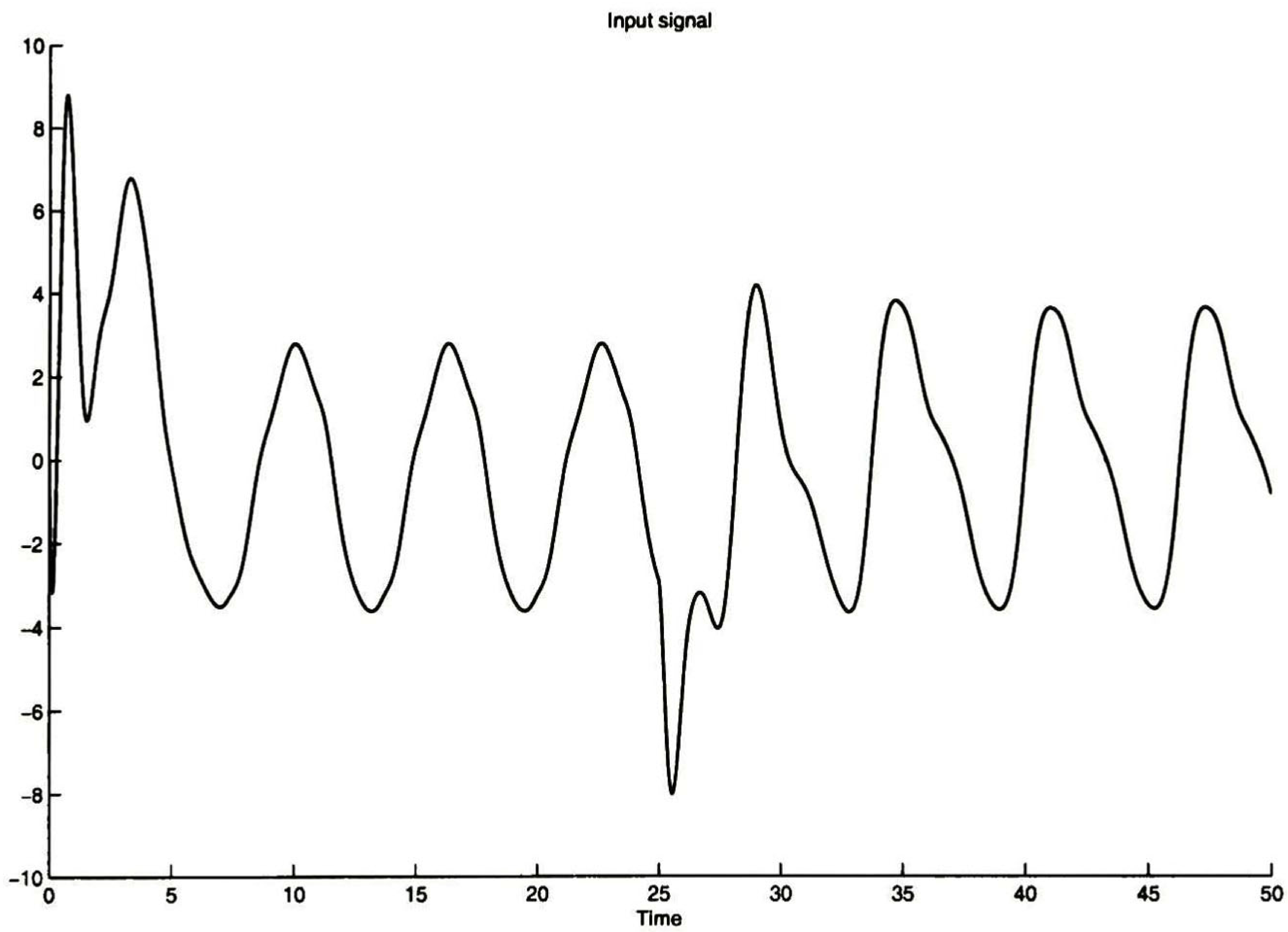

Figure A.4: Input signal for the time-delay system when the fuzzy controller is designed from linear robust regulators.

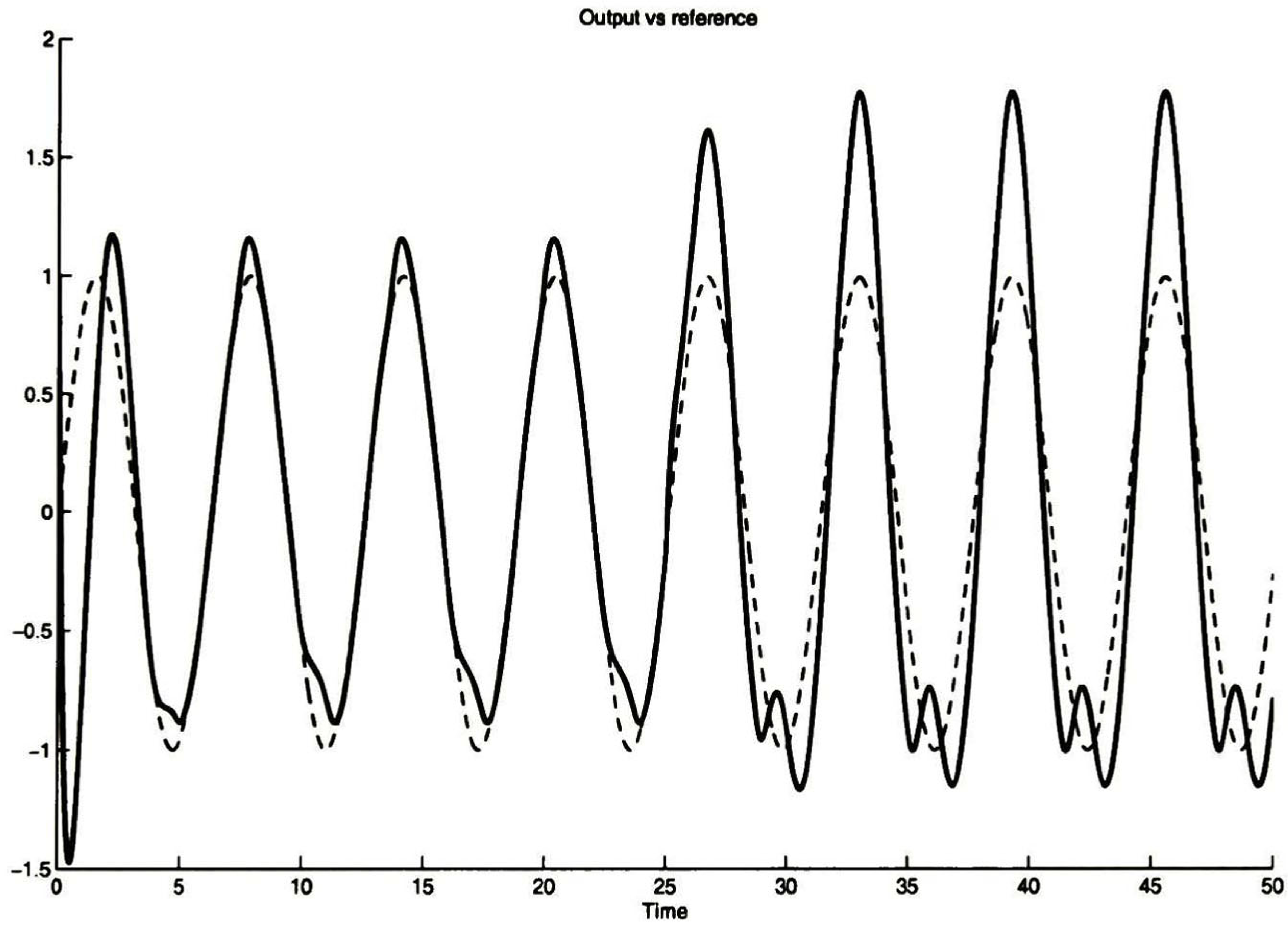

Figure A.5: Output versus reference for the time-delay system when the fuzzy controller is designed from static regulators.

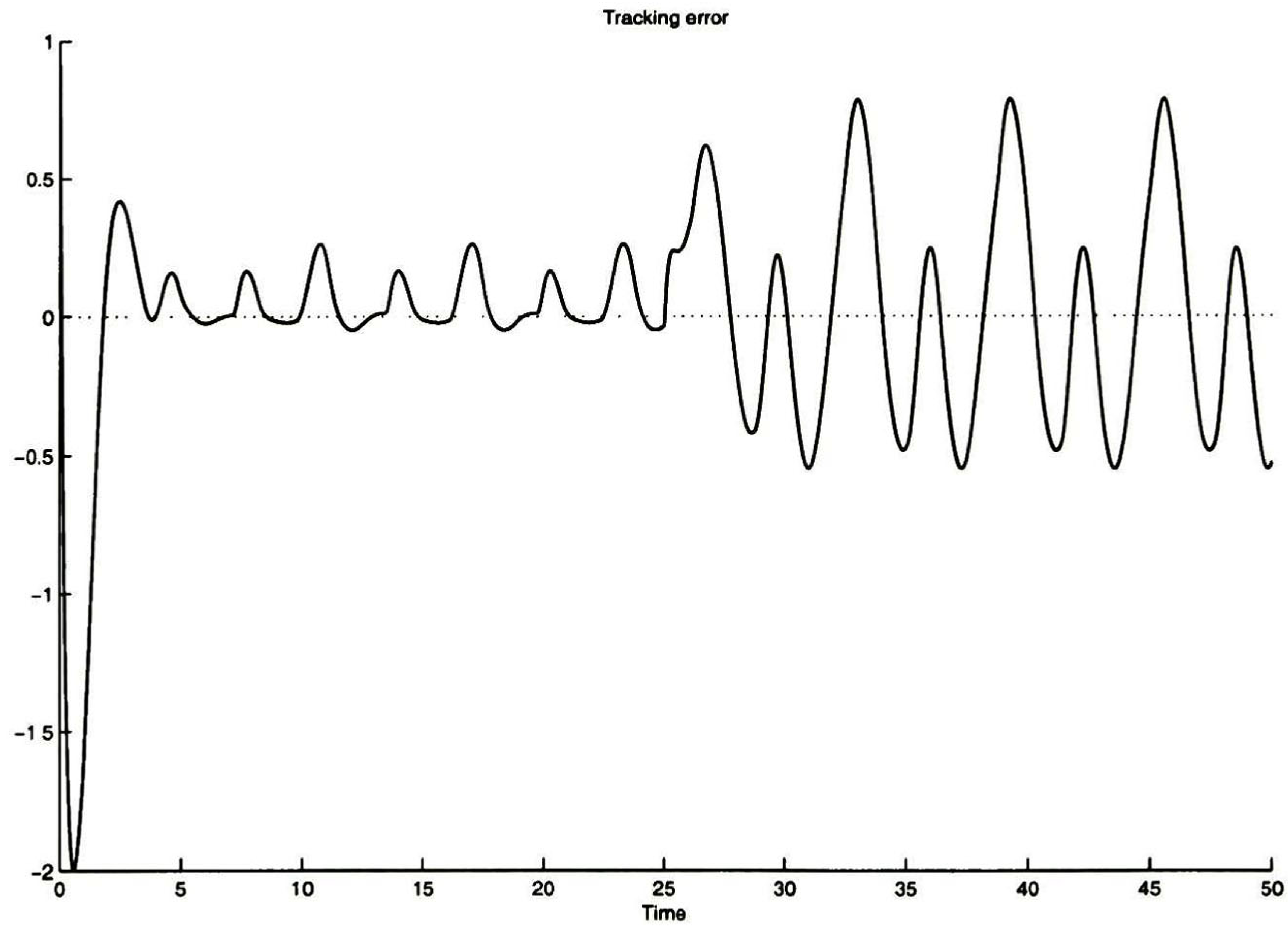

Figure A.6: Tracking error for the time-delay system when the fuzzy controller is designed from static regulators.

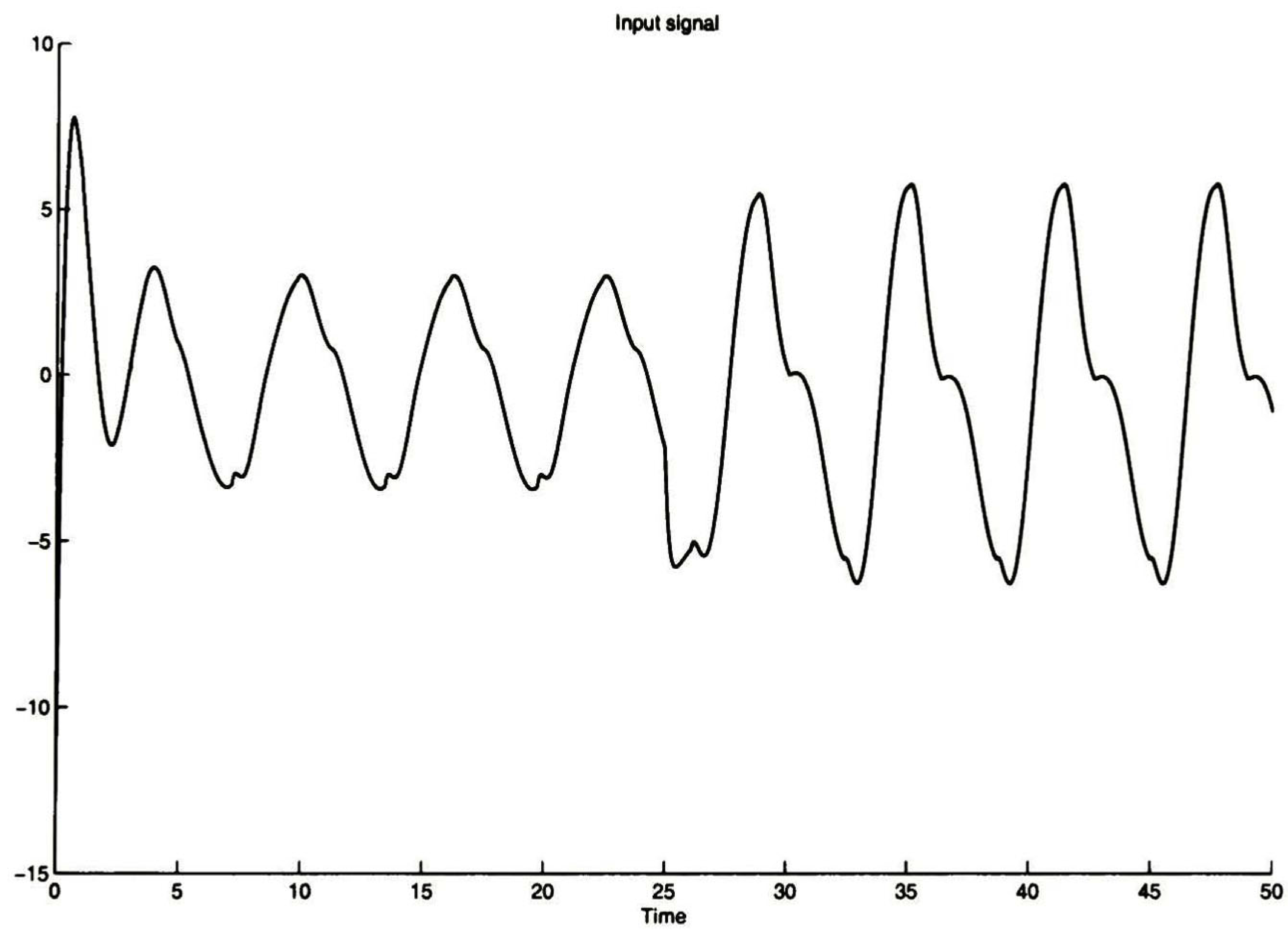

Figure A.7: Input signal for the time-delay system when the fuzzy controller is designed from static regulators.

Appendix B

Publications

" Genius is one percent inspiration
and ninety-nine percent perspiration.'

— Thomas Alva Edison

B.1 Published papers

B.1.1 Journals

- *The fuzzy discrete-time robust regulation problem: an LMI approach.* Castillo-Toledo, B.; Meda-Campaña, J.A. IEEE Transactions on Fuzzy Systems. Volume: 12. Issue: 3. June 2004. Pages: 360 – 367.

B.1.2 Symposiums and conferences

- *On the Nonlinear Fuzzy Regulation for under-actuated systems.* Meda-Campaña, J. A.; Castillo-Toledo, B.; Zúñiga, Victor. Proceedings of the 2006 IEEE International Conference on Fuzzy Systems. July 16–21, 2006. Vancouver, Canada. Pages: 10241–10248.

- *An algorithm to reduce the tracking error in TS fuzzy models: A numerical approach.* Meda-Campaña, J. A.; Castillo-Toledo, B. Preprints of the 16th IFAC World Congress. July 4–8,2005. Prague, Czech Republic.

- *The regulation problem for nonlinear time-delay systems using Takagi-Sugeno fuzzy models: An LMI approach.* Meda-Campaña, J. A.; Castillo-Toledo, B. Proceedings of the 2005 IEEE International Symposium on Intelligent Control. June 27–29, 2005. Limassol, Cyprus. Pages: 1045 – 1050.

- *On the output regulation for TS fuzzy models using sliding modes.* Meda-Campaña, J. A.; Castillo-Toledo, B. Proceedings of the 2005 American Control Conference. June 8–10, 2005. Portland, OR, USA. Pages: 4062 – 4067.

- *The optimal fuzzy robust regulator for Takagi-Sugeno discrete-time systems.* Meda-Campaña, J.A.; Castillo-Toledo, B.; Titli, A. Proceedings of the 2003 IEEE International Symposium on Intelligent Control. October 5 – 8, 2003. Houston, TX, USA. Pages: 235 – 240.

- *A fuzzy output regulator for Takagi-Sugeno fuzzy models.* Castillo-Toledo, B.; Meda-Campaña, J.A.; Titli, A. Proceedings of the 2003 IEEE International Symposium on Intelligent Control. October 5 – 8, 2003. Houston, TX, USA. Pages: 310 – 315.

- *The fuzzy discrete-time robust regulation problem: a LMI approach.* Castillo-Toledo, B.; Meda-Campaña, J.A. Proceedings of the 41st IEEE Conference on Decision and Control. December 10–13, 2002. Las Vegas, NV, USA. Pages: 2159 – 2164.

B.2 Submitted works

- *The optimal fuzzy robust regulator for T-S discrete-time systems: an LMI approach.* Submitted to International Journal of Adaptive Control and Signal Processing. Second Revision.

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CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL I.P.N.
UNIDAD GUADALAJARA

El Jurado designado por la Unidad Guadalajara del Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional aprobó la tesis

El Problema de la Regulación no Lineal con Enfoque de Lógica Difusa

del (la) C.

Jesús Alberto MEDA CAMPAÑA

el día 24 de Agosto de 2006.

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