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Observabilidad en una Clase de Sistemas Lineales Híbridos

Tesis que presenta:

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Directores de Tesis

Dr. Antonio Ramírez Treviño
Dr. José Javier Ruíz León

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Observability in a Class of Linear Hybrid Systems

A thesis presented by:
David Gómez Gutiérrez

to obtain the degree of:
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in the subject of:
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Thesis Advisors:
Dr. Antonio Ramírez Treviño
Dr. José Javier Ruíz León

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Observability of Linear Hybrid Systems

Master of Science Thesis
In Electrical Engineering

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Contents

RESUMEN

En este trabajo se aborda el problema de la observabilidad de los sistemas lineales híbridos, donde la dinámica continua está representada por una familia de sistemas lineales y la discreta por una red de Petri. El problema de la observabilidad consiste en calcular los estados del sistema, tanto continuos como discretos, y el uso de sistemas lineales y redes de Petri facilita el análisis de esta propiedad en los sistemas híbridos. En particular, este trabajo muestra cómo la información obtenida de la familia de sistemas lineales puede ser usada en el análisis de observabilidad de la red de Petri y viceversa, para sí obtener una carcterización de la observabilidad en sistemas híbridos. Además se presenta el diseño de un observador híbrido cuando la red de Petri es evento detectable es presentado.

ABSTRACT

This work is concerned with the observability of linear hybrid systems. It deals with the possibility of recovering the discrete as well as the continuous states of the hybrid system. Observability is a fundamental property of dynamic systems since it allows to implement state feedback controllers or fault diagnosers.

Through this work, the continuous part of the systems is represented by a non autonomous Linear System family, and the discrete part is modeled by an Interpreted Petri Net (IPN). Based on this model, a novel characterization of the observability in linear hybrid systems, that exploit the information of the input and the output of the IPN as well as the structure of the linear systems, is presented. It is shown that the information of the continuous systems can be used in the computation of observability of the IPN and vice versa.

This work also presents necessary and sufficient conditions for the observability of the linear hybrid systems when the continuous and discrete states are observed before the first commutation and the design of hybrid observer when the IPN is event detectable.

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Este trabajo está dedicado a Kary por su amor, su amistad y por hacer cada momento divertido.

Chapter 1

INTRODUCTION

Hybrid Linear Systems (HLS) are those dynamic systems composed by a family $\mathcal{F}$ of linear systems (LS) and a Petri Net (PN). Roughly speaking, the HLS evolves according to the following rules: a) depending on PN marking M_k, just one Linear System $LS(M_k)$ is evolving, b) depending on the Linear System state, one enabled transition is selected to fire.

HLS appear in most of the human made systems. For instance, when the normal and failing behaviors of a system must be studied gather together, where all normal and possible failure models are represented by LS; or when a non linear system is approached as several linear systems, among other interesting cases.

The study of observability [1] in HLS is primordial, since in the HLS exhibiting this property several control architectures can be used, such as feedback pole placement, regulation control, fault diagnosers, etc. When a HLS exhibits the observability property, it is possible to compute the whole system state, i.e. the continuous and discrete states. Unfortunately, the current results characterizing this property for this type of systems are restricted to autonomous systems, lead to computational complex algorithms or compute the continuos state, regarding the discrete state.

Observability in HS has been addressed by different authors using different classes of HS (*e.g.* [2], [3] and [4]). [4] addresses the observability and controllability of switched linear systems with known and periodic transitions. In [5], the observability of Jump Linear Systems was characterized. Although it is a structural property and can be analyzed efficiently, it is focused in autonomous systems (i.e. systems without inputs) and the dynamic part of the discrete system is not studied. In [3], the study of observability of piecewise affine systems is addressed. The authors show through examples that the observability of the individual subsystems has no relation with the incremental observability. The main result of that work is the definition of incremental observability and the introduction of a mixed-integer linear problem to test the observability. In [6] the authors deal with the observability of switching linear systems and they find the conditions under which there exists a control law $u(\cdot)$, depending on the observable output at time $t = t_0$, such that the hybrid state evolution can be recovered from the observed output. These are necessary and sufficient conditions. In [7], characterizations of observability, reachability and controllability were developed for a class of HS where the discrete evolution is governed by controllable discrete transitions. The authors derived the conditions to reconstruct the initial continuous state from measuring the continuos outputs and continuous inputs with a discrete control strategy design. The observability result is based on the duality property where the system is observable if and only if its dual is reachable, however it is not proved that the duality holds for this kind of nonlinear systems.

A problem related with the observability in HS is the design of observers for HS where the continuous part is linear. In [8] the design of a family of Luenberger observers for the continuous systems under the assumption that the discrete state is known is presented. In [2] the design of observers for the discrete part is combined with the design of Luenberger observers for the continuous systems that converges exponentially to the continuous state. In [9] it is presented the design of a Luenberger-Like Observer for a class of SlS, called detectable switching systems, under the assumption that the discrete state of the switching system is known.

In many works, the study of observability is limited since the discrete dynamic is not included in the analysis.

In [10] the IPN was used to model the discrete part of the SLS. That paper presents sufficient conditions for the observability of the SLS, where it is assumed that the final state of the current evolving linear system is the initial state of the next linear system selected during the switching of the system, and that the IPN is event detectable. In [11], an extension of the model of [10] is used, where previous assumption on final states is not longer needed. Also, an observer design is presented, using the IPN observer of [12] and a set of Luenberger observers for linear systems.

1.1 Objectives and goals.

The objectives and goals stated for this work are the following:

Objectives:

1. To propose a framework for studying the observability property in HLS.

2. To extend the current results in observability in HLS in two directions: one extending the type of HLS that can be analyzed, and the other extending the characterizations.

Goals:

1. We are going to use geometrical tools to analyze HLS and then we are going to translate the analysis into structural characterizations.

2. We are going to show how the continuos system information can be used to gain event detectability or observability in Petri Nets, or how the discrete system information can be used to gain observability in continuos systems.

3. We are going to propose a method to design observers for a class of observable HLS.

This work is organized as follows.

Chapter 2 presents basic concepts related with linear systems, Petri nets and the observability in these types of systems. Moreover,it presents the geometric theorem of observability presented in [10] since this work extends this theorem to derive important structural results in observability.

Chapter 3 presents how the continuous system information can be used in the observability of the Discrete system. Mainly, it presents the concept of distinguishable linear system and how using this concept, the Interpreted Petri net could become event detectable. In addition, this section also characterizes the case when the commutation time can be computed. This time can be used to know when the discrete marking changes.

Chapter 4 presents how the discrete system information can be used in the observability of the continuous system. It shows that when the discrete marking is know, then the structure of the current evolving linear system is know, then form the knowledge of the discrete marking sequence, the continuous state can be computed, even in the case when the linear systems are not observable.

Chapter 5 presents the main results derived for observability in hybrid systems. The results herein presented are derived from the observations presented in the previous chapters.

In Chapter 6 the results of Chapters 3, 4 and 5 are used to characterize the observability of Linear Hybrid Systems.

Chapter 7 presents the design of an asymptotic observer for a class of SLS.

Finally in Chapter 8 the conclusions and future work are presented.

Chapter 2

BASIC CONCEPTS

This chapter presents the definitions of linear systems [1], [13], [14] and Interpreted Petri nets [15], [16], [17] since they conform the structure of hybrid systems [10] [11] studied in this work. The chapter also presents the main results in observability for these types of systems.

2.1 Linear Systems

A linear time-invariant dynamic system (LS) is represented by the dynamic state equation:

$$\Sigma : \begin{cases} \dot{x} = Ax(t) + Bu(t) \\ y = Cx(t) \end{cases} \tag{2.1}$$

where A, B, and C are $n \times n$, $n \times p$ and $q \times n$ constants matrices. The state space generated by all possible solutions to $x(t)$ of the LS (2.1) is $\mathcal{X}$.

Remark 2.1 *Through this work the notation $\Sigma(A, B, C)$ will be used to denote a particular LS represented by the state equation (2.1).*

The solution to the linear time invariant dynamic equation given in (2.1) is:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{2.2}$$

and:

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau. \tag{2.3}$$

Previous solution can also be expressed in the frequency domain. Using the Laplace transform of (2.2) and (2.3):

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s) \tag{2.4}$$

and:

$$Y(s) = C(sI - A)^{-1} x_0 + C(sI - A)^{-1} BU(s) \tag{2.5}$$

respectively, where $U(s)$ and $Y(s)$ are the Laplace transform of $u(t)$ and $y(t)$ respectively.

The rational-function matrix:

$$G(s) = C(sI - A)^{-1} B \tag{2.6}$$

is named the transfer function matrix of the dynamic equation $\Sigma(A, B, C)$.

2.1.1 Observability of Linear Dynamic Equations

Definition 2.2 *A LS, $\Sigma(A, B, C)$, is said to be observable at t_0 if there exists a finite $t_1 > t_0$ such that for any state x_0 at time t_0, the knowledge of the input $u[t_0, t_1]$ and the output $y[t_0, t_1]$ over the time interval $[t_0, t_1]$ suffices to determine the state x_0. Otherwise, the dynamic equation $\Sigma(A, B, C)$ is said to be unobservable at t_0.*

Theorem 2.3 *Let $\Sigma(A, B, C)$ be a LS. $\Sigma(A, B, C)$ is observable if and only if any of the following conditions is satisfied:*

1. *All columns of Ce^{At} are linearly independent over the time period $[0, \infty]$ and $\mathbb{C}$, the field of complex numbers.*

1' *All columns of $C(sI - A)^{-1}$ are linearly independent over $\mathbb{C}$.*

2. *The observability grammian:*

$$W_{ot} = \int_0^t e^{A^*\tau} C^* C e^{A\tau} d\tau \tag{2.7}$$

is nonsingular for any $t > 0$.

3. *The $nq \times n$ observability matrix:*

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{2.8}$$

has rank n.

4. *For every eigenvalue λ of A (and consequently for any λ in $\mathbb{C}$) the $(n+q) \times n$ complex matrix:*

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \tag{2.9}$$

has rank n, or equivalently $sI - A$ and C are right coprime.

5. *The unobservable subspace $\mathcal{N}$ of (C, A) is the trivial subspace 0, i.e.:*

$$\mathcal{N} = \bigcap_{i+1}^{n} \ker\left(CA^{i-1}\right) = 0 \tag{2.10}$$

or equivalently, there exists a nontrivial subspace of states $\mathcal{V}$, such that $\mathcal{V}$ is $A-$invariant $(A\mathcal{V} \subset \mathcal{V})$ and $\mathcal{V} \subset \ker C$.

Proof. *The proofs of statements 1 − 4 are presented in [1] and [13], the proof of statement 5 is presented in [illegible].* ■

Remark 2.4 *In this work it is assumed that the linear equation in 2.1 is scalar, that is, it has only one input and one output.*

2.2 Interpreted Petri Nets

Interpreted Petri nets are an extension of the Petri nets. Next definitions present the Petri nets and, afterwards the Interpreted Petri nets are presented.

Definition 2.5 *A Petri Net structure G is a bipartite digraph represented by the 4-tuple $G = (P, T, I, O)$ where:*

- *$P = \{p_1, p_2, \ldots, p_r\}$ is a finite set of vertices called places,*
- *$T = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_m\}$ is a finite set of vertices called transitions,*
- *$I : P \times T \longrightarrow \mathbb{Z}^+$ is a function representing the weighted arcs going from places to transitions.*
- *$O : P \times T \longrightarrow \mathbb{Z}^+$ is a function representing the weighted arcs going from transitions to places, where $\mathbb{Z}^+$ is the set of nonnegative integers.*

The symbol ${}^\bullet\mathbf{t}_j$ denotes the set of all places p_i such that $I(p_i, \mathbf{t}_j) \neq 0$ and $\mathbf{t}_j{}^\bullet$ the set of all places p_i such that $O(p_i, \mathbf{t}_j) \neq 0$. Analogously, ${}^\bullet p_i$ denotes the set of all transitions $\mathbf{t}_j$ such that $O(p_i, \mathbf{t}_j) \neq 0$ and $p_i{}^\bullet$ the set of all transitions $\mathbf{t}_j$ such that $I(p_i, \mathbf{t}_j) \neq 0$. Pictorially, places are represented by circles, transitions are represented by rectangles, and arcs are depicted as arrows.

The pre-incidence matrix of G is $\mathcal{C}^- = [c_{ij}^-]$, where $c_{ij}^- = I(p_i, \mathbf{t}_j)$; the post-incidence matrix of G is $\mathcal{C}^+ = [c_{ij}^+]$, where $c_{ij}^+ = O(p_i, \mathbf{t}_j)$; and the incidence matrix of G is $\mathcal{C} = \mathcal{C}^+ - \mathcal{C}^-$ The marking function $M : P \longrightarrow \mathbb{Z}^+$ is a mapping from each place to the nonnegative integers representing the number of tokens (depicted as dots) residing inside each place. The marking of a PN is usually expressed as an $n-$entry vector.

Definition 2.6 *A Petri Net system or Petri Net (PN) is the pair $N = (G, M_0)$, where G is a PN structure and M_0 is an initial token distribution.*

In a PN system, a transition $\mathbf{t}_j$ is enabled at marking M_k if $\forall p_i \in P, M_k(p_i) \geqslant I(p_i, \mathbf{t}_j)$; an enabled transition $\mathbf{t}_j$ can be fired reaching a new marking M_{k+1} which can be computed using the petri net state equation

$$M_{k+1} = M_k + Cv_k, \tag{2.11}$$

where $v_k(i) = 0$, $i \neq j$, $v_k(j) = 1$, this equation is called the PN state equation.

The reachability set of a PN is the set of all possible reachable marking from M_0 firing only enabled transitions; this set is denoted by $R(G, M_0)$.

Definition 2.7 *An Interpreted Petri Net (IPN) is the 4-tuple $Q = (N, \Sigma, \lambda, \varphi)$ where:*

- $N = (G, M_0)$ *is a PN system.*
- $\Sigma = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ *is the input alphabet of the net, where α_i is an input symbol.*
- $\lambda : T \rightarrow \Sigma \cup \{\varepsilon\}$ *is a labelling function of transitions with the following constraint: $\forall \mathbf{t}_j, \mathbf{t}_k \in T$, $j \neq k$, if $\forall p_i$ $I(p_i, \mathbf{t}_j) = I(p_i, \mathbf{t}_k) \neq 0$ and both $\lambda(\mathbf{t}_j) \neq \varepsilon$, $\lambda(\mathbf{t}_k) \neq \varepsilon$, then $\lambda(\mathbf{t}_j) \neq \lambda(\mathbf{t}_k)$. In this case ε represents an internal system event.*
- $\varphi : R(Q, M_0) \rightarrow (\mathbb{Z}^+)^q$ *is an output function, that associates to each marking in $R(Q, M_0)$ an output vector. Here q is the number of outputs.*

Remarks:

1. In this work (Q, M_0) will be used instead of $Q = (N, \Sigma, \lambda, \varphi)$ to emphasize the fact that there is an initial marking in an IPN.
2. This is focused on the case when function φ is a $q \times r$ matrix, where q is the number of places representing measurable states in the DES and r is the number of places in the model (G, M_0). Each column of this matrix is an elementary or null vector. If the output symbol i is present (turned on) every time that $M(p_j) \geq 1$, then $\varphi(i, j) = 1$, otherwise $\varphi(i, j) = 0$.
3. Equivalent transitions are not allowed, i.e. it is assumed that $\forall \mathbf{t}_i, \mathbf{t}_j$ such that $\mathbf{t}_i \neq \mathbf{t}_j, \lambda(\mathbf{t}_i) = \lambda(\mathbf{t}_j)$, it holds that $C(\cdot, i) \neq C(\cdot, j)$. This is not a major constraint because those transitions are redundant.
4. Notice that by definition of λ, IPN are deterministic [18] over labeled transitions, i.e. two transitions with the same associated input symbol (different from symbol ε) cannot have the same input places. However, they can be non deterministic [18] over unlabeled transitions (those t_j such that $\lambda(t_j) = \varepsilon$).

A transition $\mathbf{t}_j \in T$ of an IPN is enabled at marking M_k if $\forall p_i \in P$, $M_k(p_i) \geqslant I(p_i, \mathbf{t}_j)$. If $\lambda(\mathbf{t}_j) = a_i \neq \varepsilon$ is present and $\mathbf{t}_j$ is enabled, then $\mathbf{t}_j$ must fire. If $\lambda(\mathbf{t}_j) = \varepsilon$ and $\mathbf{t}_j$ is enabled then $\mathbf{t}_j$ can be fired. When an enabled transition $\mathbf{t}_j$ is fired in a marking M_k, then a new marking M_{k+1} is reached. This fact is represented as $M_k \xrightarrow{t_j} M_{k+1}$ and M_{k+1} can be computed using the dynamic part of the IPN state equation:

$$\begin{aligned} M_{k+1} &= M_k + Cv_k \\ y_k &= \varphi(M_k) \end{aligned} \tag{2.12}$$

where C and v_k are defined as in PN and $y_k \in (\mathbb{Z}^+)^q$ is the $k-th$ output vector of the IPN.

According to functions λ and φ, transitions and places of an IPN (Q, M_0) can be classified as follows.

Definition 2.8 *If $\lambda(t_i) \neq \varepsilon$ the transition t_i is said to be manipulated. Otherwise it is non manipulated. A place $p_i \in P$ is said to be measurable if the $i-th$ column vector of φ is not null, i.e. $\varphi(\bullet, i) \neq 0$. Otherwise it is non measurable. A place p_i is said to be computable if it is measurable and $\forall j$, $i \neq j$, $\varphi(\bullet, i) \neq \varphi(\bullet, j)$. Otherwise it is non computable.*

Notice that computable places are measurable and the marking of these places can be computed from the output (no other place, when it is marked, generates the same output value of function φ).

2.2.1 Languages and structures of Petri nets

Definition 2.9 *Let $\sigma = \mathbf{t}_i\mathbf{t}_j\mathbf{t}_k...$ be a firing transition sequence. The Parikh vector $\vec{\sigma}$: $T \longrightarrow (\mathbb{Z}^+)^m$ of σ maps every transition $\mathbf{t} \in T$ to the number of occurrences of $\mathbf{t}$ in σ.*

Definition 2.10 *A P-semiflow of a Petri Net G is a rational-valued solution of the equation* $\mathbf{Y}^T \cdot \mathcal{C} = 0$

The following proposition presents a fundamental property of P-semiflows

Proposition 2.11 *Let (G, M_0) be a Petri Net system, and let I be a P-semiflow of G then $\forall M \in R(G, M_0)$ $I \cdot M = I \cdot M_0$*

Definition 2.12 *A T-semiflow of a Petri Net G is a rational-valued solution of the equation* $\mathcal{C} \cdot \mathbf{Y} = 0$

Proposition 2.13 *Let σ be a finite sequence of transitions of a net G which is enabled at a marking M. Then the Parikh vector $\vec{\sigma}$ is a T-semiflow iff $M \xrightarrow{\sigma} M$ (i.e. iff the occurrence of σ reproduces the marking M).*

Definition 2.14 *A sequence of input-output symbols of (Q, M_0) is a sequence $\omega = (\alpha_0, y_0)$ $(\alpha_1, y_1) \cdots$ (α_n, y_n) , where $\alpha_j \in \Sigma \cup \{\varepsilon\}$ and α_{i+1} is the current input of the IPN when the output changes from y_i to y_{i+1}. It is assumed that $\alpha_0 = \varepsilon$, $y_0 = \varphi(M_0)$ and (α_{i+1}, y_{i+1}) belongs to the sequence when:*

- (α_i, y_i) *belongs to the sequence,*
- $y_{i+1} \neq y_i$, *and*
- *there exists no* $y_j \neq y_i$, $y_j \neq y_{i+1}$ *occurring after the occurrence of* y_i *and before the occurrence of* y_{i+1}.

Definition 2.15 *Let* (Q, M_0) *be an IPN. The set* $\Lambda(Q, M_0) = \{\omega | \omega$ *is a sequence of input-output symbols*$\}$ *denotes the set of all sequences of input-output symbols of* (Q, M_0). *The set of all input-output sequences of length greater or equal than* k *will be denoted by* $\Lambda^k(Q, M_0)$, *i.e.* $\Lambda^k(Q, M_0) = \{\omega \in \Lambda(Q, M_0) \mid |\omega| \geq k\}$.

Definition 2.16 *If* $\omega = (\alpha_0, y_0)(\alpha_1, y_1)\cdots(\alpha_n, y_n)$ *is a sequence of input-output symbols, then the firing transition sequence* $\sigma \in \pounds(Q, M_0)$ *whose firing actually generates* ω *is denoted by* σ_ω. *The set of all possible firing transition sequences that could generate the word* ω *is defined as* $\Omega(\omega) = \{\sigma | \sigma \in \pounds(Q, M_0) \ \wedge$ *the firing of* σ *produces* $\omega\}$.

Definition 2.17 *The set of all input-output sequences leading to an ending marking in the IPN (markings enabling no transition or only self-loop transitions) is denoted by* $\Lambda_B(Q, M_0)$, *i.e.,* $\Lambda_B(Q, M_0) = \{\omega \in \Lambda(Q, M_0) \mid \exists\sigma \in \Omega(\omega)$ *such that* $M_0 \xrightarrow{\sigma} M_j \wedge$ *if* $M_j \xrightarrow{\mathbf{t}_i}$ *then* $C(\bullet, \mathbf{t}_i) = \vec{0}\}$.

The prefix of a sequence s is another sequence s' such that there exists a sequence s'' fulfilling that $s = s's''$ The set of all prefixes of s is denoted by $\bar{s}$.

Definition 2.18 *Let* $\omega = (\alpha_0, y_0)(\alpha_1, y_1)\cdots(\alpha_n, y_n) \in \Lambda(Q, M_0)$ *be a sequence of input-output symbols. The marking sequences set corresponding to* ω *is defined as* $S_\omega = \{M_0 M_1 \cdots M_k | \ M_i \in R(Q, M_0) \ \wedge M_0 \xrightarrow{\mathbf{t}_i} M_1 \xrightarrow{\mathbf{t}_j} \cdots \xrightarrow{\mathbf{t}_m} M_k \wedge \sigma_\omega = \mathbf{t}_i \mathbf{t}_j \cdots \mathbf{t}_m \in \Omega(\omega)\}$.

2.2.2 Subclasses and properties of Petri Nets

Definition 2.19 *A Petri Net system is a state machine if* $|\mathbf{t}^\bullet| = 1 = |{}^\bullet\mathbf{t}|$ *for every transition* $\mathbf{t}$.

The fundamental property of state machines is that all reachable markings contain exactly the same number of tokens. In other words, the total number of tokens of the system remains invariant under the occurrence of transitions.

Definition 2.20 *A Petri Net systems is a marked Graph if* $|p^\bullet| = 1 = |{}^\bullet p|$ *for every place* p.

The fundamental property of marked graphs is that the token counts of circuits remain invariant under the occurrence of transitions.

Definition 2.21 *A Petri Net system (G, M_0) is live if, for every reachable marking M and every transition $\mathbf{t}$ there exists a marking $M' \in R(G, M_0)$ which enables $\mathbf{t}$. If (G, M_0) is a live system, then it is said that M_0 is a live marking of G.*

Definition 2.22 *A Petri Net system (G, M_0) is deadlock-free if every reachable marking enables at least one transition.*

Definition 2.23 *A system is bounded if for every place p there is a natural number b such that $M(b) \leq b$ for every reachable marking M.*

Definition 2.24 *A marking $M \in R(G, M_0)$ is a home marking if $\forall M_k \in R(G, M_0)$ there is a firing sequence such that $M_k \xrightarrow{\sigma} M$ it is reachable from every reachable marking. In other words, it is a home marking if it is reachable from every reachable marking.*

Definition 2.25 *A Petri Net is cyclic if its initial marking is a home marking.*

Example 2.26 *The IPN of figure 2.1 is composed by five places and four transitions. The output function φ is represented by the matrix:*

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{2.13}$$

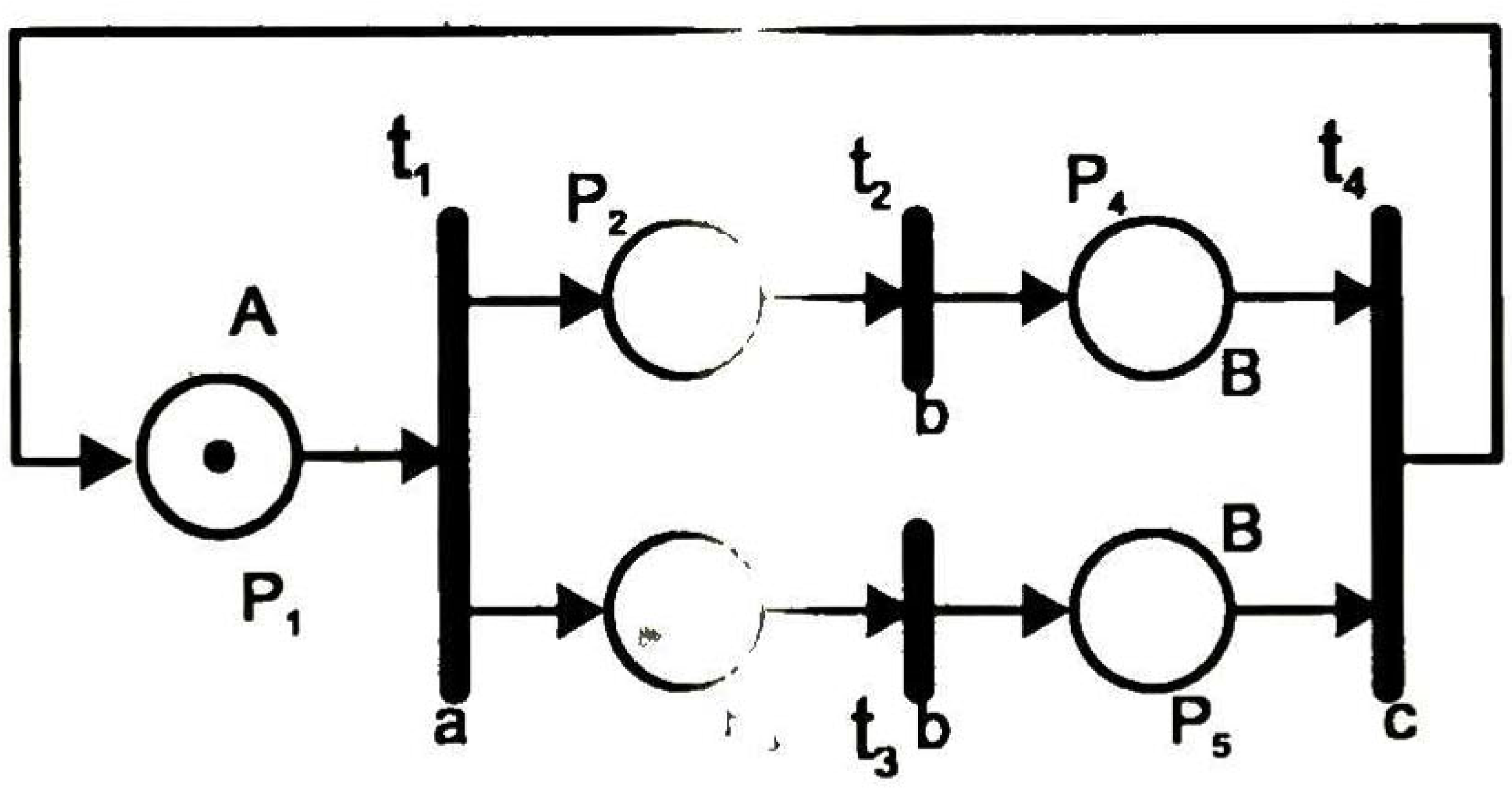


Figure 2.1: A non deterministic IPN.

Notice that places, p_1, p_4 and p_5 are measurable, however, place p_1 is the only computable place. Since all transitions are labeled, then all of them are manipulated. We obtain the following languages.

$$\mathcal{L}(Q, M_0) = \{\mathbf{t}_1, \mathbf{t}_1\mathbf{t}_2, \mathbf{t}_1\mathbf{t}_3, \mathbf{t}_1\mathbf{t}_2\mathbf{t}_3, \mathbf{t}_1\mathbf{t}_3\mathbf{t}_2, \mathbf{t}_1\mathbf{t}_2\mathbf{t}_3\mathbf{t}_4, \mathbf{t}_1\mathbf{t}_3\mathbf{t}_2\mathbf{t}_4, \mathbf{t}_1\mathbf{t}_2\mathbf{t}_3\mathbf{t}_4\mathbf{t}_5, ..\} \tag{2.14}$$

$$\Lambda(Q, M_0) = \left\{ \left(\varepsilon, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \left(\varepsilon, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \left(a, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right), \left(\varepsilon, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \left(a, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \left(b, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right), \ldots \right\} \tag{2.15}$$

$$\Lambda^2(Q,M_0)=\left\{\left(\varepsilon,\begin{bmatrix}1\\0\end{bmatrix}\right)\left(a,\begin{bmatrix}0\\0\end{bmatrix}\right)\ \left(\varepsilon,\begin{bmatrix}1\\0\end{bmatrix}\right)\left(a,\begin{bmatrix}0\\0\end{bmatrix}\right)\left(b,\begin{bmatrix}0\\1\end{bmatrix}\right),\ldots\right\} \tag{2.16}$$

$$\Lambda_B(Q,M_0)=\{\} \tag{2.17}$$

If $\omega=\left(\varepsilon,\begin{bmatrix}1\\0\end{bmatrix}\right)\left(a,\begin{bmatrix}0\\0\end{bmatrix}\right)\left(b,\begin{bmatrix}0\\1\end{bmatrix}\right)$ *then:*

$$\Omega(\omega)=\{\mathbf{t_1t_2},\mathbf{t_1t_3}\} \tag{2.18}$$

and:

$$S_\omega=\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix}\} \tag{2.19}$$

2.2.3 Observability in Interpreted Petri nets

Now, the following concept will be needed to characterize observable $IPNs$ and establishes that, even when the precise marking of a place is unknown, it can belongs to a conservative marking law. In other words, the location or state of the entities (resources, machines, buffer capacities, etc.) that constitute the DES may be unknown; however the amount of those entities is known. This concept is analogous to that of "macro markings" used in [19].

Definition 2.27 *Let* (Q,M_0) *be an* IPN *structure and* $M(p_j)$ *be any marking of a place* p_j *in* (Q,M_0). *The set of* s *equations* $CML=\{\sum_{j=1}^{n}\gamma_j^i M(p_j)=k_i | i\in[1,\ldots,s]\wedge\gamma_j^i\in\mathbb{Z}^+\}$ *form a set of conservative marking laws* (CML) *if* $\forall\gamma_k^i\neq 0$ *it holds that* k_i/γ_k^i *is an integer value and all non computable places* p_n *are contained in at least one equation of the* CML *set. A* CML *is said to be binary* ($BCML$) *if it holds that* $\forall i,j,\ \gamma_j^i\in\{0,1\}$ *and* $k_i=1$. *In addition, the* CML *can be rewritten as:*

$$\Gamma M=K \tag{2.20}$$

where M *is the marking vector,* Γ *is the matrix* $\Gamma[i,j]=\gamma_j^i$ *and* K *is the vector* $K(i)=k_i$.

Remarks:

5. Hereafter $\mathcal{M}_0$ will denote the set of all possible initial markings fulfilling the stated CML, i.e.

$$\mathcal{M}_0=\{M_0|\text{such that any } M\in R(Q,M_0) \text{ fulfills the } CML \text{ constraints}\}. \tag{2.21}$$

6. The notation $p_l \in e_i$, where $e_i \in CML$ ($e_i \in BCML$) means that there exists an equation $\sum_j \gamma_j^i \cdot M(p_j) = k_i$, named e_i in the CML (in the $BCML$), such that $\gamma_l^i \neq 0$.

Also, $(Q, \mathcal{M}_0)$ will denote an IPN where $M_0 \in \mathcal{M}_0$ and could be unknown. Notation $\mathcal{M}_0^{\mathrm{B}}$ will be used for a $BCML$.

Definition 2.28 *An IPN given by $(Q, \mathcal{M}_0)$ is event-detectable if any transition firing can be uniquely determined by the knowledge of the input given to $(Q, \mathcal{M}_0)$ and output signals that it produces.*

The following lemma provides a structural characterization of the IPN exhibiting event-detectability.

Lemma 2.29 *The IPN given by $(Q, \mathcal{M}_0)$ is event-detectable if and only if*

1. *$\forall \mathbf{t}_i, \mathbf{t}_j \in T$ such that $\lambda(\mathbf{t}_i) = \lambda(\mathbf{t}_j)$ or $\lambda(\mathbf{t}_i) = \varepsilon$ it holds that $\varphi C(\bullet, \mathbf{t}_i) \neq \varphi C(\bullet, \mathbf{t}_j)$ and*

2. *$\forall \mathbf{t}_k \in T$ it holds that $\varphi C(\bullet, \mathbf{t}_k) \neq 0$.*

Proof. The proof of this result is presented in [12]. ■

As stated previously, a state representation of a dynamic system is said to be observable if the knowledge of its inputs, outputs, and structure suffices to uniquely determine its state, for instance, $\Sigma(A, B, C)$, is said to be observable, at t_0, if there exists a finite time t_1 such that the knowledge of the model structure (A, B, C), the input signal $(u(t))$ and the output signal $(y(t))$ over the interval $t_0 \leq t \leq t_1$ suffices to uniquely determine the initial state $(x(t_0))$. Moreover, since the system is a deterministic one, then $x(t)$ for all $t \geq t_0$, can also be uniquely determined using the knowledge of $x(t_0)$ and $u(t)$, over the interval $t_0 \leq t \leq t_1$.

When the dynamic model is non deterministic (i.e. when the solution of the model is not unique [20]), however, the knowledge of $x(t_0)$ and $u(t)$ over the interval $t_0 \leq t \leq t_1$ does not guarantee the computation of $x(t)$ for all $t \geq t_0$. For instance, even if it is known that the initial marking of the IPN depicted in figure 2.1 is $M_0 = [10000]^T$ and that the input sequence $\sigma = abbcabbcabbc...ab$ is fired, it is not possible to determine if the reached marking is $[01001]^T$ or $[00110]^T$ In this case, there exists no finite transition sequence σ allowing to know the reached marking and all future reached markings of the IPN.

This fact leads to that, in the general case, the observability definition must be changed to ensure that the initial state $x(t_0)$ and all states $x(t)$, $t > t_0$ can be computed in a finite time or length of input firing words. Because of that, in the general case, the following intuitive definition is presented.

A non deterministic dynamic model, for instance an IPN, is observable, at k_0, if there exists a finite integer k_1 such that the knowledge of the model structure (C, λ, φ) and the sequence of input-output symbols ω_k for any $k \geq k_1$ suffices to uniquely determine the state sequence over $k_0 \leq l \leq k$ $(M_{k_0}...M_k)$. The observability definition in IPN can be formally proposed as follows.

Definition 2.30 *An IPN given by (Q, M_0), where M_0 may be unknown, is observable if there exists an integer $k < \infty$ such that $\forall\omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ it holds that the information provided by ω and (Q, M_0) suffices to uniquely determine the initial marking M_0 and the marking M_i reached by the firing of the underlaying firing transition sequence σ_ω.*

Therefore an IPN is observable if for any sequence of input-output signals of length equal or greater than k and for any blocking sequence, the marking sequence reached by the system can be uniquely determined.

Since the set S_ω contains the marking sequences generated by the same input-output sequence $\omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$, then when $|S_\omega| = 1$ there exists only one marking sequence for the word ω. Thus the initial and the actual marking can be computed from these marking sequence, leading to an observable IPN. This fact is formalized in the following theorem.

Theorem 2.31 *An IPN given by (Q, M_0) is observable if and only if there is an integer $k < \infty$ such that $\forall\omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ it holds that $|S_\omega| = 1$, where S_ω is the marking sequences set corresponding to ω.*

Proof. *(Sufficiency) Assume that there exists an integer $k < \infty$ such that $\forall\omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ it holds that $|S_\omega| = 1$, then a function $\Psi : \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0) \longrightarrow R(Q, M_0) \times R(Q, M_0)$ can be computed, where Ψ fulfills the following: $\forall\omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ it holds that $\Psi(\omega, (Q, M_0)) = (M_0, M_i)$ where M_0 is the initial marking and M_i is the marking reached by the firing of the underlaying firing transition sequence σ_ω.*

(Necessity) Suppose that there is no integer $k < \infty$ such that $\forall\omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ it holds that $|S_\omega| = 1$, then for any k there is at least one $\omega \in \Lambda^k(Q, M_0) \cup \Lambda_B(Q, M_0)$ such that $|S_\omega| \neq 1$, therefore $|S_\omega| > 1$. Assume further without lost of generality that $S_\omega = \{\ \gamma_1 = M_i\ M_j \cdots\ M_k \cdots\ M_n,\ \gamma_2 = M_i'\ M_j' \cdots\ M_k' \cdots\ M_n' \}$.

Since these sequences are different, then there must exist markings M_k, M_k' such that $M_k \neq M_k'$ in γ_1, γ_2 respectively. Notice that when the initial marking of (Q, M_0) is M_k or M_k' then there exist two different values to assign to M_0, or the function Ψ cannot be obtained, a contradiction. ∎

2.3 Switched Linear Systems

The definition of a SLS is presented in the following.

Definition 2.32 *Let $\mathcal{F} = \{\Sigma_1, ..., \Sigma_s\}$ be a family of linear systems, all of them of the same dimension, and let (Q, M_0) be an IPN. The 2-tupla $\langle\mathcal{F}, (Q, M_0)\rangle$ is a SLS if:*

1. *There exists a function $\Phi : R(Q, M_0) \to \mathcal{F}$, such that if the current marking of (Q, M_0) is M_k, then the linear system $\Phi(M_k)$ is the only linear system that it is evolving.*

2. *There exist bijective functions $\delta_{M_i,M_j} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\delta_{M_i,M_j}(x_f^i) = x_0^j$ indicates how the SLS state is changed when the SLS commutes from Σ_i to Σ_j, $\Sigma_i, \Sigma_j \in \mathcal{F}$*

3. *The elapsed time to change from M_k to M_{k+1} is finite and different from zero.*

4. *The IPN is live, binary and cyclic, an IPN is binary if $M(p_i) \leq 1$ for any place and any reachable marking.*

5. *It is assumed that every $\Sigma_i \in \mathcal{F}$ evolves for a finite $\tau > 0$ before Σ_i commutes to another Σ_j.*

Then, a SLS is described by the following state equation:

$$\begin{aligned} \dot{x}(t) &= A_{M_k}x(t) + B_{M_k}u(t) \\ y_c(t) &= C_{M_k}x(t) \\ M_{k+1} &= M_k + Cv_k \\ y_{d_k} &= \varphi(M_k) \end{aligned} \tag{2.22}$$

where v_k is the firing vector and there exist functions Φ and δ_{M_i,M_j} as described above.

Hereafter continuous and discrete state vectors will be represented as $x(t)$ and $M_k(t)$ respectively.

Let us define the state vector $X(t)$, the output vector $Y(t)$ and the input vector $U(t)$ of the switched linear system as:

$$X(t) = \begin{bmatrix} x(t) \\ M_k(t) \end{bmatrix}, \quad Y(t) = \begin{bmatrix} y_c(t) \\ y_{d_k}(t) \end{bmatrix}, \quad U(t) = \begin{bmatrix} u(t) \\ v_k(t) \end{bmatrix}$$

Condition 2 of previous definition is not restrictive at all, it is fulfilled in all SLS. For instance, some classes of nonlinear systems (NLS) can be approximated by SLS, where each LS is a linearization of the NLS in different operation points. Notice that each LS has its own tangent state space, the functions δ_{M_i,M_j} couple the respective tangent state spaces.

2.3.1 Observability in Switched Linear Systems

Definition 2.33 *The SLS $\mathfrak{S} = \langle \mathcal{F}, (Q, M_0) \rangle$ is said to be observable at t_0 if there exists a finite $t_1 > t_0$ such that for any state X_0 at time t_0, the knowledge of the input $U[t_0, t_1]$ and the output $Y[t_0, t_1]$ over the finite time interval $[t_0, t_1]$, and the system structure, suffices to uniquely determine the initial and current state of $\mathfrak{S}$.*

It can be observed from previous definitions that if there exist two state trajectories $X_1(t)$, $X_2(t)$, with different initial conditions $X_1(0)$, $X_2(0)$, generating for any time (t) the same output $Y_1(t) = Y_2(t)$, when the same input $U(t)$ is given to the system in both trajectories, then the SLS is unobservable. It occurs since the information of $Y_1(t) = Y_2(t) = Y(t)$ and $U(t)$ is not enough to distinguish between initial conditions $X_1(0)$, $X_2(0)$, i.e. there exists no function $\Psi(Y(t), U(t)) = X_1(0)$, $\Psi(Y(t), U(t)) = X_2(0)$.

Definition 2.34 *Let $\mathcal{Q}$ be a dynamic system, then $\mathbf{X}_t(\mathcal{Q})$ denotes the set of finite time t state trajectories that are obtained applying all possible input u_i to every possible initial state condition.*

For $s_i \in \mathbf{X}_t(\Im)$, the notation $\boldsymbol{I}(s_i) = u_i[0,t]$ means that the state trajectory s_i was generated applying the input $u_i[0,t]$ to $\Im$ in the time interval $[0,t]$, and $\boldsymbol{O}(s_i) = y_i[0,t]$ denotes the output $y_i[0,t]$ generated by s_i.

Definition 2.35 *Let $\Im$ be a SLS. The state trajectories s_i and $s_j \in \mathbf{X}_\tau(\Im)$ are said to be Input Related, denoted as (IR) $s_i \sim s_j$, if $\boldsymbol{I}(s_i) = \boldsymbol{I}(s_j)$, and they are said to be Output Related, denoted as (OR) $s_i \sim s_j$, if $\boldsymbol{O}(s_i) = \boldsymbol{O}(s_j)$. Notice that both are equivalence relationships.*

The following equivalence classes can be defined in the set $\mathbf{X}_\tau(\Im)$

$$\begin{array}{c} C_{u_i}^{\Im} = \{s_p \mid \boldsymbol{I}(s_p) = u_i\}, \\ \text{and} \\ C_{y_i}^{\Im} = \{s_p \mid \boldsymbol{O}(s_p) = y_i\} \end{array} \tag{2.23}$$

Up to now, we have grouped the equivalent hybrid state trajectories related to the same input signal into the coset $\mathbf{X}_\tau(\Im)/IR = \Pi_{input}^{\Im} = \{C_{u_i}^{\Im}\}$ and to the same output signal into the coset $\mathbf{X}_\tau(\Im)/OR = \Pi_{output}^{\Im} = \{C_{y_i}^{\Im}\}$. Now, the observability of the SLS will be characterized using these partitions.

Theorem 2.36 *The SLS $\Im$ is observable if and only if there exists a time t such that $\forall i,j$, $\pi_{ij}^{\Im} = C_{u_i}^{\Im} \cap C_{y_j}^{\Im}$, $C_{u_i}^{\Im} \in \Pi_{input}^{\Im}$ and $C_{y_j}^{\Im} \in \Pi_{output}^{\Im}$, it holds that $\left|\pi_{ij}^{\Im}\right| = 1$, where $\left|\pi_{ij}^{\Im}\right|$ denotes the cardinality of the set $\pi_{ij}^{\Im}$.*

Proof. (Sufficiency). Assume that $\forall \pi_{ij}^{\Im} = C_{u_i}^{\Im} \cap C_{y_j}^{\Im}$ it holds that $\left|\pi_{ij}^{\Im}\right| = 1$. Then $s_k \in \pi_{ij}^{\Im}$ is the only sequence in $\pi_{ij}^{\Im}$. Thus a function $\Psi : U \times Y \longrightarrow \mathbf{X}_t(\Im)$ can be found, such that $\Psi(u_i, y_j) = s_k$, where $\boldsymbol{I}(s_k) = u_i$, $\boldsymbol{O}(s_k) = y_j$, and the initial and final states $x_k(0)$, $x_k(\tau)$ of s_k can be computed, i.e. there exists a function $\Lambda(s_k) = (x_k(0), x_k(\tau))$, then $\Lambda \circ \Psi(u_i, y_j) = (x_k(0), x_k(\tau))$.

(Necessity). Assume that there exists $\pi_{ij}^{\Im} = C_{u_i}^{\Im} \cap C_{y_j}^{\Im}$ such that $\left|\pi_{ij}^{\Im}\right| > 1$. Then there exist at least two sequences $s_k, s_l \in \pi_{ij}^{\Im}$, such that $\boldsymbol{I}(s_k) = \boldsymbol{I}(s_l)$ and $\boldsymbol{O}(s_k) = \boldsymbol{O}(s_l)$. Thus for the same u_i, y_j two s_k, s_l (or even more) state sequences are generated. Then, the function $\Psi : U \times Y \longrightarrow \mathbf{X}_t(\Im)$ does not exist. Then, using the input and output sequences is not enough to determine neither the sequence nor the initial and final states of the system. Thus the SLS is unobservable. ■

The previous result characterizes the observability in SLS. This characterization, however, uses all the possible state trajectories leading to complex analysis. Next chapters are devoted to show how to use previous theorem to derive a structural characterization in SLS.

Chapter 3

CONTINUOUS TO DISCRETE SYSTEM INFORMATION

This chapter presents three cases when the IPN is not observable. The cases are classified into the following categories: a) several markings producing the same output, b) several discrete markings sequences producing the same output information. The first case deals when several marking produce the same output, thus the discrete output information is not enough to determine the discrete state; the second case addresses the problem when several marking sequences produce the same discrete output information, thus the IPN information is not enough to determine which marking sequence was fired (i.e. the continuous systems sequence), thus it is impossible to compute, using the IPN information, which is the discrete marking.

Fortunately, if the continuous information meets some properties, then it will be possible to distinguish the discrete marking.

3.1 Several places producing the same output

Let us consider the IPN of Figure 3.1. This IPN is binary, live, and cyclic. Unfortunately, two markings produce the output B. Thus if symbol B is produced as output, for instance, it is impossible to determine if the system is in marking $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ or $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$

If this IPN is representing the discrete part of a SLS, then the only possibility to

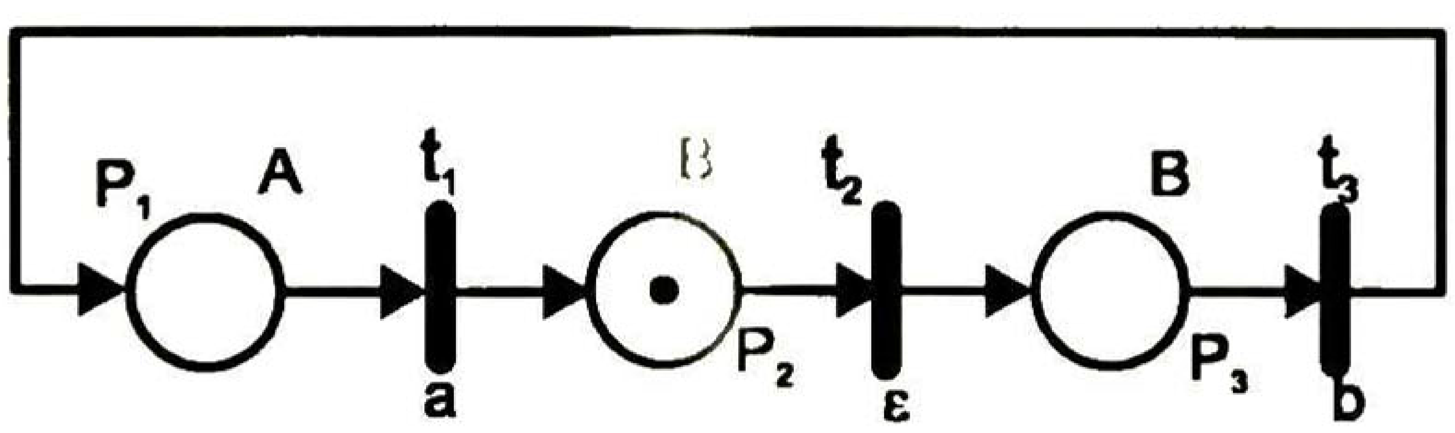


Figure 3.1: Several places producing the same output

distinguish each marking is using the continuous information. In order to do that, the outputs of the continuous systems must contain information to distinguish which linear system is working and afterwards, this information can be used to compute the discrete marking. System identification can be used to distinguish which LS is evolving, however this approach assumes that some white noise can be added to the LS inputs. If this hypothesis cannot be satisfied, then there still exists the possibility to distinguish both continuous systems. We named distinguishability to the property allowing to distinguish both systems without using an identification scheme.

Definition 3.1 *The linear systems $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ are said to be distinguishable from each other if the knowledge of the input $u[t_0, t_1]$ and the output $y[t_0, t_1]$ over the finite time interval $[t_0, t_1]$ suffices to determine which LS is evolving.*

Similar to the concept of observability two linear systems are indistinguishable from each other using the input and the output iff both systems generate the same output when the same input is applied.

Notation 3.2 *Let $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ be two SISO linear systems, the linear system $\bar{\Sigma}\{\bar{A}, \bar{B}, \bar{C}\}$ denotes the extended LS form with the pair of matrices*

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \\ \bar{B} &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \bar{C} &= \begin{bmatrix} C_1 & -C_2 \end{bmatrix}. \end{aligned} \tag{3.1}$$

Lemma 3.3 *Let $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ be two SISO LS where $A_1 \in \mathbb{R}^n$ and $A_2 \in \mathbb{R}^m$ Then the linear systems $\Sigma_1(A_1, B_1, C_1)$ and $\Sigma_2(A_2, B_2, C_2)$ are distinguishable from each other if and only if the only solution to the equation*

$$\bar{C}e^{\bar{A}t}\left[x_0 + \int_0^t e^{-\bar{A}\tau}\bar{B}u(\tau)d\tau\right] = 0 \tag{3.2}$$

is $x_0 = 0$ and $u(t) = 0$.

Proof. *If the linear systems $\Sigma_1(A_1, B_1, C_1)$ and $\Sigma_2(A_2, B_2, C_2)$ are indistinguishable from each other then there exists an input $u(t)$ such that the same output $y(t)$ is produced by both systems when $u(t)$ is applied, i.e. for two different initial conditions x_0^1, x_0^2 it holds that:*

$$y(t) = C_1 e^{A_1 t}\left[x_0^1 + \int_0^t e^{-A_1\tau}B_1 u(\tau)d\tau\right] \tag{3.3}$$

and

$$y(t) = C_2 e^{A_2 t}\left[x_0^2 + \int_0^t e^{-A_2\tau}B_2 u(\tau)d\tau\right] \tag{3.4}$$

then combining equations (3.3) and (3.4):

$$C_1 e^{A_1 t}\left[x_0^1 + \int_0^t e^{-A_1\tau} B_1 u(\tau)d\tau\right] = C_2 e^{A_2 t}\left[x_0^2 + \int_0^t e^{-A_2\tau} B_2 u(\tau)d\tau\right] \tag{3.5}$$

this equation can be written as

$$C_1 e^{A_1 t} x_0^1 - C_2 e^{A_2 t} x_0^2 = \int_0^t \left[-C_1 e^{-A_1(t-\tau)} B_1 + C_2 e^{-A_2(t-\tau)} B_2\right] u(\tau)d\tau. \tag{3.6}$$

Now, since 3.6 is equivalent to:

$$\begin{array}{c} \left[\begin{array}{cc} C_1 & -C_2 \end{array}\right] \left[\begin{array}{cc} e^{A_1 t} & 0 \\ 0 & e^{A_2 t} \end{array}\right] \left[\begin{array}{c} x_0^1 \\ x_0^2 \end{array}\right] = \\ -\int_0^t \left[\begin{array}{cc} C_1 & -C_2 \end{array}\right] \left[\begin{array}{cc} e^{A_1(t-\tau)} & 0 \\ 0 & e^{A_2(t-\tau)} \end{array}\right] \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] u(\tau)d\tau \end{array} \tag{3.7}$$

equation (3.7) can be written in terms of the matrices (3.1), with $x_0 = \left[\begin{array}{cc} x_0^1 & x_0^2 \end{array}\right]^T$, *then*

$$\bar{C} e^{\bar{A}t}\left[x_0 + \int_0^t e^{-\bar{A}\tau} \bar{B} u(\tau)d\tau\right] = 0. \tag{3.8}$$

Since $\Sigma_1(A_1, B_1, C_1)$ *is indistinguishable from* $\Sigma_2(A_2, B_2, C_2)$, *thus there exist solutions* $x_0 \neq 0$ *and* $u(t) \neq 0$ *to equation (3.8). The converse is also true, then* $\Sigma_1(A_1, B_1, C_1)$ *and* $\Sigma_2(A_2, B_2, C_2)$ *are indistinguishable from each other iff the only solution to equation (3.8) is* $x_0 = 0$ *and* $u(t) = 0$. ■

Remark 3.4 *The analysis presented in this work is restricted to inputs that can transformed into Laplace functions.*

Remark 3.5 *Taking the Laplace transform of equation (3.8) we obtain:*

$$\bar{C}\left(sI - \bar{A}\right)^{-1} x_0 + \bar{C}\left(sI - \bar{A}\right)^{-1} \bar{B}U(s) = 0 \tag{3.9}$$

which always has solutions $x_0 = -\alpha\bar{B}$ *and* $U(s) = \alpha$ *for any real constant* α *i.e. when the input is an impulse function. However, impulse functions cannot be implemented and therefore, if the analysis is restricted to inputs that can transformed into Laplace functions, then equation (3.8) has only solution* $x_0 = 0$, $u(t) = 0$ *iff the unique solution space of equation (3.9) is* $\bar{B}$.

Theorem 3.6 *Let* $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ *be two linear systems, where* $A_1 \in \mathbb{R}^{n\times n}$ *and* $A_2 \in \mathbb{R}^{m\times m}$ *Then the linear systems* $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ *are distinguishable from each other if and only if the realization* $\bar{\Sigma}\left\{\bar{A}, \bar{B}, \bar{C}\right\}$ *is minimal and its transfer function does not have transmission zeros (i.e.* $\bar{C}Adj\left(sI - \bar{A}\right)\bar{B} = \alpha$ *for some constant* $\alpha \neq 0$*), where* $\bar{A}$, $\bar{B}$, $\bar{C}$ *are the matrices of (3.1).*

Proof. *(Sufficiency) Assume that the realization* $\bar{\Sigma}\left\{\bar{A},\bar{B},\bar{C}\right\}$ *is minimal and has no transmission zeros, but the LS* $\Sigma_1(A_1,B_1,C_1)$ *and* $\Sigma_2(A_2,B_2,C_2)$ *are indistinguishable from each other, then there exist* $x_0 \neq 0$ *or* $u(t) \neq 0$ *such that equation (3.8) holds.*

Equation (3.9) can be written as:

$$\frac{\bar{C}Adj\left(sI-\bar{A}\right)x_0}{\det\left(sI-\bar{A}\right)}+\frac{\bar{C}Adj\left(sI-\bar{A}\right)\bar{B}}{\det\left(sI-\bar{A}\right)}U(s)=0, \tag{3.10}$$

since the realization $\left\{\bar{A},\bar{B},\bar{C}\right\}$ *is minimal, there is no cancellation of terms between*

$$\bar{C}Adj\left(sI-\bar{A}\right)\bar{B} \text{ and } \det\left(sI-\bar{A}\right), \tag{3.11}$$

and $U(s)$ *is:*

$$U(s)=-\frac{\bar{C}Adj\left(sI-\bar{A}\right)x_0}{\bar{C}Adj\left(sI-\bar{A}\right)\bar{B}} \tag{3.12}$$

however, since the transfer function $\bar{C}\left(sI-\bar{A}\right)^{-1}\bar{B}$ *has no transmission zeros,*

$$\bar{C}Adj\left(sI-\bar{A}\right)\bar{B} \tag{3.13}$$

does not contain terms in s. Thus in order to $U(s)$ *be a proper transfer function,* x_0 *needs to be equal to* $\alpha\bar{B}$ *for any* α*. Therefore* $U(s)=\alpha$*, i.e.* $U(s)$ *is an impulse function or Dirac delta function. Then according to Lemma 3.3 and remark 3.5 the linear systems* $\Sigma_1(A_1,B_1,C_1)$, $\Sigma_2(A_2,B_2,C_2)$ *are distinguishable from each other. A contradiction.*

(Necessity) We show that if the realization $\left\{\bar{A},\bar{B},\bar{C}\right\}$ *is not minimal (noncontrollable or nonobservable) or has transmission zeros then the linear systems* $\Sigma_1(A_1,B_1,C_1)$ *and* $\Sigma_2(A_2,B_2,C_2)$ *are indistinguishable from each other. The proof is divided in three parts.*

a) The pair $\left\{\bar{A},\bar{C}\right\}$ *is nonobservable.*

Then, for input $u(t)=0$*, the Laplace transform of equation (3.6), becomes:*

$$C_1\left(sI-A_1\right)^{-1}x_0^1-C_2\left(sI-A_2\right)^{-1}x_0^2=0 \tag{3.14}$$

i.e.

$$\bar{C}\left(sI-\bar{A}\right)^{-1}x_0=0. \tag{3.15}$$

Since the pair $\left\{\bar{A},\bar{C}\right\}$ *is nonobservable then the columns of* $\bar{C}\left(sI-\bar{A}\right)^{-1}$ *are linearly dependent [1]. Hence, there exists a vector* $x_0 \neq 0$ *such that the equation (3.15) holds. Thus, for input* $u(t)=0$*, there exist initial conditions* $x_0^1 \neq 0$ *or* $x_0^2 \neq 0$ *such that the LS* $\Sigma_1(A_1,B_1,C_1)$, $\Sigma_2(A_2,B_2,C_2)$ *are indistinguishable from each other [5].*

b) There is a transmission zero $\lambda \in \mathbb{C}$ *of* $\bar{C}\left(sI-\bar{A}\right)^{-1}\bar{B}$*.*

Then the output due to the initial state:

$$x_0=-\left(\bar{A}-\lambda I\right)^{-1}Bk \tag{3.16}$$

and the input

$$u(t) = ke^{\lambda t} \tag{3.17}$$

is identically zero [1]. Hence, equation (3.8) is satisfied, and the LS $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ *are indistinguishable from each other.*

c) *The pair* $\{\bar{A}, \bar{C}\}$ *is noncontrollable.*

Since

$$H(s) = \frac{N(s)}{D(s)} = \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B} \tag{3.18}$$

and

$$\bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B} = \frac{\bar{C}Adj\left(sI - \bar{A}\right)\bar{B}}{\det\left(sI - \bar{A}\right)} \tag{3.19}$$

then

$$\bar{C}Adj\left(sI - \bar{A}\right)\bar{B}/\det\left(sI - \bar{A}\right) \tag{3.20}$$

are not irreducible and

$$\deg(D(s)) < \deg(\det\left(sI - \bar{A}\right)), \tag{3.21}$$

because terms in $\bar{C}Adj\left(sI - \bar{A}\right)\bar{B}$ *were cancelled with terms in* $\det\left(sI - \bar{A}\right)$, *thus there exist constants* $\lambda_i \in \mathbb{C}$ *and an input*

$$U(s) = \frac{\alpha}{(s - \lambda_1)\cdots(s - \lambda_p)} \tag{3.22}$$

with $p < \hat{n} = n + m$, *such that*

$$D(s)(s - \lambda_1)\cdots(s - \lambda_p) = \det\left(sI - \bar{A}\right) \tag{3.23}$$

where $\lambda_i, i = 1, \ldots, p$ *are the noncontrollable modes, where the modes of* $\bar{A}$ *are the roots of* $\det\left(sI - \bar{A}\right)$. *And therefore,* $H(s)U(s)$ *is:*

$$H(s)U(s) = \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B}U(s) = \frac{\alpha N(s)}{\det\left(sI - \bar{A}\right)} \tag{3.24}$$

since $H(s)$ *has no transmission zeros, then* $N(s)$ *is constant, and there exists a constant* β *such that* $\beta = \alpha N(s)$. *And therefore, there exist a constant vector* x_0 *such that the equation*

$$\bar{C}Adj\left(sI - \bar{A}\right)x_0 = \beta, \tag{3.25}$$

holds. Since the left hand side of equation (3.25) is [13]:

$$\bar{C}Adj\left(sI - \bar{A}\right) = C\left(S_1 s^{\hat{n}-1} + \cdots + S_{\hat{n}}\right) \tag{3.26}$$

where S_i can be recursively computed as:

$$\begin{aligned} S_1 &= I, \\ S_k &= S_{k-1}A + a_{k-1}I \text{ for } k = 2, \ldots, \hat{n} \\ 0 &= S_{\hat{n}} + a_{\hat{n}} I \end{aligned} \tag{3.27}$$

with a_i, $i = 1, \ldots, \hat{n}$ being the coefficient of the characteristic polynomial of $\bar{A}$, i.e..

$$\det\left(sI - \bar{A}\right) = s^{\hat{n}} + a_1 s^{\hat{n}-1} + \cdots + a_{\hat{n}} \tag{3.28}$$

and

$$\det\left(sI - \bar{A}\right) = \det\left(sI - A_1\right)\det\left(sI - A_2\right). \tag{3.29}$$

Therefore, the equation (3.25) holds. if the following matrix equation is satisfied

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & 0 \\ a_{\hat{n}-1} & a_{\hat{n}-2} & a_{\hat{n}-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \vdots \\ \bar{C}\bar{A}^{\hat{n}-1} \end{bmatrix} x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -\beta \end{bmatrix} \tag{3.30}$$

since the pair $\left\{\bar{A}, \bar{C}\right\}$ is observable $x_0 \neq 0$ can be computed. Thus, there exist $x_0 \neq 0$ and $u(t) \neq 0$ satisfying equation (3.8), thus the linear systems $\Sigma_1\left(A_1, B_1, C_1\right)$ and $\Sigma_2\left(A_2, B_2, C_2\right)$ are indistinguishable from each other.

■

Notation 3.7 *In this work $\mathbf{T}(A)$ represents the matrix*

$$\mathbf{T}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{bmatrix} \tag{3.31}$$

where a_i is the $i-th$ coefficient of the characteristic polynomial of A i.e.

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n, \tag{3.32}$$

and $\mathbf{T}^{-1}(A)$ is the inverse of the matrix $\mathbf{T}(A)$, note that $\mathbf{T}^{-1}(A)$ is also a lower triangular matrix with ones over the diagonal. The matrix $\mathbf{T}(A)$ is known as a Toeplitz matrix.

Notation 3.8 *If $N(s) = \beta\left(b_0 s^p + b_1 s^{p-1} + \cdots + b_p\right)$ then the n-dimensional vector $\mathbf{v}_n(N(s))$ is*

$$\mathbf{v}_n(D(s)) = \begin{bmatrix} 0 & \cdots & 0 & \beta b_0 & \cdots & \beta b_p \end{bmatrix}^T \tag{3.33}$$

where $p < n$.

Remark 3.9 *Notice that if the linear systems $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ are indistinguishable then the class of inputs $u(t)$ that force two linear systems $\Sigma_1(A_1, B_1, C_1)$ and $\Sigma_2(A_2, B_2, C_2)$ to exhibit the same output $y(t)$ are those with Laplace transform*

$$U(s) = \frac{N(s)}{D(s)} \tag{3.34}$$

with

$$\begin{array}{c} N(s) = \beta\left(b_0 s^p + b_1 s^{p-1} + \cdots + b_p\right) \\ D(s) = \bar{C}Adj\left(sI - \bar{A}\right)\bar{B} \end{array} \tag{3.35}$$

where $p < \deg(\bar{C}Adj\left(sI - \bar{A}\right)\bar{B})$, β is a real constant, and the parameters b_i $i = 0, \ldots, p$ are such that $\mathbf{T}^{-1}(\bar{A})\mathbf{v}_{\hat{n}}(N(s)) \in \operatorname{Im}\left(\mathcal{O}\left(\bar{A}, \bar{C}\right)\right)$.

Example 3.10 *Consider the LS $\Sigma_1(A_1, B_1, C_1)$ and $\Sigma_2(A_2, B_2, C_2)$ where*

$$A_1 = -\frac{1}{RC}, \quad B_1 = \frac{1}{RC}, \quad C_1 = 1 \tag{3.36}$$

and

$$\begin{array}{c} A_2 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} \\ C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array} \tag{3.37}$$

with R, L, C constants.

Since $\bar{C}Adj\left(sI - \bar{A}\right)\bar{B} = (RC)^{-1} s^2$, with $\bar{A}, \bar{B}, \bar{C}$ computed as in equation (3.1), then the output of both systems due to the input $u(t) = 1$ and the initial conditions $x_0^1 = 1$, $x_0^2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is $y(t) = 1$, Thus the dynamic equations are indistinguishable from each other.

Notice that the only class of inputs making both systems to have the same output is:

$$U(s) = \frac{\alpha(s + \beta)}{s^2} \tag{3.38}$$

for any real constants β and α.

The concept of indistinguishability between state realizations can also be stated in terms of the Markov parameters as follows.

Corollary 3.11 *The linear systems $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ are distinguishable from each other if and only $C_1 A_1^i B_1 = C_2 A_2^i B_2$ for $i = 0, \ldots, \hat{n} - 2$ and $C_1 A_1^{\hat{n}-1} B_1 \neq C_2 A_2^{\hat{n}-1} B_2$, with $\hat{n} = n + m$.*

Proof. *It is easy to see that a realization $\left\{\bar{A}, \bar{B}, \bar{C}\right\}$ is minimal and has no transmission zeros if and only if $\bar{C}Adj\left(sI - \bar{A}\right)\bar{B} = \alpha$ for some real constant $\alpha \neq 0$. That is*

$$\begin{array}{c} \bar{C}S_k\bar{B} = 0 \text{ for } k = 1, \ldots, \hat{n} - 1 \\ \text{and} \\ \bar{C}S_{\hat{n}}\bar{B} = \alpha \text{ with } \alpha \neq 0 \end{array}, \tag{3.39}$$

thus, writing the conditions of (3.39) as a matrix equation yield

$$\mathbf{T}(\bar{A})\begin{bmatrix} \bar{C}\bar{B} \\ \vdots \\ \bar{C}\bar{A}^{\hat{n}-2}\bar{B} \\ \bar{C}\bar{A}^{\hat{n}-1}\bar{B} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\alpha \end{bmatrix} \tag{3.40}$$

since

$$\mathbf{T}^{-1}(\bar{A})\begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\alpha \end{bmatrix} \tag{3.41}$$

then

$$\begin{bmatrix} \bar{C}\bar{B} \\ \vdots \\ \bar{C}\bar{A}^{\hat{n}-2}\bar{B} \\ \bar{C}\bar{A}^{\hat{n}-1}\bar{B} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\alpha \end{bmatrix} \tag{3.42}$$

since

$$\bar{C}\bar{A}^k\bar{B} \doteq C_1A_1^kB_1 - C_2A_2^kB_2 \tag{3.43}$$

it follows that $C_1A_1^iB_1 = C_2A_2^iB_2$ *for* $i = 0, \hat{n}-2$ *and* $C_1A_1^{\hat{n}-1}B_1 \neq C_2A_2^{\hat{n}-1}B_2$. ∎

The terms

$$h_i = CA^{i-1}B \quad i = 1, 2, \ldots$$

are known as the Markov parameters of the system.

Corollary 3.12 *The linear systems* $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ *are distinguishable from each other iff the system matrix*

$$\begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} \tag{3.44}$$

has full rank for all $s \in \mathbb{C}$.

Proof. *Because* $\bar{C}Adj\left(sI - \bar{A}\right)\bar{B}$ *can be written as [13]:*

$$\bar{C}Adj\left(sI - \bar{A}\right)\bar{B} = \det\begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix}, \tag{3.45}$$

then since the LS $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ *are distinguishable from each other iff* $\bar{C}Adj\left(sI - \bar{A}\right)\bar{B} = \alpha$ *for some* $\alpha \neq 0$. *Then the LS* $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ *are distinguishable from each other iff the matrix*

$$\begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} \tag{3.46}$$

has full rank for all $s \in \mathbb{C}$. ∎

Corollary 3.13 *If the linear systems $\Sigma_1(A_1,B_1,C_1)$, $\Sigma_2(A_2,B_2,C_2)$ are distinguishable then the individual realizations are observable and controllable.*

Proof. *Without loss of generality assume that the pair $\{A_1,B_1\}$ is noncontrollable, then there exists a vector* $v=\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \neq 0$ *with $v_2=v_3=0$ and $v_1 \neq 0$, such that*

$$v_1 \begin{bmatrix} sI - A_1 & B_1 \end{bmatrix} = 0 \tag{3.47}$$

and

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} sI - A_1 & 0 & B_1 \\ 0 & sI - A_2 & B_2 \\ C_1 & -C_2 & 0 \end{bmatrix} = 0 \tag{3.48}$$

and therefore the linear systems Σ_1 and Σ_2 are indistinguishable. The proof showing that the individual systems must be observable is very similar. ∎

In order to present the distinguishability concept in the geometrical context the indistinguishability subspace will be presented next and it will be used to give a useful characterization of distinguishability between linear systems.

Definition 3.14 *Let $\Sigma_1(A_1,B_1,C_1)$, $\Sigma_2(A_2,B_2,C_2)$ be two LS of dimension n and m respectively. The subspace $\bar{\mathcal{W}} \subseteq \bar{\mathcal{X}}$ denotes the set of $x_0 \in \bar{\mathcal{X}}$ such that there exists an input $u(t)$ making both linear systems $\Sigma_1(A_1,B_1,C_1)$ and $\Sigma_2(A_2,B_2,C_2)$ to be indistinguishable from each other, where $\bar{\mathcal{X}}$ is the external direct sum of $\mathcal{X}_1$ and $\mathcal{X}_2$, written $\mathcal{X}_1 \tilde{\oplus} \mathcal{X}_2$ and defined as [14]:*

$$\bar{\mathcal{X}} = \mathcal{X}_1 \tilde{\oplus} \mathcal{X}_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 \in \mathcal{X}_1 \text{ and } x_2 \in \mathcal{X}_2 \right\}.$$

$\bar{\mathcal{W}}$ is named the indistinguishability subspace of $\Sigma_1(A_1,B_1,C_1)$, $\Sigma_2(A_2,B_2,C_2)$. The subspaces $\mathcal{W}_i \in \mathcal{X}_i$ for $i=1,2$ are defined as:

$$\mathcal{W}_i = \left\{ x_0^i : x_0^i \in \mathcal{X}_i \text{ and } \begin{bmatrix} x_0^1 & x_0^2 \end{bmatrix}^T \in \bar{\mathcal{X}} \right\} \tag{3.49}$$

Proposition 3.15 *If* $\deg(\bar{C}Adj(sI-\bar{A})\bar{B}) = q$ *then the indistinguishability subspace $\bar{\mathcal{W}}$ of $\Sigma_1(A_1,B_1,C_1)$, $\Sigma_2(A_2,B_2,C_2)$ is*

$$\bar{\mathcal{W}} = \bar{\mathcal{N}} + \bar{\mathcal{V}}_1 + \cdots + \bar{\mathcal{V}}_{q-1} \tag{3.50}$$

where $\bar{\mathcal{N}}$ is the unobservable subspace of $(\bar{C},\bar{A})$, $\bar{\mathcal{V}}_1 + \bar{\mathcal{V}}_2 \subset \bar{\mathcal{X}}$ is defined as:

$$\bar{\mathcal{V}}_1 + \bar{\mathcal{V}}_2 = \{a+b : a \in \mathcal{V}_1, b \in \mathcal{V}_2\}. \tag{3.51}$$

$$\bar{\mathcal{V}}_i = \left\{ x : \exists k \ \mathbf{T}(\bar{A})\mathcal{O}(\bar{C},\bar{A})\, x = k e_i,\ i = 1,\ldots,q-1 \right\} \tag{3.52}$$

with e_i is being a $\hat{n}$ dimensional vector where the $j-th$ element is zero if $j \neq i$ and one otherwise.

Proposition 3.16 *If $\bar{B} = 0$ then the indistinguishability subspace $\bar{\mathcal{W}}$ of $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ is equal to the unobservable subspace of $(\bar{C}, \bar{A})$, that is $\bar{\mathcal{W}} = \bar{\mathcal{N}}$.*

Corollary 3.17 *The linear systems $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ are distinguishable from each other if and only $\bar{\mathcal{W}}$ =0.*

Proof. *(Sufficiency) Suppose that $\bar{\mathcal{W}}$ =0, that is, there is no $x_0 \neq 0$ such that for some $u(t)$ equation (3.8) holds. It is next proved that, if $\bar{\mathcal{W}}$ =0 then $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ are distinguishable from each other, that is , with $x_0 = 0$,the only solution to equation (3.9) is $U(s) = 0$, i.e.*

$$\bar{C}Adj\,(sI - \bar{A})\,\bar{B}U(s) = 0. \tag{3.53}$$

Equation (3.53) holds if a) $U(s) = 0$ or b) $\bar{C}Adj\,(sI - \bar{A})\,\bar{B}$. It only remains to prove that b) does not occurs with $B \neq 0$ (because if $B \neq 0$, then both equations are autonomous and $\bar{\mathcal{W}} = \bar{\mathcal{N}}$ =0 means that $\Sigma_1(A_1, B_1, C_1)$ and $\Sigma_2(A_2, B_2, C_2)$ are distinguishable from each other [5]).

Condition b) can be written as

$$\mathbf{T}(\bar{A})\mathcal{O}\,(\bar{C}, \bar{A})\,\bar{B} = 0 \tag{3.54}$$

or

$$\mathcal{O}\,(\bar{C}, \bar{A})\,\bar{B} = 0. \tag{3.55}$$

Since $\bar{\mathcal{W}}$ =0 then $\bar{\mathcal{N}}$ =0, i.e. rank $\left(\mathcal{O}\left(\bar{C}, \bar{A}\right)\right) = \hat{n}$, and therefore condition b) only holds with $B = 0$. And $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ are distinguishable from each other.

(Necessity) According to Theorem 3.6 if Σ_1 and Σ_2 are distinguishable from each other then $\bar{\mathcal{N}} = 0$, and the only solution of equation (3.8) is $x_0 \neq 0$ and $u(t) \neq 0$, thus by definition of $\bar{\mathcal{W}}$, $\bar{\mathcal{W}}$ =0. ■

3.2 Several marking sequences producing the same output

The *IPN* depicted in Figure 3.2 represents the case when several places produce the same output information, as in the previous case, however, the systems may be indistinguishable, thus previous results cannot be used in order to gain observability in the discrete system. Moreover, the sequence of markings M_1M_3 produce the same input-output information that the sequences of markings M_2M_4, thus it is impossible, using the discrete information, to distinguish between both discrete sequences.

However, if some information is obtained from the continuous system commutation, then it is possible to know which discrete sequence was fired. The idea is to use the characterization of indistinguishability subspace for two *LS* and extend this idea to continuous system commutations and use it to distinguish between firing sequences presenting the same discrete output.

Definition 3.18 *The sequence of LS $\Phi(M_i) \cdots \Phi(M_k)$ and $\Phi(M_i') \cdots \Phi(M_k')$ where $M_i \cdots M_k \neq M_i' \cdots M_k'$, are said to be distinguishable from each other, if the information of the input and the output of the LS suffices to determine which sequence of LS evolves.*

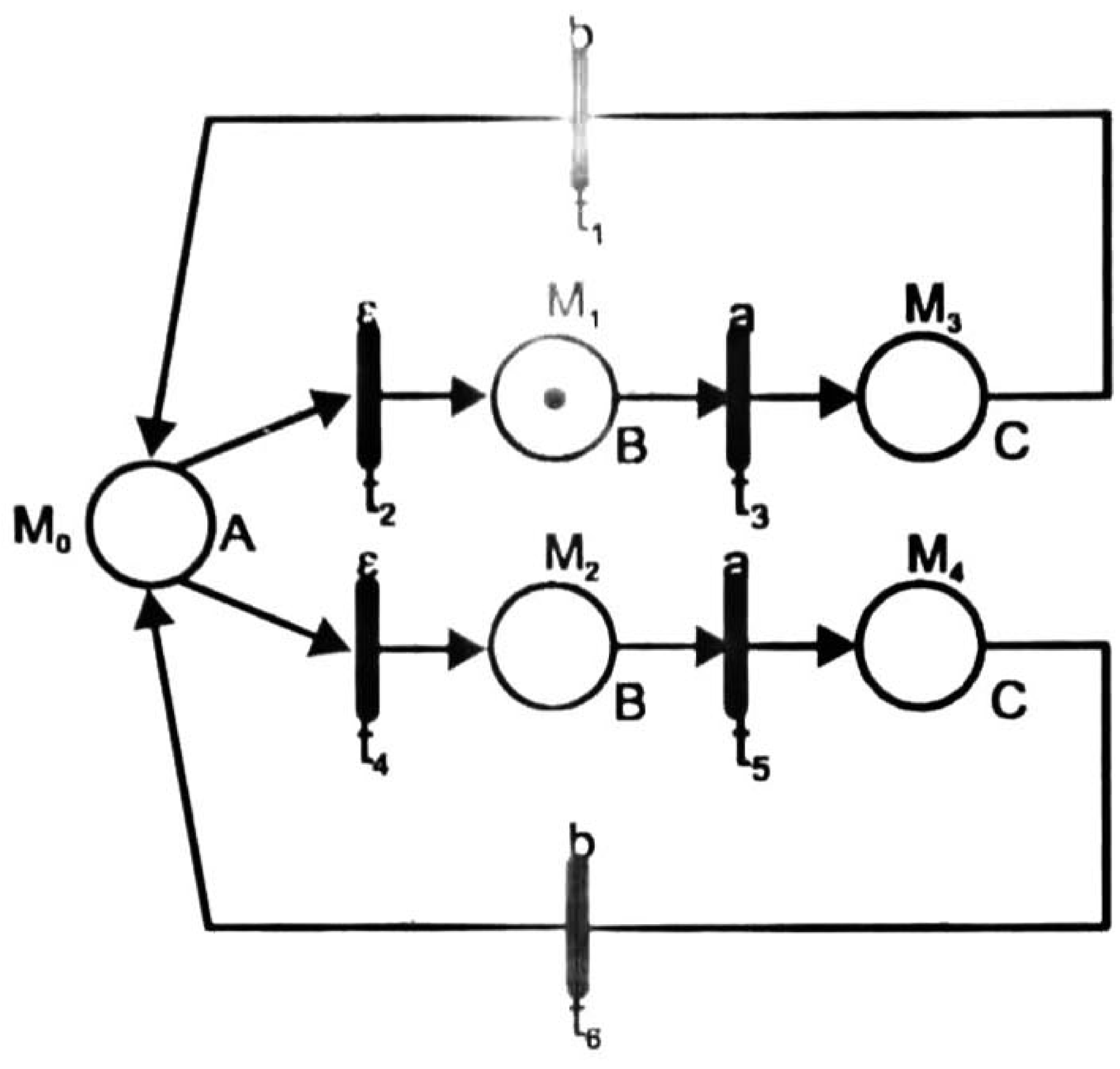


Figure 3.2: An IPN where two making sequences produce the same output

It follows from Theorem 2.36 that the sequence of LS $\Phi(M_i)\cdots\Phi(M_k)$ and $\Phi(M_i')\cdots\Phi(M_k')$ are distinguishable from each other, iff the sequence produce different outputs when the same input is applied.

Notation 3.19 *Hereafter the realization* $\{\bar{A}_k, \bar{B}_k, \bar{C}_k\}$ *denotes the realization form with the matrices*

$$\begin{aligned}\bar{A}_k &= \begin{bmatrix} A_{M_k} & 0 \\ 0 & A_{M_k'} \end{bmatrix} \\ \bar{B}_k &= \begin{bmatrix} B_{M_k} \\ B_{M_k'} \end{bmatrix} \\ \bar{C}_k &= \begin{bmatrix} C_{M_k} & -C_{M_k'} \end{bmatrix}\end{aligned} \tag{3.56}$$

for some markings M_k *and* M_k' *where* $\Phi(M_k) = \Sigma_{M_k}(A_{M_k}, B_{M_k}, C_{M_k})$ *and* $\Phi(M_k') = \Sigma_{M_k'}\left(A_{M_k'}, B_{M_k'}, C_{M_k'}\right)$.

Lemma 3.20 *Let* $\Phi(M_i)\cdots\Phi(M_k)$ *and* $\Phi(M_i')\cdots\Phi(M_k')$ *where* $M_i\cdots M_k \neq M_i'\cdots M_k'$ *be two sequence of* LS *generating the same discrete input-output information, then* $\Phi(M_i)\cdots\Phi(M_k)$ *and* $\Phi(M_i')\cdots\Phi(M_k')$ *are distinguishable from each other iff the only solution to the equation*

$$\begin{aligned}\bar{C}_i e^{\bar{A}_i(t-t_0)}\left[x(t_0) + \int_{t_0}^{t} e^{-\bar{A}_i(t-t_0-\tau)}\bar{B}_i u(\tau)d\tau\right] &= 0 \text{ for } t \in [t_0, t_1] \\ &\vdots \\ \bar{C}_k e^{\bar{A}_k(t-t_{\hat{k}})}\left[x(t_1) + \int_{t_1}^{t} e^{-\bar{A}_k(t-t_{\hat{k}}-\tau)}\bar{B}_k u(\tau)d\tau\right] &= 0 \text{ for } t \in \left(t_{\hat{k}}, t_{\hat{k}+1}\right)\end{aligned} \tag{3.57}$$

is $x_0 = 0$ *and* $u(t) = 0$, *where* $x(t_i) = \bar{\delta}_{i,j} e^{A_i(t_1 - t_0)} x(t_{i-1})$ *with* $\bar{\delta}_{i,j} = \begin{bmatrix} \delta_{M_i,M_j} & 0 \\ 0 & \delta_{M_i',M_j'} \end{bmatrix}$

Proof. *For simplicity the proof is made for sequence of LS of length two.*

(Sufficiency) Assume that there exist two sequence of LS $\Phi(M_i)\Phi(M_j)$ *and* $\Phi(M_i')$ $\Phi(M_j')$ *with* $M_iM_j \neq M_i'M_j'$ *such that* M_iM_j *and* $M_i'M_j'$ *produce the same discrete input-output sequence* w. *Now suppose that the corresponding continuous trajectories* $\Phi(M_i)\Phi(M_j)$ *and* $\Phi(M_i')\Phi(M_j')$ *produce the same continuous output for some input* $u(t)$, *that is*

$$y_1(t) = \begin{cases} C_{M_i} e^{A_{M_i}(t-t_0)} x_1(t_0) + \int_{t_0}^{t} C_{M_i} e^{A_{M_i}(t-\tau)} B_{M_i} u(\tau) d\tau & t \in [t_0, t_1] \\ C_{M_j} e^{A_{M_j}(t-t_1)} x_1(t_1) + \int_{t_1}^{t} C_{M_j} e^{A_{M_j}(t-t_1-\tau)} B_{M_j} u(\tau) d\tau & t \in (t_1, t_2) \end{cases} \tag{3.58}$$

and

$$y_2(t) = \begin{cases} C_{M_i'} e^{A_{M_i'}(t-t_0)} x_2(t_0) + \int_{t_0}^{t} C_{M_i'} e^{A_{M_i'}(t-\tau)} B_{M_i'} u(\tau) d\tau & t \in [t_0, t_1] \\ C_{M_j'} e^{A_{M_j'}(t-t_1)} x_2(t_1) + \int_{t_1}^{t} C_{M_j'} e^{A_{M_j'}(t-t_1-\tau)} B_{M_j'} u(\tau) d\tau & t \in (t_1, t_2) \end{cases} \tag{3.59}$$

with $y_{c_1} = y_{c_2}$.*Now, substracting* $y_{c_2}(t)$ *from* $y_{c_1}(t)$ *and writing the equation in terms of the matrices (3.56):*

$$\begin{array}{c} \bar{C}_i e^{\bar{A}_i(t-t_0)} \left[x(t_0) + \int_{t_0}^{t} e^{-\bar{A}_i(t-\tau)} \bar{B}_i u(\tau) d\tau \right] = 0 \ for\ t \in [t_0, t_1] \\ and \\ \bar{C}_j e^{\bar{A}_j(t-t_1)} \left[x(t_1) + \int_{t_1}^{t} e^{-\bar{A}_j(t-t_1-\tau)} \bar{B}_j u(\tau) d\tau \right] = 0 \ for\ t \in (t_1, t_2) \end{array} \tag{3.60}$$

therefore equation (3.60) holds for some $u(t)$ *and* $x(t_0)$, *where* $u(t) \neq 0$ *or* $x(t_0) \neq 0$.

The necessity follows using a similar procedure. ■

Lemma 3.21 *Let* $\Phi(M_i)\cdots\Phi(M_k)$ *and* $\Phi(M_i')\cdots\Phi(M_k')$ *where* $M_i\cdots M_k \neq M_i'\cdots M_k'$ *be two sequence of LS generating the same discrete input-output information, then* $\Phi(M_i)\cdots$ $\Phi(M_k)$ *and* $\Phi(M_i')\cdots\Phi(M_k')$ *are distinguishable from each other if*

$$\begin{array}{c} (\bar{\mathcal{W}}_i \cap \bar{\mathcal{W}}_j \bar{\delta}_{i,j} = 0) \vee (\bar{\mathcal{W}}_j \cap \bar{\mathcal{W}}_{j+1} \bar{\delta}_{j,j+1} = 0) \\ \vee \cdots \vee (\bar{\mathcal{W}}_{k-1} \cap \bar{\mathcal{W}}_k \bar{\delta}_{k-1,k} = 0) \end{array} \tag{3.61}$$

where $\bar{\mathcal{W}}_i$ *is the indistinguishability subspace of* Σ_{M_i}, $\Sigma_{M_i'}$.

Proof. *For simplicity the proof is made for sequence of LS of length equal to two.*

Assume that $M_iM_j, M_i'M_j' \in S_w$ *where* $M_iM_j \neq M_i'M_j'$ *and* $\bar{\mathcal{W}}_i \cap \bar{\mathcal{W}}_j \bar{\delta}_{i,j} = 0$ *then is easy to see that equation*

$$\begin{array}{c} \bar{C}_i e^{\bar{A}_i(t-t_0)} \left[x(t_0) + \int_{t_0}^{t} e^{-\bar{A}_i(t-\tau)} \bar{B}_i u(\tau) d\tau \right] = 0 \ for\ t \in [t_0, t_1] \\ and \\ \bar{C}_j e^{\bar{A}_j(t-t_1)} \left[x(t_1) + \int_{t_1}^{t} e^{-\bar{A}_j(t-t_1-\tau)} \bar{B}_j u(\tau) d\tau \right] = 0 \ for\ t \in (t_1, t_2) \end{array} \tag{3.62}$$

does not hold. Notice that, if

$$\bar{C}_i e^{\bar{A}_i(t-t_0)}\left[x(t_0)+\int_{t_0}^{t} e^{-\bar{A}_i\tau}\bar{B}_i u(\tau)d\tau\right]=0 \tag{3.63}$$

for $t\in[t_0,t_1)$, *then the continuous state belongs to the indistinguishability subspace of* $\Sigma_{M_i}(A_{M_i},B_{M_i},C_{M_i})$, $\Sigma_{M_i'}\left(A_{M_i'},B_{M_i'},C_{M_i'}\right)$ *i.e.* $x(t)\in\bar{\mathcal{W}}_i$ *for* $t\in[t_0,t_1)$ *because there exists an input* $u(t)$ *that make the LS systems* $\Sigma_{M_i}(A_{M_i},B_{M_i},C_{M_i})$ *to be indistinguishable from* $\Sigma_{M_i'}\left(A_{M_i'},B_{M_i'},C_{M_i'}\right)$ *for some initial condition* $x(t_0)$. *In a similar way* $x(t)\in\bar{\mathcal{W}}_j$ *for* $t\in[t_1,t_2)$. *Since* $x(t_1)=\bar{\delta}_{i,j}e^{\bar{A}_1(t_1-t_0)}x(t_0)$ *then if equation (3.62) holds for* $x_0\neq 0$ *or* $u(t)\neq 0$ *there exists a vector* $z\neq 0\in\bar{\mathcal{W}}_i\cap\bar{\mathcal{W}}_j\bar{\delta}_{i,j}$. ■

3.3 Detection of the commutation time

In *IPN* as in *SLS* it is necessary to be able to detect the time when the system commute to another state as well as to be able to compute this new state to characterize the system observability If this is not possible then the system is not sequence- nor marking-detectable, and therefore is unobservable. In the *IPN* of Figure 3.3 the time when the system commute from marking $\begin{bmatrix}0&1&0\end{bmatrix}^T$ to marking $\begin{bmatrix}0&1&0\end{bmatrix}^T$ cannot be detected using the input-output information of the *IPN* and in the *IPN* of Figure 3.4 is not possible to compute the new state when the system commute from marking $\begin{bmatrix}0&1&1&0&0\end{bmatrix}^T$ to marking $\begin{bmatrix}0&0&1&1&0\end{bmatrix}^T$, since this commutation generates the same input-output information than the commutation from marking $\begin{bmatrix}0&1&1&0&0\end{bmatrix}^T$ to marking $\begin{bmatrix}0&1&0&0&1\end{bmatrix}^T$; i.e. both *IPN* are unobservable.

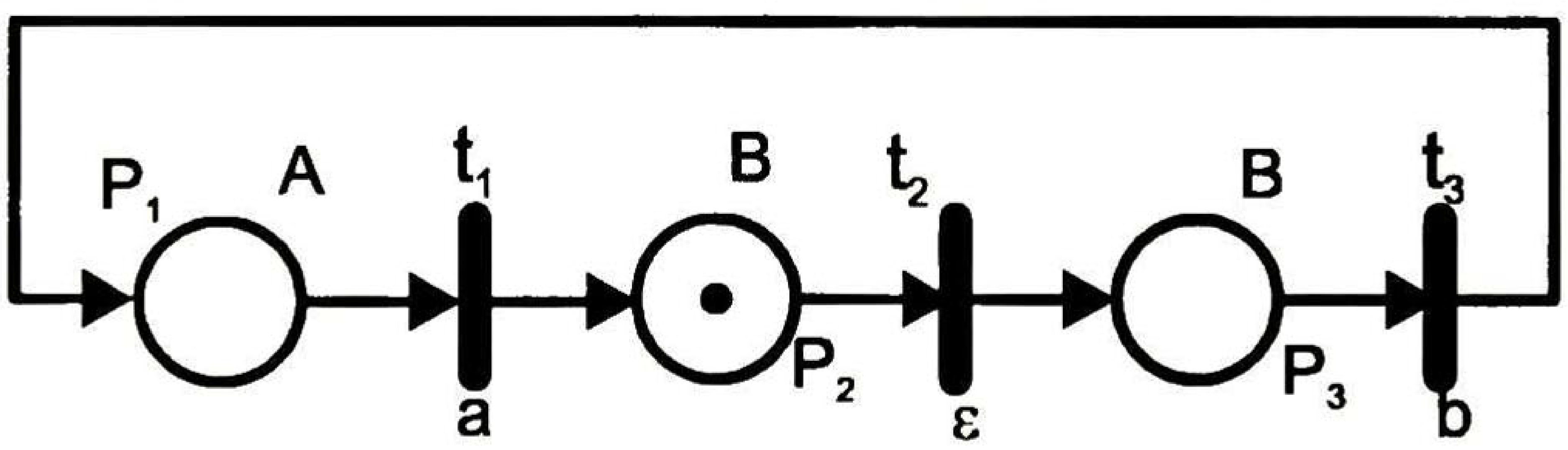


Figure 3.3: An *IPN* where the commutation time cannot be detected

Next, the conditions under which a commutation in the *SLS* can be detected are presented.

Proposition 3.22 *The time when the SLS commute from a state* M_j *to a state* M_k *when the transition* $\mathbf{t}_a$ *is fired (i.e.* $M_j\xrightarrow{\mathbf{t}_a}M_k$*), is detected using the input-output information of the IPN iff*

$$\lambda(\mathbf{t}_a)\neq\varepsilon \text{ or } \varphi C(\bullet,\mathbf{t}_a)\neq 0 \tag{3.64}$$

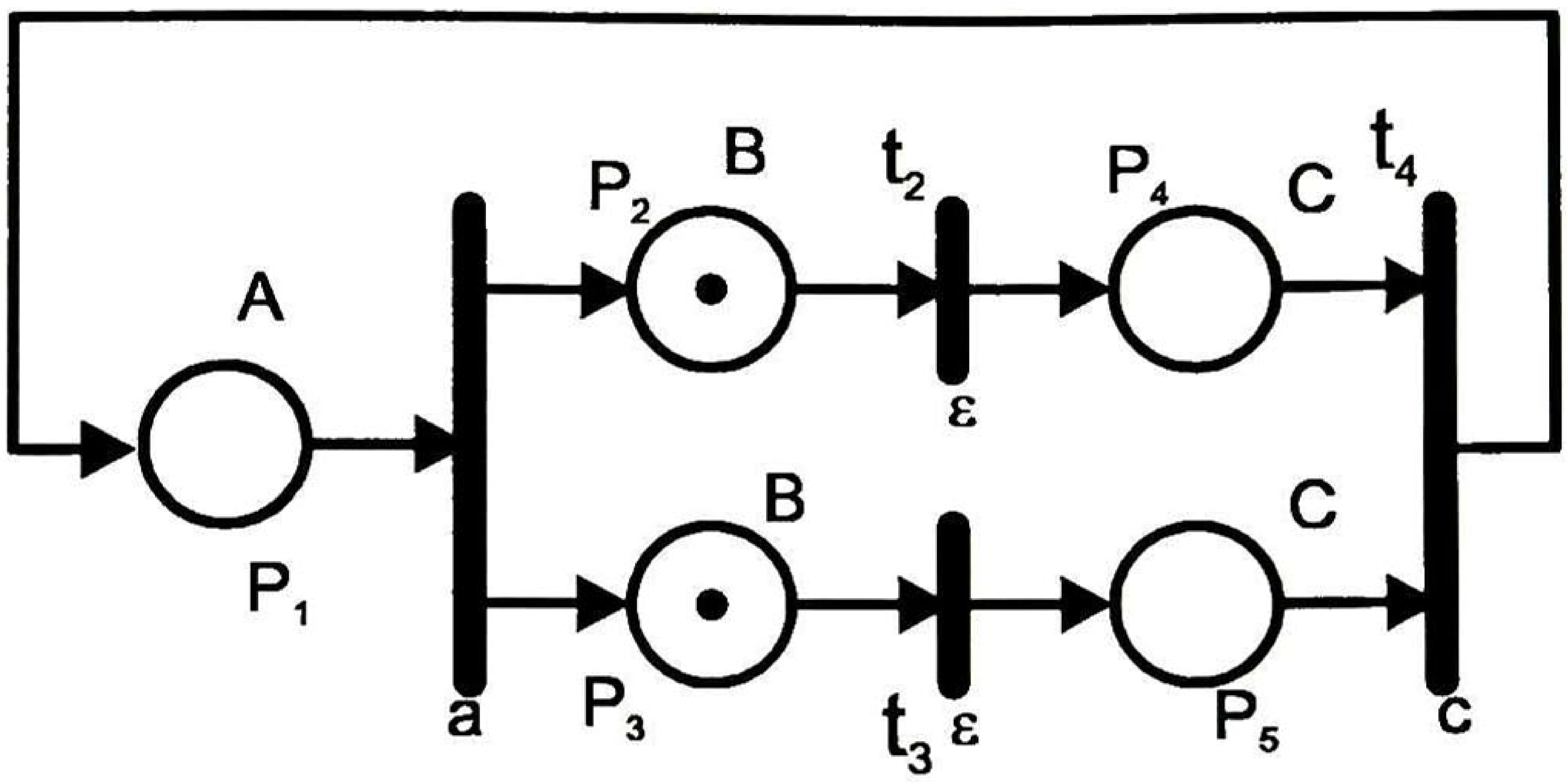


Figure 3.4: An IPN where the new state cannot be computed

If the commutation time cannot be detected using the information of the discrete systems it is possible to detect the commutation time using the information provided by the continuous states, as follows.

Proposition 3.23 *The time t_1 when the SLS commute from Σ_{M_j} to Σ_{M_k} (with $\varphi(M_j) = \varphi(M_k)$) is not detectable from the continuous system if there exists an input $u(t)$, an initial condition $x(t_0)$ and a time $t_1 \in [t_0, t]$ such that the following output trajectories*

$$y_1(t) = \left\{ C_{M_i} e^{A_{M_i}(t-t_0)} x_1(t_0) + \int_{t_0}^{t} C_{M_i} e^{A_{M_i}(t-\tau)} B_{M_i} u(\tau) d\tau \;\; t \in [t_0, t_2) \right. \tag{3.65}$$

and

$$y_2(t) = \begin{cases} C_{M_i} e^{A_{M_i}(t-t_0)} x_2(t_0) + \int_{t_0}^{t} C_{M_i} e^{A_{M_i}(t-\tau)} B_{M_i} u(\tau) d\tau & t \in [t_0, t_1] \\ C_{M_j} e^{A_{M_j}(t-t_1)} x_2(t_1) + \int_{t_1}^{t} C_{M_j} e^{A_{M_j}(t-\tau)} B_{M_j} u(\tau) d\tau & t \in (t_1, t_2) \end{cases} \tag{3.66}$$

are the same for both state trajectories, in other words a state trajectory that remains in $\Phi(M_i)$ cannot be distinguish from a state trajectory that in a time t_1 commutes from $\Phi(M_i)$ to $\Phi(M_j)$.

Now, consider the state trajectories $x_1(t)$ and $x_2(t)$ (notice that $x_1(t)$ does not commute) given by

$$x_1(t) = \begin{cases} e^{A_{M_j}(t-t_0)} x_1(t_0) + \int_{t_0}^{t} e^{A_{M_j}(t-\tau)} B_{M_j} u(\tau) d\tau & t \in [t_0, t_1] \\ e^{A_{M_j}(t-t_1)} x_1(t_1) + \int_{t_1}^{t} e^{A_{M_j}(t-\tau)} B_{M_j} u(\tau) d\tau & t \in (t_1, t_2) \end{cases} \tag{3.67}$$

and

$$x_2(t) = \begin{cases} e^{A_{M_j}(t-t_0)} x_2(t_0) + \int_{t_0}^{t} e^{A_{M_j}(t-\tau)} B_{M_j} u(\tau) d\tau & t \in [t_0, t_1] \\ e^{A_{M_k}(t-t_1)} x_2(t_1) + \int_{t_1}^{t} e^{A_{M_k}(t-\tau)} B_{M_k} u(\tau) d\tau & t \in (t_1, t_2) \end{cases} \tag{3.68}$$

where $x_1(t_0) = x_2(t_0)$, their corresponding output trajectories are:

$$y_1(t) = \begin{cases} C_{M_j} e^{A_{M_j}(t-t_0)} x_1(t_0) + \int_{t_0}^{t} C_{M_j} e^{A_{M_j}(t-\tau)} B_{M_j} u(\tau) d\tau & t \in [t_0, t_1] \\ C_{M_j} e^{A_{M_j}(t-t_1)} x_1(t_1) + \int_{t_1}^{t} C_{M_j} e^{A_{M_j}(t-\tau)} B_{M_j} u(\tau) d\tau & t \in (t_1, t_2) \end{cases} \tag{3.69}$$

and

$$y_2(t) = \begin{cases} C_{M_j} e^{A_{M_j}(t-t_0)} x_2(t_0) + \int_{t_0}^{t} C_{M_j} e^{A_{M_j}(t-\tau)} B_{M_j} u(\tau) d\tau & t \in [t_0, t_1] \\ C_{M_k} e^{A_{M_k}(t-t_1)} x_2(t_1) + \int_{t_1}^{t} C_{M_k} e^{A_{M_k}(t-\tau)} B_{M_k} u(\tau) d\tau & t \in (t_1, t_2) \end{cases} \tag{3.70}$$

then the commutation between systems can be detected iff the equation

$$\begin{aligned} & C_{M_j} e^{A_{M_j}(t-t_1)} x_1(t_1) - C_{M_k} e^{A_{M_k}(t-t_1)} x_2(t_1) = \\ & -\int_{t_1}^{t} C_{M_j} e^{A_{M_j}(t-\tau)} B_{M_j} u(\tau) d\tau + \int_{t_1}^{t} C_{M_k} e^{A_{M_k}(t-\tau)} B_{M_k} u(\tau) d\tau \end{aligned} \tag{3.71}$$

has unique solution $x_1(t_1) = 0$, $x_2(t_1) = 0$ and $u(t) = 0$. Where $x_2(t_1)$ is the initial state of the system $\Phi(M_k)$ after the commutation, i.e. $x_2(t_1) = \delta_{M_j,M_k} x_1(t_1)$.

Equation (3.71) can be written as:

$$\hat{C} e^{\hat{A}(t-t_1)} \left[x(t_1) + \int_{t_1}^{t} e^{-\hat{A}\tau} \hat{B} u(\tau) d\tau \right] = 0 \tag{3.72}$$

where

$$\begin{aligned} \hat{A} &= \begin{bmatrix} A_{M_j} & 0 \\ 0 & A_{M_k} \end{bmatrix} \\ \hat{B} &= \begin{bmatrix} B_{M_j} \\ B_{M_k} \end{bmatrix} \\ \hat{C} &= \begin{bmatrix} C_{M_j} & -C_{M_k} \end{bmatrix} \end{aligned} \tag{3.73}$$

and $x(t_1) = \begin{bmatrix} x_1(t_1) & x_2(t_1) \end{bmatrix}^T$ Then the time when the SLS commutes from the LS $\Sigma_{M_j}\left(A_{M_j}, B_{M_j}, C_{M_j}\right)$ to the LS system $\Sigma_{M_k}\left(A_{M_k}, B_{M_k}, C_{M_k}\right)$ is detectable iff the only solution to equation (3.72) is $u(t) = 0$ and $x(t_1) = 0$. Since equation (3.72) is similar to equation (3.8) with the only difference that $x(t_1)$ is restricted to be of the form $x(t_1) = \begin{bmatrix} x_1(t_1) & \delta_{M_j,M_k} x_1(t_1) \end{bmatrix}^T$ for some $x_1(t_1) \in \mathcal{X}_{M_j}$, then the time when the SLS commutes from Σ_{M_j} to Σ_{M_k} is detectable if $\begin{bmatrix} x_1(t_1) & \delta_{M_j,M_k} x_1(t_1) \end{bmatrix}^T \in \bar{\mathcal{W}}$ for some $x_1(t_1)$.

Notation 3.24 *Hereafter $\mathcal{O}_{M_i}$ denotes the observability matrix of the pair $\{A_{M_i}, C_{M_i}\}$, that is $\mathcal{O}_{M_i} = \mathcal{O}\left(A_{M_i}, C_{M_i}\right)$.*

Theorem 3.25 *Let $\Sigma_{M_j}\left(A_{M_j}, B_{M_j}, C_{M_j}\right)$, $\Sigma_{M_k}\left(A_{M_k}, B_{M_k}, C_{M_k}\right)$ be two linear systems. If $\varphi(M_j) = \varphi(M_k)$ then the time when the SLS commutes from $\Sigma_{M_j}\left(A_{M_j}, B_{M_j}, C_{M_j}\right)$ to $\Sigma_{M_k}\left(A_{M_k}, B_{M_k}, C_{M_k}\right)$ is detectable iff*

1. $\hat{B} \notin \ker \mathcal{O}\left(\hat{A}, \hat{C}\right)$ *and*

2. There is no $x(t) \neq 0$ *such that* $\begin{bmatrix} x(t) & \delta_{M_j,M_k} x(t) \end{bmatrix}^T \in \bar{\mathcal{W}}$.

Proof. *Follows easy from the fact that, if* $\begin{bmatrix} x(t) & \delta_{M_j,M_k} x(t) \end{bmatrix}^T \in \bar{\mathcal{W}}$ *then there exists a continuous state* $x(t_1) \in \mathcal{X}_{M_j}$ *such that equation (3.71) holds with* $x(t_1) \neq 0$ *or* $u(t) \neq 0$. ∎

Definition 3.26 *Let* $\Sigma_1(A_1, B_1, C_1)$, $\Sigma_2(A_2, B_2, C_2)$ *be two linear systems of dimension* n *and* m *respectively. The subspace* $\bar{\mathcal{W}}_{t_i} \subseteq \bar{\mathcal{X}}$ *denotes the set of* $x \in \bar{\mathcal{X}}$ *such that there exists an input* $u(t) = u_1(t)$ *such that the time* t_i *when the SLS commute from* Σ_1 *to* Σ_2 *cannot be detected if* $x(t_i) = x$ *and* $u(t) = u_1(t)$.

Proposition 3.27 *If* $\deg(\hat{C} Adj\left(sI - \hat{A}\right)\hat{B}) = q$ *then* $\bar{\mathcal{W}}_{t_i}$ *is*

$$\bar{\mathcal{W}}_{t_i} = \check{\mathcal{V}}_0 + \cdots + \check{\mathcal{V}}_{q-1} \tag{3.74}$$

and

$$\check{\mathcal{V}}_i = \left\{ x : \exists k \text{ such that } \mathbf{T}(\hat{A})\mathcal{O}\left(\hat{C}, \hat{A}\right) \begin{bmatrix} I_n \\ \delta_{M_j,M_k} \end{bmatrix} x = k e_i,\ i = 0, \ldots, q-1 \right\} \tag{3.75}$$

with e_i *a* $\hat{n}$*-dimenssional vector where* $e_0 = 0$ *and* e_i *a vector where the* $j-th$ *element is zero if* $j \neq i$ *and one otherwise.*

Corollary 3.28 *The time when the SLS commute from* Σ_1 *to* Σ_2 *is detectable if and only* $\bar{\mathcal{W}}_{t_i} = 0$.

Corollary 3.29 *Let* $\langle \mathcal{F}, (Q, M_0) \rangle$ *be a SLS Every commutation in* $\langle \mathcal{F}, (Q, M_0) \rangle$ *is detectable if and only if* $\forall M_i, M_j \in R(Q, M_0)$ *such that* $M_i \xrightarrow{\mathbf{t}_a} M_j$ *for some* $\mathbf{t}_a \in Q$, *one of the following conditions hold*

1. $\varphi(M_i) \neq \varphi(M_j)$
2. *The time when the SLS commutes from* Σ_{M_j} *to* Σ_{M_k} *is detectable.*

Proposition 3.30 *Let* $\langle \mathcal{F}, (Q, M_0) \rangle$ *be a SLS if*

$$t_i = \min\left\{ t > t_{i-1} : \mathcal{Y}\left(t^-\right) \neq \mathcal{Y}\left(t^+\right) \right\} \tag{3.76}$$

then the commutation time t_i *can be detected, where*

$$\mathcal{Y}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ \overset{(\hat{n}-1)}{y}(t) \end{bmatrix} \tag{3.77}$$

Once the conditions for the detectability of the commutation time have been stated, the conditions that the systems must fulfill in order to be able to compute the new state after the commutation will be presented. As mentioned above this is necessary for the observability of the SLS.

Proposition 3.31 *Assume that the SLS commute from a state M_j to a state M_k when the transition $\mathbf{t}_a$ is fired (i.e. $M_j \xrightarrow{\mathbf{t}_a} M_k$), and that the state M_j is known. The state M_k can be computed using the input-output information of the IPN iff for every transition $\mathbf{t}_b$ enabled at marking M_j holds that, if $\lambda(\mathbf{t}_a) = \lambda(\mathbf{t}_b) = \varepsilon$ then $\varphi C(\bullet, \mathbf{t}_a) \neq \varphi C(\bullet, \mathbf{t}_b)$, $\varphi C(\bullet, \mathbf{t}_a) \neq 0$ and $\varphi C(\bullet, \mathbf{t}_b) \neq 0$*

Proposition 3.32 *Assume that the SLS commute from a state M_j to a state M_k when the transition $\mathbf{t}_a$ is fired (i.e. $M_j \xrightarrow{\mathbf{t}_a} M_k$), and that the state M_j is known. The state M_k can be computed using the input and the output of the continuous system iff*

1. For every $\mathbf{t}_b$ enabled at marking M_j such that $M_j \xrightarrow{\mathbf{t}_b} M_i$ then:

   a) $\hat{B} \notin \ker \mathcal{O}\left(\hat{A}, \hat{C}\right)$ (with $\hat{\Sigma}\left(\hat{A}, \hat{B}, \hat{C}\right)$ the extended LS of $\Sigma_{Mi}(A_{M_i}, B_{M_i}, C_{M_i})$ and $\Sigma_{M_k}(A_{M_k}, B_{M_k}, C_{M_k})$) and

   b) There is no $x \neq 0$ such that $\left[\begin{array}{cc} \delta_{M_i,M_j}x & \delta_{M_i,M_k}x \end{array}\right]^T \in \mathcal{W}_{\Sigma_{M_k}}^{\Sigma_{M_j}}$

Example 3.33 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS where (Q, M_0) is depicted in figure 3.5and*

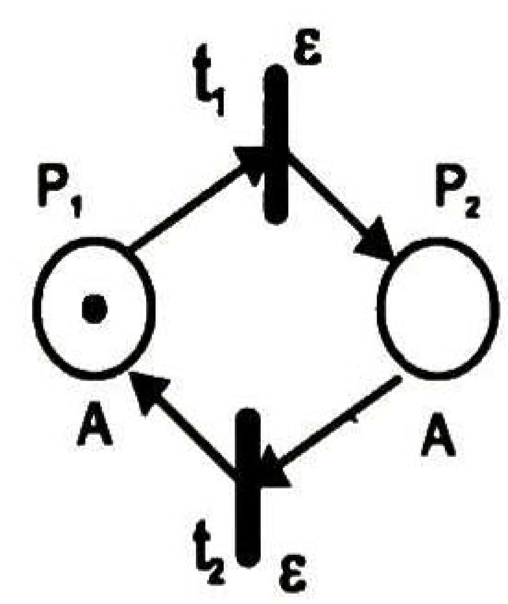


Figure 3.5: Interpreted Petri Net

$\mathcal{F} = \{\Sigma_1, \Sigma_2\}$ where $\Phi(M_1) = \Phi\left(\left[\begin{array}{cc} 1 & 0 \end{array}\right]^T\right) = \Sigma_1$ and $\Phi(M_2) = \Phi\left(\left[\begin{array}{cc} 1 & 0 \end{array}\right]^T\right) = \Sigma_2$ are two dynamic equations given by the realizations $\{A_1, C_1, B_1\}$ and $\{A_2, C_2, B_2\}$ respectively, with

$$A_1 = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right],\ C_1 = \left[\begin{array}{cc} 1 & 0 \end{array}\right],\ B_2 = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \tag{3.78}$$

and

$$A_2 = \left[\begin{array}{cc} -1 & 0 \\ 2 & -2 \end{array}\right],\ C_2 = \left[\begin{array}{cc} 1 & 0 \end{array}\right],\ B_2 = \left[\begin{array}{c} -1 \\ 1 \end{array}\right] \tag{3.79}$$

Since

$$\bar{C}\left(sI-\bar{A}\right)^{-1}x\left(t_1\right)=-\bar{C}\left(sI-\bar{A}\right)^{-1}\bar{B}U\left(s\right) \tag{3.80}$$

holds with

$$x\left(t_1\right)=\left[\begin{array}{c}-1\\1\end{array}\right] \text{ and } U\left(s\right)=-\frac{2s+4}{s^3+2s^2+s+2} \tag{3.81}$$

then the commutation time t_i cannot be detected iff

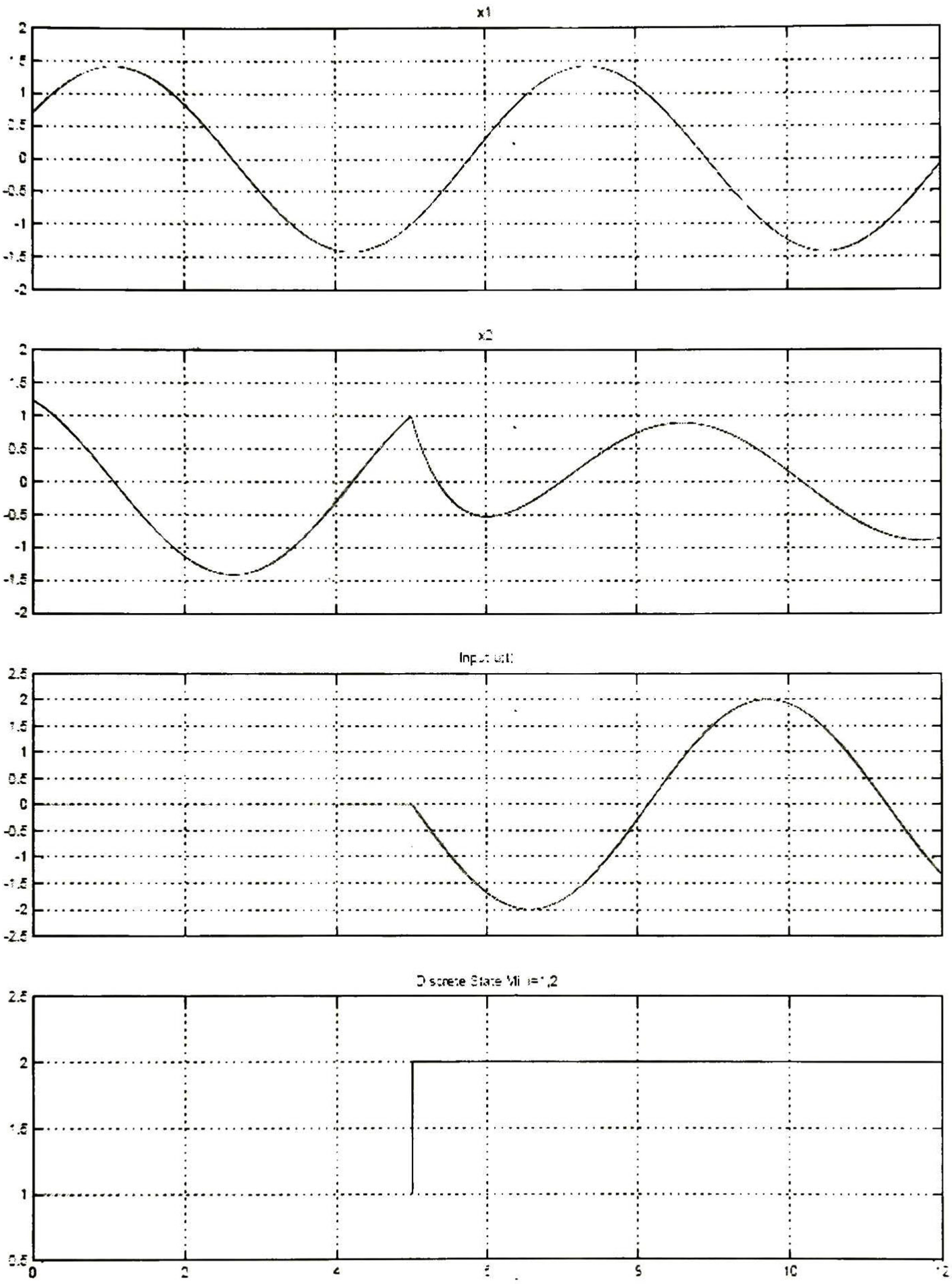
$$x\left(t_i\right)=\alpha\left[\begin{array}{c}-1\\1\end{array}\right] \text{ and } U\left(s\right)=-\alpha\frac{2s+4}{s^3+2s^2+s+2}, \tag{3.82}$$

for some real constant α.

In Figure 3.3 the commutation time cannot be detected, the commutation form Σ_1 to Σ_2 occurs in the time $t=5$, with $x\left(5\right)=\left[\begin{array}{cc}-1 & 1\end{array}\right]^T$, $U\left(s\right)=-e^{-5s}\frac{2s+4}{s^3+2s^2+s+2}$

If $C_1=C_1=\left[\begin{array}{cc}0 & 1\end{array}\right]$ then the commutation time t_i can be detected, since $\mathcal{Y}\left(5^-\right)\neq\mathcal{Y}\left(5^+\right)$.

Notice that, $x_1(t)$ in Figure 3.3, it would be the same if the SLS does not commute, that is, if evolves only in Σ_1, the outputs if $x\left(5\right)=\left[\begin{array}{cc}-1 & 1\end{array}\right]^t$ are $y_1\left(5,\infty\right)=-\cos\left(t\right)+\sin\left(t\right)$ and $y_2\left(5,\infty\right)=-\cos\left(t\right)+\sin\left(t\right)$



An example where the commutation time cannot be detected if $y(t) = x_1(t)$, and detected if $y(t) = x_2(t)$

Chapter 4

DISCRETE TO CONTINUOUS INFORMATION

In this chapter the case where the discrete system provides information to the continuous part, in such a way that the value of the continuous variables can be computed, even when every LS of the family is unobservable, is presented. This case occurs when the IPN marking sequence can be computed using only the input-output IPN information. Using the IPN marking sequence, the linear system sequence generated by this marking sequence is computed. Now, using this information, better results can be derived to obtain observability characterizations of the continuous part. This chapter also presents that, if the intersection of the unobservable subspaces of the sequence of linear systems is null, then the continuous state can be computed.

4.1 Using the Discrete Information in the observability

Definition 4.1 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS and $\sigma_f = \mathbf{t}_a \mathbf{t}_b ... \mathbf{t}_g$ a sequence in $\pounds(Q, M_0)$ such that $M_0 \xrightarrow{\mathbf{t}_a} M_1 \xrightarrow{\mathbf{t}_b} ... M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$. For each sequence σ_f there exists a linear system sequence $\Phi(M_0)\Phi(M_1)...\Phi(M_{k-1})\Phi(M_k)$ with $k = |\sigma_f|$. Then the observability matrix of a sequence associated with σ_f is:*

$$\mathcal{O}_{\sigma_f} = \begin{bmatrix} \mathcal{O}_{M_0} \\ \mathcal{O}_{M_1} \delta_{M_0,M_1} e^{A_{M_0} t_1 - t_0} \\ \vdots \\ \mathcal{O}_{M_k} \delta_{M_{k-1},M_k} e^{A_{M_{k-1}} t_k - t_{k-1}} \cdots \delta_{M_0,M_1} e^{A_{M_0} t_1 - t_0} \end{bmatrix} \tag{4.1}$$

where $\mathcal{O}_{M_0}, ..., \mathcal{O}_{M_k}$ are the observability matrices of $\Phi(M_0), ..., \Phi(M_k)$ respectively, i.e.

$$\mathcal{O}_{M_i} = \begin{bmatrix} C_{M_i} \\ \vdots \\ C_{M_i} A_{M_i}^{n-1} \end{bmatrix} \tag{4.2}$$

where t_i the ith switching time.

Theorem 4.2 *Let $\langle \mathcal{F}, (Q, M_0)\rangle$ be a SLS, and let $M_0 M_1 ... M_{k-1} M_k$ be a marking sequence evolving in the SLS when the sequence of fired transitions $\sigma = \mathbf{t}_a \mathbf{t}_b ... \mathbf{t}_g$ is fired. Assume that the marking sequence can be computed using the input-output information of the IPN, then the continuos initial state can be recovered iff the observability matrix $\mathcal{O}_\sigma$ of the sequence σ has full rank.*

Proof. *Since the marking evolution $M_0 \xrightarrow{\mathbf{t}_a} M_1 \xrightarrow{\mathbf{t}_b} ... M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$ can be computed then the sequence of linear systems $\Phi(M_0)\Phi(M_1)...\Phi(M_k)$ can be also be computed.*

(Sufficiency) Assume that the continuous state cannot be recovered then there exist two sequences generated by the same input producing the same output., i.e.:

$$y_{c_1} = \begin{cases} C_{M_0}(e^{A_{M_0} t - t_0} x_1^0(t_0) + \int_{t_0}^{t} e^{A_{M_0}(t-\tau)} B_{M_0} u(\tau) d\tau) \; t \in [t_0, t_1] \\ \vdots \\ C_{M_k}(e^{A_{M_k} t - t_k} x_1^k(t_k) + \int_{t_k}^{t} e^{A_{M_k}(t - t_k - \tau)} B_{M_k} u(\tau) d\tau) \; t \in (t_k, t_{k+1}) \end{cases} \tag{4.3}$$

and

$$y_{c_2} = \begin{cases} C_{M_0}(e^{A_{M_0} t - t_0} x_2^0(t_0) + \int_{0}^{t} e^{A_{M_0}(t - t_0 - \tau)} B_{M_0} u(\tau) d\tau) \; t \in [t_0, t_1] \\ \vdots \\ C_{M_k}(e^{A_{M_k} t - t_k} x_2^k(t_k) + \int_{t_k}^{t} e^{A_{M_k}(t - t_k - \tau)} B_{M_k} u(\tau) d\tau) \; t \in (t_k, t_{k+1}) \end{cases} \tag{4.4}$$

with $y_{c_1} = y_{c_2}$.

Now, substracting $y_{c_2}(t)$ from $y_{c_1}(t)$:

$$y_{c_1} - y_{c_2} = 0 = \begin{cases} C_{M_0} e^{A_{M_0} t - t_0} \left(x_1^0(t_0) - x_2^0(t_0)\right) \quad t \in [t_0, t_1] \\ \vdots \\ C_{M_k} e^{A_{M_k} t - t_k} \left(x_1^k(t_k) - x_2^k(t_k)\right) \quad t \in (t_k, t_{k+1}) \end{cases} \tag{4.5}$$

Equation (4.5) implies that $\left(x_1^i(t_i) - x_2^i(t_i)\right)$ belongs to the unobservable subspace of (A_{M_i}, C_{M_i}) [14]. Now, since:

$$\begin{aligned} x_k^i(t_i) &= \delta_{M_{i-1}, M_i} x_k^{i-1}(t_{i-1}) \\ &= \delta_{M_{i-1}, M_i} e^{A_{M_{i-1}} t_k - t_{k-1}} \cdots \delta_{M_0, M_1} e^{A_{M_0} t_1 - t_0} x_k^0(t_0) \end{aligned} \tag{4.6}$$

then

$$0 = \begin{cases} C_{M_0} e^{A_{M_0} t - t_0} (x_1^0(t_0) - x_2^0(t_0)) \qquad t \in [t_0, t_1] \\ \vdots \\ C_{M_k} e^{A_{M_k} t - t_k} \delta_{M_{k-1}, M_k} e^{A_{M_{k-1}} t_k - t_{k-1}} \cdots \\ \cdots \delta_{M_0, M_1} e^{A_{M_0} t_1 - t_0} (x_1^0(t_0) - x_2^0(t_0)) \quad t \in (t_k, t_{k+1}) \end{cases} \tag{4.7}$$

Using Taylor series and Cayley-Hamilton theorem, equation 4.7 can be rewritten as:

$$0=\begin{cases} C_{M_0}\left(I+A_{M_0}(t-t_0)+\cdots\frac{1}{n-1!}A_{M_0}^{n-1}(t-t_0)^{n-1}+\cdots\right) \\ (x_1^0(t_0)-x_2^0(t_0)), \qquad t\in[t_0,t_1] \\ \vdots \\ C_{M_k}\left(I+A_{M_k}(t-t_k)+\cdots\frac{1}{n-1!}A_{M_k}^{n-1}(t-t_k)^{n-1}+\cdots\right) \\ \delta_{k,k-1}e^{A_{M_{k-1}}t_k-t_{k-1}}\cdots\delta_{M_0,M_1}e^{A_{M_0}t_1-t_0} \\ (x_1^0(t_0)-x_2^0(t_0)), t\in(t_k,t_{k+1}). \end{cases} \tag{4.8}$$

from equation (4.8) the following matrix equation can be established:

$$\begin{bmatrix} \mathcal{O}_{M_0} \\ \mathcal{O}_{M_1}\delta_{M_0,M_1}e^{A_{M_0}t_1-t_0} \\ \vdots \\ \mathcal{O}_{M_k}\delta_{M_{k-1},M_k}e^{A_{M_{k-1}}t_k-t_{k-1}}\cdots\delta_{M_0,M_1}e^{A_{M_0}t_1-t_0} \end{bmatrix}\left(x_1^0(t_0)-x_2^0(t_0)\right)=0 \tag{4.9}$$

i.e.:

$$\mathcal{O}_\sigma\left(x_1^0(t_0)-x_2^0(t_0)\right)=0 \tag{4.10}$$

since $x_1^0(t_0)-x_2^0(t_0)\neq 0$ *then the rank of* $\mathcal{O}_\sigma$ *is not full.*

(Necessity) Following a similar procedure is easy to show that if the continuous state can be computed then the rank of the observability matrix O_σ *is full.* ■

A sufficient condition for the observability in terms of the unobservable subspace of every linear system can be stated as follows:

Corollary 4.3 *Let* $\langle\mathcal{F},(Q,M_0)\rangle$ *be a SLS, and let* $M_0M_1...M_{k-1}M_k$ *be a marking sequence evolving in the SLS when the sequence of fired transitions* $\sigma=\mathbf{t}_a\mathbf{t}_b...\mathbf{t}_g$ *is fired. Assume that the marking sequence can be computed using the input-output information of the IPN, then the continuos initial state can be recovered if holds that*

$$\begin{gathered}(\mathcal{N}_{M_0}\cap\mathcal{N}_{M_1}\delta_{M_0,M_1}=0)\vee(\mathcal{N}_{M_1}\cap\mathcal{N}_{M_2}\delta_{M_1,M_2}=0)\\ \vee\cdots\vee\left(\mathcal{N}_{M_{k-1}}\cap\mathcal{N}_{M_k}\delta_{M_{k-1},M_k}=0\right)\end{gathered} \tag{4.11}$$

where $\mathcal{N}_{M_j}=\bigcap_{i=1}^{n}\ker C_{M_j}A_{M_j}^{i-1}$ *is the unobservable subspace of* $\left(C_{M_j},A_{M_j}\right)$.

Proof. *Assume that the continuous state cannot be recovered, now according to theorem (4.2) then the rank of the observability matrix* $\mathcal{O}_{\sigma_f}$ *is not full. And therefore there exist vectors*

$$z_i = \left(x_1^i(t_i)-x_2^i(t_i)\right)\neq 0, \tag{4.12}$$

$$z_{i+1} = \left(x_1^{i+1}(t_{i+1})-x_2^{i+1}(t_{i+1})\right)\neq 0 \tag{4.13}$$

such that

$$\begin{array}{c} z_i \in \mathcal{N}_{M_i},\ z_{i+1} \in \mathcal{N}_{M_{i+1}} \\ \textit{and} \\ z_{i+1} = \delta_{M_i,M_{i+1}} e^{A_{M_i}t} z_i. \end{array} \tag{4.14}$$

Furthermore, the vector $v = e^{A_{M_i}t} z_i \in \mathcal{N}_{M_i}$ *(since* $e^{A_{M_i}t}\mathcal{N}_{M_i} \subseteq \mathcal{N}_{M_i}$ *i.e. the unobservable subspace of* (C_{M_i}, A_{M_i}) *is* $e^{A_{M_i}t}$ *– invariant),* $\delta_{M_i,M_{i+1}} v = z_{i+1} \in \mathcal{N}_{M_{i+1}}$. *Thus* $v \in \mathcal{N}_{M_i} \cap \mathcal{N}_{M_{i+1}} \delta_{M_i,M_{i+1}}$, *hence:*

$$\begin{array}{c} (\mathcal{N}_{M_0} \cap \mathcal{N}_{M_1} \delta_{M_0,M_1} \neq 0) \wedge (\mathcal{N}_{M_1} \cap \mathcal{N}_{M_2} \delta_{M_1,M_2} \neq 0) \\ \wedge \cdots \wedge \left(\mathcal{N}_{M_{k-1}} \cap \mathcal{N}_{M_k} \delta_{M_{k-1},M_k} \neq 0\right) \wedge \cdots \end{array} \tag{4.15}$$

and thus condition (4.11) is a sufficient condition for the observability of the continuous system after the sequence σ_f *has been fired.* ■

Chapter 5

JOINT OBSERVABILITY

In the joint observability of SLS, the aim is to recover the continuous and discrete states of the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$.

Theorem 5.1 *The system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable iff*

1. *Every commutation in the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is detectable and after a commutation the new state can be recovered assuming that previous state was known and*

2. *There exists a finite integer k such that every sequence $\sigma_w = \mathbf{t}_a \mathbf{t}_b \cdots \mathbf{t}_g \in \mathcal{L}(Q, M_0)$ of length k such that $M_i \xrightarrow{\mathbf{t}_a} M_j \xrightarrow{\mathbf{t}_b} \cdots M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$ the following conditions hold:*

   (a) If $S_w = \{M_i \cdots M_k\}$ where w is the input-output sequence of some σ_w then the observability matrix $\mathcal{O}_\sigma$ of the sequence σ_w has full rank.

   (b) If $M_i \cdots M_k, M'_i \cdots M'_k \in S_w$ where $M_i \cdots M_k \neq M'_i \cdots M'_k$ then the only solution to the equation

$$\begin{aligned} \bar{C}_i e^{\bar{A}_i(t-t_0)} \left[x(t_0) + \int_{t_0}^{t} e^{-\bar{A}_i(t-t_0-\tau)} \bar{B}_i u(\tau) d\tau \right] &= 0 \text{ for } t \in [t_0, t_1] \\ &\vdots \\ \bar{C}_k e^{\bar{A}_k(t-t_{\hat{k}})} \left[x(t_1) + \int_{t_1}^{t} e^{-\bar{A}_k(t-t_{\hat{k}}-\tau)} \bar{B}_k u(\tau) d\tau \right] &= 0 \text{ for } t \in \left(t_{\hat{k}}, t_{\hat{k}+1}\right) \end{aligned} \tag{5.1}$$

   is $x_0 = 0$ and $u(t) = 0$, where $x(t_i) = \bar{\delta}_{i,j} e^{\bar{A}_i(t_1-t_0)} x(t_{i-1})$ with

$$\bar{\delta}_{i,j} = \begin{bmatrix} \delta_{M_i, M_j} & 0 \\ 0 & \delta_{M'_i, M'_j} \end{bmatrix} \tag{5.2}$$

Theorem 5.2 *The system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if*

1. *Every commutation in the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is detectable and after a commutation the new state can be recovered assuming that previous state was known and*

2. *There exists a finite integer k such that every sequence $\sigma_w = \mathbf{t}_a\mathbf{t}_b\cdots\mathbf{t}_g \in \pounds(Q, M_0)$ of length k such that $M_i \xrightarrow{\mathbf{t}_a} M_j \xrightarrow{\mathbf{t}_b} \cdots M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$ the following conditions hold:*

   (a) *If $S_w = \{M_iM_j\cdots M_k\}$ where w is the input-output sequence of some σ_w then*

$$\begin{array}{c}(\mathcal{N}_{M_i}\cap \mathcal{N}_{M_j}\delta_{M_i,M_j}=0) \vee (\mathcal{N}_{M_j}\cap \mathcal{N}_{M_{j+1}}\delta_{M_j,M_{j+1}}=0)\\ \vee\cdots\vee(\mathcal{N}_{M_{k-1}}\cap\mathcal{N}_{M_k}\delta_{M_{k-1},M_k}=0)\end{array} \tag{5.3}$$

   (b) *If $M_i\cdots M_k, M_i'\cdots M_k' \in S_w$ where $M_i\cdots M_k \neq M_i'\cdots M_k'$ then*

$$\begin{array}{c}(\bar{\mathcal{W}}_i\cap\bar{\mathcal{W}}_j\bar{\delta}_{i,j}=0) \vee (\bar{\mathcal{W}}_j\cap\bar{\mathcal{W}}_{j+1}\bar{\delta}_{j,j+1}=0)\\ \vee\cdots\vee(\bar{\mathcal{W}}_{k-1}\cap\bar{\mathcal{W}}_k\bar{\delta}_{k-1,k}=0)\end{array} \tag{5.4}$$

   where $\bar{\mathcal{W}}_i$ is the indistinguishability subspace of Σ_{M_i}, $\Sigma_{M_i'}$.

Notice that condition 2 *a.* of Theorem 5.2 is the case where the discrete part provides information to the continuous systems, and condition 2 *b.*of Theorem 5.2 is the case where the continuous systems gives information to the discrete part in such a way that the discrete system becomes observable, if these conditions hold then it is always possible to recover the continuous and the discrete states of the SLS.

Based on the previous joint observability characterization some sufficient conditions for the observability of particular cases in SLS will be presented. Some of these results can be tested from the structure of the SLS

5.1 Results in Observability

5.1.1 Observability of SLS before the first commutation

As stated before, the concept of system distinguishability can be useful in SLS in order to identify the marking of the IPN. Next definition presents how to relabel an IPN using the continuous system information and then a useful proposition is presented.

Definition 5.3 *Let $SH = \langle\mathcal{F}, (Q, M_0)\rangle$ be a SLS and let $\Phi(M_i) = \Sigma_{M_i}(A_{M_i}, B_{M_i}, C_{M_i})$ and $\Phi(M_j) = \Sigma_{M_j}(A_{M_j}, B_{M_j}, C_{M_j})$ be two linear systems, where $\Phi(M_i)$, $\Phi(M_j) \in \mathcal{F}$ Then the IPN of SH can be relabeled adding new symbols if the linear systems $\Sigma_{M_i}(A_{M_i}, B_{M_i}, C_{M_i})$ and $\Sigma_{M_j}(A_{M_j}, B_{M_j}, C_{M_j})$ are distinguishable from each other. Thus the marking M_i must be labeled different from the marking M_j. The $SH^{\blacklozenge} = \langle\mathcal{F}, (Q^{\blacklozenge}, M_0)\rangle$ denotes the obtained SLS from relabeling $SH = \langle\mathcal{F}, (Q, M_0)\rangle$*

The obtained SLS after relabeling is the same than the original one, but the information that the continuos systems aport to the discrete part is added explicitly as virtual sensors (labels) attached to IPN marking.

Proposition 5.4 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS. $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable in $t \in [t_0, t_1)$ (where t_1 is the first commutation time) if and only if each LS is observable and the set of labels of every pair of markings of $(Q^\blacklozenge, M_0)$ are different from each other.*

Proof. (Sufficiency) The fact that every pair of markings of $(Q^\blacklozenge, M_0)$ are different from each other implies that:

1. If $\varphi(M_i) = \varphi(M_j)$, for some $M_i, M_j \in R(Q, M_0)$ then the linear systems $\Phi(M_i)$ and $\Phi(M_j)$ are distinguishable from each other, thus the initial marking can be computed using the continuous information. Now, according to corollary (3.13) the individual systems are observable hence the continuous initial condition can also be computed and the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable.

2. If $\varphi(M_i) \neq \varphi(M_j)$, for some $M_i, M_j \in R(Q, M_0)$ then the initial marking can be computed from the discrete output. Since every LS is observable then because the linear system is known the initial condition of the continuous system can also be computed and the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable.

(Necessity) The proof is divided in two parts

1. If not every LS is observable then the initial condition of the continuous system cannot be computed, thus the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is unobservable

2. If there exists a pair of markings of $(Q^\blacklozenge, M_0)$ that are not different from each other, then there exist $M_i, M_j \in R(Q, M_0)$ such that $\varphi(M_i) = \varphi(M_j)$ and for some input $u(t)$ both linear systems produce the same output, that is $\Phi(M_i)$ and $\Phi(M_j)$ are indistinguishable from each other, therefore according to theorem (2.36) the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is unobservable.

■

Proposition 5.5 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS that is observable in $t \in [t_0, t_1)$, then the initial condition of the SLS is:*

$$M_0 = \left\{ M_k : y(t) - \int_{t_0}^{t} C_{M_i} e^{A_{M_i}(t-\tau)} B_{M_i} u(\tau) d\tau \in \operatorname{Im} \left(C_{M_k} e^{A_{M_k}(t-t_0)} \right) \ t \in [t_0, t_1) \right\} \tag{5.5}$$

and

$$x_0 = \left(C_{M_k} e^{A_{M_k}(t-t_0)} \right)^{-1} \left(y(t) - \int_{t_0}^{t} C_{M_i} e^{A_{M_i}(t-\tau)} B_{M_i} u(\tau) d\tau \right) \tag{5.6}$$

5.1.2 Observability in SLS when the relabeled IPN is observable

Corollary 5.6 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS where $(Q^{\blacklozenge}, M_0)$ is observable then the system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if and only if there exists an integer $k < \infty$ such that the observability matrix $\mathcal{O}_\sigma$ of every sequence $|\sigma| \geq k$ has full rank.*

A sufficient condition for the observability in terms of the unobservable subspace of every linear system can be stated as follows:

Corollary 5.7 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS where $(Q^{\blacklozenge}, M_0)$ is observable then the system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if there exists an integer k such that every sequence $|\sigma_f| \geq k, \sigma_f = \mathbf{t}_a \mathbf{t}_b ... \mathbf{t}_g \in \pounds(Q, M_0)$, such that $M_0 \xrightarrow{\mathbf{t}_a} M_1 \xrightarrow{\mathbf{t}_b} ... M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$ it holds that:*

$$(\mathcal{N}_{M_0} \cap \mathcal{N}_{M_1} \delta_{M_0, M_1} = 0) \vee (\mathcal{N}_{M_1} \cap \mathcal{N}_{M_2} \delta_{M_1, M_2} = 0) \\ \vee \cdots \vee \left(\mathcal{N}_{M_{k-1}} \cap \mathcal{N}_{M_k} \delta_{M_{k-1}, M_k} = 0\right)$$

where $\mathcal{N}_{M_j} = \bigcap_{i=1}^{n} \ker C_{M_j} A_{M_j}^{i-1}$ *is the unobservable subspace of* $\left(C_{M_j}, A_{M_j}\right)$

Example 5.8 *Let $\mathcal{H} = \langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS represented as in Definition 2.32, where the IPN model is shown in Fig.5.1, the functions Φ, Π and δ_{M_i, M_j} are given below.*

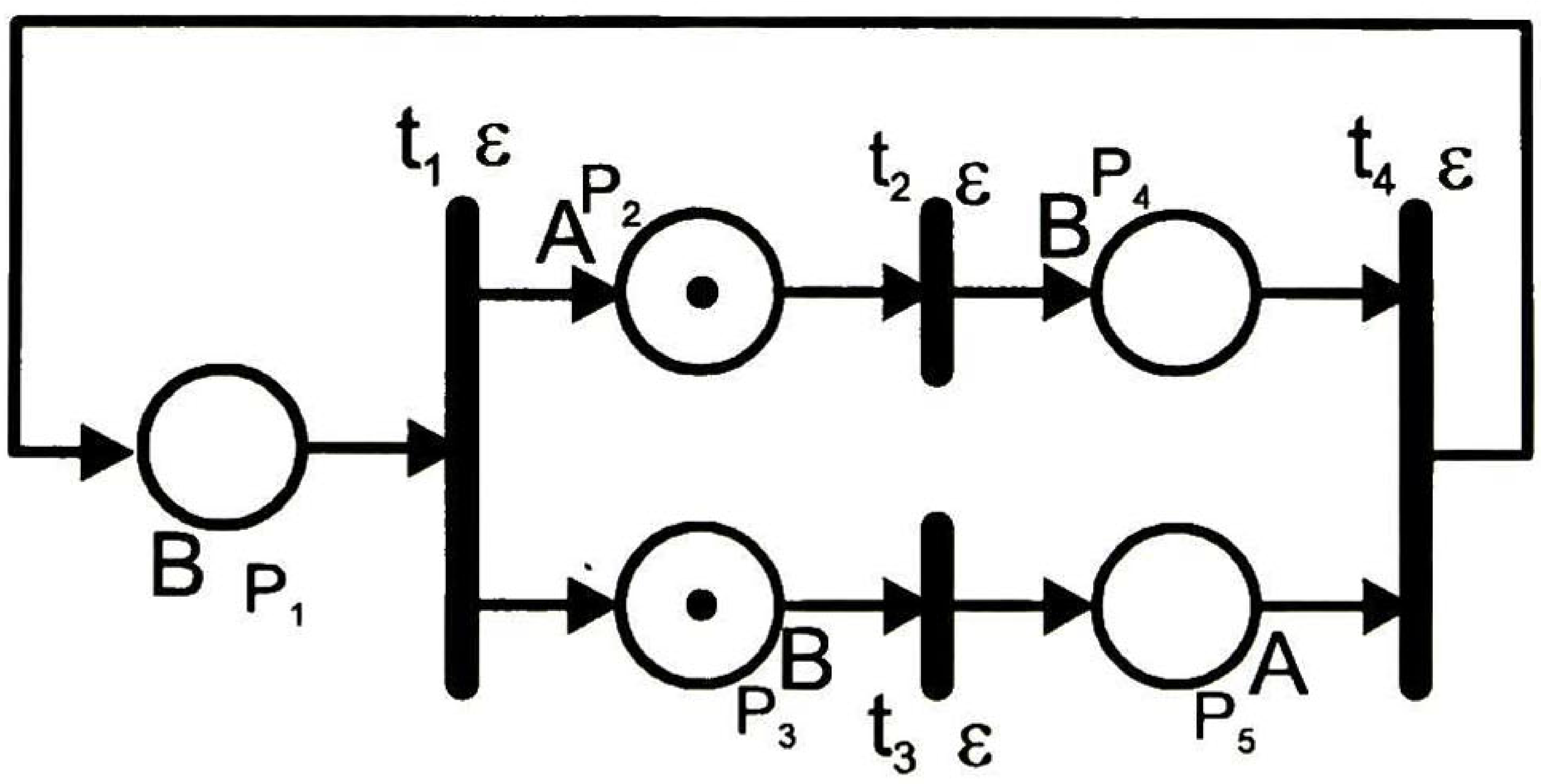


Figure 5.1: IPN model of the DES

The function Φ mapping from IPN markings to the linear systems is:

$$\Phi\left(M_k\right) = \Sigma_{M_k} : \begin{cases} \dot{x}(\tau) = A_{M_k} x(\tau) + B_{M_k} u(\tau) \\ y = C_{M_k} x(\tau) \end{cases} \tag{5.7}$$

where matrices A_{M_k}, B_{M_k} and C_{M_k} for each marking M_k, are described next:

k	$\Phi(M_k) = \Sigma_{M_k}$	A_{M_k}	B_{M_k}	C_{M_k}
0	$\Phi\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$	$\begin{bmatrix} -2 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}^T$
1	$\Phi\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$	$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T$
2	$\Phi\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$	$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 0 \\ 2 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$
3	$\Phi\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right)$	$\begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T$
4	$\Phi\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$	$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$

For simplicity the function δ_{M_i,M_j} is the identity matrix.

Since every column in

$$\varphi C = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

is different from each other, then the IPN is event detectable, and the IPN (Q, M_0) is observable.

Now, notice that every discrete sequence σ starting from any marking of $\mathcal{M}_0$ with $|\sigma| \geq 4$, can be written as $\sigma = xwy$ where $w = \mathbf{t}_2\mathbf{t}_3$ or $w = \mathbf{t}_3\mathbf{t}_2$, the linear systems sequence associated to each w are $\Phi(M_0)\Phi(M_1)\Phi(M_3)$ and $\Phi(M_0)\Phi(M_2)\Phi(M_3)$ respectively.

Notice that, every linear system is nonobservable, where the observability matrices are

$$\mathcal{O}_{M_0} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad \mathcal{O}_{M_1} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 6 & 1 & 0 \end{bmatrix} \quad \mathcal{O}_{M_2} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -5 & 0 \end{bmatrix}$$
$$\mathcal{O}_{M_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix} \quad \mathcal{O}_{M_4} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \tag{5.8}$$

However, $rank \left[\begin{array}{cc} \mathcal{O}_{M_1}^T & \mathcal{O}_{M_3}^T \end{array} \right]^T = 3$ and $rank \left[\begin{array}{cc} \mathcal{O}_{M_2}^T & \mathcal{O}_{M_3}^T \end{array} \right]^T = 3$, that is $\mathcal{N}_{M_1} \cap \mathcal{N}_{M_3} = 0$ and $\mathcal{N}_{M_2} \cap \mathcal{N}_{M_3} = 0$, thus according to Corollary 5.7, the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable.

Corollary 5.9 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS where $(Q^\blacklozenge, M_0)$ is observable and a binary state machine Petri net then the system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if for each sequence σ generating a minimal $T - semiflow$ of $(Q^\blacklozenge, M_0)$, O_σ has full rank.*

5.1.3 Observability of SLS in the first commutation

Corollary 5.10 *The system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable in $t \in [t_1, t_2)$ (where t_i is the ith commutation time) if*

Theorem 5.11 *1. Every commutation in the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is detectable and*

2. For all $M_i, M_j \in R(Q, M_0)$ such that $M_i \xrightarrow{\mathbf{t}_a} M_j$ for some $\mathbf{t}_a \in Q$, the following conditions hold:

(a) If $S_w = \{M_i M_j\}$ where w is the input-output sequence of some $|\sigma_w| = 1$ then

$$\mathcal{N}_{M_i} \cap \mathcal{N}_{M_j} \delta_{M_i, M_j} = 0 \tag{5.9}$$

(b) If $M_i M_j, M_i' M_j' \in S_w$ where $M_i M_j \neq M_i' M_j'$ then

$$\bar{\mathcal{W}}_i \cap \bar{\mathcal{W}}_j \bar{\delta}_{1,2} = 0 \tag{5.10}$$

where $\bar{\mathcal{W}}_i$ is the indistinguishability subspace of Σ_{M_i}, $\Sigma_{M_i'}$ and $\bar{\mathcal{W}}_j$ is the indistinguishability subspace of Σ_{M_j}, $\Sigma_{M_j'}$.

Chapter 6

OBSERVABILITY IN LINEAR HYBRID SYSTEMS

Unlike SLS where the commutation between linear systems depends on controllable and noncontrollable transitions, in Linear Hybrid Systems (LHS) the commutation depends only on the continuous states, where the transition $\mathbf{t}_a$ is fired and the LHS commutes from M_i to M_j (i.e. $M_i \xrightarrow{\mathbf{t}_a} M_j$) if and only if $\mathbf{t}_a$ is enabled in M_i and the continuous state $x(t) \in \mathcal{X}_{M_i}$ satisfies the hyperplane equation $\mathcal{H}_{M_i,\mathbf{t}_a} \in \mathbf{H}$, that is $H_{M_i,\mathbf{t}_a} \cdot x(t) = \varrho_{M_i,\mathbf{t}_a}$

In LHS $\lambda(\mathbf{t}) = \varepsilon$ $\forall \mathbf{t} \in T$ i.e. there are no controllable transition.

6.1 Definition of Linear Hybrid Systems

The definition of a LHS is presented in the following:

Definition 6.1 *Let $\mathcal{F} = \{\Sigma_1, ..., \Sigma_s\}$ be a family of linear systems, all of them of the same dimension, and let (Q, M_0) be an IPN. The 2-tupla $\langle \mathcal{F}, (Q, M_0) \rangle$ is a LHS if:*

1. *There exists a function $\Phi : R(Q, M_0) \to \mathcal{F}$, such that if the current marking of (Q, M_0) is M_k, then the linear system $\Phi(M_k)$ is the only linear system that is evolving.*

2. *There exist functions $\delta_{M_i,M_j} : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection such that $\delta_{M_i,M_j}(x_f^i) = x_0^j$ indicates how the SLS state is changed when the SLS commutes from Σ_i to Σ_j, $\Sigma_i, \Sigma_j \in \mathcal{F}$*

3. *There is a function $\Pi : \xi \to \mathbf{H}$ mapping from the set ξ to the set of hyperplane equations $\mathbf{H}$, where $\xi \subseteq (R(Q, M_0) \times T)$ such that $(M_k, \mathbf{t}_j) \in \xi$ if the transition $\mathbf{t}_j$ is enabled at marking M_k. A transition $\mathbf{t}_j \in T$ is fired if it is enabled and the continuous state satisfies the hyperplane equation $\mathcal{H}_{M_k,\mathbf{t}_j} \in \mathbf{H}$ defined as $H_{M_k,\mathbf{t}_j} \cdot x(t) = \varrho_{M_k,\mathbf{t}_j}$, where $H_{M_k,\mathbf{t}_j}$ is an $1 \times n$ row vector and $\varrho_{M_k,\mathbf{t}_j} \in \mathbb{R}$ are constant and known.*

4. *The elapsed time to change from M_k to M_{k+1} is finite and different from zero.*

5. *The IPN is live, binary and cyclic.*
6. *All transitions are noncontrollable that is $\lambda(\mathbf{t}_i) = \varepsilon \ \forall \mathbf{t}_i \in T$*

The results presented for the observability in SLS are sufficient conditions for the observability of LHS. It is because, given a LHS it is possible to associate a SLS where the information of the hyperplane equations has not been considered and every transition in the SLS is noncontrollable.

Next the necessary and sufficient conditions for the observability in LHS are presented:

6.2 Detectability of the Commutation Time in LHS

Theorem 6.2 *Let $\Phi(M_j) = \Sigma_{M_j}(A_{M_j}, B_{M_j}, C_{M_j})$, $\Phi(M_k)\Sigma_{M_k}(A_{M_k}, B_{M_k}, C_{M_k})$ be two linear systems. If $\varphi(M_j) = \varphi(M_k)$ then the time when the LHS commutes from the LS $\Sigma_{M_j}(A_{M_j}, B_{M_j}, C_{M_j})$ to the LS $\Sigma_{M_k}(A_{M_k}, B_{M_k}, C_{M_k})$ is detectable iff there is no $x(t) \in \mathcal{X}_{M_j}$ such that $H_{M_j,\mathbf{t}_a} \cdot x(t) = \varrho_{M_j,\mathbf{t}_a}$ and $\begin{bmatrix} x(t) & \delta_{M_j,M_k} x(t) \end{bmatrix}^T \in \bar{\mathcal{W}}$.*

Proof. *(Sufficiency) Suppose that there is no $x(t) \in \mathcal{X}_{M_j}$ such that $H_{M_j,\mathbf{t}_a} \cdot x(t) = \varrho_{M_j,\mathbf{t}_a}$ and $\begin{bmatrix} x(t) & \delta_{M_j,M_k} x(t) \end{bmatrix}^T \in \bar{\mathcal{W}}$. then since in the switching time $x(t_i) \notin \bar{\mathcal{W}}_{t_i}$ it follows from the definition of $\bar{\mathcal{W}}_{t_i}$ that the time when the LHS commutes from Σ_{M_j} to Σ_{M_k} is always detected.*

(Necessity) Assume there exists a transition $\mathbf{t}_a$ such $M_j \xrightarrow{\mathbf{t}_a} M_k$ and that the time when the LHS commutes from $\Sigma_{M_j}(A_{M_j}, B_{M_j}, C_{M_j})$ to $\Sigma_{M_k}(A_{M_k}, B_{M_k}, C_{M_k})$ is non detectable then there exists $x(t_1) \neq 0$ such that

$$\begin{bmatrix} x(t_1) & \delta_{M_j,M_k} x(t_1) \end{bmatrix}^T \in \bar{\mathcal{W}} \tag{6.1}$$

since in LHS the commutation from Σ_{M_j} to Σ_{M_k} occurs iff the continuous state $x(t_1)$ satisfies the hyperplane equation $\mathcal{H}_{M_i,\mathbf{t}_a}$, i.e. there exists and $x(t)$ such that $H_{M_j,\mathbf{t}_a} \cdot x(t) = \varrho_{M_j,\mathbf{t}_a}$. ■

6.3 The case when the relabeled IPN is Observable

Definition 6.3 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a LHS and $\sigma_f = \mathbf{t}_a\mathbf{t}_b...\mathbf{t}_g \in \pounds(Q, M_0)$ such that $M_0 \xrightarrow{\mathbf{t}_a} M_1 \xrightarrow{\mathbf{t}_b} ...M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$. For each sequence σ_f there exists a linear system sequence $\Phi(M_0)\Phi(M_1)...\Phi(M_{k-1})\Phi(M_k)$ with $k = |\sigma_f|$. Then the observability matrix of a sequence associated with σ_f is:*

$$\mathcal{O}_{\sigma_f} = \begin{bmatrix} \mathcal{O}_{M_0} \\ H_{M_0,\mathbf{t}_a} e^{A_{M_0} t_1 - t_0} \\ \mathcal{O}_{M_1} \delta_{M_0,M_1} e^{A_{M_0} t_1 - t_0} \\ \vdots \\ H_{M_{k-1},\mathbf{t}_g} e^{A_{M_{k-1}} t_k - t_{k-1}} \delta_{M_{k-2},M_{k-1}} \cdots \delta_{M_0,M_1} e^{A_{M_0} t_1 - t_0} \\ \mathcal{O}_{M_k} \delta_{M_{k-1},M_k} e^{A_{M_{k-1}} t_k - t_{k-1}} \cdots \delta_{M_0,M_1} e^{A_{M_0} t_1 - t_0} \end{bmatrix} \tag{6.2}$$

Notation 6.4 *Hereafter $\mathcal{H}_{i,a}$ denotes the hyperplane equation*

$$H_{i,a} \cdot x(t) = \varrho_{j,a} \tag{6.3}$$

where

$$H_{i,a} = \left[\begin{array}{cc} H_{M_i,\mathbf{t}_a} & -H_{M_i',\mathbf{t}_a'} \end{array} \right] \tag{6.4}$$

and

$$\varrho_{j,a} = \left[\varrho_{M_i,\mathbf{t}_a} - \varrho_{M_i',\mathbf{t}_a'} \right] \tag{6.5}$$

Theorem 6.5 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a LHS where $(Q^\blacklozenge, M_0)$ is observable then the system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if and only if there exists an integer $k < \infty$ such that the observability matrix $\mathcal{O}_\sigma$ of every sequence $|\sigma| \geq k$ has full rank.*

Proof. *The proof follows from the fact that, since the sequence of linear systems and the commutation times are known, then if the systems commute from M_k to M_k' with some $\mathbf{t}_a$, i.e. $M_k \xrightarrow{\mathbf{t}_a} M_k'$, then it is known that in the commutation time t_i the hyperplane equation $\mathcal{H}_{M_k,\mathbf{t}_a}$ was satisfied, i.e. $H_{M_k,\mathbf{t}_a} \cdot x(t_i) = \varrho_{M_k,\mathbf{t}_a}$*

This information can be used in an analysis similar to the one presented in proof of Theorem 4.2, and therefore to get full rank in the observability matrix $\mathcal{O}_\sigma$. ■

A sufficient condition for the observability in terms of the unobservable subspace of every linear system can be stated as follows:

Corollary 6.6 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS where $(Q^\blacklozenge, M_0)$ is observable then the system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if there exists an integer k such that every sequence $|\sigma_f| \geq k, \sigma_f = \mathbf{t}_a\mathbf{t}_b...\mathbf{t}_g \in \mathcal{L}(Q, M_0)$, such that $M_0 \xrightarrow{\mathbf{t}_a} M_1 \xrightarrow{\mathbf{t}_b} ...M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$ it holds that:*

$$\begin{array}{c}(N_{M_0} \cap \ker H_{M_0,\mathbf{t}_a} \cap N_{M_1}\delta_{M_0,M_1} = 0) \vee (\mathcal{N}_{M_1} \cap \ker H_{M_1,\mathbf{t}_b} \cap \mathcal{N}_{M_2}\delta_{M_1,M_2} = 0) \\ \vee \cdots \vee \left(\mathcal{N}_{M_{k-1}} \cap \ker H_{M_{k-1},\mathbf{t}_g} \cap \mathcal{N}_{M_k}\delta_{M_{k-1},M_k} = 0\right).\end{array} \tag{6.6}$$

Corollary 6.7 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a SLS where $(Q^\blacklozenge, M_0)$ is observable and a state machine Petri net then the system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if for each sequence σ generating a minimal $T-semiflow$ of $(Q^\blacklozenge, M_0)$, $\mathcal{O}_\sigma$ has full rank.*

Using the fact that, the LHS commute if and only if the continuous state met one active hyperplane, less conservative conditions can be derived for the observability of LHS. These results are presented next.

6.4 Observability of LHS in the first commutation

Theorem 6.8 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a LHS. The system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable in $t \in [t_1, t_2)$ (where t_i is the ith commutation time) if and only if*

1. *Every commutation in the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is detectable and*

2. *For all $M_i, M_j \in R(Q, M_0)$ such that $M_i \xrightarrow{\mathbf{t}_a} M_j$ for some $\mathbf{t}_a \in Q$, the following conditions hold:*

   (a) *If $S_w = \{M_i M_j\}$ where w is the input-output sequence of some $|\sigma_w| = 1$ then the observability matrix $\mathcal{O}_{\sigma_w}$ of the sequence σ_w has full rank.*

   (b) *If $M_i M_j, M_i' M_j' \in S_w$ where $M_i M_j \neq M_i' M_j'$ then the only solution to the equation*

$$\begin{aligned} \bar{C}_i e^{\bar{A}_i(t-t_0)} \left[x(t_0) + \int_{t_0}^{t} e^{-\bar{A}_i(t-\tau)} \bar{B}_i u(\tau) d\tau\right] &= 0 \text{ for } t \in [t_0, t_1] \\ H_{i,a} \cdot x(t_1) &= \varrho_{i,a} \\ \bar{C}_j e^{\bar{A}_j(t-t_1)} \left[x(t_1) + \int_{t_1}^{t} e^{-\bar{A}_j(t-t_1-\tau)} \bar{B}_j u(\tau) d\tau\right] &= 0 \text{ for } t \in (t_1, t_2) \end{aligned} \tag{6.7}$$

   is $x_0 = 0$ and $u(t) = 0$.

Theorem 6.9 *Let $\langle \mathcal{F}, (Q, M_0)\rangle$ be a LHS. The system $\langle \mathcal{F}, (Q, M_0)\rangle$ is observable in $t \in [t_1, t_2)$ (where t_i is the ith commutation time) if*

1. *Every commutation in the SLS $\langle \mathcal{F}, (Q, M_0)\rangle$ is detectable and*

2. *For all $M_i, M_j \in R(Q, M_0)$ such that $M_i \xrightarrow{\mathbf{t}_a} M_j$ for some $\mathbf{t}_a \in Q$, the following conditions hold:*

   (a) *If $S_w = \{M_i M_j\}$ where w is the input-output sequence of some $|\sigma_w| = 1$ then*

$$\mathcal{N}_{M_i} \cap \ker H_{M_i,\mathbf{t}_a} \cap \mathcal{N}_{M_j} \delta_{M_i,M_j} = 0 \tag{6.8}$$

   (b) *If $M_i M_j, M_i' M_j' \in S_w$ where $M_i M_j \neq M_i' M_j'$ then there is no $x(t)$ such that $H_{i,a} \cdot x(t) = \varrho_{i,a}$ and $x(t) \in \bar{\mathcal{W}}_i \cap \bar{\mathcal{W}}_j \bar{\delta}_{1,2}$.*

6.5 Joint Observability

Theorem 6.10 *Let $\langle \mathcal{F}, (Q, M_0)\rangle$ be a LHS. The system $\langle \mathcal{F}, (Q, M_0)\rangle$ is observable if and only if*

1. *Every commutation in the SLS $\langle \mathcal{F}, (Q, M_0)\rangle$ is detectable and*

2. *For all $M_i, M_j, \cdots, M_k \in R(Q, M_0)$ such that $M_i \xrightarrow{\mathbf{t}_a} M_j \xrightarrow{\mathbf{t}_b} \cdots M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$ for some $\sigma_w = \mathbf{t}_a \mathbf{t}_b \cdots \mathbf{t}_g \in \mathcal{L}(Q, M_0)$ with $|\sigma_w| = \hat{k}$, the following conditions hold:*

   (a) *If $S_w = \{M_i \cdots M_k\}$ where w is the input-output sequence of some σ_w then the observability matrix $\mathcal{O}_\sigma$ of the sequence σ_w has full rank.*

(b) If $M_i \cdots M_k, M_i' \cdots M_k' \in S_w$ where $M_i \cdots M_k \neq M_i' \cdots M_k'$ then the only solution to the equation

$$\begin{array}{c} \bar{C}_i e^{\bar{A}_i(t-t_0)} \left[x(t_0) + \int_{t_0}^{t} e^{-\bar{A}_i(t-t_0-\tau)} \bar{B}_i u(\tau) d\tau \right] = 0 \ for\ t \in [t_0, t_1] \\ H_{i,a} \cdot x(t_1) = \varrho_{i,a} \\ \vdots \\ H_{k-1,g} \cdot x\left(t_{\hat{k}}\right) = \varrho_{i,a} \\ \bar{C}_k e^{\bar{A}_k(t-t_{\hat{k}})} \left[x(t_1) + \int_{t_1}^{t} e^{-\bar{A}_k(t-t_{\hat{k}}-\tau)} \bar{B}_k u(\tau) d\tau \right] = 0 \ for\ t \in \left[t_{\hat{k}}, t_{\hat{k}+1}\right] \end{array} \tag{6.9}$$

is $x_0 = 0$ and $u(t) = 0$, where $x(t_i) = \bar{\delta}_{i,j} e^{\bar{A}_i(t_1-t_0)} x(t_{i-1})$.

Theorem 6.11 *Let $\langle \mathcal{F}, (Q, M_0) \rangle$ be a LHS. The system $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable if*

1. *Every commutation in the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is detectable and*

2. *For all $M_i, M_j, \cdots, M_k \in R(Q, M_0)$ such that $M_i \xrightarrow{\mathbf{t}_a} M_j \xrightarrow{\mathbf{t}_b} \cdots M_{k-1} \xrightarrow{\mathbf{t}_g} M_k$ for some $\sigma_w = \mathbf{t}_a \mathbf{t}_b \cdots \mathbf{t}_g \in \mathcal{L}(Q, M_0)$ with $|\sigma_w| = \hat{k}$, the following conditions hold:*

   (a) If $S_w = \{M_i M_j \cdots M_k\}$ where w is the input-output sequence of some σ_w then

$$\begin{array}{c} (\mathcal{N}_{M_0} \cap \ker H_{M_0,\mathbf{t}_a} \cap \mathcal{N}_{M_1} \delta_{M_0,M_1} = 0) \vee (\mathcal{N}_{M_1} \cap \ker H_{M_1,\mathbf{t}_b} \cap \mathcal{N}_{M_2} \delta_{M_1,M_2} = 0) \\ \vee \cdots \vee \left(\mathcal{N}_{M_{k-1}} \cap \ker H_{M_{k-1},\mathbf{t}_g} \cap \mathcal{N}_{M_0} \delta_{M_{k-1},M_0} = 0\right). \end{array} \tag{6.10}$$

   (b) If $M_i \cdots M_k, M_i' \cdots M_k' \in S_w$ where $M_i \cdots M_k \neq M_i' \cdots M_k'$ then there is no $x \in \mathbb{R}^n$ such that

$$\begin{array}{c} (H_{i,a} \cdot x = \varrho_{i,a} \ and\ x \in \bar{\mathcal{W}}_i \cap \bar{\mathcal{W}}_j \bar{\delta}_{i,j}) \vee (H_{j,b} \cdot x = \varrho_{j,b} \ and\ x \in \bar{\mathcal{W}}_j \cap \bar{\mathcal{W}}_{j+1} \bar{\delta}_{j,j+1}) \\ \vee \cdots \vee \left(H_{k-1,g} \cdot x = \varrho_{k-1,g} \ and\ x \in \bar{\mathcal{W}}_{k-1} \cap \bar{\mathcal{W}}_k \bar{\delta}_{k-1,k}\right) \end{array} \tag{6.11}$$

   where $\bar{\mathcal{W}}_i$ is the indistinguishability subspace of $\Sigma_{M_i}, \Sigma_{M_i'}$.

Chapter 7

OBSERVER DESIGN

In this chapter the design of asymptotic observers for observables SLS, where the discrete part is event detectable, is presented. The main idea is to design the observer for the discrete part using the asymptotic observer presented in [12] and a family of continuous observers, like the sliding mode observer based on the equivalent control method presented in [21], taking into account the information provided by the geometric characterization.

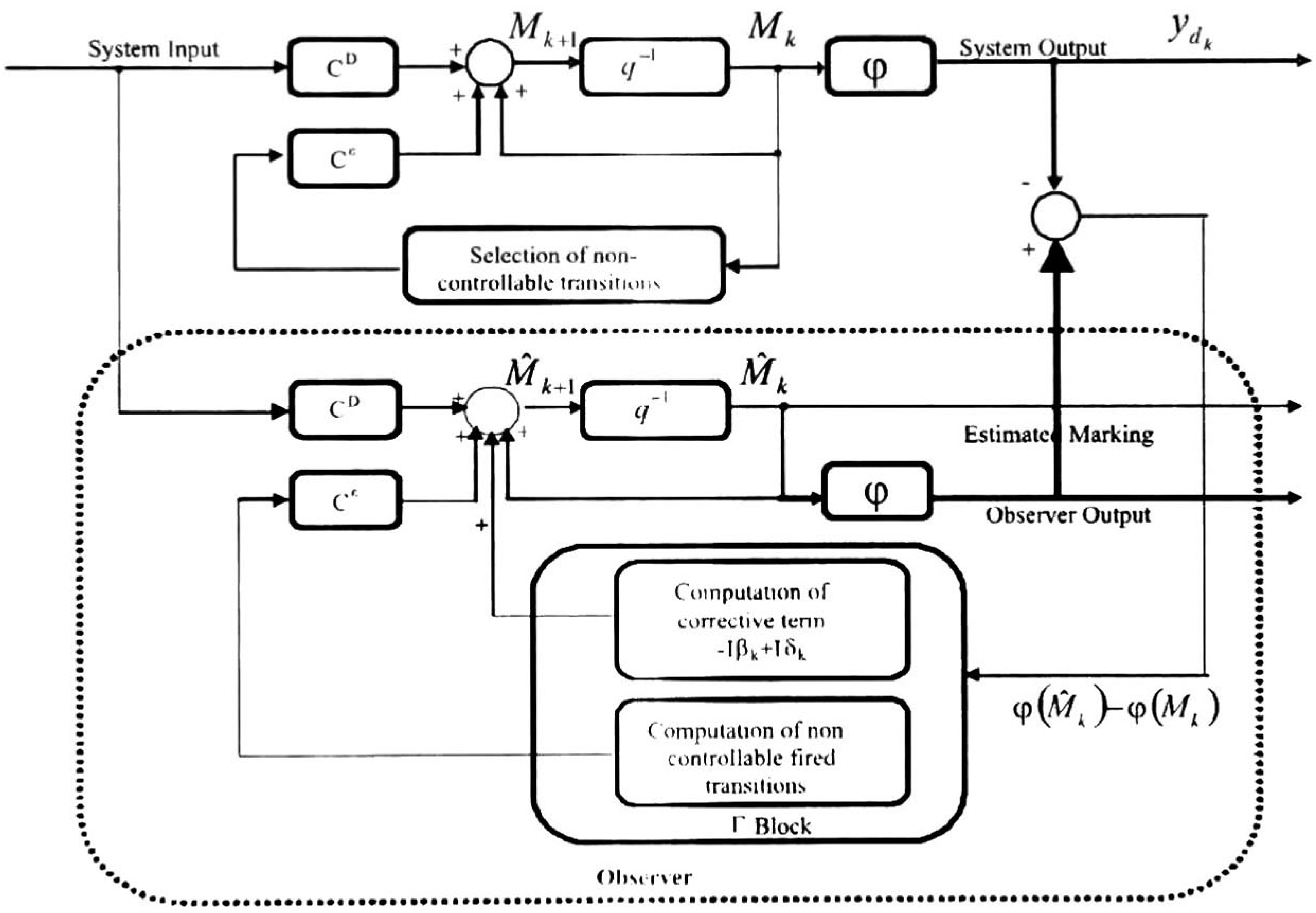


Figure 7.1: IPN Observer

The discrete observer is adopted from [22] and [12]. It is depicted in Fig. 7.1. In this figure, matrices C^D and C^ε represent, respectively, the incidence matrix columns of the manipulated and non manipulated transitions.

Definition 7.1 *Let $\Im$ be an observable SLS described by (2.22). Then an observer for the discrete part of the SLS is defined as follows.*

- *Observer structure:*

$$\hat{M}_{k+1} = \hat{M}_k + C\gamma_k + \begin{bmatrix} -I & I \end{bmatrix} \begin{bmatrix} \beta_k \\ \delta_k \end{bmatrix} \tag{7.1}$$

- *Initial marking $\hat{M}_0$ is any marking $M \in R(Q, M_0)$ such that $\varphi M = \varphi \bar{M}_0$ where $\bar{M}_0$ is the initial marking of (Q, M_0)*

- *Firing rule: when t_j is fired in the SLS then:*

$$\begin{aligned} \gamma_k &= \begin{cases} \overrightarrow{\mathbf{t}_j} & \text{if } \mathbf{t}_j \text{ is enabled at } \hat{M}_k \\ 0 & \text{otherwise} \end{cases} \\ \beta_k &= \begin{bmatrix} \varpi_1 & \cdots & \varpi_n \end{bmatrix}^T, \delta_k = \begin{bmatrix} \upsilon_1 & \cdots & \upsilon_n \end{bmatrix}^T \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} \varpi_i = &\begin{cases} 1 & \text{if } (\gamma_k = \overrightarrow{\mathbf{t}_j}, \text{ and } \hat{M}_k(p_i) + C(p_i, \cdot)\gamma_k > 1), \\ & \text{or } (\gamma_k \neq \overrightarrow{\mathbf{t}_j}, p_i \in^{\bullet} (\mathbf{t}_j) \text{ and } \hat{M}_k(p_i) > 0) \\ 0 & \text{otherwise} \end{cases} \\ \upsilon_i = &\begin{cases} 1 & \text{if } (\gamma_k = \overrightarrow{\mathbf{t}_j}, \text{ and } \hat{M}_k(p_i) + C(p_i, \cdot)\gamma_k < 0) \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{7.3}$$

For the continuos part a sliding mode observer similar to the one presented in [21], is proposed. For this kind of observers it is known that the state is estimated in finite time which is important since the switching time is finite and the observable states must be estimated before the system commute. It will be proven that the observer is robust to parametric variations. The observer design for each linear systems of the family $\mathcal{F}$ is described in the following lines.

Let

$$\Sigma_{M_k}: \begin{aligned} \dot{x}(\tau) &= A_{M_k} x(\tau) + B_{M_k} u(\tau) \\ y &= C_{M_k} x(\tau) \end{aligned} \tag{7.4}$$

be a LS where $rank(O_{M_k}) = r < n$, then there exists a similar transformation T_{M_k} that transform the system into the observable/unobservable form [13]

$$\begin{aligned} \bar{A}_{M_k} = \begin{bmatrix} \bar{A}_{M_k,o} & 0 \\ \bar{A}_{M_k,21} & \bar{A}_{M_k,no} \end{bmatrix}, \bar{B}_{M_k} = \begin{bmatrix} \bar{B}_{M_k o} \\ \bar{B}_{M_k,no} \end{bmatrix} \\ \text{and } \bar{C}_{M_k} = \begin{bmatrix} \bar{C}_{M_{k,o}} & 0 \end{bmatrix} \end{aligned} \tag{7.5}$$

where the $\bar{A}_{M_{k,o}} \in \mathbb{R}^{r \times r}, \bar{A}_{M_k,co} \in \mathbb{R}^{(n-r) \times r}, \bar{A}_{M_k,no} \in \mathbb{R}^{(n-r) \times (n-r)}$ where the subsystem $(A_{M_{k,},o}, B_{M_{k,},o}, C_{M_{k,o}})$ is observable and in the canonical observability realization

$$\begin{aligned}
&\dot{\bar{x}}_1 = \bar{x}_2 + \dot{\bar{b}}_1 u \\
&\vdots \\
&\dot{\bar{x}}_{r-1} = \bar{x}_r + \bar{b}_2 u \\
&\dot{\bar{x}}_r = -a_1\bar{x}_1 - \cdots \\
&\cdots - a_r\bar{x}_r + \bar{b}_r u \\
&\text{and } y = x_1.
\end{aligned} \tag{7.6}$$

An sliding mode observer for this subsystem is designed as

$$\begin{aligned}
&\dot{\hat{x}}_1 = \hat{x}_2 + \bar{b}_1 u + l_1 sign(x_1 - \hat{x}_1) \\
&\vdots \\
&\dot{\hat{x}}_{r-1} = \hat{x}_r + \bar{b}_2 u + l_{r-1} sign((x_{r-1} - \hat{x}_{r-1})_{eq}) \\
&\dot{\hat{x}}_r = -a_1\hat{x}_1 - \cdots - a_r\hat{x}_r + \bar{b}_r u + l_r sign((x_r - \hat{x}_r)_{eq})
\end{aligned} \tag{7.7}$$

where the equivalent control signal $(x_i - \hat{x}_i)_{eq}$ for the unmeasurable states $(x_i - \hat{x}_i)$ can be calculated by low pass filtering the signal $(x_i - \hat{x}_i)$ [21].

The following dynamic of the observation error is obtained from equations 7.6 and 7.7

$$\begin{aligned}
&\dot{e}_1 = e_2 - l_1 sign(e_1) \\
&\vdots \\
&\dot{e}_{r-1} = e_r - l_{r-1} sign((e_{r-1})_{eq}) \\
&\dot{e}_r = -a_1 e_1 - \cdots - a_r e_r - l_r sign((e_r)_{eq})
\end{aligned} \tag{7.8}$$

therefore if $l_1 > |e_2|$, e_1 goes to zero in finite time t_1, hereafter

$$e_2 = (l_1 sign(e_1))_{eq} = \check{e}_2 \tag{7.9}$$

and e_2 goes to zero in finite time t_2, if $l_2 > |e_3|$, following this procedure after time $t_{r-1} < \infty$ where $e_{r-1} = 0$, $\dot{e}_r$ is:

$$\dot{e}_r = -a_r e_r - l_r sign(e_r) \tag{7.10}$$

and e_r goes to zero in finite time t_r, if:

$$l_r > |a_r e_r| \tag{7.11}$$

Hence the continuos part of the *SLS* observer, shown in the Figure 7.2, is a set of observers where $\Gamma_{\hat{M}_k} = T^{-1}_{\hat{M}_k} L_{\hat{M}_k} sign_{\hat{M}_k}(e)$

$$\dot{\hat{x}}(\tau) = A_{\hat{M}_k}\hat{x}(\tau) + B_{\hat{M}_k}u(\tau) + T^{-1}_{\hat{M}_k} L_{\hat{M}_k} sign_{\hat{M}_k}(e)$$

where $L_{\hat{M}_k}$ and $sign_{\hat{M}_k}(e)$ are in the for,

$$L_{\hat{M}_k} = \begin{bmatrix} \bar{L}_{\hat{M}_k} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } sign_{\hat{M}_k}(e) = \begin{bmatrix} \bar{S}_{\hat{M}_k} \\ 0 \end{bmatrix}$$

with $\bar{L}_{\hat{M}_k} = diag\{l_1, \ldots, l_r\}$ and $\bar{S}_{\hat{M}_k}$is

$$\bar{S}_{\hat{M}_k} = \begin{bmatrix} sign(e_1) \\ sign(\breve{e}_2) \\ \vdots \\ sign(\breve{e}_r) \end{bmatrix}$$

where $\breve{e}_i = (l_i sign(\breve{e}_{i-1}))_{eq}$ for $1 < i \leq r$ and r is the rank of the observability matrix O_{M_k}. Thus the observers are designed to estimate the observable variables of each linear system in finite time and before the system commute, note that no assumption is made on the unobservable states.

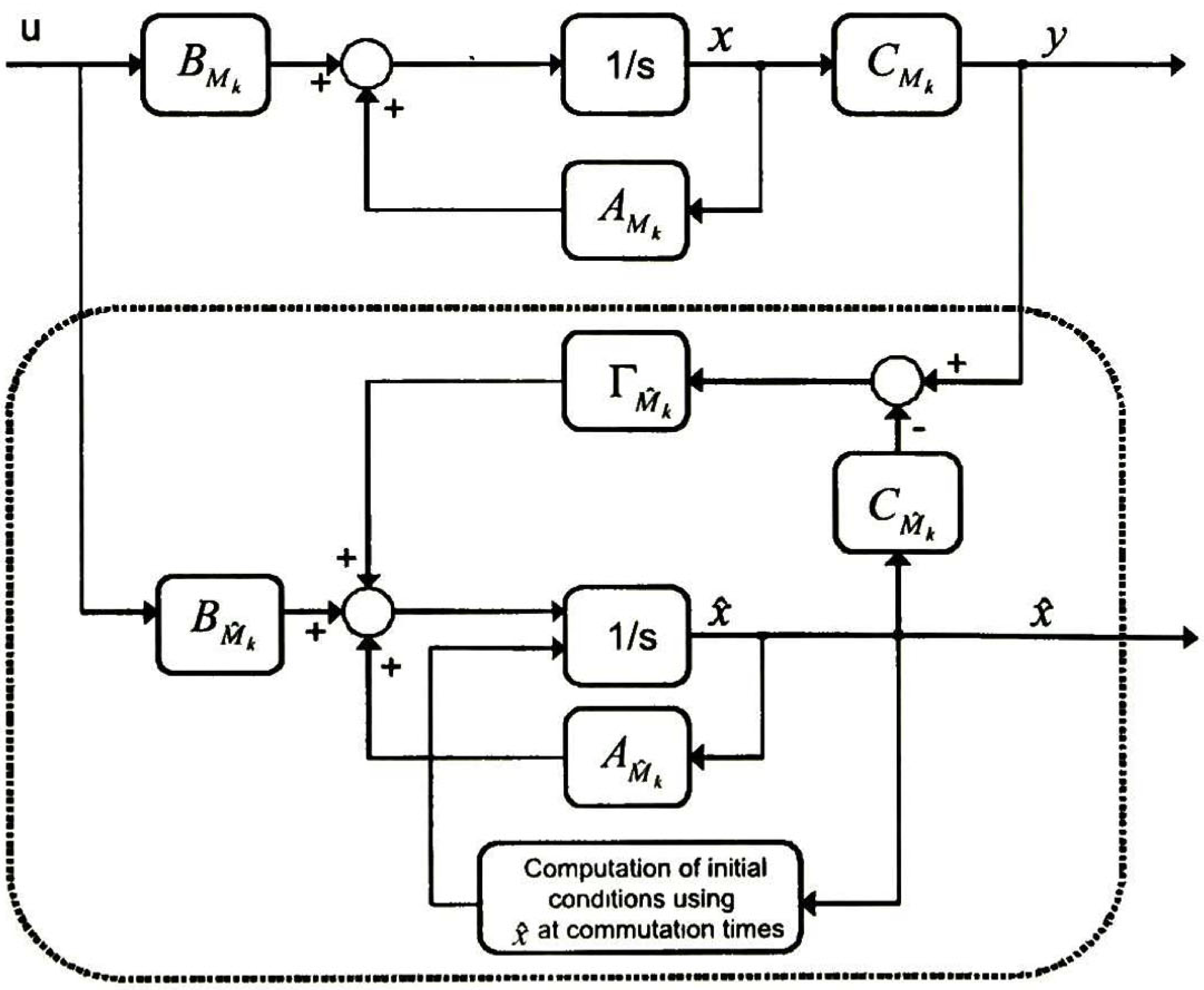


Figure 7.2: Continouos Observer

Example 7.2 *Let $\mathcal{H} = \langle \mathcal{F}, (Q, M_0) \rangle$ be the SLS of given in the Example 5.8, now since the IPN (Q, M_0) is event-detectable and according to Corollary 5.7, the SLS $\langle \mathcal{F}, (Q, M_0) \rangle$ is observable, and a hybrid observer can be designed.*

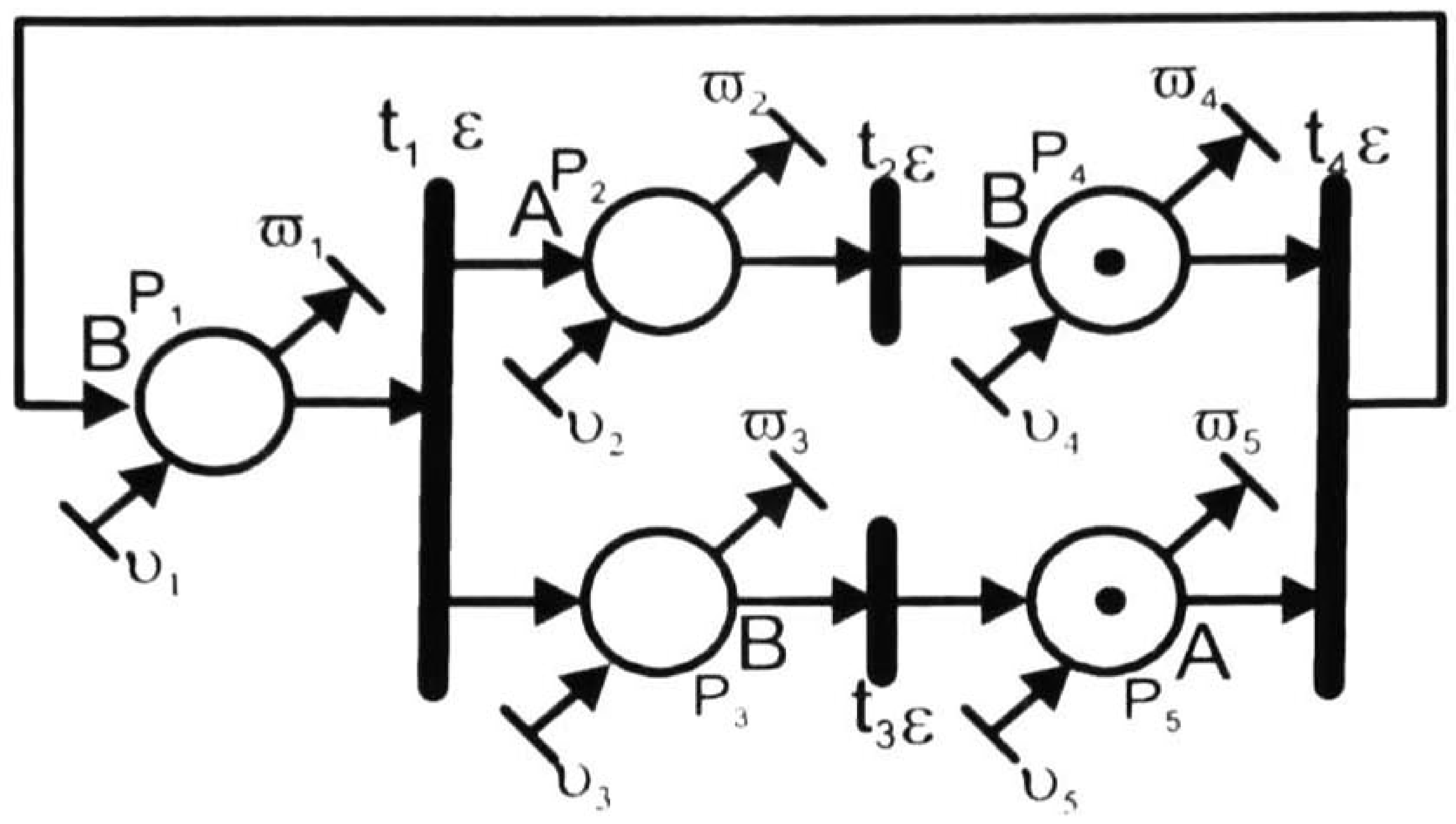


Figure 7.3: Observer of the Discrete State

The Observer is a *SLS*, the discrete part is designed as in [12] which is shown in Fig. 7.3 and depicted as in Section 7.

The state observer for the continuous part is

$$\Phi\left(\hat{M}_k\right) = \Sigma_{\hat{M}_k} \begin{cases} A_{\hat{M}_k}\hat{x}(\tau) + B_{\hat{M}_k}u(\tau) + T_{\hat{M}_k}^{-1}L_{\hat{M}_k}sign_{\hat{M}_k}(e) \\ \hat{y}(\tau) = C_{\hat{M}_k}\hat{x}(\tau) \end{cases}$$

where marking $\hat{M}_k$ is named as follows:

$\hat{M}_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \end{bmatrix}^T$ $\hat{M}_1 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \end{bmatrix}^T$ $\hat{M}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \end{bmatrix}^T$
$\hat{M}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T$ $\hat{M}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$

And *LS* matrices $(A_{M_k}, B_{M_k}, C_{M_k})$ of the system Σ_{M_k} are the same as the matrices $\left(A_{\hat{M}_k}, B_{\hat{M}_k}, C_{\hat{M}_k}\right)$ of the system $\Sigma_{\hat{M}_k}$ Evolving *LS* observer $\Sigma_{\hat{M}_k}$ may not be the same as Σ_{M_k} at every time instant.

The observer parameter $\bar{L}_{\hat{M}_k} = diag\{500\}$, where $\dim(\bar{L}_{\hat{M}_k}) = \dim(O_{\hat{M}_k})$, the similar transformations $T_{\hat{M}_k}$ are

$$T_{\hat{M}_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_{\hat{M}_1} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix} \quad T_{\hat{M}_2} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

$$T_{\hat{M}_3} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad T_{\hat{M}_4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

And the initial marking of the *IPN* observer is $\hat{M}_3$.

According to the firing rule of Section 7, the *IPN* marking of the observer is equal to the *IPN* marking of the system after two commutations, hereafter the continuous observer evolving corresponds to the evolving continuous system, the observable variables of each linear systems are estimated (Fig. 7.4) and the estimation error converges to zero (Fig. 7.4 and 7.5).

The observer herein presented is asymptotic. Once the discrete state has been observed

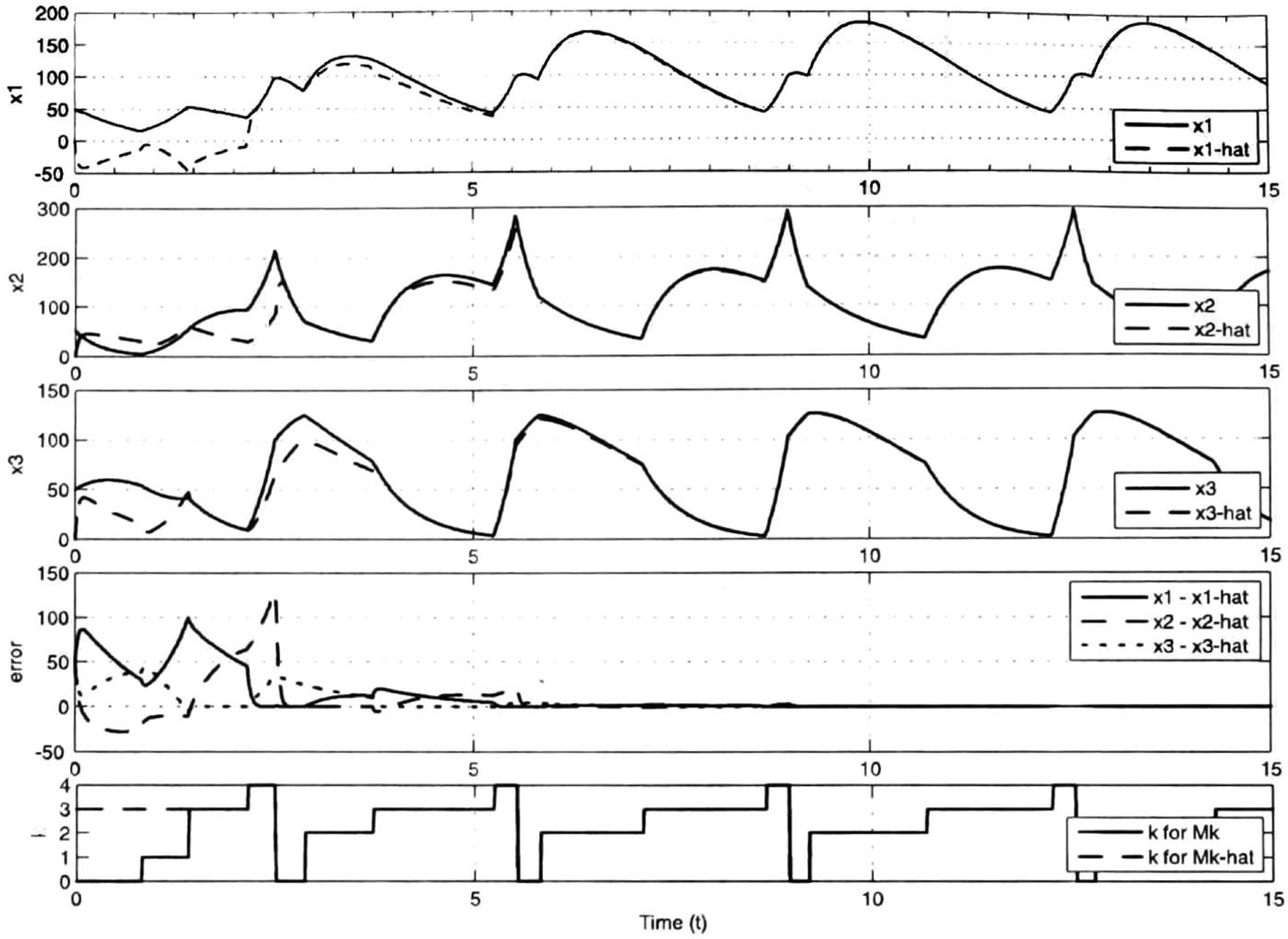


Figure 7.4: *SLS* states, observed states and observation error..

and it has been executed a sequence σ such that the observability matrix O_σ has full rank, then, current state can be calculated at each commutation time. This computed state vector can be used to set the initial condition for the continuous observer for the next commutation.

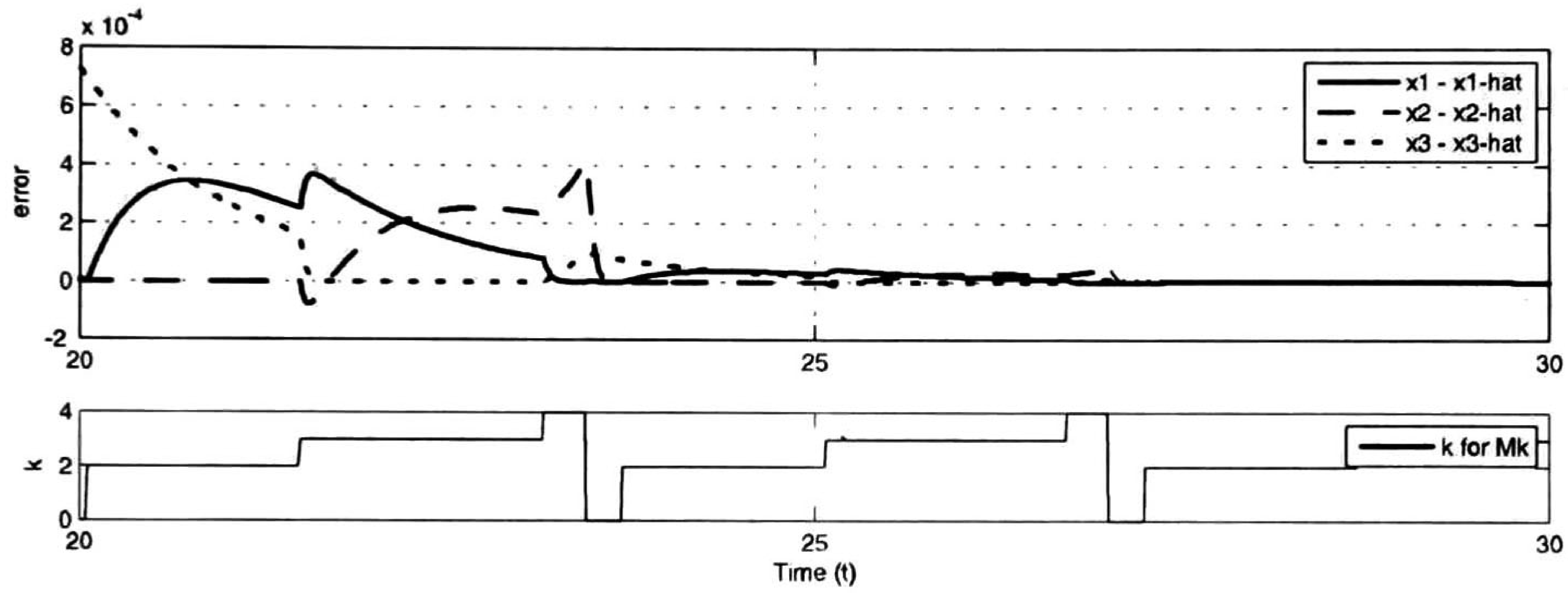


Figure 7.5: SLS Observer error for state vector for interval $t = [20, 30]$

Chapter 8

CONCLUSIONS AND FUTURE WORK

This work presented necessary and sufficient conditions for observability of two classes of hybrid systems, named Switched Linear Systems (SLS) and Linear Hybrid Systems (LHS).

The analysis herein presented exploits the information provided by the discrete system as well as the structure of the continuous system to determine when an LHS is observable. Several contributions were derived using this approach. The next lines numerates some of them.

1. The hypothesis that the IPN of the SLS should be sequence detectable was dropped by relabeling the IPN. In order to properly relabel an IPN, the distinguishability of LS was proposed. When two linear systems are distinguishable from each other, some places of the IPN are relabeled so the IPN could become sequence and marking detectable and the entire SLS becomes observable.
2. Necessary and sufficient conditions to determine the switching time between LS were proposed. This property is used to characterize the observabiity of the entire SLS.
3. Necessary and sufficient conditions to determine when a SLS is observable were presented. These results can be applied in both cases, when the switching time is finite or infinite.
4. It was concluded that the conditions for observability for SLS are sufficient for observability in LHS.
5. It was noticed that previous observability results reported in literature are particular cases of the theory herein exposed.

As future work the concept of distinguishability between linear systems used in this work can be also applied in fault detection in HS in order to distinguish between a normal state and a state of fault using just the input and the output of the continuous system, that is, the continuous information can be used to detect a state of fault even when the discrete systems is not diagnosticable. And thus conditions in the field of fault detection in HS can be derived using a similar analysis like the one presented for the observability.

Also the results presented for scalar systems can be easily extended for multivariable systems.

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Observabilidad en una Clase de Sistemas Lineales Híbridos
"Observability in a Class of Linear Hybrid Systems"

del (la) C.

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