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“Un Enfoque Perturbativo de las Masas y Mezclas de Neutrinos: Un Modelo de Dirac-Taylor”

Tesis que presenta

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A Perturbative Approach for Neutrino Masses and Mixing: A Dirac-Taylor Model

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Abstract

Based on observations of neutrino oscillations in solar, reactor, and atmospheric experiments, a relationship between measurements emerges: the ratio of squared mass differences: $\lambda = \sqrt{(\Delta m^2_{sol})/(\Delta m^2_{atm})} \ll 1$. If we assume that the mass terms are functions that can be expanded in a series using λ as the sole expansion parameter, a matrix structure of neutrino masses emerges, expressed in a general form as $M_\nu \approx M_0 + \delta M(\lambda)$, where M_0 represents the zeroth-order matrix of the expansion, and δM is a matrix formed by smooth complex functions of λ. In this work, I introduced a general method for diagonalizing the neutrino mass matrix, which is applicable independently of the nature and hierarchy of neutrinos. Utilizing the Takagi's lemma as the basis for this diagonalization process, I successfully derived the squared mass differences, Δm^2_{sol} and Δm^2_{atm}, as well as determined the mixing angles and the CP-violating phase. All these observables are expressed in terms of the expansion parameter λ. As an application of this perturbative model, I considered Dirac-type neutrinos, which imply a hermitian mass matrix. By considering $\Delta m^2_{sol} \to 0$, the neutrino spectrum becomes doubly degenerate, resulting in a matrix structure M_0. Through the corrections $\delta M(\lambda)$ and the proposed method, successfully determining the masses and angles mentioned earlier was achieved. Finally, I conducted a numerical analysis, confirming the validity of the method.

Resumen

A partir de las observaciones de las oscilaciones de neutrinos en experimentos solares, de reactores y atmosféricos, emerge una relación entre las mediciones: la razón de las diferencias de masas al cuadrado: $\lambda = \sqrt{\Delta m^2_{sol}/\Delta m^2_{atm}} \ll 1$. Si asumimos que los términos de masa son funciones que pueden expandirse en serie usando a λ como único parámetro de expansión, surge una estructura matricial de masas de neutrinos, expresada en forma general como $M_\nu \approx M_0 + \delta M(\lambda)$, donde M_0 representa la matriz a orden cero de la expansión y δM es una matriz formada por funciones complejas suaves de λ. En este trabajo introduje un método general para diagonalizar la matriz de masas de neutrinos, el cual es aplicable independientemente de la naturaleza de los neutrinos y de la jerarquía que ellos poseen. Utilizando el lema de Takagi como base para este proceso de diagonalizacion, se logro derivar las diferencias de masas al cuadrado, Δm^2_{sol} y Δm^2_{atm}, como también determinar los ángulos de mezcla y la fase de violación de CP. Todos estos observables en términos del parámetro de expansión λ. Como aplicación de este modelo perturbativo, se considero neutrinos del tipo Dirac, el cual implica una matriz de masas hermitiana. Al considerar $\Delta m^2_{sol} \to 0$, se degenera doblemente el espectro de neutrinos dando como resultado una estructura de la matriz M_0, que mediante las correcciones $\delta M(\lambda)$, se aplico el método propuesto el cual permitió determinar con éxito las masas y ángulos antes mencionados. Finalmente, realice un análisis numérico el cual confirma la validez del método.

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Introduction

Three of the four fundamental forces of nature are studied by the Standard Model, namely, the strong, weak, and electromagnetic forces. This model has successfully described a large number of phenomenological observations of interactions among elementary particles, and a vast amount of experimental analyses support the theory. However, the Standard Model is not a complete theory, as it does not allow for the prediction of certain experimental observations. For instance, the case of the dark components of the universe suggests fundamental elements not accounted for in the model, as well as observations of neutrino oscillations, indicating that neutrinos have mass and mix.

To attempt to provide an explanation for the masses of neutrinos, as well as their nature, mixing angles, matter-antimatter asymmetry, and dark matter, the possible extensions of the Standard Model are often explored. The Standard Model is constituted by the gauge symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$. A minimal extension involves the addition of right-handed neutrinos. It is important to note that this does not introduce an extra symmetry group, rather, the mere consideration of right-handed neutrinos introduces new Yukawa couplings that allow for the generation of mass terms for neutrinos. There are two possible natures for these particles: Dirac-type and Majorana-type. Neutrinos are said to be of the Dirac type if the antiparticle and the particle in question are distinct, meaning the antiparticle $\overline{\nu}$ is different from ν, and of Majorana nature if the particle and its corresponding antiparticle are the same. This latter nature of neutrinos allows for the implementation of the seesaw mechanism, which explains the relative measurements of the masses of observed neutrinos compared to quarks and charged leptons, which are much heavier [1]. However, although this seesaw mechanism explains the smallness of neutrino masses, it does not provide an explanation for their mixing.

The unitary matrix known as the PMNS matrix (Pontecorvo-Maki, Nakagawa-Sakata) appears in the vertices of the electroweak flavor-changing interaction through the W bosons when expressed in the basis of mass eigenstates rather than the weak eigenstate basis [2, 3]. This occurs in the Lagrangian of charged current. Likewise, these weak and mass states are a linear combination weighted by the elements of the PMNS matrix, which results in neutrino mixing. For this reason, it is said that this matrix plays the same role in neutrino mixing as the CKM matrix does for quarks. The mixing matrix in the PMNS parametrization can be written as the product of three rotations, two real and one complex, namely: $U = R_{23}(\theta_{23})R_{13}(\theta_{13},\delta)R_{12}(\theta_{12})$, where θ_{ij} are the mixing angles, and δ is the Dirac CP phase [4, 5].

Neutrino oscillations indicate that experimentally, it is only possible to determine the squared mass differences of the three standard neutrinos [6]. Experimental data provide values for the two independent mass scales, $\Delta m^2_{sol} = 7.41^{+0.21}_{-0.20} \times 10^{-5}\mathrm{eV}^2$ and $\Delta m^2_{atm} = 2.507^{+0.21}_{-0.20}\left(2.486^{+0.026}_{-0.028}\right) \times 10^{-3}\mathrm{eV}^2$, corresponding to the neutrino oscillation differences for solar and atmospheric neutrinos, respectively, for the normal (inverted) hierarchy.

By definition, hierarchy refers to the order of the eigenvalues of mass, $m_i, i = 1,2,3$. It is called the normal hierarchy (NH) when it satisfies $m_3^2 \gg m_2^2 > m_1^2$, and it is called the inverted hierarchy (IH) when it satisfies the relation $m_2^2 > m_1^2 \gg m_3^2$. In this same context, the following identification is made: $\Delta m^2_{sol} = \Delta m^2_{21}$ and $\Delta m^2_{atm} = \Delta m^2_{31}\left(\left|\Delta m^2_{32}\right|\right)$, for the normal (inverted) hierarchy, where in general $\Delta m^2_{ij} = m_i^2 - m_j^2$.

For the three mixing angles, we have the following values [6]: $\sin\theta_{12} = 0.303^{+0.023}_{-0.012}$, $\sin\theta_{23} = 0.451^{+0.019}_{-0.016}$, and $\sin\theta_{13} = 0.02225^{+0.00056}_{-0.00059}$. Additionally, $\delta/^\circ = 232^{+36}_{-26}\left(276^{+22}_{-29}\right)$.

From all of the above, certain peculiarities can be observed. The first one to note is that by taking the ratio of the atmospheric and solar mass scales, we have $\lambda = \sqrt{\Delta m^2_{sol}/\Delta m^2_{atm}} = 0.1719$. The tangent of the angle θ_{13} has a value of 0.1616, which is of the order of λ. Similarly, the last peculiarity that is noticed is that the difference between $\tan\theta_{atm}$ and its maximum value is also of the same order. Consequently, there are two scenarios here, either there is some relationship between these, or it is simply a numerical coincidence. Apart from this, if it is assumed that the mass terms are functions of λ, these can be expanded into a Taylor series, giving rise to a certain matrix structure for the neutrino masses (either for Dirac or Majorana neutrinos) where the aforementioned apparent connections allow the construction of the zeroth-order term in the expansion. In other words, using the parameter λ, the neutrino mass matrix can be expanded in a Taylor series as $M_\nu = M_0 + \delta M(\lambda)$, which can then be diagonalized to reproduce all the characteristics of the oscillation observations [5]. In this approach, at zeroth order, a maximal θ_{23} is required with a spectrum of two degenerate neutrinos, while keeping other mixings null. Once this degeneracy is lifted to provide the solar scale, all mixings are linked to this single parameter.

In this thesis work, a general technique for the analytical diagonalization of complex matrices has been developed, explicitly taking into account the observational fact that the reactor mixing angle, θ_{13}, is relatively small. Using this approach, the possible connection between atmospheric and reactor mixings was investigated, suggesting a rather specific structure of neutrino mass matrices. As part of the exploration of this mechanism, the case of Dirac neutrinos was examined. All of this also allows for accommodating the expected value of the CP violation phase.

With this in mind, the structure of this document is as follows. In the first chapter, I study the Standard Model with a deep emphasis on the electroweak sector, which consists of the gauge symmetry group $SU(2)_L \times U(1)_Y$. Similarly, I explore how the implementation of the Higgs Mechanism leads to the masses of the bosons Z and $W^{\pm}$ through spontaneous symmetry breaking, as well as the generation of fermion masses as a consequence of the Higgs mechanism and Yukawa couplings. In this chapter, I also address the issue of the absence of neutrino masses, which arises due to the total lack of right-handed neutrinos.

In the second chapter, I investigate the possibility of neutrino masses by adding three right-handed neutrinos, corresponding to the minimal extension of the Standard Model. Due to this extension, I discuss the two natures of neutrinos, namely, Dirac and Majorana neutrinos. Based on the exploration in the central part of my work, I describe the addition of a new $U(1)_{B-L}$ gauge symmetry, which allows to consider only Dirac neutrinos and study the absence of anomalies in this symmetry. To conclude this chapter, I describe the parametrization of a unitary matrix, the diagonalization of a complex $N \times N$ matrix through a biunitary transformation, and discuss CP violation and the Jarlskog invariant.

In the third chapter, I study the phenomenology of neutrinos, focusing on the primary observations of neutrino oscillations. I then present the formalism for neutrino oscillations, applying it to both two and three flavors. I also examine the effects of matter and how they alter oscillation observables. Finally, I provide a brief overview of the experiments involved.

The central part of the thesis lies in Chapter 4. In this final chapter, I provide a general analytical treatment for the perturbative diagonalization of complex mass matrices, explicitly taking into account that the reactor mixing is relatively small. This diagonalization technique is closely based on the Takagi factorization, where the matrix to be diagonalized is the square Hermitian of the mass matrix, that is, a technique is developed for the matrix $M_\nu M_\nu^\dagger$. This approach allows us to consider both natures of neutrinos in a general manner. Using the Takagi factorization lemma, the matrix is diagonalized. Furthermore, this approach permits exploring the connection between atmospheric and reactor mixings, revealing that neutrino mass matrices possess a pretty specific structure. Similarly, this method allows for an analysis of CP violation and the accommodation of the expected value of that phase, noting that, to enhance the latter, Dirac neutrinos were considered.

To conclude the chapter and my thesis work, I proceed to provide a numerical analysis of diagonalization on the mass matrix structure, thereby verifying the correctness of the proposed method, taking into account the oscillation parameter values reported at the 1σ and 3σ confidence levels.

We present conclusions and some perspectives in Chapter 5. Additionally, we have included a couple of appendices to enhance technical clarifications regarding the analytical approaches we are presenting.

CHAPTER 1

The Standard Model

The gauge groups determine (from the physical point of view) the interactions and the number of vector bosons corresponding to the generators of the group. In this sense, the standard model is a Gauge theory based on the local symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$, which is a direct product of the symmetry groups of strong interactions, $SU(3)_C$, and electroweak interactions, $SU(2)_L \times U(1)_Y$, while the symmetry group of electromagnetic interactions, $U(1)_{em}$, is a subgroup of the electroweak interactions. This implies that weak and electromagnetic interactions are unified.

1.1 The Electroweak Lagrangian

The Standard Model is essentially a gauge theory associated with a specific symmetry group. As a result, it is possible to determine the generators of the group. In the present case, we have 8 generators for $SU(3)_C$ called gluons, which are massless and mediate strong interactions. There are also 4 gauge bosons, out of which 3 ($W^{\pm}$ and Z) are massive, while one is massless (the photon). These gauge bosons correspond to specific combinations of the 3 generators of $SU(2)_L$ and one of $U(1)_Y$.

Electroweak interactions can be studied independently from color interactions due to the unbroken symmetry of the strong interaction. The group $SU(3)_C$ acts trivially on the $\mathbb{C}^2$ space, which is the Higgs field space, and there is no mixing between the strong and weak sectors [7]. The weak interaction is the one involved in neutrino interactions. This force or interaction is very short-range, around 10^{-17} meters, due to the mass of the W and Z bosons, which have a mass of approximately 90 GeV.

	ν_{eL}	e_L	e_R	u_L	d_L	u_R	d_R
I_3	1/2	-1/2	0	1/2	-1/2	0	0
Y	-1	-1	-2	1/3	1/3	4/3	-2/3
Q	0	-1	-1	2/3	-1/3	2/3	-1/3

Table 1.1 First family of leptons and quarks, right-handed and left-handed. Their third component of isospin, I_3, hypercharge, Y, and charge Q are listed.

The symmetry group $SU(2)_L$ is known as weak isospin, which acts non-trivially on the left-handed chirality components of fermionic fields, i.e., they are doublets, while the right-handed chirality components are singlets under the action of elements of the group $SU(2)_L$. The generators of this group (three generators, I_α, where $\alpha = 1,2,3$) satisfy the commutation rule $[I_\alpha, I_\beta] = i\varepsilon_{\alpha\beta\gamma}I_\gamma$. We also have the group $U(1)_Y$, known as hypercharge, which is generated by the hypercharge operator Y, which is related to I_3 and the charge operator Q through the Gell-Mann-Nishijima relation given by

$$Q = I_3 + \frac{Y}{2}.$$

This relation is useful as it allows us to fix the value of Y for the fermionic fields. On the other hand, the commutation relation between Y and I_α satisfies the identity $[I_\alpha, Y] = 0$. The quantum numbers of the fermionic fields (first generation) are listed in Table 1.1. For the second and third generation, the values are completely identical.

It is necessary for the theory to be locally gauge invariant, which requires the introduction of three gauge boson vector fields associated with the three generators I_α of the $SU(2)_L$ group, which we denote as Λ^μ_α. Additionally, a gauge boson vector field associated with the generator of the $U(1)_Y$ group, denoted as M_μ, is also necessary.

Furthermore, to construct the gauge invariant Lagrangian, it is necessary to define the covariant derivative D_μ, which replaces the ordinary derivative ∂_μ in the Lagrangian. In general, the covariant derivative is given by

$$D_\mu = \partial_\mu + ikT_\alpha A^\alpha_\mu, \tag{1.1}$$

where k is a constant, the T_α are the generator matrices for the group $SU(N)$ and A^α_μ are a new set of fields. Defining $\mathbf{A}_\mu = T_\alpha A^\alpha_\mu$. Thus, acting on the fields, the covariant derivative is

$$\left(D_\mu \Phi\right)_j = \partial_\mu \Phi_j + ig\left[\mathbf{A}_\mu\right]_{jk}\Phi_k.$$

In the case of a doublet, the derivative is given by

$$D_\mu = \partial_\mu + ig\Lambda_\alpha^\mu I_\alpha + ig' M_\mu \frac{Y}{2},$$

where the constants g and g' are two independent coupling constants associated with the $SU(2)_L$ and $U(1)_Y$ groups, respectively [8]. While in the case of a singlet q_j, we have

$$D_\mu q_j = \left[\partial_\mu + ig' y_j M_\mu\right] q_j.$$

Let us choose the representation in which we will arrange the fermions, which is to take the left-handed chiral components of the fermionic field grouped in a weak isospin doublet, that is

$$L'_{\alpha L} \equiv \begin{pmatrix} \nu'_{\alpha L} \\ \alpha'_L \end{pmatrix}, \tag{1.2}$$
$$Q'_{1L} \equiv \begin{pmatrix} u'_L \\ d'_L \end{pmatrix}, \quad Q'_{2L} \equiv \begin{pmatrix} c'_L \\ s'_L \end{pmatrix}, \quad Q'_{3L} \equiv \begin{pmatrix} t'_L \\ b'_L \end{pmatrix},$$

where $L'_{\alpha L}$ represents the leptons, $\alpha = \{e, \mu, \tau\}$, and Q'_{iL} represents the quarks, $i = \{1, 2, 3\}$. Meanwhile, the right-handed chiral components are singlets, for example $\ell'_{eR} \equiv e'_R$, similarly for μ and τ[1]. While for the quarks, we have the following configuration $q'^U_{\beta R} = \beta'_R$, where $\beta = u, c, t$, similarly $q'^D_{kR} = k'_R$, where $k = d, s, b$. With

[1]It is convenient to define the following arrays of charged lepton fields: $\ell'_L \equiv \begin{pmatrix} e'_L & \mu'_L & \tau'_L \end{pmatrix}^T$ and $\ell'_R \equiv \begin{pmatrix} e'_R & \mu'_R & \tau'_R \end{pmatrix}^T$, which will be used later. The prime symbol indicates that the fields are weak eigenstates.

this, it is possible to write the most general Lagrangian that symmetries allow [9]:

$$\begin{aligned}
\mathscr{L} =& i\bar{L}'_{\alpha L}\, \not{D}\, L'_{\alpha L} + i\bar{Q}'_{jL}\, \not{D}\, Q'_{jL} \\
&+ i\bar{\ell}'_{\alpha R}\, \not{D}\, \ell'_{\alpha L} + i\bar{q}'^{D}_{\beta R}\, \not{D}\, q'^{U}_{\beta R} + i\bar{q}'^{U}_{kR}\, \not{D}\, q'^{U}_{kR} \\
&- \frac{1}{4}\Lambda_{\mu\nu}\Lambda^{\mu\nu} - \frac{1}{4}M_{\mu\nu}M^{\mu\nu} \\
&+ \left(D_\rho\Phi\right)^\dagger \left(D^\rho\Phi\right) - \mu^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2 \\
&- \sum_{\alpha,\beta=e,\mu,\tau} \left(Y'^{\ell}_{\alpha\beta}\bar{L}'_{\alpha L}\Phi\ell'_{\beta R} + h.c\right) \\
&- \sum_{\alpha=1,2,3}\sum_{\beta=d,s,b} \left(Y'^{D}_{\alpha\beta}\bar{Q}'_{\alpha L}\Phi q'^{D}_{\beta R} + h.c\right) \\
&- \sum_{\alpha=1,2,3}\sum_{\beta=u,c,t} \left(Y'^{U}_{\alpha\beta}\bar{Q}'_{\alpha L}\widetilde{\Phi} q'^{U}_{\beta R} + h.c\right),
\end{aligned} \tag{1.3}$$

where we have Φ as the Higgs doublet, $\widetilde{\Phi} = i\tau_2\Phi^*$ with τ_k is the Pauli matrices and Y'^{ℓ} are the elements of the matrix that determines the Yukawa couplings with the Higgs doublet, which is generally a complex 3×3 matrix.
The Lagrangian (1.3) yields the electromagnetic interaction in the first two lines, the third line includes the kinetic terms and self-couplings of the gauge fields, the fourth line represents the Lagrangian for the Higgs fields responsible for spontaneous symmetry breaking, and the last three lines contain the Yukawa couplings with the Higgs that generate fermion masses and quark mixing.

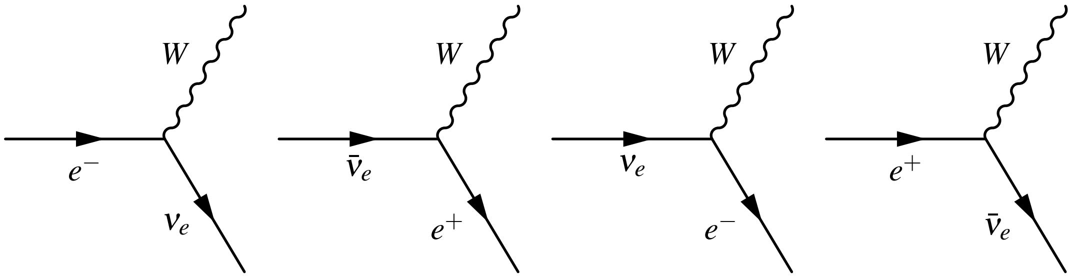
If we focus solely on the first two lines, we can obtain the Lagrangian for weak charged current (CC) and neutral current (NC) interactions, which are given by

$$\mathscr{L}_I^{CC} = -\frac{g}{2\sqrt{2}} j^\rho_W W_\rho + H.C \tag{1.4}$$

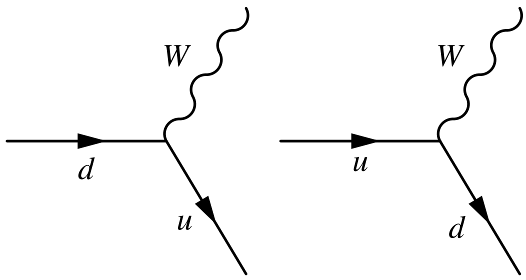
$$\mathscr{L}_I^{CN} = -\frac{g}{2\cos\theta_W} j^\rho_Z Z_\rho \tag{1.5}$$

where j^ρ_W is the fermionic charged current, which is given in terms of the charged lepton current, $J^\rho_{W,L}$, and the quark current, $J^\rho_{W,Q}$ [8, 9]. On the other hand, we have that j^ρ_Z is the neutral current, which is also given in terms of the neutral lepton current, $J^\rho_{Z,L}$, and the quark current, $J^\rho_{Z,Q}$. The angle θ_W represents the weak mixing angle or Weinberg angle. Furthermore, we have defined the field W^μ, which possesses the property of annihilating W^+ bosons and creating W^- bosons, in a manner analogous to the creation and annihilation operators of angular momentum. The expression is given by $W_{-}^{\mu} = \left(A_1^{\mu} - iA_2^{\mu}\right)/\sqrt{2}$.

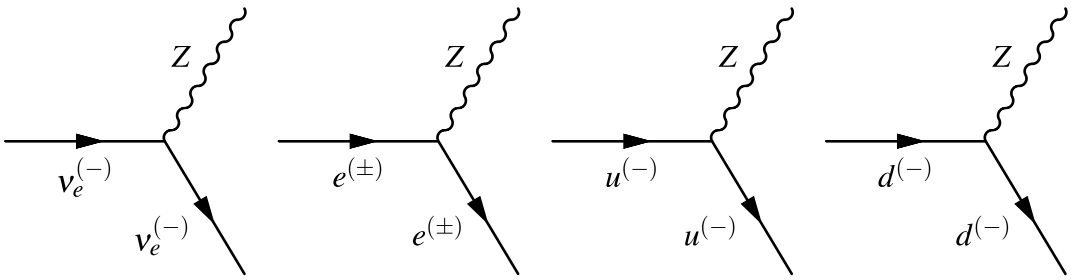
The interaction Lagrangian in Eq. (1.4) gives rise to trilinear couplings represented in the following diagrams. It should be mentioned that these diagrams only depict the case of an electron interacting with its corresponding neutrino; for the cases of muon or tau, the situation is analogous:



Considering only the first generation of quarks, Eq. (1.4) yields the following diagrams



There are two other options that arise from the application of the crossing symmetry. On the other hand, the Lagrangian of neutral current interaction,Eq. (1.5), yields the following trilinear couplings:



Analogously for the remaining generations of charged fermions[2]. It is important to note that for these expressions, an orthogonal linear combination defines the field of vector bosons Z^μ as follows

$$\begin{aligned} A^\mu &= \sin\theta_W A_3^\mu + \cos\theta_W M^\mu, \\ Z^\mu &= \cos\theta_W A_3^\mu - \sin\theta_W M^\mu. \end{aligned}$$

Then, considering only the first family, we obtain the Lagrangian for neutral current interactions as follows

$$\begin{aligned} \mathscr{L}^{CN} = &-\frac{1}{2}(\bar{\nu}_{eL}\left(\{g\cos\theta_W + g'\sin\theta_W\}\not{Z} + \{g\sin\theta_W - g'\cos\theta_W\}\not{A}\right)\nu_{eL} \\ &-\bar{e}_L\left(\{g\cos\theta_W - g'\sin\theta_W\}\not{Z} + \{g\sin\theta_W + g'\cos\theta_W\}\not{A}\right)e_L \\ &-2g'\bar{e}_R(-\sin\theta_W\not{Z} + \cos\theta_W\not{A})e_R). \end{aligned}$$

Since neutrinos do not couple to electromagnetic fields, it follows that the Lagrangian for neutral current interactions leads to

$$g\sin\theta_W = g'\cos\theta_W.$$

This implies that the coupling constants are connected to the weak mixing angle, which is an important relationship in particle physics [10].

With this, we understand how the $W^\pm$ and Z bosons interact, but the origin of their masses and the masses of charged leptons has not been explained yet. In the following sections, we will study the origin of masses and address the question of neutrino masses, which will be further explored in Chapter 2.

1.2 Higgs Mechanism

In the Standard Model, the Higgs mechanism is the responsible for the masses of the W and Z bosons, as well as the fermions. For this pupose, the following Higgs

[2]From all of this, it is observed that in the charged current, quarks change flavor, while for leptons their respective neutrino is involved. In the case of the neutral current, the particles interact via the Z boson, with only an exchange of spin and/or momentum.

doublet is typically defined

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix}, \tag{1.6}$$

where the superscript + indicates the charged complex scalar field, while the superscript 0 indicates the neutral complex scalar field. In the electroweak model, it is common to introduce the doublet, given in the Eq. (1.6), because this is the simplest way to induce symmetry breaking. It should be noted that we are working in the Hermitian basis. The Higgs Lagrangian is

$$\mathscr{L}_{Higgs} = \left(D_\rho \Phi\right)^\dagger \left(D^\rho \Phi\right) - \mu^2 \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi\right)^2, \tag{1.7}$$

which is invariant under the $U(1)$ group of global transformations [11]. In this basis, the potential is

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi\right)^2,$$

it is assumed that the coefficient μ^2 is negative, in order to generate the spontaneous symmetry breaking: $SU(2)_L \times U(1)_Y \to U(1)_{em}$, see Figure 1.1. Neglecting the term μ^4/λ^2, the potential can be rewritten as follows

$$V(\Phi) = \lambda \left(\Phi^\dagger \Phi - \frac{1}{2}\left(-\frac{\mu^2}{\lambda}\right)\right)^2.$$

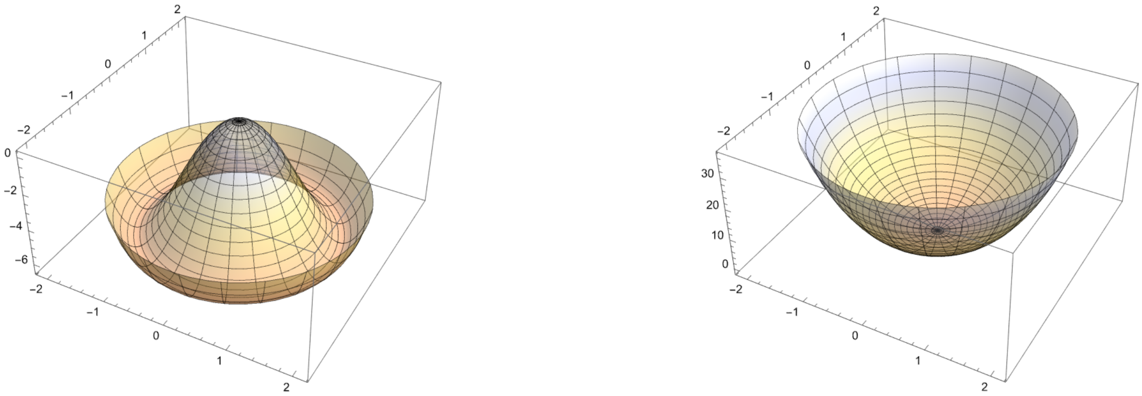
We define

$$\mathrm{v} = \sqrt{-\frac{\mu^2}{\lambda}}, \tag{1.8}$$

therefore, the Higgs potential can be written in terms of this new constant

$$V(\Phi) = \lambda \left(\Phi^\dagger \Phi - \frac{1}{2}\mathrm{v}^2\right)^2. \tag{1.9}$$

On the other hand, a vacuum vector is defined as an element of the Higgs vector space $\mathbb{W}$ that is a minimun of a real-valued function V. The set of vacuum vectors in $\mathbb{W}$ is called the vacuum space or vacuum manifold of V. From Eq.(1.9), it is easy to



(a) Higgs potential for $\mu^2 < 0$ (b) Higgs potential for $\mu^2 > 0$

Fig. 1.1 Higgs potential for the two cases of μ^2, always assuming λ to be positive. In (a) spontaneous symmetry breaking occurs, while in (b) it does not occur.

see that the minimun value of the potential occurs when

$$\Phi^{\dagger}\Phi = \frac{\mathrm{v}^2}{2},$$

so, a vector will be an empty vector for the Higgs potential, $\mathrm{v} \in W$, if and only if its magnitude has the expression given in Eq. (1.8). This potential minimum corresponds to the vacuum value (lowest energy state). When we are in the case of $\mu^2 < 0$, we see from Figure 1.1a that there are a large number of degenerate ground states, but none of them exhibit the original symmetry of the Lagrangian given in Eq.(1.7), it is said that symmetry is spontaneously broken. When we have $\mu^2 > 0$, we see from Figure 1.1b that the minimum occurs at the origin, so in the ground state, the symmetry is present. There is no symmetry breaking.

The vacuum expectation value (VEV) of the Higgs field $\langle\Phi\rangle$ is nonzero and is due to the neutral scalar field since the vacuum has no electric charge, meaning its electric charge is identically zero. Therefore, the charged scalar field Φ^+ must have a value of zero in the vacuum, while the neutral scalar field Φ^0 can have a nonzero value in the vacuum. Thus, the VEV is defined as

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \mathrm{v} \end{pmatrix}. \tag{1.10}$$

The field $\Phi(x)$ can be expanded around the vacuum. Let $H(x)$ be a Hermitian scalar field, which can be interpreted as a perturbation (excitation) of the neutral Higgs field about the vacuum. This perturbation represents the physical Higgs bosons, therefore it can be written as follows (see Eq.1.8)

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mathrm{v} + H(x) \end{pmatrix}. \tag{1.11}$$

Furthermore, it is common to use the following expression for the Higgs doublet [12]

$$\Phi = \frac{1}{\sqrt{2}} \exp\left\{ \frac{i}{2v} \vec{\xi} \cdot \vec{\tau} \right\} \begin{pmatrix} 0 \\ \mathrm{v} + H(x) \end{pmatrix},$$

where $\vec{\xi}(x) = \{\xi_1(x), \xi_2(x), \xi_3(x)\}$ are real scalar fields, which represent nonphysical states and $\vec{\tau} = \{\tau_1, \tau_2, \tau_3\}$ where $\tau_\alpha, \alpha = \{1,2,3\}$, are the three Pauli matrices. Since the theory is invariant under $SU(2)_L$, it is possible to write (1.11). Note that under this transformation, the physical states of the theory appear explicitly, which is called the unitary gauge.

The concept of the unitary gauge is present in the electroweak theory. Let us consider the following theorem regarding the existence of the unitary gauge in the theory.

Theorem 1.2.1 *Consider the electroweak theory with gauge group $G = SU(2)_L \times U(1)_Y$ and the Higgs field is given in the form of 1.6, where each component of the doublet is a function from $M \to \mathbb{C}$ (where M is a connected and simply connected manifold). Let us assume that the second entry is nonzero $\forall x \in M = \mathbb{R}^4$. Then there exists a physical gauge transformation $\eta : M \to G$ such that*

$$\eta \cdot \Phi = \begin{pmatrix} 0 \\ \psi \end{pmatrix},$$

where $\psi : M \to \mathbb{R}$ is a real-valued function. The transformed Higgs field $\eta \cdot \Phi$ is then in unitary gauge with respect to the vacuum vector $\langle \Phi \rangle$ (see 1.10)[3]*.*

[3]For more information about the theorem and demostration see [13].

Considering the expression (1.7) and (1.11), we obtain

$$\begin{aligned}
\mathscr{L}_{Higgs} =& \frac{1}{2}(\partial H)^2 - \lambda \mathrm{v}^2 H^2 - \lambda \mathrm{v} H^3 - \frac{\lambda}{4} H^4 + \frac{g^2 \mathrm{v}^2}{4} W_\mu^\dagger W^\mu + \frac{g^2 \mathrm{v}^2}{8 \cos{\theta_W}^2} Z_\mu Z^\mu \\
&+ \frac{g^2 \mathrm{v}}{2} W_\mu^\dagger W^\mu H^2 + \frac{g^2 \mathrm{v}}{4 \cos{\theta_W}^2} Z_\mu Z^\mu H \\
&+ \frac{g^2}{4} W_\mu^\dagger W^\mu H^2 + \frac{g^2}{8 \cos{\theta_W}^2} Z_\mu Z^\mu H^2 .
\end{aligned} \tag{1.12}$$

The fifth and sixth terms are of significant importance since they correspond to the mass terms of the W and Z bosons respectively:

$$m_W = \frac{g\mathrm{v}}{2}, \qquad\qquad m_Z = \frac{g\mathrm{v}}{2\cos\theta_W}. \tag{1.13}$$

The second term of Eq.(1.12) corresponds to the mass term of the Higgs boson, with a mass given by $m_H = \sqrt{2\lambda \mathrm{v}^2}$. The third and fourth terms generate the self-interactions of the Higgs field, the last four terms generate the couplings of the Higgs field with the gauge bosons (trilinear and quadrilinear couplings).

1.2.1 Nambu-Goldstone Bosons

The Goldstone bosons (also known as Nambu-Goldstone bosons) are bosons that arise from the spontaneous breaking of symmetry. These bosons are associated with the generators of the broken symmetry. It should be noted that spontaneous symmetry breaking occurs when the ground state is not invariant under the transformations of the group. In other words, the Nambu-Goldstone bosons correspond to perturbations of the Higgs field along the symmetry group orbit. On the other hand, the Higgs bosons correspond to perturbations orthogonal to the orbit. The Goldstone's theorem can be stated as follows [14].

Theorem 1.2.2 ***Goldstone Theorem:*** *If a Lagrangian is invariant under a continuous symmetry group that has n generators, and if its ground state is symmetric under a continuous group containing n′ generators, there should be n − n′ massless states*[4].

Based on what has been said before and the Goldstone's theorem, we have the following corollary: For every spontaneously broken continuous symmetry, the theory must contain massless particles called Nambu-Goldstone bosons, which are

[4]For a proof of this theorem, see the reference [15].

scalar fields with zero mass. On the other hand, Higgs bosons are scalar fields with non-zero mass [16].

1.3 Fermion masses in the Standard Model

A consequence of the Higgs Mechanism is the generation of fermion masses through the Yukawa coupling of fermion fields with the Higgs doublet. Let us consider the Lagrangian in Eq. (1.3), particularly the fifth line, as it contains the masses of the leptons, while the sixth and seventh lines contain the masses of the quarks, which are not very relevant for the present thesis. Thus, we can derive the masses of leptons from:

$$-\sum_{\alpha,\beta=e,\mu,\tau}\left(Y'^{\ell}_{\alpha\beta}\bar{L}'_{\alpha L}\Phi\ell'_{\beta R}+Y'^{\ell *}_{\alpha\beta}\bar{\ell}'_{\beta R}\Phi^{\dagger}L'_{\alpha L}\right)=-\sum_{\alpha,\beta=e,\mu,\tau}Y'^{\ell}_{\alpha\beta}\bar{L}'_{\alpha L}\Phi\ell'_{\beta R}+H.c. \tag{1.14}$$

From this, it can be seen that in the standard model neutrinos are massless because their fields do not have a right-handed component[5]. On the other hand, the matrix of Yukawa couplings Y'^{ℓ} is, in general, a non-diagonal 3×3 complex matrix. Now, let us consider the doublet Higgs as in Eq.(1.11), to have

$$\mathscr{L}_{Y,F}=-\left(\frac{v+H}{\sqrt{2}}\right)\sum_{\alpha,\beta=e,\mu,\tau}Y'^{\ell}_{\alpha\beta}\bar{\ell}'_{\alpha L}\ell'_{\beta R}+H.c, \tag{1.15}$$

where the index Y,F indicate the Yukawa coupling of leptons sector. In the matrix form the Higgs-lepton Yukawa Lagrangian (1.15) is

$$\mathscr{L}_{Y,F}=-\left(\frac{v+H}{\sqrt{2}}\right)\bar{\ell}'_{L}Y'^{\ell}\ell'_{R}+H.c. \tag{1.16}$$

Since the matrix Y'^{ℓ} is not diagonal, the charged lepton fields do not have definite masses. To obtain these fields with definite masses, it is necessary for Y'^{ℓ} to be diagonal. For this, it can be diagonalized through the biunitary transformation (see Theorem 2.3.1)

$$V_L^{\ell\dagger}Y'^{\ell}V_R^{\ell}=Y^{\ell},$$

[5] A fermion mass term must involve a left-handed and right-handed fields, a coupling of type: $\bar{f}f=\bar{f}_L f_R+\bar{f}_R f_L$

with $Y^{\ell}_{\alpha\beta} = y^{\ell}_{\alpha}\delta_{\alpha\beta}$. Where V^{ℓ}_{L} and V^{ℓ}_{R} are two appropriate 3×3 unitary matrices. This diagonalization, applied to expression (1.16), leads to

$$\mathscr{L}_{Y,F} = -\left(\frac{v+H}{\sqrt{2}}\right)\bar{\ell'}_{L}\left(V^{\ell}_{L}Y^{\ell}V^{\ell\dagger}_{R}\right)\ell'_{R} + H.c = -\left(\frac{v+H}{\sqrt{2}}\right)\left(\bar{\ell'}_{L}V^{\ell}_{L}\right)Y^{\ell}\left(V^{\ell\dagger}_{R}\ell'_{R}\right) + H.c.$$

We then define

$$\ell_L = V^{\ell\dagger}_{L}\ell'_L = \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} \quad , \quad \ell_R = V^{\ell\dagger}_{R}\ell'_R = \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}. \tag{1.17}$$

They are the mass eigenstates obtained via the rotation. Therefore, the Higgs-lepton Yukawa Lagrangian can be rewritten as

$$\mathscr{L}_{Y,F} = -\left(\frac{v+H}{\sqrt{2}}\right)\bar{\ell}_L Y^{\ell}\ell_R + H.c. \tag{1.18}$$

Now, writing the fields of the charged leptons with definite masses as $\ell_\alpha = \ell_{\alpha L} + \ell_{\alpha R}$ where $\alpha = e, \mu, \tau$ we have

$$-\mathscr{L}_{Y,F} = \sum_{\alpha}\frac{y^{\ell}_{\alpha}v}{\sqrt{2}}\bar{\ell}_\alpha \ell_\alpha + \sum_{\alpha}\frac{y^{\ell}_{\alpha}}{\sqrt{2}}H\bar{\ell}_\alpha \ell_\alpha. \tag{1.19}$$

The first term of Eq.(1.19) is the mass term for the charged leptons [9]:

$$m_\alpha = \frac{y^{\ell}_{\alpha}v}{\sqrt{2}}, \tag{1.20}$$

and the second term, proportional to the Higgs Boson field, gives trilinear couplings between the charged leptons and the Higgs boson, whereas the neutrinos do not couple to the Higgs boson. On the other hand, the leptonic charged weak current can be written as

$$j^{\rho}_{W,L} = 2\bar{\nu}'_L\gamma^{\rho}\ell'_L,$$

where

$$\nu'_L \equiv \begin{pmatrix} \nu'_{eL} \\ \nu'_{\mu L} \\ \nu'_{\tau L} \end{pmatrix}. \tag{1.21}$$

Using the expression of the mass eigenstates defined in Eq.(1.17), we have

$$j^{\rho}_{W,L} = 2\bar{\nu}'_L \gamma^{\rho} \left(V^{\ell}_L \ell_L \right).$$

Let us perform the transformation to the mass eigenstate basis (in a manner analogous to the equation 1.17) using the following relation

$$\nu_L \equiv V^{\ell\dagger}_L \nu'_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}. \tag{1.22}$$

So, the leptonic charged weak current can be rewritten as

$$j^{\rho}_{W,L} = 2 \left(\bar{\nu}_L V^{\ell\dagger}_L \right) \gamma^{\rho} \left(V^{\ell}_L \ell_L \right) = 2\bar{\nu}_L \gamma^{\rho} \ell_L. \tag{1.23}$$

From this expression, it can be seen that neutrinos ν_α, where $\alpha = e, \mu, \tau$, couples only with the corresponding charged lepton, that is, ν_e couples only with e, for example. These neutrinos ν_α defined in Eq.(1.22) are called flavor neutrino fields. Furthermore, any linear combination of massless fields is a massless field. Therefore, flavor neutrino fields are also mass eigenstates. However, in physics beyond the Standard Model, the flavor neutrino fields are not mass eigenstates, this phenomenon is known as neutrino mixing [17].

CHAPTER 2

Massive Neutrino

In the previous chapter, I have studied the Higgs mechanism and Yukawa couplings, which allow for the generation of mass for charged fermions. In the current chapter, I will discuss the existence of massive neutrinos, as well as the Dirac and Majorana mass terms and the mass matrix that arises from them.

2.1 On the Hints of Neutrino Mass

We have seen that through Yukawa couplings with the Higgs doublet, it is possible to generate fermionic masses. For this, both left-handed and right-handed fields of fermions are necessary. However, in the choice of fermion doublets and singlets in Eq.(1.2), the possible existence of the neutrino right-handed components, ν_R, has not been taken into account, as there was no phenomenological reason to do so. If the right-handed component of neutrinos is introduced, the left-handed and right-handed neutrino fields could couple through the Higgs mechanism to generate a mass term.

Theoretically, there are not sufficiently valid reasons to not consider the right-handed components ν_R, since there is no preserved gauge symmetry (as in the case of the photon) that would indicate the absence of neutrino mass.

As a motivation for the theoretical search for neutrino masses, particle physics aims to be a unifying theory [1]. In this regard, we must ask the question: How can we extend the standard model to incorporate massive neutrinos? However, we must also consider other aspects related to the previous question. It is expected that the masses of the neutrinos ν_e, ν_μ, and ν_τ differ by orders of magnitude, much like the case of the e, μ, and τ. Moreover, it is known that the masses of neutrinos are much smaller than the mass of the electron[1]. So, the question arises: Is there any model capable of explaining the smallness of neutrino masses? The answer is yes. Currently, there are extended theories of the standard model that provide a certain degree of response to these questions.

There are two main categories of experiments involving the search for neutrino masses: kinematic and exclusive [18]. In the first type, any process allowed by the standard model is considered, with the only condition that neutrinos are present in the final state. An example of such processes can be the decay of the τ or the pion. In the case of pion decay, the energy of the muon can be measured, allowing the determination of the mass of the muonic neutrino, ν_μ, as the energy depends on the neutrinos mass.

The second exclusive experiments type essentially revolves around experiments that seek processes that are not allowed (under the assumption that neutrinos have no mass). If such a process is observed, it is a clear indication of neutrino mass. However, not everything is as straightforward, since assumptions need to be made, and one of the most common assumptions is neutrino mixing. A clear example is neutrino oscillations and magnetic properties, among others.

Although these two types of experiments, the kinematic one, which we could say is a direct search, is quite obsolete. This is because, if we observe the final spectrum of muon momentum, it does not generate monochromatic lines corresponding to mass eigenstates. However, considering the experimental resolution, upper limits on the neutrino mass can be obtained (under the assumption of neutrino mixing) [1].

If we consider the standard solar model proposed by Bahcall [19], a certain amount of neutrino flux is expected to come from the Sun. However, experiments have shown a deficit in the flux of neutrinos emitted by the Sun, favoring the consideration of neutrino mixing. To explain this more clearly, let us consider a beam of pions that decay into muons and muonic neutrinos, which after a certain time encounter a detector that searches for neutrino interactions. In this detector, an interaction with an electron in the final state is detected. As we know, muonic neutrinos do not interact with electrons; it is the electronic neutrinos that do it. The only way to explain what is detected is if the ν_μ neutrino had transformed into an ν_e neutrino. This is known as neutrino oscillation, and the possibility of having this phenomenon finds its fundamental basis within the framework of Quantum theory, where it is possible to write a neutrino produced by the source in terms of a superposition of mass eigenstates. The evidence of this phenomenon provides two

[1] In general, it is known that the mass of neutrinos is smaller than that of quarks and fermions.

very important parameters: the squared mass differences of the neutrinos, Δm^2, and the mixing angles [20].

2.2 Dirac and Majorana Masses

The mass of neutrinos is a topic of paramount importance in the physics of the Standard Model and its extended theories. As I have mentioned, the origin of neutrino mass and its smallness are questions that are currently being actively explored. There are indications that the neutrino masses are manifestations of low energy physics beyond the Standard Model, and their smallness is attributed to a suppression generated by a new high-energy scale. Furthermore, as we saw in Chapter 1, even the origin of the values of masses for charged quarks and leptons remains a mystery within the standard model. All of this leads us to believe that the standard model should be considered as an effective theory derived from the low-energy limit of a more comprehensive theory, where quark and lepton masses can be derived from fundamental principles. In the following, we will delve into the study of Dirac and Majorana neutrino masses.

2.2.1 Dirac Masses

A Dirac neutrino mass can be generated using the same Higgs mechanism as in the Standard Model. Let us consider the Dirac Lagrangian

$$\mathscr{L}_{Dirac} = \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} + m_D\right)\psi, \tag{2.1}$$

where the first term is the kinetic term and the second term is the mass term:

$$\mathscr{L}_{m_D} = \overline{\psi} m_D \psi = m_D \overline{\psi}\psi. \tag{2.2}$$

Now, let us consider ψ arbitrary spinor. Considering the properties of Weyl spinors and the anticommutation relation of gamma matrices, we can demonstrate that $\overline{\psi}\psi = \overline{\psi}_L\psi_L + \overline{\psi}_R\psi_R + \overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L$ and $\overline{\psi}_L\psi_L = 0 = \overline{\psi}_R\psi_R$, so

$$\overline{\psi}\psi = \overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L. \tag{2.3}$$

Therefore, we can write Eq.(2.2) as

$$\mathscr{L}_{m_D} = m_D \left(\overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L\right). \tag{2.4}$$

From this, we can observe that a Dirac mass connects two distinct Weyl fields. On the other hand, it is evident that the only extension required for the standard model is the introduction of the right-handed component, ν_R, of neutrino fields. All of this leads to the following definitions [21].

Definition 2.2.1 ACTIVE NEUTRINO: *We refer to active neutrinos as left-chiral Weyl neutrinos that transform as doublets under $SU(2)$ and are paired with a charged lepton, resulting in weak interactions.*

This type of neutrinos appears in the representation chosen to order the fermions in doublets as Eq.(1.2)

Definition 2.2.2 STERILE NEUTRINOS: *We refer to Sterile Neutrinos (also know as right-handed neutrinos, ν_R) as singlets under $SU(2)$, which do not interact weakly, strongly, or electromagnetically. They only interact through mixing or interactions beyond the Standard Model.*

Applying this to the Dirac mass term, we notice that it connects an active neutrino with sterile neutrino:

$$\mathscr{L}_{m_D} = m_D \left(\overline{\nu}_L \nu_R + \overline{\nu}_R \nu_L\right). \tag{2.5}$$

The number of right-handed neutrinos is not constrained by the theory. However, only three will be considered, corresponding to e, μ, τ. With this minimal extension, the Higgs-lepton Yukawa Lagrangian, Eq.(1.14), can be rewritten as follows [9]:

$$\mathscr{L}_{Y,F} = -\sum_{\alpha,\beta=e,\mu,\tau} Y'^{\ell}_{\alpha\beta} \overline{L}'_{\alpha L} \Phi \ell'_{\beta R} - \sum_{\alpha,\beta=e,\mu,\tau} Y'^{\nu}_{\alpha\beta} \overline{L}'_{\alpha L} \widetilde{\Phi} \nu'_{\beta R} + H.c. \tag{2.6}$$

where Y'^{ν} is a new matrix of Yukawa couplings and ν'_R is a vector column with entries $\nu'_{eR}, \nu'_{\mu R}, \nu'_{\tau R}$. Let us consider that sterile neutrinos are invariant under the symmetry of the standard model; that is, they are singlets of $SU(3)_C \times SU(2)_L$ and have hypercharge equal to zero. For this reason, the structure of the second term on the right-hand side of equation (2.6) includes $\widetilde{\Phi}$, which has a hypercharge of -1. Thus, the entire second term remains invariant under $SU(2)_L \times U(1)_Y$.

The diagonalization of the matrices (analogously to the procedure in Section 1.3) Y'^{ℓ} and Y'^{ν}, then proceeds through the following definition of chiral massive neutrinos arrays

$$n_L = V_L^{\nu\dagger} \nu_L' = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{2L} \end{pmatrix}, \qquad n_R = V_R^{\nu\dagger} \nu_R' = \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{2R} \end{pmatrix}. \tag{2.7}$$

This leads to a neutrino mass term proportional to the VEV, as in Eq. (1.20) with the only change of $y_\alpha^\ell \to y_k^\nu$, and also generates couplings with the Higgs field and the Dirac massive neutrinos. However, this neutrino mass term obtained through the Higgs mechanism with the mentioned extension is not able to account for the very small values of the eigenvalues of the new Yukawa matrix, y_k^ν, leaving, like the mass of charged quarks and leptons, an unknown in the Standard Model.

The leptonic charged-current interaction can be expressed as follows

$$j_{W,L}^\rho = 2\overline{\nu'}_L \gamma^\rho \ell_L' = 2\overline{n_L} V_L^{\nu\dagger} \gamma^\rho V_L^\ell \ell_L = 2\overline{n_L} V_L^{\nu\dagger} V_L^\ell \gamma^\rho \ell_L \tag{2.8}$$

Notice that the previous equation depends on the product of two unitary matrices that were used to diagonalize the corresponding Yukawa matrices (see Eq.1.17). Since the product of two matrices is unitary, the following definition is relevant.

Definition 2.2.3 *The unitary Matrix*

$$U_{PMNS} = V_L^{\ell\dagger} V_L^\nu \in U(3),$$

is called the Pontecorvo-Maki Nakaga-Sakata (PMNS) matrix. This matrix fulfills the same function as the one in the quark mixing (CKM matrix) [22]. We designate U as the mixing matrix within the leptonic sector. The PMNS matrix is frequently represented in the subsequent manner

$$U_{PMNS} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}. \tag{2.9}$$

It is customary to define the left-handed flavor neutrino fields as

$$\nu_L = U n_L, \qquad \text{with} \qquad \nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}.$$

Now we will briefly examine the case of Majorana mass term.

2.2.2 Majorana Masses

When neutrinos are of Majorana nature, the mass terms for these types of neutrinos cannot be formed in dimensions less than or equal to 4. Neutrinos acquire Majorana masses through an effective Lagrangian of at least dimension 5. This dim-5 operator is suppressed by a factor, Λ, from the mass scale [23, 24]. Thus, the lowest effective operator that generates Majorana neutrino masses is the dim-5 Weinberg operator

$$\delta L = \frac{1}{\Lambda}\left(\overline{L^C}_{L\alpha}\widetilde{\Phi}^*\right)\left(\widetilde{\Phi}^\dagger L_{L\beta}\right) + h.c, \tag{2.10}$$

If one considers that the Higgs doublet develops vacuum expectation values (VEVs), that is, after electroweak symmetry breaking, the Weinberg operator induces Majorana masses for the light neutrinos. There are only three ways to generate the dim-5 operator at tree level. These are known as the see-saw mechanism type I, type II, and type III, with the mediator being an $SU(2)$ singlet fermion, triplet scalar, and triplet fermion, respectively.

The Majorana mass terms only necessitate a single Weyl field. To comprehend this, consider the Dirac equation with $\psi = \psi_L + \psi_R$, leading to the following expression:

$$i\gamma^\mu \partial_\mu \psi_L = m\psi_R, \tag{2.11}$$

$$i\gamma^\mu \partial_\mu \psi_R = m\psi_L. \tag{2.12}$$

Considering a massless fermion, this equation experiences decoupling

$$i\gamma^\mu \partial_\mu \psi_L = 0, \tag{2.13}$$

$$i\gamma^\mu \partial_\mu \psi_R = 0. \tag{2.14}$$

A massless fermion can be described by a single chiral field containing only two independent components. However, a 4-component spinor is not sufficient to describe a massive particle. To account for a massive particle, we can assume that ψ_R and ψ_L are not independent; there exists a relationship connecting ψ_R with ψ_L that satisfies Eq. (2.11) and Eq. (2.12). This implies that the two equations are two ways of expressing the same equation for an independent field, namely ψ_L. This is achieved through the Majorana relation that connects the left and right fields,

$$\psi_R = \eta C \overline{\psi_L}^T, \tag{2.15}$$

where η is an arbitrary phase factor that can be eliminated by rephasing the field ψ_L as follows: $\psi_L \to \eta^{1/2}\psi_L$ and C is the charge conjugation matrix. By considering Eq. (2.11), we obtain the Majorana equation for the chiral field ψ_R,

$$i\gamma^\mu \partial_\mu \psi_L = mC\overline{\psi_L}^T. \tag{2.16}$$

The Majorana condition for the field ψ is:

$$\psi = C\overline{\psi}^T. \tag{2.17}$$

As neutrinos exclusively interact through weak interactions, the charge parity of neutrino fields can be chosen arbitrarily. Taking $\psi_L^C = C\overline{\psi}_L^T$, the Majorana field can be expressed as

$$\psi = \psi_L + \psi_L^C. \tag{2.18}$$

The Majorana condition can be written as

$$\psi = \psi^C. \tag{2.19}$$

Given that neutrinos are neutral, it follows from Equation (2.19) that neutrinos are the only fermions capable of being described by a Majorana field[2]. A Majorana mass term it can be written as

$$\mathscr{L}^M_{mass} = -\frac{1}{2}m\overline{\nu}_L^C\nu_L + h.c. \tag{2.20}$$

The total Lagrangian is

$$\mathscr{L}^M = \frac{1}{2}\left(\overline{\nu}_L i\overleftrightarrow{\partial\!\!\!/}\,\nu_L + \overline{\nu}_L^C i\overleftrightarrow{\partial\!\!\!/}\,\nu_L^C - m\left(\overline{\nu}_L^C\nu_L + \overline{\nu}_L\nu_L^C\right)\right), \tag{2.21}$$

[2]The Majorana condition necessarily implies the equivalence between particle and antiparticle.

where the superscript M indicates the Majorana case[3]. The factor of 1/2 is used to avoid double counting since ν_L^C and ν_L are not independent [25, 9].

2.2.3 B-L Symmetry

As I previously stated, within the standard model, neutrinos are devoid of mass due to the absence of right-handed neutrinos. However, as illustrated in section 2.2, upon the inclusion of right-handed neutrinos, neutrino mass emerges naturally since no symmetry prohibits it. Moreover, as highlighted initially, the standard model hinges on Gauge symmetries, a theory that has adeptly addressed core inquiries about matter's essence. Nevertheless, despite its accomplishments, it remains incomplete, as it inherently fails to predict neutrino mass. This prompts us to delve into extensions of the standard model, grounded in its Gauge symmetry, such as the scenario involving the global symmetry $U(1)_{B-L}$, where the quantum number $B-L$ signifies the difference between baryon and lepton numbers.[4]. So, one type of minimal extension of the standard model is based on the Gauge group

$$K = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L} \tag{2.22}$$

This can be broken to $SU(3)_C \times SU(2)_L \times U(1)_Y$ by a vacuum expectation value, v$'$, which is assumed to be larger than the Higgs vacuum expectation value, v. This group K can be written as $SU(4') \times SU(2)_L \times U(1)_Y$, where the fourth color is considered to be $B-L$, which can be seen as a subalgebra of the Pati-Salam model, which is free from anomalies [26–29]. Therefore, the global symmetry $U(1)_{B-L}$ is also anomaly-free[5]. However, it is worth noting that if the $U(1)_{B-L}$ symmetry is considered as a global symmetry, it is anomaly-free within the standard model with massless neutrinos, whereas if it is considered as a gauge symmetry (which implies this symmetry is gauged), the anomalies will be canceled due to the existence of right-handed neutrinos [30–32]. Thus, it is more useful to consider a $U(1)_{B-L}$ gauge symmetry in the extension of the standard model, as given in Eq.(2.22).

The invariance of the Lagrangian under this extension of the standard model gives rise to a new gauge boson denoted as C_μ, in addition to having a new coupling

[3]Since neutrinos are left-handed, we use the left-handed chiral field ν_L

[4]Due to anomaly cancellations, this symmetry predicts the existence of the three right-handed neutrinos.

[5]It is worth mentioning that under this minimal extension symmetry, Majorana mass terms are not allowed as they do not preserve the $B-L$ symmetry. Later, this fact will be justified, resulting in only Dirac-type mass terms remaining.

constant, g'', associated with $U(1)_{B-L}$. On the other hand, to induce the spontaneous breaking of $U(1)_{B-L}$, a complex scalar field η is required in addition to the Higgs doublet Φ already present [33, 34].

2.3 On a Treatment of Three-Generation Mixing

I have mentioned in Chapter 1 that the Yukawa matrix is not diagonal, and therefore, the fields of charged leptons do not possess a definite mass. For this reason, a diagonalization process is necessary. The same applies when we have the Lagrangian for Dirac masses (Eq. (2.5)) and the Lagrangian for Majorana masses (Eq. (2.21)). In other words, the aim is to transform the mass terms from their original form to extract the physical masses. To achieve this goal, two unitary matrices are used, which I will describe next.

2.3.1 Diagonalization of the Mass Matrix

Theorem 2.3.1 *Let* M *be a complex $N \times N$ matrix, and let Q and Γ be two $N \times N$ unitary matrices. Then, $Q^\dagger \mathrm{M} \Gamma = m$, where $m_{ij} = m_i \delta_{ij}$.*

Proof. Let us consider the matrix $\mathrm{M}\mathrm{M}^\dagger$, which is obviously a Hermitian matrix, and its eigenvalues are positive. This matrix can be diagonalized through a unitary transformation (This is an immediate implication of Theorem A.1.1 from Appendix A)

$$Q^\dagger \mathrm{M}\mathrm{M}^\dagger Q = m^2,$$

where $Q^\dagger Q = \mathbf{I}$ and $(m^2)_{ij} = m_i^2 \delta_{ij}$. Let us rewrite the above expression as

$$\mathrm{M}\mathrm{M}^\dagger = Q m^2 Q^\dagger.$$

Now, observe that

$$Qm \left(m^{-1} Q^\dagger \mathrm{M} \right) = Q Q^\dagger \mathrm{M} = M.$$

If defined $\Gamma^\dagger = m^{-1} Q^\dagger \mathrm{M}$, then

$$M = Q m \Gamma^\dagger,$$

therefore

$$Q^{\dagger}\mathrm{M}\Gamma = m.$$

By construction Γ is indeed unitary, as can be deduced from:

$$\begin{aligned}\Gamma^{\dagger}\Gamma &= \left(m^{-1}Q^{\dagger}\mathrm{M}\right)\left(m^{-1}Q^{\dagger}\mathrm{M}\right)^{\dagger}\\ &= m^{-1}Q^{\dagger}\mathrm{M}\mathrm{M}^{\dagger}Q\left(m^{-1}\right)^{\dagger} = m^{-1}Q^{\dagger}\left(Qm^{2}Q^{\dagger}\right)Qm^{-1} = \mathbf{I}\blacksquare\end{aligned}$$

In general, an $N \times N$ unitary matrix has N^2 independent parameters, which are $N(N-1)/2$ mixing angles and $N(N+1)/2$ phases. However, not all phases are physical observables[6] [35, 9].

On the other hand. A unitary matrix, U, can be expressed as

$$\begin{aligned}U &= D(\omega)\Upsilon_{23}(\theta_{23},\eta_{23})\Upsilon_{12}(\theta_{12},\eta_{12})\Upsilon_{13}(\theta_{13},\eta_{13})\\ &= D(\omega)\left[\prod_{i<j}\Upsilon_{ij}\left(\theta_{ij},\eta_{ij}\right)\right] \qquad \forall i,j = 1,2,3, \end{aligned} \tag{2.23}$$

where $D(\omega) = \mathrm{diag}\left(e^{i\omega_1}, e^{i\omega_2}, e^{i\omega_3}\right)$, ω being the set of phases[7] and Υ_{ij} are unitary and unimodular matrices. The elements of this matrix are [36, 37]:

$$\begin{aligned}\left[\Upsilon_{ij}\left(\theta_{ij},\eta_{ij}\right)\right]_{\alpha\beta} = \delta_{\alpha\beta} &+ \left(c_{ij}-1\right)\left(\delta_{\alpha i}\delta_{\beta i} + \delta_{\alpha j}\delta_{\beta j}\right)\\ &+ s_{ij}\left(e^{i\eta_{ij}}\delta_{\alpha i}\delta_{\beta j} - e^{-i\eta_{ij}}\delta_{\alpha j}\delta_{\beta i}\right),\end{aligned} \tag{2.24}$$

with $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$.

It is important to note that the order of matrix products in Eq.(2.23) can be chosen arbitrarily. However, different choices of order result in different parameterizations. Following the idea discussed earlier from equations (2.23) and (2.24), we have that there are $N(N-1)/2$ mixing angles, while there are $N(N+1)/2$ phases (considering the general case). However, these latter phases can still be divided into two forms: N phases ω_k and the remaining $N(N-1)/2$ phases η_{ij}. By doing this, we can rewrite

[6] We have $(N-1)(N-2)/2$ physical phases.

[7] This form of Eq.(2.23) and $D(\omega)$ can be generalized to the N-dimensional case. In this work, I consider only the case where $N = 3$, which is why I restrict the indices to $i, j = 1,2,3$. Otherwise, in a general case, the indices would be $i, j = 1, \ldots, N$ and ω the set of N phases.

equation (2.23) in another form[8] that allows us to extract $N-1$ phases from the $N(N-1)/2$ phases η_{ij} in the product [9, 38].

For simplicity, we consider $N=2$. In this case of two generations, a unitary matrix can be written as

$$\begin{aligned} U &= \begin{pmatrix} \cos\theta e^{i\omega_1} & \sin\theta e^{i(\omega_2+\eta)} \\ -\sin\theta e^{i(\omega_1-\eta)} & \cos\theta e^{i\omega_2} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\omega_2} & 0 \\ 0 & e^{i\omega 1} \end{pmatrix} \begin{pmatrix} e^{i\eta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{-i\eta} & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Notice that we have one mixing angle, θ, and three phases. However, in this simple case, we can observe that the three phases, ω_1, ω_2, η, can be removed through a rephasing of the fields, leaving only the matrix of physical importance:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.25}$$

As we can see, this case is quite simple. However, for the three-generation case, it becomes more complicated. As I mentioned earlier, there are various ways to parameterize the mixing matrix U. Nevertheless, the standard parametrization chosen by the Particle Data Group (PDG) is the following [2]:

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \tag{2.26}$$

This form of expressing the unitary matrix U is a more convenient parametrization known as the symmetric form [4]. In the case of three neutrinos, it is given as

$$U_{PMNS} = \Upsilon_{23}(\theta_{23}, \eta_{23}) \Upsilon_{13}(\theta_{13}, \eta_{13}) \Upsilon_{12}(\theta_{12}, \eta_{12}), \tag{2.27}$$

where Υ_{ij} is defined by Eq.(2.24).

[8] $U = D(\omega - \varphi)\left[\prod_{ij} \Omega_{ij}(\theta_{ij}, \varphi_i + \eta_{ij} - \varphi_j)\right] D(\varphi)$, where φ is an arbitrary set of phases that allow us to extract the $N-1$ phases from η_{ij}.

2.3.2 CP Violation

The union of parity and charge transformations is known as the CP transformation. In general, when the action is not invariant under this CP transformation, the theory violates CP. Gravitational, electromagnetic, and strong interactions do not violate CP symmetry. However, in the electroweak theory, there are cases where CP symmetry is violated, such as in the decays of the K0 meson and the B meson, as well as in neutrino interactions, while it is respected in many other processes. This CP violation is significant because it is related to the matter-antimatter asymmetry in the universe, namely, the CP violation is a necessary condition for baryogenesis.

The Standard Model allows for CP violation if a complex phase appears in the CKM matrix (or the analogous matrix for lepton mixing). Let us delve into this in more detail. When all manipulations are performed to rephase in such a way that allows us to eliminate all non-physical phases in the Yukawa couplings, we find that there is only one complex parameter left. This is where the CP violation parameter resides. In the basis of mass states, this phase appears in the unitary matrix (see Eq.(2.26), we can say, but with great caution, that the delta phase is the CP violation phase). This type of rephasing is the form [9]:

$$U_{\alpha j} = e^{-i\psi_\alpha} U_{\alpha j} e^{i\psi_j}. \tag{2.28}$$

On the other hand, physical observables do not depend on the parametrization of the mixing matrix. It is both possible and necessary to work with quantities that remain invariant under rephasing transformations, as specified in Eq.(2.28). The squared moduli of the amplitudes are always observables, and therefore, any observable is invariant under the rephasing transformation in Eq. (2.28). We are only interested in measurable quantities that are invariant under these rephasing transformations; for example,

$$\left|U_{\alpha j}\right|^2 = U_{\alpha j} U^*_{\alpha j}. \tag{2.29}$$

However, the next invariant term is [39, 40]

$$^{\alpha k}\Box_{\beta j} \equiv U_{\alpha k} U_{\beta j} U^*_{\alpha j} U^*_{\beta k}, \qquad \forall \alpha \neq \beta; k \neq j. \tag{2.30}$$

This follows from the fact that (as we will see in Chapter 3), the oscillation probability depends on the product of four elements of the mixing matrix (which is precisely the product given in Eq. 2.30). Since the oscillation probability is an observable, it follows that it is independent of the parametrization of the mixing matrix (angles and phases). Since, in general, U is complex, then ${}^{\alpha k}\square_{\beta j}$ is also complex. Their imaginary parts provide us with a tool that is very useful for assessing the complexity of matrix U and, consequently, CP violation in an invariant manner. Due to the unitarity of the matrix U, the imaginary part of ${}^{\alpha k}\square_{\beta j}$ is unique [9].

We have the following definition.

Definition 2.3.1 *The Jarlskog invariant is*[9]*[42]:*

$$J = \mathbf{Im}\left[{}^{11}\square_{22}\right] = \mathbf{Im}\left[U_{11}U_{22}U^*_{12}U^*_{21}\right]. \tag{2.31}$$

Using the parameterization given in Eq. (2.26), it follows that the Jarlskog invariant is given by

$$J = \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta. \tag{2.32}$$

Hence, $J = 0$ if and only if $e^{i\delta} = \pm 1$ (which happens if and only if the PMNS (or CKM) matrix can be brought into real form) or if any of the mixing angles is null.

On the contrary, we can assess the level of CP violation at low-energy by examining the following invariant in the weak basis [43, 44]:

$$Tr\left[\mathscr{H}_\nu, H_\ell\right]^3 = 6i\Delta_{21}\Delta_{32}\Delta_{31}\mathbf{Im}\left[(\mathscr{H}_\nu)_{12}(\mathscr{H}_\nu)_{23}(\mathscr{H}_\nu)_{31}\right], \tag{2.33}$$

where $\mathscr{H}_\nu = M_\nu M_\nu^\dagger$, $H_\ell = m_\ell m_\ell^\dagger$, with M_ν the usual light neutrio effective mass matrix[10] and $\Delta_{21} = (m_\mu^2 - m_e^2)$, analogous expressions for Δ_{31} and Δ_{32}. On the other hand, the right side of Eq. (2.33) can be written in terms of observables [45], particularly in terms of the Jarlskog invariant

$$\mathbf{Im}\left[(\mathscr{H}_\nu)_{12}(\mathscr{H}_\nu)_{23}(\mathscr{H}_\nu)_{31}\right] = -\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}J, \tag{2.34}$$

where the Δm^2_{ij} represents the usual neutrino mass squared difference, and J denotes the Jarlskog invariant. Alternatively

$$J = -\frac{\mathbf{Im}\left[(\mathscr{H}_\nu)_{12}(\mathscr{H}_\nu)_{23}(\mathscr{H}_\nu)_{31}\right]}{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}}. \tag{2.35}$$

[9]Without loss of generality, I have particularized the invariant to the case that is most useful in my development and also the most commonly used in general. The Jarlskog invariant is a multi-linear map [41].

[10]In this weak bases invariant, m_ℓ denote the charged lepton mass matrix.

It is important to note that even though Eqs. (2.32) and (2.35) represent the same thing, the latter offers more advantages when performing calculations, as there is no need to rely on the mixing matrix. In the model presented in Chapter 4, I use this fact to carry out my research as promised.

CHAPTER 3

Neutrino Phenomenology

A peculiarity of neutrinos is their difficulty in detection. They are only detected through their weak interactions. The different flavors of neutrinos can only be distinguished by the flavors of the corresponding charged lepton produced in the weak charged current interaction. Furthermore, despite the fact that neutrino interactions have a very small cross-section, it is possible to observe them using sufficiently massive detectors. However, the study of the solar neutrino problem through the SNO experiment allowed for the strong postulation of neutrino oscillations.

3.1 In Relation to the Observations and Oscillations

The solar nuclear fusion produces a flux of electron neutrinos on the order of $10^{38}\nu_e/s$. These neutrinos are produced according to the following reaction (PP cycle) [46]:

$$P + P \rightarrow D + e^{+} + \nu_e.$$

Then, we have the reaction $D + P \rightarrow^3_2 H_e + \gamma$, wich, in turn, leads to one last reaction: $^3_2H_e +^3_2 H_e \rightarrow^4_2 H_e + P + P$, thus completing the PP cycle. The energy of the neutrinos produced in this process is low, less than 0.5 MeV, making their detection challenging. Consequently, experiments involve the detection of high-energy solar neutrinos originating from the β decay processes of boron-8, ^8B, which generates neutrinos[1] with energies exceeding 15 MeV.

The first detection experiments used radiochemical techniques to measure the solar neutrino flux. The Homestake experiment, located in the Homestake Mine in

[1]The ^8B is produced through the fusion of two helium nuclei

South Dakota, USA, consisted of a 615-ton tank of C_2Cl_4. It determined the solar neutrino flux by observing the process $\nu_e +^{37}_{17}Cl \rightarrow^{37}_{18} Ar + e^-$ and counting the number of ^{37}Ar atoms. This count yielded an observed rate of 0.48 neutrino interactions per day. However, an expected rate of 1.7 interactions per day was anticipated, resulting in a deficit. This deficit is known as the solar neutrino problem. Following this experiment, two other experiments, SAGE and GALLEX, were conducted. They used gallium as the detector, providing sensitivity to low-energy neutrinos. Nevertheless, these experiments also detected the same deficit in solar neutrinos.

In 1989, the Super-Kamiokande experiment was conducted, which is essentially a 50,000-ton water Cherenkov detector. Here, solar neutrinos are detected through the proccess of elastic scattering $\nu_e e^- \rightarrow \nu_e e^-$ [46, 10]. The final state electron is relativistic and can be detected from the Cherenkov radiation photons emitted at a fixed angle relative to its direction of motion. The number of detected photons provides a measure of the neutrino's energy and the direction of the electron.This allows the observation of elastic scattering interactions of electronic neutrinos. It is worth noting that this experiment is sensitive to the flux of neutrinos from ^8B. The measurement indicated that the flux of electronic neutrinos is approximately half of what was expected.

Due to these observations, three explanations were proposed for this. The first possible explanation is that the theoretical calculations were incorrect. The second possibility was an error in the experimental measurements, and finally, the third possible explanation is that neutrinos were not fully understood.

A decade later, the Sudbury Neutrino Observatory experiment took on the task of measuring both the total solar neutrino flux and the solar electron neutrinos. This experiment consisted of 1000 tonnes of heavy water in a 12-meter diameter vessel. The use of heavy water provides the opportunity to detect solar neutrinos through three different processes[2]. The charged current is kinematically possible and is sensitive only to the electron neutrino flux. On the other hand, all flavor neutrinos can interact with the deuteron through the neutral current, thus, neutral current processes are sensitive to all three flavor neutrinos [46, 10]. Finally, neutrinos can interact with atomic electrons through elastic scattering, which is sensitive to all flavor neutrinos but more significantly to the electron neutrino.

The different angular and energy distributions of Cherenkov radiation from these interactions allow for the determination of the rates for each process. The observed charged current (CC) rates were consistent with the electron neutrino (ν_e) flux, and

[2]The binding energy of the Deuteron is similar to that boron-8.

the rates for neutral current (NC) interactions were also consistent with the total neutrino flux. This provided evidence for an unexpected flux of ν_μ and ν_τ from the Sun. In conclusion, SNO's data demonstrated that the total neutrino flux from the Sun is consistent with theoretical expectations. Moreover, it revealed that not only ν_e is detected but also contributions from muon and tau neutrinos. Since these cannot be produced in the Sun, this provided clear evidence for neutrino oscillation, implying that neutrinos have non-zero mass.

3.2 Neutrino Oscillation

The so called mass states are physical states that are stationary states of the free particle Hamiltonian, satisfying $H\psi = i\partial_t\psi$. The mass eigenstates of neutrinos are defined in Eq.(2.7) and the weak eigenstates in Eq.(1.22). It is important to mention that these states do not correspond to the weak flavor states ν_α, with $\alpha = e, \mu, \tau$, which are produced along with their respective flavor charged lepton in weak interactions. In contrast, flavor neutrinos are produced in a charged current interaction along with the charged lepton (or its antilepton), which are generated by the Lagrangian given in Eq.(1.4), where

$$J^{\rho}_{W,L} = 2\sum_{\alpha}\sum_{k} U^{*}_{\alpha k}\overline{\nu_{kL}}\gamma^{\rho}\ell_{\alpha L}. \tag{3.1}$$

This is a clear distinction between the mass eigenstates and the weak eigenestates [46, 9]. However, the basis of weak eigenstates is related to the basis of mass eigenstates through the unitary matrix U, as defined in Definition 2.2.3. This implies that, for example, an electron neutrino, which is a state produced along with a positron in a weak charged current interaction, is a linear conbination of the mass eigenstates. In general, if the masses of the neutrinos ν_k with $k = 1, 2, 3$ are not the same, a phase difference arises between the components of the wavefunction, leading to neutrino oscillations.

3.2.1 General Formalism

Based on the above, neutrino oscillations are a consequence of the neutrino mixing matrix, which in turn implies that the left-handed components of flavor neutrino fields, ν_α are superpositions of the left-handed components of neutrino fields with defined masses, ν_k:

$$|\nu_\alpha\rangle = \sum_k U^*_{\alpha k} |\psi_k\rangle . \tag{3.2}$$

Let us keep in mind that the matrix U is unitary. Therefore, the following natural conditions follow

$$U_{\alpha k} U^*_{\beta k} = \delta_{\alpha\beta} \qquad \& \qquad U^*_{\alpha k} U_{\alpha j} = \delta_{kj}, \tag{3.3}$$

Let us assume that the orthonormality conditions for the massive neutrino states are satisfied: $\langle \nu_k | \nu_j \rangle = \delta_{ij}$. On the other hand, from the condition in Eq.(3.3), it follows that the flavor states are also orthonormal, that is, it satisfies: $\langle \nu_\alpha | \nu_\beta \rangle = \delta_{\alpha\beta}$. This is clear considering the fact that

$$\langle \nu_\alpha | = \sum_k U_{\alpha j} \langle \nu_k | . \tag{3.4}$$

As I mentioned before, the massive states are eigenstates that satisfy

$$H |\nu_k\rangle = E_k |\nu_k\rangle , \tag{3.5}$$

with eigenenergies $E_k^2 = |p|^2 + m_k^2$. Invoking the Schrödinger equation, it is clear that we obtain the time evolution, which is

$$|\nu_k(t)\rangle = e^{-iE_k t} |\nu_k\rangle . \tag{3.6}$$

It is possible to express the mass eigenstates in terms of the flavor eigenstates:

$$|\nu_k\rangle = \sum_\alpha U_{\alpha k} |\nu_\alpha\rangle .$$

Substituting this into the Eq.(3.6) and Eq.(3.2), we obtain

$$|\nu_\alpha(t)\rangle = \sum_\beta \left[\sum_k U^*_{\alpha k} e^{-iE_k t} U_{\beta k} \right] |\nu_\beta\rangle . \tag{3.7}$$

This indicates that the initial flavor state becomes a superposition of different flavor states [47].

The transition amplitudes from one flavor state to another are the coefficients in Eq.(3.7), namely

$$A(t) \equiv \langle \nu_\beta | \nu_\alpha(t) \rangle = \sum_k U^*_{\alpha k} U_{\beta k} e^{-iE_k t}. \tag{3.8}$$

Thus, the transition probability is simply the square of the amplitude, which is given by

$$P_{\nu_\alpha - \nu_\beta}(t) = \sum_{kj} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} e^{-i(E_k - E_j)t}. \tag{3.9}$$

Neutrinos in oscillation experiments are ultrarelativistic, so we can approximate their energy as $E_k = E + m_k^2/2E$, and therefore, the energy difference in the exponential can be replaced by

$$E_k - E_j \approx \frac{m_k^2 - m_j^2}{2E} \equiv \frac{\Delta m^2_{kj}}{2E}. \tag{3.10}$$

So, using Eq.(3.10) with Eq.(3.9) yields

$$\begin{aligned} P_{\nu_\alpha - \nu_\beta} =& \sum_{kj} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} e^{-i\frac{\Delta m^2_{kj}}{2E} L} \\ =& \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\mathbf{Re} \sum_{k>j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} e^{-i\Delta m^2_{kj} L/2E}, \end{aligned} \tag{3.11}$$

where I have used the fact that in experimentation, the propagation time is not measured, but due to the known distance between the source and the detector, L, and the fact that they are ultrarelativistic, we can make the approximation $t = L$. Thus, we see that the distance L and the energy E are the quantities on which the experiment depends [47, 46, 9]. Likewise, the oscillation amplitude depends on the elements of the matrix U, and the phases are determined by the squared mass differences. Therefore, neutrino oscillation allows us to determine the squared mass differences and the elementos of the U matrix. It is worth noting that it only provides information about the squared mass differences and not the absolute values of the neutrino masses themselves.

3.2.2 Oscillation Formulas for Two and Three Flavors

Let us consider the case of the mixing of two flavor neutrinos, let us say ν_α and ν_β. In this case, the two flavor states of neutrinos are a linear combination of the two massive neutrinos, ν_1 and ν_2, with coefficients given by a 2×2 unitary matrix expressed in terms of a mixing angle. This matrix is given by Eq.(2.25). Invoking Eq.(3.9), we have

$$P_{\nu_\alpha - \nu_\beta} = \sin^2 2\theta \sin^2 \frac{\Delta m^2 L}{4E}, \tag{3.12}$$

where $\Delta m^2 = m_2^2 - m_1^2$. The survival probability is

$$P_{\nu_\alpha - \nu_\alpha} = 1 - P_{\nu_\alpha - \nu_\beta} = 1 - \sin^2 2\theta \sin^2 \frac{\Delta m^2 L}{4E}. \tag{3.13}$$

This two-neutrino case is very useful because many experiments are not sensitive to the influence of three neutrino mixing, and the data can be analyzed using this two-neutrino mixing model. This defines two types of experiments: so-called appearance experiments, which involve measuring transitions between different neutrino flavors, and disappearance experiments, which involve measuring the survival probability of a neutrino flavor by counting the number of interactions in the detector and comparing it with the expected value. However, this type of experiment is not very efficient for small values of the mixing angle.

The oscillation length is defined as

$$L_{kj}^{\text{osc}} = \frac{4\pi E}{\Delta m_{kj}^2}. \tag{3.14}$$

For the case of two flavor neutrinos, this is

$$L_{\text{osc}} = \frac{4\pi E}{\Delta m^2}.$$

Therefore, the transition probability is very small if $L \ll L_{\text{osc}}$ and oscillates very rapidly if $L \gg L_{\text{osc}}$ [9]. Finally, to observe neutrino oscillations, Δm^2 must satisfy $\Delta m^2 \geq E/L$, meaning that the experiment is sensitive to the squared mass difference.

Practically all of physics is already addressed in the case of two-flavor neutrinos, which includes the relationship between the weak and mass eigenstates, and oscillations arising from the phase difference between the mass eigenstates. However, it is known that there is no upper limit on the number of massive neutrinos. Nevertheless, it is well-known that the number of active flavor neutrinos is three: ν_e, ν_μ, ν_τ. Therefore, the lower bound for massive neutrinos is three, and we are interested in studying the case of oscillations among three flavors.

In this latter case, the mixing matrix is a complex 3×3 matrix (see Eq.2.26). There are many parameterization alternatives; however, the most commonly used one is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) parameterization, given by Eq.(2.26) (Note that here we have the presence of 3 mixing angles and one physical phase, the Dirac phase δ). Unlike the previous case, here we are dealing with three squared mass differences, which are

$$\Delta m_{21}^2 = m_2^2 - m_1^2, \qquad \Delta m_{31}^2 = m_3^2 - m_1^2, \qquad \Delta m_{32}^2 = m_3^2 - m_2^2. \tag{3.15}$$

However, only two of these are independent because

$$0 = \Delta m_{21}^2 + \Delta m_{32}^2 - \Delta m_{31}^2.$$

There are two possible hierarchies for neutrino masses. The first is the normal hierarchy: $m_3^2 \gg m_2^2 > m_1^2$, and the second is the inverted hierarchy, in which $m_2^2 > m_1^2 \gg m_3^2$ is satisfied. In most experiments, $|\Delta m_{32}^2| = \Delta m_{31}^2 - \Delta m_{21}^2 \approx \Delta m_{31}^2$. This allows for the following identification: $\Delta m_{sol}^2 = \Delta m_{21}^2$ and $\Delta m_{atm}^2 = \Delta m_{31}^2$, or for the inverted hierarchy, $m_{atm}^2 = |\Delta m_{32}^2|$. Current experiments are not capable of distinguishing between these two possibilities, meaning they lack sufficient sensitivity to differentiate between NH and IH [46, 9].

3.2.3 Matter Effects in Neutrino Oscillations

When neutrinos propagate through matter, they are subject to a potential due to their interaction with the medium's fermions through coherent forward elastic scattering. Consequently, neutrino flavor oscillations are affected by the coherent scattering of charged current[3], generating an effective potential denoted as V_{CC}^{eff}. This effective potential can be interpreted as a refractive index that modifies neutrino mixing, resulting in an effective mixing angle in the matter [48, 9].

Therefore, the Hamiltonian of a system of neutrinos propagating through matter is given by $H_m = H_0 + H_{int}$, where H_0 is the Hamiltonian in vacuum, see Eq.(3.5),

[3]It can also be affected by incoherent scattering with particles in the medium, but in many cases, this scattering can be extremely small and can be neglected.

and H_{int} is the Hamiltonian associated with the neutrino's interaction with matter, meaning that $H_{int}\left|\nu_\alpha\right\rangle = V_\alpha \left|\nu_\alpha\right\rangle$, where V_α is the effective potential.

The effective Hamiltonian of CC is

$$H_{eff}^{cc} = \frac{G_F}{\sqrt{2}}\left[\overline{\nu_e}\gamma^\rho(1-\gamma^5)\nu_e\right]\left[\overline{e}\gamma_\rho(1-\gamma^5)e\right]. \tag{3.16}$$

The average of the effective Hamiltonian with the electron background in the medium is given by

$$\overline{H_{eff}^{cc}} = \sqrt{2}G_F N_e \overline{\nu_{eL}}\gamma^0\nu_{eL}, \tag{3.17}$$

where N_e is the electron density of the medium. The effective potential is defined as $V_{cc} = \sqrt{2}G_F N_e$. Similarly, it is possible to obtain the effective potential for neutral current interactions: $V_{NC} = -\sqrt{2}G_F N_n/2$. Thus, the more general effective potential is given by $V_\alpha = V_{cc}\delta_{\alpha e} + V_{NC}$ [9].

Now, for the evolution equation, we have

$$i\frac{d}{dt}\left|\nu_\alpha(t)\right\rangle = H_m\left|\nu_\alpha(t)\right\rangle, \tag{3.18}$$

with $\left|\nu_\alpha(0)\right\rangle = \left|\nu_\alpha\right\rangle$. The amplitude of transition is $\psi_{\alpha\beta} \equiv \left\langle \nu_\beta \middle| \nu_\alpha(t)\right\rangle$ with the condition $\psi_{\alpha\beta}(0) = \delta_{\alpha\beta}$. So, the evolution equation becomes

$$i\frac{d}{dx}\psi_{\alpha\beta} = \left(p + \frac{m_1^2}{2E} + V_{NC}\right)\psi_{\alpha\beta} + \sum_\eta\left(\sum_k U_{\beta k}\frac{\Delta m_{k1}^2}{2E}U_{\eta k}^* + \delta_{\beta e}\delta_{\eta e}V_{CC}\right)\psi_{\alpha\eta}.$$

The first term above is irrelevant for the flavor transition, becouse it can be eliminated by the phase shift. Therefore, the relevant evolution equation is

$$i\frac{d}{dx}\psi_{\alpha\beta} = \sum_\eta\left(\sum_k U_{\beta k}\frac{\Delta m_{k1}^2}{2E}U_{\eta k}^* + \delta_{\beta e}\delta_{\eta e}V_{CC}\right)\psi_{\alpha\eta}. \tag{3.19}$$

For the case of two neutrino mixing between, ν_e, ν_μ, the Eq.(3.19) can be written as

$$i\frac{d}{dx}\begin{pmatrix}\psi_{ee}\\ \psi_{e\mu}\end{pmatrix} = \underbrace{\frac{1}{4E}\begin{pmatrix}-\Delta m^2\cos\theta + 2EV_{CC} & \Delta m^2\sin 2\theta\\ \Delta m^2\sin 2\theta & \Delta m^2\cos 2\theta - 2EV_c c\end{pmatrix}}_{H_{eff}}\begin{pmatrix}\psi_{ee}\\ \psi_{e\mu}\end{pmatrix}.$$

This H_{eff} can be diagonalided by the orthogonal transformation: $U_M^T H_{eff} U_M = \mathrm{diag}(-\Delta m_M^2, \Delta m_M^2)/4E$, where U_M is the effective mixing matrix in matter, given by

$$U_M = \begin{pmatrix} \cos\theta_M & \sin\theta_M \\ -\sin\theta_M & \cos\theta_M \end{pmatrix}.$$

Therefore, the effective mixing angle in matter θ_M is given by

$$\tan 2\theta_M = \frac{\tan 2\theta}{1 - \frac{2EV_{CC}}{\Delta M \cos 2\theta}}. \tag{3.20}$$

The transition probability then becomes

$$P_{\nu_e \to \nu_\mu} = \sin^2 2\theta_M \sin^2\left(\frac{\Delta m_M^2 x}{4E}\right). \tag{3.21}$$

Comparing Eq.(3.12) with Eq.(3.21), we can see that they have the same structure, with the mixing angle and the squared-mass difference replaced by their effective values in matter. Furthermore, in Eq.(3.20), we can observe that the mixing angle in matter is different from the mixing angle in vacuum [48, 9].

3.3 Neutrino Oscillations Experiments

In neutrino oscillation experiments, there are various types, which include Short-Baseline experiments (SBL), Long-Baseline experiments (LBL), Very Long-Baseline experiments, Atmospheric neutrino experiments (ATM), and finally, Solar neutrino experiments (SOL) [9].

For Short-Baseline (SBL) experiments, there are two types. The first is reactor SBL, which involves using a high isotropic flux of antineutrinos produced by the beta decay of heavy nuclei. The neutrinos have energies in the MeV range, and the distance between the source and the detector is several tens of meters, such that $L/E \leq 10m/MeV \to \Delta m^2 \geq 0.1\mathrm{eV}^2$. Since the energy is very low, this type only measures the survival probability. The second type is accelerator SBL, which uses neutrino beams produced by the decay of pions, kaons, and muons created by a proton beam hitting a target. For this type, the ratio L/E is $L/E \leq 1km/GeV \to \Delta m^2 \geq 1\mathrm{eV}^2$.

In LBL experiments, the sources are the same nature as in the SBL case, with the only difference being that the distance between the source and the detector is two or three orders of magnitude greater. Similarly, there are reactor-type LBL experiments where the distance L is on the order of kilometers, and thus $L/E \leq 10^3 m/MeV \to \Delta m^2 \geq 10^{-3} eV^2$. On the other hand, there are accelerator-based LBL experiments that produce beams of neutrinos or antineutrinos from the decays of pions and kaons. Here, we have $L/E \leq 10^3 Km/GeV \to \Delta m^2 \geq 10^{-3} eV^2$.

A final type of these experiments is Very Long-Baseline (VLB) experiments, which, as the name suggests, involve even greater distances than the previous ones. Regarding reactor VLB, we have $L/E \leq 10^5 Km/MeV \to \Delta m^2 \geq 10^{-5}\text{eV}^2$. Meanwhile, for accelerator VLB experiments, the ratio is on the order of $L/E \leq 10^4 Km/GeV \to \Delta m^2 \geq 10^{-4}\text{eV}^2$.

Atmospheric neutrino experiments involve detecting neutrinos that originate from the decays of pions and kaons produced by primary cosmic rays interacting with the upper atmosphere. The detectable energy of these neutrinos ranges from 500 MeV to 100 GeV. The distances between the source and detector are typically around 20 km to 1.3×10^4Km. Typical values are $L/E \leq 10^4 Km/GeV \to \Delta m^2 \geq 10^{-4}\text{eV}^2$. Finally, we have solar neutrino experiments, which detect neutrinos generated in the core of the Sun due to thermonuclear reactions. Here, we have $L/E \leq 10^{12} m/MeV \to \Delta m^2 \geq 10^{-12}\text{eV}^2$

From these experiments, the following experimental values for the oscillation parameters are reported [6]:

	NH		IH	
	1σ	3σ	1σ	3σ
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \to 51.0$	$49.0^{+1.0}_{-1.2}$	$39.9 \to 51.2$
$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.94$
$\delta_{cp}/^\circ$	232^{+36}_{-26}	$144 \to 350$	276^{+22}_{-29}	$194 \to 344$
$\Delta m^2_{21}/(10^{-5}\text{eV}^2)$	$7.41^{+0.21}_{-0.2}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.2}$	$6.82 \to 8.03$
$\Delta M_{3\ell}/10^{-3}\text{eV}^2$	$2.507^{+0.026}_{-0.027}$	$2.427 \to 2.590$	$-2.486^{+0.026}_{-0.028}$	$-2.570 \to -2.406$

Table 3.1 Experimental values for the oscillation parameters. Note that for the normal hierarchy, $\ell = 1$, while for the inverted hierarchy $\ell = 2$.

CHAPTER 4

A Perturbative Approach for a Dirac-Taylor Model

4.1 The Model of Masses and Mixing

As seen in previous chapters, there is a need to diagonalize the mass matrices, which are generally complex matrices that describe the masses and mixings of neutrinos. Diagonalizing these matrices is done with the aim of identifying the physical masses (the massive neutrino fields) of the neutrinos. There are various ways to achieve this goal; however, these methods tend to be tedious and complicated to carry out. Therefore, I will now present a construction of a general perturbative technique[1] for analytically diagonalizing complex mass matrices, using the observational fact that the reactor mixing angle θ_{13} is relatively small. The construction of this technique is based on the Takagi factorization, which, as we can see from Theorems A.1.2 and Corollaries 1 and 2 of the Appendix A, allows for the diagonalization of a complex matrix through unitary matrices (see Eq. (A.1)). This implies that the technique to be presented is a perturbative generalization of Theorem 2.3.1[2].

To attain this, let us consider the relationship between the solar and atmospheric scales (central values), which can be expressed as

$$\kappa = \sqrt{\frac{\Delta m^2_{sol}}{\Delta m^2_{atm}}} = 0.17192. \tag{4.1}$$

[1]By perturbative technique we mean the method applied in quantum mechanics in which a hamiltonian of the form $H = H_0 + \lambda H_1$ is assumed

[2]Indeed, this very theorem is a somewhat immediate consequence of Corollary 1.

On the other hand, for the reactor mixing, we have

$$\tan\theta_{13} = 0.15087 \sim \mathscr{O}(\kappa). \tag{4.2}$$

The deviation of the atmospheric mixing from its maximal value is

$$|\tan\theta_{23} - 1| = 0.0932 \sim \mathscr{O}(\kappa). \tag{4.3}$$

The reason for this similarity among these three quantities lies in the smallness of κ, or equivalently, in the smallness of the solar scale[3]. This can be understood by assuming that if the physical masses are subject to loop corrections from an unknown interaction with a coupling parameter of the order of κ, then the mass terms can be regarded as series of functions. In other words, they can be expanded in a Taylor series, where κ serves as the expansion parameter. In this sense, at zeroth order of the coupling parameter, we have a spectrum of degenerate neutrino masses, where the atmospheric mixing is maximal, and the others are zero. Consequently, the κ-order corrections must be sufficient to reproduce all the characteristics of the mixing. In summary, the scale ratio κ can be utilized to perform a Taylor expansion of the general neutrino mass matrix. Therefore, we express the most general mass matrix (whether Dirac or Majorana) as

$$M_\nu = M_0 + \delta M(\kappa), \tag{4.4}$$

where M_0 is the zeroth-order matrix in the expansion, δM are smooth functions of κ, which are generally complex and at least of order one. In the Dirac case, M_ν must be Hermitian, and in the Majorana case, it must be symmetric.

At this point, I must mention the hypotheses that, together with considerations about numerical relations in Eqs. (4.1) to (4.3), allow me to sucessfully reproduce the mixings and mass scales observed in neutrino oscillations. It is assumed that the charged lepton sector should be diagonalized. Additionally, if V is a maximal matrix, it allows diagonalizing the matrix M_0 such that the $\mu - \tau$ symmetry is satisfied. The matrix M_0 represents a two degenerate neutrino spectrum (where other mixings are null), so I need to fix one entry of the M_0 matrix, in particular, the entry $(M_0)_{11}$. Finally, the order κ corrections, that is, $\delta M(\kappa)$ must be sufficient to predict all characteristics of the mixings.

[3]These numerical coincidences had previously been investigated in the context of Majorana neutrinos [5, 49].

Let us consider the complex matrix (squared hermitian matrix[4])

$$A = \begin{pmatrix} \alpha & \eta & \varepsilon \\ \eta^* & \beta & \rho \\ \varepsilon^* & \rho^* & \gamma \end{pmatrix}, \tag{4.5}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\eta, \varepsilon, \rho \in \mathbb{C}$.

Let κ be a dimensionless parameter, this parameter governs the strength of the perturbation, as I discussed earlier, see Eq.(4.1). So, to construct our perturbative model, we assume that our matrix A parameters are actually smooth functions of κ. In this way[5], we have: $\alpha(\kappa), \beta(\kappa), \gamma(\kappa) \in \mathbb{R}$ and $\eta(\kappa), \varepsilon(\kappa), \rho(\kappa) \in \mathbb{C}$.

Let us proceed to diagonalize, which will be done in blocks (For a better understanding of the algebraic development, refer to section A.2). First, let us consider the 2-3 block. This block diagonalization gives rise to eigenvalues $m_\pm$, where $m_+ > m_-$.Without loss of generality, considering the eigenvalue m_-, we obtain the first eigenvector, and by ensuring orthogonality, it is possible to construct the second vector. The matrix that performs the first rotation is

$$R_{23} = \frac{1}{N_{23}} \begin{pmatrix} m_- - \beta & \rho \\ -\rho^* & m_- - \beta \end{pmatrix}, \tag{4.6}$$

where N_{23} is the normalization constant. Invoking Corollary 1, the matrix provided in Eq.(4.5) can be diagonalized for the 2-3 block. It is possible to compare this with Eq.(2.24), whit $\alpha = 2$ and $\beta = 3$. Specifically,

$$\Upsilon_{23}(\theta_{23}, \eta_{23}) = \begin{pmatrix} \cos\theta_{23} & \sin\theta_{23} e^{i\eta_1} \\ -\sin\theta_{23} e^{-i\eta_1} & \cos\theta_{23} \end{pmatrix}.$$

We immediately see that

$$\tan_{\theta_{23}} = \frac{|\rho|}{m_- - \beta} \qquad \text{or} \qquad \tan\theta_{23} = \frac{|\rho|}{m_+ - \beta}, \tag{4.7}$$

It is necessary to consider the mass hierarchy. Therefore, it is convenient to define a parameter, σ, which indicates either the normal hierarchy (NH) or inverted hierarchy

[4]This is formed by the product of M_ν with its respective conjugate transpose matrix, as shown in Appendix A.2.

[5]In the development of the diagonalization method, I will use κ as the expansion parameter. However, in the application, I make the substitution $\kappa \to \lambda$

(IH). Let $\sigma = \pm$, such that $-$ indicates IH and $+$ indicates NH. Thus

$$\tan_{\theta_{23}} = \frac{|\rho|}{m_\sigma - \beta}. \tag{4.8}$$

After this first rotation, $R_{23}^\dagger A R_{23}$, we obtain

$$R_{23}^\dagger A R_{23} = \begin{pmatrix} \alpha & \Delta & \Xi \\ \Delta^* & m_+ & 0 \\ \Xi^* & 0 & m_- \end{pmatrix}. \tag{4.9}$$

where

$$\Delta = \frac{\eta(m_\sigma - \beta) - \varepsilon\rho^*}{N_{23}} \qquad \& \qquad \Xi = \frac{\eta\rho + \varepsilon(m_\sigma - \beta)}{N_{23}}, \tag{4.10}$$

Proceeding analogously to the first diagonalization by block, we will have the second rotation that diagonalizes the 1-3 block:

$$R_{13} = \frac{1}{N_{13}} \begin{pmatrix} \mu_\sigma - \alpha & \Xi \\ -\Xi^* & \mu_\sigma - \alpha \end{pmatrix}, \tag{4.11}$$

where μ_σ is the eigenvalue corresponding to the 1-3 block, and N_{13} is the normalization constant. It is important to note that, once again, this depends on whether we consider NH or IH. Comparing this with Eq.(2.24), now with $\alpha = 1$ and $\beta = 3$, we obtain:

$$\tan\theta_{13} = \frac{|\Xi|}{\mu_\sigma - \alpha}. \tag{4.12}$$

Performing the rotation as given in Eq.(4.11) on the first rotation given in Eq.(4.9), we consequently obtain the second rotation that form

$$R_{13}^\dagger R_{23}^\dagger A R_{23} R_{13} = \begin{pmatrix} \mu_{-\sigma} & \Pi & 0 \\ \Pi^* & m_{-\sigma} & \Theta \\ 0 & \Theta^* & \mu_\sigma \end{pmatrix}, \tag{4.13}$$

where

$$\Pi = \frac{\Delta(\mu_\sigma - \alpha)}{N_{13}} \qquad \& \qquad \Theta = \frac{-\Xi\Delta^*}{N_{13}}. \tag{4.14}$$

The eigenvectors are found in the same way as in the previous cases. Similarly, when compared with Υ_{12}, we have

$$\tan\theta_{12} = \frac{|\Pi|}{\lambda_+ - \mu_{-\sigma}}, \tag{4.15}$$

where λ is the eigenvalue of the 1-2 block. The rotation matrix for this block is

$$R_{12} = \frac{1}{N_{12}}\begin{pmatrix} \lambda_+ - \mu_{-\sigma} & \Pi \\ -\Pi^* & \lambda_+ - \mu_{-\sigma} \end{pmatrix}. \tag{4.16}$$

Making this last rotation again, we have

$$R_{12}^\dagger R_{13}^\dagger R_{23}^\dagger A R_{23} R_{13} R_{12} = \begin{pmatrix} \lambda_+ & 0 & E \\ 0 & \lambda_- & H \\ E^* & H^* & \mu_\sigma \end{pmatrix} = d, \tag{4.17}$$

where

$$E = \frac{-\Pi\Theta}{N_{12}} \qquad \& \qquad H = \frac{\Theta\left(\lambda_+ - \mu_{-\sigma}\right)}{N_{12}}. \tag{4.18}$$

Finally, it remains to prove that the matrix d is, to a reasonable extent, diagonal. To do this, let us notice that the parameters E and H are functions of Θ and Π, which in turn are functions of Ξ and Δ, and these are functions of the entries of the square Hermitian matrix (see equations 4.14 and 4.10). This matrix A, as seen in Appendix A, is constructed through the product $\Omega\Omega^\dagger$. Therefore, in reality, $\alpha, \beta, \gamma, \eta, \varepsilon, \rho$ are the product of fundamental parameters of the matrix Ω, which by construction are functions of κ. Consequently, this assures us that Δ, Ξ and N_{12} are functions of order $\mathcal{O}\left(\kappa^1\right)$, whereas Π and Θ are functions of order $\mathcal{O}\left(\kappa^2\right)$. Therefore, the matrix given in 4.17, is diagonalized to at least the order of $\mathcal{O}\left(\kappa^2\right)$, an order that is sufficient to provide a good description of the observational data from neutrino oscillation experiments. Furthermore, this implies that the corrections induced in the eigenvalues are of the order $\mathcal{O}\left(\kappa^4\right)$, and the corrections in mixing are of the order $\mathcal{O}\left(\kappa^2\right)$.

As a final remark, in the quark sector, the quark mixing matrix, also known as the CKM matrix (Cabibbo-Kobayashi-Maskawa), has a parametrization introduced by Wolfenstein, namely

$$\begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix},$$

where $\lambda = s_{12}, A\lambda^2 = s_{23}, A\lambda^3(\rho - i\eta) = s_{13}e^{-i\delta}$. This Wolfenstein parametrization is an approximation to the standard parametrization (similar to the PMNS, but for the quark sector). It can be interpreted that λ serves as a perturbative expansion parameter, which also provides motivation for a perturbative treatment of the PMNS matrix.

4.2 An Application to Dirac Neutrinos

Since we are considering Dirac-type neutrinos, it is necessary to extend the gauge symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$ by adding the gauge symmetry $U(1)_{B-L}$, see section 2.2.3. As mentioned earlier, the addition of $U(1)_{B-L}$ is anomaly-free. On the other hand, this also allows us to eliminate the Majorana mass term. To see the latter, consider that in general, the mass Lagrangian is of the form:

$$-\mathscr{L}_m = \underbrace{\sum_{\alpha,\beta=e\mu,\tau} Y'^{\ell}_{\alpha\beta}\overline{L}'_{\alpha L}\Phi\ell'_{\beta R}}_{\text{Lepton mass}} + \underbrace{\sum_{\alpha,\beta=e,\mu,\tau} Y'^{\nu}_{\alpha\beta}\overline{L}'_{\alpha L}\widetilde{\Phi}\nu'_{\beta R}}_{\text{Dirac mass}} - \underbrace{\frac{1}{2}\overline{\nu}^C_R M\nu_R}_{\text{Majorana mass}}, \tag{4.19}$$

where L'_L is a singlet under $SU(3)_C$, a doublet under $SU(2)_L$, with hypercharge $Y = -1$, zero baryon number (B), and a lepton number (L) of 1. Thus, the $B-L$ quantum number is -1. Similarly, for right-handed leptons, ℓ_R is a singlet under both $SU(3)_C$ and $SU(2)_L$, with hypercharge $Y = -2$, and $B-L = -1$. In the case of Φ, it is a singlet under $SU(3)_C$, a doublet under $SU(2)_L$, with hypercharge 1, and $B-L = 0$. On the other hand, we have added right-handed neutrinos, which, as mentioned earlier, are singlets under both $SU(3)_C$ and $SU(2)_L$, do not interact, have zero hypercharge, but $B-L = -1$. In summary, we have the following quantum numbers in the table 4.1. Having said that, it follows that

$$\underbrace{\sum_{\alpha,\beta=e\mu,\tau} Y'^{\ell}_{\alpha\beta}\overline{L}'_{\alpha L}\Phi\ell'_{\beta R}}_{\Delta(B-L)=1+0-1=0} + \underbrace{\sum_{\alpha,\beta=e,\mu,\tau} Y'^{\nu}_{\alpha\beta}\overline{L}'_{\alpha L}\widetilde{\Phi}\nu'_{\beta R}}_{\Delta(B-L)=1+0-1=0} - \underbrace{\frac{1}{2}\overline{\nu}^C_R M\nu_R}_{\Delta(B-L)=-1-1=-2}. \tag{4.20}$$

	Y	B	L	B-L
L_L	-1	0	1	-1
ℓ_R	-2	0	1	-1
Φ	1	0	0	0
ν_R	0	0	1	-1

Table 4.1 The representations L_L, ℓ_R, ν_R, and the Higgs doublet, along with their quantum numbers (hypercharge, baryon number, lepton number, and the new charge $B-L$).

So, under $U(1)_{B-L}$, the Majorana term can be omitted. Thus, to apply our technique described in Section 4.1 to Dirac neutrinos, we will use the $B-L$ extension of the Standard Model.

In this case, since the Dirac mass term induces a Hermitian mass matrix, the matrix M_ν given by Eq.(4.4) must necessarily be Hermitian. Consequently, both M_0 and $\delta M(\lambda)$ are also Hermitian. First, let us calculate the zeroth-order matrix, M_0, which has a spectrum of degenerate neutrino masses, with θ_{23} being maximal ($\theta_{23} = \pi/4$), and $\theta_{13} = 0$. These considerations mentioned are based on Equations (4.1) to (4.3). When we take the limit as κ tends to zero, or equivalently as $\Delta m^2_{21} \to 0$, this limit of $\kappa \to 0$ allows us to construct the zeroth-order term in Eq.(4.4), that is, M_0. To construct this matrix, let us consider the biunitary transformation provided by Theorem 2.3.1. Let M_d be a diagonal matrix, namely $M_d = (m_1^2, m_2^2, m_3^2)$, and let both V and O_R be unitary matrices, such that

$$\begin{aligned} M_d =& V^\dagger M O_R, \\ \therefore MM^\dagger =& V M_d {O_R}^\dagger O_R M_d^\dagger V^\dagger = V M_d^2 V^\dagger. \end{aligned}$$

For this consideration, we need to diagonalize the $2-3$ sector. Using the expression from (2.24), with $\alpha = 2$ and $\beta = 3$, along with the corresponding rephasing, $\theta_{13} = 0$ and $\theta_{23} = \pi/4$, we obtain the matrix that diagonalizes to zeroth-order with respect to λ:

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

So, performing the corresponding operations, we have

$$MM^{\dagger} = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & \frac{m_2^2+m_3^2}{2} & \frac{m_3^2-m_2^2}{2} \\ 0 & \frac{m_3^2-m_2^2}{2} & \frac{m_2^2+m_3^2}{2} \end{pmatrix}. \tag{4.21}$$

The idea is to parametrize the square root of the square Hermitian matrix $MM^{\dagger}$. Which is nothing more than parametrizing M in such a way that (4.21) is satisfied. There are various ways to parameterize this matrix; however, several are ruled out considering that we want the degeneracy of m_1^2 with m_2^2 and the Hermiticity of the matrix. Therefore, by imposing these requirements, the ideal parameterization is as follows

$$M \equiv M_0 \propto \begin{pmatrix} A & 0 & 0 \\ 0 & B & r \\ 0 & r & B \end{pmatrix}, \tag{4.22}$$

where $A, B, r \in \mathbb{R}$. By demanding both the inverted hierarchy (IH) and the normal hierarchy (NH) to be met, the parameterization that fulfills all of the above is as follows

$$M_0 \propto \begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & C & B \end{pmatrix}, \tag{4.23}$$

where $C = \sigma r$, which will indicate whether we have NH or IH depending on the sign of sigma ($\sigma = 1$ indicates NH and the opposite for IH). Additionally, to meet the aforementioned conditions, it is necessary that $A = B - \sigma r$.

On the other hand, the matrix $\delta M(\lambda)$ must be Hermitian and a function of λ of at least first order, meaning

$$\delta M(\lambda) \propto \begin{pmatrix} a\lambda & b\lambda & c\lambda \\ b^*\lambda & d\lambda & e\lambda \\ c^*\lambda & e^*\lambda & f\lambda \end{pmatrix}, \tag{4.24}$$

where $a, d, f \in \mathbb{R}$, while the others are complex.

Without loss of generality, let us take $B = 1$ since this only indicates the location of our degeneracy. Furthermore, for the sake of simplicity in illustrating the diagonalization, we can take, $r = 1$, so that $A = 1-\sigma$. Thus

$$M_\nu = \begin{pmatrix} \Gamma & b\lambda & c\lambda \\ b^*\lambda & \xi & \sigma\omega \\ c\lambda & \sigma\omega & 1 \end{pmatrix} m_0, \tag{4.25}$$

where $\Gamma = (1-\sigma)+g\lambda, \xi = 1+h\lambda, \omega = 1+k\lambda$, and c are real. With this, we are fully prepared to diagonalize the matrix. Furthermore, note that we naturally expect the real overall mass scale $m_0 \sim \sqrt{\Delta m^2_{atm}}$. This is the matrix that will be diagonalized (see Appendix B, Eq.B.3).

To conclude the problem setup, I want to clarify what all of the above represents. The matrix M_0 provides the form of the mass matrix to which perturbative corrections will be applied (recall that, for the construction of M_0, I started with the assumption of having one zero squared mass difference and another non-zero, Δm^2_{12} and Δm^2_{13}, respectively). After adding the matrix $\delta M(\lambda)$, that is, the corrections, this should generate the solar scale. Thus, the ansatz of the work is the matrix M_ν, which, by introducing the scale m_0, allows writing the equality.

4.2.1 Masses and Mixing

In order to diagonalize M_ν, let us consider $H = M_\nu M_\nu^\dagger$ as given by

$$H = \begin{pmatrix} \Gamma^2 + (|b|^2+c^2)\lambda^2 & \Gamma b\lambda + b\xi\lambda + c\omega\sigma\lambda & \Gamma c\lambda + b\sigma\omega\lambda + c\lambda \\ \Gamma b^*\lambda + b^*\xi\lambda + c\omega\sigma\lambda & |b|^2\lambda^2+\xi^2+\sigma^2\omega^2 & b^*c\lambda^2+\sigma\xi\omega+\sigma\omega \\ \Gamma c\lambda + \sigma\omega b^*\lambda + c\lambda & cb\lambda^2+\xi\omega\sigma+\sigma\omega & c\lambda^2+\omega^2+1 \end{pmatrix} m_0^2,$$

To simplify, the structure of this matrix is in the following form

$$H \equiv M_\nu M_\nu^\dagger = \begin{pmatrix} \alpha & \eta & \varepsilon \\ \eta^* & \beta & \rho \\ \varepsilon^* & \rho^* & \gamma \end{pmatrix}.$$

According to section 4.1, the matrix H can be diagonalized by the product $V^\dagger H V$, with $V = R_{23}(\theta_{23},\delta_1)R_{13}(\theta_{13},\delta_2)R_{12}(\theta_{12},\delta_3)$, where each unitary matrix has an explicit form given by Eq.(2.24). Furthermore, from the product of the matrices, we have explicitly that

$$\begin{aligned}
\alpha =&\lambda^2(g^2+|b|^2+c^2)+\lambda(2(1-\sigma)g)+(1-\sigma)^2 \equiv \alpha_2\lambda^2+\alpha_1\lambda+\alpha_0,\\
\beta =&\lambda^2(h^2+|b|^2+k^2)+\lambda(2h+2k)+2 \equiv \beta_2\lambda^2+\beta_1\lambda+\beta_0,\\
\gamma =&\lambda^2(c^2+k^2)+2k\lambda+2 \equiv \gamma_2\lambda^2+\gamma_1\lambda+\gamma_0,\\
\rho =&\lambda^2(\sigma kh+b^*c)+\lambda(2k+h)\sigma+2\sigma \equiv \rho_2\lambda^2+\rho_1\lambda+\rho_0,\\
\eta =&\lambda^2(bh+\sigma ck+bg)+\lambda(b(2-\sigma)+\sigma c) \equiv \eta_2\lambda^2+\eta_1\lambda,\\
\varepsilon =&\lambda^2(cg+\sigma bk)+\lambda[c(2-\sigma)+\sigma b] \equiv \varepsilon_2\lambda^2+\varepsilon_1\lambda.
\end{aligned}$$

Let us calculate $|\rho|^2$, as this is used consecutively, for example, in the calculation of the eigenvalue of the 2-3 block.

$$\begin{aligned}
|\rho|^2 =&\left|\rho_0+\rho_1\lambda+\rho_2\lambda^2\right|^2 \approx |\rho_0|^2+(\rho_0\rho_1^*+\rho_1\rho_0^*)\lambda+(|\rho_1|^2+(\rho_0\rho_2^*+\rho_2\rho_0^*))\lambda^2\\
\equiv&|\rho_0|^2+\rho_{01}\lambda+\rho_{02}\lambda^2=|\rho_0|^2\left(1+\frac{\rho_{01}}{|\rho_0|^2}\lambda+\frac{\rho_{02}}{|\rho_0|^2}\lambda^2\right). \qquad (4.26)
\end{aligned}$$

Likewise, the normalization constants N_{23} and N_{13} are

$$N_{23} \approx\sqrt{2}|\rho_0|\left(1+\frac{\rho_{01}+2|\rho|(m_\sigma-\beta)}{2|\rho_0|^2}\lambda\right) \equiv \sqrt{2}|\rho_0|(1+k\lambda) \approx \sqrt{2}|\rho_0|, \qquad (4.27)$$

$$N_{13} =\sqrt{(\mu_\sigma-\alpha)^2+|\Xi|^2} \approx \sqrt{16+8\sigma(\mu_\sigma-\alpha)_1\lambda} \approx 4\left[1+\frac{1}{4}\sigma(\mu_\sigma-\alpha)_1\lambda\right]. \qquad (4.28)$$

This is because we have neglected terms of λ^2, and Ξ is given by

$$\Xi=\frac{\eta_1\rho_0+\varepsilon_1(\pm|\rho_0|)}{\sqrt{2}|\rho_0|}\lambda=\frac{\eta_1\rho_0+\sigma\varepsilon_1|\rho_0|}{\sqrt{2}|\rho_0|}\lambda=\frac{\eta_1 2\sigma+\varepsilon_1 2\sigma}{\sqrt{2}|\rho_0|}\lambda=\sigma\frac{\eta_1+\varepsilon_1}{\sqrt{2}}\lambda. \qquad (4.29)$$

The eigenvalue of the 2-3 block is then

$$\begin{aligned} m_\pm =&\frac{1}{2}\left[(\beta+\gamma)\pm\sqrt{(\beta-\gamma)^2+4|\rho|^2}\right] \\ \approx&\frac{1}{2}\left[(\beta_0+\gamma_0)+(\beta_1+\gamma_1)\lambda+(\beta_2+\gamma_2)\lambda^2\pm\sqrt{(\gamma_1-\beta_1)^2\lambda^2+4|\rho|^2}\right] \qquad (4.30)\\ \approx&2\pm|\rho_0|+\frac{\lambda}{2}\left(\beta_1+\gamma_1\pm\frac{\rho_{01}}{|\rho_0|}\right)+ \\ &+\frac{\lambda^2}{2}\left[\beta_2+\gamma_2\pm\frac{1}{4}|\rho_0|\left[4\left(\frac{\rho_{02}}{|\rho_0|^2}+\frac{(\gamma_1-\beta_1)^2}{4|\rho_0|^2}\right)-\frac{\rho_{01}^2}{|\rho_0|^4}\right]\right], \qquad (4.31)\end{aligned}$$

where we have used the expansion for both $|\rho|^2$ and $\sqrt{ax^2+bx+c}$. Furthermore, neglecting terms of order λ^2, we have

$$m_\pm-\beta\approx\frac{1}{2}\left[(\gamma_0-\beta_0)\pm 2|\rho_0|+\lambda\left(\gamma_1-\beta_1\pm\frac{\rho_{01}}{|\rho_0|}\right)\right]=\pm|\rho_0|+\frac{\lambda}{2}\left(\gamma_1-\beta_1\pm\frac{\rho_{01}}{|\rho_0|}\right). \qquad (4.32)$$

Then, the value of $\tan\theta_{23}$ is determined by Eq.(4.8). Utilizing Eq.(4.26) and Eq.(4.32), we obtain

$$\begin{aligned}\tan\theta_{23}\approx&\frac{|\rho_0|\sqrt{1+\frac{\rho_{01}}{|\rho_0|^2}\lambda}}{\pm|\rho_0|+\frac{\lambda}{2}\left(\gamma_1-\beta_1\pm\frac{\rho_{01}}{|\rho_0|}\right)} \\ =&\pm 1\mp\frac{\gamma_1-\beta_1}{2|\rho_0|}\lambda=\pm\left(1+\frac{h}{|\rho_0|}\lambda\right).\end{aligned}$$

Eliminating the phase, thus

$$\tan\theta_{23}\approx 1+\frac{h}{2}\lambda. \qquad (4.33)$$

Let us now calculate the eigenvalues of the 1-3 block

$$\begin{aligned}\mu_q =&\frac{1}{2}\left[\alpha+m_\sigma+q\sqrt{(\alpha-m_\sigma)^2+4|\Xi|^2}\right] \\ \approx&\frac{1}{2}[(\alpha+m_\sigma)_0+(\alpha+m_\sigma)_1\lambda+(\alpha+m_\sigma)_2\lambda^2, \qquad (4.34)\end{aligned}$$

where the sign q has been defined. To obtain the tangent of the mixing angle θ_{13}, it is sufficient to expand the eigenvalues to order λ^1 and consequently the difference

$\mu_q - \alpha$. Doing this, we get

$$\tan\theta_{13} = \frac{|\eta_1 + \varepsilon_1|}{\sqrt{2}\,[2 + |\rho_0|]}\lambda.$$

Using the expressions for η, ρ, and ε, it immediately follows that

$$\tan\theta_{13} = \frac{|b+c|}{2\sqrt{2}}\lambda. \tag{4.35}$$

Invoking the normalization constant N_{13} and the expression for Π given in Eq.(4.14), we have.

$$\begin{aligned}\Pi \approx& \frac{1}{4}\left(1 - \frac{1}{4}\sigma(\mu_\sigma - \alpha)_1\lambda\right)\left[(\mu_\sigma - \alpha)_1\Delta_1\lambda + (\mu_\sigma - \alpha)_1\Delta_1\lambda^2 + (\mu_\sigma - \alpha)_0\Delta_2\lambda^2\right]\\ =& \frac{1}{4}\left[4\sigma\Delta_1\lambda + 4\sigma\Delta_2\lambda^2 + (\mu_\sigma - \alpha)_1\Delta_1\lambda^2 - (\mu_\sigma - \alpha)_1\Delta_1\Lambda^2\right] = \sigma(\Delta_1\lambda + \Delta_2\lambda^2),\end{aligned}$$

where Δ_i is the coeficient of λ^i in the expansion. Likewise, defining $\Pi_i = \sigma\Delta_i$ with $i = 1,2$, we obtain

$$\Pi_1 = \frac{\eta_1 - \varepsilon_1}{\sqrt{2}} \tag{4.36}$$

$$\Pi_2 = \frac{\sigma}{2\sqrt{2}}\left[2\sigma((\eta_2 - \varepsilon_2) - k(\eta_1 - \varepsilon_1)) + (m_\sigma - \beta)_1\eta_1 - \varepsilon_1\rho_1^*\right]. \tag{4.37}$$

On the other hand, the difference of eigenvalues λ_+ with λ_- is Δm^2_{sol}, that is

$$\Delta m^2_{sol} = m_0^2\sqrt{(m_{-\sigma} - \mu_{-\sigma})^2 + 4|\Pi|^2} \propto m_0^2\lambda^2. \tag{4.38}$$

The latter is because, at zeroth order, we have a degeneracy with m_1^2 and m_2^2, so it is necessary to have a sufficient order of perturbation to separate these squared masses and thus create a gap between them. In other words, we need $\Delta m^2_{21} \propto m_0^2\lambda^2$. Therefore, defining $\psi = m_{-\sigma} - \mu_{-\sigma}$ and $D = |\Pi|$, we have that both ψ and D must be of order $\mathcal{O}(\lambda^2)$. The tangent of the angle θ_{12} and Δm^2_{sol} is therefore

$$\tan\theta_{12} = \frac{1}{x + \sqrt{4 + x^2}}, \tag{4.39}$$

$$\Delta m^2_{sol} = m_0^2\sqrt{\psi^2 + 4D^2}, \tag{4.40}$$

where $x = \psi/D$. It is necessary to consider two cases: first, the NH (Normal Hierarchy), and second, the IH (Inverted Hierarchy).

For NH, we have $\eta_1 - \varepsilon_1 = 0$, which implies $\Pi_1 = 0$. For Π_2, we have:

$$D = |\Pi_2| = \frac{1}{2\sqrt{2}}|[2(b-c)(g-k)+bg-hc]| = \frac{1}{2\sqrt{2}}|(b-c)(3g-2(k+c))|. \tag{4.41}$$

Similarly, for ψ in the case of normal hierarchy, we have

$$\psi = \frac{1}{4}\left[(2k-h)^2 - 4g^2\right]\lambda^2. \tag{4.42}$$

Now let us consider the case of inverted hierarchy (IH). Here, we have $\eta_1 - \varepsilon_1 = 2b - 2c \neq 0$. So we have

$$\Pi = \frac{2}{\sqrt{2}}[b-c]\lambda + \mathscr{O}(\lambda^2).$$

However, as I mentioned, the quantities involved in equation 5.38 must be of order λ^2, so we must impose an additional condition, namely:

$$b = c - \varsigma\lambda \qquad \forall \varsigma \in \mathbb{C}. \tag{4.43}$$

This results in

$$D = |\Pi| = \frac{1}{\sqrt{2}}|2\varsigma + hb|\lambda^2. \tag{4.44}$$

Likewise, we also have

$$\psi = (2h + 2k - 4g)\lambda + \mathscr{O}\left(\lambda^2\right).$$

So, similarly, it is necessary to introduce a new condition to ensure that this parameter is of the required order for the perturbation. Let $\vartheta \in \mathbb{R}$ such that

$$h + 2(k-g) = \vartheta\lambda. \tag{4.45}$$

Then, ψ becomes

$$\psi = \left[2\vartheta + h(h-4k) - k^2 - |b+c|^2\right]\lambda^2. \tag{4.46}$$

Finally, we need to obtain Δm^2_{atm}, which, according to our diagonalization process, is given by

$$\Delta m^2_{atm} \begin{cases} \text{NH: } m_3^2 - m_1^2 \to (m_1^2, m_2^2, m_3^2) \equiv (\lambda_-, \lambda_+, \mu_+) \\ \text{IH: } m_2^2 - m_3^2 \to (m_1^2, m_2^2, m_3^2) \equiv (\lambda_-, \lambda_+, \mu_-) \end{cases}$$

Making the corresponding approximations for both cases and taking into account the added conditions for the inverted hierarchy case, we obtain that for both the normal hierarchy (NH) and the inverted hierarchy (IH), the mass difference $\Delta m^2_{31}(|\Delta m^2_{23}|)$ is

$$\Delta m^2_{atm} \approx [4 + (4k + 2h)\lambda]\, m_0^2. \tag{4.47}$$

From Eq.(4.33), the parameter h is completely determined by data from atmospheric mixing observations. Therefore, the model has 5 free parameters, of which c, k, and m_0 are common to both hierarchies. For the normal hierarchy, the remaining parameters are b and g, while for the inverted hierarchy, the free parameters are ς, ϑ.

4.2.2 CP Violation

The Jarlskog invariant can be used to quantify the amount of CP violation in any neutrino mass model. As I mentioned in section 2.3.2, the invariant is given by

$$J = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos\theta_{13} \sin\delta_{cp}. \tag{4.48}$$

On the other hand, by invoking Eq.(2.35) and making an identification: $\Delta m^2_{sol} \equiv \Delta m^2_{21}$ and $\Delta m^2_{atm} = \Delta m^2_{31}\left(|\Delta m^2_{32}|\right)$, $(H)_{21} = \eta^*, (H)_{32} = \rho^*, (H)_{13} = \varepsilon$, we have

$$J = -\frac{m_0^6}{\Delta m^2_{sol}\left(\Delta m^2_{ATM}\right)^2} \mathbf{Im}\left[\eta^* \rho^* \varepsilon\right].$$

For hypothesis $\lambda^2 = \Delta m^2_{sol}/\Delta m^2_{atm}$, so

$$J = \left(\frac{m_0^2}{\Delta m^2_{ATM}}\right)^3 \mathbf{Im}\left[\eta^* \rho^* \varepsilon\right] \lambda^{-2}. \tag{4.49}$$

Let $\kappa = \mathbf{Im}\left[\eta^* \rho^* \varepsilon\right]/\lambda^2$, which can be expanded in the form $\kappa = \kappa_0 + \kappa_1 \lambda + \kappa_2 \lambda^2 + \mathscr{O}(\lambda^3)$. We have two cases: NH and IH, which I will discuss below. Before going into detail for the hierarchies, I should mention that from Eqs.(4.48) and (4.49) and from the settings of the mixing angles, it follows that

$$\begin{aligned}\sin\delta_{CP} =& \frac{8}{\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}}\left(\frac{m_0^2}{\Delta m_{ATM}^2}\right)^3 \kappa \\ \approx& 0.463\left(4\frac{m_0^2}{\Delta m_{ATM}^2}\right)^3 (\kappa_0 + \kappa_1\lambda)\,. \end{aligned} \tag{4.50}$$

Normal Hierarchy

For this case, it is easy to see that $\kappa_0 = 0$, but κ_1 is different. By performing the corresponding algebra, we can observe that

$$\kappa_1 = 2|b|\,[(1-h+k)c-g]\sin\phi_b. \tag{4.51}$$

Therefore, from Eqs.(4.50) and (4.51) we have

$$\sin\delta_{CP} \approx 0.463\left(4\frac{m_0^2}{\Delta m_{ATM}^2}\right)^3 (2|b|\,[(1-h+k)c-g]\sin\phi_b)\,\lambda\,. \tag{4.52}$$

Inverted Hierarchy

In this case, the expressions for κ_0 and κ_1, both of which are non-zero, can be easily determined. Their corresponding expressions are

$$\kappa_0 = 16c|b|\sin\phi_b, \tag{4.53}$$

$$\kappa_1 = 2|b|\,[3(g+hc)+2c(g-k)]\sin\phi_b. \tag{4.54}$$

Using Eq.(4.50), the following expression arises for the CO phase,

$$\sin\delta_{CP} \approx 0.463\left(4\frac{m_0^2}{\Delta m_{ATM}^2}\right)^3 2|b|\,[8c+(3(g+hc)+2c(g-k))\,\lambda]\sin\phi_b. \tag{4.55}$$

As we can see from the expressions in (4.52) and (4.55), the CP phase is controlled by a single phase, ϕ_b. This was expected because, due to the rephasing of the M_ν matrix, the only remaining physical phase is that of b (see Eq.B.3).

4.2.3 Numerical Verification

To conclude, it is necessary to validate the diagonalization process (described in Section 4.1 and applied in this Section 4.2) through a numerical procedure to explore the behavior of the model's parameters such that they reproduce the observed mixings and oscillation scales within the current level of precision. To achieve this, we numerically solve the diagonalization (i.e., solve an eigensystem) of the Hermitian square matrix $H = M_\nu M_\nu^\dagger$, where M_ν is given by Eq.(4.25) and random values are assigned to the parameters of this matrix. Subsequently, by calculating the expressions for the squared mass differences and the CP phase, we retain only those parameters that are within one and three standard deviations of the reported mixings and CP phase values. On the other hand, from Eq.(4.25), the factor m_0 plays the role of a global scaling factor. Therefore, an appropriate choice for a discriminative factor is to use the scale relation given in Eq.(4.1), thus reducing the number of parameters for the analysis by one unit.

A numerical analysis was carried out for the normal hierarchy (NH) case. To do this, the model parameters, namely, h, c, k, b, g, were randomly varied within a range that reasonably maintains the perturbation hypothesis. From Figure 4.1, we notice a correlation between the quantities b and c, which is in complete agreement with Eq.(4.35). Similarly, in the other parameters illustrated in the same figure, an apparent correlation is observed, which again aligns with our analytical expressions (4.41) and (4.42). The red points represent parameters that satisfy the conditions to reproduce the observables obtained from neutrino oscillations at 1 sigma, while the blue points satisfy them at 3 sigma (this convention is maintained for the subsequent figures). As we can see from this, there is no strong distinction between the 1 sigma and 3 sigma regions.

For the case of the inverted hierarchy, it is necessary to explicitly consider the tuning conditions: $b = c - \varsigma\lambda$ and $h + 2(k - g) = \vartheta\lambda$ to focus the parameter search. Similarly, random values were assigned to these tuning conditions within the intervals: $\vartheta \in [-2.5, 2]$ and $\varsigma \in [1.6, 2.3]$. For φ_c, the values ranged from 0 to 2π, while for h and k, they were between $[-3, 3]$, and for b, it was within $[-1, 1]$. This resulted in the parameter space shown in Figure 4.2. From this figure, it is possible to notice that, unlike the normal hierarchy, not all parameters exhibit a pronounced correlation. This is evident, for example, in the parameters $h - b$ in Figure 4.2. On the other hand, we can see that there is a shift to the left between the values for 1σ and 3σ in the parameter ς, overlapping in the region $\varsigma_1 \equiv [2, 2.2] \subset \varsigma$. This shift is due to the lack of control over this parameter, as it was introduced "artificially" to provide the mass

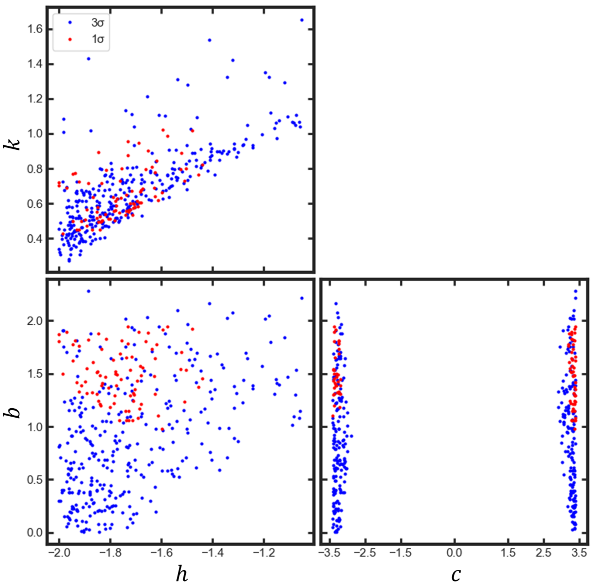


Fig. 4.1 The allowed parameter space for the normal hierarchy shows b and k as functions of h, and b also as a function of c. The blue points indicate the points that satisfy the required constraints on the angles, CP phase and $\Delta m^2_{sol}/\Delta m^2_{atm}$ at 3σ. The same applies to the red points, representing the points at 1σ.

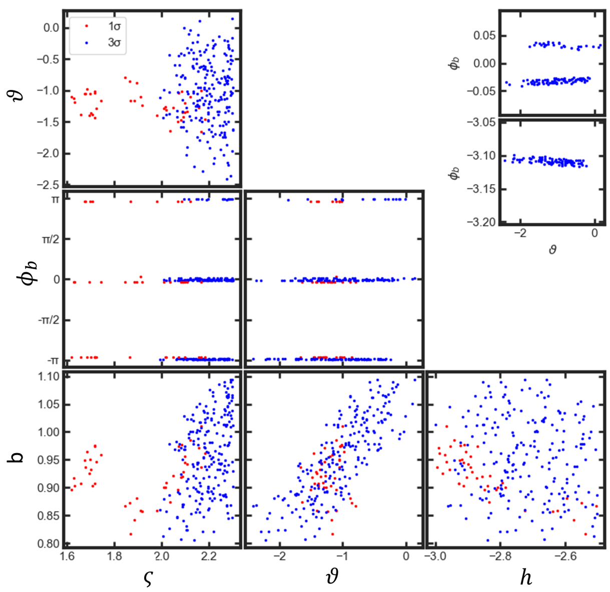


Fig. 4.2 The allowed parameter space for the Inverted Hierarchy is observed. We are looking at the subspace of $\vartheta - \varsigma - |b| - h$.

gap. In the figures of Appendix C, I illustrate the region obtained from the physical parameters based on the values obtained for both hierarchy cases.

To use the CP phase as a filter for obtaining these parameters, it was necessary to take into account the existence of a rephasing symmetry (Appendix B.2, Eq.B.14). In Figure 4.3, the CP phase is illustrated as a function of the parameter k for the normal hierarchy.

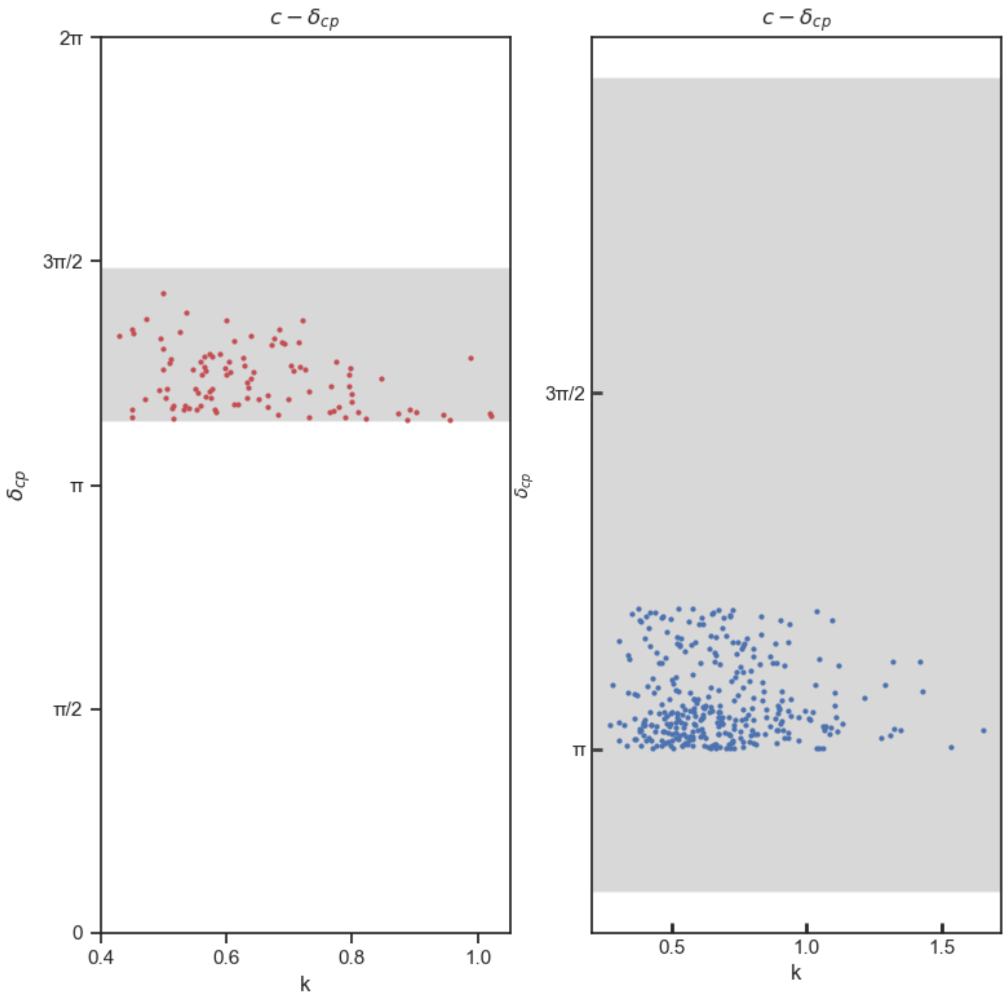


Fig. 4.3 Allowed values for the CP phase in the case of the normal hierarchy. In red, we have the values at 1σ, while the blue points are at 3σ.

In Figure 4.4, similarly, the CP phase is shown as a function of the parameter c for the inverted hierarchy. From it, we observe that the concentration of points for both 1σ and 3σ is in the interval $c \in [-0.74,-0.5] \cup [0.5,0.74] \equiv \Omega_c$. However, from these concentrations, we can see that in the case of the normal hierarchy, all our points fall within the accepted range for the CP phase, whereas in the case of the inverted hierarchy, at 1σ, the concentration of points is below the accepted CP range. This means, for any $\varepsilon > 0$, then $\forall c \in \Omega_c$, we have $E(c_i,\varepsilon) \cap \mathrm{Fr}(\delta_{cp}^{1\sigma}) = 0 \wedge E(c_i,\varepsilon) \cap \delta_{cp}^{1\sigma} = 0$, where $E(x,r)$ denotes the disk centered at x with radius r y Fr denotes the boundary of the accepted CP interval. On the other hand, for the case of 3σ, we have $E(c_i,\varepsilon) \cap \mathrm{Fr}(\delta_{cp}^{3\sigma}) \neq 0 \wedge E(c_i,\varepsilon) \cap \delta_{cp}^{3\sigma} = 0$, that is to say, the points at 3σ are well below the accepted interval. All of this leads to the indication that through the process of diagonalization, the mass matrix parametrized under the assumption of Dirac neutrinos is not sensitive to the inverted hierarchy. In other words, the inverted hierarchy is marginally, and the 1σ points would be acceptable if the phase falls within 3σ.

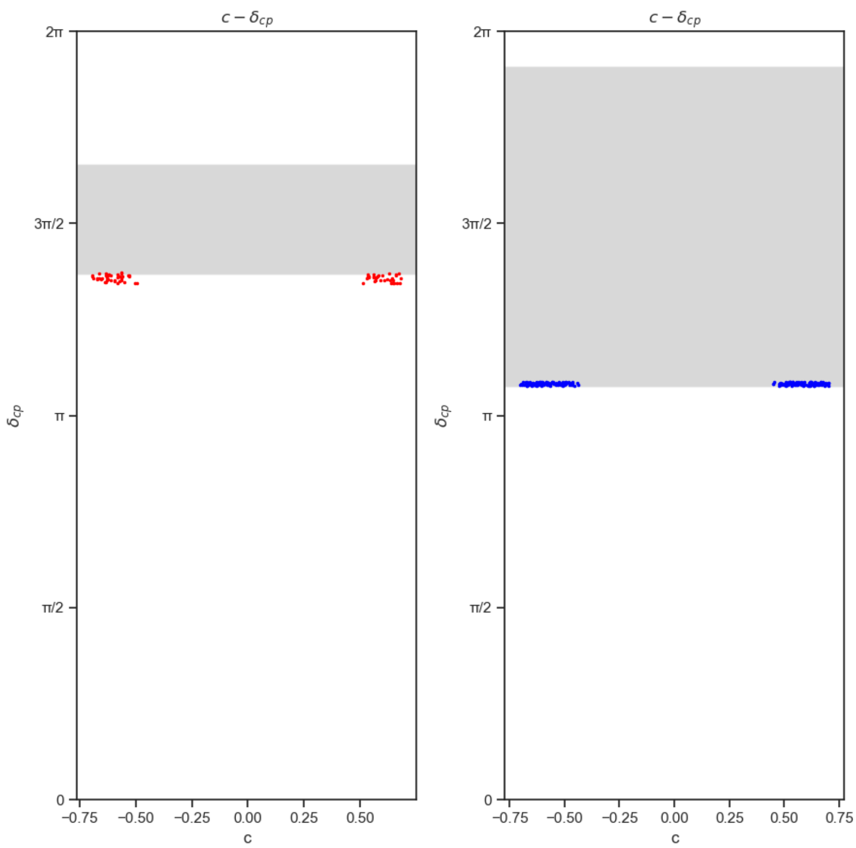


Fig. 4.4 Allowed values for the CP phase in the case of the inverted hierarchy. In red, we have the values at 1σ, while the blue points are at 3σ

CHAPTER 5

Conclusions and Perspectives

In this work, a diagonalization process focused on a perturbative technique has been studied due to the smallness of the angle θ_{13}. We employ this technique with a model that constructs a neutrino mass matrix of the form $M_0 + \delta M(\lambda)$, where λ arises from the ratio of neutrino scales. As an application of the method, we consider Dirac-type neutrinos.

Observation of oscillations suggest that the tangent of the angle θ_{13} is of the order of λ, similar to the deviation from the maximum value of $\tan\theta_{23}$, where $\lambda \sim \sqrt{\Delta M^2_{sol}/\Delta M^2_{atm}}$. This apparent relationship can be understood by considering the series expansion on λ of the mass matrix which could suggest a perturbative approach for the origin of neutrino mass terms. Indeed, the diagonalization method presented in this thesis allows us to express all observables as functions of λ, successfully reproducing the parameter's dependence on the respective observable. It was found that both the tangent of the angle θ_{23} and θ_{13} have the same structure for both hierarchies (NH and IH), as well as for the difference of masses Δm^2_{atm}. However, in the case of the angle θ_{12} and the squared mass difference Δm^2_{sol}, it was naturally obtained that for the normal hierarchy, the parameters Ψ and D are of order λ^2 as described in the previous text. While for the inverted hierarchy, these parameters are not naturally of the same order, so to satisfactorily reproduce the mass gap between m_1^2 and m_2^2, it is necessary to introduce new parameters that allow us to model this quadratic dependence artificially, thus raising some level of fine-tuning for such a hierarchy.

Similarly, a numerical analysis was carried out to verify the analytical method. From this analysis, it was found that despite the normal hierarchy being very natural and there being an apparent correlation between the parameters, there is still an additional degeneracy that is not very evident. This is illustrated by the points in the parameter space shown in Figure 4.1, where a tiny variation in the parameters can move us from the 3σ regime to the 1σ regime and vice versa. On the other hand, in the case of the inverted hierarchy, it was found that not all parameters are correlated (see Figure 4.2). However, it is also observed that similar to the normal hierarchy case, there is still an additional degeneracy (in addition to the parameters ς and ϑ introduced to reproduce the mass gap. Some of these observations coincide with the case of Majorana-type neutrinos, as discussed in [5]), as slight variations in the parameters can transition us from the 1σ regime to the 3σ regime and vice versa.

From observing the CP acceptance bands, it is evident that, in the case of the inverted hierarchy, the procedure illustrated here is not sensitive to it. This is evident from Figure 4.4, where no points are within the acceptance interval at 1σ, and only a concentration near the boundary of the CP angle band is observed at 3σ. On the other hand, the acceptance of these red points is possible if the phase is allowed to fall within the 3-sigma range. This restriction arises from the requirements related to the other parameters. In other words, the Inverted Hierarchy (IH) could still marginally conform to the observations. Thus, we conclude that by leveraging the smallness of the angle θ_{13}, it is possible to work with a perturbative method to diagonalize the mass matrix using a parameter λ, which satisfactorily reproduces the experimental observations of oscillations for the Dirac case. However, the model proposed here is only sensitive to the normal hierarchy, thus suggesting that the proposed correlation among observables may not work for the inverted hierarchy case.

There is still work to do. As a first point, we should consider the parameter c not being fixed at one, just like the parameter B as described in the previous text. This would allow us to play with the value where we fix the squared neutrino mass scale, thus providing an additional constraint that could improve the distribution of all parameters. As a second point, we might expect that if we extend the expansion to order λ^2 instead of λ, we could explore the possibility of improving the predictions of all mixing characteristics and, in doing so, delve into searching for a new fine-tuning that would eliminate the degeneracy between 1σ and 3σ in the parameters. Finally, from the observation of the structure of the mass matrix M_ν, the following questions arise, which we must then provide an answer to. Is this peculiar structure of the matrix a result of some symmetry? And if so, what is this symmetry?.

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APPENDIX A

Matrix Manipulation

In this appendix, the mathematical tools necessary for a complete understanding of the development of this thesis are provided. Section A.1 presents the definitions and conventions required, followed by section A.2 which illustrates the general process of diagonalizing 3×3 matrices of masses.

A.1 Basic Concepts

A matrix A over a field $\mathbb{K}$ is defined as an ordered table of scalars a_{ij} in the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdot & a_{mn} \end{pmatrix}.$$

More formally, a matrix is an array of size $m \times n$ of scalars over the field $\mathbb{K}$. The set of all $m \times n$ matrices over $\mathbb{K}$ is denoted as $\mathscr{M}[\mathbb{K}, m \times n]$. In general, the field $\mathbb{K}$ is often taken to be $\mathbb{C}$, that is, the field of complex numbers. Here, we have $a_{ij} \in \mathbb{C}$. The main diagonal of A consists of the elements $a_{11}, a_{22}, \cdots, a_{pp}$, where $p = \min\{m, n\}$.

The transpose of a matrix $A = [a_{ij}] \in \mathscr{M}[\mathbb{R}, m \times n]$, denoted by A^T, is given by $A^T = [a_{ji}]$. The conjugate transpose of a matrix $A = [a_{ij}] \in \mathscr{M}[\mathbb{C}, m \times n]$, denoted by $A^\dagger$, is defined as $A^\dagger = (A^*)^T$, where A^* is the conjugate of each element of the matrix.

There exists a wide range of matrix classes defined based on transpose or conjugate transpose operations. However, the three most important matrix classes are Hermitian matrices, symmetric matrices, and orthogonal matrices, which are defined as follows

Definition A.1.1 *Let $A \in \mathscr{M}[\mathbb{K}]$. A matrix is said to be symmetric if $A^T = A$, and antisymmetric if $A^T = -A$.*

Definition A.1.2 *Let $A \in \mathscr{M}[\mathbb{C}]$. A matrix is said to be Hermitian if $A^\dagger = A$.*

Definition A.1.3 *Let $A \in \mathscr{M}[\mathbb{K}]$. A matrix is said to be orthogonal if $AA^T = I$.*

here I have omitted the dimension of the matrices A. The most relevant case in this thesis is the case of Hermitian matrices, so only this type of matrices is considered. On the other hand, an important aspect of matrices is the eigenvectors and eigenvalues, as they allow for the diagonalization of a matrix.

Definition A.1.4 *Let $A \in End(\mathbb{K})$ be a linear transformation and v be a non-zero vector such that*

$$Av = \lambda v$$

with $\lambda \in \mathbb{C}$. Then we say that v is an eigenvector and λ an eigenvalue of A.

The abbreviation "End" has been used for endomorphism: $\mathrm{End}(\mathbb{K})$. It is important to note that a matrix can be the representation of a linear transformation in a respective basis. With these eigenvalues, we have the following theorem.

Theorem A.1.1 *A square matrix A of dimension n is similar to a diagonal matrix D if and only if A has n linearly independent eigenvectors. In such case, the elements of D are the eigenvalues, and $D = P^{-1}AP$, where P is the matrix whose columns are the eigenvectors.*

In the particular case where P is an orthogonal matrix, then we have $D = P^T AP$.

Furthermore, let us consider the following theorems about diagonalization of matrices, which aid in the diagonalization of the mass matrix [50, 51]:

Theorem A.1.2 (***Autonne's uniqueness theorem***). *Let $A \in \mathscr{M}[\mathbb{C}, n \times m] \equiv \mathscr{M}_{n,m}$ be given with **rank** $A = r$. Let $s_1, \dots, s_d$ be the distinct positive singular values of A, in any order, with respective multiplicities $n_1, \dots, n_d$, and let $\Sigma_d = s_1^2 I_{n_1} \oplus \cdots \oplus s_d^2 I_{n_d} \in \mathscr{M}_r$. Let $A = V\Sigma W^*$ be a singular value decomposition with $\Sigma = \begin{pmatrix} \Sigma_d & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{M}_{n,m}$, so that $\Sigma^T \Sigma =$*

$s_1^2 I_{n_1} \oplus \cdots \oplus s_d^2 I_{n_d} \oplus 0_{n-r}$ and $\Sigma\Sigma^T = s_1^2 I_{n_1} \oplus \cdots \oplus s_d^2 I_{n_d} \oplus 0_{m-r}$. Let $\hat{V} \in \mathscr{M}_n$ and $\hat{W} \in \mathscr{M}_m$ be unitary. Then $A = \hat{V}\Sigma\hat{W}^*$ if and only if there are unitary matrices $U_1 \in \mathscr{M}_{n_1}, \ldots, U_d \in \mathscr{M}_{n_d}$, $\widetilde{V} \in \mathscr{M}_{n-r}$ and $\widetilde{W} \in \mathscr{M}_{m-r}$ such that

$$\hat{V} = V(U_1 \oplus \cdots \oplus U_d \oplus \widetilde{V}) \ \ \& \ \ \hat{W} = W(U_1 \oplus \cdots \oplus U_d \oplus \widetilde{W})$$

if A is real and the factors $V, W, \hat{V}, \hat{W}$ are real orthogonal, then the matrices $U_1, \cdots U_d$, $\widetilde{V}$ and $\widetilde{W}$ may be taken to be real orthogonal.

As consequences of this uniqueness theorem, the following corollaries are obtained:

Corollary 1 (***Autonne-Takagi factorization***). *Let $A \in \mathscr{M}_n$. If A is symmetric, there is a unitary $U \in \mathscr{M}_n$ such that $A = U\Sigma U^T$, in which Σ is a nonnegative diagonal matrix whose diagonal entries are the singular values of A , in any desired order.*

Corollary 2 (***Uniqueness***). *Suppose that A is symmetric and **rank** $A = r$. Let $s_1, \ldots, s_d$ be the distinct positive singular values of A, in any given order, with respective multiplicites $n_1, \ldots, n_d$. Let $\Sigma = s_1 I_{n_1} \oplus \cdots \oplus s_d I_{n_d} \oplus 0_{n-r}$; the zero block is missing if A is nonsingular. Let $U, V \in \mathscr{M}_n$ be unitary. Then $A = U\Sigma U^T = V\Sigma V^T$ if and only if $V = UZ$, $Z = Q_1 \oplus \cdots \oplus Q_d \oplus \widetilde{Z}$, $\widetilde{Z} \in \mathscr{M}_{n-r}$ is unitary, and each $Q_j \in \mathscr{M}_{n_j}$ is real orthogonal. If the singular values of A are distinct, that is, if $d \geq n-1$, then $V = UD$, in which $D = diag\,(d_1, \ldots, d_n)$, $d_i = \pm 1$ for each $i = 1, \ldots, n-1$, $d_n = \pm 1$ if A is nonsingular, and $d_n \in \mathbb{C}$ with $|d_n| = 1$ if A is singular.*

These theorems and corollaries imply that if Ω is a symmetric matrix and if U is a unitary matrix, such that

$$D = U^T \Omega U \tag{A.1}$$

where D is a diagonal matrix. Then

$$D^2 = D^\dagger \cdot D = \left(U^T \Omega U\right)^\dagger \left(U^T \Omega U\right) = \left(U^\dagger \Omega^\dagger (U^T)^\dagger\right) \left(U^T \Omega U\right) = U^\dagger \Omega^\dagger \Omega U \tag{A.2}$$

since $U^\dagger U = I$. The factor $H \equiv \Omega^\dagger \Omega$ is a squared Hermitian matrix, and U is the matrix that diagonalized H, where $D^2 = diag(h_1^2, \ldots, h_n^2)$ with h_k being the eigenvalues of H.

A.2 Diagonalization of the Neutrino Mass Matrix

A direct consequence of Theorem A.1.2 and Corollary 1 indicates that by using a unitary matrix, which can be written as the product of other unitary matrices[1], it is

[1]For all n, the set of all $n \times n$ unitary matrices forms a group under matrix multiplication.

possible to diagonalize a complex matrix. This is of great utility because, as we will see next, it is possible to diagonalize a squared mass matrix through sub-blocks of said matrix. Now, considering the expression A.2, we see that it is possible to obtain a squared Hermitian matrix from a symmetric matrix. This is also fulfilled when our matrix Ω is Hermitian from the outset. Therefore, without loss of generality, let us consider Ω as a complex 3×3 matrix (Hermitian or symmetric). I will illustrate the general diagonalization process for a mass matrix. For the above reasons, it is not important whether we are dealing with Dirac neutrinos or Majorana neutrinos (Hermitian or symmetric mass matrix, respectively). The following method is applicable to both cases. Let us consider the following squared hermitian matrix

$$A \equiv \Omega\Omega^{\dagger} = \begin{pmatrix} \alpha & \eta & \varepsilon \\ \eta^* & \beta & \rho \\ \varepsilon^* & \rho^* & \gamma \end{pmatrix},$$

where $\alpha,\beta,\gamma \in \mathbb{R}$ and $\eta,\varepsilon,\rho \in \mathbb{C}$. The diagonalization will be carried out by blocks. Let us consider the $2-3$ block, that is

$$A_{2-3} = \begin{pmatrix} \beta & \rho \\ \rho^* & \gamma \end{pmatrix}.$$

The eigenvalues corresponding to this block are

$$m_{\pm} = \frac{1}{2}\left[\gamma + \beta \pm \sqrt{(\gamma-\beta)^2 + 4|\rho|^2}\right], \tag{A.3}$$

We observe that the smallest value of these two eigenvalues is m_-, that is, $m_- < m_+$. As the eigenvalues of Hermitian matrix are real, and therefore the discriminant will be positive. Hence, the statement above is fulfilled.

The eigenvectors corresponding to this value are

$$\begin{pmatrix} \beta - m_{\pm} & \rho \\ \rho^* & \gamma - m_{\pm} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

Considering the eigenvalue m_-, we will have the following system of equations

$$\begin{aligned} (\beta - m_-)x_1 + \rho x_2 &= 0, \\ \rho^* x_1 + (\gamma - m_-)x_2 &= 0, \end{aligned}$$

from the first equation we obtain the first eigenvector

$$v = \frac{1}{N_{23}} \begin{pmatrix} \rho \\ m_- - \beta \end{pmatrix},$$

where N_{23} is the normalization constant given by $N_{23} = \sqrt{|\rho|^2 + (m_- - \beta)^2}$. To construct the eigenvector associated with m_+, we can proceed as follows. The eigenvector associated with m_- must be orthogonal to the first one. Let w be that vector, it must satisfy

$$v^\dagger w = 0 \to \begin{pmatrix} \rho^* & m_- - \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0,$$

this leads to the following expression

$$\rho^* y_1 + (m_- - \beta) y_2 = 0$$
$$\therefore w = \frac{1}{N_{23}} \begin{pmatrix} m_- - \beta \\ -\rho^* \end{pmatrix}.$$

Therefore, our matrix that performs the first rotation is

$$R_{23} = \frac{1}{N_{23}} \begin{pmatrix} m_- - \beta & \rho \\ -\rho^* & m_- - \beta \end{pmatrix}.$$

If we compare this matrix with the rotation matrix

$$\Upsilon(\theta_{23}, \eta_{23}) = \begin{pmatrix} \cos\theta_{23} & \sin\theta_{23} e^{i\eta_{23}} \\ -\sin\theta_{23} e^{-i\eta_{23}} & \cos\theta_{23} \end{pmatrix}.$$

We immediately see that

$$\tan\theta_{23} = \frac{|\rho|}{m_- - \beta}. \tag{A.4}$$

Using a smilar procedure, we obtain

$$\tan\theta_{23} = \frac{|\rho|}{m_+ - \beta}. \tag{A.5}$$

Now, it is necessary to consider the mass hierarchy. Let us observe that when we are diagonalizing, what we are obtaining in the entry $(3,3)$ of the squared mass matrix, A, is the squared mass m_3^2. On the other hand, for the inverted hierarchy, m_3^2 is the smallest mass, so it corresponds to m_-. However, if we are in the normal hierarchy, m_3^2 is the largest mass, so it corresponds to m_+. This implies that if we want to work in the normal hierarchy, we must use the expression A.5, otherwise we should work with A.4. Let us define $\sigma = \pm$, this way we can synthesize the expressions depending on the value of σ:

$$\tan\theta_{23} = \frac{|\rho|}{m_\sigma - \beta}. \tag{A.6}$$

The matrix R_{23} and the normalization constant remain

$$R_{23} = \frac{1}{N_{23}} \begin{pmatrix} m_\sigma - \beta & \rho \\ -\rho^* & m_\sigma - \beta \end{pmatrix} \qquad \& \qquad N_{23} = \sqrt{|\rho|^2 + (m_\sigma - \beta)^2}.$$

By performing the first rotation, we obtain

$$R_{23}^\dagger A R_{23} = \begin{pmatrix} \alpha & \Delta & \Xi \\ \Delta^* & m_+ & 0 \\ \Xi^* & 0 & m_- \end{pmatrix}, \tag{A.7}$$

where

$$\Delta = \frac{\eta(m_\sigma - \beta) - \varepsilon\rho^*}{N_{23}}, \qquad \Xi = \frac{\eta\rho + \varepsilon(m_\sigma - \beta)}{N_{23}}.$$

Let us proceed to diagonalize the $1-3$ block now:

$$A_{1-3} = \begin{pmatrix} \alpha & \Xi \\ \Xi^* & m_- \end{pmatrix}.$$

The eigenvalues are

$$\mu_\pm = \frac{1}{2}\left[\alpha + m_- \pm \sqrt{(\alpha - m_-)^2 + 4|\Xi|^2}\right].$$

Proceeding analogously to the first diagonalization, we will have

$$\begin{pmatrix} \alpha - \mu_- & \Xi \\ \Xi^* & m_- - \mu_- \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$

thus

$$\begin{aligned} (\alpha - \mu_-)x_1 + \Xi x_2 =& 0, \\ \Xi^* x_2 + (m_- - \mu_-)x_2 =& 0. \end{aligned}$$

Again, taking the first expression, we obtain the first eigenvector

$$v = \frac{1}{N_{13}} \begin{pmatrix} \Xi \\ \mu_- - \alpha \end{pmatrix}.$$

To obtain the eigenvector associated with μ_+, we impose orthogonality between the two eigenvectors, so we will have

$$w = \frac{1}{N_{13}} \begin{pmatrix} \mu_- - \alpha \\ -\Xi^* \end{pmatrix}.$$

In this way, the next rotation that diagonalizes the $1-3$ block is

$$R_{13} = \frac{1}{N_{13}} \begin{pmatrix} \mu_- - \alpha & \Xi \\ -\Xi^* & \mu_- - \alpha \end{pmatrix}.$$

Once again, taking into account the normal or inverted hierarchy, we reintroduce the previously defined parameter σ in order to obtain

$$R_{13} = \frac{1}{N_{13}} \begin{pmatrix} \mu_\sigma - \alpha & \Xi \\ -\Xi^* & \mu_\sigma - \alpha \end{pmatrix}. \tag{A.8}$$

with

$$N_{13} = \left((\mu_\sigma - \alpha)^2 + |\Xi|^2\right)^{1/2} \qquad \& \qquad \tan\theta_{13} = \frac{|\Xi|}{\mu_\sigma - \alpha}.$$

Performing the obtained rotation A.8 on A.7, we get

$$R_{13}^{\dagger}R_{23}^{\dagger}AR_{23}R_{13} = \begin{pmatrix} \mu_{+} & \Pi & 0 \\ \Pi^{*} & m_{+} & \Theta \\ 0 & \Theta^{*} & \mu_{-} \end{pmatrix}, \tag{A.9}$$

where

$$\Pi = \frac{\Delta(\mu_{\sigma} - \alpha)}{N_{13}} \qquad \& \qquad \Theta = \frac{-\Xi\Delta^{*}}{N_{13}}.$$

Finally, let us diagonalize the last block. However, we must consider that in expressions A.7 and A.9, we are explicitly considering the inverted hierarchy. To consider the normal hierarchy, we must change the sign from $+$ to $-$, meaning that the order in the diagonal changes. This implies that the eigenvalues of this last block will be functions of μ_{-}, m_{-} or μ_{+}, m_{+} for the normal or inverted hierarchy, respectively. To synthesize this into a single case, let us consider the following matrix

$$R_{13}^{\dagger}R_{23}^{\dagger}AR_{23}R_{13} = \begin{pmatrix} \mu_{-\sigma} & \Pi & 0 \\ \Pi^{*} & m_{-\sigma} & \Theta \\ 0 & \Theta^{*} & \mu_{\sigma} \end{pmatrix}.$$

Thus, the last block is

$$A_{1-2} = \begin{pmatrix} \mu_{-\sigma} & \Pi \\ \Pi^{*} & m_{-\sigma} \end{pmatrix},$$

whose eigenvalues are

$$\lambda_{\pm} = \frac{1}{2}\left[\mu_{-\sigma} + m_{-\sigma} \pm \sqrt{(\mu_{-\sigma} - m_{-\sigma})^{2} + 4|\Pi|^{2}}\right].$$

The eigenvectors are found in the same way as in the previous cases. The rotation matrix for this block is

$$R_{12} = \frac{1}{N_{12}}\begin{pmatrix} \lambda_{+} - \mu_{-\sigma} & \Pi \\ -\Pi^{*} & \lambda_{+} - \mu_{-\sigma} \end{pmatrix},$$

where $N_{12} = \left((\lambda_+ - \mu_{-\sigma})^2 + |\Pi|^2 \right)^{1/2}$. Therefore, the tangent of the angle θ_{12} is

$$\tan\theta_{12} = \frac{|\Pi|}{\lambda_+ - \mu_{-\sigma}}.$$

APPENDIX B

About the rephasing of complex matrices

As we well know, in general, the complex matrices in question have both physical and non-physical phases. Therefore, we are interested in determining which of all the phases contained in the mass matrix or the PMNS matrix are physical. To achieve this, it is necessary to rephase the matrices in such a way that we can eliminate as many phases as possible from the corresponding matrices. Let us start with the mass matrix, denoted as M_ν, discussed in Chapter 4.2 and finalizing with the rephasing of the PMNS matrix.

B.1 In Relation to the Mass Matrix Rephasing

In Chapter 4.2, we encounter the following mass matrix

$$M_\nu \propto \begin{pmatrix} \Gamma & |b|e^{i\phi_b}\lambda & |c|e^{i\phi_c}\lambda \\ |b|e^{-i\phi_b}\lambda & \xi & \sigma\omega \\ |c|e^{-i\phi_c}\lambda & \sigma\omega^* & 1 \end{pmatrix}.$$

Let us eliminate the phase of ω

$$M_\nu \propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi_\omega} \end{pmatrix} \begin{pmatrix} \Gamma & |b|e^{i\phi_b}\lambda & |c|e^{i(\phi_c-\phi_\omega)}\lambda \\ |b|e^{-i\phi_b}\lambda & \xi & \sigma|\omega| \\ |c|e^{-i(\phi_c-\phi_\omega)}\lambda & \sigma|\omega| & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi_\omega} \end{pmatrix}. \quad \text{(B.1)}$$

Let us define θ_c as $\phi_c - \phi_\omega$. Similarly, we can eliminate the phase of b

$$M_\nu \propto \begin{pmatrix} e^{i\phi_b} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi_\omega} \end{pmatrix} \begin{pmatrix} \Gamma & |b|\lambda & |c|e^{i(\theta_c-\phi_b)}\lambda \\ |b|\lambda & \xi & \sigma|\omega| \\ |c|e^{-i(\theta_c-\phi_b)}\lambda & \sigma|\omega| & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi_b} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi_\omega} \end{pmatrix}.$$

Let us define φ_c as $\theta_c - \phi_b$. In this form, the final matrix is

$$M_\nu \propto \begin{pmatrix} e^{i\phi_b} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi_\omega} \end{pmatrix} \begin{pmatrix} \Gamma & |b|\lambda & |c|e^{i\varphi_c}\lambda \\ |b|\lambda & \xi & \sigma|\omega| \\ |c|e^{-i\varphi_c}\lambda & \sigma|\omega| & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi_b} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi_\omega} \end{pmatrix}. \tag{B.2}$$

In this way, we have been able to absorb two phases from the original matrix, leaving only the phase of c, and thus, b and ω become real. Similarly, it is possible to keep b complex and c real. To achieve this, let us reconsider Eq.(B.1) with $\theta_c = \phi_c - \phi_\omega$:

$$\begin{aligned} M_\nu &\propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi_\omega} \end{pmatrix} \begin{pmatrix} \Gamma & |b|e^{i\phi_b}\lambda & |c|e^{i\theta_c}\lambda \\ |b|e^{-i\phi_b}\lambda & \xi & \sigma|\omega| \\ |c|e^{-i\theta_c}\lambda & \sigma|\omega| & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi_\omega} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\theta_c} & 0 \\ 0 & 0 & e^{-i\Theta} \end{pmatrix} \begin{pmatrix} \Gamma & |b|e^{i\varphi_b} & |c|\lambda \\ |b|e^{-i\varphi_b} & \xi & \sigma|\omega| \\ |c|\lambda & \sigma|\omega| & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta_c} & 0 \\ 0 & 0 & e^{i\Theta} \end{pmatrix}. \end{aligned} \tag{B.3}$$

where $\Theta = \theta_c + \phi_\omega$ and $\varphi_b = \phi_b - \theta_c$. As we can see from Eq. (B.2) and Eq. (B.3), it is possible to eliminate two of the three phases that are originally present. This can be done by either keeping c real and making b complex, or by keeping c complex while making b real. In this work, we chose, without loss of generality, the latter case, meaning we kept $b \in \mathbb{C}$ and $c \in \mathbb{R}$.

B.2 Of the PMNS Matrix and its Rephasing

In the procces of diagonalization , we have using the following descomposition of an unitary matriz (see 4.17)

$$R = R_{23}R_{13}R_{12}, \tag{B.4}$$

where using the notation $\cos\theta_{ij} = c_{ij}$ and $\sin\theta_{ij} = s_{ij}$, we have

$$U_{ij} \equiv U_{ij}\left(\theta_{ij}, \delta_{ij}\right) = \begin{pmatrix} c_{ij} & s_{ij}e^{i\delta_{ij}} \\ -s_{ij}e^{-i\delta_{ij}} & c_{ij} \end{pmatrix}. \tag{B.5}$$

Performing the product as dictated by Eq.(B.4), we have

$$U = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12}e^{i\phi} & s_{13}e^{i\psi} \\ -s_{13}s_{23}c_{12}e^{i(\theta-\psi)} - c_{23}s_{12}e^{-i\phi} & -s_{12}s_{13}s_{23}e^{i(\theta-\psi+\phi)} + c_{12}c_{23} & c_{13}s_{23}e^{i\theta} \\ -s_{13}c_{23}c_{12}e^{-i\psi} + s_{23}s_{12}e^{-i(\theta+\phi)} & -s_{12}s_{13}c_{23}e^{-i(\psi-\phi)} - s_{23}c_{12}e^{-i\theta} & c_{23}c_{13} \end{pmatrix},$$

which can be rewritten as the product of two diagonal matrices and a phased matrix, namely:

$$U = F_1^{\dagger} \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{i\psi} \\ -s_{13}s_{23}c_{12}e^{i(\theta-\psi+\phi)} - c_{23}s_{12} & -s_{12}s_{13}s_{23}e^{i(\theta-\psi+\phi)} + c_{12}c_{23} & c_{13}s_{23}e^{i(\theta+\phi)} \\ -s_{13}c_{23}c_{12}e^{-i\psi} + s_{23}s_{12}e^{-i(\theta+\phi)} & -s_{12}s_{13}c_{23}e^{-i(\psi)} - s_{23}c_{12}e^{-i(\theta+\phi)} & c_{23}c_{13} \end{pmatrix} F_1.$$

where $F_1 = diag(1, e^{i\phi}, 1)$. Simarly, making the same:

$$U = F_2^{\dagger} \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{i(\psi-\theta-\phi)} \\ -s_{13}s_{23}c_{12}e^{i(\theta-\psi+\phi)} - c_{23}s_{12} & -s_{12}s_{13}s_{23}e^{i(\theta-\psi+\phi)} + c_{12}c_{23} & c_{13}s_{23} \\ -s_{13}c_{23}c_{12}e^{-i(\psi-\theta-\phi)} + s_{23}s_{12} & -s_{12}s_{13}c_{23}e^{-i(\psi-\theta-\phi)} - s_{23}c_{12} & c_{23}c_{13} \end{pmatrix} F_2.$$

where $F_2 = diag(1, e^{i\phi}, e^{i(\theta+\phi)})$. Let $\delta = \theta + \phi - \psi$. Finally, the following

$$U = F_2^{\dagger} \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -s_{13}s_{23}c_{12}e^{i\delta} - c_{23}s_{12} & -s_{12}s_{13}s_{23}e^{i\delta} + c_{12}c_{23} & c_{13}s_{23} \\ -s_{13}c_{23}c_{12}e^{i\delta} + s_{23}s_{12} & -s_{12}s_{13}c_{23}e^{i\delta} - s_{23}c_{12} & c_{23}c_{13} \end{pmatrix} F_2. \tag{B.6}$$

This last equation reveals that we can identify the PMNS matrix through rephasing. However, it is not possible extract unambiguously the angle of CP this expresion Eq.(B.6). This is because when we want to extract the phase from a unitary matrix unambiguously, it is not sufficient to take just one element from the input since any rephasing on the eigenvectors (a column of the unitary matrix) will still be an eigenvector. In the case of the PMNS matrix, this would be equivalent to extracting the phase of entry 13. This is why the problem of ambiguity arises in determining the CP phase, considering only the PMNS parametrization. To prove this statement, consider the following: Let $Q = diag(e^{i\omega}, e^{-i\sigma}, e^{-i\sigma})$ and $P = diag(e^{-i\omega}, e^{-i\omega}, e^{i\sigma})$ be two matrices that tranform as $U \to V = QU_{PMNS}P$:

$$V = \begin{pmatrix} U_{11} & U_{12} & U_{13}e^{i(\omega+\sigma)} \\ e^{-i(\sigma+\omega)}U_{21} & e^{-i(\sigma+\omega)}U_{22} & U_{23} \\ e^{-i(\sigma+\omega)}U_{31} & e^{-i(\sigma+\omega)}U_{32} & U_{33} \end{pmatrix}$$

This means that this transformation causes a redefinition of the phases of the element U_{e3} and the block $U_{\mu i}, U_{\tau j}$ for $i, j = 1, 2$. Note that, in general, for all the elements in the $\mu - \tau$ block, the phase change is the same when we compare this new expression with the PMNS matrix in Eq.(B.6), the phase change in U_{e3} conserves the complex nature of the $\mu - \tau$; 1-2 block. For this reason, the Dirac phase cannot be determined by considering a single element of the matrix, as it is not invariant under the transformation, but it has to be determined by invariant structures. To do this, let us consider the unitary conditions

$$\sum_k V_{ik}V^*_{jk} = 0, \qquad \sum_k |V_{ik}|^2 = 1.$$

And let us consider $i = 2, j = 3$, so

$$V_{21}V^*_{31} + V_{22}V^*_{32} + V_{23}V^*_{33} = 0 \qquad \& \qquad |V_{21}|^2 + |V_{22}|^2 + |V_{23}|^2 = 1. \tag{B.7}$$

Using the elements of the latest transformed matrix,V, we obtain the following expressions

$$V_{21}V^*_{31} = -s^2_{12}s_{23}c_{23} + s_{12}s_{13}c_{12}c^2_{23}e^{-i\delta} - s_{12}s^2_{23}s_{13}c_{12}e^{i\delta} + c^2_{12}c_{23}s^2_{13}s_{23}, \tag{B.8}$$

$$V_{22}V^*_{32} = -c^2_{12}c_{23}s_{23} - c_{12}c^2_{23}s_{12}s_{13}e^{-i\delta} + c_{12}s_{13}s_{12}s^2_{23}e^{i\delta} + s^2_{12}s^2_{13}c_{23}s_{23}, \tag{B.9}$$

$$V_{23}V^*_{33} = c^2_{13}s_{23}c_{23}. \tag{B.10}$$

From these expressions, it can be observed that we eliminate the extra phase resulting from the transformation, $e^{i(\sigma+\omega)}$, which gives us the possibility to obtain the phase δ without any ambiguity. Using the first condition in Eq.(B.7), the term containing the CP phase is nullified, making this first condition not very useful. From the second condition in Eq.(B.7), we notice that due to the requirement of squared norm, the phases are annihilated. Finally, if we subtract Eq.(B.8) from Eq.(B.9), we have:

$$\begin{aligned}
V_{21}V_{31}^* - V_{22}V_{32}^* =& -2s_{12}s_{23}^2s_{13}c_{12}e^{i\delta} + 2s_{12}s_{13}c_{12}c_{23}^2e^{-i\delta} \\
& -s_{12}^2s_{23}c_{23} + c_{12}^2c_{23}s_{23} + c_{12}^2s_{13}^2c_{23}s_{23} - s_{12}^2s_{13}^2c_{23}s_{23} \\
= -2s_{12}s_{23}^2s_{13}c_{12}e^{i\delta} &+ 2s_{12}s_{13}c_{12}c_{23}^2e^{-i\delta} + c_{23}s_{23}(c_{12}^2 - s_{12}^2) + c_{23}s_{23}s_{13}^2(c_{12}^2 - s_{12}^2) \\
=& 2s_{12}s_{13}c_{12}(c_{23}^2e^{-i\delta} - s_{23}^2e^{i\delta}) + c_{23}s_{23}\cos 2\theta_{12}(1+s_{13}^2).
\end{aligned}$$

Thus, by reorganizing we get

$$\frac{V_{21}V_{31}^* - V_{22}V_{32}^* - c_{23}s_{23}\cos 2\theta_{12}(1+s_{13}^2)}{2s_{12}s_{13}c_{12}} = c_{23}^2e^{-i\delta} - s_{23}^2e^{i\delta}. \tag{B.11}$$

Expanding the right side of the expression

$$\begin{aligned}
c_{23}^2e^{-i\delta} - s_{23}^2e^{i\delta} =& c_{23}^2(\cos\delta - i\sin\delta) - s_{23}^2(\cos\delta + i\sin\delta) \\
=& (c_{23}^2 - s_{23}^2)\cos\delta - i(c_{23}^2 + s_{23}^2)\sin\delta,
\end{aligned}$$

and substituting in Eq.(B.11), we then have

$$(c_{23}^2 - s_{23}^2)\cos\delta - i(c_{23}^2 + s_{23}^2)\sin\delta = \frac{V_{21}V_{31}^* - V_{22}V_{32}^* - c_{23}s_{23}\cos 2\theta_{12}(1+s_{13}^2)}{2s_{12}s_{13}c_{12}}.$$

Taking the imaginary part of above expression yields

$$\begin{aligned}
\sin\delta =& -\mathbf{Im}\left(\frac{V_{21}V_{31}^* - V_{22}V_{32}^* - c_{23}s_{23}\cos 2\theta_{12}(1+s_{13}^2)}{2s_{12}s_{13}c_{12}}\right) \\
=& -\frac{\mathbf{Im}\left(V_{21}V_{31}^* - V_{22}V_{32}^*\right)}{2s_{12}c_{12}s_{13}}.
\end{aligned} \tag{B.12}$$

Let us note the following

$$\begin{aligned}
|V_{12}||V_{11}| &= c_{12}s_{12}c_{13}^2 \\
\therefore c_{12}s_{12} &= \frac{|V_{12}||V_{11}|}{c_{13}^2},
\end{aligned}$$

and so

$$s_{12}c_{12}s_{13} = \frac{|V_{12}||V_{11}|}{c_{13}^2}s_{13} = \frac{|V_{12}||V_{11}|}{c_{13}^2s_{13}}s_{13}^2 = \tan^2\theta_{13}\frac{|V_{12}||V_{11}|}{s_{13}}$$

Finally, from Eq.(B.12), we have

$$\begin{aligned}\sin\delta &= -\frac{\mathbf{Im}\left(V_{21}V_{31}^{*} - V_{22}V_{32}^{*}\right)}{2|V_{12}||V_{11}|}\left(\frac{\sin\theta_{13}}{\tan^{2}\theta_{13}}\right), \\ &= -\frac{\mathbf{Im}\left(V_{21}V_{31}^{*} - V_{22}V_{32}^{*}\right)}{2\tan^{2}\theta_{13}}\left(\frac{|V_{13}|}{|V_{12}||V_{11}|}\right).\end{aligned} \tag{B.13}$$

However, we can express all of this in terms of the elements of the PMNS matrix without the need for the mixing angle θ_{13}. For this, let us observe that

$$\frac{\sin\theta_{13}}{\tan^{2}\theta_{13}} = \frac{c_{13}^{2}}{s_{13}} = \frac{c_{13}^{2}(s_{23}^{2}+c_{23}^{2})}{|V_{13}|} = \frac{c_{13}^{2}s_{23}^{2}+c_{13}^{2}c_{23}^{2}}{|V_{13}|} = \frac{|V_{23}|^{2}+|V_{33}|^{2}}{|V_{13}|}.$$

Finally, the expression that allows an unambiguous calculation of the CP phase from a given unitary mixing matrix is given by

$$\sin\delta = -\frac{\mathbf{Im}\left(V_{21}V_{31}^{*} - V_{22}V_{32}^{*}\right)}{2|V_{13}||V_{12}||V_{11}|}\left(|V_{23}|^{2}+|V_{33}|^{2}\right). \tag{B.14}$$

APPENDIX C

About the observables in the parametrization

In this appendix, I will present, for both normal and inverted hierarchy cases, the graphs of the physical observables generated from the parameters of the M_ν matrix obtained through the numerical diagonalization process illustrated in Section 4.2.3.

C.1 Normal Hierarchy

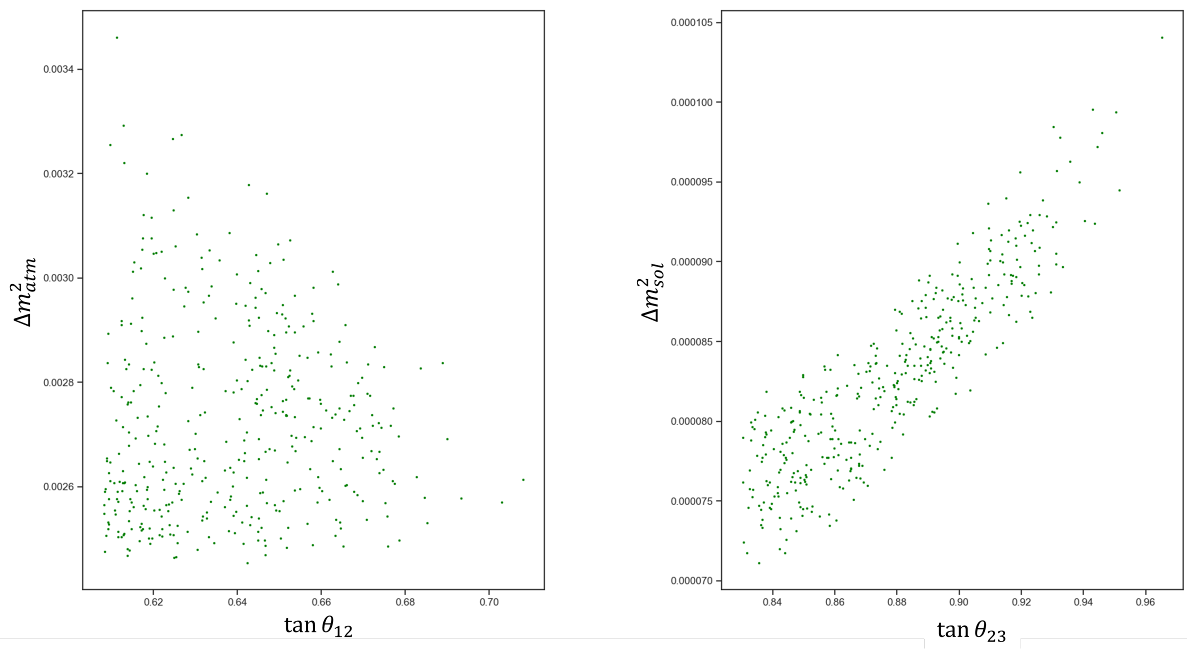

Fig. C.1 Mass scales as a function of the angles θ_{12} and θ_{23}.

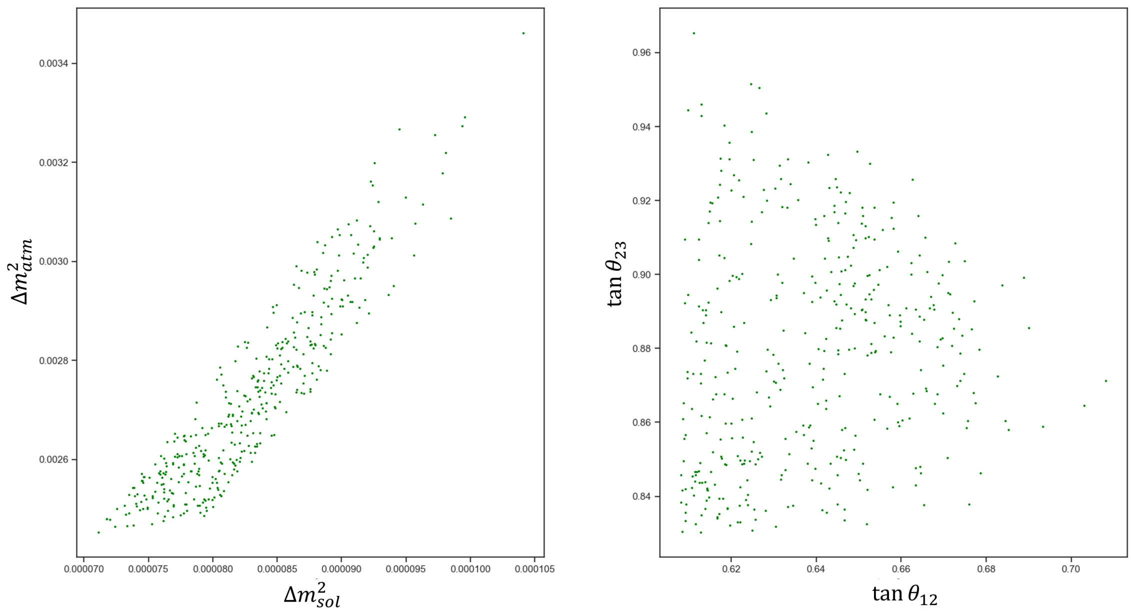

Fig. C.2 Physical observables obtained from the parameters acquired in the diagonalization process.

C.2 Inverted Hierarchy

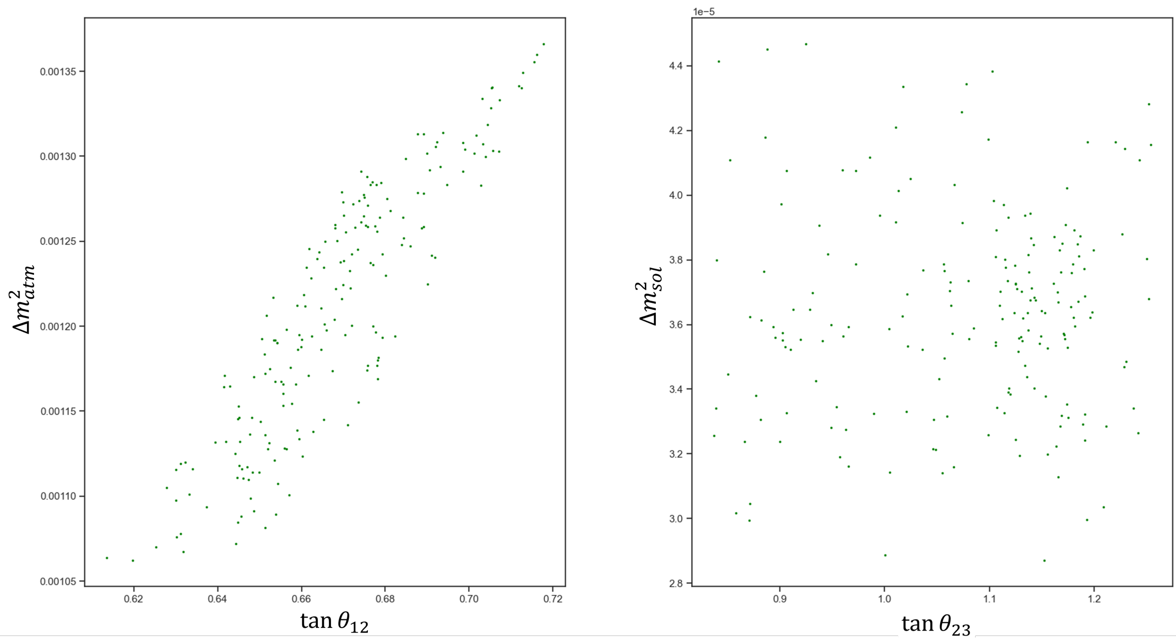

Fig. C.3 Mass scales as a function of the angles θ_{12} and θ_{23}.

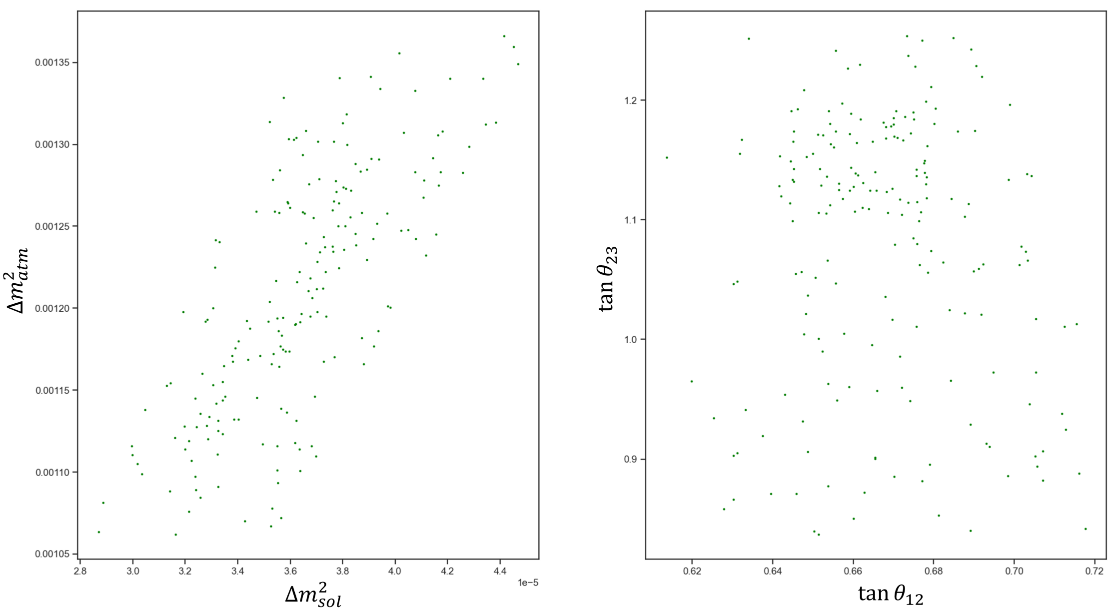

Fig. C.4 Physical observables obtained from the parameters acquired in the diagonalization process.