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Dedicatorias

A mis padres, Margarita Trujillo Salazar y Héctor Ángel Aguilera Camacho, que sin su apoyo durante toda mi vida no hubiera podido seguir un camino en la ciencia.

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Resumen

La compactificación es una característica importante de la Teoría de Cuerdas (tanto de Cuerdas Bosónicas como de Supercuerdas). Las variedades de Calabi-Yau son los candidatos más relevantes para la compactificación, dado que la teoría 4-dimensional resultante coincide, no sólo con la Relatividad General, sino con el Modelo Estándar de las Partículas. Bajo esta compactificación, se encuentra el Mecanismo Atractor, un conjunto de ecuaciones que describen la métrica 4-dimensional de un agujero negro supersimétrico y esféricamente simétrico, y el espacio modular, de dimensión $h^{2,1}$, de la respectiva variedad de Calabi-Yau en términos de la carga central del agujero negro 10-dimensional.

Los flujos sobre una Teoría de Supercuerdas Tipo IIA compactificada sobre una variedad de Calabi-Yau son equivalentes a una Teoría de Supercuerdas Tipo IIB compactifacada sobre una variedad Half-flat sin flujos. El espacio modular de una variedad Half-flat tiene dimensión $h^{2,1} + 1$, y puede ser descrita por el producto cartesiano de dos espacios modulares de ciertas variedades de Calabi-Yau, de dimensiones 1 y $h^{2,1}$, respectivamente. La dimensión extra genera una carga fraccional, conocida como pelo cuántico.

Redefiniendo el Mecanismo Atractor para variedades Half-flat, surge un nuevo conjunto de ecuaciones que describe tanto el espacio modular como el pelo cuántico. Analisando el comportamiento asintótico de las soluciones, notamos que no sólo las cargas clásicas se encuetran en el horizonte de eventos sino que también el pelo cuántico.

Abstract

Compactification is an important feature of String Theory (both Bosonic and Super). Calabi-Yau manifolds are the most relevant candidates for compactification as the resulting 4-dimensional theory agrees not only with General Relativity but also with the Standard Model of Particles. Under this compactification, it is find the Attractor Mechanism, a set of equation that describes the 4-dimensional metric of a supersymmetric spherically symmetrical black hole and the moduli space, of dimension $h^{2,1}$, of the respective Calabi-Yau manifold in terms of the central charge of the 10-dimensional black hole.

Fluxes over a Type IIA Superstring Theory compactified on a Calabi-Yau manifold are equivalent to a Type IIB Superstring Theory compactified on a Half-flat manifold without fluxes. The moduli space of a Half-flat manifold has dimension $h^{2,1}+1$, and can be described as the cartesian product of two moduli spaces of some Calabi-Yau manifolds, with dimensions 1 and $h^{2,1}$, respectively. That extra dimension generates an extra fractional charge, known as quantum hair.

Redefining the Attractor Mechanism to Half-flat manifolds, a new set of equations surges that describes both the moduli space and the quantum hair. Analyzing the asymptotic behavior of the solutions, we note that not only the classical charges are found on the event horizon but also the quantum hair.

Introduction

Since their discovery in 1916, black holes have been object of study. Starting from being a solution to the Einstein equation in a void Universe that has a singularity, they called the attention of several physicists around the World. In Chapter 1, we are doing a review of General Relativity and the most relevant black-hole solutions to the Einstein equation. For further information about the subject, you can check on textbooks such as [1], [2].

The fact that black holes generates strong gravitational fields in small scales made people think that they were the key to prove Unification theories. The strong gravitational fields made the relativistic effects impossible to neglect, and the small scales were responsible for feeling the quantum effects.

The first approach to describe the quantum effects in a black hole was made in the 1960s in papers written by Hawking, Penrose, Bekenstein, etc. about black hole thermodynamics, where they were able to compute the temperature and entropy of a black hole macroscopically. Several references and reviews on black hole thermodynamics can be found in [3], [4], [5], [6], [7], [8].

In Chapter 2, we review the basics of the Bosonic String Theory and the Superstring Theory, that currently is one of the most relevant theories in physics and is the starting point of this thesis. There are some textbooks and reviews on it that the reader may consider for further information, such as [9], [10], [11].

When String (and Superstring) Theory got considered as the most important and promising candidate for a unification theory, the study of black holes in this framework gained importance, as well as the comparison of its results with the ones obtained in classical and semiclassical approaches.

The problem with Superstring Theory is that it needs to live in 10-dimensional spacetimes. So the way to relate it to our 4-dimensional world is through compactification of six of those dimensions. The natural candidate for a manifold that compactifies these dimesions are the Calabi-Yau manifolds, that have been studied in the String theory framework in the last decades. The reason of this is that after compactifying a Type II superstring theory, we got a 4-dimensional supersymmetrical theory with $\mathcal{N} = 2$, that is precisely what it seems to be our World. Appart from the String Theory references, the reader can learn more about Calabi-Yau manifolds in [12], [13].

In Chapter 3, we study the black holes treated as Superstring Theory phenomena. Here, black holes are gotten by wrapping D3-branes on certain 3-cycles of the Calabi-Yau 3-fold. The first attempt to a black hole in String Theory was proposed by Strominger and Vafa [14]. Some current studies focus on the count of microstates in order to reproduce the results gotten in the semiclassical approach. We are not doing this in this work, but to learn more about it, the reader can check references such as [15].

Nevertheless, in order to describe more general systems, we need to include fluxes on those manifolds. The most simple way to deal with these fluxes is to change our Calabi-Yau manifold for a generalised Calabi-Yau manifold, specifically in this work we are going to study Half-flat manifolds, and how this changes the well-known Attractor Mechanism.

In Chapter 4, this Half-flat manifolds are studied and characterized. We find its Kähler potential $\mathcal{K}$, its holomorphic 3-forms Ω and the central charge of a black hole in a theory compactified in this kind of manifolds $Z(\Gamma)$, among other relevant quantities. However, the reader is may check on reviews and papers on the subject for further information on Half-flat manifolds and fluxes on Calabi-Yau manifolds, like [16], [17], [18], [19].

Finally, in chapter 5, we study the attractor equations on Half-flat manifolds, and compare this with the attractor mechanism on Calabi-Yau manifolds. For further information about attractor mechanism on regular or generalised Calabi-Yau manifolds, you may check on [20], [21], [22], [23], [24].

Contents

Abstract **ix**

Introduction **xi**

1 Black holes in General Relativity **1**

1.1 General Relativity 1

1.1.1 Comparison between General Relativity and Newtonian Gravity 2

1.1.2 Einstein-Hilbert Action 3

1.2 Schwarzschild metric 4

1.3 Reissner-Nordström metric 7

1.4 Kerr metric 10

1.4.1 Killing horizon 10

1.5 No-hair theorem 12

1.6 Black hole thermodynamics 12

1.6.1 Black hole temperature 13

2 Superstring theory **15**

2.1 Nambu-Goto action 15

2.2 Green-Schwarz action 16

2.3 Free superstring spectrum 17

2.4 Compactification over Calabi-Yau manifolds 18

2.4.1 Calabi-Yau manifolds 18

2.4.2 Field content of a Type II Superstring Theory compactified on a Calabi-Yau manifold 19
2.4.3 Special geometry 20
2.4.4 Special coordinates 22

3 Black holes in string theory 25
3.1 Strominger-Vafa black hole 25
3.2 Attractor mechanism 26
3.3 Electric and magnetic charges 28

4 Half-flat manifolds 31
4.1 Central charge 33
4.2 Fractional charges 34

5 Attractor mechanism on half-flat manifolds 37
5.1 Equation on the metric 37
5.2 Equation on the moduli space 39
5.3 Asymptotic behaviour of the quantum hair 41

Conclusion 43

A Special Relativity 47
A.1 Lorentz transformation 48
A.2 Relativistic dynamics 50
A.3 Relativistic action 51

B Geometric tools 55
B.1 Manifolds 55
B.2 Homology and cohomology 56
B.3 Covariant deraivative 58
B.4 Parallel transport and geodesics 59
B.5 Curvature 60

Chapter 1

Black holes in General Relativity

The main characters of this study are the black holes. In order to study them, we first have to know where they come from. These systems were discovered while studying the recently born General Theory of Relativity

1.1 General Relativity

Before the **General Theory of Relativity** was published, Albert Einstein studied a particular case of it, the **Special Theory of Relativity**, that is developed in the Appendix A.

Ten years took to Einstein to get to that General Theory of Relativity. In 1915, he published an article [25], in which he established his famous equation. But, where does it come from.

General Relativity is based on three principles:

1. **General covariance**: The laws of physics take the same form in every reference frame.

2. **Equivalence principle**: The laws of special relativity apply locally for all inertial observers.

3. **Spacetime curvature**: Gravitation is manifested as curvature of the spacetime, and particles that are not affected by forces (other than gravity) will travel through spacetime in geodesics.

1.1.1 Comparison between General Relativity and Newtonian Gravity

In order to be able to compare this new theory with the Newtonian Mechanics, we have to consider the case of static low gravitational field, that is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu} << 1$. Let us calculate the geodesics for this metric. In general, we have the geodesic equation B.4.3 for massive particles as:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{1.1.1}$$

Note that for small velocities, $\frac{dx^i}{d\tau} << \frac{dt}{d\tau} \approx 1$ for $i = 1, 2, 3$. Then, 1.1.1 results:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{00} = 0 \tag{1.1.2}$$

According with B.3.4

$$\begin{aligned}
\Gamma^\mu_{00} &= \frac{1}{2}g^{\mu\nu}\left(\partial_0 g_{0\nu} + \partial_0 g_{\nu 0} - \partial_\nu g_{00}\right) \\
&= -\frac{1}{2}g^{\mu\nu}\partial_\nu g_{00} \\
&= -\frac{1}{2}\left(\eta^{\mu\nu} + h^{\mu\nu}\right)\partial_\nu\left(\eta_{00} + h_{00}\right) \\
&= -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}
\end{aligned}$$

Then, the spatial components of 1.1.2 results:

$$\frac{d^2x^i}{d\tau^2} - \frac{1}{2}\eta^{i\nu}\partial_\nu h_{00} = 0$$
$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\eta^{i\nu}\partial_\nu h_{00}$$
$$\overrightarrow{a} = \frac{1}{2}\nabla h_{00}$$

Comparing with Newton's second law, $\overrightarrow{a} = -\nabla\Phi$, we get:

$$h_{00} = -2\Phi = 2\frac{GM}{r} \quad (1.1.3)$$

then, $g_{00} = -(1 + 2\Phi)$

1.1.2 Einstein-Hilbert Action

As well as Newtonian (equivalently, Lagrangian and Hamiltonian) mechanics and the Special Theory of Relativity, the General Theory of Relativity can be obtained through an action. The equation we need to get from this action is the Einstein equationB.5.5, which can be separated into a geometrical part and a dynamical part. We can do the same for the action.

The geometric part, known as **Einstein-Hilbert action**, was proposed in 1915 (published in 1924[26]) by David Hilbert, and is:

$$S_H = \frac{1}{16\pi G}\int R\sqrt{-g}d^4x \quad (1.1.4)$$

where R is the Ricci scalar (the trace of the Ricci tensor) but g is the determinant of the metric $g_{\mu\nu}$.

Note that the factor $\sqrt{-g}d^4x$ is altogether the (hyper)volume element, according to the change of variables theorem. Then, the dynamic part of the action has to have the following form:

$$S_M = \int \mathcal{L}_M\sqrt{-g}d^4x \quad (1.1.5)$$

Knowing the equation B.5.5 which we want to get, and that R and $\mathcal{L}_M$ should only depend on the metric $g_{\mu\nu}$, we are able to know the form of the stress-energy tensor in terms of the lagrangian density $\mathcal{L}_M$:

$$T_{\mu\nu} = -2\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_M \tag{1.1.6}$$

Note that B.5.5 is a differential equation, then it can be very complicated. The easiest scenario is the void ($T_{\mu\nu} = 0$), that is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{1.1.7}$$

Now, note that

$$\begin{aligned} g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) &= 0 \\ R - \frac{1}{2}\delta^{\mu}{}_{\mu}R &= 0 \\ R - 2R &= 0 \\ R &= 0 \end{aligned}$$

Therefore, any void solution has to have $R = 0$. Then, 1.1.7 is equivalent to:

$$R_{\mu\nu} = 0 \tag{1.1.8}$$

1.2 Schwarzschild metric

In order to simplify it even more, let us suppose that the metric is spherically symmetric and static. Under these assumptions, we can reparametrize any metric into

$$g_{\mu\nu} = \begin{pmatrix} -f(r) & 0 & 0 & 0 \\ 0 & g(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{1.2.1}$$

Using a program we can calculate the Riemann tensor of this metric. The non-zero components are

$$\begin{aligned} R_{00} &= \frac{f'(r)}{rh(r)} - \frac{f'^2(r)}{4f(r)h(r)} - \frac{f'(r)h'(r)}{4h^2(r)} + \frac{f''(r)}{2h(r)} \\ R_{11} &= \frac{f'^2(r)}{4f^2(r)} + \frac{h'(r)}{rh(r)} + \frac{f'(r)h'(r)}{4f(r)h(r)} - \frac{f''(r)}{2f(r)} \\ R_{22} &= 1 - \frac{1}{h(r)} - \frac{rf'(r)}{2f(r)h(r)} + \frac{rh'(r)}{2h^2(r)} \\ R_{33} &= \sin^2\theta \left(1 - \frac{1}{h(r)} - \frac{rf'(r)}{2f(r)h(r)} + \frac{rh'(r)}{2h^2(r)}\right) = \sin^2\theta R_{22} \end{aligned}$$

Then, we got three independent equations:

$$\frac{f'(r)}{rh(r)} - \frac{f'^2(r)}{4f(r)h(r)} - \frac{f'(r)h'(r)}{4h^2(r)} + \frac{f''(r)}{2h(r)} = 0 \tag{1.2.2}$$

$$\frac{f'^2(r)}{4f^2(r)} + \frac{h'(r)}{rh(r)} + \frac{f'(r)h'(r)}{4f(r)h(r)} - \frac{f''(r)}{2f(r)} = 0 \tag{1.2.3}$$

$$1 - \frac{1}{h(r)} - \frac{rf'(r)}{2f(r)h(r)} + \frac{rh'(r)}{2h^2(r)} = 0 \tag{1.2.4}$$

By adding $h(r) \times 1.2.2 + f(r) \times 1.2.3$, we get:

$$\frac{f'(r)}{r} - \frac{f'^2(r)}{4f(r)} - \frac{f'(r)h'(r)}{4h(r)} + \frac{f''(r)}{2} + \frac{f'^2(r)}{4f(r)} + \frac{h'(r)f(r)}{rh(r)} + \frac{f'(r)h'(r)}{4h(r)} - \frac{f''(r)}{2} = 0$$
$$\frac{f'(r)}{r} + \frac{h'(r)f(r)}{rh(r)} = 0$$

with solution $f(r)h(r) = 1$. Substituting $h(r) = \frac{1}{f(r)}$ in 1.2.2, we get:

$$\frac{f'(r)f(r)}{r} - \frac{f'^2(r)}{4} + \frac{f'^2(r)}{4} + \frac{f''(r)f(r)}{2} = 0$$
$$rf''(r) + 2f'(r) = 0$$

with solution $f(r) = \frac{c_1}{r} + c_2$. Same for 1.2.3 and 1.2.4.

Far away from the center, the metric should tend to the metric given by 1.1.3. So,

$$f(r) = 1 - \frac{2GM}{r} \tag{1.2.5}$$

Then, we get the same metric as Schwarzschild did in 1916[27]:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.2.6}$$

Note that this metric has two singularities, one at $r = 0$, and the other at $r = 2GM$ (known as the **Schwarzschild radius**). The sphere defined by the Schwarzschild radius is called the **event horizon**, and it is a removable singularity. Through a change of coordinates, one can find an equivalent metric that is well defined at this event horizon. Nevertheless, we cannot get rid of the singularity at the origin. We can see this through the scalar:

$$R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} = \frac{48G^2M^2}{r^6} \tag{1.2.7}$$

that diverges as $r \to 0$.

This is the simplest case of a black hole. Nevertheless, we study some phenomena that occurs in this metrics. For example, any particle (massive or massless) that falls inside the event horizon cannot escape.

1.3 Reissner-Nordström metric

In the case of a charged black hole, we can do the same geometric assumptions1.2.1, but a (electric or magnetic) charge generates electromagnetic fields. Then the stress-energy tensor is not zero. It has the following form:

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{1.3.1}$$

And assuming that the charged black hole is not moving, it only produces electric and magnetic fields in the radial direction, $F_{01} = -F_{10} = -\frac{Q}{r^2}$, $F_{23} = -F_{32} = P\sin\theta$, where Q and P are the electric and magnetic charges, respectively. Then,

$$\begin{aligned}
F_0{}^1 &= g^{11}F_{01} = (g(r))^{-1}\left(-\frac{Q}{r^2}\right) = -\frac{Q}{r^2 g(r)}, F_1{}^0 = -\frac{Q}{r^2 f(r)}, \\
F_2{}^3 &= \frac{P}{r^2\sin\theta}, F_3{}^2 = -\frac{P\sin\theta}{r^2}, F^{01} = \frac{Q}{r^2 f(r)g(r)}, \\
F^{10} &= -\frac{Q}{r^2 f(r)g(r)}, F^{23} = \frac{P}{r^4\sin\theta}, F^{32} = -\frac{P}{r^4\sin\theta} \\
F_{01}F_0{}^1 &= \frac{Q^2}{r^4 g(r)}, F_{10}F_1{}^0 = -\frac{Q^2}{r^4 f(r)}, F_{23}F_2{}^3 = \frac{P^2}{r^2} \\
F_{32}F_3{}^2 &= \frac{P^2\sin^2\theta}{r^2} \\
F_{\rho\sigma}F^{\rho\sigma} &= -2\frac{Q^2}{r^4 f(r)g(r)} + 2\frac{P^2}{r^4} = \frac{2}{r^4}\left(P^2 - \frac{Q^2}{f(r)g(r)}\right)
\end{aligned}$$

The stress-energy tensor is given by

$$
\begin{aligned}
T_{00} &= \frac{Q^2}{r^4 g(r)} + \frac{1}{4} f(r) \frac{2}{r^4}\left(P^2 - \frac{Q^2}{f(r)g(r)}\right) = \frac{1}{2r^4}\left(P^2 f(r) + \frac{Q^2}{g(r)}\right) \\
T_{11} &= -\frac{Q^2}{r^4 f(r)} - \frac{1}{4} g(r) \frac{2}{r^4}\left(P^2 - \frac{Q^2}{f(r)g(r)}\right) = -\frac{1}{2r^4}\left(P^2 g(r) + \frac{Q^2}{f(r)}\right) \\
T_{22} &= \frac{P^2}{r^2} - \frac{1}{4} r^2 \frac{2}{r^4}\left(P^2 - \frac{Q^2}{f(r)g(r)}\right) = \frac{1}{2r^2}\left(P^2 + \frac{Q^2}{f(r)g(r)}\right) \\
T_{33} &= \frac{P^2 \sin^2\theta}{r^2} - \frac{1}{4} r^2 \sin^2\theta \frac{2}{r^4}\left(P^2 - \frac{Q^2}{f(r)g(r)}\right) = \frac{\sin^2\theta}{2r^2}\left(P^2 + \frac{Q^2}{f(r)g(r)}\right)
\end{aligned}
$$

All other components are zero.

Then, we got three independent equations, similar to 1.2.2-1.2.4, but equal to T_{00}, T_{11}, T_{22} instead of zero. These new equations have solution: $f(r) = 1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2}, g(r) = \left(1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2}\right)^{-1}$. This gives the metric gotten independently by Hans Reissner[28] (1916), Hermann Weyl [29] (1917), Gunnar Nordström[30] (1918) and George Barker Jeffery[31] (1921).

Note that, besides the origin, there are two event horizon-like singularities, at $r_\pm = GM \pm \sqrt{G^2M^2 - G(Q^2+P^2)}$, that is precisely $r_s = 2GM$ when $Q, P = 0$, but if $Q^2 + P^2 > M$ there is no event horizon. We then have three cases:

1. **Case** $Q^2 + P^2 > M^2$**:** As mentioned above, there is no event horizon. Then, the physical singularity at the origin is called a **naked singularity**. Therefore, any particle is able to approach the singularity and return to their initial position.

2. **Case** $Q^2 + P^2 < M^2$**:** Here, we got two event horizon r_+, r_-. In the case of the Schwarzschild metric, r_s divides the spacetime into two regions: Outside r is spacelike, but inside r is timelike. Something similar happens for the Reissner-Nordström metric, r_+, r_- divide the spacetime into three regions. Outside of r_+ and inside of r_-, r is spacelike, but between them, r is timelike. So, if a particle comes from outside the black hole and passes through $r = r_+$ will fall to the singularity, at least until it crosses $r = r_-$. After that, the

particle will move freely and will be able to pass the horizon r_- again and "fall" outside the horizon r_+.

3. **Case $Q^2 + P^2 = M^2$:** This **extremal** case gives the metric:

$$ds^2 = -\left(1 - \frac{GM}{r}\right)^2 dt^2 + \left(1 - \frac{GM}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \qquad (1.3.2)$$

with one only event horizon at $r_0 = GM$. Nevertheless, this is not the same metric as Schwarzschild. Here, t is always timelike, and r is always spacelike. Then, in or out the event horizon any particle is free to move.

Note that for an extremal Reissner-Nordström black hole, the metric near the horizon takes the form:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{r_0}{r_0 + \varepsilon}\right)^2 dt^2 + \left(1 - \frac{r_0}{r_0 + \varepsilon}\right)^{-2} d\varepsilon^2 + r_0^2 d\Omega^2 \\ &= -\left(\frac{\varepsilon}{r_0 + \varepsilon}\right)^2 dt^2 + \left(\frac{\varepsilon}{r_0 + \varepsilon}\right)^{-2} d\varepsilon^2 + r_0^2 d\Omega^2 \\ &= -\left(1 + \frac{r_0}{\varepsilon}\right)^{-2} dt^2 + \left(1 + \frac{r_0}{\varepsilon}\right)^2 d\varepsilon^2 + r_0^2 d\Omega^2 \\ &\approx -\left(\frac{r_0}{\varepsilon}\right)^{-2} dt^2 + \left(\frac{r_0}{\varepsilon}\right)^2 d\varepsilon^2 + r_0^2 d\Omega^2 \end{aligned}$$

Making the change of variable $\varepsilon = \frac{{r_0}^2}{R}, d\varepsilon = -\left(\frac{r_0}{R}\right)^2 dR$, we get:

$$\begin{aligned} ds^2 &= -\left(\frac{R}{r_0}\right)^{-2} dt^2 + \left(\frac{R}{r_0}\right)^2 \left(\frac{r_0}{R}\right)^4 dR^2 + r_0^2 d\Omega^2 \\ &= -\left(\frac{r_0}{R}\right)^2 dt^2 + \left(\frac{r_0}{R}\right)^2 dR^2 + r_0^2 d\Omega^2 \\ &= \left(\frac{r_0}{R}\right)^2 (-dt^2 + dr^2) + r_0^2 d\Omega^2 \end{aligned}$$

that is the metric of an spacetime $AdS_2 \times S^2$.

1.4 Kerr metric

We have studied the cases of a static metric in void (Schwarzschild), and in the presence of an electromagnetic field (Reissner-Nordström). But, there is one more scenario to consider, a rotating black hole. This results to be much more complex than the other ones. It was almost fifty years after Einstein paper about General Relativity (1963) that Roy Kerr got his metric[32]:

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \\ &+ \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2 \end{aligned} \tag{1.4.1}$$

with $\Delta = r^2 - 2GMr + a^2, \rho^2 = r^2 + a^2\cos^2\theta$ and $a = \frac{J}{M}$.

The structure of the Kerr black hole is quite different to the structure of the Schwarzschild and Reissner-Nordström black holes. This metric has also a double event horizon at $r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$

1.4.1 Killing horizon

Before we continue the analysis of the structure of the Kerr black hole, we need to introduce some concepts. First, having a vector $V = \partial_x^1$ for certain frame, it is defined the **Lie derivative** $\mathcal{L}_V$ in that frame as:

$$\mathcal{L}_V u^\mu = \partial_x^1 u^\mu \tag{1.4.2}$$

Note that in this frame ($V^\mu = (1, 0, 0, 0)$):

$$[V, u] = V^\mu \partial_\mu u - u^\mu \partial_\mu V$$
$$[V, u]^\mu = \partial_\mu u^\mu = \mathcal{L}_V u^m$$

Then,

$$\mathcal{L}_V u^\mu = [V, u]^\mu \tag{1.4.3}$$

For other type of tensors, this Lie derivative is defined in a similar way than the covariant derivativeB.3.3. Then, if a vector ξ satisfies:

$$\mathcal{L}_\xi g_{\alpha\beta} \tag{1.4.4}$$

is called a **Killing vector**. Note that eq. 1.4.4 is equivalent to:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

Note that $K^\alpha = \partial_t$ is a Killing vector in Schwarzschild, Reissner-Nordström and Kerr. Let us define the **Killing horizon** as the hypersurface where the Killing vectors are null. In the case of Schwarzschild and Reissner-Nordström, this coincides with the event horizon. But this is different for the Kerr black hole. In this case, the norm of the Killing vector is given by:

$$K_\mu K^\mu = -\frac{1}{\rho^2}(\Delta - a^2 \sin^2 \theta) \tag{1.4.5}$$

that, if is equal to zero, defines a hyperellipsoid, that intersects the outer event horizon ($\Delta = 0$) at the poles ($\theta = 0, \pi$). The region between these two surfaces is called **ergosphere**. Inside this ergosphere, a particle is able to move in any radial direction (toward or away from the singularity), but is forced to rotate in the same direction as the black hole. What happens between the two event horizons is similar to Reissner-Nordström.

1.5 No-hair theorem

This three cases look very specific, but in 1967 Werner Israel showed that Schwarzschild metric 1.2.6 was the only static void black-hole solution [33]. One year after that, the same Werner Israel showed that the only static charged black-hole solution was the Reissner-Nordström metric [34]. Then, in 1971, Brandon carter demonstrated that the Kerr metric 1.4.1 is the only axisymmetric black-hole solution [35].

These three theorems are known as **No-hair theorems**, as a reference to the fact that there are no other variables that an external observer can see, *black holes have no hair*. A modern version of this theorem can be found in the book Gravitation[36] and states that Schwarzschild, Reissner-Nordström and Kerr (and a combination of these last two, known as Kerr-Newman) were the only stationary black-hole solution.

Then a stationary black hole is fully characterized by four charges: mass M, electric charge Q, magnetic charge P and angular momentum J. As we will see later, this is only valid in a classical theory.

1.6 Black hole thermodynamics

Several papers were published in the 1970s about black holes, specially by Jacob Bekenstein [3]-[4] and Stephen Hawking [5]-[6] that got to two formulas that relate some characteristics of black holes to thermodynamic functions:

$$kT = \frac{\hbar\kappa}{2\pi} \tag{1.6.1}$$

$$S = \frac{A}{4\hbar G} \tag{1.6.2}$$

where we can see classical black holes quantities, like surface gravity κ, the area of the event horizon A, the surface gravity κ or just the Newton's constant G, thermodynamic functions, like temperature T, entropy S and the Boltzmann's constant k, but also the Planck's constant $\hbar$. For further information about black-hole thermodynamics, the reader may check [37], [38], [39], [40], [41].

1.6.1 Black hole temperature

One way to get 1.6.1 is through the so called **Euclidean continuation**. First, recall the partition function Z in quantum mechanics is given by:

$$Z = \mathrm{Tr} e^{-\beta \hat{H}} \tag{1.6.3}$$

with $\beta = \frac{1}{kT}$ the inverse temperature and $\hat{H}$ the Hamiltonian operator. Comparing with the evolution operator $e^{-it\hat{H}/\hbar}$, we get

$$Z = \mathrm{Tr} e^{-it\hat{H}/\hbar} = \mathrm{Tr} e^{\tau \hat{H}/\hbar} \tag{1.6.4}$$

with $it/\hbar = \beta$ and $t = i\tau$. Here τ is called the **Euclidean time**, and $\tau = \beta\hbar$.

On the other hand, the Schwarzschild metric 1.2.6 is equivalent for radial displacements and near the horizon to:

$$ds^2 = -\rho^2 \kappa^2 dt^2 + d\rho^2 = \rho^2 \kappa^2 d\tau^2 + d\rho^2 \tag{1.6.5}$$

Note that, for $\theta = \kappa\tau$, we get the metric of a flat 2-space in polar coordinates. In this case, θ has period 2π. Then, the Euclidean time has period $2\pi/\kappa$. And

$$kT = \frac{1}{\beta} = \frac{\hbar}{\tau} = \frac{\hbar\kappa}{2\pi} \tag{1.6.6}$$

Chapter 2

Superstring theory

Although General Relativity works for most phenomena that are observed, it is not compatible with the other great revolutionary theory of the XX Century, the quantum mechanics. It is not possible to quantize the General Theory of Relativity. So we need a new theory of gravity that can be quantized and in the low energy limit is General Relativity.

2.1 Nambu-Goto action

A first approach is known as the **Bosonic String Theory**, that is described with the **Nambu-Goto action**:

$$S_{NG} = -\frac{1}{\pi}\int d\sigma^2 \sqrt{-det(\partial_\alpha X^\mu \partial_\beta X_\mu)} \tag{2.1.1}$$

Note that this action describe the area of a surface with metric $\partial_\alpha X^\mu \partial_\beta X_\mu$. This surface is the **worldsheet**, that is the analog to the worldline that describes the path of particle, of a string.

This worldsheet is parametrized naturally by the proper time τ, as the worldline, and an intrinsic parameter of the string σ, $\sigma^\alpha = (\tau, \sigma)$. The topology of this space can be a plane (if the string is open) or a cylinder (if the string is closed). Nevertheless, this space is not the physical one, the string lives in a (D-dimensional) spacetime. So, in order to describe the string in this spacetime, we need to embed the parameters σ^α to this, $X^\mu(\sigma^\alpha)$.

In the case of (Special/General) Relativity, we worked on a ($D = 4$)-dimensional spacetime, but in the case of String Theory, we need more dimensions so it to be compatible. In the case of the Bosonic String Theory, it goes up to $D = 26$, but for the Superstring Theory, that is going to be studied later, it is $D = 10$.

2.2 Green-Schwarz action

Although the Bosonic String Theory can be quantized, it fails to describe fermions, that are fundamental for Quantum Mechanics. Therefore, it surges the need to modify the action of Nambu-Goto, and then the Bosonic String Theory.

A first modification that can be made is on the metric of the string spacetime:

$$\begin{aligned} G_{\alpha\beta} &= \Pi_\alpha \cdot \Pi_\beta \\ \Pi_\alpha^\mu &= \partial_\alpha X^\mu - \bar{\Theta}^A \Gamma^\mu \partial_\alpha \Theta^A \end{aligned}$$

where Θ^A are the **Weyl-Majorana spinors** and $A = 1, \cdots, \mathcal{N}$, with $\mathcal{N}$ the number of supersymmetries of the theory. The standard model, that has proven its value during the second half of the XX Century, has a $\mathcal{N} = 2$ supersymmetry, so we want our theory to have it too.

In order to have a κ-symmetry in this theory, we need to add another term to the action, let us say:

$$\begin{aligned} S &= S_1 + S_2 \\ &= -\frac{1}{\pi} \int d\sigma^2 \sqrt{-G} + \int \Omega_2 \end{aligned}$$

where Ω_2 is an exact two-form with

$$\Omega_2 = c(\bar{\Theta}^1\Gamma_\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2)dX^\mu - c\bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\Theta}^2\Gamma^\mu d\Theta^2$$

This is known as the **Green-Schwarz action**, and is the action of the so-called **Type II Superstring Theory**. Depending on the chirality of the spinors there are two Type II theories:

1. Type IIA: $\Gamma_{11}\Theta^A = (-1)^{A+1}\Theta^A$

2. Type IIB: $\Gamma_{11}\Theta^A = \Theta^A$

2.3 Free superstring spectrum

For an open string, the spectrum consists of a massless vector 8_V and a massless spinor 8_C. Note that the only superstring theory that has open strings is Type I. So, we need to see the spectrum of the closed string in order to study the Type II (A or B).

In a open string we have a single vibrating mode, it is equivalent if the wave propagates to the left or to the right, but in a closed string we have to consider the left-movers and the right-movers as different modes. Recall that Type IIA superstring has opposite chiralities in its modes, then we have to do the tensorial product of two supermultiplets with opposite chirality:

$$(8_V + 8_C) \otimes (8_V + 8_S) = 8_V \otimes 8_V \oplus 8_V \otimes 8_S \oplus 8_C \otimes 8_V \oplus 8_C \otimes 8_S \tag{2.3.1}$$

Note that the first and last terms corresponds to bosonic fields, with $8_V \otimes 8_V = 1 + 28 + 35$, i.e., a scalar (dilaton), an antisymmetric rank-two tensor and a symmetric traceless tensor (graviton), and $8_S \otimes 8_C = 8_V + 56_t$. While the middle terms are the corresponding fermionic superpartners.

In contrast, the Type IIB superstring modes have same chirality:

$$(8_V + 8_C) \otimes (8_V + 8_C) = 8_V \otimes 8_V \oplus 8_V \otimes 8_C \oplus 8_C \otimes 8_V \oplus 8_C \otimes 8_C \qquad (2.3.2)$$

with similar decomposition as Type IIA. We can simply denote:

$$8_V \otimes 8_V = \phi \oplus B_{\mu\nu} \oplus G_{\mu\nu} \qquad (2.3.3)$$

Note that 2.3.1 and 2.3.2 have the form:

$$(NS - NS) \oplus (NS - R) \oplus (R - NS) \oplus (R - R) \qquad (2.3.4)$$

where the different terms define the different field sector. NS stands for *Neveu-Schwarz* and R stands for *Ramond*.

$NS - NS$ and $R - R$ represents the bosonic fields, and $NS - R$ and $R - NS$ represents the fermionic fields.

2.4 Compactification over Calabi-Yau manifolds

Either the Bosonic or the Superstring theory live in a spacetime with more dimensions than ours. For the Bosonic String Theory $D = 26$, and for the Type II Superstring theories $D = 10$. It then surges a question, what can we do about the extra dimensions? The most logic solution to this is compactification.

To illustrate the compactification, let us see an example. When we look close at a wire it is cyllindrical 2-dimensional surface, but the wire is getting away from us, it will start looking as a 1-dimensional line. Maybe that is what is happening, maybe we live in a 10 dimensional spacetime, but it has six dimensions extremely small that we cannot see.

2.4.1 Calabi-Yau manifolds

But, how are those extra dimensions? One promising candidate is the so called **Calabi-Yau manifolds**, that are Kähler manifolds of n complex dimensions with

$SU(n)$ holonomy. Note that 3 complex dimensions are equivalent to 6 real dimensions, that are precisely the ones that we need to compactify the 10 dimensions of the Type IIB Superstring theory into our 4 dimensions. So that, our spacetime is:

$$X_{10} = X_4 \times CY_3 \tag{2.4.1}$$

One of the reasons of choosing a Calabi-Yau manifold is that when we compactify a Type IIB Superstring Theory with it, we get a $\mathcal{N} = 2$ supersymmetric theory in 4 dimensions. One property that Calabi-Yau manifolds have is that there is a non-vanishing holomorphic n-form Ω.

There are several topological characteristics that we can use to describe a manifold M. One of them is the **Betti number** b_p that is the dimension of the pth de Rham cohomology of M, $H^p(M)$, defined by B.2.6. For Kähler manifolds, we also have the **Hodge numbers** $h^{p,q}$, that are the number of harmonic (p, q)-forms on M, and they relate with the Betti number through:

$$b_p = \sum_{q=0}^{p} h^{q,p-q} \tag{2.4.2}$$

By some relations that satisfy the Hodge numbers, we are able to compute the **Euler characteristic** of a Calabi-Yau 3-fold CY_3 as:

$$\chi(CY_3) = 2(h^{1,1} - h^{2,1}) \tag{2.4.3}$$

that is an important topological invariant. Informally, this Euler characteristic counts the number of holes of CY_3.

2.4.2 Field content of a Type II Superstring Theory compactified on a Calabi-Yau manifold

As seen in section 2.3, the NS-NS sector can be seen as:

$$\phi \oplus B_{\mu\nu} \oplus G_{\mu\nu} \tag{2.4.4}$$

a dilaton, an antisymmetric 2-form and a graviton. In four dimensions, the graviton acts as gravitational multiplet, and the antisymmetric 2-form and the dilaton act as tensor multiplets

However, the R-R sector cosists of gauge fields that depends on what Type II theory, we are working on. For Type IIA, they are 1 and 3-forms, and for Type IIB, they are 2 and 4-forms:

$$A_1 \oplus A_3, \quad \text{for Type IIA} \tag{2.4.5}$$

$$A_2 \oplus A_4, \quad \text{for Type IIB} \tag{2.4.6}$$

In Type IIA, the A_1 acts on the four-dimensional spacetime as another gravitational multiplet, while A_3 acts as a linear combination of hypermultiplets and vector multiplets. In Type IIB, A_2 acts as a linear combination of tensor multiplets, and A_4 acts as a linear combination of hypermultiplets and vector multiplets, just like A_3 in Type IIA.

Another difference between Type IIA and Type IIB Superstring Theory is that Type IIA has $h^{2,1}+1$ hypermultiplets and $h^{1,1}$ vector multiplets, while Type IIB has $h^{1,1}+1$ hypermultiplets and $h^{2,1}$ vector multiplets. This tells us that there may be a relationship between Type IIA hypermultiplets and Type IIB vector multiplets, and viceversa. This will be crucial when we get to Chapter 4.

For more information about this topic, the reader is refered to [13].

2.4.3 Special geometry

Although the Hodge numbers are topological invariants, they are not enough to totally describe the Calabi-Yau manifolds. You can define an equivalence relation using the Hodge numbers, and move inside the equivalence classes with deformations of the parameters that characterize the size and shape of the manifold, this is called **moduli**. The moduli space is often called a **Calabi-Yau space**.

Let us study the deformations in the metric of the Calabi-Yau manifold g_{mn}. These deformations are given by smooth fluctuations:

$$g_{mn} \longrightarrow g_{mn} + \delta g_{mn} \tag{2.4.7}$$

This new manifold (the one that has the new metric), in order to be Calabi-Yau, must have vanishing first Chern class, so that,

$$R = R_{mn}(g + \delta g)^{mn} = 0 \tag{2.4.8}$$

This gives a differential equation for δg. The space of solutions define the moduli space. On this moduli space there is metric naturally defined by[9] the deformations of the complex structure and the deformations of the Kähler form:

$$ds^2 = \frac{1}{2V}\int g^{a\bar{b}}g^{c\bar{d}}[\delta g_{ac}\delta_{\bar{b}\bar{d}} + (\delta g_{a\bar{b}}\delta g_{c\bar{d}} - \delta B_{a\bar{d}}\delta B_{c\bar{b}})]\sqrt{g}d^6x \tag{2.4.9}$$

with V, g the volume and metric of the CY_3, and B is the NS-NS 2-form. We can see this moduli space locally as the product of two spaces $\mathcal{M}(M) = \mathcal{M}^{2,1}(M) \times \mathcal{M}^{1,1}(M)$.

The first part (deformations of the complex structure) can be described by the base of (2,1)-forms

$$\chi_\alpha = \frac{1}{2}(\chi_\alpha)_{ab\bar{c}}dx^a \wedge dx^b \wedge dx^{\bar{c}} \tag{2.4.10}$$

$$(\chi_\alpha)_{ab\bar{c}} = -\frac{1}{2}\Omega_{ab}{}^{\bar{d}}\frac{\partial g_{\bar{c}\bar{d}}}{\partial t^\alpha} \tag{2.4.11}$$

where $t^\alpha, \alpha = 1,\cdots, h^{2,1}$ are the coordinates of the complex-structure moduli space. Knowing this, we can find the relation:

$$\delta g_{\bar{a}\bar{b}} = -\frac{1}{\|\Omega\|^2}\bar{\Omega}_{\bar{a}}^{cd}(\chi_\alpha)_{cd\bar{b}}\delta t^\alpha \tag{2.4.12}$$

where $\|\Omega\|^2 = \frac{1}{6}\Omega_{abc}\bar{\Omega}^{abc}$, and we raise and low indices with the metric g_{ab} and its inverse g^{ab}.

In order to see the components of the metric, we need to find an expression of the form:

$$ds^2 = 2G_{\alpha\bar{\beta}}\delta t^{\alpha}\delta\bar{t}^{\bar{\beta}} \tag{2.4.13}$$

And integrating 2.4.9 we get:

$$G_{\alpha\bar{\beta}}\delta t^{\alpha}\delta\bar{t}^{\bar{\beta}} = -\left(\frac{i\int \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}}}{i\int \Omega \wedge \bar{\Omega}}\right)\delta t^{\alpha}\delta\bar{t}^{\bar{\beta}} \tag{2.4.14}$$

We then define the **Kähler potential** $\mathcal{K}$ so that $G_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}}\mathcal{K}$, and we find:

$$\mathcal{K} = -\ln i \int \Omega \wedge \bar{\Omega} \tag{2.4.15}$$

2.4.4 Special coordinates

Over the Calabi-Yau 3-fold CY_3, we can define a basis for the 3-cycles (A^I, B_J), where $I, J = 0, \cdots, h^{2,1}$, chosen so that their intersection numbers are:

$$A^I \cap B_J = -B_J \cap A^I = \delta^I_J \quad A^I \cap A^J = B_I \cap B_J = 0 \tag{2.4.16}$$

and the dual cohomology basis (α_I, β^J) are chosen so that:

$$\int_{A^J} \alpha_I = \int_{CY_3} \alpha_I \wedge \beta^J = \delta^J_I, \quad \int_{B_J} \beta^I = \int_{CY_3} \beta^I \wedge \alpha_J = -\delta^I_J \tag{2.4.17}$$

Using these coordinates, we define a new ones, called the **holomorphic symplectic basis** (X^I, F_J) as:

$$X^I = \int_{A^I} \Omega, \quad F_J = \int_{B_J} \Omega \tag{2.4.18}$$

We then find that:

$$\Omega = X^I \alpha_I - F_I \beta^I \tag{2.4.19}$$

Chapter 3

Black holes in string theory

Probing the value of the String Theory has been quite difficult. It is a theory whose effects are clear at very large energy levels, that no lab on Earth can reach. This is why we need an indirect approach to see if it is correct.

String Theory, in a modern vision, is thought as a candidate for a theory that unifies the General Relativity (gravity) and the Quantum Mechanics (low-scale space), so the most naturally is to study the black holes.

3.1 Strominger-Vafa black hole

The first attempt to do this was made by Andrew Strominger and Cumrun Vafa in 1996[14]. Starting from Type II string theory compactified on $K3 \times S^1$. Note that this manifold has 5 real dimensions, then we get an extremal Reissner-Nordström metric in five dimensions:

$$ds^2 = -\left(1 - \left(\frac{r_0}{r}\right)^2\right)^2 dt^2 + \left(1 - \left(\frac{r_0}{r}\right)^2\right)^{-2} dr^2 + r^2 d{\Omega_3}^2 \tag{3.1.1}$$

with one particular change from the extremal Reissner-Nordström metric in four dimensions 1.3.2, $r \longrightarrow r^2$. Here, $r_0 = \left(\frac{8Q_H {Q_F}^2}{\pi^2}\right)^{1/6}$, $r^2 = (x^1)^2 + \cdots + (x^4)^2$ and $d{\Omega_3}^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2$. The charges Q_H, Q_F are defined by:

$$Q_H = \frac{1}{4\pi^2} \int *e^{-4\phi/3} \tilde{H} d{\Omega_3}^2 \tag{3.1.2}$$

$$Q_F = \frac{1}{16\pi} \int *e^{2\phi/3} F d{\Omega_3}^2 \tag{3.1.3}$$

with ϕ is the dilaton, F is a Ramond-Ramond 2-form field strength, commonly related to the Faraday tensor $F_{\mu\nu}$, and therefore, with the electromagnetic fields, and $\tilde{H}$ is a Neveu Schwarz-Neveu Schwarz 2-form axion field strength.

Stromiger and Vafa found that near the horizon, the geometry of the five-dimensional spacetime is $AdS \times S^3$ with a constant dilaton $\phi = \phi_h$, given by

$$e^{2\phi_h} = \frac{1}{2}\left(\frac{4Q_F}{\pi Q_H}\right)^2 \tag{3.1.4}$$

From 3.1.1, we can see the spherical symmetry, and then the Bekenstein-Hawking entropy is given by:

$$S_{BH} = \frac{2\pi^2 r_0^4}{4G_5\hbar} = \frac{2\pi}{G_5\hbar}\sqrt{\frac{Q_H {Q_F}^2}{2}} \tag{3.1.5}$$

In that paper, they counted the microstates of such a black hole, and through the Boltzmann's equation $S_{stat} = \ln W$, with W the number of microstates, computed the statistical entropy, for large charges:

$$S_{stat} \approx \frac{2\pi}{G_5\hbar}\sqrt{Q_H\left(\frac{1}{2}{Q_F}^2 + 1\right)} \approx \frac{2\pi}{G_5\hbar}\sqrt{\frac{Q_H {Q_F}^2}{2}} \tag{3.1.6}$$

that agrees with 3.1.5, precisely for large charges.

3.2 Attractor mechanism

In string theory, a black hole is obtained by wrapping D3-branes on some special Lagrangian 3-cycle $\mathcal{C} \subset CY_3$. Let Γ be the Poincaré-dual 3-form. In order to describe the electric and magnetic charges as $h^{2,1}U(1)$ gauge fields that come from the self-dual 5-form F_5, and the graviphoton from the $\mathcal{N}=2$ supergravity multiplet, let $(A^I, B_J), I, J = 1,\cdots, h^{2,1}+1$ be the basis of the 3-cycles, same as in 2.4.16, and (α_I, β^J) the basis of the 3-forms, as in 2.4.17. For the holomorphic symplectic basis, let us do a rescalation:

$$X^I = e^{\mathcal{K}/2}\int_{A^I}\Omega, \quad F_J = e^{\mathcal{K}/2}\int_{B_J}\Omega \tag{3.2.1}$$

Using this basis, we can get the electric and magnetic charges q_I, p^J as:

$$\mathcal{C} = p^J B_J - q_I A^I \tag{3.2.2}$$
$$\Gamma = p^I \alpha_I - q_J \beta^J \tag{3.2.3}$$

Then, we get:

$$p^I = \int_{A^I}\Gamma = \int_{CY_3}\Gamma\wedge\beta^I, \quad q_J = \int_{B_J}\Gamma = \int_{CY_3}\Gamma\wedge\alpha_J \tag{3.2.4}$$

And, therefore, the central charge, that depends on the choice of Γ, is:

$$Z(\Gamma) = e^{\mathcal{K}/2}\left(\int_{A^I}\Gamma\int_{B_I}\Omega - \int_{B_J}\Gamma\int_{A^J}\Omega\right) = p^I F_I - q_J X^J \tag{3.2.5}$$

Now, for a 4-dimensional spherically symmetrical supersymmetrical black hole, whose metric is:

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}d\vec{x}\cdot d\vec{x} \tag{3.2.6}$$

where r is the distance to the event horizon. Let us assume that also $t^\alpha = X^\alpha/X^0$ depends only on r, and also, let $\tau = \frac{1}{r}$. Then $\tau \to 0$ corresponds to the spatial infinity and $\tau\to\infty$ corresponds to the event horizon.

Through the supersymmetry conditions:

$$\delta\psi_\mu = 0, \quad \delta\lambda^\alpha = 0 \tag{3.2.7}$$

where ψ_μ is the gravitino and λ^α are the gauginos, we get to equations for $U(\tau), t^\alpha(\tau)$ [22]:

$$\frac{dU}{d\tau} = -e^{U(\tau)}|Z| \tag{3.2.8}$$

$$\frac{dt^\alpha}{d\tau} = -2e^{U(\tau)}G^{\alpha\bar{\beta}}\partial_{\bar{\beta}}|Z| \tag{3.2.9}$$

with $G^{\alpha\bar{\beta}}$ is the inverse of $G_{\alpha\bar{\beta}} = \partial_\alpha\partial_{\bar{\beta}}\mathcal{K}$. The analogy to classical dynamical systems is clear, where the parameter τ plays the role of the time. This differential equations are known as the **attractor mechanism**.

In the book by Becker-Becker-Schwarz [9], it is proved that 3.2.8, 3.2.9 are equivalent to:

$$2\frac{d}{d\tau}\left[e^{-U(\tau)+\mathcal{K}/2}Im\left(e^{-i\alpha}\Omega\right)\right] \sim -\Gamma \tag{3.2.10}$$

where $\sim$ means that both expressions differs at much by an exact 3-form. This equation is easy to integrate, given that Γ does not depend on τ:

$$2e^{-U(\tau)+\mathcal{K}/2}Im\left(e^{-i\alpha}\Omega\right) \sim -\Gamma\tau + 2\left[e^{-U(\tau)+\mathcal{K}/2}Im\left(e^{-i\alpha}\Omega\right)\right]_{\tau=0} \tag{3.2.11}$$

In their book [9], Becker-Becker-Schwarz prove that this equations implies that Ω can be computed as a function of the charges of the black hole.

3.3 Electric and magnetic charges

In String theory, electric and magnetic charges are related to the gauge field $\mathcal{A}_1$, that is constructed from the **self-dual Ramond-Ramond field stregth** $\mathcal{F}_5 = d\mathcal{A}_1 \wedge F_3$. Expanding the 3-form, $F_3 = e^I\alpha_I - m_I\beta^I$, we find the non-self-dual part, that is the electric part, as:

$$F_5 = F_2 \wedge F_3 = F_2 \wedge (e^I \alpha_I - m_I \beta^I) \tag{3.3.1}$$

We find that $\mathcal{C}$ is the Poincaré dual of $*F_3$, we then denote the Poincaré dual of F_3 as $*\mathcal{C}$. The electric and magnetic charges can be computed by integrating the 5-forms $*F_5$ and F_5, respectively, over the over the 5-cycle $*\mathcal{C}_5 = S^2 \times *\mathcal{C}$ and $\mathcal{C}_5 = S^2 \times \mathcal{C}$:

$$Q_e = \int_{*\mathcal{C}_5} *F_5 = -qN \tag{3.3.2}$$

$$Q_m = \int_{\mathcal{C}_5} F_5 = pN \tag{3.3.3}$$

where $\mathcal{C} \cap *\mathcal{C} = N$ and $q = \int_{S^2} *F_2, p = \int_{S^2} F_2$. We find that the coefficients e^I, m_I are related to p^I, q_I through:

$$p^I = e^J A^I_J - m_J C^{IJ} \tag{3.3.4}$$

$$q_I = -e^J B_{IJ} - m_J A^J_I \tag{3.3.5}$$

where the elements of the matrices A, B, C are defined by:

$$A^I_J = -\int \alpha_J \wedge *\beta^I = -\int \beta^I \wedge *\alpha_J \tag{3.3.6}$$

$$B_{IJ} = \int \alpha_I \wedge *\alpha_J = \int \alpha_J \wedge *\alpha_I \tag{3.3.7}$$

$$C^{IJ} = -\int \beta^I \wedge *\beta^J = -\int \beta^J \wedge *\beta^I \tag{3.3.8}$$

Finally, the total charge is:

$$Q = Q_e + Q_m = N(p - q) \tag{3.3.9}$$

that is, as a matter of fact, a integer multiple of $p - q$.

Chapter 4

Half-flat manifolds

Some phenomena in four dimensions can be modeled as fluxes on compactification Calabi-Yau manifolds. Then, for an electric Neveu-Schwarz flux on a Calabi-Yau 3-fold X_3 of a Type IIA superstring theory, this Calabi-Yau manifold X_3 is mapped through mirror symmetry into a generalized Calabi-Yau 3-fold Y_3, known as **Half-flat manifold**.

As it was mentioned above, mirror symmetry is the connection between a Calabi-Yau manifold X_3 compactification on a Type IIA superstring theory and a Half-flat manifold Y_3 compactification on a Type IIB superstring theory [17]. In order to have this mirror symmetry, there are two conditions that Y_3 must satisfy, $d\text{Im}\Omega = 0, d\text{Re}\Omega = e_i\tilde{\omega}_i, i = 1,\cdots, h^{2,1}$, where e_i has to do with the electric part of the NS-NS field strength and $\tilde{\omega}_i$ is the basis of $H^4(Y_3, \mathbb{Z})$.

Appart from (α_i, β^i), we also have (α_0, β^0), as part of the basis of the cohomology classes, satisfying:

$$d\alpha_0 = e_i\tilde{\omega}^i \tag{4.0.1}$$

$$d\tilde{\omega}_i = e_i\beta^0 \tag{4.0.2}$$

Let us write $e_i\tilde{\omega}^i$ as $k(n_i\tilde{\omega}^i)$, with $k = \gcd(e_1,\cdots, e_{h^{1,1}}), n_i \in \mathbb{Z}$. Therefore,

$$d\alpha_0 = k(n_1\tilde{\omega}^i + n_a\tilde{\omega}^a) \tag{4.0.3}$$

Then, $n_1\tilde{\omega}^1 + n_a\tilde{\omega}^a, a = 2, \ldots, h^{1,1}$ is said to be **torsional**. And $(n_1\tilde{\omega}^i + n_a\tilde{\omega}^a, \tilde{\omega}^a)$ is the basis of $H^{2,2}(Y_3;\mathbb{Z})$. Then,

$$H^{2,2}(Y_3;\mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}^{h^{1,1}-1} \tag{4.0.4}$$

Let $\hat{\Omega}^p(Y_3)$ be the set of nonclosed p-forms, and $\mathrm{Tor}H^p(Y_3;\mathbb{Z})$ the set of torsional $(p+1)$-forms, i.e., if $\sigma \in \Lambda^p(Y_3), \lambda \in \Lambda^{p+1}(Y_3)$, such that $d\sigma = k\lambda$, then $\sigma \in \hat{\Omega}^p(Y_3), \lambda \in \mathrm{Tor}H^p(Y_3;\mathbb{Z})$.

Now, note that 4.0.2 is equivalent to:

$$d(n^i\tilde{\omega}_i) = n^i k(n_i\beta^0) = k(n^i n_i\beta^0) \tag{4.0.5}$$

Then, $\hat{\alpha}_0 \equiv \alpha_0 \in \hat{\Omega}^p(Y_3), \hat{\beta}^0 \equiv n^i n_i\beta^0 \in \mathrm{Tor}H^p(Y_3;\mathbb{Z})$ are the only elements of their respective sets. So that,

$$H^3(Y_3;\mathbb{Z}) = \mathbb{Z}^{2h^{2,1}} \otimes \mathbb{Z}_k \tag{4.0.6}$$

In a similar way, we can find all the cohomology groups for Y_3. To see it explicitly, check [16].

Just as we have the basis of 3-forms $(\hat{\alpha}_0, \hat{\beta}^0)$ on the Half-flat manifold, we have a basis for the 3-cycles $\hat{\Sigma}_3$ and 3-chains $\hat{\Pi}_3$. This basis is such as:

$$k\hat{\Sigma}_3 = \partial\hat{\Pi}_4, \quad \partial\hat{\Pi}_3 = k\hat{\Sigma}_2 \tag{4.0.7}$$

According to this same article [16], we find that the holomorphic 3-form of the manifold is given by:

$$\Omega = \Omega^0 + \tilde{\Omega} = X^i\alpha_i - F_i\beta^i + \hat{\alpha}_0 - F_0\hat{\beta}^0 \tag{4.0.8}$$

where Ω^0 is a Calabi-Yau manifold-like holomorphic 3-form. The holomorphic symplectic basis of this manifold is:

$$\begin{aligned} F_I &= (F_0, F_i) = \left(\int_{\hat{\Pi}_3} \tilde{\Omega}, \int_{B_i} \Omega^0\right) \\ X^I &= \left(X^0, X^i\right) = \left(\int_{\hat{\Sigma}_3} \tilde{\Omega}, \int_{A^i} \Omega^0\right) \end{aligned} \tag{4.0.9}$$

Knowing the holomorphic 3-form Ω for this manifold, we are able to compute the Kähler potential:

$$\begin{aligned} \mathcal{K} &= -\ln i \int \Omega \wedge \bar{\Omega} \\ &= -\ln i \int (\Omega^0 + \tilde{\Omega}) \wedge (\bar{\Omega^0} + \bar{\tilde{\Omega}}) \\ &= -\ln i \left(\int \Omega^0 \wedge \bar{\Omega^0} + \int \Omega^0 \wedge \bar{\tilde{\Omega}} + \int \tilde{\Omega} \wedge \bar{\Omega^0} + \int \tilde{\Omega} \wedge \bar{\tilde{\Omega}} \right) \\ &= -\ln i \left(\int \Omega^0 \wedge \bar{\Omega^0} + \int 2\mathrm{Im}(\Omega^0 \wedge \bar{\tilde{\Omega}}) + \int \tilde{\Omega} \wedge \bar{\tilde{\Omega}} \right) \\ &= -\ln i \left(\int \Omega^0 \wedge \bar{\Omega^0} + \int \tilde{\Omega} \wedge \bar{\tilde{\Omega}} \right) \\ e^{-\mathcal{K}} &= e^{-\mathcal{K}_0} + e^{-\tilde{\mathcal{K}}} \end{aligned} \tag{4.0.10}$$

Note that the two terms in the middle of the third line vanish because Ω^0 and $\tilde{\Omega}$ corresponds to different spaces.

4.1 Central charge

As it was mentioned in the section 3.2, black holes are modeled in string theory as D3-branes wrapped on certain 3-cycle, $\mathcal{C} \subset Y_3$. $\mathcal{C}$ and its dual 3-form can be expanded as:

$$\mathcal{C} = \tilde{\mathcal{C}} \cup \mathcal{C}_0 = p^0 \hat{\Pi}_3 - q_o \hat{\Sigma}_3 + p^i B_i - q_i A^i \tag{4.1.1}$$

$$\Gamma = \tilde{\Gamma} + \Gamma_0 = p^0 \hat{\alpha}_0 - q_0 \hat{\beta}^0 + p^i \alpha_i - q_i \beta^i \tag{4.1.2}$$

where

$$p^0 = \int_{\hat{\Pi}_3} \Gamma = \int \tilde{\Gamma} \wedge \hat{\beta}^0, \quad q_0 = \int_{\hat{\Sigma}_3} \Gamma = \int \tilde{\Gamma} \wedge \hat{\alpha}^0 \tag{4.1.3}$$

And then, the central charge $Z(\Gamma)$ is given by:

$$Z(\Gamma) = e^{\mathcal{K}/2} \left(\int_{\mathcal{C}_0} \Omega^0 + \int_{\tilde{\mathcal{C}}} \tilde{\Omega} \right) = Z^0(\Gamma_0) + \tilde{Z}(\tilde{\Gamma}) \tag{4.1.4}$$

4.2 Fractional charges

To study the effects of treating a black hole on a half-flat manifold (instead of a Calabi-Yau), let us wrap a D3-brane over a chain $\tilde{\mathcal{C}} = p^0 \hat{\Pi}_3 - q_o \hat{\Sigma}_3$. Then, in analogy to the Section 3.3, we got the 3-cycles:

$$\tilde{\mathcal{C}} = p^0 \hat{\Pi}_3 - q_0 \hat{\Sigma}_3 \tag{4.2.1}$$

$$*\tilde{\mathcal{C}} = \frac{e^0}{k} \hat{\Pi}_3 - m_0 \hat{\Sigma}_3 \tag{4.2.2}$$

where the $\frac{1}{k}$ factor is due to 4.0.7. And, therefore, the Poincaré dual to these 3-cycles are the 3-forms:

$$*F_3 = p^0 \hat{\alpha}_0 - q_0 \hat{\beta}^0 \tag{4.2.3}$$

$$F_3 = \frac{e^0}{k} \hat{\alpha}_0 - m_0 \hat{\beta}^0 \tag{4.2.4}$$

with the coefficients relating each other through:

$$p^0 = \frac{e^0}{k} A_0^0 - m_0 C^{00} \tag{4.2.5}$$

$$q_0 = -\frac{e^0}{k} B_{00} - m_o A_0^0 \tag{4.2.6}$$

and the matrix elements:

$$A_0^0 = -\int \hat{\alpha}_0 \wedge *\hat{\beta}^0 = -\int \hat{\beta}^0 \wedge *\hat{\alpha}_0 \tag{4.2.7}$$

$$B_{00} = \int \alpha_0 \wedge *\alpha_0 \tag{4.2.8}$$

$$C^{00} = -\int \beta^0 \wedge *\beta^0 \tag{4.2.9}$$

However, according to [17], when working on a half-flat compactification, we need to consider a nontrivial NS-NS flux $H_3 = e_0 \hat{\beta}^0$. In order to get rid of this flux, and the problems that come with it, known as the **Freed-Witten anomaly**, it is necessary that:

$$\int_{\tilde{C}} e_o \hat{\beta}^0 = 0 \tag{4.2.10}$$

and, therefore, $p^0 = 0$. Then, by 4.2.5, we get:

$$m_o = \frac{e^0}{k} \frac{A_0^0}{C^{00}} \tag{4.2.11}$$

After eliminating the Freed-Witten anomaly, the 3-cycles are[16]:

$$\tilde{C} = -\left\{ \frac{e^0}{k} \right\} \frac{1}{C^{00}} \hat{\Sigma}_3 \tag{4.2.12}$$

$$*\tilde{C} = \left\{ \frac{e^0}{k} \right\} \left(\hat{\Pi}_3 - \frac{A_0^0}{C^{00}} \hat{\Sigma}_3 \right) \tag{4.2.13}$$

where {} denotes the fractional part. Knowing this, the electric and magnetic charges are:

$$Q_e = \frac{Q}{C^{00}} \left\{ \frac{e^0}{k} \right\}^2 \tag{4.2.14}$$

$$Q_m = -\frac{P}{C^{00}} \left\{ \frac{e^0}{k} \right\}^2 \tag{4.2.15}$$

and then the total charge is:

$$Q = \frac{Q-P}{C^{00}} \left\{ \frac{e^0}{k} \right\}^2 \tag{4.2.16}$$

that clearly is not an integer multiple of $Q - P$. Then, the torsion cycle adds a fractional charge that is not considered in a Calabi-Yau manifold. This is known as **Quantum hair**, as a reference to the No-hair theorem discussed in Section 1.5.

Chapter 5

Attractor mechanism on half-flat manifolds

In the section 3.2, we found an set of differential equations that describes the metric of a supersymmetric black hole and the moduli space of the compactification manifold. The most general expression was given by 3.2.10. In order to make it simpler, we are going to change the $\sim$ symbol for the $=$ symbol, but we have to remember that both sides of the equation may differ by closed form. Then, we part from the equation:

$$2\frac{d}{d\tau}\left[e^{-U(\tau)+\mathcal{K}/2} Im\left(e^{-i\alpha}\Omega\right)\right] = -\Gamma \tag{5.0.1}$$

In the chapter 4, we described the Half-flat manifolds, finding the expressions for its Kähler potential $\mathcal{K}$, its holomorphic 3-form Ω, and the 3-form Γ related to the 3-cycle $\mathcal{C}$ around which D3-branes are wrapped.

5.1 Equation on the metric

Let us project both sides of 5.0.1 on $e^{-i\alpha+\mathcal{K}/2}\Omega$:

$$\int e^{-i\alpha+\mathcal{K}/2}\Omega\wedge\frac{d}{d\tau}\left[e^{-U(\tau)}\left(e^{\mathcal{K}/2-i\alpha}\Omega-e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\right]=\int e^{-i\alpha+\mathcal{K}/2}\Omega\wedge(-i\Gamma)$$
$$\int e^{-i\alpha+\mathcal{K}/2}\Omega\wedge\frac{d}{d\tau}\left[e^{-U(\tau)}\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\right]=i\int e^{-i\alpha+\mathcal{K}/2}\Omega\wedge\Gamma$$

This is due to the fact that $\Omega\wedge\Omega=\Omega\wedge\frac{d}{d\tau}\Omega=0$.

Let us focus on the left-hand side:

$$\begin{aligned}&\int e^{-i\alpha+\mathcal{K}/2}\Omega\wedge\frac{d}{d\tau}\left[e^{-U(\tau)}\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\right]=\\&=\int\left[e^{\mathcal{K}}\frac{d}{d\tau}e^{-U(\tau)}\Omega\wedge\bar{\Omega}+e^{-i\alpha+\mathcal{K}/2}e^{-U(\tau)}\Omega\wedge\frac{d}{d\tau}\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\right]\end{aligned}$$

Now, note from the definition of the Kähler potential 2.4.15 that:

$$\begin{aligned}e^{-\mathcal{K}}&=i\int\Omega\wedge\bar{\Omega}\\-i&=e^{\mathcal{K}}\int\left(e^{-i\alpha}\Omega\right)\wedge\left(e^{i\alpha}\bar{\Omega}\right)\\-i&=\int\left(e^{\mathcal{K}/2-i\alpha}\Omega\right)\wedge\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\\\frac{d}{d\tau}(-i)&=\frac{d}{d\tau}\int\left(e^{\mathcal{K}/2-i\alpha}\Omega\right)\wedge\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\\0&=\int\frac{d}{d\tau}\left(e^{\mathcal{K}/2-i\alpha}\Omega\right)\wedge\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)+\left(e^{\mathcal{K}/2+i\alpha}\Omega\right)\wedge\frac{d}{d\tau}\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\\\int\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\wedge\frac{d}{d\tau}\left(e^{\mathcal{K}/2-i\alpha}\Omega\right)&=\int\left(e^{\mathcal{K}/2-i\alpha}\Omega\right)\wedge\frac{d}{d\tau}\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\end{aligned}\tag{5.1.1}$$

Then it is real, and do not contribute to $\mathrm{Im}(e^{-i\alpha}\Omega)$, and we get:

$$\int e^{-i\alpha+\mathcal{K}/2}\Omega\wedge\frac{d}{d\tau}\left[e^{-U(\tau)}\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\right]=\int e^{\mathcal{K}}\frac{d}{d\tau}e^{-U(\tau)}\Omega\wedge\bar{\Omega}$$

And note that $e^{\mathcal{K}}\frac{d}{d\tau}e^{U(\tau)}$ do not depends on the compactification manifold, it only depends on the variable τ that corresponds to the distance in the resulting 4-dimensional spacetime. Then,

$$\int e^{-i\alpha+\mathcal{K}/2}\Omega \wedge \frac{d}{d\tau}\left[e^{-U(\tau)}\left(e^{\mathcal{K}/2+i\alpha}\bar{\Omega}\right)\right] = e^{\mathcal{K}}\frac{d}{d\tau}e^{-U(\tau)}\int \Omega \wedge \bar{\Omega} \tag{5.1.2}$$

For the right-hand side, we get:

$$\begin{aligned} i\int e^{-i\alpha+\mathcal{K}/2}\Omega \wedge \Gamma &= ie^{-i\alpha+\mathcal{K}/2}\int \Omega \wedge \Gamma \\ &= ie^{-i\alpha}Z \\ &= i|Z| \end{aligned}$$

We then get:

$$-ie^{\mathcal{K}}\frac{d}{d\tau}e^{-U(\tau)}\int \Omega \wedge \bar{\Omega} = |Z| \tag{5.1.3}$$

Comparing with 2.4.15 and 4.1.4, it results:

$$\frac{d}{d\tau}e^{-U} = |Z| = e^{\mathcal{K}/2}\left|\int_{\mathcal{C}_0} \Omega^0 + \int_{\tilde{\mathcal{C}}} \tilde{\Omega}\right| \tag{5.1.4}$$

that is equivalent to 3.2.8, for a manifold with a central charge given by 4.1.4.

5.2 Equation on the moduli space

In order to get the equations on the moduli coordinates X^I, we project 5.0.1 on $e^{-i\alpha}e^{\mathcal{K}}D_I\Omega$. After this, we get the equation:

$$\frac{dX^I}{d\tau} = -2e^{U(\tau)}G^{I\bar{J}}\partial_{\bar{J}}|Z| \tag{5.2.1}$$

Note that the metric $G_{I\bar{J}}$ is given by:

$$
\begin{aligned}
G_{I\bar{J}} &= \partial_I \partial_{\bar{J}} \mathcal{K} \\
&= \partial_I \partial_{\bar{J}} \left[-\ln \left(e^{-\mathcal{K}_0} + e^{-\tilde{\mathcal{K}}} \right) \right] \\
&= -\partial_I \frac{\partial_{\bar{J}} \left(e^{-\mathcal{K}_0} + e^{-\tilde{\mathcal{K}}} \right)}{e^{-\mathcal{K}_0} + e^{-\tilde{\mathcal{K}}}} \\
&= \partial_I \frac{e^{-\mathcal{K}_0} \partial_{\bar{J}} \mathcal{K}_0 + e^{-\tilde{\mathcal{K}}} \partial_{\bar{J}} \tilde{\mathcal{K}}}{e^{-\mathcal{K}_0} + e^{-\tilde{\mathcal{K}}}} \\
&= \frac{\partial_I \left(e^{-\mathcal{K}_0} \partial_{\bar{J}} \mathcal{K}_0 + e^{-\tilde{\mathcal{K}}} \partial_{\bar{J}} \tilde{\mathcal{K}} \right) \left(e^{-\mathcal{K}_0} + e^{-\tilde{\mathcal{K}}} \right)}{\left(e^{-\mathcal{K}_0} + e^{-\tilde{\mathcal{K}}} \right)^2} \\
&\quad - \frac{\left(e^{-\mathcal{K}_0} \partial_{\bar{J}} \mathcal{K}_0 + e^{-\tilde{\mathcal{K}}} \partial_{\bar{J}} \tilde{\mathcal{K}} \right) \partial_I \left(e^{-\mathcal{K}_0} + e^{-\tilde{\mathcal{K}}} \right)}{\left(e^{-\mathcal{K}_0} + e^{-\tilde{\mathcal{K}}} \right)^2} \\
&= \frac{e^{-\mathcal{K}_0} \partial_I \mathcal{K}_0 \partial_{\bar{J}} \mathcal{K}_0 + e^{-\mathcal{K}_0} \partial_I \partial_{\bar{J}} \mathcal{K}_0 + e^{-\tilde{\mathcal{K}}} \partial_I \tilde{\mathcal{K}} \partial_{\bar{J}} \tilde{\mathcal{K}} + e^{-\tilde{\mathcal{K}}} \partial_I \partial_{\bar{J}} \tilde{\mathcal{K}}}{e^{-\mathcal{K}}} \\
&\quad + \frac{\left(e^{-\mathcal{K}_0} \partial_I \mathcal{K}_0 + e^{-\tilde{\mathcal{K}}} \partial_I \tilde{\mathcal{K}} \right) \left(e^{-\mathcal{K}_0} \partial_{\bar{J}} \mathcal{K}_0 + e^{-\tilde{\mathcal{K}}} \partial_{\bar{J}} \tilde{\mathcal{K}} \right)}{e^{-2\mathcal{K}}}
\end{aligned}
$$

But, recall that $\tilde{\mathcal{K}}$ depends only on X^0 and $\mathcal{K}_0$ depends only on $X^i, i = 1, \cdots, h^{2,1}$. Then, $G_{0i} = G_{i0} = 0$, and

$$
\begin{aligned}
G_{0\bar{0}} &= \frac{e^{-\tilde{\mathcal{K}}} \partial_0 \tilde{\mathcal{K}} \partial_{\bar{0}} \tilde{\mathcal{K}} + e^{-\tilde{\mathcal{K}}} \partial_0 \partial_{\bar{0}} \tilde{\mathcal{K}}}{e^{-\mathcal{K}}} + \frac{\left(e^{-\tilde{\mathcal{K}}} \partial_0 \tilde{\mathcal{K}} \right) \left(e^{-\tilde{\mathcal{K}}} \partial_{\bar{0}} \tilde{\mathcal{K}} \right)}{e^{-2\mathcal{K}}} \\
&= \frac{e^{-\tilde{\mathcal{K}}}}{e^{-\mathcal{K}}} \left(\partial_0 \tilde{\mathcal{K}} \partial_{\bar{0}} \tilde{\mathcal{K}} + \partial_0 \partial_{\bar{0}} \tilde{\mathcal{K}} \right) + \left(\frac{e^{-\tilde{\mathcal{K}}}}{e^{-\mathcal{K}}} \right)^2 \partial_0 \tilde{\mathcal{K}} \partial_{\bar{0}} \tilde{\mathcal{K}} \qquad (5.2.2) \\
&= \frac{e^{-\tilde{\mathcal{K}}}}{e^{-\mathcal{K}}} \left(1 + \frac{e^{-\tilde{\mathcal{K}}}}{e^{-\mathcal{K}}} \right) \partial_0 \tilde{\mathcal{K}} \partial_{\bar{0}} \tilde{\mathcal{K}} + \frac{e^{-\tilde{\mathcal{K}}}}{e^{-\mathcal{K}}} \partial_0 \partial_{\bar{0}} \tilde{\mathcal{K}}
\end{aligned}
$$

$$
\begin{aligned}
G_{i\bar{j}} &= \frac{e^{-\mathcal{K}_0}\partial_i\mathcal{K}_0\partial_{\bar{j}}\mathcal{K}_0 + e^{-\mathcal{K}_0}\partial_i\partial_{\bar{j}}\mathcal{K}_0}{e^{-2\mathcal{K}}} + \frac{\left(e^{-\mathcal{K}_0}\partial_i\mathcal{K}_0\right)\left(e^{-\mathcal{K}_0}\partial_{\bar{j}}\mathcal{K}_0\right)}{e^{-2\mathcal{K}}} \\
&= \frac{e^{-\mathcal{K}_0}}{e^{-\mathcal{K}}}\left(1 + \frac{e^{-\mathcal{K}_0}}{e^{-\mathcal{K}}}\right)\partial_i\mathcal{K}_0\partial_{\bar{j}}\mathcal{K}_0 + \frac{e^{-\mathcal{K}_0}}{e^{-\mathcal{K}}}\partial_i\partial_{\bar{j}}\mathcal{K}_0
\end{aligned} \tag{5.2.3}
$$

On the other side, we get:

$$\partial_0|Z| = \partial_0|Z^0 + \tilde{Z}| = \partial_0|\tilde{Z}| \tag{5.2.4}$$

$$\partial_i|Z| = \partial_i|Z^0 + \tilde{Z}| = \partial_i|Z^0| \tag{5.2.5}$$

Knowing the form of the metric of the moduli space, 5.2.1 can be divided into two equations:

$$\frac{dX^0}{d\tau} = -2e^{U(\tau)}G^{0\bar{0}}\partial_{\bar{0}}|\tilde{Z}| \tag{5.2.6}$$

$$\frac{dX^i}{d\tau} = -2e^{U(\tau)}G^{i\bar{j}}\partial_{\bar{j}}|Z^0| \tag{5.2.7}$$

Clearly, 5.2.7 is equivalent to 3.2.9, not only on the form, but with the same index set $\{1,\cdots,h^{2,1}\}$. And we get and independent equation 5.2.6 for the new symplectic coordinate.

5.3 Asymptotic behaviour of the quantum hair

Note that 5.2.6 implies:

$$\begin{aligned}\frac{d|\tilde{Z}|}{d\tau} &= \frac{dX^0}{d\tau}\partial_0|\tilde{Z}| + \frac{d\bar{X}^{\bar{0}}}{d\tau}\partial_{\bar{0}}|\tilde{Z}| \\ &= -2e^{U(\tau)}G^{0\bar{0}}\partial_{\bar{0}}|\tilde{Z}|\partial_0|\tilde{Z}| - 2e^{U(\tau)}G^{\bar{0}0}\partial_0|\tilde{Z}|\partial_{\bar{0}}|\tilde{Z}| \\ &= -4e^{U(\tau)}G^{0\bar{0}}\partial_{\bar{0}}|\tilde{Z}|\partial_0|\tilde{Z}| \leq 0\end{aligned}$$

Then, the function $|\tilde{Z}|(\tau)$ is monotonically decreasing. Nevertheless, the fact that $\tau \longrightarrow \infty$ corresponds to the event horizon makes any function of τ convergent at $\tau \longrightarrow \infty$. From this we may conclude:

$$\frac{d|\tilde{Z}|}{d\tau} \longrightarrow 0, \quad \text{as} \quad \tau \longrightarrow \infty \tag{5.3.1}$$

The same happens to $\tilde{Z}$. Let be

$$|\mathring{Z}| = \lim_{\tau\rightarrow\infty} |Z| \tag{5.3.2}$$

Then, for large τ's, 5.1.4 results:

$$\frac{d}{d\tau}e^{-U(\tau)} = |\mathring{Z}| \tag{5.3.3}$$

that can be integrated to:

$$\frac{e^{-U(\tau)}}{\tau} = |\mathring{Z}| \tag{5.3.4}$$

Substituting this into the metric of an supersymmetric spherically symmetrical black hole 3.2.6, we get:

$$
\begin{aligned}
ds^2 &= -e^{2U(r)}dt^2 + e^{-2U(r)}\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right) \\
&= -\frac{r^2}{|\mathring{Z}|^2}dt^2 + \frac{|\mathring{Z}|^2}{r^2}dr^2 + |\mathring{Z}|^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \\
&= \frac{|\mathring{Z}|^2}{R^2}\left(-dt^2 + dR^2\right) + |\mathring{Z}|^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)
\end{aligned}
$$

that is the $AdS_2 \times S^2$ spacetime, just like in the section 1.3 for extremal black holes. With the metric in this form, the area of the event horizon is easily computed as:

$$A = 4\pi|\mathring{Z}|^2 \tag{5.3.5}$$

Under this same approximation, 5.0.1 has solution:

$$2e^{\mathcal{K}/2}\mathrm{Im}\bar{\mathring{Z}}\Omega \sim -\Gamma \tag{5.3.6}$$

Then, in the near-horizon approximation, we are able to calculate the holomorphic 3-form $\Omega = \Omega^0 + \tilde{\Omega}$ as a function of the central charge $\mathring{Z}$, i.e., the charges (electric, magnetic, integer, fractional) at the horizon.

Conclusions

1. String theory is a proposal for a quantum theory of gravity that is able to reproduce Supergravity and General Relativity phenomena. Thus, the problems of the classical General Relativity, such as the existence of singularities and the information loss, can be reformulated in terms of the String Theory.

2. Calabi-Yau manifolds are the main candidates for compactification, because the resulting 4-dimensional spacetime has supersymmetry $\mathcal{N} = 2$.

3. In general terms, to preserve the No-hair theorem, we need to define the Attractor mechanism to localise the charges of the black hole to the event horizon.

4. In order to study fluxes over the compactification manifold, we need to consider generalised Calabi-Yau manifolds, such as Half-flat manifolds. Half-flat manifolds has a structure similar to CY manifolds, with its own moduli space. A new charge, that is additional to the classic black hole charges, is found for Half-flat manifolds, known as quantum hair.

5. Extending the definition of the Attractor mechanism to a theory compactified with a Half-flat manifold, it results that quantum hair is also localised in the event horizon.

Appendix A

Special Relativity

In 1864, Maxwell achieved to summarize and correct over a century of research work in electrodynamics in four equations:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{A.0.1}$$

$$\nabla \cdot \vec{B} = 0 \tag{A.0.2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{A.0.3}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{A.0.4}$$

known as the Maxwell equations [42]. A certain combination of these equations gives a wave behaviour for the electric and magnetic fields, both with the same velocity:

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{A.0.5}$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{B}}{\partial t} \tag{A.0.6}$$

That value of velocity agreed unexpectedly with the value of the velocity of light, $\frac{1}{\sqrt{\mu_0\epsilon_0}} = c$, largely measured by people like Galileo, Huygens and Foucault. This suggested that light actually is an electromagnetic wave.

Two questions surge after this assumption: Which is the frame in which the velocity of light is measured?, and is there any medium through which the electromagnetic waves propagate? Some people suggested the existence of a fluid called "aether" that covered the universe to answer both questions.

To prove the existence of this aether, in 1887, Michaelson and Morley [43] measured the speed of light in two directions, one perpendicular to the movement of Earth and the other parallel to it. Surprisingly, both velocities were the same.

A.1 Lorentz transformation

After the conclusions given by Michaelson and Morley, in 1895, Lorentz got a new transformation of coordinates:

$$t' = \gamma(t - \frac{V}{c^2}x) \tag{A.1.1}$$

$$x' = \gamma(x - Vt) \tag{A.1.2}$$

with

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{A.1.3}$$

that, if the second frame is moving slowly respect to first one, are similar to the ones proposed by Galileo in the XVII Century:

$$t' = t \tag{A.1.4}$$

$$x' = x - Vt \tag{A.1.5}$$

Ten years later, Albert Einstein taking the invariance of the speed of light as postulate (and then the transformation of Lorentz), published an article [44] with a new mechanical theory, that would be called the **Special Theory of Relativity**.

The fact that in this theory, the time also transforms is seen easier if we work with vector in four dimensions (or 4-vectors), as well as tensors in four dimensions, so that, the **Lorentz transformation** looks like:

$$x^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{A.1.6}$$

with

$$\Lambda^{\mu}{}_{\nu} = \frac{\partial x^{\mu}}{\partial x^{\nu}} \tag{A.1.7}$$

Note that this expression only involves rotations and boosts, not translations. Also, it defines a group, known as the **Lorentz group**. If we add the translations, i.e.,

$$x^{\mu} = \Lambda^{\mu}{}_{\bar{\nu}} x^{\bar{\nu}} + a^{\mu} \tag{A.1.8}$$

It defines the **Poincaré Group**.

A.2 Relativistic dynamics

We need a 4-vector definition for the classical quantities studied in Newtonian mechanics, such as momentum, energy, etc. Knowing the position of a massive particle, x^μ, we can define its velocity

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{A.2.1}$$

with τ the **proper time** of the particle, that is, the time that measures the particle itself. It happens to have the components:

$$u^\mu = (\gamma c, \gamma \vec{v}) \tag{A.2.2}$$

Then, we can easily define the momentum of a massive particle as:

$$p^\mu = m u^\mu \tag{A.2.3}$$

Note that it does not agree with our classical definition of momentum, even in the spatial components. But, if the speed of the particle is small compared to the speed of light, the extra γ factor tends to 1.

This definition of momentum seems to work for massive particles, but what about photons and other massless particles? We cannot define a proper time for massless particles (as they travel at the speed of light). Even if we could do it, see that A.2.2 diverges as $v \to c$.

In order to define a momentum for massless particles, note that the 0-component of the momentum of a massive particle is $p^0 = m\gamma c$, that indeed is the total energy of the particle up to a c factor, as Einstein wrote in his 1905 article [44]. So that, the components of the momentum are

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \tag{A.2.4}$$

Recall that the energy of a quantum of light (photon) is $E = h\nu = \frac{hc}{\lambda}$ [45] and the de Broglie hypothesis for the momentum of a photon, $p = \frac{h}{\lambda}$ [46]. So that, we can use A.2.4 as the definition of the (4-)momentum. This definition let us have two conservation laws (energy and momentum) into just one, the conservation of the 4-momentum.

Note that if the energy of a photon (and then its wavelength) are part of a vector, we can see how it transforms as the source is moving from (or to) the observer, this is known as **Doppler effect**. Consider a source of light that emits a photon with wavelength λ and an observer that moves from the source with speed V in some certain direction x. Acting A.1.1 over A.2.4, we get that the wavelength that the observer measures is

$$\lambda' = \sqrt{\frac{c+v}{c-v}}\lambda \tag{A.2.5}$$

A.3 Relativistic action

All this is summarized in the action:

$$S = \int \mathcal{L} = -mc \int_{s_1}^{s_2} \sqrt{\eta_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}} ds \tag{A.3.1}$$

where $\eta_{\alpha\beta}$ is the metric of the **Minkowski spacetime**, and it is given by

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.3.2}$$

This action works for free particles, but if we add an electromagnetic interaction, there is an extra term to the lagrangian function, $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$:

$$\mathcal{L}_{int} = \frac{e}{c}\frac{dx_\alpha}{ds}A^\alpha \tag{A.3.3}$$

If we apply the **Hamilton Principle** to this action[47], we get the Lorentz force

$$\frac{du^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}u_\beta \tag{A.3.4}$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{A.3.5}$$

that is, the force that feels a particle with electric charge e in the presence of a (4-)potential A^α, whose components are:

$$A^\alpha = (\phi, \vec{A}) \tag{A.3.6}$$

where, ϕ and $\overrightarrow{A}$ are the classical electrodynamics scalar and vector potentials, respectively.

Note that the components of the **tensor of Faraday** F are

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{A.3.7}$$

and that, it is indeed a tensor, i.e., it transform as

$$F'^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu} \tag{A.3.8}$$

where the Λ's are the same as in A.1.7.

There is one more action that gives us an interesting equation, that is

$$S = \int \left[-\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha \right] \tag{A.3.9}$$

where J is the (4-)current, whose components are

$$J^\alpha = (c\rho, \vec{J}) \tag{A.3.10}$$

From this action we get

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\alpha \tag{A.3.11}$$

that are the inhomogeneous Maxwell equations. The homogeneous ones can be get from Jacobi identity.

After all this, Special Relativity seems to be very useful, but it fails to describe accelerated frames. We need a generalisation of this theory to do that.

Appendix B

Geometric tools

B.1 Manifolds

Spacetime, as well as our planet, is not plane. Note that Earth cannot be described by a single coordinate system. Latitude and longitude try, but they fail at the International Date Line, that is a "discontinuity" talking about the local time. This proves that there are geometrical spaces that, in general, cannot be described by a single coordinate system, but locally we can find it. For example, a map of Mexico, in the case of the Earth. These spaces are the so called **manifolds**.

A manifold can be locally identify with a flat space $\mathbb{R}^n$. We can characterize a manifold M using its metric $g_{\mu\nu}$. The classic concept of scalar product in $\mathbb{R}^n$:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + \cdots + a_n b_n \tag{B.1.1}$$

is sustituted in a manifold, by

$$\vec{a} \cdot \vec{b} = g_{\mu\nu} a^{\mu} b^{\nu} \tag{B.1.2}$$

Note that we are using the **Einstein summation convention**.

In Special Relativity, the components of the metric is the one given in A.3.2, but in General Relativity, it only has to be a function $g : M \times M \longrightarrow \mathbb{K}$ (the field over which the vector space is defined) bilinear ($g(\lambda a, b) = \lambda g(a, b), g(a, \lambda b) = \lambda g(a, b)$), symmetric ($g(a, b) = g(b, a)$) and non-degenerate ($g(a, b) = 0, \forall b \in M \Rightarrow a = 0$).However, it is customary that every component of the metric is a "well-behaved" function (continuous, differentiable, etc.) of the coordinates.

In this manifold, tensors (including vectors and one-forms) transform in the same way as in Special Relativity:

$$A'^{\alpha_1\alpha_2\cdots}_{\beta_1\beta_2\cdots} = \frac{\partial x^{\alpha_1}}{\partial x^{\mu_1}}\frac{\partial x^{\alpha_2}}{\partial x^{\mu_2}}\cdots\frac{\partial x^{\nu_1}}{\partial x^{\beta_1}}\frac{\partial x^{\nu_2}}{\partial x^{\beta_2}}\cdots A^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots} \tag{B.1.3}$$

B.2 Homology and cohomology

We can define the concept of **differential forms** over a manifold M as an anti-symmetric $(0, p)$ tensor[2].

$$\omega = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_p} \tag{B.2.1}$$

The set of p-forms is denoted as Λ^p, and we can define an exterior product, known as the **Wedge product**, as $\wedge : \Lambda^p \times \Lambda^q \longrightarrow \Lambda^{p+q}$, with

$$(A \wedge B)_{\mu_1\cdots\mu_{p+q}} = A_{[\mu_1\cdots\mu_p} B_{\mu_{p+1}\cdots\mu_{p+q}]} \tag{B.2.2}$$

As well as we can define a **differential operator**, $d : \Lambda^p \longrightarrow \Lambda^{p+1}$, with

$$(dA)_{\mu_1\cdots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]} \tag{B.2.3}$$

Now, note that

$$d^2 A = d(dA) = 0, \forall A \in \Lambda^p \tag{B.2.4}$$

With this in mind, we can define **closed** and **exact** forms. A closed form is the one that $dA = 0$, and an exact form is the one that $A = d\omega$. Note that every exact form is closed but not every closed form is exact.

Let us define the space of closed p-forms Z^p and the space of exact p-forms

B^p. We have $B^p \subset Z^p$. Moreover, we can define an **equivalence relation** in Z^p as

$$A \sim A + d\omega \tag{B.2.5}$$

Then, the quotient space

$$H^p(M) = \frac{Z^p(M)}{B^p(M)} \tag{B.2.6}$$

is known as the **de Rham cohomology space** (with cohomology classes as elements).

We shall define one last operator over forms, the **Hodge star operator**, $* : \Lambda^p \longrightarrow \Lambda^{n-p}$, where n is the dimension of M (that is, locally we can associate M with $\mathbb{R}^n$). This Hodge star operator acts like::

$$(*A)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \cdots \nu_p}_{\mu_1 \cdots \mu_{n-p}} A_{\nu_1 \cdots \nu_p} \tag{B.2.7}$$

$*A$ is known as the **Hodge dual of** A.

As well as we have a cohomology space $H^p(M)$, we also have a **homology space** $H_p(M)$. M is a manifold, therefore it is a topological space. So there are subspaces N in M of dimension $p < n$.

The operator $\partial : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$, that acts over N giving its boundary, works as a differential operator. Following this, we call a submanifold A closed, if $\partial A = \varnothing$, and exact if there is a N, so that $A = \partial N$. If N is a non-closed $(p+1)$-submanifold of M, then $A = \partial N \neq \varnothing$ is called a p-cycle. For example, if N is a circle (that is a non-closed 2-submanifold), then $A = \partial N$ is its circunference (1-cycle).

Just as in the caso of the p-forms, we may now construct the sets Z_p of closed p-submanifolds and B_p of exact p-submanifolds. And then, the **homology space** is defined as:

$$H_p(M) = \frac{Z_p(M)}{B_p(M)} \tag{B.2.8}$$

These spaces are related through the **Poincaré Duality** and the **Stoke's Theorem**:

$$\int_N d\omega = \int_{\partial N} \omega \tag{B.2.9}$$

B.3 Covariant deraivative

For forms, we saw that the differential operator transforms p-forms into $(p+1)$-forms. Following this we define the differential operator for tensors $\nabla : T(M)^k_l \longrightarrow T(M)^k_{l+1}$ as:

$$\nabla_{\nu_1} A^{\mu_1 \cdots \mu_k}_{\nu_2 \cdots \nu_{l+1}} = \frac{\partial}{\partial x^{\nu_1}} A^{\mu_1 \cdots \mu_k}_{\nu_2 \cdots \nu_{l+1}} \tag{B.3.1}$$

This equation is only valid in one certain reference frame. Let us see how it transforms for a vector, for example. We want this derivative to be a tensor, so

$$\nabla_\beta v^\alpha = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\beta} \nabla_\nu v^\mu = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\beta} \frac{\partial v^\mu}{\partial x^\nu} \tag{B.3.2}$$

On the other side, note that $v^\mu = \frac{\partial x^\mu}{\partial x^\alpha} v^\alpha$, $\frac{\partial}{\partial x^\nu} = \frac{\partial x^\beta}{\partial x^\nu} \frac{\partial}{\partial x^\beta}$. So

$$\begin{aligned}
\nabla_\beta v^\alpha &= \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^\beta} \frac{\partial x^\delta}{\partial x^\nu} \frac{\partial}{\partial x^\delta} \left(\frac{\partial x^\mu}{\partial x^\gamma} v^\gamma \right) \\
&= \frac{\partial x^\alpha}{\partial x^\mu} \delta^\delta_\beta \frac{\partial}{\partial x^\delta} \left(\frac{\partial x^\mu}{\partial x^\gamma} v^\gamma \right) \\
&= \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial}{\partial x^\beta} \left(\frac{\partial x^\mu}{\partial x^\gamma} v^\gamma \right) \\
&= \frac{\partial x^\alpha}{\partial x^\mu} \left(v^\gamma \frac{\partial}{\partial x^\beta} \frac{\partial x^\mu}{\partial x^\gamma} + \frac{\partial x^\mu}{\partial x^\gamma} \frac{\partial v^\gamma}{\partial x^\beta} \right) \\
&= v^\gamma \frac{\partial}{\partial x^\beta} \frac{\partial x^\mu}{\partial x^\gamma} \frac{\partial x^\alpha}{\partial x^\mu} + \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\gamma} \frac{\partial v^\gamma}{\partial x^\beta} \\
&= \frac{\partial v^\alpha}{\partial x^\beta} + \left(\frac{\partial}{\partial x^\beta} \frac{\partial x^\mu}{\partial x^\gamma} \frac{\partial x^\alpha}{\partial x^\mu} \right) v^\gamma
\end{aligned}$$

or,

$$\nabla_\beta v^\alpha = \partial_\beta v^\alpha + \Gamma^\alpha_{\beta\gamma} v^\gamma \tag{B.3.3}$$

where $\partial_\beta = \frac{\partial}{\partial x^\beta}, \Gamma^\alpha_{\beta\gamma} = \frac{\partial}{\partial x^\beta}\frac{\partial x^\mu}{\partial x^\gamma}\frac{\partial x^\alpha}{\partial x^\mu}$. The Γ's are called the Christoffel symbols, and they can be easily calculated through the metric $g_{\mu\nu}$, as

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right) \tag{B.3.4}$$

B.4 Parallel transport and geodesics

While studying curved manifolds, the concept of parallel transport is important. In these manifolds, the Euclid's fifth postulate is not satisfied, so two parallel lines can intersect in a finite distance. Therefore, they are the best way to study curvature.

Let $f : \mathbb{R} \longrightarrow M$ be a curve in M, parametrized by λ, and $\mathcal{P} \in M$ a point which this path f passes through. For this point $\mathcal{P}$ ($= x^\mu(\lambda)$ in certain frame), we can define a tangent vector space, and a vector $\overrightarrow{v}$ ($= v^\nu$). Let us see how $\overrightarrow{v}$ changes as λ changes. If we want $\overrightarrow{v}$ to be transported parallely through the path f, it has to remain the same, that is

$$\frac{dv^\nu}{d\lambda} = 0 \tag{B.4.1}$$

$$\frac{dx^\mu}{d\lambda}\nabla_\mu v^\nu = 0 \tag{B.4.2}$$

The derivative defined by B.3.3 and B.3.4 is the only one that preserves angles while parallel-transporting (Actually this is how you get B.3.4).

Note in B.4.2 that $\frac{dx^\mu}{d\lambda}$ is also a vector, the vector tangent to the curve f indeed. So what if we parallely transpart this vector? We get a differential equation for the path f, that is then called geodesic.

$$\frac{dx^\mu}{d\lambda}\nabla_\mu \frac{dx^\nu}{d\lambda} = 0 \tag{B.4.3}$$

B.5 Curvature

One of the consequences of the curvature of the manifold is that covariant derivatives do not commute with each other. Then we can define a new tensor in the following way:

$$R^\alpha{}_{\beta\mu\nu} V^\beta = [\nabla_\mu, \nabla_\nu] V^\alpha \tag{B.5.1}$$

This tensor $R^\alpha{}_{\beta\mu\nu}$ is called the **Riemann tensor**. This tensor can be calculated using the Christoffel symbols as

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\sigma\mu}\Gamma^\sigma{}_{\beta\nu} - \Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\beta\mu} \tag{B.5.2}$$

Having this Riemann tensor, we can define the **Ricci tensor** as

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} \tag{B.5.3}$$

as well as the **Ricci scalar**, also known as the curvature scalar:

$$R = g^{\mu\nu} R_{\mu\nu} \tag{B.5.4}$$

After defining all these new tensors, we can write down the equation that Einstein got in his 1915 paper [25], that would be called later the **Einstein equation**:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{B.5.5}$$

where $T_{\mu\nu}$ is the **stress-energy tensor**, and describes the content of the spacetime whose curvature is described by the Ricci tensor, Ricci scalar and metric.

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