Centro de Investigación y de Estudios
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Departamento de Matemáticas

Sistemas p-Ádicos del tipo FitzHugh-Nagumo y Patrones de Turing

T E S I S

Que presenta

M. en C. Carlos Alberto Garcia Bibiano

para obtener el Grado de

Doctor en Ciencias

en la Especialidad de Matemáticas

Directores de la Tesis:

Dr. Wilson Álvaro Zúñiga Galindo

Dr. Carlos Gabriel Pacheco González

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Resumen

Esta tesis gira entorno a ecuaciones pseudo-diferenciales del tipo Nagumo, sistemas de ecuaciones pseudo-diferenciales del tipo FitzHugh-Nagumo, y patrones de Turing en el contexto p-ádico. La tesis consiste en dos partes. En la primera parte, presentamos una nueva familia de ecuaciones de evolución p-ádicas no lineales tipo Nagumo. Establecemos el buen planteamiento local del problema de Cauchy para estas ecuaciones en espacios tipo Sobolev. Para cierta subfamilia, mostramos que ocurre el fenómeno de la explosión en tiempo finito y proporcionamos simulaciones numéricas que muestran este fenómeno. En la segunda parte, presentamos versiones continuas p-ádicas y discretas de un sistema de FitzHugh-Nagumo en la bola unidimensional p-ádica. Damos criterios para la existencia de patrones de Turing. Presentamos extensas simulaciones de algunos de estos sistemas. Las simulaciones muestran que los patrones de Turing son ondas viajeras en la bola unitaria p-ádica. Esta tesis está basada en las publicaciones [12, 13] escritas en colaboración con mis directores de tesis Dr. Wilson Álvaro Zúñiga Galindo (University of Texas Rio Grande Valley, USA) y Dr. Leonardo Fabio Chacón Cortés (Pontificia Universidad Javeriana, Colombia).

Abstract

This dissertation revolves around pseudo-differential equations of the Nagumo type, systems of pseudo-differential equations of the FitzHugh-Nagumo-type, and Turing patterns in the p-adic context. The thesis consists of two parts. We present a new Nagumo-type nonlinear p-adic evolution equation family in the first part. We establish the local well-posedness of the Cauchy problem for these equations in Sobolev-type spaces. For a particular subfamily, we show that the explosion phenomenon occurs in finite time and provide numerical simulations showing this phenomenon. In the second part, we present discrete and p-adic continuous versions of a FitzHugh-Nagumo system on the p-adic one-dimensional ball. We give criteria for the existence of Turing patterns. We show extensive simulations of some of these systems. The simulations present that the Turing patterns are traveling waves on the p-adic unit ball. This thesis is based on the publications [12, 13] written in collaboration with my thesis supervisors, Dr. Wilson Álvaro Zúñiga Galindo (University of Texas Rio Grande Valley, USA) and Dr. Leonardo Fabio Chacón Cortés (Pontificia Universidad Javeriana, Colombia).

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Overview

Nowadays, the theory of linear partial pseudo-differential equations for complex-valued functions over p-adic fields is a well-established branch of mathematical analysis; see, e.g., [1, 15, 24, 34, 46, 53, 56, 65], and references therein. Meanwhile, more is needed to know about non-linear p-adic equations. We can mention some semilinear evolution equations solved using p-adic wavelets [1, 51], a kind of equations of reaction-diffusion type and Turing patterns studied in [65, 66], a p-adic analog of one of the porous medium equation [29, 46], the blow-up phenomenon studied in [14], and non-linear integro-differential equations connected with p-adic cellular networks [61].

In this work, we present the results obtained in the research article "Local Well-Posedness of the Cauchy Problem for a p-adic Nagumo-Type Equation". This work was carried out in collaboration with Dr. Wilson Álvaro Zúñiga Galindo and Dr. Leonardo Fabio Chacón Cortés, and it was published in p-Adic Numbers, Ultrametric Analysis and Applications, [12].

Among the ultrametric spaces, the field of p-adic numbers $\mathbb{Q}_p$ plays a central role. Any p-adic number is represented as a series of the form:

$$x = x_{-k}p^{-k} + x_{-k+1}p^{-k+1} + \cdots + x_0 + x_1 p + \cdots, \text{ with } x_{-k} \neq 0, \qquad (0.0.1)$$

where p is a prime number, the x_js are p-adic digits, i.e., numbers in the set $\{0, 1, \ldots, p-1\}$ and $k \in \mathbb{Z}$. The set of all the possible series of forms (0.0.1) constitutes the field of p-adic numbers, $\mathbb{Q}_p$. There are natural field operations, sum, and multiplication, on series of form (0.0.1), see, e.g., [32]. There is a natural norm in $\mathbb{Q}_p$ defined as $|x|_p = p^k$ (p-adic norm), for a non-zero p-adic number x of the form (0.0.1). The field of p-adic numbers with the distance induced by $|\cdot|_p$ is a complete ultrametric space. The ultrametric property refers to the fact that $|x-y|_p \leq \max\{|x-z|_p, |z-y|_p\}$ for any $x, y, z \in \mathbb{Q}_p$ (strong triangle inequality).

We set $[\xi]_p := \max\left\{1, \|\xi\|_p\right\}$ for $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^n$, where $\|\xi\|_p = \max_{1\leq i\leq n} |\xi_i|_p$. Given $\varphi, \varrho \in \mathcal{D}(\mathbb{Q}_p^n)$ (the Bruhat-Schwartz space, see section 1.1.4) and $s \in \mathbb{R}$, we define the scalar product:

$$\langle \varphi, \varrho \rangle_s = \int_{\mathbb{Q}_p^n} [\xi]_p^s \, \widehat{\varphi}(\xi) \overline{\widehat{\varrho}(\xi)} d^n\xi,$$

where the bar denotes the complex conjugate. We also set $\|\varphi\|_s^2 = \langle \varphi, \varphi \rangle_s$, and denote by $\mathcal{H}_s := \mathcal{H}_s(\mathbb{Q}_p^n, \mathbb{C}) = \mathcal{H}_s(\mathbb{C})$ the completion of $\mathcal{D}(\mathbb{Q}_p^n)$ with respect to $\langle \cdot, \cdot \rangle_s$.

We introduce a new family of non-linear evolution equations that we have named p-adic Nagumo-type equations:

$$u_t = -\gamma \boldsymbol{D}_x^\alpha u - u^3 + (\beta+1)\, u^2 - \beta u + P(\boldsymbol{D}_x)\,(u^m)\,,\ x \in \mathbb{Q}_p^n,\ \ t \in [0,T]\,,$$

where $\gamma > 0$, $\beta \geq 0$, $\boldsymbol{D}_x^\alpha$, $\alpha > 0$, is the Taibleson operator, m is a positive integer and $P(\boldsymbol{D}_x)$ is an operator of degree δ of the form $P(\boldsymbol{D}) = \sum_{j=0}^{k} C_j \boldsymbol{D}^{\delta_j}$, where the $C_j \in \mathbb{R}$ and $\delta_k = \delta$. We establish the local well-posedness of the following Cauchy problem:

$$\begin{cases} u \in C^1\left([0,T]\,;\mathcal{H}_s\right); \\ \\ u_t = -\gamma \boldsymbol{D}_x^\alpha u - u^3 + (\beta+1)\, u^2 - \beta u + P(\boldsymbol{D}_x)\,(u^m)\,, \quad x \in \mathbb{Q}_p^n,\ \ t \in [0,T]\,; \\ \\ u(0) = f_0 \in \mathcal{H}_s, \end{cases}$$

where T, γ, α, $\beta > 0$, and m is a positive integer, see Theorem 2.1.1. We show that the blow-up phenomenon occurs for a certain subfamily, see Theorem 2.2.1, and provide numerical simulations showing this phenomenon.

The theory of Sobolev-type spaces used here was developed in [64], see also [30, 53]. This theory is based on the theory of countably Hilbert spaces of Gel'fand-Vilenkin [20]. Some generalizations are presented in [21, 22]. We use classical techniques of operator semigroups, see, e.g., [11, 40]. The family of evolution equations studied here contains, as a particular case, equations of the form:

$$u_t = -\gamma \boldsymbol{D}_x^\alpha u - u^3 + (\beta+1)\, u^2 - \beta u, \tag{0.0.2}$$

where $x \in \mathbb{Q}_p^n$, $t \in [0,T]$, $\boldsymbol{D}_x^\alpha$ is the Taibleson operator, that resembles the classical Nagumo-type equations, see, e.g., [44].

In [39], the authors study the equations

$$u_t = Du_{xx} - u\,(u-\kappa)\,(u-1) - \varepsilon u_x^m, \tag{0.0.3}$$

where $D > 0$, $\kappa \in \left(0, \frac{1}{2}\right)$, $\varepsilon > 0$, $x \in \mathbb{R}$, $t > 0$. They establish the local well-posedness of the Cauchy problem for these equations in standard Sobolev spaces. There are several crucial differences between (0.0.2) and (0.0.3). The operators u_{xx}, u_x^m are local while the operators $\boldsymbol{D}_x^\alpha$, $P(\boldsymbol{D}_x)\,(\cdot^m)$ are non-local. The p-adic heat equation $u_t = -\gamma \boldsymbol{D}_x^\alpha u$ has an arbitrary order of pseudo-differentiability $\alpha > 0$ in the spatial variable, while in the classical fractional heat equation $u_t = D\frac{\partial^\mu u}{\partial x^\mu}$, the degree of pseudo-differentiability $\mu \in (0, 2]$. This implies that the Markov processes attached to $u_t = -\gamma \boldsymbol{D}_x^\alpha u$ are completely different to the ones attached to $u_t = Du_{xx}$. In other words, the diffusion mechanisms in (0.0.2) and (0.0.3) are completely different, since the Markov processes associated with equations (0.0.2) and (0.0.3), respectively, are different. Notice that our non-linear term involves pseudo-derivatives of arbitrary order $P(\boldsymbol{D}_x)(u^m)$, while in [39] only of first order u_x^m. Of course, the p-adic Sobolev spaces behave completely differently from their real counterparts, but the semigroup techniques are the same in both cases since time is a non-negative real variable.

The following expression gives the Cauchy problem for which we study the blow-up phenomenon:

$$\begin{cases} u_t = -\gamma \boldsymbol{D}_x^\alpha u + F(u) + \boldsymbol{D}_x^{\alpha_1} u^3, \quad x \in \mathbb{Q}_p^n, \ t \in [0, T]; \\ \\ u(0) = f_0 \in \mathcal{H}_\infty, \end{cases} \tag{0.0.4}$$

where $F(u) = -u^3 + (\beta + 1) u^2 - \beta u$. For the Cauchy problem (0.0.4), we establish Theorem 2.2.1. We present two numerical simulations for the solution of problem (0.0.4) (in dimension one) for a suitable initial datum.

Several models involving parabolic equations have been used in neuroscience to propagate nerve impulses. Among these models, the one of FitzHugh-Nagumo plays a central role. Proposed in the 1950s by FitzHugh, this model accurately explains the propagation of electric impulses along the nerve axon of the giant squid. See [43, 50] and the references therein. Nowadays, the FitzHugh-Nagumo system is the simplest model to describe pulse propagation in a spatial region. The simplest version of this system is

$$\begin{cases} \partial_t u(x, t) = mu - u^3 - v + L_u \nabla^2 u; \\ \\ \partial_t v(x, t) = c(u - av - b) + L_v \nabla^2 v, \end{cases} \tag{0.0.5}$$

where the system parameters a, b, c, m, L_u, and L_v, are assumed to be positive, and the functions u and v depend on time $t \geq 0$ and the position $x \in \mathbb{R}$ on the domain of interest. The variable u promotes the self-growth of u and, simultaneously, the growth of v and can thus be named an activator, while v plays the role of an inhibitor that annuls the growth of u.

We present the results obtained in the research article “Turing Patterns in a p-adic FitzHugh-Nagumo System on the Unit Ball”. These results were obtained in collaboration with Dr. Wilson Álvaro Zúñiga Galindo and Dr. Leonardo Fabio Chacón Cortés, and it was published in p-Adic Numbers, Ultrametric Analysis and Applications, [13].

We introduce a p-adic counterpart of system (0.0.5). In the new model, x runs through the ring of p-adic integers $\mathbb{Z}_p$; here, p is a fixed prime number, and t is a real variable. Geometrically, $\mathbb{Z}_p$ is an infinite rooted tree; analytically, $\mathbb{Z}_p$ is a locally compact topological additive group with a very rich mathematical structure. The system takes the following form:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = f(u,v) - \left(\boldsymbol{D}_0^{\alpha} - \lambda\right) u(x,t); \\ \\ \frac{\partial v}{\partial t}(x,t) = g(u,v) - d\left(\boldsymbol{D}_0^{\alpha} - \lambda\right) v(x,t), \quad x \in \mathbb{Z}_p, \quad t \geq 0, \end{cases} \tag{0.0.6}$$

where $\boldsymbol{D}_0^{\alpha} - \lambda$ is the Vladimirov operator on $\mathbb{Z}_p$, and $f(u,v) = \mu u - u^3 - v$, $g(u,v) = \gamma(u - \delta v - \beta)$, where μ, β are real numbers, and γ, δ, d are positive real numbers. This system admits a natural discretization of the form:

$$\begin{cases} \frac{\partial}{\partial t}\left[u_L(I,t)\right]_{I\in G_L} = \left[\mu u_L(I,t) - u_L^3(I,t) - v_L(I,t)\right]_{I\in G_L} - A_L^{\alpha}\left[u_L(I,t)\right]_{I\in G_L}; \\ \\ \frac{\partial}{\partial t}\left[v_L(I,t)\right]_{I\in G_L} = \left[\gamma\left(u_L(I,t) - \delta v_L(I,t) - \beta\right)\right]_{I\in G_L} - dA_L^{\alpha}\left[v_L(I,t)\right]_{I\in G_L}, \end{cases} \tag{0.0.7}$$

where G_L is a finite rooted tree with L levels, and matrix $A_L^{\alpha} = \left[A_{K,I}^{\alpha}\right]_{K,I\in G_L}$ is a discretization of operator $\boldsymbol{D}_0^{\alpha} - \lambda$, where

$$A_{K,I}^{\alpha} = \begin{cases} p^{-\frac{L}{2}} \frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \frac{1}{|K-I|_p^{\alpha+1}} & \text{if} \quad K \neq I; \\ \\ -p^{-\frac{L}{2}} \frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \sum_{K\neq I} \frac{1}{|K-I|_p^{\alpha+1}} & \text{if} \quad K = I. \end{cases}$$

We present Turing instability criteria for systems (0.0.6) and (0.0.7), see Theorems 3.1.1, 3.2.1. The conditions for the existence of Turing patterns for both systems are essentially the same, except for one condition which involves a subset Γ of the eigenvalues of $\boldsymbol{D}_0^{\alpha} - \lambda$, in the case of the system (0.0.6), and a subset Γ_L of the eigenvalues of matrix A_L^{α}, in the case of (0.0.7). We provide extensive numerical simulations of some systems of type (0.0.7); in particular, these experiments show that the Turing patterns are traveling waves inside the unit ball $\mathbb{Z}_p$. Our numerical experiments also show that the eigenvalues of matrix A_L^{α} approximate the eigenvalues of $\boldsymbol{D}_0^{\alpha} - \lambda$. We conjecture that the Turing patterns of (0.0.7) converge, in some sense, to the Turing patterns of (0.0.6). The results of Digernes and his collaborators on the problem of approximation of spectra of Vladimirov operator $\boldsymbol{D}^{\alpha}$ by matrices of type A_L^{α}, [5, 17] provide strong support to our conjecture.

Nowadays, the study of Turing patterns on networks is a relevant area. In the 70s, Othmer and Scriven pointed out that Turing instability can occur in network-organized systems [47, 48]. Since then, reaction-diffusion models on networks have been studied intensively, see, e.g., [2,8,9,16,27,41,42,45,48,54,62], and the references therein. In particular, Turing patterns of discrete FitzHugh-Nagumo systems have also been studied [10]. In [66, 67], the last author established the existence of Turing patterns for specific p-adic systems of reaction-diffusion equations. Still, these papers do not consider the problem of the numerical approximation of the Turing patterns. Digernes and his collaborators have studied extensively the problem of approximation of spectra of Vladimirov operator $\boldsymbol{D}^{\alpha}$ by matrices of type A_L^{α}, [5, 17].

This work presents numerical approximations of Turing patterns associated with specific p-adic FitzHugh-Nagumo systems. For these systems, we give various visualizations of the solutions, intending to show several aspects of the Turing patterns.

This thesis is structured as follows. In Chapter 1, we review some essential aspects of the p-adic analysis, the p-adic fractional operators and the p-adic heat equation on the unit ball, technical results about Sobolev-type spaces and p-adic pseudo-differential operators, the definition of the spaces $\mathcal{D}_M^{-L}$, and fix the notation. Chapter 2 is organized as follows. Section 2.1 shows the local well-posedness of the p-adic Nagumo-type equations. See Theorem 2.1.1. Section 2.2 shows a subfamily of p-adic Nagumo-type equations whose solutions blow up in finite time. See Theorem 2.2.1. Section 2.3 presents a numerical simulation showing the blow-up phenomenon. Chapter 3 is organized as follows. Section 3.1 introduces our p-adic FitzHugh-Nagumo system and gives a Turing instability criterion, see Theorem 3.1.1. In Section 3.2, we study a discrete version of our p-adic FitzHugh-Nagumo system and give a Turing instability criterion, see Theorem 3.2.1. Finally, Section 3.3 provides extensive numerical simulation for some discrete FitzHugh-Nagumo systems and their Turing patterns. In Chapter 4, we give conclusions about the work done in Chapters 2 and 3.

Contents

Chapter 1

Mathematical preliminaries

1.1 Essential Ideas of p-Adic Analysis

In this section, we collect some basic results on p-adic analysis that we use throughout the work. For a detailed exposition, the reader may consult [1, 33, 55, 58].

1.1.1 The field of p-adic numbers

Along this work, p will denote a prime number. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0; \\ \\ p^{-\gamma} & \text{if} \quad x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *p-adic order of* x. We extend the p-adic norm to $\mathbb{Q}_p$ by taking

$$\|x\|_p := \max_{1\leq i\leq n} |x_i|_p, \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p. \tag{1.1.1}$$

We define $ord(x) : \min_{1\leq i\leq n}\{ord(x_i)\}$, then $\|x\|_p = p^{-ord(x)}$. The metric space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a complete ultrametric space. As a topological space $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line, see, e.g., [1, 58].

Any p-adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, \dots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if} \quad x = 0 \text{ or } ord(x) \geq 0; \\ \\ p^{ord(x)} \sum_{j=0}^{-ord_p(x)-1} x_j p^j & \text{if} \quad ord(x) < 0. \end{cases}$$

1.1.2 Basic topology of $\mathbb{Q}_p^n$

For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n : \|x - a\|_p \leq p^r\}$ *the ball of radius* p^r *with center at* $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(0) := B_r^n$. Note that $B_r^n(a) = B_r(a_1) \times \dots \times B_r(a_n)$, where $B_r(a_i) := B_r^1(a_i) = \{x \in \mathbb{Q}_p : |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$, *the ring of p-adic interger* . We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n : \|x - a\|_p = p^r\}$ *the sphere of radius* p^r *with center at* $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) := S_r^n$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$). The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. In addition, two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other.
As a topological space $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is totally disconnected, i.e., the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$, see, e.g., [58, Section 1.3], or [1, Section 1.8].

Notation 1.1.1. *We will use* $\Omega(p^{-r}\|x - a\|_p)$ *to denote the characteristic function of the ball* $B_r^n(a)$. *For more general sets, we will use the notation* 1_A *for the characteristic function of a set* A.

1.1.3 Integration in $\mathbb{Q}_p^n$

We review Haar's theorem for locally compact topological groups, which allow us to develop an integration theory in $\mathbb{Q}_p^n$. For further details, the reader may consult [1, Chapter 3] and [58, Chapter 4].

Theorem 1.1.1 ([23, Theorem B, Section 58])**.** *Let* $(G, +)$ *be a locally compact topological group. There exists a Borel measure* dx, *unique up to multiplication by a positive constant, such that* $\int_U dx > 0$ *for every non empty Borel open set* U, *and* $\int_{x+E} dx = \int_E dx$, *for every Borel set* E.

The measure dx is called a Haar measure of G. Since $(\mathbb{Q}_p^n, +)$ is a locally compact topological group, by Theorem 1.1.1 there exists a Haar measure of $\mathbb{Q}_p$, which we denote by $d^n x$. We normalize this measure using the condition $\int_{\mathbb{Z}_p^n} d^n x = 1$, then $d^n x$ is unique. The measure $d^n x$ grees with the product measure $dx_1 \cdots dx_n$ and also is invariant under translations, i.e., $d^n(x+a) = d^n x$.

1.1.4 The Bruhat-Schwartz space

A complex-valued function φ defined on $\mathbb{Q}_p^n$ is *called locally constant* if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for any } x' \in B_{l(x)}^n. \tag{1.1.2}$$

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Bruhat-Schwartz function (or a test function)* if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^n) := \mathcal{D}$. We denote by $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^n) := \mathcal{D}_{\mathbb{R}}$ the $\mathbb{R}$-vector space of Bruhat-Schwartz functions. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, we have that φ has compact support, which implies that there exists the largest number $l = l(\varphi)$ satisfying (1.1.2) called *the exponent of local constancy (or the parameter of constancy) of* φ.
We denote by $\mathcal{D}_m^l(\mathbb{Q}_p^n)$ the finite-dimensional space of test functions from $\mathcal{D}(\mathbb{Q}_p^n)$ having supports in the ball B_m and with parameters of constancy $\geq l$. We now define a topology on $\mathcal{D}$ as follows. We say that a succession $\{\varphi_j\}_{j\in\mathbb{N}}$ of functions in $\mathcal{D}$ converges to zero, if the two following conditions hold:

(1) there are two fixed integers k_0 and m_0 such that each $\varphi_j \in \mathcal{D}_{m_0}^{k_0}$;

(2) $\varphi_j \to 0$ uniformly. $\mathcal{D}$ endowed with the above topology becomes a topological vector space.

1.1.5 L^ρ spaces

Given $\rho \in [1, \infty)$, we denote by $L^\rho := L^\rho\left(\mathbb{Q}_p^n\right) := L^\rho\left(\mathbb{Q}_p^n, dx\right)$, the $\mathbb{C}$-vector space of all the complex-valued functions g satisfying

$$\int_{\mathbb{Q}_p^n} |g(x)|^\rho \, d^n x < \infty.$$

The corresponding $\mathbb{R}$-vector spaces are denoted as $L^\rho_{\mathbb{R}} := L^\rho_{\mathbb{R}}\left(\mathbb{Q}_p^n\right) = L^\rho_{\mathbb{R}}\left(\mathbb{Q}_p^n, d^n x\right)$, $1 \leq \rho < \infty$.

If U is an open subset of $\mathbb{Q}_p^n$, $\mathcal{D}(U)$ define the space of test functions with supports contained in U, then $\mathcal{D}(U)$ is dense in

$$L^\rho\left(U\right) = \left\{\varphi : U \to \mathbb{C}; \left\|\varphi\right\|_\rho = \left\{\int\limits_U \left|\varphi\left(x\right)\right|^\rho dx\right\}^{\frac{1}{\rho}} < \infty\right\},$$

where dx is the normalized Haar measure on $\left(\mathbb{Q}_p^n, +\right)$, for $1 \leq \rho < \infty$, see, e.g., [1, Section 4.3]. We denote by $L^\rho_{\mathbb{R}}\left(U\right)$ the real counterpart of $L^\rho\left(U\right)$.

1.1.6 The Fourier transform

Set $\chi_p(y) = \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e., a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$. The isomorphism is given by $\kappa \to \chi_p(\kappa x)$, see, e.g., [1, Section 2.3].

Given $\xi = (\xi_1, \ldots, \xi_n)$ and $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{j=1}^n \xi_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int\limits_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x)\varphi(x)dx^n \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where dx^n is the normalized Haar measure on $\mathbb{Q}_p^n$. We will also use the notation $\mathcal{F}_{x\to\xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

The Fourier transform extends to L^2. If $f \in L^2$, its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \lim_{k\to\infty} \int\limits_{\|x\|_p \leq p^k} \chi_p(\xi \cdot x) f(x) dx^n, \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where the limit is taken in L^2.

1.1.7 Distributions

The $\mathbb{C}$-vector space $\mathcal{D}'\left(\mathbb{Q}_p^n\right) := \mathcal{D}'$ of all continuous linear functionals on $\mathcal{D}(\mathbb{Q}_p^n)$ is called the *Bruhat-Schwartz space of distributions*. Every linear functional on $\mathcal{D}$ is continuous, i.e.,

$\mathcal{D}'$ agrees with the algebraic dual of $\mathcal{D}$, see, e.g., [58, Chapter 1, VI.3, Lemma]. We denote by $\mathcal{D}'_{\mathbb{R}}\left(\mathbb{Q}_p^n\right) := \mathcal{D}'_{\mathbb{R}}$ the dual space of $\mathcal{D}_{\mathbb{R}}$.

We endow $\mathcal{D}'$ with the weak topology, i.e., a sequence $\{T_j\}_{j\in\mathbb{N}}$ in $\mathcal{D}'$ converges to T if $\lim_{j\to\infty} T_j(\varphi) = T(\varphi)$ for any $\varphi \in \mathcal{D}$. The map

$$\begin{array}{lcl} \mathcal{D}' \times \mathcal{D} & \to & \mathbb{C} \\ (T,\varphi) & \to & T(\varphi) \end{array}$$

is a bilinear form which is continuous in T and φ separately. We call this map the pairing between $\mathcal{D}'$ and $\mathcal{D}$. From now on we will use (T,φ) instead of $T(\varphi)$.

Every f in L^1_{loc} defines a distribution $f \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ by the formula

$$(f,\varphi) = f(x)\,\varphi(x)\,dx.$$

Such distributions are called *regular distributions*. Notice that for $f \in L^2_{\mathbb{R}}$, $(f,\varphi) = \langle f,\varphi\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2_{\mathbb{R}}$.

1.1.8 The Fourier transform of a distribution

The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ is defined by

$$(\mathcal{F}[T],\varphi) = (T,\mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

The Fourier transform $T \to \mathcal{F}[T]$ is a linear (and continuous) isomorphism from $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ onto $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

1.2 p-Adic fractional operators

1.2.1 The Taibleson operator

Let $\alpha > 0$, the Taibleson operator is defined as

$$(\boldsymbol{D}^\alpha\varphi)(x) = \mathcal{F}^{-1}_{\xi\to x}(\|\xi\|_p^\alpha\,(\mathcal{F}_{x\to\xi}\varphi)),$$

for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$. This operator admits the extension

$$(\boldsymbol{D}^\alpha f)(x) = \frac{1-p^\alpha}{1-p^{-\alpha-n}}\int_{\mathbb{Q}_p^n}\|y\|_p^{-\alpha-n}\{f(x-y)-f(x)\}d^n y$$

to locally constant functions satisfying

$$\int_{\|x\|_p>1} \|x\|_p^{-\alpha-n} |f(x)| \, d^n x < \infty.$$

The Taibleson operator $\boldsymbol{D}^\alpha$ is the p-adic analog of the fractional derivative. If $n = 1$, $\boldsymbol{D}^\alpha$ agrees with the Vladimirov operator. The operator $\boldsymbol{D}^\alpha$ does not satisfy the chain rule neither Leibniz formula. We also use the notation $\boldsymbol{D}_x^\alpha$, when the Taibleson operator acts on functions depending on the variables $x \in \mathbb{Q}_p^n$ and $t \geq 0$.

Given $0 = \delta_0 < \delta_1 < \cdots < \delta_{k-1} < \delta_k = \delta$, we define

$$P(\boldsymbol{D}) = \sum_{j=0}^{k} C_j \boldsymbol{D}^{\delta_j}, \text{ where the } C_j \in \mathbb{R}.$$

1.2.2 The Vladimirov operator

The one-dimensional fractional operator $\boldsymbol{D}_x^\alpha : \varphi(x) \to \boldsymbol{D}_x^\alpha \varphi(x)$ is defined on $\mathcal{D}(\mathbb{Q}_p)$ as a convolution operator

$$(\boldsymbol{D}_x^\alpha \varphi)(x) = f_{-\alpha}(x) * \varphi(x), \quad x \in \mathbb{Q}_p, \quad \mathrm{Re}(\alpha) \neq -1. \tag{1.2.1}$$

The operator $\boldsymbol{D}_x^\alpha$ is called the operator of (fractional) differentiation of order α with respect to x, for $\mathrm{Re}(\alpha) > 0$; the operator of (fractional) integration of order α with respect to x, for $\mathrm{Re}(\alpha) < 0$, $\mathrm{Re}(\alpha) \neq -1$; for $\alpha = 0$, $\boldsymbol{D}_x^0 \varphi(x) = \delta(x) * \varphi(x) = \varphi(x)$ is the identity operator.

For $\mathrm{Re}(\alpha) > 0$, the relation (1.2.2) is an extension of (1.2.1)

$$\begin{aligned}(\boldsymbol{D}_x^\alpha \varphi)(x) &= \frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{\varphi(x) - \varphi(\xi)}{|x - \xi|_p^{\alpha+1}} d\xi \\ &= \int_{\mathbb{Q}_p} |\xi|_p^\alpha \mathcal{F}_{x\to\xi}\varphi(x) \chi_p(-\xi x) d\xi, \quad x \in \mathbb{Q}_p.\end{aligned} \tag{1.2.2}$$

If $\mathrm{Re}(\alpha) < 0$, $\mathrm{Re}(\alpha) \neq -1$, the relation (1.2.1) implies that

$$(\boldsymbol{D}_x^\alpha \varphi)(x) = \frac{1 - p^\alpha}{1 - p^{-\alpha-1}} \int_{\mathbb{Q}_p} |x - \xi|_p^{-\alpha} \varphi(\xi) d\xi$$

$$= \begin{cases} \int_{\mathbb{Q}_p} |\xi|_p^{\alpha_j} \mathcal{F}_{x\to\xi}\varphi(x) \chi_p(-\xi x) d\xi, & \mathrm{Re}(\alpha) > -1; \\ \\ \int_{\mathbb{Q}_p} |\xi|_p^\alpha \left(\mathcal{F}_{x\to\xi}\varphi(x) \chi_p(-\xi x) - \mathcal{F}_{x\to\xi}(0)\right) d\xi, & \mathrm{Re}(\alpha) < -1. \end{cases} \tag{1.2.3}$$

1.2.3 The p-adic heat equation

For $\alpha > 0$, the Vladimirov-Taibleson operator $\boldsymbol{D}^{\alpha}$ is defined as

$$\begin{array}{ccc} \mathcal{D}(\mathbb{Q}_p) & \to & L^2(\mathbb{Q}_p) \cap \mathcal{C}(\mathbb{Q}_p) \\ \\ \varphi & \to & \boldsymbol{D}^{\alpha}\varphi, \end{array}$$

where

$$(\boldsymbol{D}^{\alpha}\varphi)(x) = \frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{\varphi(x-y)-\varphi(x)}{|y|_p^{\alpha+1}} dy.$$

We use the notation $\boldsymbol{D}_x^{\alpha}$, in cases where the operator acts on functions depending on several variables, to indicate that the Vladimirov-Taibleson derivative is with respect to the variable x.

The set of functions $\{\Psi_{rnj}\}$ defined as

$$\Psi_{rnj}(x) = p^{-\frac{r}{2}} \chi_p\left(p^{-1}j\left(p^r x - n\right)\right) \Omega\left(\left|p^r x - n\right|_p\right),$$

where $r \in \mathbb{Z}$, $j \in \{1, \ldots, p-1\}$, and n runs through a fixed set of representatives of $\mathbb{Q}_p/\mathbb{Z}_p$, is an othonormal basis of $L^2(\mathbb{Q}_p)$ consisting of eigenvectors of operator $\boldsymbol{D}^{\alpha}$:

$$\boldsymbol{D}^{\alpha}\Psi_{rnj} = p^{(1-r)\alpha}\Psi_{rnj} \text{ for any } r, n, j,$$

[1, Theorem 9.4.2], [30, Theorem 3.29]. We set $\mathcal{L}(\mathbb{Z}_p)$ to be the $\mathbb{C}$-vector space generated by the functions $\Psi_{rnj}(x)$ with support in $\mathbb{Z}_p$, which are exactly those satisfying

$$r \leq 0, \ n \in p^r\mathbb{Z}_p \cap \mathbb{Q}_p/\mathbb{Z}_p. \ j \in \{1, \ldots, p-1\}. \tag{1.2.4}$$

Note that $\mathcal{L}(\mathbb{Z}_p)$ is a closed subspace of $L^2(\mathbb{Z}_p)$.

The p-adic analogue of the heat equation is

$$\frac{\partial u(x,t)}{\partial t} + a\boldsymbol{D}^{\alpha}u(x,t) = 0, \text{ with } a > 0.$$

The solution of the Cauchy problem attached to the heat equation with initial datum $u(x,0) = \varphi(x) \in \mathcal{D}(\mathbb{Q}_p)$ is given by

$$u(x,t) = \int_{\mathbb{Q}_p} Z(x-y,t)\varphi(y)\,dy,$$

where $Z(x,t)$ is the p-*adic heat kernel* defined as

$$Z(x,t) = \int_{\mathbb{Q}_p} \chi_p(-x\xi)\, e^{-at|\xi|_p^\alpha} d\xi, \tag{1.2.5}$$

where $\chi_p(-x\xi)$ is the standard additive character of the group $(\mathbb{Q}_p, +)$. The p-adic heat kernel is the transition density function of a Markov stochastic process with space state $\mathbb{Q}_p$, see, e.g., [33, 63].

1.2.4 The p-adic heat equation on the unit ball

We define the operator $\boldsymbol{D}_0^\alpha$, $\alpha > 0$, by restricting $\boldsymbol{D}^\alpha$ to $\mathcal{D}(\mathbb{Z}_p)$ and considering $(\boldsymbol{D}^\alpha\varphi)(x)$ only for $x \in \mathbb{Z}_p$. The operator $\boldsymbol{D}_0^\alpha$ satisfies

$$(\boldsymbol{D}_0^\alpha - \lambda)\,\varphi(x) = \frac{1-p^\alpha}{1-p^{-\alpha-1}} \int_{\mathbb{Z}_p} \frac{\varphi(x-y)-\varphi(x)}{|y|_p^{\alpha+1}} dy, \tag{1.2.6}$$

for $\varphi \in \mathcal{D}(\mathbb{Z}_p)$, with

$$\lambda = \frac{p-1}{p^{\alpha+1}-1} p^\alpha.$$

Consider the Cauchy problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + (\boldsymbol{D}_0^\alpha - \lambda)\, u(x,t) = 0, & x \in \mathbb{Z}_p, \quad t > 0; \\ \\ u(x,0) = \varphi(x), & x \in \mathbb{Z}_p, \end{cases}$$

where $\varphi \in \mathcal{D}(\mathbb{Z}_p)$. The solution of this problem is given by

$$u(x,t) = \int_{\mathbb{Z}_p} Z_0(x-y,t)\varphi(y)\, dy,\ x \in \mathbb{Z}_p,\ t > 0,$$

where

$$Z_0(x,t) := e^{\lambda t} Z(x,t) + c(t),\ x \in \mathbb{Z}_p,\ t > 0,$$
$$c(t) := 1 - (1-p^{-1}) e^{\lambda t} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n \frac{1}{1-p^{-n\alpha-1}},$$

and $Z(x,t)$ is given (1.2.5). The function $Z_0(x,t)$ is non-negative for $x \in \mathbb{Z}_p$, $t > 0$, and

$$\int_{\mathbb{Z}_p} Z_0(x,t) dx = 1,$$

[33]. Furthermore, $Z_0(x,t)$ is the transition density function of a Markov process with space state $\mathbb{Z}_p$.

1.2.5 p-Adic wavelets supported in balls

The set of functions $\{\Psi_{rnj}\}$ defined as

$$\Psi_{rnj}(x) = p^{\frac{-r}{2}} \chi_p\left(p^{-1} j\left(p^r x - n\right)\right) \Omega\left(\left|p^r x - n\right|_p\right), \tag{1.2.7}$$

where $r \in \mathbb{Z}$, $j \in \{1, \cdots, p-1\}$, and n runs through a fixed set of representatives of $\mathbb{Q}_p/\mathbb{Z}_p$, is an orthonormal basis of $L^2(\mathbb{Q}_p)$ consisting of eigenvectors of operator $\boldsymbol{D}^\alpha$:

$$\boldsymbol{D}^\alpha \Psi_{rnj} = p^{(1-r)\alpha} \Psi_{rnj} \text{ for any } r, n, j,$$

see, e.g., [1, Theorem 9.4.2], [30, Theorem 3.29]. By using this basis, it is possible to construct an orthonormal basis for $L^2(\mathbb{Z}_p)$:

Proposition 1.2.1 ([68, Propositions 1, 2])**.** *The set of functions*

$$\left\{\Omega\left(|x|_p\right)\right\} \bigcup \bigcup_{j \in \{1, \ldots, p-1\}} \bigcup_{r \leq 0} \bigcup_{\substack{np^{-r} \in \mathbb{Z}_p \\ n \in \mathbb{Q}_p/\mathbb{Z}_p}} \left\{\Psi_{rnj}(x)\right\} \tag{1.2.8}$$

is an orthonormal basis of $L^2(\mathbb{Z}_p)$. Furthermore,

$$L^2(\mathbb{Z}_p) = \mathbb{C}\Omega\left(|x|_p\right) \bigoplus L_0^2(\mathbb{Z}_p), \tag{1.2.9}$$

where

$$L_0^2(\mathbb{Z}_p) = \left\{ f \in L^2(\mathbb{Z}_p); \int_{\mathbb{Z}_p} f \; dx = 0 \right\}.$$

Now, by using (1.2.6), (2.2.2), (1.2.8), the functions in (1.2.8) are eigenfunctions of $\boldsymbol{D}_0^\alpha - \lambda$:

$$\left(\boldsymbol{D}_0^\alpha - \lambda\right) \Psi_{rnj} = \left(p^{(1-r)\alpha} - \lambda\right) \Psi_{rnj} \tag{1.2.10}$$

for any $r \leq 0$, $n \in p^r \mathbb{Z}_p \cap \mathbb{Q}_p/\mathbb{Z}_p$, $j \in \{1, \ldots, p-1\}$, and

$$\left(\boldsymbol{D}_0^\alpha - \lambda\right) \Omega\left(|x|_p\right) = 0, \text{ for } x \in \mathbb{Z}_p. \tag{1.2.11}$$

An eigenvalue problem in the unit ball

We now consider the following eigenvalue problem:

$$\begin{cases} \left(\boldsymbol{D}_0^\alpha - \lambda\right) \theta(x) = \kappa \theta(x), \quad \kappa \in \mathbb{R} \\ \\ \theta \in L_{\mathbb{R}}^2\left(\mathbb{Z}_p\right). \end{cases} \tag{1.2.12}$$

By using (1.2.10)-(1.2.11), the functions $\Psi_{rnj}(x)$ given in (1.2.8) are complex-valued eigenfunctions of (1.2.12) with eigenvalues $\kappa \in \left\{p^{(1-r)\alpha} - \lambda; r \leq 0\right\}$. Therefore

$$\begin{aligned} &p^{\frac{-r}{2}} \cos\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right) \Omega\left(|p^r x - n|_p\right), \\ &p^{\frac{-r}{2}} \sin\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right) \Omega\left(|p^r x - n|_p\right), \end{aligned}$$

with $|p^{-r}n|_p \leq 1$ and $r \leq 0$, $n \in p^r\mathbb{Z}_p \cap \mathbb{Q}_p/\mathbb{Z}_p$, $j \in \{1, \ldots, p-1\}$, are real-valued eigenfunctions of (1.2.12) with $\kappa = p^{(1-r)\alpha} - \lambda$. Furthermore, any $f(x) \in L^2_{\mathbb{R}}(\mathbb{Z}_p)$ admits an expansion of the form

$$\begin{aligned} f(x) = &\sum_{rnj} p^{\frac{-r}{2}} \mathrm{Re}(A_{rnj}) \cos\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right) \Omega\left(|p^r x - n|_p\right) \\ &- \sum_{rnj} p^{\frac{-r}{2}} \mathrm{Im}(A_{rnj}) \sin\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right) \Omega\left(|p^r x - n|_p\right) \\ &+ A_0 \Omega\left(|x|_p\right) \end{aligned} \tag{1.2.13}$$

where

$$\mathrm{Re}(A_{rnj}) = p^{\frac{-r}{2}} \int_{\mathbb{Z}_p} f(x) \cos\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right) \Omega\left(|p^r x - n|_p\right) dx,$$

$$\mathrm{Im}(A_{rnj}) = p^{\frac{-r}{2}} \int_{\mathbb{Z}_p} f(x) \sin\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right) \Omega\left(|p^r x - n|_p\right) dx,$$

and

$$A_0 = \int_{\mathbb{Z}_p} f(x) dx.$$

1.3 Sobolev-Type Spaces

The Sobolev-type spaces used here were introduce in [53, 64]. We follow here closely the presentation given in [30, Sections 10.1, 10.2].

We set $[\xi]_p := \max\left\{1, \|\xi\|_p\right\}$ for $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^n$. Given $\varphi, \varrho \in \mathcal{D}(\mathbb{Q}_p^n)$ and $s \in \mathbb{R}$, we define the scalar product:

$$\langle \varphi, \varrho \rangle_s = \int_{\mathbb{Q}_p^n} [\xi]_p^s \, \widehat{\varphi}(\xi) \overline{\widehat{\varrho}(\xi)} d^n\xi,$$

where the bar denotes the complex conjugate. We also set $\|\varphi\|_s^2 = \langle \varphi, \varphi \rangle_s$, and denote by $\mathcal{H}_s := \mathcal{H}_s(\mathbb{Q}_p^n, \mathbb{C}) = \mathcal{H}_s(\mathbb{C})$ the completion of $\mathcal{D}(\mathbb{Q}_p^n)$ with respect to $\langle \cdot, \cdot \rangle_s$. Notice that if

$r, s \in \mathbb{R}$, with $r \leq s$, then $\|\cdot\|_r \leq \|\cdot\|_s$ and $\mathcal{H}_s \hookrightarrow \mathcal{H}_r$ (continuous embedding). In particular,

$$\cdots \supset \mathcal{H}_{-2} \supset \mathcal{H}_{-1} \supset \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \cdots,$$

where $\mathcal{H}_0 = L^2$. We set

$$\mathcal{H}_\infty(\mathbb{Q}_p^n, \mathbb{C}) = \mathcal{H}_\infty := \bigcap_{s \in \mathbb{N}} \mathcal{H}_s.$$

Since $\mathcal{H}_{[s]+1} \subseteq \mathcal{H}_s \subseteq \mathcal{H}_{[s]}$ for $s \in \mathbb{R}_+$, where $[\cdot]$ is the integer part function, then $\mathcal{H}_\infty = \bigcap_{s \in \mathbb{R}_+} \mathcal{H}_s$. With the topology induced by the family of seminorms $\{\| \cdot \|_l\}_{l \in \mathbb{N}}$, $\mathcal{H}_\infty$ becomes a locally convex space, which is metrizable. Indeed,

$$d(f, g) := \max_{l \in \mathbb{N}} \left\{ 2^{-l} \frac{\|f - g\|_l}{1 + \|f - g\|_l} \right\}, \text{ with } f, g \in \mathcal{H}_\infty,$$

is a metric for the topology of $\mathcal{H}_\infty$ considered as a convex topological space. The metric space $(\mathcal{H}_\infty, d)$ is the completion of the metric space $(\mathcal{D}(\mathbb{Q}_p^n), d)$, cf. [30, Lemma 10.4]. Furthermore, $\mathcal{H}_\infty \subset L^\infty \cap C^{\text{unif}} \cap L^1 \cap L^2$, and $\mathcal{H}_\infty(\mathbb{Q}_p^n, \mathbb{C})$ is continuously embedded in $C_0(\mathbb{Q}_p^n, \mathbb{C})$. This is the non-Archimedean analog of the Sobolev embedding theorem, cf. [30, Theorem 10.15].

Lemma 1.3.1. *If $s_1 \leq s \leq s_2$, with $s = \theta s_1 + (1-\theta) s_2$, $0 \leq \theta \leq 1$, then $\|f\|_s \leq \|f\|_{s_1}^{\theta} \|f\|_{s_2}^{(1-\theta)}$.*

Proof. Take $f \in \mathcal{H}_s$, then by using the Hölder inequality for the exponents $\frac{1}{q} = \theta, \frac{1}{q'} = 1 - \theta$,

$$\begin{aligned}
\|f\|_s^2 &= \int_{\mathbb{Q}_p^n} [\xi]_p^s \left| \widehat{f}(\xi) \right|^2 d^n\xi = \int_{\mathbb{Q}_p^n} [\xi]_p^{\theta s_1 + (1-\theta) s_2} \left| \widehat{f}(\xi) \right|^{2(\theta + (1-\theta))} d^n\xi \\
&= \int_{\mathbb{Q}_p^n} \left([\xi]_p^{s_1} \left| \widehat{f}(\xi) \right|^2 \right)^{\theta} \left([\xi]_p^{s_2} \left| \widehat{f}(\xi) \right|^2 \right)^{1-\theta} d^n\xi \\
&\leq \left(\int_{\mathbb{Q}_p^n} [\xi]_p^{s_1} \left| \widehat{f}(\xi) \right|^2 d^n\xi \right)^{\theta} \left(\int_{\mathbb{Q}_p^n} [\xi]_p^{s_2} \left| \widehat{f}(\xi) \right|^2 d^n\xi \right)^{1-\theta} d^n\xi.
\end{aligned}$$

□

The following characterization of the spaces $\mathcal{H}_s$ and $\mathcal{H}_\infty$ is useful:

Lemma 1.3.2 ([30, Lemma 10.8]). *(i) $\mathcal{H}_s = \{f \in L^2; \|f\|_s < \infty\} = \{T' \in \mathcal{D}; \|T\|_s < \infty\}$, (ii) $\mathcal{H}_\infty = \{f \in L^2; \|f\|_s < \infty$ for any $s \in \mathbb{R}_+\} = \{T' \in \mathcal{D}; \|T\|_s < \infty$ for any $s \in \mathbb{R}_+\}$. The equalities in (i)-(ii) are in the sense of vector spaces.*

Proposition 1.3.1. *If $s > n/2$, then $\mathcal{H}_s$ is closed for the product of functions. That is, if $f, g \in \mathcal{H}_s$, then $fg \in \mathcal{H}_s$ and $\|fg\|_s \leq C(n, s) \|f\|_s \|g\|_s$, where $C(n, s)$ is a positive constant.*

Proof. By the ultrametric property of $\|\cdot\|_p$, $\|\xi\|_p \leq \max\left\{\|\xi-\eta\|_p, \|\eta\|_p\right\}$ for $\xi, \eta \in \mathbb{Q}_p^n$, we have $\max\left\{1, \|\xi\|_p\right\} \leq \max\left\{1, \|\xi-\eta\|_p, \|\eta\|_p\right\}$, which implies that

$$\left[\max\left\{1, \|\xi\|_p\right\}\right]^s \leq \max\left\{1, \|\xi-\eta\|_p^s, \|\eta\|_p^s\right\} = \max\left\{1, \|\xi-\eta\|_p, \|\eta\|_p\right\}^s$$

for $s > 0$. Therefore

$$[\xi]_p^s \leq [\xi-\eta]_p^s + [\eta]_p^s. \tag{1.3.1}$$

Now, for $f, g \in L^2$, by using (1.3.1),

$$\begin{aligned}
[\xi]_p^{\frac{s}{2}} \left|\widehat{fg}(\xi)\right| &= \left|[\xi]_p^{\frac{s}{2}} \int_{\mathbb{Q}_p^n} \widehat{f}(\xi-\eta)\widehat{g}(\eta) d^n\eta\right| \\
&\leq \int_{\mathbb{Q}_p^n} [\xi-\eta]_p^{\frac{s}{2}} \left|\widehat{f}(\xi-\eta)\right| |\widehat{g}(\eta)| \, d^n\eta + \int_{\mathbb{Q}_p^n} [\eta]_p^{\frac{s}{2}} |\widehat{g}(\eta)| \left|\widehat{f}(\xi-\eta)\right| d^n\eta \\
&= [\xi]_p^{\frac{s}{2}} \left|\widehat{f}(\xi)\right| * |\widehat{g}(\xi)| + \left|\widehat{f}(\xi)\right| * [\xi]_p^{\frac{s}{2}} |\widehat{g}(\xi)|.
\end{aligned}$$

Then

$$\begin{aligned}
\|fg\|_s &\leq \left\|[\xi]_p^{\frac{s}{2}} \left|\widehat{f}(\xi)\right| * |\widehat{g}(\xi)| + \left|\widehat{f}(\xi)\right| * [\xi]_p^{\frac{s}{2}} |\widehat{g}(\xi)|\right\|_2 \\
&\leq \left\|[\xi]_p^{\frac{s}{2}} \left|\widehat{f}(\xi)\right| * |\widehat{g}(\xi)|\right\|_2 + \left\|\left|\widehat{f}(\xi)\right| * [\xi]_p^{\frac{s}{2}} |\widehat{g}(\xi)|\right\|_2.
\end{aligned}$$

Since $[\xi]_p^{\frac{s}{2}} \left|\widehat{f}(\xi)\right|$, $[\xi]_p^{\frac{s}{2}} |\widehat{g}(\xi)| \in L^2$, by using the Cauchy-Schwarz inequality with $s > n/2$, we have $\| |\widehat{g}(\xi)| \|_1 \leq A(n,s) \|g\|_s$, $\left\| \left|\widehat{f}(\xi)\right| \right\|_1 \leq A(n,s) \|f\|_s$, i.e., $|\widehat{g}(\xi)|$, $\left|\widehat{f}(\xi)\right| \in L^1$. Now, by a Young-type inequality, see [55, Chapter II, Theorem 1.7], we obtain that

$$\|fg\|_s \leq \|f\|_s \|\widehat{g}\|_1 + \|g\|_s \left\|\widehat{f}\right\|_1 \leq 2A(n,s) \|f\|_s \|g\|_s.$$

□

Lemma 1.3.3 ([30, Lemma 10.13 and Theorem 10.15])**.** *For $s \in \mathbb{R}_+$, the mapping $P(\boldsymbol{D})$: $\mathcal{H}_{s+2\delta} \longrightarrow \mathcal{H}_s$ is a well-defined continuous mapping.*

Lemma 1.3.4. *Take $s - 2\delta > n/2$ and $f, g \in \mathcal{H}_{s+2\delta}$. Then*

$$\|P(\boldsymbol{D})(fg)\|_s \leq C(n,s,\delta) \|f\|_{s+2\delta} \|g\|_{s+2\delta},$$

where $C(n,s,\delta)$ is a positive constant that depends of n, s and δ.

Proof. Since $s > n/2$ and $f, g \in \mathcal{H}_{s+2\delta}$, by Proposition 1.3.1, $fg \in \mathcal{H}_{s+2\delta}$, and by Lemma 1.3.3, $P(\boldsymbol{D})(fg) \in \mathcal{H}_s$. Now by using Proposition 1.3.1,

$$\|P(\boldsymbol{D})(fg)\|_s \leq \sum_{j=0}^{k} |C_j| \left\|\boldsymbol{D}^{\delta_j}(fg)\right\|_s$$

$$= \sum_{j=0}^{k} |C_j| \left(\int_{\mathbb{Q}_p^n} [\xi]_p^s \|\xi\|_p^{2\delta_j} \left|\widehat{fg}(\xi)\right|^2 d^n\xi \right)^{\frac{1}{2}} \leq \sum_{j=0}^{k} |C_j| \left(\int_{\mathbb{Q}_p^n} [\xi]_p^{s+2\delta_j} \left|\widehat{fg}(\xi)\right|^2 d^n\xi \right)^{\frac{1}{2}}$$

$$= \sum_{j=0}^{k} |C_j| \|fg\|_{s+2\delta_j} \leq \sum_{j=0}^{k} |C_j| C(n, s, \delta_j) \|f\|_{s+2\delta_j} \|g\|_{s+2\delta_j}$$

$$\leq \left(\sum_{j=0}^{k} |C_j| C(n, s, \delta_j) \right) \|f\|_{s+2\delta} \|g\|_{s+2\delta}.$$

□

1.4 The Spaces $\mathcal{D}_M^{-L}$

We fix $M \in \mathbb{Z}$ and $L \in \mathbb{N}$, with $L \geq -M$, and define

$$G_{L,M} = p^{-M}\mathbb{Z}_p / p^L\mathbb{Z}_p.$$

Then, $G_{L,M}$ is a finite ring, with $\#G_{L,M} = p^{L+M}$ elements. We define the following set of representatives for $G_{L,M}$:

$$I = I_{-M}p^{-M} + I_{-M+1}p^{-M+1} + \ldots + I_{L-1}p^{L-1},$$

where the I_j's are p-adic digits, i.e., elements from $\{0, 1, \ldots, p-1\}$.
We define $\mathcal{D}_M^{-L}$ to be the $\mathbb{R}$-vector space formed by test functions φ supported in the ball $p^{-M}\mathbb{Z}_p$, having the form

$$\varphi(x) = p^{\frac{L}{2}} \sum_{I \in G_{L,M}} \varphi(I) \Omega\left(p^L |x - I|_p\right), \text{ with } \varphi(I) \in \mathbb{R}.$$

Since $\Omega\left(p^L|x - I|_p\right) \Omega\left(p^L|x - J|_p\right) = 0$ if $I \neq J$, the set

$$\left\{ p^{\frac{L}{2}} \Omega\left(p^L|x - I|_p\right) : I \in G_{L,M} \right\}$$

is an orthonormal basis for $\mathcal{D}_M^L$. Then, by using that

$$\|\varphi\|_{L^2} = \sqrt{p^L \sum_{I \in G_{L,M}} |\varphi(I)|^2 \int\limits_{p^{-M}\mathbb{Z}_p} \Omega\left(p^L |x - I|_p\right) dx}$$
$$= \sqrt{\sum_{I \in G_{L,M}} |\varphi(I)|^2},$$

we have

$$\left(\mathcal{D}_M^{-L}, \|\cdot\|_{L^2}\right) \simeq \left(\mathbb{R}^{\# G_{L,M}}, \|\cdot\|_{\mathbb{R}}\right) \text{ as Banach spaces,}$$

where $\|\cdot\|_{\mathbb{R}}$ denotes the usual norm of $\mathbb{R}^{\# G_{L,M}}$.

Chapter 2

Local Well-Posedness of the Cauchy Problem for a p-Adic Nagumo-Type Equation

We introduce a new family of p-adic nonlinear evolution equations. We establish the local well-posedness of the Cauchy problem for these equations in Sobolev-type spaces. For a certain subfamily, we show that the blow-up phenomenon occurs and provide numerical simulations showing this phenomenon.

2.1 Main result

Consider the following Cauchy problem:

$$\begin{cases} u \in C^1\left([0,T]\,;\mathcal{H}_s\right); \\ \\ u_t = -\gamma \boldsymbol{D}_x^{\alpha} u - u^3 + (\beta+1)\,u^2 - \beta u + P(\boldsymbol{D}_x)\left(u^m\right), \quad x \in \mathbb{Q}_p^n,\ t \in [0,T]\,; \\ \\ u(0) = f_0 \in \mathcal{H}_s, \end{cases} \tag{2.1.1}$$

where T, γ, α, $\beta > 0$, and m is a positive integer. The main result of this work is the following:

Theorem 2.1.1. *For $s > n/2 + 2\delta$, the Cauchy problem (2.1.1) is locally well-posed in $\mathcal{H}_s$.*

2.1.1 Preliminary results

We denote by $\boldsymbol{V}(t) = e^{-(\gamma \boldsymbol{D}^{\alpha}+\beta \boldsymbol{I})t}$, $t \geq 0$, the semigroup in L^2 generated by the operator $\boldsymbol{A} = -\gamma \boldsymbol{D}^{\alpha} - \beta \boldsymbol{I}$, that is,

$$\boldsymbol{V}(t)f\,(x) = \mathcal{F}^{-1}_{\xi \to x}\left(e^{-(\gamma \|\xi\|_p^{\alpha}+\beta)t}\mathcal{F}_{x\to\xi}f\right), \text{ for } f \in L^2,\ t \geq 0.$$

Lemma 2.1.1. *$\{\boldsymbol{V}(t)\}_{t\geq 0}$ is a C^0-semigroup of contractions in $\mathcal{H}_s$, $s \in \mathbb{R}$, satisfying $\|\boldsymbol{V}(t)\|_s \leq e^{-\beta t}$ for $t \geq 0$. Moreover, $u(x,t) = \boldsymbol{V}(t)f_0(x)$ is the unique solution to the following Cauchy problem:*

$$\begin{cases} u \in C^1\left([0,T]\,;\mathcal{H}_s\right); \\ \\ u_t = -\gamma \boldsymbol{D}^{\alpha}u - \beta u,\ t \in [0,T]\,; \\ \\ u(x,0) = f_0(x) \in \mathcal{H}_s, \end{cases} \tag{2.1.2}$$

where T is an arbitrary positive number.

Proof. We just verify the strongly continuity of the semigroup. Since

$$\left\|\mathcal{F}^{-1}_{\xi\to x}\left(e^{-(\gamma\|\xi\|_p^{\alpha}+\beta)t}\mathcal{F}_{x\to\xi}f\right) - f\,(x)\right\|_s^2 = \int_{\mathbb{Q}_p^n} [\xi]_p^s \left|\widehat{f}\,(\xi)\right|^2 \left\{1 - e^{-(\gamma\|\xi\|_p^{\alpha}+\beta)t}\right\}^2 d^n\xi \leq \|f\|_s^2\,,$$

it follows from the Dominated Convergence Theorem that

$$\lim_{t\to 0+}\|\boldsymbol{V}(t)f - f\|_s = 0.$$

The existence and uniqueness of a solution for the Cauchy problem (2.1.2) follows from a well-known result, see, e.g., [40, Theorem 4.3.1]. □

Lemma 2.1.2. *Let $f_0 \in \mathcal{H}_s$, $s \in \mathbb{R}$, $\lambda \geq 0$. Then, there exists a positive constant $C(\lambda,\alpha)$ that depends of λ and α such that*

$$\|\boldsymbol{V}(t)f_0\|_{s+\lambda} \leq e^{-\beta t}\left(1 + C(\lambda,\alpha)\left(\frac{\lambda}{2\alpha\gamma t}\right)^{\frac{\lambda}{2\alpha}}\right)\|f_0\|_s \quad \textit{for } t > 0. \tag{2.1.3}$$

Proof. We first notice that

$$\|\boldsymbol{V}(t)\,f_0\|_{s+\lambda}^2 = \int_{\mathbb{Q}_p^n} [\xi]_p^{s+\lambda} e^{-2(\gamma\|\xi\|_p^\alpha+\beta)t}\,|f_0(\xi)|^2\,d^n\xi$$

$$\leq e^{-2\beta t}\left(\sup_{\xi\in\mathbb{Q}_p^n}[\xi]_p^\lambda e^{-2\gamma\|\xi\|_p^\alpha t}\right)\|f_0\|_s^2 \leq e^{-2\beta t}\left(1+\sup_{\xi\in\mathbb{Q}_p^n\smallsetminus\mathbb{Z}_p^n}\|\xi\|_p^\lambda\,e^{-2\gamma\|\xi\|_p^\alpha t}\right)\|f_0\|_s^2$$

$$\leq e^{-2\beta t}\left(1+\sup_{\xi\in\mathbb{Q}_p^n}\|\xi\|_p^\lambda\,e^{-2\gamma\|\xi\|_p^\alpha t}\right)\|f_0\|_s^2.$$

We now set $y=\|\xi\|_p$ and $h(y)=y^\lambda e^{-2\gamma y^\alpha t}$. By using the fact that $h(y)$ reaches its maximum at $y_{\max}=\left(\frac{\lambda}{2\alpha\gamma t}\right)^{\frac{1}{\alpha}}$, we conclude that

$$\sup_{\xi\in\mathbb{Q}_p^n}\|\xi\|_p^\lambda\,e^{-2\gamma\|\xi\|_p^\alpha t} \leq \left(\frac{\lambda}{2\alpha\gamma t}\right)^{\frac{\lambda}{\alpha}} e^{-\frac{\lambda}{\alpha}} \leq \mathrm{C}(\lambda,\alpha)\left(\frac{\lambda}{2\alpha\gamma t}\right)^{\frac{\lambda}{\alpha}}.$$

□

Proposition 2.1.1. *Let $s>n/2+2\delta$ and $F(u)=(\beta+1)u^2-u^3+P(\boldsymbol{D})(u^m)$. Then $F:\mathcal{H}_s\longrightarrow\mathcal{H}_{s-2\delta}$ is a continuous function satisfying*

$$\|F(u)-F(w)\|_{s-2\delta}\leq L(\|u\|_s,\|w\|_s)\|u-w\|_s, \tag{2.1.4}$$

for $u,w\in\mathcal{H}_s$, here $L(\cdot,\cdot)$ is a continuous function, which is not decreasing with respect to each of their arguments. In particular,

$$\|F(u)\|_{s-2\delta}\leq L(\|u\|_s,0)\|u\|_s. \tag{2.1.5}$$

Proof. We first notice that

$$F(u)-F(w)=(\beta+1)(u^2-w^2)-(u^3-w^3)+P(\boldsymbol{D})(u^m-w^m)$$

$$=(\beta+1)(u-w)(u+w)-(u-w)(u^2+uw+w^2)+P(\boldsymbol{D})((u-w)q(u,w)),$$

where $q(u,w)=\sum_{k=0}^{m-1}u^k w^{m-1-k}$. By using Proposition 1.3.1 and Lemma 1.3.4, the condition $s>n/2$ implies that if $u,w\in\mathcal{H}_s$, then any polynomial function in u,w belongs to $\mathcal{H}_s$, and

$$\|F(u)-F(w)\|_{s-2\delta}\leq C\left\{(\beta+1)\|u-w\|_{s-2\delta}\|u+w\|_{s-2\delta}+\right.$$
$$\left.\|u-w\|_{s-2\delta}\|u^2+uw+w^2\|_{s-2\delta}+\|u-w\|_s\|q(u,w)\|_s\right\},$$

where $C=C(n,s,\delta)$. Then

$$\|F(u)-F(w)\|_{s-2\delta}\leq A(\|u\|_s,\|w\|_s)\|u-w\|_s,$$

where

$$
\begin{aligned}
A(\|u\|_s, \|w\|_s) &= C\left\{(\beta+1)\|u+w\|_s + \|u^2+uw+w^2\|_s + \|q(u,w)\|_s\right\} \\
&\le C\left\{(\beta+1)\|u\|_s + (\beta+1)\|w\|_s + \|u^2\|_s + \|uw\|_s + \|w^2\|_s + \sum_{k=0}^{m-1}\left\|u^k w^{m-1-k}\right\|_s\right\} \\
&\le C(\beta+1)\|u\|_s + C(\beta+1)\|w\|_s + C^2\|u\|_s^2 + C^2\|u\|_s\|w\|_s + C^2\|w\|_s^2 + \\
&\qquad C^{m+1}\sum_{k=0}^{m-1}\|u\|_s^k\,\|w\|_s^{m-1-k} =: L(\|u\|_s, \|w\|_s).
\end{aligned}
$$

□

For $M, T > 0$ and $f_0 \in \mathcal{H}_s$, we set

$$
\mathcal{X}(M,T,f_0) := \left\{u(t) \in C\left([0,T];\mathcal{H}_s\right);\ \sup_{t\in[0,T]}\|u(t) - V(t)f_0\|_s \le M\right\}.
$$

We endow $\mathcal{X}(M,T,f_0)$ with the metric $d(u(t),v(t)) = \sup_{t\in[0,T]}\|u(t)-v(t)\|_s$. The resulting metric space is complete.

Proposition 2.1.2. *Take $f_0 \in \mathcal{H}_s$ with $s > n/2 + 2\delta$, $\delta > 0$. Then, there exists $T = T(\|f_0\|_s, M) > 0$ and a unique function $u \in C([0,T];\mathcal{H}_s)$ satisfying the integral equation*

$$
u(t) = \boldsymbol{V}(t)f_0 + \int_0^t \boldsymbol{V}(t-\tau)F(u(\tau))d\tau, \tag{2.1.6}
$$

such that $u(0) = f_0$. Here $F(u) = (\beta+1)u^2 - u^3 + P(\boldsymbol{D})(u^m)$ as before.

Remark 2.1.1. *Since $F(u)$ is not a locally Lipschitz function because inequality (2.1.5) involves two different norms, the existence of mild solutions of type (2.1.6) does not follow directly from standard results in semigroup theory, see, e.g., [40, Theorem 5.2.2].*

Proof. Given $u \in \mathcal{X}(M,T,f_0)$, we set

$$
\boldsymbol{N}u(t) = \boldsymbol{V}(t)f_0 + \int_0^t \boldsymbol{V}(t-\tau)F(u(\tau))d\tau.
$$

Claim 1. $\boldsymbol{N} : \mathcal{X}(M,T,f_0) \longrightarrow C([0,T];\mathcal{H}_s)$.

Take $u \in \mathcal{X}(M,T,f_0)$, then

$$
\begin{aligned}
\|\boldsymbol{N}u(t_1) - \boldsymbol{N}u(t_2)\|_s &\le \|(\boldsymbol{V}(t_1) - \boldsymbol{V}(t_2))f_0\|_s \\
&+ \left\|\int_0^{t_1}\boldsymbol{V}(t_1-\tau)F(u(\tau))d\tau - \int_0^{t_2}\boldsymbol{V}(t_2-\tau)F(u(\tau))d\tau\right\|_s.
\end{aligned} \tag{2.1.7}
$$

Since $\{\boldsymbol{V}(t)\}_{t\geq 0}$ is a C_0-semigroup in $\mathcal{H}_s$, cf. Lemma 2.1.1, the first term on the right-hand side of the inequality (2.1.7) tends to zero when $t_2 \to t_1$. To study the second term, we assume without loss of generality that $0 < t_1 < t_2 < T$. Then

$$\begin{aligned}
&\left\| \int_0^{t_1} \boldsymbol{V}\,(t_1-\tau)\, F(u(\tau)) d\tau - \int_0^{t_2} \boldsymbol{V}\,(t_2-\tau)\, F(u(\tau)) d\tau \right\|_s \\
&\leq \int_0^{t_1} \|\{\boldsymbol{V}\,(t_1-\tau) - \boldsymbol{V}\,(t_2-\tau)\} F(u(\tau))\|_s \, d\tau + \int_{t_1}^{t_2} \|\boldsymbol{V}\,(t_2-\tau)\, F(u(\tau))\|_s \, d\tau.
\end{aligned}$$

By using Lemma 2.1.2 with $\lambda = \alpha$ and Proposition 2.1.1,

$$\begin{aligned}
&\|(\boldsymbol{V}\,(t_1-\tau) - \boldsymbol{V}\,(t_2-\tau))\, F(u(\tau))\|_s \\
&\leq \|\boldsymbol{V}\,(t_1-\tau)\, F(u(\tau))\|_s + \|\boldsymbol{V}\,(t_2-\tau)\, F(u(\tau))\|_s \\
&\leq \left\{ 2 + C_0 \left(\frac{1}{2\gamma(t_1-\tau)} \right)^{\frac{1}{2}} + C_0 \left(\frac{1}{2\gamma(t_2-\tau)} \right)^{\frac{1}{2}} \right\} \|F(u(\tau))\|_{s-\alpha} \\
&\leq 2 \left\{ 1 + C_0 \left(\frac{1}{2\gamma(t_1-\tau)} \right)^{\frac{1}{2}} \right\} \sup_{\tau\in[0,T]} \|F\,(u(\tau))\,\|_{s-\alpha} \\
&= A(T,s,\alpha) \left\{ 1 + C_0 \left(\frac{1}{2\gamma(t_1-\tau)} \right)^{\frac{1}{2}} \right\} \in L^1([0,t_1]).
\end{aligned}$$

Now, by applying the Dominated Convergence Theorem,

$$\lim_{t_2\to t_1} \int_0^{t_1} \|(\boldsymbol{V}\,(t_1-\tau) - \boldsymbol{V}\,(t_2-\tau))\, F(u(\tau))\|_s \, d\tau = 0.$$

By a similar argument, one shows that

$$\|\boldsymbol{V}\,(t_2-\tau)\, F(u(\tau))\|_{s-2\delta} \leq 1 + C_0 \left(\frac{1}{2\gamma(t_2-\tau)} \right)^{\frac{1}{2}} L(\|u\,(\tau)\|_s, 0)\|u\,(\tau)\|_s,$$

and since

$$\|u(\tau)\|_s \leq \|u(\tau) - \boldsymbol{V}(\tau) f_0\|_s + \|\boldsymbol{V}(\tau) f_0\|_s \leq M + \|f_0\|_s, \text{ for all } \tau \in [0,T], \tag{2.1.8}$$

we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \|\boldsymbol{V}(t_2-\tau) F(u(\tau))\|_s \, d\tau \qquad (2.1.9) \\
&\leq L(M + \|f_0\|_s, 0)(M + \|f_0\|_s) \left(\int_{t_1}^{t_2} \left(1 + C_0 \left(\frac{1}{2\gamma(t_2-\tau)} \right)^{\frac{1}{2}} \right) d\tau \right) \\
&= L\,(M + \|f_0\|_s, 0)\,(M + \|f_0(\cdot)\|_s) \left((t_2-t_1) + C_0 \left(\sqrt{\frac{2(t_2-t_1)}{\gamma}} \right) \right),
\end{aligned}$$

and consequently, by applying the Dominated Convergence Theorem,

$$\lim_{t_2 \to t_1} \int_{t_1}^{t_2} \|\boldsymbol{V}(t_2-\tau)F(u(\tau))\|_s d\tau = 0.$$

Claim 2. There exists T_0 such that $\boldsymbol{N}(\mathcal{X}(M,T_0,f_0)) \subseteq \mathcal{X}(M,T_0,f_0)$.

By using a reasoning similar to the one used to established inequality (2.1.9), one gets

$$\begin{aligned}\|(\boldsymbol{N}u)(t)-\boldsymbol{V}(t)f_0\|_s &\leq \int_0^t \|\boldsymbol{V}(t-\tau)F(u(\tau))\|_s d\tau \\ \leq L\left(M+\|f_0\|_s,0\right)\left(M+\|f_0\|_s\right) & \left(\int_0^t \left(1+C_0\left(\frac{1}{2\gamma(t-\tau)}\right)^{\frac{1}{2}}\right)d\tau\right) \\ \leq L\left(M+\|f_0\|_s,0\right)\left(M+\|f_0\|_s\right) & \left(T+C_0\left(\sqrt{\frac{2T}{\gamma}}\right)\right).\end{aligned}$$

Now taking T_0 such that

$$L\left(M+\|f_0\|_s,0\right)\left(M+\|f_0\|_s\right)\left(T_0+C_0\left(\sqrt{\frac{2T_0}{\gamma}}\right)\right) \leq M, \tag{2.1.10}$$

we conclude that $\boldsymbol{N}u \in \mathcal{X}(M,T_0,f_0)$, for all $u(t) \in \mathcal{X}(M,T_0,f_0)$.

Claim 3. There exists T_0' such that $\boldsymbol{N}$ is a contraction on $\mathcal{X}(M,T_0',f_0)$.

Given $u(t)$, $v(t) \in \mathcal{X}(M,T_0,f_0)$, by using Proposition 2.1.1, with

$$C_0' = L\left(M+\|f_0\|_s, M+\|f_0\|_s\right),$$

see (2.1.8), we have

$$\begin{aligned}\|\boldsymbol{N}u(t)-\boldsymbol{N}v(t)\|_s &\leq \int_0^t \|\boldsymbol{V}(t-\tau)[F(u(\tau))-F(v(\tau))]\|_s d\tau \\ &\leq \int_0^t \left(1+C_0\left(\frac{1}{2\gamma(t-\tau)}\right)^{\frac{1}{2}}\right)\|F(u(\tau))-F(v(\tau))\|_{s-\alpha}\, d\tau \\ &\leq C_0' \int_0^t \left(1+C_0\left(\frac{1}{2\gamma(t-\tau)}\right)^{\frac{1}{2}}\right)\|u(\tau)-v(\tau)\|_s d\tau \\ &\leq C_0'\left(\sup_{\tau\in[0,T_0]}\|u(\tau)-v(\tau)\|_s\right)\int_0^t \left(1+C_0\left(\frac{1}{2\gamma(t-\tau)}\right)^{\frac{1}{2}}\right)d\tau \\ &\leq C_0'\left(T_0+C_0\left(\sqrt{\frac{2T_0}{\gamma}}\right)\right)d(u(t),v(t)).\end{aligned}$$

Thus, taking T_0' such that

$$C := C_0' \left(T_0' + C_0 \left(\sqrt{\frac{2T_0'}{\gamma}} \right) \right) < 1, \tag{2.1.11}$$

we obtain that $d(\boldsymbol{N}u(t), \boldsymbol{N}v(t)) \leq Cd(u(t), v(t))$, that is, $\boldsymbol{N}$ is a strict contraction in $\mathcal{X}(M, T_0', f_0)$. We pick T such that the inequalities (2.1.10) and (2.1.10) hold true, and apply the Banach's Fixed Point Theorem to get $u(t) \in \mathcal{X}(M, T, f_0)$ a unique fixed point of $\boldsymbol{N}$, which satisfies the integral equation (2.1.6), where $T = T(\|f_0\|_s, M) > 0$. □

Lemma 2.1.3 ([40, Theorem 5.1.1])**.** *If $h \in L^1(0, T)$, with $T > 0$, is a real-valued function such that*

$$h(t) \leq a + b \int_0^t h(s) ds,$$

for $t \in (0, T)$ a.e., where $a \in \mathbb{R}$ and $b \in [0, \infty)$ then $h(t) \leq ae^{bt}$ for almost all t in $(0, T)$.

Proposition 2.1.3. *Let f_0, $f_1 \in \mathcal{H}_s$ and $u(t), v(t) \in C([0, T]; \mathcal{H}_s)$ be the corresponding solutions of equation (2.1.6) with initial conditions $u(0) = f_0$ and $v(0) = f_1$, respectively. If $s > n/2 + 2\delta$, then*

$$\|u(t) - v(t)\|_s \leq e^{L(W,W)} \|f_0 - f_1\|_s,$$

where L is given in Proposition 2.1.1 and

$$W := \max \left\{ \sup_{t \in [0,T]} \|u(t)\|_s, \sup_{t \in [0,T]} \|v(t)\|_s \right\}.$$

Proof. By using (2.1.6), we have

$$u(t) - v(t) = \boldsymbol{V}(t)(f_0 - f_1) + \int_0^t \boldsymbol{V}(t - \tau)\{F(u(\tau)) - F(v(\tau))\} d\tau.$$

By using Proposition 2.1.1, we get

$$\begin{aligned} \|u(t) - v(t)\|_s &\leq \|f_0 - f_1\|_s + \int_0^t \|\boldsymbol{V}(t - \tau)\{F(u(\tau)) - F(v(\tau))\}\|_s d\tau \\ &\leq \|f_0 - f_1\|_s + \int_0^t \|F(u(\tau)) - F(v(\tau))\|_{s-\alpha} \, d\tau \\ &\leq \|f_0 - f_1\|_s + L(W, W) \int_0^t \|u(\tau) - v(\tau)\|_s d\tau. \end{aligned}$$

Now the result follow from Lemma 2.1.3, by taking $h(t) = \|u(t) - v(t)\|_s$, $a = \|f_0 - f_1\|_s$, $b = L(W, W)$. □

Proposition 2.1.4. *Let $s > n/2 + 2\delta$ and $\delta \geq 0$. Then, the map $f_0 \mapsto u(t)$ is continuous in the following sense: if $f_0^{(n)} \to f_0$ in $\mathcal{H}_s$ and $u_n(t) \in C([0, T_n]; \mathcal{H}_s)$, with $T_n = T\left(\left\|f_0^{(n)}\right\|_s, M\right) > 0$, are the corresponding solutions to the Cauchy problem (2.1.1) with $u_n(0) = f_0^{(n)}$. Then, there exist $T > 0$ and a positive integer $N = N(\gamma, f_0, T)$ such that $T_n \geq T$ for all $n \geq N$ and*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t) - u(t)\|_s = 0. \tag{2.1.12}$$

Proof. By Proposition 2.1.2, the $T_n = T\left(\left\|f_0^{(n)}\right\|_s, M\right) > 0$ are continuous functions of $\left\|f_0^{(n)}\right\|_s$, then, given $T^* > 0$ there exists $N \in \mathbb{N}$ such that $T^* \leq T_n$ for all $n \geq N$. We set $T := \min\{T^*, T_1, T_2, \ldots, T_{N-1}\}$. Therefore, all the $u_n(t)$ are defined on $[0, T]$, furthermore, $u \in \mathcal{X}\left(M, T, f_0^{(n)}\right)$ for all n, and

$$\|u_n(t)\|_s \leq \left\|f_0^{(n)}\right\|_s + M \leq \delta + M,$$

where $\delta = \sup_{n\in\mathbb{N}} \left\|f_0^{(n)}\right\|_s$. Now

$$\sup_{t\in[0,T]} \|u_n(t)\|_s \leq \delta + M \text{ for all } n, \text{ and } \sup_{t\in[0,T]} \|u(t)\|_s \leq \delta + M.$$

On the other hand, by reasoning as in the proof of Proposition 2.1.3, we have

$$\|u_n(t) - u(t)\|_s \leq \left\|f_0^{(n)} - f_0\right\|_s + L(\delta + M, \delta + M) \int_0^t \|u_n(\tau) - u(\tau)\|_s d\tau,$$

and by applying Lemma 2.1.3

$$\|u_n(t) - u(t)\|_s \leq e^{TL(\delta+M,\delta+M)} \left\|f_0^{(n)} - f_0\right\|_s,$$

which in turns implies (2.1.12). □

2.1.2 Proof of the main result

The local well-posedness of the Cauchy problem (2.1.1) in $\mathcal{H}_s$, $s > n/2 + 2\delta$, follows from Propositions 2.1.2, 2.1.3 and 2.1.4.

2.2 The Blow-up phenomenon

In this section, we study the blow-up phenomenon for the solution of the equation

$$\begin{cases} u_t = -\gamma \boldsymbol{D}_x^{\alpha} u + F(u) + \boldsymbol{D}_x^{\alpha_1} u^3, \quad x \in \mathbb{Q}_p^n, \ t \in [0, T]\,; \\ \\ u(0) = f_0 \in \mathcal{H}_\infty, \end{cases} \tag{2.2.1}$$

where $F(u) = -u^3 + (\beta + 1)\, u^2 - \beta u$. We will say that a non-negative solution $u(x,t) \geq 0$ of (2.2.1) blow-up in a finite time $T > 0$, if $\lim_{t \to T^-} \sup_{x \in \mathbb{Q}_p^n} u(x,t) = +\infty$. This limit makes sense since $\mathcal{H}_\infty(\mathbb{Q}_p^n, \mathbb{C})$ is continuously embedded in $C_0(\mathbb{Q}_p^n, \mathbb{C})$, [30, Theorem 10.15].

2.2.1 Pseudo-differential operators and p-adic wavelets

We denote by $C(\mathbb{Q}_p, \mathbb{C})$ the $\mathbb{C}$-vector space of continuous $\mathbb{C}$-valued functions defined on $\mathbb{Q}_p$. We fix a function $\mathfrak{a} : \mathbb{R}_+ \to \mathbb{R}_+$ and define the pseudo-differential operator

$$\begin{array}{ccc} \mathcal{D} & \to & C(\mathbb{Q}_p, \mathbb{C}) \cap L^2 \\ \\ \varphi & \to & \boldsymbol{A}\varphi, \end{array}$$

where $(\boldsymbol{A}\varphi)(x) = \mathcal{F}^{-1}_{\xi \to x} \left\{ \mathfrak{a}\left(|\xi|_p \right) \mathcal{F}_{x \to \xi} \varphi \right\}$.
The set of functions $\{\Psi_{rnj}\}$ defined as

$$\Psi_{rnj}(x) = p^{\frac{-r}{2}} \chi_p \left(p^{-1} j \left(p^r x - n \right) \right) \Omega \left(|p^r x - n|_p \right), \tag{2.2.2}$$

where $r \in \mathbb{Z}$, $j \in \{1, \ldots, p-1\}$, and n runs through a fixed set of representatives of $\mathbb{Q}_p/\mathbb{Z}_p$, is an orthonormal basis of $L^2(\mathbb{Q}_p)$ consisting of eigenvectors of operator $\boldsymbol{A}$:

$$\boldsymbol{A}\Psi_{rnj} = \mathfrak{a}(p^{1-r})\Psi_{rnj} \text{ for any } r,\ n,\ j, \tag{2.2.3}$$

see, e.g., [30, Theorem 3.29], [1, Theorem 9.4.2]. Notice that

$$\widehat{\Psi}_{rnj}(\xi) = p^{\frac{r}{2}} \chi_p \left(p^{-r} n \xi \right) \Omega \left(\left| p^{-r} \xi + p^{-1} j \right|_p \right),$$

and then

$$\mathfrak{a}\left(|\xi|_p \right) \widehat{\Psi}_{rnj}(\xi) = \mathfrak{a}(p^{1-r}) \widehat{\Psi}_{rnj}(\xi).$$

In particular, $\boldsymbol{D}_x^\alpha \Psi_{rnj} = p^{(1-r)\alpha}\Psi_{rnj}$, for any r, n, j and $\alpha > 0$, and since $p^{(1-r)\alpha}$,

$$\boldsymbol{D}_x^\alpha \operatorname{Re}(\Psi_{rnj}) = p^{(1-r)\alpha} \operatorname{Re}(\Psi_{rnj}), \; \boldsymbol{D}_x^\alpha \operatorname{Im}(\Psi_{rnj}) = p^{(1-r)\alpha} \operatorname{Im}(\Psi_{rnj}).$$

And,

$$\{\Psi_{rn1}(x)\}^2 = p^{-r}\chi_p\left(2p^{-1}(p^r x - n)\right)\Omega\left(|p^r x - n|_p\right) = p^{\frac{-r}{2}}\Psi_{rn2}(x),$$

then

$$\boldsymbol{D}_x^\alpha \operatorname{Re}\left(\{\Psi_{rn1}(x)\}^2\right) = p^{\frac{-r}{2}} p^{(1-r)\alpha} \operatorname{Re}(\Psi_{rn2}(x)) = p^{(1-r)\alpha} \operatorname{Re}\left(\{\Psi_{rn1}(x)\}^2\right).$$

Proposition 2.2.1 ([58, VII.2.])**.** *The Parseval-Steklov equality*

$$\int_{\mathbb{Q}_p} \varphi(x)\overline{\psi(x)}d^n x = \int_{\mathbb{Q}_p} (\mathcal{F}\varphi)(\chi)\overline{(\mathcal{F}\psi)(\chi)}d^n\chi, \;\; \varphi, \psi \in \mathcal{D}(\mathbb{Q}_p), \tag{2.2.4}$$

and the equivalent equality

$$\int_{\mathbb{Q}_p} \varphi(x)(\mathcal{F}\psi)(\chi)d^n x = \int_{\mathbb{Q}_p} (\mathcal{F}\varphi)(\chi)\psi(\chi)d^n\chi, \;\; \varphi, \psi \in \mathcal{D}(\mathbb{Q}_p), \tag{2.2.5}$$

hold.

2.2.2 The Blow-up

In this section, we assume that $u(x,t)$ is real-valued non-negative solution of the Cauchy problem (2.1.1) in $\mathcal{H}_\infty$. We set $w(x) := \operatorname{Re}\left(\{\Psi_{rn1}(x)\}^2\right)$, so $\boldsymbol{D}_x^\alpha w(x) = p^{(1-r)\alpha}w(x)$. Thus $w(x)dx$ defines a (positive) measure. We also set $G(t) := \int_{\mathbb{Q}_p} u(x,t)w(x)dx$, where $u(x,t)$ is a positive solution of (2.2.1), then

$$\begin{aligned} G'(t) &= \int_{\mathbb{Q}_p} u_t(x,t)w(x)dx = -\gamma \int_{\mathbb{Q}_p} (\boldsymbol{D}_x^\alpha u)(x,t)w(x)dx \\ &+ \int_{\mathbb{Q}_p} F(u(x,t))w(x)dx + \int_{\mathbb{Q}_p} (\boldsymbol{D}_x^{\alpha_1} u^3)(x,t)w(x)dx. \end{aligned} \tag{2.2.6}$$

Now, by using that $\boldsymbol{D}_x^\alpha u(\cdot,t)$, $w \in L^2$, and $F(u(\cdot,t))$, $\boldsymbol{D}_x^{\alpha_1}u^3(\cdot,t) \in L^2$ since for $s > n/2$, $\mathcal{H}_s$ is a Banach algebra contained in L^2 cf. Proposition 1.3.1, and applying Proposition 2.2.1, we get (2.2.6) can be rewritten as

$$G'(t) = \int_{\mathbb{Q}_p} \left(-\gamma p^{(1-r)\alpha}u(x,t) + F(u(x,t)) + p^{(1-r)\alpha_1}u^3(x,t)\right) w(x)dx.$$

Since the function $H(y) = -\gamma p^{(1-r)\alpha} y + F(y) + p^{(1-r)\alpha_1} y^3$ is convex because

$$H''(y) = -6y + 2(\beta+1) + p^{(1-r)\alpha_1} 6y = 6y\left(p^{(1-r)\alpha_1} - 1\right) + 2(\beta+1) \geq 0,$$

for $y \geq 0$, and $r \leq 0$, we can use the Jensen's inequality to get $G'(t) \geq H(G(t))$, then the function $G(t)$ can not remain finite for every $t \in [0, \infty)$. Then there exists $T \in (0, \infty)$ such that $\lim_{t \to T^-} G(t) = +\infty$, hence $u(x,t)$ blow ups at the time T. Then we have established the following result:

Theorem 2.2.1. *Let $u(x,t)$ be a positive solution of (2.2.1). Then there $T \in (0, +\infty)$ depending on the initial datum such that* $\lim_{t \to T^-} \sup_{x \in \mathbb{Q}_p^n} u(x,t) = +\infty$.

2.3 Numerical Simulations

In this section, we present two numerical simulations for the solution of problem (2.2.1) (in dimension one) for a suitable initial datum. We solve and visualize (using a heat map) the radial profiles of the solution of (2.2.1). We consider equation (2.2.1) for radial functions $u(x, \cdot)$. In [35], Kochubei obtained a formula for $\boldsymbol{D}_x^{\alpha} u(x,t)$ as an absolutely convergent real series, we truncate this series and then we apply the classic Euler Forward Method (see, e.g., [52]) to find the values of $u(p^{-ord(x)}, t)$, when $-20 \leq ord(x) \leq 20$ (vertical axis) and when $t = \{t_k : \ t_k = 1/k, k = 1, \ldots, 300\}$ (horizontal axis). In Figure 2.1, on the left, the heat map of the numerical solution of the homogeneous equation $u_t(x,t) = -\boldsymbol{D}_x^{\alpha} u(x,t)$ with initial data $u(x,0) = 4e^{-p^{|ord(x)|}/100}$ (Gaussian bell type), and parameters $p = 3$, $\alpha = 0.2$, $\gamma = 1$. On the right side, we have the numerical solution of the equation $u_t(x,t) = -\boldsymbol{D}_x^{\alpha} u(x,t) - u^3(x,t) + (\beta+1)u^2(x,t) - \beta u(x,t) + \boldsymbol{D}_x^{\alpha_1} u^3(x,t)$, with $p = 3$, $\alpha = 0.2$, $\alpha_1 = 0.1$, and $\beta = 0.7$.

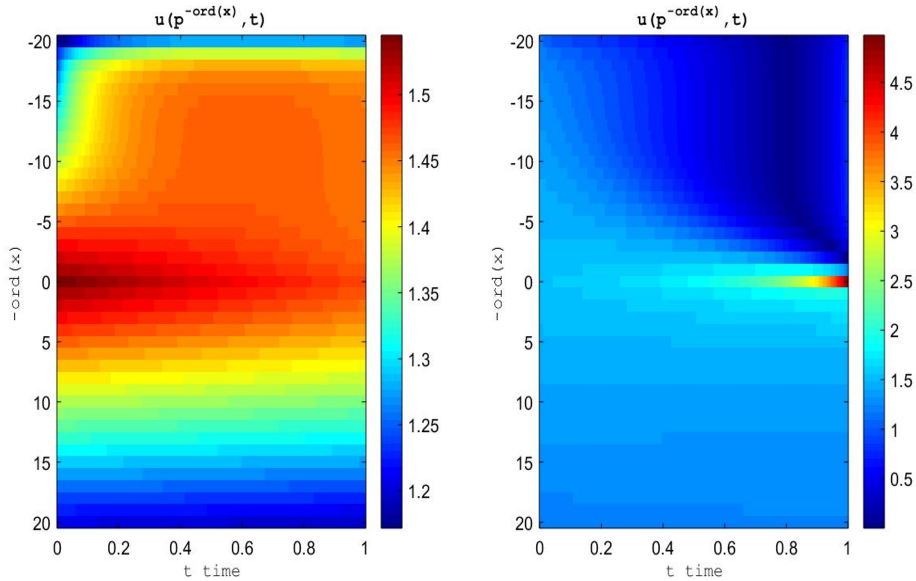

Figure 2.1: Numerical Simulations.

On the left side of Figure 2.1, we observe that the solution u is uniformly decreasing with respect to the variable t. This behavior is typical for solutions of diffusion equations. These equations have been extensively studied, see, e.g., [30], [63] and the references therein.

On the right side of Figure 2.1, we see that the evolution of $u(x,t)$ is controlled by the diffusion term $-\boldsymbol{D}_x^{\alpha} u(x,t)$, up to a time T (blow-up time), this behavior is similar to that described above. When $t > T$, the reactive term $-u^3(x,t)+(\beta+1)u^2(x,t)-\beta u(x,t)+\boldsymbol{D}_x^{\alpha_1} u^3(x,t)$ takes over and $u(x,t)$ grows rapidly towards infinity. The method converges quite fast, but still lacks a mathematical formalism to support it, for this reason, we refer to it as a numerical simulation of the solution.

Chapter 3

Turing Patterns in a p-Adic FitzHugh-Nagumo System on the Unit Ball

In this chapter, we introduce discrete and p-adic continuous versions of the FitzHugh-Nagumo system on the one-dimensional p-adic unit ball. We provide criteria for the existence of Turing patterns. We present extensive simulations of some of these systems. The simulations show that the Turing patterns are traveling waves in the p-adic unit ball.

3.1 Homogeneous steady states

A reaction-diffusion system exhibits diffusion-driven instability, or Turing instability, if the homogeneous steady state is stable to small perturbations in the absence of diffusion but unstable to small spatial perturbations when diffusion is present. The main process driving the spatially inhomogeneous instability is diffusion: the mechanism determines the spatial pattern that evolves. For Turing instability, we require that the system is stable in the absence of diffusion. For a detailed exposition, the reader may consult [43, Section 2.2, Section 2.3]. From now on, we set $u(x,t), v(x,t) : \mathbb{Z}_p \times [0,\infty) \to \mathbb{R}$. We consider the following FitzHugh-Nagumo system with p-adic diffusion:

$$
\begin{cases}
u(\cdot,t), v(\cdot,t) \in L^2_{\mathbb{R}}(\mathbb{Z}_p), \text{ for } t \geq 0; \\
\\
\frac{\partial u}{\partial t}(x,t) = f(u,v) - \left(\boldsymbol{D}_0^\alpha - \lambda\right) u(x,t), \quad x \in \mathbb{Z}_p,\, t \geq 0; \\
\\
\frac{\partial v}{\partial t}(x,t) = g(u,v) - d\left(\boldsymbol{D}_0^\alpha - \lambda\right) v(x,t), \quad x \in \mathbb{Z}_p,\;\; t \geq 0,
\end{cases}
\tag{3.1.1}
$$

where

$$
f(u,v) = \mu u - u^3 - v,\; g(u,v) = \gamma(u - \delta v - \beta), \tag{3.1.2}
$$

and μ, β, $\gamma \neq 0$, $\delta \neq 0$, d are real numbers.

We now consider a homogeneous steady state (also called an equilibrium point) of (3.1.1) which is a positive (u_0, v_0) solution of

$$
\begin{cases}
\frac{\partial u}{\partial t} = f(u,v), \quad t \geq 0; \\
\\
\frac{\partial v}{\partial t} = g(u,v), \quad t \geq 0.
\end{cases}
\tag{3.1.3}
$$

The equilibrium points associated with (3.1.3) are given by

$$
\begin{cases}
\mu u - u^3 - v = 0; \\
\\
\gamma(u - \delta v - \beta) = 0.
\end{cases}
\tag{3.1.4}
$$

Using the substitution method on (3.1.4), we have that (3.1.4) is equivalent to

$$
u^3 + \eta u + \tau = 0, \tag{3.1.5}
$$

where $\eta := \frac{1-\delta\mu}{\delta}$ and $\tau := -\frac{\beta}{\delta}$. Here we use the hypothesis that $\gamma \neq 0$, $\delta \neq 0$. We denote by u_0 a real solution of (3.1.5). Then (u_0, v_0), with $v_0 = \frac{u_0 - \beta}{\delta}$, is the real equilibrium point of (3.1.3).

We denote by $\sigma_{\text{eigen}}\left(\boldsymbol{D}_0^\alpha - \lambda\right)$ to the set of eigenvalues of $\boldsymbol{D}_0^\alpha - \lambda$. We also set

$$
\kappa_1 = \frac{1}{2d}\left\{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right) - \sqrt{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2 - 4d\det(A)}\right\}, \tag{3.1.6}
$$

$$
\kappa_2 = \frac{1}{2d}\left\{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right) + \sqrt{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2 - 4d\det(A)}\right\}, \tag{3.1.7}
$$

where

$$
A = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}_{u=u_0, v=v_0} := \begin{bmatrix} f_{u_0} & f_{v_0} \\ g_{u_0} & g_{v_0} \end{bmatrix}.
$$

Notice that A is the Jacobian matrix of the mapping $(u,v) \to (f(u,v), g(u,v))$. A straightforward calculation shows that

$$A = \begin{bmatrix} \mu - 3u_0^2 & -1 \\ \gamma & -\gamma\delta \end{bmatrix}. \tag{3.1.8}$$

Theorem 3.1.1. *Consider the reaction-diffusion system (3.1.1). The steady state (u_0, v_0) is linearly unstable (Turing unstable), if the following conditions hold:*

1. $Tr(A) = \mu - 3u_0^2 - \gamma\delta < 0$;
2. $\det(A) = -\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma > 0$;
3. $d(\mu - 3u_0^2) - \gamma\delta > 0$;
4. *The derivatives $\mu - 3u_0^2$ and $-\gamma\delta$ must have opposite signs;*
5. $(d(\mu - 3u_0^2) - \gamma\delta)^2 - 4d(-\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma) > 0$;
6. $\Gamma = \{\kappa \in \sigma_{eigen}(\boldsymbol{D}_0^\alpha - \lambda); \kappa_1 < \kappa < \kappa_2\} \neq \emptyset$.

Furthermore, there are infinitely many unstable eigenmodes, and the Turing pattern has the form (3.1.21).

Proof. We observe that (3.1.3) is a system of ordinary differential equations in $\mathbb{R}^2$. In order to linearize about the steady state (u_0, v_0), we set

$$w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix},$$

then the linear approximation is

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} \mu - 3u_0^2 & -1 \\ \gamma & -\gamma\delta \end{bmatrix} \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}.$$

By taking $w_t := \begin{bmatrix} u_t \\ v_t \end{bmatrix}$, the linearization can be written as

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = A \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}.$$

The steady state $w = 0$ is linearly stable, if $\mathrm{Re}(\rho_{1,2}) < 0$, where $\rho_{1,2}$ are the eigenvalues of A:

$$\det(A-\rho I) = \det\begin{pmatrix} \mu - 3u_0^2 - \rho & -1 \\ \gamma & -\gamma\delta - \rho \end{pmatrix} = 0.$$

Thus, we have that

$$\rho^2 - \left(\mu - 3u_0^2 - \gamma\delta\right)\rho + \left(-\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma\right) = 0. \tag{3.1.9}$$

Which are given by

$$\rho_{1,2} = \frac{\pm\sqrt{\left(\mu - 3u_0^2 - \gamma\delta\right)^2 - 4\left(-\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma\right)}}{2} + \frac{\left(\mu - 3u_0^2 - \gamma\delta\right)}{2}.$$

The condition $\mathrm{Re}(\rho_{1,2}) < 0$ is guaranteed, if $\mathrm{Tr}(A) < 0$ and $\det(A) > 0$.

$$\mathrm{Tr}(A) < 0,\ \det(A) > 0. \tag{3.1.10}$$

Now, we linearize the entire reaction-ultradiffusion system close to the steady state $w = 0 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\begin{aligned} \frac{\partial w}{\partial t} &= Aw - D\left(\boldsymbol{D}_0^\alpha - \lambda\right)w, \text{ where } D = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}; \\ \left(\boldsymbol{D}_0^\alpha - \lambda\right)w &:= \begin{bmatrix} \left(\boldsymbol{D}_0^\alpha - \lambda\right)w_1 \\ \left(\boldsymbol{D}_0^\alpha - \lambda\right)w_2 \end{bmatrix}. \end{aligned} \tag{3.1.11}$$

We now find a solution of the system (3.1.11) satisfying $u(\cdot,t),\ v(\cdot,t) \in L^2_{\mathbb{R}}(\mathbb{Z}_p)$, for $t \geq 0$. First, we determine a solution w_κ of the following eigenvalue problem:

$$\begin{cases} \left(\boldsymbol{D}_0^\alpha - \lambda\right)w_\kappa(x) = \kappa w_\kappa(x); \\ \\ w_\kappa \in L^2_{\mathbb{R}}(\mathbb{Z}_p), \end{cases} \tag{3.1.12}$$

where $w_\kappa = \begin{bmatrix} w_{1,\kappa} \\ w_{2,\kappa} \end{bmatrix}$. By using the results of section 1.2.5, the solutions to the eigenvalue problem (3.1.12) are

$$\begin{aligned} w_{1,\kappa}, w_{2,\kappa} \in &\bigsqcup_{rnj}\left\{p^{-\frac{r}{2}}\cos\left(\left\{p^{-1}j\left(p^r x - n\right)\right\}_p\right)\Omega\left(|p^r x - n|_p\right)\right\} \bigsqcup \\ &\bigsqcup_{rnj}\left\{p^{-\frac{r}{2}}\sin\left(\left\{p^{-1}j\left(p^r x - n\right)\right\}_p\right)\Omega\left(|p^r x - n|_p\right)\right\} \bigsqcup \left\{\Omega\left(p^{-M}|x|_p\right)\right\}, \end{aligned}$$

where the parameters r, j and n as in section 1.2.5. We look for a solution of the form $w(x,t) = \sum_{\kappa,\rho} C_{\kappa,\rho} e^{\rho t} w_\kappa(x)$. The function $e^{\rho t} w_\kappa(x)$ is a non-trivial solution of (3.1.11), if ρ satisfies

$$\det(\rho I - A + \kappa D) = 0, \tag{3.1.13}$$

that is,

$$\rho^2 + [\kappa(1+d) - \mathrm{Tr}(A)]\rho + h(\kappa) = 0, \tag{3.1.14}$$

where

$$h(\kappa) = d\kappa^2 - \kappa\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right) + \det(A). \tag{3.1.15}$$

Since $\kappa = 0$, isn't an eigenvalue of the operator $\boldsymbol{D}_0^\alpha - \lambda$, the conditions (3.1.13) and (3.1.9) are independent. For the steady state to be unstable for spatial perturbations, we need that $\mathrm{Re}(\rho(\kappa)) > 0$, for some $\kappa \neq 0$, this can happen either if the coefficient of ρ in (3.1.14) is negative or if $h(\kappa) < 0$, for some $\kappa \neq 0$ in (3.1.15). For being $\mathrm{Tr}(A) < 0$ of the conditions (3.1.10) and the coefficient of ρ in (3.1.14) is $\kappa(1+d) - \mathrm{Tr}(A)$, which is positive, so the only way that $\mathrm{Re}(\rho(\kappa))$ can be positive is if $h(\kappa) < 0$ for some $\kappa \neq 0$. Then, we have that $\det(A) > 0$ of (3.1.10), in order for that $h(\kappa)$ to be negative, it is necessary that $d\left(\mu - 3u_0^2\right) - \gamma\delta > 0$. Now, since $\mu - 3u_0^2 - \gamma\delta = \mathrm{Tr}(A) < 0$, necessarily $d \neq 1$ and $\mu - 3u_0^2$ and $-\gamma\delta$ must have opposite signs. Thus, we have that an additional requirement to (3.1.10) is that $d \neq 1$. This is a necessary, but not sufficient, condition for that $\mathrm{Re}(\rho(\kappa)) > 0$. For that $h(\kappa)$ to be negative for some non-zero κ, the minimum $h_{\min}$ of $h(\kappa)$ must be negative. Using elementary calculations, we show that

$$h_{\min} = \det(A) - \frac{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2}{4d}, \tag{3.1.16}$$

and the minimum is reached at

$$\kappa_{\min} = \frac{d\left(\mu - 3u_0^2\right) - \gamma\delta}{2d}. \tag{3.1.17}$$

Therefore, the condition $h(\kappa) < 0$ for some $\kappa \neq 0$ is

$$\frac{\left(d(\mu - 3u_0^2) - \gamma\delta\right)^2}{4d} > \det(A). \tag{3.1.18}$$

A bifurcation occurs when $h_{\min} = 0$, this happens when the condition

$$\det(A) = \frac{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2}{4d}, \tag{3.1.19}$$

is verified. This condition defines a critical diffusion d_c, which is given as an appropriate root of

$$\left(\mu - 3u_0^2\right)^2 d_c^2 + 2\left(-2\gamma + \mu\gamma\delta - 3\gamma\delta u_0^2\right) d_c + \gamma^2\delta^2 = 0.$$

The model (3.1.1) for $d > d_c$ exhibits Turing instability, while for $d < d_c$ it does not. Note that $d_c > 1$. A critical 'wavenumber' is obtained using (3.1.17)

$$\kappa_c = \frac{d_c\left(\mu - 3u_0^2\right) - \gamma\delta}{2d_c} = \sqrt{\frac{\det(A)}{d_c}}. \tag{3.1.20}$$

When $d > d_c$, there is a range of number of unstable positive waves $\kappa_1 < \kappa < \kappa_2$, where κ_1, κ_2 are the zeros of $h(\kappa) = 0$, see (3.1.6)-(3.1.7). We call to function $\rho(\kappa)$ the dispersion relation. We note that, within the unstable range, $\mathrm{Re}(\rho(\kappa)) > 0$ has a maximum for the wavenumber $\kappa_{min}^{(0)}$ obtained from (3.1.17) with $d > d_c$. Then as t it increases, the behavior of $w(x,t)$ is controlled by the dominant mode, that is, those $e^{\rho(\kappa)t} w_\kappa(x)$ with $\mathrm{Re}(\rho(\kappa)) > 0$, since the other modes go to zero exponentially. Then,

$$\begin{aligned} w(x,t) \sim & \sum_{\kappa_1<\kappa<\kappa_2} \sum_{r,n} A_{rn} e^{\rho(\kappa)t} \Omega\left(|p^r x - n|_p\right) \\ & + \sum_{\kappa_1<\kappa<\kappa_2} \sum_{r,n,j} A_{rnj} e^{\rho(\kappa)t} p^{-\frac{r}{2}} \cos\left(\left\{p^{-1} j \left(p^r x - n\right)\right\}_p\right) \Omega\left(|p^r x - n|_p\right) \\ & + \sum_{\kappa_1<\kappa<\kappa_2} \sum_{r,n,j} B_{rnj} e^{\rho(\kappa)t} p^{-\frac{r}{2}} \sin\left(\left\{p^{-1} j \left(p^r x - n\right)\right\}_p\right) \Omega\left(|p^r x - n|_p\right), \end{aligned} \tag{3.1.21}$$

for $t \to \infty$, where $\rho(\kappa)$ are eigenvalues of matrix A depending on $\kappa \in \sigma_{\text{eigen}}(\boldsymbol{D}_0^\alpha - \lambda)$, with $\mathrm{Re}(\rho(\kappa)) > 0$. In the above expansion in all series the r's and j's take only finite numbers of values. Thus, all the κ's are, but one has the form $p^{(1-r)\alpha} - \lambda$, the condition $\kappa_1 < \kappa < \kappa_2$ implies that there is only one finite number of the r's. But the n's run through an infinite set, which is $p^r\mathbb{Z}_p \cap \mathbb{Q}_p/\mathbb{Z}_p$. Now, fixing r, then, for a given $x \in \mathbb{Z}_p$ there exist only finite numbers of balls of type $B_r\left(p^{-r}n\right)$ that contains x. This fact implies that the value in (x,t) of $w(x,t)$ in the expansion (3.1.21) is determined only by a finite number of the n's, and consequently the series in expansion (3.1.21) is convergent. □

3.2 A discrete FitzHugh-Nagumo system on $\mathbb{Z}_p$

3.2.1 Discretization of the operator $\boldsymbol{D}_0^{\alpha} - \lambda$

A natural discretization of $\boldsymbol{D}_0^{\alpha} - \lambda$ is obtained by taking its restriction to $\mathcal{D}_L$. We denote this restriction by $\boldsymbol{D}_L^{\alpha} - \lambda$. Since $\mathcal{D}_L$ is a finite vector space, $\boldsymbol{D}_L^{\alpha} - \lambda$ is represented by a matrix $A_L^{\alpha} = \left[A_{K,I}^{\alpha}\right]_{K,I\in G_L}$, where

$$A_{K,I}^{\alpha} = \begin{cases} p^{-\frac{L}{2}} \frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \frac{1}{|K-I|_p^{\alpha+1}} & \text{if} \quad K \neq I; \\ \\ -p^{-\frac{L}{2}} \frac{1-p^{\alpha}}{1-p^{-\alpha-1}} \sum_{K\neq I} \frac{1}{|K-I|_p^{\alpha+1}} & \text{if} \quad K = I, \end{cases} \tag{3.2.1}$$

see [67].

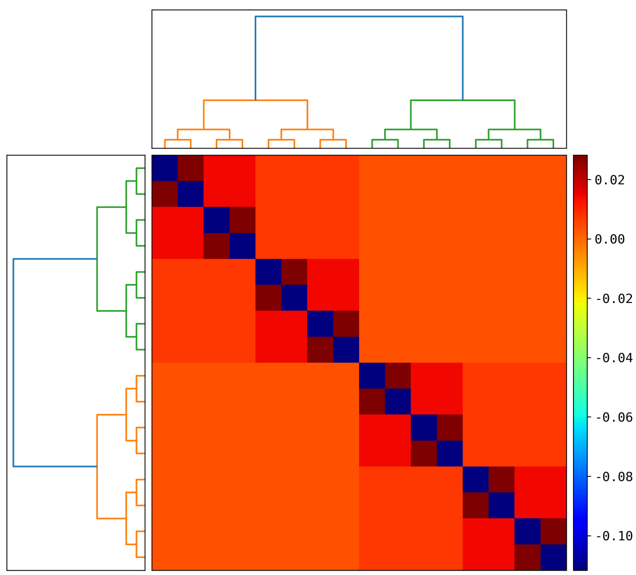

Figure 3.1: The heat map for matrix A_L^{α}; $p = 2, L = 4, M = 0, \alpha = 0.01$. Here the parameters L and M are taken as in section 1.4. The vertical and horizontal scales run through the points of tree G_4.

3.2.2 Discretization of the p-adic Turing System (3.1.1)

A discretization of the Turing system (3.1.1) is obtained by approximating the functions $u(x,t)$, $v(x,t)$ as

$$u_L(x,t) = \sum_{I \in G_L} u_L(I,t) \Omega \left(p^L |x - I|_p \right)$$

and

$$v_L(x,t) = \sum_{I \in G_L} v_L(I,t) \Omega \left(p^L |x - I|_p \right),$$

where $u_L(I,\cdot)$, $v_L(I,\cdot) \in C^1([0,T])$ for some fixed positive T. We set

$$u_L(x,t) = [u_L(I,t)]_{I \in G_L}, \; \; v_L(x,t) = [v_L(I,t)]_{I \in G_L}.$$

Notice that

$$\begin{aligned} f\left(\sum_{I \in G_L} u_L(I,t) \Omega \left(p^L |x - I|_p \right), \sum_{J \in G_L} u_L(J,t) \Omega \left(p^L |x - J|_p \right) \right) \\ = \sum_{I \in G_L} f\left(u_L(I,t), v_L(I,t) \right) \Omega \left(p^L |x - I|_p \right) \\ = \sum_{I \in G_L} \left\{ \mu u_L(I,t) - u_L^3(I,t) - u_L(I,t) \right\} \Omega \left(p^L |x - I|_p \right). \end{aligned}$$

A similar formula holds for function g. Then, using (3.1.2), the discretization of the p-adic Turing system (3.1.1) has the form:

$$\begin{cases} \frac{\partial}{\partial t} [u_L(I,t)]_{I \in G_L} = & [\mu u_L(I,t) - u_L^3(I,t) - v_L(I,t)]_{I \in G_L} - A_L^\alpha [u_L(I,t)]_{I \in G_L} \\ \\ \frac{\partial}{\partial t} [v_L(I,t)]_{I \in G_L} = & [\gamma \left(u_L(I,t) - \delta v_L(I,t) - \beta \right)]_{I \in G_L} - d A_L^\alpha [v_L(I,t)]_{I \in G_L}, \end{cases} \tag{3.2.2}$$

where $A_L^\alpha = \left[A_{K,I}^\alpha \right]_{K,I \in G_L}$.

We now rewrite system (3.2.2) in a matrix form. We denote by $diag\left(a_I; I \in G_L\right)$, a diagonal matrix of size $\#G_L \times \#G_L$. Now, by using (3.2.1) and (3.2.2), we have

$$\frac{\partial}{\partial t}\begin{bmatrix} [u_L(I,t)]_{I\in G_L} \\ \\ [v_L(I,t)]_{I\in G_L} \end{bmatrix} = \tag{3.2.3}$$

$$\begin{bmatrix} diag\left(f\left(u_L(I,t), v_L(I,t)\right); I \in G_L\right) & 0_{\#G_L\times\#G_L} \\ \\ 0_{\#G_L\times\#G_L} & diag\left(g\left(u_L(I,t), v_L(I,t)\right); I \in G_L\right) \end{bmatrix}$$

$$-\begin{bmatrix} I_{\#G_L\times\#G_L} & 0_{\#G_L\times\#G_L} \\ \\ 0_{\#G_L\times\#G_L} & dI_{\#G_L\times\#G_L} \end{bmatrix}\begin{bmatrix} A_L^{\alpha} & 0_{\#G_L\times\#G_L} \\ \\ 0_{\#G_L\times\#G_L} & A_L^{\alpha} \end{bmatrix}\begin{bmatrix} [u_L(I,t)]_{I\in G_L} \\ \\ [v_L(I,t)]_{I\in G_L} \end{bmatrix},$$

where $0_{\#G_L\times\#G_L}$ denotes a matriz of size $\#G_L \times \#G_L$ with all its entries equal to zero, and $I_{\#G_L\times\#G_L}$ denotes the identity matrix of size $\#G_L \times \#G_L$.

3.2.3 Discrete homogeneous steady states

We study the equilibrium points of the system

$$\frac{\partial}{\partial t}\begin{bmatrix} [u_L(I,t)]_{I\in G_L} \\ \\ [v_L(I,t)]_{I\in G_L} \end{bmatrix} = \tag{3.2.4}$$

$$\begin{bmatrix} diag\left(f\left(u_L(I,t), v_L(I,t)\right); I \in G_L\right) & 0_{\#G_L\times\#G_L} \\ \\ 0_{\#G_L\times\#G_L} & diag\left(g\left(u_L(I,t), v_L(I,t)\right); I \in G_L\right) \end{bmatrix}.$$

The equilibrium points are the solutions of the following system of algebraic equations:

$$\begin{cases} f\left(u_L(I), v_L(I)\right) = 0 \\ \\ g\left(u_L(I), v_L(I)\right) = 0, \end{cases} \tag{3.2.5}$$

where $I \in G_L$. Notice that if $f(u_0, v_0) = g(u_0, v_0) = 0$, then $u_L(I) = u_0$, $v_L(I) = v_0$ is a solution of (3.2.5) for any $I \in G_L$.

Take $\eta = \frac{1-\delta\mu}{\delta}$ and $\tau = -\frac{\beta}{\delta}$, as before. Then,

$$\begin{bmatrix} [u_0]_{I \in G_L} \\ \\ [v_0]_{I \in G_L} \end{bmatrix} \tag{3.2.6}$$

is one equilibrium point.

3.2.4 The Jacobian matrix

We now consider the following polynomial mapping:

$$\begin{array}{ccc} \mathbb{R}^{2\#G_L} & \to & \mathbb{R}^{2\#G_L} \\ \\ \begin{bmatrix} [u_L(I)]_{I \in G_L} \\ \\ [v_L(I)]_{I \in G_L} \end{bmatrix} & \to & \begin{bmatrix} [f(u_L(I), v_L(I))]_{I \in G_L} \\ \\ [g(u_L(I), v_L(I))]_{I \in G_L} \end{bmatrix}. \end{array} \tag{3.2.7}$$

We denote by $\nabla f(u_0, v_0)$, the 1×2 matrix $\begin{bmatrix} \frac{\partial f(u_0,v_0)}{\partial u} & \frac{\partial f(u_0,v_0)}{\partial v} \end{bmatrix}$, and by

$$diag\left(\nabla f(u_0, v_0); I \in G_L\right),$$

the block diagonal matrix

$$\begin{bmatrix} \nabla f(u_0, v_0) & & 0 \\ & \ddots & \\ 0 & & \nabla f(u_0, v_0) \end{bmatrix}$$

of size $\#G_L \times 2\#G_L$. In a similar form, we define the block diagonal matrix

$$diag\left(\nabla g(u_0, v_0); I \in G_L\right).$$

The Jacobian matrix $\mathcal{A}$ of mapping (3.2.7) at the equilibrium point (3.2.6) is the $2\#G_L \times 2\#G_L$ matrix

$$\mathcal{A} = \begin{bmatrix} \nabla f(u_0, v_0) & & 0 \\ & \ddots & \\ 0 & & \nabla f(u_0, v_0) \\ \nabla g(u_0, v_0) & & 0 \\ & \ddots & \\ 0 & & \nabla g(u_0, v_0) \end{bmatrix} = \begin{bmatrix} diag\left(\nabla f(u_0, v_0); I \in G_L\right) \\ \\ diag\left(\nabla g(u_0, v_0); I \in G_L\right) \end{bmatrix}.$$

We now set

$$A = \begin{bmatrix} \frac{\partial f(u_0,v_0)}{\partial u} & \frac{\partial f(u_0,v_0)}{\partial v} \\ \frac{\partial g(u_0,v_0)}{\partial u} & \frac{\partial g(u_0,v_0)}{\partial v} \end{bmatrix} = \begin{bmatrix} \nabla f(u_0, v_0) \\ \nabla g(u_0, v_0) \end{bmatrix}$$

as before, and by a finite sequence of swapings of rows, matrix $\mathcal{A}$ can be writen as

$$\mathcal{A}' = \begin{bmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{bmatrix}, \tag{3.2.8}$$

which is a $\#G_L \times \#G_L$ block matrix.

We denote by $\sigma(A_L^\alpha)$ the spectrum of A, and use the κ_1, κ_2 defined in (3.1.6)-(3.1.7).

Theorem 3.2.1. *Let us consider the reaction-diffusion system (3.2.3). The discrete steady state* $\begin{bmatrix} [u_0]_{I\in G_L} \\ \\ [v_0]_{I\in G_L} \end{bmatrix}$ *is linearly unstable (Turing unstable), if the following conditions hold:*

1. $Tr(A) = \mu - 3u_0^2 - \gamma\delta < 0$;
2. $\det(A) = -\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma > 0$;
3. $d(\mu - 3u_0^2) - \gamma\delta > 0$;
4. *The derivatives* $\mu - 3u_0^2$ *and* $-\gamma\delta$ *must have opposite signs;*
5. $(d(\mu - 3u_0^2) - \gamma\delta)^2 - 4d(-\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma) > 0$;
6. $\Gamma_L = \{\kappa_L \in \sigma(A_L^\alpha); \kappa_1 < \kappa_L < \kappa_2\} \neq \emptyset$.

Furthermore, the Turing pattern has the form (3.2.18).

Proof. We first linearize sytem (3.2.3) about the steady state (3.2.6). Set

$$\begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix} := \begin{bmatrix} [u_L(I,t) - u_0]_{I\in G_L} \\ \\ [v_L(I,t) - v_0]_{I\in G_L} \end{bmatrix}.$$

Then the linear approximation is

$$\begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix} = \mathcal{A} \begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix}.$$

The equilibrium point

$$\begin{bmatrix} [0]_{I\in G_L} \\ \\ [0]_{I\in G_L} \end{bmatrix} \tag{3.2.9}$$

is linearly stable, if the eigenvalues of $\mathcal{A}$ have negative real parts. By a suitable sequence of swapings of the rows of $\mathcal{A}$, we have

$$\det(\mathcal{A}-\rho I)=\pm\det(\mathcal{A}'-\rho I)=\pm\det(A-\rho I)^{\#G_L}.$$

Then the eigenvalues of $\mathcal{A}$ are exactly the eigenvalues of A counted with multiplicity G_L:

$$\det(A-\rho I)=\det\begin{bmatrix} \mu-3u_0^2-\rho & -1 \\ \gamma & -\lambda\delta-\rho \end{bmatrix}=\rho^2-\rho Tr(A)+\det(A)=0. \tag{3.2.10}$$

Then

$$\rho_{1,2}=\frac{\pm\sqrt{(\mu-3u_0^2-\gamma\delta)^2-4\left(-\mu\gamma\delta+3\gamma\delta u_0^2+\gamma\right)}}{2}+\frac{\mu-3u_0^2-\gamma\delta}{2}.$$

The condition $\mathrm{Re}(\rho_{1,2})<0$ is guaranteed, if the trace and the determinant of matrix A satisfy

$$\mathrm{Tr}(A)<0,\ \det(A)>0. \tag{3.2.11}$$

Now, we linearize the entire reaction-ultradiffusion system close to the steady state (3.2.9):

$$\frac{\partial}{\partial t}\begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix}=(\mathcal{A}-D_L\mathcal{A}_L^{\alpha})\begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix}, \tag{3.2.12}$$

where

$$D_L:=\begin{bmatrix} I_{\#G_L\times\#G_L} & 0_{\#G_L\times\#G_L} \\ \\ 0_{\#G_L\times\#G_L} & dI_{\#G_L\times\#G_L} \end{bmatrix},\ \mathcal{A}_L^{\alpha}:=\begin{bmatrix} A_L^{\alpha} & 0_{\#G_L\times\#G_L} \\ \\ 0_{\#G_L\times\#G_L} & A_L^{\alpha} \end{bmatrix}.$$

The matrices A_L^{α}, $\mathcal{A}_L^{\alpha}$ are real symmetric, and consequently they are diagonalizable. Then, there exists a basis $\{\boldsymbol{e}_\kappa\}$ of $\mathbb{R}^{\#G_L}$ such that

$$A_L^{\alpha}\boldsymbol{e}_\kappa=\kappa\boldsymbol{e}_\kappa,$$

where $\kappa=\kappa(L)$. Then

$$\mathcal{A}_L^{\alpha}\begin{bmatrix} \boldsymbol{e}_\kappa \\ \boldsymbol{e}_\kappa \end{bmatrix}=\kappa\begin{bmatrix} \boldsymbol{e}_\kappa \\ \boldsymbol{e}_\kappa \end{bmatrix}.$$

We now look for a solution of system (3.2.12) of the form

$$\begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix},$$

where $w_L^{(j)}(I,t) = \sum_{\kappa,\rho} C_{\kappa,\rho} e^{\rho t} \boldsymbol{e}_\kappa$, where $\rho = \rho(j, I, L)$, $\kappa = \kappa(j, I, L)$. The function $e^{\rho t}\boldsymbol{e}_\kappa$ is a non-trivial solution of (3.2.12), if ρ satisfies

$$\det(\rho I - \mathcal{A} + \kappa D_L) = 0. \tag{3.2.13}$$

By a finite sequence of swapings of rows, we have

$$\begin{aligned} \det(\rho I - \mathcal{A} + \kappa D_L) &= \pm \det \begin{bmatrix} \rho I_{2\times 2} - A + \kappa D & & 0 \\ & \ddots & \\ 0 & & \rho I_{2\times 2} - A + \kappa D \end{bmatrix} \\ &= \pm \det\left(\rho I_{2\times 2} - A + \kappa D\right)^{\# G_L} \\ &= \rho^2 + [\kappa(1+d) - \mathrm{Tr}(A)]\rho + h(\kappa) = 0, \end{aligned} \tag{3.2.14}$$

where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix},$$

and

$$h(\kappa) := d\kappa^2 - \kappa\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right) + \det(A). \tag{3.2.15}$$

Since $\kappa = 0$ is not an eigenvalue of the matrix A_L^α, the conditions (3.2.10) and (3.2.14) are independent. For that the steady state to be unstable for spatial perturbations, we need that $\mathrm{Re}(\rho(\kappa)) > 0$, for some $\kappa \neq 0$, this can happen either if the coefficient of ρ in (3.2.14) is negative or if $h(\kappa) < 0$, for some $\kappa \neq 0$ in (3.2.15). For being $\mathrm{Tr}(A) < 0$ of the conditions (3.2.11) and the coefficient of ρ in (3.2.14) is $\kappa(1+d) - \mathrm{Tr}(A)$, which is positive, so the only way that $\mathrm{Re}(\rho(\kappa))$ can be positive is if $h(\kappa) < 0$ for some $\kappa \neq 0$. As $\det(A) > 0$ of (3.2.11), in order for $h(\kappa)$ to be negative, it is necessary that $d\left(\mu - 3u_0^2\right) - \gamma\delta > 0$. Now, since $\mathrm{Tr}(A) = \mu - 3u_0^2 - \gamma\delta < 0$, necessarily $d \neq 1$ and $\mu - 3u_0^2$ and $-\gamma\delta$ must have opposite signs. Thus, we have that an additional requirement to (3.2.11) is that $d \neq 1$. This is a necessary, but not sufficient, condition for that $\mathrm{Re}(\rho(\kappa)) > 0$. For that $h(\kappa)$ to be negative for some non zero κ, the minimum $h_{\min}$ of $h(\kappa)$ must be negative. Using elementary calculations, we show that

$$h_{\min} = \det(A) - \frac{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2}{4d},$$

and the minimum is reached at

$$k_{\min} = \frac{d\left(\mu - 3u_0^2\right) - \gamma\delta}{2d}. \tag{3.2.16}$$

Therefore, the condition $h(\kappa) < 0$ for some $\kappa \neq 0$ is

$$\frac{\left(d(\mu - 3u_0^2) - \gamma\delta\right)^2}{4d} \det(A).$$

A bifurcation occurs when $h_{\min} = 0$, this happens when the condition

$$\det(A) = \frac{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2}{4d},$$

is verifiesd. This condition defines a critical diffusion d_c, which is given as an appropriate root of

$$\left(\mu - 3u_0^2\right)^2 d_c^2 + 2\left(-2\gamma + \mu\gamma\delta - 3\gamma\delta u_0^2\right) d_c + \gamma^2\delta^2 = 0.$$

The model (3.2.3) for $d > d_c$ exhibits Turing instability, while for $d < d_c$ it does not. Note that $d_c > 1$. A critical 'wavenumber' is obtained using (3.2.16)

$$\kappa_c = \frac{d_c\left(\mu - 3u_0^2\right) - \gamma\delta}{2d_c} = \sqrt{\frac{\det(A)}{d_c}}. \tag{3.2.17}$$

When $d > d_c$, there is a range of number of unstable positive waves $\kappa_1 < \kappa < \kappa_2$, where κ_1, κ_2 are the zeros of $h(\kappa) = 0$, see (3.1.6)-(3.1.7). We call to function $\rho(\kappa)$ the dispersion relation. We note that, within the unstable range, $\mathrm{Re}(\rho(\kappa)) > 0$ has a maximum for the wavenumber $\kappa_{min}^{(0)}$ obtained from (3.2.16) with $d > d_c$. Then as t it increases, the behavior of $\begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix}$ is controlled by the dominant mode, that is, those $e^{\rho(\kappa)t}\begin{bmatrix} \boldsymbol{e}_\kappa \\ \boldsymbol{e}_\kappa \end{bmatrix}$ with $\mathrm{Re}(\rho(\kappa)) > 0$, since the other modes go to zero exponentially. We recall that $\kappa = \kappa(L)$. For this reason, we use the notation $\kappa = \kappa_L$. With this notation,

$$w_L^{(j)}(I,t) \sim \sum_{\kappa_1 < \kappa_L < \kappa_2} A_\kappa(j, I)\, e^{\rho(\kappa_L)t} \boldsymbol{e}_\kappa, \text{ for } t \to \infty, \tag{3.2.18}$$

where $j = 1, 2$.

□

Digernes and his collaborators have studies extensively the problem of approximation of spectra of Vladimirov operator $\boldsymbol{D}^{\alpha}$ by matrices of type A_L^{α}, [5, 17]. By using the fact that the eigenvalues $\varsigma \neq \lambda$ and eigenfuntions Ψ_{rnj} of $\boldsymbol{D}_0^{\alpha}$ are also eigenvalues and eigenfunctions of $\boldsymbol{D}^{\alpha}$, and Theorem 4.1 in [4], one concludes that for L sufficiently large, the eigenvalues of matrix A_L^{α} approximate the eigenvalues $\varsigma \neq \lambda$ of $\boldsymbol{D}_0^{\alpha} - \lambda$, in a symbolic form $\Gamma_L \approx \Gamma \smallsetminus \{\lambda\}$.

3.3 Numerical approximations of Turing patterns

In this section, we present numerical approximations of Turing patterns associated with specific p-adic FitzHugh-Nagumo systems. By suitable choosing of the parameters $(\mu, \gamma, \delta, \beta, d,$ with $d > 1)$, we find a region where the conditions (1)-(5) of Theorem 3.2.1 are satisfied. Then, we solve numerically the system of ODEs (3.2.3). Finally, we give various visualizations of the solutions intending to show several aspects of the Turing patterns. To construct a region (called the Turing unstable region), we use an (f_{u_1}, g_{v_1}) plane, i.e., we set

$$x = f_{u_1} = \mu - 3u_1^2, \quad y = g_{v_1} = -\gamma\delta.$$

Figure 3.2 shows a Turing unstable region associated with a steady state of system (3.2.3). The parameters $(\mu, \gamma, \delta, \beta, d,$ with $d > 1)$ that give rise to green points in Figure 3.2 correspond to some Turing pattern.

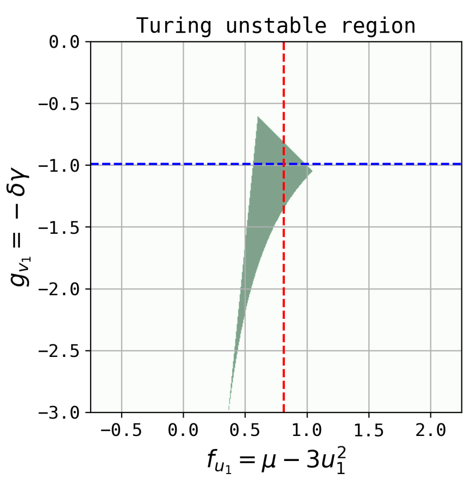

Figure 3.2: All the points in the green region of the $(f_{u_1} g_{v_1})$ -plane, which satisfies the conditions (1)−(5) of Theorem 3.2.1. The parameters are $p = 2, \mu = 1.26, \beta = 0, \delta = 0.9, \gamma = 1.1, d = 10, L = 9$, and $(u_1, v_1) = (0.3858, 0.4287)$.

The last condition in Theorem 3.2.1 is shown in the left part of Figure 3.3. More precisely, the eigenvalues of matrix A_L^α between the dotted lines (which represent the values κ_1, κ_2) satisfy condition (6) in Theorem 3.2.1. The right part of Figure 3.3 shows the eigenvalues of operator $\boldsymbol{D}_0^\alpha - \lambda$, see Section 3.2.1. For L sufficiently large, the eigenvalues of A_L^α approach to the ones of $\boldsymbol{D}_0^\alpha - \lambda$, such it was discussed at the end of Section 3.2.

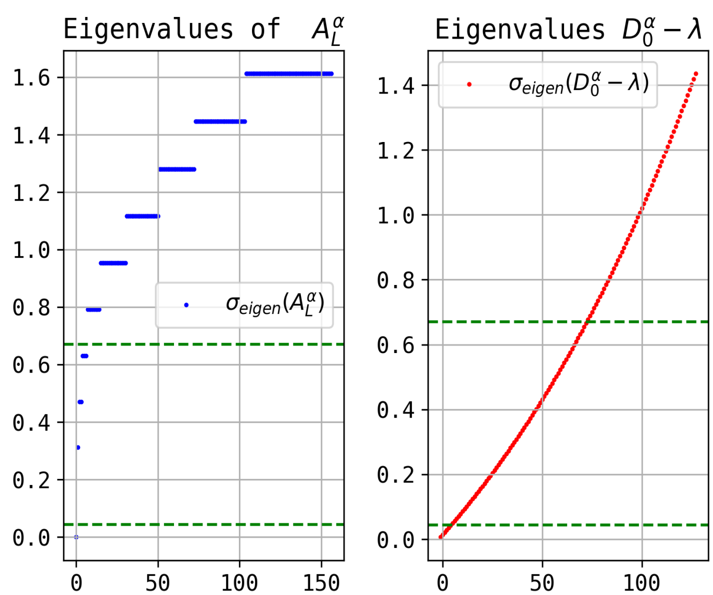

Figure 3.3: The left part of the figure shows the first 150 eigenvalues of the matrix A_L^α, which is a discretization of the Vladimirov operator $\boldsymbol{D}_0^\alpha - \lambda$. The right part of the figure shows the first 20 eigenvalues of $\boldsymbol{D}_0^\alpha - \lambda$. Notice that eigenvalue λ is very close to 1.

Figure 3.4 and 3.5 show the Turing patterns, which are solutions of the Cauchy problem associated with system (3.2.3), with an initial datum close to (u_1, v_1), for t sufficiently large.

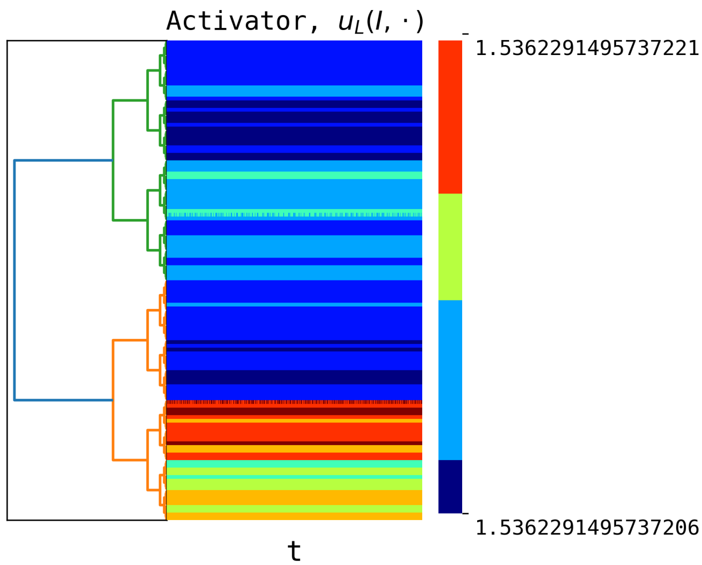

Figure 3.4: The activator states $u_L(I,\cdot)$ for $731 < t < 2000$, and $L = 9$. The vertical scale runs through the points of tree G_9.

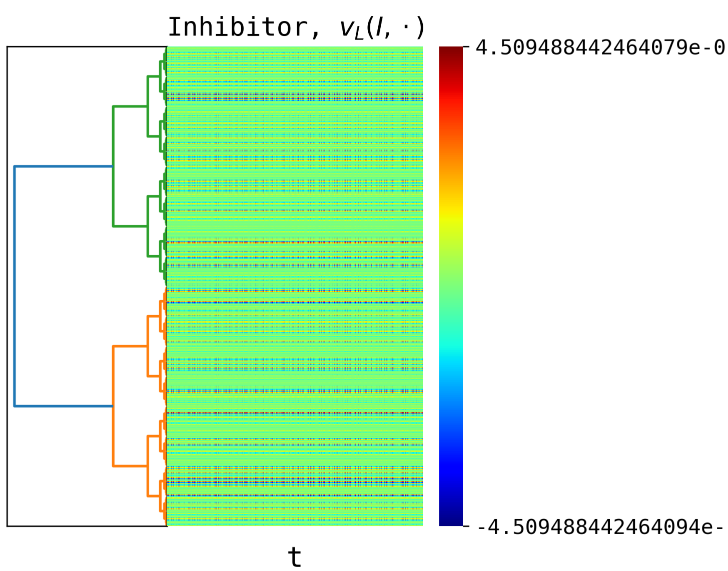

Figure 3.5: The inhibitor states $v_L(I,\cdot)$ for $731 < t < 2000$, and $L = 9$. The vertical scale runs through the points of tree G_9.

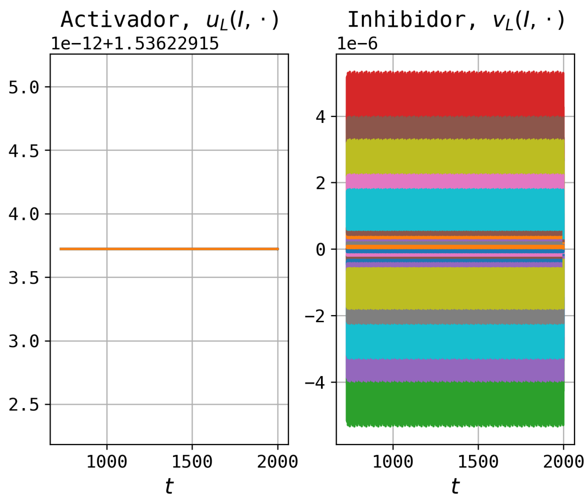

Figure 3.6: The left side of the figure shows the activator $u_L(I, \cdot)$, while the right side shows the inhibitor $v_L(I, \cdot)$. This figure shows the evolution of all the system states (3.2.3) for time $731 < t < 2000$. At time $t = 0$, the initial datum for the Cauchy problem is $(\widetilde{u_1}, \widetilde{v_1})$, where $\widetilde{u_1}$ is a sample of a Gaussian variable with mean u_1 and variance 0.01, and $\widetilde{v_1}$ is a sample of a Gaussian variable with mean v_1 and variance 0.01. For any initial state $(\widetilde{u_1}, \widetilde{v_1})$, the system (3.2.3) develops the Turing pattern shown in this figure.

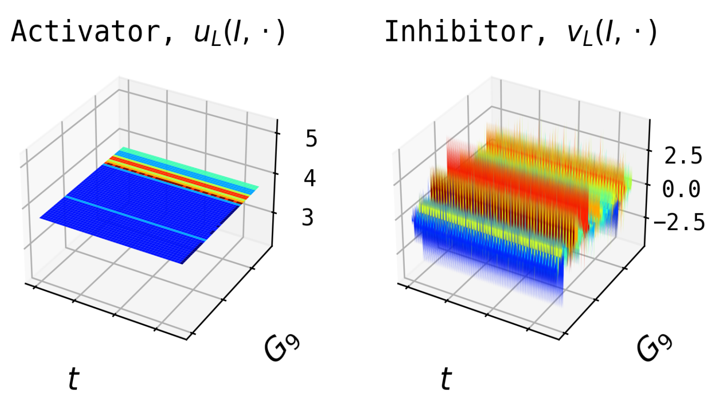

Figure 3.7: This Figure is a $3D$ version of Figure 3.6. The left side of the figure shows the activator $u_L(I, \cdot)$, while the right side shows the inhibitor $v_L(I, \cdot)$. It shows the evolution of all the states of the system (3.2.3) for time $731 < t < 2000$. The initial datum for the Cauchy problem is the same as in Figure 3.6. The Turing patterns are traveling waves.

Chapter 4

Conclusions

In this thesis, in Chapter 2, for a suitable $s \in \mathbb{R}$, we obtained results related to Sobolev-type spaces $\mathcal{H}_s$. We establish the local well-posedness of the Cauchy problem for a family of p-adic Nagumo-type equations in Sobolev-type spaces $\mathcal{H}_s$, for when $s > n/2 + 2\delta$. Also, we show that the blow-up phenomenon occurs in finite time and provide numerical simulations showing this phenomenon. Our results can serve to study the local or global well-posedness of the Cauchy problem for other parabolic equations in the p-adic context on Sobolev-type spaces $\mathcal{H}_s$ for a suitable $s \in \mathbb{R}$.

In Chapter 3, we considered the discrete and p-adic continuous versions of the FitzHugh-Nagumo system defined on $\mathbb{Z}_p$ and G_L, respectively. This system on G_L can be identified as a network, cf. [67, Section 5], the activator and inhibitor species react on the nodes and spread (in two directions) across the available links. In this context, we proved that Turing patterns can develop. Analytically, we show the conditions and the set of parameters associated with the onset of the Turing instability in the system (3.1.1). The simulations show that the Turing patterns are traveling waves on the p-adic unit ball. The results obtained can be used to study Turing patterns in other reaction-diffusion systems in the p-adic context.

Appendix A

Evolution equations and the blow-up phenomenon: basic aspects

A.1 A locally well-posed Cauchy problem

Definition A.1.1. *Let X, Y Banach spaces, $T_0 \in (0,\infty)$ and $F : [0,T_0] \times Y \longrightarrow X$ a continuous function. The Cauchy problem*

$$\begin{cases} \partial_t u(t) = F\left(t, u\left(t\right)\right) \\ \\ u(0) = \phi \in Y \end{cases} \tag{A.1.1}$$

is locally well-posed in Y, if the following conditions are satisfied.

(i) There is $T \in (0,T_0]$ and a function $u \in C([0,T];Y)$, with $u(0) = \phi$, satisfying the differential equation in the following sense:

$$\lim_{h\to 0} \left\| \frac{u(t+h)-u(t)}{h} - F(t,u(t)) \right\|_X = 0,$$

where the derivatives at $t = 0$ and $t = T$ are calculated from the right and left, respectively.

(ii) The initial value problem (A.1.1) has at most one solution in $C([0,T];Y)$.

(iii) The function $\phi \to u$ is continuous. That is, let $\{\phi_n\}$ be a sequence in Y such that $\phi_n \to \phi_\infty$ in Y and let $u_n \in C\left([0,T_n];Y\right)$, resp. $u_\infty \in C\left([0,T_\infty];Y\right)$, be the corresponding solutions. Let $T \in (0,T_\infty)$, then the solutions u_n are defined in $[0,T]$ for all n big enough and

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| u_n(t) - u_\infty(t) \right\|_Y = 0.$$

A.2 Essential Ideas of Semigroup Theory

We will give definitions and basic results about Semigroup Theory. This section is based in [40, Chapter 4].

A.2.1 Semigroups of Linear Operators

We shall study problems related to the abstract differential equation of the form

$$u' + Au = 0,$$

where A is a linear operator in a Banach space X and u' denotes the derivative of $u : [0, \infty) \to X$ in the Banach space, i.e.,

$$\lim_{h\to 0} \left\| \frac{1}{h}(u(t+h) - u(t)) - u'(t) \right\| = 0.$$

Assuming that for given $u(0)$ the equation has a unique solution on $[0, \infty)$ implies that there exist linear operators $Q(t)$, for $t \geq 0$, such that

$$u(t) = Q(t)u(0)$$

and $Q(t)Q(s) = Q(t+s)$. Formally, $Q(t) = e^{At}$. Assuming continuous dependence on initial conditions gives us that $Q(t)$ should be a bounded linear map from its domain into X. It can be arranged in applications so that the domain of $Q(t)$ is the whole X. The map $t \to Q(t)$ has to have some continuity properties. This map and some technical considerations suggest the following definitions.

Definition A.2.1. *A family of bounded linear operators $\{Q(t)\}_{t\geq 0}$ on a Banach space X is called a* **strongly continuous semigroup** *(or C_0* **semigroup***) if*
(a) $Q(0) = 1$ *(identity map)*
(b) $Q(t)Q(s) = Q(t+s)$ *for all* $t \geq U, s \geq 0$
(c) $\lim_{t\to 0+} \|Q(t)x - x\| = 0$ *for all* $x \in X$.

Definition A.2.2. *Let $\{Q(t)\}_{t\geq 0}$ be a strongly continuous semigroup of operators on a Banach space X. Define $\mathcal{D}(A)$ to be the set of all $x \in X$ for which there exists $y \in X$ such that*

$$\lim_{t\to 0+} \left\| \frac{1}{t}(x - Q(t)x) - y \right\| = 0;$$

for such x and y define $Ax = y$. The linear operator, $-A$, is called the generator (or the infinitesimal generator) of the semigroup.

The following example is taken directly from [40, EXAMPLE 4.1.3].

Example A.2.1. *Let the Banach space X be $C_u(\mathbb{R})$. For $t \geq 0$ define $Q(t) \in \mathfrak{B}(X)$, where $\mathfrak{B}(X)$ to be the set of all linear operators of X on X, by*

$$(Q(t)f)(x) = f(x-t) \text{ for } f \in X, x \in \mathbb{R}.$$

One can easily verify that $\{Q(t)\}_{t\geq 0}$ is a strongly continuous (but not continuous, [40, Exercise 1]) semigroup of operators on X. Let $-A$ be its generator. If $f \in \mathcal{D}(A)$ and

$$e(t) = \frac{f - Q(t)f}{t} - Af \text{ for } t > 0,$$

then $\lim_{t\to 0+} \|e(t)\| = 0$ and, since for all $x \in \mathbb{R}$,

$$\left|\frac{f(x-t) - f(x)}{-t} - (Af)(x)\right| \leq \|e(t)\|, \quad \left|\frac{f(x+t) - f(x)}{t} - (Af)(x+t)\right| \leq \|e(t)\|,$$

we have that $Af = f'$ and $f \in C_u^1(\mathbb{R})$. If $f \in C_u^1(\mathbb{R})$, then

$$\left(\frac{f - Q(t)f}{t} - f'\right)(x) = \int_0^1 (f'(x-st) - f'(x))\, ds \overset{t\to 0}{\longrightarrow} 0$$

uniformly in x, which implies that $f \in \mathcal{D}(A)$. Thus

$$Af = f' \text{ for } f \in \mathcal{D}(A) = C_u^1(\mathbb{R}).$$

If $f \in \mathcal{D}(A)$ and $u(t) = Q(t)f$, then $u(t) \in \mathcal{D}(A)$ and

$$\left\|\frac{u(t+h) - u(t)}{h} + Au(t)\right\| = \sup_{x\in\mathbb{R}} \left|\frac{f(x-h) - f(x)}{h} + f'(x)\right| \overset{h\to 0}{\longrightarrow} 0;$$

hence

$$u' + Au = 0 \text{ on } [0, \infty),$$

where u' denotes the derivative of u in the Banach space X. Note also that if $v(x,t) = (u(t))(x) = f(x-t)$, then

$$v_t(x,t) + v_x(x,t) - 0 \text{ for } t \geq 0, x \in \mathbb{R},$$

where v_t, v_x denote the classical partial derivatives.

A.2.2 Basic properties of semigroups

Theorem A.2.1 ([40, Theorem 4.3.1]). *Suppose $\{Q(t)\}_{t\geq 0}$ is a strongly continuous semigroup on a Banach space X and let $-A$ be the generator of the semigroup. Then*

(1) There exist $M \in [0,\infty), a \in \mathbb{R}$ such that $\|Q(t)\| \leq Me^{-at}$ for all $t \geq 0$.

(2) $t \to Q(t)x$ is a continuous mapping of $[0,\infty)$ into X for every $x \in X$.

(3) If $x \in X, t \geq 0$, then $\int_0^t Q(s)xds \in \mathcal{D}(A)$ and $x - Q(t)x = A\int_0^t Q(s)xds$.

(4) If $x \in \mathcal{D}(A), u(t) = Q(t)x$ for $t \geq 0$, then for each $t \geq 0$ we have that

$$\frac{du}{dt}(t) \text{ exists, } u(t) \in \mathcal{D}(A), \frac{du}{dt}(t) = -Au(t) = -Q(t)Ax.$$

(5) $(A-\lambda)^{-1}Q(t) = Q(t)(A-\lambda)^{-1}$ for every $\lambda \in \rho(A), t \geq 0$.

(6) A is a closed linear operator.

(7) $\cap_{n=1}^{\infty}\mathcal{D}(A^n)$ is dense in X.

(8) If $\tau \in (0,\infty], u : [0,\tau) \to X$ is continuous and such that

$$\frac{du}{dt}(t) \text{ exists, } u(t) \in \mathcal{D}(A) \text{ and } \frac{du}{dt}(t) + Au(t) = 0 \text{ for } t \in (0,\tau),$$

then $u(t) = Q(t)u(0)$ for all $t \in [0,\tau)$.

(9) If $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup on X and the generator of this semigroup is equals $-A$, then $T(t) = Q(t)$ for all $t \geq 0$.

(10) If λ is any scalar, then $\left\{e^{\lambda t}Q(t)\right\}_{t\geq 0}$ is a strongly continuous semigroup on X and the generator of this semigroup is $\lambda - A$.

Theorem A.2.2 ([40, Theorem 4.3.2]). *Suppose that $-A$ is the generator of a strongly continuous semigroup $\{Q(t)\}_{t\geq 0}$ on a Banach space X. Let $M \in [0,\infty), a \in \mathbb{R}$ be such that $\|Q(t)\| \leq Me^{-at}$ for all $t > 0$. Then every scalar λ, with $\mathrm{Re}(\lambda) < a$, belongs to the resolvent set of A and, moreover,*

$$\left\|(A-\lambda)^{-n}\right\| \leq M(a - \mathrm{Re}(\lambda))^{-n} \text{ for } n \geq 1;$$
$$(A-\lambda)^{-n}x = \frac{1}{(n-1)!}\int_0^{\infty} s^{n-1}e^{\lambda a}Q(s)xds \text{ for } n \geq 1, x \in X.$$

Corollary A.2.1 ([40, Corollary 4.3.3]). *Suppose $-A$ is the generator of a strongly continuous semigroup $\{Q(t)\}_{t\geq 0}$ on a Banach space X. If λ is in the spectrum of A and if*

$b \in \mathbb{R}, b > \operatorname{Re}\lambda$, *then*

$\sup_{t>0} \|Q(t)x\|e^{bt} = \infty$ *for some* x *in* X.

Theorem A.2.3 ([40, Theorem 4.3.5]). *Suppose that* A *is a densely defined linear operator in a Banach space* X *and that for some* $M \in [0, \infty)$, $a \in \mathbb{R}$, *we have that* $(-\infty, a) \subset \rho(A)$ *and*

$$\left\|(A-\lambda)^{-n}\right\| \leq M(a-\lambda)^{-n} \text{ for } \lambda \in (-\infty, a), n = 1, 2, \ldots \tag{A.2.1}$$

Then there exists a strongly continuous semigroup $\{Q(t)\}_{t\geq 0}$ *on* X *whose generator is equal to* $-A$. *Moreover.*

$$\|Q(t)\| \leq Me^{-nt} \text{ for } t \geq 0, \tag{A.2.2}$$

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\|Q(t)x - (1+(t/n)A)^{-n}x\right\| = 0 \text{ for } T \in (0, \infty), x \in X. \tag{A.2.3}$$

The implicit **Euler method** for approximating the solution of

$$u' + Au = 0, \quad u(0) = u_0$$

is

$$\frac{u(t+h) - u(t)}{\hbar} + Au(t+h) \approx 0.$$

hence

$$u(t+h) \approx (1+hA)^{-1}u(t)$$

$$u(nh) \approx (1+hA)^{-n}u_0$$

$$u(t) \approx (1+(t/n)A)^{-n}u_0.$$

Note that Theorem A.2.3 implies convergence of the approximations whenever $-A$ is the generator of a strongly continuous semigroup.

Theorem A.2.4 ([40, Theorem 4.3.6]). *Suppose that* $-A$ *is the generator of a strongly continuous semrgroup* $\{Q(t)\}_{t\geq 0}$ *on a reflexive Banach space* X. *Then* $\{Q(t)^{\star}\}_{t\geq 0}$ *is a strongly continuous semigroup on* X^* *whose generator is* $-A^*$.

Thus, the definition of the Hilbert space adjoint and [40, Theorem 2.2.5] imply

Corollary A.2.2. *Suppose* $-A$ *is the generator of a strongly continuous semigroup* $\{Q(t)\}_{t\geq 0}$ *on a Hilbert space* H. *Then* $\{Q(t)^*\}_{t\geq 0}$ *is a strongly continuous semigroup on* H *whose generator is* $-A^*$.

A strongly continuous semigroup $\{Q(t)\}_{t\geq 0}$ is said to be a **contraction semigroup** if $\|Q(t)\| \leq 1$ for $t \geq 0$.

Theorem A.2.5 ([40, Theorem 4.3.8])**.** *Suppose that A is a linear operator in a Banach space X. Then $-A$ is the generator of a contraction semigroup if and only if A is a densely defined accretive linear operator and $\mathcal{R}(A+\lambda) = X$ for some $\lambda > 0$.*

A.3 Finite Time Blow-Up for Evolution Equations

We will emphasize the methods and techniques for studying blow-up problems. This section is based in [26, Chapter 5]. Many methods for studying blow-up issues apply to more general equations. We shall use the simplest model in our explication to avoid lengthy computations. The equation

$$u_t - \Delta u = f(u) \text{ in } \Omega_T \equiv \Omega \times (0,T), \tag{A.3.1}$$

$$u = 0 \text{ on } \partial\Omega \times (0,T), \tag{A.3.2}$$

$$u(x,0) = u_0(x) \text{ for } x \in \Omega \tag{A.3.3}$$

can be used to model solid fuel ignition (see [7] for details). The function $f(u)$ is typically a nonlinear function such as up $(p > 1)$, $exp(u)$, etc. If the source term is on the boundary, we then have the following system:

$$u_t - \Delta u = 0 \text{ in } \Omega_T \equiv \Omega \times (0,T), \tag{A.3.4}$$

$$\frac{\partial u}{\partial n} = f(u) \text{ on } \partial\Omega \times (0,T), \tag{A.3.5}$$

$$u(x,0) = u_0(x) \text{ for } x \in \Omega \tag{A.3.6}$$

where $\mathbf{n}$ is the unit exterior normal vector.

The first question is the existence and uniqueness for either (A.3.1)–(A.3.3) or (A.3.4)–(A.3.6). For the corresponding linear problem, the existence and uniqueness are stated in [26, Theorem 3.5] for the Dirichlet problem. The Neumann problem can be studied similarly. For the nonlinear problem, the questions about existence (locally in time) and uniqueness have been answered in a similar manner. One can, of course, study the problem of combined heat sources in the interior and on the boundary, the problem of systems of more than one equation, etc. In recent decades, many results have been obtained on these problems.

In this Appendix, we want to study whether a solution exists globally in time or has a finite time blow-up. We want to determine the blow-up rate and the asymptotic behavior near the blow-up point if the blow-up occurs.

Definition A.3.1 (L^∞ **blow-up).** *We say that a solution u blows up (or thermal runaway) at $t = T$ if there exists (x_n, t_n), $t_n \nearrow T$, such that $|u(x_n, t_n)| \to +\infty$. In this case, we say that the solution blows up in finite time if T is finite; if there exists a sequence y_n and $t_n \nearrow T$ such that $y_n \to x$ and $|u(y_n, t_n)| \to +\infty$, then we say that x is a blow-up point. The collection of all blow-up points is called the blow-up set.*

For the finite time blow-up, Osgood [49] gave a criterion, namely the right-hand side nonlinear term must satisfy

$$\int^{\infty} \frac{ds}{f(s)} < \infty.$$

The earliest blow-up results on parabolic equations are due to Kaplan [28] and Fujita [19]. Some early papers appeared in the 1970s: Tsutsumi [57], Hayakawa [25], Levine [36], Levine and Payne [37, 38], Walter [59], Ball [6], Kobayashi et al. [31], Aronson and Weinberger [3]. We begin by presenting some of their results.

A.3.1 Finite Time Blow-Up: Kaplan's First Eigenvalue Method

If we drop the diffusion term in (A.3.1), the positive solution of the ordinary differential equation (ODE) $u_t = f(u)$ will blow up in finite time for any positive initial data, provided f is defined for all $u \in \mathbb{R}$, and satisfies

$$f(u) > 0 \text{ for } u > 0, \quad \int_M^{\infty} \frac{du}{f(u)} < \infty, \tag{A.3.7}$$

for some $M > 0$. So, a natural question is whether the diffusion is strong enough to diffuse the energy to prevent a finite time blow-up.

Remark A.3.1 (necessary condition)**.** *(A.3.7) is a necessary condition for blow-up to occur. In fact, if $\int_M^{\infty} \frac{du}{f(u)} = \infty$, one can then obtain global existence of (A.3.1)–(A.3.3) by comparing its solution with an ODE solution.*

Here, we shall introduce the first eigenvalue method introduced in 1963 by Kaplan [28]. As we shall see from the proof, it is a straightforward method that applies to a large class of equations.

Theorem A.3.1 ([26, Theorem 5.1]). *Let Ω be a bounded domain with $\partial\Omega \in C^1$. Assume that f is convex (i.e., $f'' \geq 0$), and (A.3.7) is satisfied. Let $u \in C(\overline{\Omega}_T) \cap C^2(\Omega_T)$ be a solution of (A.3.1)–(A.3.3). If $\int_\Omega u_0(x)dx$ is sufficiently large, then the solution u must blow up in a finite time.*

The assumption that the initial datum is "large" cannot be dropped. The solution can exist globally in time if this assumption is dropped. We use $f(u) = u^p$ to illustrate this in the following theorem.

We take ϕ to be the solution of [26, (5.8)–(5.10)] and take $\psi(x) = \eta\phi(x)$. If $0 < \eta \ll 1$, then $-\Delta\psi - \psi^p = \eta(\lambda_1\phi - \eta^{p-1}\phi^p) \geq 0$. It follows that ψ is a supersolution and $u(x,t) \leq \psi(x)$ for all t if initially $0 \leq u_0(x) \leq \psi(x)$. We proved

Theorem A.3.2 (**Global existence**). *In the case $f(u) = u^p$, if $0 \leq u_0(x) \leq \psi(x)$, then the solution exists globally in time.*

A.3.2 Finite Time Blow-Up: Concavity Method

In this section, we introduce another method to establish finite time blow-up. This method, introduced by Levine–Payne in the papers [37], [38] and Levine [36] in the 1970s, uses the concavity of an auxiliary function.

For simplicity, we shall only consider (A.3.1)–(A.3.3) with

$$f(t) = |u|^{p-1}u \ (p > 1). \tag{A.3.8}$$

This concavity method is powerful enough to apply to many other types of second parabolic and evolution equations. It does not use maximum principles, and the following theorem is only a particular case discussed in [36].

Theorem A.3.3 ([26, Theorem 5.3 (Levine)]). *Let Ω be a smooth domain. If f is given by (A.3.8) and $u_0(x)$ satisfies*

$$-\frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 dx + \frac{1}{p+1}\int_\Omega |u_0(x)|^{p+1} dx > 0. \tag{A.3.9}$$

then the solution of (A.3.1)–(A.3.3) must blow-up in finite time.

Remark A.3.2. *The domain Ω can be either bounded or unbounded.*

Definition A.3.2. *Let $s \in \mathbb{R}$. We define the Sobolev space of order s, denoted by $H^s(\mathbb{R}^n)$, as:*

$$H^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \Lambda^s f(x) = ((1+|\xi|^2)^{s/2}\widehat{f}(\xi))^{\vee}(x) \in L^2(\mathbb{R}^n)\}, \tag{A.3.10}$$

with norm $\|\cdot\|_{s,2}$ defined as:

$$\|f\|_{s,2} = \|\Lambda^s f\|_2.$$

Remark A.3.3. *The condition (A.3.9) is explicit on the initial datum. Such u_0 always exists. One can pick any non-trivial function $\psi \in H^1(\Omega) \cap L^{p+1}(\Omega)$ and let $u_0(x) = \lambda\psi(x)$, $\lambda \gg 1$.*

Remark A.3.4. *Note that if the solution of (A.3.1)–(A.3.3) (with $f(u) = |u|^{p+1}u$) is global, then we must have*

$$-\frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \frac{1}{p+1}\int_\Omega |u(x,t)|^{p+1} dx < 0 \text{ for all } t > 0,$$

this estimate is useful in establishing a global bond for global solutions.

A.3.3 Finite Time Blow-Up: A Comparison Method

If the system under consideration has a comparison principle, the solution must blow up if a subsolution blows up in a finite time. However, there are no general rules on how to construct comparison functions. Here, we give one example.

Theorem A.3.4 ([26, Theorem 5.4])**.** *Consider the system (A.3.4) and (A.3.6) in a bounded smooth domain with $f(u) = u^p$ $(p > 1)$ and $u_0(x) \geq 0$. In this case, all nontrivial solutions blow up in a finite time.*

A.3.4 Fujita Types of Results on Unbounded Domains

One of the earliest results is Fujita's critical exponent (Fujita [19]) on the simple system

$$u_t - \Delta u = u^p \text{ x } \in \mathbb{R}^n, t > 0, \quad (p > 1) \tag{A.3.11}$$

$$u(x,0) = u_0(x) \geq 0, \text{ x } \in \mathbb{R}^n. \tag{A.3.12}$$

We can compare the solution with a blow-up solution of (A.3.1)–(A.3.3) (with $f(u) = u^p$), so that the solution of (A.3.11) and (A.3.12) always blows up in finite time if the initial datum

$u_0(x)$ is large enough in a certain sense. The question is whether or not there are global solutions for all time. If u is *small*, then u^p is very *small* if p is large. As Fujita observed, if p is *large* and u_0 is small, global solutions exist. On the other hand, if p is close to 1, then all positive solutions will blow up in a finite time. In the case of the nonlinearity given by up on the whole space, the cutoff value, or the *critical exponent*, is $1 + \frac{2}{n}$. Fujita's result does not include the critical exponent itself. The critical exponent itself actually belongs to the blow-up case (see Hayakawa [25], Kobayashi et al. [31], Aronson and Weinberger [3], Weissler [60]). The proof represents the solution in an integral equation regarding its fundamental solution. This approach does not need a separate proof to include the critical exponent in the blow-up case. The solution of (A.3.11) and (A.3.12) has an integral representation [18, p. 51, (17)]:

$$u(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-\tau)u^p(y,\tau)dyd\tau, \tag{A.3.13}$$

where

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Gamma(x,t) = 0 \quad \text{for } (x,t) \neq (0,0).$$

Theorem A.3.5 ([26, Theorem 5.5])**.** *(i) If $p > 1 + \frac{2}{n}$, then the solution of (A.3.11) and (A.3.12) is global in time, provided the initial datum satisfies, for some small $\varepsilon > 0$,*

$$u_0(x) \leq \varepsilon\Gamma(x,1) \ for \ x \in \mathbb{R}^n.$$

(ii) If $p \leq 1 + \frac{2}{n}$, then all nontrivial solutions of (A.3.11) and (A.3.12) blow-up in finite time.

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