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To my parents, Yrma Elena Dávila Morales and J. Isabel Hernández Crúz.

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Abstract

In this work it is given for the first time an explicit description of the transversally locally constant Teichmüller space of certain minimal laminations fibering over an infinite type hyperbolic surface. In particular, it is shown that it is a contractible and separable space. For this it is defined an Ahlfors-Bers model for the Teichmüller space of laminations fibering over hyperbolic surfaces analogously to the known Ahlfors-Bers model for Riemann surfaces.

Resumen

En este trabajo se da por primera vez una descripción explícita del espacio Teichmüller transversal-localmente constante de ciertas laminaciones minimales fibrando sobre superficies hiperbólicas de tipo infinito. En particular se muestra que este espacio es contraíble y separable. Para esto se define un modelo de Ahlfors-Bers para el espacio Teichmüller de laminaciones que fibran sobre superficies hiperbólicas análogamente al conocido modelo de Ahlfors-Bers para superficies de Riemann.

Introduction

In this work it is given an explicit description of the Teichmüller space of laminations fibering over hyperbolic surfaces. A hyperbolic surface is a Riemannian surface of constant sectional curvature -1. Only metrically complete surfaces will be considered here and a hyperbolic surface will be denoted by Σ. Given the well-known relation between the group of all biholomorphisms on the unit disc $\mathrm{Bihol}(\Delta)$ and the set of all positively oriented isometries on the unit disc $\mathrm{Isom}^+(\Delta)$,

$$\mathrm{Bihol}(\Delta) = \mathrm{Isom}^+(\Delta),$$

from now on it will be considered a hyperbolic surface as a Riemann surface.

A lamination is a compact and metrizable topological space L locally modeled on the product of the unit disc by a topological space. It comes with an atlas, whose transition functions preserve the disc factor of the product structure. When transition functions are holomorphic along the disc coordinate, L is called a Riemann surface lamination.

Teichmüller spaces have various applications. For example, in physics they have been used within string theory (see [32] and [9]). In other areas of mathematics they have also been used, for example in Sullivan's proof of the non-wandering domain theorem (see [31]). Furthermore, conformal structures have been used in the digitization of three-dimensional scenes, (see [33]).

The Teichmüller space of a Riemann surface lamination was defined by Sullivan in [29], see also [15]. It was defined as the space of all transversally continuous conformal structures along the space of leaves up to the action of the group of quasiconformal isotopies tangent to the leaves. In [30] he proved that this Teichmüller space is a Banach manifold and he considered those objects which are continuous and locally constant in the transverse direction, these objects define the transversal-locally constant Teichmüller space.

Considering this definition, unless the lamination is fibering over a Riemann surface with discrete fiber, the conformal structures of L cannot be defined as the union of the conformal structures of the sheets.

Regarding the Teichmüller space of laminations, there are general results such as [13] and [5].

For explicit descriptions of the Teichmüller space, there are recent works in which have been considered specific laminations. For example in [3], Alvarez and Lessa considered the Hirsch foliation. They give a description of the Teichmüller space for the

Hirsch foliation as the space of closed curves in the plane. In [10], Burgos and Verjovsky consider the universal hyperbolic lamination Σ_∞ of a compact hyperbolic surface Σ_g of genus g. This lamination is defined as the inverse limit of the category of unramified holomorphic connected coverings of a Riemann surface Σ_g. They show that the Teichmüller space of the universal hyperbolic lamination is biholomorphic to a space of continuous functions. In [24], Penner and Šărić considered the punctured solenoid which is defined as the inverse limit of all finite covers of any fixed punctured surface with negative Euler characteristic and they give a description of the decorated Teichmüller space of this lamination as a space of continuous functions.

Then with the idea of considering more general laminations whose Teichmüller space is described explicitly, we proceed as follows. Consider a minimal lamination L fibering over a hyperbolic surface Σ with fiber F, that is to say $\pi : L \to \Sigma$ is a locally trivial fibration with fiber F such that π restricted to any leaf is a local diffeomorphism. It is assumed that F is a Hausdorff compact space. With this hypotheses L is a hyperbolic surface lamination that is each leaf is a hyperbolic surface. Since the lamination L is minimal, thus each sheet of the lamination is densely immersed in L.

Note that if the surface Σ is not hyperbolic, according to the uniformization theorem Σ is the Riemann sphere or a surface with a universal covering biholomorphic to the plane $\mathbb{C}$. In the first case, since the sphere is simply connected, any minimal lamination fibering on Σ is a sphere. The Teichmüller space of the sphere is a single point (see page 44), then this case is not considered. In the case of surfaces with universal covering biholomorphic to the complex plane, it is a work in progress.

Denoting by G the fundamental group of Σ, it can be considered the right holonomic action on the fiber

$$Hol : G \to Homeo(F)^{op}, \quad g \longmapsto \varphi_g, \text{ where } \varphi_g : F \to F, \ \varphi_g(k) = k \cdot g. \tag{1}$$

We can identify the pullback of the fibration π by the uniformization $u : \Delta \to \Sigma$ with the product $F \times \Delta$, then it can be shown that the lamination L is isometric to $(F \times \Delta)/G$, where the diagonal action of G is defined by $g \cdot (k, a) = (k \cdot g^{-1}, g \cdot a)$.

In this work it is given a mathematically tractable definition of the space $T_{TLC}(L)$. For this, in Chapter 3, it is defined an Ahlfors-Bers model for the Teichmüller space of the lamination L. The Beltrami differential space of the lamination L is defined as the set of continuous functions $\mu : F \to Bel(\Delta)$ such that for every $k \in F$ and each $g \in G$, we have $l_g^*(\mu(k)) = \mu(k \cdot g)$, and it is denoted by $Bel(L)$. Here $Bel(\Delta)$ is the set of L_∞ section of $\omega^* \otimes \overline{\omega}$ here ω is the canonical bundle on the disc whose supremum norm is less than one.

It also is defined on $Bel(L)$ the following equivalence relation:

$$\mu \sim \eta \quad if \quad \mu(k) \sim \eta(k), \quad \forall k \in F.$$

Then, analogously to the case of Riemann surfaces, the Teichmüller space of the lamination L can be defined as the quotient space

$$T(L) = Bel(L)/\sim .$$

Let $\mathcal{B} : Bel(L) \to T(L)$ be the quotient map. The transversally locally constant Beltrami differential space on the lamination L is by definition

$$Bel_{TLC}(L) = C_{LC}(F, Bel(\Delta))_{eq(G)}$$

and it is defined the transversally locally constant Teichmüller space as

$$T_{TLC}(L) = \mathcal{B}(Bel_{TLC}(L)).$$

Now, suppose that Σ is a complete hyperbolic surface without boundary and the lamination is minimal with Hausdorff compact fiber and holonomy action (1). Given a pair of pants decomposition of the surface Σ by generalized hyperbolic pair of pants (see Section 2.6, Chapter 2), it will be said that the fibration *has trivial holonomy on pants* if the interior P of every pair of pants in the decomposition verifies

$$\mathrm{Hol}(\iota_*(\pi_1(P))) = \{id_F\}, \quad \iota : P \to \Sigma.$$

In this case it will said that *the pants decomposition has trivial holonomy.* Then, in the Chapter 3, it is given the following explicit description of the space $T_{TLC}(L)$ of the lamination L.

It is shown that given a hyperbolic surface Σ without boundary obtained by gluing a (possibly finite) sequence of a pair of generalized hyperbolic pair of pants, each glued to the next along a common boundary geodesic, such that the length of these geodesic boundaries is uniformly upper bounded and given a minimal lamination L fibering over Σ with a Hausdorff compact fiber F whose holonomy action continuously extends to the profinite completion of the fundamental group G and has a trivial holonomy on pants, then we have the following homeomorphism

$$T_{TLC}(L) \cong C_{LC}(F, T(\Sigma)),$$

where the left hand is the transversally locally constant Teichmüller space of L and the right hand is the space of locally constant functions valued on the Teichmüller space of Σ.

This result provides an explicit description of the transversally locally constant Teichmüller space of laminations fibering over infinite conformal type hyperbolic surfaces (see Example 7 in Chapter 3). This work is the first time that an explicit description of a Teichmüller space of laminations fibering over an infinite conformal type surface is given. Also, laminations are not required to be compact, here it is only asked for the fiber to be compact.

Let me briefly mention the structure of this thesis. In Chapter 1, notation and definitions of terms that will be used in the following chapters are given. In Chapter 2, it is given the definition of the Teichmüller space for Riemann surfaces and some known results for these spaces are included.

Chapter 3 is original work, the Ahlfors-Bers model for laminations is defined and an explicit description of the transversally locally constant Teichmüller space for specific laminations is given (also see [11]). To demonstrate the central result, it was necessary to prove certain key Lemmas, which although I do not dare to proclaim originality about them, I could not have these results available since I only found similar but not exact versions. Such is the case of Lemmas 1.3.7, 1.3.13 and 1.3.21.

Chapter 1

Preliminaries

In this chapter it is introduced the notation that will be used in the following chapters. It will be use the term *domain* to mean a connected open subset of the complex plane $\mathbb{C}$ and unless it will be indicated otherwise, the letter D will denote a domain. Also, throughout this text the Riemann sphere will be denoted by $\hat{\mathbb{C}}$, that is to say, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathbb{C}^*$ will denote the punctured plane $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$. A homeomorphism (or biholomorphism) on a domain D onto its image will be called simply homeomorphism (or biholomorphism) on D. Also, it will be called measurable subsets to the Lebesgue measurable subsets.

1.1 Quasiconformal Mappings

In this section, the Beltrami coefficients are introduced which will play a very important role throughout this text.

Definition 1.1.1. Let $D \subset \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$ be a homeomorphism which preserves orientation. It will be said that f is a *quasiconformal mapping* (abbreviated as qc mapping) on D if f satisfies the following conditions:

(1) The distributional partial derivatives of f with respect to z and $\bar{z}$ can be represented by functions $f_z, f_{\bar{z}} \in L_1^{loc}(D)$, respectively.

(2) There exists $k \in \mathbb{R}$ such that $0 \leq k < 1$ and $|f_{\bar{z}}| \leq k|f_z|$ a.e. on D.

Let D and D' be subset of $\mathbb{C}$, it will be denoted by $QC(D, D')$ the set of quasiconformal maps from D to D', with the topology of uniform convergence on compact subsets. If $D = D'$, it will simply be written $QC(D)$.

Remark 1.1.2. *If a homeomorphism $f : D \to \mathbb{C}$ has the derivatives f_x and f_y a.e. on D, then f is totally differentiable a.e. on D, see [17, Proposition 4.1]. Now, let f be a quasiconformal mapping, then it is totally differentiable a.e. on D and the Jacobian determinant of f is $J(f) = |f_z|^2 - |f_{\bar{z}}|^2$. By the second item of the above definition, we have that $J(f) > 0$.*

If f is quasiconformal mapping on D and k is a constant for which the second item of the definition 1.1.1 is satisfied, setting $K = \frac{1+k}{1-k}$, it will be said that f is a *K-qc mapping* on D and $K(f)$ will be called *maximal dilatation*, where it is defined by:

$$K(f) = \inf\left\{\frac{1+k}{1-k} : \ |f_{\bar{z}}| \leq k|f_z| \text{ a.e. on } D\right\}.$$

Now, let's see some examples of quasiconformal mappings.

Example 1. If f is a biholomorphism on D, then f is a quasiconformal mapping. In fact, since f is infinitely differentiable, it is enough verify the item 2 of the definition 1.1.1. But, in this case $f_{\bar{z}} = 0$, then we can take $k = 0$. Also, f is a 1-qc mapping, moreover $K(f) = 1$.

Example 2. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = az + b\bar{z} + c$ with $a, b, c \in \mathbb{C}$. Notice that if $b = 0$, then the Weyl's lemma implies that f is holomorphic. In general, the distributional partial derivatives of f are given by $f_z = a$ and $f_{\bar{z}} = b$, thus f is quasiconformal if $|b| < |a|$ and in this case $K(f) = \frac{|a|+|b|}{|a|-|b|}$.

Example 3. Let $f : \mathbb{C} \to \mathbb{C}$ be the function defined by $f(z) = x + iKy$ with $z = x + iy$ and $K \geq 1$. Notice that f can be written as:

$$f(z) = \frac{z+\bar{z}}{2} + K\frac{z-\bar{z}}{2} = \frac{1+K}{2}z + \frac{1-K}{2}\bar{z}.$$

Since $f_z = (1+K)/2$ and $f_{\bar{z}} = (1-K)/2$, for the Weyl's lemma again, f is holomorphic iff $K = 1$ and f is quasiconformal iff $(K-1)/(K+1) < 1$, the latter happens iff $K < +\infty$. In this case $K(f) = K$.

Now, let's see an example of a homeomorphism which isn't a quasiconformal mapping.

Example 4. Consider the unit disk Δ and let $f : \Delta \to \mathbb{C}$ be the function defined by $f(z) = \frac{z}{1-|z|^2}$. Let us see that f isn't a quasiconformal mapping[1]. A direct calculation show that

$$f_z(z) = \frac{1}{1-|z|^2} + \frac{|z|^2}{(1-|z|^2)^2} = \frac{1}{(1-|z|^2)^2}, \ f_{\bar{z}}(z) = \frac{z^2}{(1-|z|^2)^2}.$$

Thus $\frac{|f_{\bar{z}}|}{|f_z|} = z^2$, which converges to 1 when $|z| \to 1$. So, we can't find a constant k such that the item 2 of the definition 1.1.1 be satisfied.

In the above example, the item 2 of the definition 1.1.1 fails, let's see an example in which the condition 1 of that definition is not satisfied.

Remark 1.1.3. *In the definition 1.1.1 we can change the item 1 by let f be ACL* [2]*, and we get an equivalent definition, see [17, Theorem 4.4].*

[1] There exist no quasiconformal mapping of Δ onto $\mathbb{C}$, (See [17, Proposition 4.32]).

[2] Absolutely continuous on lines.

Example 5. Let $f : \mathbb{C} \to \mathbb{C}$ be the function defined by

$$f(z) = \begin{cases} \frac{g(x)+x}{2} + iy & if \quad x \in [0,1] \\ \\ z & if \quad x \notin [0,1], \end{cases}$$

where $g : [0,1] \to [0,1]$ is the function known as "Devil's staircase", which assigns to each x the value $g(x)$ determined as follow:

- Let $[x]_3$ the representation of x as number ternary.
- If $[x]_3$ contains its first 1 at position n, $[x]_3 = 0.x_1x_2...x_{n-1}1x_{n+1}...$, we replace every ternary digit following the 1 by a 0, then we consider $[T(x)]_3 = 0.x_1x_2...x_{n-1}2$. Otherwise, if $[x]_3$ doesn't have $1's$, then we let $[T(x)]_3 = 0.x_1x_2...x_{n-1}1x_{n+1}...$.
- Finally, we remplace all $2's$ as $1's$ and we interpreted the string that remains as a binary number which gives $g(x)$.

The function g is not absolutely continuous because $g' = 0$ almost everywhere on $[0,1]$ [3]. Therefore the function f is not ACL and by the observation 1.1.3, f does not satisfy the item 1 of the quasiconformal definition. So, f is a homeomorphism, but it is not a quasiconformal mapping.

Now, consider $f : D \to \mathbb{C}$ be a quasiconformal mapping. For every Borel set $E \subset D$, we define $A(E)$ as the area of $f(E)$. This define a locally finite additive measure and by Lebesgue's Theorem, see [19, Page 120], we have A has almost everywhere on D a finite derivative J_f which is measurable as a function of z. Also, for every measurable subset E, we have:

$$\int_E J_f(z)\, dxdy < A(E).$$

(See [19, Lemma 3.3]). Since f is differentiable at almost every $z \in D$, by the observation 1.1.2 and [19, Lemma 3.2], it result that at almost every $z \in D$,

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2.$$

By the item 2 of the definition of quasiconformal mapping, we have the inequality $|f_z(z)|^2 - |f_{\bar{z}}(z)|^2 \geq |f_z|^2(1-k^2)$, then:

$$|f_{\bar{z}}|^2 \leq |f_z|^2 \leq \frac{J_f}{1-k^2}$$

almost everywhere on D. Since J_f is locally integrable, It has been shown the following result:

Proposition 1.1.4. Let $f : D \to \mathbb{C}$ be a quasiconformal mapping. Then the partial derivatives f_z and $f_{\bar{z}}$ are locally square integrable on D.

[3] A absolutely continuous function $h : (a,b) \to \mathbb{R}$ such that $h' = 0$ a.e. on (a,b) is constant.

Let me notice that even though the partial derivatives of a quasiconformal mapping are in $L_2^{loc}(D)$, in the definition of quasiconformal mapping, it is only asked that they are in $L_1^{loc}(D)$.

In the example 1 it is seen that a biholomorphism is a 1-qc mapping, now it will be seen the converse, for this, a geometric definition of quasiconformal mapping is given.

First, a quadrilateral is a pair (Q, q_1, q_2, q_3, q_3) of a Jordan closed domain Q and four points q_1, q_2, q_3, q_4 on the boundary ∂Q of Q which are mutually distinct and located in this order with respect to the positive orientation of ∂Q.

Proposition 1.1.5.[17, Proposition 4.7] For every quadrilateral $(Q; q_1, q_2, q_3, q_4)$, there is a homeomorphism h of Q onto some rectangle $R = [0, a] \times [0, b]$ $(a, b > 0)$ which is conformal in the interior $Int(Q)$ of Q, and satisfies

$$h(q_1) = 0, h(q_2) = a, h(q_3) = a + ib, h(q_4) = ib.$$

Moreover, a/b is independent of h.

It will be said that a/b is the module of the quadrilateral $(Q; q_1, q_2, q_3, q_4)$, and it will be denoted by $M(Q)$.

Definition 1.1.6. Let f be a homeomorphism of a domain D into $\mathbb{C}$ which preserves orientation. It will be said that f is *quasiconformal* on D if f satisfies the following condition:
There is a constant $K \geq 1$ *such that*

$$M(f(Q)) \leq KM(Q) \tag{1.1}$$

holds for every quadrilateral Q in D.

Now, it is shown that the condition 1.1 is equivalent to the following condition:
There is a constant $K \geq 1$ *such that*

$$\frac{1}{K}M(Q) \leq M(f(Q)) \leq KM(Q) \tag{1.2}$$

holds for every quadrilateral Q in D.

Let $f : D \to \mathbb{C}$ be a orientation preserving homeomorphism that satisfies 1.1. Let Q be a quadrilateral, then there exist conformal homeomorphisms $h : Q \to R_1 = [0, a] \times [0, b]$ and $\hat{h} : f(Q) \to R_2 = [0, \hat{a}] \times [0, \hat{b}]$ such that

$$h(q_1) = 0,\ \ h(q_2) = a,\ \ h(q_3) = a + ib,\ \ h(q_4) = ib \quad \text{and}$$

$$\hat{h}(f(q_1)) = 0,\ \ \hat{h}(f(q_2)) = \hat{a},\ \ h(f(q_3)) = \hat{a} + i\hat{b},\ \ h(f(q_4)) = i\hat{b}.$$

By the definition of module we have that $M(Q) = \frac{a}{b}$ and $M(f(Q)) = \frac{\hat{a}}{\hat{b}}$. Now, if we consider the conformal maps $ih + b$ and $i\hat{h} + \hat{b}$, by the proposition 1.1.5 and 1.1, we have $\frac{\hat{b}}{\hat{a}} \leq K\frac{b}{a}$. Thus $\frac{1}{k}\frac{a}{b} \leq \frac{\hat{a}}{\hat{b}}$ that is to say $\frac{1}{K}M(Q) \leq M(f(Q))$.

Theorem 1.1.7. *The definition 1.1.1 is equivalent to the definition 1.1.6.*

Proof. See [17, Lemma 4.8 and Lemma 4.14]. □

It will be needed the following lemma, the proof can be consulted in [17, Theorem 4.10].

Lemma 1.1.8. *If $f : D \to \mathbb{C}$ is a $K-qc$ mapping, then f^{-1} is also a $K-qc$ mapping.*

Proof. By the above observation we have

$$\frac{1}{K}M(f^{-1}(f(Q))) \leq M(f(Q)).$$

Therefore, $M(f^{-1}(f(Q))) \leq KM(f(Q))$, that is to say f^{-1} is K-qc. □

Theorem 1.1.9. *Let $f : D \to \mathbb{C}$ be a function, then f is a 1-qc mapping if and only if f is a conformal map.*

The proof of the above theorem follows from the Weyl's lemma which we include below, Lemma 1.1.10. In fact, if f is a 1-qc mapping, then the distributional partial derivative satisfies $f_{\bar{z}} = 0$, by Lemma 1.1.10, f is holomorphic. Also, by Lemma 1.1.8, f^{-1} is also a holomorphic function.

Lemma 1.1.10. *(Weyl's lemma). Let $f : D \to \mathbb{C}$ be a continuous function with distributional partial derivative $f_{\bar{z}} \in L_1^{loc}(D)$. If $f_{\bar{z}} = 0$, then f is holomorphic.*

Proof. Let $K \subset D$ be a compact set and consider a positive real function $\varphi \in C_c^{\infty}(D)$ such that $||\varphi||_1 = 1$, its support is a disc of radius r centered at 0 and $K + supp(\varphi) \subset D$. For each $0 < \varepsilon < 1$, we define the function $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^2}\varphi(x/\varepsilon)$. Then $supp(\varphi_\varepsilon) = \varepsilon \cdot supp(\varphi)$. Define $f_\varepsilon = f * \varphi_\varepsilon$, then f_ε is defined on K and since $\varphi_\varepsilon \in C_c^{\infty}(D)$, $f_\varepsilon \in C_c^{\infty}(D)$. By properties of the derivatives of the convolution we have:

$$(f_\varepsilon)_{\bar{z}} = (f * \varphi_\varepsilon)_{\bar{z}} = f_{\bar{z}} * \varphi_\varepsilon = 0, \text{ on the compact subset } K.$$

Therefore, f_ε is holomorphic on K for every ε. Also, $(f_\varepsilon)_\varepsilon$ converge uniformly to f on K when ε tends to zero. We conclude that f is holomorphic on K and because the compact set was arbitrary, f is holomorphic on D. □

The quasiconformal mappings are less rigid than conformal mappings, but they have good properties as show the following proposition.

Proposition 1.1.11.[17, Proposition 4.11] If f is quasiconformal on D, then $f_z \neq 0$ a.e. on D.

It will be denoted by $L_\infty(D)$ the complex Banach space of all bounded measurable functions on a domain D. It will be considered the norm

$$||\mu||_\infty = ess.sup_{z\in D}|\mu(z)|, \ \mu \in L_\infty(D). \tag{1.3}$$

$B(D)_1$ will denote the set of functions in $L_\infty(D)$ with norm less than 1 with respect to the norm 1.3. By the Proposition 1.1.11, for every quasiconformal mapping f on D, we can consider the following quantity called *complex dilatation*:

$$\mu_f = \frac{f_{\bar z}}{f_z} \quad \text{a.e. on } D.$$

Since f_z and $f_{\bar z}$ are measurable functions, we have that μ_f is a bounded measurable function a.e. on D, and satisfies:

$$\operatorname*{ess\,sup}_{x \in D} |\mu_f(z)| \leq \frac{K(f)-1}{K(f)+1} < 1.$$

Therefore we have the following result.

Theorem 1.1.12. *Let $f : D \to \mathbb{C}$ a homeomorphism. Then f is a quasiconformal mapping iff there exists $\mu \in B(D)_1$ such that*

$$f_{\bar z} = \mu f_z \text{ a.e. on } D \tag{1.4}$$

where $f_z, f_{\bar z} \in L_1^{loc}(D)$ represent the distributional partial derivatives of f.

Equation 1.4 is called *Beltrami equation* and μ is a *Beltrami coefficient* on D. By the Theorem 1.1.12, a Beltrami coefficient let us "measure" the deformation of the complex structure realized by a quasiconformal mapping, so the Beltrami coefficients can be thought of as deformation parameters.

Consider f^μ and f^ν be quasiconformal mappings with μ and ν their respective Beltrami coefficients. Then the Beltrami coefficient of $f^\mu \circ (f^\nu)^{-1}$ is given by (see [17]):

$$\mu_{f^\mu \circ (f^\nu)^{-1}} = \frac{f_z^\nu}{\overline{f_z^\nu}} \frac{\mu_{f^\mu} - \mu_{f^\nu}}{1 - \overline{\mu_{f^\nu}}\mu_{f^\mu}}. \tag{1.5}$$

1.1.1 Solutions of the Beltrami Equation

In this section solutions of the Beltrami equation are studied. First, it will be seen how to get many solutions from a given solution. Then necessary and sufficient conditions will be obtained for the Beltrami equation to have a solution.

Proposition 1.1.13. Let $f : D \to D'$ be a solution of the Beltrami equation 1.4. If $g : D' \to \mathbb{C}$ is a holomorphic function, then $g \circ f$ is a solution of the equation 1.4 too.

Proof. By a direct calculation it follows that

$$(g \circ f)_z = (g_w \circ f) f_z + (g_{\bar w} \circ f)\bar f_z = (g_w \circ f)\bar f_z,$$
$$(g \circ f)_{\bar z} = (g_w \circ f) f_{\bar z} + (g_{\bar w} \circ f)\bar f_{\bar z} = (g_{\bar w} \circ f) f_{\bar z}.$$

Therefore, taking the quotient:

$$\frac{(g \circ f)_{\bar z}}{(g \circ f)_z} = \left(\frac{g_{\bar w} \circ f}{g_w \circ f}\right) \frac{f_{\bar z}}{f_z} = \mu.$$

□

In the Proposition 1.1.13, the function g is not necessarily a homeomorphism, let's see the following example.

Example 6. Let $f : D \to D'$ be a qc mapping and consider the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = (z-a)^2$ with $a \notin Im(f)$. Then g is not a homeomorphism because it is not injective, but $g_{\bar{z}} = 0$. Thus $(g \circ f)_z(z) = 2(f(z)-a)f_z$ and $(g \circ f)_{\bar{z}} = 2(f(z)-a)f_{\bar{z}}$. This mean that

$$\frac{(g \circ f)_{\bar{z}}}{(g \circ f)_z} = \frac{2(f(z)-a)f_{\bar{z}}}{2(f(z)-a)f_z} = \mu.$$

Notice that given a solution of the Beltrami equation 1.4, $f : D' \to \mathbb{C}$, if $g : D \to D'$ is a biholomorphism onto a domain D', then $f \circ g$ is a solution of the Beltrami equation where the Beltrami coefficient is given by

$$(\mu \circ g)\frac{\bar{g}_{\bar{z}}}{g_z}. \tag{1.6}$$

Because if g is a biholomorphism and $\varphi \in C_c^\infty(D')$, then $\varphi \circ g^{-1} \in C_c^\infty(D)$. This implies that there exist the distributional derivatives $(f \circ g)_z$ and $(f \circ g)_{\bar{z}}$. Then we have $(f \circ g)_z = (f_w \circ g)\bar{g}_z$ and $(f \circ g)_{\bar{z}} = (f_{\bar{w}} \circ g)\bar{g}_{\bar{z}}$. Therefore

$$\frac{(f \circ g)_{\bar{z}}}{(f \circ g)_z} = \left(\frac{f_{\bar{w}} \circ g}{f_w \circ g}\right)\frac{\bar{g}_{\bar{z}}}{g_z} = (\mu \circ g)\frac{\bar{g}_{\bar{z}}}{g_z}.$$

Now, necessary and sufficient conditions will be put for that solutions to the Beltrami equation there exist. Some results that will be necessary to show the existence are included, its proof can be consulted in [17, Chapter 4].

Consider the following operators on $L^p(\mathrm{C})$:

$$Ph(\zeta) = -\frac{1}{\pi}\iint_{\mathrm{C}} h(z)\left(\frac{1}{z-\zeta} - \frac{1}{z}\right)dxdy \quad h \in L^p(\mathrm{C}), \zeta \in \mathrm{C}.$$

$$Th(\zeta) = \lim_{\epsilon \to 0}\left\{-\frac{1}{\pi}\iint_{|z-\zeta|>\epsilon} \frac{h(z)}{(z-\eta)^2}dxdy\right\} \quad h \in C_c^\infty(\mathrm{C}).$$

Lemma 1.1.14. *For every p with $2 < p < \infty$ and for every $h \in L^p(\mathbb{C})$, Ph is a uniformly Hölder continuous function on $\mathbb{C}$, with exponent $(1-2/p)$, and satisfies $Ph(0) = 0$. Moreover, Pf satisfies $(Ph)_{\bar{z}} = h$ on $\mathbb{C}$ in the sense of distribution.*

Lemma 1.1.15. *For an arbitrarily given $p > 2$ and every $h \in L^p(\mathbb{C})$,*

$$(Ph)_z = Th$$

on $\mathbb{C}$ in the sense of distribution.

Theorem 1.1.16. *For every p with $2 \le p < \infty$,*

$$C_p = \sup_{h \in C_0^\infty(\mathbb{C}), ||h||_p = 1} ||Th||_p$$

is finite. Hence, the operator T is extended to a bounded linear operator of $L^p(\mathbb{C})$ into itself with norm C_p. Moreover, C_p is continuous with respect to p. In particular, C_p satisfies $\lim_{p\to 2} C_p = 1$.

First, it is supposed that μ has compact support. Let $K \subset \mathbb{C}$ be a compact set. We denote by $B_K(\mathbb{C})$ the subset of $L_\infty(\mathbb{C})_1$ consisting of functions with support in K.

Theorem 1.1.17. *Fix k such that $0 \le k < 1$ arbitrarily. Take a positive $p > 2$ with $kC_p < 1$. The for every $\mu \in B(\mathbb{C})_1$ with $||\mu||_\infty \le k$ and with compact support, there exists a continuous function f such that $f(0) = 0$, $f_z - 1$ belongs to $L^p(\mathbb{C})$, and f satisfies*

$$f_{\bar{z}} = \mu f_z$$

on $\mathbb{C}$ in the sense of distribution. Moreover, such an f is determined uniquely by these conditions.

Proof. First, we will obtain a condition that the partial derivative f_z has to satisfy. Since $f_{\bar{z}} = \mu f_z$ has a compact support, and since $f_z - 1$ belongs to $L_p(\mathbb{C})$, then $f_{\bar{z}}$ also belongs to $L^p(\mathbb{C})$. Thus $P(f_{\bar{z}})$ is defined. Let $F(z) = f(z) - P(f_{\bar{z}})(z)$, $z \in \mathbb{C}$. Then by the Lemma 1.1.14 we have that F is continuous and $F(0) = 0$. Moreover, since $(Pf_{\bar{z}})_{\bar{z}} = f_{\bar{z}}$, then $F_{\bar{z}} = 0$ in the sense of the distribution. By the Weyl´s Lemma, F is holomorphic on the whole $\mathbb{C}$. On the other hand, since $f_z - 1$ and $(Pf_{\bar{z}})_z = T(f_{\bar{z}})$ belong to $L^p(\mathbb{C})$, so does $F' - 1$. Thus we can conclude that $F'(z) = 1$, i.e., $F(z) = z + a$. Since $f(0) = 0$, we have $a = 0$, and hence

$$f(z) = P(f_{\bar{z}})(z) + z, z \in \mathbb{C}.$$

Taking the derivative with respect to z we obtain the equation that must satisfy f:

$$f_z = P(f_{\bar{z}})_z + 1 = T(f_{\bar{z}}) + 1 = T(\mu f_z) + 1.$$

Using the above equation, we shall show the uniqueness of the solution. Suppose that there is another solution g. Then, we have $g_z = T(\mu g_z) + 1$. By the Calderón and Zygmund Theorem we obtain

$$||f_z - g_z||_p = ||T(\mu f_z) - T(\mu g_z)||_p \le C_p||\mu||_\infty||f_z - g_z||_p \le C_p k||f_z - g_z||_p.$$

Since $kC_p < 1$ by the assumption, we get $f_z = g_z$ a.e. on $\mathbb{C}$. Hence, again by Weyl´s lemma we conclude that $f - g$ and $\bar{f} - \bar{g}$ are holomorphic on $\mathbb{C}$, which in turn implies that $f - g$ should be a constant. Since $f(0) = g(0) = 0$, we conclude that $f = g$, which implies the uniqueness of the solution. Finally, the existence of the normal solution follows also from the obtained equation. In fact, repeat substituting the whole right hand side for f_z on the right hand side. Then, we have the following formal series for $f_z - 1$:

$$f_z - 1 = T\mu + T(\mu T\mu) + T(\mu T(\mu T\mu)) + \cdots . \tag{1.7}$$

The series actually converges in $L^p(\mathbb{C})$, since the linear operator which sends $h \in L^p(\mathbb{C})$ to $T(\mu h) \in L^p(\mathbb{C})$ has the operator norm not greater than $kC_p < 1$. We set $h = T\mu + T(\mu T\mu) + \cdots$. Then h belongs to $L^p(\mathbb{C})$. We shall show that

$$f(z) = P(\mu(h+1))(z)$$

is a desired solution. In fact, $\mu(h+1)$ belongs to $L^p(\mathbb{C})$, for μ has a compact support. Hence, f is continuous, $f(0) = 0$, and $f_{\bar{z}} = \mu(h+1)$. Moreover, we have $f_z = T(\mu(h+1)) + 1 = h + 1$. Hence, f satisfies the Beltrami equation $f_{\bar{z}} = \mu f_z$, and $f_z - 1$ belongs to $L^p(\mathbb{C})$. □

The function f is called the normal solution of the Beltrami equation for μ.
By the Theorem 1.1.17, the following map is well defined.

$$\mathcal{M} : B_K(\mathbb{C}) \to QC(\mathbb{C}, \mathbb{C}), \tag{1.8}$$
$$\mu \mapsto f^{\mu}.$$

Proposition 1.1.18. Let k and p be as in Theorem 1.1.17. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence in $B(\mathbb{C})_1$ satisfying the following conditions:

- $||\mu_n||_\infty < k$ for every n,
- every μ_n has a support contained in $z \in \mathbb{C} : |z| < M$ with a suitable constant M independent of n, and
- μ_n converges to some $\mu \in B(\mathbb{C})_1$ a.e. on $\mathbb{C}$ as $n \to \infty$.

Let f_n be the normal solution for μ_n, and f be the normal solution for μ. Then $f_n \to f$ uniformly on $\mathbb{C}$ as $n \to \infty$, and

$$\lim_{n\to\infty} ||(f_n)_z - f_z||_p = 0. \tag{1.9}$$

Proof. As in the prove of the Theorem 1.1.17, we have:

$$f_z = T(\mu f_z) + 1 \text{ y } (f_n)_z = T(\mu_n (f_n)_z) + 1,$$

then by the Calderón and Zygmund

$$\begin{aligned} ||f_z - (f_n)_z||_p &= ||T(\mu f_z) - T(\mu_n (f_n)_z)||_p \\ &\leq ||T(\mu_n (f_z - (f_n)_z))||_p + ||T(\mu_n f_z) - T(\mu f_z)||_p \\ &= ||T(\mu_n (f_z - (f_n)_z))||_p + ||T(\mu_n - \mu) f_z||_p \\ &\leq kC_p ||f_z - (f_n)_z||_p + C_p ||(\mu_n - \mu) f_z||_p. \end{aligned}$$

Thus, we have the following inequality

$$||f_z - (f_n)_z||_p \leq \frac{C_p ||(\mu_n - \mu) f_z||_p}{1 - kC_p}.$$

Since that every μ_n is uniformly bounded and since μ_n converges to μ a.e. on $\mathbb{C}$, we have that 1.9 is satisfied. Now, since $f(z) = P(f_{\bar{z}})(z) + z$ and $f_n(z) = P((f_n)_{\bar{z}})(z) + z$, we obtain

$$\begin{aligned} |f(\delta) - f(\delta)| = |P(f_{\bar{z}})(\delta) - P((f_n)_{\bar{z}})(\delta)| &\leq K_p||f_{\bar{z}} - (f_n)_{\bar{z}}||_p|\delta|^{1-2/p} \\ &\leq K_p\{||\mu f_z - \mu_n f_z||_p + k||f_z - (f_n)_z||_p\}|\delta|^{1-2/p} \end{aligned}$$

for every $\delta \in \mathbb{C}$. Since the right side converges to zero and the unique factor which depends of δ is $|\delta|^{1-2/p}$, we have that $f_n \to f$ locally uniform on $\mathbb{C}$. Since $f_n - f$ is holomorphic in a fixed neighborhood of ∞ for every n, we conclude that f_n converges to f uniformly on $\mathbb{C}$. □

Now, it will be proved that the normal solution is a qc mapping, but we will need some lemmas, their proof can be consulted in [17].

Lemma 1.1.19. *Let u and v be continuous functions on a simply connected domain D whose distributional partial derivatives can be represented by locally integrable functions. Further, suppose that $u_{\bar{z}} = v_z$. Then there exist a function f which is continuously differentiable and satisfies $f_z = u$ and $f_{\bar{z}} = v$.*

Let $\mu \in L_\infty(\mathbb{C})$ with compact support and let M be a fix constant such that $\{z \in \mathbb{C} : |z| < M\}$ contains the support of μ. Fix also a sequence $\{\mu_n\}_{n=1}^\infty$ in $C_c^\infty(\mathbb{C})$ with $||\mu_n||_\infty \leq k$ such that the support of μ_n is contained in $\{z \in \mathbb{C} : |z| < M\}$ for every n, and that $\mu_n \to \mu$ a.e. on $\mathbb{C}$ as $n \to \infty$. It will be denoted by f_n the normal solution for μ_n for every n.

Lemma 1.1.20. *If a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is locally homeomorphism, then f is actually a homeomorphism of $\hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$.*

Proof. Denote by $\hat{\mathbb{C}}_z$ and by $\hat{\mathbb{C}}_w$ the Riemann spheres which are the domains and the target of f, respectively. Since f is an open mapping and $\hat{\mathbb{C}}_z$ is compact, $f(\hat{\mathbb{C}}_z)$ is open and compact. We conclude that $f(\hat{\mathbb{C}}_z) = \hat{\mathbb{C}}_w$. Next, one way to show that f is a homeomorphism is to consider f as a holomorphic function. This can be done by introducing a new complex structure on $\hat{\mathbb{C}}_z$ by pulling back the structure on $\hat{\mathbb{C}}_w$. We have that f^{-1} has holomorphic branches in a neighborhood of any point of $\hat{\mathbb{C}}_w$. Since $\hat{\mathbb{C}}_w$ is simply connected, the classical monodromy theorem implies that f^{-1} has a single valued branch on the whole $\hat{\mathbb{C}}_w$. Thus, f is a homeomorphism. □

Lemma 1.1.21. *f_n is a qc mapping.*

Proof. Consider a function g with $g_{\bar{z}} = \mu g_z$. Set

$$u = g_z \quad \text{and} \quad v = g_{\bar{z}} = \mu_n u.$$

By Lemma 1.1.19 if we show that u is a continuous function which satisfies $u_{\bar{z}} = (\mu_n u)_z$ then we will have that g is a quasiconformal mapping. If we set $\sigma = \log u$, then

$$\sigma_{\bar{z}} = u_{\bar{z}} e^{-\sigma} = [(\mu_n)_z u + \mu_n u_z] e^{-\sigma} = (\mu_n)_z + \mu_n \sigma_z.$$

Thus, it is enough to see that σ is a continuous function and satisfies $\sigma_{\bar{z}} = (\mu_n)_z + \mu_n \sigma_z$. Now, the above differential equation is solved in a similar way to the case of the Beltrami equation and as in the prove of the existence Theorem, we can construct a solution $\tilde{\mathrm{h}}$ in $L^p(\mathbb{C})$ defined by

$$\tilde{\mathrm{h}} = T(\mu_n \tilde{\mathrm{h}}) + T((\mu_n)_z).$$

Then, we define $\sigma = P(\mu_n \tilde{\mathrm{h}} + (\mu_n)_z) + C$. We choose C such that $\lim_{z\to\infty} \sigma(z) = 0$. Note that σ is holomorphic in a neighborhood of $z = \infty$. Since $\sigma_z = T(\mu_n \tilde{\mathrm{h}} + (\mu_n)_z) = \acute{\mathrm{h}}$ and $\sigma_{\bar{z}} = \mu_n \acute{\mathrm{h}} + (\mu_n)_z$, then σ is a solution of the equation:

$$\sigma_{\bar{z}} = \mu_n \sigma_z + (\mu_n)_z.$$

Now, consider $\tau = e^\sigma$. Then τ is continuous and we have $\tau_{\bar{z}} = (e^\sigma)_{\bar{z}} = \sigma_{\bar{z}} e^\sigma = (\mu_n \tilde{\mathrm{h}} + (\mu_n)_z) e^\sigma = (\mu_n \sigma_z + (\mu_n)_z) e^\sigma = (e^\sigma \mu_n)_z = (\mu_n \tau)_z$. By the Lemma 1.1.19, since τ and $\mu_n \tau$ are continuous and their distributional derivatives are locally integrable functions, we have that there exist a function $g \in C^1(\mathbb{C})$ such that $g_{\bar{z}} = \mu_n \tau$ and $g_z = \tau$, therefore $g_{\bar{z}} = \mu_n g_z$. We can suppose that $g(0) = 0$. Since $g_z = \tau$ is holomorphic in a neighborhood of $z = \infty$ and $\lim_{z\to\infty} \sigma(z) = 0$ implies $\lim_{z\to\infty} \tau(z) = 1$, we have that $g_z - 1$ is a continuous function and vanishes at infinity, therefore it belongs to $L^p(\mathbb{C})$. The uniqueness of the normal solution implies that $g = f_n$. Then f_n is of class C^1. Finally, if we consider the Jacobian, we have

$$|(f_n)_z|^2 - |(f_n)_{\bar{z}}|^2 = |\tau|^2 - |\mu_n \tau|^2 = |e^\sigma|^2 - |\mu_n e^\sigma|^2 = (1 - |\mu_n|^2)|e^{2\sigma}|,$$

which is positive, then f_n is a local homeomorphism in $\mathbb{C}$. Since f_n has a simple pole in $z = \infty$, then f_n is a local homeomorphism in $\hat{\mathbb{C}}$. Thus, by the above lemma, we have that f_n is a homeomorphism of $\hat{\mathbb{C}}$. ☐

It will be used the inequality of the following lemma. The proof of the lemma can be consulted in [17].

Lemma 1.1.22. *Every f_n satisfy the following inequality:*

$$|z_1 - z_2| \leq \frac{K_p}{(1 - kC_p)^{1+2/p}} ||\mu_n||_p |f_n(z_1) - f_n(z_2)|^{1-2/p} + |f_n(z_1) - f_n(z_2)|,$$

for every $z_1, z_2 \in \mathbb{C}$.

Theorem 1.1.23. *A normal solution of the Beltrami equation is a quasiconformal map on $\mathbb{C}$, and satisfies $\mu_f = \mu$ a.e. on $\mathbb{C}$.*

Proof. Consider μ_{n_n} and f_{n_n} as before, then by the Corollary 1.1.18, we have $f_n \to f$ uniformly in $\mathbb{C}$. Since $||\mu_n||_p \to ||\mu||_p$, then the inequality in the Lemma 1.1.22 is satisfied by f and μ. Therefore $f : \mathbb{C} \to \mathbb{C}$ is a continuous bijection and thus, it is a homeomorphism. Since $f_z - 1$ belongs to $L^p(\mathbb{C})$, then $f_{\bar{z}} = \mu f_z$. By the definition of quasiconformal mapping we have that f is a quasiconformal mapping. ☐

Following theorem gives conditions for the existence of solutions to the Beltrami equation where an arbitrarily Beltrami coefficient is taken.

Now, it will be seen the existence of a quasiconformal mapping with complex dilatation μ for a general $\mu \in L_\infty(\mathbb{C})_1$. It will be called the canonical mapping with complex dilatation μ and it will be denoted by f^μ.

Theorem 1.1.24. *For every Beltrami coefficient $\mu \in B(\mathbb{C})_1$, there exists a homeomorphism f of $\hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$ which is a quasiconformal mapping of $\mathbb{C}$ with complex dilatation μ. Moreover, f is uniquely determined by the following normalization conditions:*

$$f(0) = 0,\ f(1) = 1 \text{ and } f(\infty) = \infty.$$

Proof. Uniqueness: Let g another quasiconformal mapping with complex dilatation μ and such that g satisfies the normalization conditions. Then, $g \circ f^{-1}$ is a conformal map in $f(\mathbb{C}) = \mathbb{C}$. Therefore, $g \circ f^{-1}(z) = \alpha z + \beta$. Using the normalization conditions, we have:

$$g \circ f^{-1}(0) = \beta = 0,\ g \circ f^{-1}(1) = \alpha = 1, \text{ then } f = g.$$

Existence: First, we suppose that μ has compact support. In this case, let F^μ be the normal solution for μ. Then, $\mu_f = \mu$ and the wanted homeomorphism is $F^\mu(z)/F^\mu(1)$, because $F^\mu(0) = 0$ and $F^\mu(1)/F^\mu(1) = 1$. Now, we suppose that $\mu = 0$ a.e. in some neighborhood of the origin. In this case, we consider the pullback of μ by a Möbius transformation and define:

$$\hat{\mu}(z) = \mu\left(\frac{1}{z}\right)\frac{z^2}{\bar{z}^2},\ z \in \mathbb{C}.$$

Since $||\hat{\mu}||_\infty = ||\mu(\frac{1}{z})\frac{z^2}{\bar{z}^2}||_\infty \leq ||\mu(\frac{1}{z})||_\infty < 1$, then $\hat{\mu}$ belongs to $B(\mathbb{C})_1$ and it has compact support because μ has compact support. Thus, there exists a canonical mapping $\hat{\mu} - qc$ of $\mathbb{C}$, which is denoted by $f^{\hat{\mu}}$. Since $f^{\hat{\mu}}$ is a homeomorphism and it has partial derivatives a.e. on $\mathbb{C}$, then $f^{\hat{\mu}}$ is totally differentiable a.e. on $\mathbb{C}$. For each point $1/z$, the quasiconformal map defined by

$$f(z) = \frac{1}{f^{\hat{\mu}}\left(\frac{1}{z}\right)}$$

is also totally differentiable a.e. on $\mathbb{C}$. Then, we can applied the usual chain rule and we have

$$\mu_f(z) = \frac{f_{\bar{z}}(z)}{f_z(z)} = \frac{\left(f^{\hat{\mu}}_{\bar{z}}\left(\frac{1}{z}\right)\right)\left(-\frac{1}{\bar{z}^2}\right)}{\left(f^{\hat{\mu}}_{z}\left(\frac{1}{z}\right)\right)\left(-\frac{1}{z^2}\right)} = \frac{z^2}{\bar{z}^2}\hat{\mu}\left(\frac{1}{z}\right) = \mu(z), \text{ a.e. on } \mathbb{C}.$$

Also, $f(0) = 0,\ f(1) = 1,\ f(\infty) = \infty$. Therefore f is the wanted homeomorphism. Finally, suppose that μ is a general Beltrami coefficient. In this case, we define:

$$\mu_1(z) = \begin{cases} \mu(z) & \text{if } z \in \mathbb{C} - \overline{\Delta} \\ 0 & \text{if } z \in \overline{\Delta}. \end{cases} \tag{1.10}$$

Then, by the above case, there exists f^{μ_1}. Now, we define:

$$\mu_2 = \left(\frac{\mu - \mu_1}{1 - \bar{\mu}_1 \mu} \frac{(f^{\mu_1})_z}{\overline{(f^{\mu_1})_z}} \right) \circ (f^{\mu_1})^{-1}.$$

Since μ_2 has compact support, there exists f^{μ_2}. Moreover $g = f^{\mu_2} \circ f^{\mu_1}$ is a quasiconformal mapping. Also, $\mu_g = \mu$, because $f^{\mu_2} = g \circ (f^{\mu_1})^{-1}$ and, by 1.5, the Beltrami coefficient of $g \circ (f^{\mu_1})^{-1}$ is given by:

$$\mu_{g\circ(f^{\mu_1})^{-1}} \circ f^{\mu_1} = \left(\frac{\mu_g - \mu_1}{1 - \bar{\mu}_g \mu_1} \frac{(f^{\mu_1})_z}{\overline{(f^{\mu_1})_z}} \right).$$

Since f^{μ_2} and f^{μ_1} satisfy the normalization conditions, then g satisfies the normalization conditions and g is the wanted function. □

Now, if $f : \Delta \to D$ is a quasiconformal mapping with $\mu_f = \mu$, then f has an extension of $\overline{\Delta}$ onto $\overline{D}$. In fact, we can define $\mu = 0$ in $\mathbb{C} - \Delta$ and thus there exists a canonical mapping f^μ defined on $\hat{\mathbb{C}}$. Now, if we consider $g = f^\mu \circ f^{-1}$, then g is a quasiconformal mapping on D. Moreover, g is a conformal mapping because $\mu_g = 0$ a.e. on D. Since $f^\mu(\Delta)$ is a Jordan domain, by the Caratheodory theorem, g can be extended to a homeomorphism of $\overline{D}$ onto $\overline{f^\mu(\Delta)}$. Since $f = g^{-1} \circ f^\mu$, then f has an extension of $\overline{\Delta}$ onto $\overline{D}$.

In the Example 4 of the Chapter 1 we saw that there are orientation preserving diffeomorphisms defined of Δ onto $\mathbb{C}$ which are not quasiconformal mappings. In general, there are not quasiconformal mappings of Δ to $\mathbb{C}$. In fact, if there exists a quasiconformal map $f : \Delta \to \mathbb{C}$, then $\mu = \mu_{f^{-1}}$ is defined on $\mathbb{C}$. If we consider $g = f^\mu \circ f$, then $\mu_g = 0$ a.e. on Δ. Therefore g is a conformal map. In the other hand, since $g^{-1}(\mathbb{C}) = \Delta$, then g is a bounded entire function and by the Liouville Theorem, g^{-1} should be a constant, a contradiction.

We can consider Beltrami coefficients defined on the upper half plane H and we can show analogously to the Theorem 1.1.24 the following result.

Proposition 1.1.25. Let μ be an arbitrary element of $B(\Delta)_1$. Then there exists a quasiconformal mapping w of Δ onto Δ with complex dilatation μ. Moreover, such a mapping w is uniquely determined by the following normalization conditions:

$$w(i) = i, \; w(1) = 1, \text{ and } w(-1) = -1.$$

There is a dependence of quasiconformal solutions on the Beltrami coefficients as show the following proposition. For the proof, it can be consulted in [17].

Proposition 1.1.26. *If μ converges to 0 in $B(\mathbb{C})_1$, then the canonical μ-qc mappings f^μ converges to the identity mapping locally uniformly on $\mathbb{C}$.*

1.1.2 Analytical Dependence on Beltrami Coefficients

In this subsection it is described the dependence of the solutions of the Beltrami equation on the Beltrami coefficients. For this, it will be used the Lemma 1.1.27 and some definitions to show the Theorem 1.1.28. The proof of this theorem give us an explicit solution of the Beltrami equation but following this proof we do not ensure that the solution is quasiconformal.

Lemma 1.1.27. *Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function with distributional partial derivatives represented by $f_z, f_{\bar{z}} \in L_p^{loc}(\mathbb{C})$, $1 \leq p < \infty$. Suppose that $f_{\bar{z}}$ has compact support and $f(z) = O(1/|z|)$, then f satisfies:*

$$\frac{1}{\pi z} * f_{\bar{z}} = f. \tag{1.11}$$

In particular, $f \in L_p^{loc}(\mathbb{C})$.

Proof. Let $(f_n)_n$ be a sequence L_p-smoothing. We will see that f_n satisfies the equation 1.11. By definition of convolution product we have:

$$\begin{aligned}\left(\frac{1}{\pi z} * (f_n)_{\bar{z}}\right)(\xi) &= \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{(f_n)_{\bar{z}}(z)}{\xi - z}\, d\bar{z} \wedge dz \\ &= \frac{1}{2\pi i}\lim_{r\to 0^+}\iint_{D(\xi,1/r)-D(\xi,r)} \frac{(f_n)_{\bar{z}}(z)}{\xi - z}\, d\bar{z} \wedge dz.\end{aligned}$$

Since that we can write the integrand as:

$$\begin{aligned}\frac{(f_n)_{\bar{z}}(z)}{\xi - z} d\bar{z} \wedge dz &= \left(\frac{(f_n)(z)}{\xi - z}\right)_{\bar{z}} d\bar{z} \wedge dz \\ &= \left(\partial_z\left(\frac{(f_n)(z)}{\xi - z}\right) dz + \partial_{\bar{z}}\left(\frac{(f_n)(z)}{\xi - z}\right) d\bar{z}\right) dz \\ &= d\left(\frac{(f_n)(z)}{\xi - z} dz\right),\end{aligned}$$

we can simplify the above formula and we obtain:

$$\left(\frac{1}{\pi z} * (f_n)_{\bar{z}}\right)(\xi) = \frac{1}{2\pi i}\lim_{r\to 0^+}\iint_{D(\xi,1/r)-D(\xi,r)} d\left(\frac{(f_n)(z)}{\xi - z}\, dz\right),$$

then applying the Stokes's theorem it result that

$$\begin{aligned}\left(\frac{1}{\pi z} * (f_n)_{\bar{z}}\right)(\xi) &= \frac{1}{2\pi i}\lim_{r\to 0^+}\int_{\partial(D(\xi,1/r)-D(\xi,r))} \frac{(f_n)(z)}{\xi - z} dz \\ &= \frac{1}{2\pi i}\lim_{r\to 0^+}\left(\int_{\partial D(\xi,1/r)} \frac{(f_n)(z)}{\xi - z} dz - \int_{\partial D(z,r)} \frac{(f_n)(z)}{\xi - z} dz\right).\end{aligned}$$

Because the first term does not contribute to the integral, applying a change of variable to the second term and using the continuity of f_n, we have:

$$\begin{aligned}\left(\frac{1}{\pi z} * (f_n)_{\bar{z}}\right)(\xi) &= -\frac{1}{2\pi i}\lim_{r\to 0^+}\int_{\partial D(z,r)}\frac{(f_n)(z)}{\xi - z}dz \\ &= -\frac{1}{2\pi i}\lim_{r\to 0^+}\int_0^{2\pi}\frac{(f_n)(\xi - re^{i\theta})ire^{i\theta}}{-re^{i\theta}}d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} i(f_n)(\xi)d\theta = f_n(\xi).\end{aligned}$$

Therefore $\left(\frac{1}{\pi z} * (f_n)_{\bar{z}}\right)_n$ converges uniformly on compact sets to f. Also, since $((f_n)_{\bar{z}})_n$ converge to $f_{\bar{z}}$ with respect to the norm L_p, then $\left(\frac{1}{\pi z} * (f_n)_{\bar{z}}\right)_n$ converges to $\frac{1}{\pi z} * (f)_{\bar{z}}$. Hence, $\frac{1}{\pi z} * (f)_{\bar{z}} = f$. □

Now, it is considered the Fourier transform on $L_1(\mathbb{C})$ which is given by the formula

$$\hat{f}(\zeta) := \hat{f}(\xi + i\eta) = \int_{\mathbb{C}} f(x+iy)e^{-2\pi i(x\xi + y\eta)}dxdy.$$

It is well known that for every $f \in L_2(\mathbb{C}) \cap L_1(\mathbb{C})$, we have $||f||_2 = ||\hat{f}||_2$. Also the Fourier transform can be extend to an isometry $L_2(\mathbb{C}) \to L_2(\mathbb{C})$.

By properties of the Fourier transform, we have

$$\widehat{\frac{\partial f}{\partial z}}(\zeta) = 2\pi i\bar{\zeta}\hat{f}(\zeta), \qquad \widehat{\frac{\partial f}{\partial \bar{z}}}(\zeta) = 2\pi i\zeta\hat{f}(\zeta).$$

Theorem 1.1.28. *The map $\mathcal{M}$ defines in 1.8 is analytic, in the sense that it is continuous, and for each $z \in \mathbb{C}$ the map $\mu \mapsto f^\mu(z)$ is analytic.*

Proof. Considering the multiplication by $\bar{\zeta}/\zeta$ which is an isometry on $L_2(\mathbb{C})$, we can take the unique isometry

$$\mathcal{L} : L_2(\mathbb{C}) \to L_2(\mathbb{C}) \text{ such that } \widehat{\mathcal{L}(f)} = \frac{\bar{\zeta}}{\zeta}\hat{f}.$$

Therefore, if a function $F \in L_2(\mathbb{C})$ has distributional derivatives in L_2, then

$$\frac{\partial F}{\partial z} = \mathcal{L}\left(\frac{\partial F}{\partial \bar{z}}\right).$$

In particular, if $\mu \in B_K(\mathbb{C})$ and we write the solution of Beltrami equation as $f^\mu(z) = z + g^\mu(z)$ with $g^\mu(z) \in O(1/|z|)$, then Beltrami equation becomes

$$\frac{\partial g^\mu}{\partial \bar{z}} = \mu\left(1 + \mathcal{L}\left(\frac{\partial g^\mu}{\partial \bar{z}}\right)\right). \tag{1.12}$$

Since $||\mu\mathcal{L}|| = ||\mu||_\infty < 1$, $id - \mu\mathcal{L}$ is invertible and the equation 1.12 can be rewritten

$$(id - \mu\mathcal{L})^{-1}h = \mu.$$

The inverse of $id-\mu\mathcal{L}$ is the sum of the convergent geometric series. Applied to equation 1.12 this gives

$$\frac{\partial g^\mu}{\partial \bar{z}} = \mu + \mu\mathcal{L}\mu + \mu\mathcal{L}\mu\mathcal{L}\mu + \cdots$$

The sum of this series depends analytically on μ, since it is the sum of a uniformly convergent series of analytic functions of μ. Thus $\partial g^\mu/\partial \bar{z}$ depends analytically on μ, and so does

$$g^\mu = \frac{1}{\pi z} * \frac{\partial g^\mu}{\partial \bar{z}}.$$

This convolution is well defined, since $\partial g^\mu/\partial \bar{z}$ has compact support and we have used the Lemma 1.1.27 to get the equality; f^μ also depends analytically on μ, since $f^\mu = z + g^\mu(z)$. □

1.2 Riemann Surfaces and Complex Structures

In this section some terms that will be used throughout this text are introduced. Let X be a connected Hausdorff space, and let $\{U_i\}_{i\in\mathbb{N}}$ be an open covering of X consisting of domains. Suppose that on each $U_j \subset X$, we have a homeomorphism $z_j : U_j \to D_j$ defined by:

$$z_j : p \to z_j(p) = \left(z_j^1(p), ..., z_j^n(p)\right),\ p \in U_j,$$

where $D_j \subset \mathbb{C}^n$ is a domain. Then for each j, k such that $U_j \cap U_k \neq \emptyset$, the map

$$\tau_{jk} : z_k(p) \to z_j(p),\ p \in U_j \cap U_k,$$

is a homeomorphism of the open set $D_{kj} = \{z_k(p) : p \in U_j \cap U_k\} \subset D_k$ in $\mathbb{C}^n$ onto the set $D_{jk} = \{z_j(p) : p \in U_j \cap U_k\} \subset D_j$.

If τ_{jk} is biholomorphic for any j, k such that $U_j \cap U_k \neq \emptyset$, each $z_j : p \to z_j(p)$ is called a *local complex coordinates defined on* U_j. Each domain U_j is called a *coordinate neighbourhood*. The collection $\{z_1, ..., z_j, ...\}$ is called a *system of local complex coordinates on* X. If a system of local complex coordinates $\{z_1, ..., z_j, ...\}$ is defined on X, it will be said that a *complex structure is defined on* X. A connected Hausdorff space X with a complex structure defined on it is called a *complex manifold*. Usually a complex manifold will be denoted by the letter M. The dimension or complex dimension of X is defined to be n.

Definition 1.2.1. A *Riemann surface* is a 1-dimensional connected complex manifold. Usually a Riemann surface will be denoted by Σ.

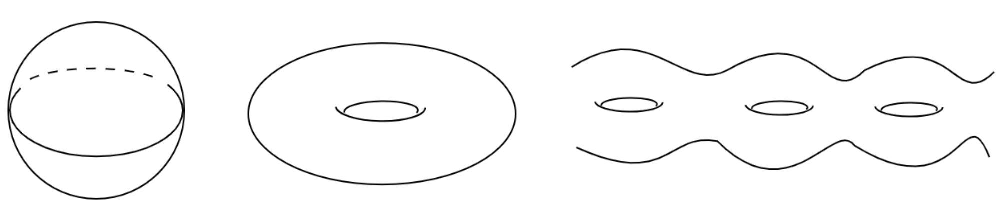


Figure 1.1: Examples of Riemann surfaces.

Consider a Riemann surface Σ. Let $p \in \Sigma$, the point $z_j(p) = \left(z_j^1(p)\right)$ of $\mathbb{C}$ is called the local coordinate of p. If we choose a coordinate neighbourhood U_j with $p \in U_j$, p is determined uniquely by its local coordinate $z_j = \left(z_j^1\right) = z_j(p)$. Identifying U_j with D_j via z_j, we can consider that $\Sigma = \bigcup_j D_j$. Then $z_j \in D_j$ and $z_k \in D_k$ are the same point on Σ if and only if $z_j = \tau_{jk}(z_k)$.

Any compact Riemann surface is homeomorphic to a sphere with a certain number, say $g \geq 0$, of handles attached. This nonnegative integer g is the *genus* of the surface.

Definition 1.2.2. A Riemann surface is of *finite conformal type* if it is obtained from a compact Riemann surface by removing a finite number of points, and otherwise it is said to be of *infinite conformal type*.

1.2.1 Covering Surfaces and Uniformization

In this part it will be seen that any Riemann surface is covered by $\hat{\mathbb{C}}$, $\mathbb{C}$ or H, then we can study the complex structures of a Riemann surface, studying the Beltrami coefficients defined in the first chapter.

Let B and X be topological spaces and let p be a map from X over B. The couple (X, p) will be called a *covering space of* B if the following property is satisfied:

> for each $b \in B$ there exists an open neighborhood U of b in B, a non-empty discrete space F and a homeomorphism $\phi : p^{-1}(U) \to U \times F$ such that the following diagram commutes:
>
> $$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\ \phi\ } & U \times F \\ & {\scriptstyle p|_{p^{-1}(U)}} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & U & \end{array} \tag{1.13}$$
>
> where $pr_1 : U \times F \to U$ is the first projection map.

With the above notation, it is said that B is the *base* of the covering, X is the *total space* and $p^{-1}(b)$ the *fiber over* b. If for every $b \in B$, the fiber over b is a finite set of X, it will be said that (X, p) is a finite covering space and if all fibers have the same cardinality, say d, then it will be said that (X, p) is a finite covering space of B of degree d.

We are interested in the special case in which B and X are Riemann surfaces, then we have the following definition.

Definition 1.2.3. Let $\hat{\Sigma}$ and Σ be Riemann surfaces. Let $(\hat{\Sigma}, \pi)$ be a covering space of Σ and let U be an open subset satisfying 1.13. If π is a surjective holomorphic mapping and $\pi : V \to U$ is biholomorphic, for each connected component V of the inverse image $\pi^{-1}(U)$ of U, it is said that $\hat{\Sigma}$ is a *covering surface* of Σ and π is called the projection of $\hat{\Sigma}$ onto Σ. If $\hat{\Sigma}$ is simply connected, $\hat{\Sigma}$ is called a *universal covering surface* of R.

The universal covering surface of a Riemann surface Σ is unique, that is to say, for any two universal coverings $\hat{\Sigma}$ and $\hat{\Sigma}'$ of Σ, with projections π and π' respectively, there exists a biholomorphic mapping φ of $\hat{\Sigma}$ to $\hat{\Sigma}'$ with $\pi' \circ \varphi = \pi$.

Now, functions between covering spaces of a same space B can be defined. Let B be a topological space and (X, p) and (Y, q) two covering spaces of B. A continuous map $f : X \to Y$ will be called *morphism of covering spaces of* B *from* (X, p) *to* (Y, q) if the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

It will be said that a morphism f of covering spaces of B is an *isomorphism* if there exists a morphism of covering spaces $g : (Y, q) \to (X, p)$ such that the topological maps $g \circ f$ and $f \circ g$ are the identity maps of X and Y respectively. In the particular case in which X is a Riemann surface and $X = Y$ we have the following definition.

Definition 1.2.4. Let Σ be a Riemann surface and let $\hat{\Sigma}$ be a universal covering surface of Σ with projection π. Any biholomorphism $\gamma : \hat{\Sigma} \to \hat{\Sigma}$ with $\pi \circ \gamma = \pi$ is called a *covering transformation* of $\hat{\Sigma}$.

For a given covering $\hat{\Sigma}$, denote by Γ the set of all its covering transformations. By composition of mappings, Γ forms a group, which is called the *covering transformation group* of $\hat{\Sigma}$. In particular, if $\hat{\Sigma}$ is a universal covering surface of Σ, Γ is called the *universal covering transformation group* of $\hat{\Sigma}$.

Theorem 1.2.5. *Let Σ be a Riemann surface, $\pi_1(\Sigma)$ its fundamental group and Γ its universal covering transformation group. Then we have the following isomorphism:*

$$\pi_1(\Sigma) \simeq \Gamma.$$

Riemann surfaces can be represented as quotient spaces. Let $\hat{\Sigma}$ be a Riemann surface and let Γ' be a subgroup of the covering transformation group of $\hat{\Sigma}$. Suppose that Γ' acts properly discontinuously on $\hat{\Sigma}$, that is to say, for every $\hat{p} \in \hat{\Sigma}$, there exists a suitable neighborhood $\hat{U}$ of $\hat{p}$ in $\hat{\Sigma}$ such that $\gamma\big(\hat{U}\big) \cap \hat{U} = \emptyset$ for every $\gamma \in \Gamma' - \{id\}$.

Two points $\hat{p}, \hat{q} \in \hat{\Sigma}$ are said to be Γ'-equivalent if there exists an element $\gamma \in \Gamma'$ satisfying $\hat{q} = \gamma(\hat{p})$. Denote by $[\hat{p}]$ the equivalence class of $\hat{p}$. Let $\hat{\Sigma}/\Gamma'$ be the set of all these equivalence classes $[\hat{p}]$, which is called the quotient space of $\hat{\Sigma}$ by Γ. Define the projection $\pi : \hat{\Sigma} \to \hat{\Sigma}/\Gamma'$ by $\pi(\hat{p}) = [\hat{p}]$. Consider the quotient topology on $\hat{\Sigma}/\Gamma'$, since $\hat{\Sigma}$ is connected, so is $\hat{\Sigma}/\Gamma'$. Also, $\hat{\Sigma}/\Gamma'$ is a Hausdorff space, for Γ' acts properly discontinuously on $\hat{\Sigma}$.

Now, we define a complex structure on $\hat{\Sigma}/\Gamma'$ as follow: for any point $\hat{p} \in \hat{\Sigma}$, take a neighborhood $\hat{U}_{\hat{p}}$ of $\hat{p}$ satisfying the hypothesis $\gamma(\hat{U}) \cap \hat{U} = \emptyset$ for every $\gamma \in \Gamma' - \{id\}$. We may assume that there exists a local coordinate $z_{\hat{p}}$ on $\hat{U}_{\hat{p}}$. Then, putting $p = \pi(\hat{p})$, $U_p = \pi(\hat{U}_{\hat{p}})$, we see that $\pi : \hat{U}_{\hat{p}} \to U_p$ is a homeomorphism. Hence, setting $z_p = z_{\hat{p}} \circ \pi^{-1}$, we conclude that $\{(U_p, z_p)\}_{p\in\hat{\Sigma/\Gamma'}}$ defines a complex structure so that $\hat{\Sigma}$ is a covering of $\hat{\Sigma}/\Gamma'$. This Riemann surface $\hat{\Sigma}/\Gamma'$ will be called the quotient Riemann surface of $\hat{\Sigma}$ by Γ.

Theorem 1.2.6. *Let Σ be a Riemann surface with universal covering transformation group Γ. Then the quotient Riemann surface $\hat{\Sigma}/\Gamma$ of $\hat{\Sigma}$ by Γ is biholomorphically equivalent to Σ.*

Now, it will be studied the biholomorphisms on the universal covering of a Riemann surface.

A *Möbius transformation* is a map $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$T(z) = \frac{az+b}{cz+d} \tag{1.14}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. The set of all Möbius transformations will be denoted by Möb($\hat{\mathbb{C}}$). Also, if T and S are Möbius transformations, then T^{-1} and $S \circ T$ are Möbius transformations too, consequently, the set Möb($\hat{\mathbb{C}}$) is a group under composition.

Remark 1.2.7. *Consider the Möbius transformation 1.14 such that $T \neq Id$, then its fixed points are given by:*

- *If $c = 0$, $z = \infty$ is a fixed point. Also, if $a/b \neq 1$, T has a finite fixed point given by $z = \frac{b}{d-a}$. If $a/d = 1$ and $b \neq 0$, then T has a double fixed point in $z = \infty$.*
- *If $c \neq 0$, the two fixed points of T are obtained solving the equation $cz^2 + (d - a)z - b = 0$.*

Therefore any Möbius transformation, $T \neq Id$, has two fixed points. That is to say, if T is a Möbius transformation with more than two fixed points, then $T = Id$.

We denote by Möb($\mathbb{C}$), Möb(H) and Möb(Δ) the subgroups of Möb($\hat{\mathbb{C}}$) consisting of all Möbius transformations that preserve $\mathbb{C}$, H and Δ, respectively. Möbius transformations of canonical domains $\mathbb{C}$, Δ and H have the following forms:

Proposition 1.2.8.

- Every element of Möb($\mathbb{C}$) has a form

$$\gamma(z) = az + b, \quad \text{where} \quad a, b \in \mathbb{C} \quad \text{with} \quad a \neq 0.$$

- Every element of Möb(Δ) has a form

$$\gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}, \quad \text{where} \quad a, b \in \mathbb{C} \quad \text{with} \quad |a|^2 - |b|^2 = 1.$$

- Every element of Möb(H) has a form

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \text{where} \quad a, b, c, d \in \mathbb{R} \quad \text{with} \quad ad - bc = 1.$$

Each element of $\text{Möb}(\hat{\mathbb{C}})$ can be represented by a matrix of non-zero determinant where the matrix entries are the complex coefficients a, b, c and d. Then there is a surjective map $SL(2, \mathbb{C}) \to \text{Möb}(\hat{\mathbb{C}})$. Notice that if A is a matrix representation of $\gamma \in \text{Möb}(\hat{\mathbb{C}})$, then $-A$ is also a matrix representation of γ. Therefore the map $SL(2, \mathbb{C})/\{\pm I\} \to \text{Möb}(\hat{\mathbb{C}})$ is an isomorphism and we have the following identifications

$$\text{Möb}(\hat{\mathbb{C}}) \cong SL(2, \mathbb{C})/\{\pm I\} \cong PSL(2, \mathbb{C}). \tag{1.15}$$

Now, by Proposition 1.2.8 we have the following group isomorphisms:

$$\text{Möb}(H) \simeq PSL(2, \mathbb{R}), \ \text{Möb}(\Delta) \simeq PSU(1, 1), \tag{1.16}$$

where $PSL(2, \mathbb{R})$ is the real projective special linear group of degree 2 and $PSU(1, 1)$ is the projective special unitary group of signature $(1, 1)$.

Now, it is defined the boundary of a Riemann surface, for this, the following definitions are given ([22, Subsection 1.1.4, pag. 12]).

We consider in $SL(2, \mathbb{C})$ the topology whose base is generated as follows: if we take $\gamma \in SL(2, \mathbb{C})$ and $\varepsilon > 0$, $B(\gamma, \varepsilon)$ is the set of elements of $SL(2, \mathbb{C})$ whose distance to γ is, entry by entry, less than ε. Then this topology passes to the quotient $PSL(2, \mathbb{C})$, and given the identification in 1.15, the group $\text{Möb}(\hat{\mathbb{C}})$ inherits a topology. We consider this topology on $\text{Möb}(\hat{\mathbb{C}})$. Let G be a discrete subgroup of $\text{Möb}(\hat{\mathbb{C}})$. The set

$$\Omega(G) = \{z \in \hat{\mathbb{C}} \mid G \text{ acts properly discontinuously at } z\}$$

will be called the *region of discontinuity of* G. The complementary set $L(G) = \hat{\mathbb{C}} - \Omega(G)$ is called the *limit set of* G. It is the set of accumulation points of orbits of G. The *orbit* of $z \in \mathbb{C}$ is the set $G(z) = \{g(z) \mid g \in G\}$.

If $\Omega(G)$ is nonempty, it will be said that G is a Kleinian group. A Kleinian group G is called *Fuchsian* if its limit set lies on a circle C on the Riemann sphere and G preserves each of the two disks into which $\hat{\mathbb{C}}$ is separated by C. We may always take $C = \hat{\mathbb{R}}$ and in this case, we can assume G operates properly discontinuously on H and H^*. Therefore G can be considered as a subgroup of $\text{Möb}(H)$ up to a global conjugation. The proof of the following theorem can be consulted in [17, Theorem 2.17, page 43].

Theorem 1.2.9. *For a subgroup* Γ *of Möb*(H) *the following are equivalent:*

(1) Γ *is a discrete subgroup of Möb*(H)*.*
(2) Γ *acts properly discontinuously on* H*.*

Considering that a Fuchsian group acts properly discontinuous in H and by the identification in 1.16, we can define a Fuchsian group as follow.

Definition 1.2.10. A *Fuchsian group* is a discrete subgroup of $PSL(2, \mathbb{R})$.

Definition 1.2.11. Let $\Sigma = \Delta/\Gamma$ be a Riemann surface. The *boundary* of Σ is defined as

$$\partial\Sigma = \partial\Sigma^* = (\Omega(\Gamma) \cap \hat{\mathbb{R}})/\Gamma.$$

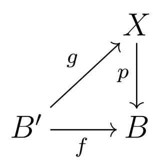
Theorem 1.2.12 (Uniformization Theorem)**.** *Every simply connected Riemann surface is biholomorphically equivalent to one of the three Riemann surfaces $\hat{\mathbb{C}}$, $\mathbb{C}$, or H.*

Corollary 1.2.13. *For every Riemann surface Σ, there exists a universal covering surface $\hat{\Sigma}$ of Σ, which is biholomorphic to one of the three Riemann surfaces $\hat{\mathbb{C}}$, $\mathbb{C}$, or H.*

Corollary 1.2.14. *Each Riemann surface Σ with universal covering biholomorphic to H can be represented as a quotient H/Γ, where Γ is a Fuchsian group.*

Proof. By Theorem 1.2.6, we have $\Sigma \cong H/\Gamma$, where Γ is the covering transformations group. Since Γ acts properly discontinuously on H, by Theorem 1.2.9, Γ is a discrete subgroup of $\text{Möb}(H)$. Therefore, Γ is a Fuchsian group. □

Now, if $f : B' \to B$ is a continuous map between topological spaces B' and B and (X, p) is a covering space of B, we can "*lift*" f to a continuous map $g : B' \to X$ such that the following diagram commutes.



The mapping g is called a *lifting* of f for p. A *section* of the covering (X, p) is a lifting of the identity of B for p.

Also, for Riemann surfaces Σ and Σ', let $(\hat{\Sigma}, \pi)$ and $(\hat{\Sigma}', \pi')$ be their universal covering, respectively. Let $f : \Sigma \to \Sigma'$ be a continuous mapping if $\hat{f} : \hat{\Sigma} \to \hat{\Sigma}'$ satisfies $f \circ \pi = \pi' \circ \hat{f}$, we call $\hat{f}$ a lift of f.

Theorem 1.2.15. *For Riemann surfaces Σ and Σ' and an arbitrary continuous mapping $f : \Sigma \to \Sigma'$ there exists a lift of f, $\hat{f} : \hat{\Sigma} \to \hat{\Sigma}'$, which is uniquely determined under the condition that $\hat{f}(\hat{p}_1) = \hat{q}_1$, where $\hat{p}_1 \in \hat{\Sigma}$ and $\hat{q}_1 \in \hat{\Sigma}'$ are such that $\pi'(\hat{q}_1) = f(\pi(\hat{p}_1))$. Moreover, if f is differentiable or holomorphic, then $\hat{f}$ is also differentiable or holomorphic.*

1.2.2 Quasiconformal Mappings on Surfaces

In this subsection a definition of quasiconformal mappings is given and it will be seen another definition in the second section of the following chapter.

Let Σ be a Riemann surface, $\{z_1, ..., z_j, ...\}$ the system of local complex coordinates, U_j the domain of z_j, and $\mathcal{U}_j = z_j(U_j)$. It will be defined a holomorphic mapping from the Riemann surface Σ to another Riemann surface Σ'. Let $\{w_1, ..., w_\lambda, ...\}$ be the system of local complex coordinates of Σ', W_λ the domain of w_λ, and $\mathcal{W}_\lambda = w_\lambda(W_\lambda) \subset \mathbb{C}$. Let $\Phi : p \to q = \Phi(p)$ be a continuous map from a domain $D \subset \Sigma$ into Σ'. Since $z_j : p \to z_j(p)$ maps U_j homeomorphically onto $\mathcal{U}_j$, and $w_\lambda : q \to w_\lambda(q)$ maps W_λ homeomorphically onto $\mathcal{W}_\lambda$ for λ, j such that $\Phi^{-1}(W_\lambda) \cap U_j \neq \emptyset$,

$$\Phi_{\lambda j} : z_j(p) \to w_\lambda(q), \; q = \Phi(p), \; p \in \Phi^{-1}(W_\lambda) \cap U_j,$$

is a continuous map from the domain $\mathcal{U}_{j\lambda} = \{z_j(p) \mid p \in \Phi^{-1}(W_\lambda) \cap U_j\} \subset \mathbb{C}$ into $\mathcal{W}_\lambda$:

$$\begin{array}{ccc} \Phi^{-1}(W_\lambda) \cap U_j & \xrightarrow[\Phi]{} & W_\lambda \\ {\scriptstyle z_j}\downarrow & & \downarrow{\scriptstyle w_\lambda} \\ \mathcal{U}_{j\lambda} & \xrightarrow[\Phi_{\lambda j}]{} & \mathcal{W}_\lambda \end{array}$$

Definition 1.2.16. If $\Phi_{\lambda j}$ is holomorphic, quasiconformal or of class C^r for each λ and j such that $U_j \cap \Phi^{-1}(W_\lambda) \cap D \neq \emptyset$, $\Phi : D \to \Sigma'$ is said to be *holomorphic, quasiconformal or of class* C^r ($r = 1, 2, ..., \infty$) map on D, respectively. If $D = \Sigma$, then we simply say that Φ is a holomorphic, quasiconformal or of class C^r map.

Two surfaces Σ and Σ' are called biholomorphically equivalent if there exist a biholomorphic map Φ from Σ onto Σ'.

1.2.3 Riemannian Metrics and hyperbolic surfaces

In this section, it is briefly seen the relation between Riemann surfaces and Riemannian surfaces.

Consider a differentiable manifold M. A *Riemannian metric* on M is a correspondence which associates to each point $p \in M$ an inner product $\langle , \rangle_p$ (that is, a symmetric, bilinear, positive-definite form) on the tangent space T_pM, which varies differentiably in the following sense: If $x : U \subset \mathbb{R}^n \to M$ is a system of coordinates around q, with $x(x_1, ..., x_n) = q \in x(U)$ y $\frac{\partial}{\partial x_i}(q) = dx_q(0, ..., 1, ..., 0)$, then $\left\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \right\rangle_q = g_{ij}(x_1, ..., x_n)$ is a differentiable function on U.

Definition 1.2.17. A differentiable manifold M with a given Riemannian metric will be called a *Riemannian manifold.* If M is 2-dimensional, M is called a *Riemannian surface.*

Let M and N be Riemannian manifolds. A diffeomorphism $f : M \to N$ is called an *isometry* if:

$$\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}, \text{ for all } p \in M, \ u, v \in T_pM.$$

It is said that a 2-dimensional manifold Σ is a *hyperbolic surface* if it is a Riemannian surface of constant sectional curvature -1. Then the universal covering of a hyperbolic surface is biholomorphic to Δ and we can identify the fundamental group of Σ with the set of all positively oriented isometries on the unit disc $\text{Isom}^+(\Delta)$. Here, only metrically complete surfaces will be considered.

Since $\text{Isom}^+(\Delta)$ acts on Δ such that we have the identification $\Sigma \cong \Delta/\text{Isom}^+(\Delta)$ and $\text{Isom}^+(\Delta)$ coincides with the group of all biholomorphisms on the unit disc $\text{Bihol}(\Delta)$, then $\Sigma \cong \Delta/\text{Bihol}(\Delta)$. Therefore Σ has a complex structure and it can be considered as a Riemann surface.

1.2.4 Deformations of Complex Structures

In this section Beltrami coefficients on surfaces are defined, they will play a very important role in the following chapters. Let $f : \Sigma \to \Sigma'$ be a diffeomorphism. Consider (U, z) a coordinate neighborhood of Σ and let (V, w) be a coordinate neighborhood of Σ' with $f(U) \subset V$. Define $F = w \circ f \circ z^{-1}$. Then we can consider the Beltrami coefficient

$$\mu = \frac{F_{\bar{z}}}{F_z}$$

which is a smooth complex function on $z(U)$.

The function μ is transformed under a holomorphic change of coordinates. Let (U_j, z_j) and (U_k, z_k) be coordinate neighborhoods of Σ and let (V_j, w_j) and (V_k, w_k) be coordinate neighborhoods of Σ' such that $f(U_j) \subset V_f$ and $f(U_j) \subset V_k$. Let μ_k and μ_j be Beltrami coefficients defined wit respect to (U_k, z_k) and (U_j, z_j). When $U_j \cap U_k \neq \emptyset$, we have

$$\mu_j = (\mu_k \circ z_{kj}) \cdot \left(\overline{\frac{dz_{kj}}{dz_j}}\right) \Big/ \left(\frac{dz_{kj}}{dz_j}\right) \text{ on } z_j(U_j \cap U_k), \text{ where } z_{kj} = z_k \circ z_j^{-1}.$$

The set of Beltrami differentials defines a differential form of type $(-1, 1)$ on Σ. Denote this $(-1, 1)$-differential form by

$$\mu_f = \mu \frac{d\bar{z}}{dz},$$

this is called the Beltrami differential of f on Σ. Let $\{(V_\alpha, w_\alpha)\}_{\alpha \in \Lambda}$ be a system of coordinates on Σ' and for a diffeomorphism which preserve orientation $f : \Sigma \to \Sigma'$ consider a system of coordinates $\{(f^{-1}(V_\alpha), w_\alpha \circ f)\}_{\alpha \in \Lambda}$ which defines a complex structure on Σ. Then we have a new Riemann surface Σ_f with system of coordinates neighborhood $\{(f^{-1}(V_\alpha), w_\alpha \circ f)\}_{\alpha \in \Lambda}$. Also, the identity map $id : \Sigma \to \Sigma_f$ is a diffeomorphism and $f : \Sigma_f \to \Sigma'$ is a biholomorphism.

Now, suppose that $f : \Sigma \to \Sigma'$ is a quasiconformal mapping between Riemann surfaces. If $f(z) = w$ represents the mapping in terms of the z and w local coordinates on Σ and Σ', respectively, then $\mu_f = f_{\bar{z}}/f_z$ is a L_∞ section of the complex line bundle $\omega^* \otimes \overline{\omega}$, where ω is the holomorphic cotangent bundle of Σ. Since f is quasiconformal, we obtain that μ_f takes values in the unit disk bundle with respect to absolute value norm on the fibres. Then we can consider $||\mu_f||_\infty$ and this is a number strictly less than 1. Thus μ_f lies in the open unit ball of the Banach space of sections $L_\infty(\omega^* \otimes \overline{\omega})$ for any quasiconformal f.

Definition 1.2.18. A Beltrami differential on the disc is an L_∞, $(-1, 1)$-form on the disc, that is to say an L_∞ section of $\omega^* \otimes \overline{\omega}$. The space of these differentials will be denoted by $Bel(\Delta)$.

A Beltrami differential on the disc can be thought of as an element μ in the unit ball of $L_\infty(\Delta)$ such that

$$\gamma^*(\mu) = \mu \circ \gamma \frac{\bar{\gamma}'}{\gamma'}$$

where γ is any Möbius transformation on the disc.

Now, to represent the Beltrami differential of any hyperbolic surface in terms of the Beltrami differentials, we proceed as follow: consider a hyperbolic surface Σ and its Poincaré uniformization $u : \Delta \to \Sigma$, (see Theorem 1.2.12). Then the group of deck transformations G is a Fuchsian group.

The group of deck transformations defines an action on the disc

$$Deck : G \times \Delta \to \Delta,\ (g, a) \longmapsto g \cdot a \tag{1.17}$$

whose orbit space is isometric with the surface Σ, that is to say, $\Sigma \cong \Delta/G$.

Definition 1.2.19. A Beltrami differential on Σ is a Beltrami differential on the disc μ invariant under the corresponding deck action, that is to say

$$l_g^*(\mu) = \mu,\ \forall g \in G. \tag{1.18}$$

where l_g denotes the deck action $(a \longmapsto g \cdot a)$ on the disc. The space of these differentials will be denoted by $Bel(\Sigma)$.

By the existence Theorem 1.1.24, we can find quasiconformal mappings $w_\alpha(z_\alpha)$ on each holomorphic coordinate patch (U_α, z_α) such that the Beltrami coefficient of w_α on U_α is $\mu(z_\alpha)$. These w_α provide a complex analytic atlas on Σ. Consequently any μ in $L_\infty(\omega^* \otimes \overline{\omega})_1$ assigns a complex analytic atlas on Σ whose local charts are quasiconformal with respect to the original complex structure of Σ. Denote by Σ_μ the Riemann surface Σ with the new structure. The identity map $Id : \Sigma \to \Sigma_\mu$ is a quasiconformal map with Beltrami differential μ. Also the Beltrami differentials on Σ_μ are given by definition 1.2.19 replacing G for $w^\mu G (w^\mu)^{-1}$.

1.3 Notions of Topology

In this section some definitions and results on topology that will be used in the following chapters are included. Here it is defined the notion of a lamination, this is one of the main objects of study in this work.

1.3.1 Laminations and Foliations

A p-dimensional *lamination*, $\mathcal{L}$, is a separable, locally compact, metrizable space covered by an atlas $\{(U_i, \varphi_i)\}_{i \in I}$. Every $\varphi_i : U_i \to T_i \times D_i$ is a homeomorphism whose target space is the product of some topological space T_i with a set D_i homeomorphic to an open subset of $\mathbb{C}^p$ in such a way that the transition maps preserve the first factor, that is to say, the maps $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_j \cap U_i) \to \varphi_j(U_j \cap U_i)$ satisfy:

(1) $\varphi_j \circ \varphi_i^{-1}(k, a) = (h_{ji}(k), f_{ji}(k, a))$,

where each map f_{ji} is smooth in the second variable and all its partial derivatives with respect to the second variable are continuous functions of all the variables.

A *Riemann surface lamination*, denoted by L, is a lamination 1-dimensional whose transition maps are holomorphic with respect to the second factor, that is

(2) $\varphi_j \circ \varphi_i^{-1}(k, \cdot)$ is holomorphic for all $k \in T_i$ and $\forall i, j \in I$.

Laminations can have very interesting and complex dynamic behavior. In this regard, see the works [2] and [3].

A *foliation* $\mathcal{F}$ of codimension q (and class C^r) is defined as a lamination where $T_i = \mathbb{R}^q$, D_i is an open subset of $\mathbb{R}^p$ and which is differentiable (class C^r), that is to say, the transition maps $\varphi_j \circ \varphi_i^{-1}$ are differentiable (class C^r).

Given a lamination $\mathcal{L}$, $\varphi_i^{-1}(t_0 \times D_i)$ is called a *plaque* and the maximal joins of plaques will be called the *leaves* of the lamination. In particular, a Riemann surface lamination has well defined the concept of leaves. In this case, leaves are Riemann surfaces immersed in the lamination. It will be said that the lamination is *minimal* if all the leaves are dense.

Now, consider a Riemann surface lamination L fibering over a complete hyperbolic surface Σ with fiber F, that is to say $\pi : L \to \Sigma$ is a locally trivial fibration with fiber F such that π restricted to any leaf is a local diffeomorphism. Since we can consider on Σ the atlas whose charts are given by the maps $\varphi_j \circ \pi^{-1}$ restricted to the domains where π is a diffeomorphism, then the map π is a local holomorphic homeomorphism. This implies that every leaf is a covering of the base and thus it is a hyperbolic surface. Hence L is a hyperbolic surface lamination.

By the Theorem 1.2.12, we can consider the uniformization of Σ:

$$u : \Delta \to \Sigma.$$

Consider the pullback of the fibration π by the uniformization u, that is to say

$$\hat{L} = \{(l, a) \in L \times \Delta \mid \pi(l) = u(a)\},$$

where the hatted morphism are the restrictions of the respective projection on $\hat{L}$.

$$\begin{array}{ccc} \hat{L} & \xrightarrow{\hat{u}} & L \\ {\scriptstyle \hat{\pi}}\downarrow & p.b. & \downarrow{\scriptstyle \pi} \\ \Delta & \xrightarrow[u]{} & \Sigma \end{array} \tag{1.19}$$

Lemma 1.3.1. *(1) $\hat{L}$ is a lamination.*
(2) $\hat{u}$ is a morphism of laminations.
(3) $\hat{\pi}$ is a locally trivial fibration.
(4) $\hat{L}$ is a lamination fibering over Δ.

Proof. Let $\mathcal{C}$ be a collection of open sets of the unit disk Δ such that $u|_A$ is a homeomorphism for each $A \in \mathcal{C}$ and $u(A)$ is a trivializing neighborhood of π. Thus, each $A \in \mathcal{C}$ is a trivializing neighborhood of $\hat{\pi}$. In particular, $\hat{\pi}$ is a locally trivial fibration, this proves item 3. Also, $\mathcal{A} = \{\hat{\pi}^{-1}(A) \mid A \in \mathcal{C}\}$ is an atlas of $\hat{L}$ as lamination, this proves items 1 and 2. Item 4 is trivial and follows from the previous ones. ☐

Since Δ is simply connected, $\hat{\pi}$ is a trivial fibration, hence it is assumed that the lamination $\hat{L}$ is $F \times \Delta$ and $\hat{\pi}$ is the projection on the second factor denoted by p_2. The following commutative diagram will be called the *lamination uniformization.*

$$\begin{array}{ccc} F \times \Delta & \xrightarrow{\hat{u}} & L \\ {\scriptstyle p_2}\downarrow & {\scriptstyle p.b.} & \downarrow{\scriptstyle \pi} \\ \Delta & \xrightarrow[u]{} & \Sigma \end{array} \tag{1.20}$$

1.3.2 Holonomy

In this part it is defined the concept of holonomy. Consider a lamination L. We will call *coordinate disc* of L to any open set of L homeomorphic to a coordinate chart $T_i \times \Delta$ for some $i \in I$. Consider a sheet $\mathcal{L}_x$ of a lamination L at the point $x \in F$ and let $\gamma : [0,1] \to \mathcal{L}_x$ be a continuous path. We will say that a collection of coordinate discs $\{U_j\}_{j=0}^k$ is a subordinate chain to γ if it satisfies

- There exists a partition of $I = [0,1]$, $0 = t_0 < t_1 < \cdots < t_{i+1} = 1$ such that $\gamma([t_i, t_{i+1}]) \subset U_i$, $0 \leq i \leq k$.
- If $U_i \cap U_j \neq \emptyset$, then $U_i \cup U_j$ is contained in a coordinate disc V_{ij}.

A transversal is a Borel subset of L which intersects each leaf in a countable subsets. The standard ones are those of the form $T_i \times x$ for some chart $T_i \times D_i$. Regular transversals are those contained in some standard transversal.

We can define local homeomorphism between regular transversals. Let us describe the construction of such homeomorphisms. consider $\gamma : [0,1] \to F$ a continuous path and let $\hat{D}_0 = T_r \times \{\gamma(0)\}$ and $\hat{D}_1 = T_s \times \{\gamma(1)\}$ be regular transversals. Fix a subordinate chain to γ, $\{U_j\}_{j=0}^k$ and a partition $0 = t_0 < t_1 < \cdots < t_{k+1} = 1$ which satisfies the item 1 of the definition of subordinate chain. For each $i \in \{1, ..., k\}$ we fix an embedded regular transversal $D_i = T_i' \times \{\gamma(t_i)\} \subset U_{i-1} \cap U_i$. Consider $D_0 \subset \hat{D}_0 \cap U_0$ and $D_{k+1} \subset \hat{D}_1 \cap U_k$. For each $x \in D_i$ sufficiently near of x_i, a plaque that passes through x intersects D_{i+1} at a single point $f_i(x)$. Then we have a homeomorphism defined in a domain $D_i' \subset D_i$ onto $f_i(D_i')$. Hence we can consider the function $f_\gamma = f_k \circ f_{k-1} \circ \cdots f_0$ which is a homeomorphism of a domain D onto $f_\gamma(D)$.

The function f_γ is called a *holonomy transformation* and it is independent of the regular transversal chosen, of the partition and of the subordinate chain. Let γ_1 and γ_2 be continuous paths and x a point in the domain of f_{γ_1} and f_{γ_2}, we say that f_{γ_1} and f_{γ_2} are equivalent if there exist an open set containing to x in which coincide. The equivalence class of a function f_γ is denominated the *germ* of f_γ at x. The set of germs that fix a point x is denoted by $G(x, D)$ and it is a group with the multiplication

$$germ(f) \cdot germ(g) = germ(f \circ g).$$

The proof of the following proposition can be consulted in [12, Proposition 1.1]

Proposition 1.3.2. Let $\gamma_i : I \to \mathcal{L}_{y_0}$, $i = 0, 1$ be continuous paths on the leaf $\mathcal{L}_x$ such that $\gamma_i(0) = y_0$, $\gamma_i(1) = y_1$, $i = 0, 1$. Let D_0, D_1 be regular transversals of $\mathcal{L}_{y_0}$ in y_0, y_1, $D_0 \supset V_i \to D_1$ holonomy transformations associate to γ_i and Φ_{γ_i} the germ of f_{γ_i} at y_0.

(1) If $\gamma_0 \simeq \gamma_1$ rel(0,1) then $\Phi_{\gamma_0} = \Phi_{\gamma_1}$.

(2) If $y_0 = y_1$ and $D_0 = D_1$, then the transformation $\gamma \mapsto \Phi_\gamma$ induces a homomorphism
$$\Phi : \pi_1(\mathcal{L}_{y_0}, y_0) \to G(y_0, D_0),\ \Phi([\gamma]) = \Phi_\gamma.$$

Note that by the first item of the Proposition 1.3.2, in this case, the notions of monodromy and the holonomy coincide.

Definition 1.3.3. The group $\mathrm{Hol}(\mathcal{L}_{y_0}, y_0) = \Phi(\pi_1(\mathcal{L}_{y_0}, y_0))$ is called the *holonomy group* of $\mathcal{L}_{y_0}$ at y_0.

Now, since the lamination L fiber over the surface Σ, we can consider the action of the fundamental group G on the fiber F of the lamination. This action will be called the *holonomy action*.

$$\mathrm{Hol} : G \to \mathrm{Homeo}(F)^{op}, \quad G = \pi_1(\Sigma). \tag{1.21}$$

If Σ' is a Riemann surface such that $\Sigma' \subset \Sigma$ we can consider the embedding $\iota : \Sigma' \to \Sigma$. This map induces a homomorphism in the fundamental groups $\iota_* : \pi_1(\Sigma') \to \pi_1(\Sigma)$. Then it is considered the holonomy of Σ' as $\mathrm{Hol}(\iota_*(\pi_1(\Sigma')))$.

Henceforth the following hypotheses on the lamination L are considered.

Hypotheses 1.3.4. Let $L \to \Sigma$ be a fibration, such that L is minimal, the fiber F is a Hausdorff compact space and the holonomy action continuously extends to the profinite completion $\widehat{G}$ and has trivial holonomy on pants[4].

For the proof of the following proposition see [27].

Proposition 1.3.5. The map $\hat{u}$ in the diagram (1.20) is the canonical map of the orbit space of the diagonal action on $F \times \Delta$ such that the following is an isometry of laminations
$$L \cong (F \times \Delta)/G,\ g \cdot (k, a) = (k \cdot g^{-1}, g \cdot a) \tag{1.22}$$
where the action on the first factor is the holonomy action 1.21 and the one in the second factor is the deck action 1.17.

Let L be a lamination satisfying the hypotheses 1.3.4. Considering the identification between L and the quotient space $(F \times \Delta)/G$ we can define the leaf of the lamination L at $x \in F$ as
$$\mathcal{L}_x = \pi\left(\{x\} \times \Delta\right) = \{[(x, k)]\ \mid k \in \Delta\} \tag{1.23}$$
where π is the canonical morphism of the quotient by the right diagonal action with the identification 1.22.

[4]Pants are defined in the following chapter.

Remark 1.3.6. *On the leaf $\mathcal{L}_x$ we will consider the final topology of the map $\widetilde{\pi}$: $\{x\} \times \Delta \to \mathcal{L}_x$ with the usual topology on the disc Δ. Whit this topology the leaf is a densely immersed manifold in L. Since an open set in $\{x\} \times \Delta$ is not necessarily open in $F \times \Delta$, unless the fiber F is discrete, the topology in $\mathcal{L}_x$ is strictly finer the relative one induced by L.*

The leaf $\mathcal{L}_x$ is a covering of Σ hence $\pi_1(\mathcal{L}_x)$ is a subgroup of $G = \pi_1(\Sigma)$. Concretely,

$$\pi_1(\mathcal{L}_x) = \{g \in G \mid (x, g) \sim (x, g \cdot k),\ \forall k \in \Delta\}. \tag{1.24}$$

For every x in the fiber F, denote by G_x the isotropy group of the holonomy action 1.21,

$$G_x = \{g \in G \mid x \cdot g = x\}. \tag{1.25}$$

Lemma 1.3.7. *For every $x \in F$, the isotropy of the holonomy at x coincides with the fundamental group of the leaf at x,*

$$G_x = \pi_1(\mathcal{L}_x),\ \forall x \in F. \tag{1.26}$$

Proof. Consider $g \in \pi_1(\mathcal{L}_x)$. By definition of the fundamental group of the leaf, for every $k \in \Delta$, $(x, k) \sim (x, g \cdot k)$. Let's fix a $k \in \Delta$, by definition of the diagonal action 1.22, there is $h \in G$ such that

$$(x, g \cdot k) = h \cdot (x, k) = (x \cdot h^{-1}, h \cdot k).$$

Then $g \cdot k = h \cdot k$, this implies that $g = h$ because every element $g \in G - \{id\}$ acts on Δ without fix points. Therefore $x \cdot h^{-1} = x \cdot g^{-1} = x$ and we conclude that $g \in G_x$. We have proved that $\pi_1(\mathcal{L}_x) \subset G_x$. For the reverse inclusion, consider $g \in G_x$, by definition of G_x, $x = x \cdot g$. Then we have for every $k \in \Delta$:

$$(x, k) \sim g \cdot (x, k) \sim (x \cdot g^{-1}) \sim (x \cdot g^{-1}, g \cdot k) \sim (x, g \cdot k).$$

That is to say, $g \in \mathcal{L}_x$. We have proved $G_x \subset \pi_1(\mathcal{L}_x)$ and we have the result. □

1.3.3 Collars

In this section it is considered a connected Riemannian manifold (M, g). Let $N \subset M$ be a submanifold with the Riemannian metric induced by M. Since the set of rectifiable curves joining the points q_1 and q_2 in N is contained in the set of the respective curves in M, d_M is dominated by d_N in N, that is

$$d_M(q_1, q_2) \leq d_N(q_1, q_2),\ \forall q_1, q_2 \in N.$$

It will be said that the submanifold N is a *length space* if the canonical inclusion $(N, d_N) \to (M, d_M)$ is an isometric embedding as metric spaces, that is

$$d_M(q_1, q_2) = d_N(q_1, q_2),\ \forall q_1, q_2 \in N.$$

Definition 1.3.8. A subset $S \subset M$ is strongly convex if for every pair of points q_1 and q_2 in the closure $\bar{S}$, there is a unique minimizing geodesic joining these points whose interior is contained in S.

As a consequence of the definition we have that any strongly convex submanifold is a length space. The converse is false in general, for example if we consider the unit disc without the zero $\Delta\backslash\{0\}$ in Euclidean space, then this is a length space but there is no a minimizing geodesic contained in $\Delta\backslash\{0\}$ joining antipodal points in S^1.

A careful inspection of Whitehead's Theorem [14, Proposition 4.2, Chapter 3] concerning strongly convex balls allows the following statement.

Lemma 1.3.9. *For every point p in M, there is $\beta > 0$ such that $B_{\beta'}(p)$ is strongly convex for every $\beta' \leq \beta$.*

Lemma 1.3.10. *[14, Corollary 3.9, Chapter 3]. If a piecewise differential curve $\gamma : [a, b] \to M$ with parameter proportional to arc length, has length less or equal to the length of any other piecewise differential curve joining $\gamma(a)$ to $\gamma(b)$ then γ is a geodesic. In particular, γ is regular.*

Definition 1.3.11. A connected Riemannian manifold (M, g) is *uniquely geodesic* if for every pair of points q_1 and q_2 in M, there is a unique minimizing geodesic in M joining these points.

By definition strongly convex submanifolds are uniquely geodesic and uniquely geodesic submanifolds are length spaces.

Definition 1.3.12. Consider an oriented Riemannian surface Σ and a closed minimizing geodesic C with a neighbourhood U such that $U - C$ has two connected components. A collar of C is $\mathfrak{c} = C \cup U'$ where U' is a connected component of $U - C$.

In general, $\mathfrak{c} - C$ will not be a length space for a collar $\mathfrak{c}$ of C. However, it will be so locally on C in the sense of the next lemma.

Lemma 1.3.13. *Consider an oriented Riemannian surface Σ and a collar $\mathfrak{c}$ of a closed minimizing geodesic C in Σ. Then, for every point $y \in C$, there is a neighbourhood B of y in Σ such that $B \cap (\mathfrak{c} - C)$ is uniquely geodesic. In particular, $B \cap (\mathfrak{c} - C)$ is a length space.*

Proof. Consider $y \in C$ and $B = B_{\beta'}(y)$ as in Lemma 1.3.9. Recall that, by construction, B is a normal neighbourhood of y and because C is a minimizing geodesic passing at y, $B \cap C$ is a proper segment of C and $B - C$ has only two distinct connected components B' and B''. Take β' small enough such that

$$B' \subset \mathfrak{c}, \quad B'' \cap \mathfrak{c} = \emptyset.$$

It is enough to show that B' is uniquely geodesic. Consider a pair of points q_1 and q_2 in B'. There is a unique minimizing geodesic γ from q_1 and q_2 such that it is contained in B. It rest to show that γ is contained in B'.

Suppose that $\gamma : [a, b] \to B$ is not contained in B'. Then, there is $t \in (a, b)$ such that $\gamma(t) \in C$. Define

$$t_- = \min\{t \in [a,b] \mid \gamma(t) \in C\},$$
$$t_+ = \max\{t \in [a,b] \mid \gamma(t) \in C\}.$$

Define the segments $\alpha_1 = \gamma|_{[a,t_-]}$, $\alpha_3 = \gamma|_{[t_+,b]}$ and α_2 as the minimizing geodesic segment of C from $q_1' = \gamma(t_-)$ to $q_2' = \gamma(t_+)$. Because C is a minimizing geodesic, we have

$$\begin{aligned} l(\alpha_1 * \alpha_2 * \alpha_3) &= l(\alpha_1) + l(\alpha_2) + l(\alpha_3) \\ &\leq l(\alpha_1) + l(\gamma|_{[t_-,t_+]}) + l(\alpha_3) = l(\gamma). \end{aligned} \tag{1.27}$$

Parametrizing the concatenated curve proportional to the arc length, by Lemma 1.3.10 we have that the concatenated curve is a geodesic and in particular a regular curve. By 1.27, the concatenated curve is a minimizing geodesic joining the point q_1 and q_2 and because the minimizing geodesic joining these point is unique, the concatenated curve coincides with γ up to reparametrization. We have proved that γ is contained in $\mathfrak{c}$ and reaches the curves the curve C at the point q_1'. Because γ is regular, it must be tangent to C at q_1' hence γ coincides with C up to reparametrization. This is absurd since neither q_1 nor q_2 belong to C by hypothesis and we have proved that B' is uniquely geodesic. This concludes the proof. □

1.3.4 Profinite Completion

This section contains some results needed in the following chapters about the profinite completion of groups. For details, see [25, Chapter 1]. Consider a group G and let $\mathcal{C}$ be the set of finite index normal subgroups. For every $S, S' \in \mathcal{C}$ such that $S' \leq S$, there is a canonical epimorphism

$$\eta_{S'S} : G/S' \to G/S.$$

The collection $(\eta_{S'S})_{S' \leq S}$ is an inverse system, that is, if S'' is another finite index normal subgroup of G such that $S'' \leq S \leq S$, then $\eta_{S''S} = \eta_{S''S'} \circ \eta_{S'S}$.

Definition 1.3.14. The profinite completion of G is the inverse limit $\widehat{G}$ of the inverse system $(\eta_{S'S})_{S' \leq S}$, that is,

$$\widehat{G} = \varprojlim_{S \in \mathcal{C}} G/S.$$

Example 7. Consider $G = \mathbb{Z}$, then its profinite completion is

$$\widehat{\mathbb{Z}} = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}.$$

By the Chinese Remainder theorem, we can identify with the product of rings of p-adic integers:

$$\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p.$$

Since every G/S with S a normal finite index subgroup is a finite group with the discrete topology, the inverse limit $\widehat{G}$ is a topological group with the following properties.

Proposition 1.3.15. The profinite completion $\widehat{G}$ is a compact, Hausdorff and totally disconnected topological group.

Also we can define the profinite completion as follow: we define the *profinite topology* on G as the topology generated by the basis of cosets of normal finite index subgroups of G. We will denote the topological group G with this topology by G_{pf}. This topology is inherited from a metric whose completion is the topological group $\widehat{G}$.

There is a canonical morphism of groups

$$\eta : G \to \widehat{G}$$

whose image is dense.

Definition 1.3.16. The group G is *residually finite* if the intersection of all the normal finite index subgroups is the trivial subgroup.

Proposition 1.3.17. The following assertions are equivalents.

(1) The group G is residually finite.
(2) The topological group G_{pf} is Hausdorff.
(3) The canonical map $\eta : G \to \widehat{G}$ is injective.

Theorem 1.3.18. *The profinite completion $\widehat{G}$ is a topological group with a fundamental system $\mathcal{U}$ of open neighbourhoods U of the identity element such that each $U \in \mathcal{C}$ and*

$$\widehat{G} = \varprojlim_{U \in \mathcal{C}} G/U.$$

Corollary 1.3.19. *The profinite completion $\widehat{G}$ is residually finite. In particular, $\mathcal{C}$ is a fundamental system of open neighbourhoods of the identity.*

Proof. By the previous theorem there is a fundamental system $\mathcal{U}$, then we have:

$$\{1\} \subset \bigcap_{S \in \mathcal{C}} S \subset \bigcap_{U \in \mathcal{U}} U = \{1\}.$$

Therefore, we have the result. □

Corollary 1.3.20. *Every subgroup $S \in \mathcal{C}$ is open and closed.*

Proof. Let S be a finite index normal subgroup of $\widehat{G}$. By the Corollary 1.3.19, S is open. However, $\widehat{G} - S$ is a finite union of cosets of S which are open as well since $\widehat{G}$ is a topological group, hence $\widehat{G} - S$ is open and we conclude that U is closed. □

Lemma 1.3.21. *Consider a group morphism $f : G_1 \to G_2$ such that the groups G_1 and G_2 are residually finite. Then there is a continuous extension of f to the profinite completions of the groups, that is there is a commutative diagram*

$$\begin{array}{ccc} \widehat{G}_1 & \xrightarrow{\widehat{f}} & \widehat{G}_2 \\ \uparrow & & \uparrow \\ G_1 & \xrightarrow{f} & G_2 \end{array} \tag{1.28}$$

such that $\hat{f}$ is a continuous group morphism.

Proof. For every finite index subgroup S of G_2, we have a monomorphism

$$G_1/f^{-1}(S) \hookrightarrow G_2/S. \tag{1.29}$$

In particular, $f^{-1}(S)$ is a finite index subgroup of G_1 and we have the isomorphism

$$\widehat{G}_1/\widehat{f^{-1}(S)} \cong G_1/f^{-1}(S). \tag{1.30}$$

Define the group morphism f_S as the composition of the top row in the next commutative diagram

$$f_S : \quad \begin{array}{ccccccc} \widehat{G}_1 & \longrightarrow & \widehat{G}_1/\widehat{f^{-1}(S)} & \xrightarrow{\cong} & G_1/f^{-1}(S) & \hookrightarrow & G_2/S \\ \uparrow & & & & \uparrow & & \uparrow \\ G_1 & & \xrightarrow{=} & & G_1 & \xrightarrow{f} & G_2 \end{array} \tag{1.31}$$

By the universal property of the inverse limit and Proposition 1.3.17, taking the inverse limit of diagram 1.31 with respect to the finite index subgroups S gives diagram 1.28 and the morphism $\widehat{f}$. It is clear that $\widehat{f}$ is continuous since by construction, for every finite index subgroup U of $\widehat{G}_2$, we have

$$\widehat{f}^{-1}(U) = \widehat{f^{-1}(U \cap G_2)},$$

a finite index subgroup of $\widehat{G}_1$. Then the result is proved. □

Theorem 1.3.22. *The fundamental group of a surface is residually finite.*

If the surface is non compact, then its fundamental group is free [28] with a countable set of generators hence residually finite [7]. In the case the surface is compact, this was proved in [20] and there is also a one page proof in [16].

1.4 Riemann-Roch Theorem

In this section it is presented the Riemann-Roch Theorem and it will be used to show that the space of holomorphic quadratic differentials of a closed Riemann surface of genus g is $3g-3$-dimensional. This result will be used in the following chapter.

Let Σ be a compact Riemann surface and let ω be the holomorphic cotangent bundle of Σ. If we have any nontrivial meromorphic section of a holomorphic line bundle over Σ, there can be only finitely many zeros and poles. Then we consider the *divisor of the section* as the formal sum of the points where there are zeros minus the points where there are poles (each point being weighted by the multiplicity of the zero or pole there at). The *degree f a divisor* $D = n_1P_1 + \cdots + n_kP_k$, $n_i \in \mathbb{Z}$, $P_i \in \Sigma$, is defined as the integer $\sum_1^k n_i$.

The degree of the divisor of any nontrivial meromorphic section of a holomorphic line bundle over a compact Riemann surface Σ is independent of which meromorphic section is chosen. Since any holomorphic line bundle φ on Σ always has nontrivial meromorphic sections, we may associate with φ the degree of the divisor of any nontrivial meromorphic section of φ. This integer is called the degree of φ and it is denoted $\deg(\varphi)$.

Denote by dim hol(φ) the dimension of the vector space of holomorphic sections of the holomorphic line bundle φ on Σ. When Σ is a compact surface, dim hol(φ) is finite. Notice that

$$\deg(\varphi) < 0 \text{ implies that } \dim \text{hol}(\varphi) = 0. \tag{1.32}$$

Because if there exists an holomorphic section f, then by definition f is a meromorphic section. Since the definition of $\deg(\varphi)$ is independent of the meromorphic section chosen, we can consider the section f and we have $\deg(\varphi) = 0$.

A family $\varphi = \{\varphi_j\}_j$ of holomorphic functions φ_h on $z_j(U_j)$ for all coordinate neighborhoods (U_j, z_j) of a Riemann surface Σ is called a *holomorphic quadratic differential* on Σ if it satisfies

$$\varphi_k(z_k) = \varphi_j \circ z_{jk}(z_k) \cdot \left(z'_{jk}(z_k)\right)^2, \text{ on } U_j \cap U_k, \text{ where } z_{jk} = z_j \circ z_k^{-1}.$$

A holomorphic quadratic differential will be denoted by $\varphi = \varphi(z)dz^2$ and $A_2(\Sigma)$ will denote the complex vector space of all holomorphic quadratic differentials on Σ. The holomorphic quadratic differentials are holomorphic sections of the bundle $\omega \otimes \omega$.

Now, it will be used the Riemann-Roch Theorem to show that the dimension of $A_2(\Sigma_g)$ the space of holomorphic quadratic differentials of a closed Riemann surface of genus g, is $3g-3$ $(g \geq 2)$.

Theorem 1.4.1 (Riemann-Roch)**.** *Let Σ_g be a compact Riemann surface of genus g and let φ be any holomorphic line bundle over Σ_g. Then*

$$\dim \text{hol}(\varphi) - \dim \text{hol}(\omega \otimes \varphi^{-1}) = deg(\varphi) - g + 1. \tag{1.33}$$

Let Σ_g be a compact Riemann surface of genus $g \geq 2$ and consider the holomorphic line bundle $\varphi = \omega \otimes \omega$, where ω is the holomorphic cotangent bundle over Σ_g. Since

$\deg(\omega) = 2g-2$, $\deg(\omega \otimes \varphi^{-1}) = 2-2g < 0$. By 1.32, we have dim $\mathrm{hol}(\omega \otimes \varphi^{-1}) = 0$. Using the Riemann-Roch Theorem we have

$$\dim \mathrm{hol}(\varphi) = deg(\varphi) - g + 1 = 2(2g-1) - g + 1 = 3g - 3.$$

Then we have the following result.

Proposition 1.4.2. The space of holomorphic quadratic differentials of a closed Riemann surface of genus g, denoted by $A_2(\Sigma_g)$, is a complex vector space of dimension $3g-3$.

Chapter 2

Teichmüller Theory for Riemann Surfaces

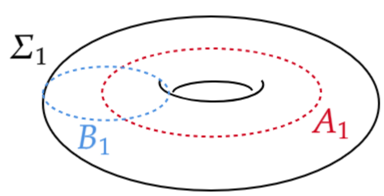
In this chapter different ways, in which the Teichmüller space for Riemann surfaces has been defined, are presented. This definitions coincide in the case of compact surfaces of finite genus. In the next chapter, based on one of these Teichmüller representations, it is defined the Teichmüller space for laminations.

2.1 Marked Riemann Surfaces

We can think the Teichmüller space as a parametrization of all the complex structures on a given surface and before giving a precise definition, let's see an example. Let Σ_1 be a closed Riemann surface of genus 1. For any point $p = (z(p), w(p))$ on Σ, we define the elliptic integral Φ selecting a branch of the algebraic function $w(z) = \sqrt{z(z-1)(z-\lambda)}$ and a path from ∞ to $z(p)$, and by setting

$$\Phi(p) = \int_{\infty}^{z(p)} \frac{dz}{\sqrt{z(z-1)(z-\lambda)}},$$

The value of this elliptic integral Φ is not determined uniquely. It depends on a path joining p_∞ and p.



The values of Φ along the simple closed curves A_1 and B_1 (see figure above) are represented by

$$\pi_1 = 2\int_0^1 \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \text{ and } \pi_2 = 2\int_0^{\lambda} \frac{dz}{\sqrt{z(z-1)(z-\lambda)}},$$

respectively. Setting

$$\Gamma = \{m\pi_1 + n\pi_2 \mid \ m, n \in \mathbb{Z}\},$$

then, we have that the function $\Phi(p)$ has infinitely many values which differ from each other by elements of Γ. Every element of Γ is called a period of Φ. Since the periods π_1, π_2 satisfy $Im(\pi_1/\pi_2) > 0$, they are linearly independent over the real number field $\mathbb{R}$. Also, every $\gamma = m\pi_1 + n\pi_2 \in \Gamma$ can be identified with a translation $\gamma(z) = z + m\pi_1 + n\pi_2$ of $\mathbb{C}$. Then we say that two points $z, z' \in \mathbb{C}$ are equivalent under Γ if there exists an element $\gamma \in \Gamma$ with $z' = \gamma(z)$. This equivalent relation defines a quotient space $\mathbb{C}/\Gamma$ and we have that Σ_1 is biholomorphic to $\mathbb{C}/\Gamma$. Therefore each torus is represented by a Riemann surface $\mathbb{C}/\Gamma$ for a lattice group Γ.

We may assume that the generators π_1 and π_2 for Γ are the canonical ones 1 and τ with $Im(\tau) > 0$, respectively. Then, we consider a lattice group

$$\Gamma_\tau = \{\gamma = m + n\tau \mid \ m, n \in \mathbb{Z}\}$$

where $\tau \in H$. In this case, we will denote the surface $\mathbb{C}/\Gamma_\tau$ by Σ_1^τ and we have the following theorem. Its proof can be consulted in [17, Theorem 1.1].

Theorem 2.1.1. *For any two points τ and τ' in the upper half-plane H, two tori Σ_1^τ and $\Sigma_1^{\tau'}$ are biholomorphically equivalent if and only if τ and τ' satisfy the relation*

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

where a, b, c and d are integers with $ad - bc = 1$.

Considering the Theorem 2.1.1 we have a parametrization of complex structures of tori up to biholomorphism given by $H/PSL(2,\mathbb{Z})$ where $PSL(2,\mathbb{Z})$ is the modular group defined by

$$PSL(2,\mathbb{Z}) = \left\{\gamma(z) = \frac{az + b}{cz + d} \ \mid \ a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1\right\}.$$

This parametrization is the moduli space of tori, and it is denoted by M_1. Then the Teichmüller space will be a universal covering of this space. In this case the Teichmüller space is identify with the upper half plane H.

Now, let Σ_g be a closed Riemann surface of genus g. Consider simple closed curves $A_1, B_1, ..., A_g, B_g$ with base point p such that the fundamental group $\pi_1(\Sigma_g, p)$ of Σ_g with base point p is generated by the homotopy classes $[A_1], [B_1], ..., [A_g], [B_g]$. Then $S_p = \{[A_j], [B_j]\}_{j=1}^g$ is a canonical system of generators of $\pi(\Sigma_g, p)$ and we call it a *marking* on Σ_g. We will say that two markings $S_p = \{[A_j], [B_j]\}_{j=1}^g$ and $S_{p'} = \{[A'_j], [B'_j]\}_{j=1}^g$ on Σ_g are equivalents if there exists a continuous curve C_0 on Σ_g such that $[A'_j] = T_{C_0}([A_j])$ and $[B'_j] = T_{C_0}([B_j])$ for $j = 1, ..., g$, where T_{C_0} is the isomorphism of $\pi_1(\Sigma_g, p)$ to $\pi_1(\Sigma_g, p')$ sending any $[C]$ to $[C_0^{-1} \cdot C \cdot C_0]$. Let S_p and S_q be markings on closed Riemann surfaces Σ_g and Σ'_g of genus g, respectively. Two pairs (Σ, S_p) and (Σ', S_q) are said to be equivalent if there exists a biholomorphic mapping $h : \Sigma \to \Sigma'$ such that the marking $h_*(S_q) = \{h_*([A'_j], h_*([B'_j]))\}_{j=1}^{g}$ is equivalent to $S_p = \{[A_j],[B_j]\}_{j=1}^{g}$. The equivalence class of (Σ, S_p) is denoted by $[\Sigma, S_p]$ and called a *marked closed Riemann surface of genus g*. Then, the *Teichmüller space T_g of genus g* is the set of all marked closed Riemann surfaces of genus g.

2.2 Quasiconformal Teichmüller

In this section it will be given a general representation of the Teichmüller space using quasiconformal maps and also, it will be defined the reduced Teichmüller space.

Consider an arbitrary, not necessarily closed, Riemann surface Σ. For every quasiconformal mapping f of Σ onto another Riemann surface Σ', consider a pair (Σ', f). We say that two pairs (Σ', f_1) and (Σ'', f_2) are equivalent if there is a conformal map $h : \Sigma' \to \Sigma''$ such that $f_2^{-1} \circ h \circ f_1 : \Sigma \cup \partial\Sigma \to \Sigma \cup \partial\Sigma$ is homotopic to the identity by a homotopy (via continuous mappings) that keeps every point of $\partial\Sigma$ fixed throughout. Denote by $[\Sigma, f]$ the equivalence class of (Σ, f). We call the set of all such equivalence classes the *quasiconformal Teichmüller space of* Σ or simply *Teichmüller space of* Σ and denote it by $T(\Sigma)$. We will call to $[\Sigma, id]$ the base point of $T(\Sigma)$, where id is the identity on Σ.

Remark 2.2.1. *The extension to the boundary of the quasiconformal mappings f_1 and f_2 in this definition is well defined, see discussion after Theorem 1.1.24. The boundary of a Riemann surface was defined in 1.2.11.*

Now, we say that two pairs (Σ', f_1) and (Σ'', f_2) are weakly equivalent if there is a conformal map $h : \Sigma' \to \Sigma''$ such that $f_2^{-1} \circ h \circ f_1 : \Sigma \to \Sigma$ is homotopic (via continuous mappings) to the identity. Denote by $[\Sigma, f]$ the equivalence class of (Σ, f). We call the set of all such equivalence classes the *reduced Teichmüller space of* Σ and denote it by $T^{\#}(\Sigma)$. Analogously to the Teichmüller space, we will call to $[\Sigma, id]$ the base point of $T^{\#}(\Sigma)$.

Remark 2.2.2. *Note that if $\Sigma = \Delta/\Gamma$ is a hyperbolic surface with $\partial\Sigma = \emptyset$, then $T(\Sigma) = T^{\#}(\Sigma)$. That is to say, if $L(\Gamma) = \hat{\mathbb{R}}$, then $T(\Sigma) = T^{\#}(\Sigma)$.*

As it is mentioned in [21, Page 4], there is a substantial difference between the two theories obtained by consider Teichmüller space or reduced Teichmüller space. For instance, the Teichmüller space of the unit disk in $\mathbb{C}$ is, in the non-reduced theory, infinite-dimensional (and it is called the universal Teichmüller space), whereas in the reduced theory, this space is reduced to a point.

Since any orientation-preserving diffeomorphism with compact domain is quasiconformal, for any closed Riemann surface of genus g, it is enough to take the pairs (Σ'_g, f) where Σ'_g is a closed Riemann surface and $f : \Sigma_g \to \Sigma'_g$ is an orientation-preserving diffeomorphism in the definitions above.

Let $\hat{\Sigma}$ the universal covering of Σ. By the Uniformization Theorem 1.2.12 we can suppose that $\hat{\Sigma}$ is $\hat{\mathbb{C}}$, $\mathbb{C}$ or the upper half plane H.

Let $f : \Sigma \to \Sigma'$ be a homeomorphism, by the Lifting Theorem 1.2.15, there exists a homeomorphism $\hat{f}$ of $\hat{\Sigma}$ to $\hat{\Sigma}'$. We say that f is a quasiconformal map if the lifting $\hat{f}$ is quasiconformal, here we say that a map f on $\hat{\mathbb{C}}$ is quasiconformal if it is a canonical mapping of $\mathbb{C}$ compound with a Möbius transformation. This definition is independent to the lifting and it coincides with the definition 1.2.16.

Now, we want to obtain the reduced Teichmüller space of a surface depending on its cover. First, we suppose that $\hat{\Sigma} = \hat{\mathbb{C}}$. Then $\Sigma = \hat{\mathbb{C}}$ and each quasiconformal map of Σ is homotopic to the identity *id*. Therefore, $T^{\#}(\hat{\mathbb{C}})$ consists of only one point.

Then, we suppose that $\hat{\Sigma} = \mathbb{C}$. In this case Σ is conformally equivalent to $\mathbb{C}$, $\mathbb{C}-\{0\}$ or a tori. If $f : \Sigma \to \Sigma'$ is a quasiconformal mapping between surfaces and $\Sigma = \mathbb{C}$ or $\Sigma = \mathbb{C} - \{0\}$, then $f(\Sigma)$ is conformally equivalent to $\mathbb{C}$ or $\mathbb{C} - \{0\}$, respectively. Moreover, each quasiconformal map of $\mathbb{C}$ is homotopic to the identity *id*. Therefore $T^{\#}(\mathbb{C})$ consists on only one point. Each quasiconformal map of $\mathbb{C} - \{0\}$ is homotopic to the identity map or to the conformal map $z \longmapsto 1/z$. Thus, $T^{\#}(\mathbb{C} - \{0\})$ consists of only one point. Finally, if Σ is the tori, then we saw that $T(\Sigma_1)$ is identified with the upper half plane H. Since $\partial\Sigma_1 = \emptyset$ we have $T^{\#}(\Sigma_1) = H$.

Thus, from here on, it is assumed that $\hat{\Sigma}$ is the upper half plane. In this case we have that the fundamental group of Σ is abelian if and only if Σ is conformally equivalent to Δ, $\Delta-\{0\}$ or ring domains $\{z \in \mathbb{C} \ : \ 1 < |z| < r(< \infty)\}$. Suppose that $\Sigma = \Delta$ or $\Sigma = \Delta - \{0\}$. Then the image under a quasiconformal map is conformally equivalent to Σ. Also, each quasiconformal map of Σ is homotopic to *id*. Therefore, $T(\Sigma)$ consists of only one point. Now, if $\Sigma = \{z \in \mathbb{C} \ : \ 1 < z < r\}$ then the image under a quasiconformal map is conformally equivalent to another ring $\Sigma' = \{z \in \mathbb{C} : 1 < |z| < r'\}$ and each quasiconformal map of Σ is homotopic to *id* or to the map $z \longmapsto s/z$. Moreover, by the reflection principle, we have that ring domains corresponding to different r' are not conformally equivalent. Hence, $T(\Sigma)$ is identified with the open interval $(1, +\infty)$.

Now, we suppose that Σ is a Riemann surface which has no-abelian fundamental group and its covering is conformally equivalent to the upper half plane H. The group of deck transformations of Σ, Γ, will be called *Fuchsian model*. In this case Γ is a subgroup of $\mathrm{Aut}(H)$. By the Theorem 1.2.5, we can suppose that the Fuchsian model is not abelian. Also, the set of fixed points of elements of $\Gamma - \{id\}$ contains at least three points.

We can suppose that $0, 1$ and ∞ are fixed points of some element of $\Gamma - \{id\}$. Let $\hat{f} : H \to H$ be a lifting of a quasiconformal map $f : \Sigma \to \Sigma'$, which fix $0, 1$ and ∞. This map is uniquely determined and it is the restriction of a quasiconformal map defined on $\hat{\mathbb{C}}$. We will call to $\hat{f}$ the *canonical lifting*.

Remark 2.2.3. *Notice that the Beltrami differential of this map $\hat{f}$ satisfies 1.18.*

Now, considering the Remark 2.2.3, we can define the following map:

$$\Theta_{\hat{f}} : \Gamma \to PSL(2, \mathbb{R})$$

$$\Theta_{\hat{f}}(\gamma) = \hat{f} \circ \gamma \circ \hat{f}^{-1}, \ \gamma \in \Gamma.$$

This map is well defined because if we consider the Beltrami coefficient of $\Theta_{\hat{f}}$, by the properties of the coefficient Beltrami of a composition 1.5 and the Remark 2.2.3, we have $\mu_{\Theta_{\hat{f}}} = 0$, therefore $\Theta_{\hat{f}}$ is an element in $PSL(2,\mathbb{R})$. Let us see that Θ is injective: If $\Theta_{\hat{f}}(\gamma) = id$, then $\hat{f} \circ \gamma \circ \hat{f}^{-1} = id$, since $\hat{f}$ is bijective we have $\gamma = id$. Now, let us see that $\Theta_{\hat{f}}$ is a homomorphism.

$$\Theta_{\hat{f}}(\gamma_1 \circ \gamma_2) = \hat{f} \circ (\gamma_1 \circ \gamma_2) \circ \hat{f}^{-1} = \Theta_{\hat{f}}(\gamma_1) \cdot \Theta_{\hat{f}}(\gamma_2).$$

Then we have an isomorphism $\Theta_{\hat{f}}$ of Γ onto another Fuchsian group Γ_1 and another Riemann surface $\Sigma' = H/\Gamma_1$.

Notice that for any quasiconformal map $\hat{f} : H \to H$ with Beltrami differential μ_f, if $\Theta_{\hat{f}}(\Gamma)$ is an automorphism group, then we have

$$\Theta_{\hat{f}}(\gamma) \circ \hat{f} = \hat{f} \circ \gamma, \quad \gamma \in \Gamma.$$

Differentiating with respect to z, since γ and $\Theta_{\hat{f}}(\gamma)$ are Möbius transformations, we have:

$$(\Theta_{\hat{f}}(\gamma) \circ \hat{f})_z = \Theta_{\hat{f}}(\gamma)_z \circ \hat{f} \cdot \hat{f}_z + \Theta_{\hat{f}}(\gamma)_{\bar{z}} \circ \hat{f} \cdot \hat{f}_z = \Theta_{\hat{f}}(\gamma)_z \circ \hat{f} \cdot \hat{f}_z,$$

$$(\hat{f} \circ \gamma)_z = \hat{f}_z \circ \gamma \cdot \gamma_z + \hat{f}_{\bar{z}} \circ \gamma \cdot \overline{\gamma_z} = \hat{f}_z \circ \gamma \cdot \gamma_z.$$

Analogously, differentiating with respect to $\bar{z}$, we have:

$$(\Theta_{\hat{f}}(\gamma) \circ \hat{f})_{\bar{z}} = \Theta_{\hat{f}}(\gamma)_{\bar{z}} \circ \hat{f} \cdot \overline{\hat{f}_z} + \Theta_{\hat{f}}(\gamma)_z \circ \hat{f} \cdot \hat{f}_{\bar{z}} = \Theta_{\hat{f}}(\gamma)_z \circ \hat{f} \cdot \hat{f}_{\bar{z}},$$

$$(\hat{f} \circ \gamma)_{\bar{z}} = \hat{f}_{\bar{z}} \circ \gamma \cdot \overline{\gamma_z} + \hat{f}_z \circ \gamma \cdot \gamma_{\bar{z}} = \hat{f}_{\bar{z}} \circ \gamma \cdot \overline{\gamma_z}.$$

Therefore, for almost every $z \in H$, we have:

$$\Theta_{\hat{f}}(\gamma)_z \circ \hat{f} \cdot \hat{f}_z = \hat{f}_z \circ \gamma \cdot \gamma_z$$

$$\Theta_{\hat{f}}(\gamma)_z \circ \hat{f} \cdot \hat{f}_{\bar{z}} = \hat{f}_{\bar{z}} \circ \gamma \cdot \overline{\gamma_z}.$$

Taking the quotient between the two equations above, we obtain:

$$\mu_f = (\mu_f \circ \gamma)\overline{\gamma_z}/\gamma_z, \quad a.e. \text{ on } H, \ \gamma \in \Gamma. \tag{2.1}$$

Conversely, if (2.1) is satisfied, then $\Theta_{\hat{f}}(\Gamma)$ is an automorphism group. That is to say, $\Theta_{\hat{f}}(\Gamma)$ is an automorphism group if and only if 2.1 is satisfied.

Lemma 2.2.4. *Two points $[\Sigma', f_1]$, $[\Sigma'', f_2] \in T^{\#}(\Sigma)$ satisfy $[\Sigma', f_1] = [\Sigma'', f_2]$ in $T^{\#}(\Sigma)$ if and only if $\Theta_{\hat{f}_1} = \Theta_{\hat{f}_2}$, where $\hat{f}_j$ is the canonical lift of f_j for each $j = 1, 2$.*

Proof. First, we suppose that $[\Sigma', f_1] = [\Sigma'', f_2]$. By definition of the equivalence relation, there exists a conformal map $h : \Sigma' \to \Sigma''$ such that $f_2 \circ f_1^{-1}$ is homotopic to h. By composing with a conformal mapping, if it is necessary, we can assume that $\Sigma' = \Sigma''$ and f_1 is homotopic to f_2. Let $\{f_t\}_{1\le t\le 2}$ a homotopy between f_1 and f_2. Let $\hat{f}_1$ the canonical lift of f_1 with respect to Γ. Then the homotopy $\{f_t\}$ has a unique continuous lift $\{\hat{F}_t\}$ such that $\hat{F}_1 = \hat{f}_1$ and $\{\hat{F}_t\}$ give a homotopy between $\hat{f}_1$ and a lift $\hat{F}_2$ of f_2. Fix $\gamma \in \Gamma$ and $z \in H$ arbitrarily. Then the following two paths have the same initial point $\hat{f}_1 \circ \gamma(z)$:

$$\left\{\hat{F}_t \circ \gamma(z) \ : \ 1 \le t \le 2\right\},$$

$$\left\{\hat{f}_1 \circ \gamma \circ \hat{f}_1^{-1}\left(\hat{F}_t(z)\right) \ : \ 1 \le t \le 2\right\}$$

and they also have the same projection on Σ', $\{f_t \ : \ 1 \le t \le 2\}$. Therefore both paths coincide with each other. In particular, the terminal point $\hat{F}_2 \circ \gamma(z)$ is equal to $\hat{f}_1 \circ \gamma \hat{f}_1(\hat{F}_2(z))$. Since z was arbitrarily, we conclude that $\hat{F}_2 \circ \gamma \circ \hat{F}_2^{-1} = \Theta_{\hat{f}_1}(\gamma)$. Since γ was chosen arbitrarily and $0, 1$ and ∞ are fixed by some element of Γ, then $\hat{F}_2$ fixes $0, 1$ and ∞. Thus, $\hat{F}_2$ coincides with the canonical lift of f_2 with respect to Γ and hence $\Theta_{\hat{f}_1} = \Theta_{\hat{f}_2}$.

Now, we suppose that $\Theta_{\hat{f}_1} = \Theta_{\hat{f}_2} = \Theta$. Then for every $\gamma \in \Gamma$, we have

$$\hat{f}_j \circ \gamma = \Theta(\Gamma) \circ \hat{f}_j, \ \ j = 1, 2.$$

For every $t \in [0, 1]$ and every $z \in H$, let g_z be the geodesic (with respect to the Poincaré metric) connecting $\hat{f}_1$ and $\hat{f}_2$. We denote by $\hat{f}(z, t)$ the point which divides g_z in the ratio $t : (1 - t)$. Then $\{\hat{f}_t = \hat{f}(z, t - 1) : 1 \le t \le 2\}$ is a homotopy between $\hat{f}_1$ and $\hat{f}_2$. Now, we have

$$\hat{f}_t \circ \gamma = \Theta(\gamma) \circ \hat{f}_t, \ \ \gamma \in \Gamma, t \in [1, 2].$$

Therefore every $\hat{f}_t$ is projected to a continuous map f_t of Σ onto $\Sigma' = \Sigma''$ and we have a homotopy between f_1 and f_2. Hence $[\Sigma', f_1] = [\Sigma'', f_2]$. □

Remark 2.2.5. *Two quasiconformal maps $f_j : R \to S_j$ $(j = 1, 2)$ satisfy $\Theta_{\hat{f}_1} = \Theta_{\hat{f}_2}$ if and only if $\hat{f}_1 = \hat{f}_2$ on the limit set $L(\Gamma)$ of Γ, see [17, Remark, page 123].*

Now, it is defined the *reduced Teichmüller of a Fuchsian model* as follow:

$$T^{\#}(\Gamma) = \{\Theta_{\hat{f}} \ : \ \hat{f} \textit{ is a canonical qc map of } \hat{\mathbb{C}} \textit{ such that } \Theta_{\hat{f}}(\Gamma) \textit{ is a Fuchsian group}\}.$$

Let me note that in the definition of $T^{\#}(\Gamma)$, it is considered qc mappings defined on $\hat{\mathbb{C}}$, then $\Theta_{\hat{f}}$ is not necessarily a Fuchsian group and it does not necessarily act properly discontinuously on $\hat{\mathbb{C}}$.

Now, it is defined the *Teichmüller space of a Fuchsian group*, for this it is considered the following set:

$$QC(\Gamma) = \{w \ : \ w \textit{ is a canonical qc map of } \hat{\mathbb{C}} \textit{ and } \Theta_w(\Gamma) \textit{ is a Fuchsian group}\}.$$

It will be said that two elements $w_1, w_2 \in QC(\Gamma)$ are equivalent if $w_1 = w_2$ on $\mathbb{R}$. We denote by $[w]$ the equivalence class of w. Let

$$T(\Gamma) = \{[w] \, : \, w \in QC(\Gamma)\}.$$

$T(\Gamma)$ is called the Teichmüller space of Γ.

Let me note that it can be defined the reduced Teichmüller space in terms of $QC(\Gamma)$ defining that two elements $w_1, w_2 \in QC(\Gamma)$ are equivalent if $\Theta_{\hat{w_1}} = \Theta_{\hat{w_2}}$. Then $T^{\#}(\Gamma)$ is the quotient of $QC(\Gamma)$ by this equivalence relation. For this and the Remark 2.2.5, we have that if two mappings determining the same point of $T(\Gamma)$, they determine the same point of $T^{\#}(\Gamma)$. That is to say, the equivalence classes in $T^{\#}(\Gamma)$ are larger than in $T(\Gamma)$, thus in general, $T^{\#}(\Gamma)$ has fewer elements. Moreover, if $\Sigma = H/\Gamma$ is a closed Riemann surface, then it will be seen that $T(\Gamma) = T^{\#}(\Gamma)$. To show this, the two following results will be used, their proof can be consulted in [17, Subsection 2.4.3].

Lemma 2.2.6. *Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of $\mathrm{Aut}(H)$ which converges uniformly on compact subsets of H to a holomorphic function f defined in H. Here f admits a constant function with value ∞. Then either one of the following holds:*

(1) f is an element of $\mathrm{Aut}(H)$.

(2) f is a constant function c with $c \in \hat{\mathbb{R}}$.

Lemma 2.2.7. *Let Γ be a Fuchsian model of a closed Riemann surface of genus $g \geq 2$. For an arbitrary point $\zeta \in \hat{\mathbb{R}}$, there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of Γ such that $\{\gamma_n(z_0)\}_{n=1}^{\infty}$ converges to ζ for any point $z_0 \in H$.*

Proposition 2.2.8. Let Σ be a compact Riemann surface. Two quasiconformal maps $f_j : \Sigma \to S_j$ (j=1,2) satisfy $\Theta_{\hat{f}_1} = \Theta_{\hat{f}_2}$ if and only if $\hat{f}_1 = \hat{f}_2$ on $\mathbb{R}$.

Proof. Suppose that $\hat{f}_1 = \hat{f}_2$ on $\mathbb{R}$. Then for every $\gamma \in \Gamma$, $\Theta_{\hat{f}_1}(\gamma) = \Theta_{\hat{f}_2}(\gamma)$ on $\mathbb{R}$. Since $\Theta_{\hat{f}_1}(\gamma)$ and $\Theta_{\hat{f}_2}(\gamma)$ are Möbius transformations and they coincide on more than two points, by Remark 1.2.7, we have $\Theta_{\hat{f}_1} = \Theta_{\hat{f}_2}$ for every $\gamma \in \Gamma$, that is to say, $\Theta_{\hat{f}_1} = \Theta_{\hat{f}_2}$. Conversely, suppose that $\Theta_{\hat{f}_1} = \Theta_{\hat{f}_2} = \Theta$. Let z_0 be a fixed point in H. By the Lemma 2.2.7, for every $\zeta \in \hat{\mathbb{R}}$ there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of Γ such that $\{\gamma_n(z_0)\}_{n=1}^{\infty}$ converges to ζ. Moreover, by the Lemma 2.2.6, such a sequence converges locally uniformly on H to a constant function ζ. Since

$$\hat{f}_j \circ \gamma_n(z_0) = \Theta(\gamma_n) \circ \hat{f}_j(z_0),$$

we have $\hat{f}_1 \circ \gamma_n \circ \hat{f}_1^{-1}(\hat{f}_1(z_0)) = \hat{f}_2 \circ \gamma_n \circ \hat{f}_2^{-1}(\hat{f}_2(z_0))$ and taking the limit when $n \to \infty$, we conclude that $\hat{f}_1(\zeta) = \hat{f}_2(\zeta)$. Since ζ was arbitrarily, we have $\hat{f}_1 = \hat{f}_2$ on $\mathbb{R}$. $\square$

By definition of the Teichmüller space and the reduced Teichmüller space, the Proposition 2.2.8 implies that $T(\Gamma) = T^{\#}(\Gamma)$ for compact surfaces. Now, we want to see the relation of this Teichmüller spaces with the quasiconformal Teichmüller spaces defined in last section.

Remark 2.2.9. *By Lemma 2.2.6, the limit set $L(\Gamma)$ of a Fuchsian group Γ, is the set of all accumulation points of the set $\{\gamma(z_0) \mid \gamma \in \Gamma\}$ for any $z_0 \in \Delta$. Since $L(\Gamma)$ is a closed set and by Lemma 2.2.7, if $\Sigma = \Delta/\Gamma$ is a compact surface, then $L(\Gamma) = \hat{\mathbb{R}}$.*

Proposition 2.2.10. Let Γ be a Fuchsian model of a Riemann surface Σ. Then the reduced Teichmüller space $T^{\#}(\Sigma)$ of Σ is identified with $T^{\#}(\Gamma)$.

Proof. Consider the map $\eta : T^{\#}(\Sigma) \to T^{\#}(\Gamma)$ defined by

$$[\Sigma', f] \longmapsto \Theta_{\hat{f}}. \tag{2.2}$$

By the Lemma 2.2.4, this map is well defined and it is injective. For every quasiconformal map $\hat{f}$ of $\hat{\mathbb{C}}$ such that $\Theta_{\hat{f}}(\Gamma) = \hat{f}\Gamma\hat{f}^{-1}$ is a Fuchsian group, consider the group $\Gamma_1 = \Theta_{\hat{f}}(\Gamma)$. Then $\hat{f}$ can be projected to a quasiconformal map f of $\Sigma = H/\Gamma$ onto $\Sigma' = H/\Gamma_1$. Therefore each quasiconformal map $\hat{f}$ determines a point $[\Sigma', f] \in T^{\#}(\Sigma)$, that is to say, the map (2.2) is surjective. □

Corollary 2.2.11. *If Σ is a compact surface, then $T(\Sigma) \cong T^{\#}(\Sigma) \cong T^{\#}(\Gamma) \cong T(\Gamma)$.*

Proof. By the Lemma 2.2.8, if Σ is a compact Riemann surface, then two maps determine the same point in $T^{\#}(\Gamma)$ if and only if they determine the same point in $T(\Gamma)$. Therefore, if Σ is compact, $T^{\#}(\Sigma)$ is identified with $T(\Gamma)$. Also, by Remarks 2.2.9 and 2.2.2, if Σ is compact, $T(\Sigma) = T^{\#}(\Sigma)$. By 2.2.10, we have the result. □

2.3 Completeness of Teichmüller Spaces

In this section it is considered a compact Riemann surface Σ of genus $g \geq 2$, then it will not be distinguish between Teichmüller space and reduced Teichmüller space. Let $p_1 = [\Sigma, f_1]$ and $p_2 = [\Sigma', f_2]$ be points in $T(\Sigma)$. Let $\mathcal{F}_{f_1,f_2}$ be the set of quasiconformal maps of Σ onto Σ' which are homotopic to $f_2 \circ f_1^{-1}$. Define

$$d(p_1, p_2) = \inf_{g \in \mathcal{F}_{f_1,f_2}} \log K(g), \quad K(g) = \inf\{K \mid g \text{ is a K-qc map}\}. \tag{2.3}$$

We call to d the Teichmüller distance on $T(\Sigma)$. Let's see that this function is independent of the choice of representatives at p_1 and p_2. Suppose that $[\Sigma, f_1] = [\Sigma'', f_1']$, then $\hat{f}_1 \circ (\hat{f'}_1)^{-1}$ is homotopic to a conformal mapping h. Let g a quasiconformal mapping homotopic to $\hat{f}_2 \circ (\hat{f}_1)^{-1}$ with complex dilatation $K(g)$, then $g \circ h$ is a quasiconformal map homotopic to $\hat{f}_2 \circ (\hat{f'}_1)^{-1}$ with complex dilatation $K(g \circ h) = K(g)$.

Now, we will see that d is a metric. If $p_1 = p_2$, then by definition of infimum there exists a sequence $\{g_n\}_n$ in $\mathcal{F}_{f_1,f_2}$ such that $\log K(g_n) \to 0$, that is to say $K(g_n) \to 1$ when $n \to \infty$. Let $\hat{g}_n$ the canonical lift of g_n with respect to $\Theta_{\hat{f}_1}(\Gamma)$ for each n. Since $\mu_{\hat{g}_n} \to 0$, by Proposition 1.1.26, $\hat{g}_n$ converges to id locally uniform on H. On the other hand, since $g_n \in \mathcal{F}_{f_1,f_2}$, then $\widehat{f_2 \circ f_1^{-1}} \simeq \hat{g}_n$. Therefore $[\Sigma', \widehat{f_2 \circ f_1^{-1}}] = [\Sigma', \hat{g}_n]$ and by Lemma 2.2.4, we have $\Theta_{\widehat{f_2 \circ f_1^{-1}}}(\gamma) = \Theta_{\hat{g}_n}(\gamma)$, $\gamma \in \Gamma$, $n \in \mathbb{N}$. Taking the limit when

$n \to \infty$, we have $\Theta_{\widehat{f_2 \circ f_1^{-1}}}(\gamma) = \gamma,\ \gamma \in \Gamma$. This implies that $[\Sigma, f_1] = [\Sigma, f_2]$, that is to say $p_1 = p_2$. If $p_1 = p_2$, the $K(g) = 1$ for every $g \in \mathcal{F}_{f_1,f_2}$ and thus $d(p_1, p_2) = 0$. Since $K(g) = K(g^{-1})$ for every quasiconformal map g, then d is symmetric. The triangle inequality follows from $K(g_1 \circ g_2) \leq K(g_1) \cdot K(g_2)$.

Theorem 2.3.1. *The Teichmüller space $T(\Sigma)$ is complete with respect to the Teichmüller distance.*

Proof. Let $\{p_n = [\Sigma^n, f_n]\}_n$ be a Cauchy sequence in $T(\Sigma)$ with respect to the Teichmüller distance. By definition of d, for every $\epsilon > 0$ we can find a sufficiently large N_ϵ such that for every $n, m \geq N_\epsilon$ there exists a quasiconformal map $f_{n,m}$ homotopic to $f_m \circ f_n^{-1}$ and such that $||\mu_{n,m}||_\infty < \epsilon$, where $\mu_{n,m} = \mu_{f_{n,m}}$. In particular, we can find a subsequence $\{p_{n_j}\}_j$ and a sequence $\{f_{n_j,n_{j+1}}\}$ of quasiconformal maps such that:

$$||\mu_{n_j,n_{j+1}}||_\infty < 2^{-1},\ j = 1, 2, 3, \ldots.$$

Then, let p_0 the base point in $T(\Sigma)$. Since $\{d(p_0, p_n)\}_n$ is a bounded sequence, we can suppose that $K(f_n) < K$ for every $n \in \mathbb{N}$ with K sufficiently large. Since

$$1 + \frac{1}{2^j} \leq \left(1 + 4 \cdot 2^{-j}\right)\left(1 - 2^{-j}\right) = 1 + \left(3 - \frac{4}{2^j}\right) \cdot \frac{1}{2^j},$$

we have

$$K\left(f_{n_j,n_{j+1}}\right) \leq \frac{1 + 2^{-j}}{1 - 2^{-j}} \leq 1 + 4 \cdot 2^{-j},\ j \geq 1.$$

Then $g_j = f_{n_{j-1},n_j} \circ f_{n_{j-2},n_{j-1}} \circ \cdots \circ f_{n_1,n_2} \circ f_{n_1}$ is a quasiconformal map of Σ onto Σ^{n_j}. Also it is homotopic to f_{n_j}, because $f_{n_{j-1},n_j} \simeq f_{n_j} \circ f_{n_{j-1}}^{-1}$ and it satisfies

$$K(g_j) \leq K(f_{n_{j-1},n_j}) \cdots K(f_{n_1}) \leq K \cdot \prod_{j=1}^{j-1}\left(1 + 4 \cdot 2^{-j}\right).$$

Therefore $\{K(g_j)\}_j$ is a bounded sequence. Denote by K_1 the supreme of $\{K(g_j)\}_j$. Let $\hat{g}_j$ the canonical left of g_j with respect to Γ for each j. Then $\mu_j := \mu_{\hat{g}_j}$ belongs to $Bel(\Sigma)$ and $||\mu_j||_\infty \leq k_1 := \frac{1-K_1}{1+K_1} < 1$, because $(K_j + 1)(K_1 - 1) \geq (K_j - 1)(K_1 + 1)$.

Also, we have

$$\frac{1}{2}\,||\mu_j - \mu_{j+1}||_\infty \leq \left|\left|\frac{\mu_{j+1} - \mu_j}{1 - \overline{\mu_j}\mu_{j+1}}\right|\right|_\infty = \left|\left|\mu_{n_j,n_{j+1}}\right|\right|_\infty < 2^{-j}$$

for every $j \in \mathbb{N}$. In particular, $\{\mu_j\}_j$ is a Cauchy sequence in $Bel(\Sigma)$. Therefore $\mu = \lim_{j\to\infty} \mu_j$ there exists in $Bel(\Sigma)$ and satisfies $||\mu||_\infty \leq k_1$. Let $\hat{f}$ be the canonical map μ- quasiconformal of H. Then $\hat{f} \in QC(\Gamma)$. Let $p = [\Sigma', p]$ be the point determined by $\Theta_{\hat{f}}$. Since

$$\tanh\left(\frac{d(p_{n_j}, p)}{2}\right) \leq \left|\left|\frac{\mu - \mu_j}{1 - \overline{\mu_j}\mu}\right|\right|_\infty \leq \frac{1}{1 - (k_1)^2}||\mu_j - \mu||_\infty,$$

then $p_{n_j} \to p$. Since the limit of a Cauchy sequence is unique, $p_n \to p$. Hence, $T(\Sigma)$ is a complete metric space. □

Now, it will be seen that the Teichmüller space is independent of the base point. Let $[\Sigma', f_1] \in T(\Sigma)$ be a arbitrary point. Define

$$[f_1]_*([\Sigma'', f]) = [\Sigma'', f \circ f_1^{-1}],\ [\Sigma'', f] \in T(\Sigma),$$

then we have a mapping of $T(\Sigma)$ onto $T(\Sigma')$ with base point $[\Sigma', id]$ and we have the following result.

Proposition 2.3.2. The map $[f_1]_* : T(\Sigma) \to T(\Sigma')$ is an isometric homeomorphism with respect to the Teichmüller distance. In particular $T(\Sigma)$ is homeomorphic to $T(\Sigma')$.

Proof. Since $[f_1^{-1}]_* : T(\Sigma') \to T(\Sigma)$ gives the inverse map, then $[f_1]_*$ is a bijection. Now, for any two points $p = [\Sigma^n, f]$ and $q = [\Sigma^m, g]$ in $T(\Sigma)$, the set $\mathcal{F}_{f,g}$ coincides with $\mathcal{F}_{f\circ f_1^{-1}, g\circ f_1^{-1}}$, because $f \circ g^{-1} = (f \circ f_1^{-1}) \circ (f_1 \circ g^{-1}) = (f \circ f_1^{-1}) \circ (g \circ f_1^{-1})^{-1}$. Then

$$d(p, q) = d([\Sigma^n, f \circ f_1^{-1}], [\Sigma^m, g \circ f_1^{-1}]).$$

Therefore $[f_1]_*$ is an isometry. □

2.4 $T(\Sigma_g)$ as a Holomorphic Quadratic Differential Space

In this section, it will be identified the Teichmüller space with a quadratic differential space. Consider $k \in \mathbb{R}$ such that $0 < k < 1$ and $\varphi \in A_2(\Sigma) - \{0\}$. It will be said that a quasiconformal mapping f is a *formal Teichmüller mapping* of Σ for the pair (k, φ) if the Beltrami differential μ_f of f is equal to $k\overline{\varphi}/|\varphi|$. It is considered that the conformal mappings are formal Teichmüller mapping corresponding to the case $k = 0$ or $\varphi = 0$.

Denote by $A_2(\Sigma)_1$ the unit ball of holomorphic quadratic differential space, that is to say,

$$A_2(\Sigma)_1 = \{\varphi \in A_2(\Sigma) \mid ||\varphi||_1 < 1\},$$

where $\varphi = \varphi(z)dz^2,\ ||\varphi||_1 = 2\iint_\Sigma |\varphi(z)|dxdy$.

Consider $\varphi \in A_2(\Sigma)$ and suppose that $k = ||\varphi||_1$. Then we simply will call to a formal Teichmüller mapping of the pair (k, φ) a *Teichmüller mapping* for φ.

Now, let Σ_g be a closed Riemann surface of genus $g \geq 2$ with Fuchsian model Γ. Then for every $\varphi \in A_2(\Sigma)$ we have $||\varphi||_1 < \infty$ and $A_2(\Sigma)$ is a complex Banach space of dimension $3g - 3$, see 1.4.2.

Define

$$\mathcal{T} : A_2(\Sigma_g)_1 \to T(\Sigma_g)$$

$$\mathcal{T}(\varphi) = [\Sigma', f],\ \varphi \in A_2(\Sigma_g)_1$$

where $f : \Sigma_g \to \Sigma' = f(\Sigma_g)$ is a Teichmüller mapping for $\varphi \neq 0$ and $f = id$ for $\varphi = 0$.

Now, we want to show the following result:

Theorem 2.4.1. *The mapping $\mathcal{T}$ is a surjective homeomorphism. In particular, $T(\Sigma_g)$ is homeomorphic to $\mathbb{R}^{6g-6}$.*

For the proof we will show the followings lemmas and we will use the Theorem 2.4.2, whose proof can be consulted in [17, Section 5.3].

Theorem 2.4.2. *Let f be a Teichmüller mapping for an element $\varphi \in A_2(\Sigma_g)_1$ and let $\mathcal{T}(\varphi) = [\Sigma', f]$. Then every quasiconformal mapping f_1 of Σ_g to Σ' which is homotopic to f satisfies*

$$||\mu_{f_1}||_\infty \geq ||\mu_f||_\infty.$$

Moreover, the equality holds if and only if $f_1 = f$.

Lemma 2.4.3. *The mapping $\mathcal{T}$ is injective.*

Proof. Suppose the $\mathcal{T}(\varphi_1) = \mathcal{T}(\varphi_2)$ with $\varphi_1, \varphi_2 \in A_2(\Sigma_g)_1$. Consider f_j the Teichmüller mappings for φ_j and $\mathcal{T}(\varphi_j) = [\Sigma^j, f_j]$ for every j. Then there exists a conformal map $h : \Sigma^1 \to \Sigma^2$ such that $h \circ f_1 \simeq f_2$. By the Theorem 2.4.2, we have:

$$||\mu_{f_1}||_\infty = ||\mu_{h\circ f_1}||_\infty \geq ||\mu_{f_2}||_\infty.$$

Analogously, $h^{-1} \circ f_2 \simeq f_1$, then we have:

$$||\mu_{f_2}||_\infty = ||\mu_{h^{-1}\circ f_2}||_\infty \geq ||\mu_{f_1}||_\infty.$$

By the Theorem 2.4.2, we obtain that $h \circ f_1 = f_2$ and therefore $\mu_{f_1} = \mu_{f_2}$.

If $\varphi_1 = 0$, $\varphi_2 = 0$. If $\varphi_1 \neq 0$, then $||\varphi_1||_1 = ||\varphi_2||_1$ and $\varphi_1/|\varphi_1| = \varphi_2/|\varphi_1|$ a.e. on $\mathbb{R}$. Therefore φ_2/φ_1 is positive a.e. on $\mathbb{R}$. Since φ_2/φ_1 is a meromorphic function, it must be constant. Then there exists a positive constant C with $\varphi_1 = C\varphi_2$, since $||\varphi_1||_1 = ||\varphi_2||_2$, we conclude that $C = 1$, that is to say, $\varphi_1 = \varphi_2$. □

It can be shown that for a point $[\Sigma, S]$ in T_g, the Fuchsian model Γ has a canonical system of generators $\{\alpha_j, \beta_j\}_{j=1}^g$. This canonical system is written uniquely in the form

$$\alpha_j = \frac{a_j z + b_j}{c_j z + d_j},\ a_j, b_j, c_j \in \mathbb{R},\ c_j > 0,\ a_j d_j - b_j c_j = 1,$$
$$\beta_j = \frac{a'_j z + b'_j}{c'_j z + d'_j},\ a'_j, b'_j, c'_j \in \mathbb{R},\ c'_j > 0,\ a'_j d'_j - b'_j c'_j = 1,$$

for each $j = 1, 2, ..., g-1$. Now, we define the Fricke coordinates $\widetilde{\mathcal{F}}_g : T_g \to \mathbb{R}^{6g-6}$ by

$$\widetilde{\mathcal{F}}_g([\Sigma, S]) = (a_1, c_1, d_1, a'_1, c'_1, d'_1, ..., a_{g-1}, c_{g-1}, d_{g-1}, a'_{g-1}, c'_{g-1}, d'_{g-1}).$$

The image $F_g = \widetilde{\mathcal{F}}_g(T_g)$ is called the *Fricke space* of closed Riemann surfaces of genus g.

Lemma 2.4.4. *The mapping $\mathcal{T}$ is a homeomorphism onto its image.*

For the proof of Lemma 2.4.4, we will use that there exists a continuous bijection $\mathcal{F}_g$ between $T(\Sigma_g)$ and the Fricke space F_g, see [17, Lemmas 5.6 and 5.8] and $\widetilde{\mathcal{T}} := \mathcal{F}_g \circ \mathcal{T}$ is a continuous map, [17, Lemma 5.7].

Proof. First, we are going to show that $\mathcal{T}$ is continuous. Let $\varphi \in A_2(\Sigma_g)_1$ and $p = \mathcal{T}(\varphi) = [\Sigma', f_1]$ Consider the translation $[f_1]_* : T(\Sigma_g) \to T(\Sigma')$ of the base point, which maps p to $[\Sigma', Id]$. Then $[f_1]_*$ is a surjective isometry. Define

$$\mathcal{T}_1 : A_2(\Sigma') \to T(\Sigma') \text{ such that } \varphi \longmapsto [\Sigma'', f]$$

and consider $\widetilde{\mathcal{T}_1} = \mathcal{F}_g \circ [f_1]_*^{-1} \circ \mathcal{T}_1 : A_2(\Sigma')_1 \to F_g$, then we have

$$\widetilde{\mathcal{T}}(\varphi) = \mathcal{F}_g(p) = \mathcal{F}_g \circ [f_1]^{-1}([\Sigma', Id]) = \mathcal{F}_g \circ [f_1]^{-1} \circ \mathcal{T}_1(0) = \widetilde{\mathcal{T}_1}(0).$$

Therefore $(\widetilde{\mathcal{T}_1})^{-1} \circ \widetilde{\mathcal{T}} = (\widetilde{\mathcal{T}_1})^{-1} \circ [f_1]_* \circ \mathcal{T} : A_2(\Sigma)_1 \to A_2(\Sigma')_1$ is well defined in a neighborhood of φ and it is a homeomorphism onto its image. Hence, $\mathcal{T}$ is continuous at φ if and only if $\mathcal{T}_1$ is continuous at the origin. Let $\{\psi_n\}_n$ be a sequence in $A_2(\Sigma')_1$ such that $||\psi_n||_1 \to 0$. Since the maximal dilatation of Teichmüller mapping for ψ_n is equal to $(1 + ||\psi_n||_1)/(1 - ||\psi_n||_1)$ for each n, we have:

$$d(\mathcal{T}_1(0), \mathcal{T}_1(\psi_n)) \leq \log \frac{1 + ||\psi_n||_1}{1 - ||\psi_n||_1} \to 0.$$

Then, $\mathcal{T}_1(0)$ is continuous at the origin. Since φ was arbitrarily, $\mathcal{T}$ is continuous. □

Lemma 2.4.5. *The mapping $\mathcal{T}$ is surjective.*

Proof. Since $\mathcal{T}$ is injective and $\mathcal{F}_g$ is bijective, then $\widetilde{\mathcal{T}}$ is injective, also it is continuous. Since $A_2(\Sigma_g)_1$ is homeomorphic to $\mathbb{R}^{6g-6}$, then by the Invariance of domain Theorem we have that $\widetilde{\mathcal{T}}(A_2(\Sigma_g)_1)$ is an open set.

Consider $E = \mathcal{T}(A_2(\Sigma_g)_1) = (\mathcal{F}_g)^{-1}(\widetilde{\mathcal{T}}(A_2(\Sigma_g)_1))$, since $T(\Sigma_g)$ is a connected space (see [17, Lemma 5.12]) to show the surjectivity of $\mathcal{T}$ it is enough to show that $\partial E = \emptyset$. Suppose that $\partial E \neq \emptyset$ and let $[\Sigma', f]$ be a point in ∂E. Then there exists a sequence $\{\varphi_n\}_n$ in $A_2(\Sigma_g)$ such that $\mathcal{T}(\varphi_n) \to [\Sigma', f]$ and $||\varphi_n||_1 \to 1$. Let f_n be a Teichmüller mapping for φ_n and $\mathcal{T}(\varphi_n) = [\Sigma^n, f_n]$. By the hypotheses, there exists a quasiconformal mapping $h_n : \Sigma^n \to \Sigma'$ which is homotopic to $f \circ f_n^{-1}$ for every n and such that $||\mu_{h_n}||_\infty \to 0$. In particular, for some $k < 1$ we have $||\mu_{g_n}||_\infty \leq k$, where $g_n = h_n^{-1} \circ f$. On the other hand, since g_n is homotopic to f_n, by the Theorem 2.4.2, we obtain

$$||\mu_{g_n}||_\infty \geq ||\mu_{f_n}||_\infty = ||\varphi_n||_\infty \to 1.$$

This is a contradiction, therefore $\partial E = \emptyset$. □

The following theorem is a corollary of the Lemma 2.4.5.

Theorem 2.4.6. *For every quasiconformal mapping $f : \Sigma_g \to \Sigma'$ there exists a Teichmüller mapping homotopic to f.*

Since we are considering Riemann surfaces of genus $g \geq 2$, we have the following result.

Corollary 2.4.7. *The spaces $T^{\#}(\Sigma)$, $T(\Sigma)$, $T^{\#}(\Gamma)$, $T(\Gamma)$, F_g, $\mathbb{R}^{6g-6}$ are mutually homeomorphic to each other.*

2.5 $T(\Sigma_g)$ as a Derivative Space

In this section it will be considered a closed Riemann surface of genus g. Set $\mu \in Bel(\Sigma_g)$ then by the Proposition 1.1.25 there exists a canonical quasiconformal mapping w^μ of Δ. Now, we define:

$$\widetilde{\mu}(z) = \begin{cases} \mu(z) & if \quad z \in \Delta \\ 0 & if \quad z \in \mathbb{C} - \Delta. \end{cases}$$

By Theorem 1.1.24, for $\widetilde{\mu}$ there exists a quasiconformal mapping on $\hat{\mathbb{C}}$ which will be denoted by w_μ.

Remark 2.5.1. *Note that w_μ, is holomorphic on the outer of the unit disc, Δ^*.*

Lemma 2.5.2. *Let μ, ν be elements of $Bel(\Sigma_g)$, then the following statements are equivalent:*

(1) $w^\mu = w^\nu$ on S^1.

(2) $w_\mu = w_\nu$ on Δ^*.

Proof. First, we suppose that $w^\mu = w^\nu$ on $\mathbb{R}$, then we have a homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by:

$$f(z) = \begin{cases} (w^\mu)^{-1} \circ w^\nu(z) & if \quad z \in \Delta \\ z & if \quad z \in \Delta^* \cup S^1. \end{cases}$$

Since $(w^\mu)^{-1} \circ w^\nu$ is a quasiconformal mapping on $\mathbb{C}$, then it is ACL on $\mathbb{C}$ and the identity function is ACL too. Therefore f is a ACL function on $\mathbb{C}$. By the Observation 1.1.3 and since the composition of quasiconformal maps is quasiconformal, we have that the function f is a quasiconformal map. Hence, by the Observation 2.5.1 and since w_μ and w_ν coincide with w^μ and w^ν respectively on Δ, we have that $g := w_\mu \circ f \circ (w_v)^{-1}$ is a conformal mapping on $\mathbb{C}$. Namely, g is a Möbius transformation and since g fixes $0, 1, \infty$ then g is the identity function. We conclude that $w_\mu = w_\nu$ on Δ^* because f is the identity on Δ^*. Conversely, if $w_\mu = w_\nu$ on Δ^*, then $w_\mu = w_\nu$ on $\Delta^* \cup S^1$, because w_μ and w_ν are continuous and the codomain is Hausdorff, then the set where they coincide is closed. Therefore, we obtain a quasiconformal mapping $h = w^\mu \circ (w_\mu)^{-1} \circ w_\nu \circ (w^\nu)^{-1} : \Delta \to \Delta$. The mapping h is conformal on $\mathbb{C}$ and since it fixes $0, 1, \infty$, h has to be the identity function on $\mathbb{C}$. Thus $w^\nu = w^\mu \circ (\mu_\mu)^{-1} \circ w_\nu = w^\mu$ on $\mathbb{R}$. ☐

Now, we can define that w_μ and w_ν are equivalent, with $\mu, \nu \in Bel(\Sigma_g)$, if $w_\mu = w_\nu$ on Δ^*. Let $T_\beta(\Gamma)$ be the set of equivalence classes and let

$$\beta : Bel(\Sigma_g) \to T_\beta(\Gamma) \tag{2.4}$$

the projection map. By the Lemma 2.5.2, we can identify $T(\Gamma)$ with $T_\beta(\Gamma)$.

Note that analogously to the case of closed hyperbolic surfaces, for every hyperbolic surface without boundary, we can define the Teichmüller space as follow: we say that μ and ν in $Bel(\Sigma)$ are equivalent, if $w^\mu = w^\nu$ on S^1. The quotient space

$$T(\Sigma) = Bel(\Sigma)/\sim \tag{2.5}$$

with this equivalence relation is called *Ahlfors-Bers model* of the Teichmüller space of the surface Σ. By Remark 2.2.2, it coincides with the reduced Teichmüller space.

Now, we associate a derivative to each conformal map. Let f be a conformal mapping on $\mathbb{C}$, then its *Schwarzian derivative* is given by

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{2.6}$$

Lemma 2.5.3. *If f and g are conformal maps of D and $f(D)$, respectively, then*

$$\{g \circ f, z\} = \{g, f(z)\} \cdot f'(z)^2 + \{f, z\},\ \ z \in D.$$

Moreover, a conformal map of D is a Möbius transformation if and only if $\{f, z\} = 0$ on D.

Proof. By a direct calculation we have:
$(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$,
$(g \circ f)''(z) = g''(f(z)) \cdot f'(z)^2 + g'(f(z)) \cdot f''(z)$,
$(g \circ f)'''(z) = g'''(f(z)) \cdot f'(z)^3 + 3 \cdot g''(f(z)) \cdot f'(z) \cdot f''(z) + g'(f(z)) \cdot f'''(z)$.
Then we have

$$\begin{aligned}\frac{(g \circ f)'''(z)}{(g \circ f)'(z)} - \frac{3}{2}\left(\frac{(g \circ f)''(z)}{(g \circ f)'(z)}\right)^2 &= \\ &= \left(\frac{g'''(f(z))}{g'(f(z))} - \frac{3}{2}\left(\frac{g''(f(z)}{g'(f(z))}\right)^2\right) \cdot f'(z)^2 + \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \\ &= \{g, f(z)\} \cdot f'(z)^2 + \{f, z\}.\end{aligned}$$

Therefore $\{g \circ f, z\} = \{g, f(z)\} \cdot f'(z)^2 + \{f, z\}$.

Let γ be a Möbius transformation. Then it is of the form:

$$\gamma(z) = \frac{a \cdot z + b}{c \cdot z + d} \quad \text{with } a, b, c, d \in \mathbb{C} \quad \text{and} \quad ad - bc = 1.$$

Taking the derivative, we have:

$$\gamma'(z) = \frac{1}{(c \cdot z + d)^2} \quad \text{and} \quad \gamma''(z) = \frac{-2c}{(c \cdot z + d)^3}.$$

Then, we obtain $\frac{\gamma'''(z)}{\gamma'(z)} - \frac{3}{2}\left(\frac{\gamma''(z)}{\gamma'(z)}\right)^2 = 0$ and therefore $\{\gamma, z\} = 0$.

Now, suppose that f is a conformal mapping such that $\{f, z\} = 0$. Note that

$$\{f, z\} = (\log(f'(z)))'' - \frac{1}{2}\{(\log f'(z))'\}^2.$$

Then we have the differential equation $y'' - \frac{1}{2}(y')^2$ where $y = \log f'(z)$. Solving this equation we have that f is of the form $\frac{C}{z+C_1}$, with C and C_1 constants, that is to say, f is a Möbius transformation. □

For every $\mu \in Bel(\Sigma_g)$, we consider

$$\varphi_\mu(z) = \{w_\mu, z\}, \; z \in \Delta^*.$$

Proposition 2.5.4. If $\gamma \in \Gamma$, then

$$\varphi_\mu(\gamma(z)) \cdot \gamma'(z)^2 = \varphi_\mu(z), \; z \in \Delta^*.$$

That is to say, φ_μ is regarded as a holomorphic quadratic differential on a Riemann surface Δ^*/Γ. Moreover, for any two elements $\mu, \nu \in Bel(\Sigma_g)$, $[w_\mu] = [w_\nu]$ in $T_\beta(\Gamma)$ if and only if $\varphi_\mu = \varphi_\nu$ on Δ^*.

Proof. Let γ be an element of Γ. Since $\mu \in Bel(\Sigma_g)$, then $\gamma_\mu = w_\mu \circ \gamma \circ (w_\mu)^{-1}$ is Möbius transformation. Taking the Schwarzian derivative on both sides of $w_\mu \circ \gamma = \gamma_\mu \circ w_\mu$ and considering the Lemma 2.5.3 we obtain:

$$\{w_\mu \circ \gamma, z\} = \{w_\mu, \gamma(z)\} \cdot \gamma'(z)^2 + \{\gamma, z\} = \{w_\mu, \gamma(z)\} \cdot \gamma'(z)^2,$$
$$\{\gamma_\mu \circ w_\mu, z\} = \{\gamma_\mu, w_\mu(z)\} \cdot w_\mu'(z)^2 + \{w_\mu, z\} = \{w_\mu, z\}.$$

Therefore, $\{w_\mu, \gamma(z)\} \cdot \gamma'(z)^2 = \{w_\mu, z\}$ on Δ^*. Since $\varphi_\mu(z) = \{w_\mu, z\}$, we have

$$\varphi_\mu(\gamma(z)) \cdot \gamma'(z)^2 = \varphi_\mu(z).$$

If $[w_\mu] = [w_\nu]$ in $T_\beta(\Gamma)$, then by definition of the equivalence relation we have $w_\mu = w_\nu$ on Δ^*. This implies that $\varphi_\mu = \varphi_\nu$ on Δ^*. Now, suppose that $\varphi_\mu = \varphi_\nu$ on Δ^*. Consider the map $F : w_\mu(\Delta^*) \to w_\nu(\Delta^*)$ defined by $F = w_\nu \circ (w_\mu)^{-1}$. F is a conformal map because w_μ is a conformal map on Δ^* and thus w_μ^{-1} so is. By Lemma 2.5.3, we have:

$$\varphi_\nu(z) = \{F \circ w_\mu, z\} = \{F, w_\mu(z)\} \cdot w_\mu'(z)^2 + \varphi_\mu(z), \text{ on } \Delta^*.$$

By the hypotheses, we conclude that $\{F, z\} = 0$ on $w_\mu(\Delta^*)$. Again, by the Lemma 2.5.3, F is a Möbius transformation. Since F fixes $0, 1, \infty$, F is the identity. Therefore $w_\mu = w_\nu$ on Δ^*, that is to say, $[w_\mu] = [w_\nu]$ in $T_\beta(\Gamma)$. □

An holomorphic automorphic form of weight -4 on Δ^* with respect to Γ is a holomorphic function φ on Δ^* such that

$$\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z), \; z \in \Delta^*, \gamma \in \Gamma. \tag{2.7}$$

The space of all holomorphic automorphic forms will be denoted by $A_2(\Delta^*,\Gamma)$ and it is identified with $A_2(\Delta^*/\Gamma)$. By Riemann-Roch Theorem 1.4.1, $A_2(\Delta^*,\Gamma)$ is a $(3g-3)$-dimensional complex vector space.

Now, we define the following map:

$$B : T_\beta(\Gamma) \to A_2(\Delta^*,\Gamma)$$

$$[w_\mu] \longmapsto \varphi_\mu,$$

where φ_μ is the Schwarzian derivative. By the Proposition 2.5.4, the map B is well defined and is injective. It is called *Bers' embedding*. The map $\Phi : Bel(\Sigma_g) \to A_2(\Delta^*,\Gamma)$ defined by $\Phi(\mu) = B \circ \beta(\mu)$ is called *Bers' projection*.

It can be show that B and Φ are continuous maps, (see [17, Proposition 6.5]). Then, we have the following Remark.

Remark 2.5.5. *Notice that the Teichmüller space of a Riemann surface Σ_g can be defined as the set*

$$\{\varphi_\mu \mid \mu \in Bel(\Sigma_g)\}.$$

Now, we define a metric in $A_2(\Delta^*,\Gamma)$ as follow. Consider the Poincaré metric on Δ^* given by

$$ds^2 = \lambda(z)^2|dz|^2, \tag{2.8}$$

where $\lambda(z) = \frac{|z|^2}{|z|^2-1}$. This metric is invariant under transformation in $PSL(2,\mathbb{R})$, that is to say,

$$\lambda(\gamma(z))^2 \cdot |\gamma'(z)|^2 = \lambda(z)^2, \quad z \in \Delta.$$

By 2.7, we have

$$|\gamma'(z)|^2|\varphi(\gamma(z))| = |\varphi(z)|$$

and by the invariance of ds^2 under $PSL(2,\mathbb{R})$, we have

$$(\lambda(\gamma(z))^2)^{-1}|\varphi(\gamma(z))| = (\lambda(z)^2)^{-1}|\varphi(z)|.$$

Therefore $(\lambda(z)^2)^{-1}|\varphi(z)|$ can be consider as a function on $\Sigma^* = \Delta^*/\Gamma$. Then, the L_∞-norm in $A_2(\Delta^*,\Gamma)$ is defined by

$$||\varphi||_\infty = \sup_{z\in\Delta^*}(\lambda(z)^2)^{-1}|\varphi(z)|.$$

Since we are considering closed Riemann surfaces, we can take the supremum over only a relatively compact fundamental domain, then $||\varphi||_\infty$ is finite for every $\varphi \in A_2(\Delta^*,\Gamma)$, and therefore $A_2(\Delta^*,\Gamma)$ is a complex Banach space. Also, we have the following theorem, (its proof can be consulted in [17, Theorem 6.6]).

Theorem 2.5.6. *The Teichmüller space $T_B(\Gamma)$ is contained in the open ball in $A_2(\Delta^*,\Gamma)$ with center 0 and radius $3/2$.*

Now, we want to associate a measurable function μ_φ satisfying (1.18) to each $\varphi \in A_2(\Delta^*, \Gamma)$. For this, consider φ an element of $A_2(\Delta^*, \Gamma)$ and define $\psi \in A_2(\Delta, \Gamma)$ as

$$\psi(z) = (\overline{\varphi(\bar{z}^{-1})})^{-1}, \text{ for all } z \in \Delta.$$

Consider the Poincaré metric $\lambda(1/\bar{z})^2|dz|^2$ with $z \in \Delta$, then

$$\psi(z)dz^2/ds^2 = \lambda(\bar{z}^{-1})^2\overline{\varphi(\bar{z}^{-1})}^{-1}d\bar{z}/dz$$

is a Beltrami differential on Δ/Γ because, by the invariance of the Poincaré metric we have

$$\psi(z)dz^2/ds^2 = \lambda(\gamma(\bar{z}^{-1}))|\gamma'(\bar{z}^{-1})|^2\overline{\varphi(\bar{z}^{-1})}^{-1}d\bar{z}/dz,$$

and by the condition 2.7, we obtain

$$\psi(z)dz^2/ds^2 = \lambda(\gamma(\bar{z}^{-1}))\overline{\varphi(\bar{z}^{-1})}^{-1}\frac{\gamma'(\bar{z}^{-1})}{\overline{\gamma'(\bar{z}^{-1})}^{-1}}\frac{d\bar{z}}{dz}.$$

Then, we define $\mu_\varphi(z) = \lambda(\gamma(\bar{z}^{-1}))|\gamma'(\bar{z}^{-1})|^2\overline{\varphi(\bar{z}^{-1})}^{-1}d\bar{z}/dz$. This Beltrami differential is called a harmonic Beltrami differential. Set $V = \{\varphi \in A_2(\Delta^*, \Gamma) \mid ||\varphi||_\infty < 1/2\}$, then μ_φ with $\varphi \in V$ belongs to $Bel(\Sigma)$.

Now, we can define a continuous mapping $\Psi : V \to T_\beta(\Gamma)$ by $\Psi(\varphi) = [w_{\mu_\varphi}]$. Then we have the Ahlfors and Weill Theorem (see [17, Theorem 6.9]).

Theorem 2.5.7. *For any element $\varphi \in A_2(\Delta^*, \Gamma)$ with $||\varphi||_\infty < 1/2$, the harmonic Beltrami differential μ_φ constructed from φ satisfies $B([w_{\mu_\varphi}]) = \varphi$.*

Corollary 2.5.8. *For any $\varphi \in V$, there exists $\mu \in Bel(\Sigma_g)$ such that w_μ is real-analytic on Δ and $B^{-1}(\varphi) = [w_\mu]$.*

Moreover, each point $[\Sigma', f]$ of the Teichmüller space $T(\Sigma_g)$ of $\Sigma_g = \Delta/\Gamma$ is represented by a real-analytic quasiconformal mapping g of Σ onto Σ'.

By the Theorem 2.5.7, we have that the Teichmüller space of a compact surface $\Sigma_g = \Delta/\Gamma$ can be identified with the open ball in $A_2(\Delta^*, \Gamma)$ with center 0 and radius 3/2. That is to say, the Teichmüller space of a Riemann surface of genus g is a $3g-3$-dimensional complete space.

2.6 Fenchel-Nielsen Coordinates

Let's fix a Riemann surface Σ. It will be said that Σ is of (topological) *finite type* if its fundamental group is finitely generated. Otherwise the surface is said of *infinite type* and in this case its fundamental group is free with a countable number of generators.

Remark 2.6.1. *A surface of finite type need not be of finite conformal type (see definition 1.2.2); for instance, the open unit disk is of (topological) finite type, but it is not a surface of finite conformal type. A surface of infinite type is also of infinite conformal type.*

A *pair of pants* is a surface whose interior is homeomorphic to a sphere with three distinct points deleted and whose boundary is a (possibly empty) disjoint union of circles, (see figure 2.1).

Figure 2.1: Examples of pair of pants.

Remark 2.6.2. *In the definition of pair of pants, the boundary is considered in the topological sense. We call pair of pants without boundary to a pair of pants such that it does not intersect to its topological boundary.*

Let Σ be a Riemann surface without boundary. A (topological) *pair of pants decomposition* of Σ is a family of pairwise disjoint simple closed curves $\mathcal{C} = \{C_i\}_{i\in I}$ in Σ, such that

- $\Sigma\setminus \bigcup_{i\in I} C_i$ is a disjoint union of pairs of pants without boundary;
- it is possible to find a family of pairwise disjoint tubular neighborhoods of these curves C_i in Σ.

Example 1. Consider the closed Riemann surface Σ_2 of genus 2. Then two pair of pants decomposition of Σ_2 had been represented in Figure 2.2. Thus the pair of pants decomposition is not unique.

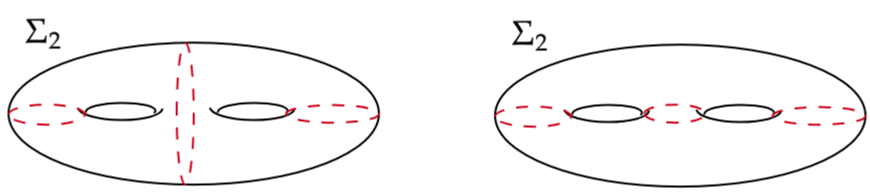


Figure 2.2: Example of a pair of pants decomposition of the surface Σ_2.

Definition 2.6.3. A *generalized hyperbolic pair of pants* is a hyperbolic sphere with three geometric holes, where a geometric hole is either a geodesic boundary component or a puncture whose neighborhood is a cusp.

The different generalized hyperbolic pair of pants had been represented in Figure 2.3.

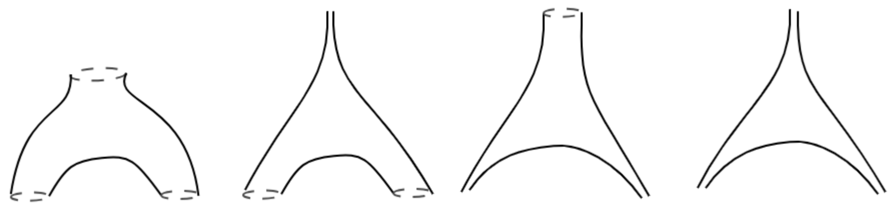


Figure 2.3: Representation of generalized hyperbolic pair of pants.

We call a decomposition of a hyperbolic surface into generalized hyperbolic pair of pants glued along their boundary components a *geometric pair of pants decomposition.*

Given a geometric pair of pants decomposition of a hyperbolic surface Σ consider the boundary components simple closed geodesics

$$\mathcal{G} = \{\gamma_i \mid i \in \Lambda\}$$

of the respective pair of pants. We will identify a geometric pair of pants decomposition with the family of curves $\mathcal{G}$. Every geodesic γ_i has associates two parameters. These are length l_i and twist θ_i coordinates respectively in $\mathbb{R}^+$ and $\mathbb{R}$, and we define them as follow.

Let H_0 be a hyperbolic structure on Σ, the length parameter of each $\gamma_i \in \mathcal{G}$ is the length of the geodesic γ_i with respect to the hyperbolic structure H_0 and it is denoted by $l(\gamma_i)$. If γ_i is not a boundary curve of Σ, the twist parameter of γ_i is defined as the relative twist amount along the geodesic between the two generalized pairs of pants that have this geodesic in common and it is denoted by $\theta(\gamma_i)$.

The collection of these coordinates $(l(\gamma_i), \theta(\gamma_i))_{i\in\mathcal{G}_i}$ will be called *Fenchel-Nielsen coordinates* of the geometric decomposition $\mathcal{G}$. Also, if H_0 and H_0' are given hyperbolic structures we say that they are Fenchel-Nielsen equivalent relative to the pair of pants decomposition $\mathcal{G}$ if their Fenchel-Nielsen parameters are equal.

The Fenchel-Nielsen distance with respect to $\mathcal{G}$ between two hyperbolic metrics H_0 and H_0' is defined by:

$$d_{FN}(H_0, H_0') = \sup_{i=1,2,\ldots} \max\left(\left|\log\frac{l_{H_0}(\gamma_i)}{l_{H_0'}(\gamma_i)}\right|, \left|l_{H_0}(\gamma_i)\theta_{H_0}(\gamma_i) - l_{H_0'}(\gamma_i)\theta_{H_0'}(\gamma_i)\right|\right).$$

If γ_i is in the homotopy class of a boundary component of Σ, then there is not twist parameter to be considered. Also, the distance d_{FN} depends on the geometric pair of pants decomposition $\mathcal{G}$.

A homeomorphism $f : (\Sigma, H_0) \to (\Sigma, H_0')$ that is isotopic to the identity is *Fenchel-Nielsen bounded* if $d_{FN}(H_0, H_0')$ is finite.

Now, consider a Riemann surface Σ. We fix a hyperbolic structure H_0 on Σ and let $\mathcal{C}$ be a (topological) pair of pants decomposition on Σ. Let $\mathcal{G}$ be a geometric pair of pants decomposition isotopic to $\mathcal{C}$. Consider the collection of marked hyperbolic structures (f, H) relative to H_0, with the property that the marking $f : H_0 \to H$ is Fenchel-Nielsen bounded with respect $\mathcal{G}$.

We say that two hyperbolic structures (f, H) and (f', H') are equivalent if there exists an isometry $f'' : H \to H'$ which is homotopic to $f' \circ f^{-1}$.

Definition 2.6.4. Let Σ be a Riemann surface and let $\mathcal{C}$ be a (topological) pair of pants decomposition on Σ. Let $\mathcal{G}$ be a geometric pair of pants decomposition isotopic to $\mathcal{C}$. The *Fenchel-Nielsen Teichmüller space* with respect to $\mathcal{C}$ and H_0, denoted by $\mathcal{T}_{FN}(H_0, \mathcal{C})$, is the space of equivalence classes $[f, H]$ of Fenchel-Nielsen bounded marked hyperbolic structures (f, H), where $f : H_0 \to H$ is Fenchel-Nielsen bounded with respect $\mathcal{G}$.

The function d_{FN} is a distance function on $\mathcal{T}_{FN}(H_0)$ and the base point in $\mathcal{T}_{FN}(H_0)$ is $[id, H_0]$.

The map defined as follow

$$\mathcal{T}_{FN}(H_0, \mathcal{C}) \to l_\infty(\mathcal{C}), \quad [f, H] \longmapsto (\log l(\gamma), l(\gamma)\theta(\gamma))_{\gamma \in \mathcal{G}} \tag{2.9}$$

defines a isometric homeomorphism.

Definition 2.6.5. Consider a surface Σ with hyperbolic structure H_0 and pair of pants decomposition $\mathcal{C} = \{C_i |\ i \in \Lambda\}$. The hyperbolic structure H_0 is *upper-bounded with respect to* $\mathcal{C}$ if there is a constant M such that length$(\gamma_i) \leq M$ for $i \in \Lambda$, where γ_i is the simple closed geodesic freely homotopic to C_i.

The proof of the following theorem can be consulted in [6, Theorem 8.10].

Theorem 2.6.6. *Let H_0 be a complete hyperbolic structure on Σ, and suppose that H_0 is upper-bounded with respect to some pair of pants decomposition $\mathcal{C}$. Then, the natural map*

$$j : T^{\#}(\Sigma) \to \mathcal{T}_{FN}(H_0, \mathcal{C}),\ [\Sigma', f] \longmapsto (l_i(f, H), \theta_i(f, H))_{i \in \Lambda} \tag{2.10}$$

is a locally bi-Lipschitz homeomorphism. (Here H in the image is the hyperbolic structure of Σ').

Remark 2.6.7. *If Σ is a hyperbolic surface without boundary, then the Theorem 2.6.6, gives us a locally bi-Lipschitz homeomorphism between the quasiconformal Teichmüller space $T(\Sigma)$ and the Fenchel Nielsen Teichmüller space.*

Corollary 2.6.8. *Consider a complete hyperbolic surface Σ without boundary and suppose that it is upper-bounded with respect to some pair of pants decomposition $\mathcal{G}$. Then, there is a locally bi-Lipschitz homeomorphism*

$$FN_{\mathcal{C}} : T(\Sigma) \to l_\infty(\mathcal{G}).$$

Let us show the construction of the mapping in the Corollary 2.6.8. Let $[\mu]$ be a point in $T(\Sigma)$. Considering the Beltrami equation for μ, by the Ahlfors- Bers Theorem, there is a unique quasiconformal homeomorphism f^μ of the disc whose continuous quasisymmetric extension to the boundary fixes the points $1, i, -1$.

Consider the Fuchsian model of the hyperbolic surface Σ by the Fuchsian group $\Gamma <$Möb(Δ), that is

$$\Sigma \cong \Delta/\Gamma. \tag{2.11}$$

Define the group

$$\Gamma_{[\mu]} = \Theta_{[\mu]}(\Gamma) = \{f^\mu \circ \gamma \circ (f^\mu)^{-1} | \gamma \in \Gamma\}.$$

The group Γ_μ is a Fuchsian group and defines a hyperbolic surface

$$\Sigma_{[\mu]} = \Delta/\Gamma_{[\mu]}.$$

Consider the geometric pair of pants decomposition $\mathcal{G}_{[\mu]}$ isotopic to $\mathcal{C}$ whose geodesic are with respect to the hyperbolic structure of $\Sigma_{[\mu]}$. The sequence of length and twist coordinates of $\mathcal{G}_{[\mu]}$ gives a sequence in $l_\infty(\mathcal{C})$ via 2.9.

Chapter 3

Teichmüller Theory for Laminations

In this chapter it is considered the Teichumüller space of a lamination. As it was mentioned at the introduction, in recent works, specific laminations have been considered and they have been explicitly described as function spaces. Then, it is wanted to extend the description of the Teichmüller space of more general laminations as a function space.

In [30] Sullivan defined the Teichmüller space of a Riemann surface lamination, denoted by $T(L)$, as the space of all transversally continuous conformal structures along the space of leaves up to the action of the group of quasiconformal isotopies tangent to the leaves.

Sullivan also considered the set of those objects which are continuous and locally constant in the transverse direction and he denoted it by $T_{TLC}(L)$, (see [30]). Taking this definition into account I want to give a mathematically tractable definition of the space $T_{TLC}(L)$. Then, in the first section of this chapter it is defined an Ahlfors-Bers model to the space $T_{TLC}(L)$, in the second section it is compared the lamination that is being considered with the universal solenoid and in the following sections some constructions are done to give an explicit description of this space for a special class of hyperbolic surface laminations.

Throughout this chapter it is considered a complete hyperbolic surface without boundary Σ and L a minimal lamination fibering over Σ with Hausdorff compact fiber F. Also it will be considered the right holonomy action on the fiber

$$\mathrm{Hol} : G \to \mathrm{Homeo}(F)^{op}, \quad G = \pi_1(\Sigma). \tag{3.1}$$

Remark 3.0.1. *It will be considered some results in [6] and this work is in the context of reduced Teichmüller spaces. Since our surfaces do not have boundary by hypothesis, the reduced and non reduced Teichmüller spaces coincide and no distinction is made (see Remark 2.2.2).*

3.1 Ahlfors-Bers Model for Laminations

As in the case of surfaces, it is wanted to define a Ahlfors-Bers model to a lamination, then it is given the following definition of Beltrami differentials on the lamination.

Definition 3.1.1. A Beltrami differential on the lamination L is a continuous function

$$\mu : F \to Bel(\Delta)$$

invariant under the diagonal action 1.22, that is to say, for every k in the fiber F and every g in the group G we have the equivariance

$$l_g^*(\mu(k)) = \mu(k \cdot g). \tag{3.2}$$

The space of Beltrami differentials on the lamination will be denoted by $Bel(L)$ and we have by definition

$$Bel(L) = C(F, Bel(\Delta))_{eq(G)}. \tag{3.3}$$

Given a Beltrami differential μ, by definition, $\mu(k)$ is a Beltrami differential on the disc for every $k \in F$. Then by the Ahlfors-Bers Theorem, there exists a unique quasiconformal mapping $f^{\mu(k)}$ on the disc which is a solution of the respective Beltrami equation fixing the boundary points $1, i, -1$ such that these maps vary continuously along the fiber, (Theorem 1.1.28). The normalization condition is well defined since these maps uniquely extend as quasiconformal maps on the boundary of the disc (see the discussion after the existence Theorem 1.1.24). We have a continuous function

$$f^\mu : F \to QC(\Delta),\ \ k \mapsto f^{\mu(k)},$$

where $QC(\Delta)$ is the space of quasiconformal maps on the disc.

It also is defined on $Bel(L)$ an equivalence relation:

Definition 3.1.2. Let μ, ν Beltrami differentials on the lamination L. We will say that they are Teichmüller equivalent if for every $k \in F$, $\mu(k)$ and $\nu(k)$ are equivalents as Beltrami differentials on the disc

$$\mu \sim \eta \quad if \quad \mu(k) \sim \eta(k), \quad \forall k \in F.$$

Therefore by the equivalence relation between Beltrami differentials on the disc 2.5, we have:

$$\mu(k) \sim \eta(k) \Longleftrightarrow f^{\mu(k)}|_{S^1} = f^{\eta(k)}|_{S^1}.$$

Then, analogously to the case of Riemann surfaces, the Teichmüller space of the lamination L can be defined as the quotient space

$$T(L) = Bel(L)/\sim. \tag{3.4}$$

Let $\mathcal{B} : Bel(L) \to T(L)$ the quotient map, it will be called *laminated Bers map*. The transversally locally constant Beltrami differential space on the lamination L is by definition the space of locally constant continuous functions verifying 3.2, that is to say

$$Bel_{TLC}(L) = C_{LC}(F, Bel(\Delta))_{eq(G)} \tag{3.5}$$

and the transversally locally constant Teichmüller space is defined as

$$T_{TLC}(L) = \mathcal{B}(Bel_{TLC}(L)).$$

Henceforth, laminations satisfying the hypotheses 1.3.4 are considered and surfaces which satisfy the following hypotheses. In the next section we will see the relation of the laminations satisfying this hypotheses with the universal solenoid.

Hypotheses 3.1.3. Σ is a hyperbolic surface without boundary obtained by gluing a sequence of generalized hyperbolic pair of pants, each glued to the next along a common boundary geodesic such that the length of these geodesic boundaries is uniformly upper bounded.

3.2 Universal solenoid

Before beginning the construction to give the explicit description of the space $T_{TLC}(L)$, let me describe the relation of the laminations that are being considered (see hypotheses 1.3.4) with the universal solenoid. For a thorough description of the universal solenoid see [26].

Consider an arbitrary marked compact hyperbolic surface (Σ', p) where p is a point in Σ'. Define the universal solenoid Σ_∞ as the inverse limit of the inverse system of finite marked coverings of (Σ', p). There is a canonical projection from the inverse limit construction

$$\pi'_\infty : \Sigma_\infty \to \Sigma'$$

which is a locally trivial fibration.

If (Σ'', q) is another marked compact hyperbolic surface, then there is a cofinal subsystem for both surfaces hence the inverse limit are isomorphic, that is the universal solenoid Σ_∞ is independent of the original chosen marked compact hyperbolic surface and fibers over any one of them.

Every finite covering of a marked surface (Σ', p) corresponds to a subgroup of $\pi_1(\Sigma', p)$. Analogous to Proposition 1.3.5, it can be proved that

$$\Sigma_\infty \cong (\widehat{G} \times \Delta)/G, \ \widehat{G} = \widehat{\pi_1(\Sigma', p)}. \tag{3.6}$$

Under this description, the projection onto Σ' is given by $\pi_\infty([(h, k)]) = G \cdot k$ where we have identified G-orbits in Δ with points in Σ', recall expression 2.11.

Because the right action of G on $\widehat{G}$ is faithful, the leaves of Σ_∞ are densely immersed discs and it fibers over Σ' with fiber $\widehat{G}$.

Recall that a laminated map sends leaves to leaves, that is it is compatible with the leaf structure of the laminations.

Proposition 3.2.1. Let Σ be a surface and let L be a lamination fibering over Σ which satisfy the hypotheses 3.1.3 and 1.3.4. Suppose that Σ is compact. Then, there is a continuous laminated surjective map

$$\phi : \Sigma_\infty \to L$$

compatible with the fibrations, that is the following diagram commutes

$$\begin{array}{ccc} \Sigma_\infty & \xrightarrow{\ \phi\ } & L. \\ & \searrow^{\pi_\infty} \quad {}^{\pi}\swarrow & \\ & \Sigma & \end{array} \tag{3.7}$$

Proof. Denote by F the fiber of the fibration L. Consider $x \in F$ and define the map $\rho_x : \widehat{G} \to F$ by $\rho_x(g) = x \cdot g$. By definition, this map is right $\widehat{G}$-equivariant. By hypothesis, ρ_x is continuous and by the minimality of the lamination, $\rho_x(G) = x \cdot G$ is dense in F. Therefore, the image of ρ is closed since $\widehat{G}$ is compact and contains a dense subset hence it is surjective. Define the map

$$\widehat{\phi} : \widehat{G} \times \Delta \to F \times \Delta, \quad \widehat{\phi}(h, k) = (\rho_x(h), k).$$

This map is $\widehat{G}$-equivariant by the diagonal action. Indeed,

$$\widehat{\phi}(g \cdot (h, k)) = \widehat{\phi}((h \cdot g^{-1}, g \cdot k)) = (\rho_x(h \cdot g^{-1}), g \cdot k) = (\rho_x(h) \cdot g^{-1}, g \cdot k) = g \cdot \widehat{\phi}(h, k).$$

By the description 3.6 of the universal solenoid, $\widehat{\phi}$ defines a map

$$\phi : \Sigma_\infty \to L.$$

By construction this is a continuous laminated surjective map and makes the diagram 3.7 commute since the classes $[(h, k)]$ and $[(\rho_x(h), k)]$ are mapped to the same G-orbit $G \cdot k$ and this finishes the proof. □

Corollary 3.2.2. *There is an isometric embedding of the Teichmüller space of the lamination into the Teichmüller space of the universal solenoid.*

Proof. Recall definition 3.3. Continuing with the proof of Proposition 3.2.1, there is a monomorphism

$$\rho_x^* : C(F, Bel(\Delta))_{eq(G)} \to C(\widehat{G}, Bel(\Delta))_{eq(G)},$$

given by $(\rho_x^* \mu)(g) = \mu(\rho_x(g)) = \mu(x \cdot g)$ for every g in $\widehat{G}$. Then ρ_x^* is a monomorphism since ρ_x is an epimorphism. Moreover, it is well defined, that is $\rho_x^* \mu$ is G-equivariant. Indeed,

$$l_s^*((\rho_x^* \mu)(g)) = l_s^*(\mu(x \cdot g)) = \mu(x \cdot gs) = (\rho_x^* \mu)(gs),$$

for every $g \in \widehat{G}$ and every $s \in G$, where it has been used the G-equivariance of μ in the second equality. Moreover,

$$\mu \sim \nu \text{ iff } \rho_x^* \nu \sim \rho_x^* \mu.$$

In effect, this follows from the chain of equivalences

$$\mu \sim \nu \Longleftrightarrow \mu(k) \sim \nu(k), \ \forall k \in F \Longleftrightarrow \mu(\rho_x(g)) \sim \nu(\rho_x(g)), \ \forall g \in \widehat{G}$$
$$\Longleftrightarrow (\rho_x^* \mu)(g) \sim (\rho_x^* \nu)(g), \ \forall g \in \widehat{G} \Longleftrightarrow \rho_x^* \nu \sim \rho_x^* \mu$$

where it has been used the definition 3.6 and the fact that ρ is surjective.

Recalling definitions of Beltrami differentials and Teichmüller space of the lamination L, it has been proved that ρ_x^* induce an inclusion of Teichmüller spaces

$$\iota : T(L) \to T(\Sigma_\infty),\ [\mu] \to [\rho_x^* \mu].$$

In the model described in definition 3.4, the Teichmüller distance reads as follows

$$d_{T(L)}([\mu], [\nu]) = \sup_{k \in F} d_{T(\Delta)}([\mu(k)], [\nu(k)]).$$

Therefore, by the surjectivity of the map ρ_x, we have that the inclusion ι is an isometry,

$$\begin{aligned} d_{T(L)}([\mu], [\nu]) &= \sup_{k \in F} d_{T(\Delta)}([\mu(k)], [\nu(k)]) \\ &= \sup_{g \in \widehat{G}} d_{T(\Delta)}([(\rho_x^* \mu)(g)], [(\rho_x^* \nu)(g)]) = d_{T(\Sigma_\infty)}([\rho_x^* \mu], [\rho_x^* \nu]). \end{aligned}$$

In particular, the inclusion ι is an isometric embedding and we have the result. □

3.3 Canonical Tower of Coverings

In this section it will be defined a tower of finite coverings of the Riemann surface Σ such that they are larger and larger until they approximate L. In this line, to ensure the convergence of the tower, it will be considered the following hypothesis.

Hypotheses 3.3.1. The holonomy action 1.21 continuously extends to the profinite completion $\widehat{G}$ of the fundamental group G.

Let $\mathcal{S}$ be the set of normal finite index subgroups of the profinite completion of the fundamental group of Σ, (see section 1.3.4). Let S be an element of $\mathcal{S}$. Then, we can consider the restriction of the extended holonomic action to the subgroup S and we have a continuous right action of the subgroup S on the fiber F with respect to the profinite topology on S,

$$\mathrm{Hol}_S : S \hookrightarrow \widehat{G} \to \mathrm{Homeo}(F)^{op}. \tag{3.8}$$

Now, we can consider the orbits generated by the action 3.8 and we define the set of S-orbits as follow

$$\mathcal{O}_S(F) = \{x \cdot S \mid x \in F\}.$$

Then, sending an element of the fiber F to its corresponding orbit we can define a canonical map ψ_S as follow:

$$\psi_S : F \to \mathcal{O}_S(F),\ k \in \psi_S(k). \tag{3.9}$$

Also, we can consider a canonical right action of $\widehat{G}$ on $\mathcal{O}_S(F)$ given by

$$(x \cdot S) \cdot g := x \cdot (Sg). \tag{3.10}$$

Let us see that this action is well defined. By definition of the action we have $(x\cdot S)\cdot g := x\cdot(Sg)$ and using that S is a normal subgroup, we have $x\cdot(Sg) = (x\cdot g)\cdot(g^{-1}Sg)$. By definition again, $(x\cdot g)\cdot(g^{-1}Sg) = (x\cdot g)\cdot S$. Therefore $(x\cdot S)\cdot g = (x\cdot g)\cdot S$.

Considering the previous calculation we have

$$\psi_S(k\cdot g) = (k\cdot g)\cdot S = (k\cdot S)\cdot g = \psi_S(k)\cdot g, \tag{3.11}$$

that is to say, the map ψ_S is right $\widehat{G}$-equivariant.

Lemma 3.3.2. *For every $S \in \mathcal{S}$, the map ψ_S is continuous and the set of S-orbits is a finite discrete set.*

Proof. Consider the final topology on $\mathcal{O}_{\mathcal{S}}(F)$ induced by the map ψ_S, that is the finest topology on $\mathcal{O}_{\mathcal{S}}(F)$ such that ψ_S is continuous. By Proposition 1.3.15 and Corollary 1.3.20, the subgroup S is compact in $\widehat{G}$ hence, by hypothesis 3.3.1 and the fact that every point in F is compact, every S-orbit in F is compact. Since the fiber is a Hausdorff space, the final topology on the set of S-orbits is T_1, that is points in $\mathcal{O}_S(F)$ are closed. Since the holonomy action is transitive, we have that if $x\cdot S$ and $y\cdot S$ are S-orbits, then there exists an element $h \in F$ such that $x\cdot h = y$ and therefore $(x\cdot S)\cdot h = (x\cdot h)\cdot S = y\cdot S$. Then the action 3.10 of $\widehat{G}$ on $\mathcal{O}_S(F)$ is transitive. Also the number of elements of $\mathcal{O}_S(F)$ is at most the index of S because the action of $\widehat{G}$ on $\mathcal{O}_S(F)$ factors through the quotient group $\widehat{G}/S$. Since S has finite index, $\mathcal{O}_S(F)$ is a finite set with the discrete topology. ◻

Now, consider normal finite index subgroups S' and S'' of $\widehat{G}$ such that

$$S'' \leq S' \leq S \leq \widehat{G}.$$

We define the map $\psi_{S'S} : \mathcal{O}_{S'}(F) \to \mathcal{O}_S(F)$ such that each S'-orbit is sending to the S-orbit that contains it, that is to say,

$$\psi_{S'S}(x\cdot S') = x\cdot S,\ \forall x\cdot S' \in \mathcal{O}_{S'}(F).$$

The map $\psi_{S'S}$ is right $\widehat{G}$-equivariant because

$$\psi_{S'S}((x\cdot S')\cdot g) = \psi_{S'S}((x\cdot g)\cdot S') = (x\cdot g)\cdot S = (x\cdot S)\cdot g = \psi_{S'S}(x\cdot S)\cdot g.$$

Then we can construct the following commutative diagrams of continuous and right $\widehat{G}$-equivariant maps,

$$\begin{array}{ccc} F & \xrightarrow{\psi_{S'}} & \mathcal{O}_{S'}(F) \\ & {\scriptstyle\psi_S}\searrow & \downarrow{\scriptstyle\psi_{S'S}} \\ & & \mathcal{O}_S(F) \end{array} \qquad \begin{array}{ccc} \mathcal{O}_{S''}(F) & \xrightarrow{\psi_{S''S'}} & \mathcal{O}_{S'}(F) \\ & {\scriptstyle\psi_{S''S}}\searrow & \downarrow{\scriptstyle\psi_{S'S}} \\ & & \mathcal{O}_S(F) \end{array} \tag{3.12}$$

These maps define a continuous right $\widehat{G}$-equivariant map

$$\psi : F \to \varprojlim_{S\in\mathcal{S}} \mathcal{O}_S(F),\ \ \psi(k) = (\psi_S(k))_{S\in\mathcal{S}},$$

with the inverse limit topology on the inverse limit.

Lemma 3.3.3. *The map ψ is a homeomorphism. In particular, the fiber F is either a finite set or a Cantor set.*

Proof. Since $\mathcal{O}_S(F)$ is finite for every $S \in \mathcal{S}$, by the Lemma 3.3.2, the inverse limit is a Hausdorff, compact and totally disconnected space. Because the fiber F is a compact space, $\varprojlim_{S\in\mathcal{S}} \mathcal{O}_S(F)$ is a Hausdorff space and ψ is a continuous function, it is enough to prove that the map is bijective.

Let's see the map ψ is injective. Suppose that $\psi(K) = \psi(k')$. Then for every $S \in \mathcal{S}$, $\psi_S(k) = \psi_S(k')$, that is to say $k \cdot S = k' \cdot S$. Because $\widehat{G}$ is residually finite, (see Corollary 1.3.19), the intersection of all elements of $\mathcal{S}$ is trivial, then

$$\{k\} = k \cdot \bigcap_{S\in\mathcal{S}} S = \bigcap_{S\in\mathcal{S}} k \cdot S = \bigcap_{S\in\mathcal{S}} k' \cdot S = k' \cdot \bigcap_{S\in\mathcal{S}} S = \{k'\}.$$

Finally, let's see that ψ is surjective. Let $(m_S)_{S\in\mathcal{S}}$ be a sequence in the inverse limit. Because F is compact and Hausdorff, $(m_S)_{S\in\mathcal{S}}$ is a nested sequence of compact sets in F hence it has non empty intersection with some element $k \in F$. By the definition of the maps ψ_S and ψ we have $\psi(k) = (m_S)_{S\in\mathcal{S}}$. We conclude that ψ is surjective and therefore a homeomorphism. □

For every $S \in \mathcal{S}$ define the hyperbolic surface

$$\Sigma_S = (\mathcal{O}_S(F) \times \Delta)/G,\ g \cdot (m, a) = (m \cdot g^{-1}, g \cdot a). \tag{3.13}$$

On the other hand, for each $S \in \mathcal{S}$ we have an action of $S \cap G$ on Δ given by restriction of the action of $\widehat{G}$ and we can consider the Riemann surface $\Delta/(S \cap G)$.

Lemma 3.3.4. *For every $S \in \mathcal{S}$, there is an isometric homeomorphism*

$$\Sigma_S = \Delta/(S \cap G). \tag{3.14}$$

Proof. Consider an S-orbit $m \in \mathcal{O}_S(F)$. Let's see that up to this choice, the morphism is canonical. Define the function

$$\alpha : \Delta \to \mathcal{O}_S(F) \times \Delta,\ \alpha(k) = (m, k).$$

Since the space $\mathcal{O}_S(F)$ is discrete, by Lemma 3.3.2, we have that α is an isometrically embedding of the disk and for every $s \in S \cap G$ we have $\alpha(s \cdot k) = (m, s \cdot k)$. Also, m is an S-orbit, then $(m, s \cdot k) = (m \cdot s^{-1}, s \cdot k)$ and by definition of the action 3.13, $(m \cdot s^{-1}, s \cdot k) = s \cdot (m, k)$. Therefore $\alpha(s \cdot k) = s \cdot \alpha(k)$. In particular, α induce an isometric morphism

$$\hat{\alpha} : \Delta/(S \cap G) \to \Sigma_S,\ (S \cap G) \cdot k \mapsto G \cdot (m, k).$$

Because it is an isometry, it rest to show that $\hat{\alpha}$ has an inverse. Let $G \cdot (m_1, k_1)$ be an arbitrary G-orbit. Since the holonomy action is transitive, there exists $g \in G$ such that $m_1 = m \cdot g$, where m is the S-orbit fixed to the beginning of the proof. Then,

$$G \cdot (m_1, k_1) = G \cdot (m \cdot g, k_1) = G \cdot (m, g \cdot k_1) = G \cdot (m, k)$$

where we have defined $k = g \cdot k_1$. Restricting this orbit to the subgroup $S \cap G$ gives the orbit $(S \cap G) \cdot (m, k) = \{m\} \times (S \cap G) \cdot k$ and forgetting the first coordinate gives the orbit $(S \cap G) \cdot k$. Then we can define the map

$$\Sigma_S \to \Delta/(S \cap G),\ G \cdot (m, k) \mapsto (S \cap G) \cdot k$$

and this is an inverse of the map $\hat{\alpha}$. This conclude the proof. □

For every $S' \in \mathcal{S}$ such that $S' \leq S$ define

$$\hat{\psi}_{S'S} : \Sigma_{S'} \to \Sigma_S,\ \hat{\psi}_{S'S}\left[(m, a)\right] = \left[(\psi_{S'S}(m), a)\right]. \tag{3.15}$$

Since the map $\psi_{S'S}$ is right G-equivariant then $\psi_{S'S}(m \cdot g^{-1}) = \psi_{S'S}(m) \cdot g^{-1}$. Therefore $\hat{\psi}_{S'S}$ is well defined, because we have

$$\hat{\psi}_{S'S}\left[g \cdot (m, a)\right] = \left[(\psi_{S'S}(m \cdot g^{-1}), g \cdot a)\right] = \left[(\psi_{S'S}(m) \cdot g^{-1}, g \cdot a)\right] = \left[g \cdot (\psi_{S'S}(m), a)\right].$$

The map $\hat{\psi}_{S'S}$ is a locally isometric finite covering. Remembering the isometry 1.22, we also can define a map

$$\hat{\psi}_S : L \to \Sigma_S,\ \ [(m, a)] \mapsto [(\psi_S(m), a)] \tag{3.16}$$

This map is also a locally isometric covering. Now, consider S' and S'' in $\mathcal{S}$ such that

$$S'' \leq S' \leq S \leq \widehat{G}.$$

Since the maps $\psi_{S'S}$ define commutative diagrams 3.12, we have the following commutative diagrams of locally isometric coverings

$$\begin{array}{ccc} L & \xrightarrow{\hat{\psi}_{S'}} & \Sigma_{S'} \\ & \searrow^{\hat{\psi}_S} & \downarrow \hat{\psi}_{S'S} \\ & & \Sigma_S \end{array} \qquad \begin{array}{ccc} \Sigma_{S''} & \xrightarrow{\hat{\psi}_{S''S'}} & \Sigma_{S'} \\ & \searrow^{\hat{\psi}_{S''S}} & \downarrow \hat{\psi}_{S'S} \\ & & \Sigma_S \end{array} \tag{3.17}$$

Considering the maps $\hat{\psi}_{S'S}$ as bonding maps, $\left(\hat{\psi}_{S'S}\right)_{S' \leq S}$ defines an inverse system.

Definition 3.3.5. The inverse system $\left(\hat{\psi}_{S'S}\right)_{S' \leq S}$ with S and S' running over all the elements of $\mathcal{S}$ is the *canonical tower*.

Proposition 3.3.6. The following diagram commutes

$$\begin{array}{ccc} L & \xrightarrow{\hat{\psi}, \cong} & \varprojlim_{S \in \mathcal{S}} \Sigma_S \\ & \searrow \quad \swarrow^{\hat{\psi}_{S'S}} & \\ & \Sigma \cong \Sigma_{\widehat{G}} & \end{array} \qquad \hat{\psi}(l) = (\hat{\psi}_S(l))_{S \in \mathcal{S}}. \tag{3.18}$$

Proof. By Lemma 3.3.3, we have the isometric isomorphisms

$$F \times \Delta \cong \left(\varprojlim_{S \in \mathcal{S}} \mathcal{O}_S(F) \right) \times \Delta, \tag{3.19}$$

Also, we can consider the following map

$$((m_S)_{S \in \mathcal{S}}, a)) \mapsto ((m_S, a))_{S \in \mathcal{S}}\,.$$

Then we have the isometric isomorphism

$$\left(\varprojlim_{S \in \mathcal{S}} \mathcal{O}_S(F) \right) \times \Delta \cong \varprojlim_{S \in \mathcal{S}} (\mathcal{O}_S(F) \times \Delta). \tag{3.20}$$

Denote by Ψ the isometric isomorphism resulting from the composition of the isomorphisms in 3.19 and 3.20. There is a natural action by G on the right hand side of 3.20 given by

$$g \cdot ((m_S, a))_{S \in \mathcal{S}} = (g \cdot (m_S, a))_{S \in \mathcal{S}} = \left((m_S \cdot g^{-1}, g \cdot a)\right)_{S \in \mathcal{S}}\,. \tag{3.21}$$

Let's see that the isomorphism Ψ is G-equivariant with respect to the action just defined.

$$\begin{aligned}\Psi(g \cdot (k, a)) = \Psi((k \cdot g^{-1}, g \cdot a)) &= \left((\psi_S(k \cdot g^{-1}), g \cdot a)\right)_{S \in \mathcal{S}} = \left((\psi_S(k) \cdot g^{-1}, g \cdot a)\right)_{S \in \mathcal{S}} \\ &= (g \cdot (\psi_k, a))_{S \in \mathcal{S}} = g \cdot ((\psi_S(k), a))_{S \in \mathcal{S}} = g \cdot \Psi((k, a)).\end{aligned}$$

In particular, the map Ψ induces an isometric isomorphism between the respective spaces of G-orbits and because of the model 1.22 for the lamination L and the definition of the action 3.21, this is the isometric isomorphism $\hat{\psi}$ in 3.18. The projection $\pi_{\widehat{G}}$ of the inverse system onto the $\widehat{G}$-coordinate gives the diagram commutative 3.18, where $\hat{\psi}_{\widehat{G}}$ is the map 3.16 with S evaluated on $\widehat{G}$. Since $\mathcal{O}_{\widehat{G}}(F)$ consists of only one point for the holonomy action on the fiber is transitive, by the model 2.11 of the surface we have $\Sigma = \Sigma_{\widehat{G}}$ and the map $\hat{\psi}_{\widehat{G}}$ is the fibration of the lamination onto the surface. This concludes the proof. □

Therefore there is an isometric isomorphism between the inverse limit of the canonical tower and the lamination L and the map is also an isomorphism of fibrations over the surface Σ,

$$L \cong \varprojlim_{S \in \mathcal{S}} \Sigma_S.$$

3.4 Canonical Tower of Teichmüller Spaces

In this section it will be used the canonical tower of coverings constructed in the previous section. Here, it is assumed that the lamination L satisfies Hypotheses 1.3.4. Another tower with the Teichmüller spaces associated to each covering will be constructed. Then it is shown that the direct limit of this tower of Teichmüller spaces can be identify with the space $T_{TLC}(L)$, L satisfying hypotheses 1.3.4.

The pullback by the map ψ_S is defined by the formula $\psi_S^*(\nu)(k) = \nu(\psi_S(k))$,

$$\psi_S^* : C(\mathcal{O}_S(F), Bel(\Delta))_{eq(G)} \to C_{LC}(F, Bel(\Delta))_{eq(G)}. \tag{3.22}$$

Let's see that ψ_S^* is well defined. Since ψ_S is constant on the S-orbits, the image of ψ_S^* is the subspace of functions which are constant on the S-orbits. Let f be an element of $C(\mathcal{O}_S(F), Bel(\Delta))_{eq(G)}$, then by definition we have

$$l_g^*(\psi_S^*(f)(k)) = l_g^*(f(\psi_S(k))).$$

Using the equivariance 1.18 with respect to the right action on the S-orbits, we obtain $l_g^*(f(\psi_S(k))) = f(\psi_S(k) \cdot g)$. Also by the right equivariance of the map ψ_S, we have $f(\psi_S(k) \cdot g) = f(\psi_S(k \cdot g))$. Then we conclude that $l_g^*(f(\psi_S(k))) = \psi_S^*(f)(k \cdot g)$. That is to say, the image of a G-equivariant f under ψ_S^* is G-equivariant.

Analogously we define the pullbacks $(\psi_{S'S}^*)_{S' \leq S}$ as follow

$$\psi_{S'S}^* : C(\mathcal{O}_S(F), Bel(\Delta))_{eq(G)} \to C(\mathcal{O}_{S'}(F), Bel(\Delta))_{eg(G)}$$

$$f \longmapsto \psi_{S'S}^*(f)(x \cdot S') = f(\psi_{S'S}(x \cdot S')) = f(x \cdot S).$$

Then $(\psi_{S'S}^*)_{S' \leq S}$ defines a direct system. The respective images of this direct system constitute a nested sequence whose direct limit equals to the union of them.

Lemma 3.4.1. *Consider $S \in \mathcal{S}$ and $m \in \mathcal{O}_S(F)$. Then, for every $g \in G$ such that $m \cdot g = m$, there is $s \in S \cap G$ such that the corresponding left actions on the disk coincide, that is $l_g = l_s$.*

Proof. Consider $g \in G$ such that $m \cdot g = m$. Then $m \cdot g^{-1} = m$ and for every $k \in \Delta$ we have

$$g \cdot (m, k) = (m \cdot g^{-1}, g \cdot k) = (m, g \cdot k).$$

Fixing $k \in \Delta$, by definition of the action in 3.13, we have $[(m, k)] = [(m, g \cdot k)]$ in Σ_S and by the isomorphism 3.14, we obtain $g \cdot k \in (S \cap G) \cdot k$. In particular, there is an element $s \in S \cap G$ such that $l_g(k) = l_s(k)$. Since every element $g \in G - \{id_\Delta\}$ acts on the disc without fix points, we conclude $l_g = l_s$. □

As in definition 1.2.19, it can be considered the set of Beltrami differentials $Bel(\Sigma_S)$ on each surface Σ_S. Then a differential on Σ_S is a Beltrami differential on the disk Δ such that

$$l_s^*(\mu) = \mu, \ \forall s \in S \cap G. \tag{3.23}$$

Lemma 3.4.2. *There is an isometric homeomorphism*

$$Bel(\Sigma_S) \cong C(\mathcal{O}_S(F), Bel(\Delta))_{eq(G)}.$$

Proof. Consider a Beltrami differential $\mu \in Bel(\Sigma_S)$ and let m be an S-orbit in $\mathcal{O}_S(F)$. Let's see that the right action of G on $\mathcal{O}_S(F)$ is transitive. Let $x \cdot S$ and $y \cdot S$ be S-orbits in $\mathcal{O}_S(F)$. Because the lamination is minimal, the holonomy action on the fiber F by the group G is transitive, then there exists $g \in G$ such that $x \cdot g = y$.

Hence $(x \cdot S) \cdot g = (x \cdot g) \cdot S = y \cdot S$, that is to say, the right action of G on $\mathcal{O}_S(F)$ is transitive. Therefore, every S-orbit has the form $m \cdot g$ for some $g \in G$. Define the function $h_\mu : \mathcal{O}_S(F) \to Bel(\Delta)$ by the formula

$$h_\mu(m \cdot g) = l_g^*(\mu), \ \forall g \in G.$$

This function is well defined since $m \cdot g = m \cdot g'$ implies $m = m \cdot (g' \cdot g^{-1})$ and by Lemma 3.4.1, there is an element $s \in S \cap G$ such that $l_{g'g^{-1}} = l_s$. In particular,

$$l_{g'}^*(\mu) = (l_s l_g)^*(\mu) = l_g^*(l_s^*(\mu)) = l_g^*(\mu),$$

where we have used 3.23 in the last equality. Also h_μ is equivariant because $h_\mu(m \cdot g \cdot g_0) = l_{g \cdot g_0}^*(\mu) = l_{g_0}^*(h_\mu(m \cdot g))$.

Thus, we have constructed a map

$$F : Bel(\Sigma_S) \to C(\mathcal{O}_S(F), Bel(\Delta))_{eq(G)}, \ \mu \to h_\mu$$

This is a continuous map, moreover it is an isometry. In fact,

$$||F(\mu)||_\infty = \max_{m \in \mathcal{O}_S(F)} ||F(\mu)(m)||_\infty = ||\mu||_\infty,$$

because on every S-orbit $m \cdot g$ we have

$$||F(\mu)(m \cdot g)||_\infty = ||l_g^*(\mu)||_\infty = ||\mu||_\infty.$$

Now, we can define a continuous inverse of F. In effect, consider the map

$$f \to \mu_f = f(m), \ \forall f \in C(\mathcal{O}_S(F), Bel(\Delta))_{eq(G)},$$

where m is the S-orbit as in the previous part of the proof. This map is well define since

$$l_s^*(\mu_f) = l_s^*(f(m)) = f(m \cdot s) = f(m) = \mu_f, \ \forall s \in S \cap G,$$

where we have used the equivariance of f in the second equality and the fact that $m \cdot s = m$ since $s \in S \cap G$. In particular, $\mu_f \in Bel(\Sigma_S)$. This map is an inverse of F and it is an isometry since F is so. In particular, the inverse is continuous. This finishes the proof. □

Corollary 3.4.3. *The map ψ_S^* is an embedding of Beltrami differentials*

$$\psi_S^* : Bel(\Sigma_S) \to Bel_{TLC}(L).$$

Proof. By the embedding ψ_S^* of $C(\mathcal{O}_S(F), Bel(\Delta))_{eq(G)}$ in $C_{LC}(F, Bel(\Delta))_{eq(G)}$ and by the Lemma 3.4.2, we have an embedding of $Bel(\Sigma_S)$ in $C_{LC}(F, Bel(\Delta))_{eq(G)}$. By definition of the space $Bel_{TLC}(L)$ (see definition 3.5), we have the result. □

Lemma 3.4.4. *Every ψ_S^* induces an embedding of Teichmüller spaces*

$$\eta_S : T(\Sigma_S) \to T_{TLC}(L),$$

and the union of the images is the space

$$\bigcup_{S \in \mathcal{S}} \eta_S(T(\Sigma_S)) = T_{TLC}(L).$$

Proof. By Corollary 3.4.3, every map psi_S^* is an embedding. Then to prove the embedding between Teichmüller spaces, it is enough to show that

$$\nu \sim \nu' \ iff \ \psi_S^*(\nu) \sim \psi_S^*(\nu').$$

In effect, this follows from the chain of equivalences

$$\begin{aligned} \nu \sim \nu' &\iff f^{\nu(m)}|_{S^1} = f^{\nu'(m)}|_{S^1}, \ \forall m \in \mathcal{O}_S(F) \\ &\iff f^{\nu(\psi_S(k))}|_{S^1} = f^{\nu'(\psi_S(k))}|_{S^1}, \ \forall k \in F \\ &\iff f^{\psi_S^*(\nu)(k)}|_{S^1} = f^{\psi_S^*(\nu')(k)}|_{S^1}, \ \forall k \in F \\ &\iff \psi_S^*(\nu) \sim \psi_S^*(\nu'). \end{aligned} \tag{3.24}$$

Thus we have an embedding $\eta_S : T(\Sigma_S) \to T_{TLC}(L)$ defined by

$$[\mu] \longmapsto [\psi_S^*(\mu)].$$

In particular, we have

$$\bigcup_{S \in \mathcal{S}} \eta_S(T(\Sigma_S)) \subseteq T_{TLC}(L). \tag{3.25}$$

Now, consider $[\mu] \in T_{TLC}(L)$, that is to say, μ is locally constant on the fiber F. Because the fiber is compact, the number of open sets where μ is constant is finite and because of the residual finiteness of $\widehat{G}$, there is a normal finite index subgroup S such that every S-orbit is contained in one of these open sets. Hence, μ is the image of some μ_S by ψ_S^* and we have $[\mu] = \eta_S([\mu_S])$. Then, we have $\bigcup_{S \in \mathcal{S}} \eta_S(T(\Sigma_S)) = T_{TLC}(L)$. □

Lemma 3.4.5. *For S and S' in $\mathcal{S}$ such that $S' \leq S$, the canonical inclusion $Bel(\Sigma_S) \subset Bel(\Sigma_{S'})$ induces an inclusion in Teichmüller spaces $T(\Sigma_S) \subset T(\Sigma_{S'})$.*

Proof. Note that since $S' \leq S$, the condition $l_g^*(\mu), \ \forall g \in S \cap G$ implies that $l_g^*(\mu), \ \forall g \in S' \cap G$. Then $Bel(\Sigma_S) \subset Bel(\Sigma_{S'})$. By the Teichmüller equivalence relation, for every $[\mu] \in T(\Sigma_S)$, we can consider the corresponding class $[\mu]$ in $T(\Sigma_{S'})$. This define an injective function, and we have the inclusion $T(\Sigma_S) \subset T(\Sigma_{S'})$. □

Lemma 3.4.6. *There is a canonical homeomorphism*

$$\eta : \varinjlim_{S \in \mathcal{S}} T(\Sigma_S) \to T_{TLC}(L)$$

induced by the embeddings η_S in Lemma 3.4.4.

Proof. By the inclusion between Teichmüller spaces, Lemma 3.4.5, $(T(\Sigma_S))_{S\in\mathcal{S}}$ is a directed sequence. By Lemma 3.4.4, there is a canonical embedding

$$\eta : \varinjlim_{S\in\mathcal{S}} T(\Sigma_S) \to T_{TLC}(L)$$

whose image is the union of the nested sequence $\eta_S(T(\Sigma_S))_{S\in\mathcal{S}}$. By the second part of the Lemma 3.4.4, this union is $T_{TLC}(L)$ hence η is a homeomorphism and we have the result. □

3.5 Canonical Tower of Fenchel-Nielsen Coordinates

In this section he towers defined in the previous sections will be considered. Now, we want to identify the canonical tower of Teichmüller spaces with a canonical tower of Fenchel-Nielsen coordinates. It will be denoted by π_S the finite locally isometric covering $\hat{\psi}_{SG} : \Sigma_S \to \Sigma$ defined in the previous section. From now on, it will be considered the following hypotheses on Σ:

Hypotheses 3.5.1. Let Σ be a hyperbolic surface without boundary obtained by gluing a (possibly finite) sequence $\mathcal{C}$ of generalized hyperbolic pair of pants, each glued to the next along a common boundary geodesic such that the length of these geodesic boundaries is uniformly upper bounded. Also every pair of pants P in $\Sigma - \bigcup_{C\in\mathcal{C}} C$ verifies

$$\mathrm{Hol}(\iota_*(\pi_1(P))) = \{id_F\}, \quad \iota : P \to \Sigma.$$

Let $\mathcal{C}$ be a pair of pants decomposition of Σ as in the hypothesis 3.5.1. Since the length of geodesic boundaries is uniformly upper bounded, then we can consider the Fenchel-Nielsen Teichmüller space of each surface Σ_S defined in 2.6.4. After a quasiconformal deformation, $\mathcal{C}$ will be a topological pair of pants decomposition but no longer a geometric one in general.

Define the set $\mathcal{G}$ of geodesic representatives of isotopic classes of the curves in $\mathcal{C}$. Since there is a unique geodesic for each isotopic class, $\mathcal{G}$ is well defined and it is a geometric pair of pants decomposition corresponding to the new deformed hyperbolic structure. Here, it will be abused of notation and it is said that $\mathcal{G}$ is isotopic to $\mathcal{C}$.

By Wolpert's inequality, the lengths of the geodesics in $\mathcal{G}$ will be uniformly upper-bounded ([6, Lemma 8.1]) hence the new deformed hyperbolic structure will be metric complete [6, Lemma 4.7]. Since $\mathcal{G}$ is isotopic to $\mathcal{C}$, the pair of pants decomposition $\mathcal{G}$ has trivial holonomy as well.

Lemma 3.5.2. *Consider the pair of pants decomposition $\mathcal{C}$. Then, for every $S \in \mathcal{S}$, the set $\mathcal{C}_S = \pi_S^{-1}(\mathcal{C})$ is a pair of pants decomposition of Σ_S.*

Proof. By definition, $\mathcal{C} = \{C_i \mid i \in \Lambda\}$ is a family of pairwise disjoint simple curves in Σ such that removing these curves gives a disjoint union of pair of pants without boundary and there is a family of pairwise disjoint tubular neighbourhoods $\mathcal{U} = \{U_i | i \in \Lambda\}$ of these curves in Σ such that $C_i \subset U_i$.

Consider a curve $C_i \in \mathcal{C}$. Since the covering π_S is finite, then it is a proper map hence $\pi_S^{-1}(C_i)$ is compact. Also, π_S is a local homeomorphism, then $\pi_S^{-1}(C_i)$ is a one dimensional submanifold without boundary. By the classification of compact one-manifolds without boundary, we conclude that $\pi_S^{-1}(C_i)$ is a finite disjoint union of simple closed curves in Σ_S. Therefore $\mathcal{C}_S$ is a disjoint union of simple closed curves in Σ_S.

Consider a pair of pants P in $\Sigma - \bigcup_{C \in \mathcal{C}} C$. By the definition of the right action of $\widehat{G}$ on the set of S-orbits, we have

$$(x \cdot S) \cdot g := x \cdot (Sg) = (x \cdot g) \cdot (g^{-1} \cdot S \cdot g) = x \cdot S, \ \forall g \in \iota_*(\pi_1(P)),$$

where we have used in the last equality the normality of the group S and the fact that the right action on the fiber is the identity since $\mathcal{C}$ has trivial holonomy. Then, we have the holonomy action of the covering π_S is trivial on the subgroup $\iota_*(\pi_1(P))$. Therefore, the locally isometric covering π_S restricted to the pair of pants P is trivial hence its preimage under π_S is a disjoint union of homeomorphic copies of P. In particular, we have proved that $\Sigma_S - \bigcup_{C \in \mathcal{C}} C$ is a disjoint union of pair of pants without boundary.

Because every C_i is compact and the covering π_S is finite and locally isometric, we can take every neighbourhood U_i small enough such that $\pi_S^{-1}(U_i)$ is a disjoint union of tubular neighbourhoods of the preimage curves in $\pi_S^{-1}(U_i)$. In particular, $\bigcup_{U \in \mathcal{U}} \pi_S^{-1}(U)$ is a disjoint union of tubular neighbourhoods of the curves in $\mathcal{C}_S$. This conclude the proof. □

Recall that if C is a closed minimizing geodesic in Σ_S with a neighbourhood U in Σ_S such that $U - C$ has two connected components, then a collar of C is $\mathfrak{c} = C \cup U'$ where U' is a connected component of $U - C$. Also all of the geodesics isotopic to curves in $\mathcal{C}$ are minimizing.

Lemma 3.5.3. *Consider the geometric pair of pants decomposition $\mathcal{G}$ isotopic to $\mathcal{C}$ in Σ. Then, for every normal finite index subgroup S of $\widehat{G}$, the set $\mathcal{G} = \pi_S^{-1}(\mathcal{G})$ is the geometric pair of pants decomposition in Σ_S isotopic to $\mathcal{C}_S$. Moreover,*

(1) For every collar $\mathfrak{c}''$ of a geodesic in $\mathcal{G}_S$, there is a collar $\mathfrak{c}' \subset \mathfrak{c}''$ of the same geodesic which is isometrically homeomorphic to a collar $\mathfrak{c}$ of a geodesic in $\mathcal{G}$ via π_S.

(2) Every pair of pants in $\Sigma_S - \bigcup_{C \in \mathcal{G}_S} C$ is isometrically homeomorphic to its image, a pair of pants in $\Sigma - \bigcup_{C \in \mathcal{G}} C$, via π_S.

Proof. Since π_S is a finite locally isometric covering and $\mathcal{G}$ is isotopic to $\mathcal{C}$, a similar argument as in Lemma 3.5.2 shows that

(1) $\mathcal{G}_S$ is a disjoint union of simple closed geodesics in Σ_S and each one of them is locally isometric to its image via π_S.

(2) $\Sigma_S - \bigcup_{C \in \mathcal{G}_S} C$ is a disjoint union of a pair of pants without boundary and each one of them is isometrically homeomorphic to its image via π_S.

(3) There is a family of pairwise disjoint tubular neighbourhoods in Σ of the geodesic in $\mathcal{G}$ whose preimage is a family of pairwise disjoint tubular neighbourhoods in Σ_S of the geodesics in $\mathcal{G}_S$.

In particular, $\mathcal{G}_S$ is a geometric pair of pants decomposition of Σ_S. Consider a geodesic γ' in $\mathcal{G}_S$ and the geodesic $\gamma = \pi_S(\gamma')$ in $\mathcal{G}$. Let C be the simple closed curve in $\mathcal{C}$ isotopic to γ. By the unique lifting property of the covering π_S, there is a curve C' in Σ_S isotopic to γ' whose image by π_S is C. Thus, by definition of $\mathcal{C}_S$ and Lemma 3.5.2, C' is a simple closed curve in $\mathcal{C}_S$ isotopic to γ'. It has been proved that $\mathcal{G}_S$ is isotopic to $\mathcal{C}_S$.

Now, let C' be a minimizing closed geodesic in $\mathcal{G}_S$ and let $\mathfrak{c}''$ be a collar of the curve C'. Since π_S is a finite locally isometric covering, there is a small enough collar $\mathfrak{c}' \subset \mathfrak{c}''$ of C' which is locally isometric to a collar $\mathfrak{c} = \pi_S(\mathfrak{c}')$ of the geodesic $C = \pi_S(C')$ in $\mathcal{G}$ via π_S and $\mathfrak{c}' - C'$ is contained in a pair of pants P' in $\Sigma_S - \bigcup_{C \in \mathcal{G}_S} C$. Let´s see that $\mathfrak{c}'$ is isometrically homeomorphic to $\mathfrak{c}$. For this, it is enough to show that there is a continuous inverse map.

Let y be a point in C and consider a sequence $(y_n)_n$ in $\mathfrak{c} - C$ converging to y. Since $\pi_S|_{\mathfrak{c}'-C'}$ is an isometric homeomorphism, we can consider the sequence $x_n = \pi_S|^{-1}_{\mathfrak{c}'-C'}(y_n)$. Denote by $\omega(y_n)$ the set of limit points of the sequence $(x_n)_n$. We claim that $\omega(y_n)$ consists of just one point. Indeed, since $\omega(y_n) \subset C'$ and C' is compact, $\omega(y_n)$ is non-empty. Suppose that $\omega(y_n)$ hast two distinct points x and x'. In particular $d = d_{\Sigma_S}(x, x') > 0$, (here the distance between two points is given by the length of the geodesic that joins them). Then there are subsequences $(z_m)_m$ and $(z'_m)_m$ converging to x and x' respectively such that

$$d_{\mathfrak{c}'-C'}(z_m, z'_m) \geq d_{\Sigma_S}(z_m, z'_m) \geq d/2 > 0,\ \forall m \in \mathbb{N}.$$

Therefore,

$$d_{\mathfrak{c}-C}(\pi_S(z_m), \pi_S(z'_m)) \geq d/2 > 0,\ \forall m \in \mathbb{N}.$$

Now, $(y_n)_n$ is a Cauchy sequence with respect to the distance d_Σ, but by Lemma 1.3.13, $(y_n)_n$ is a Cauchy sequence with respect to $d_{\mathfrak{c}-C}$. Then

$$\lim_{m\to+\infty} d_{\mathfrak{c}-C}(\pi_S(z_m), \pi_S(z'_m)) = 0$$

because $(\pi_S(z_m))_m$ and $(\pi_S(z'_m))_m$ are subsequences of $(y_n)_n$. This is a absurd and this proves the claim. Because the sequence $(y_n)_n$ was arbitrary, defining

$$\{\pi_S|^{-1}_{\mathfrak{c}'}(y)\} = \omega(y_n)$$

gives the continuous inverse to $\pi_S|_{\mathfrak{c}'}$ we were looking for. In effect, if (y'_n) is another sequence in $\mathfrak{c} - C$ converging to y, then the sequence

$$(y_1, y'_1, y_2, y'_2, ...)$$

has the same properties and the limit set of its preimage has only one point. In particular, the preimages of the sequences $(y_n)_n$ and $(y'_n)_n$ have the same limit and the mentioned inverse map is well defined.

Since $\pi_S|_{\mathcal{C}'}$ is an invertible locally isometric map, the inverse is continuous and we have proved that $\pi_S|_{\mathcal{C}'}$ is an isometric homeomorphism. This concludes the proof. □

In general, the map π_S will not be a homeomorphism on the closure $\overline{P}$ of a pair of pants P in $\Sigma_S - \bigcup_{C \in \mathcal{G}_S} C$. Let's see an example.

Example 1. Let Σ be the pinched torus with a cusp and $\mathcal{C}$ its pair of pants decomposition. Then $\mathcal{C}$ consists of a single curve C. Consider Σ_S the covering resulting from two copies of the surface but interchanging the boundary components of the pair of pants in the gluing process, see figure 3.1. Here the map π_S is not a homeomorphism in $\overline{P}$ because an open set containing points on the geodesic boundary **a** is not sent under $\pi_S|_P$ to an open set in Σ.

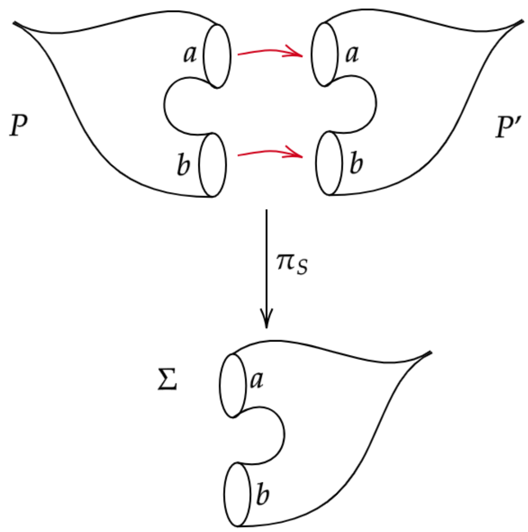


Figure 3.1: Representation of the double covering of pinched torus.

Corollary 3.5.4. *For every $S \in \mathcal{S}$, the lengths of the geodesic in $\mathcal{G}_S$ are uniformly upper bounded and the hyperbolic surface Σ_S is complete without boundary.*

Proof. Since π_S is a locally isometric finite covering, then Σ_S is also hyperbolic and neither has boundary. By the first item in the Lemma 3.5.3, the lengths of the geodesics in $\mathcal{G}_S$ are uniformly upper bounded since the same holds for $\mathcal{G}$. In particular, Σ_S is complete, [6, Lemma 4.7]. □

Corollary 3.5.5. *There is an isometric homeomorphism*

$$T_{FN}(\Sigma_S, \mathcal{C}_S) \cong C(\mathcal{O}_S(F), T_{FN}(\Sigma, \mathcal{C})).$$

Proof. From Lemma 3.5.3, it follows that π_S is trivial on $\mathcal{C}$ hence $\mathcal{C}_S \cong \mathcal{O}_S(F) \times \mathcal{C}$ since $\mathcal{O}_S(F)$ is the fiber of the covering. In particular, we have the isometric homeomorphisms

$$l_\infty(\mathcal{C}_S) \cong l_\infty(\mathcal{O}_S(F) \times \mathcal{C}) \cong C(\mathcal{O}_S(F), l_\infty(\mathcal{C})).$$

By the isometric homeomorphism between the Fenchel-Nielsen Teichmüller T_{FN} space and the space of l_∞ sequences 2.9, we have the result. □

Corollary 3.5.6. *For S and S' in $\mathcal{S}$ such that $S' \leq S$, consider the finite locally isometric covering $\hat{\psi}_{S'S} : \Sigma_{S'} \to \Sigma_S$ defined in 3.15. Then,*

(a) For every collar $\mathfrak{c}''$ of a geodesic in $\mathcal{G}_S$, there is a collar $\mathfrak{c}' \subset \mathfrak{c}$ of the same geodesic which is isometrically homeomorphic to a collar $\mathfrak{c}$ of a geodesic in $\mathcal{G}_S$ via $\hat{\psi}_{S'S}$.

(b) Every pair of pants in $\Sigma_S - \bigcup_{C \in \mathcal{G}} C$, via $\hat{\psi}_{S'S}$.

(c) For every geodesic in $\mathcal{G}_S$, there is a neighborhood U in $\Sigma_{S'}$ of the geodesic where $\hat{\psi}_{S'S}$ is an isometric homeomorphism.

Proof. The first two items follow from Lemma 3.5.3 and the factorization

$$\pi_{S'} = \pi_S \circ \hat{\psi}_{S'S}.$$

The third item follows from the first. □

Now, we will define maps between the Fenchel-Nielsen Teichmüller spaces of the surfaces Σ_S.

Corollary 3.5.7. *For S and S' in $\mathcal{S}$ such that $S' \leq S$, there is an embedding of Fenchel-Nielsen Teichmüller spaces*

$$\Delta_{SS'} : T_{FN}(\Sigma_S, \mathcal{C}_S) \to T_{FN}(\Sigma_{S'}, \mathcal{C}_{S'}) \tag{3.26}$$

defined through the length and twist coordinates by

$$\Delta_{SS'}((l_C, \theta_C)_{C \in \mathcal{C}_S}) = (l_{C'}, \theta_{C'})_{C' \in \mathcal{C}_{S'}}$$

where $l_{C'} = l_{\hat{\psi}_{S'S}(C')}$, $\theta_{C'} = \theta_{\hat{\psi}_{S'S}(C')}$.

Proof. Consider Σ_S with another complete hyperbolic structure and denote it by Σ'_S. This is equivalent to consider another Fuchsian group Γ'_S in the Fuchsian model and all of the previous results in this chapter regarding Σ_S apply the same for Σ'_S. In particular, by Corollary 3.5.6, the maps $\hat{\psi}_{S'S}$ sends isometrically geodesics in $\mathcal{G}_{S'}$ to geodesics in $\mathcal{G}_S$. Then by definition of the Fenchel-Nielsen coordinates, we have the embedding 3.26. □

For every $S \in \mathcal{S}$ it is defined the following isometric embedding

$$\Psi^*_S : C(\mathcal{O}_S(F), T_{FN}(\Sigma, \mathcal{C})) \to C_{LC}(F, T_{FN}(\Sigma, \mathcal{C})) \tag{3.27}$$

by the formula $\Psi^*_S(f)(k) = f(\psi_S(k))$. In the following lemma, it will be considered the identification in Corollary 3.5.5.

Lemma 3.5.8. *The maps 3.27 define an isometric homeomorphism*

$$\Psi^* : \varinjlim_{S \in \mathcal{S}} T_{FN}(\Sigma_S, \mathcal{C}_S) \to C_{LC}(F, T_{FN}(\Sigma, \mathcal{C})). \tag{3.28}$$

Proof. Since $T_{FN}(\Sigma_S, \mathcal{C}_S)$ is identified with $C(\mathcal{O}_S(F), T_{FN}(\Sigma, \mathcal{C}))$, we can consider the embedding $\Delta_{SS'} = C(\mathcal{O}_S(F), T_{FN}(\Sigma, \mathcal{C})) \to C(\mathcal{O}_{S'}(F), T_{FN}(\Sigma, \mathcal{C}))$. Also, by definition of the maps $\Delta_{SS'}$ we have for $S'' \leq S' \leq S$:

$$\Delta_{SS''} = \Delta_{S'S''} \circ \Delta_{SS'}, \ \Psi^*_S = \Psi^*_{S'} \circ \Delta_{SS'}. \tag{3.29}$$

Since every $\Delta_{SS''}$ is an isometrical embedding with respect to the metric induced by the identification 2.9 with the l_∞ sequences, expressions 3.29 show that there is a well defined metric in the direct limit and the following is an isometrical embedding

$$\Psi^* : \varinjlim_{S \in \mathcal{S}} T_{FN}(\Sigma_S, \mathcal{C}_S) \to C_{LC}(F, T_{FN}(\Sigma, \mathcal{C})).$$

It rest to show that the image of Ψ^* is the whole space. In effect, consider a locally constant function f. Since F is compact and $\widehat{G}$ is residually finite, there is a normal finite index subgroup S of $\widehat{G}$ and a function f_S such that $\Psi^*(f_S) = \Psi^*_S(f_S) = f$ and we have the result. □

3.6 Description of the Leaves

Given the hypotheses that we are considering (hypotheses 3.5.1 and 1.3.4), the lamination L is not necessarily compact. Let's see what the laminations that are being considered look like. Let $\mathcal{C}$ be the set of geodesic boundaries of the pair of pants decomposition of the surface Σ.

Lemma 3.6.1. *For every x in the fiber, the subgroup generated by the freely homotopic classes of curves in $\mathcal{C}$ is contained in the isotropy of the holonomy at x,*

$$\langle [C] \in G \mid C \in \mathcal{C} \rangle \subset G_x, \ \forall x \in F.$$

Proof. Every curve C in $\mathcal{C}$ has a tubular neighbourhood in the surface Σ hence there is a homotopically equivalent curve C' entirely contained in the interior P of a pair of pants in the decomposition. Since the fibration has trivial holonomy on pants, we have

$$\mathrm{Hol}([C]) = \mathrm{Hol}([C']) \in \mathrm{Hol}(\iota_*(\pi_1(P))) = \{id_F\}.$$

We conclude that $\mathrm{Hol}([C]) = id_F$ for every curve $C \in \mathcal{C}$ therefore

$$x \cdot [C] = \mathrm{Hol}([C])(x) = x, \ \forall C \in \mathcal{C}, \ \forall x \in F$$

and we have the result. □

Corollary 3.6.2. *For every x in the fiber, the subgroup generated by the freely homotopic classes of curves in $\mathcal{C}$ is contained in the fundamental group of the leaf at x,*

$$\langle [C] \in G \mid C \in \mathcal{C} \rangle \subset \pi_1(\mathcal{L}_x), \ \forall x \in F.$$

In particular, the fundamental group of any leaf of the lamination L has infinite elements and none of the leaves is simply connected.

Proof. By Lemmas 1.3.7 and 3.6.1, it rest to show the last statement. If Σ is the sphere with three cusps, then the surface is a single pair of pants and since the lamination has trivial holonomy on pants, the fibration is trivial. Because the lamination is minimal, the lamination consists of a single leaf homeomorphic to Σ whose fundamental group has infinite elements. If Σ is not the sphere with three cusps, the set $\mathcal{C}$ is non-empty because Σ is obtained by gluing a sequence of generalized hyperbolic pair of pants. Then we have the result. □

Now, we want to associate a graph to each hyperbolic surface considered. Then it is considered the following. Since the set $\mathcal{C}$ is a topological pants decomposition, every curve C in $\mathcal{C}$ has a tubular neighbourhood U_C. We can take these tubular neighbourhoods small enough such that every one of them is foliated by simple closed curves γ_i^C equidistant to the corresponding geodesic, then $U_C = \bigcup_{i\in I} \gamma_i^C$.

On the other hand, by definition, every cusp B has a neighbourhood isometric to the quotient of the hyperbolic upper-half plane H by the isometry group generated by the translation $z \mapsto z+1$,

$$U_B = H/(z \mapsto z+1) = \{z = x+iy \mid a < y\}/(z \mapsto z+1). \tag{3.30}$$

Also, $\mathcal{A} = \Sigma - \left(\bigcup_{C\in\mathcal{C}} U_C \cup \bigcup_B U_B\right)$ is a disjoint union of topological pair of pants, where the second union running over all cusps of Σ.

Then the following map is defined:

$$f : \Sigma \to \sigma$$

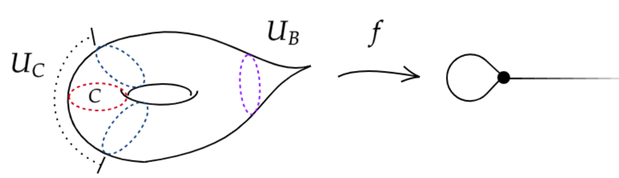
(1) For every $C \in \mathcal{C}$ contract each curve γ_i^C to a point.

(2) For every cusp B, contract each of the y-constant circles in 3.30 to a point.

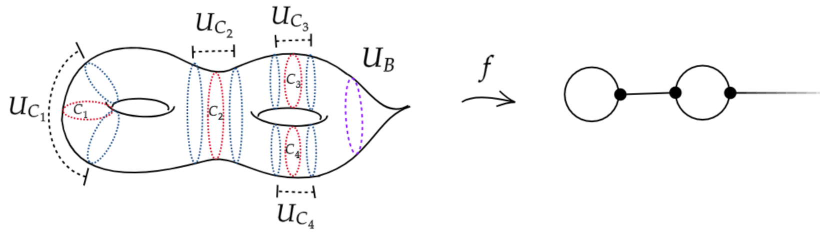
(3) Contract every component of $\mathcal{A}$ to a point.

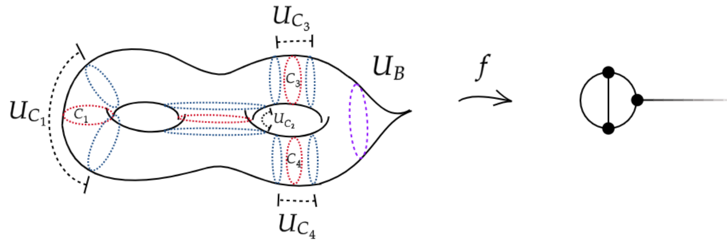
We will call the map f and the graph σ as the *contraction* of Σ. Let's see some examples. The Example 3 show that the graph σ depends of the decomposition $\mathcal{C}$ of Σ.

Example 2. Let Σ_1^* be a torus with one cusp. Then the graph associate to Σ_1^* is represented in the following figure where C represent the geodesic in the pair of pants decomposition.



Example 3. Let Σ_2^* be a double torus with one cusp. Then two graph associate to Σ_2^* are represented in the following figure where C_i represent a geodesic in each of the pair of pants decompositions.





Since each pair of pants in Σ is contracted to a point and it contains three geometric holes, then σ is the underlying topological space of a trivalent graph which has "infinite legs" corresponding to cusps of Σ.

We define a *graph with infinite legs* as a graph with half real lines attached to their vertices. Also, as topological space σ has a natural CW-decomposition. We will identify a graph with its underlying topological space along with its natural CW-decomposition.

In this sense, every interior of either an edge or infinite leg of the graph is a one-cell of the CW-decomposition, then it has a canonical differential structure. In the following we will denote by e a one-cell of the CW-decomposition. Considering this, the following is well- defined.

Definition 3.6.3. We say $\epsilon : \sigma \to \mathbb{R}^3$ is a *smooth embedding of the graphic* σ if it is a continuous embedding and restricted to each one-cell of the CW-decomposition of σ, it is a smooth embedding.

From now on, we will always consider graphs smoothly embedded in the euclidean three space such that every infinite leg goes to infinity. We will identify the graph with its embedded image as usual.

Definition 3.6.4. Consider a smooth embedding of σ in the euclidean three space such that every infinite leg goes to infinity. A *tubular neighbourhood of the graph* σ, denoted by N, is an open set in the euclidean three space containing the graph such that

(1) There is a continuous surjective map $\pi_N : N \to \sigma$.

(2) For every e of the CW-decomposition, the map π_N restricted to $\pi_N^{-1}(e)$ is a tubular neighbourhood.

We can take N small enough such that N has the structure of a tubular neighbourhood. Now, we can recover the surface Σ from the contraction graph Σ up to homeomorphism.

Lemma 3.6.5. *There is a tubular neighbourhood N of the embedding of the contraction graph $\sigma = f(\Sigma)$ such that there is a homeomorphism from Σ to the boundary of N.*

Proof. Let N be a small enough tubular neighbourhood of σ such that

$$\pi_N : N \to \sigma$$

extends to the closure of N. Consider the restriction of π_N to the boundary

$$\partial\pi_N : \partial N \to \sigma.$$

Every component of $\partial N - \bigcup_{C\in\mathcal{C}} \partial\pi_N^{-1}(f(C))$ is homeomorphic to a sphere with three punctures. In particular, every one of these components is homeomorphic to the interior of a pair of pants P in the decomposition of Σ and the correspondence is bijective. Also, every curve $C \in \mathcal{C}$ is a simple closed curve homeomorphic to $\partial\pi_N^{-1}(f(C))$. Then we can deform each of these homeomorphisms in such a way that they glue together into a global homeomorphism. We conclude that Σ is homeomorphic to ∂N. □

Lemma 3.6.6. *The holonomy action 1.21 factors through $f_* : G \to \pi_1(\sigma)$, that is there is a commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{\quad \text{Hol} \quad} & \text{Homeo}(F)^{op} \\ & {\scriptstyle f_*}\searrow \quad \nearrow{\scriptstyle \text{hol}_\sigma} & \\ & \pi_1(\sigma) & \end{array} \tag{3.31}$$

Proof. Let P be the interior of a pair of pants in the decomposition of Σ. By definition of the function f, $f(P)$ retracts to a point hence $f(\gamma)$ is contractible for every closed curve γ contained in P and we have

$$f_*\iota_*\pi_1(P) = \{e\}, \ \iota : P \hookrightarrow \Sigma$$

where ι is the canonical inclusion. Since the lamination has trivial holonomy on pants and P was arbitrary, we have $\pi_1(\sigma) \cong G/\ker(\text{Hol})$. By the first isomorphism theorem there is a unique homomorphism hol_σ such that the diagram 3.31 is commutative. □

Every locally trivial fibration is determined by its holonomy action

$$\text{Hol} : G \to \text{Homeo}(F)^{op}, \ G = \pi_1(\Sigma)$$

and in this formalism, morphisms of these fibrations become right G-equivariant continuous maps between the fibers

$$\zeta : F \to F', \ \zeta(k \cdot g) = \zeta(k) \cdot g, \ \forall g \in G$$

where the right action is defined by $k \cdot g = \text{Hol}(g)(k)$.

Since the holonomy action extends to the profinite completion $\widehat{G}$, then the map ζ is right $\widehat{G}$-equivariant. The same applies for the group $\pi_1(\sigma)$, that is to say, the map

$$\zeta' : F \to F', \ \zeta'(k \cdot g) = \zeta'(k) \cdot g, \ \forall g \in \pi_1(\sigma)$$

extends to the profinite completion of $\pi_1(\sigma)$ and it is $\widehat{\pi_1(\sigma)}$-equivariant.

Lemma 3.6.7. *Consider a hyperbolic surface Σ as in the hypotheses 3.5.1. The category of laminations verifying the hypotheses 1.3.4 is equivalent to the category of locally trivial fibrations on the contraction graph $\sigma = f(\Sigma)$ whose holonomy continuously extends to the profinite completion $\widehat{\pi_1(\sigma)}$.*

Proof. Denote by $\mathcal{C}_\sigma$ the small category whose objects consist of the set of continuous group morphism $\widehat{\pi_1(\sigma)} \to \mathrm{Homeo}(F)^{op}$ for every Hausdorff compact space and given two objects, the morphisms between them is the set of right $\widehat{\pi_1(\sigma)}$-equivariant continuous maps.

Denote by $\mathcal{C}_\Sigma$ the category whose objects consist of the set of continuous group morphism $\widehat{G} \to \mathrm{Homeo}(F)^{op}$ such that the diagram 3.31 commute, F is a Hausdorff compact space and given two objects, the morphisms between them is the set of right $\widehat{G}$-equivariant continuous maps.

Consider the continuous extension of the map f_* in 3.31 to the profinite completions (see 1.3.21), which will be denote by $\widehat{f_*}$, and define the functor $\mathbf{F} : \mathcal{C}_\sigma \to \mathcal{C}_\Sigma$ on objects as

$$\mathbf{F}(h) = h \circ \widehat{f_*}, \quad \text{where } h : \widehat{\pi_1(\sigma)} \to \mathrm{Homeo}(F)^{op}, \quad \widehat{f_*} : \widehat{G} \to \widehat{\pi_1(\sigma)}$$

This is well defined since precomposing an object in $\mathcal{C}_\sigma$ with $\widehat{f_*}$ gives an object in $\mathcal{C}_\Sigma$. And we define $\mathbf{F}$ on morphisms simply as

$$\mathbf{F}(\zeta) = \zeta.$$

This is well defined because for every right $\widehat{\pi_1(\sigma)}$-equivariant continuous map ζ, it is $\widehat{G}$-equivariant. This follows from

$$\zeta(k) \cdot g = \zeta(k) \cdot \widehat{f_*}(g) = \zeta(k \cdot \widehat{f_*}(g)) = \zeta(k \cdot g)$$

where we have used the right $\widehat{\pi_1(\sigma)}$-equivariance in the second equality and the commutativity of diagram 3.31 in the first and last equality.

Now, consider an object H in $\mathcal{C}_\Sigma$. By hypothesis, $\ker(f_*) \subset \ker(H|_G)$ hence

$$\ker(\widehat{f_*}) = \widehat{\ker(f_*)} \subset \widehat{\ker(H|_G)} = \ker(H)$$

and by the first isomorphism theorem, there is an object h in $\mathcal{C}_\sigma$ such that $H = h \circ \widehat{f_*}$. Define the functor $\mathbf{G} : \mathcal{C}_\Sigma \to \mathcal{C}_\sigma$ on objects as

$$\mathbf{G}(H) = h$$

and on morphism simply as $\mathbf{G}(\zeta) = \zeta$. Analogously as before, this define a functor and it is well defined. By construction, the functors $\mathbf{F}$ and $\mathbf{G}$ define an equivalence of categories and the result is proved. □

By Lemma 3.6.7, considering the lamination L fibering over Σ with fiber F, we denote by L_σ the corresponding lamination fibering over the contraction graph $\sigma = f(\Sigma)$ with the same fiber F. We are abusing of the term lamination since the leaves are not immersed manifolds. Every leaf of L_σ is a normal covering[1] of the graph σ. σ_x will denote an arbitrary leaf of L_σ through x with the underlying topology of a graph. This topology, unless the fiber F is discrete, will be strictly finer than the relative one induced by L_σ.

Corollary 3.6.8. *Consider a lamination L as in the hypotheses 1.3.4. Then, for every $x \in F$, the graph $f(\mathcal{L}_x)$ is homeomorphic to σ_x.*

Proof. By Lemma 3.6.7, the locally trivial fibration on the graph σ corresponding to $\mathcal{L}_x$ is σ_x. Also, by Lemma 3.6.6, such fibration must be homeomorphic to $f(\mathcal{L}_x)$. □

Corollary 3.6.9. *Any leaf of L is homeomorphic to the boundary of a tubular neighbourhood of some normal covering of $\sigma = f(\Sigma)$ whose fiber is dense in F. In particular, if the fiber F is infinite, then every leaf of L is of infinite type.*

Proof. The proof is analogous to the proof of the Lemma 3.6.5, changing Σ by L. □

Conversely, we have the following result.

Corollary 3.6.10. *Consider a hyperbolic surface Σ as in hypotheses 3.5.1 and a normal covering σ' of the graph $\sigma = f(\Sigma)$. Then, there is a locally trivial fibration $L \to \Sigma$ verifying the hypotheses of 1.3.4 such that there is a leaf homeomorphic to the boundary of a tubular neighbourhood of σ'.*

Proof. Denote by $\widetilde{F}$ the fiber of the covering $\sigma' \to \sigma$ in the hypotheses. Because the covering is normal, the right action of the group $\pi_1(\sigma)$ on the fiber is transitive. In particular, there is a subgroup S of $\pi_1(\sigma)$ such that the right cosets are in bijective correspondence with the fiber $\widetilde{F}$,

$$\pi_1(\sigma)/S \cong \widetilde{F}.$$

Taking the profinite completion of $\pi_1(\sigma)$, define the new fiber F as the set of right cosets

$$F = \widehat{\pi_1(\sigma)}/S.$$

Then, there is a canonical inclusion of $\widetilde{F}$ into F. The fiber F has a natural right action by $\widehat{\pi_1(\sigma)}$ in such a way that every orbit of $\pi_1(\sigma)$ is dense in F. In particular, this action defines a locally trivial fibration $L_\sigma \to \sigma$ such that L_σ is a minimal lamination and its holonomy action continuously extends to $\widehat{\pi_1(\sigma)}$. By construction, this lamination has the property that its leaf through $[e] \in F$ coincides with the graph σ',

$$\sigma_{[e]} = \sigma', \tag{3.32}$$

where e denotes the neutral element in $\pi_1(\sigma)$.

[1]Here it is said that a covering is normal if the covering transformation group acts transitively on fibers.

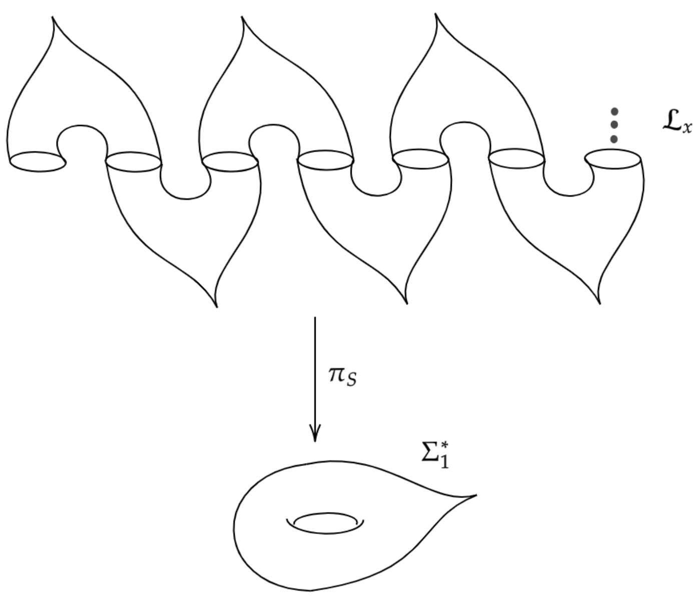
Precomposing the resulting holonomy action with f_* as in diagram 3.31, we have a new holonomy action which defines a locally trivial fibration $L \to \Sigma$ with the required properties.

In effect, since the fundamental group of a graph is free hence residually finite [7], by Lemma 1.3.21 and Theorem 1.3.22 the map f_* continuously extends to the respective profinite completions. In particular, the new holonomy action continuously extends to $\widehat{G}$. By construction, it verifies that it has trivial holonomy on pants. Finally, because of 3.32 and Lemma 3.6.6 again, by a similar argument as in Lemma 3.6.5 we have that the leaf of L through $[e] \in F$ is homeomorphic to the boundary of a tubular neighbourhood of σ',

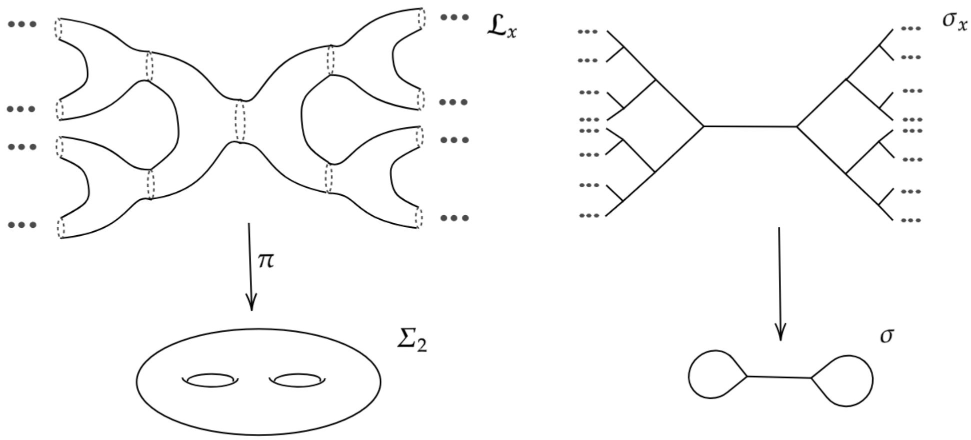
$$\mathcal{L}_{[e]} \cong \partial N(\sigma'),$$

where $N(\sigma')$ denotes a tubular neighbourhood of σ'. This finished the proof. □

Example 4. Consider the surface Σ_1^* in the example 2. The following figure represents a sheet of any lamination with the hypotheses considered. The leaf is homeomorphic to a cylinder with infinite punctures.



Example 5. Consider the genus two surface Σ_2 with the unique decomposition consisting of two pants such that each pair of pants has a common boundary and both of them glue along the remaining boundary. The following figure represents a sheet of any lamination fibering over Σ_2 with the considered hypotheses, and their respective graphs.



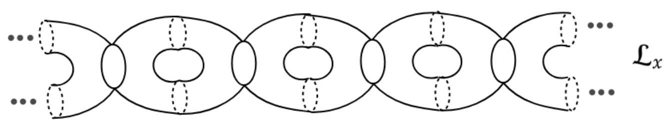
Example 6. Given an orientation on the graph σ of the Example 5 in such a way the concatenation cb is defined. The fundamental group $\pi_1(\sigma)$ is the free group generated by the elements a and cbc^{-1},

$$\pi_1 = \langle a, cbc^{-1} \rangle.$$

Consider the normal covering σ' corresponding to the subgroup of $\pi_1(\sigma)$ generated by the elements

$$(cbc^{-1})^n a^{-n}, \quad n \in \mathbb{Z}.$$

The graph σ' is the infinite ladder. By Corollary 3.6.10, there is a locally trivial fibration L as in the hypotheses 1.3.4 with a leaf homeomorphic to the infinite ladder surface represented in the following figure.



3.7 Explicit Description of $T_{TLC}(L)$

In this section it will be used the canonical towers constructed in Sections 3.3, 3.4 and 3.5 to give the desired explicit description of the space $T_{TLC}(L)$. Before to give the description, note that under the considered hypotheses we can include infinite conformal type surface laminations. This work is the first time that an explicit description of a Teichmüller space of laminations fibering over an infinite conformal type surface is given.

In the following lemma the functions $\Delta_{SS'}$ defined in Corollary 3.5.7 are considered. Recall that the identification (2.9) of the Fenchel-Nielsen Teichmüller space T_{FN} with the space of l_∞ sequences was used in such a corollary.

Lemma 3.7.1. *For S and S' in $\mathcal{S}$ such that $S' \leq S$, the following diagram commutes*

$$\begin{array}{ccc} T(\Sigma_{S'}) & \xrightarrow{\quad FN_{\mathcal{C}_{S'}} \quad} & T_{FN}(\Sigma_{S'}, \mathcal{C}_{S'}) \\ \big\downarrow \varepsilon_{SS'} & & \big\downarrow \Delta_{S'S} \\ T(\Sigma_S) & \xrightarrow{\quad FN_{\mathcal{C}_S} \quad} & T_{FN}(\Sigma_S, \mathcal{C}_S) \end{array}$$

where $\varepsilon_{SS'}$ is the canonical inclusion (see Lemma 3.4.5), $\Delta_{SS'}$ is the embedding in Corollary 3.5.7 and FN is the locally bi-Lipschitz homeomorphism in Corollary 2.6.8.

Proof. Consider $\Gamma_S = \{l_g \mid g \in G \cap S\} < \text{Möb}(\Delta)$, this is a Fuchsian group of Σ_S. We have an analogous definition of the Fuchsian group $\Gamma_{S'}$. Then we have $\Gamma_{S'} \leq \Gamma_S$.

Consider a point $[\mu] \in T(\Sigma_S)$ and the Fuchsian groups

$$\Gamma_{S,[\mu]} = \Theta_{f^\mu}(\Gamma_S),\ \Gamma_{S',[\mu]} = \Theta_{f^\mu}(\Gamma_{S'}),$$

along with the corresponding hyperbolic surfaces

$$\Sigma_{S,[\mu]} = \Delta/\Gamma_{S,[\mu]},\ \Sigma_{S',[\mu]} = \Delta/\Gamma_{S',[\mu]}.$$

Since $\Gamma_{S'} \leq \Gamma_S$, we have by definition $\Gamma_{S',[\mu]} \leq \Gamma_{S,[\mu]}$ hence there is a finite locally isometric covering

$$\Sigma_{S',[\mu]} \to \Sigma_{S,[\mu]}.$$

By the embedding between Teichmüller spaces, we have $[\mu] \in T(\Sigma_{S'})$ and the corresponding hyperbolic surface is $\Sigma_{S',[\mu]}$. Considering the map $\hat{\psi}_{S'S} : \Sigma_{S',[\mu]} \to \Sigma_{S,[\mu]}$ defined as before, and by construction of $\Delta_{SS'}$, we have

$$FN_{\mathcal{C}_{S'}}(\epsilon_{SS'}([\mu])) = \Delta_{SS'}(FN_{\mathcal{C}_S}([\mu])).$$

Because the point $[\mu]$ was arbitrary, we have the result. □

Theorem 3.7.2. *Consider a hyperbolic surface Σ without boundary obtained by gluing a (possibly finite) sequence of generalized hyperbolic pair of pants, each glued to the next along a common boundary geodesic such that the length of these geodesic boundaries is uniformly upper bounded. Consider a minimal lamination L fibering over Σ with Hausdorff compact fiber F whose holonomy action continuously extends to the profinite completion of the fundamental group G and has trivial holonomy on pants. Then, there is a homeomorphism*

$$T_{TLC}(L) \cong C_{LC}(F, T(\Sigma)),$$

where the left hand side is the transversally locally constant Teichmüller space of L and the right hand side is the space of locally constant functions valued on the Teichmüller space of Σ. In particular, $T_{TLC}(L)$ is contractible since $T(\Sigma)$ is so.

Proof. Taking direct limits in the commutative diagram of Lemma 3.7.1 gives the homeomorphism

$$FN : \varinjlim_{S \in \mathcal{S}} T(\Sigma_S) \to \varinjlim_{S \in \mathcal{S}} T_{FN}(\Sigma_S, \mathcal{C}_S).$$

Composing this maps with the homeomorphisms η and Ψ^* in Lemmas 3.4.6 and 3.28 respectively gives a homeomorphism

$$T_{TLC}(L) \cong C_{LC}(F, T_{FN}(\Sigma, \mathcal{C})).$$

Finally, by Corollary 2.6.8 we have

$$T_{TLC}(L) \cong C_{LC}(F, T(\Sigma))$$

and this concludes the proof. □

We have that although the transversally locally constant Teichmüller space is in general infinite dimensional, if we consider a surface with separable Teichmüller space, the Theorem 3.7.2 gives a family of separable contractible infinite dimensional transversally locally constant Teichmüller spaces.

Example 7. Consider the Jacob´s ladder surface Σ which is the boundary of a small enough neighbourhood $\mathcal{S}$ of a ladder Γ that is infinitely long in both directions. The canonical inclusion $\iota : \Sigma \to \bar{\mathcal{S}}$ induce an epimorphism

$$\iota_* : \pi_1(\Sigma) \to \pi_1(\bar{\mathcal{S}}) \cong \pi_1(\Gamma).$$

Consider the lamination L that is the inverse limit of all finite coverings corresponding to the preimages by ι_* of the normal finite index subgroups of $\pi_1(\Gamma)$. Since the ladder Γ is homotopically equivalent to the infinite wedge product of circles,

$$\Gamma \simeq (S^1)^{\wedge \mathbb{Z}},$$

the fundamental group of the ladder is the free group with generating set $\mathbb{Z}$ and the fiber of the laminations L is its profinite completion. Then, by Theorem 3.7.2 we have a homeomorphism

$$T_{TLC}(L) \cong C_{LC}\left(\overline{F(\mathbb{Z})}^{p.f}, T(\Sigma)\right).$$

Corollary 3.7.3. *Under the hypotheses of Theorem 3.7.2, we have an embedding*

$$C(F, T(\Sigma)) \to T(L).$$

Proof. If we take the Teichmüller distance on the left hand side of the homeomorphism, then it is dominated by the supremum distance on the right. Therefore, taking closures on both sides we have the result. □

Corollary 3.7.4. *Consider the lamination L as in the hypotheses of Theorem 3.7.2. Then the Teichmüller space of L is infinite dimensional if and only if there is a leaf in L of infinite type.*

Proof. Consider a surface Σ and a lamination L fibering over Σ with fiber F as in the hypotheses of Theorem 3.7.2.

Suppose that $T(L)$ is infinite dimensional. If the fiber F is finite, then, by the minimality of the lamination, L consists of a single leaf $\mathcal{L}$ that is a normal covering of Σ and verifies $T(\mathcal{L}) \cong T(L)$. The leaf $\mathcal{L}$ must be of infinite type otherwise $T(\mathcal{L})$ would be finite dimensional which is absurd. If the fiber F is infinite, then, by the minimality of the lamination again, every G-orbit $x \cdot G$ has infinite elements hence, by the characterization in Corollary 1.23, every leaf $\mathcal{L}_x$ is of infinite type. Conversely, suppose that L has a leaf of infinite type. Then, the fiber F is infinite or Σ is of infinite type. In either case, $C(F, T(\Sigma))$ is infinite dimensional hence, by Corollary 3.7.3, $T(L)$ is infinite dimensional as well and the result follows. □

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