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Criterios de Normalidad para Ideales Monomiales

Tesis que presenta

Humberto Muñoz George

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Center for Research and Advanced Studies
of the National Polythechnic Institute
Campus Zacatenco
Department of Mathematics

Normality Criteria for Monomial Ideals

A dissertation presented by

Humberto Muñoz George

to obtain the Degree of

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in the Speciality of

Mathematics

Thesis Advisor: Dr. Rafael Heraclio Villarreal Rodríguez

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To my oldest brother, Daniel.

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Co-Authorship

Chapters 3, 4 and 5 are based on the joint work with Rafael H. Villarreal and Luis A. Dupont in [35]. The rest of the chapters constitute literature review and results from other autors (see references).

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Resumen

En esta tesis estudiamos la normalidad de ideales monomiales utilizando programación lineal y teoría de grafos. Damos criterios de normalidad para ideales monomiales, para ideales generados por monomios de grado dos y para ideales de aristas de grafos, clutters y sus ideales de coberturas.

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Abstract

In this thesis we study the normality of monomial ideals using linear programming and graph theory. We give normality criteria for monomial ideals, for ideals generated by monomials of degree two, and for edge ideals of graphs and clutters and their ideals of covers.

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Introduction

In this thesis we study the normality of monomial ideals, and the normality of edge ideals of graphs and clutters and their ideals of covers. A main problem in this area is the characterization of the normality of the ideal of covers $I_c(G)$ of a graph G in terms of the combinatorics of G. If G is an odd cycle or a perfect graph, then $I_c(G)$ is normal, see [1] and [32], respectively.

The contents of this thesis are as follows. In Section 1, we introduce the notation requeried, we also introduce definitions as well as some background knowledge of these areas that will be used later to describe the problem investigated in this job.

In Section 2, we introduce a few results from polyhedral geometry, commutative algebra, and linear programming.

As we now recall there are two well known characterizations of the normality of I, one comes from commutative algebra and uses Rees algebras, and the other comes from integer programming and uses Hilbert bases.

The Rees algebra of I can be written as

$$S[Iz] = S \oplus Iz \oplus \cdots \oplus I^n z^n \oplus \cdots \subset S[z].$$

It is well known [29, p. 168] that the integral closure of $S[Iz]$ is given by

$$\overline{S[Iz]} = S \oplus \overline{I}z \oplus \cdots \oplus \overline{I^n} z^n \oplus \cdots \subset S[z].$$

Thus, the ring $S[Iz]$ is normal if and only if the ideal I is normal. Normal monomial subrings arise in the theory of toric varieties [6].

Let I be a monomial ideal of S and let $\mathcal{G}(I) = \{t^{v_1}, \ldots, t^{v_q}\}$ be the minimal set of generators of I. Let $\mathcal{A}' := \{e_1, \ldots, e_s, (v_1, 1), \ldots, (v_q, 1)\}$, where e_i is the i-th unit vector in $\mathbb{R}^{s+1}$. The ideal I is normal if and only if $\mathcal{A}'$ is a Hilbert basis (Lemma 2.6.2).

In Section 3 we give normality criteria for monomial ideals and membership tests to determine whether or not a given monomial lies in the integral closure of I^n or is a minimal generator of the integral closure of I^n. Let I_1 and I_2 be ideals of S generated by monomials in disjoint sets of variables, we show that $I_1 I_2$ is normal if and only if I_1 and I_2 are normal (Proposition 3.1.1).

We arrive at our first normality criterion.

Proposition 3.1.3 *Let I be a monomial ideal of S and let A be its incidence matrix. The following conditions are equivalent.*

(a) I is a normal ideal.

(b) For each pair of vectors $\alpha \in \mathbb{N}^s$ and $\lambda \in \mathbb{Q}_+^q$ such that $A\lambda \leq \alpha$, there is $m \in \mathbb{N}^q$ satisfying $Am \leq \alpha$ and $|\lambda| = |m| + \epsilon$ with $0 \leq \epsilon < 1$.

Given a monomial ideal I and a monomial t^α, a linear programming membership test for the question "is t^α a member of $\overline{I}$?" was shown in [7, Proposition 3.5], [19, Proposition 1.1]. The following proposition gives a linear algebra membership test that complement these results.

Proposition 3.2.1 (Membership test) *Let I be a monomial ideal of S, let A be its incidence matrix, and let t^α be a monomial in S. The following are equivalent.*

(a) $t^\alpha \in \overline{I^n}$, $n \geq 1$.

(b) $A\lambda \leq \alpha$ for some $\lambda \in \mathbb{Q}_+^q$ with $|\lambda| \geq n$.

(c) $\max\{\langle y, 1\rangle \mid y \geq 0;\ Ay \leq \alpha\} = \min\{\langle \alpha, x\rangle \mid x \geq 0;\ xA \geq 1\} \geq n$.

As a byproduct we obtain a minimal generators test for the integral closure of the powers of a monomial ideal.

Proposition 3.2.2 *Let I be a monomial ideal of S and let A be its incidence matrix. A monomial $t^\alpha \in S$ is a minimal generator of $\overline{I^n}$ if and only if the following two conditions hold.*

$$\max\{\langle y, 1\rangle \mid y \geq 0;\, Ay \leq \alpha\} = \min\{\langle \alpha, x\rangle \mid x \geq 0;\, xA \geq 1\} \geq n.$$
$$\max\{\langle y, 1\rangle \mid y \geq 0;\, Ay \leq \alpha - e_i\} = \min\{\langle \alpha - e_i, x\rangle \mid x \geq 0;\, xA \geq 1\} < n$$

for each e_i for which $\alpha - e_i \geq 0$.

The equality $\overline{I^n} = \overline{(t^{nv_1}, \ldots, t^{nv_q})}$ for $n \geq 1$ comes from [7, Examples 1.4, 3.7]. We use the membership test to give a short proof of this equality (Corollary 3.2.3).

The normality of I is also related to integer rounding properties [9, Corollary 2.5]. Systems with the integer rounding property are well studied; see [2, 3], [26, Chapter 22], [27, Chapter 5], and references therein.

As an application we give a short proof of the fact that I is normal if and only if the system $x \geq 0; xA \geq 1$ has the integer rounding property [9, Corollary 2.5] (Corollary 3.3.2). This fact was shown in [9] using the theory of blocking and antiblocking polyhedra [2], [27, p. 82].

The notions of *contraction*, *deletion*, and *minor* come from combinatorial optimization [27].

If $\mathcal{C}$ is a clutter and $I(\mathcal{C})$ is a normal ideal, then $I(\mathcal{H})$ is a normal ideal for any minor $\mathcal{H}$ of $\mathcal{C}$ [11, Proposition 4.3]. The following result complements this fact.

Proposition 3.4.3 *Let $\mathcal{C}$ be a clutter and let $I_c(\mathcal{C})$ be its ideal of covers. If $I_c(\mathcal{C})$ is normal, then $I_c(\mathcal{H})$ is normal for any minor $\mathcal{H}$ of $\mathcal{C}$.*

In Section 4 we use Hilbert basis, Ehrhart rings, and a duality for integer rounding properties, to examine the normality of ideals generated by monomials of degree 2, and generalize some results that were previously known to be valid for edge ideals of graphs.

In what follows $I = (t^{v_1}, \ldots, t^{v_q})$ is an ideal generated by monomials of degree 2 and A is the incidence matrix of I. We set

$$
\begin{aligned}
&\mathcal{B} := \{e_{s+1}\} \cup \{e_i + e_{s+1}\}_{i=1}^{s} \cup \{(v_i, 1)\}_{i=1}^{q},\\
&\mathcal{Q} := \mathrm{conv}(0, e_1, \ldots, e_s, v_1, \ldots, v_q),\\
&R := K[\mathbb{N}\mathcal{B}] = K[z, t_1 z, \ldots, t_s z, t^{v_1} z, \ldots, t^{v_q} z], \text{ the } \textit{semigroup ring} \text{ of } \mathbb{N}\mathcal{B},\\
&\mathrm{Er}(\mathcal{Q}) := K[t^{\alpha} z^{b} \mid \alpha \in \mathbb{Z}^s \cap b\mathcal{Q}] \subset S[z], \text{ the } \textit{Ehrhart ring} \text{ of } \mathcal{Q}.
\end{aligned}
$$

The following is one of our main results.

Theorem 4.1.1 *Let $I = (t^{v_1}, \ldots, t^{v_q})$ be an ideal of S generated by monomials of degree 2. Then, I is a normal ideal if and only if $\mathcal{B}$ is a Hilbert basis.*

As an application we recover the fact that if I is the edge ideal of a connected graph, then I is normal if and only if $K[\mathbb{N}\mathcal{B}]$ is normal [8, Theorem 3.3] (Corollary 4.1.2).

We characterize when the Ehrhart ring of $\mathcal{Q}$ is the monomial subring $K[\mathbb{N}\mathcal{B}]$ using the integer rounding property.

Theorem 4.2.1 *Let I be an ideal of S generated by monomials of degree 2 and let A be the incidence matrix of I. Then, $\overline{K[\mathbb{N}\mathcal{B}]} = \mathrm{Er}(\mathcal{Q})$, and the equality*

$$K[\mathbb{N}\mathcal{B}] = \mathrm{Er}(\mathcal{Q})$$

holds if and only if the system $x \geq 0; xA \leq 1$ has the integer rounding property.

In particular, we recover the fact that if I is the edge ideal of a connected graph, then the semigroup ring $K[\mathbb{N}\mathcal{B}]$ is normal if and only if the system $x \geq 0; xA \leq 1$ has the integer rounding property [8, Theorem 3.3].

If A and A^* are the incidence matrices of I and I^*, respectively, then the system $x \geq 0; xA \geq 1$ has the integer rounding property if and only if the system $x \geq 0; xA^* \leq 1$ has the integer rounding property [3, Theorem 2.11]

(Theorem 2.4.7). We will use this duality to characterize the normality of the dual of the edge ideal of a graph using Hilbert bases.

If I is the edge ideal of a graph, we prove that I^* is normal if and only if the set $\mathcal{B}$ is a Hilbert basis (Corollary 4.3.2), and furthermore we prove that I^* is normal if and only if I is normal (Proposition 4.3.3). These two results follow from [3, Theorem 2.12] and [8, Theorem 3.3] when I is the edge ideal of a connected graph.

In Section 5, we study the normality of the ideal of covers $I_c(G)$ of a graph G and give a combinatorial criterion in terms of Hochster configurations for the normality of $I_c(G)$ when the independence number of G is at most two.

Let v be a vertex of a graph G. Recall that if $I_c(G)$ is normal, then $I_c(G \setminus v)$ is normal. The following results shows that the converse holds under a certain condition.

Theorem 5.1.1 *Let v be a vertex of a graph G. If the neighbor set $N_G(v)$ of v is a minimal vertex cover of G, then $I_c(G \setminus v)$ is normal if and only if $I_c(G)$ is normal.*

As a consequence, we recover a result of Al-Ayyoub, Nasernejad and Roberts showing that the ideal of covers of the cone over the graph G is normal if the ideal of covers of the graph G is normal [1, Theorem 1.6] (Corollary 5.1.3).

Let G be a graph and let $G_1, \ldots, G_r$ be its connected components. If the edge ideal $I(G)$ is normal, then the edge ideal $I(G_i)$ is normal for $i = 1, \ldots, r$ [11, Proposition 4.3] but the converse is not true (Example 6.0.8). Using Proposition 3.1.1, we show that $I_c(G)$ is normal if and only if $I_c(G_i)$ is normal for $i = 1, \ldots, r$ (Corollary 5.1.4).

Hochster gave an example of a connected graph whose edge ideal is not normal [33, p. 457] (cf. [28, Example 4.9]).

It was conjectured in [28, Conjecture 6.9] that the edge ideal of a graph G is normal if and only if the graph has no Hochster configurations. This conjecture was proved in [16, Corollary 5.8.10], [33, Corollary 10.5.9]. We give a direct proof of this conjecture using Vasconcelos's description of the integral closure of the Rees algebra of $I(G)$ [16, p. 265] (Theorem 5.2.3).

Using Proposition 4.3.3 and Theorem 5.2.3, we prove that if $\overline{G}$ is the disjoint union of two odd cycles of length at least 5, then $I_c(G)$ is not normal, and if $\overline{G}$ is an odd cycle of length at least 5, then $I_c(G)$ is normal (Corollary 5.2.4). Furthermore, we show that if $I_c(G)$ is normal, then $\overline{G}$ has no Hochster configuration with induced odd cycles C_1, C_2 of length at least 5 (Corollary 5.2.5).

The following result gives a combinatorial description of the normality of the ideal of covers of graphs with independence number at most two.

Theorem 5.2.7 (Duality criterion) *Let G be a graph with $\beta_0(G) \leq 2$. The following hold.*

(a) $I_c(G)$ is normal if and only if $I(\overline{G})$ is normal.

(b) $I_c(G)$ is normal if and only if $\overline{G}$ has no Hochster configurations.

In Section 6 and Appendix A, we show some examples and procedures for *Normaliz* [4] and *Macaulay*2 [18]. We give an example showing that Theorem 4.1.1 does not extend to ideals generated by monomials of degree greater than 2 (Example 6.0.6). If $\overline{G}$ is a cycle of length 7, then the ideals $I(G)$, $I(\overline{G})$, $I_c(G)$, and $I_c(\overline{G})$, are all normal (Example 6.0.7). In Examples 6.0.9–6.0.11, we illustrate an auxiliary result (Lemma 5.1.8) and the duality criterion (Theorem 5.2.7).

The graph G in Example 6.0.12 was introduced by Kaiser, Stehlík, and Škrekovski [23]. They show that the ideal of covers of G satisfies

$$\operatorname{depth}(S/I_c(G)^3) = 0 < 4 = \operatorname{depth}(S/I_c(G)^4),$$

i.e., the depth function of the cover ideal of G is not non-increasing, answering a question of Herzog and Hibi. They also show that $\operatorname{Ass}(S/I_c(G)^3)$ is not a subset of $\operatorname{Ass}(S/I_c(G)^4)$, i.e., the ideal of covers $I_c(G)$ of G does not satisfy the persistence property of associated primes (see [12, 24] and references therein). The graph G is denoted by H_4 in [23]. Using the normality test of Procedure 6.0.16 and *Macaulay*2 [18], we obtain that $I(G)$ and $I_c(G)$ are not normal whereas $I(\overline{G})$ and $I_c(\overline{G})$ are normal. This example shows that none of the implications of the duality criterion of Theorem 5.2.7(a) hold for graphs with independence number greater than 2.

Let G be the graph of Example 6.0.13. This graph has independence number equal to 3. Using the normality test of Procedure 6.0.16 and *Macaulay*2 [18], we obtain that $I(G)$ is normal, $I_c(G)$ is not normal, and furthermore $I_c(G)^5$ is not integrally closed. This example shows that the Hochster configurations of $\overline{G}$, with C_1, C_2 induced odd cycles of length at least five, are not the only obstructions to the normality of $I_c(G)$ (see Corollary 5.2.5).

For unexplained terminology and additional information, we refer to [7, 22, 29, 30] for the theory of integral closure, [21, 33] for the theory of edge ideals and monomial ideals, and [20, 26, 27] for the theory of Hilbert bases and polyhedral geometry.

Chapter 1

Definitions

In this chapter we introduce the notation requeried, we also introduce definitions as well as some background knowledge of these areas that will be used later to describe the problem investigated in this job.

1.1 Clutters

Definition 1.1.1. A *clutter* $\mathcal{C}$ is a pair (V, E) where V is a finite set and E is a family of subsets of V none of which is included in another.

The elements of V are called the *vertices* of $\mathcal{C}$ and those of E are the *edges* of $\mathcal{C}$. V and E are denoted by $V(\mathcal{C})$ and $E(\mathcal{C})$ respectively. See 6.0.1, 6.0.2, 6.0.3 to ilustrate examples of clutters.

Definition 1.1.2. A *graph* is a clutter $\mathcal{C}$ such that

$$E(\mathcal{C}) \subset \{\{x, y\} \mid x, y \in V(\mathcal{C}); x \neq y\}$$

See 6.0.3 to ilustrate an example of a graph.

Definition 1.1.3. A set of vertices C of $\mathcal{C}$ is called a *vertex cover* if every edge of $\mathcal{C}$ contains at least one vertex of C.

Definition 1.1.4. A *minimal vertex cover* of $\mathcal{C}$ is a vertex cover which is minimal with respect to inclusion.

Example 6.0.3 shows us a clutter and its 3 minimal vertex covers. In example 6.0.1 we have that $\{t_1, t_2\}$ is a vertex cover and $\{t_1\}, \{t_2, t_4\}$ are minimal vertex covers.

Definition 1.1.5. The clutter of minimal vertex cover of a clutter $\mathcal{C}$, denoted $\mathcal{C}^\vee$, is called the *blocker* of $\mathcal{C}$, where

$$V(\mathcal{C}^\vee) = V(\mathcal{C}) \quad and \quad E(\mathcal{C}^\vee) = \{C : C \text{ is a minimal vertex cover of } \mathcal{C}\}.$$

Example 6.0.1 shows the difference between a clutter $\mathcal{C}$ and its blocker.

1.2 Monomial ideals

Let K be a field and $S = K[\mathbf{x}]$ be the polynomial ring over K in n indeterminates $\mathbf{x} = (x_1, ..., x_n)$.

Definition 1.2.1. A *monomial* of S is an element of the form

$$\lambda x_1^{a_1}, ..., x_n^{a_n} \qquad\qquad (0 \neq \lambda; a_i \in \mathbb{N} = \{0, 1, ...\}).$$

Throughout this tesis, with very few exceptions, by a *monomial* we really mean a monomial with $\lambda = 1$. To make notation simpler we set

$$x^a = x_1^{a_1} ... x_n^{a_n} \qquad\qquad \text{for } (a_1, ..., a_n) \in \mathbb{N}^n.$$

In particular $t_i = t_{e_i}$, where e_i is the i-th unit vector in $\mathbb{R}^n$.

Definition 1.2.2. An ideal I of S is called a *monomial ideal* if there $\mathcal{A} \subset \mathbb{N}^n$ such that I is generated by $\{x^a \mid a \in \mathcal{A}\}$. If I is a monomial ideal the quotient ring R/I is called a *monomial ring*.

Theorem 1.2.3. *Every monomial ideal has a unique minimal set of monomial generators, and this set is finite.*

Proof. The *Hilbert Basis Theorem* says that every ideal I in S is finitely generated. A polynomial belongs to I if only if all its monomials belong to I. It implies that if I is a monomial ideal, then $I = \langle x^{a_1}, ..., x^{a_r} \rangle$. Now a monomial belongs to I if only if it is divisible by one of the generators of I so it follows that this set it must be unique. $\square$

Let I a monomial ideal. We denoted this unique minimal set of monomial generators by $\mathcal{G}(I)$

Definition 1.2.4. Let I be a monomial ideal of S and let $\mathcal{G}(I) = \{t^{v_1}, \ldots, t^{v_q}\}$ be the minimal set of generators of I. The *incidence matrix* of I, denoted by A, is the $s \times q$ matrix with column vectors $v_1, \ldots, v_q$.

Definition 1.2.5. A monomial x^a is *squarefree* if every coordinate of a is 0 or 1. An ideal is *squarefree* if it is generated by squarefree monomials.

1.3 Polyhedral sets and cones

An *affine space* or *linear variety* in $\mathbb{R}^n$ is by definition a translation of a linear subspace of $\mathbb{R}^n$.

Definition 1.3.1. Let A be a subset of $\mathbb{R}^n$. The *affine space generated* by A, denoted by aff(A), is the set of all *affine combinations* of points in A

$$\text{aff(A)} := \{a_1v_1 + ... + a_rv_r | v_i \in A, a_i \in \mathbb{R}, a_1 + ... + a_r = 1\}$$

There is a unique linear subspace V of $\mathbb{R}^n$ such that aff(A) $= x_0 +$ V, for some $x_0 \in \mathbb{R}^n$. The dimension of A is defined as dim A $= dim_{\mathbb{R}}(V)$.

A point $x \in \mathbb{R}^n$ is called a *convex combination* of $v_1, ..., v_r \in \mathbb{R}^n$ if there are $a_1, ..., a_r \in \mathbb{R}$ such that $a_i \geq 0$ for all i, $x = \sum_i a_iv_i$ and $\sum_i a_i = 1$.

Definition 1.3.2. Let A be a subset of $\mathbb{R}^n$. The *convex hull* of A, denoted by conv(A) is the set of all convex combinations of points in A. If A $=$ conv(A) we say that A is a convex set.

The *convex hull* of a subset A of $\mathbb{R}^n$ is the intersection of all convex sets containing it.

Definition 1.3.3. A *convex polytope* $P \subset \mathbb{R}^n$ is the convex hull of a finite set of points $v_1, ..., v_r \in \mathbb{R}^n$, that is, P $=$conv$(v_1, ..., v_r)$.

Definition 1.3.4. The *inner product* of two vector $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n) \in \mathbb{R}^n$ is defined by

$$\langle x, y \rangle = xy = x_1 y_1 + ... + x_n y_n$$

An affine space of $\mathbb{R}^n$ of dimension $n - 1$ is called an *affine hyperplane*.

Definition 1.3.5. Given $a \in \mathbb{R}^s \setminus \{0\}$ and $c \in \mathbb{R}$, the *affine hyperplane* $H(a, c)$ is defined as

$$H(a, c) := \{x \in \mathbb{R}^s | \, \langle x, a \rangle = c\}.$$

Notice that any affine hyperplane of $\mathbb{R}^n$ has this form and notice that there are two spaces bounded by $H(a, c)$.

Definition 1.3.6. Given $a \in \mathbb{R}^s \setminus \{0\}$ and $c \in \mathbb{R}$, the *two closed halfspaces* are defined as

$$H^+(a, c) := \{x \in \mathbb{R}^s | \, \langle x, a \rangle \geq c\}. \qquad \textit{and} \qquad H^-(a, c) := H^+(-a, -c)$$

If a and c are rational, the closed halfspace $H^+(a, c)$ (resp. affine hyperplane) is called a *rational closed halfspace* (resp. *rational affine hyperplane*). If $c = 0$, for simplicity the set H_a will denote the hyperplane of $\mathbb{R}^n$ through the origin with normal vector a, that is,

$$H_a := \{x \in \mathbb{R}^s | \langle x, a\rangle = c\}.$$

The two *closed halfspaces* bounded by H_a are denoted by

$$H_a^+ := \{x \in \mathbb{R}^s | \langle x, a\rangle \geq c\} \qquad \textit{and} \qquad H_a^- := \{x \in \mathbb{R}^s | \langle x, a\rangle \leq c\}.$$

Definition 1.3.7. A *polyhedral set* or *convex polyhedron* is a subset of $\mathbb{R}^n$ which is the intersection of a finite number of closed halfspaces of $\mathbb{R}^n$. The set $\mathbb{R}^n$ is considered a polyhedron.

Definition 1.3.8. A *rational polyhedron* is a subset of $\mathbb{R}^n$ which is the intersection of a finite number of rational closed halfspaces of $\mathbb{R}^n$.

Since the intersection of an arbitrary family of convex sets in $\mathbb{R}^n$ is convex, one derives that any polyhedral set is convex and closed.

The transpose of a matrix A (resp. vector x) will be denoted by A^t or $A^\top$ (resp. x^t or $x^\top$). Often a vector x will denote a column vector or a row vector, from the context the meaning should be clear. Thus, a polyhedral set $\mathcal{Q}$ can be represented as

$$\mathcal{Q} = \{x \in \mathbb{R}^n | Ax \leq b\}$$

for some matrix A and for some vector b. As usual, if $a = (a_1, ..., a_q)$ and $c = (c_1, ..., c_q)$ are vectors in $\mathbb{R}^q$, then $a \leq c$ means $a_i \leq c_i$ for all i.

Definition 1.3.9. A *hyperplane* H of $\mathbb{R}^n$ is an affine subspace of codimension 1 in an affine space.

Definition 1.3.10. Let $\mathcal{Q}$ be a closed convex set in $\mathbb{R}^n$. A hyperplane H of $\mathbb{R}^n$ is called a *supporting hyperplane* of $\mathcal{Q}$ if $\mathcal{Q}$ is contained in one of the two closed halfspaces bounded by H and $\mathcal{Q} \cap H = \emptyset$.

Definition 1.3.11. A *proper face* of a polyhedral set $\mathcal{Q}$ is a set $F \subset Q$ such that there is a supporting hyperplane $H(a,c)$ satisfying the conditions

a) $F = \mathcal{Q} \cap H(a,c) \neq \emptyset$

b) $\mathcal{Q} \not\subset H(a,c)$, and $\mathcal{Q} \not\subset H^+(a,c)$ or $\mathcal{Q} \not\subset H^-(a,c)$.

The *improper faces* of a polyhedral set $\mathcal{Q}$ are $\mathcal{Q}$ itself and $\emptyset$.

Definition 1.3.12. A proper face F of a polyhedral set $\mathcal{Q} \subset \mathbb{R}^n$ is called a *facet* of $\mathcal{Q}$ if $dim(F) = dim(\mathcal{Q}) - 1$.

Definition 1.3.13. The *cone generated*by a finite subset Γ of $\mathbb{R}^s$, denoted $\mathbb{R}_+\Gamma$, is the set of all finite linear combinations of Γ with coefficients in $\mathbb{R}_+$. If Γ is finite we say that $C = \mathbb{R}_+\Gamma$ is a *finitely generated cone*.

Definition 1.3.14. If $C \subset \mathbb{R}^n$ is closed under linear combinations with non negative real coefficients, we say that C is a *convex cone*. A *polyhedral cone* is a convex cone which is also a polyhedral set.

An affine space on dimension 1 is called a *line*. The following result is quite useful in determining the facets of a polyhedral cone.

Proposition 1.3.15. *[33, proposition 1.1.23] Let $\mathcal{A}$ be a finite set of points in $\mathbb{Z}^n$ and let F be a face of $\mathbb{R}_+\mathcal{A}$. The following hold.*

a) If $F \neq \{0\}$, then $F = \mathbb{R}_+\mathcal{V}$ for some $\mathcal{V} \subset \mathcal{A}$.

b) If $dim(F) = 1$ and $\mathbb{R}_+\mathcal{A}$ contains no lines, then $F = \mathbb{R}_+\alpha$ with $\alpha \in \mathcal{A}$.

c) If $dim(\mathbb{R}_+\mathcal{A}) = n$ and F F is a facet defined by the supporting hyperplane H_a then H_a a is generated by a linearly independent subset of $\mathcal{A}$.

Proof. Let $F = \mathbb{R}_+\mathcal{A}$ with $\mathbb{R}_+\mathcal{A} \subset H_a^-$. Then F is equal to the cone generated by the set $\mathcal{V} = \{\alpha \in \mathcal{A} \mid \langle \alpha, a \rangle\}$. This proves a). Parts b) and c) follow from a). $\square$

Definition 1.3.16. Let $\mathcal{Q}$ be a polyhedral set and $x_0 \in \mathcal{Q}$. The point x_0 is called a *vertex* or an *extreme point* of $\mathcal{Q}$ if $\{x_0\}$ is a proper face of $\mathcal{Q}$.

Definition 1.3.17. A polyhedron containing no lines is called *pointed*.

1.4 Edge ideals

Let $\mathcal{C}$ be a clutter with vertex set $V(\mathcal{C}) = \{t_1, \ldots, t_s\}$, we consider each vertex t_i as a variable, and we consider the polynomial ring $S = K[t_1, \ldots, t_s]$ over a field K.

Definition 1.4.1. The *edge ideal* of $\mathcal{C}$, denoted $I(\mathcal{C})$, is the ideal of S given by

$$I(\mathcal{C}) := (\{\textstyle\prod_{t_i \in e} t_i \mid e \in E(\mathcal{C})\}).$$

The minimal set of generators of $I(\mathcal{C})$, denoted $\mathcal{G}(I(\mathcal{C}))$, is the set of all squarefree monomials $t_e := \prod_{t_i \in e} t_i$ such that $e \in E(\mathcal{C})$. Any squarefree monomial ideal I of S is the edge ideal $I(\mathcal{C})$ of a clutter $\mathcal{C}$ with vertex set $V(\mathcal{C}) = \{t_1, \ldots, t_s\}$.

Definition 1.4.2. Let I be a monomial ideal of S. If I is the edge ideal of a clutter $\mathcal{C}$, the incidence matrix of I is called the *incidence matrix* of $\mathcal{C}$.

The example 6.0.3 show us a clutter and its incidence matrix.

Definition 1.4.3. The edge ideal $I(\mathcal{C}^\vee)$ of $\mathcal{C}^\vee$ is called the *ideal of covers* of $\mathcal{C}$ and is denoted by $I_c(\mathcal{C})$

1.5 Two special polyhedrons

1.5.1 The covering polyhedron

Definition 1.5.1. Let $I = I(\mathcal{C})$. The *covering polyhedron* of I, denoted by $\mathcal{Q}(I)$, is the rational polyhedron

$$\mathcal{Q}(I) := \{x \mid x \geq 0; xA \geq 1\},$$

where $1 = (1, \ldots, 1)$ and A is the incidence matrix of I.

Theorem 1.5.2. *[33, proposition 13.1.2] Let $\mathcal{Q}(I) := \{x \mid x \geq 0; xA \geq 1\}$ be the covering polyhedron of $I = I(\mathcal{C})$. The following are equivalent*

a) $C = \{t_1, ..., t_r\}$ is a minimal vertex cover of $\mathcal{C}$.

b) $\alpha = e_1 + ... + e_r$ is a vertex of $\mathcal{Q}(I)$.

Corollary 1.5.3. *[33, Corollary 13.1.3] A vector $\alpha \in \mathbb{R}^n$ is an integer vertex of $Q(I)$ if only if $\alpha = e_{i_1} + ... + ei_s$ for some minimal vertex cover $\{t_{i_1}, ..., t_{i_s}\}$ of $\mathcal{C}$*

Proof. By theorem 1.5.2 it suffices to observe that any integral vertex of $\mathcal{Q}(A)$ has entries in $\{0, 1\}$. □

We observe that exist a bijection between $E(\mathcal{C}^\vee)$ and the set of integral vertices of $\mathcal{Q}(I)$ (see example 6.0.3). More specifically, the map $E(\mathcal{C}^\vee) \to \{0, 1\}^s$, $C \mapsto \sum_{t_i \in C} e_i$, induces these bijection.

1.5.2 The Newton polyhedron

Definition 1.5.4. The *Minkowski sum* of subsets $A, B \subset \mathbb{R}^n$ is

$$\mathrm{A} + \mathrm{B} := \{a + b \mid a \in A, b \in B\}$$

Definition 1.5.5. Let I a monomial ideal with minimal set of generators $\{t^{v_1}, \ldots, t^{v_q}\}$. The *Newton polyhedron* of I, denoted $\mathrm{NP}(I)$, is the rational polyhedron

$$\mathrm{NP}(I) = \mathbb{R}^s_+ + \mathrm{conv}(v_1, \ldots, v_q),$$

where $\mathbb{R}_+ = \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$.

This polyhedron is the convex hull of the vectors exponents of monomials in I [34, p. 141] and is equal to $\{x \mid x \geq 0; xB \geq 1\}$ for some rational matrix B with non-negative entries [15, Proposition 3.5 (b)].

1.6 Rees algebras and Rees cones

Definition 1.6.1. The *Rees algebra* of an monomial ideal I with minimal set of generators $\{t^{v_1}, \ldots, t^{v_q}\}$ is the monomial subring

$$S[Iz] = K[t_1, \ldots, t_s, t^{v_1}z, \ldots, t^{v_q}z],$$

where $z = t_{s+1}$ is a new variable.

Following [11], we have the next definition.

Definition 1.6.2. The *Rees cone* of an ideal I with minimal set of generators $\{t^{v_1}, \ldots, t^{v_q}\}$, denoted $\mathrm{RC}(I)$, is defined as the rational cone

$$\mathrm{RC}(I) := \mathbb{R}_+\mathcal{A}' \tag{1.1}$$

generated by the set $\mathcal{A}' := \{e_1, \ldots, e_s, (v_1, 1), \ldots, (v_q, 1)\}$, where e_i is the i-th unit vector in $\mathbb{R}^{s+1}$.

Chapter 2

Preliminaries

In this section we introduce a few results from polyhedral geometry, commutative algebra, and linear programming. We continue to employ the notations and definitions used in chapter 1.

2.1 The fundamental theorem of linear inequalities

The fundamental theorem is due to Farkas and Minkowski, with sharpenings by Caratheodory and Weyl. Its geometric content is easily understood in three dimensions.

Theorem 2.1.1. (Fundamental theorem of Linear inequalities) [26, Theorem 7.1] *Let $a_1, ..., a_m, b$ be vectors in n-dimensional space. Then one and only one of the following happens:*

a) b is a nonnegative linear combination of linearly independent vectors from $a_1, ..., a_m$;

b) there exists a hyperplane $\{x|cx = 0\}$, containing $t-1$ linearly independent vectors from $a_1, ..., a_m$ such that $cb < 0$ and $ca_1, ..., ca_m \geq 0$, *where* $t := rank\{a_1, ..., a_m, b\}$.

Proof. We may asume that $a_1, ..., a_m$ span the n-dimensional space. Clearly, a) and b) exclude each other, as otherwise, if $b = \lambda_1 a_1 + .. + \lambda_m a_m$ with $\lambda_1, ..., \lambda_m \geq 0$, we would have the contradiction

$$0 > cb = \lambda_1(ca_1) + .. + \lambda_m(ca_m) \geq 0$$

To see that at least one of a) and b) holds, choose linearly independent $a_{i_1}, ..., a_{i_n}$ from $a_1, ..., a_m$ and set $D := \{a_{i_1}, ..., a_{i_n}\}$. Next apply the following iteration:

i) Write $b = \lambda_{i_1} a_{i_1} + ... + \lambda_{i_n} a_{i_n}$. If $\lambda_{i_1}, ..., \lambda_{i_n} \geq 0$, we are in case a).

ii) Otherwise, choose the smallest h among $i_1, ..., i_n$ with $i_h < 0$. Let $\{x|cx = 0\}$ be the hyperplane spanned by $D \setminus \{a_h\}$. We normalize c so that $ca_h = 1$. [Hence $cb = \lambda_h < 0$.]

iii) If $ca_1, ..., ca_m \geq 0$ we are in case b).

iv) Otherwise, choose the smallest s such that $ca_c < 0$. Then replace D by $(D \setminus \{a_h\}) \cup \ \{a_s\}$, and start the iteration anew.

We are finished if we have shown that this process terminates. Let D_k denote the set D as it is in the kth iteration. If the process does not terminate, then $D_k = D_1$ for some $k < l$ (as there are only finitely many choices for D). Let r be the highest index for which U, has been removed from D at the end of one of the iterations $k, k+1, ..., l-1$, say in iteration p. As $D_k = D_1$, we know that a_r also has been added to D in some iteration q with $k \leq q < l$. So

$$D_p \cap \{a_{r+1}, ..., a_m\} = D_q \cap \{a_{r+1}, ..., a_m\}$$

Let $D_p = \{a_{i_1}, ..., a_{i_n}\}$, $b = \lambda_{i_1} a_{i_1} + ... + \lambda_{i_n} a_{i_n}$, and let c' be the vector c found in ii) of iteration q. Then we have the contradiction:

$$0 > c'b = c'(\lambda_{i_1} a_{i_1} + ... + \lambda_{i_n} a_{i_n}) = \lambda_{i_1} c' a_{i_1} + ... + \lambda_{i_n} c' a_{i_n} > 0$$

The first inequality was noted in ii) iteration above. The last inequality follows from:

$$\begin{cases} i_j < r \quad then \quad \lambda_{i_j} \geq 0, c' a_{i_j} \geq 0 \\ \\ i_j < r \quad then \quad \lambda_{i_j} < 0, c' a_{i_j} < 0 \\ \\ i_j < r \quad then \quad \qquad c' a_{i_j} = 0 \end{cases}$$

□

2.2 The finite basis theorem

Most of the notions and results considered thus far make sense if we replace $\mathbb{R}$ by an intermediate field $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$, i.e., we can work in the affine space $\mathbb{K}^n$. However with very few exceptions, like for instance Theorem 2.2.1, we will always work in the Euclidean space $\mathbb{R}^n$ or $\mathbb{Q}^n$.

From the last section, we state the *fundamental theorem of linear inequalities* (see last section theorem 2.1.1):

Theorem 2.2.1. *[26, Theorem 7.1] Let $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$ be an intermediate field and let $C \subset \mathbb{K}^n$ be a cone generated by $\mathcal{A} = \{\alpha_1, ..., \alpha_q\}$. If $\alpha \in \mathbb{K}^n \setminus C$ and $t = rank\{\alpha_1, ..., \alpha_q, \alpha\}$, then there exists a hyperplane H_a containing $t - 1$ linearly independent vectors from $\mathcal{A}$ such that $\langle a, \alpha \rangle$ and $\langle a, \alpha_i \rangle \leq 0$ for $i = 1, ..., q$*

Theorem 2.2.2. (Farkas's Lemma) [33, Theorem 1.1.25] *Let A be an $s \times q$ matrix with entries in a field $\mathbb{K}$ and let $\alpha \in \mathbb{K}^s$. Assume $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$. Then,* *either there exists $x \in \mathbb{K}^q$ with $Ax = \alpha$ and $x \geq 0$, or there exists $\beta \in \mathbb{K}^s$ with $\beta A \geq 0$ and $\langle \beta, \alpha \rangle < 0$, but not both.*

Proof. Let $\mathcal{A} = \{\alpha_1, ..., \alpha_q\}$ be the set of column vectors of A. Assume that there is no $x \in \mathbb{K}^q$ with $Ax = \alpha$ and $x \geq 0$, i.e., α is not in $\mathbb{R}_+\mathcal{A}$. By theorem there is a hyperplane H_β such that $\langle \beta, \alpha \rangle < 0$ and $\langle \beta, \alpha_i \rangle \geq 0$ for all i. Hence $\beta A \geq 0$, as required. If both conditions hold, then $0 > \langle \beta, \alpha \rangle = \langle \beta, Ax \rangle = \langle \beta A, x \rangle \geq 0$, a contradiction. □

The theorem 2.2.2 is a general result but basically tell us that given a point and a cone then the point belong to cone or can separate it from the cone. The next result tells us how to separate a point from a cone (see example 6.0.9).

Corollary 2.2.3. *[33, Theorem 1.1.26] Let $C \subset \mathbb{K}^n$ be a cone generated by $\mathcal{A} = \{a_1, ..., a_m\}$. If $\gamma \in \mathbb{K}^n$ and $\gamma \notin C$, then there is a hyperplane H through the origin such that $\gamma \in H^{-1} \diagdown H$ and $C \subset H^+$.*

Proof. Let A be the matrix with column vectors $a_1, ..., a_m$. By Farkas's lemma (see Theorem) there exists $\mu \in \mathbb{K}^n$ such that $\mu A \geq 0$ and $\langle \gamma, \mu \rangle < 0$. Thus $\langle \mu, a_i \rangle \geq 0$ for all i. If H is the hyperplane through the origin with normal vector μ we get $C \subset H^+$, as required. □

Corollary 2.2.4. *Let $\mathcal{A}$ be a finite set in $\mathbb{Z}^n$, then*

$$\mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_+\mathcal{A} \qquad\qquad \mathbb{Z}^n \cap \mathbb{R}_+\mathcal{A} = \mathbb{Z}^n \cap \mathbb{Q}_+\mathcal{A}$$

where $\mathbb{Z}\mathcal{A}$ is the subgroup of $\mathbb{Z}^n$ spanned by $\mathcal{A}$ and $\mathbb{Q}_+ = \{x \in \mathbb{Q} \mid x \leq 0\}$.

Proof. It follows at once from Theorem 2.2.3. □

Theorem 2.2.5. *[33, Theorem 1.1.29] If $C \subset \mathbb{R}^n$ then C is a finitely generated cone (resp. finitely generated cone by rational vectors) if and only if C is a polyhedral cone (resp. rational polyhedral cone) in $\mathbb{R}^n$.*

Proof. $\Rightarrow$) Assume that $C \neq (0)$ is a cone generated by $\mathcal{A} = \{\alpha_1, ..., \alpha_m\}$. We set $r = \dim(\mathbb{R}_+\mathcal{A})$. Notice that $\mathrm{aff}(C)$ is the real vector space generated by $\mathcal{A}$, because $0 \in C$. If $\mathrm{aff}(C) = C$, then C is a polyhedral cone, because C is the intersection of n-r hyperplanes of $\mathbb{R}^n$ through the origin. Now assume that $C \subsetneq \mathrm{aff(C)}$. Consider the family

$$\mathcal{F} = \{F \mid F = H_a \cap C; \mathrm{dim(F) = r - 1; C \subset H_a^-}\}$$

By theorem 2.2.1 the family $\mathcal{F}$ is non-empty. Notice that $\mathcal{F}$ is a finite set because eache F in $\mathcal{F}$ is a cone generated by a subset of $\mathcal{A}$; see proposition 1.3.15. Assume that $\mathcal{F} = \{F_1, ..., F_s\}$, where $F_i = H_a \cap C$. We claim that the following equality holds

$$C = H_{a_1}^- \cap ... \cap H_{a_s}^- \cap \mathrm{aff(C)} \tag{2.1}$$

The inclusion "$\subset$" is clear. To show the other inclusion, we proceed by contradiction. Assume that there exits $\alpha \notin C$ such that α belongs to the right-hand side of Eq. 2.1. By theorem 2.2.1 and by reordering the elements of $\mathcal{A}$ if necessary, there is a hyperplane H_a containing linearly independent vectors $\alpha_1, ..., \alpha_{r-1}$ such that

i) $\langle a, \alpha \rangle > 0$

ii) $\langle a, \alpha_i \rangle \leq 0$ for $i = 1, 2, ..., m$.

Thus $F = H_a \cap C = H_{a_k} \cap C$ for some $a \leq k \leq s$. We may assume that $\alpha_1, ..., \alpha_r$ form a basis of aff(C). Since $\alpha \in$ aff(C) there are scalars $\lambda_1, ..., \lambda_r$ with $\alpha = \lambda_1\alpha_1 + ... + \lambda_r\alpha_r$. Using ii) we get $\langle a, \alpha \rangle = \lambda_r \langle \alpha_r, a \rangle > 0$. Hence $\lambda_r < 0$. On the other hand $\langle a_k, \alpha_r \rangle < 0$ because $C \subset H_{a_k}^-$ and $\dim(F_k)$=$r-1$. Therefore $\langle a_k, \alpha \rangle = \lambda_r \langle a_k, \alpha_r \rangle$ with $\lambda_r < 0$ and $\langle a_k, \alpha_r \rangle < 0$ a contradiction to the fact that $\alpha \in H_{a_k}^-$.
$\Leftarrow$) Assume that $C = H_{b_1}^- \cap ... \cap H_{b_r}^-$, where $b_1, ..., b_r \in \mathbb{R}^n$. Consider the cone

C' generated by $b_1, ..., b_r$. From the first part of the proof we can write

$$C' = \mathbb{R}_+\{b_1, ..., b_r\} = H_{\alpha_1}^- \cap ... \cap H_{\alpha_m}^- \tag{2.2}$$

for some set of vectors $\mathcal{A} = \{\alpha_1, ..., \alpha_m\}$ in $\mathbb{R}^n$. Next we show the equality $C = \mathbb{R}_+\mathcal{A}$. Notice that $\langle b_i, \alpha_j \rangle \leq 0$ for all i,j, because $b_i \in C'$ for all i. Thus $\mathbb{R}_+\mathcal{A} \subset C$. Assume that there is $\alpha \in C \setminus \mathbb{R}_+\mathcal{A}$. By corollary 2.2.3, there exists a hyperplane H_a such that $\mathbb{R}_+\mathcal{A} \subset H_a^-$ and $\langle a, \alpha \rangle > 0$. Hence $\langle \alpha_i, a \rangle \leq 0$ for all i, and by Eq. 2.2 we conclude that $a \in \mathbb{R}_+\{b_1, ..., b_r\}$. Therefore, we can write $a = \lambda_1 b_1 + ... + \lambda_r b_r$, $\lambda_i \geq 0$ for all i. Since $\alpha \in C$, we have $\langle \alpha, a \rangle = \lambda_1 \langle \alpha, b_1 \rangle + ... + \lambda_r \langle \alpha, b_r \rangle \leq 0$, contradicting $\langle a, \alpha \rangle > 0$. Thus $C = \mathbb{R}_+\mathcal{A}$. The respective statement about the rationality character of the representations is left as an exercise. $\square$

Theorem 2.2.6. *(Finite basis theorem)[33, Theorem 1.1.33] If $\mathcal{Q}$ is a set in $\mathbb{R}^n$, then $\mathcal{Q}$ is a polyhedron (resp. rational polyhedron) if and only if $\mathcal{Q}$ can be expressed as $\mathcal{Q} = \mathcal{P} + C$, where $\mathcal{P}$ is a convex polytope (resp. rational polytope) and C is a finitely generated cone (resp. finitely generated rational cone).*

Proof. $\Rightarrow$) Let $\mathcal{Q} = \{x | Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$, where A is a matrix and b is a vector. Considere the set

$$\left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \mid x \in \mathbb{R}^n; \lambda \in \mathbb{R}_+; Ax - \lambda b \leq 0 \right\}.$$

Notice that C' can be written as

$$\left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \mid x \in \mathbb{R}^n; \lambda \in \mathbb{R}; \begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} \leq 0 \right\},$$

where $-b$ is a column vector. Thus C' is a polyhedral cone in $\mathbb{R}^{n+1}$. Using corollary 2.2.5 we get that C' can be expressed as

$$C' = \mathbb{R}_+ \left\{ \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ \lambda_m \end{pmatrix} \right\} \qquad\qquad (\lambda_i \geq 0; x_i \in \mathbb{R}^n).$$

We may assume that $\lambda_i \in \{0, 1\}$ for all i. Set

$$\mathcal{A} = \{x_i \mid \lambda_i = 1\} = \{x_1, ..., x_r\}, \quad \mathcal{B} = \{x_i \mid \lambda_i = 0\} = \{x_{r+1}, ..., x_m\},$$

$\mathcal{P} = \mathrm{conv}(\mathcal{A})$, and $C = \mathbb{R}_+\mathcal{B}$. Notice that $x \in \mathcal{Q}$ if and only if $(x, 1) \in C'$. Thus $x \in \mathcal{Q}$ if and only if $(x, 1)$ can be written as

$$\begin{pmatrix} x_1 \\ 1 \end{pmatrix} = \mu_1 \begin{pmatrix} x_1 \\ 1 \end{pmatrix} + ... + \mu_r \begin{pmatrix} x_r \\ 1 \end{pmatrix} + \mu_{r+1} \begin{pmatrix} x_{r+1} \\ 0 \end{pmatrix} + ... + \mu_m \begin{pmatrix} x_m \\ 0 \end{pmatrix}$$

with $\mu_i \geq 0$ for all i. Consequently $x \in \mathcal{Q}$ if and only if $x \in \mathcal{P} + C$. Therefore we obtain $\mathcal{Q} = \mathcal{P} + C$.

$\Leftarrow$) Assume that $\mathcal{Q}$ is equal to $\mathcal{P} + C$ with $\mathcal{P} = \mathrm{conv}(\mathrm{x}_1, ..., \mathrm{x}_\mathrm{r})$ and $C = \mathbb{R}_+(x_{r+1}, ..., x_m)$. Consider the following finitely generated cone

$$C' = \mathbb{R}_+ \left\{ \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} x_r \\ 1 \end{pmatrix}, \begin{pmatrix} x_{r+1} \\ 0 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ 0 \end{pmatrix} \right\}.$$

By corollary 2.2.5 the cone C' is a polyhedron. Thus there exists a matrix A and a vector b such that C' can be written as

$$\left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \mid x \in \mathbb{R}^n; \lambda \in \mathbb{R}; Ax + \lambda b \leq 0 \right\}.$$

Since $x \in \mathcal{Q}$ if and only if $(x, 1) \in C'$ we conclude that $\mathcal{Q} = \{x \mid Ax \leq -b\}$, that is $\mathcal{Q}$ is a polyhedron. $\square$

The *finite basis theorem* asserts that a subset $\mathcal{Q}$ of $\mathbb{R}^s$ is a rational polyhedron if and only if

$$\mathcal{Q} = \mathbb{R}_+\Gamma + \mathcal{P},$$

where Γ is a finite set of rational points and $\mathcal{P}$ is the convex hull conv$(\mathcal{A})$ of a finite set $\mathcal{A}$ of rational points (see example 6.0.4). In particular the Newton polyhedron of a monomial ideal is a rational polyhedron.

Corollary 2.2.7. *A set $\mathcal{Q} \subset \mathbb{R}^n$ is a convex polytope if and only if $\mathcal{Q}$ is a bounded polyhedral set.*

Proof. If $\mathcal{Q} = \text{conv}(\alpha_1, ..., \alpha_{\text{m}})$ is a polytope, then by the triangle inequality for all $x \in \mathcal{Q}$ we have $\|x\| \leq \|\alpha_1\| + ... + \|\alpha_m\|$. Thus $\mathcal{Q}$ is bounded.

Conversely if $\mathcal{Q}$ is a bounded polyhedron, then by Theorem 2.2.6 we can descompose $\mathcal{Q}$ as $\mathcal{Q} + \mathcal{P} + C$, with $\mathcal{P}$ a polytope and C a finitely generated cone. Notice that $C = \{0\}$, otherwise fixing $p_0 \in \mathcal{P}$ and $c_0 \in C \setminus \{0\}$ we get $p_0 + \lambda c_0 \in \mathcal{Q}$ for all $\lambda > 0$, a contradiction because $\mathcal{Q}$ is bounded. Thus $\mathcal{P} = \mathcal{Q}$, as required. □

2.3 Hilbert basis

Definition 2.3.1. A *Hilbert basis* is a finite set of vectors $\mathcal{B} \subset \mathbb{R}^n$ with the property that every integer vector in the cone generated by this set is also a nonnegative integer combination of its elements ,i.e, if $\mathbb{R}_+\mathcal{B} \cap \mathbb{Z}^n = \mathbb{N}\mathcal{B}$.

Notice that all vectors in a Hilbert basis are integral.

Definition 2.3.2. Consider a cone $\mathbb{R}_+\Gamma \subset \mathbb{R}^n$ containing no lines and generated by a finite set Γ of rational points. A finite set $\mathcal{B}$ is called a *Hilbert basis* of $\mathbb{R}_+\Gamma$ if $\mathbb{R}_+\Gamma = \mathbb{R}_+\mathcal{B}$ and $\mathcal{B}$ is a Hilbert basis.

The program *Normaliz* [4] can be used to compute the Hilbert basis of $\mathbb{R}_+\Gamma$. Hilbert bases of rational polyhedral cones always exist. For a pointed cone there is only one minimal Hilbert basis. To prove these two facts, we need Gordan's lemma. Below we give two versions of this lemma.

Definition 2.3.3. Given a vector $c = (c_1, \ldots, c_n)$ in $\mathbb{R}^n$, we define the *value* of c as $|c| := \sum_{i=1}^{n} c_i$.

Lemma 2.3.4. *(Gordan, version 1) If $\mathcal{A} = \{v_1, \ldots, v_q\} \subset \mathbb{Z}^n$ and $\mathbb{Z}\mathcal{A}$ is the subgroup generated by $\mathcal{A}$, then there exist $\gamma_1, \ldots, \gamma_m$ in $\mathbb{Z}^n$ such that*

$$\mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A} = \mathbb{N}\{\gamma_1, \ldots, \gamma_m\} := \mathbb{N}\gamma_1 + \ldots + \mathbb{N}\gamma_m$$

and $\gamma_i \in [N, M]^n$ for all i, where $N = -\mathrm{q} \max_{1 \le \mathrm{i} \le \mathrm{q}} |v_i^-|$ and $M = \mathrm{q} \max_{1 \le \mathrm{i} \le \mathrm{q}} |v_i^+|$.

Proof. Recall that $C = \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_+\mathcal{A}$: see corollary 2.2.4. Let $\beta \in C$. Then one can write

$$\beta = \sum_{i=1}^{q} \left(\frac{x_i}{y_i}\right) v_i$$

Where $x_i \in \mathbb{N}$ and $0 \neq y_i \in \mathbb{N}$. By the division algorithm there are r_i, n_i in $\mathbb{N}$ such that $x_i = n_i y_i + r_i$ and $0 \le r_i < y_i$. Therefore one can write

$$\beta = \sum_{i=1}^{q} n_i v_i + \sum_{i=1}^{q} a_i v_i,$$

with $a_i \in [0, 1] \cap \mathbb{Q}$. As $\sum_{i=1}^{q} a_i v_i \in C \cap [N, M]^n$, the set $\mathcal{A} \cup (C \cap [N, M]^n)$ is a generating set for C with the required property. $\square$

Definition 2.3.5. A semigroup $(S, +, 0)$ of $\mathbb{Z}^n$ is said to be *finitely generated* if there exists a finite set $\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subset \mathcal{S}$ such that:

$$\mathcal{S} = \mathbb{N}\Gamma := \mathbb{N}\gamma_1 + \ldots + \mathbb{N}\gamma_r$$

A set of generators Γ of $\mathcal{S}$ is called minimal if $\gamma_i = 0$ $\forall i$ and none of its elements is a linear combination with coefficients in $\mathbb{N}$ of the others.

Any subsemigroup of $\mathbb{N}$ is finitely generated but there are examples of subsemigroups of $\mathbb{N}^n$, with $n \ge 2$, wich are not finitely generated.

Lemma 2.3.6. *(Gordan, version 2) If $\mathbb{R}_+\mathcal{A}$ is a cone in $\mathbb{R}^n$ generated by a finite set $\mathcal{A} \subset \mathbb{Z}^n$, the semigroup $\mathbb{Z}^n \cap \mathbb{R}_+\mathcal{A}$ is finitely generated.*

Proof. If $\mathcal{A} = \{v_1, \ldots, v_q\}$, consider the finite set of integral points:

$$\{a_1 v_1 + \cdots + a_q v_q v \mid 0 \leq a_i \leq 1\} \cap \mathbb{Z}^n = \{\gamma_1, \ldots, \gamma_r\}$$

It is left to the reader to prove that $\gamma_1, \ldots, \gamma_r$ is the required set of generators for $\mathbb{Z}^n \cap \mathbb{R}_+\mathcal{A}$. See [26, p. 233]. $\square$

Let $\mathcal{A}$ be a finite set in $\mathbb{Z}^n$ and let $G = \mathbb{Z}^n$ or $G = \mathbb{Z}\mathcal{A}$. Then, by Gordan's lemma (versions 1 and 2) there exists $\gamma_1, \ldots, \gamma_r \in \mathbb{Z}^n$ such that:

$$G \cap \mathbb{R}_+ = \mathbb{N}\gamma_1 + \ldots + \mathbb{N}\gamma_r,$$

Therefore Hilbert bases of rational polyhedral cones always exist:

Proposition 2.3.7. *Let $\mathcal{A}$ be a finite set in $\mathbb{Z}^n$. Then there exist $\gamma_1, \ldots, \gamma_r$ such that*

$$\mathbb{R}_+\mathcal{A} \cap \mathbb{Z}^n = \mathbb{N}\gamma_1 + \ldots + \mathbb{N}\gamma_r$$

and $\{\gamma_1, \ldots, \gamma_r\}$ is a Hilbert basis of $\mathbb{R}_+\mathcal{A}$.

Proof. The existence follows from Lemma 2.3.6. That $\mathcal{A}$ is a Hilbert basis of $\mathbb{R}_+\mathcal{A}$ follows from the equality $\mathbb{R}_+\mathcal{A} = \mathbb{R}_+\gamma_1 + \ldots + \mathbb{R}_+\gamma_r$. $\square$

In Definition 1.3.17 we considered pointed polyhedra. For cones we have the following equivalent definition.

Definition 2.3.8. A polyhedral cone $C = \{x \mid Ax \leq 0\}$ is called *pointed* if the lineality space $C = \{x \mid Ax = 0\}$ is equal to $\{0\}$.

Lemma 2.3.9. *Let C be a rational polyhedral cone. If C is pointed, then there exists an integral vector b such that $\langle b, x\rangle > 0$ for all $0 \neq x \in C$.*

Proof. There is a rational matrix A such that $C = \{x \mid Ax \leq 0\}$; see Theorem 2.2.5. We may assume that A is integral. If $u_1, \ldots, u_t$ are the rows of A, then $b = -u_1 - \ldots - u_t$ satisfies the required condition. $\square$

Theorem 2.3.10. *Let $\mathcal{A}$ be a finite set in $\mathbb{Z}^n$ and let $C = \mathbb{R}_+\mathcal{A}$. If $\mathcal{C}$ is pointed, then there exists a unique minimal Hilbert basis of C given by*

$$\mathcal{H} = \{x \in C \cap \mathbb{Z}^n \mid 0 \neq x \notin \mathbb{N}y_1 + \mathbb{N}y_2; \forall y_1, y_2 \in (C \setminus \{0\}) \cap \mathbb{Z}^n\}$$

Proof. Let $- = \{\gamma, \ldots, \gamma\}$ be a generating set of C. We claim that $\langle \subset \Gamma$. Let $x \in \mathcal{H}$. Since $\mathbb{Z}^n \cap C = \mathbb{N}\Gamma$ we can write $x = \sum_{i=1}^{r} a_i\gamma_i$, $0 \neq a_i \in \mathbb{N}$, Thus, by construction of $\mathcal{H}$, one has $x = \gamma$ for some i. To finish the proof it suffices to prove $\mathbb{N}\mathcal{H} = \mathbb{N}\Gamma$, because this equality implies $\mathbb{R}_+\mathcal{H} = \mathbb{R}_+\Gamma$ and consequently $\mathbb{Z}^n \cap \mathbb{R}_+\mathcal{H} = \mathbb{N}\mathcal{H}$. We argue by contradiction by assuming that the set:

$$\mathcal{V} = \mathbb{N}\Gamma \setminus \mathbb{N}\mathcal{H}$$

is not empty. Since the cone $\mathbb{R}_+\mathcal{A}$ is pointed, by theorem 2.3.9, there exists $b \in \mathbb{Z}^s$ such that $\langle b, x\rangle > 0$ for all $0 \neq x \in C$. Let $x_0 \in \mathcal{V}$ such that

$$\langle x_0, b\rangle = \min\{\langle x, b\rangle | \, x \in \mathcal{V}\}.$$

Since $x_0 \notin \mathcal{H}$, we can write $x_0 = x_1 + x_2$ with x_1, x_2 in $C \setminus \{0\} \cap \mathbb{Z}^n$. Thus, since $\langle x_i, b\rangle < \langle x_0, b\rangle$ for $i = 1, 2$, we have $x_1, x_2 \in \mathbb{N}\mathcal{H}$ and $x_0 = x_1 + x_2 \in \mathbb{N}\mathcal{H}$, a contradiction. $\square$

It may be verified easily that if C is not pointed there are several minimal bases.

2.4 Complete and normal ideals

Let R be a ring and let I be an ideal of R, an element $z \in R$ is *integral* over I if z satisfies an equation

$$z^l + a_1 z^{l-1} + ... + a_{l-1} z + a_l = 0, \qquad\qquad a_i \in I^i.$$

The *integral closure* of I is the set of all elements $z \in R$ which are integral over I. This set will be denoted by $\overline{I}$

Definition 2.4.1. If $I = \overline{I}$, I is said to be *integrally closed* or *complete*.

Definition 2.4.2. If I^n is integrally closed for all $n \geq 1$, I is said to be *normal*.

Given an integer $n \geq 1$, the *integral closure* of I^n, denoted $\overline{I^n}$, can be described as

$$\overline{I^n} = (\{t^a \mid a/n \in \mathrm{NP}(I)\}) = (\{t^a \mid \langle a, u_i \rangle \geq n \text{ for } i = 1, \ldots, p\}), \tag{2.3}$$

where $\langle\ ,\ \rangle$ denotes the ordinary inner product and $u_1, \ldots, u_p$ are the vertices of $\mathcal{Q}(I)$ [15, Theorem 3.1, Proposition 3.5]. The edge ideal of a clutter is integrally closed [33, p. 153].

Theorem 2.4.3. *[33, Proposition 4.3.7] If A is an integral domain and S is a multiplicatively closed subset of A, then*

$$\overline{S^{-1}(A)} = S^{-1}(\overline{A}) \tag{2.4}$$

Proof. Note that A and $S^{-1}(A)$ have the same field of fractions K. First we prove $\overline{S^{-1}(A)} \subset S^{-1}(\overline{A})$. Take any x in K integral over $S^{-1}(A)$. There is an equation

$$x^n + \frac{a_1}{s_1} x^{n-1} + ... + \frac{a_{n-1}}{s_{n-1}} x + \frac{a_n}{s_n} = \frac{0}{1} \tag{2.5}$$

where $a_i \in A$ and $s_i \in S$ for all i. Set $s = s_1 ... s_n$ If we multiply by s^n, it follows that $sx \in \overline{a}$ and $x \in S^{-1}(\overline{A})$.
Conversely take $x \in S^{-1}(\overline{A})$. There is $s \in S$ such that sx is integral over A. Hence sx satisfies

$$(sx)^n + a_1(sx)^{n-1} + ... + a_{n-1}sx + a_n = 0 \tag{2.6}$$

for some $a_1, ..., a_n$ in A, dividing by sn immediately yields that x is integral over $S^{-1}(A)$, as required. □

Corollary 2.4.4. *[33, Corollary 4.3.8] If A is a normal domain and S is a multiplicatively closed subset of A, then $S^{-1}(A)$ is a normal domain.*

Corollary 2.4.5. *[33, Corollary 4.3.9] Let $R = K[x_1, ..., x_n]$ be a polynomial ring over a field K and let K_R be its field of fractions. If $R' = K[x^{\pm 1}, ..., x^{\pm 1}] \subset K_R$ is the ring of Laurent polynomials , then R' is a normal domain.*

Proof. Notice that R' is the localization of R at the multiplicative set of monomials of R. Hence, by the last Corollary, we get that R' is normal. □

Lemma 2.4.6. *[29, p. 169] If I is a monomial ideal of S and $n \in \mathbb{N}_+$, then*

$$\overline{I^n} = (\{t^a \in S \mid (t^a)^p \in I^{pn} \textit{ for some } p \geq 1\}).$$

Proof. This follows using Eq. (2.3) and Farkas's lemma (Theorem 2.2.2). □

The following duality is valid for incidence matrices of clutters (See definition 3.3.1).

Theorem 2.4.7. *[3, Theorem 2.11] Let $A = (a_{i,j})$ be a $\{0, 1\}$-matrix and let $A^* = (a^*_{i,j})$ be the matrix whose (i, j)-entry is $a^*_{i,j} = 1 - a_{i,j}$. Then, the system $x \geq 0; xA \geq 1$ has the integer rounding property if and only if the system $x \geq 0; xA^* \leq 1$ has the integer rounding property.*

2.5 Monomial subrings

Let $F = \{t^{w_1}, \ldots, t^{w_r}\}$ be a set of monomials of S and let $\mathcal{F}$ be the family of all subrings D of S such that $K \cup F \subset D$. The *monomial subring generated* F is defined by

$$K[F] := \bigcap_{D \in \mathcal{F}} D.$$

The elements of $K[F]$ have the form $\sum c_a \, (t^{w_1})^{a_1} \cdots (t^{w_r})^{a_r}$ with $c_a \in K$ and all but a finite number of c_a's are zero. Let $\mathcal{A}$ be the set of vectors $\{w_1, \ldots, w_r\} \subset \mathbb{N}^s$. As a K-vector space $K[F]$ is generated by the set of monomials of the form t^a, with a in the semigroup $\mathbb{N}\mathcal{A}$ generated by $\mathcal{A}$. That is, $K[F] = K[\{t^a \mid a \in \mathbb{N}\mathcal{A}\}]$. This means that $K[F]$ coincides with $K[\mathbb{N}\mathcal{A}]$, the *semigroup ring* of the semigroup $\mathbb{N}\mathcal{A}$ (see [14]).

2.6 Integral closure of monomial subrings

If R is an integral domain with field of fractions K_R, recall that the *normalization* or *integral closure* of R is the subring $\overline{R}$ consisting of all the elements of K_R that are integral over R. If $R = \overline{R}$ we say that R is *normal*. Normal monomial subrings arise in the theory of toric varieties [6].

Theorem 2.6.1. *([13, pp. 29–30], [33, Theorem 9.1.1]) If $\Gamma \subset \mathbb{N}^s$ is a finite set of points and $\mathcal{S} = \mathbb{Z}\Gamma \cap \mathbb{R}_+\Gamma$, then the following hold:*

a) $K[\mathcal{S}] := K[\{t^a \mid a \in \mathcal{S}\}]$ *is normal*

b) $\overline{K[F]} = K[\mathcal{S}]$, *where* $F = \{t^a \mid a \in \Gamma\}$.

Proof. (a): By Theorem 2.2.5, there are non-zero vectors $a_1, ..., a_p$ in $\mathbb{Z}^n$ such that $\mathbb{R}_+\Gamma = H^+_{a_1} \cap ... \cap H^+_{a_p}$. We set $\mathcal{S}_i = \mathbb{Z}\Gamma \cap H^+_{a_i}$. Since $K[\mathcal{S}]$ is equal to $K[\mathcal{S}_1] \cap ... \cap K[\mathcal{S}_p]$ and since the intersection of normal domains is a normal domain, it suffices to show that $K[\mathcal{S}_i]$ is normal for all i. If $\mathbb{Z}\Gamma \subset H_{a_i}$, then $\mathcal{S}_i = \mathbb{Z}\Gamma \simeq \mathbb{Z}^r$. Hence $K[\mathcal{S}_i] \simeq K[\mathbb{Z}^r] = k[x_1^{\pm}, ..., x_n^{\pm}]$ which is normal by Corollary 2.4.5. Thus, we may assume that $\mathbb{Z}\Gamma \not\subset H_{a_i}$. Setting $r = rank(\mathbb{Z}\Gamma)$ and $\mathcal{L} = \mathbb{Z}\Gamma \cap H_a$, one has $rank(\mathcal{L}) = r - 1$. This follows by noticing that $\mathbb{Q}\mathcal{L} = \mathbb{Q}\Gamma \cap H_a$, and using the equality

$$n = dim_{\mathbb{Q}}(\mathbb{Q}\Gamma + H_a) = dim_{\mathbb{Q}}(\mathbb{Q}\Gamma) + dim_{\mathbb{Q}}(H_a) - dim_{\mathbb{Q}}(\mathbb{Q}\Gamma \cap H_a) \qquad (2.7)$$

together with the fact that the ranks of are equal to $\mathbb{Z}\Gamma$ and $\mathcal{L}$ are equal to $dim_{\mathbb{Q}}(\mathbb{Q}\Gamma)$ and $dim_{\mathbb{Q}}(\mathbb{Q}\mathcal{L})$, respectively. The quotient group $H = \mathbb{Z}\Gamma/\mathcal{L}$ is torsion-free of rank 1. Thus, H is a free abelian group of rank 1 and we can write $H = \mathbb{Z}\overline{\alpha}$ for some is $0 \neq \alpha \in \mathbb{Z}\Gamma$ such that $\langle \alpha, a_i \rangle > 0$. As a consequence, we get that $\mathcal{S}_i = \mathcal{L} \oplus \mathbb{N}\alpha \simeq \mathbb{Z}^{r-1} \oplus \mathbb{N}$ and $K[\mathcal{S}_i] \simeq K[\mathbb{Z}^{r-1} \oplus \mathbb{N}]$. Therefore $\mathcal{S}_i$ is normal because $K[\mathbb{Z}^{r-1} \oplus \mathbb{N}]$ is equal to $K[x_1^{\pm}, ..., x_{r-1}^{\pm}, x_r]$ and this ring is normal again by Corollary 2.4.5.
(b): Since $K[F] \subset K[\mathcal{S}]$ taking integral closures and using part (a) gives the inclusion $\overline{K[F]} \subset K[\mathcal{S}]$. To show the reverse inclusion, note the equality $\mathbb{Z}\Gamma \cap \mathbb{R}_+\Gamma = \mathbb{Z}\Gamma \cap \mathbb{Q}_+\Gamma$ (see corollary 2.2.4). A straightforward calculation shows that $x\alpha$ is in the field of fractions of $K[F]$ if $\alpha \in \mathbb{Z}\Gamma$, and x^{α} is an integral element over $K[F]$ if $\alpha \in \mathbb{Q}_+\Gamma$. Hence $K[\mathcal{S}] \subset \overline{K[F]}$. $\square$

Lemma 2.6.2. *Let $I = (t^{v_1}, \ldots, t^{v_q})$ be a monomial ideal of S and let $\mathcal{A}'$ be the subset of $\mathbb{R}^{s+1}$ given by $\{e_i\}_{i=1}^{s} \cup \{(v_i, 1)\}_{i=1}^{q}$. Then, I is normal if and only if $\mathbb{R}_+\mathcal{A}' \cap \mathbb{Z}^{s+1} = \mathbb{N}\mathcal{A}'$.*

Proof. The Rees algebra of I is the monomial subring

$$S[Iz] = K[t_1, \ldots, t_s, t^{v_1}z, \ldots, t^{v_q}z]$$

where $z = t_{s+1}$ is a new variable. The Rees algebra of I can be written as

$$S[Iz] = S \oplus Iz \oplus \cdots \oplus I^n z^n \oplus \cdots \subset S[z]. \qquad (2.8)$$

It is well known [29, p. 168] that the integral closure of $S[Iz]$ is given by

$$\overline{S[Iz]} = S \oplus \overline{I}z \oplus \cdots \oplus \overline{I^n}z^n \oplus \cdots \subset S[z]. \tag{2.9}$$

By Eqs. (2.8)–(2.9), the ring $S[Iz]$ is normal if and only if the ideal I is normal. On the other hand the Rees algebra of I can also be written as

$$S[It] = K[\{t^a z^b \mid (a,b) \in \mathbb{N}\mathcal{A}'\}] = K[\mathbb{N}\mathcal{A}'], \tag{2.10}$$

and since $\mathbb{Z}\mathcal{A}' = \mathbb{Z}^{s+1}$, by Theorem 2.6.1, one has

$$\overline{S[It]} = K[\{t^a z^b \mid (a,b) \in \mathbb{Z}^{s+1} \cap \mathbb{R}_+\mathcal{A}'\}] = K[\mathbb{Z}^{s+1} \cap \mathbb{R}_+\mathcal{A}']. \tag{2.11}$$

Hence, by Eqs. (2.10)–(2.11),

$$S[It] \text{ is normal if and only if } \mathbb{N}\mathcal{A}' = \mathbb{Z}^{s+1} \cap \mathbb{R}_+\mathcal{A}'.$$

$\square$

Chapter 3

Normality Criteria for Monomial Ideals

In this section we give normality criteria for monomial ideals and membership tests to determine whether or not a given monomial lies in the integral closure of I^n or is a minimal generator of the integral closure of I^n. To avoid repetitions, we continue to employ the notations and definitions used in Sections 1 and 2.

3.1 First normality criterion

Proposition 3.1.1. *Let I_1 and I_2 be ideals of S generated by monomials in disjoint sets of variables. The following hold.*

(a) $I_1I_2 = I_1 \cap I_2$. (b) $\overline{I_1}\ \overline{I_2} \subset \overline{I_1I_2}$.

(c) $\overline{(I_1I_2)^n} = \overline{I_1^n}\ \overline{I_2^n}$ *for all* $n \geq 1$.

(d) I_1I_2 *is normal if and only if* I_1 *and* I_2 *are normal.*

Proof. (a) Clearly $I_1I_2 \subset I_1 \cap I_2$. To show the reverse inclusion take $t^a \in I_1 \cap I_2$. Then, we can write $t^a = t^bt^\gamma$ and $t^a = t^ct^\delta$ with $t^b \in \mathcal{G}(I_1)$ and

$t^c \in \mathcal{G}(I_2)$. Since the monomials in $\mathcal{G}(I_1)$ do not have any common variable with the monomials in $\mathcal{G}(I_2)$, from the equality $t^b t^\gamma = t^c t^\delta$ it follows that $t^a \in I_1 I_2$.

(b) Take $t^a t^b \in \overline{I_1}\,\overline{I_2}$ with $t^a \in \mathcal{G}(\overline{I_1})$ and $t^b \in \mathcal{G}(\overline{I_2})$. Then, by Lemma 2.4.6, we get that $(t^a)^k \in I_1^k$ and $(t^b)^\ell \in I_2^\ell$ for some positive integers k, ℓ. Thus

$$(t^a t^b)^{k\ell} = (t^a)^{k\ell}(t^b)^{\ell k} \in I_1^{k\ell} I_2^{\ell k} = (I_1 I_2)^{k\ell},$$

and consequently $(t^a t^b)^{k\ell} \in (I_1 I_2)^{k\ell}$. Hence $t^a t^b \in \overline{I_1 I_2}$.

(c) It suffices to show the case $n = 1$ because $(I_1 I_2)^n = I_1^n I_2^n$ for all $n \geq 1$. By parts (a) and (b), one has $\overline{I_1 I_2} \subset \overline{I_1} \cap \overline{I_2} = \overline{I_1}\ \overline{I_2} \subset \overline{I_1 I_2}$, and equality holds everywhere.

(d) $\Rightarrow$) Assume that $I_1 I_2$ is normal. To show that I_1 is normal we need only show the inclusion $\overline{I_1^n} \subset I_1^n$ for all $n \geq 1$. Take $t^a \in \mathcal{G}(\overline{I_1^n})$ and fix $t^b \in \mathcal{G}(\overline{I_2^n})$. From (c), one has

$$(I_1 I_2)^n = \overline{(I_1 I_2)^n} = \overline{I_1^n}\ \overline{I_2^n},$$

and consequently $t^a t^b \in (I_1 I_2)^n$. Thus, we can write

$$t^a t^b = (t^{c_1} t^{d_1}) \cdots (t^{c_n} t^{d_n}) t^\epsilon, \tag{3.1}$$

with $t^{c_i} \in \mathcal{G}(I_1)$ and $t^{d_i} \in \mathcal{G}(I_2)$ for $i = 1, \ldots, n$. Since the monomials in $\mathcal{G}(I_1)$ do not have any common variable with the monomials in $\mathcal{G}(\overline{I_2^n})$, from Eq. (3.1) it follows that t^a is a multiple of $t^{c_1} \cdots t^{c_n}$, and $t^a \in I_1^n$. By a similar argument we obtain that I_2 is normal.

$\Leftarrow$) Assume that I_1 and I_2 are normal. Then, by part (c), $I_1 I_2$ is normal.

$\square$

Definition 3.1.2. Given a vector $c = (c_1, \ldots, c_p)$ in $\mathbb{R}^p$, we denote the integral part of c by $\lfloor c \rfloor$ and the ceiling of c by $\lceil c \rceil$. We denote the nonnegative rational numbers by $\mathbb{Q}_+$.

We come to our first normality criterion.

Proposition 3.1.3. *Let I be a monomial ideal of S and let A be its incidence matrix. The following conditions are equivalent.*

(a) I is a normal ideal.

(b) For each pair of vectors $\alpha \in \mathbb{N}^s$ and $\lambda \in \mathbb{Q}_+^q$ such that $A\lambda \le \alpha$, there is $m \in \mathbb{N}^q$ satisfying $Am \le \alpha$ and $|\lambda| = |m| + \epsilon$ with $0 \le \epsilon < 1$.

Proof. (a)$\Rightarrow$(b): Pick $r \in \mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ such that $A(r\lambda) \le r\alpha$ and $r\lambda \in \mathbb{N}^q$. Let $\lambda_1, \ldots, \lambda_q$ be the entries of λ. Then, regarding the v_i's as column vectors, one has

$$A(r\lambda) = (r\lambda_1)v_1 + \cdots + (r\lambda_q)v_q \le r\alpha \quad \therefore \quad (t^\alpha)^r \in I^{r|\lambda|} \subset I^{r\lfloor |\lambda| \rfloor}.$$

Thus, by Lemma 2.4.6, $t^\alpha \in \overline{I^{\lfloor |\lambda| \rfloor}} = I^{\lfloor |\lambda| \rfloor}$ because I is a normal ideal. Then, we can write $t^\alpha = (t^{v_1})^{m_1} \cdots (t^{v_q})^{m_q} t^\delta$, with $m_i \in \mathbb{N}$ for all i, $\sum_{i=1}^q m_i = \lfloor |\lambda| \rfloor$, and $\delta \in \mathbb{N}^s$. If m is the vector with entries $m_1, \ldots, m_q$, then $Am \le \alpha$ and $|\lambda| = \lfloor |\lambda| \rfloor + \epsilon = |m| + \epsilon$ with $0 \le \epsilon < 1$.

(b)$\Rightarrow$(a): To show that I is a normal ideal take $t^\alpha \in \overline{I^n}$. Then, by Lemma 2.4.6, $(t^\alpha)^r \in I^{rn}$ for some $r \in \mathbb{N}_+$ and we can write

$$t^{r\alpha} = (t^{v_1})^{n_1} \cdots (t^{v_q})^{n_q} t^\delta,$$

with $n_i \in \mathbb{N}$ for all i, $\sum_{i=1}^q n_i = nr$, and $\delta \in \mathbb{N}^s$. Setting $\lambda = (n_1, \ldots, n_q)/r$, one has

$$\alpha = \left(\sum_{i=1}^q (n_i/r)v_i \right) + (\delta/r) = A\lambda + (\delta/r) \ge A\lambda.$$

Hence, by hypothesis, there is $m \in \mathbb{N}^q$ satisfying $Am \le \alpha$ and $|\lambda| = |m| + \epsilon$ with $0 \le \epsilon < 1$. Note that $|\lambda| = n$, and consequently $\epsilon = 0$ because $n - |m| = \epsilon$ is an integer. Thus, $n = |\lambda| = |m|$ and from the inequality $Am \le \alpha$ it follows readily that $t^\alpha \in I^n$. $\square$

3.2 Membership test

Given a monomial ideal I and a monomial t^α, a linear programming membership test for the question "is t^α a member of $\overline{I}$?" was shown in [7, Proposition 3.5], [19, Proposition 1.1]. The following proposition gives a linear algebra membership test that complement these results.

Proposition 3.2.1. (Membership test) *Let $I = (t^{v_1}, \ldots, t^{v_q})$ be a monomial ideal of S, let A be its incidence matrix, and let t^α be a monomial in S. The following are equivalent.*

(a) $t^\alpha \in \overline{I^n}$, $n \geq 1$.

(b) $A\lambda \leq \alpha$ *for some* $\lambda \in \mathbb{Q}_+^q$ *with* $|\lambda| \geq n$.

(c) $\max\{\langle y, 1\rangle \mid y \geq 0;\, Ay \leq \alpha\} = \min\{\langle \alpha, x\rangle \mid x \geq 0;\, xA \geq 1\} \geq n$.

Proof. The sets $A_1 = \{y \mid y \geq 0;\, Ay \leq \alpha\}$ and $A_2 = \{x \mid x \geq 0;\, xA \geq 1\}$ are not empty because $0 \in A_1$ and, since $v_i \in \mathbb{N}^s \setminus \{0\}$ for all i, one can choose the entries of x large enough so that $x \in A_2$, and furthermore the maximum in the left hand side of part (c) is finite. Hence, by linear programming duality [26, Corollary 7.1g, p. 91, Eq. (19)], the equality in part (c) holds.

(a)$\Rightarrow$(b): By Lemma 2.4.6, $(t^\alpha)^r \in I^{rn}$ for some $r \in \mathbb{N}_+$. Hence,

$$r\alpha = p_1 v_1 + \cdots + p_q v_q + \delta = Ap + \delta \geq Ap,$$

where $p = (p_1, \ldots, p_q) \in \mathbb{N}^q$, $|p| = rn$ and $\delta \in \mathbb{N}^s$. Therefore, making $\lambda = p/r$, one obtains that $\lambda \in \mathbb{Q}_+^q$, $A\lambda \leq \alpha$ and $|\lambda| = n$.

(b)$\Rightarrow$(c): This is clear because $\max\{\langle y, 1\rangle \mid y \geq 0;\, Ay \leq \alpha\} \geq n$ and, as noted above, the equality of part (c) holds.

(c)$\Rightarrow$(a): Pick an optimal feasible solution $\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{Q}_+^q$ where the maximum in the linear programming duality equation is attained. Then, $|\lambda| \geq n$ and $A\lambda \leq \alpha$. Choose $r \in \mathbb{N}_+$ such that $r\lambda \in \mathbb{N}^q$. Hence

$$A(r\lambda) = (r\lambda_1)v_1 + \cdots + (r\lambda_q)v_q \leq r\alpha,$$

and consequently $(t^\alpha)^r \in I^{r|\lambda|} \subset I^{rn}$. Thus, by Lemma 2.4.6, $t^\alpha \in \overline{I^n}$. $\Box$

As a byproduct we obtain a minimal generators test for the integral closure of the powers of a monomial ideal.

Proposition 3.2.2. *Let I be a monomial ideal of S and let A be its incidence matrix. A monomial $t^\alpha \in S$ is a minimal generator of $\overline{I^n}$ if and only if the following two conditions hold.*

$$\max\{\langle y, 1\rangle \mid y \geq 0;\ Ay \leq \alpha\} = \min\{\langle \alpha, x\rangle \mid x \geq 0;\ xA \geq 1\} \geq n. \tag{3.2}$$

$$\max\{\langle y, 1\rangle \mid y \geq 0;\ Ay \leq \alpha - e_i\} = \min\{\langle \alpha - e_i, x\rangle \mid x \geq 0;\ xA \geq 1\} < n \tag{3.3}$$

for each e_i for which $\alpha - e_i \geq 0$.

Proof. $\Rightarrow$) By Proposition 3.2.1, Eq. (3.2) holds. If Eq. (3.3) does not hold for some i for which $\alpha - e_i \geq 0$, then by Proposition 3.2.1 one has $t^{\alpha - e_i} \in \overline{I^n}$ and $t^\alpha = t_i t^{\alpha - e_i}$, a contradiction.

$\Leftarrow$) By Eq. (3.2) and Proposition 3.2.1, one has that $t^\alpha \in \overline{I^n}$. We argue by contradiction assuming that $t^\alpha \notin \mathcal{G}(\overline{I^n})$. Then, there is $t^\beta \in \mathcal{G}(\overline{I^n})$ such that $t^\alpha = t^\delta t^\beta$ and t_i divides t^δ for some i. Hence $t^{\alpha - e_i} \in \overline{I^n}$ and, by Proposition 3.2.1, Eq. (3.3) does not hold, a contradiction. $\Box$

The equality $\overline{I^n} = \overline{(t^{nv_1}, \ldots, t^{nv_q})}$ for $n \geq 1$ comes from [7, Examples 1.4, 3.7]. We use the membership test to give a short proof of this equality.

Corollary 3.2.3. *If $I = (t^{v_1}, \ldots, t^{v_q})$ is a monomial ideal, then*

$$\overline{I^n} = \overline{(t^{nv_1}, \ldots, t^{nv_q})}$$

for $n \geq 1$.

Proof. Let A be the incidence matrix of I. We set $J = (t^{nv_1}, \ldots, t^{nv_q})$. The inclusion $\overline{I^n} \supset \overline{J}$ is clear because $I^n \supset J$. To show the reverse inclusion take

$t^\alpha \in \overline{I^n}$. Then, by Proposition 3.2.1, $A\lambda \leq \alpha$ for some $\lambda \in \mathbb{Q}_+^q$ with $|\lambda| \geq n$. Hence $(nA)(\lambda/n) \leq \alpha$ and, by Proposition 3.2.1, we get that $t^\alpha \in \overline{J}$ because nA is the incidence matrix of J and $|\lambda/n| \geq 1$. $\square$

3.3 Integer rounding properties

The normality of I is also related to integer rounding properties [9, Corollary 2.5].

Definition 3.3.1. We have the two next definitions

a) The linear system $x \geq 0; xA \geq 1$ has the *integer rounding property* if

$$\max\{\langle y, 1\rangle \mid y \in \mathbb{N}^q;\ Ay \leq \alpha\} = \lfloor \max\{\langle y, 1\rangle \mid y \geq 0;\ Ay \leq \alpha\}\rfloor \quad (3.4)$$

for each integral vector α for which the right-hand side is finite.

b) The linear system $x \geq 0; xA \leq 1$ has the *integer rounding property* if

$$\lceil \min\{\langle y, 1\rangle \mid y \geq 0;\ Ay \geq \alpha\}\rceil = \min\{\langle y, 1\rangle \mid y \in \mathbb{N}^q;\ Ay \geq \alpha\} \quad (3.5)$$

for each integral vector α for which the left hand side is finite.

Systems with the integer rounding property are well studied; see [2, 3], [26, Chapter 22], [27, Chapter 5], and references therein.

As an application we give a short proof of the fact that I is normal if and only if the system $x \geq 0; xA \geq 1$ has the integer rounding property [9, Corollary 2.5] (Corollary 3.3.2). This fact was shown in [9] using the theory of blocking and antiblocking polyhedra [2], [27, p. 82].

Corollary 3.3.2. *[9, Corollary 2.5] Let $I = (x^{v_1}, \ldots, x^{v_q})$ be a monomial ideal and let A be the matrix with column vectors $v_1, \ldots, v_q$. Then, I is a normal ideal if and only if the system $x \geq 0; xA \geq 1$ has the integer rounding property.*

Proof. $\Rightarrow$) Assume that I is normal. Let α be an integral vector for which the right-hand side of Eq. (3.4) is finite. Therefore

$$\max\{\langle y, 1\rangle \mid y \geq 0; Ay \leq \alpha\} = \langle \lambda, 1\rangle = |\lambda|$$

for some $\lambda \in \mathbb{Q}_+^q$ with $A\lambda \leq \alpha$. Note that $\alpha \in \mathbb{N}^s$. In general one has

$$\max\{\langle y, 1\rangle \mid y \in \mathbb{N}^q;\ Ay \leq \alpha\} \leq \lfloor \max\{\langle y, 1\rangle \mid y \geq 0; Ay \leq \alpha\} \rfloor = \lfloor |\lambda| \rfloor. \tag{3.6}$$

Then, by Proposition 3.1.3, there is $m \in \mathbb{N}^q$ satisfying $Am \leq \alpha$ and $|\lambda| = |m| + \epsilon$ with $0 \leq \epsilon < 1$. Thus, the left-hand side of Eq. (3.6) is at least $|m|$, the right-hand side of Eq. (3.6) is $|m|$, and equality holds in Eq. (3.6).

$\Leftarrow$) Assume that the system $x \geq 0; xA \geq 1$ has the integer rounding property. To prove that I is normal take $t^\alpha \in \overline{I^n}$, $n \geq 1$. Then, by Proposition 3.2.1, $A\lambda \leq \alpha$ for some $\lambda \in \mathbb{Q}_+^q$ with $|\lambda| \geq n$. Therefore

$$\max\{\langle y, 1\rangle \mid y \in \mathbb{N}^q;\ Ay \leq \alpha\} = \lfloor \max\{\langle y, 1\rangle \mid y \geq 0; Ay \leq \alpha\} \rfloor \geq \lfloor |\lambda| \rfloor \geq n.$$

Hence, there is $m \in \mathbb{N}^q$ such that $Am \leq \alpha$, $|m| \geq n$, and consequently $t^\alpha \in I^{|m|} \subset I^n$. $\square$

3.4 Blocking and antiblocking polyhedral

The following notions of *contraction*, *deletion*, and *minor* come from combinatorial optimization [27].

Definition 3.4.1. Given a clutter $\mathcal{C}$ and a vertex $t_i \in V(\mathcal{C})$, the *contraction* $\mathcal{C}/t_i$ and *deletion* $\mathcal{C} \setminus t_i$ are the clutters constructed as follows:
Both have $V(\mathcal{C}) \setminus \{t_i\}$ as vertex set, $E(\mathcal{C}/t_i)$ is the set of minimal elements of $\{e \setminus \{t_i\} | \, e \in E(\mathcal{C}\}$, minimal with respect to inclusion, and $E(\mathcal{C} \setminus t_i)$ is the set $\{e | \, t_i \notin e \in E(\mathcal{C})\}$. A *minor* of $\mathcal{C}$ is a clutter obtained from $\mathcal{C}$ by a sequence of deletions and contractions in any order.

Definition 3.4.2. The *support* of a monomial $t^a \in S$, $a = (a_1, \ldots, a_s)$, denoted by $\mathrm{supp}(t^a)$, is the set of all variables t_i such that $a_i > 0$.

If $\mathcal{C}$ is a clutter and $I(\mathcal{C})$ is a normal ideal, then $I(\mathcal{H})$ is a normal ideal for any minor $\mathcal{H}$ of $\mathcal{C}$ [11, Proposition 4.3]. The following result complements this fact.

Proposition 3.4.3. *Let $\mathcal{C}$ be a clutter and let $I_c(\mathcal{C})$ be its ideal of covers. If $I_c(\mathcal{C})$ is normal, then $I_c(\mathcal{H})$ is normal for any minor $\mathcal{H}$ of $\mathcal{C}$.*

Proof. It suffices to show that $I_c(\mathcal{C} \setminus v)$ and $I_c(\mathcal{C}/v)$ are normal for any $v \in V(\mathcal{C})$. We set $V(\mathcal{C}) = \{t_1, \ldots, t_s\}$, $\mathcal{H} = \mathcal{C} \setminus v$, $\mathcal{D} = \mathcal{C}/v$, and $v = t_s$. We may assume that t_s is not an isolated vertex of $\mathcal{C}$, i.e., there is at least one edge of $\mathcal{C}$ that contains t_s.

To prove that $I_c(\mathcal{C} \setminus t_s)$ is normal, we show that $I_c(\mathcal{H})^n = \overline{I_c(\mathcal{H})^n}$ for all $n \geq 1$. The inclusion $I_c(\mathcal{H})^n \subset \overline{I_c(\mathcal{H})^n}$ holds in general. To show the reverse inclusion take $t^a = t_1^{a_1} \cdots t_{s-1}^{a_{s-1}} \in \overline{I_c(\mathcal{H})^n}$, $a = (a_1, \ldots, a_{s-1}, 0)$. Then, there is $k \geq 1$ such that $(t^a)^k \in I_c(\mathcal{H})^{kn}$ and we can write

$$(t^a)^k = t^{b_1} \cdots t^{b_{nk}} t^{\delta},$$

with $t^{b_i} \in \mathcal{G}(I_c(\mathcal{H}))$ for $i = 1, \ldots, kn$. We may assume that $t^{b_1}, \ldots, t^{b_r}$ are in $\mathcal{G}(I_c(\mathcal{C}))$ and $t^{b_{r+1}}, \ldots, t^{b_{kn}}$ are not in $\mathcal{G}(I_c(\mathcal{C}))$. Note that $t^{b_{r+1}} t_s, \ldots, t^{b_{kn}} t_s$ are in $\mathcal{G}(I_c(\mathcal{C}))$. Therefore,

$$(t^a t_s^n)^k = t^{b_1} \cdots t_r^{b_r} (t^{b_{r+1}} t_s) \cdots (t^{b_{kn}} t_s) t_s^r t^{\delta},$$

and consequently $(t^a t_s^n)^k \in I_c(\mathcal{C})^{kn}$, that is, $t^a t_s^n \in \overline{I_c(\mathcal{C})^n} = I_c(\mathcal{C})^n$. Then, $t^a t_s^n = t^{c_1} \cdots t^{c_n} t^{\gamma}$ with $t^{c_i} \in \mathcal{G}(I_c(\mathcal{C}))$ for $i = 1, \ldots, n$. Any minimal vertex cover of $\mathcal{C}$ contains a minimal vertex cover of $\mathcal{H}$. Hence, we can write each t^{c_i} as $t^{c_i} = t^{d_i} t^{\epsilon_i}$ with $t^{d_i} \in \mathcal{G}(I_c(\mathcal{H}))$. From the equality

$$t^a t_s^n = (t^{d_1} \cdots t^{d_n})(t^{\epsilon_1} \cdots t^{\epsilon_n}) t^{\gamma},$$

we obtain $t^a = t^{d_1}\cdots t^{d_n}t^{\epsilon}$ because $t_s \notin \mathrm{supp}(t^{d_i})$ for $i = 1,\ldots,n$, and $t^a \in I_c(\mathcal{H})^n$.

To prove that $I_c(\mathcal{C}/t_s)$ is normal, we show that $I_c(\mathcal{D})^n = \overline{I_c(\mathcal{D})^n}$ for all $n \geq 1$. The inclusion $I_c(\mathcal{D})^n \subset \overline{I_c(\mathcal{D})^n}$ holds in general. To show the reverse inclusion take $t^a = t_1^{a_1}\cdots t_{s-1}^{a_{s-1}} \in \overline{I_c(\mathcal{D})^n}$, $a = (a_1,\ldots,a_{s-1},0)$. Then, there is $k \geq 1$ such that $(t^a)^k \in I_c(\mathcal{D})^{kn}$ and we can write

$$(t^a)^k = t^{b_1}\cdots t^{b_{nk}}t^{\delta},$$

with $t^{b_i} \in \mathcal{G}(I_c(\mathcal{D}))$ for $i = 1,\ldots,kn$. Let f_i be the support of t^{b_i}. Then, either $f_i \in E(\mathcal{C}^\vee)$ or $f_i = e \setminus t_s$ for some $e \in E(\mathcal{C}^\vee)$ for which $t_s \in e$. Thus, either $t^{b_i} \in \mathcal{G}(I_c(\mathcal{C}))$ or $t_s t^{b_i} \in \mathcal{G}(I_c(\mathcal{C}))$. Hence, $(t^a t_s^n)^k \in I_c(\mathcal{C})^{kn}$, that is, $t^a t_s^n \in \overline{I_c(\mathcal{C})^n} = I_c(\mathcal{C})^n$. Then, $t^a t_s^n = t^{c_1}\cdots t^{c_n}t^{\gamma}$ with $t^{c_i} \in \mathcal{G}(I_c(\mathcal{C}))$ for $i = 1,\ldots,n$. Making $t_s = 1$, it follows readily that $t^a \in I_c(\mathcal{D})^n$ because each t^{c_i} is divisible by some monomial t^{u_i} in $\mathcal{G}(I_c(\mathcal{D}))$. $\square$

Chapter 4

Ideals Generated by Monomials of Degree 2

In this section we use Hilbert basis, Ehrhart rings, and a duality for integer rounding properties, to examine the normality of ideals generated by monomials of degree 2, and generalize some results that were previously known to be valid for edge ideals of graphs.

In what follows $I = (t^{v_1}, \ldots, t^{v_q})$ is an ideal generated by monomials of degree 2 and A is the incidence matrix of I. We set

$$\mathcal{B} := \{e_{s+1}\} \cup \{e_i + e_{s+1}\}_{i=1}^{s} \cup \{(v_i, 1)\}_{i=1}^{q}, \tag{4.1}$$

$$\mathcal{Q} := \operatorname{conv}(0, e_1, \ldots, e_s, v_1, \ldots, v_q), \tag{4.2}$$

$$R := K[\mathbb{N}\mathcal{B}] = K[z, t_1 z, \ldots, t_s z, t^{v_1} z, \ldots, t^{v_q} z], \text{ the } \textit{semigroup ring} \text{ of } \mathbb{N}\mathcal{B}, \tag{4.3}$$

$$\operatorname{Er}(\mathcal{Q}) := K[t^{\alpha} z^{b} \mid \alpha \in \mathbb{Z}^s \cap b\mathcal{Q}] \subset S[z], \text{ the } \textit{Ehrhart ring} \text{ of } \mathcal{Q}. \tag{4.4}$$

4.1 Hilbert basis

The following is one of our main results.

Theorem 4.1.1. *Let $I = (t^{v_1}, \ldots, t^{v_q})$ be an ideal of S generated by monomials of degree 2. Then, I is a normal ideal if and only if $\mathcal{B}$ is a Hilbert basis.*

Proof. $\Rightarrow$) Assume that I is normal. The inclusion $\mathbb{R}_+\mathcal{B} \cap \mathbb{Z}^{s+1} \supset \mathbb{N}\mathcal{B}$ holds in general. To show the reverse inclusion take $(\alpha, b) \in \mathbb{R}_+\mathcal{B} \cap \mathbb{Z}^{s+1}$, $\alpha \in \mathbb{N}^s$, $b \in \mathbb{N}$. By Farkas's lemma (Theorem 2.2.2), we obtain that (α, b) is in $\mathbb{Q}_+\mathcal{B}$, that is, we can write

$$(\alpha, b) = \tau_1 e_{s+1} + \sum_{i=1}^{s} \mu_i(e_i + e_{s+1}) + \sum_{i=1}^{q} \lambda_i(v_i, 1), \tag{4.5}$$

where τ_1, μ_i, and λ_i are in $\mathbb{Q}_+$. Then, setting $\lambda = (\lambda_1, \ldots, \lambda_q)$ and $\mu = (\mu_1, \ldots, \mu_s)$, one has $A\lambda \leq \alpha$ and $b \geq |\mu| + |\lambda|$. Hence, by Proposition 3.1.3, there is $m = (m_1, \ldots, m_q) \in \mathbb{N}^q$ satisfying $Am \leq \alpha$ and $|\lambda| = |m| + \epsilon$ with $0 \leq \epsilon < 1$. Thus

$$\alpha = \sum_{i=1}^{s} c_i e_i + \sum_{i=1}^{q} m_i v_i, \tag{4.6}$$

where $c_1, \ldots, c_s$ are in $\mathbb{N}$. Setting $c = (c_1, \ldots, c_s)$, from Eqs. (4.5) and (4.6), we get

$$|\mu| + 2|\lambda| = |c| + 2|m|. \tag{4.7}$$

Therefore, using that $|\lambda| = |m| + \epsilon$, $b \geq |\mu| + |\lambda|$, and Eq. (4.7), one has

$$b + (|m| + \epsilon) = b + |\lambda| \geq (|\mu| + |\lambda|) + |\lambda| = |c| + 2|m|,$$

and consequently $b \geq |c| + |m| - \epsilon$. We claim that $b \geq |c| + |m|$. We argue by contradiction assuming that $b < |c| + |m|$. Then, $|c| + |m| - \epsilon \leq b \leq |c| + |m| - 1$, and we obtain that $\epsilon \geq 1$, a contradiction. Then, by Eq. (4.6), we can write

$$(\alpha, b) = (b - |c| - |m|)e_{s+1} + \sum_{i=1}^{s} c_i(e_i + e_{s+1}) + \sum_{i=1}^{q} m_i(v_i, 1),$$

and $(\alpha, b) \in \mathbb{N}\mathcal{B}$. This completes the proof that $\mathcal{B}$ is a Hilbert basis.

$\Leftarrow$) Assume that $\mathcal{B}$ is a Hilbert basis and set $\mathcal{A}' = \{e_i\}_{i=1}^s \cup \{(v_i, 1)\}_{i=1}^q$. To show the normality of I we need only show that $\mathbb{R}_+\mathcal{A}' \cap \mathbb{Z}^{s+1} = \mathbb{N}\mathcal{A}'$ (see Lemma 2.6.2). The inclusion $\mathbb{R}_+\mathcal{A}' \cap \mathbb{Z}^{s+1} \supset \mathbb{N}\mathcal{A}'$ holds in general. To show the reverse inclusion take $(\alpha, b) \in \mathbb{R}_+\mathcal{A}' \cap \mathbb{Z}^{s+1}$, $\alpha \in \mathbb{N}^s$, $b \in \mathbb{N}$. Then, we can write

$$(\alpha, b) = \sum_{i=1}^{s} \mu_i e_i + \sum_{i=1}^{q} \lambda_i (v_i, 1), \tag{4.8}$$

where μ_i and λ_i are in $\mathbb{R}_+$. Hence, setting $\mu = (\mu_1, \ldots, \mu_s)$ and $\lambda = (\lambda_1, \ldots, \lambda_q)$, one has $|\alpha| = |\mu| + 2|\lambda| = |\mu| + 2b$. Noticing that $|\mu| = |\alpha| - 2b$ is in $\mathbb{N}$, by Eq. (4.8), we obtain that $(\alpha, b) + |\mu| e_{s+1}$ is in $\mathbb{R}_+\mathcal{B} \cap \mathbb{Z}^{s+1} = \mathbb{N}\mathcal{B}$. Therefore, we can write

$$(\alpha, b) + |\mu| e_{s+1} = n e_{s+1} + \sum_{i=1}^{s} c_i (e_i + e_{s+1}) + \sum_{i=1}^{q} m_i (v_i, 1), \tag{4.9}$$

where n, c_i, and m_i are in $\mathbb{N}$. Setting $c = (c_1, \ldots, c_s)$ and $m = (m_1, \ldots, m_q)$, from Eqs. (4.8) and (4.9), we get the equalities

$$|\alpha| = |\mu| + 2b = |c| + 2|m|, \tag{4.10}$$

$$b = |\lambda| = n + |c| + |m| - |\mu|. \tag{4.11}$$

Then, from Eq. (4.10) and (4.11), one has

$$|\mu| + 2b = |c| + 2|m| = (b - n - |m| + |\mu|) + 2|m| = b - n + |\mu| + |m|,$$

and consequently $|\mu| + 2b = b - n + |\mu| + |m|$. Thus, $b + n = |m|$. Therefore, adding n to both sides of Eq. (4.11), we obtain

$$|m| = b + n = 2n + |c| + |m| - |\mu| \;\therefore\; 2n + |c| - |\mu| = 0. \tag{4.12}$$

Consider the multiset

$$\mathcal{F}=\{\underbrace{(v_1,1),\ldots,(v_1,1)}_{m_1 \text{ times}},\ldots,\underbrace{(v_q,1),\ldots,(v_q,1)}_{m_q \text{ times}}\}.$$

This multiset has $|m|=n+b$ elements. Pick a multiset $\mathcal{F}_1=\{(v_{\ell_1},1),\ldots,(v_{\ell_b},1)\}$ of b vectors in $\mathcal{F}$ and let $\mathcal{F}_2=\mathcal{F}\setminus\mathcal{F}_1=\{(v_{j_1},1),\ldots,(v_{j_n},1)\}$ be the complement of $\mathcal{F}_1$. Then, by Eqs. (4.9) and (4.12), one has

$$\begin{aligned}(\alpha,b)&=(n-|\mu|)e_{s+1}+\sum_{i=1}^{s}c_i(e_i+e_{s+1})+\sum_{i=1}^{q}m_i(v_i,1)\\&=(n-|\mu|+|c|)e_{s+1}+\sum_{i=1}^{s}c_ie_i+\sum_{i=1}^{n}(v_{j_i},1)+\sum_{i=1}^{b}(v_{\ell_i},1)\\&=(2n-|\mu|+|c|)e_{s+1}+\sum_{i=1}^{s}c_ie_i+\sum_{i=1}^{n}v_{j_i}+\sum_{i=1}^{b}(v_{\ell_i},1)\\&=\sum_{i=1}^{s}c_ie_i+\sum_{i=1}^{n}v_{j_i}+\sum_{i=1}^{b}(v_{\ell_i},1).\end{aligned}$$

Thus, (α,b) is in $\mathbb{N}\mathcal{A}'$ and the proof is complete. □

As an application we have the next corollary.

Corollary 4.1.2. *[8, Theorem 3.3] If $I=(t^{v_1},\ldots,t^{v_q})$ is the edge ideal of a connected graph, then I is normal if and only if the subring $R=K[z,t_1z,\ldots,t_sz,t^{v_1}z,\ldots,t^{v_q}z]$ is normal.*

Proof. The subgroup $\mathbb{Z}\mathcal{B}$ spanned by $\mathcal{B}$ is $\mathbb{Z}^{n+1}$. Then, by Theorem 2.6.1, one has

$$R=K[\{t^az^b|\,(a,b)\in\mathbb{N}\mathcal{B}\}]\subset\overline{R}=K[\{t^az^b|\,(a,b)\in\mathbb{Z}^{s+1}\cap\mathbb{R}_+\mathcal{B}\}].$$

Hence, R is normal if and only if $\mathbb{N}\mathcal{B}$ is equal to $\mathbb{Z}^{s+1}\cap\mathbb{R}_+\mathcal{B}$. Thus, the result follows from Theorem 4.1.1 because I is generated by squarefree monomials of degree 2. $\square$

Lemma 4.1.3. *Let $I=(t^{v_1},\ldots,t^{v_q})$ be an ideal of S generated by monomials of degree 2 and let $\mathcal{A}$ be the set of vectors $\{e_i\}_{i=1}^s\cup\{v_i\}_{i=1}^q$. The following hold.*

(a) If α is in $\mathbb{N}^s\setminus\{0\}$, $\beta_1,\ldots,\beta_k$ are in $\mathcal{A}$ and $\sum_{i=1}^k\beta_i\geq\alpha$, then there are $\gamma_1,\ldots,\gamma_\ell$ in $\mathcal{A}$ such that $\alpha=\sum_{i=1}^\ell\gamma_i$ and $k\geq\ell$.

(b) If $a\in\mathbb{Q}_+^s$, $b\in\operatorname{conv}(\{0\}\cup\mathcal{A})\cap\mathbb{Q}^s$ and $a\leq b$, then $a\in\operatorname{conv}(\{0\}\cup\mathcal{A})$.

Proof. (a) Consider the following procedure. Assume that $\sum_{i=1}^k\beta_i\neq\alpha$. Then, the j-th entry of $\sum_{i=1}^k\beta_i$ is greater than the j-th entry of α for some j, and consequently $\sum_{i=1}^k\beta_i\geq\alpha+e_j$. Hence, $\beta_p\geq e_j$ for some p, and either $\beta_p=e_j$ or $\beta_p-e_j=e_r$ for some $1\leq r\leq s$. Thus

$$\left(\sum_{i=1}^k\beta_i\right)-e_j=\left(\sum_{i\neq p}\beta_i\right)+(\beta_p-e_j)\geq\alpha,$$

where $\beta_p-e_j=0$ or $\beta_p-e_j\in\mathcal{A}$. If $\left(\sum_{i\neq p}\beta_i\right)+(\beta_p-e_j)\neq\alpha$, we repeat the procedure. Since

$$\left|\left(\sum_{i\neq p}\beta_i\right)+(\beta_p-e_j)\right|<\left|\sum_{i=1}^k\beta_i\right|,$$

applying this procedure recursively, we get that $\alpha=\sum_{i=1}^\ell\gamma_i$ for some $\gamma_1,\ldots,\gamma_\ell$ in $\mathcal{A}$ and $k\geq\ell$.

(b) One can write $b=\sum_{i=1}^s\mu_ie_i+\sum_{i=1}^q\lambda_iv_i$, $\sum_{i=1}^s\mu_i+\sum_{i=1}^q\lambda_i\leq 1$, μ_i, λ_j in $\mathbb{Q}_+$ for all i,j. If $a=0$, there is nothing to prove. Assume that $a\neq 0$. Choose $r\in\mathbb{N}_+$ such that $r\mu_i$, $r\lambda_j$ are in $\mathbb{N}$ for all i,j and $ra\in\mathbb{N}^s\setminus\{0\}$. Then

$$ra\leq rb=\sum_{i=1}^s(r\mu_i)e_i+\sum_{i=1}^q(r\lambda_i)v_i,$$

and $k := \sum_{i=1}^{s}(r\mu_i) + \sum_{i=1}^{q}(r\lambda_i) \leq r$. Then, by part (a), there are $\gamma_1, \ldots, \gamma_\ell$ in $\mathcal{A}$ such that $ra = \sum_{i=1}^{\ell}\gamma_i$, $\ell \leq k \leq r$, and $a = \sum_{i=1}^{\ell}(\gamma_i/r)$. Hence, as $\ell/r \leq 1$, we get $a \in \mathrm{conv}(\{0\} \cup \mathcal{A})$. $\square$

We characterize when the Ehrhart ring of $\mathcal{Q}$ is the monomial subring $K[\mathbb{N}\mathcal{B}]$ using the integer rounding property.

4.2 Ehrhart ring

Theorem 4.2.1. *Let I be an ideal of S generated by monomials of degree 2 and let A be the incidence matrix of I. Then, $\overline{K[\mathbb{N}\mathcal{B}]} = \mathrm{Er}(\mathcal{Q})$, and the equality*

$$K[\mathbb{N}\mathcal{B}] = \mathrm{Er}(\mathcal{Q})$$

holds if and only if the system $x \geq 0; xA \leq 1$ has the integer rounding property.

Proof. We may assume that $\{t_1, \ldots, t_s\} = \bigcup_{i=1}^{q} \mathrm{supp}(t^{v_i})$, i.e., each variable t_i occurs in at least one minimal generator of I. The equality $\overline{K[\mathbb{N}\mathcal{B}]} = \mathrm{Er}(\mathcal{Q})$ follows readily from [10, Theorem 3.9].

$\Rightarrow$) Assume that $K[\mathbb{N}\mathcal{B}] = \mathrm{Er}(\mathcal{Q})$. Let α be an integral vector for which the left hand side of Eq. (3.5) is finite. By replacing α by its positive part α_+, we may assume that $\alpha \in \mathbb{N}^s \setminus \{0\}$. In general one has

$$\lceil|\lambda|\rceil = \lceil\min\{\langle y, 1\rangle | \, y \geq 0; \; Ay \geq \alpha\}\rceil \leq \min\{\langle y, 1\rangle \mid y \in \mathbb{N}^q; \; Ay \geq \alpha\}, \tag{4.13}$$

where $\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{Q}_+^q$ and $A\lambda \geq \alpha$. Then

$$\frac{\alpha}{\lceil|\lambda|\rceil} \leq \frac{\alpha}{|\lambda|} \leq \sum_{i=1}^{q}\frac{\lambda_i}{|\lambda|}v_i$$

and, by Lemma 4.1.3, we obtain that $\alpha/\lceil|\lambda|\rceil \in \mathcal{Q}$, that is, $t^\alpha z^{\lceil|\lambda|\rceil} \in \mathrm{Er}(\mathcal{Q}) =$

$K[\mathbb{N}\mathcal{B}]$. Therefore, there are nonnegative integers $\tau_1, n_i, m_j \in \mathbb{N}$ such that

$$t^\alpha z^{\lceil|\lambda|\rceil} = z^{\tau_1}(t_1 z)^{n_1} \cdots (t_s z)^{n_s}(t^{v_1} z)^{m_1} \cdots (t^{v_q} z)^{m_q}, \tag{4.14}$$

$$\lceil|\lambda|\rceil = \tau_1 + n_1 + \cdots + n_s + m_1 + \cdots + m_q, \tag{4.15}$$

$$\alpha = n_1 e_1 + \cdots + n_s e_s + m_1 v_1 + \cdots + m_q v_q. \tag{4.16}$$

For each e_i there is $v_{j_i} \in \{v_1, \ldots, v_q\}$ satisfying $e_i \leq v_{j_i}$. Then, by Eq. (4.16), $\alpha \leq Aw$ for some $w \in \mathbb{N}^q$ with $|w| = \sum_{i=1}^s n_i + \sum_{i=1}^q m_i$. From Eqs. (4.13) and (4.15), we get

$$|w| \leq \lceil|\lambda|\rceil = \lceil\min\{\langle y, 1\rangle \mid y \geq 0;\ Ay \geq \alpha\}\rceil \leq \min\{\langle y, 1\rangle \mid y \in \mathbb{N}^q;\ Ay \geq \alpha\} \leq |w|,$$

and we have equality everywhere. Thus, the system $x \geq 0$; $xA \leq 1$ has the integer rounding property and the proof of this implication is complete.

$\Leftarrow$) Assume that the linear system $x \geq 0$; $xA \leq 1$ has the integer rounding property. The inclusion $K[\mathbb{N}\mathcal{B}] \subset \mathrm{Er}(\mathcal{Q})$ is clear because $\overline{K[\mathbb{N}\mathcal{B}]} = \mathrm{Er}(\mathcal{Q})$. To show the reverse inclusion take $t^\alpha z^b \in \mathrm{Er}(\mathcal{Q})$, that is, $\alpha \in b\mathcal{Q}$, $\alpha \in \mathbb{N}^s$, $b \in \mathbb{N}_+$. Then,

$$\alpha/b = \sum_{i=1}^{s} \mu_i e_i + \sum_{i=1}^{q} \lambda_i v_i,$$

where μ_i, λ_j are in $\mathbb{R}_+$, and $\sum_{i=1}^s \mu_i + \sum_{i=1}^q \lambda_i \leq 1$. For any vector x that satisfies $x \geq 0$; $xA \leq 1$, one has $\langle x, e_i\rangle \leq 1$ for $i = 1, \ldots, s$ because any t_i occurs in at least one of the minimal generators of I. Hence, for any such x, we obtain

$$\langle \alpha/b, x\rangle = \sum_{i=1}^{s} \mu_i \langle e_i, x\rangle + \sum_{i=1}^{q} \lambda_i \langle v_i, x\rangle \leq \sum_{i=1}^{s} \mu_i + \sum_{i=1}^{q} \lambda_i \leq 1.$$

Thus, $\langle \alpha, x\rangle \leq b$ and, by linear programming duality [26, Corollary 7.1g], one has

$$b \geq \max\{\langle \alpha, x\rangle \mid x \geq 0;\ xA \leq 1\} = \min\{\langle y, 1\rangle \mid y \geq 0;\ Ay \geq \alpha\}, \tag{4.17}$$

and since system $x \geq 0;\ xA \leq 1$ has the integer rounding property, we get

$$b \geq \lceil \min\{\langle y, 1\rangle \mid y \geq 0;\ Ay \geq \alpha\}\rceil = \min\{\langle y, 1\rangle \mid y \in \mathbb{N}^q;\ Ay \geq \alpha\}. \tag{4.18}$$

Hence, we can choose $m \in \mathbb{N}^q$ such that $b \geq |m|$ and $Am \geq \alpha$. Setting $k = |m|$, by Lemma 4.1.3, there are $\gamma_1, \ldots, \gamma_\ell$ in $\{e_i\}_{i=1}^s \cup \{v_i\}_{i=1}^q$, $|m| \geq \ell$, such that $\alpha = \sum_{i=1}^{\ell} \gamma_i$. Thus,

$$t^\alpha z^b = (t^{\gamma_1} z) \cdots (t^{\gamma_\ell} z) z^{b-\ell},$$

and consequently $t^\alpha z^b \in K[\mathbb{N}\mathcal{B}]$. $\square$

In particular, we recover the fact that if I is the edge ideal of a connected graph, then the semigroup ring $K[\mathbb{N}\mathcal{B}]$ is normal if and only if the system $x \geq 0; xA \leq 1$ has the integer rounding property [8, Theorem 3.3].

Corollary 4.2.2. *Let I be an ideal of S generated by monomials of degree 2. Then, the following conditions are equivalent.*

(a) $K[\mathbb{N}\mathcal{B}] = \mathrm{Er}(\mathcal{Q})$; (b) $K[\mathbb{N}\mathcal{B}]$ *is normal* ; (c) $\mathcal{B}$ *is a Hilbert basis.*

Proof. (a)⇒(b) This follows from the fact that the Ehrhart ring of a lattice polytope is a normal domain [33, Theorem 9.3.6].

(b)⇒(c) Noticing that $\mathbb{Z}\mathcal{B}$ is $\mathbb{Z}^{s+1}$, by the description of the integral closure of $K[\mathbb{N}\mathcal{B}]$ given in Theorem 2.6.1, one has

$$K[\mathbb{N}\mathcal{B}] = \overline{K[\mathbb{N}\mathcal{B}]} = K[\mathbb{Z}\mathcal{B} \cap \mathbb{R}_+\mathcal{B}] = K[\mathbb{Z}^{s+1} \cap \mathbb{R}_+\mathcal{B}].$$

Thus, $\mathbb{N}\mathcal{B} = \mathbb{Z}^{s+1} \cap \mathbb{R}_+\mathcal{B}$ and $\mathcal{B}$ is a Hilbert basis.

(c)⇒(a) By Theorem 4.2.1, we get $\overline{K[\mathbb{N}\mathcal{B}]} = \mathrm{Er}(\mathcal{Q})$. Hence, using that $\mathcal{B}$ is a Hilbert basis and the description of the integral closure of $K[\mathbb{N}\mathcal{B}]$ given in Theorem 2.6.1, one has

$$\overline{K[\mathbb{N}\mathcal{B}]} = K[\mathbb{Z}\mathcal{B} \cap \mathbb{R}_+\mathcal{B}] = K[\mathbb{Z}^{s+1} \cap \mathbb{R}_+\mathcal{B}] = K[\mathbb{N}\mathcal{B}].$$

Thus, $K[\mathbb{N}\mathcal{B}] = \mathrm{Er}(\mathcal{Q})$ and the proof is complete. □

4.3 Duality for integer rounding properties

Definition 4.3.1. The *dual* of the edge ideal I of a clutter $\mathcal{C}$, denoted I^*, is the ideal of S generated by all monomials $t_1 \cdots t_s/t_e$ such that e is an edge of $\mathcal{C}$.

If A and A^* are the incidence matrices of I and I^*, respectively, then the system $x \geq 0; xA \geq 1$ has the integer rounding property if and only if the system $x \geq 0; xA^* \leq 1$ has the integer rounding property (Theorem 2.4.7). We will use this duality to characterize the normality of the dual of the edge ideal of a graph using Hilbert bases.

Corollary 4.3.2. *Let I be the edge ideal of a graph and let I^* be the dual of I. Then, I^* is normal if and only if $\mathcal{B}$ is a Hilbert basis.*

Proof. Let A be the incidence matrix of I. By Corollary 3.3.2, I^* is normal if and only if the system $xA^* \geq 1; x \geq 0$ has the integer rounding property. Then, by Theorem 2.4.7, I^* is normal if and only if the system $xA \leq 1; x \geq 0$ has the integer rounding property. Hence, by Theorem 4.2.1, I^* is normal if and only if $K[\mathbb{N}\mathcal{B}] = \mathrm{Er}(\mathcal{Q})$. Thus, by Corollary 4.2.2, I^* is normal if and only if $\mathcal{B}$ is a Hilbert basis. □

Proposition 4.3.3. *Let I be the edge ideal of a graph and let I^* be the dual of I. Then, I^* is normal if and only if I is normal.*

Proof. This follows from Theorem 4.1.1 and Corollary 4.3.2. □

The last two, Corollary 4.3.2 and Proposition 4.3.3 follow from [3, Theorem 2.12] and [8, Theorem 3.3] when I is the edge ideal of a connected graph.

Chapter 5

Normality of Ideals of Covers of Graphs

In this section, we study the normality of the ideal of covers $I_c(G)$ of a graph G and give a combinatorial criterion in terms of Hochster configurations for the normality of $I_c(G)$ when the independence number of G is at most two.

Lemma 5.0.1. *Let v be a non isolated vertex of a graph G. If the neighbor set $N_G(v)$ of v is a minimal vertex cover of G, then a set $C \subset V(G)$ is a minimal vertex cover of G if and only if $C = N_G(v)$ or $C = \{v\} \cup D$ with D a minimal vertex cover of $G \setminus v$ such that $N_G(v) \not\subset D$.*

Proof. $\Rightarrow$) Assume that C is a minimal vertex cover of G. If $v \notin C$, then $N_G(v) \subset C$ and, by the minimality of C, one has the equality $C = N_G(v)$. Now assume that $v \in C$ and set $D = C \setminus \{v\}$. If $N_G(v) \subset D$, we get that D is a vertex cover of G with $D \subsetneq C$, a contradiction. Thus, $N_G(v) \not\subset D$ and the proof reduces to showing that D is a minimal vertex cover of $H = G \setminus v$. Take $f \in E(H)$, then $v \notin f$ and $f \cap C \neq \emptyset$. Thus, $f \cap D \neq \emptyset$, and D is a vertex cover of H. To show that D is minimal take $t_k \in D$. Then $t_k \neq v$ and, by the minimality of C, there is $e \in E(G)$ such that $e \cap (C \setminus \{t_k\}) = \emptyset$. Then, $v \notin e$, $e \in E(H)$, and $e \cap (D \setminus \{t_k\}) = \emptyset$.

$\Leftarrow$) Assume that $C = \{v\} \cup D$ with D a minimal vertex cover of $H = G \setminus v$ such that $N_G(v) \not\subset D$. Take $e \in E(G)$. If $v \in e$, then, $e \cap C \neq \emptyset$ and if $v \notin e$, then $e \cap D \neq \emptyset$. Thus, C is a vertex cover of G. Next we show that C is minimal. As $N_G(v) \not\subset D$, it follows that $D = C \setminus \{v\}$ is not a vertex cover of G. Indeed, pick $t_k \in N_G(v) \setminus D$, then $e = \{v, t_k\} \in E(G)$ and $e \cap (C \setminus \{v\}) = \emptyset$. Now take $t_i \in C$, $t_i \neq v$. Then, there is $f \in E(H)$ such that $f \cap (D \setminus \{t_i\}) = \emptyset$ because D is a minimal vertex cover of H. Then, $f \cap (C \setminus \{t_i\}) = \emptyset$. $\square$

5.1 Ideal of covers

Let v be a vertex of a graph G. Recall that if $I_c(G)$ is normal, then $I_c(G \setminus v)$ is normal. The following results shows that the converse holds under a certain condition.

Theorem 5.1.1. *Let v be a vertex of a graph G. If the neighbor set $N_G(v)$ of v is a minimal vertex cover of G, then $I_c(G \setminus v)$ is normal if and only if $I_c(G)$ is normal.*

Proof. Setting $V(G) = \{t_1, \ldots, t_s\}$ and $H = G \setminus v$, we may assume that v is not an isolated vertex, $v = t_s$, and $N_G(t_s) = \{t_1, \ldots, t_r\}$.

$\Rightarrow$) Assume that $I_c(H)$ is a normal ideal. To prove that $I_c(G)$ is normal, we will show that $I_c(G)^n = \overline{I_c(G)^n}$ for all $n \geq 1$. We argue by induction on n. The case $n = 1$ is clear because $I_c(G)$ is squarefree [33, p. 153]. Assume that $n > 1$. The inclusion $I_c(G)^n \subset \overline{I_c(G)^n}$ holds in general. To show the reverse inclusion take $t^a \in \overline{I_c(G)^n}$. Then, by Lemma 2.4.6, there is $k \in \mathbb{N}_+$ such that $(t^a)^k \in I_c(G)^{kn}$ and we can write

$$(t^a)^k = t^{b_1} \cdots t^{b_{nk}} t^{\delta},$$

where $t^{b_1}, \ldots, t^{b_{nk}}$ are minimal generators of $I_c(G)$. Then, by Lemma 5.0.1, we get

$$(t^a)^k = (t_1 \cdots t_r)^m (t_s t^{d_{m+1}}) \cdots (t_s t^{d_{nk}}) t^\delta, \tag{5.1}$$

where $t^{d_{m+1}}, \ldots, t^{d_{nk}}$ are minimal generators of $I_c(H)$. As $\{t_1, \ldots, t_r\}$ contains a minimal vertex cover of H, one has $(t^a)^k \in I_c(H)^{nk}$, and consequently $t^a \in \overline{I_c(H)^n}$. By the normality of $I_c(H)$, we obtain that $t^a \in I_c(H)^n$. Then

$$t^a = t^{c_1} \cdots t^{c_n} t^\gamma, \tag{5.2}$$

where $t^{c_1}, \ldots, t^{c_n}$ are minimal generators of $I_c(H)$ and $t^\gamma \in S = K[t_1, \ldots, t_s]$.

Case (I) $n \leq a_s$, $a = (a_1, \ldots, a_s)$. By Eq. (5.2), $t_s^{a_s}$ divides t^γ, and we get

$$t^a = (t_s t^{c_1}) \cdots (t_s t^{c_n})(t^\gamma / t_s^{a_s}) t^{a_s - n}.$$

Then, $t^a \in I_c(G)^n$ because $t_s t^{c_i} \in I_c(G)$ for $i = 1, \ldots, n$.

Case (II) $n > a_s$. By Eq. (5.1), one has $ka_s \geq kn - m$ and $ka_i \geq m$ for $i = 1, \ldots, r$. Therefore, $m \geq k(n - a_s)$ and $a_i \geq n - a_s$ for $i = 1, \ldots, r$. Thus, we can write

$$t^a = (t_1 \cdots t_r)^{n - a_s} t^\epsilon. \tag{5.3}$$

Using Eqs. (5.1) and (5.3), we obtain

$$(t^a)^k = (t_1 \cdots t_r)^m (t_s t^{d_{m+1}}) \cdots (t_s t^{d_{nk}}) t^\delta = (t_1 \cdots t_r)^{k(n - a_s)} (t^\epsilon)^k,$$

and consequently

$$(t_1 \cdots t_r)^{m - k(n - a_s)} (t_s t^{d_{m+1}}) \cdots (t_s t^{d_{nk}}) t^\delta = (t^\epsilon)^k.$$

As $m - k(n - a_s) + kn - m = ka_s$, we obtain that $(t^\epsilon)^k \in I_c(G)^{ka_s}$, that is, $t^\epsilon \in \overline{I_c(G)^{a_s}}$. By induction $I_c(G)^{a_s}$ is equal to $\overline{I_c(G)^{a_s}}$. Thus $t^\epsilon \in I_c(G)^{a_s}$ and, by Eq. (5.3), we get $t^a \in I_c(G)^n$. Hence, $I_c(G)^n$ is equal to $\overline{I_c(G)^n}$ and the proof is complete.

$\Leftarrow$) This implication follows at once from Proposition 3.4.3. $\square$

Definition 5.1.2. Let G be a graph. The *cone* over G with apex v, denoted $C(G)$, is obtained by adding a new vertex v to G and joining every vertex of G to v.

Corollary 5.1.3. *[1, Theorem 1.6] Let G be a graph and let $C(G)$ the cone over G with apex v. Then, $I_c(G)$ is normal if and only if $I_c(C(G))$ is normal.*

Proof. This follows readily from Theorem 5.1.1, by noticing that $V(G)$ is a minimal vertex cover of $C(G)$, $N_{C(G)}(v) = V(G)$, and $C(G) \setminus v = G$. $\square$

Let G be a graph and let $G_1, \dots, G_r$ be its connected components. If the edge ideal $I(G)$ is normal, then the edge ideal $I(G_i)$ is normal for $i = 1, \dots, r$ [11, Proposition 4.3] but the converse is not true (Example 6.0.8).

Corollary 5.1.4. *Let G be a graph and let $G_1, \dots, G_r$ be its connected components. Then*

(a) $I_c(G) = I_c(G_1) \cdots I_c(G_r)$,

(b) $\overline{I_c(G)^n} = \overline{I_c(G_1)^n} \cdots \overline{I_c(G_r)^n}$ *for all* $n \geq 1$, *and*

(c) $I_c(G)$ *is normal if and only if* $I_c(G_i)$ *is normal for* $i = 1, \dots, r$.

Proof. To show part (a), let C be a set of vertices of G. Note that C is a minimal vertex cover of G if and only if $C = C_1 \cup \cdots \cup C_r$ with C_i a minimal vertex cover of G_i for $i = 1, \dots, r$. Hence, $I_c(G)$ is equal to $I_c(G_1) \cdots I_c(G_r)$. Parts (b) and (c) follow from part (a) and Proposition 3.1.1. $\square$

Definition 5.1.5. A *clique* of a graph G is a set of vertices inducing a complete subgraph.

Definition 5.1.6. The *clique clutter* of G, denoted by $\mathrm{cl}(G)$, is the clutter on $V(G)$ whose edges are the maximal cliques of G (maximal with respect to inclusion).

We also call a complete subgraph of G a clique and denote a complete subgraph of G with r vertices by $\mathcal{K}_r$. Then

$$I(\mathrm{cl}(G))^* = (\{(t_1 \cdots t_s)/t_e \mid e \in E(\mathrm{cl}(G))\}).$$

If G is a discrete graph, by convention

$$I_c(G) = S, I(G) = (0), \qquad \textit{and} \qquad I(G)^* = (0).$$

Definition 5.1.7. The *complement* of a graph G, denoted $\overline{G}$, has the same vertex set as G, and $\{t_i, t_j\}$ is an edge of $\overline{G}$ if and only if $\{t_i, t_j\}$ is not an edge of G.

Lemma 5.1.8. *Let $\overline{G}$ be the complement of a graph G, let $\mathrm{cl}(\overline{G})$ be the clique clutter of $\overline{G}$, and let $\mathrm{Isol}(\overline{G})$ be the set of isolated vertices of $\overline{G}$. The following hold.*

(a) $I_c(G) = I(\mathrm{cl}(\overline{G}))^*$.

(b) If $\overline{G}$ has no triangles, then $I_c(G) = (\{(t_1 \cdots t_s)/t_i \mid t_i \in \mathrm{Isol}(\overline{G})\}) + I(\overline{G})^*$.

(c) If $\overline{G}$ is a discrete graph, then $I_c(G) = (\{(t_1 \cdots t_s)/t_i \mid t_i \in V(G)\})$.

(d) If $\overline{G}$ has no triangles and no isolated vertices, then $I_c(G) = I(\overline{G})^*$.

Proof. (a) To show the inclusion "$\subset$" take t^a a minimal generator of $I_c(G)$, i.e., the support U of t^a is a minimal vertex cover of G, and consequently $V(G) \setminus U$ is a maximal clique of $\overline{G}$. Thus, U is the complement of a maximal clique of $\overline{G}$, and consequently $t^a \in I(\mathrm{cl}(\overline{G}))^*$. The inclusion "$\supset$" is also easy to prove.

(b) As the graph $\overline{G}$ has no triangles, the edges of the clique clutter $\mathrm{cl}(\overline{G})$ are either isolated vertices of $\overline{G}$ (i.e., maximal cliques that correspond to $\mathcal{K}_1$)

or edges of $\overline{G}$ (i.e., maximal cliques that correspond to $\mathcal{K}_2$). Hence

$$I(\mathrm{cl}(\overline{G})) = (\mathrm{Isol}(\overline{G})) + I(\overline{G}),$$

and using part (a) we obtain the equalities

$$I_c(G) = I(\mathrm{cl}(\overline{G}))^* = (\mathrm{Isol}(\overline{G}))^* + (I(\overline{G}))^* = (\{(t_1 \cdots t_s)/t_i \mid t_i \in \mathrm{Isol}(\overline{G})\}) + I(\overline{G})^*.$$

Parts (c) and (d) follow from part (b). $\Box$

Lemma 5.1.9. *If G is a graph and $U \subset V(\overline{G})$, then*

$$\text{(a)}\ \ \overline{G} \setminus U = \overline{G \setminus U}; \quad \text{(b)}\ \ \overline{G} \setminus \mathrm{Isol}(\overline{G}) = \overline{G \setminus \mathrm{Isol}(\overline{G})}.$$

Proof. (a) To show equality we need to show that the vertex set and edge set of the two graphs $\overline{G} \setminus U$ and $\overline{G \setminus U}$ are equal. From the equalities

$$V(\overline{G} \setminus U) = V(\overline{G}) \setminus U = V(G) \setminus U,$$
$$V(\overline{G \setminus U}) = V(G \setminus U) = V(G) \setminus U,$$

the vertex sets of the two graphs are equal. We set

$$H_0 = \overline{G} \setminus U \ \text{ and } \ H = G \setminus U.$$

Note that $V(H_0) = V(\overline{H}) = V(H)$. To show the inclusion $E(H_0) \subset E(\overline{H})$ take $e \in E(H_0)$. Then, $e \in E(\overline{G})$ and $e \cap U = \emptyset$. If $e \in E(H)$, then $e \in E(G)$ and $e \cap U = \emptyset$, a contradiction. Thus, $e \in E(\overline{H})$. To show the inclusion $E(H_0) \supset E(\overline{H})$ take $e \in E(\overline{H})$. Then, $e \notin E(H)$ and $e \subset V(\overline{H})$. Hence, $e \in E(\overline{G})$ and $e \cap U = \emptyset$. Thus, $e \in E(H_0)$.

(b) Setting $U = \mathrm{Isol}(\overline{G})$, this part follows from (a). $\Box$

5.2 Hochster configurations

Hochster gave an example of a connected graph whose edge ideal is not normal [33, p. 457] (cf. [28, Example 4.9]). This example leads to the following concept [28, Definition 6.7].

Definition 5.2.1. A *Hochster configuration* of a graph G consists of two odd cycles C_1, C_2 of G satisfying the following two conditions:

(i) $C_1 \cap N_G(C_2) = \emptyset$, where $N_G(C_2)$ is the neighbor set of C_2.

(ii) No chord of C_i, $i = 1, 2$, is an edge of G, i.e., C_i is an induced cycle of G.

Lemma 5.2.2. *Let I be the edge ideal of a graph G, let C_1, C_2 be two odd cycles of G with at most one common vertex, and let*

$$M_{C1,C_2} := (\prod_{t_i \in C_1} t_i \prod_{t_i \in C_2} t_i) z^{(|C_1|+|C_2|)/2}.$$

The following hold

(a) If $|C_1 \cap C_2| = 1$, then $M_{C_1,C_2} \in S[Iz]$.

(b) If $C_1 \cap C_2 = \emptyset$ and there is $e \in E(G)$ intersecting C_1 and C_2, then $M_{C_1,C_2} \in S[Iz]$.

(c) If C_1, C_2 form a Hochster configuration, then $M_{C_1,C_2} \notin S[Iz]$.

Proof. We may assume that $C_1 = \{t_1, \ldots, t_{\ell_1}\}$ and $C_2 = \{t_{\ell+1}, \ldots, t_{\ell_1+\ell_2}\}$ are odd cycles of lengths ℓ_1 and ℓ_2, respectively.

(a) Assume that $C_1 \cap C_2 = \{t_{\ell_1}\}$ and $t_{\ell_1} = t_{\ell_1+1}$. Then

$$\begin{aligned} M_{C_1,C_2} &= (t_1 \cdots t_{\ell_1})(t_{\ell_1+1} \cdots t_{\ell_1+\ell_2}) z^{(\ell_1+\ell_2)/2} \\ &= (t_1 \cdots t_{\ell_1-1} t_{\ell_1}^2 z^{(\ell_1+1)/2})(t_{\ell_1+2} \cdots t_{\ell_1+\ell_2} z^{(\ell_2-1)/2}). \end{aligned}$$

Thus, $M_{C_1,C_2} \in I^{(\ell_1+1)/2}z^{(\ell_1+1)/2}I^{(\ell_2-1)/2}z^{(\ell_2-1)/2} = I^{(\ell_1+\ell_2)/2}z^{(\ell_1+\ell_2)/2} \subset S[Iz]$.

(b) Assume that $C_1 \cap C_2 = \emptyset$ and $\{t_{\ell_1}, t_{\ell_1+1}\}$ is an edge of G. Then

$$\begin{aligned} M_{C_1,C_2} &= (t_1 \cdots t_{\ell_1})(t_{\ell_1+1} \cdots t_{\ell_1+\ell_2})z^{(\ell_1+\ell_2)/2} \\ &= (t_1 \cdots t_{\ell_1} t_{\ell_1+1} z^{(\ell_1+1)/2})(t_{\ell_1+2} \cdots t_{\ell_1+\ell_2} z^{(\ell_2-1)/2}). \end{aligned}$$

Thus, $M_{C_1,C_2} \in I^{(\ell_1+1)/2}z^{(\ell_1+1)/2}I^{(\ell_2-1)/2}z^{(\ell_2-1)/2} = I^{(\ell_1+\ell_2)/2}z^{(\ell_1+\ell_2)/2} \subset S[Iz]$.

(c) We argue by contradiction assuming that $M_{C_1,C_2} \in S[Iz]$. Then, $M_{C_1,C_2} \in I^m z^m$ for some $m \geq 1$, that is, $\prod_{t_i \in C_1 \cup C_2} t_i \in I^m$ and $m = (\ell_1 + \ell_2)/2$. Thus

$$(t_1 \cdots t_{\ell_1})(t_{\ell_1+1} \cdots t_{\ell_1+\ell_2}) = t_{e_1} \cdots t_{e_m} t^\delta$$

for some edges $e_1, \ldots, e_m$ of G. Hence, for each $1 \leq i \leq m$, either t_{e_i} divides $t_1 \cdots t_{\ell_1}$ or t_{e_i} divides $t_{\ell_1+1} \cdots t_{\ell_1+\ell_2}$. Thus, we may assume that $t_{e_1} \cdots t_{e_r}$ divides $t_1 \cdots t_{\ell_1}$, $r \geq 1$, and $t_{e_{r+1}} \cdots t_{e_m}$ divides $t_{\ell_1+1} \cdots t_{\ell_1+\ell_2}$. Therefore, $\ell_1 \geq 2r$ and $\ell_2 \geq 2(m-r)$. As ℓ_1 is odd, one has $\ell_1 > 2r$, and consequently $m = (\ell_1 + \ell_2)/2 > r + (m-r) = m$, a contradiction. $\square$

It was conjectured in [28, Conjecture 6.9] that the edge ideal of a graph G is normal if and only if the graph has no Hochster configurations. This conjecture was proved in [16, Corollary 5.8.10], [33, Corollary 10.5.9]. We give a direct proof of this conjecture using Vasconcelos's description of the integral closure of the Rees algebra of $I(G)$ [16, p. 265].

Theorem 5.2.3. ([28, Conjecture 6.9], [16, Corollary 5.8.10]) *The edge ideal $I(G)$ of a graph G is normal if and only if G admits no Hochster configurations.*

Proof. To show this result we use the following description of the integral closure of the Rees algebra $S[Iz]$ of the ideal $I = I(G)$ [16, Proposition 5.8.13]:

$$\overline{S[Iz]} = S[Iz][\mathcal{B}'],$$

where $\mathcal{B}'$ is the set of all monomials $M_{C1,C_2} := (\prod_{t_i\in C_1} t_i \prod_{t_i\in C_2} t_i)z^{(|C_1|+|C_2|)/2}$ such that C_1 and C_2 are two induced odd cycles of G with at most one common vertex. If C_1 and C_2 intersect at a point or C_1 and C_2 are joined by at least one edge of G, then M_{C_1,C_2} is in $S[It]$ by Lemma 5.2.2. Hence, if $\mathcal{U}$ is the set of all monomials M_{C_1,C_2} such that C_1, C_2 is a Hochster configuration of G, one has the equality

$$\overline{S[Iz]} = S[Iz][\mathcal{U}].$$

Therefore, by Lemma 5.2.2(c), $S[Iz]$ is normal if and only if G has no Hochster configurations, and the result follows from the fact that I is normal if and only if $S[Iz]$ is normal. □

Corollary 5.2.4. *Let G be a graph. The following hold.*

(a) If $\overline{G}$ is the disjoint union of two odd cycles of length at least 5, then $I_c(G)$ is not normal.

(b) If $\overline{G}$ is an odd cycle of length at least 5, then $I_c(G)$ is normal.

Proof. (a) As $\overline{G}$ has no triangles and no isolated vertices, by Lemma 5.1.8, one has $I_c(G) = I(\overline{G})^*$. Then, by Proposition 4.3.3, $I_c(G)$ is not normal if and only if $I(\overline{G})$ is not normal. As $\overline{G}$ is a Hochster configuration, by Theorem 5.2.3, $I(\overline{G})$ is not normal. Thus, $I_c(G)$ is not normal.

(b) As $\overline{G}$ has no triangles and no isolated vertices, by Lemma 5.1.8, one has the equality $I_c(G) = I(\overline{G})^*$. Then, by Proposition 4.3.3, $I_c(G)$ is normal if and only if $I(\overline{G})$ is normal. As $\overline{G}$ is an odd cycle, by Theorem 5.2.3, $I(\overline{G})$ is normal. Thus, $I_c(G)$ is normal. □

Corollary 5.2.5. *Let G be a graph. If $I_c(G)$ is normal, then $\overline{G}$ has no Hochster configuration with induced odd cycles C_1, C_2 of length at least* 5.

Proof. We argue by contradiction assuming that $\overline{G}$ has a Hochster configuration with induced odd cycles C_1, C_2 of length at least 5. Let U be the set of vertices of $\overline{G}$ not in $V(C_1) \cup V(C_2)$. Then, by Proposition 3.4.3, $I_c(G \setminus U)$ is normal. The subgraph $\overline{G} \setminus U$ is the union $C_1 \cup C_2$ of the cycles C_1 and C_2 because $C_1 \cup C_2$ is an induced subgraph of $\overline{G}$. Hence, by Lemma 5.1.9, $\overline{G \setminus U} = C_1 \cup C_2$. Therefore, by Corollary 5.2.4, $I_c(G \setminus U)$ is not normal, a contradiction. □

Definition 5.2.6. Let G be a graph with vertex set $V(G) = \{t_1, \ldots, t_s\}$. A subset B of $V(G)$ is called *independent* or *stable* if $e \not\subset B$ for any $e \in E(G)$. The *independence number* of G, denoted $\beta_0(G)$, is the number of vertices in any largest stable set of vertices.

The following result gives a combinatorial description of the normality of the ideal of covers of graphs with independence number at most two.

Theorem 5.2.7 (Duality criterion)**.** *Let G be a graph with $\beta_0(G) \leq 2$. The following hold.*

(a) $I_c(G)$ is normal if and only if $I(\overline{G})$ is normal.

(b) $I_c(G)$ is normal if and only if $\overline{G}$ has no Hochster configurations.

Proof. (a) $\Rightarrow$) Assume that $I_c(G)$ is normal. We proceed by induction on $s = |V(G)|$. If $s = 1$, then $I_c(G) = S$ and $I(\overline{G}) = (0)$, and if $s = 2$, then either G is a discrete graph with two vertices, $I_c(G) = S$ and $I(\overline{G}) = (t_1t_2)$, or $G = \mathcal{K}_2$, $I_c(G) = (t_1, t_2)$ and $I(\overline{G}) = (0)$. Thus, $I(\overline{G})$ is normal in these cases. Assume that $s \geq 3$.

Case (I) t_i is an isolated vertex of $\overline{G}$ for some i. By Proposition 3.4.3, $I_c(G \setminus t_i)$ is normal. Then, by induction, $I(\overline{G \setminus t_i})$ is normal. As $\overline{G \setminus t_i} = \overline{G} \setminus t_i$ and t_i is isolated in $\overline{G}$, one has $I(\overline{G} \setminus t_i) = I(\overline{G})$. Thus, $I(\overline{G})$ is normal.

Case (II) $\overline{G}$ has no isolated vertices. The graph $\overline{G}$ has no triangles because $\beta_0(G) \leq 2$. Then, by Lemma 5.1.8, one has $I_c(G) = I(\overline{G})^*$. Hence, the ideal $I(\overline{G})^*$ is normal and, by Proposition 4.3.3, the ideal $I(\overline{G})$ is normal.

$\Leftarrow$) Assume that $I(\overline{G})$ is normal. By Lemmas 5.1.8 and 5.1.9, one has

$$I_c(G \setminus \mathrm{Isol}(\overline{G})) = I\left(\overline{G \setminus \mathrm{Isol}(\overline{G})}\right)^* = I\left(\overline{G} \setminus \mathrm{Isol}(\overline{G})\right)^* = I\left(\overline{G}\right)^*.$$

Hence, the ideal $I_c(G \setminus \mathrm{Isol}(\overline{G}))$ is normal because $I\left(\overline{G}\right)^*$ is normal by Proposition 4.3.3. We set $H = G \setminus \mathrm{Isol}(\overline{G})$. We may assume that $t_1, \ldots, t_r$ are the isolated vertices of $\overline{G}$, then

$$G = H \cup H_1 \cup \cdots \cup H_r,$$

where H_i is the subgraph of G given by

$$V(H_1) = V(H) \cup \{t_1\} \text{ and } E(H_1) = \{\{t_1, t_j\} \mid t_j \in V(H)\} \text{ if } i = 1,$$
$$V(H_i) = V(H) \cup \{t_1, \ldots, t_i\} \text{ and } E(H_i) = \{\{t_i, t_j\} \mid t_j \in V(H) \cup \{t_1, \ldots, t_{i-1}\} \text{ if } i \geq 2.$$

Setting $G_0 = H$ and $G_i = H \cup H_1 \cup \cdots \cup H_i$ for $i = 1, \ldots, r$, note that G_i is the cone over G_{i-1} with apex t_i for $i = 1, \ldots, r$, i.e., G_i is obtained from G_{i-1} by joining every vertex of G_{i-1} to t_i. As $I_c(H)$ is normal, by successively applying Corollary 5.1.3, we obtain that $I_c(G)$ is normal.

(b) This part follows from part (a) and Theorem 5.2.3. $\square$

Chapter 6

Examples

In this section we give some examples to illustrate and complement our results. In particular, we show that Theorem 4.1.1 does not extend to ideals generated by monomials of degree greater than 2 (Example 6.0.6). In Examples 6.0.9–6.0.11, we illustrate Lemma 5.1.8 and the duality criterion given in Theorem 5.2.7. Then, we show that none of the implications of the duality criterion hold for arbitrary graphs (Example 6.0.12).

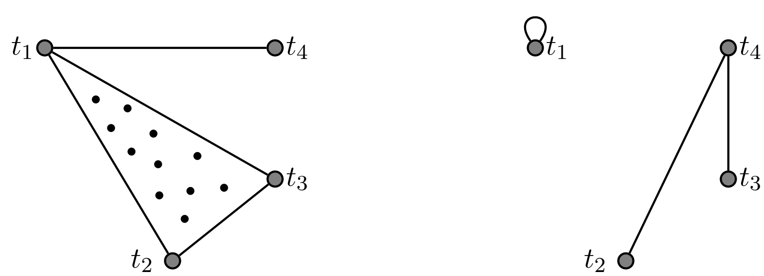

Figure 6.1: Clutter $\mathcal{C}$ (left) and clutter $\mathcal{C}^\vee$ (right)

Example 6.0.1. Let $\mathcal{C}$ the clutter (see figure 6.1) given by

$$V(\mathcal{C}) = \{t_1, t_2, t_3, t_4\} \qquad \textit{and} \qquad E(\mathcal{C}) = \{\{t_1, t_2, t_3\}, \{t_1, t_4\}\}$$

Then

$$V(\mathcal{C}^\vee) = V(\mathcal{C}) \qquad \textit{and} \qquad E(\mathcal{C}^\vee) = \{\{t_1\}, \{t_2, t_4\}, \{t_3, t_4\}\}$$

We can observe that $\mathcal{C}$ and $\mathcal{C}^\vee$ have different edges.

Example 6.0.2. Let $\mathcal{C}$ the clutter (see figure 6.2) given by

$V(\mathcal{C}) = \{t_1, t_2, t_3, t_4, t_5, t_6\}$

$E(\mathcal{C}) = \{\{t_1, t_2, t_5\}, \{t_1, t_3, t_4\}, \{t_2, t_3, t_6\}, \{t_4, t_5, t_6\}\}$

This clutter, usually is denoted by $\mathcal{Q}_6$ in the literature and plays an important role in combinatorial optimization. In this case then

$$I(\mathcal{C}) = (\{t_1t_2t_5, t_1t_3t_4, t_2t_3t_6, t_4t_5t_6\})$$

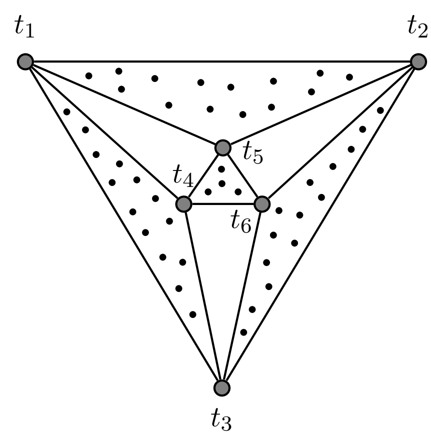

Figure 6.2: Clutter $\mathcal{Q}_6$.

Example 6.0.3. Let $\mathcal{C}$ the clutter (see figure 6.3) given by

$V(\mathcal{C}) = \{t_1, t_2, t_3\}$

$E(\mathcal{C}) = \{\{t_1, t_2\}, \{t_2, t_3\}, \{t_3, t_1\}\}$

This clutter (graph), usually is denoted by C_3 or K_3 in the literature. In this case then

$$I(\mathcal{C}) = (\{t_1t_2, t_2t_3, t_3t_1\}) \qquad \textit{and} \qquad A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Moreover $\mathcal{C}$ has 3 minimal vertex covers $\{\{t_1, t_2\}, \{t_2, t_3\}, \{t_1, t_3\}\}$, by corollary 1.5.3 the covering polyhedron $\mathcal{Q}(I) := \{x \mid x \geq 0;\ xA \geq 1\}$ has 3 integer vertices $\{a = (1, 1, 0), b = (0, 1, 1), c = (1, 0, 1)\}$ (see figure 6.4).

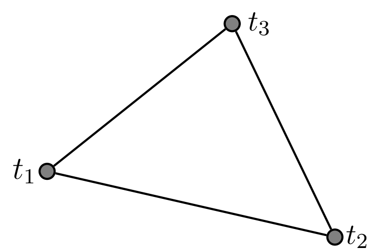

Figure 6.3: Clutter K_3.

Example 6.0.4. Consider the *rational polyhedron* $\mathcal{Q}$ which is the intersection of the rational closed halfspaces $H^+((-2, -3), -1)$, $H^+((-1, -1), -1)$ and $H^+((1, 0), 0)$, in this case $\mathcal{Q} = \mathbb{R}_+\Gamma + \mathcal{P}$ where $\Gamma = \{(0, -1), (1, -1)\}$ and $\mathcal{P}$ is the convex hull of $\{(0, 1/3), (2, -1)\}$ (see figure **??**)

Example 6.0.5. Let $C \subset \mathbb{R}^2$ be a cone generated by $\mathcal{A} = \{a_1, a_2\}$ where $a_1 = (1, 2)$ and $a_2 = (2, 1)$ and let $\gamma = (-2, 1) \notin C$. Let A be the matrix whose set of column vectors is $\mathcal{A}$, by theorem **??** there exists $\mu \in \mathbb{R}^2$ such that $\mu A \geq 0$ and $\langle \gamma, \mu \rangle < 0$. In this case we can take for example $\mu = (4, -1)$ .Thus $\langle \mu, a_i \rangle \geq 0$ for $i = 1, 2$ then $C \subset H^+ = H^+(\mu, 0)$ and $\langle \gamma, \mu \rangle < 0$ then $\mu \in H^- \setminus H$ where H is the hyperplane through the origin with normal vector μ (Line red).We can see that H separates γ from C (see figure 6.6).

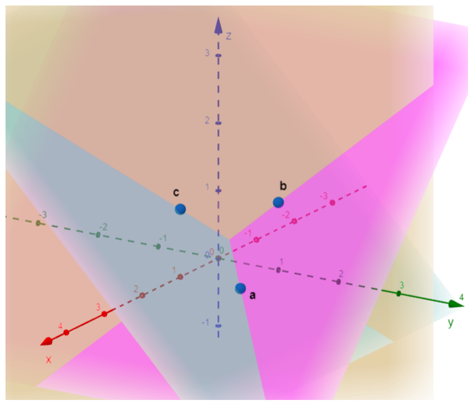

Figure 6.4: Covering polyhedron of example 6.0.3.

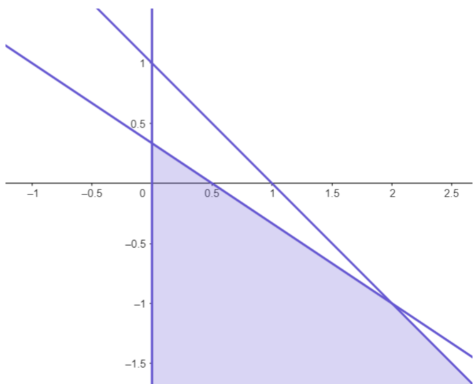

Figure 6.5: The rational polyhedron $\mathcal{Q}$ of example 6.0.4.

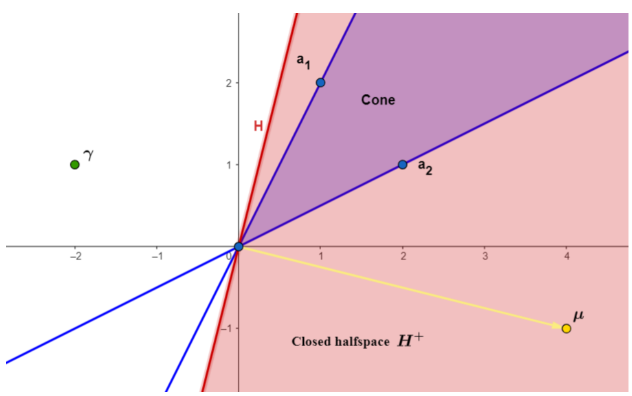

Figure 6.6: Ilustration of example 6.0.5.

Example 6.0.6. Let $S = \mathbb{Q}[t_1, \ldots, t_{10}]$ be a polynomial ring and let $I = (t^{v_1}, \ldots, t^{v_{10}})$ be the monomial ideal of S generated by the set

$$\begin{aligned}\mathcal{G}(I) =\{&t_1t_2t_3t_4t_5t_6t_7,\ t_1t_2t_3t_4t_5t_7t_8,\ t_1t_2t_3t_4t_5t_8t_9,\ t_1t_2t_3t_4t_5t_8t_{10},\ t_1t_2t_3t_4t_7t_8t_{10},\\ &t_2t_3t_5t_7t_8t_9t_{10},\ t_1t_2t_6t_7t_8t_9t_{10},\ t_2t_3t_6t_7t_8t_9t_{10},\ t_3t_4t_6t_7t_8t_9t_{10},\ t_3t_5t_6t_7t_8t_9t_{10}\}.\end{aligned}$$

Then, using Procedures 6.0.14 and 6.0.15, we get that $\mathcal{B} = \{e_{11}\} \cup \{e_i + e_{11}\}_{i=1}^{10} \cup \{(v_i, 1)\}_{i=1}^{10}$ is not a Hilbert basis and I is a normal ideal. Thus, Theorem 4.1.1 does not extend to ideals generated by monomials of degree greater than 2.

Example 6.0.7. Let $S = \mathbb{Q}[t_1, \ldots, t_7]$ be a polynomial ring and let

$$I = I(G) = (t_1t_3, t_1t_4, t_2t_4, t_1t_5, t_2t_5, t_3t_5, t_1t_6, t_2t_6, t_3t_6, t_4t_6, t_2t_7, t_3t_7, t_4t_7, t_5t_7)$$

be the edge ideal of the graph G of Figure 6.7. The complement $\overline{G}$ of G is a cycle of length 7. The graph G is called an *odd antihole* in the theory of perfect graphs [17, p. 71].

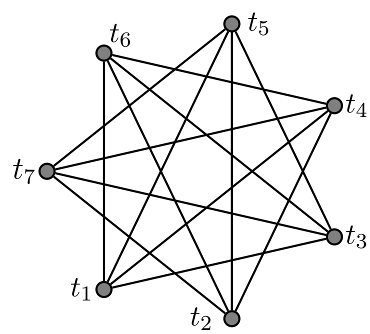

Figure 6.7: Graph G is an odd antihole with 7 vertices.

Using Procedure 6.0.16 for *Macaulay*2 [18], we obtain the following information. The incidence matrix B of $I_c(G)$ is

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The ideals $I_c(G)$ and $I(G)$ are normal, and so are the ideals $I_c(\overline{G})$ and $I(\overline{G})$. The normality of $I(G)$, $I(\overline{G})$, and $I_c(G)$ also follow from Theorem 5.2.3 and Corollary 5.2.4, and the normality of $I_c(\overline{G})$ also follows from [1, Theorem 1.10].

Example 6.0.8. Let G be a graph whose connected components are two triangles G_1 and G_2. Then, by Theorem 5.2.3, $I(G_i)$ is normal for $i = 1, 2$ but $I(G)$ is not normal by the same theorem.

Example 6.0.9. If G is the bipartite graph $\mathcal{K}_{1,2}$ with edges $\{t_1, t_2\}$ and $\{t_1, t_3\}$. Then, t_1 is an isolated vertex of $\overline{G}$, $E(\overline{G}) = \{\{t_2, t_3\}\}$ and, by Lemma 5.1.8, one has

$$I_c(G) = (t_2t_3) + I(\overline{G})^* = (t_2t_3) + (t_2t_3)^* = (t_2t_3,\ t_1).$$

Example 6.0.10. If G is a graph whose independence number $\beta_0(G)$ is 1, then $G = \mathcal{K}_s$ is a complete graph, $I(\overline{G}) = (0)$, $I_c(G)$ is normal (Theorem 5.2.7) and, by Lemma 5.1.8, one has

$$I_c(G) = (\{(t_1 \cdots t_s)/t_i \mid 1 \leq i \leq s\}).$$

The normality of $I_c(G)$ also follows from the fact that the ideal generated by all squarefree monomials of S of fixed degree $k \geq 1$ is normal [31, Proposition 2.9].

Example 6.0.11. Let G be the cone with apex t_6 over the cycle $C_5 = \{t_1, \ldots, t_5\}$. Then, $\beta_0(G) = 2$, $I_c(G)$ is normal (Theorem 5.2.7) and, by Lemma 5.1.8, one has

$$I_c(G) = (t_1t_2t_3t_4t_5,\ t_2t_4t_5t_6,\ t_1t_2t_4t_6,\ t_1t_3t_4t_6,\ t_1t_3t_5t_6,\ t_2t_3t_5t_6).$$

Example 6.0.12. [23, Fig. 1, p. 241] Let $I_c(G)$ be the ideal of covers of the graph G defined by the generators of the following ideal

$$\begin{aligned} I = (&t_1t_2, t_2t_3, t_3t_4, t_4t_5, t_5t_6, t_6t_7, t_7t_8, t_8t_9, t_9t_{10}, t_1t_{10}, && (6.1)\\ &t_2t_{11}, t_8t_{11}, t_3t_{12}, t_7t_{12}, t_1t_9, t_2t_8, t_3t_7, t_4t_6, t_1t_6, t_4t_9, && (6.2)\\ &t_5t_{10}, t_{10}t_{11}, t_{11}t_{12}, t_5t_{12}). \end{aligned}$$

The graph G is denoted by H_4 in [23]. Using the normality test of Procedure 6.0.16 and *Macaulay*2 [18], we get that $I(G)$ and $I_c(G)$ are not normal whereas $I(\overline{G})$ and $I_c(\overline{G})$ are normal. This example shows that none of the implications of the duality criterion of Theorem 5.2.7(a) hold for graphs with independence number $\beta_0(G)$ greater than 2 because $\beta_0(G) = 4$ and $\beta_0(\overline{G}) = 3$.

Example 6.0.13. Let G be the graph whose complement $\overline{G}$ is the graph depicted in Figure 6.8. The graph G has 50 edges and $\beta_0(G) = 3$. Using the normality test of Procedure 6.0.16 for *Macaulay*2 [18], we obtain that $I(G)$ is normal, $I_c(G)$ is not normal, and furthermore $I_c(G)^5$ is not integrally closed because one has

$$f = t_1^4\, t_2^4\, t_3^4\, t_4^4\, t_5^4\, t_6^4\, t_7^2\, t_8^4\, t_9^4\, t_{10}^4\, t_{11}^4\, t_{12}^4\, t_{13}^4 \in \overline{I_c(G)^5} \setminus I_c(G)^5.$$

This example shows that the Hochster configurations of $\overline{G}$, with C_1, C_2 cycles of length at least five, are not the only obstructions for the normality of $I_c(G)$ (see Corollary 5.2.5).

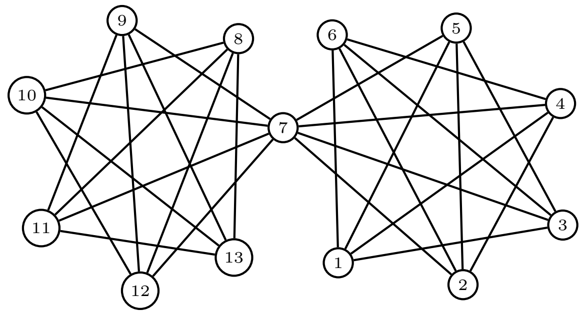

Figure 6.8: Graph $\overline{G}$ consists of two antiholes joined by a vertex.

Appendix A: Procedures

In this appendix we give procedures for *Normaliz* [4] and *Macaulay*2 [18] to determine the normality of a monomial ideal, the minimal generators of the ideal of covers of a clutter, the Hilbert basis of the Rees cone $\mathbb{R}_+\mathcal{A}'$ defined in Eq. (1.1), and the Hilbert basis of the cone $\mathbb{R}_+\mathcal{B}$ generated by the set $\mathcal{B}$ defined in Eq. (4.1). The sets $\mathcal{A}'$ and $\mathcal{B}$ are used to characterize the normality of a monomial ideal (Lemma 2.6.2) and the normality of an ideal generated by monomials of degree two (Theorem 4.1.1).

Procedure 6.0.14. Let $I = (t^{v_1}, \ldots, t^{v_q})$ be a monomial ideal of S. The following procedure for *Normaliz* [4] computes the Hilbert basis of the cone generated by

$$\mathcal{B} = \{e_{s+1}\} \cup \{e_i + e_{s+1}\}_{i=1}^{s} \cup \{(v_i, 1)\}_{i=1}^{q},$$

and determines whether or not $\mathcal{B}$ is a Hilbert basis. The input is the matrix whose rows are the vectors in the set $\mathcal{B}$. This procedure corresponds to Example 6.0.6.

```
amb_space 11
normalization 21
0 0 0 0 0 0 0 0 0 0 1
1 0 0 0 0 0 0 0 0 0 1
```

```
0 1 0 0 0 0 0 0 0 0 1
0 0 1 0 0 0 0 0 0 0 1
0 0 0 1 0 0 0 0 0 0 1
0 0 0 0 1 0 0 0 0 0 1
0 0 0 0 0 1 0 0 0 0 1
0 0 0 0 0 0 1 0 0 0 1
0 0 0 0 0 0 0 1 0 0 1
0 0 0 0 0 0 0 0 1 0 1
0 0 0 0 0 0 0 0 0 1 1
0 0 1 1 0 1 1 1 1 1 1
0 0 1 0 1 1 1 1 1 1 1
0 1 1 0 0 1 1 1 1 1 1
1 1 0 0 0 1 1 1 1 1 1
0 1 1 0 1 0 1 1 1 1 1
1 1 1 1 1 0 0 1 1 0 1
1 1 1 1 1 0 0 1 0 1 1
1 1 1 1 1 0 1 1 0 0 1
1 1 1 1 1 1 1 0 0 0 1
1 1 1 1 0 0 1 1 0 1 1
```

Procedure 6.0.15. Let $I = (t^{v_1}, \ldots, t^{v_q})$ be a monomial ideal of S. The following procedure for *Normaliz* [4] computes the Hilbert basis of the Rees cone of I defined in Eq. (1.1) and determines whether or not the set $\mathcal{A}' = \{e_i\}_{i=1}^{s} \cup \{(v_i, 1)\}_{i=1}^{q}$ is a Hilbert basis. In particular, by Lemma 2.6.2, we can determine whether or not I is a normal ideal. The input is the matrix with rows $v_1, \ldots, v_q$. This procedure corresponds to Example 6.0.6.

```
amb_space 11
rees_algebra 10
0 0 1 1 0 1 1 1 1 1
0 0 1 0 1 1 1 1 1 1
0 1 1 0 0 1 1 1 1 1
```

```
1 1 0 0 0 1 1 1 1 1
0 1 1 0 1 0 1 1 1 1
1 1 1 1 1 0 0 1 1 0
1 1 1 1 1 0 0 1 0 1
1 1 1 1 1 0 1 1 0 0
1 1 1 1 1 1 1 0 0 0
1 1 1 1 0 0 1 1 0 1
```

Procedure 6.0.16 (Normality test)**.** Let I be a monomial ideal. We implement a procedure—that uses the interface of *Macaulay*2 [18] to *Normaliz* [4]—to determine the normality of I, and the normality and minimal generators of the ideal of covers of a clutter. This procedure corresponds to Example 6.0.7. To compute other examples, in the next procedure simply change the polynomial rings R and S, and the generators of I.

```
restart
loadPackage("Normaliz",Reload=>true)
loadPackage("Polyhedra", Reload => true)
R=QQ[t1,t2,t3,t4,t5,t6,t7];
--antihole, complement of C7
I=monomialIdeal(t1*t3,t1*t4,t1*t5,t1*t6,t2*t4,t2*t5,t2*t6,t2*t7,
t3*t5,t3*t6,t3*t7,t4*t6,t4*t7,t5*t7)
dim I
--Ideal of covers of clutter associated to I
J=dual(I)
--transpose incidence matrix of I
A = matrix flatten apply(flatten entries gens I , exponents)
--transpose incidence matrix of J
AJ=matrix flatten apply(flatten entries gens J , exponents)
--generators of Rees cone of I
M = id_(ZZ^(numcols(A)+1))^{0..numcols(A)-1}||
(A|transpose matrix {for i to numrows A-1 list 1})
```

```
--generators of Rees cone of J
MJ = id_(ZZ^(numcols(AJ)+1))^{0..numcols(AJ)-1}||
(AJ|transpose matrix {for i to numrows AJ-1 list 1})
--rows of M
l= entries M
--rows of MJ
lJ= entries MJ
--Next we compute the normalization of Rees algebras
S=QQ[t1,t2,t3,t4,t5,t6,t7,t8];
L=for i in l list S_i
LJ=for i in lJ list S_i
--Normalization of the Rees algebra of I
ICL=intclToricRing L
gens ICL
#gens ICL
flatten \ exponents \  gens ICL
--Normalization of the Rees algebra of J
ICLJ=intclToricRing LJ
gens ICLJ
flatten \ exponents \  gens ICLJ
--Normality test for ideal I
sort L==sort gens ICL
--Normality test for ideal J
sort LJ==sort gens ICLJ
```

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