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Funciones Armónicas y Monogénicas en el Toro

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Directores de la tesis:

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Abstract

This work studies the structure of real-valued harmonic functions and quaternion-valued monogenic functions on the solid torus in Euclidean space.

An explicit expression is obtained for the Dirichlet-to-Neumann mapping for the Laplace operator with respect to series expansions in toroidal harmonics, thereby reducing the calculation of the operator to algebraic manipulations on the coefficients. Using these expansions, a method for computing the numerical solutions of the corresponding Neumann problem is presented, and numerical illustrations are provided [8].

The second significant contribution presented in this work is devoted to the construction of a reverse-Appell basis of toroidal harmonic functions. This is used to obtain the principal contribution, which is the construction of bases in the real L^2-Hilbert spaces of reduced quaternion and quaternion-valued monogenic functions on toroidal domains.

Resumen

Este trabajo estudia la estructura de las funciones armónicas real-valuadas y las funciones monogénicas cuaternio-valuadas en el toro sólido en el espacio euclidiano.

Se obtiene una expresión explícita del mapeo de Dirichlet-a-Neumann para el operador de Laplace con respecto a expansiones en serie en armónicos toroidales, así reduciendo el cálculo del operador a manipulaciones algebraicas en los coeficientes. Por medio de estas expansiones, se presenta un método para calcular las soluciones numéricas del problema de Neumann correspondiente y se proporcionan ilustraciones numéricas.

La segunda contribución significativa que se presenta en este trabajo está dedicada a la construcción de una base de Appell inversa de funciones armónicas toroidales. Esto se emplea para obtener la contribución principal, que es la construcción de bases en los espacios de Hilbert L^2 reales consistentes de funciones monogénicas en dominios toroidales con valores en los cuaterniones reducidas y cuaterniones en los cuaternios completos.

Dedication

In memory of

my parents:

Maliheh and Mohammad.

Also to

Batoul and Mike

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Contents

Introduction

This thesis divides naturally into two parts. The first part, a study of toroidal harmonics, has been included as a necessary prelude to the second part, which deals with the main topic of toroidal monogenics.

Toroidal harmonics. Square-integrable harmonic functions defined on a torus are represented in a natural way by expansions in basic functions known as toroidal harmonics, in a manner analogous to power series expansions. There have been numerous applications of toroidal harmonic functions in solving elliptic boundary-valued problems of partial differential equations in analysis and physics (e.g., in magnetostatic field problems [24], electrical engineering and electromechanics [91], and electrostatics [85]). For certain types of problems, such as boundary problems involving normal derivatives, these series often require manipulation by termwise differentiation, which produces a new series which is not immediately recognizable in terms of the basic toroidal harmonics.

In the case of spherical harmonics, the derivative with respect to the axial variable produces a constant multiple of another spherical harmonic. This characteristic is called the "Appell property" after [7], and is used to great advantage in the study of harmonic functions on the ball.

To facilitate a similar comparison of coefficients in the case of the torus, it would be desirable for toroidal harmonics to satisfy an identity with respect to a real derivative with a similar Appell property. We will show that they do not, and in fact, the application of $\partial/\partial x_0$, $\partial/\partial x_1$, or $\partial/\partial x_2$ increases the index (or degree) of the toroidal harmonics. However, with an appropriate change of basis, we can create a new collection of toroidal harmonics that satisfy what we may call a "reverse Appell property" with respect to $\partial/\partial x_0$. The proposed basis, which is easily handled from computational and theoretical considerations, is fundamental to studying the collection of monogenic functions on the torus in the second part of the thesis.

Quaternionic function theory. Although methods of classical complex analysis have a wide range of applications, they are intrinsically restricted to two-dimensional problems, which has led to an increasing need for higher-dimensional counterparts of such methods.

INTRODUCTION

Two main ways exist to generalize the theory of functions of one complex variable to higher dimensions. One is the function theory of several complex variables, which is restricted to an even number of real variables. The other can be realized using the Hamiltonian quaternion algebra (and its generalization to Clifford numbers), which leads to quaternionic function theory. In both approaches, the starting point lies in considering null solutions of particular underdetermined systems of first-order constant coefficient partial differential equations in Euclidean spaces.

Quaternionic function theory has several advantages compared with the theory of several complex variables. One advantage is that it does not limit the dimension of the real vector space used as the domain. Another advantage is that higher-order differential operators can be factored into products of lower-order operators (for example, the Laplace operator can be factorized by two first-order quaternionic differential operators like in the plane case for complex variables). In this thesis, we are concerned with the Euclidean space of dimension 3. For those reasons, our approach fits into the framework of quaternionic analysis.

The main objects of study in the quaternionic analysis setting are the classes of harmonic functions and solutions of the generalized (reduced)

Cauchy-Riemann (or Fueter) equation $\overline{\partial} f = (\partial/\partial x_0 + \sum_{i=1}^{2} \mathbf{e}_i \partial/\partial x_i) f = 0$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are two of the three basic quaternionic units, subject to the Hamiltonian multiplication rules (this is described in detail in Subsection 4.4 below). The elements of the latter class are usually called monogenic (or regular, holomorphic, or hyperholomorphic) functions. See [14, 35, 36, 39, 40, 42, 51, 52, 64, 74, 88] and references therein.

Although quaternionic analysis generalizes the most important features of complex analysis, monogenic functions do not enjoy all properties of holomorphic functions of one complex variable. For instance, due to the non-commutativity of the quaternion algebra, the product of two monogenic functions is generally not monogenic; the same holds for the composition. It is natural, therefore, that new techniques have been sought for constructing monogenic functions.

In the literature, one technique which has been applied in different contexts is based on the factorization of the Laplacian using $\overline{\partial}$ and its conjugate. In this way orthogonal sets have been constructed and studied spanning the Hilbert spaces of square-integrable monogenic functions in the interior and exterior of the ball [13, 15, 16, 17], spheroidal domains (both prolate and oblate) [66, 69, 72, 73, 74, 75, 77], as well as over finite and infinite cylinders [67, 71]. To the best of our knowledge, complete sets of monogenic functions have not previously been built in the context of the torus.

Each part of this thesis is comprised of three chapters. The outline of the contents of each chapter is as follows:

In Chapter 1, we summarize the basic facts about the associated Legendre functions of the first and second kinds, emphasizing those of half-an-odd integer degree and argument greater than 1. For context, we also describe the spherical harmonics. We then give the classical definition of the family Ω_{η_0} of tori depending on one real parameter η_0 and the toroidal harmonics and their properties. Some results on Hilbert spaces and Fourier series are collected at the end of the chapter. None of the material in this chapter is new.

Chapter 2 is devoted to introducing a doubly-indexed reverse-Appell basis of harmonic functions expressed in toroidal coordinates as independent variables. This basis will be fundamental to the construction of bases in the real L^2-Hilbert spaces of reduced quaternion and quaternion-valued monogenic functions on the torus in Chapters 5 and 6. For the construction of the reverse-Appell basis, new formulas for the partial derivatives of the classical toroidal harmonics are produced.

Chapter 3 introduces new techniques for studying the Dirichlet-to-Neumann mapping and the Neumann problem for the Laplace operator on a torus. We show that the Dirichlet-to-Neumann mapping is expressed with respect to series expansions in toroidal harmonics and thereby reduced to algebraic manipulations on the coefficients. Unlike the case for a sphere, the Dirichlet-to-Neumann mapping for a torus turns out to be much more complicated, and the numerical solutions of the corresponding Neumann problem involve solving an infinite system of linear equations. We express the well-known necessary and sufficient condition for the solvability of the Neumann problem (compatibility condition), as well as the normalization condition, in terms of the Fourier coefficients. The solution to the Neumann problem involves a special twist: the free parameter in the undetermined linear system cannot be found algebraically, as far as we know. Therefore we express it as a limit of easily calculated algebraic expressions. The analysis is illustrated through numerical examples. We then combine the results for interior and exterior domains to solve the Neumann problem for a toroidal shell.

Chapter 4 gives basic concepts and terminology concerning quaternions, including its insertion as a particular case of a Clifford algebra. We start by recalling the fundamental notions and results from quaternionic function theory, which provides us with the basic tools for our analysis in the following chapters. The remainder of this chapter is devoted to a special class of quaternion-valued functions named monogenic. Furthermore, properties of monogenic functions which hold in arbitrary domains in $\mathbb{R}^3$ are discussed. None of the material in this chapter is new.

In Chapter 5, we construct a complete independent set in the real linear Hilbert space of $\mathcal{A}$-valued monogenic functions on the torus. Underlying our manipulations is a cohomology coefficient associated with an arbitrary monogenic function. We calculate this coefficient for monogenic constants (i.e., functions in $\operatorname{Ker}\overline{\partial} \cap \operatorname{Ker}\partial$, where $\partial = \partial/\partial x_0 - \sum_{i=1}^{2} \mathbf{e}_i \partial/\partial x_i$). Unlike the sphere or other simply-connected domains, non-exact monogenics exist on the torus, although the application of the operator ∂ cannot produce them. A second difference comes from the already mentioned fact that $\partial/\partial x_0$ has a reverse-Appell property, which means that the application of ∂ cannot produce those monogenics on Ω_{η_0} whose scalar part contains the 0-level toroidal harmonics. These two differences make the study of monogenics on Ω_{η_0} very different from domains, such as the ball, prolate and oblate spheroids, and cylinders.

In Chapter 6, we use the basis of $\mathcal{A}$-valued monogenic functions on the torus built in the previous chapter to construct a basis for the space of $\mathbb{H}$-valued monogenics in the torus.

At the end, some open problems for future research are stated.

The results of Chapter 3 were published in [8]. Further results of this thesis have been submitted for publication [9].

Important Symbols

Part I

<u>First Part: Toroidal Harmonics</u>

Chapter 1

Toroidal harmonic functions

The main object of study of this thesis is the space of quaternion-valued monogenic functions on a torus, which will be introduced in Chapter 4. Since the real components of monogenic functions are harmonic functions, we will need as much information as possible about harmonic functions on the torus. In this chapter, we will give the classical definition of the family Ω_{η_0} of tori depending on one real parameter η_0 and the toroidal harmonics in Section 1.3. As prerequisites, in Section 1.1, we summarize the basic facts about the classical associated Legendre functions of the first and second kinds with particular emphasis on those of half-an-odd integer degree and argument greater than 1 and, for context, the spherical harmonics in Section 1.2. Some results we will need on Hilbert spaces and Fourier series are collected at the end of the chapter. None of the results in this chapter are original.

1.1 Legendre functions

In this section, we collect the results we will need concerning the associated Legendre functions of the first and second kinds. The primary references are [11, 12, 46, 93].

1.1.1 Definition of the associated Legendre functions

The Legendre polynomials of integral degree n are given by Rodrigues' formula

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n, \;\; t \in \mathbb{R}, \; n = 0, 1, \ldots, \tag{1.1}$$

and the Legendre functions of the second kind are

$$Q_n(t) = \frac{1}{2}P_n(t)\log\frac{t+1}{t-1} - \sum_{k=0}^{n-1}\frac{P_n(t)P_{n-k}(t)}{n-k}, \;\; |t| > 1. \tag{1.2}$$

The functions P_n and Q_n are the particular cases for $m = 0$ of the following definition.

Definition 1.1. Let n, m be nonnegative integers such that $0 \leq m \leq n$. The *associated Legendre functions of the first and second kind* (also known as

Ferrer's functions), $P_n^m(t)$ and $Q_n^m(t)$, are given respectively by

$$P_n^m(t) = \begin{cases} (-1)^m(1-t^2)^{m/2}\frac{d^m P_n(t)}{dt^m}, & t \in [-1,1], \\ (t^2-1)^{m/2}\frac{d^m P_n(t)}{dt^m}, & |t| > 1, \end{cases}$$

and

$$Q_n^m(t) = \begin{cases} (1-t^2)^{m/2}\frac{d^m Q_n(t)}{dt^m}, & t \in [-1,1], \\ (t^2-1)^{m/2}\frac{d^m Q_n(t)}{dt^m}, & |t| > 1. \end{cases}$$

The index n is called the degree, and m is the order of the associated Legendre function. P_n^m and Q_n^m are linearly independent solutions of the ordinary differential equation known as the *Legendre differential equation*,

$$(1-t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + \left(n(n+1) - \frac{m^2}{1-t^2}\right)y = 0. \tag{1.3}$$

We will mostly need the associated Legendre functions for which the degree is half an odd integer. They are given as follows:

Definition 1.2. [46, pp. 437, 438]) Let $n, m \in \mathbb{Z}$ and $n, m \geq 0$. Define

$$P^m_{n-\frac{1}{2}}(\cosh\eta) = \\ (-1)^m \frac{1}{2\pi}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})}\int_0^{2\pi}\frac{\cos m\varphi}{(\cosh\eta + \sinh\eta\cos\varphi)^{n+1/2}}\,d\varphi, \tag{1.4}$$

and

$$Q^m_{n-\frac{1}{2}}(\cosh\eta) = (-1)^m 2^m \frac{\Gamma(n+m+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n+1)}\sinh^m\eta\, e^{-(n+m+\frac{1}{2})\eta} \\ \times {}_2F_1(\frac{1}{2}+m, n+m+\frac{1}{2}; n+1; e^{-2\eta}), \tag{1.5}$$

where Γ denotes the (complete) gamma function defined by $\Gamma(n) = (n-1)!$ (see, e.g., [2, pp. 255–258]), and

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{1.6}$$

with $|z| < 1$ is the Gaussian hypergeometric function and the Pochhammer symbol is $(q)_n = q(q+1)\cdots(q+n-1)$ with $(q)_0 = 1$ by convention.

We give these definitions explicitly because the values of $P_n^m(t)$ and $Q_n^m(t)$ vary from author to author, by a change of sign or, in some cases, by a factor of the complex number i.

We need the following results related to the Legendre functions of the second kind.

Proposition 1.3. *For all $n, m \in \mathbb{Z}$, $n, m \geq 0$ and $\eta \in \mathbb{R}^+$ we have the following inequality:*

$$ (-1)^m Q_{n-1/2}^m(\cosh\eta) > 0. \tag{1.7}$$

Proof. It is shown in [46, p. 195] that

$$ Q_{n-1/2}^m(t) = \frac{(-1)^m}{2^{n+1/2}}\frac{\Gamma(n+m+1/2)}{\Gamma(n+1/2)}\,(t^2-1)^{m/2}\int_{-1}^{1}\frac{(1-s^2)^{n-1/2}}{(t-s)^{n+m+1/2}}\,ds. $$

The inequality follows immediately from this, because all of the factors except $(-1)^m$ are positive. $\square$

Proposition 1.4. *Let $\eta > 0$ and let m be any integer. Then*

$$\lim_{n\to\infty} \frac{Q^m_{n-1/2}(\cosh\eta)}{Q^m_{n-3/2}(\cosh\eta)} = e^{-\eta}, \tag{1.8}$$

which does not depend on the value m.

Proof. In [46, p. 305] it is shown that for fixed η and m,

$$Q^m_n(\cosh\eta) \;\sim (-1)^m \frac{\Gamma(n+m+1)}{\Gamma(n+1)} \Big(\frac{\pi}{n}\Big)^{1/2} \frac{e^{-(n+1/2)\eta}}{(2\sinh\eta)^{1/2}} \tag{1.9}$$

as $n \to \infty$, where the symbol $\sim$ means that the ratio of the two expressions tends to 1. By dividing two of these expressions, we obtain the desired limit. □

1.1.2 Recursion formulas for Legendre functions

There are many formulas connecting three associated Legendre functions whose indices differ by no more than 1. The best known of these formulas hold when the functions are applied in the range $|t| < 1$ [11, 46, 93] in part because of their close relationship to spherical harmonics (see Section 1.2 below). The Legendre functions of half-integer degree have a singularity at $t = 1$, and can be continued analytically to find a recurrence formula for $t \in \mathbb{C} - [-1, 1]$. Since we will often need to use $Q^m_{n-1/2}(t)$ in the following chapters, we give a list of the main formulas with emphasis on $|t| > 1$.

The notation is as follows. The formulas are valid for any value of n, which in later applications will be substituted for $n-1/2$. The symbol $L_n^m(t)$ represents $P_n^m(t)$ or $Q_n^m(t)$.

Proposition 1.5 ([11, 46]). *Let $n, m \in \mathbb{Z}$ and $t \in \mathbb{R}$. Then*

$$L_{-n}^m(t) = L_{n-1}^m(t). \tag{1.10}$$

$$(1-t^2)(L_{n+1}^m)'(t) = (n+m+1)L_n^m(t) - (n+1)tL_{n+1}^m(t), \tag{1.11}$$

$$(n-m+1)L_{n+1}^m(t) = (2n+1)tL_n^m(t) - (n+m)L_{n-1}^m(t). \tag{1.12}$$

Proposition 1.6 ([11]). *Let $n \geq 0$, $0 \leq m \leq n$, and let $|t| > 1$. Then*

$$(t^2-1)(L_{n+1}^m)'(t) = (t^2-1)^{1/2}L_{n+1}^{m+1}(t) + mtL_{n+1}^m(t), \tag{1.13}$$

$$(t^2-1)^{1/2}L_{n+1}^m(t) = \frac{1}{2n+3}(L_{n+2}^{m+1}(t) - L_n^{m+1}(t)), \tag{1.14}$$

$$\begin{aligned} &2mtL_{n+1}^m(t) \\ &= (t^2-1)^{1/2}\Big(- L_{n+1}^{m+1}(t) + (n+m+1)(n-m+2)L_{n+1}^{m-1}(t)\Big), \end{aligned} \tag{1.15}$$

$$(n-m)tL_n^m(t) = (t^2-1)^{1/2}L_n^{m+1}(t) + (n+m)L_{n-1}^m(t). \tag{1.16}$$

The following recursive formulas are deduced similarly to the corresponding ones for argument $|t| < 1$. Since we have not found them in the literature, we include the proof for completeness.

Proposition 1.7. *Let $n \geq 0$, $0 \leq m \leq n$, and let $|t| > 1$. Then*

$$\begin{aligned} &2mL_n^m(t) \\ &= (t^2-1)^{1/2}\Big(- L_{n-1}^{m+1}(t) + (n+m-1)(n+m)L_{n-1}^{m-1}(t)\Big), \end{aligned} \tag{1.17}$$

$$L_{n+1}^m(t) = t\, L_n^m(t) + (n+m)\,(t^2-1)^{1/2}\, L_n^{m-1}(t). \tag{1.18}$$

$$\begin{aligned} &L_{n-1}^{m+1}(t) - (n+m)(n+m-1)L_{n-1}^{m-1}(t) \\ &\qquad = L_{n+1}^{m+1}(t) - (n-m+1)(n-m+2)L_{n+1}^{m-1}(t). \end{aligned} \tag{1.19}$$

Proof. The formulas follow by analytic continuation of the corresponding recurrence formulas for $t \in [-1,1]$, using the following properties. Let $s = 1-\epsilon$, $t = 1+\epsilon$. The function $(1-s^2)^{1/2}$ continued in the upper half-plane from s to t in a semicircle around $1+0i$ gives $(t^2-1)^{1/2} + O(\epsilon)$. In [46], it is shown that the continuation of $P_n^m(s)$ gives $iP_n^m(t) + O(\epsilon)$ while the continuation of $Q_n^m(s)$ gives $-iQ_n^m(t) + O(\epsilon)$. The imaginary factors $\pm i$ combined with those of Definition 1.1 give the desired formulas for $t > 1$. □

Whipple's transformation for Legendre functions, named after Francis John Welsh Whipple, arises from a general expression concerning associated Legendre functions [92]. The Whipple formulas for Legendre functions are as follows:

Proposition 1.8 ([2]).

$$P^m_{n-\frac{1}{2}}(\cosh\eta) = \frac{(-1)^n}{\Gamma(n-m+\frac{1}{2})}\sqrt{\frac{2}{\pi\ \sinh\eta}}\, Q^n_{m-\frac{1}{2}}(\coth\eta), \tag{1.20}$$

$$Q^m_{n-\frac{1}{2}}(\cosh\eta) = \frac{(-1)^n\pi}{\Gamma(n-m+\frac{1}{2})}\sqrt{\frac{\pi}{2\ \sinh\eta}}\, P^n_{m-\frac{1}{2}}(\coth\eta). \tag{1.21}$$

The following result shows how the Legendre functions of the second kind appear in the Fourier series of powers of a constant minus $\cos\theta$.

Proposition 1.9 ([26]). *For all $\alpha \in \mathbb{C}$,*

$$\begin{aligned}&(\cosh\eta - \cos\theta)^{-\alpha}\\ &= \frac{1}{\Gamma(\alpha)}\sqrt{\frac{2}{\pi}}\frac{e^{-i\pi(\alpha-1/2)}}{(\sinh\eta)^{\alpha-1/2}}\sum_{n=0}^{\infty}\varepsilon_n\, Q^{\alpha-1/2}_{n-1/2}(\cosh\eta)\ \cos(n\theta)\end{aligned}$$

with $\varepsilon_n = 1 + \delta_{0,n}$, where $\delta_{i,j}$ is the Kronecker delta function defined by $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ if $i \neq j$.

A particular case is known as *Heine's formula* due to Heinrich Eduard Heine:

$$\frac{1}{\sqrt{\cosh\eta - \cos\theta}} = \frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\varepsilon_n\, Q^0_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta). \tag{1.22}$$

1.2 Spherical harmonics

Solid spherical harmonics are homogeneous harmonic functions. Therefore a spherical harmonic is determined in all of $\mathbb{R}^3$ by its degree of homogeneity and its values on the surface of the sphere S^2. These restrictions to S^2 are known simply as spherical harmonics. Although we will not present new research on spherical harmonics in this thesis, it is important to summarize some basic facts which can be found in [6, 12, 84] because they will serve for comparison with results on toroidal harmonics.

We consider the spherical coordinate system (r, θ, φ) in $\mathbb{R}^3 \setminus \{0\}$, defined by

$$x_0 = r\cos\theta, \quad x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi, \tag{1.23}$$

where $r \in [0, \infty)$ is equal to $|x|$, $\theta \in [0, \pi]$ is the polar angle of x, and $\varphi \in [0, 2\pi)$ is the azimuthal angle.

The three-dimensional Laplacian is expressed in spherical coordinates as follows,

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right).$$

The procedure of separation of variables leads to the following solutions.

Definition 1.10. The *spherical harmonics* of degree $n = 0, 1, \ldots,$ and order $m = 0, 1, \ldots, n$, are defined by

$$Y^{\pm}_{n,m}(\theta, \varphi) = P^m_n(\cos\theta)\Phi^{\pm}_m(\varphi),$$

where

$$\Phi^{+}_m(\varphi) = \cos(m\varphi), \quad \Phi^{-}_m(\varphi) = \sin(m\varphi). \tag{1.24}$$

Further, the *interior solid spherical harmonics* are

$$r^n Y^{\pm}_{n,m}(\theta, \varphi), \tag{1.25}$$

and the *exterior solid spherical harmonics* are

$$\frac{1}{r^{n+1}} Y^{\pm}_{n,m}(\theta, \varphi), \;\; r > 0. \tag{1.26}$$

The interior (resp., exterior) solid spherical harmonics are harmonic functions and are homogeneous polynomials (resp., functions) of degree n (resp., $-(n+1)$) in the variables x_0, x_1, x_2.

The spherical harmonics are orthogonal in $L^2(\{x : |x| = 1\})$, i.e.,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi} Y^{\pm}_{n_1,m_1}(\theta, \varphi) Y^{\pm}_{n_2,m_2}(\theta, \varphi) \sin\theta \, d\theta \, d\varphi \\
&= \frac{2\pi}{(2n_1+1)} \frac{(n_1+m_1)!}{(n_1-m_1)!} \, \delta_{n_1,n_2} \, \delta_{m_1,m_2},
\end{aligned}$$

and in fact they form an orthogonal basis of $L^2(\{x : |x| = 1\})$.

It is well-known that the internal solid spherical harmonics (1.25) form an orthogonal basis of $L^2(\{x : |x| < 1\})$ and the external solid spherical harmonics (1.26) form an orthogonal basis of $L^2(\{x : |x| > 1\})$.

Clearly, the partial derivatives of $r^n Y^{\pm}_{n,m}$ (resp., $(1/r^{n+1}) Y^{\pm}_{n,m}$) with respect to x_0, x_1, x_2 are also harmonic, and they are polynomials (resp., functions) of degree $n-1$ (resp., $-(n+2)$). It further follows that they can be expressed as linear combinations of interior (resp., exterior) solid spherical harmonics of degree $n-1$ (resp., $-(n+2)$). The expressions for the partial derivatives with respect to x_0 are particularly simple.

Proposition 1.11 ([68, 74, 75])**.**

$$\frac{\partial}{\partial x_0} r^n Y^{\pm}_{n,m} = (n+m) r^{n-1} Y^{\pm}_{n-1,m} \tag{1.27}$$

for $m = 0, 1, \ldots, n-1$, *and*

$$\frac{\partial}{\partial x_0} \frac{1}{r^{n+1}} Y^{\pm}_{n,m} = -(n+1-m) \frac{1}{r^{n+2}} Y^{\pm}_{n+1,m} \tag{1.28}$$

for $m = 0, 1, \ldots, n$.

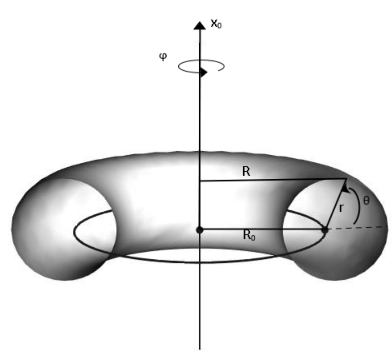
Equality (1.27) (resp., (1.28)) is known as an "Appell property" (resp., "reverse-Appell property") for the interior (resp., exterior) solid spherical harmonics with respect to $\partial/\partial x_0$, and will be discussed in Chapter 2.

1.3 Toroidal harmonics

1.3.1 Elementary coordinates on torus

In geometry, a torus (plural tori, colloquially donut, or doughnut) is a surface of revolution generated by revolving a circle in three-dimensional space about an axis that is coplanar with the circle. The distance R_0 from the center of the tube to the axis of the torus is called the major radius. The distance r, $0 \leq r \leq R_0$, is the radius of the tube and is called the minor radius. $\theta, \varphi \in [0, 2\pi)$ are called the poloidal and toroidal angles, respectively. The *elementary toroidal coordinates* (r, θ, φ) are then given by

$$x_0 = r \sin\theta, \;\; x_1 = (R_0 + r\cos\theta)\cos\varphi, \;\; x_2 = (R_0 + r\cos\theta)\sin\varphi,$$



where

$$
\begin{aligned}
r &= \sqrt{((x_1^2 + x_2^2)^{1/2} - R_0)^2 + x_0^2},\\
\theta &= \tan^{-1}\Big(\frac{x_0}{(x_1^2 + x_2^2)^{1/2} - R_0^2}\Big),\\
\varphi &= \tan^{-1}\Big(\frac{x_2}{x_0}\Big).
\end{aligned}
$$

In these coordinates the Laplacian is

$$
\begin{aligned}
\Delta v &= \frac{\partial^2 v}{\partial x_0^2} + \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2}\\
&= \left(\frac{\cos\theta}{R + r\cos\theta} + \frac{\cos^2\theta}{r^2} + \frac{\sin^2\theta}{r}\right)\frac{\partial v}{\partial r} + \left(\frac{-\sin\theta}{r(R + r\cos\theta)}\right)\frac{\partial v}{\partial \theta}\\
&\quad - \left(\frac{\sin\varphi\cos\varphi(\sin^2\theta + r)}{r(R + r\cos\theta)^2}\right)\frac{\partial v}{\partial \varphi} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\\
&\quad + \frac{1}{(R + r\cos\theta)^2}\frac{\partial^2 v}{\partial \varphi^2} + \left(\frac{2\sin\theta\cos\theta\cos^2\varphi}{r}\right)\frac{\partial^2 v}{\partial r \partial \theta}
\end{aligned}
$$

whenever $v \in C^2$.

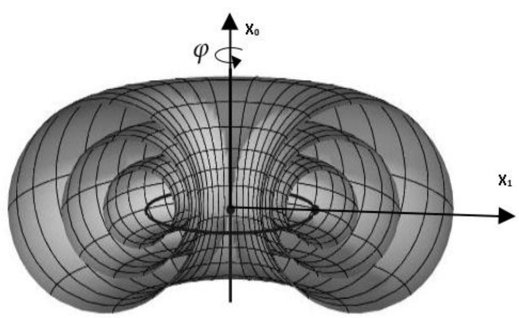
While the elementary toroidal coordinates may appear natural and intuitive, they have limitations. In the first place, they do not fill up $\mathbb{R}^3$ in a natural way and depend on the auxiliary constant R_0. These coordinates are not commonly used to solve the Laplace equation because it does not lend itself easily to separating the variables. We have only presented them here to highlight the contrast with classical toroidal coordinates, the coordinate system given in the following subsection.

1.3.2 Toroidal coordinates

Our basic reference for properties of toroidal coordinates is [46]. See also [56].

Definition 1.12. Let $0 \leq \eta < \infty$, $-\pi \leq \theta < \pi$, and $-\pi \leq \varphi < \pi$. The relationship between the toroidal coordinates (η, θ, φ) and the Cartesian coordinates $x = (x_0, x_1, x_2)$ is given by

$$x_0 = \frac{\sin\theta}{\cosh\eta - \cos\theta}, \quad x_1 = \frac{\sinh\eta\ \cos\varphi}{\cosh\eta - \cos\theta}, \quad x_2 = \frac{\sinh\eta\ \sin\varphi}{\cosh\eta - \cos\theta}. \tag{1.29}$$



The surface $\eta = \eta_0$ is a two-dimensional torus with axial circle of radius $\coth\eta_0$ centered at the origin in the x_1, x_2-plane, having a circular cross section of radius $\operatorname{csch}\eta_0$. The surface $\theta = \theta_0$ is that part of the sphere of radius $\csc\theta_0$, with center at $x_0 = \cot\theta_0$, $x_1 = x_2 = 0$, which is above the x_1, x_2-plane, and the rest of the same sphere below the x_1, x_2-plane, is the surface $\theta = \pi - \theta_0$. The limiting torus $\eta = \infty$ is a circle of radius 1. The toroidal coordinates are not defined in the small subsets

$$S^1 = \{x \in \mathbb{R}^3\colon\ x_0 = 0,\ x_1^2 + x_2^2 = 1\},$$

$$\mathbb{R}_0 = \{x \in \mathbb{R}^3\colon\ x_1 = x_2 = 0\}.$$

Definition 1.13. For any fixed $\eta_0 > 0$, we define the *interior and exterior toroidal domains*

$$\Omega_{\eta_0} = \{x\colon\ \eta > \eta_0\} \cup S^1,$$

$$\Omega^*_{\eta_0} = \{x\colon\ \eta < \eta_0\} \cup \mathbb{R}_0.$$

Proposition 1.14. *The three coordinate tangent vectors to $\partial\Omega_{\eta_0} = \partial\Omega^*_{\eta_0}$ are*

$$\begin{aligned}
x_\eta &= \frac{\partial x}{\partial \eta} = \frac{1}{(\cosh\eta_0 - \cos\theta)^2}\times \\
&\qquad \big(-\sinh\eta_0 \sin\theta,\ \cos\varphi(1-\cosh\eta_0\cos\theta),\ \sin\varphi(1-\cosh\eta_0\cos\theta)\big),\\
x_\theta &= \frac{\partial x}{\partial \theta} = \frac{1}{(\cosh\eta_0 - \cos\theta)^2}\times \\
&\qquad \big(\cos\theta\cosh\eta_0 - 1,\ -\sin\theta\cos\varphi\sinh\eta_0,\ -\sin\theta\sin\varphi\sinh\eta_0\big),\\
x_\varphi &= \frac{\partial x}{\partial \varphi} = \frac{1}{\cosh\eta_0 - \cos\theta}\big(0,\ -\sin\varphi\sinh\eta_0,\ \cos\varphi\sinh\eta_0\big).
\end{aligned} \tag{1.30}$$

Proposition 1.15. *Toroidal coordinates are an orthogonal coordinate system; that is, x_η, x_θ, x_φ are orthogonal vectors at every point of $\mathbb{R}^3 - (S^1 \cup \mathbb{R}_0)$.*

Proof. The proof follows directly from Proposition 1.14. □

1.3.3 Definition of toroidal harmonics

The Laplacian in toroidal coordinates is given by [46]

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x_0^2} + \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \\ &= \frac{\cosh\eta - \cos\theta}{\sinh\eta}\left(\sinh\eta \frac{\partial}{\partial\theta}\left(\frac{1}{\cosh\eta - \cos\theta}\frac{\partial u}{\partial\theta}\right)\right. \\ &\quad \left. + \frac{\partial}{\partial\eta}\left(\frac{\sinh\eta}{\cosh\eta - \cos\theta}\frac{\partial u}{\partial\eta}\right) + \frac{1}{\sinh\eta(\cosh\eta - \cos\theta)}\frac{\partial^2 u}{\partial\varphi^2}\right). \qquad (1.31)\end{aligned}$$

Thus $\Delta u = 0$ if and only if

$$\begin{aligned}&\sinh\eta \frac{\partial}{\partial\theta}\left(\frac{1}{\cosh\eta - \cos\theta}\frac{\partial u}{\partial\theta}\right) + \frac{\partial}{\partial\eta}\left(\frac{\sinh\eta}{\cosh\eta - \cos\theta}\frac{\partial u}{\partial\eta}\right) \\ &\qquad + \frac{1}{\sinh\eta(\cosh\eta - \cos\theta)}\frac{\partial^2 u}{\partial\varphi^2} = 0. \qquad (1.32)\end{aligned}$$

Equation (1.32) is not separable in this form. However, one can obtain a separable equation by a change of variable:

$$u = \sqrt{\cosh\eta - \cos\theta}\, v. \qquad (1.33)$$

The derivatives of this new function v satisfy

$$\begin{aligned}\frac{\partial u}{\partial\theta} &= \frac{1}{2}\frac{\sin\theta}{\sqrt{\cosh\eta - \cos\theta}}\, v + \sqrt{\cosh\eta - \cos\theta}\,\frac{\partial v}{\partial\theta}, \\ \frac{\partial u}{\partial\eta} &= \frac{1}{2}\frac{\sinh\eta}{\sqrt{\cosh\eta - \cos\theta}}\, v + \sqrt{\cosh\eta - \cos\theta}\,\frac{\partial v}{\partial\eta}.\end{aligned}$$

Inserting these into equation (1.32) converts it into

$$\begin{aligned}
&\sinh\eta\frac{\partial}{\partial\theta}\left(\frac{1}{\cosh\eta-\cos\theta}\left(\frac{1}{2}\frac{\sin\theta}{\sqrt{\cosh\eta-\cos\theta}}\,v+\sqrt{\cosh\eta-\cos\theta}\,\frac{\partial v}{\partial\theta}\right)\right)\\
&\quad+\frac{\partial}{\partial\eta}\left(\frac{\sinh\eta}{\cosh\eta-\cos\theta}\left(\frac{1}{2}\frac{\sinh\theta}{\sqrt{\cosh\eta-\cos\theta}}\,v+\sqrt{\cosh\eta-\cos\theta}\,\frac{\partial v}{\partial\eta}\right)\right)\\
&\quad+\frac{1}{\sinh\eta\sqrt{\cosh\eta-\cos\theta}}\frac{\partial^2 v}{\partial\varphi^2}=0.
\end{aligned}$$

After calculating the partial derivatives and making some cancellations, one finds that u is harmonic when

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\left(\frac{1}{2}\frac{\sinh\eta\sin\theta}{(\cosh\eta-\cos\theta)^{3/2}}\right)v+\frac{\sinh\eta}{\sqrt{\cosh\eta-\cos\theta}}\frac{\partial^2 v}{\partial\theta^2}\\
&\quad+\frac{\partial}{\partial\theta}\left(\frac{1}{2}\frac{\sinh^2\eta}{(\cosh\eta-\cos\theta)^{3/2}}\right)v+\frac{1}{\sqrt{\cosh\eta-\cos\theta}}\frac{\partial}{\partial\eta}(\sinh\eta\frac{\partial v}{\partial\eta})\\
&\quad+\frac{1}{\sinh\eta\sqrt{\cosh\eta-\cos\theta}}\frac{\partial^2 v}{\partial\varphi^2}=0.
\end{aligned}$$

Multiply this equation by $\sinh\eta(\cosh\eta-\cos\theta)^{1/2}$:

$$\begin{aligned}
&\sinh^2\eta\frac{\partial^2 v}{\partial\theta^2}+\sinh\eta\frac{\partial}{\partial\eta}(\sinh\eta\frac{\partial v}{\partial\eta})+\frac{\partial^2 v}{\partial\varphi^2}+\sqrt{\cosh\eta-\cos\theta}\times\\
&\left(\frac{\partial}{\partial\theta}(\frac{1}{2}\frac{\sinh^2\eta\sin\theta}{(\cosh\eta-\cos\theta)^{3/2}})+\sinh\eta\frac{\partial}{\partial\eta}(\frac{1}{2}\frac{\sinh^2\eta}{(\cosh\eta-\cos\theta)^{3/2}})\right)v=0.
\end{aligned}$$

We can make the simplification in the last term,

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\left(\frac{1}{2}\frac{\sinh^2\eta\sin\theta}{(\cosh\eta-\cos\theta)^{3/2}}\right)+\sinh\eta\frac{\partial}{\partial\eta}\left(\frac{1}{2}\frac{\sinh^2\eta}{(\cosh\eta-\cos\theta)^{3/2}}\right)\\
&\quad=\frac{\sinh^2\eta}{4\sqrt{\cosh\eta-\cos\theta}},
\end{aligned}$$

to find that the condition for u to be harmonic is that v satisfies

$$\sinh^2\eta \frac{\partial^2 v}{\partial\theta^2} + \sinh\eta \frac{\partial}{\partial\eta}\left(\sinh\eta \frac{\partial v}{\partial\eta}\right) + \frac{\partial^2 v}{\partial\varphi^2} + \left(\frac{\sinh^2\eta}{4}\right)v = 0. \tag{1.34}$$

Equation (1.34) is separable. Suppose that v can be expressed as a product

$$v = H(\eta)\Theta(\theta)\Phi(\varphi). \tag{1.35}$$

We substitute formula (1.35) into (1.34) and find

$$\sinh^2\eta \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \sinh\eta \frac{1}{H}\frac{d}{d\eta}(\sinh\eta \frac{dH}{d\eta}) + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} + \frac{\sinh^2\eta}{4} = 0,$$

which we can write as

$$\sinh^2\eta \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \sinh\eta \frac{1}{H}\frac{d}{d\eta}(\sinh\eta \frac{dH}{d\eta}) + \frac{\sinh^2\eta}{4} = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = m^2,$$

where m must be a constant because a function of (η, θ) is equated to a function of φ. Also, m has to be an integer because $\Phi(\varphi)$ is a periodic function. Thus

$$\Phi(\varphi) = c_1\cos(m\varphi) + c_2\sin(m\varphi). \tag{1.36}$$

We will assume that $m \geq 0$ because negative values of m will give no new functions. Similarly, we see that n is constant in

$$\frac{1}{\sinh\eta}\frac{1}{H}\frac{d}{d\eta}(\sinh\eta \frac{dH}{d\eta}) + \frac{1}{4} - \frac{m^2}{\sinh^2\eta} = \frac{-1}{\Theta}\frac{d^2\Theta}{d\theta^2} = n^2 \tag{1.37}$$

since it equates a function of η to a function of θ. We have

$$\Theta(\theta) = c_3 \cos(n\theta) + c_4 \sin(n\theta), \tag{1.38}$$

where n is a positive integer. Finally, $H(\eta)$ satisfies the differential equation

$$\frac{1}{\sinh \eta} \frac{d}{d\eta}(\sinh \eta \frac{dH}{d\eta}) - (\frac{m^2}{\sinh^2 \eta} + (n^2 - \frac{1}{4}))H = 0.$$

By comparison with the Legendre equation (1.3), this equation is satisfied when

$$H(\eta) = c_5 P^m_{n-\frac{1}{2}}(\cosh \eta) + c_6 Q P^m_{n-\frac{1}{2}}(\cosh \eta), \tag{1.39}$$

where $P^m_{n-\frac{1}{2}}$ or $Q^m_{n-\frac{1}{2}}$ are given in Definition 1.2.

The solution v of (1.35) is obtained by combining (1.36), (1.38), and (1.39). According to (1.33), one only needs to multiply the solution by $\sqrt{\cosh \eta - \cos \theta}$ to find solutions u for Laplace's equation in toroidal coordinates. In the following definition, recall that $\Phi^{\pm}_n$ is defined in (1.24).

Definition 1.16. Let n and m be nonnegative integers. Let $\nu, \mu = \pm 1$ serve as superscripts in (1.24). For $x \notin \mathbb{R}_0 \cup S^1$, the *interior toroidal harmonics* $I^{\nu,\mu}_{n,m}$ are defined as follows:

$$I^{\nu,\mu}_{n,m}(x_0, x_1, x_2) = (\cosh \eta - \cos \theta)^{1/2} Q^m_{n-1/2}(\cosh \eta) \Phi^{\nu}_m(\theta) \Phi^{\mu}_m(\varphi). \tag{1.40}$$

The *exterior toroidal harmonics* $E^{\nu,\mu}_{n,m}$ are defined as follows:

$$E^{\nu,\mu}_{n,m}(x_0, x_1, x_2) = (\cosh\eta - \cos\theta)^{1/2} P^{m}_{n-1/2}(\cosh\eta)\Phi^{\nu}_{m}(\theta)\Phi^{\mu}_{m}(\varphi). \quad (1.41)$$

The combinations of indexes $(n, m, \nu, \mu) = (0, m, -1, \mu)$ or $(n, 0, \nu, -1)$ will never be considered because $\Phi^{-}_{0} = 0$ identically.

For convenience, we will often write superscripts as "$+$" in place of 1 and "$-$" in place of -1.

1.3.4 Asymptotic values of toroidal harmonics

Definition 1.16 below is not applicable to points in $\mathbb{R}_0 \cup S^1$ because toroidal coordinates are singular at $\eta = 0$ and $\eta = \infty$, and the solution of a differential equation at such points is meaningless. We must extend the definitions of $I^{\nu,\mu}_{n,m}$ and $E^{\nu,\mu}_{n,m}$ to the full domains Ω_{η_0} and $\Omega^{*}_{\eta_0}$.

We will use the following asymptotic properties of the Legendre functions:

Lemma 1.17. *For fixed n and m, $e^{(n+1/2)\eta} Q^{m}_{n-1/2}(\cosh\eta)$ is bounded as $\eta \to \infty$; $P^{m}_{n-1/2}(\cosh\eta)$ is bounded as $\eta \to 0$.*

Proof. In [46, p. 436], it is shown that

$$Q^m_{n-1/2}(\cosh\eta) \sim$$
$$e^{-(n+1/2)\eta}(1-e^{-2\eta})^m F(m+1/2,\ n+m+1/2;\ n+1;\ e^{-2\eta})$$

as $\eta \to \infty$. Since $(1-e^{-2\eta})^m \to 1$ and the hypergeometric function tends to $F(m+1/2,\ n+m+1/2;\ n+1;0)$, we have the desired asymptotic statement about $Q^m_{n-1/2}$. Similarly, from [46, p. 255 (110)]

$$P^m_{n-1/2}(t) = \frac{\pi\,\Gamma(n+m+1/2)}{\Gamma(n+1/2)} \int_0^\pi (t+\sqrt{t^2-1}\cos\varphi)^{n-1/2}\,\Phi^+_m(\varphi)\,d\varphi.$$

The Γ factors are constant, and the integral is bounded as $t = \cosh\eta \to 1$. □

Proposition 1.18. *The values of $I^{\nu,\mu}_{n,m}(x)$ are bounded for x near S^1, and the values of $E^{\nu,\mu}_{n,m}(x)$ are bounded for x near $\mathbb{R}_0$.*

Proof. We look at the limiting values of $\sup_{\theta,\varphi}|I^{\nu,\mu}_{n,m}(\eta,\theta,\varphi)|$ for $\eta \to \infty$, and the limiting values of $\sup_{\theta,\varphi}|E^{\nu,\mu}_{n,m}(\eta,\theta,\varphi)|$ for $\eta \to 0$. The factors $\Phi^\pm_m(\theta)\Phi^\pm_n(\varphi)$ in Definition 1.16 are bounded by 1, so one only has to look at

$$\sqrt{\cosh\eta-\cos\theta}\,Q^m_{n-1/2}(\cos\theta), \quad \sqrt{\cosh\eta-\cos\theta}\,P^m_{n-1/2}(\cos\theta).$$

As $\eta \to \infty$ or $\eta \to 0$,

$$\sqrt{\cosh \eta - \cos \theta} \sim \frac{1}{\sqrt{2}} e^{\eta/2} \quad \text{or} \quad \sqrt{\cosh \eta - \cos \theta} \sim \sqrt{1 - \cos \theta},$$

respectively. The result now follows from Lemma 1.17. $\square$

A further remarkable result is the following:

Proposition 1.19. *The interior toroidal harmonics extend to harmonic functions on $\mathbb{R}^3 \setminus \mathbb{R}_0$. The exterior toroidal harmonics extend to harmonic functions on $\mathbb{R}^3 \setminus S^1$.*

Proof. In [10, p. 202, Thm. 9.13] it is shown that any subset of $\mathbb{R}^n$ which is a C^1 image of a closed ball of dimension no greater than $n - 2$ is a removable set for harmonic functions; that is, any bounded harmonic function in the complement of such a set in a domain Ω is extendable to a harmonic function in Ω. Since S^1 as well as finite intervals in $\mathbb{R}_0$, are submanifolds of dimension 1 in $\mathbb{R}^3$, this result applies to $I_{n,m}^{\nu,\mu}(x)$ and $E_{n,m}^{\nu,\mu}(x)$ by Proposition 1.18. $\square$

It may be verified by similar asymptotic expansions that the interior toroidal harmonics (1.40) tend to infinity near the axis $\mathbb{R}_0$, while the exterior toroidal harmonics (1.41) tend to infinity near the circle S^1. We will not need to use this fact explicitly.

1.4 Contrast between interior harmonics in the torus and the sphere

In Section 1.2, we gave a brief summary of basic interior harmonic functions on the sphere. The following are some similarities and differences between these function spaces and the torus.

Spherical	Toroidal
Homogeneous polynomials.	Not polynomials
Finite subbasis for each degree	Doubly infinite family

In Chapter 2, we will find another important distinction:

Appell property	Reverse Appell property after change of basis

The transition from the expansion in interior and exterior spherical harmonics to that in interior toroidal harmonics (and vice-versa) is worked out in [58]. One of these fundamental formulas relevant to the sequel is the following.

Proposition 1.20. *Let $k \geq 0$ and $0 \leq m \leq k$. Then*

$$|x|^n Y^{\pm}_{k,m}(x)$$
$$= (-1)^m \frac{\sqrt{2}}{\pi} \begin{cases} P_k^m(0) \sum_{n=0}^{\infty} \varepsilon_n \frac{\Gamma(n-m+1/2)}{\Gamma(n+m+1/2)} c^m_{nk} I^{+,\pm}_{n,m}(x), & n+m \text{ even}, \\ -2P^m_{k+1}(0) \sum_{n=1}^{\infty} \frac{\Gamma(n-m+1/2)(n-m+1)}{\Gamma(n+m+1/2)(n+m+1)} s^m_{nk} I^{-,\pm}_{n,m}(x), & n+m \text{ odd}, \end{cases} \tag{1.42}$$

where c^m_{nk} and s^m_{nk} are rational numbers that satisfy certain recurrence relations. Here ε_n has the same meaning as in Proposition 1.9.

1.5 Limits of harmonic functions

In this section, we introduce some basic results on harmonic functions and their associated function theory. In all cases, we refer to real-valued functions of three variables x_0, x_1, x_2 unless specified otherwise. For detailed information, we refer, for instance, to [10].

1.5.1 Basic properties

Harmonic functions are well-known to have a powerful connection with complex function theory [10].

Proposition 1.21. *Let f_n be harmonic functions, and let $f_n \to f$ uniformly on every compact subset of the common domain $\Omega \subseteq \mathbb{R}^3$. Then f is harmonic.*

Proposition 1.22. *Every harmonic function in $\Omega \subseteq \mathbb{R}^3$ is real-analytic in Ω,* *that is, in a sufficiently small neighborhood of every point* $(\widetilde{x_0}, \widetilde{x_1}, \widetilde{x_2}) \in \Omega$ *it can be expressed as a uniformly convergent power series*

$$\sum_{n=0}^{\infty} \sum_{i_0+i_1+i_2=n} a_{i_0,i_1,i_2}(x_0 - \widetilde{x_0})^{i_0}(x_1 - \widetilde{x_1})^{i_1}(x_2 - \widetilde{x_2})^{i_2}.$$

The polynomial $\sum_{i_0+i_1+i_2=n} a_{i_0,i_1,i_2}(x_0 - \widetilde{x_0})^{i_0}(x_1 - \widetilde{x_1})^{i_1}(x_2 - \widetilde{x_2})^{i_2}$ is homogeneous of degree n, and is called the *homogeneous part of degree n* of the analytic function. If

$$f(x) = \sum_{n=0}^{\infty} p_n(x),$$

where p_n is a homogeneous polynomial of degree n, then the homogeneous part p_n is uniquely determined by f.

Proposition 1.23. *A uniformly convergent series on a compact set of analytic functions may be differentiated term by term, that is,*

$$\frac{\partial}{\partial x_i} \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} \frac{\partial}{\partial x_i} f_n$$

for $i = 0, 1, 2$.

As a consequence, *the homogeneous part of any degree n of a harmonic function is harmonic.*

In later chapters one needs to take the partial derivatives of uniformly convergent series (on compact subsets) of the form

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\nu=\pm1}\sum_{\mu=\pm1} a_{n,m}^{\nu,\mu}\, I_{n,m}^{\nu,\mu}(x)$$

with $a_{n,m}^{\nu,\mu} \in \mathbb{R}$.

1.5.2 Hilbert spaces

Now we state a few standard definitions relating to real Hilbert spaces [45, 55]. Here $\Omega \subseteq \mathbb{R}^3$ is an open set.

Definition 1.24. $L^2(\Omega) = \{f\colon \Omega \to \mathbb{R}\colon \text{f is measurable and } \|f\|_2 < \infty\}$ with

$$\|f\|_2 = \left(\int_\Omega |f(x)|^2\, dV_x\right)^{1/2}, \tag{1.43}$$

where $dV = dV_x = dx_0 dx_1 dx_2$ is the volume element on $\mathbb{R}^3$. Two functions in $L^2(\Omega)$ are considered identical when their difference is zero outside of a set of measure zero.

We may write $L^2(\Omega, \mathbb{R}) = L^2(\Omega)$ when we are also considering functions taking nonreal values. The norm (1.43) is induced by the inner product

$$\langle f, g\rangle_2 = \int_\Omega f(x)\, g(x)\, dV_x. \tag{1.44}$$

With this inner product, $L^2(\Omega)$ is a real Hilbert space, that is, a complete inner product space.

Definition 1.25. The *Sobolev space* $H^1(\Omega)$ consists of $f \in L^2(\Omega)$ which have derivatives in the generalized sense that are also in $L^2(\Omega)$. The norm on $H^1(\Omega)$ is defined by

$$\|f\|_{1,2} = \left((\|f\|_2)^2 + \sum_{i=0}^{2} (\|\frac{\partial f}{\partial x_i}\|_2)^2 \right)^{1/2}.$$

Definition 1.26. $H_0^1(\Omega)$ is the closure of the subspace of $H^1(\Omega)$ comprised of infinitely differentiable functions with compact support in Ω. $H^{1/2}(\partial\Omega)$ is the quotient space

$$H^{1/2}(\partial\Omega) = H^1(\Omega)/H_0^1(\Omega)$$

endowed with the quotient norm.

One thinks of elements of $H_0^1(\Omega)$ as those functions in $H^1(\Omega)$ which vanish on the boundary. In this sense, $H^{1/2}(\partial\Omega)$ can be thought of as the collection of boundary values of functions in $H^1(\Omega)$. The boundary value of $f \in H^1(\Omega)$ is called the *trace* and is denoted by $\mathrm{tr} f$.

Proposition 1.27. *The trace map* $tr\colon H^1(\Omega) \to H^{1/2}(\partial\Omega)$ *is a bounded linear function with respect to the corresponding Sobolev norms.*

The larger space $H^{-1/2}(\partial\Omega)$ is defined in terms of Fourier transforms; we refer to [45] for details.

The following result states that the harmonic functions in $L^2(\Omega)$ form a closed subspace of $L^2(\Omega)$.

Proposition 1.28 ([10])**.** *Let f_n be harmonic functions and let $f_n \to f$ in $L^2(\Omega)$. Then $f_n \to f$ uniformly on compact subsets of Ω, so f is also harmonic.*

Now we look at the specific case of the torus Ω_{η_0}. Using the weight function

$$w(\eta, \theta, \varphi) = \frac{(\cosh\eta - \cos\theta)^2}{\sinh\eta}, \tag{1.45}$$

one may define a weighted L^2 inner product on real-valued functions in the torus Ω_{η_0} by

$$\langle f, g\rangle_{\eta_0} = \int_{\Omega_{\eta_0}} f g\, w\, dV. \tag{1.46}$$

Proposition 1.29. *(a) The interior toroidal harmonics $\{I^{\nu,\mu}_{n,m}\}$ form a complete orthogonal system in* $\mathrm{Har}(\Omega_{\eta_0}) \cap L^2(\Omega_{\eta_0}, w)$*. Their norms are*

$$\|I^{\nu,\mu}_{n,m}\|^2_{\eta_0} = \varepsilon_n \varepsilon_m \int_{n_0}^{\infty} \Big(Q^m_{n-1/2}(\cosh\eta) \Big)^2 d\eta$$

where $\varepsilon_n = 1 + \delta_{n,0}$ and $\delta_{n,m}$ is the Kronecker delta function.

(b) The restrictions on the boundary $\{\, I^{\nu,\mu}_{n,m}\big|_{\partial\Omega_{\eta_0}}\}$ are complete in $L^2(\partial\Omega_{\eta_0})$ and $L^2(\partial\Omega_{\eta_0}, w)$.

Proof. First we prove (b). It is well known that $\{\Phi^\nu_n(\theta)\, \Phi^\mu_m(\varphi)\}$ is a complete set in $L^2([-\pi,\pi]^2)$. Given a boundary function expressed as $h(\theta,\varphi)$ in toroidal coordinates, we can write

$$\frac{h(\theta,\varphi)}{\sqrt{\cosh\eta_0-\cos\theta}}=\sum_n\sum_m a^{\nu,\mu}_{n,m}\Phi^\nu_n(\theta)\,\Phi^\mu_m(\varphi), \tag{1.47}$$

convergent in $L^2(\Omega_{\eta_0})$, which gives

$$h(\theta,\varphi)=\sum_n\sum_m\frac{a^{\nu,\mu}_{n,m}}{Q^m_{n-1/2}(\cosh\eta_0)}\,I^{\nu,\mu}_{n,m}(x_0,x_1,x_2) \tag{1.48}$$

showing the completeness on the boundary.

Now we consider (a). The orthogonality of the toroidal harmonics with respect to the weight function w follow from the fact that this function admits separating the three integrals:

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{\eta_0}^{\infty}(\cosh\eta-\cos\theta)^{1/2}Q^m_{n-1/2}(\cosh\eta)\Big)^2\frac{\sinh\eta}{(\cosh\eta-\cos\theta)^3}$$
$$\cdot\frac{(\cosh\eta-\cos\theta)^2}{\sinh\eta}\cos^2(n\theta)\cos^2(m\varphi)\,d\eta\,d\theta\,d\varphi=0.$$

Let $h\in L^2(\partial\Omega_{\eta_0})$. We are interested in the harmonic extension of h to the interior of Ω_{η_0}. First suppose that h is of class $C^3(\partial\Omega_{\eta_0})$. Then the Fourier series (1.47) converges uniformly, i.e., the tails of sums for $n,m\geq N$ converge uniformly to zero as $N\to\infty$. (See Proposition 1.31 below, and recall that $\sum_{n,m}(m+n+1)^{-3}<\infty$.) Further, this argument shows that the condition $h\in C^3(\partial\Omega_{\eta_0})$ may be weakend to the assumption that the Fourier series of the boundary values converges absolutely at every point of $\partial\Omega_{\eta_0}$. By the Maximum Principle, these tails converge uniformly to zero in Ω_{η_0}. Thus by (1.48) the partial sums

$$u_N = \sum_{n,m \leq N} \frac{a_{n,m}^{\nu,\mu}}{Q_{n-1/2}^{m}(\cosh \eta_0)} \, I_{n,m}^{\nu,\mu}(x_0, x_1, x_2)$$

are harmonic functions which converge uniformly to u in Ω_{η_0} as $N \to \infty$. Therefore the toroidal harmonics $I_{n,m}^{\nu,\mu}$ are complete in the subspace $\mathrm{Har}(\Omega_{\eta_0}) \cap C^3(\partial\Omega_{\eta_0})$ with respect to the norm $\|\cdot\|_\infty$, and hence also in the weighted norm $\|\cdot\|_{2,w}$. We omit the detailed proof that they are complete in $L^2(\Omega, w)$; see [11, 29, 34, 37, 45, 56, 58, 76, 87]. □

Because the weight function w is bounded above and below on Ω_{η_0}, the completeness holds for the unweighted Hilbert space as well (although not the orthogonality):

Corollary 1.30. $\{I_{n,m}^{\nu,\mu}\}$ *is a complete system in the unweighted space* $L^2(\Omega_{\eta_0})$.

By Proposition 1.28, the L^2 representation of a harmonic function in terms of a series of toroidal harmonics $I_{n,m}^{\nu,\mu}$ converges uniformly on compact subsets of Ω_{η_0}.

We will also use the following fact about Fourier coefficients [37, Corollary 3.3.10, Proposition 3.3.12] as per Proposition 1.29:

Proposition 1.31. *Let $f = f(\theta,\varphi)$ be represented as*

$$f(\theta,\varphi) = \sum_{n,m,\nu,\mu} a^{\nu,\mu}_{n,m} I^{\nu,\mu}_{n,m}(\eta_0,\theta,\varphi) \tag{1.49}$$

with $a^{\nu,\mu}_{n,m} \in \mathbb{R}$. If $f(\theta,\varphi)$ is of class C^r, then

$$|a^{\nu,\mu}_{n,m}| \leq \frac{C}{(m+n+1)^r} \tag{1.50}$$

for some constant $C > 0$. Conversely, if (1.50) holds and $r \geq 2$, then f is of class C^{r-2}.

The indices of summation in the expansion (1.49) are as in Definition 1.16.

Chapter 2

Appell systems of harmonic functions

In 1880, P. Appell [7] generalized the property of classical monomials which states that the derivative of a basis function is a multiple of another basis function (from the same system). He studied sequences $\{p_n(t)\}_{n\in\mathbb{N}_0}$ of polynomials $p_n(t)$ of one real variable, such that $p_0(t) \neq 0$, which satisfy two conditions: the degree of p_n is exactly n, and

$$\frac{d}{dt}p_n(t) = n\, p_{n-1}(t),\ n = 1, 2, \ldots .$$

Such a sequence of functions is said to have the *Appell property*. The original example is $p_n(t) = t^n$. The coefficient n is not important in the notion of an Appell system, since one may replace $p_n(t)$ with $c_n p_n(t)$ for suitable constants c_n and thus replace n by any other desired multiplier.

The classical Bernoulli, Hermite and Euler polynomials are well-known examples of Appell sequences. H. Malonek et al. generalized this idea for monogenic polynomials [31, 32, 33] (see definitions in Chapter 4). They defined a system of homogeneous monogenic polynomials, $P_k(x)$ with $x \in \mathbb{R}^d$, of exact degree k having the property that

$$\partial P_k(x) = k\, P_{k-1}(x), \quad k = 1, 2, \ldots,$$

where here "∂" denotes the Clifford-Fueter operator appropriate to the dimension d under consideration. These generalized Appell polynomials were applied to the study of elementary functions [1, 19, 20, 32, 61, 62], the computation of combinatorial identities [4, 21, 22], and the study of generalized Joukowski transformations in Euclidean spaces of arbitrary higher dimension [5, 28]. These generalized polynomials were used to prove a higher-dimensional counterpart of Hadamard's three-hyperballs Theorem [3, 94].

As commented in Section 1.2, the classical spherical harmonics satisfy a natural Appell property with respect to $\partial/\partial x_0$. (They do not satisfy such a property with respect to the other variables.) Therefore it is natural to ask whether the toroidal harmonics also satisfy an Appell property. We will see in Section 2.2 that they do not, and in fact, the application of $\partial/\partial x_0$ increases the index (or degree) of the toroidal harmonics. However, with an appropriate change of basis, it is possible to create a collection of functions which satisfy what we may call a "reverse Appell property." This basis will be fundamental for our study of the collection of monogenic functions on the torus in Chapters 5 and 6.

2.1 Partial derivatives of toroidal harmonics

We begin by calculating the partial derivatives of the toroidal harmonics $I_{n,m}^{\nu,\mu}$ defined by (1.40). The following results directly from (1.29) and the Chain Rule.

Proposition 2.1.

$$\begin{aligned}\frac{\partial x_0}{\partial \eta} &= \frac{-\sinh\eta\sin\theta}{(\cosh\eta-\cos\theta)^{1/2}},\\ \frac{\partial x_0}{\partial \theta} &= \frac{\cos\theta\cosh\eta-1}{(\cosh\eta-\cos\theta)^{1/2}},\\ \frac{\partial x_0}{\partial \varphi} &= 0,\end{aligned} \tag{2.1}$$

$$\begin{aligned}\frac{\partial x_1}{\partial \eta} &= \frac{(1-\cosh\eta\cos\theta)\cos\varphi}{(\cosh\eta-\cos\theta)^{1/2}},\\ \frac{\partial x_1}{\partial \theta} &= -\frac{\sin\theta\sinh\eta\cos\varphi}{(\cosh\eta-\cos\theta)^{1/2}},\\ \frac{\partial x_1}{\partial \varphi} &= -\frac{\sin\varphi\sinh\eta(\cosh\eta-\cos\theta)}{(\cosh\eta-\cos\theta)^{1/2}},\end{aligned} \tag{2.2}$$

$$\frac{\partial x_2}{\partial \eta} = \frac{(1-\cosh\eta\cos\theta)\sin\varphi}{(\cosh\eta-\cos\theta)^{1/2}},$$

$$\frac{\partial x_2}{\partial \theta} = -\frac{\sin\theta \sinh\eta \sin\varphi}{(\cosh\eta - \cos\theta)^{1/2}},$$
$$\frac{\partial x_2}{\partial \varphi} = \frac{\cos\varphi \sinh\eta(\cosh\eta - \cos\theta)}{(\cosh\eta - \cos\theta)^{1/2}}. \tag{2.3}$$

These are the components of the Jacobian matrix of the change of coordinates $\partial(x_0, x_1, x_2)/\partial(\eta, \theta, \varphi)$, and the Jacobian determinant is

$$\left|\frac{\partial(x_0, x_1, x_2)}{\partial(\eta, \theta, \varphi)}\right| = \frac{\sinh\eta}{(\cosh\eta - \cos\theta)^3}. \tag{2.4}$$

The inverse matrix is

$$\frac{\partial(\eta, \theta, \varphi)}{\partial(x_0, x_1, x_2)} = \begin{pmatrix} -\sin\theta\sinh\eta & -\cos\varphi(\cos\theta\cosh\eta - 1) & -\sin\varphi(\cos\theta\cosh\eta - 1) \\ \cosh\eta\cos\theta - 1 & -\sinh\eta\sin\theta\cos\varphi & -\sin\varphi\sin\theta\sinh\eta \\ 0 & \dfrac{-\sin\varphi(\cosh\eta - \cos\theta)}{\sinh\eta} & \dfrac{\cos\varphi(\cosh\eta - \cos\theta)}{\sinh\eta} \end{pmatrix} \tag{2.5}$$

which gives us the partial derivatives of η, θ, φ with respect to x_0, x_1, x_2.

2.2 Basis expression of derivatives of toroidal harmonics

Since $I^{\nu,\mu}_{n,m}$ of Definition 1.16 is harmonic, all of its partial derivatives are also harmonic. By Proposition 1.29, it must be expressible in terms of the full collection of toroidal harmonics. The following calculations determine the explicit expressions. The formulas for $n = 0$ or $m = 0$ are somewhat different from the general case. Together with the four combinations of the signs $(\pm, \pm)$, the number of cases to be considered is rather large.

Proposition 2.2. *Let* $m = n = 0$ *in* (1.40). *Then we have the following:*

$$\begin{aligned}
\frac{\partial}{\partial x_0} I_{0,0}^{+,+} &= -\frac{1}{2} I_{1,0}^{-,+}, \\
\frac{\partial}{\partial x_1} I_{0,0}^{+,+} &= I_{0,1}^{+,+} - I_{1,1}^{+,+}, \\
\frac{\partial}{\partial x_2} I_{0,0}^{+,+} &= I_{0,1}^{+,-} - I_{1,1}^{+,-}.
\end{aligned}$$

Proof. Note that $I_{0,0}^{+,+} = \sqrt{\cosh\eta - \cos\theta}\, Q^0_{-1/2}(\cosh\eta)$. This contains the Legendre function $Q^0_{-1/2}$, with a negative index, which is defined by the same formulas as all other Legendre functions of the second kind, and satisfies the same general properties. First we consider $\partial/\partial x_0$. By (1.40) and the Chain Rule, we have

$$\begin{aligned}
\frac{\partial}{\partial x_0} I_{0,0}^{+,+} &= \frac{\partial}{\partial x_0}\Big((\cosh\eta - \cos\theta)^{1/2}\, Q^0_{-\frac{1}{2}}(\cosh\eta) \Big) \\
&= -\sin\theta \sinh\eta \Bigg(\frac{\sinh\eta}{2(\cosh\eta - \cos\theta)^{1/2}}\, Q^0_{-\frac{1}{2}}(\cosh\eta) \\
&\qquad + \sinh\eta\, (Q^0_{-\frac{1}{2}})'(\cosh\eta)(\cosh\eta - \cos\theta)^{1/2} \Bigg) \\
&\quad + (\cosh\eta\cos\theta - 1)\Bigg(\frac{\sin\theta}{2(\cosh\eta - \cos\theta)^{1/2}}\, Q^0_{-\frac{1}{2}}(\cosh\eta) \Bigg).
\end{aligned}$$

After simplifying and using the fact that $\cosh^2\eta - \sinh^2\eta = 1$, this reduces to

$$-(\cosh\eta-\cos\theta)^{1/2}\bigg(\frac{1}{2}\sin\theta\cosh\eta\,Q^0_{-\frac{1}{2}}(\cosh\eta)$$
$$+\sin\theta\sinh^2\eta\,(Q^0_{-\frac{1}{2}})'(\cosh\eta)\bigg).$$

Next we apply the recursion formulas (1.13) and (1.16) respectively, to find that $\partial I^{+,+}_{0,0}/\partial x_0$ is equal to

$$-\frac{1}{2}(\cosh\eta-\cos\theta)^{1/2}\,Q^0_{-\frac{3}{2}}(\cosh\eta)\,\sin\theta=-\frac{1}{2}I^{-,+}_{1,0}$$

as required.

Next, from the Chain Rule and (2.2), one obtains in a similar manner that

$$\begin{aligned}\frac{\partial}{\partial x_1}I^{+,+}_{0,0}&=\frac{\partial}{\partial x_1}\Big((\cosh\eta-\cos\theta)^{1/2}Q^0_{\frac{1}{2}}(\cosh\eta)\Big)\\&=-\cos\varphi(\cosh\eta\cos\theta-1)\bigg(\frac{\sinh\eta}{2(\cosh\eta-\cos\theta)^{1/2}}\,Q^0_{-\frac{1}{2}}(\cosh\eta)\\&\qquad+\sinh\eta\,(Q^0_{-\frac{1}{2}})'(\cosh\eta-\cos\theta)^{1/2}\bigg)\\&\quad-\sinh\eta\sin\theta\cos\varphi\bigg(\frac{\sin\theta}{2(\cosh\eta-\cos\theta)^{1/2}}\,Q^0_{-\frac{1}{2}}(\cosh\eta)\bigg).\end{aligned}$$

Once again, we use trigonometric identities and simplify to reduce this to

$$-(\cosh\eta-\cos\theta)^{1/2}\cos\varphi\bigg(\frac{1}{2}\cos\theta\sinh\eta\,Q^0_{-\frac{1}{2}}(\cosh\eta)$$
$$+\cos\theta\sinh\eta\cosh\eta\,(Q^0_{-\frac{1}{2}})'(\cosh\eta)$$

$$- \sinh\eta \, (Q^0_{-\frac{1}{2}})'(\cosh\eta)\Big).$$

Now apply the recursion formulas (1.13) with $m = 0$, $n = -3/2$, (1.15) when $m = 1$, $n = -3/2$, and (1.17) with $m = 1$, $n = 1/2$, respectively, to see that $\partial I^{+,+}_{0,0}/\partial x_1$ is equal to

$$-(\cosh\eta - \cos\theta)^{1/2}\Big(- Q^1_{-\frac{1}{2}}(\cosh\eta) + \cos\theta \, Q^1_{\frac{1}{2}}(\cosh\eta)\Big)\cos\varphi$$

which by (1.40), is equal to $I^{+,+}_{0,1} - I^{+,+}_{1,1}$. The calculation of $\partial I^{+,+}_{0,0}/\partial x_2$ is similar and will be omitted. □

Recall that the functions $I^{+,-}_{0,0}$ and $I^{-,\pm}_{0,0}$ do not exist because the formula (1.40) produces 0 identically. We proceed to the remaining cases for $m = 0$.

Proposition 2.3. *Let $m = 0$. For every $n > 0$ we have the following:*

$$\begin{aligned}
\frac{\partial}{\partial x_0} I^{+,+}_{n,0} &= -\frac{2n+1}{4}\, I^{-,+}_{n+1,0} + n\, I^{-,+}_{n,0} - \frac{2n-1}{4}\, I^{-,+}_{n-1,0}, \\
\frac{\partial}{\partial x_1} I^{+,+}_{n,0} &= -\frac{1}{2}\, I^{+,+}_{n+1,1} + I^{+,+}_{n,1} - \frac{1}{2}\, I^{+,+}_{n-1,1}, \\
\frac{\partial}{\partial x_2} I^{+,+}_{n,0} &= -\frac{1}{2}\, I^{+,-}_{n+1,1} + I^{+,-}_{n,1} - \frac{1}{2}\, I^{+,-}_{n-1,1}, \\
\frac{\partial}{\partial x_0} I^{-,+}_{n,0} &= \frac{2n+1}{4}\, I^{+,+}_{n+1,0} - n\, I^{+,+}_{n,0} + \frac{2n-1}{4}\, I^{+,+}_{n-1,0}, \\
\frac{\partial}{\partial x_1} I^{-,+}_{n,0} &= -\frac{1}{2}\, I^{-,+}_{n+1,1} + I^{-,+}_{n,1} - \frac{1}{2}\, I^{-,+}_{n-1,1}, \\
\frac{\partial}{\partial x_2} I^{-,+}_{n,0} &= \frac{1}{2}\, I^{-,-}_{n+1,1} - I^{-,-}_{n,1} + \frac{1}{2}\, I^{-,-}_{n-1,1}.
\end{aligned}$$

Proof. First we calculate for $(\nu,\mu)=(+1,+1)$. From (1.40),

$$\begin{aligned}
\frac{\partial}{\partial x_0} I^{+,+}_{n,0} &= \frac{\partial}{\partial x_0}\Big((\cosh\eta-\cos\theta)^{1/2}Q^0_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta)\Big) \\
&= -\sin\theta\sinh\eta\Bigg(\frac{\sinh\eta}{2(\cosh\eta-\cos\theta)^{1/2}}\,Q^0_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta) \\
&\qquad\qquad + \sinh\eta\,(Q^0_{n-\frac{1}{2}})'(\cosh\eta)(\cosh\eta-\cos\theta)^{1/2}\cos(n\theta)\Bigg) \\
&\quad + (\cosh\eta\cos\theta-1)\Bigg(\frac{\sin\theta}{2(\cosh\eta-\cos\theta)^{1/2}}\,Q^0_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta) \\
&\qquad\qquad - \,n\,(\cosh\eta-\cos\theta)^{1/2}\,Q^0_{n-\frac{1}{2}}(\cosh\eta)\sin(n\theta)\Bigg) \\
&= (\cosh\eta-\cos\theta)^{1/2}\Bigg(-\frac{1}{2}\,Q^0_{n-\frac{1}{2}}(\cosh\eta)\cosh\eta\cos(n\theta)\sin\theta \\
&\qquad\qquad\qquad - (\cosh\eta\cos\theta-1)\,Q^0_{n-\frac{1}{2}}(\cosh\eta)\sin(n\theta) \\
&\qquad\qquad\qquad - \sinh^2\eta\,(Q^0_{n-\frac{1}{2}})'(\cosh\eta)\cos(n\theta)\sin\theta\Bigg).
\end{aligned}$$

Applying the recursion formula (1.11) and some trigonometric identities simplifying, we obtain

$$\begin{aligned}
(\cosh\eta-\cos\theta)^{1/2}\Bigg(&-n\,\cosh\eta\,Q^0_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta)\sin\theta \\
&- n\,(\cosh\eta\cos\theta-1)\,Q^0_{n-\frac{1}{2}}(\cosh\eta)\sin(n\theta) \\
&+ (n-\frac{1}{2})\,Q^0_{n-\frac{3}{2}}(\cosh\eta)\cos(n\theta)\sin\theta\Bigg).
\end{aligned}$$

Now use the recursion formula (1.12) with $n-1/2$ in place of n and with

$m = 0$ to convert the expression into

$$(\cosh\eta - \cos\theta)^{1/2}\Big(-\frac{2n+1}{4}\, Q^0_{n+\frac{1}{2}}(\cosh\eta)\sin(n\theta) + n\, Q^0_{n-\frac{1}{2}}(\cosh\eta)\sin(n\theta) - \frac{2n-1}{4}\, Q^0_{n-\frac{3}{2}}(\cosh\eta)\sin(n-1)\theta\Big),$$

which is the desired value of $\partial I^{+,+}_{n,0}/\partial x_0$. The calculation for $\partial I^{-,+}_{n,0}/\partial x_0$ is essentially the same.

Next,

$$\begin{aligned}\frac{\partial}{\partial x_1} I^{+,+}_{n,0} = &- \cos\varphi(\cos\theta\cosh\eta - 1)\Big(\frac{\sinh\eta}{2(\cosh\eta - \cos\theta)^{1/2}}\, Q^0_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta) \\ &\qquad + (\cosh\eta - \cos\theta)^{1/2}\, \sinh\eta\, (Q^0_{n-\frac{1}{2}})'(\cosh\eta)\cos(n\theta)\Big) \\ &- \sinh\eta\sin\theta\cos\varphi\Big(\frac{\sin\theta}{2(\cosh\eta - \cos\theta)^{1/2}}\, Q^0_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta) \\ &\qquad - n\,(\cosh\eta - \cos\theta)^{1/2} Q^0_{n-\frac{1}{2}}(\cosh\eta)\sin(n\theta)\Big) \\ = &\cos\varphi(\cosh\eta - \cos\theta)^{1/2}\Big(-\frac{1}{2}\sinh\eta\, Q^0_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta) \\ &\qquad - (\cos\theta\cosh\eta - 1)\sinh\eta\,(Q^0_{n-\frac{1}{2}})'(\cosh\eta)\cos(n\theta) \\ &\qquad + n\, \sinh\eta\, Q^0_{n-\frac{1}{2}}(\cosh\eta)\sin\theta\sin(n\theta)\Big).\end{aligned}$$

By the recursion formulas (1.13) and (1.14), this is equal to

$$\begin{aligned}(\cosh\eta - \cos\theta)^{1/2}\cos\varphi\Big(&-\frac{1}{4n}\, Q^1_{n+\frac{1}{2}}(\cosh\eta)\cos\theta\cos(n\theta) \\ &+ \frac{1}{4n}\, Q^1_{n-\frac{3}{2}}(\cosh\eta)\cos\theta\cos(n\theta)\end{aligned}$$

$$
\begin{aligned}
&- \cosh\eta\, Q^1_{n+\frac{1}{2}}(\cosh\eta)\cos\theta\cos(n\theta) + Q^1_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta) \\
&+ \frac{1}{2}\, Q^1_{n+\frac{1}{2}}(\cosh\eta)\sin\theta\sin\, n\theta - \frac{1}{2}\, Q^1_{n-\frac{3}{2}}(\cosh\eta)\sin\theta\sin\,(n\theta)\Big).
\end{aligned}
$$

With the recursion formula (1.12) the trigonometric identity which reduces products to sums, we obtain

$$
\begin{aligned}
(\cosh\eta - \cos\theta)^{1/2}\cos\varphi\Big(&-\frac{1}{2} Q^1_{n+\frac{1}{2}}(\cosh\eta)\cos(n+1)\theta \\
&- \frac{1}{2} Q^1_{n-\frac{3}{2}}(\cosh\eta)\cos(n-1)\theta + Q^1_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta)\Big),
\end{aligned}
$$

which is the formula claimed for $\partial I^{+,+}_{n,0}/\partial x_1$. The calculation for $\partial I^{+,-}_{n,0}/\partial x_1$ is essentially the same.

Finally we calculate

$$
\begin{aligned}
\frac{\partial}{\partial x_2} I^{+,+}_{n,0} = &- \sin\varphi(\cos\theta\cosh\eta - 1)\Bigg(\frac{\sinh\eta}{2(\cosh\eta - \cos\theta)^{1/2}}\, Q^0_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta) \\
&+ (\cosh\eta - \cos\theta)^{1/2}\sinh\eta\,(Q^0_{n-\frac{1}{2}})'(\cosh\eta)\cos(n\theta)\Bigg) \\
&- \sin\varphi\sin\theta\sinh\eta\Bigg(\frac{\sin\theta}{2(\cosh - \cos\theta)^{1/2}}\,(Q^0_{n-\frac{1}{2}})'(\cosh\eta)\cos(n\theta) \\
&- \, n\,(\cosh\eta - \cos\theta)^{1/2}\, Q^0_{n-\frac{1}{2}}(\cosh\eta)\sin(n\theta)\Bigg)\Bigg).
\end{aligned}
$$

After simplification we have

$$
\begin{aligned}
(\cosh\eta - \cos\theta)^{1/2}\sin\varphi\Big(&-\frac{1}{2}\, Q^0_{n-\frac{1}{2}}(\cosh\eta)\sinh\eta\cos\theta\cos(n\theta) \\
&- (\cos\theta\cosh\eta - 1)\sinh\eta\,(Q^0_{n-\frac{1}{2}})'(\cosh\eta)\cos(n\theta)
\end{aligned}
$$

$$+ \, n \sinh\eta \, Q_{n-\frac{1}{2}}(\cosh\eta) \sin\theta \sin(n\theta)\Big).$$

Now use the recursion formulas (1.13) and (1.14) to obtain

$$\begin{aligned}
&(\cosh\eta - \cos\theta)^{1/2} \sin\varphi\Big(- \frac{1}{4\,n} Q^1_{n+\frac{1}{2}}(\cosh\eta)\cos\theta\cos(n\theta) \\
&\quad + \frac{1}{4\,n} Q^1_{n-\frac{3}{2}}(\cosh\eta)\cos\theta\cos(n\theta) - \cosh\eta \, Q^1_{n-\frac{1}{2}}(\cosh\eta)\cos\theta\cos(n\theta) \\
&\quad + Q^1_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta) + \frac{1}{2} Q^1_{n+\frac{1}{2}}(\cosh\eta)\sin\theta\sin(n\,\theta) \\
&\quad - \frac{1}{2} Q^1_{n-\frac{3}{2}}(\cosh\eta)\sin\theta\sin(n\theta)\Big).
\end{aligned}$$

Now use $n - 1/2$ for n, $m = 1$ in the recursion formula (1.12) and the trigonometric identities for $\cos\theta\cos(n\theta)$ and $\sin\theta\sin(n\theta)$, to arrive at

$$\begin{aligned}
(\cosh\eta - \cos\theta)^{1/2} \sin\varphi\Big(&- \frac{1}{2} Q^1_{n+\frac{1}{2}}(\cosh\eta)\cos(n+1)\theta \\
&+ \frac{1}{4\,n} Q^1_{n-\frac{3}{2}}(\cosh\eta)\cos(n-1)\theta \\
&+ Q^1_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta)\Big),
\end{aligned}$$

which is the desired formula for $\partial I^{+,+}_{0,0}/\partial x_1$. The formula for $\partial I^{+,+}_{0,0}/\partial x_2$ is obtained in the same way. $\square$

Proposition 2.4. *Let $n = 0$. Then for every $m > 0$ we have the following:*

$$\frac{\partial}{\partial x_0} \, I^{+,+}_{0,m} = (m - \frac{1}{2}) I^{-,+}_{1,m},$$

$$\frac{\partial}{\partial x_1} \, I^{+,+}_{0,m} = -\frac{1}{2}(m - \frac{1}{2})(m - \frac{3}{2}) \, I^{+,+}_{1,m-1} - \frac{1}{2} \, I^{+,+}_{1,m+1}$$

$$
\begin{aligned}
&\quad + \frac{1}{2}(m+\frac{1}{2})(-m+\frac{1}{2})\, I^{+,+}_{0,m-1} + \frac{1}{2}\, I^{+,+}_{0,m+1}, \\
\frac{\partial}{\partial x_2}\, I^{+,+}_{0,m} &= \frac{1}{2}(m-\frac{1}{2})(m-\frac{3}{2})\, I^{+,-}_{1,m-1} - \frac{1}{2}\, I^{+,-}_{1,m+1} \\
&\quad - \frac{1}{2}(m+\frac{1}{2})(-m+\frac{1}{2})\, I^{+,-}_{0,m-1} + \frac{1}{2}\, I^{+,-}_{0,m+1}, \\
\frac{\partial}{\partial x_0}\, I^{+,-}_{0,m} &= (m-\frac{1}{2}) I^{-,-}_{1,m}, \\
\frac{\partial}{\partial x_1}\, I^{+,-}_{0,m} &= -\frac{1}{2}(m-\frac{1}{2})(m-\frac{3}{2})\, I^{+,-}_{1,m-1} - \frac{1}{2}\, I^{+,-}_{1,m+1} \\
&\quad + \frac{1}{2}(m+\frac{1}{2})(-m+\frac{1}{2})\, I^{+,-}_{0,m-1} + \frac{1}{2}\, I^{+,-}_{0,m+1}, \\
\frac{\partial}{\partial x_2}\, I^{+,-}_{0,m} &= \frac{1}{2}(m-\frac{1}{2})(m-\frac{3}{2})\, I^{+,-}_{1,m-1} - \frac{1}{2}\, I^{+,-}_{1,m+1} \\
&\quad - \frac{1}{2}(m+\frac{1}{2})(-m+\frac{1}{2})\, I^{+,-}_{0,m-1} + \frac{1}{2}\, I^{+,-}_{0,m+1}.
\end{aligned}
$$

Proof. First we consider $\partial I^{+,+}_{0,m}/\partial x_0$:

$$
\begin{aligned}
\frac{\partial}{\partial x_0}\, I^{+,+}_{0,m} =& \frac{\partial}{\partial\, x_0}\Big((\cosh\eta-\cos\theta)^{1/2}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\cos\theta\Big) \\
=& -\sin\theta\sinh\eta\Big(\frac{\sinh\eta}{2(\cosh\eta-\cos\theta)^{1/2}}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\cos m\varphi \\
&+ (\cosh\eta-\cos\theta)^{1/2}(Q^m_{-\frac{1}{2}})'(\cosh\eta)\sinh\eta\cos m\varphi\Big) \\
&+ (\cosh\eta\cos\theta-1)\Big(\frac{\sin\theta}{2(\cosh\eta-\cos\theta)^{1/2}}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\cos m\varphi\Big).
\end{aligned}
$$

Using $\cosh^2\eta - 1 = \sinh^2\eta$, this becomes

$$
(\cosh\eta-\cos\theta)^{1/2}\cos m\varphi\Big(-\frac{1}{2}\cosh\eta\, Q^m_{-\frac{1}{2}}(\cosh\eta)\sin\theta
$$

$$- \sinh^2 \eta \, (Q^m_{-\frac{1}{2}})'(\cosh \eta) \sin \theta \Big).$$

Now we can apply the recursion relation (1.13) with $n = -3/2$, giving $(m - 1/2)\, I^{-,+}_{1,m}$ as required. The proof for $\partial I^{+,-}_{0,m} / \partial x_0$ is similar.

In the following, we compute $(\partial/\partial x_1) I^{+,-}_{0,m}$ rather than $(\partial/\partial x_1) I^{+,+}_{0,m}$ to illustrate slight differences between $\cos(m\varphi)$ and $\sin(m\varphi)$. We have

$$\begin{aligned}
\frac{\partial}{\partial x_1} I^{+,-}_{0,m} = & \frac{\partial}{\partial x_1}\Big((\cosh \eta - \cos \theta)^{1/2}\, Q^m_{-\frac{1}{2}}(\cosh \eta) \sin(m\varphi) \Big) \\
= & -\cos\varphi(\cos\theta\cosh\eta - 1)\Big(\frac{\sinh\eta}{2(\cosh\eta - \cos\theta)^{1/2}}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\sin(m\varphi) \\
& \qquad + (\cosh\eta - \cos\theta)^{1/2} \sinh\eta\, (Q^m_{-\frac{1}{2}})'(\cosh\eta)\sin(m\varphi)\Big) \\
& - \sinh\eta\sin\theta\cos\varphi\Big(\frac{\sin\theta}{2(\cosh\eta - \cos\theta)^{1/2}}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\sin(m\varphi)\Big) \\
& - \frac{\sin\varphi(\cosh\eta - \cos\theta)}{\sinh\eta}\Big(m\,(\cosh\eta - \cos\theta)^{1/2}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\cos(m\varphi)\Big) \\
= & (\cosh\eta - \cos\theta)^{1/2}\Big(-\frac{1}{2}\sinh\eta\, Q^m_{-\frac{1}{2}}(\cosh\eta)\cos\theta\cos\varphi\sin(m\varphi) \\
& \qquad - (\cos\theta\cosh\eta - 1)(Q^m_{-\frac{1}{2}})'(\cosh\eta)\sinh\eta\cos\varphi\sin(m\varphi) \\
& \qquad - \frac{m}{\sinh\eta}(\cosh\eta - \cos\theta) Q^m_{-\frac{1}{2}}(\cosh\eta)\sin\varphi\cos(m\varphi)\Big),
\end{aligned}$$

which by trigonometric identities is equal to

$$\begin{aligned}
(\cosh\eta - \cos\theta)^{1/2}\Big(\cos\theta\cos(m-1)\varphi\Big(& -\frac{1}{4}\sinh\eta\, Q^m_{-\frac{1}{2}}(\cosh\eta) \\
& - \frac{1}{2}\cosh\eta\sinh\eta\, (Q^m_{-\frac{1}{2}})'(\cosh\eta)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - \frac{1}{2}\frac{m}{\sinh\eta}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\Big) \\
&+ \cos\theta\cos(m+1)\varphi\Big(- \frac{1}{4}\sinh\eta\, Q^m_{-\frac{1}{2}}(\cosh\eta) \\
&\qquad\qquad - \frac{1}{2}\cosh\eta\sinh\eta\,(Q^m_{-\frac{1}{2}})'(\cosh\eta) \\
&\qquad\qquad - \frac{1}{2}\frac{m}{\sinh\eta}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\Big) \\
&+ \cos(m-1)\varphi\left(\frac{1}{2}\sinh\eta\,(Q^m_{-\frac{1}{2}})'(\cosh\eta)\right. \\
&\qquad\qquad \left. + \frac{m}{2\sinh\eta}\cosh\eta\, Q^m_{-\frac{1}{2}}(\cosh\eta)\right) \\
&+ \cos(m+1)\varphi\left(\frac{1}{2}\sinh\eta\,(Q^m_{-\frac{1}{2}})'(\cosh\eta)\right. \\
&\qquad\qquad \left. + \frac{m}{2\sinh\eta}\cosh\eta\, Q^m_{-\frac{1}{2}}(\cosh\eta)\right)\Bigg).
\end{aligned}
$$

From this the recursion formulas (1.13), (1.17), (1.18), and (1.10) lead to the desired formula. The calculation of $\partial I^{+,+}_{0,m}/\partial x_0$ proceeds in the same way.

Finally,

$$
\begin{aligned}
\frac{\partial}{\partial x_2} I^{+,+}_{0,m} &= \frac{\partial}{\partial x_2}\Big((\cosh\eta - \cos\theta)^{1/2} Q^m_{-\frac{1}{2}}(\cosh\eta)\cos m\varphi\Big) \\
&= -\sin\varphi(\cos\theta\cosh\eta - 1)\Big(\frac{\sinh\eta}{2(\cosh\eta-\cos\theta)^{1/2}}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\cos(m\varphi) \\
&\qquad\qquad + (\cosh\eta-\cos\theta)^{1/2}\sinh\eta\,(Q^m_{-\frac{1}{2}})'(\cosh\eta)\cos(m\varphi) \\
&\quad - \sin\varphi\sin\theta\sinh\eta\Big(\frac{\sin\theta}{2(\cosh\eta-\cos\theta)^{1/2}}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\cos(m\varphi) \\
&\quad - \frac{\cos\varphi(\cosh\eta-\cos\theta)}{\sinh\eta}\Big(m\,(\cosh\eta-\cos\theta)^{1/2} Q^m_{-\frac{1}{2}}(\cosh\eta)\sin(m\varphi)\Big)
\end{aligned}
$$

$$
\begin{aligned}
= (\cosh\eta - \cos\theta)^{1/2}\Big(& -\frac{1}{2}\sinh\eta\, Q^m_{-\frac{1}{2}}(\cosh\eta)\cos\theta\sin\varphi\cos(m\varphi) \\
& - (\cos\theta\cosh\eta - 1)\sinh\eta\,(Q^m_{-\frac{1}{2}})'(\cosh\eta)\sin\varphi\cos(m\varphi) \\
& - \frac{m}{\sinh\eta}(\cosh\eta - \cos\theta)\, Q^m_{-\frac{1}{2}}(\cosh\eta)\cos\varphi\sin(m\,\varphi)\Big),
\end{aligned}
$$

which after trigonometric simplification becomes

$$
\begin{aligned}
(\cosh\eta - \cos\theta)^{1/2}\Big(& \cos\theta\,\sin(m-1)\varphi\,\Big(\frac{1}{4}\,\sinh\eta\, Q^m_{-\frac{1}{2}}(\cosh\eta) \\
& \qquad + \frac{1}{2}\cosh\eta\sinh\eta\,(Q^m_{-\frac{1}{2}})'(\cosh\eta) \\
& \qquad + \frac{m}{2\sinh\eta}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\Big) \\
& + \cos\theta\,\sin(m+1)\varphi\Big(-\frac{1}{4}\sinh\eta\, Q^m_{-\frac{1}{2}}(\cosh\eta) \\
& \qquad - \frac{1}{2}\cosh\eta\sinh\eta\,(Q^m_{-\frac{1}{2}})'(\cosh\eta) \\
& \qquad + \frac{m}{2\sinh\eta}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\Big) \\
& + \sin(m-1)\varphi\Big(-\frac{1}{2}\sinh\eta\,(Q^m_{-\frac{1}{2}})'(\cosh\eta) \\
& \qquad - \frac{m}{2\sinh\eta}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\Big) \\
& + \sin(m+1)\varphi\Big(\frac{1}{2}\sinh\eta\,(Q^m_{-\frac{1}{2}})'(\cosh\eta) \\
& \qquad - \frac{m}{2\sinh\eta}\, Q^m_{-\frac{1}{2}}(\cosh\eta)\Big)\Big).
\end{aligned}
$$

We obtain the result by the same recursion formulas we used to prove the previous formula. □

It remains to calculate the derivatives of $I^{\nu,\mu}_{n,m}$ for general values of the parameters. Due to the complexity of the formulas, we separate them into a separate proposition for each $\partial/\partial x_i$.

Proposition 2.5. *Let $n, m > 0$. Then for arbitrary $\nu, \mu \in \{-1, +1\}$, the derivatives of $I^{\nu,\mu}_{n,m}$ with respect to x_0 are as follows:*

$$\begin{aligned}
\frac{\partial}{\partial x_0}\, I^{+,+}_{n,m} &= -\frac{1}{2}(n-m+\frac{1}{2})\, I^{-,+}_{n+1,m} + n\, I^{-,+}_{n,m} - \frac{1}{2}(n+m-\frac{1}{2}) I^{-,+}_{n-1,m},\\
\frac{\partial}{\partial x_0}\, I^{+,-}_{n,m} &= -\frac{1}{2}(n-m+\frac{1}{2})\, I^{-,-}_{n+1,m} + \, n\, I^{-,-}_{n,m} - \frac{1}{2}(n+m-\frac{1}{2}) I^{-,-}_{n-1,m},\\
\frac{\partial}{\partial x_0}\, I^{-,+}_{n,m} &= \frac{1}{2}(n+m-\frac{1}{2})\, I^{-,-}_{n+1,m} + \, n\, I^{-,-}_{n,m} - \frac{1}{2}(n+m-\frac{1}{2}) I^{-,-}_{n-1,m},\\
\frac{\partial}{\partial x_0}\, I^{-,-}_{n,m} &= \frac{1}{2}(n-m+\frac{1}{2})\, I^{+,-}_{n+1,m} - n\, I^{+,-}_{n,m} + \frac{1}{2}(n+m-\frac{1}{2}) I^{+,-}_{n-1,m}.
\end{aligned}$$

Proof. By the Chain Rule and (2.5), we have

$$\begin{aligned}
\frac{\partial}{\partial x_0}\, I^{+,+}_{n,m} &= \frac{\partial I^{+,+}_{n,m}}{\partial \eta}\frac{\partial \eta}{\partial x_0} + \frac{\partial I^{+,+}_{n,m}}{\partial \theta}\frac{\partial \theta}{\partial x_0} + \frac{\partial I^{+,+}_{n,m}}{\partial \varphi}\cdot 0\\
&= -\sin\theta \sinh\eta \Bigg(\frac{\sinh\eta}{2(\cosh\eta - \sin\theta)^{1/2}}\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\cos n\theta\, \cos m\varphi\\
&\qquad\qquad + \sinh\eta\, (Q^m_{n-\frac{1}{2}})'(\cosh\eta)\cos n\theta \cos m\varphi (\cosh\eta - \sin\theta)^{1/2}\Bigg)\\
&\quad + (\cosh\eta\cos\theta - 1)\Bigg(\frac{\sin\theta}{2(\cosh\eta-\sin\theta)^{1/2}}\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\cos n\theta\, \cos m\varphi\\
&\qquad\qquad - n\,(\cosh\eta - \sin\theta)^{1/2}\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\sin n\theta\, \cos m\varphi\Bigg)\\
&= (\cosh\eta - \sin\theta)^{1/2}\cos m\varphi \Bigg(-\frac{1}{2}\cosh\eta\sin\theta\cos n\theta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)
\end{aligned}$$

$$
\begin{aligned}
&- \sin\theta \sinh^2\eta \,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)\cos n\theta \\
&- n\,(\cosh\eta - \sin\theta)^{1/2}\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\sin n\theta\Big).
\end{aligned}
$$

The recursion formula (1.11) for $n - 3/2$ says this is equal to

$$
\begin{aligned}
(\cosh\eta - \cos\theta)^{1/2}\cos m\varphi\Big(&- n\cosh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\cos n\theta\sin\theta \\
&+ (n+m-\frac{1}{2})Q^m_{n-\frac{3}{2}}(\cosh\eta)\cos n\theta\sin\theta \\
&- n\ \cosh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\cos\theta\sin n\theta \\
&+ n\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\sin n\theta\Big).
\end{aligned}
$$

Now use the recursion formula (1.12) for $n - 1/2$ and the trigonometric relations

$$
\cos n\theta\sin\theta = \frac{1}{2}\big(\sin(n+1)\theta - \sin(n-1)\theta\big)
$$

and

$$
\sin n\theta\cos\theta = \frac{1}{2}\big(\sin(n+1)\theta + \sin(n-1)\theta\big)
$$

to arrive at

$$
\begin{aligned}
(\cosh\eta - \cos\theta)^{1/2}\cos m\varphi\Big(&-\frac{1}{2}(n-m+\frac{1}{2})Q^m_{n+\frac{1}{2}}(\cosh\eta)\sin(n+1)\theta \\
&+ n\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\sin n\theta - \frac{1}{2}(n+m-\frac{1}{2})Q^m_{n-\frac{3}{2}}(\cosh\eta)\sin(n-1)\theta\Big)
\end{aligned}
$$

which is clearly equal to

$$-\frac{1}{2}(n-m+\frac{1}{2})\, I^{-,+}_{n+1,m} + n\, I^{-,+}_{n,m} - \frac{1}{2}(n+m-\frac{1}{2}) I^{-,+}_{n-1,m}$$

as required.

The same recursion formulas (1.11), (1.12) together with the identities

$$\begin{aligned}
\cos n\theta \cos\theta &= \frac{1}{2}\big(\cos(n-1)\theta + \cos(n+1)\theta\big)\\
\sin\, n\theta \sin\theta &= \frac{1}{2}\big(\cos(n-1)\theta - \cos(n+1)\theta\big)
\end{aligned}$$

lead to the formula for $\partial I^{-,-}_{n,m}/\partial x_0$. The remaining cases are similar. □

Proposition 2.6. *Let $n, m > 0$. Then for arbitrary $\nu, \mu \in \{-1, +1\}$, the derivatives of $I^{\nu,\mu}_{n,m}$ with respect to x_1 are as follows:*

$$\begin{aligned}
\frac{\partial}{\partial x_1} I^{+,+}_{n,m} =& -\frac{1}{4}(n+m-\frac{1}{2})(n+m-\frac{3}{2})\, I^{+,+}_{n-1,m-1} - \frac{1}{4}\, I^{+,+}_{n-1,m+1}\\
&+ \frac{1}{2}(n+m-\frac{1}{2})(n-m+\frac{1}{2})\, I^{+,+}_{n,m-1} + \frac{1}{2}\, I^{+,+}_{n,m+1}\\
&- \frac{1}{4}(n-m+\frac{1}{2})(n-m+\frac{3}{2})\, I^{+,+}_{n+1,m-1} - \frac{1}{4}\, I^{+,+}_{n+1,m+1},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x_1} I^{+,-}_{n,m} =& -\frac{1}{4}(n+m-\frac{1}{2})(n+m-\frac{3}{2})\, I^{+,-}_{n-1,m-1} - \frac{1}{4}\, I^{+,-}_{n-1,m+1}\\
&+ \frac{1}{2}(n+m-\frac{1}{2})(n-m+\frac{1}{2})\, I^{+,-}_{n,m-1} + \frac{1}{2}\, I^{+,-}_{n,m+1}\\
&- \frac{1}{4}(n-m+\frac{1}{2})(n-m+\frac{3}{2})\, I^{+,-}_{n+1,m-1} - \frac{1}{4}\, I^{+,-}_{n+1,m+1},
\end{aligned}$$

$$\frac{\partial}{\partial x_1} I^{-,+}_{n,m} = -\frac{1}{4}(n+m-\frac{1}{2})(n+m-\frac{3}{2})\, I^{-,+}_{n-1,m-1} - \frac{1}{4}\, I^{-,+}_{n-1,m+1}$$

$$
\begin{aligned}
&+\frac{1}{2}(n+m-\frac{1}{2})(n-m+\frac{1}{2})\, I^{-,+}_{n,m-1}+\frac{1}{2}\, I^{-,+}_{n,m+1}\\
&-\frac{1}{4}(n-m+\frac{1}{2})(n-m+\frac{3}{2})\, I^{-,+}_{n+1,m-1}-\frac{1}{4}\, I^{-,+}_{n+1,m+1},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x_1}\, I^{-,-}_{n,m} = &-\frac{1}{4}(n+m-\frac{1}{2})(n+m-\frac{3}{2})\, I^{-,-}_{n-1,m-1}-\frac{1}{4}\, I^{-,-}_{n-1,m+1}\\
&+\frac{1}{2}(n+m-\frac{1}{2})(n-m+\frac{1}{2})\, I^{-,-}_{n,m-1}+\frac{1}{2}\, I^{-,-}_{n,m+1}\\
&-\frac{1}{4}(n-m+\frac{1}{2})(n-m+\frac{3}{2})\, I^{-,-}_{n+1,m-1}-\frac{1}{4}\, I^{-,-}_{n+1,m+1}.
\end{aligned}
$$

Proof. For the first formula we calculate

$$
\begin{aligned}
\frac{\partial}{\partial x_1}\, I^{+,+}_{n,m} &= \frac{\partial}{\partial x_1}\Big((\cosh\eta-\cos\theta)^{1/2}\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\,\cos n\theta\cos m\varphi\Big)\\
&= -\cos\varphi(\cosh\eta\cos\theta-1)\Big(\frac{\sinh\eta}{2(\cosh\eta-\cos\theta)^{1/2}}\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\,\cos n\theta\cos m\varphi\\
&\qquad\qquad +\sinh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)(\cosh\eta-\cos\theta)^{1/2}\,\cos n\theta\cos m\varphi\Big)\\
&\quad -\sinh\eta\sin\theta\cos\varphi\Big(\frac{\sin\theta}{2(\cosh\eta-\cos\theta)^{1/2}}\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\,\cos n\theta\cos m\varphi\\
&\qquad\qquad -n\,(\cosh\eta-\cos\theta)^{1/2}\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\sin n\theta\cos m\varphi\Big)\\
&\quad +\frac{m\sin\varphi}{\sinh\eta}(\cosh\eta-\cos\theta)^{3/2}\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\cos n\theta\sin m\varphi\\
&= (\cosh\eta-\cos\theta)^{1/2}\Big(\cos\theta\cos n\theta\cos\varphi\cos m\varphi\,\big(-\frac{1}{2}\sinh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\\
&\qquad\qquad -\cosh\eta\sinh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)\big)\\
&\qquad +\cos n\theta\cos\varphi\cos m\varphi\big(\sinh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)\big)\\
&\qquad +\cos n\theta\sin\varphi\sin m\varphi\big(\frac{m\cosh\eta}{\sinh\eta}Q^m_{n-\frac{1}{2}}(\cosh\eta)\big)
\end{aligned}
$$

$$+\cos\theta\cos n\theta\sin\varphi\sin m\varphi\big(-\frac{m}{\sinh\eta}Q^m_{n-\frac{1}{2}}(\cosh\eta)\big)$$
$$+\sin\theta\sin n\theta\cos\varphi\cos m\varphi\big(n\ \sinh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\big)\Big).$$

After applying trigonometric relations this becomes

$$(\cosh\eta-\cos\theta)^{1/2}(A_1+A_2+A_3+A_4+A_5+A_6), \tag{2.6}$$

where

$$A_1=\frac{1}{4}\Big(-\frac{1}{2}\sinh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)-\cosh\eta\sinh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)$$
$$-\frac{m}{\sinh\eta}\,Q^m_{n-\frac{1}{2}}(\cosh\eta)+n\ \sinh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\Big)\cos(n-1)\theta\cos(m-1)\varphi,$$

$$A_2=\frac{1}{2}\Big(\sinh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)-\frac{m\ \cosh\eta}{\sinh\eta}\,Q^m_{n-\frac{1}{2}}(\cosh\eta)\Big)\cos n\theta\cos(m-1)\varphi,$$

$$A_3=-\frac{1}{4}\Big(\frac{1}{2}\sinh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)+\cosh\eta\sinh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)$$
$$+\frac{m}{\sinh\eta}\,Q^m_{n-\frac{1}{2}}(\cosh\eta)+\sinh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\Big)\cos(n+1)\theta\cos(m-1)\varphi,$$

$$A_4=\frac{1}{4}\Big(-\frac{1}{2}\sinh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta-\cosh\eta\sinh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta))$$
$$+\frac{m}{\sinh\eta}\,Q^m_{n-\frac{1}{2}}(\cosh\eta)+\ n\sinh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta))\Big)\cos(n-1)\theta\cos(m+1)\varphi,$$

$$A_5=\frac{1}{2}\Big(\sinh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)-\frac{m\ \cosh\eta}{\sinh\eta}\,Q^m_{n-\frac{1}{2}}(\cosh\eta)\Big)\cos n\theta\cos(m+1)\varphi,$$

$$A_6=\frac{1}{4}\Big(-\frac{1}{2}\sinh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)-\cosh\eta\sinh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)$$
$$+\frac{m}{\sinh\eta}\,Q^m_{n-\frac{1}{2}}(\cosh\eta)-n\sinh\eta\, Q^m_{n-\frac{1}{2}}(\cosh\eta)\Big)\cos(n+1)\theta\ \cos(m+1)\varphi.$$

Now we simplify every term separately. Applying the recurrence relations (1.13) for $n-3/2$, (1.14) for $n-3/2$, (1.12) for $n-1/2$, $m+1$, and (1.17) for $n-1/2$, we find

$$\begin{aligned}
A_1 &= Q_{n+\frac{1}{2}}^{m+1}(\cosh\eta)(n-m-\frac{1}{2})(\frac{1}{8n}-\frac{1}{8n}) \\
&\quad + Q_{n-\frac{3}{2}}^{m+1}(\cosh\eta)\Big(-\frac{1}{8n}(n-m-\frac{1}{2})-\frac{1}{8n}(n+m+\frac{1}{2})+\frac{1}{4}\Big) \\
&\quad - Q_{n-\frac{3}{2}}^{m-1}(\cosh\eta)\frac{1}{4}(n+m-\frac{3}{2})(n+m-\frac{1}{2}) \\
&= Q_{n-\frac{3}{2}}^{m-1}(\cosh\eta)\,\frac{1}{4}\,(n+m-\frac{3}{2})(n+m-\frac{1}{2}).
\end{aligned}$$

For A_2 we can apply the recurrence relations (1.13) for $n-3/2$ and (1.15) for $n-3/2$, giving

$$A_2 = \frac{1}{2}(n+m-\frac{1}{2})(n-m+\frac{1}{2})\,Q_{n-\frac{1}{2}}^{m-1}(\cosh\eta).$$

Now apply to A_3 relations (1.13), (1.14), and (1.15) for $n-3/2$, together with relations (1.12) and (1.16) for $n-1/2$. Then apply relation (1.17) for $n-1/2$. The result is

$$A_3 = (n-m+\frac{1}{2})(n-m+\frac{3}{2})\,Q_{n+\frac{1}{2}}^{m-1}(\cosh\eta).$$

Similarly, by using the recursion formulas (1.12), (1.13), and (1.14), we find

$$A_4 = -\frac{1}{4}\,Q_{n-\frac{3}{2}}^{m+1}(\cosh\eta).$$

Recursion relation (1.13) gives

$$A_5 = \frac{1}{2}\, Q^{m+1}_{n-\frac{1}{2}}(\cosh\eta).$$

Finally, applying relations (1.13) and (1.14) gives

$$A_6 = -\frac{1}{4}\, Q^{m+1}_{n+\frac{1}{2}}(\cosh\eta).$$

These equations together with (2.6) give the formula for $I^{+,+}_{n,m}$. The formulas for the remaining three combinations of signs are derived in the same way. $\square$

Proposition 2.7. *Let $n, m > 0$. Then for arbitrary $\nu, \mu \in \{-1, +1\}$, the derivatives of $I^{\nu,\mu}_{n,m}$ with respect to x_2 are as follows:*

$$\begin{aligned}
\frac{\partial}{\partial x_2}\, I^{+,+}_{n,m} &= \frac{1}{4}(n+m-\frac{1}{2})(n+m-\frac{3}{2})\, I^{+,-}_{n-1,m-1} - \frac{1}{4}\, I^{+,-}_{n-1,m+1} \\
&\quad - \frac{1}{2}(n+m-\frac{1}{2})(n-m+\frac{1}{2})\, I^{+,-}_{n,m-1} + \frac{1}{2}\, I^{+,-}_{n,m+1} \\
&\quad + \frac{1}{4}(n-m+\frac{1}{2})(n-m+\frac{3}{2})\, I^{+,-}_{n+1,m-1} - \frac{1}{4}\, I^{+,-}_{n+1,m+1}, \\
\frac{\partial}{\partial x_2}\, I^{+,-}_{n,m} &= -\frac{1}{4}(n+m-\frac{1}{2})(n+m-\frac{3}{2})\, I^{+,+}_{n-1,m-1} + \frac{1}{4}\, I^{+,+}_{n-1,m+1} \\
&\quad + \frac{1}{2}(n+m-\frac{1}{2})(n-m+\frac{1}{2})\, I^{+,+}_{n,m-1} - \frac{1}{2}\, I^{+,+}_{n,m+1} \\
&\quad - \frac{1}{4}(n-m+\frac{1}{2})(n-m+\frac{3}{2})\, I^{+,+}_{n+1,m-1} + \frac{1}{4}\, I^{+,+}_{n+1,m+1}, \\
\frac{\partial}{\partial x_2}\, I^{-,+}_{n,m} &= \frac{1}{4}(n+m-\frac{1}{2})(n+m-\frac{3}{2})\, I^{-,-}_{n-1,m-1} - \frac{1}{4}\, I^{-,-}_{n-1,m+1} \\
&\quad - \frac{1}{2}(n+m-\frac{1}{2})(n-m+\frac{1}{2})\, I^{-,-}_{n,m-1} + \frac{1}{2}\, I^{-,-}_{n,m+1}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}(n-m+\frac{1}{2})(n-m+\frac{3}{2})\,I^{-,-}_{n+1,m-1}-\frac{1}{4}\,I^{-,-}_{n+1,m+1},\\
\frac{\partial}{\partial x_2}\,I^{-,-}_{n,m} &= -\frac{1}{4}(n+m-\frac{1}{2})(n+m-\frac{3}{2})\,I^{-,+}_{n-1,m-1}+\frac{1}{4}\,I^{-,+}_{n-1,m+1}\\
&+\frac{1}{2}(n+m-\frac{1}{2})(n-m+\frac{1}{2})\,I^{-,+}_{n,m-1}-\frac{1}{2}\,I^{-,+}_{n,m+1}\\
&-\frac{1}{4}(n-m+\frac{1}{2})(n-m+\frac{3}{2})\,I^{-,+}_{n+1,m-1}+\frac{1}{4}\,I^{-,+}_{n+1,m+1}.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\frac{\partial}{\partial\, x_2} I^{+,+}_{n,m} &= \frac{\partial}{\partial\, x_2}\Big((\cosh\eta-\cos\theta)^{1/2} Q^m_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta)\cos(m\varphi)\Big)\\
&= -\sin\varphi(\cosh\eta\cos\theta-1)\Big(\frac{\sinh\eta}{2(\cosh\eta-\cos\theta)^{1/2}}\,Q^m_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta)\cos(m\varphi)\\
&\qquad +\sinh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)(\cosh\eta-\cos\theta)^{1/2}\cos(n\theta)\cos(m\varphi)\Big)\\
&\quad -\sin\varphi\sin\theta\sinh\eta\Big(\frac{\sin\theta}{2(\cosh\eta-\cos\theta)^{1/2}}\,Q^m_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta)\cos(m\varphi)\\
&\qquad -n\,(\cosh\eta-\cos\theta)^{1/2} Q^m_{n-\frac{1}{2}}(\cosh\eta)\sin(n\theta)\cos(m\varphi)\Big)\\
&\quad -\frac{\cos\varphi(\cosh\eta-\cos\theta)}{\sinh\eta}\Big(m\,(\cosh\eta-\cos\theta)^{1/2} Q^m_{n-\frac{1}{2}}(\cosh\eta)\cos(n\theta)\sin(m\varphi)\Big).
\end{aligned}
$$

After some trigonometry and simplification this is found to equal

$$
\begin{aligned}
&(\cosh\eta-\cos\theta)^{1/2}\Big(\\
&\cos(n-1)\theta\sin(m-1)\varphi\Big(\frac{1}{8}\sinh\eta\,Q^m_{n-\frac{1}{2}}(\cosh\eta)+\frac{1}{4}\sinh\eta\cosh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)\\
&\qquad\qquad -\frac{n}{4}\,\sinh\eta\,Q^m_{n-\frac{1}{2}}(\cosh\eta)+\frac{m}{4\sinh\eta}\,Q^m_{n-\frac{1}{2}}(\cosh\eta)\Big)\\
&+\cos(n-1)\theta\sin(m+1)\varphi\Big(-\frac{1}{8}\sinh\eta\,Q^m_{n-\frac{1}{2}}(\cosh\eta)-\frac{1}{4}\sinh\eta\cosh\eta\,(Q^m_{n-\frac{1}{2}})'(\cosh\eta)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \frac{n}{4}\,\sinh\eta\, Q^m_{n-\frac12}(\cosh\eta) + \frac{m}{4\sinh\eta}\, Q^m_{n-\frac12}(\cosh\eta)\Big) \\
&+ \cos(n\theta)\sin(m-1)\varphi\Big(-\frac{\sinh\eta}{2}\,(Q^m_{n-\frac12})'(\cosh\eta) - \frac{m\cosh\eta}{2\sinh\eta}\, Q^m_{n-\frac12}(\cosh\eta)\Big) \\
&+ \cos(n\theta)\sin(m+1)\varphi\Big(\frac{\sinh\eta}{2}\,(Q^m_{n-\frac12})'(\cosh\eta) - \frac{m\cosh\eta}{2\sinh\eta}\, Q^m_{n-\frac12}(\cosh\eta)\Big) \\
&+ \cos(n+1)\theta\sin(m-1)\varphi\Big(\frac{\sinh\eta}{8}\, Q^m_{n-\frac12}(\cosh\eta) + \frac{1}{4}\cosh\eta\sinh\eta\,(Q^m_{n-\frac12})'(\cosh\eta) \\
&\qquad\qquad + \frac{n}{4}\,\sinh\eta\, Q^m_{n-\frac12}(\cosh\eta) + \frac{m}{4\sinh\eta} Q^m_{n-\frac12}(\cosh\eta)\Big) \\
&+ \cos(n+1)\theta\sin(m+1)\varphi\Big(-\frac{\sinh\eta}{8}\, Q^m_{n-\frac12}(\cosh\eta) + \frac{1}{4}\cosh\eta\sinh\eta\,(Q^m_{n-\frac12})'(\cosh\eta) \\
&\qquad\qquad - \frac{n}{4}\,\sinh\eta\, Q^m_{n-\frac12}(\cosh\eta) + \frac{m}{4\sinh\eta}\, Q^m_{n-\frac12}(\cosh\eta)\Big)\Big)
\end{aligned}
$$

by applying the recursion formulas (1.12), (1.13), (1.14), (1.15), and (1.17) this becomes

$$
\begin{aligned}
&(\cosh\eta - \cos\theta)^{1/2}\Big(\\
&\frac14(n+m-\frac12)(n+m-\frac32)\, Q^{m-1}_{n-\frac32}(\cosh\eta)\cos(n-1)\theta\ \sin(m-1)\varphi \\
&-\frac14\, Q^{m+1}_{n-\frac32}(\cosh\eta)\cos(n-1)\theta\ \sin(m+1)\varphi \\
&-\frac12(n+m-\frac12)(n-m+\frac12)\, Q^{m-1}_{n-\frac12}(\cosh\eta)\cos(n\theta)\sin(m-1)\varphi \\
&+\frac12\, Q^{m+1}_{n-\frac12}(\cosh\eta)\cos(n\theta)\ \sin(m+1)\varphi \\
&+\frac14(n-m+\frac12)(n-m+\frac32) Q^{m-1}_{n+\frac12}(\cosh\eta)\cos(n+1)\theta\ \sin(m-1)\varphi \\
&-\frac14\, Q^{m+1}_{n+\frac12}(\cosh\eta)\cos(n+1)\theta\ \sin(m+1)\varphi\Big),
\end{aligned}
$$

which is the formula stated for $\partial I^{+,+}_{n,m}/\partial x_2$. The proofs of the remaining cases are analogous. □

One can see

	$\frac{\partial}{\partial x_0}$
$I^{\nu,\mu}_{0,0}$ $(\nu,\mu+1)$	$-\frac{1}{2}I^{-\nu,\mu}_{1,0}$
$I^{\nu,\mu}_{n,0}$ $(n>0,$ $\mu+1)$	$-\frac{\nu(2n-1)}{4}I^{-\nu,\mu}_{n-1,0}+\nu n I^{-\nu,\mu}_{n,0}-\frac{\nu(2n+1)}{4}I^{-\nu,\mu}_{n+1,0}$
$I^{\nu,\mu}_{0,m}$ $(m>0,$ $\nu=+1)$	$(m-\frac{1}{2})I^{-\nu,\mu}_{1,m}$
$I^{\nu,\mu}_{n,m}$ $(n,m>0)$	$-\frac{\nu}{2}(n+m-\frac{1}{2})I^{-\nu,\mu}_{n-1,m}+\nu n I^{-\nu,\mu}_{n,m}-\frac{\nu}{2}(n-m+\frac{1}{2})I^{-\nu,\mu}_{n+1,m}$

Table 2.1: Derivative of $I^{\nu,\mu}_{n,m}(x_0,x_1,x_2)$ with respect to x_0 .

	$\frac{\partial}{\partial x_1}$
$I^{\nu,\mu}_{0,0}$ $(\nu,\mu+1)$	$I^{\nu,\mu}_{0,1} - I^{\nu,\mu}_{1,1}$
$I^{\nu,\mu}_{n,0}$ $(n>0,$ $\mu=+1)$	$-\frac{1}{2}I^{\nu,\mu}_{n-1,1} + I^{\nu,\mu}_{n,1} - \frac{1}{2}I^{\nu,\mu}_{n+1,1}$
$I^{\nu,\mu}_{0,m}$ $(m>0,$ $\nu=+1)$	$-\frac{1}{2}(m-\frac{1}{2})(m-\frac{3}{2})I^{\nu,\mu}_{1,m-1} - \frac{1}{2}I^{\nu,\mu}_{1,m+1}$ $+\frac{1}{2}(m+\frac{1}{2})(\frac{1}{2}-m)I^{\nu,\mu}_{0,m-1} + \frac{1}{2}I^{\nu,\mu}_{0,m+1}$
$I^{\nu,\mu}_{n,m}$ $(n,m>0)$	$-\frac{1}{4}(n+m-\frac{1}{2})(n+m-\frac{3}{2})I^{\nu,\mu}_{n-1,m-1} - \frac{1}{4}I^{\nu,\mu}_{n-1,m+1}$ $+\frac{1}{2}(n+m-\frac{1}{2})(n-m+\frac{1}{2})I^{\nu,\mu}_{n,m-1} + \frac{1}{2}I^{\nu,\mu}_{n,m+1}$ $-\frac{1}{4}(n-m+\frac{1}{2})(n-m+\frac{3}{2})I^{\nu,\mu}_{n+1,m-1} - \frac{1}{4}I^{\nu,\mu}_{n+1,m+1}$

Table 2.2: Derivative of $I^{\nu,\mu}_{n,m}(x_0,x_1,x_2)$ with respect to x_1 .

	$\frac{\partial}{\partial x_2}$
$I^{\nu,\mu}_{0,0}$ $(\nu, \mu = +1)$	$I^{\nu,-\mu}_{0,1} - I^{\nu,-\mu}_{1,1}$
$I^{\nu,\mu}_{n,0}$ $(n > 0, \mu = +1)$	$-\frac{\nu}{2} I^{\nu,-\mu}_{n-1,1} + \nu I^{\nu,-\mu}_{n,1} - \frac{\nu}{2} I^{\nu,-\mu}_{n+1,1}$
$I^{\nu,\mu}_{0,m}$ $(m > 0, \nu = +1)$	$\frac{\mu}{2}(m - \frac{1}{2})(m - \frac{3}{2}) I^{\nu,-\mu}_{1,m-1} - \frac{\mu}{2} I^{\nu,-\mu}_{1,m+1}$ $+ \frac{\mu}{2}(m + \frac{1}{2})(m - \frac{1}{2}) I^{\nu,-\mu}_{0,m-1} + \frac{\mu}{2} I^{\nu,-\mu}_{0,m+1}$
$I^{\nu,\mu}_{n,m}$ $(n, m > 0)$	$\frac{\mu}{4}(n + m - \frac{1}{2})(n + m - \frac{3}{2}) I^{\nu,-\mu}_{n-1,m-1} - \frac{\mu}{4} I^{\nu,-\mu}_{n-1,m+1}$ $- \frac{\mu}{2}(n + m - \frac{1}{2})(n - m + \frac{1}{2}) I^{\nu,-\tau}_{n,m-1} + \frac{\mu}{2} I^{nu,-\mu}_{n,m+1}$ $+ \frac{\mu}{4}(n - m + \frac{1}{2})(n - m + \frac{3}{2}) I^{\nu,-\mu}_{n+1,m-1} - \frac{\mu}{4} I^{\nu,-\mu}_{n+1,m+1}$

Table 2.3: Derivative of $I^{\nu,\mu}_{n,m}(x_0, x_1, x_2)$ with respect to x_2 .

2.3 Construction of reverse-Appell system of toroidal harmonics

The results of Section 2.2 provide the partial derivatives of most of the $I^{\nu,\mu}_{n,m}$ as a linear combination of two or more functions in the same family. In particular, this implies that the toroidal harmonics are not an Appell system with respect to any of the three partial derivatives $\partial/\partial x_i$. More interestingly, the derivatives of $I^{\nu,\mu}_{n,m}$ with respect to x_0 involve harmonics with indices $n-1$, n, and $n+1$.

In this section, we construct a new basis for the toroidal harmonics for which $\partial/\partial x_0$ sends each element to a constant multiple of a single other element. Since the index n is increased by 1, we call this a "reverse-Appell system." We are only interested in the partial derivatives with respect to x_0 in this regard because this is what we will need for studying monogenic functions.

Definition 2.8. For every $n \geq 1$ define

$$\kappa^n_{k,m} = \begin{cases} -\frac{1}{2}(n+m-\frac{1}{2}), & k = n-1, \\ n, & k = n, \\ -\frac{1}{2}(n-m+\frac{1}{2}), & k = n+1, \end{cases}$$

with $\kappa^n_{k,m} = 0$ other than in the cases listed here. Further, for $n = 0$ define

$$\kappa^0_{0,m} = m - \frac{1}{2} \quad (m \geq 1).$$

With this notation, Propositions 2.2–2.5 it may be expressed succinctly as follows.

Proposition 2.9.

$$\partial_0 I^{\nu,\mu}_{n,m} = \sum_{k=0}^{\infty} \kappa^n_{k,m} I^{-\nu,\mu}_{k,m} = \sum_{k=(n-1)_+}^{n+1} \kappa^n_{k,m} I^{-\nu,\mu}_{k,m},$$

where we denote $(t)_+ = \max(t, 0)$.

The coefficients $\kappa^n_{k,m}$ permit us to define two infinite triangular matrices $(i^n_{n,m})$ and $(i^{*\,n}_{n,m})$:

Definition 2.10. Let $i^{*\,n}_{n,m} = 1$ and then recursively for $0 \le k \le n-1$ let

$$i^{*\,n}_{k,m} = \frac{1}{\kappa^{n-1}_{n,m}} \sum_{j=(k-1)_+}^{n-1} \kappa^j_{k,m} i^{*\,n-1}_{j,m}.$$

Also, define $i^n_{n,m} = 1$ and then recursively for $k = n-1, n-2, \ldots, 0$ let

$$i^n_{k,m} = -\sum_{j=k+1}^{n} i^{*\,j}_{k,m}\, i^n_{j,m}.$$

Definition 2.11. Let m and n be nonnegative integers, and let $\nu, \mu = \pm 1$ serve as superscripts in (1.24). We introduce a new doubly indexed collection $\{I^{*\,\nu,\mu}_{n,m}\}$ of *reverse-Appell toroidal harmonic functions* on Ω_{η_0} as follows:

$$I^{*\,\nu,\mu}_{n,m} = \sum_{k=0}^{n} i^{*\,n}_{k,m} I^{\nu,\mu}_{k,m}. \tag{2.7}$$

We obtain the following result:

Proposition 2.12. *The collection* $\{I^{*\,\nu,\mu}_{n,m}\}$ *satisfies the following reverse Appell-type property:*

$$\frac{\partial I^{*\,\nu,\mu}_{n,m}}{\partial x_0} = \kappa^n_{n+1,m} I^{*\,-\nu,\mu}_{n+1,m} \quad (n \ge 0). \tag{2.8}$$

Also, we have the inverse relation

$$I_{n,m}^{\nu,\mu} = \sum_{k=0}^{n} i_{k,m}^{n} I_{k,m}^{*\,\nu,\mu}. \tag{2.9}$$

Proof. First we note that

$$\begin{aligned}
\frac{\partial I_{n,m}^{*\,\nu,\mu}}{\partial x_0} &= \frac{\partial}{\partial x_0} \sum_{j=0}^{n} i_{j,m}^{*\,n} I_{j,m}^{\nu,\mu} \\
&= \sum_{j=0}^{n} i_{j,m}^{*\,n} \sum_{k=(j-1)_+}^{j+1} \kappa_{k,m}^{j} I_{k,m}^{-\nu,\mu} \\
&= \sum_{k=0}^{n+1} \left(\sum_{j=(k-1)_+}^{n} \kappa_{k,m}^{j} i_{j,m}^{*\,n} \right) I_{k,m}^{-\nu,\mu} \\
&= \sum_{k=0}^{n} \left(\sum_{j=(k-1)_+}^{n} \kappa_{k,m}^{j} i_{j,m}^{*\,n} \right) I_{k,m}^{-\nu,\mu} + \kappa_{n+1,m}^{n} i_{n+1,m}^{*\,n+1} I_{n+1,m}^{-\nu,\mu}.
\end{aligned} \tag{2.10}$$

Let $\lambda_{n,m} = \kappa_{n+1,m}^{n}$, so by definition

$$\lambda_{n,m} I_{n+1,m}^{*\,-\nu,\mu} = \lambda_{n,m} \sum_{k=0}^{n+1} i_{k,m}^{*\,n+1} I_{k,m}^{-\nu,\mu}.$$

Since $\lambda_{n,m} i_{k,m}^{*\,n+1} = \sum_{j=(k-1)_+}^{n} \kappa_{k,m}^{j} i_{j,m}^{*n}$ and $i_{n+1,m}^{*\,n+1} = 1$ we have

$$\begin{aligned}
\frac{\partial I_{n,m}^{*\,\nu,\mu}}{\partial x_0} &= \sum_{k=0}^{n} \lambda_{n,m} i_{k,m}^{*\,n+1} I_{k,m}^{-\nu,\mu} + \lambda_{n,m} I_{n+1,m}^{-\nu,\mu} \\
&= \lambda_{n,m} \sum_{k=0}^{n+1} i_{k,m}^{*\,n+1} I_{k,m}^{-\nu,\mu} \\
&= \lambda_{n,m} I_{n+1,m}^{*\,-\nu,\mu} \\
&= \kappa_{n+1,m}^{n} I_{n+1,m}^{*\,-\nu,\mu},
\end{aligned}$$

which is (2.8).

To verify (2.9), simply note that by Definition 2.11 the vector $(I_{n,m}^{*\,\nu,\mu})_n$ is the image of the vector $(I^{\nu,\mu}_{k,m})_k$ under the triangular matrix $(i^{*j}_{k,m})_{k,j}$, which by Definition 2.10 is the inverse of the matrix $(i^n_{j,m})_{j,n}$. $\square$

Chapter 3

Neumann and Dirichlet-to-Neumann problems on the torus

In the previous chapter, we carried out a detailed calculation of the partial derivatives of the toroidal harmonics. Our main purpose for doing so is to use them in the construction of monogenic functions on Ω_{η_0}, which we will carry out in Chapters 5 and 6. However, the formulas for these derivatives enable us to give some new results on toroidal harmonics themselves. This chapter is devoted to these results and thus is a digression from the main purpose of this thesis.

The explicit formulas for the partial derivatives of the toroidal harmonics allow us to explore their normal derivatives. Normal derivatives are the principal element in Neumann's problems. The Dirichlet-to-Neumann map for the Laplacian, which we will describe in the first section, is important in many areas of basic analysis (elliptic boundary value problems [29, 53, 63, 81], inverse problems [23, 49]), as well as physics (e.g., fluid mechanics [27], electromagnetic theory [24], electrical impedance tomography [48, 50, 89], electrical transmission [30]).

3.1 Expression of the Neumann problem in toroidal coordinates

In this section, we define the basic concepts involved in the Dirichlet-to-Neumann mapping and explicitly express the mapping for Ω_{η_0} in terms of the toroidal harmonics. We use this information to derive expressions for the compatibility and normalization conditions in terms of Fourier coefficients and give a result of independent interest (Proposition 3.13), which is a formula for the surface area of $\partial\Omega_{\eta_0}$ in terms of Legendre functions of the second kind.

3.1.1 Normal derivative on the torus

Let $\Omega \subseteq \mathbb{R}^3$ be a domain with smooth boundary.

Definition 3.1. The *normal derivative* of a function u defined in a neighborhood V of a point $x \in \partial\Omega$ (or a half-neighborhood $V \cap \overline{\Omega}$) is by definition

$$\operatorname{nor} u(x) = \frac{d}{dt} u(x + t\,\mathbf{n})|_{t=0^+} = (\operatorname{grad} u(x)) \cdot \mathbf{n}, \tag{3.1}$$

where $\mathbf{n} = \mathbf{n}(x)$ is the inward pointing unit normal vector at x.

For the torus Ω_{η_0}, the unit normal vector on $\partial\Omega_{\eta_0}$ is represented in terms of the toroidal coordinate function $x(\eta, \theta, \varphi)$ as

$$\mathbf{n} = \mathbf{n}(x) = (x_\theta \times x_\varphi)/|x_\theta \times x_\varphi|, \tag{3.2}$$

where the values of x_θ, x_φ were calculated in (1.30). From this we have an explicit representation

$$\mathbf{n} = \frac{\big(-\sinh\eta_0 \sin\theta,\ -\cos\varphi(\cosh\eta_0 \cos\theta - 1),\ -\sin\varphi(\cosh\eta_0 \cos\theta - 1)\big)}{(\cosh\eta_0 - \cos\theta)}. \tag{3.3}$$

As a consequence of Proposition 1.15, we can also express the unit normal vector as $\mathbf{n} = x_\eta/|x_\eta|$, and since η increases as points move towards the interior of Ω_{η_0}, we see that (3.3) is in fact the inward pointing normal vector on $\partial\Omega_{\eta_0}$.

In using the notation $\operatorname{nor} u$, the fixed value of η_0 will always be understood.

3.1.2 Background information on the Neumann problem

Here we summarize well-known facts about the Neumann problem for reasonably general spatial domains. In the statements of the results, Ω will be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Later we will return to Ω_{η_0}.

Given a suitable $f\colon \partial\Omega \to \mathbb{R}$, let u be the unique harmonic function in Ω with boundary values $f = u|_{\partial\Omega}$. One says that u is the solution to the *Dirichlet problem* with boundary condition f (see [10]). Let $h = \text{nor}\, u$ be the normal derivative of u on $\partial\Omega$.

Definition 3.2. The *Dirichlet-to-Neumann mapping* Λ is given by

$$\Lambda f = h. \tag{3.4}$$

The *Neumann problem* corresponding to h is to find f such that (3.4) holds.

A common setting [55] for the Neumann problem is for f to be in $H^{1/2}(\partial\Omega)$ (with u in $H^1(\Omega_{\eta_0})$) and h in the boundary space $H^{-1/2}(\partial\Omega_{\eta_0})$. These spaces were described in Subsection 1.5.2 of Chapter 1. For $v \in H^1(\Omega_{\eta_0})$ we will informally write $v|_{\partial\Omega_{\eta_0}}$ for the image of v under the trace map $\text{tr}[v]$.

The following result describes the existence of solutions to the Neumann problem in three important contexts. In any event, the formula (3.4) implies that h satisfies the *compatibility condition*

$$\int_{\partial\Omega} h \, dS = 0, \tag{3.5}$$

which therefore is a necessary hypothesis in the following result.

Proposition 3.3 ([29, 34, 60]). *(i) Suppose that $h \in H^{-1/2}(\partial\Omega)$ satisfies the compatibility condition* (3.5). *Then there exists an $f \in H^{1/2}(\partial\Omega)$ such that $\Lambda f = h$. This solution f is unique up to an additive constant.*
(ii) If $h \in L^2(\partial\Omega)$, then $f \in L^2(\partial\Omega)$.
(iii) If h is continuous on $\partial\Omega$, then f is also continuous.

The solution f can made unique by requiring the *normalization condition*

$$\int_{\partial\Omega_{\eta_0}} f \, dS = c \tag{3.6}$$

for a chosen constant c.

3.1.3 Dirichlet-to-Neumann mapping in toroidal coordinates

We return to the particular case of Ω_{η_0}. The coordinate expression of the Dirichlet-to-Neumann mapping on Ω_{η_0} will be based on the following theorem. We will abbreviate

$$q_{n,m} = Q^m_{n-\frac{1}{2}}(\cosh \eta_0), \tag{3.7}$$

so that by Definition 1.40, on the boundary we have

$$I_{n,m}^{\nu,\mu}(\eta_0, \theta, \varphi) = q_{n,m}\sqrt{\cosh\eta_0 - \cos\theta}\,\Phi_n^\nu(\theta)\Phi_m^\mu(\varphi). \tag{3.8}$$

Theorem 3.4. *The normal derivatives of the interior toroidal harmonics* (1.40) *are given by the formula*

$$\begin{aligned}\operatorname{nor} I_{n,m}^{\nu,\mu} = \Bigg(&\bigg((1+2n)q_{n,m}\cosh\eta_0 - \Big(2(n-m)+1\Big)q_{n+1,m}\bigg)\Phi_{n-1}^\nu(\theta)\\
&+\bigg(-2\Big(q_{n,m}(2n\cosh^2\eta_0+1) - \Big(2(n-m)+1\Big)\cosh\eta_0\, q_{n+1,m}\Big)\bigg)\Phi_n^\nu(\theta)\\
&+\bigg((1+2n)q_{n,m}\cosh\eta_0 - \Big(2(n-m)+1\Big)q_{n+1,m}\bigg)\Phi_{n+1}^\nu(\theta)\Bigg)\times\\
&\frac{(\cosh\eta_0-\cos\theta)^{1/2}}{4\sinh\eta_0}\,\Phi_m^\mu(\varphi).\end{aligned}$$

Proof. Apply (3.1) to (1.40) and obtain a formula for $\operatorname{nor} I_{n,m}^{\nu,\mu}$. Now eliminate all occurrences of the derivative $(Q_{n+1}^m)'$ by means of the recurrence formula (1.11). Finally, regroup the terms as multiples of the trigonometric expressions of the form Φ_n^ν. $\square$

Since every continuous function $f\colon \partial\Omega_{\eta_0} \to \mathbb{R}$ is periodic in both θ and φ in toroidal coordinates, and the function $\sqrt{\cosh\eta_0 - \cos\theta}$ is also periodic, we may express f in terms of a Fourier series as follows:

$$\frac{1}{\sqrt{\cosh\eta_0 - \cos\theta}} f(\theta,\varphi) = \sum_{n,m,\nu,\mu} a_{n,m}^{\nu,\mu}\,\Phi_n^\nu(\theta)\Phi_m^\mu(\varphi) \tag{3.9}$$

with $a_{n,m}^{\nu,\mu} \in \mathbb{R}$. This will be a convenient form because of the appearance of the factor $\sqrt{\cosh \eta_0 - \cos \theta}$ in (3.8).

Proposition 3.5. *Let $f \in H^{1/2}(\partial\Omega_{\eta_0})$. Suppose that f is given by* (3.9)*. Let u be the solution of the corresponding Dirichlet problem in Ω_{η_0}, $u|_{\partial\Omega_{\eta_0}} = f$. Then the Dirichlet-to-Neumann mapping of f is given by*

$$h(\theta, \varphi) = \Lambda f = \operatorname{nor} u = \sum_{n,m,\nu,\mu} \frac{a_{n,m}^{\nu,\mu}}{q_{n,m}} \operatorname{nor} I_{n,m}^{\nu,\mu}(\eta_0, \theta, \varphi). \tag{3.10}$$

Proof. By (3.8), it is immediate that the solution of the Dirichlet problem on Ω_{η_0} for f given by (3.9) is given by

$$u = \sum_{n,m,\nu,\mu} \frac{a_{n,m}^{\nu,\mu}}{q_{n,m}} I_{n,m}^{\nu,\mu}, \tag{3.11}$$

because this sum is harmonic and coincides with $f(\theta, \varphi)$ when $\eta = \eta_0$. Since the dominant factor in (1.9) is $e^{(-n+1/2)\eta_0}$, the sum (3.11) converges when (3.9) converges. Then by Theorem 3.4, it follows that $h = \Lambda f$ is in turn given by

$$h = \sum_{n,m,\nu,\mu} \frac{a_{n,m}^{\nu,\mu}}{q_{n,m}} \operatorname{nor} I_{n,m}^{\nu,\mu}, \tag{3.12}$$

assuming that f is sufficiently well-behaved to justify the exchange of summation and differentiation. For example, since the trace operator and the Dirichlet-to-Neumann mapping $\Lambda\colon H^{1/2}(\partial\Omega_{\eta_0}) \to H^{-1/2}(\partial\Omega_{\eta_0})$ are continuous [83],

$$
\begin{aligned}
\operatorname{nor} u = \operatorname{nor} \sum_{n,m,\nu,\mu} \frac{a_{n,m}^{\nu,\mu}}{q_{n,m}} I_{n,m}^{\nu,\mu} &= \Lambda\Big(\Big(\sum_{n,m,\nu,\mu} \frac{a_{n,m}^{\nu,\mu}}{q_{n,m}} I_{n,m}^{\nu,\mu}\Big)\Big|_{\partial\Omega_{\eta_0}}\Big) \\
&= \Lambda\Big(\sum_{n,m,\nu,\mu} \frac{a_{n,m}^{\nu,\mu}}{q_{n,m}} \big(I_{n,m}^{\nu,\mu}\big|_{\partial\Omega_{\eta_0}}\big)\Big) \\
&= \sum_{n,m,\nu,\mu} \frac{a_{n,m}^{\nu,\mu}}{q_{n,m}} \Lambda\big(I_{n,m}^{\nu,\mu}\big|_{\partial\Omega_{\eta_0}}\big) = \sum_{n,m,\nu,\mu} \frac{a_{n,m}^{\nu,\mu}}{q_{n,m}} \operatorname{nor} I_{n,m}^{\nu,\mu},
\end{aligned}
$$

with the last sum converging in the dual space $H^{-1/2}(\partial\Omega_{\eta_0})$ of $H^{1/2}(\partial\Omega_{\eta_0})$. It is also valid under the assumption that the sum in (3.11) and the sums of the partial derivatives of the terms converge uniformly on compact subsets of Ω_{η_0}. □

We will complement (3.7) with the further abbreviations

$$
t_0 = \cosh \eta_0, \;\; s_0 = \sinh \eta_0. \tag{3.13}
$$

Definition 3.6. The *toroidal Neumann constants*, $\rho_{n,m} = \rho_{n,m}(\eta_0)$, $\sigma_{n,m} = \sigma_{n,m}(\eta_0)$, and $\tau_{n,m} = \tau_{n,m}(\eta_0)$ associated to Ω_{η_0} are defined as follows:

$$
\begin{aligned}
\rho_{1,m} &= \frac{1}{2s_0}\Big(t_0 + (2m-1)\frac{q_{1,m}}{q_{0,m}}\Big), \\
\rho_{n,m} &= \frac{1}{4s_0}\Big((2n-1)t_0 + (2(m-n)+1)\frac{q_{n,m}}{q_{n-1,m}}\Big) \;\; (n \geq 2), \qquad (3.14) \\
\sigma_{n,m} &= \frac{-1}{2s_0}\Big((2nt_0^2+1) + (2(m-n)-1)t_0\frac{q_{n+1,m}}{q_{n,m}}\Big) \;\; (n \geq 0), \\
\tau_{n,m} &= \frac{1}{4s_0}\Big((2n+3)t_0 + (2(m-n)-3)\frac{q_{n+2,m}}{q_{n+1,m}}\Big) \;\; (n \geq 0), \qquad (3.15)
\end{aligned}
$$

for all $m \geq 0$.

Proposition 3.7. *For fixed m and η_0,*

$$\lim_{n\to\infty} \frac{\rho_{n,m}}{n} = \frac{1}{2},$$
$$\lim_{n\to\infty} \frac{\sigma_{n,m}}{n} = -t_0,$$
$$\lim_{n\to\infty} \frac{\tau_{n,m}}{n} = \frac{1}{2}.$$

Proof. By Proposition 1.4, we calculate that

$$\begin{aligned}\lim_{n\to\infty} \frac{\rho_{n,m}}{n} &= \lim_{n\to\infty} \frac{1}{4ns_0}\left((2n-1)t_0 + (2(m-n)+1)\frac{q_{n,m}}{q_{n-1,m}}\right)\\ &= \frac{1}{2s_0}(t_0 - e^{\eta_0}) = \frac{1}{2}.\end{aligned}$$

Further,

$$\begin{aligned}\lim_{n\to\infty} \frac{\sigma_{n,m}}{n} &= \lim_{n\to\infty} \frac{-1}{2n\, s_0}\left((2n\, t_0^2+1) + (2(m-n)-1)t_0\frac{q_{n+1,m}}{q_n}\right)\\ &= \frac{-t_0}{2s_0}(t_0 - e^{-\eta_0}) = -t_0.\end{aligned}$$

Finally,

$$\lim_{n\to\infty} \frac{\tau_{n,m}}{n} = \lim_{n\to\infty} \frac{1}{2\, s_0}(t_0 - e^{-\eta_0}) = \frac{1}{2}. \qquad \square$$

Proposition 3.8. *The values $\tau_{n,m}$ and $\rho_{n,m}$ are never zero.*

Proof. First we consider the statement for $\tau_{n,m}$. In [46, p. 195] an alternative definition of $Q_n^m(t)$, consistent with Definitions 1.1 and 1.2, is given for all $n, m \in \mathbb{C}$ as follows:

$$Q_n^m(t) = \frac{(-1)^m}{2^{n+1}} \frac{(n+m)!}{n!} (t^2-1)^{m/2} \int_{-1}^{1} \frac{(1-s^2)^n}{(t-s)^{n+m+1}}\, ds. \tag{3.16}$$

From this formula, it follows that

$$(-1)^m Q_n^m(\cosh \eta) > 0 \tag{3.17}$$

for all $n, m, \eta \in \mathbb{R}^+$, and in particular when n is half an odd integer. According to Definition 3.6, we need to show that the value

$$16 s_0\, q_{n+1,m}\, \tau_{n,m} = 4\big((2n+3)t_0\, q_{n+1,m} + \big(2(n-m)-3\big)q_{n+2,m}\big) \tag{3.18}$$

does not vanish. Recall the recursion formula (1.12) which we can write as

$$2(n+1)\, t_0\, q_{n+1,m} = (n-m+\frac{3}{2})q_{n+2,m} + (n+m+\frac{1}{2})q_{n,m}. \tag{3.19}$$

Now add and subtract $(2n+3)(2(m+n)+1)q_{n+2,m}$ and use the recursion formula as (1.14)

$$2(n+1)\, s_0\, q_{n+1,m-1} = q_{n,m} - q_{n+2,m}, \tag{3.20}$$

Finally,

$$16\, s_0\, q_{n+1,m}\, \tau_{n,m} = -2\, s_0\, (2n+3)(2n+2m+1)q_{n+1,m-1} + \frac{8\,(m+2)\,n}{(n+1)}\, q_{n+2,m},$$

which by Proposition (1.3) for all $n, m \geq 0$ is never zero. the proof for $\rho_{n,m}$ is similar, only we add and subtract $(2n-1)(n+m-\frac{3}{2})q_{n,m}$ in the step after (3.19). □

Proposition 3.9. *The following expressions vanish identically:*

$$\sigma_{0,0}\, q_{0,0} + 2\,\tau_{0,0}\, q_{1,0} = 0, \qquad (i)$$

$$\rho_{1,0}\, q_{0,0} + 2(\sigma_{1,0}\, q_{1,0} + \tau_{1,0}\, q_{2,0}) = 0, \qquad (ii)$$

while for $n \geq 2$,

$$\rho_{n,0}\, q_{n-1,0} + \sigma_{n,0}\, q_{n,0} + \tau_{n,0}\, q_{n+1,0} = 0. \qquad (iii)$$

Proof. Via the recursion formula (3.19) as well as (1.16), written as

$$(n - m - \frac{1}{2}) t_0 q_{n,m} = s_0\, q_{n,m+1} + (n + m - \frac{1}{2})\, q_{n-1,m}, \tag{3.21}$$

direct computations show that

$$\begin{aligned}\sigma_{0,0}\, q_{0,0} + 2\,\tau_{0,0}\, q_{1,0} &= \frac{-1}{2s_0}\big(q_{0,0} - t_0 q_{1,0}\big) + \frac{3}{2s_0}\big(t_0 q_{1,0} - q_{2,0}\big) \\ &= \frac{3}{8s_0}\big(q_{2,0} - q_{0,0}\big) - \frac{3}{8s_0}\big(q_{2,0} - q_{0,0}\big) = 0,\end{aligned}$$

which is (i). Similarly, we find

$$\begin{aligned}&\rho_{1,0}\, q_{0,0} + 2(\sigma_{1,0}\, q_{1,0} + \tau_{1,0}\, q_{2,0}) \\ &= \frac{1}{4s_0}\big(t_0\, q_{0,0} - q_{1,0}\big) + \frac{1}{s_0}\big(- (2t_0^2 + 1)\, q_{1,0} + \frac{11}{2}\, t_0\, q_{2,0} - \frac{5}{2}\, q_{3,0}\big) \\ &= -\frac{1}{2}\, q_{0,0} + \frac{1}{s_0}\big(\frac{1}{2} q_{1,0} + \frac{1}{2}\, s_0\, q_{0,1} - \frac{1}{2}\, q_{-1,0}\big) \\ &= 0,\end{aligned}$$

which is (ii). Finally,

$$\begin{aligned}
\rho_{n,0}\, q_{n-1,0} + \sigma_{n,0}\, q_{n,0} + \tau_{n,0}\, q_{n+1,0} = \\
& (2n-1)t_0 q_{n-1,0} + (-2n+1-4nt_0^2-2)q_{n,0} \\
& + (6n+5)t_0 q_{n+1,0} - (2n+3)q_{n+2,0} \\
& = 0
\end{aligned}$$

which shows that (iii) is valid. □

We have

Theorem 3.10. *For a fixed η_0, let $f\colon \partial\Omega_{\eta_0} \to \mathbb{R}$ given by* (3.9) *and suppose that the Fourier coefficients satisfy* (1.50) *where $r \geq 4$. for some constant C. Define, for $\mu = \pm 1$ and all $m \geq 0$,*

$$\begin{aligned}
b_{0,m}^{+,\mu} &= \sigma_{0,m}\, a_{0,m}^{+,\mu} + \tau_{0,m} a_{1,m}^{+,\mu}, \\
b_{n,m}^{+,\mu} &= \rho_{n,m}\, a_{n-1,m}^{+,\mu} + \sigma_{n,m}\, a_{n,m}^{+,\mu} + \tau_{n,m}\, a_{n+1,m}^{+,\mu} \quad (n \geq 1), \\
b_{1,m}^{-,\mu} &= \sigma_{1,m}\, a_{1,m}^{-,\mu} + \tau_{1,m}\, a_{2,m}^{-,\mu}, \\
b_{n,m}^{-,\mu} &= \rho_{n,m}\, a_{n-1,m}^{-,\mu} + \sigma_{n,m}\, a_{n,m}^{-,\mu} + \tau_{n,m}\, a_{n+1,m}^{-,\mu} \quad (n \geq 2).
\end{aligned} \tag{3.22}$$

Then the Dirichlet-to-Neumann mapping $h = \Lambda f$ is given by the formula

$$h(\theta, \varphi) = \sqrt{\cosh \eta_0 - \cos\theta} \sum_{n,m,\nu,\mu} b_{n,m}^{\nu,\mu}\, \Phi_n^{\nu}(\theta) \Phi_m^{\mu}(\varphi), \tag{3.23}$$

which converges absolutely, and h is of class $r-3$.

Proof. From (3.8) it follows that

$$\left|I_{n,m}^{\nu,\mu}(\eta_0,\theta,\varphi)\right| \le q_{n,m}\sqrt{t_0+1},$$

so the expansion (3.9) converges absolutely. Similarly, one verifies from Theorem 3.4 and (1.8) that

$$\left|\operatorname{nor} I_{n,m}^{\nu,\mu}\right| \le C_1(n+m+1)$$

for some constant C_1 (which depends on η_0), and hence by (1.50)

$$\left|a_{n,m}^{\nu,\mu}\operatorname{nor} I_{n,m}^{\nu,\mu}\right| \le \frac{CC_1}{(n+m+1)^{r-1}}.$$

This is enough to guarantee that (3.12), a double series in θ and φ, also converges absolutely. (Recall that the convergence exponents for double series are different from simple series, in fact $\sum 1/(m+n+1)^2 = \infty$.) This in turn permits us to substitute the formula for $\operatorname{nor} I_{n,m}^{\nu,\mu}$ into (3.12) and then reindex $\Phi_{n-1}^{\nu}(\theta)$ and $\Phi_{n+1}^{\nu}(\theta)$ into $\Phi_n^{\nu}(\theta)$ to obtain after some straightforward calculations that

$$\begin{aligned} h(\theta,\varphi) = \sqrt{t_0-\cos\theta}\sum_{m,n,\nu,\mu}\Phi_n^{\nu}(\theta)\Phi_m^{\mu}(\varphi)\big(&\rho_{n,m}a_{n-1,m}^{\nu,\mu} \\ &+\sigma_{n,m}a_{n,m}^{\nu,\mu}+\tau_{n,m}a_{n+1,m}^{\nu,\mu}\big). \end{aligned} \tag{3.24}$$

This is statement (3.23). The Fourier coefficients $b_{n,m}^{\nu,\mu}$ of h in (3.24) are of order no greater than $1/(m+n+1)^{r-1}$, so by Proposition (1.31) the series

(3.23) converges absolutely. $\square$

Lemma 3.11. *(a) The compatibility condition* (3.5) *applied to a boundary function h of the form* (3.23) *is equivalent to*

$$\sum_{n=0}^{\infty} \varepsilon_n^2 \, q_{n,1} \, b_{n,0}^{+,+} = 0. \tag{3.25}$$

(b) The normalization condition (3.6) *applied to f of the form* (3.9) *is equivalent to*

$$\sum_{n=0}^{\infty} \varepsilon_n^2 \, q_{n,1} \, a_{n,0}^{+,+} = -\frac{c}{4\pi\sqrt{2}}, \tag{3.26}$$

where ε_n has the same meaning as in Proposition 1.9.

Proof. (a) Since $\int_0^{2\pi} \Phi_n^-(\theta)\, d\theta = 0$, while

$$\int_0^{2\pi} \Phi_0^+(\varphi)\, d\varphi = 2\pi,$$
$$\int_0^{2\pi} \Phi_m^+(\varphi)\, d\varphi = 0$$

for $m \geq 1$, the surface integral of h is

$$\begin{aligned}
&\iint_{\partial\Omega_{\eta_0}} h(\theta,\varphi)\, dS \\
&= \int_0^{2\pi}\int_0^{2\pi} h(\theta,\varphi)\, \frac{s_0}{(t_0-\cos\theta)^2}\, d\theta d\varphi \\
&= s_0 \sum_m \sum_n b_{n,m}^{\nu,\mu} \int_0^{2\pi} (t_0-\cos\theta)^{-3/2} \Phi_n^{\nu}(\theta)\, d\theta \int_0^{2\pi} \Phi_m^{\mu}(\varphi)\, d\varphi \\
&= -4\sqrt{2}\,\pi \sum_{n=0}^{\infty} \varepsilon_n^2 b_{n,0}^{+,+} Q_{n-\frac{1}{2}}^1(\cosh\eta_0),
\end{aligned}$$

with the last equality following from Proposition (1.9) with $\alpha = 3/2$. The proof of (b) follows the same lines as (a). $\square$

3.2 Solution of the Neumann problem on the torus

To solve the Neumann problem on Ω_{η_0} is effectively to solve the system (3.22). In this section, we will show that an algebraic solution for the coefficients $a_{n,m}^{\nu,\mu}$ will not always converge, and we will give a method for finding the solution numerically.

3.2.1 Algebraic solutions for the Neumann coefficients

Definition 3.12. We will say that a collection of real numbers $\{a_{n,m}^{\nu,\mu}\}$ is an *algebraic solution* of the Neumann problem posed by $\{b_{n,m}^{\nu,\mu}\}$ when all of the equations (3.22) are satisfied.

An algebraic solution will give rise to a solution of the Neumann problem if it defines a convergent series (3.9).

The upper triangular system (3.22) can be solved almost trivially by back-substitution. It separates naturally into a separate system for each fixed set of parameters (m, ν, μ), which are called the *modes* of the problem. For

$\nu = +1$, the solution is

$$\begin{aligned} a_{1,m}^{+,\mu} &= \frac{1}{\tau_{0,m}}(b_{0,m}^{+,\mu} - \sigma_{0,m}\, a_{0,m}^{+,\mu}), \\ a_{n+1,m}^{+,\mu} &= \frac{1}{\tau_{n,m}}(b_{n,m}^{+,\mu} - \rho_{n,m}\, a_{n-1,m}^{+,\mu} - \sigma_{n,m}\, a_{n,m}^{+,\mu}); \end{aligned}$$

while for $\nu = -1$, the relations are

$$\begin{aligned} a_{2,m}^{-,\mu} &= \frac{1}{\tau_{1,m}}(b_{1,m}^{-,\mu} - \sigma_{1,m}\, a_{1,m}^{-,\mu}), \\ a_{n+1,m}^{-,\mu} &= \frac{1}{\tau_{n,m}}(b_{n,m}^{-,\mu} - \rho_{n,m}\, a_{n-1,m}^{-,\mu} - \sigma_{n,m}\, a_{n,m}^{-,\mu}). \end{aligned}$$

Proposition 3.8 guarantees that these formulas are meaningful. Thus algebraic solutions always exist, and the values $a_{0,m}^{+,\mu}$ or $a_{1,m}^{-,\mu}$, respectively, are the only free parameters.

3.2.2 Convergent solutions for the toroidal Neumann problem

In the proof of Proposition 3.5, we wrote down the formula for the harmonic function (3.11) directly from the Dirichlet data. This says that the Dirichlet problem is effectively trivial when data is expressed in terms of the basis of toroidal harmonics. This is different for the Neumann problem, and we will find that some perhaps surprising particularities occur in connection with the convergence properties of algebraic solutions.

The question is when will the algebraic solution be determined by choice of the initial coefficients $a_{0,m}^{+,\mu}$ and $a_{1,m}^{-,\mu}$ to provide a convergent series in (3.9). That is to say, one wants to know when the function

$$f_m^{\nu,\mu} = \sum_{n=0}^{\infty} \frac{a_{n,m}^{\nu,\mu}}{q_{n,m}} I_{n,m}^{\nu,\mu} \tag{3.27}$$

corresponding to the mode (m, ν, μ) will exist. The principal mode $(0, +1, +1)$ will be considered separately from the remaining modes.

3.2.3 Convergence of the algebraic solution for the principal mode

For the particular indices $(m, \nu, \mu) = (0, +1, +1)$, the Neumann constants satisfy some special relations which we will need. When $b_{n,0}^{+,+} = 0$ for all n (i.e., when h vanishes identically), the corresponding equations (3.22) are linear homogeneous, and Proposition 3.9 implies that

$$\frac{a_{n,0}^{+,+}}{q_{n,0}} = 2\frac{a_{0,0}^{+,+}}{q_{0,0}} \quad (n \geq 1); \tag{3.28}$$

i.e., $a_{n,0}^{+,+} = \varepsilon_n a_{0,0}^{+,+}$ for all n, where as usual $\varepsilon_n = 1 + \delta_{n,0}$. On the other hand, the formula of Proposition 1.9 with the exponent determined by $\alpha = 1/2$ gives

$$1 = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \varepsilon_n I_{n,0}^{+,+}. \tag{3.29}$$

Therefore the solution

$$f_0^{+,+}(\theta, \varphi) = \frac{a_{0,0}^{+,+}}{q_{0,0}} \sum_{n=0}^{\infty} \varepsilon_n q_{n,0} \cos n\theta \sqrt{t_0 - \cos\theta}$$

is indeed equal to a constant function on $\partial\Omega_{\eta_0}$ (as must be the case according to Proposition 3.3) with value $(\pi/\sqrt{2})(a_{0,0}^{+,+}/q_{0,0})$. The solution of the Dirichlet problem in Ω_{η_0} is the same constant. Thus, *given any* $a_{0,0}^{+,+} \in \mathbb{R}$, *the algebraic solution gives a convergent series* $\sum_{n=0}^{\infty} \frac{a_{n,0}^{+,+}}{q_{n,0}} I_{n,0}^{+,+}(\eta_0, \theta, \varphi)$.

These considerations also lead to the following result which is of independent interest.

Proposition 3.13. *The area of* $\partial\Omega_{\eta_0}$ *is equal to*

$$\alpha(\eta_0) = -8 \sum_{n=0}^{\infty} \varepsilon_n^3 q_{n,0} q_{n,1}.$$

Proof. Take $a_{0,0}^{+,+} = (\sqrt{2}/\pi) q_{0,0}$, which gives $f_0^{+,+} = 1$ identically. Then apply (3.26) to evaluate $\alpha(\eta_0) = \int_{\partial\Omega_{\eta_0}} f_0^{+,+} \, dS$. □

These considerations also give us a formula for normalizing any given solution of the toroidal Neumann problem:

Proposition 3.14. *Let* f *be a particular solution of* $\Lambda f = h$ *and set* $c_1 = \int_{\partial\Omega_{\eta_0}} f \, dS$. *Let* $\hat{f}$ *be obtained by replacing the coefficients* $a_{n,0}^{+,+}$ *for* f *with*

$$\hat{a}_{n,0}^{+,+} = a_{n,0}^{+,+} + \varepsilon_n \frac{\sqrt{2}}{\pi} \frac{q_{n,1}}{\alpha(\eta_0)} (c - c_1).$$

Then $\hat{f}$ is the unique solution of the Neumann problem which satisfies the normalization condition (3.26)*.*

3.2.4 Determination of parameter for convergence for non-principal modes

We assume now that $(m, \nu, \mu) \neq (0, +1, +1)$. The essence of the matter is that according to Proposition 3.3, there is exactly one algebraic solution of (3.22) for which (3.9) converges. Every value of $a_{0,m}^{+,\mu}$ (or $a_{1,m}^{-,\mu}$) defines an algebraic solution. The determination of the corresponding unique value a_{opt} of $a_{0,m}^{+,\mu}$ or $a_{1,m}^{-,\mu}$ is not an algebraic question; in fact, is natural to believe that a_{opt} is not the solution of any algebraic or transcendental equation associated with the Neumann data. In the following discussion, we will assume that $\mu = +1$ since the case $\mu = -1$ is analogous, the only difference being the start of the indexing from $n = 1$ instead of $n = 0$.
To simplify the notation, we will write $a_n = a_{n,m}^{\nu,\mu}$. Let $A_n(a)$ denote the value of a_n in the solution of equations (3.22) determined by setting the arbitrary parameter $a_0 = a_{0,m}^{+,\mu}$ (or $a_{1,m}^{-,\mu}$) equal to a.

Proposition 3.15. *The values $A_n(a)$ are given by*

$$A_n(a) = C_n a + D_n \ (n \geq 0), \tag{3.30}$$

where the coefficients C_n, D_n are defined recursively by

$$\begin{aligned}
C_0 &= 1, & D_0 &= 0, \\
C_1 &= \frac{-\sigma_0}{\tau_0}, & D_1 &= \frac{b_0}{\tau_0}, \\
C_{n+1} &= \frac{-1}{\tau_n}(\rho_n C_{n-1} + \sigma_n C_n), & D_{n+1} &= \frac{-1}{\tau_n}(\rho_n D_{n-1} + \sigma_n D_n - b_n) \ \ (n \geq 1).
\end{aligned} \tag{3.31}$$

In particular, $A_0(a) = a$.

Proof. Since by definition $A_0(a)$ is the value of a_0 determined by setting $a_0 = a$, we have $A_0(a) = a$. This gives the values of C_0, D_0. The remaining values follow by induction because of equations (3.15). □

By construction, the collection $\{A_n(a)\}$ is an algebraic solution of the system (3.22), whatever the value of a may be. According to (3.9) and Theorem 3.10, we want to find the unique value $a_{\text{opt}} \in \mathbb{R}$ provided by Theorem 3.16 for which the series

$$\sum_{n=0}^{\infty} A_n(a_{\text{opt}})\, \Phi_n^{\nu}(\theta)\Phi_m^{\mu}(\varphi) \tag{3.32}$$

converges absolutely and thus gives $f_m^{\nu,\mu}(\theta, \varphi)$. Since the summands of any convergent series tend to zero, it is necessary that $A_n(a_{\text{opt}}) \to 0$ as $n \to \infty$. By (3.30), this says $C_n a_{\text{opt}} + D_n \to 0$.

Note that the C_n's do not depend on the data $\{b_n\}$. It is clear that two consecutive terms C_n, C_{n+1} can never vanish, because otherwise by Proposition 3.8 all of the preceding C_n would also vanish, contradicting $C_0 = 1$. Under the assumption that $C_n > \epsilon > 0$ for infinitely many n, we have

$$-\frac{D_n}{C_n} \to a_{\mathrm{opt}} \tag{3.33}$$

as $n \to \infty$ on that subsequence. We will look further into this question in a moment.

We summarize our conclusions as follows.

Theorem 3.16. *Let the coefficients* $\{b^{\nu,\mu}_{n,m}\}$ *be such that the series* (3.23) *converges absolutely, defining* $h\colon \partial\Omega_{\eta_0} \to \mathbb{R}$*. Assume that* $b^{+,+}_{n,0}$ *satisfy* (3.25)*, so* h *satisfies the compatibility condition* (3.5)*. Suppose further that the continuous solution* $f\colon \partial\Omega_{\eta_0} \to \mathbb{R}$ *of* $\Lambda f = h$ *specified in Proposition 3.3 has a double Fourier series which converges absolutely. Then (i) for every value of* $a^{+,+}_{0,0} \in \mathbb{R}$*, the resulting algebraic solution for the sequence* $\{a^{+,+}_{n,0}\}$ *produces an absolutely convergent series*

$$\sum_n \frac{a^{+,+}_{n,0}}{q_{n,0}} I^{+,+}_{n,0}(\eta_0, \theta, \phi)$$

whose value is $f^{+,+}_0$ *plus a constant. Further, (ii) for* (m, ν, μ) *different from* $(0, +1, +1)$*, there exists a unique value of* $a^{\nu,\mu}_{0,m}$ *(when* $\nu = 1$*) or* $a^{\nu,\mu}_{1,m}$ *(when*

$\nu = -1$*) for which the resulting algebraic solution gives a convergent series*

$$\sum_n \frac{a^{\nu,\mu}_{n,m}}{q_{n,m}} I^{\nu,\mu}_{n,m}(\eta_0, \theta, \phi).$$

The sum of this series is the mode $f^{\nu,\mu}_m$ *given by equation* (3.27)*.*

The contrast of the behavior of the principal mode in this theorem is striking since equations (3.22) show no apparent structural difference between the principal and other modes. The algorithm is essentially as follows:

Algorithm 1. *(Solution to the Neumann problem) Given the coefficients* $\{b^{\nu,\mu}_{n,m}\}$*, calculate the coefficients* C_n, D_n*,* $0 \leq n \leq N$*, by* (3.31) *which give the approximation* $a_{opt} = a_{opt}(m, \nu, \mu) = -D_N/C_N$ *and then also* $\{a^{\nu,\mu}_{n,m}\}$ *by* (3.30) *and* (3.33) *.*

3.3 Numerical results

3.3.1 Calculation of $q_{n,m}$

From (1.9) it is clear that $q_{n,m}$ decreases exponentially as $n \to \infty$. In fact, for many values of η_0 in the range we will be considering, $|q_{100,0}|$ is less than 10^{-300} and thus effectively becomes zero when represented in standard IEEE double-precision format [47]. For many problems, values of n as large as 100 would not be necessary, but if one wants to use a large number of terms in a series solution, two ways out of this difficulty present themselves.

The first way is via asymptotic expansions. For large values of n we can estimate $q_{n,m}$ to high accuracy by means of (1.9) and Stirling's formula for the Γ function. However, it turns out there is a gap between small values of n where machine precision is applicable and the large values of n for which Stirling's estimate is sufficiently accurate. For example, for $\eta_0 = 1.5$ and $m = 2$, we find in machine precision that $q_{131,2}$ vanishes, while $q_{130,2}$ suffers a relative error of 0.0006. The estimate based on (1.9) gives a relative error of 0.003. The formula (1.9) is in fact the first term of an asymptotic expansion for Q_n^m given in [46], and can be made much more accurate by multiplying it by the factor

$$1 - \frac{1}{8n} - \frac{m^2}{n} + \frac{4m^2 - 1}{n} \frac{1}{1 - e^{-2\eta_0}}. \tag{3.34}$$

Note that by (3.15), we only need to know the ratios $q_{n+1,m}/q_{n,m}$. After cancelling several factors, one derives an approximation which can be substituted into (3.15) to find, for example, that $\tau_{136,2}$ vanishes in machine precision, while the approximations of $\tau_{n,2}$ already for $n \geq 20$ have relative error less than 0.00001. Similar statements hold for the coefficients $\rho_{n,m}$ and $\sigma_{n,m}$. The optimal cutoff value of n at which to switch from a direct calculation to the asymptotic formula would depend on the values of m

and η_0 as well as a consideration of the acceptable error in the coefficients. It is likely that other more convenient asymptotic expressions could be developed but this would lead us rather far afield.

Multiple-precision arithmetic was used in *Mathematica* to verify the correctness of the above statements. Multiple precision is also a convenient second alternative for calculating all of the Neumann constants when one is planning to solve many Neumann problems on the same torus: once the coefficients have been obtained and rounded to machine precision, the computation will then easily move forward without further recourse to higher precision arithmetic.

3.3.2 Numerical behavior of C_n

We observed that a_{opt} is given by (3.33) unless it happens that $C_n \to 0$. (Recall that C_n does not depend on the Neumann data $\{b_{n,m}^{\nu,\mu}\}$.) For small values of n, we have a priori little control over even the sign of the coefficients defined in (3.15). However, from (3.7), $\rho_{n,m}/\tau_{n,m} \to 1$ and $\sigma_{n,m}/\tau_{n,m} \to -2t_0$. Therefore if for a single large n we have

$$C_n \approx e^{\eta_0} C_{n-1},$$

Table 3.1: Sample values of C_n for $m = 1$, $(\nu, \mu) = (+, +)$. 200-digit precision was used to avoid underflow in the calculations.

	$\eta = 0.1$	$\eta = 0.3$	$\eta = 0.5$	$\eta = 1.$	$\eta = 1.5$	$\eta = 2.$
C_0	1.	1.	1.	1.	1.	1.
C_1	1.943	1.697	1.420	0.866	0.523	0.316
C_2	1.852	1.418	1.098	0.720	0.599	0.557
C_3	1.752	1.229	0.985	0.996	1.444	2.294
C_4	1.654	1.120	1.016	1.761	4.302	11.303
C_5	1.562	1.076	1.171	3.476	14.040	60.780
C_6	1.48	1.085	1.458	7.286	48.463	345.631
C_7	1.407	1.144	1.911	15.885	173.925	2043.827
C_8	1.344	1.249	2.595	35.635	642.402	12440.253
C_9	1.290	1.404	3.611	81.710	2425.889	77424.156
C_{10}	1.246	1.616	5.120	190.646	9323.354	490447.458
C_{15}	1.139	4.005	34.827	15650.902	9.298×10^6	5.953×10^9
C_{20}	1.189	11.881	279.376	1.521×10^6	1.099×10^{10}	8.571×10^{13}
C_{30}	1.722	132.742	22884.183	1.839×10^{10}	1.969×10^{16}	2.278×10^{22}
C_{40}	3.068	1751.093	2.221×10^6	2.641×10^{14}	4.196×10^{22}	7.201×10^{30}
C_{50}	6.053	25335.250	2.369×10^8	4.173×10^{18}	9.835×10^{28}	2.505×10^{39}

then by (3.31) it would follow that

$$C_{n+1} \approx -C_{n-1} + 2t_0 C_n = -e^{-\eta_0} C_n + 2t_0 C_n = e^{\eta_0} C_n;$$

i.e., the sequence $\{C_n\}$ grows exponentially. Table 3.1 lists some calculated values of C_n corresponding to $m = 1$ and a range of values of η_0. Other values of m are shown in Table 3.2. Even though the initial values can decrease, in all cases that we have examined it appears that $C_n \to \infty$ exponentially as $n \to \infty$, and we conjecture that this always holds. Then Algorithm 1 is applicable.

Table 3.2: Sample values of C_n for $\eta = 0.4$, $(\nu, \mu) = (+, +)$.

	$m = 2$	$m = 3$	$m = 4$	$m = 5$
C_0	1.	1.	1.	1.
C_1	1.333	2.112	2.711	3.119
C_2	2.125	4.111	6.103	7.835
C_3	5.013	10.308	16.235	22.035
C_4	13.765	28.988	46.956	65.655
C_{10}	16716.741	36306.992	61217.891	89729.231
C_{15}	1.014×10^7	2.211×10^7	3.749×10^7	5.533×10^7
C_{20}	7.279×10^9	1.590×10^{10}	2.701×10^{10}	$3.996 times 10^{10}$
C_{30}	4.801×10^{15}	1.050×10^{16}	1.786×10^{16}	2.648×10^{16}
C_{40}	3.764×10^{21}	8.233×10^{21}	1.402×10^{22}	2.080×10^{22}
C_{50}	3.246×10^{27}	7.102×10^{27}	1.209×10^{28}	1.795×10^{28}

3.3.3 Examples of Dirichlet-to-Neumann and Neumann problem calculations

We illustrate the solution of the Neumann problem with numerical examples. For the principal mode $(m, \nu, \mu) = (0, +1, +1)$ there is not much to be said. Every choice of $a_0 = a_{0,0}^{++}$ gives an algebraic solution which defines a series which solves the Neumann problem. Proposition 3.14 may be used for normalization. It may be worth noting that when $b_n = 0$ for all n (i.e., the coefficients for a vanishing normal derivative in the mode under consideration), since the sum is a constant, one can dispense with the series altogether for this case. But if one does wish to follow the procedure, the formulas (3.31) give $D_n = 0$ always, and by (3.30), we have $A_n(a) = aC_n$. Choosing $a_0 = 1$ without any loss of generality and fixing η_0, one obtains the values C_n by (3.31) and then the initial coefficients a_n by (3.30). This calculation amounts simply to finding values of the associated Legendre functions of the second kind via the classical recursion formulas, and the only numerical error is that which accumulates due to roundoff.

The numerical examples were calculated in *Mathematica* on a household laptop computer. As noted, all calculations were in machine precision except for a few coefficients. The calculations took a fraction of a second apart from the rendering of the graphics.

Example 3.17. To illustrate the calculation of the Dirichlet to Neumann mapping, consider $a^{+,+}_{n,m} = a_n$ where $a_n = (-1)^{(n-1)/2} n^{-2}$ for n odd, and $a_n = 0$ for n even, $0 \leq n \leq 50$. Figure 3.1 shows the function f and the image $h = \Lambda f$ for a few values of m. While f is smooth, it does not satisfy the regularity condition of Theorem 3.10, and h has jump discontinuities. Except at the jumps, the values of h agree to within 10^{-14} to 10^{-15} with the normal derivative of the f series obtained by numerical derivation.

As another example, let $a_n = (-1)^{(n-1)/2} n^{-1}$ for n odd instead. As Figure 3.2 shows, now f has a jump discontinuity while the series for the resulting h does not converge, further illustrating the tendency of the Dirichlet to Neumann mapping to reduce the degree of differentiability.

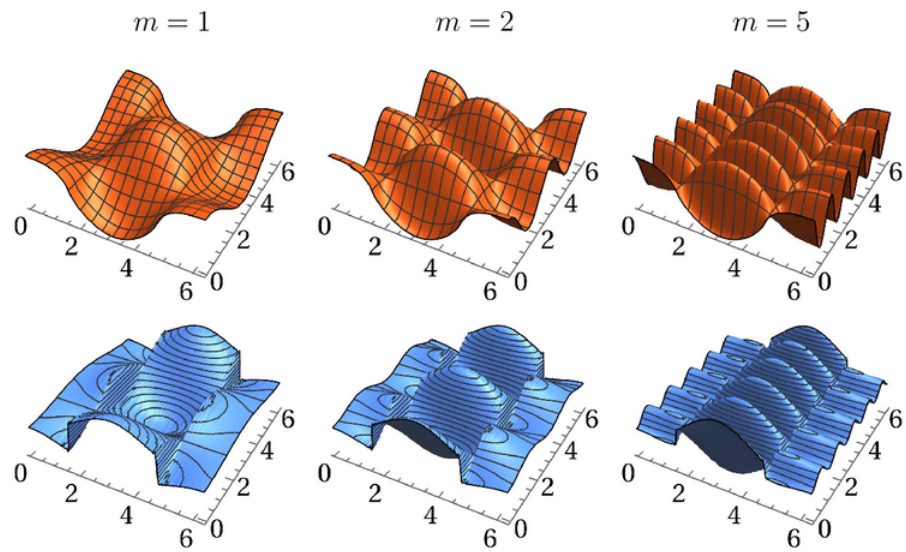


Figure 3.1: Given function $f(\theta, \varphi)$ determined by $a_n = (-1)^{(n-1)/2} n^{-2}$ for n odd (above) and calculated D-to-N mapping $h(\theta, \varphi)$ (below). $\eta_0 = 1.5$. All calculations carried out with machine precision.

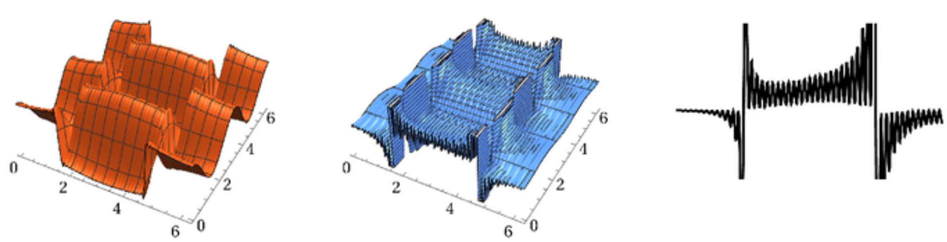


Figure 3.2: Solution determined by $a_n = (-1)^{(n-1)/2} n^{-1}$ for n odd, $m = 2$, $\eta_0 = 1.5$. The profile of h for $\varphi = 0$ (right) results from an attempt to graph a nonconvergent series.

Example 3.18. Let

$$u = \left(\frac{\sinh \eta}{\cosh \eta - \cos \theta} \right)^m \cos m\varphi. \tag{3.35}$$

It is readily checked that u is harmonic and

$$\operatorname{nor} u = m \left(\frac{\sinh \eta_0}{\cosh \eta_0 - \cos \theta} \right)^m ((\cosh \eta_0 - \cos \theta) \coth \eta_0 + \sinh \eta_0) \cos m\varphi. \tag{3.36}$$

(Clearly one also would obtain a harmonic function with $\sin m\varphi$ in place of $\cos m\varphi$ in (3.35).) By Proposition 1.9, the coefficients in the series for u are equal to

$$a_{n,m}^{\nu,\mu} = (-1)^m \frac{\sqrt{2/\pi}}{\Gamma(m+1/2)} \varepsilon_n q_{n,m}. \tag{3.37}$$

We substitute these coefficients into (3.22) to obtain numerical values for the $b_{n,m}^{\nu,\mu}$. Then we compare truncations of the series (3.24) with the true values of $h = \operatorname{nor} u$ according to (3.36). Figure 3.3 displays the base-10 logarithm of the absolute error for different combinations of m and η_0. As is expected, the error is reduced when the number of terms in the series increases. It is also seen that the error increases steadily when larger values of m and η_0 are used.

Example 3.19. We illustrate our algorithm for solving the Neumann problem using the same function u as in the previous example. The Fourier coefficients $b_{n,m}^{\nu,\mu}$ are obtained by numerical integration. Then the auxiliary coefficients C_n, D_n are obtained recursively by (3.31), and then a_{opt} is approximated by the last value of $-C_n/D_n$ according to (3.33). One would expect that the values $a_{n,m}^{\nu,\mu} = A_n(a_{\text{opt}})$ of (3.33) provide a convergent series, while for $a \neq a_{\text{opt}}$, $\{A_n(a)\}$ would not. This is confirmed by Figure 3.4, which shows the values of $A_n(a+\epsilon)$ for small values of ϵ. The error in a particular series solution h of the Neumann problem compared to (3.36) is shown in Figure 3.5. Maximum errors for combinations of η_0, m are shown in Table 3.3.

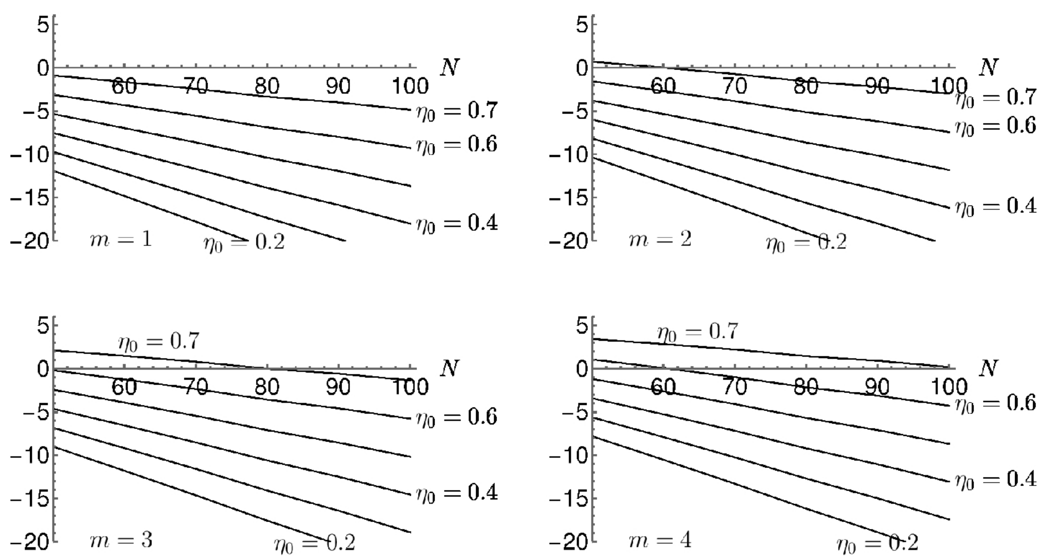


Figure 3.3: Base-10 logarithm indicating number of significant figures of approximation of the Dirichlet-to-Neumann mapping given by equations (3.22) truncating the series (3.23) to $0 \leq n \leq N$ for varying values of N. Accuracy is lost as m or η_0 increases. For comparison purposes 100-digit precision was used to obtain the Neumann constants. It was found that machine precision was sufficient for $\eta_0 > 0.5$, which is applicable for most "reasonably-shaped" toroidal domains.

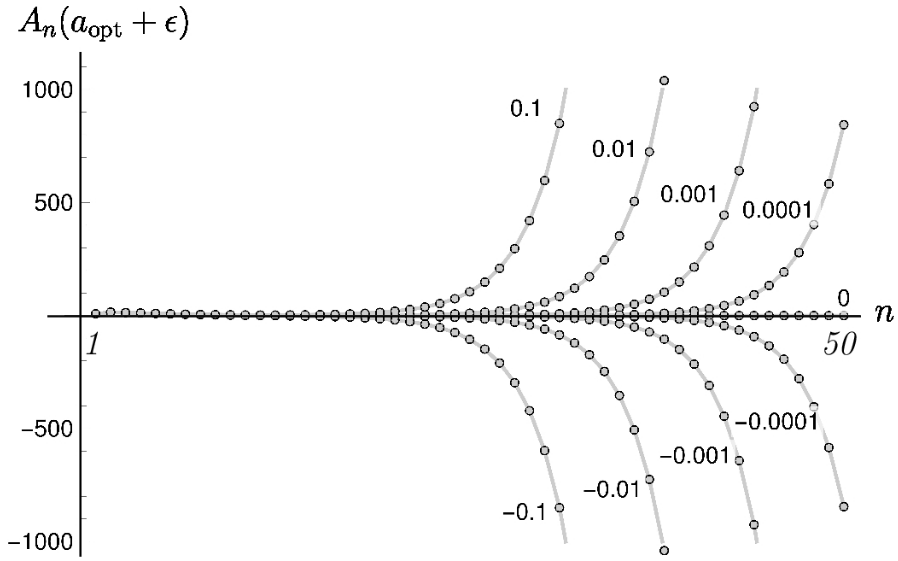


Figure 3.4: Rapid growth of the first 50 coefficients in nonconvergent algebraic solutions generated by to $a_{\mathrm{opt}} + \epsilon$, illustrated for $\eta_0 = 0.4$ and $m = 2$, with a_{opt} approximated by $-D_{50}/C_{50}$. (The graphic is truncated: for $\epsilon = .1$, the coefficients reach approximately 10^7. Even at this scale, the coefficients for $\epsilon = 0$ are virtually indistinguishable from the horizontal axis.

3.3.4 Computational complexity

In order to calculate the Dirichlet-to-Neumann mapping or solve the Neumann problem in the context of the Fourier representation, one needs the coefficients $\{a_{n,m}^{\nu,\mu}\}$ or $\{b_{n,m}^{\nu,\mu}\}$, respectively. Methods for calculating these coefficients given the values of f or h on $\partial\Omega_{\eta_0}$ are well known, depending on a choice of numerical integration procedure, can be found in many standard software packages, and will not be discussed here. In this sense, Algorithm 1 can be considered part of a larger computation.

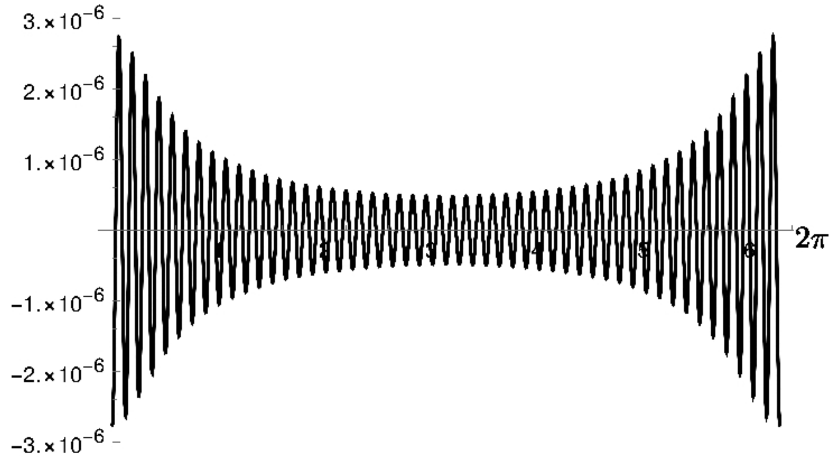


Figure 3.5: Error in solution for Neumann problem for $\eta_0 = 0.4$, $m = 2$ and 50 terms, distributed over the range $0 \leq \theta \leq 2\pi$, with $\varphi = 0$.

The amount of calculation required for Algorithm 1 is simple to estimate. For $0 \leq m \leq M$ and each choice of ν, μ, take N terms in the truncated Fourier series and use the values C_N, D_N to determine a_{opt}. Assume that the Neumann constants $\rho_{n,m}$, $\sigma_{n,m}$, $\tau_{n,m}$ have been calculated previously. This is also reasonable when one wishes to solve many Neumann problems on a single surface. Then by (3.31), approximately $4N$ multiplications in machine-precision are needed to find a_{opt}, and since we then have all of the C_n, D_n for $n < N$, by (3.30) we need N more multiplications to find the Taylor coefficients for each mode. The calculation time of the Neumann constants is also of order $O(NM)$, but one must take into account that ei-

Table 3.3: Significant figures in the numerical solution of the Neumann problem on the torus showing the increase in accuracy with the number of terms.

N	$m = 1$	$m = 2$	$m = 3$	$m = 4$
15	4.7	3.6	2.8	1.0
20	6.7	5.5	4.5	3.7
25	8.4	7.2	6.3	5.4

ther multiple precision or a slightly more complicated asymptotic formula may be necessary for a certain collection of coefficients of low index.

3.4 Exterior toroidal domain and toroidal shells

3.4.1 Exterior domain

The formula for the normal derivative of a harmonic function in the exterior $\Omega^*_{\eta_0}$ of a torus and the solution of the corresponding Neumann problem are quite analogous to that of the interior domain Ω_{η_0}, using the exterior harmonics $E^{\nu,\mu}_{n,m}$ defined by (1.41). Since the solution of the Dirichlet problem in $\Omega^*_{\eta_0}$ with boundary condition f given by (3.9) is

$$u = \sum_{n,m,\nu,\mu} \frac{a^{\nu,\mu}_{n,m}}{q_{n,m}} E^{\nu,\mu}_{n,m}(\eta,\theta,\varphi), \tag{3.38}$$

one finds that the normal derivative of f will be given by equations (3.22) when $q_{n,m}$ is replaced in (3.15) with

$$p_{n,m} = P^m_{n-1/2}(\cosh\eta_0). \tag{3.39}$$

The method we have described is then applicable with no essential changes for solving the Dirichlet-to-Neumann problem in $\Omega^*_{\eta_0}$. It is worth noting that parallel to (1.8) we have [46, p. 305] that

$$\lim_{n\to\infty} \frac{p_{n-1,m}}{p_{n,m}} = e^{\eta_0}. \tag{3.40}$$

3.4.2 Toroidal shell

Finally, we discuss how to combine our results in the interior and exterior domains for solving the Neumann problem in a toroidal shell. Let $\eta_{\text{int}} < \eta_{\text{ext}}$. Common to an interior and an exterior domain are the points of the toroidal shell

$$\Omega = \Omega_{\eta_{\text{int}},\eta_{\text{ext}}} = \Omega^*_{\eta_{\text{ext}}} \cap \Omega_{\eta_{\text{int}}}.$$

A general harmonic function u in $\Omega_{\eta_{\text{ext}},\eta_{\text{int}}}$ and continuous in the closure can be expressed via an integral of its boundary values over $\partial\Omega_{\eta_{\text{ext}},\eta_{\text{int}}}$ using the Poisson kernel for the torus [45, Ch. 1]. This integral is the difference of the integrals over $\partial\Omega_{\eta_{\text{ext}}}$ and $\partial\Omega_{\eta_{\text{int}}}$, which give a decomposition $u = u_0 + u_1$ with $u_0 \in \operatorname{Har}\Omega_{\eta_{\text{int}}}$ and $u_1 \in \operatorname{Har}\Omega^*_{\eta_{\text{ext}}}$. Consequently, we may express u as the sum of two series

$$u = \sum_{n,m,\nu,\mu} c^{\text{int}\,\nu,\mu}_{n,m}\, I^{\nu,\mu}_{n,m} + \sum_{n,m,\nu,\mu} c^{\text{ext}\,\nu,\mu}_{n,m}\, E^{\nu,\mu}_{n,m}, \tag{3.41}$$

analogous to the Laurent series for holomorphic functions in an annular domain in the complex plane, converging uniformly in proper closed subdomains. (Note, however, that the inner and outer harmonics together do not form an orthogonal system in $\Omega_{\eta_{\text{ext}},\eta_{\text{int}}}$.)

A boundary function $f\colon \partial\Omega \to \mathbb{R}$ is given collectively by its values for $\eta = \eta_{\text{int}}$ and $\eta = \eta_{\text{ext}}$ collectively, let us say by the two functions

$$\begin{aligned} f_{\text{int}}(\theta,\varphi) &= f(\eta_{\text{int}},\theta,\varphi) = \sum a_{n,m}^{\text{int}\,\nu,\mu} I_{n,m}^{\nu,\mu}[\eta_1], \\ f_{\text{ext}}(\theta,\varphi) &= f(\eta_{\text{ext}},\theta,\varphi) = \sum a_{n,m}^{\text{ext}\,\nu,\mu} E_{n,m}^{\nu,\mu}[\eta_0]. \end{aligned} \tag{3.42}$$

For u to be the solution of the Dirichlet problem for f, we combine (3.41) with (3.42) to find

$$\begin{aligned} q_{n,m}^{\text{int}} c_{n,m}^{\text{int}\,\nu,\mu} + p_{n,m}^{\text{int}} c_{n,m}^{\text{ext}\,\nu,\mu} &= a_{n,m}^{\text{int}\,\nu,\mu}, \\ q_{n,m}^{\text{ext}} c_{n,m}^{\text{int}\,\nu,\mu} + p_{n,m}^{\text{ext}} c_{n,m}^{\text{ext}\,\nu,\mu} &= a_{n,m}^{\text{ext}\,\nu,\mu}, \end{aligned} \tag{3.43}$$

where

$$\begin{aligned} q_{n,m}^{\text{int}} &= Q_{n-1/2}^{m}(\cosh\eta_{\text{int}}), \quad & q_{n,m}^{\text{ext}} &= Q_{n-1/2}^{m}(\cosh\eta_{\text{ext}}), \\ p_{n,m}^{\text{int}} &= P_{n-1/2}^{m}(\cosh\eta_{\text{int}}), \quad & p_{n,m}^{\text{ext}} &= P_{n-1/2}^{m}(\cosh\eta_{\text{ext}}). \end{aligned}$$

This can be written in matrix notation as

$$\begin{pmatrix} q^{\text{int}} & p^{\text{int}} \\ q^{\text{ext}} & p^{\text{ext}} \end{pmatrix} \begin{pmatrix} c^{\text{int}} \\ c^{\text{ext}} \end{pmatrix} = \begin{pmatrix} a^{\text{int}} \\ a^{\text{ext}} \end{pmatrix}.$$

To solve this system, one needs to verify that it is nonsingular. Instead of a direct verification as in Lemma 3.8, we simply note that if for even one combination of (n, m, ν, μ), there were more than one solution, one could easily construct a Dirichlet problem in the shell Ω with more than one solution.

We see that nor $I_{n,m}^{\nu,\mu}|_{\partial\Omega_{\text{int}}}$ is obtained from the formula of Theorem 3.4 with η_0 replaced with η_{int}, while nor $I_{n,m}^{\nu,\mu}|_{\partial\Omega_{\text{ext}}}$ is obtained by using η_{ext} instead. The boundary values nor $E_{n,m}^{\nu,\mu}|_{\partial\Omega_{\text{int}}}$ and nor $E_{n,m}^{\nu,\mu}|_{\partial\Omega_{\text{ext}}}$ are then obtained by replacing $Q_{n-1/2}^{m}$ with $P_{n-1/2}^{m}$. Once we have the harmonic function u as in (3.41), we have then

$$\begin{aligned}
\operatorname{nor} u\Big|_{\partial\Omega_{\text{int}}} &= \sum_{n,m,\nu,\mu} c_{n,m}^{\text{int}\,\nu,\mu} \operatorname{nor} I_{n,m}^{\nu,\mu}\Big|_{\partial\Omega_{\text{int}}} + \sum_{n,m,\nu,\mu} c_{n,m}^{\text{ext}\,\nu,\mu} \operatorname{nor} E_{n,m}^{\nu,\mu}\Big|_{\partial\Omega_{\text{int}}}, \\
\operatorname{nor} u\Big|_{\partial\Omega_{\text{ext}}} &= \sum_{n,m,\nu,\mu} c_{n,m}^{\text{int}\,\nu,\mu} \operatorname{nor} I_{n,m}^{\nu,\mu}\Big|_{\partial\Omega_{\text{ext}}} + \sum_{n,m,\nu,\mu} c_{n,m}^{\text{ext}\,\nu,\mu} \operatorname{nor} E_{n,m}^{\nu,\mu}\Big|_{\partial\Omega_{\text{ext}}}.
\end{aligned}$$

When the convergence of the series is absolute, one may apply the same rearranging and reindexing as described in the proof of Theorem 3.4 to obtain the coefficients in the Dirichlet-to-Neumann mapping $h = \Lambda f$,

$$\begin{aligned}
h(\eta_{\text{int}}, \theta, \varphi) &= \sqrt{\cosh \eta_0 - \cos \theta} \sum b_{n,m}^{\text{int}\,\nu,\mu} \Phi_n^{\nu}(\theta) \Phi_m^{\mu}(\varphi), \\
h(\eta_{\text{ext}}, \theta, \varphi) &= \sqrt{\cosh \eta_0 - \cos \theta} \sum b_{n,m}^{\text{ext}\,\nu,\mu} \Phi_n^{\nu}(\theta) \Phi_m^{\mu}(\varphi). \qquad (3.44)
\end{aligned}$$

As in the solution of the Neumann problem for the interior domain, the equations for a fixed value of (m, ν, μ) are independent of those for another value of these indices. They can be solved recursively. The only difference will be that one must solve a pair of equations at each step.

Part II

Second Part: Toroidal Monogenics

Chapter 4

Quaternionic analysis

In this chapter, we summarize the basic concepts and terminology concerning quaternions and quaternionic operators that will be used in the thesis. We begin by reviewing the algebraic properties of the Hamiltonian quaternions and their embedding in more general systems of Clifford numbers. Then we give the definition of the generalized Cauchy-Riemann operator as conceived by W. R. Hamilton and R. Fueter, which generalizes the classical two-dimensional Cauchy-Riemann operator to quaternionic analysis. The null solutions of this operator are called monogenic. It is possible to factor the Laplace operator in terms of the generalized Cauchy-Riemann operator and its quaternionic conjugate, similar to the complex case. This factorization allows one to generate classes of monogenic functions from classes of harmonic functions, a fact which we will apply in

Chapter 5.

The main topic of this thesis is to investigate monogenic functions defined on a torus. In this chapter, we describe properties of monogenic functions which hold in arbitrary domains in $\mathbb{R}^3$. All of the material in this chapter may be found in [14, 39, 40, 41, 42, 70, 74, 88].

4.1 The Hamiltonian quaternion algebra

The geometric properties induced on the plane by complex numbers strongly motivated Hamilton to look for a higher dimensional generalization of the complex number system. Searching in vain for a three-dimensional vector system with a proper multiplication operation, in 1843, Hamilton discovered the quaternions, usually denoted by $\mathbb{H}$. Although quaternions form a four-dimensional associative algebra over the real numbers, they are not commutative.

For the standard basis system of the Hamiltonian quaternion algebra, one often uses the notation $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. In this thesis, we prefer to use the notation $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ instead. The basis elements satisfy the following multiplication rules, where in some places we write 1 in place of $\mathbf{e}_0$:

$$\mathbf{e}_0^2 = \mathbf{e}_0 = 1; \quad \mathbf{e}_i\mathbf{e}_0 = \mathbf{e}_0\mathbf{e}_i = \mathbf{e}_i,\ i = 1, 2, 3,$$

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1, \quad \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\,\delta_{ij}, \tag{4.1}$$

with the Kronecker symbol δ_{ij} and the relation

$$\mathbf{e}_1\,\mathbf{e}_2 = \mathbf{e}_3.$$

Every element $\mathbf{a}$ of $\mathbb{H}$ is represented in the form

$$\mathbf{a} = \sum_{i=0}^{3} \mathbf{e}_i a_i,$$

where the a_i are real numbers. We will write $[\mathbf{a}]_i = a_i$. For the particular case of $[a]_0$ we denote by

$$\mathrm{Sc}(\mathbf{a}) = a_0,$$

the *scalar part* of $\mathbf{a}$, and by

$$\mathrm{Vec}(\mathbf{a}) = \sum_{i=1}^{3} \mathbf{e}_i a_i$$

its *vector part*.

Consistently with (4.1), we define the product of two quaternions $\mathbf{a} = \sum_{i=0}^{3} \mathbf{e}_i a_i$ and $\mathbf{b} = \sum_{i=0}^{3} \mathbf{e}_i b_i$ as follows:

Definition 4.1.

$$\begin{aligned}\mathbf{ab} = {} & (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + \mathbf{e}_1(a_1b_0 + a_0b_1 + a_2b_3 - a_3b_2) \\ & + \mathbf{e}_2(a_2b_0 + a_0b_2 + a_3b_1 - a_1b_3) + \mathbf{e}_3(a_3b_0 + a_0b_3 + a_1b_2 - a_2b_1).\end{aligned}$$

The *quaternionic conjugate* of a quaternion $\mathbf{a}$ is

$$\overline{\mathbf{a}} = \mathrm{Sc}(\mathbf{a}) - \mathrm{Vec}(\mathbf{a}) = a_0 - \sum_{i=1}^{3} \mathbf{e}_i a_i.$$

It is immediate that

$$\mathrm{Sc}(\mathbf{a}) = \frac{1}{2}(\mathbf{a} + \overline{\mathbf{a}})$$

and

$$\mathrm{Vec}(\mathbf{a}) = \frac{1}{2}(\mathbf{a} - \overline{\mathbf{a}}).$$

The (algebraic) *norm* of a quaternion $\mathbf{a} = \sum_{i=0}^{3} \mathbf{e}_i a_i$ is defined by

$$|\mathbf{a}| = (\mathbf{a}\overline{\mathbf{a}})^{1/2} = (\overline{\mathbf{a}}\mathbf{a})^{1/2} = \left(\sum_{i=0}^{3} a_i^2 \right)^{1/2}.$$

It follows that $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$ and that every nonzero quaternion $\mathbf{a}$ possesses an inverse defined by $\mathbf{a}^{-1} := \overline{\mathbf{a}}/|\mathbf{a}|^2$. This inverse satisfies $\mathbf{a}\mathbf{a}^{-1} = \mathbf{a}^{-1}\mathbf{a} = 1$. Furthermore, $|\mathbf{a}|^{-1} = |\mathbf{a}^{-1}|$.

The existence of inverse signifies that the quaternions form a noncommutative division algebra, the skew-field $\mathbb{H}$ of quaternions. The quaternions remain the most straightforward algebra after the real and complex numbers.

4.2 Clifford algebras

Inspired by the work of Hamilton and combining ideas of geometric algebra developed by H. Grassmann in 1878, W. K. Clifford [25] introduced the notion of what is now known as the universal Clifford algebra, which includes generalizations of the scalar and vector products to higher dimensions. A Clifford algebra is an associative but usually noncommutative algebra over the real or the complex field. For more information on the history of Clifford algebras, we refer to [57, 80].

4.2.1 Definition of Clifford algebra

We denote by $\mathcal{C}\ell_{0,n}$ $(n \in \mathbb{N}_0)$ the (universal) real Clifford algebra constructed over the orthonormal basis $\{\mathbf{i}_0, \mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_n\}$ of the Euclidean vector space $\mathbb{R}^{n+1}$, where the elements of the basis satisfy the following multiplication rules:

$$\mathbf{i}_k\mathbf{i}_l + \mathbf{i}_l\mathbf{i}_k = -2\delta_{k,l}\mathbf{i}_0,$$

$$\mathbf{i}_0\mathbf{i}_k = \mathbf{i}_k\mathbf{i}_0 = \mathbf{i}_k, \quad (k, l = 1, 2, \ldots, n),$$

where the element $\mathbf{i}_0$ is regarded as the usual unit, that is, $\mathbf{i}_0 = 1$.

A basis for $\mathcal{C}\ell_{0,n}$ is given by the elements

$$\mathbf{i}_A = \mathbf{i}_{j_1}\mathbf{i}_{j_2}\cdots\mathbf{i}_{j_k},$$

where $A = \{j_1, j_2, \ldots, j_k\} \subseteq \{1, 2, \ldots, n\}$ is such that $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. For the empty set $\emptyset$, we put $\mathbf{i}_\emptyset = \mathbf{i}_0 = 1$. It then follows that the dimension of $\mathcal{C}\ell_{0,n}$ is 2^n.

Any Clifford number $\mathbf{a} \in \mathcal{C}\ell_{0,n}$ may thus be written as

$$\mathbf{a} = \sum_{A \subseteq \{1,\ldots,n\}} \mathbf{i}_A a_A, \quad a_A \in \mathbb{R}.$$

The *addition and multiplication of elements of $\mathcal{C}\ell_{0,n}$ by real numbers* are defined componentwise. In this way, the multiplication between two elements of $\mathcal{C}\ell_{0,n}$ turns out to be associative, anticommutative, and has distributive properties.

As $\mathcal{C}\ell_{0,n}$ is isomorphic to $\mathbb{R}^{2^n}$, we may provide it with the $\mathbb{R}^{2^n}$-norm $|\mathbf{a}|$, and one can verify that for any $\mathbf{a}, \mathbf{b} \in \mathcal{C}\ell_{0,n}$, $|\mathbf{a}\,\mathbf{b}| \leq 2^{n/2}\,|\mathbf{a}|\,|\mathbf{b}|$, where $\mathbf{a} = \sum_{A \subseteq \{1,\ldots,n\}} \mathbf{i}_A a_A$ and $\mathbf{b} = \sum_{A \subseteq \{1,\ldots,n\}} \mathbf{i}_A b_A$.

The collection $\mathcal{C}\ell_{0,n}^k$ of *k-vectors in $\mathcal{C}\ell_{0,n}$* is the real linear subspace of $\mathcal{C}\ell_{0,n}$, defined as

$$\mathcal{C}\ell_{0,n}^k = \left\{\mathbf{a} \in \mathcal{C}\ell_{0,n} : \mathbf{a} = \sum_{|A|=k} \mathbf{i}_A a_A\right\},$$

where $|A|$ denotes the cardinality of the set A. The elements of $\mathcal{C}\ell_{0,n}^2$ are called *bivectors*, while the elements of $\mathcal{C}\ell_{0,n}^n$ are called *pseudoscalars*.

We define the *even subalgebra* by

$$\mathcal{C}\ell_{0,n}^{+} = \bigoplus_{k\,\text{even}} \mathcal{C}\ell_{0,n}^{k}.$$

According to the fact that the set $\mathcal{C}\ell_{0,n}^{+}$ contains the identity 1 and the product of two elements of even order forms an element of even order, it is clear that $\mathcal{C}\ell_{0,n}^{+}$ is again an algebra, but not a Clifford algebra.

4.2.2 Quaternions seen as Clifford numbers

$\mathbb{H}$ can be interpreted as a Clifford algebra in two different ways. On the one hand, $\mathbb{H}$ is isomorphic to the four-dimensional, even subalgebra $\mathcal{C}\ell_{0,3}^{+}$ of the universal Clifford algebra $\mathcal{C}\ell_{0,3}$ of dimension 8 with the generators $\mathbf{i}_1\mathbf{i}_2, \mathbf{i}_3$. The subalgebra is generated by

$$\{1,\ -\mathbf{i}_1\mathbf{i}_2,\ -\mathbf{i}_1\mathbf{i}_3,\ \mathbf{i}_2\mathbf{i}_3\}$$

The identification of this subalgebra with $\mathbb{H}$ is

$$\mathbf{e}_1 \to -\mathbf{i}_1\mathbf{i}_2, \quad \mathbf{e}_2 \to -\mathbf{i}_1\mathbf{i}_3, \quad \mathbf{e}_3 \to \mathbf{i}_2\mathbf{i}_3.$$

In this context the basis elements $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ have an interpretation as bivectors in a Clifford algebra.

On the other hand, $\mathbb{H}$ can be also realized as the universal Clifford algebra

$$\mathcal{C}\ell_{0,2} = \langle\{1, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_1\mathbf{i}_2\}\rangle$$

with the identification

$$\mathbf{e}_1 \to \mathbf{i}_1, \quad \mathbf{e}_2 \to \mathbf{i}_2, \quad \mathbf{e}_3 \to \mathbf{i}_1\mathbf{i}_2.$$

From a geometrical point of view, this identification is not particularly interesting since $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ have different meanings according to the context: the basis elements $\mathbf{e}_1$ and $\mathbf{e}_2$ are vectors in the Clifford algebra, while in contrast, $\mathbf{e}_3$ corresponds to a bivector.

The field of complex numbers can be identified with the Clifford algebra $\mathcal{C}\ell_{0,1}$.

We proceed to consider an important subset of $\mathbb{H}$.

Definition 4.2. A quaternion is called a *reduced quaternion* when is an element of the subset

$$\mathcal{A} = \left\{ \sum_{i=0}^{3} \mathbf{e}_i a_i \in \mathbb{H} : a_i \in \mathbb{R}, a_3 = 0 \right\}. \tag{4.2}$$

Thus $\mathbf{a} \in \mathcal{A} \Leftrightarrow [\mathbf{a}]_3 = 0$. Points $(x_0, x_1, x_2) \in \mathbb{R}^3$ can be identified with reduced quaternions $\sum_{i=0}^{2} \mathbf{e}_i x_i \in \mathcal{A}$. Since $\mathcal{A}$ is not closed under the quaternionic multiplication, it is clear that $\mathcal{A}$ is only a real vector subspace and not a subalgebra of $\mathbb{H}$.

There are other ways of embedding $\mathbb{R}^3$ in $\mathbb{H}$, for example, using the subspace of pure quaternions, i.e., by considering $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Lemma 4.3. *(i) Every quaternion* $\mathbf{a} \in \mathbb{H}$ *may be expressed in the form* $\mathbf{a} = \mathbf{b} + \mathbf{c}\mathbf{e}_3$ *where* $\mathbf{b}, \mathbf{c} \in \mathcal{A}$*. (ii) If also* $\mathbf{a} = \mathbf{b}' + \mathbf{c}'\mathbf{e}_3$ *where* $\mathbf{b}', \mathbf{c}' \in \mathcal{A}$*, then* $[\mathbf{b}]_0 = [\mathbf{b}']_0$ *and* $[\mathbf{c}]_0 = [\mathbf{c}']_0$*.*

Proof. Part (i) is trivial since we can write $\mathbf{a} = (a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2) + (a_3)\mathbf{e}_3$. For part (ii), by equating the scalar parts in the equation $\mathbf{b} - \mathbf{b}' = -(\mathbf{c} - \mathbf{c}')\mathbf{e}_3$ with $\mathbf{b} - \mathbf{b}' \in \mathcal{A}$ and $\mathbf{c} - \mathbf{c}' \in \mathcal{A}$, we have $[\mathbf{b}]_0 - [\mathbf{b}']_0 = 0$, and by equating the $\mathbf{e}_3$ parts we have $[\mathbf{c}]_3 - [\mathbf{c}']_3 = 0$. □

4.3 Operators on quaternionic Hilbert spaces

4.3.1 Linear spaces of $\mathbb{H}$-valued functions

Consider functions defined on an open subset Ω of $\mathbb{R}^3$ and taking values in the quaternions (possibly in the reduced quaternions). These functions are mappings of the form

$$f\colon \Omega \to \mathbb{H}$$

such that

$$f(x) = \sum_{i=0}^{3} \mathbf{e}_i [f(x)]_i, \quad x \in \Omega,$$

where the coordinate-functions $[f]_i$ are real-valued functions in Ω. Properties such as continuity, differentiability or integrability are ascribed coordinate-wise. It is clear that $[f(x)]_3 = 0$ for all $x \in \Omega$ if and only if f is an $\mathcal{A}$-valued function.

Due to the noncommutativity of quaternions, it is necessary to distinguish between two types of linear spaces over $\mathbb{H}$, namely left-linear and right-linear spaces.

Definition 4.4. A *right-linear space* over $\mathbb{H}$ is an additive abelian group $\mathcal{L}$ in which there is defined operation of scalar multiplication by elements of $\mathbb{H}$, for which the following laws hold for all $x, y \in \mathcal{L}$, $\mathbf{a}, \mathbf{b} \in \mathbb{H}$:

(i) $(x + y)\mathbf{a} = x\mathbf{a} + y\mathbf{a}$,

(ii) $x(\mathbf{a} + \mathbf{b}) = x\mathbf{a} + x\mathbf{b}$,

(iii) $x(\mathbf{ab}) = (x\mathbf{a})\mathbf{b}$.

All $\mathbb{H}$-linear spaces mentioned in this thesis will be assumed to be right spaces over $\mathbb{H}$. We use the notation $\mathcal{L}(\Omega, \mathbb{H})$ for right-linear spaces consisting of $\mathbb{H}$-valued functions, where $\mathcal{L}$ may denote a class of differentiability as in Definition 4.5 below. The spaces $\mathcal{L}(\Omega, \mathcal{A})$ of $\mathcal{A}$-valued functions will be $\mathbb{R}$-linear but not right-linear in the sense of a $\mathbb{H}$-linear space.

We denote the partial derivative of an $\mathbb{H}$-valued function with respect to the variable x_i by $\partial/\partial x_i$, $i \in \{0, 1, 2\}$. This means that the partial derivative is applied to each $\mathbb{R}$-valued component separately. The partial derivatives of higher-order are denoted by

$$\partial^{\lambda} = \frac{\partial^{|\lambda|}}{\partial x_0^{\lambda_0} \partial x_1^{\lambda_1} \partial x_2^{\lambda_2}},$$

where $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ is a multi-index of nonnegative integers and we write $|\lambda| = \lambda_0 + \lambda_1 + \lambda_2$.

We introduce the following quaternionic spaces, which will be of use in further discussion:

Definition 4.5. We denote by

(i) $C(\Omega, \mathbb{H})$ the space of all $\mathbb{H}$-valued functions that are continuous in Ω;

(ii) $C^m(\Omega, \mathbb{H})$ the space of all $\mathbb{H}$-valued functions f such that $\partial^{\lambda} f \in C(\Omega, \mathbb{H})$ whenever $|\lambda| \leq m$;

(iii) $C^{\infty}(\Omega, \mathbb{H})$ the space of all $\mathbb{H}$-valued functions that belong to $C^m(\Omega, \mathbb{H})$ for every $m \in \mathbb{N}$.

4.3.2 Right-linear spaces of square integrable functions

Primary in our study is the following space.

Definition 4.6. The space $L^2(\Omega, \mathbb{H})$ is defined to be the class of all

Lebesgue measurable $\mathbb{H}$-valued functions defined on Ω such that $|f|^2 \in L^1(\Omega)$ for all $f \in L^2(\Omega, \mathbb{H})$; that is,

$$L^2(\Omega, \mathbb{H}) = \left\{ f \colon \Omega \to \mathbb{H} \text{ measurable} : \int_\Omega |f(x)|^2 \, dV < \infty \right\}.$$

If $f \in L^2(\Omega, \mathbb{H})$, then $f\mathbf{a}$ is also in $L^2(\Omega, \mathbb{H})$ for all $\mathbf{a} \in \mathbb{H}$. Since

$$|f + g|^2 \leq 2|f|^2 + 2|g|^2,$$

it follows that $L^2(\Omega, \mathbb{H})$ is a right-linear space over $\mathbb{H}$.

Bearing in mind that $2|f\overline{g}| = 2|f||g| \leq |f|^2 + |g|^2$, if $\int_\Omega |f(x)|^2 \, dV$ and $\int_\Omega |g(x)|^2 \, dV$ are finite, then so is $\int_\Omega f(x)\overline{g(x)} \, dV$. We are lead to the following definition:

Definition 4.7. The $\mathbb{H}$-*valued inner product* on $L^2(\Omega, \mathbb{H})$ is defined by

$$\langle f, g \rangle_2 = \int_\Omega \overline{f(x)} \, g(x) \, dV \tag{4.3}$$

for all $f, g \in L^2(\Omega, \mathbb{H})$.

The $\mathbb{H}$-valued inner product satisfies the following properties:

1. $\langle f, f \rangle_2 > 0$, $f \neq 0$;

2. $\langle f, g \rangle_2 = \overline{\langle g, f \rangle_2}$;

3. $\langle f + g, h \rangle_2 = \langle f, h \rangle_2 + \langle g, h \rangle_2$;

4. $\langle f\mathbf{a}, g\rangle_2 = \bar{\mathbf{a}}\langle f, g\rangle_2$, $\langle f, g\mathbf{a}\rangle_2 = \langle f, g\rangle_2\mathbf{a}$.

Definition 4.8. The *scalar inner product* on the $\mathbb{R}$-linear space $L^2(\Omega, \mathcal{A})$ is defined by

$$\langle f, g\rangle_0 = \frac{1}{2}\left[\langle f, g\rangle_2 + \langle g, f\rangle_2\right] = \mathrm{Sc}\int_\Omega \overline{f(x)}\, g(x)\, dV. \tag{4.4}$$

Note that $\langle f, g\rangle_0$ does not define an inner product in $L^2(\Omega, \mathbb{H})$ seen as an $\mathbb{H}$-linear space because it is does not satisfy property (4) of Definition 4.7.

Definition 4.9. Two functions $f, g \in L^2(\Omega, \mathbb{H})$ (resp., $L^2(\Omega, \mathcal{A})$) are called *orthogonal* in the L^2-sense if $\langle f, g\rangle_2 = 0$ (resp., $\langle f, g\rangle_0 = 0$).

Definition 4.10. Let Λ be an index set and $\{f_i\}_{i\in\Lambda}$ a subset of $L^2(\Omega, \mathbb{H})$ (resp., $L^2(\Omega, \mathcal{A})$). $\{f_i\}_{i\in\Lambda}$ is called an *orthonormal set* if $\langle f_i, f_j\rangle_2 = \delta_{i,j}$ (resp., $\langle f_i, f_j\rangle_0 = \delta_{i,j}$).

Definition 4.11. A set $\{f_i\}_{i\in\Lambda}$ (not necessarily orthonormal) is called *complete* in $L^2(\Omega, \mathbb{H})$ (resp., $L^2(\Omega, \mathcal{A})$) if for every element $f \in L^2(\Omega, \mathbb{H})$ (resp., $f \in L^2(\Omega, \mathcal{A})$), the assumption $\langle f, f_i\rangle_2 = 0\ \forall\, i \in \Lambda$ (resp. $\langle f, f_i\rangle_0 = 0\ \forall\, i \in \Lambda$) implies that $f = 0$.

Definition 4.12. For $f \in L^2(\Omega, \mathbb{H})$, *the L^2-norm induced by the (right) quaternionic inner product* (4.3) is defined by

$$\|f\|_2 = \left(\langle f, f\rangle_2\right)^{1/2} = \left(\int_\Omega |f(x)|^2\, dV\right)^{1/2}. \tag{4.5}$$

(The L^2-norm induced by the quaternionic inner product (4.3) coincides with the L^2-norm of f seen as a vector-valued function.)

The space $L^2(\Omega, \mathbb{H})$ furnished with the quaternionic inner product (4.3) is called a *(right) quaternionic Hilbert space*, and the norm (4.5) turns $L^2(\Omega, \mathbb{H})$ into a Banach space, that is, the metric associated to the norm is complete. The following theorems were proved by O. Teichmüller in [90].

Theorem 4.13. *Every nonzero Hilbert space over* $\mathbb{H}$ *contains an orthonormal basis.*

Theorem 4.14. *Let* $\{f_i\}_{i\in\Lambda}$ *be an orthonormal set in* $L^2(\Omega, \mathbb{H})$ *(resp.,* $L^2(\Omega, \mathcal{A})$*). The following conditions are equivalent:*

(i) $\{f_i\}_{i\in\Lambda}$ *is complete;*

(ii) if $f \in L^2(\Omega, \mathbb{H})$ *(resp.,* $f \in L^2(\Omega, \mathcal{A})$*), then* $f = \sum \langle f, f_i \rangle_2 f_i$ *(resp.* $f = \sum \langle f, f_i \rangle_0 f_i$*);*

(iii) if $f \in L^2(\Omega, \mathbb{H})$ *(resp.,* $L^2(\Omega, \mathcal{A})$*), then* $\|f\|_2^2 = \sum |\langle f, f_i \rangle_2|^2$ *(resp.* $\|f\|_2^2 = \sum |\langle f, f_i \rangle_0|^2$ *(Parseval's identity).*

4.4 Monogenic functions

Monogenic functions are the central object of study in quaternionic analysis. The concept of the monogenicity of a function is a higher dimensional counterpart of holomorphy in the complex plane.

4.4.1 Quaternionic Cauchy-Riemann operator

Definition 4.15. For real-differentiable functions $f\colon \Omega \to \mathbb{H}$, the *(reduced) quaternionic generalized Cauchy-Riemann (or Fueter) operator acting from the left* is defined via the formula

$$\overline{\partial} f = \sum_{i=0}^{2} \mathbf{e}_i \frac{\partial f}{\partial x_i}. \tag{4.6}$$

Note that $\partial f / \partial x_i$ is $\mathbb{H}$-valued, so the fact that multiplication by $\mathbf{e}_i$ is from the left in (4.6) is important. The operator acting from the right is $f \overline{\partial} = \sum_{i=0}^{2} \frac{\partial f}{\partial x_i} \mathbf{e}_i$.

The operator (4.6) is a three-dimensional extension of the classical Cauchy-Riemann operator $2\partial_{\overline{z}} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$, $z = x + iy \in \mathbb{C}$. On the other hand, (4.6) is also a reduction of the operator studied by R. Fueter [35, 36] which includes a fourth variable x_3.

When we apply $\overline{\partial}$ to a given function f of class C^1, we obtain the generalization of the areolar derivative in the sense of Pompeiu [79]. Similarly, we define the conjugate quaternionic Cauchy-Riemann operator ∂ as

$$\partial f = \sum_{i=0}^{2} \overline{\mathbf{e}_i} \frac{\partial f}{\partial x_i}, \tag{4.7}$$

which is a generalization of the complex operator $2\partial_z$.

It is well-known that for a complex functions f is holomorphic if and only if $\frac{\partial f}{\partial \overline{z}} = 0$. The extension of this concept to $\mathbb{H}$-valued functions leads to the following definition, where $\Omega \subseteq \mathbb{R}^3$:

Definition 4.16. A function $f\colon \Omega \to \mathbb{H}$ of class C^1 is called *left* (resp., *right*) *monogenic* in Ω if it satisfies

$$\overline{\partial} f = 0 \text{ in } \Omega \text{ (resp., } f\overline{\partial} = 0 \text{ in } \Omega).$$

We write $\mathcal{M}(\Omega, \mathbb{H})$ for the collection of all left-monogenic functions in Ω, and $\mathcal{M}(\Omega, \mathcal{A})$ for the subset of $\mathcal{A}$-valued functions.

Proposition 4.17. *$\mathcal{M}(\Omega, \mathbb{H})$ is a right-linear space. $\mathcal{M}(\Omega, \mathcal{A})$ is an $\mathbb{R}$-linear space.*

Proof. Since $\overline{\partial}(f\mathbf{a}) = (\overline{\partial} f)\mathbf{a}$ for all $\mathbf{a} \in \mathbb{H}$, clearly $\overline{\partial}(f\mathbf{a}) = 0 \Leftrightarrow (\overline{\partial} f)\mathbf{a} = 0$, so $f\mathbf{a}$ is monogenic whenever f is monogenic. □

In general, left (resp., right) monogenic functions are not right (resp., left) monogenic. This is visible in the following example:

Example 4.18. The function $f(x_0, x_1, x_2) = \mathbf{e}_2 x_1 - \mathbf{e}_3 x_0$ is left monogenic but not right monogenic because of

$$\overline{\partial} f = 0, \quad f\overline{\partial} = -2\mathbf{e}_3.$$

Monogenicity provides a reasonable generalization of complex analyticity to quaternionic analysis since many classical theorems from complex analysis (such as the Cauchy integral theorem and Cauchy integral formula) can be generalized to higher dimensions by this approach [41, 42]. Since these results hold for general domains Ω, they hold in particular for the torus Ω_{η_0}. We will not need these results in this thesis. Therefore we will not reproduce them here, with the following exception.

Proposition 4.19 ([41]). *A uniform limit (on compact subsets) of monogenic functions is monogenic.*

In contrast to the complex case, the composition of monogenic functions is not necessarily monogenic. In fact, the identity function $x \mapsto x$, $x \in \mathbb{R}^3$, is not monogenic.

4.4.2 $\mathcal{A}$-valued monogenic functions and monogenic constants

We restrict ourselves to $\mathcal{A}$-valued functions for the rest of this chapter and the next. They enjoy the following property which is not shared by $\mathbb{H}$-valued functions:

Proposition 4.20 ([65, 74]). *Every monogenic $\mathcal{A}$-valued function is two-sided monogenic.*

This means that when f is either left- or right- monogenic it satisfies simultaneously the equations

$$\overline{\partial} f = 0, \quad f\overline{\partial} = 0,$$

either of which is equivalent to the system

$$\left\{ \begin{array}{rcl} \dfrac{\partial [f]_0}{\partial x_0} - \dfrac{\partial [f]_1}{\partial x_1} - \dfrac{\partial [f]_2}{\partial x_2} & = & 0 \\ \dfrac{\partial [f]_0}{\partial x_1} + \dfrac{\partial [f]_1}{\partial x_0} & = & 0 \\ \dfrac{\partial [f]_0}{\partial x_2} + \dfrac{\partial [f]_2}{\partial x_0} & = & 0 \\ \dfrac{\partial [f]_1}{\partial x_2} - \dfrac{\partial [f]_2}{\partial x_1} & = & 0 \end{array} \right. \tag{4.8}$$

or, in a more compact form:

$$\left\{ \begin{array}{rcl} \operatorname{div} \overline{f} & = & 0, \\ \operatorname{curl} \overline{f} & = & 0. \end{array} \right. \tag{4.9}$$

This 3-tuple $\overline{f}$ is said to be *a system of conjugate harmonic functions* in the sense of Stein-Weiß [86, 87] and system (4.9) is called the *Riesz system* [82].

Proposition 4.21. *The generalized Cauchy-Riemann operator* (4.6) *and its conjugate* (4.7) *factor the* 3*-dimensional Laplace operator, that is,*

$$\Delta_3 f = \overline{\partial}\partial f = \partial\overline{\partial} f, \tag{4.10}$$

whenever $f \in C^2$.

This implies that every monogenic function is also a harmonic function.

(In other words, every component function is harmonic.) This factorization of the Laplace operator establishes a special relationship between quaternionic analysis and harmonic analysis in that monogenic functions refine the properties of harmonic functions.

The following is a fundamental fact in the construction of monogenic functions.

Corollary 4.22. *Let h be a real-valued harmonic function. Then ∂h is monogenic.*

Definition 4.23. A monogenic function f is called *exact* when it is of the form $f = \partial h$ for a real-valued harmonic function h.

Definition 4.24. A function $\varphi\colon \Omega \to \mathcal{A}$ is a *monogenic constant* if $\varphi \in \mathcal{M}(\Omega, \mathcal{A})$ and $\overline{\varphi} \in \mathcal{M}(\Omega, \mathcal{A})$. The collection of reduced-quaternion valued monogenic constants in Ω will be denoted $\mathcal{M}_C(\Omega, \mathcal{A})$.

According to the previous definition we have the following:

Proposition 4.25 ([65]). *Let $\varphi = \varphi_0 + \mathbf{e}_1\varphi_1 + \mathbf{e}_2\varphi_2\colon \Omega \to \mathcal{A}$ be a monogenic constant. Then $\varphi_0 \in \mathbb{R}$ is a constant, while φ_1 and φ_2 are functions of x_1 and x_2, and $\varphi_1 - i\varphi_2$ is a holomorphic function of the complex variable $x_1 + ix_2$. Further, if $f \in \mathcal{M}(\Omega, \mathcal{A})$ and $\operatorname{Sc} f$ is a constant, then $f \in \mathcal{M}_C(\Omega, \mathcal{A})$. Therefore $\mathcal{M}_C(\Omega, \mathcal{A})$ is isomorphic to the class of complex holomorphic functions in Ω.*

Proof. Suppose that $\varphi \in \mathcal{M}_{\mathbb{C}}(\Omega, \mathcal{A})$. By adding and subtracting the equations $\overline{\partial}\varphi = 0$, $\partial\varphi = 0$, we obtain $\partial_0\varphi = 0$, $(\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2)\varphi = 0$. The first equation says that $\varphi(x_1, x_2)$ does not depend on x_0. The second equation is equivalent to the system

$$\frac{\partial[\varphi]_1}{\partial x_1} + \frac{\partial[\varphi]_2}{\partial x_2} = 0,$$
$$\frac{\partial[\varphi]_1}{\partial x_2} - \frac{\partial[\varphi]_2}{\partial x_1} = 0,$$

which is satisfied by anti-holomorphic functions of $x_1 + ix_2$.

Now suppose that $\operatorname{Sc} f$ is constant where $f \in \mathcal{M}(\Omega, \mathcal{A})$. Then

$$0 = \overline{\partial} f = \overline{\partial}(\mathbf{e}_1 f_1 + \mathbf{e}_2 f_2) = \partial_0(\mathbf{e}_1 f_1 + \mathbf{e}_2 f_2) + (\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2)(\mathbf{e}_1 f_1 + \mathbf{e}_2 f_2).$$

The term $\partial_0(\mathbf{e}_1 f_1 + \mathbf{e}_2 f_2)$ is in the subspace $\mathbf{e}_1\mathbb{R} \oplus \mathbf{e}_2\mathbb{R}$ of $\mathbb{H}$, while the other term $(\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2)(\mathbf{e}_1 f_1 + \mathbf{e}_2 f_2)$ is in the complementary subspace $\mathbb{R} \oplus \mathbf{e}_3\mathbb{R}$. Therefore both terms are zero. This tells us that $\mathbf{e}_1 f_1 + \mathbf{e}_2 f_2$ is a function of x_1, x_2, and then that this function is a holomorphic function of $x_2 + ix_1$. $\square$

One of the reasons that monogenic constants are important is the following.

Proposition 4.26 ([65]). *Let $f, g \in \mathcal{M}(\Omega, \mathcal{A})$ be such that $\partial f = \partial g$ in Ω. Then f and g differ by a monogenic constant.*

Proof. Since the difference of two monogenic functions is monogenic, by hypothesis both $f - g$ and $\overline{f - g}$ are monogenic. $\square$

Chapter 5

$\mathcal{A}$-valued toroidal monogenic functions

In Chapter 4, we surveyed some basic properties of monogenic functions in general domains. Now we study the particular properties of the real linear space of monogenic functions on the torus. In this chapter, we will work exclusively with $\mathcal{A}$-valued functions. We will construct a complete independent set in the real Hilbert space

$$\mathcal{M}(\mathcal{A}) = M(\Omega_{\eta_0}, \mathcal{A}) \cap L^2(\Omega_{\eta_0}, \mathcal{A}), \tag{5.1}$$

beginning with the monogenic constants $\mathcal{M}_{\mathrm{C}}(\mathcal{A}) = \mathcal{M}_{\mathrm{C}}(\Omega_{\eta_0}, \mathcal{A}) \cap \mathcal{M}(\mathcal{A})$. This will be used in Chapter 6 to construct a complete independent set in $\mathcal{M}(\mathbb{H}) = \mathcal{M}(\Omega_{\eta_0}, \mathbb{H}) \cap L^2(\Omega_{\eta_0}, \mathbb{H})$.

To clarify the subject matter, we note beforehand some of the major differences between the spaces of monogenic functions on the torus and on the ball. The main difference is that one obtains a complete set of monogenic functions on the ball by applying ∂ to interior solid spherical harmonics, as in Corollary 4.22 [18, 65]. More exactly, the $2n+3$ harmonics $r^n Y^{\pm}_{n+1,m}(\theta,\varphi)$ defined in Section 1.2 provide $2n+3$ monogenic functions, which are a basis of the real-linear space of homogeneous monogenic polynomials of degree n. For the torus, there exist non-exact monogenics, and this process cannot produce them. A second difference comes from the fact that ∂_0 has a reverse-Appell property, which means that this process does not produce those monogenics on Ω_{η_0}, whose scalar part contains the 0-level toroidal harmonics $I^{+,\pm}_{0,m}$. These two differences make the study of monogenics on Ω_{η_0} very different from domains that have been studied previously.

5.1 Non-exactness of monogenic functions

In this section, we describe a tool for considering that the torus is not simply-connected.

Definition 5.1. Let $f = f_0 + \mathbf{e}_1 f_1 + \mathbf{e}_2 f_2 \in C(\Omega, \mathcal{A})$ with real-valued components f_0, f_1, f_2. The *real differential 1-form associated to f* is

$$\omega_f = f_0\,dx_0 - f_1\,dx_1 - f_2\,dx_2.$$

In the following proposition, we regard the differential $dx = (dx_0, dx_1, dx_2)$, $x \in \mathbb{R}^3$, as a paravector,

$$dx = dx_0 + \mathbf{e}_1(dx_1) + \mathbf{e}_2(dx_2).$$

The previous definition leads to the following results:

Proposition 5.2. *Let $h\colon \Omega \to \mathbb{R}$ be harmonic, $h \in L^2(\Omega)$. Let $f = \partial h$ be the corresponding exact monogenic function. Then $dh = \omega_f = \mathrm{Sc}(f\,dx)$.*

Proof. By Definition 5.1,

$$\begin{aligned} dh &= (\partial_0 h)\,dx_0 + (\partial_1 h)\,dx_1 + (\partial_2 h)\,dx_2 \\ &= f_0\,dx_0 - f_1\,dx_1 - f_2\,dx_2 \\ &= \mathrm{Sc}\left((f_0 + \mathbf{e}_1 f_1 + \mathbf{e}_2 f_2)(dx_0 + \mathbf{e}_1 dx_1 + \mathbf{e}_2 dx_2)\right). \\ &= \mathrm{Sc}(f\,dx). \end{aligned}$$

□

Proposition 5.3. *Consider an $\mathcal{A}$-valued monogenic function $f \in \mathcal{M}(\Omega, \mathcal{A})$. Then ω_f is a closed differential form.*

Proof. We have

$$\begin{aligned} d\omega_f &= d(f_0\,dx_0 - f_1\,dx_1 - f_2\,dx_2) \\ &= df_0 \wedge dx_0 - df_1 \wedge dx_1 - df_2 \wedge dx_2 \\ &= (\partial_0 f_0\,dx_0 + \partial_1 f_0\,dx_1 + \partial_2 f_0\,dx_2) \wedge dx_0 \end{aligned}$$

$$
\begin{aligned}
&- (\partial_0 f_1\, dx_0 + \partial_1 f_1\, dx_1 + \partial_2 f_1\, dx_2) \wedge dx_1 \\
&- (\partial_0 f_2\, dx_0 + \partial_1 f_2\, dx_1 + \partial_2 f_2\, dx_2) \wedge dx_2 \\
= &-(\partial_1 f_0 + \partial_0 f_1) dx_0 \wedge dx_1 - (\partial_2 f_0 + \partial_0 f_2) dx_0 \wedge dx_2 \\
&+ (\partial_2 f_1 - \partial_1 f_2) dx_1 \wedge dx_2.
\end{aligned}
$$

By (4.8), $d\omega_f = 0$. Therefore ω_f is a closed form. □

Proposition 5.4. *If f is monogenic and $\mathcal{A}$-valued in a simply-connected domain $\Omega \subseteq \mathbb{R}^3$, the integral*

$$\int_a^b \omega_f$$

does not depend on the curve from a to b in that domain.

Proof. Since Ω is simply connected, the closed form ω_f is exact, so we have $\omega_f = dg$ for some well-defined C^1 function g in Ω, and g is uniquely determined up to an additive real constant. Thus

$$\int_a^b \omega_f = g(b) - g(a)$$

independently of the curve in Ω from a to b. □

For simplicity, the following definition will be restricted to the torus domain. The only nontrivial topological aspect of Ω_{η_0} is that the unit circle $S^1 = \{\eta = \infty\}$ in the plane $x_0 = 0$ is not contractible.

Definition 5.5. Let $f \in \mathcal{M}(\mathcal{A})$ be a monogenic function in Ω_{η_0}. We define the *cohomology coefficient* $\operatorname{coh} f$ of f by an integration along the central curve S^1, for which we orient S^1 in the positive direction when seen from the half-space $x_0 > 0$:

$$\operatorname{coh} f = \frac{1}{2\pi} \int_{S^1} \omega_f. \tag{5.2}$$

Proposition 5.6. *Let $f \in \mathcal{M}(\mathcal{A})$. Then f is exact if and only if $\operatorname{coh} f = 0$.*

Proof. By exactness, $f = \partial h$ for some harmonic h. Then

$$\operatorname{coh} f = \frac{1}{2\pi} \int_{S^1} \omega_f = \frac{1}{2\pi} \int_{S^1} dh = 0 \tag{5.3}$$

since S^1 is a closed curve.

Conversely, consider two curves γ_1, γ_2 joining the point $(0,1,0)$ to x in Ω_{η_0}, and let γ denote the closed curve obtained by following γ_1 by the reverse of γ_2. Then γ is freely homotopic in Ω_{η_0} to S^1 traced an integral number n times. It follows from Proposition 5.4 that integrals of ω_f over homotopic curves coincide, so

$$\int_{\gamma_1} \omega_f = \int_{\gamma_2} \omega_f + n \int_{S^1} \omega_f = \int_{\gamma_2} \omega_f$$

under the supposition that $\operatorname{coh} f = 0$. It follows that the value

$$h(x) = \int_{(0,1,0)}^{x} \omega_f$$

does not depend on the curve in Ω_{η_0} joining $(0,1,0)$ to x, and that $\partial h = \omega_f$, so ω_f is exact. $\square$

We will not need to use the fact that $\operatorname{coh} f = 0$ implies that f is exact. In the following section, we will give an example of a monogenic function that is not exact.

5.2 $\mathcal{A}$-valued monogenic constants on the torus

Recall that $\mathcal{M}_C(\mathcal{A})$ denotes the subspace of $\mathcal{A}$-valued monogenic constants on Ω_{η_0} where η_0 is fixed.

Definition 5.7. Let $m \in \mathbb{Z}$. For $x \in \mathbb{R}^3$ define the scalar-valued functions

$$J_m^+(x) = \operatorname{Re}(x_1 + ix_2)^m,$$

$$J_m^-(x) = \operatorname{Im}(x_1 + ix_2)^m,$$

$$\widehat{J}(x) = -\log|x_1 + ix_2|.$$

These functions are independent of x_0. They are well-defined in Ω_{η_0} (in fact, in all of $\mathbb{R}^3 - \mathbb{R}_0$) since $(x_1, x_2) \neq (0,0)$. Clearly $\pm J_m^{\mp}$ is a harmonic conjugate of $J_m^{\pm}$ when considered as a function of the complex variable $x_1 + ix_2$, since the function $(x_1 + ix_2)^m$ is holomorphic for all $m \in \mathbb{Z}$. However, the function $\widehat{J}(x)$ does not admit a harmonic conjugate since $\arg(x_1 + ix_2)$ is not single-valued.

Proposition 5.8. *Let $h\colon \Omega_{\eta_0} \to \mathbb{R}$ be harmonic, $h \in L^2(\Omega_{\eta_0})$, and suppose $\partial h/\partial x_0 = 0$. Then $h(x_1, x_2)$ has a unique representation as*

$$h = \hat{a}\widehat{J} + \sum_{m=-\infty}^{\infty} (a_m^+ J_m^+ + a_m^- J_m^-) \tag{5.4}$$

with real coefficients $\hat{a}, a_m^{\pm}$.

Proof. This is a standard fact about complex variables. There is a harmonic conjugate v of h in a neighborhood of $x_1 + ix_2 = 1 + 0i$, and continuation of v around S^1 leads to a harmonic conjugate $v - \hat{a}$ which differs from v by a constant. Since $-\arg(x_2 + ix_1)$ is a local harmonic conjugate of $\widehat{J}$, the function $v - (\hat{a}/2\pi)\arg(x_2 + ix_1)$ continues back to its original value along S^1, and must be the real part of a holomorphic function,

$$v - \frac{\hat{a}}{2\pi}\frac{x_2}{x_1} = \operatorname{Re} g,$$

where g is defined for all $(x_1, x_2) \neq (0, 0)$. The Laurent series

$$g(x_1, x_2) = \sum_{m=-\infty}^{\infty} c_m (x_1 + ix_2)^m$$

gives

$$v - \frac{\hat{a}}{2\pi}\frac{x_2}{x_1} = \sum_{m=-\infty}^{\infty} ((\operatorname{Re} c_m) J_m^+ - (\operatorname{Im} c_m) J_m^-),$$

from which we find $a_m^+ = \operatorname{Re} c_m$, $a_m^- = -\operatorname{Im} c_m$. $\square$

In the expansion (5.4) and all similar sums, J_0^- is to be excluded since it is identically zero.

Definition 5.9. We introduce the *basic monogenic constants* on Ω_{η_0},

$$W_m^{\pm} = \mathbf{e}_1 J_m^{\pm} \mp \mathbf{e}_2 J_m^{\mp},$$

for all integers m.

We have

Proposition 5.10. *The basic monogenic constants are exact monogenic functions; in fact,*

$$\begin{aligned} W_m^{\pm} &= \frac{-1}{m+1}\,\partial J_{m+1}^{\pm}, \quad m \neq -1, \\ W_{-1}^{+} &= \partial \widehat{J}. \end{aligned}$$

Proof. We refer to Definition 5.7. Let $m \neq -1$. Then we can differentiate $(x_1 + ix_2)^{m+1}$ in the customary way,

$$\begin{aligned} \partial J_{m+1}^{\pm} &= \partial_0 J_{m+1}^{\pm} - \mathbf{e}_1 \partial_1 J_{m+1}^{\pm} - \mathbf{e}_2 \partial_2 J_{m+1}^{\pm} \\ &= 0 - \mathbf{e}_1 (m+1) J_{m+1}^{\pm} \pm \mathbf{e}_2 (m+1) J_{m+1}^{\pm} \\ &= (-1)(m+1)\big(\mathbf{e}_1 J_{m+1}^{\pm} \mp \mathbf{e}_2 J_{m+1}^{\pm}\big) \\ &= -(m+1) W_m^{\pm} \end{aligned}$$

as claimed. For the second statement,

$$\partial \widehat{J} = \partial_0 \widehat{J} - \mathbf{e}_1 \partial_1 \widehat{J} - \mathbf{e}_2 \partial_2 \widehat{J}$$

$$\begin{aligned} &= 0 - \mathbf{e}_1 \frac{-x_1}{x_1^2 + x_2^2} - \mathbf{e}_2 \frac{-x_2}{x_1^2 + x_2^2} \\ &= W_{-1}^{+}. \end{aligned}$$ □

The basic monogenic constant W_{-1}^{-} is the only one not mentioned in Proposition 5.10. In fact, even though locally W_{-1}^{-} is equal to the $\mathcal{A}$-valued function $-\partial \arctan(x_2/x_1)$, it cannot obtained globally by applying ∂ to a real-valued harmonic function, as the following result implies.

Proposition 5.11. *All of the basic monogenic constants $W_m^{\pm}$ have vanishing cohomology coefficient with the exception of*

$$\operatorname{coh} W_{-1}^{-} = 1,$$

so W_{-1}^{-} is not exact.

Proof. The vanishing cohomology follows from Proposition 5.6 together with the exactness which is given by Proposition 5.10. Further, from Definition 5.9 with $m = -1$,

$$W_{-1}^{-} = \mathbf{e}_1 \operatorname{Im} \frac{1}{x_1 + ix_2} + \mathbf{e}_2 \operatorname{Re} \frac{1}{x_1 + ix_2},$$

so by Definition 5.1,

$$\begin{aligned} \omega_{W_{-1}^{-}} &= \frac{x_2}{x_1^2 + x_2^2}\, dx_1 - \frac{x_1}{x_1^2 + x_2^2}\, dx_2 \\ &= d\big(\tan \frac{x_2}{x_1}\big). \end{aligned}$$

From this and Definition 5.5 it follows that $\operatorname{coh} W_{-1}^{-} = 1$ so W_{-1}^{-} is not exact. $\square$

Being harmonic functions on the torus, by Proposition 1.29, the basic monogenic constants can be expressed in terms of the basic toroidal harmonics $I_{n,m}^{\pm,\pm}$ introduced in Section 1.3 of Chapter 1.

We will need the explicit expression.

Proposition 5.12. *For all $m \in \mathbb{Z}$, we have the following representation in terms of the toroidal harmonics:*

$$J_m^{\pm} = \sum_{n=0}^{\infty} j_{n,m}\, I_{n,|m|}^{+,\pm} \tag{5.5}$$

where

$$j_{n,m} = \begin{cases} \varepsilon_m(-1)^m \sqrt{\frac{2}{\pi}} \frac{1}{\Gamma(m+\frac{1}{2})}, & m \geq 0, \\ \pm\varepsilon_m(-1)^m \sqrt{\frac{2}{\pi}} \frac{1}{\Gamma(m+\frac{1}{2})} \frac{\Gamma(n+m+1/2)}{\Gamma(n-m+1/2)}, & m < 0, \end{cases} \tag{5.6}$$

where ε_m has the same meaning as in Proposition 1.9.

Proof. By using the definition (1.29) of toroidal coordinates we have

$$(x_1 + ix_2)^m = \frac{\sinh^m \eta}{(\cosh \eta - \cos\theta)^m}\, e^{im\varphi},$$

so

$$J_m^{\pm} = \frac{\sinh^m \eta}{(\cosh \eta - \cos\theta)^m}\, \Phi_m^{\pm}(\varphi).$$

Now expand $(\cosh\eta - \cos\theta)^{-m-1/2}$ in a Fourier series in θ by means of Proposition 1.9 to see that

$$J_m^{\pm} = \sinh^m \eta \ \frac{1}{\Gamma(m+\frac{1}{2})}\sqrt{\frac{2}{\pi}}\frac{(-1)^m}{\sinh^m \eta}(\cosh\eta - \cos\theta)^{1/2} \\ \times \sum_{n=0}^{\infty} \varepsilon_n \ \cos(n\theta)\, Q_{n-\frac{1}{2}}^m(\cosh\eta)\, \Phi_m^{\pm}(\varphi).$$

This is valid for all integers m. First suppose that $m \geq 0$. Then one can identify elements of (1.40) to obtain the desired formula for $J_m^{\pm}$. On the other hand, if $m < 0$, we have

$$J_m^{\pm} = \sinh^m \eta \ \frac{1}{\Gamma(m+\frac{1}{2})}\sqrt{\frac{2}{\pi}}\frac{(-1)^m}{\sinh^m \eta}(\cosh\eta - \cos\theta)^{1/2} \\ \times \sum_{n=0}^{\infty} \varepsilon_n \ \cos(n\theta)\, \frac{\Gamma(n+m+1/2)}{\Gamma(n-m+1/2)} Q_{n-\frac{1}{2}}^{-m}(\cosh\eta)\, (\pm\Phi_{-m}^{\pm}(\varphi)),$$

which permits identifying terms with $I_{n,-m}^{+,\pm}$. □

By Proposition 5.12, we can express the basic monogenic constants as

$$W_m^{\pm} = \sum_{n=0}^{\infty} j_{n,m}(\mathbf{e}_1 I_{n,m}^{+,\pm} \mp \mathbf{e}_2 I_{n,m}^{+,\mp}) \tag{5.7}$$

for all $m \in \mathbb{Z}$, although the individual summands are not monogenic.

We have the further consequence of Proposition 5.8.

Corollary 5.13. *Every monogenic constant $\varphi \in \mathcal{M}(\mathcal{A})$ has a unique represen-*

tation of the form

$$\varphi = a_0 + \sum_{m=-\infty}^{\infty} (a_m^+ W_m^+ + a_m^- W_m^-),$$

where $a_0, a_m^\pm \in \mathbb{R}$ *are constants.*

Proof. By Proposition 4.25, we can write $\varphi = a_0 + \mathbf{e}_1\varphi_1 + \mathbf{e}_2\varphi_2$, where φ_1, φ_2 are harmonic functions independent of x_0. According to Proposition 5.8, we can write

$$\varphi_1 = \hat{a}\,\hat{J} + \sum_{m=-\infty}^{\infty} (a_m^+ J_m^+ + a_m^- J_m^-).$$

Since φ_1 admits $-\varphi_2$ as a harmonic conjugate and each sum $\sum_m a_m^+ J_m^+$, $\sum_m a_m^- J_m^-$ also has a harmonic conjugate, namely $\pm \sum_m a_m^\pm J_m^\mp$, necessarily $\hat{a}\hat{J}$ has a harmonic conjugate as well, thus $\hat{a} = 0$. Therefore

$$\varphi_2 = \sum_m (a_m^+ J_m^- - a_m^- J_m^+).$$

This gives the desired representation. The uniquess follows from the fact that the $J_m^\pm$ form a basis in $L^2(\Omega_{\eta_0})$ sense. $\square$

Proposition 5.14. *Let* $f_0 \colon \Omega_{\eta_0} \to \mathbb{R}$ *be harmonic,* $f_0 \in L^2(\Omega_{\eta_0})$. *Then there exists* $h \colon \Omega_{\eta_0} \to \mathbb{R}$ *harmonic such that* $f_0 = \partial_0 h$ *if and only if* $f_0 \in \overline{\mathrm{Span}}\{I_{n,m}^{\nu,\mu} \colon\ n \geq 1\}$.

Proof. Recall that from Section 2.3 of Chapter 2 we always have $I_{0,m}^{*,\nu,\mu} =$

$I_{0,m}^{\nu,\mu}$. Supposing the condition, we may write

$$f_0 = \sum_{n\geq 1} \sum_{m,\nu,\mu} a_{n,m}^{\nu,\mu} I_{n,m}^{*\,\nu,\mu}.$$

Then by the formula of Proposition 2.9, $f_0 = \partial_0 h$ where

$$h = \sum_{n\geq 1} \sum_{m,\nu,\mu} a_{n,m}^{\nu,\mu} (1/\kappa_{n,m}^{n-1}) I_{n-1,m}^{*\,-\nu,\mu}.$$

Now suppose on the other hand that $f_0 = \partial_0 h$. We can express h in toroidal coordinates as $h = \sum_{n\geq 0} \sum_{m,\nu,\mu} b_{n,m}^{\nu,\mu} I_{n,m}^{*\,\nu,\mu}$ and again apply Proposition 2.9 to see that $f_0 \in \overline{\mathrm{Span}}\{I_{n,m}^{\nu,\mu}\colon\ n \geq 1\}$. □

This statement may be surprising because an analogy with spherical harmonics might lead one to expect the range $n \geq 0$ instead of $n \geq 1$. The correct condition is a consequence of the fact that ∂_0 increases the index n rather than decreasing it.

5.3 Basis for $\mathcal{A}$-valued monogenic functions

We now have almost enough material to construct a basis for $\mathcal{M}(\mathcal{A})$, the principal result of this chapter. The first part of the construction, which is analogous to the theory of spherical monogenics, is as follows.

Definition 5.15. The *basic exact toroidal monogenic functions* are defined as

$$T_{n,m}^{\nu,\mu} = \partial I_{n-1,m}^{*\,-\nu,\mu},\ n \geq 1. \tag{5.8}$$

By Propositions 2.12 and 5.6, for $n \geq 1$, $T_{n,m}^{\nu,\mu} \in \mathcal{M}(\mathcal{A})$ and

$$\operatorname{Sc} T_{n,m}^{\nu,\mu} = \kappa_{n,m}^{n-1} I_{n,m}^{*\,\nu,\mu}, \quad \operatorname{coh} T_{n,m}^{\nu,\mu} = 0. \tag{5.9}$$

We will define monogenics with index $n = 0$ later.

Proposition 5.16. *Let $h\colon \Omega_{\eta_0} \to \mathbb{R}$ be harmonic, $h \in L^2(\Omega_{\eta_0})$. Then*

$$\partial h \in \overline{\operatorname{Span}}\{T_{n,m}^{\nu,\mu}\colon\ n \geq 1\}.$$

As usual, all indices other than n are understood to vary among all admissible combinations.

Proof. We have $h = \sum_{n\geq 0} a_{n,m}^{\nu,\mu} I_{n,m}^{*\,\nu,\mu}$, converging in $L^2(\Omega_{\eta_0})$ (where the notation does not mention the summation over the range of the indices m, ν, μ, since they will always be the same). By Proposition 1.23 and (5.8),

$$\partial h = \sum_{n\geq 0} a_{n,m}^{\nu,\mu} \partial I_{n,m}^{*\,\nu,\mu} = \sum_{n\geq 0} a_{n,m}^{\nu,\mu} T_{n+1,m}^{-\nu,\mu} = \sum_{n\geq 1} a_{n-1,m}^{\nu,\mu} T_{n,m}^{-\nu,\mu}$$

which is in the required $\overline{\operatorname{Span}}$. □

Proposition 5.14 tells us immediately that the harmonics $I_{n,m}^{\nu,\mu}$ and $I_{n,m}^{*\,\nu,\mu}$ are scalar parts of monogenic functions when $n \geq 1$. The verification of this property for $n = 0$ will follow along different lines.

Definition 5.17. The *Teodorescu operator* for a bounded domain $D \subseteq \mathbb{C}$ in the complex plane is given by

$$\mathbf{T}_D f(w) = \frac{-1}{\pi} \int_D \frac{f(z)}{z-w}\, dx\, dy.$$

It satisfies $\partial \mathbf{T}_D(z)/\partial \overline{z} = f(z)$. We take D to be the annulus with inner and outer radii $\sinh \eta_0/(\cosh \eta_0 \pm 1)$, which corresponds to the slice of Ω_{η_0} at the plane $x_0 = 0$.

Consider the following well-known construction [88] starting with a real-valued function f_0 in Ω_{η_0}. Let $w(z) = \mathbf{T}_D((\partial_0 f_0)(0, x, y)) = w_1(z) + iw_2(z)$, $z = x + iy \in \mathbb{C}$, so

$$\frac{1}{2}(\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial y}) = \partial_0 f_0, \quad \frac{1}{2}(\frac{\partial w_2}{\partial x} + \frac{\partial w_1}{\partial y}) = 0.$$

Define

$$\vec{v}(x_1, x_2) = -\mathbf{e}_1 \frac{1}{2} w_1(x_1 + ix_2) + \mathbf{e}_2 \frac{1}{2} w_2(x_1 + ix_2), \tag{5.10}$$

so $(\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2)\vec{v} = \partial_0 f_0$.

Next, let

$$\mathbf{e}_1 f_1(x) + \mathbf{e}_2 f_2(x) = -\int_0^{x_0} (\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2) f_0(t, x_1, x_2)\, dt - \vec{v}(x_1, x_2) \tag{5.11}$$

which is well defined for $x \in \Omega_{\eta_0}$.

Finally, we have the following.

Definition 5.18. The above procedure determines the operator

$$\Psi[f_0] = f_0 + \mathbf{e}_1 f_1 + \mathbf{e}_2 f_2. \tag{5.12}$$

Now we prove the following result:

Proposition 5.19. *When f_0 is harmonic, the function $\Psi[f_0]$ given by* (5.12) *is a monogenic function whose scalar part is f_0.*

Proof. By definition,

$$\begin{aligned}
\overline{\partial}\Psi[f_0] &= \partial_0 f_0 + \partial_0(\mathbf{e}_1 f_1 + \mathbf{e}_2 f_2) + (\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2) f_0 \\
&\quad + (\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2)(\mathbf{e}_1 f_1 + \mathbf{e}_2 f_2) \\
&= \partial_0 f_0 - \left(\partial_0(\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2)\int_0^{x_0} f_0(t, x_1, x_2)dt + \partial_0 \vec{v}(x_1, x_2)\right) \\
&\quad + (\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2) f_0 - \Bigg((\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2)^2 \int_0^{x_0} f_0(t, \vec{x})dt \\
&\quad + (\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2)\vec{v}(x_1, x_2)\Bigg).
\end{aligned}$$

First we note that by exchange of order of differentiation,

$$\partial_0(\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2)\int_0^{x_0} f_0(t, x_1, x_2)dt = (\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2) f_0(x).$$

Further, since f_0 is harmonic, we have

$$\begin{aligned}
(\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2)^2 \int_0^{x_0} f_0(t, x_1, x_2)dt &= -(\partial_1^2 + \partial_2^2)\int_0^{x_0} f_0(t, x_1, x_2)dt \\
&= \int_0^{x_0} \partial_0^2 f_0(t, x_1, x_2)dt
\end{aligned}$$

$$= (\partial_0 f_0)(x_0, x_1, x_2) - (\partial_0 f_0)(0, x_1, x_2).$$

Substituting, we have $\overline{\partial}(\Psi[f_0]) = 0$ as required. □

By means of the operator Ψ, we complement the monogenic functions of (5.8), which were defined for $n \geq 1$, as follows.

Definition 5.20. The *basic toroidal monogenics for index* $n = 0$ are

$$T^{+,\mu}_{0,m} = \Psi[I^{+,\mu}_{0,m}] - (\operatorname{coh}\Psi[I^{+,\mu}_{0,m}])W^{-}_{-1}. \tag{5.13}$$

(We recall that a function " $T^{-,\mu}_{0,m}$ " is not to be defined because $\Phi^{-}_{0} \equiv 0$.)

Proposition 5.21. *For all m, μ, $T^{+,\mu}_{0,m} \in \mathcal{M}(\mathcal{A})$ and*

$$\operatorname{Sc} T^{+,\mu}_{0,m} = I^{+,\mu}_{0,m} = I^{*,+,\mu}_{0,m}. \tag{5.14}$$

Further,

$$\operatorname{coh} T^{+,\mu}_{0,m} = 0. \tag{5.15}$$

Proof. By Proposition 5.19, $T^{+,\mu}_{0,m}$ is monogenic and $\operatorname{Sc} T^{+,\mu}_{0,m}$ is as stated. We now show that $T^{+,\mu}_{0,m} \in L^2(\Omega_{\eta_0}, \mathcal{A})$. Since $I^{+,\mu}_{0,m}$ and $\partial_0 I^{+,\mu}_{0,m}$ are bounded in $\overline{\Omega}_{\eta_0}$, then w_1, w_2 as well as $\vec{v}$ given by (5.10) are also bounded. Continuing the construction of Ψ we find that this is also bounded in $\overline{\Omega}_{\eta_0}$ because $\partial_1 I^{+,\mu}_{0,m}$ and $\partial_2 I^{+,\mu}_{0,m}$ in (5.11) are also bounded, and so Ψ is in $L^2(\Omega_{\eta_0}, \mathcal{A})$.

Lastly, by Proposition 5.11 and (5.13) (but not by Proposition 5.6), we see that $\operatorname{coh} T^{+,\mu}_{0,m} = 0$, which establishes the statement. $\square$

The next step requires the representation of the constant function $1 + \mathbf{e}_1 0 + \mathbf{e}_2 0$ in terms of the basic toroidal monogenics. We have a representation of the harmonic function x_0 in terms of toroidal harmonics, which is given in [58] and may be derived from Proposition 1.20 by taking $k = 1$ and $m = 0$:

$$x_0 = \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} n\, I^{-,+}_{n,0}(\eta, \theta, \varphi). \tag{5.16}$$

Substituting (2.9) this can be written

$$\begin{aligned} x_0 &= \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} n \sum_{k=0}^{n} i^n_{k,0} I^{*\,-,+}_{k,0} \\ &= \frac{4\sqrt{2}}{\pi} \sum_{k=0}^{\infty} \sum_{n=(k-1)_+}^{\infty} n\, i^n_{k,0}\, I^{*\,-,+}_{k,0} \\ &= \sum_{k=0}^{\infty} s_k I^{*\,-,+}_{k,0}, \end{aligned} \tag{5.17}$$

where

$$s_k = \frac{4\sqrt{2}}{\pi} \sum_{n=(k-1)_+}^{\infty} n\, i^n_{k,0}.$$

By (5.17) and Definition 5.15,

$$1 = \partial_0 x_0 = \partial x_0 = \partial \sum_{n=0}^{\infty} s_n I^{*\,-,+}_{n,0} = \sum_{n=0}^{\infty} s_n T^{+,+}_{n+1,0}. \tag{5.18}$$

The reason for carrying out this calculation is that it shows that the monogenic function 1, which is a monogenic constant, is not independent of the $T^{\nu,\mu}_{n,m}$.

We are now ready to give the complete independent set of toroidal monogenic functions. In the following chapter, we will use a modified form of this basis.

We have the following relationships among linear spans:

$$\overline{\text{Span}}\{J^{\pm}_m\} \subseteq \overline{\text{Span}}\{I^{\nu,\mu}_{n,m}\} = \overline{\text{Span}}\{I^{*\,\nu,\mu}_{n,m}\},$$

the latter spans are equal to the subspace of all harmonic functions in $L^2(\Omega_{\eta_0})$.

We are now ready to formulate the main result of this section.

Theorem 5.22. *The following set is a basis for the Hilbert space $\mathcal{M}(\mathcal{A})$:*

$$\{T^{\nu,\mu}_{n,m}\}_{n\geq 0} \,\cup\, \{W^{\pm}_m\}^{\infty}_{-\infty}.$$

Thus, every $f \in \mathcal{M}(\mathcal{A})$ can be written uniquely in the form

$$f = \sum_{n\geq 0} a^{\nu,\mu}_{n,m}\, T^{\nu,\mu}_{n,m} + \sum_{-\infty}^{\infty} b^{\mu}_m W^{\mu}_m$$

with real coefficients $a^{\nu,\mu}_{n,m}, b^{\mu}_m \in \mathbb{R}$.

Proof. First we verify that the proposed basis in fact generates $\mathcal{M}(\mathcal{A})$. We are working with the following subspaces of $\mathcal{M}(\mathcal{A})$,

$$E_{\mathrm{W}} = \overline{\mathrm{Span}}(\{W_m^{\pm}\}_{-\infty}^{\infty}), \quad E_{\mathrm{T}} = \overline{\mathrm{Span}}(\{T_{n,m}^{\nu,\mu}\}_{n\geq 0}. \tag{5.19}$$

Write $f = f_0 + f_1\mathbf{e}_1 + f_2\mathbf{e}_2$. Our first step is to pull out the monogenic constants. Decompose the harmonic function $f_0 \in \overline{\mathrm{Span}}(\{I_{n,m}^{\nu,\mu}\})$ as

$$f_0 = h_0 + h_1$$

with $h_0 \in \overline{\mathrm{Span}}\{I_{0,m}^{+,\mu}\}$ and $h_1 \in \overline{\mathrm{Span}}\{I_{n,m}^{\nu,\mu}\colon\ n \geq 1\}$. (Recall that there are no harmonics $I_{0,m}^{-,\mu}$.) By Proposition 5.14, take h so that $\partial_0 h = h_1$. Let

$$\varphi = f - (\Psi[h_0] + \partial h).$$

Then $\mathrm{Sc}\,\varphi = f_0 - (h_0 + h_1) = 0$, so $\varphi \in \mathcal{M}_{\mathrm{C}}(\mathcal{A})$ by Proposition 4.25. Since $h_0 \in \overline{\mathrm{Span}}\{I_{0,m}^{+,\mu}\} = \overline{\mathrm{Span}}\{I_{0,m}^{*\,+,\mu}\}$, by (5.14) we can take $g \in \mathcal{M}(\mathcal{A})$ such that

$$g \in \overline{\mathrm{Span}}(\{T_{0,m}^{+,\mu}\}) \subseteq E_{\mathrm{T}}, \quad \mathrm{Sc}\,g = h_0.$$

By (5.12), this implies that the function

$$\psi = \Psi[h_0] - g$$

is also in $\mathcal{M}_{\mathrm{C}}(\mathcal{A})$. From the above we have

$$f = (g + \partial h) + (\varphi + \psi),$$

where

$$g + \partial h \in E_{\mathrm{T}}, \quad \varphi + \psi \in \mathcal{M}_{\mathrm{C}}(\mathcal{A}) = E_{\mathrm{W}} + \overline{\mathrm{Span}}(1).$$

By (5.18), $\overline{\mathrm{Span}}(1) \subseteq E_{\mathrm{T}}$, so $\mathcal{M}_{\mathrm{C}}(\mathcal{A}) \subseteq E_{\mathrm{W}} + E_{\mathrm{T}}$, which says that $f \in E_{\mathrm{W}} + E_{\mathrm{T}}$ as required.

For the uniqueness, suppose that $f = 0$ in the series representation of the statement of the theorem. Then since $\mathrm{Sc}\, W_m^{\pm} = 0$, by (5.9), we have

$$\begin{aligned} 0 = \mathrm{Sc}\, f &= \sum_{n \geq 0} a_{n,m}^{\nu,\mu}\, \mathrm{Sc}\, T_{n,m}^{\nu,\mu} \\ &= \sum_{m,\nu,\mu} a_{0,m}^{\nu,\mu} I_{0,m}^{\nu,\mu} + \sum_{n \geq 1} \sum_{m,\nu,\mu} a_{n,m}^{\nu,\mu} \kappa_{n,m}^{n-1} I_{n,m}^{\nu,\mu}. \end{aligned}$$

Since the $I_{n,m}^{\nu,\mu}$ are orthogonal in the weighted Hilbert space, we must have $a_{n,m}^{\nu,\mu} = 0$ for all n, m, ν, μ. Therefore

$$\sum_{-\infty}^{\infty} b_m^{\mu} W_m^{\mu} = 0$$

by (corollary 5.13), we have $b_m^{\mu} = 0$ for all m, μ. Therefore the entire collection $\{T_{n,m}^{\nu,\mu}\} \cup \{W_m^{\mu}\}$ is linearly independent in the Hilbert sense. $\square$

We make no claim that the basis of Theorem 5.22 is orthogonal.

Chapter 6

$\mathbb{H}$-valued toroidal monogenic functions

Our approach to the study of monogenic functions on the torus has followed, in very general terms and insofar as possible, the progression of the study of monogenic functions on the ball. We recall that the general theory was developed by R. Fueter [35, 36] for monogenic functions in $\mathbb{H}$, including a basis for homogeneous monogenic polynomials of every given degree in four variables. Later specific bases were constructed for $\mathcal{A}$-valued monogenic functions of three variables, at first only algebraic bases [54], and then orthonormal bases [38]. Much later, an orthonormal basis was constructed for $\mathbb{H}$-valued monogenic functions of three variables in [13, 15, 18, 38, 69, 74, 77].

We will use the construction of a basis for $\mathcal{M}(\mathcal{A}) = \mathcal{M}(\Omega_{\eta_0}, \mathcal{A})$ in the previous chapter as a stepping stone for the construction of a basis for $\mathcal{M}(\mathbb{H}) = \mathcal{M}(\Omega_{\eta_0}, \mathbb{H})$, the space of $\mathbb{H}$-valued monogenics in the torus. The construction is in some ways analogous to [18, 75]. However, at present, it appears that it would be a difficult project to use this approach to obtain an orthonormal basis for these functions.

6.1 Auxiliary results

We begin with the following two auxiliary results which will help us to relate $\mathcal{M}(\mathcal{A})$ to $\mathcal{M}(\mathbb{H})$. A key concept is the subspace $\mathcal{M}_{\mathbb{C}}(\mathcal{A})$ of $\mathcal{A}$-valued monogenic functions.

Lemma 6.1. *Let $f, g \in \mathcal{M}(\mathcal{A})$ be such that $f + g\mathbf{e}_3 = 0$ identically. Then $f, g \in \mathcal{M}_{\mathbb{C}}(\mathcal{A})$.*

Proof. By Lemma 4.3 it follows that $f_0 = g_0 = 0$. By Proposition 4.25, both f and g are monogenic constants. $\square$

Lemma 6.2. *(i) Let $F \in \mathcal{M}(\mathbb{H})$. Then there exist $f, g \in \mathcal{M}(\mathcal{A})$ such that $F = f + g\mathbf{e}_3$.*

(ii) If $f + g\mathbf{e}_3 = \tilde{f} + \tilde{g}\mathbf{e}_3$ with $f, g, \tilde{f}, \tilde{g} \in \mathcal{M}(\mathcal{A})$, then $f - \tilde{f}, g - \tilde{g} \in \mathcal{M}_{\mathbb{C}}(A)$.

Proof. i) We note that construction of Lemma 4.3 is not applicable here, because dropping the $\mathbf{e}_3$ part of an element of $\mathcal{M}(\mathbb{H})$ does not produce an element of $\mathcal{M}(\mathcal{A})$. Using the operator Ψ of (5.12), Section 5.3 of the previous chapter, we construct by Proposition 5.19 monogenic functions $\tilde{f}, g \in \mathcal{M}(\mathcal{A})$ such that

$$\operatorname{Sc} \tilde{f} = F_0, \ \ \operatorname{Sc} g = F_3.$$

Let $\varphi = F - (\tilde{f} + g\mathbf{e}_3)$. Then

$$\varphi = (F_0 - \tilde{f}_0) + \mathbf{e}_1(F_1 - \tilde{f}_1 - g_2) + \mathbf{e}_2(F_2 - \tilde{f}_2 + g_1) + \mathbf{e}_3(F_3 - g_0).$$

Since $\tilde{f}_0 = F_0$ and $g_0 = F_3$, we have $\varphi \in \mathcal{M}_C(\mathcal{A})$, which leaves us $F = f + g\mathbf{e}_3$ where we define $f = \tilde{f} + \varphi$.

(ii) The hypothesis gives $(f - \tilde{f}) + \mathbf{e}_3(g - \tilde{g}) = 0 \in \mathcal{M}_C(\mathcal{A})$, so $f - \tilde{f}, g - \tilde{g} \in \mathcal{M}_C(\mathcal{A})$ by Lemma 6.2. $\square$

It will be useful to note that when $\varphi \in \mathcal{M}_C(\mathcal{A})$, we also have $\varphi e_3 \in \mathcal{M}_C(\mathcal{A})$. In particular, the basic monogenic constants introduced in Definition 5.9 satisfy

$$W_m^{\pm}\,\mathbf{e}_3 = \mp W_m^{\mp}. \tag{6.1}$$

6.2 New complete set for $\mathcal{M}(\mathcal{A})$

A basis for $\mathcal{M}(\mathcal{A})$ as a vector space over $\mathbb{R}$ was given in Theorem 5.22. We want to use Lemma 6.2 to combine a basis for $\mathcal{M}(\mathcal{A})$ with the elements of this same basis multiplied on the right by $\mathbf{e}_3$. Lemma 6.2 makes it clear that the monogenic constants cause ambiguity in such a construction. Equation (6.1) points out in particular that the basic monogenic constants $W_m^{\pm}$ would provide a duplication under this operation. For this reason, instead of the basis of Theorem 5.22, we will construct another basis in which the elements $T_{n,m}^{\nu,\mu}$ will "have the monogenic constants within them removed."

Proposition 6.3. *The collection*

$$(\{T_{n,m}^{\nu,\mu}\}_{n\geq 0} \setminus \{T_{1,0}^{+,+}\}) \cup \{1\} \cup \{W_m^{\pm}\}_{-\infty}^{\infty} \tag{6.2}$$

is a basis for $\mathcal{M}(\mathcal{A})$ over $\mathbb{R}$.

As always, in this notation all admissible combinations of signs (m, ν, μ) are intended.

Proof. First we verify that this set generates $\mathcal{M}(\mathcal{A})$. Let $E \subseteq \mathcal{M}(\mathcal{A})$ denote the closed span of (6.2). By (5.18),

$$T_{1,0}^{+,+} = \frac{1}{s_0}(1 - \sum_{n\geq 2} s_{n-1} T_{n,0}^{+,+}) \in E.$$

Therefore, in the notation of Theorem 5.22, $E_{\mathrm{T}} \subseteq E$, and since by definition $E_{\mathrm{W}} \subseteq E$, we have $E = \mathcal{M}(\mathcal{A})$.

Now we verify the independence. Suppose that

$$\sum_{(n,m,\nu,\mu)\neq(1,0,+,+)} a_{n,m}^{\nu,\mu} T_{n,m}^{\nu,\mu} \;+\; b_0 \;+ \sum_{m=-\infty}^{\infty} c_m^{\mu} W_m^{\mu} \;=\; 0.$$

The scalar part is

$$\sum_{m,\nu,\mu} a_{0,m}^{\nu,\mu} I_{0,m}^{\nu,\mu} \;+ \sum_{\substack{n\geq 1\\(n,m,\nu,\mu)\neq(1,0,+,+)}} a_{n,m}^{\nu,\mu} \kappa_{n,m}^{n} I_{n,m}^{\nu,\mu} \;+\; b_0 \;=\; 0. \tag{6.3}$$

If $b_0 \neq 0$, then we would have

$$1 = -\frac{1}{b_0}\Bigg(\sum_{m,\nu,\mu} a_{0,m}^{\nu,\mu} I_{0,m}^{\nu,\mu} \;+ \sum_{\substack{n\geq 1\\(n,m,\nu,\mu)\neq(1,0,+,+)}} a_{n,m}^{\nu,\mu} \kappa_{n,m}^{n} I_{n,m}^{\nu,\mu} \Bigg).$$

This is a series of toroidal harmonics which does not include $I_{1,0}^{+,+}$, contradicting the unique representation (3.29). Therefore $b_0 = 0$.

Now (6.3) is reduced to

$$\sum_{(n,m,\nu,\mu)\neq(1,0,+,+)} a_{n,m}^{\nu,\mu} T_{n,m}^{\nu,\mu} \;+ \sum_{m=-\infty}^{\infty} c_m^{\mu} W_m^{\mu} \;=\; 0,$$

and by the independence part of Theorem 5.22, we conclude that $a_{n,m}^{\nu,\mu} = 0$ and $c_m^{\mu} = 0$. Therefore the proposed basis has the uniqueness property as claimed. $\square$

Finally we produce the basis for $\mathcal{M}(\mathbb{H})$.

Theorem 6.4. *The set*

$$\{T_{n,m}^{\nu,\mu}\}_{(n,m,\nu,\mu)\neq(1,0,+,+)} \cup \{1\} \cup \{W_m^{\mu}\}_{-\infty}^{\infty} \cup \{T_{n,m}^{\nu,\mu}\mathbf{e}_3\}_{(n,m,\nu,\mu)\neq(1,0,+,+)} \cup \{1\mathbf{e}_3\} \tag{6.4}$$

is a basis for the Hilbert space $\mathcal{M}(\mathbb{H})$ over $\mathbb{R}$.

Proof. Let us write

$$E'_{\mathrm{T}} = \overline{\mathrm{Span}}(\{T_{n,m}^{\nu,\mu}\} \setminus \{T_{1,0}^{+,+}\}).$$

First we show that (6.4) generates $\mathcal{M}(\mathbb{H})$. Let $F \in \mathcal{M}(\mathbb{H})$. By Lemma 6.2 we can write $F = f + g\mathbf{e}_3$ for some $f, g \in \mathcal{M}(\mathcal{A})$. By Proposition 6.3, $f, g \in E'_{\mathrm{T}} + \mathbb{R} + E_{\mathrm{W}}$. By (6.1), $E_{\mathrm{W}}\mathbf{e}_3 = E_{\mathrm{W}}$. Therefore $ge_3 \in E'_{\mathrm{T}}\mathbf{e}_3 + \mathbb{R}\mathbf{e}_3 + E_{\mathrm{W}}$, and it follows that

$$F \in E'_{\mathrm{T}} + \mathbb{R} + E_{\mathrm{W}} + E'_{\mathrm{T}}\mathbf{e}_3 + \mathbb{R}\mathbf{e}_3$$

as claimed.

It only remains to show that (6.4) is a unique representation. Suppose that

$$\sum_{(n,m,\nu,\mu)\neq(1,0,+,+)} a_{n,m}^{\nu,\mu} T_{n,m}^{\nu,\mu} + \ b_0 \ + \ \sum_0^{\infty} c_m^{\mu} W_m^{\mu} \ + \sum_{(n,m,\nu,\mu)\neq(1,0,+,+)} \hat{a}_{n,m}^{\nu,\mu} T_{n,m}^{\nu,\mu}\mathbf{e}_3 \ + \ \hat{b}_0\,\mathbf{e}_3 \ = 0$$

with real coefficients, and the usual conventions concerning m, ν, μ.

Consider the following elements of $\mathcal{M}(\mathcal{A})$:

$$f = \sum_{(n,m,\nu,\mu)\neq(1,0,+,+)} a_{n,m}^{\nu,\mu} T_{n,m}^{\nu,\mu} \ + \ b_0 \ + \ \sum_0^{\infty} c_m^{\mu} W_m^{\mu},$$

$$g = \sum_{(n,m,\nu,\mu)\neq(1,0,+,+)} \hat{a}_{n,m}^{\nu,\mu} T_{n,m}^{\nu,\mu} \ + \ \hat{b}_0.$$

Since they satisfy $f + g\mathbf{e}_3 = 0$ by Lemma 6.1, $f, g \in \mathcal{M}_{\mathbb{C}}(\mathcal{A})$. Since $b_0 + \sum c_m^{\mu} W_m^{\mu} \in \mathcal{M}_{\mathbb{C}}(\mathcal{A})$, necessarily also

$$\sum_{(n,m,\nu,\mu)\neq(1,0,+,+)} a_{n,m}^{\nu,\mu} T_{n,m}^{\nu,\mu} \in \mathcal{M}_{\mathbb{C}}(\mathcal{A}) = \overline{\mathrm{Span}}(\{1\} \cup \{W_m^{\pm}\}).$$

By Proposition 6.3, $\{1\} \cup \{W_m^{\pm}\}$ is independent of $\{T_{n,m}^{\nu,\mu}\}_{(n,m,\nu,\mu)\neq(1,0,+,+)}$, so $a_{n,m}^{\nu,\mu} = 0$. Since g is a monogenic constant, $\hat{a}_{n,m}^{\nu,\mu} = 0$.

From the foregoing,

$$b_0 + \sum_0^{\infty} c_m^{\mu} W_m^{\mu} + \hat{b}_0 \mathbf{e}_3 = 0$$

Since the scalar and e_3 components vanish, $b_0 = \hat{b}_0 = 0$. Then $\sum_0^{\infty} c_m^{\mu} W_m^{\mu} = 0$, so by the unique representation of the W_m^{μ}, $c_m^{\mu} = 0$. This concludes the proof of the uniqueness of the series representation, so (6.4) is indeed a basis for $\mathcal{M}(\mathbb{H})$. $\square$

Conclusions and ideas for further study

In the first part of this thesis, a function theory related to toroidal harmonics was developed in two separate contexts. The first context concerns the construction of a reverse-Appell basis of harmonic functions expressed in terms of toroidal coordinates as independent variables to derive bases in the real L^2-Hilbert spaces of reduced quaternion and quaternion-valued monogenic functions on toroidal domains. In contrast to the classical toroidal harmonics, the reverse-Appell system is not orthogonal, and it is not known whether it is possible to construct an orthogonal system along these lines. Further, the norms of the reverse-Appell harmonics have not been calculated. The second context develops a technique for studying the Dirichlet-to-Neumann mapping and solving the Neumann problem for the Laplace operator on a torus. It would be interesting to see how this technique applies in practical situations, such as electrostatics.

In the second part of this thesis, bases for the spaces of $\mathcal{A}$-valued and $\mathbb{H}$-valued monogenic functions are constructed on the torus. Again, the bases are not orthogonal and it seems a challenging problem to orthogonalize them. A further problem would be to find a basis for $\mathcal{M}(\mathbb{H})$ over $\mathbb{H}$ rather than only over $\mathbb{R}$.

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