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“Dualidad-T No-Abeliana en Modelos Sigma Lineales Supersimétricos (0,2) Normados en Dos Dimensiones”

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“Non-Abelian Tduality in Two Dimensional Supersymmetric (0,2) Gauged Linear Sigma Models”

by

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Abstract

It is studied $U(1)$ two-dimensional $(0, 2)$ gauged linear sigma models with global symmetries, which manifest T-duality by gauging these symmetries. First, the dual theory is found using a reduction derived from $(2, 2)$ dualization. Subsequently, an abelian T-dual model of the $(0, 2)$ theory is constructed without supersymmetry reduction. Moreover, it is generalized the non-abelian T-dualization of $U(1)$ $(0, 2)$ 2D GLSMs and a specific model with $SU(2)$ global symmetry is investigated. In all cases, the supersymmetric vacua of the bosonic potential is analysed, which determine the target space geometry in both the original and dual models. For the case with abelian global symmetry, the dual model should act as the mirror. It is also described instanton corrections in these setups. Finally, an example of a non-compact Calabi-Yau manifold and its non-abelian $SU(2) \times SU(2)$ T-dual is analysed.

Resumen

Se consideran modelos sigma lineales *normados* $(0,2)$ bidimensionales con simetrías globales $U(1)$, y se realiza la T-dualidad al hacer estas simetrías locales. Primero, se encuentra el dual de la teoría utilizando la reducción obtenida de la dualización supersimétrica $(2,2)$. Luego, se construye un modelo T-dual abeliano de la teoría $(0,2)$ sin necesidad de reducción de supersimetría. Además, se lleva a cabo la dualización T no abeliana de los modelos sigma lineales $(0,2)$ 2D con simetría $U(1)$ en general, y se estudia un modelo específico con simetría global $SU(2)$. En todos los casos analizados, se examinan los vacíos supersimétricos del potencial bosónico, lo que permite determinar la geometría del espacio objetivo en el modelo original y en el dual. Para el caso con simetría global abeliana, el modelo dual debe actuar como el espejo del original. También se describen las correcciones de instantón en los diferentes escenarios. Finalmente, se analiza un ejemplo de una variedad de Calabi-Yau no compacta y su T-dual no abeliano $SU(2) \times SU(2)$.

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Contents

Introduction

The main goal of **string theory** (ST) is to provide a consistent framework for a quantum theory of the gravitational field. As a byproduct the formulation has the potential to achieve the unification of all fundamental forces of nature. This means describing gravity, electromagnetism and the weak and strong forces within the same theoretical framework. String theory has made significant progress and has led to many interesting predictions. In superstring theory (the string theory which incorporates supersymmetry[1]) one notable prediction is that space-time must have ten dimensions for the theory to be consistent. In 1985, *Candelas et al.* [1] showed that compactifying six[2] of those dimensions on a Calabi-Yau (CY) manifold[3] can preserve the $\mathcal{N} = 1$ SUSY in the four observable dimensions and this results in particle spectra and forces that could potentially match those observed in the real world, providing a foundation for constructing more realistic models. Additionally, the properties of this CY manifold affect the physics in the resulting four-dimensional theory. For a broad overview, see [3].

[1]Supersymmetry (SUSY) will be discussed in the next chapter

[2]Six real dimensions or three complex ones.

[3]A CY manifold is a complex, compact Kähler manifold with a vanishing first Chern class, which is also Ricci flat, see [2] for example.

In string theory the point-like particles are replaced by one-dimensional objects called strings that can be open or closed, and they propagate through space-time, tracing out a two-dimensional surface called the *worldsheet* as it evolves over time. On distance scales larger than the string scale, a string will look just like an ordinary particle, with its mass, charge, and other properties determined by the vibrational state of the string and some properties of the internal space, the CY manifold. Splitting and recombination of strings correspond to particle emission and absorption, giving rise to the interactions between particles[2, 4]. Its dynamics is governed by a **Non-Linear Sigma Model** (NLSM), this is a QFT which provides the mathematical framework to describe how the string's worldsheet maps into the target space, capturing the interaction between the string and the geometry of the space it inhabits. Therefore, a consistent superstring model is a non-linear supersymmetric sigma model with four flat Minkowski space-time directions and the remaining six dimensions are extended on a Calabi-Yau manifold as the internal target space [3]. CY sigma models provide one means of building $\mathcal{N} = 2$ superconformal models that can be taken as the internal part of a string theory.

Another type of two-dimensional theory related to target-space geometries is known as **Landau-Ginzburg** models (LG). Any quantum field theory with a unique classical vacuum state and a potential energy with a degenerate critical point is called a Landau-Ginzburg theory. The generalization to $\mathcal{N} = (2, 2)$ supersymmetric theories in 2D was proposed in [5]. It was noted that their massless spectra are often similar to those of non-linear sigma models. Later, it was found that these theories are related by a renormalization group flow to sigma models on Calabi-Yau manifolds [6].

Witten found in 1993 a natural relation between sigma models based on Calabi-Yau manifolds and Landau-Ginzburg models for $\mathcal{N} = 2$ SUSY gauge theories, but in $2D$ [7]. This construction also allows to include models with $(0, 2)$ world-sheet supersymmetry, until that time only the case of $(2, 2)$ supersymmetry has been considered. In that notorious work Witten constructed **Gauged Linear Sigma Models** (GLSM) and described that Landau–Ginzburg theories and sigma models on Calabi–Yau manifolds are different phases of the same GLSM, interpolating one on each other under the change of the parameters.

In 1987, Dixon *et al.* noticed that given a compactification of string theory, it is not possible to uniquely reconstruct a corresponding Calabi–Yau manifold [8]. Instead, two apparently different versions of string theory can be compactified on completely different Calabi–Yau manifolds, resulting in the same physics. Thus, both string theories are equivalent and describe two isomorphic theories from the 2D point of view. In the perspective of the target space they seem to be different theories, but they are actually equivalent. These manifolds are called mirror pairs, and the relationship between the two physical theories is referred to as **mirror symmetry** [9]. Mirror symmetry has become a valuable tool for calculations in string theory. Mathematically, mirror symmetry can also be seen as a relationship between manifolds, where two distinct Calabi-Yau manifolds, typically having different topologies, can be physically equivalent in the worldsheet perspective. The mirror map associates with almost any Calabi-Yau three-fold M its "mirror" Calabi-Yau three-fold W such that:

$$H^{p,q}(M) = H^{3-p,q}(W), \tag{1.1}$$

where $H^{p,q}(M)$ are the Hodge numbers of the complex manifold M [2].

Mirror symmetry is just a particular example of what physicists call a **duality**. In general, duality refers to a description in which two seemingly different physical theories result to be equivalent in a non-trivial way. If a theory can be transformed so it looks just like another theory, the two are said to be dual each other under a duality transformation. In other words, both theories are mathematically different descriptions of the same phenomena. Dualities play an important role in modern physics, especially in string theory. These dualities led to the realization, in the mid-1990s, that all of the five[4] consistent superstring theories are just different limiting cases of a single eleven-dimensional theory called M-theory [10].

One of the most known of such dualities is **Target-space duality** or **T-duality** noted for the first time in 1986 [11], which in the simplest case consist in one theory describing strings propagating in a space-time shaped like a circle of some radius R, and another theory describing strings propagating on a space-time shaped like a circle of radius proportional to $1/R$, they are equivalent in the sense that all observable quantities (spectrum and interactions) in one description are identified with quantities in the dual description. For example, momentum in one description takes discrete values and is equal to the number of times the string winds around the circle (winding number) in the dual description.

Dualities have significant implications for the theory. For example, it was subsequently shown that Type IIA string theory is equivalent to Type IIB string theory via T-duality, and that the two versions of heterotic string theory are also related by it.

[4]There appear to be only five different consistent (anomaly free) superstring theories known as Type I $SO(32)$, Type IIA, Type IIB, $SO(32)$ Heterotic and $E_8 \times E_8$ Heterotic.

Moreover, Type IIA superstring theory compactified on a Calabi-Yau manifold M is equivalent to Type IIB superstring theory compactified on the mirror dual Calabi-Yau manifold W. In this way **mirror symmetry is T-duality**, as established rigorously in the SYZ[5] conjecture [12].

Today, mirror symmetry continues to be an active area of research in both physics and mathematics. Mathematicians are striving to develop a more complete mathematical understanding of mirror symmetry, following the intuition provided by physicists [13]. In general, mirror symmetry is a conjecture; however, many works provide mathematical proofs of numerous results originally obtained by physicists using mirror symmetry [14]. In 2000, Kentaro Hori and Cumrun Vafa have proved mirror symmetry for supersymmetric sigma models on Kähler manifolds in $2D$ dimensions using **abelian T-duality** on GLSMs [15]. One motivation to go beyond the realm of Abelian T-duality in [15] comes from the fact that there is a large set of CY manifolds that do not constitute complete intersections but rather Grassmanians, Pfaffians or determinantal; that can be studied as Non-Abelian GLSMs [16], and a description of the symmetries in these models is of interest [17, 18, 19], in particular the study of mirror symmetry [20, 21, 22, 23]. Localization properties of the partition function have been used to test Abelian T-duality in GLSMs that lead to mirror symmetry [24].

In GLSM with $\mathcal{N} = 2$ SUSY, T-duality can be achieved by gauging abelian global symmetries and introducing Lagrange multipliers [25, 26, 27]. Unlike the approach by Hori and Vafa, this method allows for a generalization to the **non-abelian T-duality** case, as demonstrated in [28]. In that article, the authors performed the abelian du-

[5]Named after Strominger, Yau, and Zaslow.

alization of GLSM with $\mathcal{N} = (2,2)$, recovering the results of [15], and extending it to non-abelian global symmetry. This Buscher-Giveon-Roček-Verlinde target space (T-)duality algorithm involves gauging global symmetries. This procedure was successful in providing a physical proof of the mirror symmetry correspondence.

This idea can be followed for $2D$ $\mathcal{N} = (0,2)$ GLSM. Unlike their $(2,2)$ counterparts (which manifest an unbroken E_6 gauge symmetry), the $(0,2)$ world-sheet supersymmetry is of phenomenological interest as it can naturally lead, for instance, to models with space-time grand unified gauge group $SU(5)$ or $SO(10)$ and therefore of particular interest for model building [29, 30, 31, 32]. These $(0,2)$ models have similar properties as the $(2,2)$ models, but there are many features in which they differ. For instance, the $(0,2)$ models are chiral. These models have been studied actively and very good treatments can be found in [7, 33, 34, 35, 36, 37, 35].This thesis is devoted mainly to the study of those $(0,2)$ **dualities in $2D$ GLSM.**

Recently, [20, 36] proposed an abelian ansatz to relate non-abelian mirrors by examining non-linear sigma models in the topologically twisted A and B models. They also discussed the hypothesis that non-abelian T-duality is related to these mirrors. More predictions of phenomenological interest are relevant in the early universe or in microscopic aspects of black hole physics [38, 39].

As mentioned earlier, $(2,2)$ GLSMs are an important tool in proving mirror symmetry for CY manifolds, particularly in the case of complete intersections of CY manifolds and toric varieties [15, 13]. There have been numerous studies on GLSMs and their applications [40, 18, 41, 17, 42, 19, 43, 22, 23, 44, 45, 46, 47]. However, in the case of GLSMs with $(0,2)$ supersymmetry, the realization of mirror symmetry is less apparent. A specific type of mirror map can be defined for these models [48, 49, 50, 51, 52, 53]. Other notions of the $(0,2)$ mirror map are discussed in [54, 35, 36, 55, 21, 37]. In particular, [54] studies the Abelian GLSM with a gauged Abelian global symmetry. Other developments of $(0,2)$ GLSMs in different contexts can be found in Refs. [56, 57, 58, 59, 60, 61]. For a very recent overview of some important results of the GLSMs see [62, 63]. AdS/CFT solutions have been explored from the non Abelian T-dualities [64, 65, 66, 67, 68, 69].

The work by Cabo *et al.* [28] has served as the primary motivation for this thesis, which considers a $U(1)$ two-dimensional $(0,2)$ GLSM with global symmetries and realizes T-duality as a gauging of these symmetries, both for abelian and non-abelian T-duality. This thesis is primarily based on the collaborative research conducted by the author and his doctoral advisors, which was published in [70].

The thesis is organized as follows: A brief overview of supersymmetry is presented in Chapter 2. In the first section 2.1, the fundamental concepts and essential mathematical tools that will be utilized are discussed, providing notation and conventions for future reference and introduce some basic material for the reader. The second section 3.3 emphasize the case of $\mathcal{N}=2$ and the content of $(0,2)$ fields.

In Chapter 4, abelian duality is carried out. First, in Section 4.1, the method is illustrated using the $\mathcal{N}=(2,2)$ case from [28] as an example. Then, the $(0,2)$ cases are shown in a general form for an arbitrary number of fields. Later, in Section 4.4, two cases are discussed: one where $(0,2)$ fields can be viewed as a reduction of $(2,2)$ fields, and another where they cannot. In the former case, the results are compared with those obtained previously in the scientific literature.

Non-abelian T-duality is performed in chapter 5 for both cases as reduction and as a pure case. In all cases the instanton correction is discussed. Finally in section 5.2 a case with $SU(2) \times SU(2)$ global symmetry is fully analysed.

Supersymmetry

The study of *supersymmetry* is a recent field of research in theoretical physics and one of the most extensively researched areas today. Broadly speaking, supersymmetry (SUSY) is said to be a symmetry that associates fermions with bosons and vice versa; that is, for every bosonic variable there is (at least) another fermionic one and vice versa. The important thing is that the correspondence between bosons and fermions is such that the number of degree of freedom of both match. In contrast to a non-supersymmetric theory where there can be only fermions or only bosons, SUSY always includes both. The fermionic (or bosonic) variable associated with a given bosonic (or fermionic, respectively) variable is known as a "superpartner". It is important to emphasize that a theory is supersymmetric if it fulfills this "association", it is not that supersymmetry is added, or that the theory has supersymmetry, although it is often mentioned as such.

For example, the Standard Model (SM) is not a supersymmetric theory. Many models have been proposed to implement a supersymmetric extension that includes the SM, where, for instance, there is a supersymmetric partners for the electron, photon, etc. and similarly for all other particles of the SM. In general, if the particle is a fermion, its (super)partner will be called by prefixing "s" to the name of the fermion. If it is a boson, it will be called by replacing "on" with "ino" at the end of the boson's name (or simply adding it). Thus, it would be had selectron, squark, photino, gravitino and higgsino to mention a few.

In what follows, many of the "super" prefixes will be omitted for convenience when there is no ambiguity. Thus, instead of superfields, it will be said only fields, etc.

For notation and conventions on supersymmetric field theory we follow some classical references, see [71, 72, 73]. For some background material regarding $(0, 2)$ GLSMs the reader would consult Refs. [7, 33, 34].

2.1 Generalities

The symmetries of a quantum field theory (QFT) can be classified into two general categories: internal symmetries, which correspond to the transformations of the different fields while keeping the spacetime point where the field is defined fixed, and spacetime symmetries (in relativistic theories, this symmetry is the Poincaré group).

In attempts to find more general symmetry groups, in 1967 Coleman and Mandula proved a no-go theorem asserting that any larger symmetry group (that includes both) must be a direct product of the two symmetries. In other words, there was no non-trivial way to combine them. However, they considered that all symmetries had to be written in terms of Lie groups (or Lie algebras) [74].

Nevertheless, in 1975, Haag, Łopuszański, and Sohnius [75] proved that this could be resolved by taking **graded Lie algebras** instead of Lie algebras. In physics, $\mathbb{Z}_2$-graded algebras[1] are now called superalgebras, as they are relevant in supersymmetry.

2.1.1 $\mathbb{Z}_2$-graded Lie Algebras

To broadly describe $\mathbb{Z}_2$-graded Lie algebras, we start by recalling that the generators of a Lie algebra obey certain commutation rules, and we add other generators that obey anticommutation rules. That is, there are "even" and "odd" generators that satisfy:

$$[\text{even}, \text{even}] = \text{even},$$

$$[\text{even}, \text{odd}] = \text{odd},$$

$$\{\text{odd}, \text{odd}\} = \text{even},$$

similar to the addition in $\mathbb{Z}_2$. In a given theory, variables that commute are called bosonic, and those that anticommute are called fermionic.

2.1.2 Superspace

In the development of supersymmetric theories, it is common to use the construction of superspace. This means (remembering that supersymmetry is a spacetime symmetry) adding to the spacetime coordinates other "supercoordinates" which are variables that anticommute[2]. Thus, if θ^1 and θ^2 are two Graßmann numbers, they anticommute ($\theta^1\theta^2 = -\theta^2\theta^1$) and satisfy:

$$(\theta^\alpha)^2 = 0 \, , \tag{2.1}$$

[1] In mathematics, other gradations can occur. Here we only consider the group $\mathbb{Z}_2$, which can be thought of as the group of two elements, even and odd integers.

[2] These variables are usually called Graßmann numbers and are denoted by θ

with $\alpha \in \{1, 2\}$. Moreover, any higher power of each of them vanishes.

A number of Graßmann numbers are added to the coordinates according to the degree of supersymmetry in question. Thus, for example, if x^μ are the given spacetime coordinates and we add[3] θ^1, $\bar{\theta}^{\dot{1}}$, θ^2 and $\bar{\theta}^{\dot{2}}$ as Graßmann variables; we obtain a superspace with supercoordinates $(x^\mu, \theta^1, \bar{\theta}^{\dot{1}}, \theta^2, \bar{\theta}^{\dot{2}})$. Here, the coordinates x^μ are bosonic degrees of freedom and the coordinates θ^α are fermionic degrees of freedom.

All this formalism can be deepened to formally define supervarieties (or supermanifolds) or geometry in the superspace. In this way one can define functions on the superspace, which will be the superfields.

2.1.3 Superfields

A function $\Phi(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ that takes values in superspace is called a **superfield**[4] (or simply a field in the context of supersymmetry if there is no ambiguity). Each theory (or model) that is considered will have different types of (super)fields and different quantities of them; and they may or may not exhibit very diverse properties.

If for every fermionic coordinate θ^α it holds that:

$$[\theta^\alpha, \Phi] = 0, \tag{2.2}$$

the field Φ is said to be **bosonic**. And if it holds that:

$$\{\theta^\alpha, \Phi\} = 0, \tag{2.3}$$

[3]Here, the conventional QFT notation is used to denote right-handed spinors or spinors in the $(0, 1/2)$ representation with a dot over the indices and a bar over the variable, and left-handed spinors or spinors in the $(1/2, 0)$ representation without a dot over the indices, nor a bar.

[4]The codomain of the function is completely arbitrary.

the field Φ is said to be **fermionic**.

Due to (2.1), a Taylor series expansion only in the θ coordinates truncates at a finite number of terms.

For example, if we take the superspace coordinates as (x^μ, θ), then the superfield $\Phi_1(x^\mu, \theta)$ will have the expansion:

$$\Phi_1(x^\mu, \theta) = \phi_1(x^\mu) + \theta\psi_1(x^\mu) \ . \tag{2.4}$$

Or if we take a superspace as $(x^\mu, \theta, \bar{\theta})$, then the superfield $\Phi_2(x^\mu, \theta, \bar{\theta})$ will have the expansion:

$$\Phi_2(x^\mu, \theta) = \phi_2(x^\mu) + \theta\psi_2(x^\mu) + \bar{\theta}\bar{\psi}_2(x^\mu) + \theta\bar{\theta}F(x^\mu) \ . \tag{2.5}$$

Furthermore, the coefficients of such an expansion will depend only on the spacetime coordinates x^μ (they are ordinary fields) and there will be only a finite (and very small) number of them. In this way, each superfield will be completely determined by a finite collection of fields. These fields (the coefficients in the expansion) are called **component fields** (or components).

If N is the number of Graßmann coordinates, there are 2^N components. The set of components of a field is called a **multiplet**, and this multiplet shows the particle content of the theory. It is common to denote the superfield with an uppercase letter (Φ, Σ, V, ...) and the *first* component with a lowercase letter ($\Phi|_{\theta=\bar{\theta}=0} = \phi$, $\Sigma|_{\theta=\bar{\theta}=0} = \sigma$, $V|_{\theta=\bar{\theta}=0} = v$, ...).

For the $\mathcal{N} = 1$ superfield in (2.4), there are 2 components and the multiplet is $\{\phi, \psi_1\}$, which are fields that only depend on x^μ. We can identify ϕ as a bosonic field and ψ_1 as a fermionic field and each is the superpartner of the other. For the $\mathcal{N} = 2$ superfield in (2.5), there are 4 components and the multiplet is $\{\phi, \psi_2, \bar{\psi}_2, F\}$, which we can think of as two bosonic fields ϕ and F, and the fermionic fields ψ and $\bar{\psi}$.

2.2 Dimensional Reduction

In this work, two cases will be considered. Both are obtained from the dimensional reduction of a theory in 4 spacetime dimensions and an $\mathcal{N} = 1$ supersymmetric gauge theory, which is reduced to 2 dimensions. The resulting models, denoted as $(2, 2)$ and $(0, 2)$, have $\mathcal{N} = 2$ supersymmetry.

2.2.1 $\mathcal{N} = (2, 2)$ SUSY

In the first method of reduction, there are two left-handed supersymmetry generators Q_-, $\bar{Q}_-$ and two right-handed generators Q_+, $\bar{Q}_+$ in $2D$, which emerge from the 4 generators in $4D$; this is denoted as $(2, 2)$.

Starting from an $\mathcal{N} = 1$ gauge theory in $4D$, we can write the spacetime coordinates as: (x^0, x^1, x^2, x^3) with 4 supersymmetry generators.

If all fields are taken to be independent of the coordinates x^1 and x^2, then the components v^1 and v^2 of the gauge field V (along with all other fields) are functions of x^0 and x^3 only. Hence, v^1 and v^2 are free fields in the new model. With this, we adopt the new two-dimensional notation (variable change) $x^0 = \tilde{x}^0$ and $x^3 = \tilde{x}^1$.

After this, following [7] it is convenient to relabel the fermionic components as: $(\psi^1, \psi^2) = (\psi^-, \psi^+)$ and similarly for dotted components. Thus, in $(2,2)$, the 4 SUSY generators will be Q^-, $\bar{Q}^-$, Q^+, and $\bar{Q}^+$.

With all this, the coordinates of the $(2,2)$ superspace can be taken as:

$$(\tilde{x}^0, \tilde{x}^1, \theta^-, \bar{\theta}^-, \theta^+, \bar{\theta}^+).$$

2.2.2 $\mathcal{N} = (0,2)$ SUSY

In the second method, there are only 2 right-handed supersymmetries, denoted as $(0,2)$; here, only the operators Q_+ and $\bar{Q}_+$ generate SUSY in 2D.

The process is similar to the previous one, and in this case, the coordinates of the $(0,2)$ superspace are taken as:

$$(\tilde{x}^0, \tilde{x}^1, \theta^+, \bar{\theta}^+).$$

2.2.3 Differential Operators

Differential operators can be introduced to act in superspace to generalize ordinary derivatives $\partial \to D$. These operators anticommute with the supersymmetry generating charges. Their explicit form in 4 dimensions is given by[5] $D_\alpha = \partial_{\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m$.

[5]The notation σ^m refers to the Pauli matrices. In the $2D$ case, only some of them are used.

For the $(2,2)$ case, they have the explicit form:

$$\bar{D}_{\pm} = -\partial_{\bar{\theta}^{\pm}} + i\theta^{\pm}(\partial_0 \mp \partial_1) \, , \qquad D_{\pm} = \partial_{\theta^{\pm}} - i\bar{\theta}^{\pm}(\partial_0 \mp \partial_1) \, , \tag{2.6}$$

while for the $(0,2)$ case, they have the form:

$$D_{+} = \partial_{\theta^{+}} - i\bar{\theta}^{+}(\partial_0 + \partial_1) \, , \qquad \bar{D}_{+} = -\partial_{\bar{\theta}^{+}} + i\theta^{+}(\partial_0 + \partial_1) \, . \tag{2.7}$$

There are some constraints that frequently appear in SUSY.

$$\bar{D}_{\alpha}\Phi = 0 \, , \tag{2.8}$$

$$D_{\alpha}\bar{\Phi} = 0 \, , \tag{2.9}$$

$$V = \bar{V} \, . \tag{2.10}$$

Fields that satisfy these equations receive special names. A field Φ that satisfies (2.8) is called **chiral**, a field $\bar{\Phi}$ that satisfies (2.9) is called **antichiral**[6], and a field V that satisfies (2.10) is called vectorial supermultiplet. As the notation suggests, the complex conjugate of a chiral field is antichiral. The product of two (anti)chiral fields is (anti)chiral. A simple example of a vectorial field is $\bar{\Phi}\Phi$. Naturally, the multiplet of a chiral field is called a **chiral multiplet**; similarly for the other cases.

For the $(2,2)$ model, the component expansion of a chiral field $\Phi(\tilde{x}^{\mu}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}})$ is obtained by solving (2.8). This is easily solved by considering the variable change $y^{\mu} := \tilde{x}^{\mu} + i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$ [73]. In this way, Φ will not have explicit dependence on $\bar{\theta}^{\dot{\alpha}}$, implying that Φ has the expansion:

$$\Phi(y^{\mu}, \theta^{\pm}) = \phi(y^{\mu}) + \sqrt{2}\theta^{\alpha}\psi_{\alpha}(y^{\mu}) + \theta^{\alpha}\theta_{\alpha}F(y^{\mu}) \, . \tag{2.11}$$

[6] Here, the field is $\bar{\Phi}$ (with the bar). If the bar is omitted, it should also be omitted in the equation.

The factor $\sqrt{2}$ was manually added and adjusts with a simple redefinition of ψ_α. To obtain the expansion of an antichiral field, it can be done similarly or more conveniently, as the notation suggests, just take the conjugate of (2.11):

$$\bar{\Phi}(\bar{y}^\mu, \bar{\theta}^\pm) = \bar{\phi}(\bar{y}^\mu) + \sqrt{2}\bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}(\bar{y}^\mu) + \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\bar{F}(\bar{y}^\mu)\ . \tag{2.12}$$

The expansion for a vectorial field will not be provided for now, but it is generally done using the Wess-Zumino gauge, see for example page 300 of [73].

In the $(0, 2)$ model case, it is a bit simpler to operate, as there are only 2 Grassmann coordinates. The expansion for a chiral field can be obtained from its expansion as in (2.5) and the derivative in (2.7); it is quickly obtained that it should satisfy: $\bar{\lambda} = 0$ and $F = -i(\partial_0 + \partial_1)\phi$; thus, the expansion is:

$$\Phi(y^\mu, \theta^+, \bar{\theta}^+) = \phi(y^\mu) + \sqrt{2}\theta^+\psi_+(y^\mu) - i\theta^+\bar{\theta}^+(\partial_0 + \partial_1)\phi(y^\mu)\ . \tag{2.13}$$

The antichiral case is obtained by conjugation.

Sigma Models

In the 60's Gell-Mann and Lévy [76] proposed a model for a hypothetical spinless particle called sigma (σ), a scalar meson[1] previously introduced by Julian Schwinger [77]. It was an outdated QFT where the field took values in a vector space, it was called **sigma model**. Later, a modified version of the sigma model, where the scalar field takes values in a more general manifold was called the "non-linear sigma model".

3.1 Introduction

A sigma model is a FT (field theory) formed by a scalar field (or several):

$$\phi_i : \mathcal{M} \to \mathcal{X} \ ,$$

which maps from a manifold $\mathcal{M}$ called worldsheet with metric $h^{\alpha\beta}$, to another space called "target" with metric $g^{\mu\nu}$ and which is described by the Lagrangian density:

$$\mathcal{L} = \sum_{i,j=1}^{n} g_{ij} d\phi_i \wedge \star d\phi_j = \sum_{i,j=1}^{n} g_{ij} \partial_\nu \phi_i \partial^\mu \phi_j \sqrt{h}\, dx^1 \wedge \cdots \wedge dx^m$$

[1]The sigma-meson, now also called the $f_0(500)$-resonance, is the lightest Lorentz-scalar and isospin-scalar meson.

If the target $\mathcal{X}$ is a linear space, it is called a linear sigma model, otherwise it is a non-linear sigma model. $\mathcal{M}$ is usually thought of as our spacetime, and the target as some abstract space; however, with the advent of string theory, the use of the target as our spacetime gained popularity, and the domain space as some lower-dimensional space, often denoted Σ. In superstring theories, dynamics is governed by non-linear sigma models.

A trivial example can be the Newtonian particle, where the particle's position $x(t)$ is described as a field defined over $\mathbb{R}$ with the Lagrangian:

$$L = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 \ .$$

A less trivial but very direct example is the Polyakov action of the bosonic string:

$$\mathcal{L} = \sqrt{-h} h^{ab} g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \ .$$

Here, a configuration of X fields is then interpreted as the trajectory of an extended particle, where $\mathcal{M}$ the worldsheet is two-dimensional and the target is spacetime.

3.2 GLSM

Thus, a sigma model can be coupled with a connection or a gauge theory[2] that takes values in some Lie group, to make a gauged sigma model (GSM). It is common to talk about GLSM or GNLSM, gauged linear sigma model or gauged nonlinear sigma model.

[2]Mathematically, a gauge theory is one in which the fields are connections (in the sense of differential geometry).

Working with supersymmetric sigma models (or SUSY GLSM supersymmetry gauged linear sigma model), in addition to the scalar field ϕ, there will be spinorial fields ψ as their supersymmetric partners. Or grouped into a single multiplet as in (2.11):

$$\Phi(y^\mu, \theta^\pm) = \phi(y^\mu) + \sqrt{2}\theta^\alpha \psi_\alpha(y^\mu) + \theta^\alpha \theta_\alpha F(y^\mu) \ .$$

As a quick example, a supersymmetric extension with $\mathcal{N} = 1$ and $n = 1$ has the Lagrangian:

$$L = \frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}\psi\dot{\psi}.$$

In this work, the SUSY $(2,2)$ and $(0,2)$ models will be studied. It will be introduced a gauge field to give rise to a GLSM; that is, replace the differential operators D_α, $\bar{D}_\alpha$, and ∂_m with covariant derivatives $\mathcal{D}_\alpha$, $\bar{\mathcal{D}}_\alpha$, and $\mathcal{D}_m$. The following construction will adhere to the one presented by Witten in [7] for the formulation of GLSMs.

Let's start with the fields representation of GLSM with $\mathcal{N} = (2,2)$. In the next section the fields content of $(0,2)$ is reviewed since is the main topic in this work. A $(2,2)$ chiral superfield Φ in a certain representation of the gauge group is a superfield that satisfies $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$, where the covariant derivative is taken in the appropriate representation.

To allow the existence of charged chiral superfields, the integrability of the equation $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$ is needed. For this, it must be satisfied:

$$0 = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} \ . \tag{3.1}$$

This implies, with the gauge partially fixed, that $\bar{\mathcal{D}}_{\dot\alpha} = e^V \bar{D}_{\dot\alpha} e^{-V}$, where V is a vector field that takes values in the Lie algebra, so V transforms under the gauge group as:

$$V \quad \rightarrow \quad V + i\Lambda - i\bar{\Lambda} \ ,$$

where Λ is a chiral field.

A chiral superfield Φ of charge Q transforms under the residual gauge as:

$$\Phi \quad \rightarrow \quad e^{-iQ\Lambda}\Phi \ .$$

One of the novelties that appear in 2D is that in addition to chiral fields, it is possible to have "twisted chiral superfields" U, which are fields that satisfy:

$$\bar{D}_+ U = D_- U = 0 \ . \tag{3.2}$$

As mirror symmetry transforms chiral multiplets into twisted ones, it is likely that taking models with multiplets of both types will be useful to understand it. This is an exclusive feature for $(2,2)$ models, since twisted multiplets don't exist in $(0,2)$.

In this regard, the field strength Σ invariant under basic gauge is twisted chiral:

$$\Sigma = \frac{1}{2}\{\bar{\mathcal{D}}_+, \mathcal{D}_-\} \ ;$$

for the abelian case, $\Sigma_0 = \frac{1}{2}\bar{D}_+ D_- V_0$.

Supersymmetric Lagrangians can be obtained from integrations over suitable fermionic coordinates. The Lagrangian for this $(2,2)$ case is given by:

$$L = L_{kin} + L_W + L_{gauge} + L_{D,\theta} \tag{3.3}$$

In this work the superpotential term is not considered:

$$L = L_{kin} + L_{gauge} + L_{D,\theta} = \int d^2y d^4\theta(\bar{\Phi}\Phi - \frac{1}{4e^2}\bar{\Sigma}\Sigma) - \frac{1}{2}\int d^2\tilde{\theta} t \Sigma_0 + \text{h.c.} \tag{3.4}$$

3.3 Field representations of GLSM with $(0,2)$ supersymmetry

As usual, the coordinates of the $(0,2)$ superspace are $(y^0, y^1, \theta^+, \bar{\theta}^+)$. The covariant superderivatives are given by

$$D_+ = \partial_{\theta^+} - i\bar{\theta}^+\partial_+ \ , \qquad \overline{D}_+ = -\partial_{\bar{\theta}^+} + i\theta^+\partial_+ \ , \tag{3.5}$$

where $\partial_+ := \frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^1}$, $\partial_- := \frac{\partial}{\partial y^0} - \frac{\partial}{\partial y^1}$, $\partial_{\theta^+} := \frac{\partial}{\partial \theta^+}$ and $\partial_{\bar{\theta}^+} := \frac{\partial}{\partial \bar{\theta}^+}$.

The gauge covariant superderivatives $\mathcal{D}_+$, $\overline{\mathcal{D}}_+$, $\mathcal{D}_0$ and $\mathcal{D}_1$ are constructed with the following constraints:

$$\mathcal{D}_0 = D_0 \ , \qquad \mathcal{D}_1 = D_1 \ , \tag{3.6}$$

$$\mathcal{D}_+ = e^{-\Psi} D_+ e^{\Psi} \quad = \quad (D_+ + D_+\Psi), \tag{3.7}$$

$$\overline{\mathcal{D}}_+ = e^{\Psi} \overline{D}_+ e^{-\Psi} \quad = \quad (\overline{D}_+ - \overline{D}_+\Psi), \tag{3.8}$$

$$\mathcal{D}_0 - \mathcal{D}_1 \quad = \quad \partial_- + iV, \tag{3.9}$$

where Ψ and V are real functions, that constitute the gauge degrees of freedom. V is the $(0,2)$ vector superfield and in the Wess-Zumino gauge it can be expanded in components as follows:

$$\begin{aligned} V &= v_- - 2i\theta^+\overline{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+ D, \\ \Psi &= v_+\theta^+\bar{\theta}^+. \end{aligned} \tag{3.10}$$

The supersymmetric derivative has some properties related to the fermionic (bosonic) nature of the superfields[3]. Let χ be an arbitrary bosonic superfield and Λ a fermionic one, then their components expansions are:

$$\chi := x + \xi\theta^+ + \rho\bar{\theta}^+ + z\theta^+\bar{\theta}^+, \qquad \Lambda := \omega + k\theta^+ + l\bar{\theta}^+ + \varepsilon\theta^+\bar{\theta}^+. \tag{3.11}$$

[3] These properties are stated in the $(2,2)$ context, but it can be done in more general SUSY.

From their derivatives $\bar{D}_+\chi = \rho + z\theta^+ + \ldots$ and $\bar{D}_+\Lambda = -l - \varepsilon\theta^+ + \ldots$, it's clear that the supersymmetric derivative exchange the even/odd nature of the superfields; this is:

$$\text{if} \qquad \text{sgn}(A) = 0, \qquad \text{then} \qquad \text{sgn}(\bar{D}_+ A) = 1, \qquad \text{etc.} \tag{3.12}$$

And in this way, if A and B are arbitrary superfields, the following properties are fulfilled:

$$\overline{D_+ A} = (-1)^{\text{sgn}(A)}\bar{D}_+\bar{A}, \qquad D_+(AB) = D_+AB + (-1)^{\text{sgn}(A)}AD_+B. \tag{3.13}$$

However for $\mathcal{D}_+$, it is modified:

$$\overline{\bar{\mathcal{D}}_+ A} = (-1)^{\text{sgn}(A)}\mathcal{D}_+\bar{A} - (-1)^{\text{sgn}(A)}2D_+\Psi\bar{A}\ . \tag{3.14}$$

The basic gauge invariant field strength Υ is defined as the field strength of V:

$$\Upsilon = [\overline{\mathcal{D}}_+, \mathcal{D}_0 - \mathcal{D}_1]V = \overline{D}_+(iV + \partial_-\Psi) = i\overline{D}_+V + \partial_-\overline{D}_+\Psi\ . \tag{3.15}$$

In components field strength is written as:

$$\Upsilon = -2\lambda_- + [2iD + (\partial_- v_+ - \partial_+ v_-)]\theta^+ + 2i\partial_+\lambda_-\theta^+\bar{\theta}^+\ . \tag{3.16}$$

The U(1) gauge theory has a natural Lagrangian given by

$$L_{gauge} = \frac{1}{8e^2}\int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\overline{\Upsilon}\Upsilon, \tag{3.17}$$

where e is the gauge coupling constant.

3.3.1 Fields content

There are two kinds of matter fields: the chiral multiplets Φ and the Fermi multiplets Γ. The (bosonic) covariant chiral fields $\widetilde{\Phi}$ are defined by the following constraint:

$$\overline{\mathcal{D}}_+\widetilde{\Phi} = 0, \tag{3.18}$$

where $\overline{\mathcal{D}}_+$ is the covariant derivative and consequently it has the components expansion:

$$\widetilde{\Phi} = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\overline{\theta}^+(\partial_+ + iv_+)\phi, \tag{3.19}$$

which is defined with $\widetilde{\Phi} := \Phi e^{\Psi}$, where the (uncharged) chiral superfield Φ fulfils the relation $\overline{D}_+\Phi = 0$. The corresponding gauge invariant Lagrangian is given by

$$\begin{aligned} L_{chiral} &= -\frac{i}{2}\int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+ \ \widetilde{\Phi}^\dagger(\mathcal{D}_0 - \mathcal{D}_1)\widetilde{\Phi} = \bar{\phi}\phi D + i\bar{\psi}_+(\partial_- + iv_-)\psi \qquad (3.20)\\ &- \sqrt{2}i(\lambda_-\psi_+\bar{\phi} - \bar{\psi}_+\bar{\lambda}_-\phi) - \frac{1}{2}[\bar{\phi}(\partial_- + iv_-)(\partial_+ + iv_+)\phi - (\partial_+ + iv_+)\bar{\phi}(\partial_- + iv_-)\phi]. \end{aligned}$$

In order to complete the rest of matter content let us introduce $\widetilde{\Gamma}$ which is a $(0,2)$ Fermi multiplet. This multiplet satisfies the constraint:

$$\overline{\mathcal{D}}_+\widetilde{\Gamma} = \sqrt{2}\widetilde{E}\,, \qquad \text{then} \qquad \widetilde{E} = \frac{\sqrt{2}}{2}e^{\Psi}\overline{D}_+\Gamma, \tag{3.21}$$

where $\widetilde{E} = E(\widetilde{\Phi})$ is a holomorphic function of the superfield $\widetilde{\Phi}$. Similarly, we can define $\widetilde{\Gamma} := \Gamma e^{\Psi}$ and $\widetilde{E} := Ee^{\Psi}$, where $\overline{D}_+\Gamma = \sqrt{2}E$. Thus, the expansion for this Fermi multiplet and the field E are given by:

$$\widetilde{\Gamma} = \gamma - \sqrt{2}G\theta^+ - i\theta^+\bar{\theta}^+(\partial_+ + iv_+)\lambda - \sqrt{2}\widetilde{E}\bar{\theta}^+\,. \tag{3.22}$$

$$\tilde{E}(\Phi) = E(\phi) + \sqrt{2}\theta^+\frac{\partial E}{\partial\phi}\psi_+ - i\theta^+\bar{\theta}^+(\partial_+ + iv_+)E(\phi). \tag{3.23}$$

The dynamics of the Fermi field is given by the Lagrangian:

$$\begin{aligned} L_F &= -\frac{1}{2}\int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\bar{\tilde{\Gamma}}\tilde{\Gamma} \qquad (3.24)\\ &= i\bar{\gamma}(\partial_+ + iv_+)\gamma + |G|^2 - |E|^2 - \left(\bar{\gamma}\frac{\partial E}{\partial\phi}\psi_+ + \frac{\partial\bar{E}}{\partial\bar{\phi}}\bar{\psi}_+\gamma\right). \end{aligned}$$

In the case of $U(1)$ gauge theories (or non-abelian gauge theories with a gauge group with a $U(1)$ factor) we have an additional term in the Lagrangian given by the

Fayet-Iliopoulos term

$$L_{D,\theta} = \frac{t}{4}\int \mathrm{d}\theta^{+}\Upsilon|_{\bar{\theta}^{+}=0} + \text{h.c.}, \tag{3.25}$$

where $t = \frac{\theta}{2\pi} + ir$, with θ being an angular parameter and r is the Fayet-Iliopoulos parameter.

In this work it will be considered $(0,2)$ GLSMs with a $U(1)$ gauge group, with non-abelian global symmetries to be gauged up. Thus the dynamics of the addition of all these Lagrangians, i.e.

$$L = L_{gauge} + L_{chiral} + L_F + L_{D,\theta} + L_J, \tag{3.26}$$

where L_J is an interaction Lagrangian which is the $(0,2)$ analog of the superpotential term of the $(2,2)$ model. L_J is of the form

$$L_J = -\frac{1}{\sqrt{2}}\int \mathrm{d}\theta^{+}\sum_a \left(\Gamma_a J^a|_{\bar{\theta}^{+}=0}\right) - \text{h.c.}, \tag{3.27}$$

where $J^a = J^a(\widetilde{\Phi})$ are holomorphic functions of the $(0,2)$ chiral superfield $\widetilde{\Phi}$, and Γ_a are Fermi superfields. Moreover J^a satisfies the relation $\sum_a E_a J^a = 0$.

The scalar potential can be obtained by the usual procedure in supersymmetric theories (integrating in the superspace) and it is given by

$$U(\phi_i) = \frac{e^2}{2}\left(\sum_i Q_i|\phi_i|^2 - r\right)^2 + \sum_a \left(|E_a|^2 + |J_a|^2\right), \tag{3.28}$$

where it is clear the contributions coming the D-terms from the $(0,2)$ gauge multiplet and from the FI term. The last two terms come from the E field and the last one, corresponds to the contribution from the superpotential.

3.3.2 $(0,2)$ superfields from $(2,2)$ multiplets

It is very well known that certain $(0,2)$ GLSMs can be regarded as a supersymmetric reduction of $(2,2)$ GLSMs. The $(2,2)$ GLSM consists of chiral supefield $\Phi^{(2,2)}$, vector superfield $V^{(2,2)}$ and its twisted field strength $\Sigma^{(2,2)}$. It is written below how the decomposition of $(2,2)$ multiplets can be written in terms of $(0,2)$ multiplets:

- The $(2,2)$ chiral superfield $\Phi^{(2,2)}$ can be decomposed in the $(0,2)$ chiral by: $\Phi = \Phi^{(2,2)}|_{\theta^-=\bar{\theta}^-=0}$; and in the $(0,2)$ Fermi by: $\Phi = \frac{1}{\sqrt{2}}D_-\Phi^{(2,2)}|_{\theta^-=\bar{\theta}^-=0}$. Both matter fields are supersymmetry reductions of the single $(2,2)$ chiral superfield.

- The $(2,2)$ vector superfield $V^{(2,2)}$ gives the gauge field $\Upsilon = i\bar{D}_+(V - i\partial_-\Psi)$ by: $V - i\partial_-\Psi = -\bar{D}_-D_-V^{(2,2)}|_{\theta^-=\bar{\theta}^-=0}$. And also it gives a chiral superfield Σ, which is be identified as: $\bar{\theta}^+\Sigma = -\frac{1}{\sqrt{2}}D_-V|_{\theta^-=\bar{\theta}^-=0}$; which is also simply: $\Sigma = \Sigma^{(2,2)}|_{\theta^-=\bar{\theta}^-=0}$.

- It can be verified that if $\Phi^{(2,2)}$ has charge Q, then the E field can be written as $E = Q\sqrt{2}\Sigma\Phi$.

- The holomorphic function J can be obtained from the $(2,2)$ superpotential W in the form $J = \frac{\partial W}{\partial \tilde{\Phi}}$.

Abelian T-duality in $(0, 2)$ GLSMs

In this chapter it is described the T-duality for GLSMs with a $U(1)$ gauge group and a $U(1)$ global symmetry group to be gauged. The original model (action) is given and the T-dual model is found. For the sake of simplicity we consider the case when the superpotential J of the $(0, 2)$ model vanishes, thus the underlying scalar potential consists only of the D-term and the Fayet-Iliopoulos term. The equations of motion are obtained.

It is described two separate cases, the first one is the case in which the (0, 2) GLSM can be obtained by reduction from a (2, 2) model. The second case is the general case of a pure (0, 2) GLSM which cannot be obtained from a reduction. In both cases we describe their corresponding instanton corrections. First, as an example to review the process, the abelian dualization of a $(2, 2)$ model is described following the results in [28]. Then, the main results of this thesis are discussed. Initially, the most general case for $n + \tilde{n}$ fields is described, and it is solved explicitly for the particular case of a single chiral and Fermi field. This is done both for $(0, 2)$ models that do not originate from reduction and for those that do. The dual action is found and also the equations of motion and the geometry of the space of dual vacua are determined.

In the remaining part of the chapter, it is described a particular reduced model with gauge group $U(1)\times U(1)$ and an abelian global symmetry $U(1)^4$. This model was discussed in [54] and it will be analysed in the context of non-abelian duality in last chapter.

4.1 Example of $(2,2)$ fields with $U(1)$ global symmetry

In this section, the work by Cabo *et al.* [28] is followed. This model consists of a $2D$ supersymmetric $(2,2)$ GLSM with an abelian gauge group $U(1)$ and two chiral superfields Φ_1 and Φ_2. The original action for a simple $(2,2)$ model is given in (3.4). In this case there are two chiral fields, thus the original (before dualization) action is:

$$S_{(2,2)} = \int d^2y d^4\theta(\bar{\Phi}_1\Phi_1 + \bar{\Phi}_2\Phi_2 - \frac{1}{2e^2}\bar{\Sigma}\Sigma) - \frac{1}{2}\int d^2\tilde{\theta} t\Sigma + \text{h.c.} \tag{4.1}$$

This system has one global $U(1)$ symmetry realized as the phase rotations modulo the $U(1)$ gauge transformations. If the gauge is fixing to remove the phase transformation of Φ_2, then the T-Duality is implemented by gauging the phase rotation of the field Φ_1. Thus, the under the abelian global symmetry with parameter α, the fields transform as:

$$\Phi_1 \to e^{i\alpha}\Phi_1 \qquad \text{and} \qquad \Phi_2 \to \Phi_2 \tag{4.2}$$

The procedure to find the dual action involves gauging[1] the global symmetry in the Lagrangian and adding a Lagrange multiplier as shown in [25, 26, 27]. This global symmetry is transformed into a local one by taking $i\alpha = 2QV \to 2QV + 2Q_1V_1$, this means introducing a new vector superfield[2] V_1 and adding Lagrange multipliers terms

[1]In this context, “gauging” means transform the global symmetry into a local one.

[2]Which is now the relevant function on the superspace.

with the superfields Ψ and $\bar{\Psi}$.

With this, it can be written the "master" Lagrangian:

$$\begin{aligned} L_{master} = \int d^4\theta\big(& \bar{\Phi}_1 e^{2QV+2Q_1V_1}\Phi_1 + \bar{\Phi}_2 e^{2QV}\Phi_2 - \frac{1}{2e^2}\bar{\Sigma}\Sigma + \Psi\Sigma_1 + \bar{\Psi}\bar{\Sigma}_1\big) \\ - & \frac{1}{2}\int d^2\tilde{\theta} t\Sigma + \text{h.c.} \,, \end{aligned} \tag{4.3}$$

where the terms $\Psi\Sigma_1$ and $\bar{\Psi}\bar{\Sigma}_1$ were added, V and $\Sigma = \frac{1}{2}\bar{D}_+D_-V$ are the vector superfield and field strength of the $U(1)$ gauge symmetry, and V_1 and Σ_1 are the ones of the gauged symmetry. Integrating out the Lagrange multipliers Ψ and $\bar{\Psi}$ one gets the condition $\Sigma_1 = 0$, which is a pure gauge field, leading to the original GLSM of two chiral superfields coupled to a $U(1)$ vector superfield V.

The dual Lagrangian is obtained integrating out the new V_1 field. The equation of motion with respect to V_1 is $\frac{\delta S}{\delta V_1} = 0$ and is given by:

$$\bar{\Phi}_1 e^{2QV+2Q_1V_1}\Phi_1 = \frac{\Lambda + \bar{\Lambda}}{2Q_1}, \tag{4.4}$$

with $\Lambda = \frac{1}{2}\bar{D}_+D_-\Psi$. Note that Λ and $\bar{\Lambda}$ are twisted and anti-twisted chiral superfields, respectively, and this duality maps the Φ chiral superfields into the Λ twisted chiral superfields, in the same way as mirror symmetry does. Therefore using the *e.o.m.* (4.4) inside the master Lagrangian (4.3) it is obtained the dual Lagrangian:

$$\begin{aligned} L_{dual} = \int d^4\theta\Big(& - \frac{\Lambda+\bar{\Lambda}}{2Q_1}\ln\left(\frac{\Lambda+\bar{\Lambda}}{2Q_1}\right) + \bar{\Phi}_2 e^{2QV}\Phi_2 - \frac{1}{2e^2}\bar{\Sigma}\Sigma\Big) \\ + & \frac{1}{2}\Big(\int d^2\tilde{\theta}(\Lambda Q/Q_1 - t)\Sigma + \int d^2\bar{\tilde{\theta}}(\bar{\Lambda}Q/Q_1 - \bar{t})\bar{\Sigma}\Big)\,. \end{aligned} \tag{4.5}$$

This result with $Q_1 = 1$ is the same as the one obtained by Hori and Vafa in [15] with a different approach. This procedure is used to find non-abelian T dualities.

4.2 Pure $(0,2)$ GLSM with $U(1)^n$ gauge and $U(1)^k$ global

Here it is described the generic abelian T-dualization for general model $(0,2)$ (which it will be called "pure" in contrast to the case when it comes as a $(2,2)$ reduction) $U(1)^m$ GLSM with $U(1)^k$ global symmetry related to the chiral fields and $U(1)^s$ global symmetry associated to the Fermi fields. It is a theory with n chiral superfields Φ_i and $\widetilde{n}$ Fermi superfields Γ_j, and a given number m of $U(1)$ gauge symmetries.

The chiral superfields Φ_i have Q_i^a charges and the Fermi superfields Γ_i^a have charges $\widetilde{Q}_j^a$. The Lagrangian is then:

$$
\begin{aligned}
L &= \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \left\{ \sum_{a=1}^{m} \frac{1}{8e_a^2} \overline{\Upsilon}_a \Upsilon_a - \sum_{i=1}^{n} \frac{i}{2} \overline{\Phi}_i e^{2\sum_{a=1}^m Q_i^a \Psi_a} \left(\partial_- + i \sum_{a=1}^{m} Q_i^a V_a \right) \Phi_i \right\} \\
&+ \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \left\{ \sum_{i=1}^{n} \frac{i}{2} \overline{\Phi}_i \left(\overleftarrow{\partial}_- - i \sum_{a=1}^{m} Q_i^a V_a \right) e^{2\sum_{a=1}^m Q_i^a \Psi_a} \Phi_i \right\} \\
&- \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \left\{ \sum_{j=1}^{\widetilde{n}} \frac{1}{2} e^{2\sum_{a=1}^m \widetilde{Q}_j^a \Psi_a} \overline{\Gamma}_j \Gamma_j \right\} + \sum_{a=1}^{m} \frac{t_a}{4} \int \mathrm{d}\theta^+ \Upsilon_a |_{\bar{\theta}^+=0}
\end{aligned} \tag{4.6}
$$

There are m vector superfields V_a, Ψ_a with field strength Υ_a. In principle each kinetic term has a global phase symmetry, under which the chiral or the Fermi fields transform. As all the superfields are distinct, one can employ the m gauge symmetries to absorb m of these phases, giving a total of $k+s$ global symmetries where $k = n - m$ $(n > m)$ $U(1)$ global symmetries, these transformations are:

$$
\delta_\Lambda V_a = -\partial_-(\Lambda_a + \bar{\Lambda}_a)/2, \qquad \delta\Psi = -i(\Lambda_a - \bar{\Lambda}_a)/2, \tag{4.7}
$$

$$
\Phi_i \rightarrow e^{i\sum_{a=1}^m Q_i^a \Lambda_a} \Phi_i, \qquad \Gamma_i \rightarrow e^{i\sum_{a=1}^m \tilde{Q}_i^a \Lambda_a} \Gamma_i. \tag{4.8}
$$

In general it can be possible to absorb with the m abelian gauge symmetries not only the global symmetries of the chiral superfields, but the total amount of global symmetries of the chiral and Fermi fields $n + \widetilde{n}$. The master Lagrangian will remain the same with some few modifications in the sum's indices. This case will be not considered in this thesis.

In general one can consider a generic number of Fermi multiplets, this is true because the general (0,2) model, presented here, doesn't come necessarily from a SUSY reduction from the (2,2) theory. Therefore the Fermi multiplets are not necessarily related or coupled to the chiral multiplets. In the opposite case when the chiral superfields and the Fermi superfields come both from the (2,2) reduction, the number of Fermi and chiral fields and their charges need to match.

Starting from (4.6) one can construct the *master Lagrangian* (or also named intermediate Lagrangian) by gauging the global symmetries and adding terms with Lagrange multipliers Λ_b related to field strengths Υ_b

$$
\begin{aligned}
L_{\texttt{master}} \;=\; & \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \sum_{a=1}^{m} \frac{1}{8e_a^2} \overline{\Upsilon}_a \Upsilon_a \\
- & \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \left\{ \sum_{i=1}^{k} \frac{i}{2} \overline{\Phi}_i e^{2\sum_{a=1}^{m} Q_i^a \Psi_a + 2\sum_{b=1}^{k} Q_{1i}^b \Psi_{1b}} \left(\partial_- + i\sum_{a=1}^{m} Q_i^a V_a + i\sum_{b=1}^{k} Q_{1i}^b V_{1b} \right) \Phi_i \right\} \\
+ & \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \left\{ \sum_{i=1}^{k} \frac{i}{2} \overline{\Phi}_i \left(\overleftarrow{\partial}_- - i\sum_{a=1}^{m} Q_i^a V_a - i\sum_{a=1}^{k} Q_{1i}^b V_{1b} \right) e^{2\sum_{a=1}^{m} Q_i^a \Psi_a + 2\sum_{b=1}^{k} Q_{1i}^b \Psi_{1b}} \Phi_i \right\} \\
- & \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \left\{ \sum_{j=1}^{s} \frac{1}{2} e^{2\sum_{a=1}^{m} \widetilde{Q}_j^a \Psi_a + 2\sum_{c=1}^{s} \widetilde{Q}_{1j}^c \Psi_{1c}} \overline{\Gamma}_j \Gamma_j + \sum_{j=s+1}^{\widetilde{n}} \frac{1}{2} e^{2\sum_{a=1}^{m} \widetilde{Q}_j^a \Psi_a} \overline{\Gamma}_j \Gamma_j \right\} \\
+ & \sum_{a=1}^{m} \frac{t_a}{4} \int \mathrm{d}\theta^+ \Upsilon_a|_{\bar{\theta}^+=0} + \sum_{b=1}^{k} \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Lambda_b \Upsilon_{1b} + \sum_{b=k+1}^{k+s} \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Lambda_b \Upsilon_{1b} + \text{h.c.} + \text{spectators.}
\end{aligned}
\tag{4.9}
$$

For simplicity in this expression it is assumed that the chiral superfields are not charged under the global symmetries that the Fermi superfields are charged, and vice-versa. One could choose that each of the chiral superfields to dualize it is charged only under a single $U(1)$ global, such that $Q^b_{1i} = \delta_i{}^b$, as it was done by Hori and Vafa in their fundamental work on mirror symmetry as a T-duality [15]. There are $U(1)^{k+s}$ global symmetries, where $k+s = n-m+s$. For models coming from supersymmetric reduction s is zero and the Fermi superfield will be gauged with the same global symmetry implemented by the chiral superfields. In the general case there will be additional global symmetries arising due to the Fermi superfields in addition to those due to the chiral superfields in the $(2,2)$ GLSM.

Let us now analyze the equations of motion from this master Lagrangian when the gauged fields are integrated. Due to the Weiss-Zumino gauge (3.10), $e^{2\Psi} = 1+2\Psi$. In this way, the fields Ψ_1, V_1 and Γ_1 are linear and the variation is easy performed. Carrying out the variation of the Lagrangian with respect to ψ_{1b} we obtain for the field V_{1b}:

$$V_{1b} = A^{-1}_{bd}\left(-\frac{i}{2}\partial_- Y^d_- - R^d\right), \tag{4.10}$$

where

$$A_{bd} = \sum_{i=1}^{k} |\phi_i|^2 Q^d_{1i} Q^b_{1i}, \tag{4.11}$$

and

$$R^d = \sum_{i=1}^{k}\left(-\frac{i}{2}\overline{\Phi}_i \delta_- \Phi_i Q^d_{1i} + |\Phi_i|^2 \sum_{a=1}^{m} Q^a_i V_a Q^d_{1i}\right). \tag{4.12}$$

Here the new dual variable is defined by: $Y^c_\pm \equiv i\overline{D}_+\Lambda^c \pm iD_+\overline{\Lambda}^c$, and for simplicity, it has been used $\delta_- = \partial_- - \overleftarrow{\partial}_-$, this definition will be used many times in this work.

Performing the variation of the Lagrangian with respect to V_{1b}, for the component

ψ_{1b} we have

$$\psi_{1b} = A^{-1}_{bd}\Big(-\frac{i}{2}\partial_- Y^d_+ - S^d \Big), \tag{4.13}$$

and

$$S^d = \sum_{i=1}^{k} |\Phi_i|^2 Q^d_{1i} + 2\sum_{a=1}^{m} |\Phi_i|^2 Q^d_{1i} Q^a_{1i}\psi_a. \tag{4.14}$$

The variation with respect the component ψ_{1d} for $d \in \{k+1, ..., 2k\}$ yields

$$Q^d_{1j}\overline{\Gamma}_j\Gamma_j = -i\partial_- Y^d_- \to \overline{\Gamma}_j\Gamma_j = -Q^{-1}_{1jd}\partial_- Y^d_-. \tag{4.15}$$

These equations of motion are employed to find the dual model.

4.3 Reduced from $(2,2)$ model dualization

Now in the case that the model comes from a $(2,2)$ reduction, there is a Fermi superfield for every chiral superfield and there could be also extra Fermi superfields. These Fermi fields have the same charges under the gauge group than the chiral superfields related to them i.e. $Q_i = \widetilde{Q}_i$ and we consider the case $s = 0$, such that there are no extra Fermi fields[3]. Then all the global symmetries will affect equally the chiral superfields and the Fermi superfields. In this case the main difference is that the duality procedure will be carry out in the fields Φ and Γ, and there are Lagrange multipliers $\overline{\chi}$ associated to E. So, the new dual fields are $\widetilde{\mathcal{F}} = e^{\Psi}\mathcal{D}_+\chi$ and Y_a. Therefore there are two dual fields and one extra equation of motion.

We start from the following Lagrangian, with n chiral fields and then n Fermi fields (related to them) and without any extra Fermi field. This means $\widetilde{n} = n$. Thus, after

[3]That are no related to the chiral ones.

adding the Lagrange multiplier terms and gauging the global symmetries the master Lagrangian is:

$$
\begin{aligned}
L_{\texttt{master}} =& \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+ \sum_{a=1}^{m} \frac{1}{8e_a^2}\overline{\Upsilon}_a\Upsilon_a \\
-& \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+ \Big\{ \sum_{i=1}^{k} \frac{i}{2}\overline{\Phi}_i e^{2\sum_{a=1}^{m} Q_i^a\Psi_a + 2\sum_{b=1}^{k} Q_{1i}^b\Psi_{1b}} \Big(\partial_- + i\sum_{a=1}^{m} Q_i^a V_a + i\sum_{b=1}^{k} Q_{1i}^b V_{1b}\Big)\Phi_i\Big\} \\
+& \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+ \Big\{ \sum_{i=1}^{k} \frac{i}{2}\overline{\Phi}_i(\overleftarrow{\partial}_- - i\sum_{a=1}^{m} Q_i^a V_a - i\sum_{b=1}^{k} Q_{1i}^b V_{1b}) e^{2\sum_{a=1}^{m} Q_i^a\Psi_a + 2\sum_{b=1}^{k} Q_{1i}^b\Psi_{1b}}\Phi_i\Big\} \\
-& \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+ \Big\{ \sum_{j=1}^{k} \frac{1}{2} e^{2\sum_{a=1}^{m} Q_j^a\Psi_a + 2\sum_{b=1}^{k} Q_{1j}^b\Psi_{1b}} (\overline{\Gamma}_j + \overline{\Gamma}_{1j})(\Gamma_j + \Gamma_{1j})\Big\} \\
+& \sum_{a=1}^{m} \frac{t_a}{4}\int \mathrm{d}\theta^+ \Upsilon_a|_{\bar{\theta}^+=0} + \sum_{b=1}^{k}\int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+ \Lambda_b\Upsilon_b + \sum_{b=1}^{k}\int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+ \bar{\chi}_b E_b + \text{h.c.} \\
-& \frac{i}{2}\int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+ \sum_{i=k+1}^{n} \Big\{ \overline{\Phi}_i e^{2\sum_{a=1}^{m} Q_i^a\Psi_a} \Big(\partial_- + i\sum_{a=1}^{m} Q_i^a V_a\Big)\Phi_i \\
-& \overline{\Phi}_i \Big(\overleftarrow{\partial}_- - i\sum_{a=1}^{m} Q_i^a V_a\Big) e^{2\sum_{a=1}^{m} Q_i^a\Psi_a}\Phi_i\Big\} + \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+ \sum_{j=k+1}^{\widetilde{n}} \frac{1}{2} e^{2\sum_{a=1}^{k} Q_j^a\Psi_a}\overline{\Gamma}_j\Gamma_j .
\end{aligned}
\tag{4.16}
$$

The terms in the last two lines are not charged under the global (gauged) symmetry so those fields behave as spectators. The main difference with the other case (without reduction) lies on the dualized fields. In this reduced model, the Fermi fields are also gauged and new Lagrange multipliers were added, these terms are located on the 5th line of previous equation. If exists $Q \in GL(k)$ such that $(Q)_i^c := Q_{1i}^c$, then let be $X := Q^{-1}$ to find the variations, which results in:

$$
\delta_{V_{1c}} S = 0 : \qquad \Big(1 + 2\sum_{a=1}^{m} Q_j^a\Psi_a + 2\sum_{b=1}^{k} Q_{1j}^b\Psi_{1b}\Big) = -\frac{X_c^j Y_+^c}{|\Phi_j|^2}
$$

$$
\text{or} \qquad \Psi_{1d} = \sum_{j=1}^{k} X_d^j \left(\frac{X_c^j Y_+^c}{2|\Phi_j|^2} + \frac{1}{2} + Q_j^a\Psi_a\right), \tag{4.17}
$$

$$\delta_{\Psi_{1c}} S = 0: \qquad -i\overline{\Phi}_j \delta_- \Phi_j + 2\sum_{a=1}^{m} Q_j^a V_a |\Phi_j|^2 + 2\sum_{b=1}^{k} Q_{1j}^b V_{1b} |\Phi_j|^2 = -\frac{i}{2} X_c^j \partial_- Y_-^c, \tag{4.18}$$

$$\delta_{\Gamma_{1j}} S = 0: \qquad \left(1 + 2\sum_{a=1}^{m} Q_j^a \Psi_a + 2\sum_{b=1}^{k} Q_{1j}^b \Psi_{1b}\right)(\overline{\Gamma}_j + \overline{\Gamma}_{1j}) = -\sqrt{2}\widetilde{\mathcal{F}}_j^\dagger. \tag{4.19}$$

Notice that it has been solved the equations for the gauged fields, the previous notation $\delta_X S = 0$ is used to the equation of motion obtained for the field X.

In both cases, the final dual Lagrangian can be explicitly found for different values of m, n, and $\tilde{n}$. This is done in the following sections. However, before proceeding, we will develop an example case that we can compare with a previous result from the scientific literature.

$U(1) \times U(1)$ global symmetry

As an example, apply this T-dualization procedure to the case of a $(0,2)$ GLSM coming from a reduction, as discussed in Ref. [54]. This is a GLSM with two gauge groups $U(1)$. It is needed to gauge 4 global symmetries; this is 3 chiral fields Φ and 3 Fermi fields Γ charged under a $U(1)$ gauge symmetry and another 3 chiral $\widetilde{\Phi}$ and Fermi $\widetilde{\Gamma}$ charged under the other $U(1)$. In this case it has to be taken $m = 2$, $n = 6$, $\widetilde{n} = 6$, $k = 2$ and $s = 0$. The former example has to give the same dual model of [54], apart from the addition of 2 spectator fields. The master Lagrangian in general:

$$
\begin{aligned}
L_{\mathtt{master}} = \; & - \frac{i}{2} e^{2\Psi_1+2\Psi_1'} \overline{\Phi}_1 \Big(\partial_- - i(V_1+V_1') \Big) \Phi_1 + \text{h.c.} \\
& - \frac{i}{2} e^{2\Psi_1+2\Psi_2'} \bar{\Phi}_2 \Big(\partial_- - i(V_1+V_2') \Big) \Phi_2 + \text{h.c.} \\
& - \frac{i}{2} e^{2\Psi_2+2\Psi_3'} \overline{\widetilde{\Phi}_1} \Big(\partial_- - i(V_2+V_3') \Big) \widetilde{\Phi}_1 + \text{h.c.} \\
& - \frac{i}{2} e^{2\Psi_2+2\Psi_4'} \overline{\widetilde{\Phi}}_2 \Big(\partial_- - i(V_2+V_4') \Big) \widetilde{\Phi}_2 + \text{h.c.} \\
& - \frac{1}{2} (\overline{\Gamma}_1 + \overline{\Gamma'}_1) e^{2\Psi_1+2\Psi_1'} (\Gamma_1+\Gamma_1') - \frac{1}{2} (\overline{\Gamma}_2 + \overline{\Gamma'}_2) e^{2\Psi_1+2\Psi_2'} (\Gamma_2+\Gamma_2') \\
& - \frac{1}{2} (\overline{\widetilde{\Gamma}}_1 + \overline{\widetilde{\Gamma}}_1') e^{2\Psi_2+2\Psi_3'} (\widetilde{\Gamma_1} + \widetilde{\Gamma_1'}) - \frac{1}{2} (\overline{\widetilde{\Gamma}}_2 + \overline{\widetilde{\Gamma}}_2') e^{2\Psi_2+2\Psi_4'} (\widetilde{\Gamma}_2 + \widetilde{\Gamma}_2') \\
& + \Lambda_1 \Upsilon_1' + \Lambda_2 \Upsilon_2' + \Lambda_3 \Upsilon_3' + \Lambda_4 \Upsilon_4' + \text{h.c.} + \overline{\chi}_1 E_1' + \overline{\chi}_2 E_2' + \overline{\chi}_3 E_3' + \overline{\chi}_4 E_4' + \text{h.c.} \\
& - \frac{i}{2} e^{2\Psi_1} \overline{\Phi}_3 \Big(\partial_- - iV_1 \Big) \Phi_3 + \text{h.c.} - \frac{i}{2} e^{2\Psi_2} \overline{\widetilde{\Phi}}_3 \Big(\partial_- - iV_2 \Big) \widetilde{\Phi}_3 + \text{h.c.} \\
& - \frac{1}{2} e^{2\Psi_1} \overline{\Gamma}_3 \Gamma_3 - \frac{1}{2} e^{2\Psi_1} \overline{\widetilde{\Gamma}}_3 \widetilde{\Gamma}_3 + \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \left\{ \frac{1}{8e_1^2} \overline{\Upsilon}_1 \Upsilon_1 + \frac{1}{8e_2^2} \overline{\Upsilon}_2 \Upsilon_2 \right\} \\
& + \frac{t_1}{4} \int \mathrm{d}\theta^+ \Upsilon_1 |_{\overline{\theta}^+=0} + \frac{t_2}{4} \int \mathrm{d}\theta^+ \Upsilon_2 |_{\overline{\theta}^+=0} + \text{h.c.}
\end{aligned}
\tag{4.20}
$$

The dual fields to the Fermi multiplet are given by $F = \bar{D}_+ \chi$, $\mathcal{F} = e^{\psi} F$. The scalar potential, the analysis of the supersymmetric vacua and the instanton corrections will not discussed here; since for the case of the non-abelian global symmetry they will be discussed in detail in section 5.2. The Lagrangian previously obtained is exactly the one obtained by [54] excluding the spectator terms. In their work they considered mirror symmetry for $(0, 2)$ models coming from a reduction of $(2, 2)$.

4.4 GLSMs with $U(1)$ global symmetry

These general results can be applied to specific models that can be solved explicitly. Let be the simple case of the GLSM Lagrangian with one chiral multiplet and one Fermi superfield, this is given by

$$L = \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \left\{ \frac{1}{8e^2}\bar{\Upsilon}\Upsilon - \frac{i}{2}\overline{\Phi} e^{2\Psi}(\partial_- + iV)\Phi + \frac{i}{2}\overline{\Phi}(\overleftarrow{\partial}_- - iV)e^{2\Psi}\Phi - \frac{1}{2}e^{2\Psi}\overline{\Gamma}\Gamma \right\}$$
$$+\frac{t}{4}\int \mathrm{d}\theta^+ \Upsilon|_{\bar{\theta}^+=0} \ . \tag{4.21}$$

From this common Lagrangian, it will be taken the 2 cases, when the model is a reduction from $(2,2)$ and the pure $(0,2)$ case.

4.4.1 $(0,2)$ GLSM from a reduction of a $(2,2)$ GLSM

As it was mentioned before in the case when the $(0,2)$ model is obtained as a reduction from a $(2,2)$ model, the E field has a special form with the reduced fields

$$E = iQ\sqrt{2}\Sigma'\Phi' \ , \tag{4.22}$$

where $\Sigma' = \Sigma|_{\theta^-=\bar{\theta}^-=0}$, and $\Phi' = \Phi|_{\theta^-=\bar{\theta}^-=0}$. Thus the gauged Lagrangian is written as:

$$L = \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \left\{ -\frac{i}{2}\overline{\Phi} e^{2(\Psi_0+\Psi_1)}\left[\partial_- + i(V_0+V_1)\right]\Phi + \frac{i}{2}\overline{\Phi}\left[\overleftarrow{\partial}_- - i(V_0+V_1)\right]e^{2(\Psi_0+\Psi_1)}\Phi \right.$$
$$\left. -\frac{1}{2}e^{2(\Psi_0+\Psi_1)}\overline{\Gamma}\Gamma + \Lambda\Upsilon_1 + \overline{\Upsilon}_1\overline{\Lambda} + \overline{\chi}\widetilde{E}_1 + \overline{\widetilde{E}}_1\chi \right\} \ . \tag{4.23}$$

Thus, subtituting back the eqs. (4.18-4.19) for this case, the dual Lagrangian becomes:

$$L_{dual} = \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \left\{ -\frac{i}{2}\frac{Y_-\partial_- Y_+}{Y_+} + \frac{|\Phi|^2\widetilde{\mathcal{F}}\overline{\widetilde{\mathcal{F}}}}{Y_+} - (\Lambda\Upsilon_0 + \overline{\Upsilon}_0 + \chi E_0 + \overline{E}_0\overline{\chi}) \right\} \tag{4.24}$$

This dualization can be also performed in components, this is gauging each component of V and Ψ to obtain the same result. This is done in section 4.4.3.

To describe the various contributions to the scalar potential coming from the complete dual action (4.24) it is needed the component expansion of the dual fields Y and F which are:

$$Y_\pm = y_\pm + \sqrt{2}(\theta^+\overline{\upsilon}_+ + \upsilon\bar{\theta}^+) - i\theta^+\bar{\theta}^+\partial_+ y_\mp, \tag{4.25}$$

$$F = \eta_- - \sqrt{2}\theta^+ f - i\theta^+\bar{\theta}^+\partial_+\eta_-. \tag{4.26}$$

Thus as y and H are the bosonic component of each superfield respectively, the scalar potential consist of the complete Lagrangian adding Fayet-Iliopoulos term, gauge action and the spectators fields. The kinetic term of the dual variable Y in the first term of the dual action $-\frac{i}{2}\frac{Y_-\partial_- Y_+}{Y_+}$, does not contribute to the scalar potential. The third term in the Lagrangian $iY\Upsilon_0$ leads to a scalar potential of the form $-2Dy_+ + 2iv_{01}y_- + h.c.$ Moreover, the Fayet-Iliopoulos and Theta term give rise to a potential of the form $D(\frac{it}{2} - \frac{i\bar{t}}{2}) + v_{01}(\frac{t}{2} + \frac{\bar{t}}{2})$. The gauge sector L_{gauge} contributes with a term of the form $\frac{v_{01}^2}{2e^2} + \frac{D^2}{2e^2}$.We have two additional contributions from the terms $\frac{\widetilde{\mathcal{F}}\bar{\widetilde{\mathcal{F}}}}{Y_+}$ and $-\widetilde{\mathcal{F}}E + \text{h.c.}$ which lead to terms of the potential of the form $\frac{-2H\overline{H}}{y_+}$ and $-\sqrt{2}(HE+\overline{EH})$, respectively.

Therefore, the scalar potential coming from (4.24) can be written as

$$\begin{aligned} U_{\mathtt{dual}} &= D\left(\frac{i}{2}(t-\bar{t}) - 2y_+ + |\phi_2|^2\right) + \frac{D^2}{2e^2} + \frac{v_{01}^2}{2e^2} + v_{01}\left(\frac{i}{2}(t+\bar{t}) - 2iy_-\right) \\ &+ \frac{H\overline{H}}{\Re(y)} + \sqrt{2}(HE + \overline{EH}). \end{aligned} \tag{4.27}$$

After eliminating the auxiliary field D and v_{01}, the potential is:

$$U_{\texttt{dual}} = \frac{e^2}{2}\Big(- \Im(t) - \Re(y) + |\phi_2|^2 \Big)^2 \tag{4.28}$$

$$+ \frac{e^2}{2}\Big(\Re(t) + \Im(y)\Big)^2 + \frac{H\overline{H}}{\Re(y)} + \sqrt{2}(HE + \overline{EH}), \tag{4.29}$$

which minimum condition with respect to H, E and $\Re(y)$ gives $E = 0$, $H = 0$ and:

$$|\phi_2|^2 - \Re(y) = \Im(t), \tag{4.30}$$

while for the original theory, te vacua is:

$$U_{\texttt{original}} = 0 \to |\phi|^2 + |\phi_2|^2 = \Im(t). \tag{4.31}$$

From ec. (4.30) one obtains a cone with vertex at $y_+ = -r$. Considering the $U(1)$ gauge symmetry this will lead to the line $\mathbb{R}^+$, such that the dual expected vacua is $\mathbb{R}^+ \times \mathbb{R}$ while for the original model is $\mathbb{P}^1$.

This T-duality is not mirror symmetry, because the dualization is performed in a single field direction. The mirror symmetry can also be obtained by this method, by adding an spectator chiral superfield and gauging two $U(1)$ symmetries, one for each chiral superfield. Although mirror symmetry is obtained by specific dualization of abelian global symmetries, the whole set of abelian T-dualities that can be explored is wider. These dualities have been discussed in the (2,2) cases in [28].

The superpotential of the original theory is given by [34]:

$$W_{\texttt{original}} = \frac{\Upsilon}{4\pi\sqrt{2}} \ln\left(\frac{\Sigma}{q\mu}\right). \tag{4.32}$$

The following ansatz for superpotential of the dual theory can be established:

$$W_{\texttt{dual}} = iY\Upsilon - \overline{E}F + \beta F e^{\alpha Y}, \tag{4.33}$$

thus:

$$W_{\mathtt{dual}} = \frac{i\Upsilon}{\alpha}\ln\left(\frac{\overline{E}}{\beta}\right)$$
$$= \frac{i\Upsilon}{\alpha}\ln\left(\frac{-i\Upsilon_0}{\alpha\beta F}\right). \tag{4.34}$$

For $(0,2)$ theories coming from a reduction of a (2,2) model with $\overline{E} = -iQ\sqrt{2}\Sigma\Phi$, the nonperturbative dual superpotential is written as [34]

$$W_{\mathtt{dual}} = \frac{i\Upsilon}{\alpha}\ln\left(\frac{\Sigma}{\beta/(-iQ\sqrt{2}\Phi)}\right), \tag{4.35}$$

where it can be compared with (4.34) to find the choices of:

$$\alpha = 4i\pi\sqrt{2} \qquad \text{and} \qquad \beta = -iq\mu\sqrt{2}Q\Phi\ . \tag{4.36}$$

4.4.2 A pure $(0,2)$ GLSM

For a pure $(0,2)$ abelian case, model which is not coming from reduction of a $(2,2)$ case, has $m=1$, $n=2$, $\widetilde{n}=1$, $k=1$ and $s=0$. The original Lagranian is the same:

$$L = \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\left\{\frac{1}{2}(1+2\Psi)(-i\overline{\Phi}\delta_-\Phi + 2V\overline{\Psi}\Psi - \overline{\Gamma}\Gamma)\right\}; \tag{4.37}$$

But this time the field Γ is not dualized and the gauged fields are only V and Ψ. Thus the dual Lagrangian is given by:

$$\boldsymbol{\Delta} L_{\mathtt{dual}} = \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\left\{-\frac{i}{2}\frac{Y_-\partial_-Y_+}{Y_+} + \frac{Y_+\overline{\Gamma}\Gamma}{2|\Phi|^2}\right\} + \int \mathrm{d}\theta^+(iY\Upsilon) + \frac{t}{4}\int d\theta_+\Upsilon + \text{h.c.} \tag{4.38}$$

The scalar potential is found to be the same that in the previous case discussed in section 3.1.1, except for the term $\frac{Y_+\overline{\Gamma}\Gamma}{2}$ which contributes to the scalar potential with a term of the form $y_+\overline{G}G - y_+\overline{E}E$. Gathering all that, it results that the scalar potential of the dual theory after eliminating the auxiliary field D is given by:

$$U_{\texttt{dual}} = \frac{e^2}{2}\bigg(- \Im(t) - \Re(y) + |\phi_2|^2\bigg)^2 + \frac{e^2}{2}\big[\Re(t) + \Im(y)\big]^2 + 2\Re(y)(E\overline{E} - G\overline{G}), \quad (4.39)$$

which minimum condition with respect to G, E and $\Re(y)$ gives $E = G = 0$ and:

$$|\phi_2|^2 - \Re(y) = \Im(t). \quad (4.40)$$

This is precisely the same equation found in the previous case (4.30) and consequently the topology of the manifold of vacua is a also $\mathbb{R}^+ \times \mathbb{R}$. Recall that for the original model the scalar potential reads:

$$|\phi|^2 + |\phi_2|^2 = \Im(t), \quad (4.41)$$

which together with the $U(1)$ gauge symmetry it constitutes a $\mathbb{P}^1$.

Thus, the ansatz for the superpotential in the dual model is:

$$\begin{aligned} W_{\texttt{dual}} &= iY\Upsilon + \beta e^{\alpha(Y+1)}, \\ &= \frac{i\Upsilon}{\alpha}\left[\ln\left(\frac{-i\Upsilon}{\alpha\beta}\right)\right]. \end{aligned} \quad (4.42)$$

Here it has been employed an ansatz for the instanton corrections, in order to obtain the same effective potential for the $U(1)$ gauge field as in the $(0,2)$ case coming from a reduction.

4.4.3 Abelian T-dualization in superfield components

There is an alternative way to perform the dualization of a $(0,2)$ GLSM coming from a $(2,2)$ reduction in terms of superfield components. This leads to the same result as in section 4.4.1.

First the appropriate Lagrangian of a single chiral field and a Fermi one with abelian global symmetry in component fields is written with $\delta_\pm := \partial_\pm - \overleftarrow{\partial}_\pm$ and $I_\pm = \delta_\pm + 2iv_\pm$ as:

$$\begin{aligned}\mathbf{\Delta} L_{\texttt{components}} = & - \frac{1}{2}\bar{\phi} I_- I_+ \phi + \frac{i}{2}\bar{\gamma}_- I_+ \gamma_- + i\bar{\psi}_+ I_- \psi_+ + 2D\bar{\phi}\phi + 2\sqrt{2}i(\bar{\lambda}_- \bar{\psi}_- \phi - \bar{\phi}\psi_+ \lambda_-) \\ & + \bar{G}G - \bar{E}E - \bar{\gamma}_- \frac{\partial E}{\partial \phi}\psi_+ - \bar{\psi}_+ \frac{\partial \bar{E}}{\partial \bar{\phi}}\gamma_- \end{aligned} \tag{4.43}$$

To realize the T duality algorithm, gauging the global symmetry we add the fields $v_\pm$, λ_-, D and E (components of the gauged V, Ψ and Γ) as well as the Lagrange multipliers. The original fields will be denoted with a subindex 0, the gauged ones with a subindex 1 and the sum of both without any subindex. Thus, for example: $a := a_0 + a_1$, etc. thus:

$$\begin{aligned}\mathbf{\Delta} L_{\texttt{master}} = & - \frac{1}{2}\bar{\phi} I_- I_+ \phi + \frac{i}{2}\bar{\gamma}_- I_+ \gamma_- + i\bar{\psi}_+ I_- \psi_+ + 2(D_0 + D_1)\bar{\phi}\phi \\ & + 2\sqrt{2}i((\bar{\lambda}_{-0} + \bar{\lambda}_{-1})\bar{\psi}_- \phi - \bar{\phi}\psi_+ (\lambda_{-0} + \lambda_{-1})) - 2i(l - \bar{l})(D_1) \\ & + \bar{G}G - (\bar{E}_0 + \bar{E}_1)(E_0 + E_1) - \bar{\gamma}_- \frac{\partial (E_0 + E_1)}{\partial \phi}\psi_+ - \bar{\psi}_+ \frac{\partial (\bar{E}_0 + \bar{E}_1)}{\partial \bar{\phi}}\gamma_- \\ & + i(\partial_+ \bar{E}_1 x - \bar{x}\partial_+ E_1) + \sqrt{2}\left(\xi \frac{\partial E_1}{\partial \phi}\psi_+ + \bar{\psi}_+ \frac{\partial \bar{E}_1}{\partial \bar{\phi}}\xi\right) + \bar{z}E_1 + \bar{E}_1 z \\ & + 2i(\omega \partial_+ \lambda_{-1} - \partial_+ \bar{\lambda}_{-1}\bar{\omega}) - 2(\varepsilon\lambda_{-1} + \bar{\lambda}_{-1}\bar{\varepsilon}) - (l + \bar{l})(\partial_- v_{+1} - \partial_+ v_{-1}) \end{aligned} \tag{4.44}$$

Taking variations with respect to $v_{\pm,1}$, λ_1, D_1 and E_1 one obtains the corresponding equations of motion:

For $\delta_{D_1} L$:

$$i(l-\bar{l}) = |\phi|^2. \tag{4.45}$$

For $\delta_{\lambda_1} L$:

$$\varepsilon + i\partial_+\omega = -\sqrt{2}\overline{\phi}\psi_+. \tag{4.46}$$

For $\delta_{v_{-1}} L$:

$$2v_+|\phi|^2 = -2\overline{\psi}_+\psi_+ - i(\overline{\phi}\delta_+\phi) - \partial_+(l+\bar{l}). \tag{4.47}$$

For $\delta_{v_{+1}} L$:

$$2v_-|\phi|^2 = -\overline{\gamma}_-\gamma_- - i(\overline{\phi}\delta_-\phi) + \partial_-(l+\bar{l}). \tag{4.48}$$

For $\delta_{E_1} L$:

$$\bar{z} + i\partial_+\bar{x} = \bar{E} \tag{4.49}$$

For $\delta_{\partial_\phi E_1} L$:

$$\sqrt{2}\xi = \bar{\gamma}_-. \tag{4.50}$$

New variables can be defined in the form:

$$y_\pm := il\mp = i(l\mp\bar{l}), \qquad \sqrt{2}f := \bar{z} + i\partial_+\bar{x}, \qquad \sqrt{2}v = \partial_+\omega - i\varepsilon, \qquad \bar{\eta}_- = \bar{\xi}\ . \tag{4.51}$$

Thus, using Eqs. (4.45-4.50) in the Lagrangian (4.44) it results the dual Lagrangian:

$$\begin{aligned} L_{\mathtt{dual}} = & - \sqrt{2}\left(\bar{f}\bar{E}_0 + E_0 f + \xi\frac{\partial E_0}{\partial\phi}\psi_+ + \bar{\psi}_+\frac{\partial\bar{E}_0}{\partial\bar{\phi}}\bar{\xi} + 2i\bar{\lambda}_{-0}\bar{v} - 2iv\lambda_{-0}\right) \\ & + iy_-\left(\partial_+v_{-0} - \partial_-v_{+0}\right) + 2y_+D_0 - \frac{1}{2}\bar{\phi}\delta_-\delta_+\phi + \frac{i}{2}\bar{\gamma}_-\delta_+\gamma_- + i\bar{\psi}_+\delta_-\psi_+ + \bar{G}G \\ & - \frac{1}{2y_+}\left(-i\bar{\phi}\delta_+\phi - 2\bar{\psi}_+\psi_+ + i\partial_+y_-\right)\left(-i\bar{\phi}\delta_+\phi - \bar{\gamma}_-\gamma_- - i\partial_-y_-\right) + 2\bar{f}f. \end{aligned} \tag{4.52}$$

It is easy to check that this dual Lagrangian coincides with the component field expansion of the dual Lagrangian (4.24). A similar procedure could be carried over in the case of models with non-Abelian T-duality.

Non-Abelian T-duality with gauge group $U(1)^n$

In this section it is performed the non-abelian dualization with gauge group $U(1)^n$. It is when the Lagrangian has non-abelian global symmetries, the dualization algorithm is realized gauging these symmetries and adding Lagrange multipliers which has values in the Lie Algebra of the group. The only models considered in the present chapter are assumed to come from a reduction of a $(2,2)$ supersymmetric model thus the number of chiral fields $\mathbf{\Phi}_i$ and the number of Fermi fields $\mathbf{\Gamma}_i$ coincide. They are equally charged under the $U(1)^m$ gauge group and they are assumed to be also equally charged under the global group. Moreover in order to be as general as possible, there are considered models where the total non-abelian gauged group is $G = G_1 \times \cdots \times G_S$. To write the original Lagrangian, it is needed to emphasize that the $\Phi_I = (\Phi_{I1}, ...\Phi_{In_I})$ are vectors of chiral superfields, for $I = 1...s$, and $V_I = V_{Ia}T_a$, $\Psi_I = \Psi_{Ia}T_a$ are superfields for each gauged group $SU(n_I)$.

The master Lagrangian is written as:

$$\begin{aligned}
L_{\texttt{master}} &= \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \sum_{a=1}^{m} \frac{1}{8e_a^2} \overline{\Upsilon}_a \Upsilon_a \\
&- \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Bigg\{ \sum_{I=1}^{S} \frac{i}{2} \mathbf{\Phi}_I^\dagger e^{2\sum_{a=1}^m Q_I^a \Psi_a + 2\Psi_{1I}} \Bigg(\partial_- + i \sum_{a=1}^{m} Q_I^a V_a + i V_{1I} \Bigg) \mathbf{\Phi}_I \Bigg\} \\
&+ \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Bigg\{ \sum_{I=1}^{S} \frac{i}{2} \mathbf{\Phi}_I^\dagger \Bigg(\overleftarrow{\partial}_- - i \sum_{a=1}^{m} Q_I^a V_a - i V_{1I} \Bigg) e^{2\sum_{a=1}^m Q_I^a \Psi_a + 2\Psi_{1I}} \mathbf{\Phi}_I \Bigg\} \\
&- \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Bigg\{ \sum_{I=1}^{S} \frac{1}{2} \big(\mathbf{\Gamma}_I^\dagger + \mathbf{\Gamma}_{1I}^\dagger \big) e^{2\sum_{a=1}^m Q_I^a \Psi_a + 2\Psi_{1I}} \big(\mathbf{\Gamma}_I + \mathbf{\Gamma}_{1I} \big) \Bigg\} \\
&+ \int \mathrm{d}\theta^+ \sum_{a=1}^{m} \frac{t_a}{4} \Upsilon_a \big|_{\bar{\theta}^+=0} + \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \sum_{I=1}^{S} \mathrm{Tr}\big(\Lambda_I \Upsilon_I \big) + \mathrm{h.c.} \\
&+ \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \sum_{I=1}^{S} \big(\chi_I^\dagger \widetilde{\mathbf{E}}_I \big) + \mathrm{h.c.},
\end{aligned} \tag{5.1}$$

Here, $\mathcal{T}^a$ are the generators of the Lie algebra of G_I. In the notation of the Lagrangian it is understood an inner product on the vector space indexed by the number of factors of the global group, thus it is needed to sum over the S factors there. To simplify the relevant terms in the Lagrangian, it is defined as: $a_I^{ab} := \mathbf{\Phi}_I^\dagger \{\mathcal{T}^a, \mathcal{T}^b\} \mathbf{\Phi}_I$, $e_I := 1_I + 2\sum_{\alpha=1}^m Q_I^\alpha \Psi_\alpha$, $Z_I^a := \mathbf{\Phi}_I^\dagger \mathcal{T}^a \mathbf{\Phi}_I$. Thus, to implement the duality algorithm, the partial Lagrangian is given by:

$$
\begin{aligned}
\mathbf{\Delta} L_{\mathtt{master}} &= \sum_{I=1}^{S} \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Bigg\{ -\frac{i}{2} e_I \mathbf{\Phi}_I^\dagger \delta_- \mathbf{\Phi}_I + \mathbf{\Phi}^\dagger \mathbf{\Phi} e_I Q_I^\beta V_\beta + V_{1I}^b e_I Z_I^b \\
&+ \Psi_{1I}^a \Big(-i \mathbf{\Phi}_I^\dagger \mathcal{T}^a \delta_- \mathbf{\Phi}_I + 2 Q_I^\beta V_\beta Z_I^a \Big) + \Psi_{1I}^a V_{1I}^b a_I^{ab} \\
&- \frac{1}{2} \big(\mathbf{\Gamma}_I^\dagger + \mathbf{\Gamma}_{1I}^\dagger \big) \big(e_I + 2 \Psi_{1I}^a \mathcal{T}^a \big) \big(\mathbf{\Gamma}_I + \mathbf{\Gamma}_{1I} \big) \\
&+ \Big(V_{1I}^b Y_{+a} + i \Psi_{1I}^b \partial_- Y_{-I}^a \Big) \mathrm{Tr}(\mathcal{T}^a \mathcal{T}^b) \\
&- \frac{\sqrt{2}}{2} \Big(\mathbf{\Gamma}_{1I}^{a\dagger} \mathcal{T}^a \mathcal{T}^b \widetilde{\mathcal{F}}_I^b + \widetilde{\mathcal{F}}_I^{a\dagger} \mathcal{T}^a \mathcal{T}^b \mathbf{\Gamma}_{1I}^b \Big) \Bigg\} , \qquad (5.2)
\end{aligned}
$$

which is basically the sum of Lagrangians corresponding to each factor of the global group G_I.

The variations with respect to V_{1I}^c, Ψ_{1I}^c and Γ_{1I}^c, give the following equations of motion:

$$
\begin{aligned}
\delta_{V_{1I}^c} S = 0 : &\qquad \Psi_{1I}^a a^{ca} = -Y_{+Ia} \mathrm{Tr}(\mathcal{T}^a \mathcal{T}^c) - e_I Z_I^c := K_I^c, \qquad (5.3) \\
\delta_{\Psi_{1I}^c} S = 0 : &\qquad V_{1I}^b a_I^{bc} + 2 Q_I^\beta V_\beta Z_I^c - i \Phi_I^\dagger \mathcal{T}^c \delta_- \Phi_I + i \partial_- Y_{-aI} \mathrm{Tr}(\mathcal{T}^a \mathcal{T}^c) \\
&\qquad -(\Gamma_I^\dagger + \Gamma_I^{a\dagger} \mathcal{T}^a) \mathcal{T}^b (\Gamma_I + \Gamma_I^c \mathcal{T}^c) = 0, \qquad (5.4) \\
\delta_{\Gamma_{1I}} S = 0 : &\qquad -\frac{1}{2} (\Gamma_I^\dagger + \Gamma_{1I}^\dagger)(e_I + 2 \Psi_{1I}^b \mathcal{T}^b) - \frac{\sqrt{2}}{2} \widetilde{\mathcal{F}}_I^{a\dagger} \mathcal{T}^a = 0. \qquad (5.5)
\end{aligned}
$$

Thus the corresponding partial dual Lagrangian becomes

$$
\begin{aligned}
\mathbf{\Delta} L_{\mathtt{dual}} &= \sum_{I=1}^{S} \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Bigg\{ -\frac{i}{2} e_I \mathbf{\Phi}_I^\dagger \delta_- \mathbf{\Phi}_I + \mathbf{\Phi}_I^\dagger \mathbf{\Phi}_I e_I Q_I^\beta V_\beta + \widetilde{\mathcal{F}}_I^\dagger X_I^{-1} \widetilde{\mathcal{F}}_I \\
&+ \frac{\sqrt{2}}{2} (\widetilde{\mathcal{F}}_I^\dagger \mathbf{\Gamma}_I + \mathbf{\Gamma}_I^\dagger \widetilde{\mathcal{F}}_I) \Big(-i \mathbf{\Phi}_I^\dagger \delta_- \mathcal{T}^a \mathbf{\Phi}_I + 2 Q_I^\beta V_\beta Z_I^a \Big) \\
&\times \Big(-Y_{+Ib} \mathrm{Tr}(\mathcal{T}^b \mathcal{T}^c) - e_I Z_I^c \Big) b_{ac} \Bigg\}, \qquad (5.6)
\end{aligned}
$$

where $X_I := e_I 2\mathcal{T}^a K_I^a = e_I - 2\mathcal{T}^a e_I Z_I^c b^{ca} - 2\mathcal{T}^a Y_{+b}\mathrm{Tr}(\mathcal{T}^b\mathcal{T}^c)b^{ca}$. It is missed to remove the original chiral fields $\mathbf{\Phi}$, this is done by gauge fixing them.

This is written for a general model and a generic Lie group G_i. Explicit solutions can be found for each specific group G_i. In the next section, this is performed when the system has $\mathfrak{g} = SU(2)$ global symmetries for simplicity.

5.1 $\mathfrak{g} = SU(2)$ global symmetries

Before solving in a particular case, it can be considered any group G with Lie Algebra generators $\mathcal{T}^a$ that satisfies $\mathrm{Tr}(\mathcal{T}^a\mathcal{T}^b) = 2\delta^{ab}$, $\{\mathcal{T}^a, \mathcal{T}^b\} = 2\delta^{ab}\mathbf{I_d}$. Thus with $e_I := 1_I + 2\sum_{\alpha=1}^m Q_I^\alpha \Psi_\alpha$ it is obtained $X_I := e_I\mathbf{I_d} - \mathcal{T}^a\frac{e_I Z_I^a + 2Y_+^a}{|\Phi_I|^2}$. Therefore, the dual Lagrangian becomes:

$$\begin{aligned} L_{dual} = \sum_{I=1}^{s}\int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\Big\{ & (-i\mathbf{\Phi}_I^\dagger\delta_-\mathcal{T}^a\mathbf{\Phi}_I + Q_\beta V^\beta)\Big[e_I|\mathbf{\Phi}_I|^2 - \frac{e_I}{|\mathbf{\Phi}_I|^2}Z^a Z_a - Y_+^a\frac{Z^a}{|\mathbf{\Phi}_I|^2}\Big] \\ + & \widetilde{\mathcal{F}}^\dagger X^{-1}\widetilde{\mathcal{F}} + \frac{\sqrt{2}}{2}(\widetilde{\mathcal{F}}_I^\dagger\mathbf{\Gamma}_I + \mathbf{\Gamma}_I^\dagger\widetilde{\mathcal{F}}_I) - \frac{i}{2}e_I\mathbf{\Phi}_I^\dagger\delta_-\mathbf{\Phi}_I\Big\}. \end{aligned} \tag{5.7}$$

Surely this is the case of $SU(2)$. In this algebra note that $\mathbf{\Phi} = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}$ which are 2 complex fields, can be redefined in terms of new real fields Z_0, Z_1, Z_2, Z_3, with the transformation:

$$Z_0 = \overline{\Phi}_1\Phi_1 + \overline{\Phi}_2\Phi_2, \qquad Z_1 = 2\Re(\overline{\Phi}_1\Phi_2), \qquad Z_2 = 2\Im(\overline{\Phi}_1\Phi_2), \qquad Z_3 = \overline{\Phi}_1\Phi_1 - \overline{\Phi}_2\Phi_2, \tag{5.8}$$

Then, the original chiral fields can be eliminated by gauge fixing the Z's, these are 4 real constants; and with the inverse transformation, the products of the original fields are written as sums of these new fields:

$$\overline{\Phi}_1\Phi_1 = \frac{Z_0 + Z_3}{2}, \qquad \overline{\Phi}_1\Phi_2 = \frac{Z_1 + iZ_2}{2}, \qquad \overline{\Phi}_2\Phi_2 = \frac{Z_0 - Z_3}{2}. \tag{5.9}$$

Thus, with the partial gauge fixing: $\Phi_I^\dagger T^b \partial_- \Phi_I = \partial_- \Phi_I^\dagger T^b \Phi_I$, for $b \in \{0,1,2,3\}$, the partial dual Lagrangian has the following form

$$\begin{aligned}
\boldsymbol{\Delta} L_{\texttt{dual}} &= \sum_{I=1}^{s} \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Big\{ Q_\beta V^\beta \Big(e_I - e_I \frac{Z^a Z_a}{Z_0} - \frac{Y_+^a Z^a}{Z_0} \Big) \\
&+ \widetilde{\mathcal{F}}^\dagger \Big(e_I \mathbf{I_d} - \frac{\mathcal{T}^a}{Z_0} (e_I Z_I^a + 2Y_+^a) \Big)^{-1} \widetilde{\mathcal{F}} \\
&+ \frac{\sqrt{2}}{2} \Big(\widetilde{\mathcal{F}}_I^\dagger \boldsymbol{\Gamma}_I + \boldsymbol{\Gamma}_I^\dagger \widetilde{\mathcal{F}}_I \Big) \Big\} + \frac{t}{4} \int \mathrm{d}\theta^+ \Upsilon|_{\bar{\theta}^+=0}.
\end{aligned} \tag{5.10}$$

To write the scalar potential, it can be defined to simplify $u^a = 2\frac{y_+^a}{Z_0} + \frac{Z_a}{Z_0}$. So the new dual coordinate is u^a. Thus, the contribution to the scalar potential is:

$$\begin{aligned}
U &= \frac{-2}{1 - u^a u_a - 2\frac{Z^a Z_a}{Z_0^2} + 2\frac{Z_a u^a}{Z_0}} \Big[\bar{H}_1 H_1 + \bar{H}_2 H_2 + \Big(\bar{H}_1 H_1 - \bar{H}_2 H_2 \Big) u^3 + \bar{H}_2 H_1 \bar{u}^{12} + \text{c.c.} \Big] \\
&+ \sqrt{2} (\bar{H}_1 E_1(\phi) + \bar{H}_2 E_2(\phi) + H_1 \bar{E}_1(\phi) + H_2 \bar{E}_2(\phi)) \\
&- \frac{iQ v_- \partial_+ u_-^a Z_a}{Z_0^2} + 2Q^2 v_- v_+ \Big(1 - \frac{Z^a Z_a}{Z_0^2} \Big) + 2QD \Big(1 - \frac{Z^a Z_a}{2Z_0^2} - \frac{u^a Z_a}{2Z_0^2} \Big) + \frac{D^2}{2e} - Dr
\end{aligned} \tag{5.11}$$

After integrating D:

$$
\begin{aligned}
U &= \frac{-2}{1-u^a u_a - 2\frac{Z^a Z_a}{Z_0^2} + 2\frac{Z_a u^a}{Z_0}}\left[\bar{H}_1 H_1 + \bar{H}_2 H_2 + \left(\bar{H}_1 H_1 - \bar{H}_2 H_2\right) u^3 + \bar{H}_2 H_1 \bar{u}^{12} + \text{c.c.}\right] \\
&+ \sqrt{2}(\bar{H}_1 E_1(\phi) + \bar{H}_2 E_2(\phi) + H_1 \bar{E}_1(\phi) + H_2 \bar{E}_2(\phi)) \qquad (5.12) \\
&- \frac{iQ v_- \partial_+ u^a_- Z_a}{Z_0^2} + 2Q^2 v_- v_+ \left(1 - \frac{Z^a Z_a}{Z_0^2}\right) - \frac{e^2}{2}\left(1 - \frac{Z^a Z_a}{2Z_0^2} - \frac{u^a Z_a}{2Z_0^2} + r\right)^2
\end{aligned}
$$

To find the minimum through derivatives of H's it is found the vacua condition $U = 0$:

$$
\begin{aligned}
0 &= \frac{e^2}{2}(r + 1 + \frac{Z^a Z_a}{2Z_0^2} - \frac{u^a Z_a}{2Z_0^2})^2 + \qquad (5.13) \\
&+ \frac{u^c u_c - \frac{2u^c Z_c}{Z_0} + \frac{2Z^a Z_a}{Z_0^2} - 1}{1 - u^c u_c}\left[|E_1|^2(1-u_3) + |E_2|^2(1+u_3) - E_1 \bar{E}_2 \bar{u}_{12} - E_2 \bar{E}_1 u_{12}\right] .
\end{aligned}
$$

if $A = \frac{-2y^b_+ y^b_+}{1-u_c u^c}\begin{pmatrix} u_3 - 1 & u_{12} \\ \bar{u}_{12} & -1-u_3 \end{pmatrix}$, then that can be written:

$$
\frac{e^2}{2}(\Im(t) - y^a_+ Z_a)^2 + (\bar{E}_1 \quad \bar{E}_2) A \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = 0 \, . \qquad (5.14)
$$

while for the original scalar potential:

$$
U_{ori} = \frac{e^2}{2}\left(\sum_i Q_i |\phi_i|^2 - r\right)^2 + \sum_a |E_a|^2 = 0. \qquad (5.15)
$$

Notice that the dependence on the fields E is similar in the original and in the dual model. With the difference that in (5.14) this term is positive definite only in a bounded region of the moduli space. The vacua manifold $\mathcal{W}$ is characterized by the 3 coordinates:

y_+^a, and there is one equation for the vacua, thus it is a two-dimensional surface. The Y_- term does not appear on the Lagrangian, then y_- is not a coordinate in the potential. The eigenvalues of the matrix A are: $\lambda_\pm = \frac{2y_+^a y_+^a}{1 \mp \sqrt{u^a u^a}}$. So, because A is Hermitian, there exists a unitary matrix P such that $A = P^\dagger D P$, and D is the diagonal matrix with eigenvalues as entries; therefore:

$$\widetilde{\mathbf{E}}^\dagger A \widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}^\dagger P^\dagger D P \widetilde{\mathbf{E}} = (P\widetilde{\mathbf{E}})^\dagger D (P\widetilde{\mathbf{E}}) = \lambda_+ |(P\widetilde{\mathbf{E}})_+|^2 + \lambda_- |(P\widetilde{\mathbf{E}})_-|^2, \tag{5.16}$$

which is a quadratic form. Thus, the vacua manifold $\mathcal{W}$ is made up of 3 regions depending on whether $y_+^a y_+^a + Z^a y_+^a$ is greater than, less than, or equal to 0, these regions are: the inside of a sphere, the outside of it, and the shell of the sphere. Thus, there are three cases:

- Region 1:

$$y^a y^a + Z^a y^a < 0, \qquad \Im(t) = y_+^a Z^a \quad \text{and} \quad |\widetilde{\mathbf{E}}|^2 = 0. \tag{5.17}$$

- Region 2:

$$y^a y^a + Z^a y^a = 0, \qquad \Im(t) = y_+^a Z^a \quad \text{and} \quad |(P\widetilde{\mathbf{E}})_-|^2 = 0. \tag{5.18}$$

- Region 3:

$$y^a y^a + Z^a y^a > 0, \qquad \frac{e}{2}(\Im(t) - y_+^a Z_a)^2 + \lambda_- |(P\widetilde{\mathbf{E}})_-|^2 = -\lambda_+ |(P\widetilde{\mathbf{E}})_+|^2, \tag{5.19}$$

where:

$$(P\widetilde{\mathbf{E}})_\pm = \frac{\mp u_{12} E_1 + (\sqrt{u_c u^c} \pm u_3) E_2}{\sqrt{2\sqrt{u_c u^c}(\sqrt{u_c u^c} \pm u_3)}}. \tag{5.20}$$

As we have three real variables y_1, y_2, y_3, then the vacua $\mathcal{W}$ consist of a two-dimensional surface. For the regions 1 and 2, the potential is semi-definite positive and the surface $y_+^a Z_a = \Im(t)$ is a plane inside the sphere, thus this is a disk $\mathbf{D}$, where the modulus $r = \Im(t)$ determines the size of the disk (as its relative position inside the sphere). It is important to notice that if $r \notin [-1, 0]$ this disk is empty. However, for the case outside the sphere, one has a surface given by equation (5.19). As a solution of equation (5.14). In figure 5.1 the geometry of the dual vacua is represented for $r = \Im(t) = -\frac{1}{2}$ while in figure 5.2 the dual vacua is represented for $r = \Im(t) = 1$; in both cases the rest of the parameters are $e = 5$, $z_1 = 0.700629$, $z_2 = 0.509037$, $z_3 = \frac{1}{2}$, $E_1 = 1 + 1i$, $E_2 = 3 + 4i$ and $z_0 = 1$.

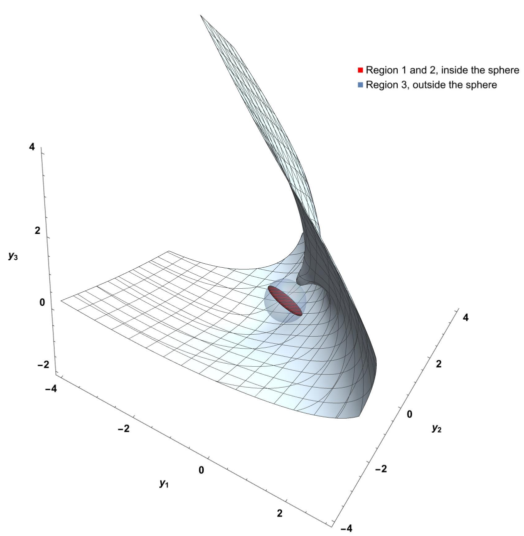

Figure 5.1: Vacua of the dual model, with parameter $r = \Im(t) = -\frac{1}{2}$. Notice that the change of this parameter changes the topology of the dual space.

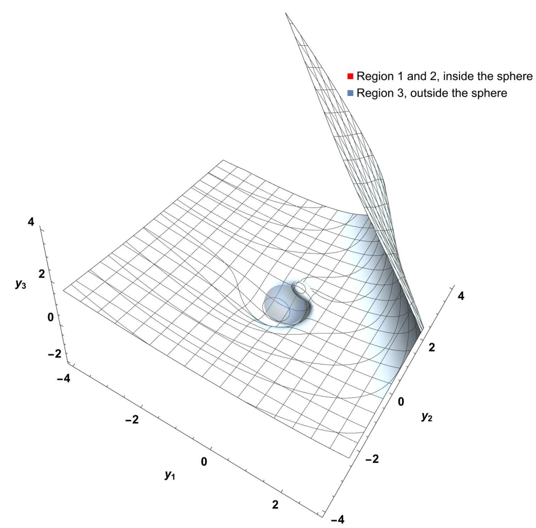

Figure 5.2: Vacua of the dual model, with $r = \Im(t) = 1$. Notice that the change of this parameter changes the topology of the dual space.

It can be seen that for $r \in [-1, 0]$ the vacua space has the topology of $\mathbb{R}^2 \cup \mathbf{D}$, while for $r \notin [-1, 0]$ the vacua has simply the topology of $\mathbb{R}^2$, although it is geometrically distinct from $\mathbb{R}^2$.

Until this point the superfield E has been arbitrary and it is a function only on Φ (which is gauge fixed) and other parameters; however, it can be chosen a particular form of it that comes from the $(2,2)$ reduction:

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \Sigma \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} , \tag{5.21}$$

where $\Sigma = \sigma + \sqrt{2}\theta^+\overline{\lambda}_+ - i\theta^+\bar{\theta}^+\partial_+\sigma$, this means for region 1: $|\sigma|^2 = 0$.

It is remarkable that in this case the equations of motion (5.3),(5.4) and (5.5) can be exactly solved, without requiring to project out to an abelian component (or to particularize to a semichiral vector field) as in the $(2,2)$ supersymmetric non-abelian T-duality, as well for the $SU(2)$ group [28].

Instanton correction

The Lagrangian with the instanton correction is given by

$$\begin{aligned} L_{\texttt{dual}} &= \sum_{I=1}^{s} \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\Big\{ Q_\beta V^\beta\Big(e_I - e_I\frac{Z^a Z_a}{Z_0} - \frac{Y_+^a Z^a}{Z_0}\Big) \\ &+ \widetilde{\mathcal{F}}^\dagger\Big(e_I\mathbf{I_d} - \frac{\mathcal{T}^a}{Z_0}(e_I Z_I^a + 2Y_+^a)\Big)^{-1}\widetilde{\mathcal{F}} + \frac{\sqrt{2}}{2}\Big(\widetilde{\mathcal{F}}_I^\dagger\boldsymbol{\Gamma}_I + \boldsymbol{\Gamma}_I^\dagger\widetilde{\mathcal{F}}_I\Big)\Big\} \\ &+ \int \mathrm{d}\theta^+\Big\{\frac{t}{4}\Upsilon|_{\bar{\theta}^+=0} + \widetilde{\mathcal{F}}^\dagger\boldsymbol{\beta}e^{\alpha^b Y_b}\Big\}, \end{aligned} \tag{5.22}$$

where the last term is the instanton correction and its contribution to the bosonic scalar potential is:

$$\int \mathrm{d}\theta^+\widetilde{\mathcal{F}}^\dagger\boldsymbol{\beta}e^{\alpha^b Y_b} = -\sqrt{2}\big(\overline{H}_0\beta^0 + \overline{H}_1\beta^1\big)e^{\alpha_b y_+^b}. \tag{5.23}$$

Then, the new vacua equation is:

$$\frac{e}{2}\bigg(\Im(t) - y_+^a Z_a\bigg)^2 + \bigg(\widetilde{\mathbf{E}} - e^{\alpha_b y_+^b}\boldsymbol{\beta}\bigg)^\dagger A\bigg(\widetilde{\mathbf{E}} - e^{\alpha_b y_+^b}\boldsymbol{\beta}\bigg) = 0 \; , \tag{5.24}$$

and similarly, when $0 > y_+^a y_+^a + y_+^a Z_a$ the solution gives:

$$\frac{e}{2}\bigg(\Im(t) - y_+^a Z_a\bigg)^2 = 0 \qquad \text{and} \qquad |\varepsilon_1|^2(1-u_3) + |\varepsilon_2|^2(1+u_3) - \varepsilon_1\overline{\varepsilon}_2\overline{u}_{12} - \varepsilon_2\overline{\varepsilon}_1 u_{12} = 0, \tag{5.25}$$

where $\boldsymbol{\varepsilon} = \widetilde{\mathbf{E}} - e^{\alpha_b y_+^b}\boldsymbol{\beta}$. Notice that the effect of the instanton in the effective potential is just a displacement of the holomorphic function E. Therefore the dual geometry coincides with the analysis performed without instanton corrections. This is a common point with observations of the dualities in the $(2,2)$ GLSMs [28].

5.2 A model with global symmetry $SU(2) \times SU(2)$

In this section a generalization of the model presented in [54] is studied which consist of a GLSM with gauge symmetry $U_1(1) \times U_2(1)$, two chiral fields Φ_1, Φ_2 and two Fermi Γ_1, Γ_2 with charge 1 under the first factor of the gauge symmetry $U_1(1)$; as well as two chiral fields $\widetilde{\Phi}_1$, $\widetilde{\Phi}_2$ and two Fermi $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$ with charge 1 under the $U_2(1)$ gauge group. This a deformation of a $(2,2)$ model into a $(0,2)$ model, so the restrictions for the fields E's are:

$$\begin{aligned}
E_1 &= \sqrt{2}\{\Phi_1\Sigma + \widetilde{\Sigma}(\alpha_1\Phi_1 + \alpha_2\Phi_2)\}, \\
E_2 &= \sqrt{2}\{\Phi_2\Sigma + \widetilde{\Sigma}(\alpha_1'\Phi_1 + \alpha_2'\Phi_2)\}, \\
\widetilde{E}_1 &= \sqrt{2}\{\widetilde{\Phi}_1\widetilde{\Sigma} + \Sigma(\beta_1\widetilde{\Phi}_1 + \beta_2\widetilde{\Phi}_2)\}, \\
\widetilde{E}_2 &= \sqrt{2}\{\widetilde{\Phi}_2\widetilde{\Sigma} + \Sigma(\beta_1'\widetilde{\Phi}_1 + \beta_2'\widetilde{\Phi}_2)\},
\end{aligned} \tag{5.26}$$

where α, α', β and β' are real parameters. In the limit when the α's and β's parameters vanish the reduced $(0,2)$ model is recovered.

The Lagrangian is:

$$
\begin{aligned}
L &= \sum_{i=1}^{2}\int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\Big\{-\frac{i}{2}\overline{\Phi}_i[e^{2\Psi}\partial_- - \overleftarrow{\partial}_- e^{2\Psi}]\Phi_i + Ve^{2\Psi}|\Phi_i|^2 - \frac{1}{2}e^{2\Psi}\overline{\Gamma}_i\Gamma_i\Big\} \\
&+ \sum_{i=1}^{2}\int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\Big\{-\frac{i}{2}\overline{\widetilde{\Phi}}_i[e^{2\widetilde{\Psi}}\partial_- - \overleftarrow{\partial}_- e^{2\widetilde{\Psi}}]\widetilde{\Phi}_i + Ve^{2\widetilde{\Psi}}|\widetilde{\Phi}_i|^2 - \frac{1}{2}e^{2\widetilde{\Psi}}\overline{\widetilde{\Gamma}}_i\widetilde{\Gamma}_i\Big\},
\end{aligned}
\tag{5.27}
$$

and the scalar potential is given by

$$
U_{\texttt{original}} = \frac{e^2}{2}\Big(|\phi_1|^2+|\phi_2|^2-r_1\Big)^2+\frac{e^2}{2}\Big(|\widetilde{\phi}_1|^2+|\widetilde{\phi}_2|^2-r_2\Big)^2+|E_1|^2+|E_2|^2+|\widetilde{E}_1|^2+|\widetilde{E}_2|^2.
\tag{5.28}
$$

The vacuum solution for this model is [54]:

$$
|\phi_1|^2+|\phi_2|^2=r_1, \qquad |\widetilde{\phi}_1|^2+|\widetilde{\phi}_2|^2=r_2,
\tag{5.29}
$$

i.e., the vacua manifold is a product of $\mathbb{P}^1\times\mathbb{P}^1$ with Kähler classes r_1 and r_2 respectively, and

$$
E_i=\widetilde{E}_i=0.
\tag{5.30}
$$

In the $SU(2)\times SU(2)$ generalization both chiral fields and Fermi fields are $SU(2)$

multiplets related to a different $SU(2)$ sector. Let us write the master Lagrangian

$$\begin{aligned}
\boldsymbol{\Delta} L_{\texttt{master}} &= \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \sum_{i=1}^{2} \frac{1}{8e_i^2} \overline{\Upsilon}_i \Upsilon_i + \int \mathrm{d}\theta^+ \sum_{i=1}^{2} \frac{t_i}{4} \Upsilon_i|_{\bar{\theta}^+=0} \\
&- \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \frac{i}{2} \overline{\Phi} e^{2\Psi_1 + 2\Psi_{1a}T_a} \Big(\partial_- + iV_1 + iV_{1a}T_a \Big) \Phi \\
&- \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \frac{i}{2} \overline{\widetilde{\Phi}} e^{2\Psi_2 + 2\Psi_{2a}T_a} \Big(\partial_- + iV_1 + iV_{2a}T_a \Big) \widetilde{\Phi} \\
&+ \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Big\{ \frac{i}{2} \overline{\Phi} \Big(\overleftarrow{\partial}_- - iV_1 - iV_{1a}T_a \Big) e^{2\Psi_1 + 2\Psi_{1a}T_a} \Phi \Big\} \\
&+ \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Big\{ \frac{i}{2} \overline{\widetilde{\Phi}} \Big(\overleftarrow{\partial}_- - iV_2 - iV_{2a}T_a \Big) e^{2\Psi_2 + 2\Psi_{2a}T_a} \widetilde{\Phi} \Big\} \\
&- \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Big\{ \frac{1}{2} \big(\overline{\Gamma} + \overline{\Gamma}_1 \big) e^{2\Psi_1 + 2\Psi_{1a}T_a} \big(\Gamma + \Gamma_1 \big) \Big\} \\
&- \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \Big\{ \frac{1}{2} \big(\overline{\widetilde{\Gamma}} + \overline{\widetilde{\Gamma}}_2 \big) e^{2\Psi_2 + 2\Psi_{2a}T_a} \big(\widetilde{\Gamma} + \widetilde{\Gamma}_2 \big) \Big\} \\
&+ \sum_{i=1}^{2} \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \mathrm{Tr}(\Lambda_i \Upsilon_i) + \sum_{i=1}^{2} \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \mathrm{Tr}(\widetilde{\Lambda}_i \Upsilon_i) + \text{h.c.} \\
&+ \sum_{i=1}^{2} \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \overline{\chi}_i E_i + \sum_{i=1}^{2} \int \mathrm{d}\theta^+ \mathrm{d}\bar{\theta}^+ \overline{\widetilde{\chi}}_i E_i + \text{h.c.}
\end{aligned} \tag{5.31}$$

Let us consider the following ansatz for the deformation of the (2,2) model in which α and β are the parameters of the deformation:

$$\begin{aligned}
\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} &= \Sigma_0 \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} + \widetilde{\Sigma}_0 \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \alpha_1 + \Sigma \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \alpha_2, \\
\begin{pmatrix} \widetilde{E}_1 \\ \widetilde{E}_2 \end{pmatrix} &= \widetilde{\Sigma}_0 \begin{pmatrix} \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \end{pmatrix} + \Sigma_0 \begin{pmatrix} \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \end{pmatrix} \beta_1 + \widetilde{\Sigma} \begin{pmatrix} \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \end{pmatrix} \beta_2,
\end{aligned} \tag{5.32}$$

This implies that (E_1, E_2) and $(\widetilde{E}_1, \widetilde{E}_2)$ are vectors under $SU_1(2)$ and $SU_2(2)$ respectively. As well (E_1,E_2) and $(\widetilde{E}_1,\widetilde{E}_2)$ are charged with charges 1 under the $U(1)_1$ and $U(1)_2$ respectively. Then, the dual Lagrangian becomes:

$$\begin{aligned}
\mathbf{\Delta} L_{\mathtt{dual}} = \int \mathrm{d}\theta^+\mathrm{d}\bar{\theta}^+\Big\{ \quad & \Big[Ve - Ve\frac{Z^aZ_a}{Z_0} - \frac{VY_+^aZ^a}{Z_0}\Big] + \widetilde{\mathcal{F}}^\dagger\Big(e\mathbf{I_d} - \frac{\mathcal{T}^a}{Z_0}(eZ^a+2Y_+^a)\Big)^{-1}\widetilde{\mathcal{F}} \\
+ \quad & \Big[\widetilde{V}\widetilde{e} - \widetilde{V}\widetilde{e}\frac{\widetilde{Z}^a\widetilde{Z}_a}{\widetilde{Z}_0} - \frac{\widetilde{V}\widetilde{Y}_+^a\widetilde{Z}^a}{\widetilde{Z}_0}\Big] + \widetilde{\mathcal{F}}^\dagger\Big(\widetilde{e}\mathbf{I_d} - \frac{\mathcal{T}^a}{\widetilde{Z}_0}(\widetilde{e}\widetilde{Z}^a+2\widetilde{Y}_+^a)\Big)^{-1}\widetilde{\mathcal{F}}\Big\} \\
+ \quad & \frac{t}{4}\int \mathrm{d}\theta^+\Upsilon|_{\bar{\theta}^+=0} - \int \mathrm{d}\theta^+\Big[(\Sigma_0+\alpha_1\widetilde{\Sigma}_0)\widetilde{\mathcal{F}}^\dagger\mathbf{\Phi} + \alpha 2\widetilde{\mathcal{F}}^\dagger\Sigma\mathbf{\Phi}\Big] \\
+ \quad & \frac{\widetilde{t}}{4}\int \mathrm{d}\theta^+\widetilde{\Upsilon}|_{\bar{\theta}^+=0} - \int \mathrm{d}\theta^+\Big[(\widetilde{\Sigma}_0+\beta_1\Sigma_0)\widetilde{\mathcal{F}}^\dagger\tilde{\mathbf{\Phi}} + \beta 2\widetilde{\mathcal{F}}^\dagger\widetilde{\Sigma}\widetilde{\mathbf{\Phi}}\Big], \qquad (5.33)
\end{aligned}$$

where the $\mathbf{\Phi}$ is fixed (in terms of the Z-parameters) with (5.9).

The scalar potential therefore is:

$$\begin{aligned}
U_{\mathtt{dual}} \;=\; & -e\Big(-y_+^aZ_a+\Im(t)\Big)^2 - \tilde{e}\Big(-\tilde{y}_+^a\tilde{Z}_a+\Im(\tilde{t})\Big)^2 \\
+ \; & \frac{1}{2y_+^ay_{a+}}\Big[\overline{H}_1H_1+\overline{H}_2H_2+\Big(\overline{H}_1H_1-\overline{H}_2H_2\Big)\Big(Z^3+2y_+^3\Big) \\
& \qquad +\overline{H}_2H_1\Big(2\overline{w}+\overline{Z}^{12}\Big)+\mathrm{h.c.}\Big] \\
+ \; & \frac{1}{2\widetilde{y}_+^a\widetilde{y}_{a+}}\Big[\overline{\widetilde{H}}_1\widetilde{H}_1+\overline{\widetilde{H}}_2\widetilde{H}_2+\Big(\overline{\widetilde{H}}_1\widetilde{H}_1-\overline{\widetilde{H}}_2\widetilde{H}_2\Big)\Big(\widetilde{Z}^3+2\widetilde{y}_+^3\Big) \\
& \qquad +\overline{\widetilde{H}}_2\widetilde{H}_1\Big(2\overline{w}+\overline{Z}^{12}\Big)+\mathrm{h.c.}\Big] \\
+ \; & \sqrt{2}\Big[(\sigma_0+\alpha_1\widetilde{\sigma}_0)(\overline{H}_1\phi_1+\overline{H}_2\phi_2)+\alpha_2\overline{H}_1(\sigma^{11}\phi_1+\sigma^{12}\phi_2) \\
& \qquad +\alpha_2\overline{H}_2(\sigma^{21}\phi_1+\sigma^{22}\phi_2)+\mathrm{h.c}\Big] \\
+ \; & \sqrt{2}\Big[(\widetilde{\sigma}_0+\beta_1\sigma_0)\Big(\overline{\widetilde{H}}_1\widetilde{\phi}_1+\overline{\widetilde{H}}_2\widetilde{\phi}_2\Big)+\beta_2\overline{\widetilde{H}}_1\Big(\widetilde{\sigma}^{11}\widetilde{\phi}_1+\widetilde{\sigma}^{12}\widetilde{\phi}_2\Big) \\
& \qquad +\beta_2\overline{\widetilde{H}}_2\Big(\widetilde{\sigma}^{21}\widetilde{\phi}_1+\widetilde{\sigma}^{22}\widetilde{\phi}_2\Big)+\mathrm{h.c}\Big]. \qquad (5.34)
\end{aligned}$$

Thus, the bosonic scalar potential depends of 6 coordinates y_+^a and $\widetilde{y}_+^a$, and the vacua $U_{\mathtt{dual}}=0$

after the minimum condition for H's gives:

$$U_{\texttt{dual}} = \frac{e}{2}\Big(\Im(t) - y_+^a Z_a\Big)^2 + (\overline{E}_1 \quad \overline{E}_2) A \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$
$$+\frac{e}{2}\Big(\Im(\tilde{t}) - \overline{y}_+^a \widetilde{Z}_a\Big)^2 + (\overline{\widetilde{E}}_1 \quad \overline{\widetilde{E}}_2)\overline{A} \begin{pmatrix} \widetilde{E}_1 \\ \widetilde{E}_2 \end{pmatrix} = 0, \tag{5.35}$$

with $A = \frac{-2y_+^b y_+^b}{1-u_c u^c}\begin{pmatrix} u_3 - 1 & u_{12} \\ \overline{u}_{12} & -1-u_3 \end{pmatrix}$, $\widetilde{A} = \frac{-2\widetilde{y}_+^b \widetilde{y}_+^b}{1-\widetilde{u}_c \widetilde{u}^c}\begin{pmatrix} \widetilde{u}_3 - 1 & \widetilde{u}_{12} \\ \overline{\widetilde{u}}_{12} & -1-\widetilde{u}_3 \end{pmatrix}$ and

$$\begin{aligned} E_1 &= \big[\sigma_0 + \alpha_1 \widetilde{\sigma}_0 + \alpha_2(\sigma_{11}+\sigma_{12})\big]\phi_1,\ E_2 = \big[\sigma_0 + \alpha_1 \widetilde{\sigma}_0 + \alpha_2(\sigma_{21}+\sigma_{22})\big]\phi_2\ . \\ \widetilde{E}_1 &= \big[\widetilde{\sigma}_0 + \beta_1 \sigma_0 + \beta_2(\widetilde{\sigma}_{11}+\widetilde{\sigma}_{12})\big]\widetilde{\phi}_1,\ \widetilde{E}_2 = \big[\widetilde{\sigma}_0 + \beta_1 \sigma_0 + \beta_2(\widetilde{\sigma}_{21}+\widetilde{\sigma}_{22})\big]\widetilde{\phi}_2\ , \end{aligned} \tag{5.36}$$

which as in the previous case, each one is positive quadratic form when $0 < 1 - u^a u_a$ and $0 < 1 - \widetilde{u}^a \widetilde{u}_a$. If it is this case, the solution is

$$\begin{aligned} r &= y_+^a Z^a, & \widetilde{r} &= \widetilde{y}_+^a \widetilde{Z}^a, \\ \sigma_0 + \alpha_1\widetilde{\sigma}_0 + \alpha_2(\sigma_{11}+\sigma_{12}) &= 0, & \sigma_0 + \alpha_1\widetilde{\sigma}_0 + \alpha_2(\sigma_{21}+\sigma_{22}) &= 0, \\ \widetilde{\sigma}_0 + \beta_1\sigma_0 + \beta_2(\widetilde{\sigma}_{11}+\widetilde{\sigma}_{12}) &= 0 \quad \text{and} & \widetilde{\sigma}_0 + \beta_1\sigma_0 + \beta_2(\widetilde{\sigma}_{21}+\widetilde{\sigma}_{22}) &= 0, \end{aligned} \tag{5.37}$$

which is simply the Cartesian product $\mathcal{W} \times \mathcal{W}'$ of two copies of the vacua manifold $\mathcal{W}$ found in the example of non-abelian duality in section 5.1. For the instanton correction, it is

$$\int \mathrm{d}\theta^+ \Big(\widetilde{\mathcal{F}}^\dagger \boldsymbol{\beta} e^{\alpha^b Y_b} + \widetilde{\widetilde{\mathcal{F}}}^\dagger \widetilde{\boldsymbol{\beta}} e^{\widetilde{\alpha}^b \widetilde{Y}_b}\Big). \tag{5.38}$$

Thus the change for the scalar potential is given by: $\widetilde{\mathbf{E}} \to \widetilde{\mathbf{E}} - e^{\alpha_b y_+^b}\boldsymbol{\beta}$. This means that the last 4 equations in (5.37) are equal to $|e^{\alpha_b y_+^b}\boldsymbol{\beta}|^2$.

For the analysed case, when the potential is positive definite, the geometry of the dual model is the one of the product of two disks $\mathbf{D}_1 \times \mathbf{D}_2$, which are the building blocks of the duality in subsection 5.1. Other possible cases involve a not positive definite matrix A or $\tilde{A}$. Notice that the inclusion of instanton corrections preserves the geometry.

Discussion and outlook

In this thesis, T-dualities of $(0,2)$ GLSMs have been discussed. After a brief review of the basics of supersymmetry and GLSMs, it was described abelian T-duality in $U(1)$ gauge theories with a $U(1)$ global symmetry, which was then gauged out. This means, the algorithm of dualization is realized when a model has a global symmetry, then this global symmetry is transform into a local one adding new "gauge" fields to the original ones and adding a Lagrange multiplier term to the original Lagrangian which then it will be called "master Lagrangian". Integration on these Lagrange multipliers results into the original action, and integration in the new gauge fields lead to the dual action.

This has been done in two $(0,2)$ cases, one when it comes from a reduction and one that doesn't come. The fundamental difference is that in the first case the Fermi multiplet is dualized (reduction), meanwhile in the second one, it is not dualized; this is shown also in the field E in the second case, where the vacua space has a dependence on E. At the beginning it has been solved for a general case with many fields, then it was explicitly solved for a model with two chiral superfields, one as an spectator (one of the csf is charged under the global $U(1)$).

It has been computed the contributions to the scalar potential for all the terms in the dual Lagrangian. From the potential the geometry of the space of supersymmetric vacua was determined. The geometry of the vacua space for the original model in both cases is $\mathbb{P}^1$. The dual model, under a single $U(1)$ T-duality, has the topology of $\mathbb{R}^+ \times \mathbb{R}$ for both cases. Notice that this is very different to the standard mirror symmetry duality, which will be a T-dualization of both chiral superfields. This model has a single $U(1)$ global symmetry. One can add an spectator superfield in order to have two global U(1)s. Mirror symmetry will be obtained by a T-dualization of both global U(1)s, and this model is obtained by a dualization of a single U(1). In general models, as there are many global symmetries, there are different dualizations that can be realized.

The instanton contributions to the superpotential are known for $(0,2)$ models coming from a $(2,2)$ reduction [54, 34]. For the case of a pure $(0,2)$ model, it was argued their structure, but to match them to the original theory is a plan for future work. From our results it seems that there is a difference of considering $(0,2)$ models and their dual counterparts, if they come from a reduction or not. In order to match it with previous results, in section 4.3 the duality algorithm for a model with two global abelian symmetries [54] was performed. This is a model which was later generalized in section 5.2 to the non-abelian T-duality case. It consists of a reduction $(0,2)$ GLSM with gauge symmetry $U(1) \times U(1)$, six chiral superfields and six Fermi fields. The global abelian symmetry is given by $U(1)^4$.

Afterward, T-dualities were constructed when the global symmetry is non-abelian, but only for the case of reduction because the number of chiral and Fermi fields should match. To be as general as possible, the analysis began with a general group, then proceeded with a group whose generators satisfy certain conditions (true for $SU(n)$), and finally focused on the case when the global symmetry is $SU(2)$. Suitable coordinates were found to write down the original Lagrangian, and the dual model and its vacua were described as a quadratic form.

This quadratic form was particularly useful in identifying the geometry of the vacua, which is composed of different regions. These regions correspond to conditions with the topology of an open ball, a two-sphere, or the outside part of the sphere, respectively. Thus, the vacua manifold for a positive semidefinite scalar potential corresponds to the closed disk D. If the potential is not positive definite, the component of the vacua manifold is simply $\mathbb{R}^2$.

Furthermore, it was discussed non-perturbative corrections to the superpotential via instantons. If the instanton corrections are incorporated in the potential $U_{\texttt{dual}}$ the effect is equivalent to shift $\widetilde{\mathbf{E}}$ function as $\widetilde{\mathbf{E}} - e^{\alpha_b y_+^b}\boldsymbol{\beta}$ in the potential without instanton corrections. This coincides with the observation in the $(2,2)$ GLSMs non-abelian T-duality were the instanton corrections preserve the dual geometry [28].

In last section it was presented a non-abelian generalization to the example of GLSM discussed in [54], which comes from a continuous $(0,2)$ deformation of a $(2,2)$ model. This model is a genuine pure $(0,2)$ GLSM. This model was worked by gauging the global non-abelian symmetry $SU(2) \times SU(2)$. It was found the dual Lagrangian, and analyzed the dual geometry of the vacua manifold. For the case of a positive definite potential the manifold is the Cartesian product of the vacua space of the $SU(2)$ simple model already discussed in section 5.1, i.e. $\mathbf{D}_1 \times \mathbf{D}_2$. There are also instanton corrections affecting both sectors by a similar shifting of $\widetilde{\mathbf{E}}$.

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