Centro de Investigación y de Estudios Avanzados
del Instituto Politécnico Nacional

Unidad Zacatenco

Departamento de Matemáticas

Funciones Polianalíticas en el Disco

Tesis que presenta

Julio Eduardo Enciso Molina

para obtener el Grado de

Maestro en Ciencias

en la Especialidad de

Matemáticas

Director de la Tesis

Dr. Nikolai Vasilevski

Ciudad de México **Julio, 2024**

Center for Research and Advanced Studies
of the National Polytechnic Institute

Campus Zacatenco

Department of Mathematics

Polyanalytic Functions on the Disk

Thesis that

Julio Eduardo Enciso Molina

presents to obtain the

Master's Degree in Science

with specialization in

Mathematics

Thesis Advisor

Ph.D. Nikolai Vasilevski

Mexico City **July, 2024**

Agradecimientos

Me gustaría expresar mi gratitud hacia el CONAHCYT. No sería una exageración decir que este proyecto de investigación no sería posible sin la beca que me fue otorgada.

De igual manera, estoy igual de agradecido hacia el CINVESTAV, y en particular al Departamento de Matemáticas, ya que el programa de maestría que ellos ofrecen resultó ser una experiencia invaluable para mi desarrollo académico.

De hecho, me gustaría expresar mi agradecimiento a todo el personal del Departamento de Matemáticas, tanto a los miembros académicos como los administrativos. Expresamente, a ambos jefes de departamento que fungieron durante mi estadía, el Dr. Héctor Jasso (jefe anterior) y el Dr. Feliú Sagols (jefe actual); y al Coordinador Académico, el Dr. Carlos G. Pacheco, quien siempre hizo todo lo posible por que la experiencia académica mía y de mis compañeros fuera la mejor. De verdad, les agradezco mucho.

Por supuesto, también menciono a nuestras secretarias, Adriana Sánchez, Ninfa Gómez, Anabel Lagos, Nancy Ortega, Laura Valencia y por encima de todo, a Roxana Martínez. Ella me brindó tanta ayuda, manejando todos mis documentos y apoyándome con cualquier trámite. Estoy endeudado con ella.

Por último, pero no menos importante, el personal auxiliar: Omar Hernández, José Luis Enríquez, Estela Hernández y Óscar Méndez. Agradezco su labor, la cual mantiene nuestras instalaciones en buena condición.

También debo dar gracias a mi asesor, el Dr. Nikolai Vasilevski. No hubiera podido llevar a cabo este proyecto sin él. Estaré siempre agradecido por haber podido ser su alumno y que me compartiera un poco de su conocimiento. Sólo espero que este trabajo satisfaga sus expectativas.

Del mismo modo, le extiendo mi agradecimiento a la Dra. Maribel Loaiza, quien me compartió su tiempo y su atención para apoyarme a terminar esta tesis. Aquí también menciono a mi compañera Diana Marcela, quien nos ayudó a la revisión de mi trabajo. Y desde luego, gracias igualmente a los demás sinodales que revisaron este documento, los doctores Carlos G. Pacheco y Egor Maximenko.

En términos más personales, quiero agradecerle a mi familia. A mi padre, Julio Enciso, quien me ha dado apoyo incondicional durante estos años; a mi hermana Karla, quien siempre me mantuvo con los pies en la tierra y me ofreció su ayuda; y mi madre Gloria Molina por todo lo que me enseñó. Oh, mamá, lo que daría por que pudieras ver esto. No encuentro las palabras para expresar que tan agradecido les estoy a ustedes.

Quisiera agradecerle a mis amigos cercanos: Oscar Díaz, Tairi Hoz, Rodrigo Luna, Miguel Arriaga, Rafael Reveles, Fernando Venegas y Rair Villagomez. Su compañía y apoyo han dado fruto y me han permitido completar este proyecto. Tienen mi gratitud por haberme acompañado todos estos años.

Asimismo, a mis amigos y conocidos del Departamento de Matemáticas: mi gran amigo Omar Alvarado, quien me ha apoyado en incontables ocasiones; mis amigos Luis y Humberto, con quienes compartí casi todos los cursos y que hacían que esas largas horas de estudio pasaran en un abrir y cerrar de ojos; mi compañero Sergio Gómez, quien me ha ofrecido su guía y ayuda en mi trayectoria académica; y mi compañero de cubículo Saúl Valdéz, quien siempre me dio su agradable compañía

A mi familia académica, mis colegas Christian Leal, Alejandro Soto, Jordi Arreortúa, Iván Avilés y Juanita Gasca.

Y a todos los demás, Mary Pérez, Eduardo Márquez, Juan Pablo Serrano, Moisés Algalán, entre otros.

Y también a todos aquellos que no pude mencionar, sinceramente les agradezco con todo mi corazón.

Resumen

En este trabajo estudiamos principalmente a los espacios Poli-Bergman con peso definidos sobre el disco unitario, los cuales son subespacios del espacio L^2 con peso definido sobre este mismo dominio. Para nosotros en particular es de importancia la teoría del formalismo extendido del espacio de Fock aplicada a este caso, así como la generalización de los resultados probados por el Dr. Nikolai Vasilevski en el caso estándar al caso con peso.

Utilizando propiedades de los elementos de una base ortonormal para el espacio L^2 con peso, el cual consiste de polinomios ortogonales, denominados polinomios del disco o polinomios de Jacobi trasladados, se expresan varios operadores, incluyendo isometrías puras, de forma independiente de dicha base, completando así la descripción del espacio de Fock extendido descrito por los operadores de escalera definidos en dicho espacio.

Abstract

In this work we deal mainly with the Poly-Bergman weighted spaces defined on the unit disk, subspaces of the weighted L^2 space over the same domain, and their elements. Particularly, we take interest in the extended Fock space formalism theory applied to this case and mean to generalize Dr. Nikolai Vasilevski's results from the standard to the weighted case.

Using properties of the elements of an orthonormal basis for the weighted L^2 space, comprised of orthogonal polynomials referred to as disk polynomials or shifted Jacobi polynomials, we express a variety of operators, including pure isometries, in a basis-independent manner, hence completing a description to the extended Fock space described by the ladder operators defined on the space.

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Introduction

Within the branch of Functional Analysis arises the theory of Hilbert spaces. One such structure that is of much importance is the notion of function spaces: a vector space of certain functions that satisfy a number of conditions. There are many examples one can choose from this class of spaces, and they have varying structures and properties, but we will restrict ourselves to the study of what is known as Bergman spaces.

Bergman spaces are function spaces that have as elements complex-valued functions of one complex variable that are defined on some region of the complex plane, and are holomorphic on it. On top of this, the functions satisfy some condition regarding a measure space defined over this same domain (e.g. square-integrability with respect to the Lebesgue measure). Here we clarify that the theory of Bergman spaces is quite vast, and we'll only look into very specific subclass of these spaces. However, we will deal with a slight generalization of this fact by not considering only holomorphic functions, but *poly-analytic* ones. These are referred to as *Poly-Bergman spaces*.

Motivated by the applications that manifest in these types of complex function spaces, like in physics (specifically in quantum mechanics), we aim to provide a deeper understanding of certain Poly-Bergman spaces by providing characterization for different objects in these spaces, that are described by the theory of the *extended Fock space formalism*, which is a supplementary structure defined for Hilbert spaces in terms of operators that act on it.

Then, to begin this work, we will proceed as follows: First, establish some background for the theories that appear in the research. Second, state the objects and problems that we aim to study. Third, define our goals and scope while adressing our limitations. And to conclude, give a structural outline of the document including a succint description of the contents of each chapter.

Having said this, we elucidate what may have become a question already: what is a poly-analytic function? And immediately after that: why are they a topic of research? Well, to elaborate on these questions, we run a quick summary on these mathematical objects.

Poly-analytic functions are complex-valued functions that try to generalize the notion of analytic, or holomorphic, functions. This is done simply by attaching non-holomorphic terms in the form of complex conjugations (implemented formally by the Wirtinger differential operators). In due time the corresponding definitions and properties of these functions will be stated accurately, but a broad analogy that can be made is this: analytic functions may be expressed as power series of the variable z;

poly-analytic functions can be expressed as series where the terms have some powers of both variables z and $\overline{z}$.

They were introduced in 1908 by the russian mathematician G. V. Kolossov in his work "Sur les probléms d'élasticité a deux dimensions". This topic was heavily researched by other russian mathematicians, led by Mark Benevich Balk. In recent years this theory became relevant within operator theory, and advances were made in applications to the theory of Hilbert spaces in mathematics, and in the theory of quantum mechanics in physics.

Some of the main applications of the theory are seen in the fields of signal analysis, wavelet theory and, as has been mentioned many times already, quantum mechanics. In the words of some other authors, the main appeal of working with poly-analytic functions, is that they are kind of a middle ground between the theory of holomorphic functions of a single variable, and the one with several complex variables. That is, they provide a wider workspace than the theory of one complex variable, but are marginally less complicated to deal with than the theory of several complex variables.

Next, we explain what we mean by the extended Fock space described by a Hilbert space and a pair of operators defined on it, another key theory for our interests.

This notion is inspired by what Berezin and Shubin establish as the *Fock space formalism* in their book "The Schrödinger Equation". In it, they speak of a way to classify certain Hilbert spaces that admit a pair of mutually adjoint operators that behave in a very specific manner. These operators take after the ladder operators that are defined in the theory of quantum mechanics when analyzing quantum systems.

The result that Berezin and Shubin present is that any two Hilbert spaces that have these operators, i.e. that are compatible with this structure, must be isomorphic. However, the conditions that this equivalence warrants are much too strict.

Thus, in a recent article by the name of "Extended Fock Space Formalism and Polyanalytic Functions", [9], Dr. Vasilevski explored a slight generalization of this notion that he called the *extended* Fock space formalism. Essentially, he relaxes some of the conditions of the original Fock space formalism and attempts a similar task of characterization of Hilbert spaces. As we need some aspects of this theory, we briefly describe it further below in the document, but refer to the original piece for the more precise formulations and results.

So, knowing a little about the main objects of study, we now turn to explaining what the research problem of this thesis is.

First, consider the following: Dr. Vasilevski's generalization to the theory of the Fock space formalism allowed for the case of the L^2 space defined on the whole complex plane using the Gaussian measure to be compatible (it is not apt for the structure of the classical Fock space formalism). Very intuitive operators (in terms of the Wirtinger differential operators) act as ladder operators in this space and the classical Poly-Fock spaces are described as a by-product of this approach.

In contrast to this, when trying to apply the theory of extended Fock spaces to the weighted L^2 space on the disk, the results are not what one might expect from the previous example.

This is because no "intuitive" or "obvious" combination of operators can be defined as ladder operators in this case and fit with the theory of extended Fock spaces. In fact, in the same article of "Extended Fock Space Formalism and Polyanalytic Functions" Dr. Vasilevski tries to fit a few pairs of operators to no avail.

However, as it will be shown later, due to a result by Dr. Vasilevski, it is known that those operators must exist. That is, the weighted L^2 space on the disk does admit two "nice" enough ladder operators as the theory describes. Furthermore, they can be computed, but only through their action upon a specific orthonormal basis for the space.

This differs a lot from the case of the L^2 space on the plane, since it does not have this restriction attached to it. So a reasonable question one might pose is the following: can these operators be described in a basis-independent manner? In an article from 2022 named "Yet Another Approach to Poly-Bergman Spaces", [11], Dr. Vasilevski answers affirmatively in the unweighted case. Then, what happens in the weighted space?

The aim of this dissertation is this: answer this question for the weigthed case and generalize Dr. Vasilevski's results from the article "Yet Another Approach to Poly-Bergman Spaces", [11].

We will mimic Dr. Vasilevski's scheme and proceed using operators that will be referred to as *unilateral shifts*. They have a very particular effect on the disk polynomials, and using their properties, we will be able to define suitable operators that have that same action on the elements of the basis for the space. To connect everything, we verify that these shifts are actually a special type of operators called *pure isometries*, and that they allow us to compute the correct ladder operators that we were looking for. This way, we will be able to remove the dependance of the basis.

As one can surmise from this brief explanation of our course of action, we do interface with some more niche concepts of the theory of Hilbert spaces, and engage rather deeply with the theory of the extended Fock space formalism. Our main tool in the whole work is the sequence of disk polynomials, so we operate with them time and time again. Here we make use of Dr. Alfred Wünsche's treatment of this polynomials from his article "Generalized Zernike or Disk Polynomials", [12].

Some positives from our research is that the expressions we find are actually a bit more general than our original goal, so perhaps they can be used for other purposes as well. Also, we see that our results do in fact generalize Dr. Vasilevski's (when considering the weight equal to zero). However, it would be disingenuous to say that there are no drawbacks in our developments. One of them is that, even if from a theoretical point of view we successfully exhibit these representations for our operators, in terms of computability we gain very little. This is because some of the tools used in the construction are difficult to deal with, numerically speaking, such as the continuous functional calculus. More of these advantages and disadvantages are discussed at the end of the work.

Lastly, we summarize the contents of the four chapter this text is divided into.

Chapter 1 has the definitions and notation for the more general concepts that are shared through the upcoming sections. It also includes a few auxiliary results that are necessary to prove several results.

Chapter 2 includes the definition of a sequence of polynomials referred to as the *disk polynomials*. A handful of properties of these functions is proved and it is shown that they constitute an orthonormal basis for the weighted L^2 space on the disk. After this, the Poly-Bergman subspaces are formally defined and orthonormal basis for these spaces are also detailed.

Chapter 3 has, as it was mentioned before, a detailed description of one of the attempts that Dr. Vasilevski made to find the ladder operators for the weighted L^2 space on the disk, so a pair of operators are defined over it, and several of their properties are proved. In the end, it is explained why they are not compatible with the structure that we're looking for.

The last chapter, Chapter 4, includes another approach to finding these operators. First, we define a pair of isometries over the weighted L^2 space on the disk. We prove some of their properties and show that these are in fact pure isometries. All of this done with the help of the disk polynomials and their condition as an orthogonal basis. Then, we exhibit a basis-independent form for these pure isometries, and using the theory of the extended Fock spaces, we reconstruct the corresponding ladder operators and also express it in a basis-independent manner.

Chapter 1

Preliminaries

This first chapter is written with the intent to introduce some of the concepts that will be used in the text. We divide it in subsections, since these notions are not all related to one another; in this way its best compared to a list of useful mathematical objects or facts.

The first few subsections detail some of the more general ideas and concepts that are more or less standard in the theory. Here we establish the notation for these objects and state any conventions or remarks about them. On the latter part of this section, we simply state some other ancillary objects that are required further in the analysis.

To begin, we first mention that, as the title of the work implies, the main object of study in this thesis is poly-analytic functions on the disk. We will get to the definition of what exactly a poly-analytic function is very shortly, but we would like to mention that throughout the text, we will deal mainly with complex functions of one complex variable. In particular, the functions will always be complex valued; and their domain will mainly be the open unit disk in the complex plane, $\mathbb{D}$, unless specified otherwise.

1.1 Poly-analytic Functions

We begin by defining a class of functions among the continuously differentiable functions of some order defined on the disk.

Definition 1.1.1. Let $n \in \mathbb{N}$. A function $f : \mathbb{D} \to \mathbb{C}$ that belongs to $C^n(\mathbb{D})$ is called *n-poly-analytic* or *poly-analytic of order n* if it satisfies the following equation:

$$\frac{\partial^n}{\partial \overline{z}^n} f = 0,$$

where $\frac{\partial^n}{\partial \overline{z}^n}$ is the n-th order Wirtinger differential operator $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

We denote by $\mathcal{O}_n(\mathbb{D})$ the set of all n-poly-analyic functions on $\mathbb{D}$. Granted the linearity of the Wirtinger operator, we get that this is in fact a linear space.

Notice that, in the previous definition, if $n = 1$, we are describing precisely the class of holomorphic functions on $\mathbb{D}$.

Essential to having a deeper understanding of these types of functions, we state and prove a characterization of the condition of being a poly-analytic function of some order, due to Balk, contained in his book "Poly-analytic Functions", [1].

This proposition reveals then, that being a poly-analytic function boils down to being a sum of powers of $\overline{z}$ paired with some holomorphic functions.

Proposition 1.1.2. *Let $n \in \mathbb{N}$ and $f : \mathbb{D} \to \mathbb{C}$. Then, f is poly-analytic of order n if and only if there exists $\varphi_0, \varphi_1, ..., \varphi_{n-1} : \mathbb{D} \to \mathbb{C}$ analytic functions on $\mathbb{D}$ such that the following equality holds:*

$$f(z) = \sum_{j=0}^{n-1} \varphi_j(z)\, \overline{z}^{\,j}, \quad \forall z \in \mathbb{D}.$$

Proof. The sufficiency of this proposition follows directly from the properties of the Wirtinger derivative with respect to $\overline{z}$, so we concern ourselves with the necessity only.

We show this condition by inducting on the order of the poly-analytic function.

In particular, for $n = 1$ the result does hold. Then, let us suppose the proposition is true for a natural number n and verify it for the case $n + 1$.

Let $f \in C^n(\mathbb{D})$. Suppose that f is poly-analytic of order $n + 1$.

Then, by definition, f must satisfy the following equality:

$$\frac{\partial^{n+1} f}{\partial \overline{z}^{\,n+1}} = 0.$$

Define $g = \frac{\partial}{\partial \overline{z}} f$. Clearly g is a poly-analytic function of order n.

By the induction hypothesis, there exist $\varphi_0, \varphi_1, ..., \varphi_{n-1} : \mathbb{D} \to \mathbb{C}$ analytic functions on $\mathbb{D}$ such that the following equality holds:

$$g(z) = \sum_{j=0}^{n-1} \varphi_j(z)\, \overline{z}^{\,j}, \quad \forall z \in \mathbb{D}.$$

Next we define the following:

$$\psi_j(z) = \frac{\varphi_{j-1}}{j}, \quad \forall j \in \{1, \ldots, n\}.$$

And let $\psi_0(z) = f(z) - \sum_{j=1}^{n} \psi_j(z)\, \overline{z}^{\,j}$.

By definition, the functions ψ_j are analytic for $j \neq 0$. But, we see that:

$$\frac{\partial}{\partial \overline{z}} \psi_0(z) = \frac{\partial}{\partial \overline{z}} \left(f(z) - \sum_{j=1}^{n} \psi_j(z)\, \overline{z}^{\,j} \right) = g(z) - \sum_{j=0}^{n-1} \varphi_j(z)\, \overline{z}^{\,j} = 0.$$

So we get that all of them are analytic, and the following equality holds:

$$f(z) = \sum_{j=0}^{n} \psi_j(z)\, \overline{z}^{\,j}.$$

□

1.2 Pure isometries

Before proceeding to the next subsection, we define a special type of isometry in Hilbert spaces. To talk about what a pure isometry is, first we introduce another concept:

Definition 1.2.1. Let H be a Hilbert space. Let $V : H \to H$ be an isometry. We say that a subspace L of H is *wandering* for V if it satisfies the following condition:

$$V^p(L) \perp V^q(L)\,, \quad \forall p, q \in \mathbb{N}, p \neq q.$$

Given a Hilbert space and an isometry defined on it, any wandering space for the isometry in question then defines a countable sequence of mutually orthogonal subspaces: the iterated images of the wandering space.

With such a sequence, one may define their orthogonal sum, which will turn out to be a subspace of the original space. Well, a pure isometry is an isometry with a wandering subspace such that this orthogonal sum coincides with the whole space. Or, more precisely:

Definition 1.2.2. Let H be a Hilbert space. Let $V : H \to H$ be an isometry. We say that V is a *pure isometry* if there exists L, a wandering subspace for V that satisfies:

$$H = \bigoplus_{k \in \mathbb{Z}_+} V^k(L)$$

This concept is defined in more specialized theory of Hilbert spaces. Pure isometries are also referred to as unilateral shifts, such as in Nagy's book "Harmonic Analysis of Operators on Hilbert Space", [6].

1.3 Extended Fock Space Formalism

In this section, we will present a very basic and outright laconic overview of the theory of the "Extended Fock Space Formalism." This was explored in Vasilevski's article "Extended Fock Space Formalism and Polyanalytic Functions", [9], published in 2022. In this paper, Dr. Vasilevski tries to generalize the concept of the "Fock Space Formalism", that Berezin introduces in his book "The Schrödinger Equation". This is an auxiliary structure that manages to characterize a certain class of Hilbert spaces through operators defined in it.

Thus we present the more general and thorough version that Dr. Vasilevski defines in his research, since we will use some results from this theory later on.

We start with the following definition:

Definition 1.3.1. Let H be a separable Hilbert space, and $\mathfrak{a} : \mathcal{D}_{\mathfrak{a}} \to H$, $\mathfrak{b} : \mathcal{D}_{\mathfrak{b}} \to H$, operators defined on their natural domains, respectively (dense in H), that satisfy:

1. There exists a subspace $\mathcal{D} \subset \mathcal{D}_{\mathfrak{a}} \cap \mathcal{D}_{\mathfrak{b}}$ dense in H and invariant under both $\mathfrak{a}$ and $\mathfrak{b}$ on which they satisfy:
$$[\mathfrak{a}, \mathfrak{b}] = I.$$
2. The subspace $L_{[1]} := \ker \mathfrak{a}|_{\mathcal{D}}$ is non-trivial with $\dim L_{[1]} > 1$.
3. The set $\mathcal{D}_0$, formed by linear combinations of elements from the subspaces $L_{[n]} := \mathfrak{b}^{n-1} L_{[1]}$, $n \in \mathbb{N}$, is dense in H.

Any Hilbert space with such operators is called the *extended Fock space* defined by H, $\mathfrak{a}$ and $\mathfrak{b}$.

We quickly describe some of the main properties and notation used for the study of this type of structure.

Proposition 1.3.2. *For all $n \in \mathbb{N}$, $\dim L_{[n]} = \dim L_{[n+1]}$.*

This implies that all subspaces $L_{[n]}$ are of the same dimension.

Lemma 1.3.3. *Let $n \in \mathbb{Z}_+$. The operators $\mathfrak{a}$ and $\mathfrak{b}$ restricted to $L_{[n+1]}$ and $L_{[n]}$, respectively act as isomorphisms between the following spaces:*

$$\mathfrak{a}|_{L_{[n+1]}} : L_{[n+1]} \to L_{[n]}$$

and,

$$\mathfrak{b}|_{L_{[n]}} : L_{[n]} \to L_{[n+1]}.$$

This lemma justifies why these operators are sometimes referred to as lowering and raising operators.

Proposition 1.3.4. *Let $n, k \in \mathbb{Z}_+$, with $n \neq k$. Then the intersection of the closed subspaces $\overline{L_{[n]}}$ and $\overline{L_{[k]}}$ is trivial.*

Corollary 1.3.5. *Any finite number of spaces $L_{[n]}$ are linearly independent.*

When we speak of "linearly independent" subspaces, we mean in the following sense:

Let V be a linear space. Consider $W_1, \ldots, W_n$, a finite amount of subspaces of V. We say they are *linearly independent* if they satisfy the next condition:

$$\forall c_1, \ldots, c_n \in \mathbb{C}, w_1 \in W_1, \ldots, w_n \in W_n : c_1 w_1 + \cdots + c_n w_n = 0 \implies c_1 = \cdots = c_n = 0.$$

Theorem 1.3.6. *Let $n \in \mathbb{N}$. An element $h \in \mathcal{D}_0$ satisfies the equation $\mathfrak{a}^n(h) = 0$ if and only if h admits the following representation:*

$$h = \sum_{j=0}^{n-1} \mathfrak{b}^j h_j \,, \quad \textit{with } h_j \in \ker \mathfrak{a}|_{\mathcal{D}_0} \,, \, \forall j \in \{0, 1, \dots, n-1\}.$$

This theorem implies that in $\mathcal{D}_0$, the condition $\mathfrak{a}^n(h) = 0$ is equivalent to h belonging to $L_{[1]} + \cdots + L_{[n]}$.

For each $n \in \mathbb{N}$, the direct sum $L_{[1]} + \cdots + L_{[n]}$ may not be closed, even if all the spaces $L_{[j]}$ are closed themselves, it depends on the minimal angle between them.

Then, for all $n \in \mathbb{N}$, we define:

$$L_n := \operatorname{clos}\,(L_{[1]} + \cdots + L_{[n]}),$$

considering $L_1 = \overline{L_{[1]}}$.

Notice that the sequence $\{L_n\}_{n\in\mathbb{N}}$ of subspaces is increasing (w.r.t. set inclusion). This gives:

$$H = \operatorname{clos}\left(\bigcup_{n\in\mathbb{N}} L_n \right).$$

Having said this, let us introduce the following spaces:

$$L_{(n)} := L_n \ominus L_{n-1} = L_n \cap L_{n-1}^{\perp} \,, \quad \forall n \in \mathbb{N}.$$

With these subspaces, we get the following representation:

$$H = \bigoplus_{n\in\mathbb{N}} L_{(n)}.$$

Before we proceed to the next subsection, we would like to show an example of an extended Fock space.

In this case, we consider $H = L^2(\mathbb{C}, \lambda)$, where λ denotes the normalized Gaussian measure defined on the complex plane:

$$d\lambda(z) = \frac{1}{\pi}\, e^{-|z|^2} d\mu(z).$$

Here, we define the operators $\mathfrak{a}$ and $\mathfrak{b}$ as follows:

$$\mathfrak{a} := \frac{\partial}{\partial \,\overline{z}}$$

and

$$\mathfrak{b} := -\frac{\partial}{\partial z} + \overline{z}\, I.$$

These operators will satisfy the conditions given in definition 1.3.1, and in particular, the subspaces $L_{(n)}$ turn out to be the true Poly-Fock spaces, i.e.:

$$L_{(n)} = \mathcal{F}_{(n)}(\mathbb{C}) \,, \quad \forall n \in \mathbb{N}.$$

1.3.1 Special Case for Formally Mutual Adjoints

Now we pay closer attention to the special case when the operators $\mathfrak{a}$ and $\mathfrak{b}$ are formally mutually adjoint.

After giving a summary of what this condition entails, we will cite the theorem we will be using in the latter part of the work. It will be paramount to the analysis given further below.

For this section, we use $\mathfrak{a}^\dagger$ to refer to the operator $\mathfrak{b}$.

First, this proposition:

Proposition 1.3.7. *Let $m, n \in \mathbb{N}$, with $m \neq n$. Then the subspaces $L_{[m]}$ and $L_{[n]}$ are orthogonal to each other.*

The mutual orthogonality of the subspaces $L_{[n]}$ implies then that:

$$L_n = \operatorname{clos}\left(L_{[1]} + \cdots + L_{[n]}\right) = \overline{L_{[1]}} \oplus \cdots \oplus \overline{L_{[n]}}$$

and $\overline{L_{[n]}} = L_{(n)}$.

Proposition 1.3.8. *The operator V defined on each $L_{(n)}$ acting as:*

$$V|_{L_{(n)}} = \frac{1}{\sqrt{n}}\mathfrak{a}^\dagger : L_{(n)} \to L_{(n+1)}$$

for all $n \in \mathbb{N}$, may be extended by continuity to a pure isometry on H.

Its adjoint V^* is defined by its action on the subspaces $L_{(n)}$ as follows:

$$V^*|_{L_{(n)}} = \begin{cases} \frac{1}{\sqrt{n-1}}\mathfrak{a}|_{L_{(n)}} : L_{(n)} \to L_{(n-1)} & n > 1, \\ \mathfrak{a}|_{L_{(1)}} & n = 1. \end{cases}$$

They satisfy $(\operatorname{Im} V)^\perp = \ker V^* = L_{(1)}$.

Corollary 1.3.9. *The operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ admit an extension to the common domain:*

$$\mathcal{D}_{ext} = \left\{ h = \sum_{n\in\mathbb{N}} h_n \in H \,\middle|\, h_n \in L_{(n)}\,, \quad \sum_{n\in\mathbb{N}} n\|h_n\|^2 < \infty \right\},$$

on which they act as:

$$\mathfrak{a} : \sum_{n\in\mathbb{N}} h_n \mapsto \sum_{n\in\mathbb{N}} \sqrt{n}V^*(h_n)$$

and,

$$\mathfrak{a}^\dagger : \sum_{n\in\mathbb{N}} h_n \mapsto \sum_{n\in\mathbb{N}} \sqrt{n}V(h_n)$$

and are mutually adjoint.

Considering the domain $\mathcal{D}_{ext}$, we can state a stronger formulation for Theorem 1.2.6.

Lemma 1.3.10. *Let $n \in \mathbb{N}$. Then the following equality holds:*

$$\ker \mathfrak{a}^n = \{h \in H \mid \mathfrak{a}^n(h) = 0\} = \left\{ h \in H \;\middle|\; h = \sum_{j=1}^{n} (\mathfrak{a}^\dagger)^{j-1} g_j \,,\; g_j \in \ker \mathfrak{a} \right\} = L_n.$$

This leads us into the main result of this section.

Theorem 1.3.11. *Let H be a separable infinite dimensional Hilbert space. Then the following are equivalent:*

1) *There is a pure isometry V in H.*

2) *The Hilbert space H admits the orthogonal sum decomposition*

$$\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_{(n)}$$

where all $\mathcal{H}_{(n)}$ have the same dimension (be it finite or infinite).

3) *There are two formally adjoint lowering and raising operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ that act invariantly on a common domain dense in H, such that the following commutation relation holds*

$$[\mathfrak{a}, \mathfrak{a}^\dagger] = I,$$

the set $L_{(1)} = \ker \mathfrak{a}$ is a closed subspace of H, and the set of finite linear combinations of elements from all spaces $L_{(n)} := (\mathfrak{a}^\dagger)^{n-1} L_{(1)}$ is dense in H.

Moreover, the subspaces $\mathcal{H}_{(n)}$ in 2 are related to the operators $V, \mathfrak{a}$ and $\mathfrak{a}^\dagger$ as follows:

$$\mathcal{H}_{(1)} = \ker V^* = \ker \mathfrak{a}$$

and

$$\mathcal{H}_{(n)} = V^{n-1}(\ker V^*) = (\mathfrak{a}^\dagger)^{n-1}(\ker \mathfrak{a})$$

for all $n \in \mathbb{N}$, $n > 1$.

Essentially, this theorem establishes the equivalence between the concepts of a pure isometry, an orthogonal sum decomposition of the space, and the admittance of an extended Fock space structure with two mutually adjoint operators.

In fact, we will be interested in the extended Fock space structure in the L^2 weighted space on the disk, and the definition of both pure isometries and the ladder operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ in it, as well as their relation with the Poly-Bergman spaces.

1.4 Weighted measure on the Disk

Now we turn to elucidate the measure space we will be working on.

Definition 1.4.1. Let $\alpha > -1$. We define the *weight function* (with parameter α) $w_\alpha : \mathbb{D} \to \mathbb{R}$ as follows:

$$w_\alpha(z) \coloneqq \frac{(\alpha+1)}{\pi}(1-|z|^2)^\alpha \,, \; \forall z \in \mathbb{D}.$$

With this function, we define the new probability measure μ_α over $\mathbb{D}$ in terms of the normalized Lebesgue measure μ defined on the disk:

$$d\mu_\alpha(z) \coloneqq w_\alpha(z) d\mu(z).$$

With this established, we get the measure space $(\mathbb{D}, l_\mathbb{D}, \mu_\alpha)$ (with $l_\mathbb{D}$ referring to the σ-algebra of Lebesgue measurable sets in $\mathbb{D}$), which will become our main interest from this point on.

Then, we consider the semi-normed space of square-integrable functions in $\mathbb{D}$ w.r.t. the measure μ_α:

$$\mathcal{L}^2(\mathbb{D}, \mu_\alpha) \coloneqq \left\{ f \in \mathbb{C}^\mathbb{D} \,\middle|\, \int_\mathbb{D} |f(z)|^2 d\mu_\alpha(z) < \infty \right\}$$

equipped with the semi-norm $\|\cdot\|_{2,\alpha} : \mathcal{L}^2(\mathbb{D}, \mu_\alpha) \to \mathbb{R}$ defined as:

$$\|f\|_{2,\alpha} \coloneqq \left(\int_\mathbb{D} |f(z)|^2 d\mu_\alpha(z) \right)^{1/2}.$$

Since we deal directly with functions themselves, strictly speaking we make use of this semi-normed space. Nevertheless, we do make use of the more robust quotient space which we introduce now:

Definition 1.4.2. Consider the measure space $(\mathbb{D}, l_\mathbb{D}, \mu_\alpha)$. We define the space of square integrable functions as the following set:

$$L^2(\mathbb{D}, \mu_\alpha) \coloneqq \mathcal{L}^2(\mathbb{D}, \mu_\alpha)/\mathcal{N}_2,$$

where we take the quotient by the subspace of elements of semi-norm zero:

$$\mathcal{N}_2 \coloneqq \big\{ f \in \mathcal{L}^2(\mathbb{D}, \mu_\alpha) \,|\, \|f\|_{2,\alpha} = 0 \big\}.$$

Claim 1.4.3. The space $L^2(\mathbb{D}, \mu_\alpha)$ is a Hilbert space, with the inner product $\langle \cdot, \cdot \rangle_\alpha : L^2(\mathbb{D}, \mu_\alpha) \times L^2(\mathbb{D}, \mu_\alpha) \to \mathbb{R}$, given by:

$$\langle f, g \rangle_\alpha = \int_\mathbb{D} f(z)\, \overline{g(z)}\, d\mu_\alpha(z) = \int_\mathbb{D} f(z)\, \overline{g(z)}\, w_\alpha(z) d\mu(z) \,, \quad \forall f, g \in L^2(\mathbb{D}, \mu_\alpha).$$

Now, from here on after, we will use the symbol $L^2(\mathbb{D}, \mu_\alpha)$ to refer to either of these spaces interchangeably, and make no distinction for the equivalence classes, in order to lessen the burden of notation.

1.5 Functions of compact support in $\mathbb{D}$

As an auxiliary concept, we define a certain sub-collection of $C(\mathbb{C})$.

Definition 1.5.1. Let $f \in C(\mathbb{C})$. We define the *support of* f as the following set:

$$\operatorname{supp}(f) := \operatorname{clos}\{z \in \mathbb{C} \,|\, f(z) \neq 0\}.$$

In particular, if this set is bounded, the set itself becomes compact in $\mathbb{C}$, and we refer to f as being *of compact support*.

Definition 1.5.2. We define the set of *functions with support contained in* $\mathbb{D}$ as:

$$C_s(\mathbb{D}) := \{f \in C(\mathbb{C}) \,|\, \operatorname{supp}(f) \subset \mathbb{D}\}$$

It is clear that all functions in $C_s(\mathbb{D})$ are of compact support.

We also have the following fact:

Claim 1.5.3. $C_s(\mathbb{D})$ is a linear space.

All we need of this class of functions is the next theorem:

Theorem 1.5.4. *The collection $C_s(\mathbb{D})$ is dense in $L^2(\mathbb{D}, \mu)$.*

This is a classical result from general measure theory on Euclidean spaces. We refer to Stronberg and Hewitt's "Real and Abstract Analysis", [5], book for a quick reference.

1.6 Unbounded Operators on Hilbert Spaces

For the latter chapters, we will need some theory about unbounded operators defined on Hilbert spaces, since we will define and handle such operators.

First, we define the spectrum of a densely defined operator.

Definition 1.6.1. Let H be a Hilbert space. Let T be a densely defined operator on H.

A complex number λ is said to be *an element of the resolvent set of* T if the densely defined operator $T - \lambda I$ is bijective and the inverse operator $(T - \lambda I)^{-1}$ is a bounded operator.

This is how one defines the resolvent set for the operator T, denoted by $\rho(T)$.

Then one can define the *spectrum set for the operator* T, $\sigma(T)$, in terms of the resolvent set as follows:

$$\sigma(T) := \mathbb{C} \setminus \rho(T)$$

Having defined this, now we talk about some properties about *self-adjoint* or *Hermitian* operators.

Proposition 1.6.2. *Let H be a Hilbert space. Let T be a densely defined operator on H. Then, the following conditions are equivalent:*

1. *There exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of real numbers such that:*

$$\lim_{n\to\infty} |\lambda_n| = \infty,$$

and an orthonormal basis for H, $\{e_n\}_{n\in\mathbb{N}}$ that satisfies:

$$T(e_n) = \lambda_n e_n \,, \forall n \in \mathbb{N}.$$

2. *The operator T has a purely discrete spectrum.*

This result tells us that the fact that a self-adjoint operator is diagonal with respect to a certain orthonormal basis for the space is both necessary and sufficient for the operator to have a purely discrete spectrum.

Next, we describe what the functional calculus for self-adjoint operators is.

This is a result that guarantees that given a self-adjoint operator T defined on a Hilbert space H, any Borel function $f : \mathbb{R} \to \mathbb{C}$ defines an operator on H denoted by $f(T)$.

The correspondence $f \mapsto f(T)$ is called the *functional calculus for the operator T*.

We state some of the properties of the functional calculus that are relevant to us in the following theorem.

Theorem 1.6.3. *Let H be a Hilbert space. Let T be a self-adjoint operator defined on H. Let $f : \mathbb{R} \to \mathbb{C}$ be a Borel function. Then the following are true:*

1. $(\overline{f}\,)(T) = (f(T))^*$. *In particular, if f is real-valued, the operator $f(A)$ is self-adjoint.*

2. *If $f \neq 0$ in $\mathbb{R}$, then the operator $f(T)$ is invertible and $(f(T))^{-1} = (1/f)(T)$.*

This is all part of the theory of unbounded operators defined on Hilbert spaces, and both proofs of the facts stated in this section, and much more detailed descriptions of the concepts touched here can be found, for example, in Schmüdgen's book "Unbounded Self-adjoint Operators on Hilbert Space", [7].

1.7 Subspaces of Hilbert Spaces

Finally, we state a criterion for the closed-ness of the direct sum of two sub-spaces of a Hilbert space.

For the result it is necessary to introduce the concept of *minimal angle* between closed subspaces of a Hilbert space.

Definition 1.7.1. Let H be a Hilbert space. Let H_1, H_2 be two closed subspaces of H. First we define the *cosine of the minimal angle between H_1 and H_2* in terms of the inner product of H as:

$$\cos\varphi^{(m)}(H_1, H_2) := \sup\left\{ |\langle x, y\rangle| \,\middle|\, x \in H_1, y \in H_2, \text{ and } \|x\| = \|y\| = 1 \right\}.$$

This way, the *minimal angle between H_1 and H_2* is defined by:

$$\varphi^{(m)}(H_1, H_2) := \arccos(\cos\varphi^{(m)}(H_1, H_2)).$$

As mentioned, we can determine if a couple of closed subspaces are closed or not depending on the minimal angle between them.

Theorem 1.7.2. *Let H be a Hilbert space. Let H_1, H_2 be two closed subspaces of H such that $H_1 \cap H_2 = \{0\}$. The direct sum of the subspaces H_1 and H_2 is closed if and only if $\varphi^{(m)}(H_1, H_2) > 0$.*

A proof of this fact is available in [3, Lemma 1].

Chapter 2

Weighted L^2 space and Poly-Bergman spaces

In this chapter we will focus in the following: establishing an orthonormal basis for the weighted L^2 space on the disk, and describing the weighted Poly-Bergman subspaces.

The first one is made possible by studying the collection of functions that will comprise our orthonormal basis, and is done in great detail. After that we define all relevant subspaces and explore some of their properties.

2.1 Construction of an Orthonormal Basis in the Weighted Measure L^2 Space

In order to construct an orthogonal basis for the Hilbert spaces relevant in our study, we must introduce the *disk polynomials*: a special family of polynomials that has very particular properties.

They are defined through the Jacobi polynomials, so we begin describing some of their characteristics first.

2.1.1 Jacobi polynomials

These are classical orthogonal polynomials. They can be defined in several ways, but here we give the following definition:

Definition 2.1.1. Let $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$. The corresponding Jacobi polynomial may be defined through the Rodrigues formula as:

$$P_n^{(\alpha,\beta)}(x) := \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\,\frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right), \quad x \in (-1,1).$$

Another expression for the Jacobi polynomials:

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}, \quad x \in (-1,1)$$

Next we list the most relevant properties of the Jacobi polynomials as a theorem.

Theorem 2.1.2. *Let $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$. The following properties hold*

a) $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$.

b) If $\alpha, \beta > -1$ the following expression holds:

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\alpha+\beta+n+k+1)}{\Gamma(\alpha+k+1)} \left(\frac{x-1}{2}\right)^k.$$

c) The polynomials obey the following orthogonality relation:

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_l^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \delta_{n,l}.$$

d) The polynomials are solutions to the following second order differential equation:

$$(1-x^2)\frac{\partial^2}{\partial x^2}y + (\beta-\alpha-(\alpha+\beta+2)x)\frac{\partial}{\partial x}y + n(n+\alpha+\beta+1)y = 0\,, \quad x \in (-1,1).$$

The previous results are available in [8, Chapter IV].

2.1.2 Shifted Jacobi Polynomials

We now detail a change of variables to couple with the Jacobi polynomials to transform them into the *shifted Jacobi polynomials*.

Definition 2.1.3. We define a C^1-class isomorphism $\phi : (0,1) \to (-1,1)$ as:

$$\phi(t) = 2t - 1\,, \quad \forall t \in (0,1).$$

Let $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$. We define the *shifted Jacobi polynomial obtained from* $P_n^{(\alpha,\beta)}$ as

$$Q_n^{(\alpha,\beta)} := P_n^{(\alpha,\beta)} \circ \phi \tag{2.1}$$

2.1.3 Disk Polynomials

After that preamble, we define the disk polynomials and describe some of their properties. We consider a constant $\alpha > -1$ that will be used from here on out. This also determines the weighted measure on $\mathbb{D}$.

Definition 2.1.4. Let $m, n \in \mathbb{Z}^+$. We define the corresponding *disk polynomial*:

$$P_{m,n}^{\alpha}(z, \bar{z}) = \frac{n!\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} z^{m-n} Q_n^{(\alpha,m-n)}(z\,\bar{z}),\ \forall z \in \mathbb{D}.$$

Here $Q_n^{(\alpha,m-n)}$ represents the corresponding shifted Jacobi polynomial, as was defined above, in equation 2.1.

Properties of the disk polynomials

We state and prove several propositions pertaining these polynomials.

Proposition 2.1.5. *Let $m, n \in \mathbb{Z}^+$. Then, it holds for any $z = re^{i\theta} \in \mathbb{D}$:*

$$P^{\alpha}_{m,n}(z, \overline{z}) = e^{i(m-n)\theta} P^{\alpha}_{m,n}(r, r).$$

Proof. We simply evaluate the polynomials in $z = re^{i\theta}$.

$$\begin{aligned}
P^{\alpha}_{m,n}(z, \overline{z}) &= \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)} z^{m-n} Q^{(\alpha,m-n)}_n(z\,\overline{z}\,) \\
&= \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)} z^{m-n} P^{(\alpha,m-n)}_n(2z\overline{z}-1) \\
&= \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)} (re^{i\theta})^{m-n} P^{(\alpha,m-n)}_n(2(re^{i\theta})(\overline{re^{\theta}})-1) \\
&= e^{i(m-n)\theta} \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)} r^{m-n} P^{(\alpha,m-n)}_n(2r^2-1) \\
&= e^{i(m-n)\theta} P^{\alpha}_{m,n}(r, r).
\end{aligned}$$

□

Now we cite a result from Wünsche's article [12, Equations 2.4, 2.5], in which he states two exact expressions for the disk polynomials. They will be very useful for the next results.

Claim 2.1.6. Let $m, n \in \mathbb{Z}^+$. Then we have the following explicit representations for $P^{\alpha}_{m,n}(z, \overline{z})$:

$$\begin{aligned}
P^{\alpha}_{m,n}(z, \overline{z}) &= \frac{m!n!\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)} \sum_{j=0}^{\min\{m,n\}} \frac{(-1)^j\Gamma(m+n+\alpha-j+1)}{j!(m-j)(n-j)} z^{m-j}\overline{z}^{n-j} \\
&= \sum_{k=0}^{\min\{m,n\}} \frac{(-1)^k m!n!\Gamma(\alpha+1)(1-z\,\overline{z}\,)^k}{k!(m-k)!(n-k)!\Gamma(k+\alpha+1)} z^{m-k}\,\overline{z}^{\,n-k}.
\end{aligned}$$

Proposition 2.1.7. *Let $m, n \in \mathbb{Z}^+$. For any $r \in \mathbb{R}$ such that $|r| < 1$, it holds that:*

$$P^{\alpha}_{m,n}(r, r) = P^{\alpha}_{n,m}(r, r) \quad \textit{and is a real number.}$$

Proof. Since $r = \overline{r}$, the expression in Claim 2.1.6 becomes symmetric with respect to m and n. This yields the desired equation. □

Proposition 2.1.8. *Let $m, n \in \mathbb{Z}^+$. Then it holds for any $z \in \mathbb{D}$:*

$$\overline{P^{\alpha}_{m,n}(z, \overline{z})} = P^{\alpha}_{n,m}(z, \overline{z}).$$

Proof. First we write z in polar coordinates and use the two previous propositions:

$$\begin{aligned}\overline{P^{\alpha}_{m,n}(z,\overline{z})} &= \overline{P^{\alpha}_{m,n}(re^{i\theta},re^{-i\theta})} = \overline{e^{i(m-n)\theta}P^{\alpha}_{m,n}(r,r)}\\ &= e^{i(n-m)\theta}P^{\alpha}_{m,n}(r,r) = e^{i(n-m)\theta}P^{\alpha}_{n,m}(r,r) = P^{\alpha}_{n,m}(z,\overline{z})\end{aligned}$$

□

Lemma 2.1.9. *Let $m, n, l \in \mathbb{Z}^+$, with $m - n > -1$. Then it holds the following equality:*

$$\int_0^1 r(1-r^2)^{\alpha}P^{\alpha}_{l+m-n,l}(r,r)P^{\alpha}_{m,n}(r,r)dr = \frac{m!l!\Gamma(\alpha+1)^2}{2(n+m+\alpha+1)\Gamma(m+\alpha+1)\Gamma(l+\alpha+1)}\delta_{n,l}.$$

Proof. We start from the orthogonality relation of the Jacobi polynomials (Theorem 2.1.2 c).

$$\int_{-1}^1 (1-x)^{\alpha}(1+x)^{\beta}P^{(\alpha,\beta)}_l(x)P^{(\alpha,\beta)}_n(x)dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}\delta_{n,l}.$$

Then, apply the change of variables $x = 2r^2 - 1$ and substitute $\beta = m - n$ to obtain the following equation:

$$\int_0^1 r(1-r^2)^{\alpha}r^{2(m-n)}P^{(\alpha,m-n)}_l(2r^2-1)P^{(\alpha,m-n)}_n(2r^2-1)dr = \frac{\Gamma(n+\alpha+1)l!}{2(n+m+\alpha+1)n!\Gamma(m+\alpha+1)}\delta_{n,l}. \quad (2.2)$$

Next, consider the two following equalities:

$$P^{(\alpha,m-n)}_l(2r^2-1) = \frac{\Gamma(l+\alpha+1)}{l!\Gamma(\alpha+1)}r^{n-m}P^{\alpha}_{l+m-n,l}(r,r).$$

$$P^{(\alpha,m-n)}_n(2r^2-1) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}r^{n-m}P^{\alpha}_{m,n}(r,r).$$

Now putting these into equation 2.2 we get the desired relation:

$$\int_0^1 r(1-r^2)^{\alpha}P^{\alpha}_{l+m-n,l}(r,r)P^{\alpha}_{m,n}(r,r)dr = \frac{m!l!\Gamma(\alpha+1)^2}{2(n+m+\alpha+1)\Gamma(m+\alpha+1)\Gamma(l+\alpha+1)}\delta_{n,l}.$$

□

To end this section, we include some recurrence relations regarding the indices of the disk polynomials.

Theorem 2.1.10. *The disk polynomials obey the following relations:*

For all $m, n \in \mathbb{Z}_+$, with $n \geq 1$

$$(m+n+1+\alpha)zP^{\alpha}_{m,n}(z,\overline{z}) = (m+1+\alpha)P^{\alpha}_{m+1,n}(z,\overline{z}) + nP^{\alpha}_{m,n-1}(z,\overline{z})$$

and with $m \geq 1$,

$$(m+n+\alpha+1)\overline{z}P^{\alpha}_{m,n}(z,\overline{z}) = (n+1+\alpha)P^{\alpha}_{m,n+1}(z,\overline{z}) + mP^{\alpha}_{m-1,n}(z,\overline{z})$$

With respect to the Wirtinger operator, they satisfy:

$$(m+n+1+\alpha)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}P^{\alpha}_{m,n}(z,\,\overline{z}\,)=m(n+1+\alpha)\big(P^{\alpha}_{m-1,n}(z,\,\overline{z}\,)-P^{\alpha}_{m,n+1}(z,\,\overline{z}\,)\big)$$

and,

$$(m+n+1+\alpha)(1-z\,\overline{z}\,)\frac{\partial}{\partial \overline{z}}P^{\alpha}_{m,n}(z,\,\overline{z}\,)=n(m+1+\alpha)\big(P^{\alpha}_{m,n-1}(z,\,\overline{z}\,)-P^{\alpha}_{m+1,n}(z,\,\overline{z}\,)\big)$$

for all $m,n\in\mathbb{Z}_+$, considering $m\geq 1$ and $n\geq 1$, respectively.

Proof. We proceed as follows: we show the first and third recurrence relations, and the other follow simply by conjugating the former ones.

For the first one, we take $m,n\in\mathbb{Z}_+$, with $n\geq 1$.
We use the following explicit representation for the disk polynomials from Claim 2.1.6:

$$P^{\alpha}_{m,n}(z,\,\overline{z}\,)=\frac{m!n!\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)}\sum_{j=0}^{\min\{m,n\}}\frac{(-1)^j\Gamma(m+n+\alpha-j+1)}{j!(m-j)(n-j)}z^{m-j}\overline{z}^{n-j} \tag{2.3}$$

For this proof only, we will assign some special notation for the "constants" in order to simplify the expressions.
Then, we define:

$$A^{\alpha}_{m,n}:=\frac{m!n!\Gamma(\alpha+1)}{\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)},$$

for all $m,n\in\mathbb{Z}_+$.
And also,

$$d^{\alpha}_{m,n}(j):=\frac{(-1)^j\Gamma(m+n+\alpha-j+1)}{j!(m-j)!(n-j)!},$$

for all $m,n,j\in\mathbb{Z}_+$, with $j\leq\min\{m,n\}$.
So equation 2.3 becomes:

$$P^{\alpha}_{m,n}=A^{\alpha}_{m,n}\sum_{j=0}^{\min\{m,n\}}d^{\alpha}_{m,n}(j)z^{m-j}\overline{z}^{n-j} \tag{2.4}$$

We start by assuming that $n\leq m$. Equivalently, we have that $n<m+1$.
This means that $\min\{m,n\}=n$, $\min\{m+1,n\}=n$, $\min\{m,n-1\}=n-1$.
Then, consider the following expressions:

$$(m+1+\alpha)\,P^{\alpha}_{m+1,n}(z,\,\overline{z}\,)=(m+1+\alpha)\,A^{\alpha}_{m+1,n}\sum_{j=0}^{n}d^{\alpha}_{m+1,n}(j)z^{m+1-j}\overline{z}^{n-j} \tag{2.5}$$

$$=(m+1)\,A^{\alpha}_{m,n}\sum_{j=0}^{n}d^{\alpha}_{m+1,n}(j)z^{m+1-j}\overline{z}^{n-j}. \tag{2.6}$$

And:

$$n\,P^{\alpha}_{m,n-1}(z,\overline{z}\,) = n\,A^{\alpha}_{m,n-1}\sum_{j=0}^{n-1} d^{\alpha}_{m,n-1}(j) z^{m-j}\overline{z}^{n-1-j} \tag{2.7}$$

$$= (n+\alpha)\,A^{\alpha}_{m,n}\sum_{j=0}^{n-1} d^{\alpha}_{m,n-1}(j) z^{m-j}\overline{z}^{n-1-j}. \tag{2.8}$$

In both expressions, the constants at the beginning already coincide, save for the terms in brackets, so they can be factored. With this, we work on the remaining terms.

For the first polynomial, $P^{\alpha}_{m+1,n}(z,\overline{z}\,)$, using equation 2.6, we separate some of the constants and get:

$$(m+1)\sum_{j=0}^{n} d^{\alpha}_{m+1,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j} = (m+1)\sum_{j=0}^{n} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j}\cdot\frac{m+n+\alpha-j+1}{m-j+1}$$

Then, we separate the zeroth term and further reduce the expression:

$$(m+1)\,d^{\alpha}_{m,n}(0) z^{m+1}\,\overline{z}^{\,n}\cdot\frac{m+n+\alpha+1}{m+1} + \sum_{j=1}^{n} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j}\cdot(m+1)\left(\frac{m+n+\alpha-j+1}{m-j+1}\right) \tag{2.9}$$

$$= (m+n+\alpha+1)\,d^{\alpha}_{m,n}(0) z^{m+1}\,\overline{z}^{\,n} + \sum_{j=1}^{n} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j}\cdot(m+1)\left(1+\frac{n+\alpha}{m-j+1}\right) \tag{2.10}$$

$$= (m+n+\alpha+1)\,d^{\alpha}_{m,n}(0) z^{m+1}\,\overline{z}^{\,n} + \sum_{j=1}^{n} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j}\cdot\left(m+1+\frac{(m+1)(n+\alpha)}{m-j+1}\right). \tag{2.11}$$

For the second polynomial, $P^{\alpha}_{m,n-1}$, using the term from equation 2.8, we shift the index by 1, so that it starts at 1 and ends at n:

$$(n+\alpha)\sum_{j=1}^{n} d^{\alpha}_{m,n-1}(j-1) z^{m+1-j}\,\overline{z}^{\,n-j}.$$

Then, we make some adjustments to the coefficients as well:

$$\sum_{j=1}^{n} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j}\cdot\left(-\frac{j(n+\alpha)}{m-j+1}\right). \tag{2.12}$$

Now we sum the expressions from equations 2.11 and 2.12:

$$\sum_{j=1}^{n} d_{m,n}^{\alpha}(j) z^{m+1-j}\,\overline{z}^{\,n-j} \cdot \left(m+1+\frac{(m+1)(n+\alpha)}{m-j+1} - \frac{j(n+\alpha)}{m-j+1} \right)$$
$$+ (m+n+\alpha+1)\, d_{m,n}^{\alpha}(0) z^{m+1}\,\overline{z}^{\,n}$$
$$= (m+n+\alpha+1) \sum_{j=1}^{n} d_{m,n}^{\alpha}(j) z^{m+1-j}\,\overline{z}^{\,n-j} + (m+n+\alpha+1)\, d_{m,n}^{\alpha}(0) z^{m+1}\,\overline{z}^{\,n}$$
$$= (m+n+\alpha+1) \sum_{j=0}^{n} d_{m,n}^{\alpha}(j) z^{m+1-j}\,\overline{z}^{\,n-j}$$
$$= (m+n+\alpha+1)\, z \sum_{j=0}^{n} d_{m,n}^{\alpha}(j) z^{m-j}\,\overline{z}^{\,n-j}.$$

Notice that the sum on the last expression coincides, without the constant $A_{m,n}^{\alpha}$, to the one corresponding to the polynomial $P_{m,n}^{\alpha}$.

Thus, returning to the original expression, using equations 2.5 and 2.7:

$$(m+1+\alpha)\, P_{m+1,n}^{\alpha}(z,\overline{z}\,) + n\, P_{m,n-1}^{\alpha}(z,\overline{z}\,)$$
$$= (m+n+\alpha+1)\, z\, A_{m,n}^{\alpha} \sum_{j=0}^{n} d_{m,n}^{\alpha}(j) z^{m-j}\,\overline{z}^{\,n-j}$$
$$= (m+n+\alpha+1)\, z\, P_{m,n}^{\alpha}.$$

Which proves the equality in this case.

Now, suppose that $n > m$. Equivalently, $n \geq m+1$.

In this case, we have that $\min\{m,n\} = m$, $\min\{m+1,n\} = m+1$, $\min\{m,n-1\} = m$. We use the explicit description for the disk polynomials and notation as in the last case. We proceed in a similar fashion, rearraging the two terms from the right hand side.

That is:

$$(m+1+\alpha)\, P_{m+1,n}^{\alpha}(z,\overline{z}\,) = (m+1+\alpha)\, A_{m+1,n}^{\alpha} \sum_{j=0}^{m+1} d_{m+1,n}^{\alpha}(j) z^{m+1-j}\overline{z}^{n-j} \tag{2.13}$$
$$= (m+1)\, A_{m,n}^{\alpha} \sum_{j=0}^{m+1} d_{m+1,n}^{\alpha}(j) z^{m+1-j}\overline{z}^{n-j}. \tag{2.14}$$

And:

$$n\, P_{m,n-1}^{\alpha}(z,\overline{z}\,) = n\, A_{m,n-1}^{\alpha} \sum_{j=0}^{m} d_{m,n-1}^{\alpha}(j) z^{m-j}\overline{z}^{n-1-j} \tag{2.15}$$
$$= (n+\alpha)\, A_{m,n}^{\alpha} \sum_{j=0}^{m} d_{m,n-1}^{\alpha}(j) z^{m-j}\overline{z}^{n-1-j}. \tag{2.16}$$

Then, like before, we work on the terms at the end without the constant $A^{\alpha}_{m,n}$.

For the first polynomial, using equation 2.14, we reduce some of the coefficients, and separate both the zeroth and $m+1$-th term:

$$\begin{aligned}(m+1)\sum_{j=0}^{m+1} d^{\alpha}_{m+1,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j} &= (m+1)\sum_{j=1}^{m} d^{\alpha}_{m+1,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j} \\ &+ (m+1)\, d^{\alpha}_{m+1,n}(0) z^{m+1}\,\overline{z}^{\,n-j} + (m+1)\, d^{\alpha}_{m+1,n}(m+1)\,\overline{z}^{\,n-m-1} \\ &= \sum_{j=1}^{m} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j}\cdot\left(m+1+\frac{(m+1)(n+\alpha)}{m-j+1}\right) \\ &+ (m+n+\alpha+1) d^{\alpha}_{m,n}(0) z^{m+1}\,\overline{z}^{\,n} - (n-m) d^{\alpha}_{m,n}(m)\,\overline{z}^{\,n-m-1}. \end{aligned} \tag{2.17}$$

For the second polynomial, using the expression from equation 2.16, we first shift the index so that it begins in 1 and ends in $m+1$, and then separate the $m+1$-th term:

$$\begin{aligned}(n+\alpha)&\sum_{j=0}^{m} d^{\alpha}_{m,n-1}(j) z^{m-j}\,\overline{z}^{\,n-1-j} \\ &= (n+\alpha)\sum_{j=1}^{m+1} d^{\alpha}_{m,n-1}(j-1) z^{m+1-j}\,\overline{z}^{\,n-j} \\ &= (n+\alpha)\sum_{j=1}^{m} d^{\alpha}_{m,n-1}(j-1) z^{m+1-j}\,\overline{z}^{\,n-j} + (n+\alpha) d^{\alpha}_{m,n-1}(m)\,\overline{z}^{\,n-1-m} \\ &= \sum_{j=1}^{m+1} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j}\cdot\left(-\frac{j(n+\alpha)}{m-j+1}\right) + (n+\alpha) d^{\alpha}_{m,n-1}(m)\,\overline{z}^{\,n-1-m} \\ &= \sum_{j=1}^{m} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j}\cdot\left(-\frac{j(n+\alpha)}{m-j+1}\right) + (n-m) d^{\alpha}_{m,n}(m)\,\overline{z}^{\,n-1-m}. \end{aligned} \tag{2.18}$$

Then, we sum the expressions from equations 2.17 and 2.18:

$$\begin{aligned}&\sum_{j=1}^{m} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j}\cdot\left(m+1+\frac{(m+1)(n+\alpha)}{m-j+1}-\frac{j(n+\alpha)}{m-j+1}\right) \\ &+ (m+n+\alpha+1) d^{\alpha}_{m,n}(0) z^{m+1}\,\overline{z}^{\,n} - (n-m) d^{\alpha}_{m,n}(m)\,\overline{z}^{\,n-m-1} \\ &+ (n-m) d^{\alpha}_{m,n}(m)\,\overline{z}^{\,n-m-1} \\ &= (m+n+\alpha+1)\sum_{j=1}^{m} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j} + (m+n+\alpha+1) d^{\alpha}_{m,n}(0) z^{m+1}\,\overline{z}^{\,n} \\ &= (m+n+\alpha+1)\sum_{j=0}^{m} d^{\alpha}_{m,n}(j) z^{m+1-j}\,\overline{z}^{\,n-j}. \end{aligned} \tag{2.19}$$

And, just as before, the sum coincides with the desired polynomial without the constant $A^{\alpha}_{m,n}$. So, using the expression from equation 2.19, and combining it with equations 2.13 and 2.15:

$$\begin{aligned}(m+1+\alpha)\, P^{\alpha}_{m+1,n}(z,\overline{z}\,) + n\, P^{\alpha}_{m,n-1}(z,\overline{z}\,) \\ &= (m+n+\alpha+1)\, z\, A^{\alpha}_{m,n} \sum_{j=0}^{m} d^{\alpha}_{m,n}(j) z^{m+1-j}\, \overline{z}^{\,n-j} \\ &= (m+n+\alpha+1)\, z\, P^{\alpha}_{m,n}(z,\overline{z}\,).\end{aligned}$$

That is, the equality holds in this case as well.

Since the equality is satisfied in both cases, we have our desired recursion formula.

Now, as it was mentioned, we can derive the second relation by conjugating this first one. Watch:

We have the statement for the first recurrence relation, but we flip the sub-indices m and n:

$$(m+n+\alpha+1)\, z\, P^{\alpha}_{n,m}(z,\overline{z}\,) = (n+1+\alpha)\, P^{\alpha}_{n+1,m}(z,\overline{z}\,) + m\, P^{\alpha}_{n,m-1}(z,\overline{z}\,).$$

So we conjugate it all, and reduce the expression:

$$\overline{(m+n+\alpha+1)\, z\, P^{\alpha}_{n,m}(z,\overline{z}\,)} = \overline{(n+1+\alpha)\, P^{\alpha}_{n+1,m}(z,\overline{z}\,) + m\, P^{\alpha}_{n,m-1}(z,\overline{z}\,)}\,.$$

$$(m+n+\alpha+1)\ \overline{z}\ P^{\alpha}_{m,n}(z,\overline{z}\,) = (n+1+\alpha)\, P^{\alpha}_{m,n+1}(z,\overline{z}\,) + m\, P^{\alpha}_{m-1,n}(z,\overline{z}\,).$$

And there we have it.

We prove the third recurring relation.

To do this, we will use the other explicit representation for the disk polynomials from Claim 2.1.6:

$$P^{\alpha}_{m,n}(z,\overline{z}\,) = \sum_{k=0}^{\min\{m,n\}} \frac{(-1)^k m! n! \Gamma(\alpha+1)(1-z\,\overline{z}\,)^k}{k!(m-k)!(n-k)!\Gamma(k+\alpha+1)} z^{m-k}\, \overline{z}^{\,n-k}. \tag{2.20}$$

Much like the previous case, we define notation for the coefficients of the polynomials just for this proof.

Let:

$$e^{\alpha}_{m,n}(k) := \frac{(-1)^k m! n! \Gamma(\alpha+1)}{k!(m-k)!(n-k)!\Gamma(k+\alpha+1)},$$

for all $m,n,k \in \mathbb{Z}_+$ with $k \leq \min\{m,n\}$.

Then, equation 2.20 becomes:

$$P^{\alpha}_{m,n}(z,\overline{z}\,) = \sum_{k=0}^{\min\{m,n\}} e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^k z^{m-k}\, \overline{z}^{\,n-k}. \tag{2.21}$$

First we suppose that $n < m$. Or, equivalently, $n \leq m-1$. In this case, $\min\{m,n\} = n$, $\min\{m-1,n\} = n$, and $\min\{m,n+1\} = n+1$.

We begin manipulating the left hand side of the recurrence relation:

$$
\begin{aligned}
&(m+n+\alpha+1)\,(1-z\,\overline{z}\,)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = (m+n+\alpha+1)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,))\\
&-(m+n+\alpha+1)\,z\,\overline{z}\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,))\\
&=(m+n+\alpha+1)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) - z\frac{\partial}{\partial z}\Big((m+n+\alpha+1)\;\overline{z}\;P^{\alpha}_{m,n}(z,\,\overline{z}\,)\Big) \qquad (2.22)\\
&=(m+n+\alpha+1)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) - z\frac{\partial}{\partial z}\Big((n+1+\alpha)P^{\alpha}_{m,n+1}(z,\,\overline{z}\,)+mP^{\alpha}_{m-1,n}(z,\,\overline{z}\,)\Big)\\
&=(m+n+\alpha+1)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) - z\,(n+1+\alpha)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n+1}(z,\,\overline{z}\,))\\
&-m\,z\,\frac{\partial}{\partial z}(P^{\alpha}_{m-1,n}(z,\,\overline{z}\,)) \qquad (2.23)
\end{aligned}
$$

Notice that one of the recurrence relations we just proved was used in the equality 2.22.

Then, we compute the three derivatives that appear in equation 2.23:

$$
\begin{aligned}
&\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = \frac{\partial}{\partial z}\left(\sum_{k=0}^{n} e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n-k}\right)\\
&=\sum_{k=0}^{n} e^{\alpha}_{m,n}(k)\frac{\partial}{\partial z}\big((1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n-k}\big)\\
&=\sum_{k=0}^{n} e^{\alpha}_{m,n}(k)\,\overline{z}^{\,n-k}\;\cdot\;\frac{\partial}{\partial z}\big((1-z\,\overline{z}\,)^k z^{m-k}\big)\\
&=\sum_{k=0}^{n} e^{\alpha}_{m,n}(k)\,\overline{z}^{\,n-k}\;\cdot\;\big((m-k)(1-z\,\overline{z}\,)^k z^{m-1-k} - k(1-z\,\overline{z}\,)^{k-1}\,\overline{z}\,z^{m-k}\big)\\
&=\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&-\sum_{k=1}^{n} k e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^{k-1} z^{m-k}\,\overline{z}^{\,n+1-k}\\
&=\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&+\sum_{k=0}^{n-1} -(k+1)e^{\alpha}_{m,n}(k+1)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}. \qquad (2.24)
\end{aligned}
$$

$$\frac{\partial}{\partial z}(P^{\alpha}_{m,n+1}(z,\,\overline{z}\,)) = \frac{\partial}{\partial z}\left(\sum_{k=0}^{n+1} e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k} z^{m-k}\,\overline{z}^{\,n+1-k}\right)$$

$$= \sum_{k=0}^{n+1} e^{\alpha}_{m,n+1}(k)\frac{\partial}{\partial z}\big((1-z\,\overline{z}\,)^{k} z^{m-k}\,\overline{z}^{\,n+1-k}\big)$$

$$= \sum_{k=0}^{n+1} e^{\alpha}_{m,n+1}(k)\,\overline{z}^{\,n+1-k}\cdot\frac{\partial}{\partial z}\big((1-z\,\overline{z}\,)^{k} z^{m-k}\big)$$

$$= \sum_{k=0}^{n+1} e^{\alpha}_{m,n+1}(k)\,\overline{z}^{\,n+1-k}\cdot\big((m-k)(1-z\,\overline{z}\,)^{k} z^{m-1-k} - k(1-z\,\overline{z}\,)^{k-1}\,\overline{z}\,z^{m-k}\big)$$

$$= \sum_{k=0}^{n+1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k} z^{m-1-k}\,\overline{z}^{\,n+1-k}$$

$$- \sum_{k=1}^{n+1} k e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k-1} z^{m-k}\,\overline{z}^{\,n+2-k}$$

$$= \sum_{k=0}^{n+1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k} z^{m-1-k}\,\overline{z}^{\,n+1-k}$$

$$+ \sum_{k=0}^{n} -(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k} z^{m-1-k}\,\overline{z}^{\,n+1-k}. \tag{2.25}$$

And finally:

$$\frac{\partial}{\partial z}(P^{\alpha}_{m-1,n}(z,\overline{z})) = \frac{\partial}{\partial z}\left(\sum_{k=0}^{n} e^{\alpha}_{m-1,n}(k)(1-z\overline{z})^{k} z^{m-1-k}\,\overline{z}^{\,n-k}\right)$$

$$= \sum_{k=0}^{n} e^{\alpha}_{m-1,n}(k)\frac{\partial}{\partial z}\big((1-z\overline{z})^{k} z^{m-1-k}\,\overline{z}^{\,n-k}\big)$$

$$= \sum_{k=0}^{n} e^{\alpha}_{m-1,n}(k)\,\overline{z}^{\,n-k}\cdot\frac{\partial}{\partial z}\big((1-z\overline{z})^{k} z^{m-1-k}\big)$$

$$= \sum_{k=0}^{n} e^{\alpha}_{m-1,n}(k)\,\overline{z}^{\,n-k}\cdot\big((m-1-k)(1-z\overline{z})^{k} z^{m-2-k} - k(1-z\overline{z})^{k-1}\,\overline{z}\,z^{m-1-k}\big)$$

$$= \sum_{k=0}^{n}(m-1-k)e^{\alpha}_{m-1,n}(k)(1-z\overline{z})^{k} z^{m-2-k}\,\overline{z}^{\,n-k}$$

$$-\sum_{k=1}^{n} k e^{\alpha}_{m-1,n}(k)(1-z\overline{z})^{k-1} z^{m-1-k}\,\overline{z}^{\,n+1-k}$$

$$= \sum_{k=0}^{n}(m-1-k)e^{\alpha}_{m-1,n}(k)(1-z\overline{z})^{k} z^{m-2-k}\,\overline{z}^{\,n-k}$$

$$+\sum_{k=0}^{n-1} -(k+1)e^{\alpha}_{m-1,n}(k+1)(1-z\overline{z})^{k} z^{m-2-k}\,\overline{z}^{\,n-k}. \tag{2.26}$$

With this, we substitute the expressions 2.24, 2.25 and 2.26 back into equation 2.23:

$$(m+n+\alpha+1)\,(1-z\overline{z})\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\overline{z}))$$

$$= (m+n+\alpha+1)\,\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\overline{z})^{k} z^{m-1-k}\,\overline{z}^{\,n-k}$$

$$+ (m+n+\alpha+1)\,\sum_{k=0}^{n-1} -(k+1)e^{\alpha}_{m,n}(k+1)(1-z\overline{z})^{k} z^{m-1-k}\,\overline{z}^{\,n-k}$$

$$- (n+1+\alpha)\,\sum_{k=0}^{n+1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\overline{z})^{k} z^{m-k}\,\overline{z}^{\,n+1-k}$$

$$- (n+1+\alpha)\,\sum_{k=0}^{n} -(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^{k} z^{m-k}\,\overline{z}^{\,n+1-k}$$

$$- m\,\sum_{k=0}^{n}(m-1-k)e^{\alpha}_{m-1,n}(k)(1-z\overline{z})^{k} z^{m-1-k}\,\overline{z}^{\,n-k}$$

$$- m\,\sum_{k=0}^{n-1} -(k+1)e^{\alpha}_{m-1,n}(k+1)(1-z\overline{z})^{k} z^{m-1-k}\,\overline{z}^{\,n-k}. \tag{2.27}$$

We will verify that all these terms reconstruct the right hand side of the desired recurrence relation.

To try and make this proof somewhat understandable, we will take the following approach: we will highlight a term in equation 2.27, make some kind of modification to it, be it separating terms or adding multiple expressions, etc, and substitute it back again and rearrange the expression.

So, we begin:

$$
\begin{aligned}
&(m+n+\alpha+1)\,(1-z\,\overline{z}\,)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,))\\
&=\left[(m+n+\alpha+1)\,\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&+(m+n+\alpha+1)\,\sum_{k=0}^{n-1}-(k+1)e^{\alpha}_{m,n}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&-(n+1+\alpha)\,\sum_{k=0}^{n+1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)\,\sum_{k=0}^{n}-(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-m\sum_{k=0}^{n}(m-1-k)e^{\alpha}_{m-1,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&-m\sum_{k=0}^{n-1}-(k+1)e^{\alpha}_{m-1,n}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}.
\end{aligned}
\tag{2.28}
$$

We expand as follows:

$$
\begin{aligned}
&(m+n+\alpha+1)\,\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&=m\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&+(n+\alpha+1)\,\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&=m\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&+m(n+\alpha+1)\,\sum_{k=0}^{n}e^{\alpha}_{m-1,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&=m\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}+m(n+\alpha+1)P^{\alpha}_{m-1,n}.
\end{aligned}
\tag{2.29}
$$

We substitute back in equation 2.28, and highlight the next term to be modified:

$$
\begin{aligned}
&(m+n+\alpha+1)\,(1-z\,\overline{z}\,)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)P^{\alpha}_{m-1,n}\\
&+m\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&+(m+n+\alpha+1)\sum_{k=0}^{n-1}-(k+1)e^{\alpha}_{m,n}(k+1)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&-(n+1+\alpha)\sum_{k=0}^{n+1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)\sum_{k=0}^{n}-(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k}\\
&\Bigg[-m\sum_{k=0}^{n}(m-1-k)e^{\alpha}_{m-1,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&-m\sum_{k=0}^{n-1}-(k+1)e^{\alpha}_{m-1,n}(k+1)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}.\Bigg]
\end{aligned}
\tag{2.30}
$$

We first separate the n-th term from the first sum and then add these two expressions together:

$$
\begin{aligned}
&-m\sum_{k=0}^{n}(m-1-k)e^{\alpha}_{m-1,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&-m\sum_{k=0}^{n-1}-(k+1)e^{\alpha}_{m-1,n}(k+1)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&= m(n+1-m)e^{\alpha}_{m-1,n}(n)(1-z\,\overline{z}\,)^n z^{m-1-n}\\
&-m\sum_{k=0}^{n-1}(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\big((k+1)e^{\alpha}_{m-1,n}(k+1)-(m-1-k)e^{\alpha}_{m-1,n}(k)\big)\\
&= m(n+1-m)e^{\alpha}_{m-1,n}(n)(1-z\,\overline{z}\,)^n z^{m-1-n}\\
&+m\sum_{k=0}^{n-1}(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\big((k+1)e^{\alpha}_{m-1,n}(k+1)-(m-1-k)e^{\alpha}_{m-1,n}(k)\big).
\end{aligned}
\tag{2.31}
$$

We substitute back and highlight the next terms:

$$
\begin{aligned}
&(m+n+\alpha+1)\,(1-z\,\overline{z}\,)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)P^{\alpha}_{m-1,n}\\
&+\left[m\sum_{k=0}^{n}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&+(m+n+\alpha+1)\sum_{k=0}^{n-1}-(k+1)e^{\alpha}_{m,n}(k+1)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&-(n+1+\alpha)\sum_{k=0}^{n+1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-\left[(n+1+\alpha)\sum_{k=0}^{n}-(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k}\right]\\
&+m\sum_{k=0}^{n-1}(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\big((k+1)e^{\alpha}_{m-1,n}(k+1)-(m-1-k)e^{\alpha}_{m-1,n}(k)\big)\\
&+m(n+1-m)e^{\alpha}_{m-1,n}(n)(1-z\,\overline{z}\,)^n z^{m-1-n}. \qquad (2.32)
\end{aligned}
$$

For the terms highlighted we separate the n-term in both cases and add them.

$$
\begin{aligned}
&m(m-n)e^{\alpha}_{m,n}(n)(1-z\,\overline{z}\,)^n z^{m-1-n}+(n+1+\alpha)(n+1)e^{\alpha}_{m,n+1}(n+1)(1-z\,\overline{z}\,)^n z^{m-n}\,\overline{z}\\
&= m\frac{(-1)^n m!n!\Gamma(\alpha+1)}{n!(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^n z^{m-1-n}\\
&-(n+\alpha+1)\frac{(-1)^n m!(n+1)!\Gamma(\alpha+1)}{n!(m-1-n)!\Gamma(n+\alpha+2)}(1-z\,\overline{z}\,)^n z^{m-n}\,\overline{z}\\
&= m\frac{(-1)^n m!n!\Gamma(\alpha+1)}{n!(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^n z^{m-1-n}\\
&-(n+1)\frac{(-1)^n m!\Gamma(\alpha+1)}{(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^n z^{m-n}\,\overline{z}\\
&= \frac{(-1)^n m!\Gamma(\alpha+1)}{(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^n z^{m-1-n}\big(m-(n+1)\,\overline{z}\,\big)\\
&= \frac{(-1)^n m!\Gamma(\alpha+1)}{(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^n z^{m-1-n}\big((n+1)(1-z\,\overline{z}\,)+(m+n-1)\big)\\
&= (n+1)\frac{(-1)^n m!\Gamma(\alpha+1)}{(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^{n+1} z^{m-1-n}\\
&+(m-n-1)\frac{(-1)^n m!\Gamma(\alpha+1)}{(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^n z^{m-1-n}\\
&= (n+1)\frac{(-1)^n m!\Gamma(\alpha+1)}{(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^{n+1} z^{m-1-n}\\
&+m(m-n-1)e^{\alpha}_{m-1,n}(n)(1-z\,\overline{z}\,)^n z^{m-1-n}. \qquad (2.33)
\end{aligned}
$$

And again, we substitute back and highlight the next terms. We put the additional terms at the end.

$$\begin{aligned}
&(m+n+\alpha+1)\,(1-z\,\overline{z}\,)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)P^{\alpha}_{m-1,n}\\
&+\left[m\sum_{k=0}^{n-1}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&+\left[(m+n+\alpha+1)\sum_{k=0}^{n-1}-(k+1)e^{\alpha}_{m,n}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&-(n+1+\alpha)\sum_{k=0}^{n+1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)\sum_{k=0}^{n-1}-(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&+\left[m\sum_{k=0}^{n-1}(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\Big((k+1)e^{\alpha}_{m-1,n}(k+1)-(m-1-k)e^{\alpha}_{m-1,n}(k)\Big)\right]\\
&+m(n+1-m)e^{\alpha}_{m-1,n}(n)(1-z\,\overline{z}\,)^{n}z^{m-1-n}\\
&+m(m-n-1)e^{\alpha}_{m-1,n}(n)(1-z\,\overline{z}\,)^{n}z^{m-1-n}\\
&+(n+1)\frac{(-1)^{n}m!\Gamma(\alpha+1)}{(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^{n+1}z^{m-1-n}. \qquad (2.34)
\end{aligned}$$

Notice that some terms cancel out.

The next highlighted terms will be all added together. Taking into account that they are all sums from 0 to $n-1$ and within this sum, they share the powers of $1-z\,\overline{z}\,$, z, and $\overline{z}\,$, so we focus on adding the other terms:

$$\begin{aligned}
&m(m-k)e^{\alpha}_{m,n}(k)-(m+n+\alpha+1)(k+1)e^{\alpha}_{m,n}(k+1)+m(k+1)e^{\alpha}_{m-1,n}(k+1)-m(m-1-k)e^{\alpha}_{m-1,n}(k)\\
&=m(m-k)\frac{(-1)^{k}m!n!\Gamma(\alpha+1)}{k!(m-k)!(n-k)!\Gamma(k+\alpha+1)}\\
&-(m+n+\alpha+1)(k+1)\frac{(-1)^{k+1}m!n!\Gamma(\alpha+1)}{(k+1)!(m-1-k)!(n-1-k)!\Gamma(k+\alpha+2)}\\
&+m(k+1)\frac{(-1)^{k+1}(m-1)!n!\Gamma(\alpha+1)}{(k+1)!(m-2-k)!(n-1-k)!\Gamma(k+\alpha+2)}\\
&-m(m-1-k)\frac{(-1)^{k}(m-1)!n!\Gamma(\alpha+1)}{k!(m-1-k)!(n-k)!\Gamma(k+\alpha+1)}\\
&=\frac{(-1)^{k}m!n!\Gamma(\alpha+1)}{k!(m-1-k)!(n-k)!\Gamma(k+\alpha+2)}\Big(m(k+\alpha+1)\\
&+(m+n+\alpha+1)(n-k)-(m-1-k)(n-k)-(m-1-k)(k+\alpha+1)\Big)\\
&=(n+1+\alpha)(n+1)\frac{(-1)^{k}m!n!\Gamma(\alpha+1)}{k!(m-1-k)!(n-k)!\Gamma(k+\alpha+2)}\\
&=-(n+1+\alpha)(k+1)e^{\alpha}_{m,n+1}(k+1). \qquad (2.35)
\end{aligned}$$

So we substitute back and highlight the next term. Again, we put the new terms at the end:

$$
\begin{aligned}
&(m+n+\alpha+1)\,(1-z\,\overline{z}\,)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)P^{\alpha}_{m-1,n} \\
&-\left[(n+1+\alpha)\sum_{k=0}^{n+1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\right] \\
&+(n+1+\alpha)\sum_{k=0}^{n-1}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k} \\
&+(n+1)\frac{(-1)^{n}m!\Gamma(\alpha+1)}{(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^{n+1}z^{m-1-n} \\
&-(n+1+\alpha)\sum_{k=0}^{n-1}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}.
\end{aligned}
\tag{2.36}
$$

This next term we only separate as follows:

$$
\begin{aligned}
&(n+1+\alpha)\sum_{k=0}^{n+1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k} \\
&= m(n+1+\alpha)\sum_{k=0}^{n+1}e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k} \\
&-(n+1+\alpha)\sum_{k=1}^{n+1}ke^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k} \\
&= m(n+1+\alpha)P^{\alpha}m,n+1(z,\,\overline{z}\,) \\
&-(n+1+\alpha)\sum_{k=1}^{n+1}ke^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}.
\end{aligned}
\tag{2.37}
$$

This time we add the new terms in place:

$$
\begin{aligned}
&(m+n+\alpha+1)\,(1-z\,\overline{z}\,)\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\,\overline{z}\,)-P^{\alpha}_{m,n+1}(z,\,\overline{z}\,)\big) \\
&+(n+1+\alpha)\sum_{k=1}^{n+1}ke^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k} \\
&+\left[(n+1+\alpha)\sum_{k=0}^{n-1}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\right] \\
&+(n+1)\frac{(-1)^{n}m!\Gamma(\alpha+1)}{(m-1-n)!\Gamma(n+\alpha+1)}(1-z\,\overline{z}\,)^{n+1}z^{m-1-n} \\
&-\left[(n+1+\alpha)\sum_{k=0}^{n-1}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}.\right]
\end{aligned}
\tag{2.38}
$$

We add the new highlighted terms:

$$(n+1+\alpha)\sum_{k=0}^{n-1}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\big(z\overline{z}-1\big)$$
$$-(n+1+\alpha)\sum_{k=0}^{n-1}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^{k+1}z^{m-1-k}\,\overline{z}^{\,n-k} \tag{2.39}$$

So we add the term back at the end:

$$\begin{aligned}(m+n+\alpha+1)\,(1-z\overline{z})\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\overline{z})) &= m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\overline{z})-P^{\alpha}_{m,n+1}(z,\overline{z})\big)\\
&+(n+1+\alpha)\sum_{k=1}^{n+1}ke^{\alpha}_{m,n+1}(k)(1-z\overline{z})^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)(n+1)e^{\alpha}_{m,n+1}(n+1)(1-z\overline{z})^{n+1}z^{m-1-n}\\
&-(n+1+\alpha)\sum_{k=0}^{n-1}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^{k+1}z^{m-1-k}\,\overline{z}^{\,n-k}.\end{aligned} \tag{2.40}$$

And we're finally at the point we can just reduce the terms:

$$\begin{aligned}(m+n+\alpha+1)\,(1-z\overline{z})\,\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\overline{z})) &= m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\overline{z})-P^{\alpha}_{m,n+1}(z,\overline{z})\big)\\
&+(n+1+\alpha)\sum_{k=1}^{n+1}ke^{\alpha}_{m,n+1}(k)(1-z\overline{z})^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)(n+1)e^{\alpha}_{m,n+1}(n+1)(1-z\overline{z})^{n+1}z^{m-1-n}\\
&-(n+1+\alpha)\sum_{k=0}^{n-1}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^{k+1}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&= m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\overline{z})-P^{\alpha}_{m,n+1}(z,\overline{z})\big)\\
&+(n+1+\alpha)\sum_{k=1}^{n+1}ke^{\alpha}_{m,n+1}(k)(1-z\overline{z})^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)\sum_{k=0}^{n}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^{k+1}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&= m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\overline{z})-P^{\alpha}_{m,n+1}(z,\overline{z})\big)\\
&+(n+1+\alpha)\sum_{k=0}^{n}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^{k+1}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&-(n+1+\alpha)\sum_{k=0}^{n}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^{k+1}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&= m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\overline{z})-P^{\alpha}_{m,n+1}(z,\overline{z})\big).\end{aligned}$$

Therefore, the equality holds true in this case.

Now we detail the other case. The general idea of the proof is the exact same, so we omit many of the computations of the terms, since they're very similar to the first case.

Then, we suppose $m \leq n$. This allows us to get the degrees of the polynomials that we need:

$$\min\{m, n\} = m, \ \min\{m-1, n\} = m-1, \ \text{and} \ \min\{m, n+1\} = m.$$

We manipulate the terms just as before, so we part from equation 2.23:

$$(m+n+\alpha+1)(1-z\overline{z})\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\overline{z})) = (m+n+\alpha+1)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\overline{z}))$$
$$-z(n+1+\alpha)\frac{\partial}{\partial z}(P^{\alpha}_{m,n+1}(z,\overline{z})) - mz\frac{\partial}{\partial z}(P^{\alpha}_{m-1,n}(z,\overline{z})). \tag{2.41}$$

Once again we compute the derivatives from the last equation. The only thing that changes is the degree of the polynomials that are used, so we take the expressions from equations 2.24, 2.25, and 2.26:

$$\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\overline{z})) = \sum_{k=0}^{m-1}(m-k)e^{\alpha}_{m,n}(k)(1-z\overline{z})^k z^{m-1-k}\,\overline{z}^{\,n-k}$$
$$+\sum_{k=0}^{m-1} -(k+1)e^{\alpha}_{m,n}(k+1)(1-z\overline{z})^k z^{m-1-k}\,\overline{z}^{\,n-k}. \tag{2.42}$$

$$\frac{\partial}{\partial z}(P^{\alpha}_{m,n+1}(z,\overline{z})) = \sum_{k=0}^{m-1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\overline{z})^k z^{m-1-k}\,\overline{z}^{\,n+1-k}$$
$$+\sum_{k=0}^{m-1} -(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^k z^{m-1-k}\,\overline{z}^{\,n+1-k}. \tag{2.43}$$

$$\frac{\partial}{\partial z}(P^{\alpha}_{m-1,n}(z,\overline{z})) = \sum_{k=0}^{m-2}(m-1-k)e^{\alpha}_{m-1,n}(m)(1-z\overline{z})^k z^{m-2-k}\,\overline{z}^{\,n-k}$$
$$+\sum_{k=0}^{m-2} -(k+1)e^{\alpha}_{m-1,n}(k+1)(1-z\overline{z})^k z^{m-2-k}\,\overline{z}^{\,n-k}. \tag{2.44}$$

We substitute the expressions 2.42, 2.43, and 2.44 back into equation 2.41:

$$
\begin{aligned}
&(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = \\
&+(m+n+\alpha+1)\sum_{k=0}^{m-1}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k} \\
&+(m+n+\alpha+1)\sum_{k=0}^{m-1}-(k+1)e^{\alpha}_{m,n}(k+1)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k} \\
&-(n+1+\alpha)\sum_{k=0}^{m-1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k} \\
&-(n+1+\alpha)\sum_{k=0}^{m-1}-(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k} \\
&-m\sum_{k=0}^{m-2}(m-1-k)e^{\alpha}_{m-1,n}(m)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k} \\
&-m\sum_{k=0}^{m-2}-(k+1)e^{\alpha}_{m-1,n}(k+1)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}. \qquad (2.45)
\end{aligned}
$$

Then, we adopt the same approach as before, we highlight a set of terms to be modified, and substitute it back in the equation from above. But, as we mentioned, we omit many of the algebraic manipulations:

So, first off:

$$
\begin{aligned}
&(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = \\
&+\left[(m+n+\alpha+1)\sum_{k=0}^{m-1}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\right] \\
&+(m+n+\alpha+1)\sum_{k=0}^{m-1}-(k+1)e^{\alpha}_{m,n}(k+1)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k} \\
&-(n+1+\alpha)\sum_{k=0}^{m-1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k} \\
&-(n+1+\alpha)\sum_{k=0}^{m-1}-(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k} \\
&-m\sum_{k=0}^{m-2}(m-1-k)e^{\alpha}_{m-1,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k} \\
&-m\sum_{k=0}^{m-2}-(k+1)e^{\alpha}_{m-1,n}(k+1)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}. \qquad (2.46)
\end{aligned}
$$

We separate the constants as before:

$$
\begin{aligned}
&(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)P^{\alpha}_{m-1,n}(z,\,\overline{z}\,)\\
&+\left[m\sum_{k=0}^{m-1}(m-k)e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&+\left[(m+n+\alpha+1)\sum_{k=0}^{m-1}-(k+1)e^{\alpha}_{m,n}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&-(n+1+\alpha)\sum_{k=0}^{m-1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)\sum_{k=0}^{m-1}-(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-m\sum_{k=0}^{m-2}(m-1-k)e^{\alpha}_{m-1,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\\
&-m\sum_{k=0}^{m-2}-(k+1)e^{\alpha}_{m-1,n}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}.
\end{aligned}
\tag{2.47}
$$

We add the highlighted terms and put the sum at the end:

$$
\begin{aligned}
&(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)P^{\alpha}_{m-1,n}(z,\,\overline{z}\,)\\
&-(n+1+\alpha)\sum_{k=0}^{m-1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)\sum_{k=0}^{m-1}-(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-\left[m\sum_{k=0}^{m-2}(m-1-k)e^{\alpha}_{m-1,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&-\left[m\sum_{k=0}^{m-2}-(k+1)e^{\alpha}_{m-1,n}(k+1)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&+(n+1+\alpha)\sum_{k=0}^{m-1}\frac{(m-k)(m+n-k)}{(k+\alpha+1)}e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^{k}z^{m-1-k}\,\overline{z}^{\,n-k}.
\end{aligned}
\tag{2.48}
$$

We do the same, add the highlighed terms and put the result at the end:

$$\begin{aligned}
&(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)P^{\alpha}_{m-1,n}(z,\,\overline{z}\,)\\
&-\left[(n+1+\alpha)\sum_{k=0}^{m-1}(m-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k}\right]\\
&-(n+1+\alpha)\sum_{k=0}^{m-1}-(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k}\\
&+(n+1+\alpha)\sum_{k=0}^{m-1}\frac{(m-k)(m+n-k)}{(k+\alpha+1)}e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&-(n+1+\alpha)\sum_{k=0}^{m-2}\frac{m(m-1-k)}{k+\alpha+1}e^{\alpha}_{m-1,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}. \qquad (2.49)
\end{aligned}$$

We separate the term distributing the $(m-k)$ part and complete the polynomial $P^{\alpha}_{m,n+1}$:

$$\begin{aligned}
&(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\,\overline{z}\,)-P^{\alpha}_{m,n+1}(z,\,\overline{z}\,)\big)\\
&+m(n+1+\alpha)e^{\alpha}_{m,n+1}(m)(1-z\,\overline{z}\,)^m z^{n+1-m}\\
&+(n+1+\alpha)\sum_{k=0}^{m-1}ke^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)\sum_{k=0}^{m-1}-(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^k z^{m-k}\,\overline{z}^{\,n+1-k}\\
&+\left[(n+1+\alpha)\sum_{k=0}^{m-1}\frac{(m-k)(m+n-k)}{(k+\alpha+1)}e^{\alpha}_{m,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&-(n+1+\alpha)\sum_{k=0}^{m-2}\frac{m(m-1-k)}{k+\alpha+1}e^{\alpha}_{m-1,n}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}. \qquad (2.50)
\end{aligned}$$

We separate the $m-1$-th term:

$$
\begin{aligned}
&(m+n+\alpha+1)(1-z\overline{z})\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\overline{z})) = m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\overline{z}) - P^{\alpha}_{m,n+1}(z,\overline{z})\big)\\
&+\left[m(n+1+\alpha)e^{\alpha}_{m,n+1}(m)(1-z\overline{z})^m z^{n+1-m}\right]\\
&+(n+1+\alpha)\sum_{k=1}^{m-1} k e^{\alpha}_{m,n+1}(k)(1-z\overline{z})^k z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)\sum_{k=0}^{m-1} -(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^k z^{m-k}\,\overline{z}^{\,n+1-k}\\
&+(n+1+\alpha)\sum_{k=0}^{m-2}\frac{(m-k)(m+n-k)}{(k+\alpha+1)}e^{\alpha}_{m,n}(k)(1-z\overline{z})^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&+\left[\frac{(n+1+\alpha)(n+1)}{(m-\alpha)}e^{\alpha}_{m,n}(m-1)(1-z\overline{z})^{m-1}\,\overline{z}^{\,n+1-m}\right]\\
&-(n+1+\alpha)\sum_{k=0}^{m-2}\frac{m(m-1-k)}{k+\alpha+1}e^{\alpha}_{m-1,n}(k)(1-z\overline{z})^k z^{m-1-k}\,\overline{z}^{\,n-k}. \qquad (2.51)
\end{aligned}
$$

We add the two terms and put the result at the end:

$$
\begin{aligned}
&(m+n+\alpha+1)(1-z\overline{z})\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\overline{z})) = m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\overline{z}) - P^{\alpha}_{m,n+1}(z,\overline{z})\big)\\
&+(n+1+\alpha)\sum_{k=1}^{m-1} k e^{\alpha}_{m,n+1}(k)(1-z\overline{z})^k z^{m-k}\,\overline{z}^{\,n+1-k}\\
&-(n+1+\alpha)\sum_{k=0}^{m-1} -(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\overline{z})^k z^{m-k}\,\overline{z}^{\,n+1-k}\\
&+\left[(n+1+\alpha)\sum_{k=0}^{m-2}\frac{(m-k)(m+n-k)}{(k+\alpha+1)}e^{\alpha}_{m,n}(k)(1-z\overline{z})^k z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&-\left[(n+1+\alpha)\sum_{k=0}^{m-2}\frac{m(m-1-k)}{k+\alpha+1}e^{\alpha}_{m-1,n}(k)(1-z\overline{z})^k z^{m-1-k}\,\overline{z}^{\,n-k}\right]\\
&-m(n+1+\alpha)e^{\alpha}_{m,n+1}(m)(1-z\overline{z})^{m-1} z\,\overline{z}^{\,n+2-m}. \qquad (2.52)
\end{aligned}
$$

We add these two terms and put the result where they were:

$$(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\,\overline{z}\,) - P^{\alpha}_{m,n+1}(z,\,\overline{z}\,)\big)$$
$$+\left[(n+1+\alpha)\sum_{k=1}^{m-1} k e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k} z^{m-k}\,\overline{z}^{\,n+1-k}\right]$$
$$-(n+1+\alpha)\sum_{k=0}^{m-1} -(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k} z^{m-k}\,\overline{z}^{\,n+1-k}$$
$$+(n+1+\alpha)\sum_{k=0}^{m-2}(m-k)(n+1-k)e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k} z^{m-1-k}\,\overline{z}^{\,n-k}$$
$$-m(n+1+\alpha)e^{\alpha}_{m,n+1}(m)(1-z\,\overline{z}\,)^{m-1} z\,\overline{z}^{\,n+2-m}. \tag{2.53}$$

We shift this sum so it begins on 0 and $m-2$:

$$(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\,\overline{z}\,) - P^{\alpha}_{m,n+1}(z,\,\overline{z}\,)\big)$$
$$+(n+1+\alpha)\sum_{k=0}^{m-2}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k+1} z^{m-1-k}\,\overline{z}^{\,n-k}$$
$$-\left[(n+1+\alpha)\sum_{k=0}^{m-1} -(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k} z^{m-k}\,\overline{z}^{\,n+1-k}\right]$$
$$+(n+1+\alpha)\sum_{k=0}^{m-2}\frac{(m-k)(n+1-k)}{k+\alpha+1}e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k} z^{m-1-k}\,\overline{z}^{\,n-k}$$
$$-m(n+1+\alpha)e^{\alpha}_{m,n+1}(m)(1-z\,\overline{z}\,)^{m-1} z\,\overline{z}^{\,n+2-m}. \tag{2.54}$$

We separate the $m-1$-th term from the highlighted sum and put it at the end, it will cancel out:

$$(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\,\overline{z}\,) - P^{\alpha}_{m,n+1}(z,\,\overline{z}\,)\big)$$
$$+(n+1+\alpha)\sum_{k=0}^{m-2}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k+1} z^{m-1-k}\,\overline{z}^{\,n-k}$$
$$+(n+1+\alpha)\sum_{k=0}^{m-2}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^{k} z^{m-k}\,\overline{z}^{\,n+1-k}$$
$$+(n+1+\alpha)\sum_{k=0}^{m-2}\frac{(m-k)(n+1-k)}{k+\alpha+1}e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^{k} z^{m-1-k}\,\overline{z}^{\,n-k}$$
$$-m(n+1+\alpha)e^{\alpha}_{m,n+1}(m)(1-z\,\overline{z}\,)^{m-1} z\,\overline{z}^{\,n+2-m}$$
$$+m(n+1+\alpha)e^{\alpha}_{m,n+1}(m)(1-z\,\overline{z}\,)^{m-1} z\,\overline{z}^{\,n+2-m}. \tag{2.55}$$

Yet again, we arrive at the stage where we can just simplify the expression:

$$
\begin{aligned}
&(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\,\overline{z}\,) - P^{\alpha}_{m,n+1}(z,\,\overline{z}\,)\big)\\
&+(n+1+\alpha)\sum_{k=0}^{m-2}(k+1)e^{\alpha}_{m,n+1}(k+1)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&+(n+1+\alpha)\sum_{k=0}^{m-2}\frac{(m-k)(n+1-k)}{k+\alpha+1}e^{\alpha}_{m,n+1}(k)(1-z\,\overline{z}\,)^k z^{m-1-k}\,\overline{z}^{\,n-k}\\
&= m(n+\alpha+1)\big(P^{\alpha}_{m-1,n}(z,\,\overline{z}\,) - P^{\alpha}_{m,n+1}(z,\,\overline{z}\,)\big).
\end{aligned}
\tag{2.56}
$$

The last equality is obtained because the coefficients in the sums are equal but have opposite signs.

And we get the desired recurrence relation once again.

As with the other recurrence relation, we quickly show that the other formula can be obtained by conjugating the one we just showed.

So, we begin with the current recurrence relation, but flipping the sub-indices:

$$(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{n,m}(z,\,\overline{z}\,)) = n(m+\alpha+1)\big(P^{\alpha}_{n-1,m}(z,\,\overline{z}\,) - P^{\alpha}_{n,m+1}(z,\,\overline{z}\,)\big)$$

So we conjugate both sides:

$$\overline{(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial z}(P^{\alpha}_{n,m}(z,\,\overline{z}\,))} = \overline{n(m+\alpha+1)\big(P^{\alpha}_{n-1,m}(z,\,\overline{z}\,) - P^{\alpha}_{n,m+1}(z,\,\overline{z}\,)\big)}$$

$$(m+n+\alpha+1)(1-z\,\overline{z}\,)\frac{\partial}{\partial \overline{z}}(P^{\alpha}_{m,n}(z,\,\overline{z}\,)) = n(m+\alpha+1)\big(P^{\alpha}_{m,n-1}(z,\,\overline{z}\,) - P^{\alpha}_{m+1,n}(z,\,\overline{z}\,)\big)$$

It's just that easy.

□

Remark 2.1.11. Actually, the recurrence relations that were proved in Theorem 2.1.10 can be generalized a bit to be valid for all non-negative integer indices if one defines the polynomials with -1 as an index (be it the first or second one) as zero. Of course, one would have to prove this, but it will not provide us with many benefits, so we abstain from showing it.

2.2 Description of the Weighted Measure L^2 Space

After the results that were laid out in the last section, we are now ready to exhibit an orthonormal basis for the weighted L^2 space on the disk.

2.2.1 Orthonormal basis for L^2

To start off, we prove the orthogonality of the sequence of disk polynomials in $L^2(\mathbb{D},\mu_\alpha)$.

We recall the orthogonality of the Fourier basis:

Claim 2.2.1. Let $m,n \in \mathbb{Z}^+$. It holds the following relation:

$$\int_0^{2\pi} e^{i(m-n)\theta} d\theta = 2\pi\delta_{m,n}.$$

Lemma 2.2.2. *The sequence* $\{P^\alpha_{m,n}\}_{m,n\in\mathbb{Z}_+}$ *is orthogonal in* $L^2(\mathbb{D},\mu_\alpha)$*. Furthermore, they satisfy the following relation:*

$$\langle P^\alpha_{k,l}(z,\overline{z}), P^\alpha_{m,n}(z,\overline{z})\rangle = \frac{\pi m! l! \Gamma(\alpha+1)^2}{(m+n+\alpha+1)\Gamma(m+\alpha+1)\Gamma(l+\alpha+1)}\delta_{k,m}\delta_{l,n}.$$

Proof. We compute the corresponding inner product. Let $k,l,m,n \in \mathbb{Z}_+$.

$$\begin{aligned}
\langle P^\alpha_{k,l}(z,\overline{z}), P^\alpha_{m,n}(z,\overline{z})\rangle &= \int_{\mathbb{D}} P^\alpha_{k,l}(z,\overline{z})\overline{P^\alpha_{m,n}(z,\overline{z})}d\mu_\alpha(z) = \int_{\mathbb{D}} P^\alpha_{k,l}(z,\overline{z})\overline{P^\alpha_{m,n}(z,\overline{z})}w(z)dxdy \\
&= \frac{\alpha+1}{\pi}\int_{\mathbb{D}} P^\alpha_{k,l}(z,\overline{z})\overline{P^\alpha_{m,n}(z,\overline{z})}(1-z\overline{z})^\alpha dxdy \\
&= \frac{\alpha+1}{\pi}\int_{\mathbb{D}} P^\alpha_{k,l}(z,\overline{z})P^\alpha_{n,m}(z,\overline{z})(1-z\overline{z})^\alpha dxdy.
\end{aligned}$$

Next, apply a change of variables to polar coordinates.

$$\begin{aligned}
\langle P^\alpha_{k,l}(z,\overline{z}), P^\alpha_{m,n}(z,\overline{z})\rangle &= \frac{\alpha+1}{\pi}\int_0^1\int_0^{2\pi} P^\alpha_{k,l}(re^{i\theta},re^{-i\theta})P^\alpha_{n,m}(re^{i\theta},re^{-i\theta})(1-r^2)^\alpha r d\theta dr \\
&= \frac{\alpha+1}{\pi}\int_0^1\int_0^{2\pi} e^{i(k-l)\theta}P^\alpha_{k,l}(r,r)e^{i(n-m)\theta}P^\alpha_{n,m}(r,r)(1-r^2)^\alpha r d\theta dr \\
&= \frac{\alpha+1}{\pi}\int_0^1 P^\alpha_{k,l}(r,r)P^\alpha_{n,m}(r,r)r(1-r^2)^\alpha\int_0^{2\pi} e^{i(k+n-(l+m))\theta}d\theta\ dr.
\end{aligned}$$

By Claim 2.2.1 we get:

$$\langle P^\alpha_{k,l}(z,\overline{z}), P^\alpha_{m,n}(z,\overline{z})\rangle = 2(\alpha+1)\delta_{k+n,l+m}\int_0^1 P^\alpha_{k,l}(r,r)P^\alpha_{n,m}(r,r)r(1-r^2)^\alpha\ dr.$$

To further reduce the expression, we may consider the condition for the Delta to be satisfied, since the expression would become zero otherwise; that is: $k+m=l+n$, or equivalently, $l=k+m-n$.

Simultaneously, by Proposition 2.1.7, we may substitute $P^\alpha_{n,m}(r,r)$ for $P^\alpha_{m,n}(r,r)$. So equation 2.2.1 becomes:

$$\langle P^\alpha_{k,l}(z,\overline{z}), P^\alpha_{m,n}(z,\overline{z})\rangle = 2(\alpha+1)\delta_{k+n,l+m}\int_0^1 P^\alpha_{l+m-n,l}(r,r)P^\alpha_{m,n}(r,r)r(1-r^2)^\alpha\ dr.$$

Which, by Lemma 2.1.9, is in turn equal to:

$$\langle P^\alpha_{k,l}(z,\overline{z}), P^\alpha_{m,n}(z,\overline{z})\rangle = \frac{(\alpha+1)m!\Gamma(\alpha+1)l!\Gamma(\alpha+1)}{(m+n+\alpha+1)\Gamma(m+\alpha+1)\Gamma(l+\alpha+1)}\delta_{k+n,m+l}\delta_{l,n}.$$

Finally, the last product of deltas may be substituted by $\delta_{k,m}\delta_{l,n}$, which yields the desired expression. □

These results prompt us to give the next definition.

Definition 2.2.3. We define a new sequence of polynomials normalizing the disk polynomials. Let $m, n \in \mathbb{N}$. We define the following function:

$$b^{\alpha}_{m,n}(z) = c^{\alpha}_{m,n} \, P^{\alpha}_{m,n}(z, \overline{z}\,), \quad \forall z \in \mathbb{D}.$$

where

$$c^{\alpha}_{m,n} = \frac{1}{\|P^{\alpha}_{m,n}\|_{2,\alpha}} = \sqrt{\frac{(m+n+\alpha+1)\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)}{(\alpha+1)\, m!\, n!\, \Gamma(\alpha+1)^2}}.$$

By construction, we get as a result:

Proposition 2.2.4. *The sequence $\{b^{\alpha}_{m,n}\}_{m,n\in\mathbb{Z}_+}$ is an orthonormal set in $L^2(\mathbb{D}, \mu_\alpha)$.*

Notice that the elements $\{b^{\alpha}_{m,n}\}_{m,n\in\mathbb{Z}_+}$ are multiples of the disk polynomials by a real constant, so they satisfy most of their properties, like their relationship under conjugation or the recurrence relations stated before. We will make use of these properties for the newly defined polynomials without mention of this fact, when appliccable.

For instance, from the explicit representation of the disk polynomials (Claim 2.1.6), we get an explicit representation for the elements $\{b^{\alpha}_{m,n}\}_{m,n\in\mathbb{Z}_+}$.

Proposition 2.2.5. *For any $m, n \in \mathbb{Z}_+$, the following equality holds:*

$$b^{\alpha}_{m,n}(z) = \sqrt{\frac{(m+n+\alpha+1)m!n!}{(\alpha+1)\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)}} \sum_{j=0}^{\min\{m,n\}} \frac{(-1)^j\Gamma(m+n+\alpha-j+1)}{j!(m-j)!(n-j)!} z^{m-j}\,\overline{z}^{\,n-j}, \quad \forall z \in \mathbb{D}.$$

Now we set out to prove that it is in fact an orthonormal basis for this space. To do so we rely heavily on the fact that the linear span of the monomials is dense in $\mathbb{D}$. This next subsection details a proof of this fact.

2.2.2 Density of monomials in the weighted L^2 space

First, we introduce some notation for the concepts to be worked on.

Definition 2.2.6. Let $p, q \in \mathbb{N}$. We define the corresponding *monomial in z and $\overline{z}$* as the polynomial:

$$m_{p,q}(z) = z^p\,\overline{z}^{\,q}, \quad \forall z \in \mathbb{D}.$$

In general, the monomials are defined on the whole complex plane $\mathbb{C}$, however, in the arguments below

As an auxiliary facts we state an important theorem.

Theorem 2.2.7. *The linear span of the sequence of monomials $\{m_{p,q}\}_{p,q\in\mathbb{Z}_+}$ is dense in the normed space $(C(\overline{\mathbb{D}}, \mathbb{C}), \|\cdot\|_\infty)$.*

This is a direct result of the complex version of the Stone-Weierstrass Theorem, since the linear span of monomials is an algebra that contains constants, separates points and is closed with respect to complex conjugation.

And next we prove:

Theorem 2.2.8. *The linear span of the sequence of monomials* $\{m_{p,q}\}_{p,q\in\mathbb{Z}_+}$ *is dense in the normed space* $L^2(\mathbb{D},\mu_\alpha)$*.*

Proof. In virtue of Theorem 2.2.7, it suffices to approximate functions in $L^2(\mathbb{D},\mu_\alpha)$ by continuous functions on $\overline{\mathbb{D}}$, or more accurately, functions defined on the disk $\mathbb{D}$ that admit a continuous extension to the circle $\mathbb{D}$.

Let $f\in L^2(\mathbb{D},\mu_\alpha)$ and $\varepsilon>0$.

Notice the following equivalence for any measurable function $f:\mathbb{D}\to\mathbb{C}$,

$$f\in L^2(\mathbb{D},\mu_\alpha) \iff |f|^2 w_\alpha\in L^1(\mathbb{D},\mu_\alpha) \iff f w_\alpha^{1/2}\in L^2(\mathbb{D},\mu).$$

So $fw_\alpha^{1/2}\in L^2(\mathbb{D},\mu)$. Hence, by Theorem 1.5.4, there exists a function $g\in C_s(\mathbb{D})$ such that:

$$\|fw_\alpha^{1/2}-g\|_2<\varepsilon\left(\frac{\pi}{\alpha+1}\right)^{1/2}.$$

Recall that, since $g\in C_s(\mathbb{D})$, we have that $\operatorname{supp}(g)\subset\mathbb{D}$.

Now define another function $h:\mathbb{C}\to\mathbb{C}$ as:

$$h(z)=\begin{cases}\left(gw_\alpha^{-1/2}\right)(z) & z\in\mathbb{D},\\ 0 & z\notin\mathbb{D}.\end{cases}$$

We get that h also belongs to $C_s(\mathbb{D})$ (in fact, h shares the same support with g). In particular, h is continuous (in all of $\mathbb{C}$, but also) in $\overline{\mathbb{D}}$.

Then we compute:

$$\begin{aligned}\int_{\mathbb{D}}|f-h|^2d\mu_\alpha &= \frac{\alpha+1}{\pi}\int_{\mathbb{D}}|f-h|^2w_\alpha d\mu\\ &=\frac{\alpha+1}{\pi}\int_{\mathbb{D}}|f-h|^2(w_\alpha^{1/2})^2d\mu\\ &=\frac{\alpha+1}{\pi}\int_{\mathbb{D}}|fw_\alpha^{1/2}-hw_\alpha^{1/2}|^2d\mu\\ &=\frac{\alpha+1}{\pi}\int_{\mathbb{D}}|fw_\alpha^{1/2}-g|^2d\mu.\end{aligned}$$

That is, $\|f-h\|_{2,\alpha}=\left(\frac{\alpha+1}{\pi}\right)^{1/2}\|fw_\alpha^{1/2}-g\|_2$. Ergo, the proposition holds. □

This last theorem basically guarantees the totality of the sequence of disk polynomials. We need only the following proposition to conclude.

Proposition 2.2.9. *Let* $\mathcal{D}$ *be the linear space of complex polynomials in* z *and* $\overline{z}$*. Then:*

$$\mathcal{D}=\operatorname{span}\{m_{p,q}\}_{p,q\in\mathbb{Z}_+}=\operatorname{span}\{P^\alpha_{k,l}\}_{k,l\in\mathbb{Z}_+}.$$

Immediately:

Corollary 2.2.10. *The sequence of normalized disk polynomials $\{b^{\alpha}_{k,l}\}_{k,l\in\mathbb{Z}_+}$ is an orthornormal basis for $L^2(\mathbb{D},\mu_\alpha)$.*

Proof. By proposition 2.2.4, we get the orthonormality of the sequence. And considering both Theorem 2.2.8 and Proposition 2.2.9, the total property of the set is satisfied. □

2.3 Poly-Bergman Spaces

After getting our basis for the weighted L^2 measure space, we now move on to the next object of study: the weighted Poly-Bergman Spaces.

Definition 2.3.1. Let $n \in \mathbb{N}$. We define the *n-Poly-Bergman weighted space*, or equivalently the *Poly-Bergman weighted space of n-th order* (with weight α) as:

$$\mathcal{A}^{\alpha,2}_n(\mathbb{D}) := \mathcal{O}_n(\mathbb{D}) \cap L^2(\mathbb{D},\mu_\alpha) = \left\{ f:\mathbb{D}\to\mathbb{C} \,\middle|\, f\in\mathcal{O}_n(\mathbb{D}), \int_{\mathbb{D}} |f|^2 d\mu_\alpha < \infty \right\}.$$

We introduce notation for the special case $n = 1$, which is the classical weighted Bergman space of holomorphic functions:

$$\mathcal{A}^{\alpha,2}_1(\mathbb{D}) := \mathcal{A}^{\alpha,2}(\mathbb{D}).$$

Since these spaces describe a monotone sequence of subspaces (w.r.t. set inclusion), the following spaces are also introduced.

Definition 2.3.2. Let $n \in \mathbb{N}$. We define the *true n-Poly-Bergman weighted spaces* (with weight α) as:

$$\mathcal{A}^{\alpha,2}_{(n)}(\mathbb{D}) = \mathcal{A}^{\alpha,2}_n(\mathbb{D}) \ominus \mathcal{A}^{\alpha,2}_{n-1}(\mathbb{D}).$$

These spaces will become some of the main objects of study of the next chapter. Since we won't deal with Bergman-type spaces defined on any other domain, the notation will be shortened by omitting the "$(\mathbb{D})$" part when denoting either the regular or pure Poly-Bergman spaces.

We state the following theorem of the description of the structure of the poly-Bergman spaces.

Theorem 2.3.3. *Let $n \in \mathbb{N}$. Then, $\mathcal{A}^{\alpha,2}_n(\mathbb{D})$ is a closed subset of $L^2(\mathbb{D},\mu_\alpha)$.*

Thus, it becomes a Hilbert space, with inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}} : \mathcal{A}^{\alpha,2}_n(\mathbb{D})\times\mathcal{A}^{\alpha,2}_n(\mathbb{D}) \to \mathbb{C}$ inherited from $L^2(\mathbb{D},\mu_\alpha)$.

Furthermore, $\mathcal{A}^{\alpha,2}_n(\mathbb{D})$ is a reproducing kernel Hilbert space.

This result can be found in [10].

Due to a result of Dr. Maximenko and their collaborators, published in their 2021 article "Radial operators on the weighted Poly-Bergman spaces", [2, Proposition 5.1, Corollary 5.2], we can also describe a basis for both the regular and pure Poly-Bergman spaces in terms of the basis for $L^2(\mathbb{D},\mu_\alpha)$.

Theorem 2.3.4. *Let $n \in \mathbb{N}$. Then, the following conditions hold:*

a) The sequence $\{b_{k,l}^{\alpha}\}_{k\in\mathbb{Z}_+,l=0}^{l=n-1}$ of normalized disk polynomials is an orthonormal basis for $\mathcal{A}_n^{\alpha,2}(\mathbb{D})$.

b) The sequence $\{b_{k,n-1}^{\alpha}\}_{k\in\mathbb{Z}_+}$ of disk polynomials is an orthonormal basis for $\mathcal{A}_{(n)}^{\alpha,2}(\mathbb{D})$.

Remark 2.3.5. Let us mention another result that this theorem implies.

Given that the sequence $\{b_{k,l}\}_{k,l\in\mathbb{Z}_+}$ is a basis for the whole space $L^2(\mathbb{D}, \mu_\alpha)$, and how the basis for the true Poly-Bergman spaces $\mathcal{A}_{(n)}^{\alpha,2}(\mathbb{D})$ are arranged, coupled with the fact that these are pairwise orthogonal subspaces, we have the following decomposition of the space as an orthogonal sum:

$$L^2(\mathbb{D}, \mu_\alpha) = \bigoplus_{n\in\mathbb{N}} \mathcal{A}_{(n)}^{\alpha,2}(\mathbb{D})$$

We will explain at the beginning of the next chapter what this decomposition entails.

Chapter 3

Extended Fock Space Structure for the Weighted L^2 Space

After the preliminaries, and having defined our main objects of study, we're now ready for the next step: applying the theory of the extended Fock space formalism to the weighted L^2 space on the disk.

For this we make a very strong comparison to the example we used before, the L^2 space on the complex plane with Gaussian measure, in which the extended Fock space was described by mutually adjoint operators, and the true Poly-Fock spaces were recovered by them.

So, first off: does the weighted L^2 space admit the extended Fock space structure? The answer may not obvious, but a previous result by Dr. Vasilevski, namely Theorem 1.3.11, combined with Remark 2.3.5 tell us that yes, one can describe an extended Fock space in the weighted L^2 space.

Furthermore, we will actually verify in the section after this one that one can define a pure isometry (two, in fact) on the space. With this, it is possible to recover the operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ in terms of this very isometry. However, as it will be shown, such description won't be very satisfactory. It will differ greatly from the simplicity of the operators displayed in the Fock case.

This brings us to the matter at hand: is there another way to find a representation for these operators? Do more "natural" or "simple" definitions for them not work in accordance to the theory of the extended Fock space formalism? This first section aims to at least partially answer these questions.

3.1 Towards a description using operators $\mathfrak{a}$ and $\mathfrak{b}$

We present an argument made by Dr. Vasilevski in his article [9, Section 3.1], Here, we begin backwards. We propose a pair of operators, describe their properties, and see if they're a good fit.

These operators are introduced in the following definition.

Definition 3.1.1. Let $\mathcal{D}_\mathfrak{a} := \{f \in L^2(\mathbb{D}, \mu_\alpha) \mid \frac{\partial f}{\partial \bar{z}} \in L^2(\mathbb{D}, \mu_\alpha)\}$, and define the operators $\mathfrak{a}, \mathfrak{b} : \mathcal{D}_{\mathfrak{a}} \to L^2(\mathbb{D}, \mu_\alpha)$ as:

$$\mathfrak{a}(f)(z) = \frac{\partial f}{\partial \overline{z}}(z), \quad \forall z \in \mathbb{D}, f \in \mathcal{D}_{\mathfrak{a}}$$

and,

$$\mathfrak{b}(f)(z) = \overline{z}\, f(z), \quad \forall z \in \mathbb{D}, f \in \mathcal{D}_{\mathfrak{a}}.$$

Proposition 3.1.2. *The set $\mathcal{D}_{\mathfrak{a}}$ is dense in $L^2(\mathbb{D}, \mu_\alpha)$, and on it the operators $\mathfrak{a}$ and $\mathfrak{b}$ satisfy the following commutation relation:*

$$[\mathfrak{a}, \mathfrak{b}] = I.$$

Proof. The density of the domain $\mathcal{D}_{\mathfrak{a}}$ follows from the fact that the orthonormal basis $\{b^\alpha_{m,n}\}_{m,n\in\mathbb{Z}_+}$ is contained in it.

Let $f \in \mathcal{D}_{\mathfrak{a}}, z \in \mathbb{D}$.

$$\begin{aligned}[\mathfrak{a}, \mathfrak{b}](f)(z) = (\mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a})(f)(z) = \mathfrak{a}\mathfrak{b}(f)(z) - \mathfrak{b}\mathfrak{a}(f)(z) &= \frac{\partial}{\partial \overline{z}}\big(\overline{z}\, f(z)\big) - \overline{z}\,\frac{\partial f}{\partial \overline{z}} \\ &= f(z) + \overline{z}\,\frac{\partial f}{\partial \overline{z}}(z) - \overline{z}\,\frac{\partial f}{\partial \overline{z}}(z) = f(z)\end{aligned}$$

□

Definition 3.1.3. Let $L_{[1]} \coloneqq \ker \mathfrak{a}$. For $n \in \mathbb{N}, n > 1$, define $L_{[n]} \coloneqq \mathfrak{b}^{n-1} L_{[1]} = \overline{z}^{\,n-1} L_{[1]}$.

Given the definition of $L_{[1]}$, we have that:

$$L_{[1]} = \ker \mathfrak{a} = \left\{ f \in L^2(\mathbb{D}, \mu_\alpha) \,\middle|\, \mathfrak{a}(f) = \frac{\partial f}{\partial \overline{z}} = 0 \right\}.$$

So this subspace coincides with $\mathcal{A}^{2,\alpha}$, the weighted Bergman space.

This in turn gives for $L_{[n]}$:

$$L_{[n]} = \overline{z}^{\,n-1} L_{[1]} = \overline{z}^{\,n-1} \mathcal{A}^{2,\alpha} = \mathcal{A}^{2,\alpha}_{(n)}.$$

Proposition 3.1.4. *Define the following set:*

$$\mathcal{D}_0 = \operatorname{span}\left(\bigcup_{n\in\mathbb{N}} L_{[n]} \right).$$

Then $\mathcal{D}_0$ is dense in $L^2(\mathbb{D}, \mu_\alpha)$. Also, the operators $\mathfrak{a}$ and $\mathfrak{b}$ act invariantly over $\mathcal{D}_0$.

Proof. Once again, the density comes from the fact that $\mathcal{D}_0$ contains the orthonormal basis $\{b^\alpha_{m,n}\}_{m,n\in\mathbb{Z}_+}$.

Let $f \in \mathcal{D}_0$. Then we can write f as:

$$f(z) = g_0(z) + \sum_{j=1}^{n} \overline{z}^{\,m_j} g_j(z), \quad \forall z \in \mathbb{D}$$

where $g_0, g_1, \ldots, g_m \in \mathcal{A}^{2,\alpha}$, and $m_1, \ldots, m_n \in \mathbb{N}$.
Then:

$$\mathfrak{a}(f)(z) = \frac{\partial f}{\partial \overline{z}}(z) = \sum_{j=1}^{n} \frac{\partial}{\partial \overline{z}}\big(\overline{z}^{\,m_j} g_j(z)\big) = \sum_{j=1}^{n} (m_j)\,\overline{z}^{\,m_j-1} g_j(z).$$

And:

$$\mathfrak{b}(f)(z) = \overline{z}\, f(z) = \overline{z}\left(g_0(z) + \sum_{j=1}^{n} \overline{z}^{\,m_j} g_j(z)\right) = \overline{z}\, g_0(z) + \sum_{j=1}^{n} \overline{z}^{\,m_j+1} g_j(z).$$

Therefore, $\mathfrak{a}(f), \mathfrak{b}(f) \in \mathcal{D}_0$. □

With this proposition, at this point, we have a bonafide extended Fock space defined by $L^2(\mathbb{D}, \mu_\alpha)$, $\mathfrak{a}$ and $\mathfrak{b}$. We proceed with the next definitions as was shown before.

In this case, even as the subspaces $L_{[n]}$ are all closed, their direct sums may not (and is not, actually), so we introduce the following subsets as described before:

Definition 3.1.5. Let $n \in \mathbb{N}$. We define the following set:

$$L_n = \operatorname{clos}(L_{[1]} + \cdots + L_{[n]})$$

considering $L_1 = \operatorname{clos}(L_{[1]})$.
These are all closed subspaces.

Proposition 3.1.6. *Let $n \in \mathbb{N}$. Then $L_n = \mathcal{A}^{2,\alpha}_n$.*

Proof. Let $n \in \mathbb{N}$.

If $n = 1$, we have that $L_1 = L_{[1]} = \mathcal{A}^{2,\alpha}$, which is closed.

Then, suppose $n \geq 2$. Consider an element $f \in L_{[1]} + \cdots + L_{[n]}$. There exist $\phi_1, \ldots, \phi_n \in L_{[1]}$ such that:

$$f(z) = \sum_{k=0}^{n-1} \overline{z}^{\,k} \phi_{k+1}(z)\,, \quad \forall z \in \mathbb{D}.$$

Having this expression, evaluating the n-th order Wirtinger derivative yields:

$$\begin{aligned}
\frac{\partial^n}{\partial \overline{z}^{\,n}}\big(f(z)\big) &= \frac{\partial^n}{\partial \overline{z}^{\,n}}\left(\sum_{k=0}^{n-1} \overline{z}^{\,k} \phi_{k+1}(z)\right) \\
&= \sum_{k=0}^{n-1} \frac{\partial^n}{\partial \overline{z}^{\,n}}\big(\overline{z}^{\,k} \phi_{k+1}(z)\big) \\
&= \sum_{k=0}^{n-1} \phi_{k+1}(z) \frac{\partial^n}{\partial \overline{z}^{\,n}}\big(\overline{z}^{\,k}\big) = 0
\end{aligned}$$

This guarantees that f belongs to $\mathcal{O}_n$.

Next, let $k \in \{1, \dots, n\}$.

$$\int_{\mathbb{D}} |\,\overline{z}^{\,k-1}\phi_k(z)|^2 d\mu_\alpha(z) \leq \int_{\mathbb{D}} |\phi_k(z)|^2 d\mu_\alpha(z) = \|\phi_k\|_{2,\alpha}^2.$$

Hence, all terms $\phi_1(z), z\phi_2(z), \dots, z_{n-1}\phi_n(z)$ belong to $L^2(\mathbb{D}, \mu_\alpha)$. Therefore, f does, too.

This gives: $L_{[1]} + \cdots + L_{[n]} \subset \mathcal{A}_n^{2,\alpha}$.

Which in turn implies: $\operatorname{clos}(L_{[1]} + \cdots + L_{[n]}) \subset \mathcal{A}_n^{2,\alpha}$.

Now, consider an element $f \in \mathcal{A}_n^{2,\alpha}$.

As it was stated before, the sequence $\{b_{k,l}^\alpha\}_{k\in\mathbb{Z}_+,l=0}^{n-1}$ constitutes an orthonormal basis for this space.

Then, there exists a sequence $\{\lambda_{k,l}\}_{k\in\mathbb{Z}_+,l=0}^{n-1}$ of complex numbers such that:

$$f = \sum_{k=0}^{\infty}\sum_{l=0}^{n-1} \lambda_{k,l} b_{k,l}^\alpha.$$

In particular, we see that:

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{l=0}^{n-1} \lambda_{k,l} b_{k,l}^\alpha &= \sum_{k=0}^{\infty} \left(\lambda_{k,0}b_{k,0}^\alpha + \lambda_{k,1}b_{k,1}^\alpha + \cdots + \lambda_{k,n-1}b_{k,n-1}^\alpha\right) \\ &= \lim_{m\to\infty} \sum_{k=0}^{m} \left(\lambda_{k,0}b_{k,0}^\alpha + \cdots + \lambda_{k,n-1}b_{k,n-1}^\alpha\right)\end{aligned}$$

Since every expression between parenthesis in the last expression belongs to the sum $L_{[1]} + \cdots + L_{[n]}$, we get that in fact the original function f is an element of $\operatorname{clos}(L_{[1]} + \cdots + L_{[n]})$. □

This last proposition implies that the structure of the extended Fock space described by $L^2(\mathbb{D}, \mu_\alpha)$, $\mathfrak{a}$, and $\mathfrak{b}$ does yield the Poly-Bergman subspaces, much like in the Fock case.

However, this is as far as we go. We will not be able to replicate any more conditions from the Fock case. First, because the operators we have are not mutually adjoint. And second, the next proposition shows that the direct sums of the subspaces $L_{[n]}$ is not closed.

But before getting to it, a lemma.

Lemma 3.1.7. *Let $n, k \in \mathbb{N}$. Then, the following holds:*

$$\|z^n\,\overline{z}^{\,k}\|_{2,\alpha} = \|z^{n+k}\|_{2,\alpha} = \sqrt{\frac{\Gamma(\alpha+2)(k+n)!}{\Gamma(k+n+\alpha+2)}}$$

Proof. We compute it directly:

$$\begin{aligned}\|z^n\,\overline{z}^{\,k}\|_{2,\alpha}^2 = \|z^{n+k}\|_{2,\alpha}^2 &= \int_{\mathbb{D}} |z|^{2(n+k)}\mu_\alpha(z) \\ &= \frac{\alpha+1}{\pi}\int_{\mathbb{D}} |z|^{2(n+k)}(1-|z|^2)^\alpha \mu_(z)\end{aligned}$$

Next, we use a change of variables to polar coordinates:

$$\begin{aligned}\|z^n\,\overline{z}^{\,k}\|_{2,\alpha}^2 &= \frac{\alpha+1}{\pi}\int_0^1\int_0^{2\pi} r^{2(n+k)}(1-r^2)^\alpha r d\theta dr\\ &= \frac{\alpha+1}{\pi}\int_0^1 r^{2(n+k)}(1-r^2)^\alpha r(2\pi)dr\\ &= (\alpha+1)\int_0^1 r^{2(n+k)}(1-r^2)^\alpha(2r)dr\end{aligned}$$

Then, we use the change of variables $u = r^2$.

$$\begin{aligned}\|z^n\,\overline{z}^{\,k}\|_{2,\alpha}^2 &= (\alpha+1)\int_0^1 u^{n+k}(1-u)^\alpha du\\ &= (\alpha+1)B(n+k+1,\alpha+1)\\ &= (\alpha+1)\frac{\Gamma(n+k+1)\Gamma(\alpha+1)}{\Gamma(n+k+\alpha+2)}\\ &= \frac{(n+k)!\Gamma(\alpha+2)}{\Gamma(n+k+\alpha+2)}\end{aligned}$$

□

Next, we cite the following limit:

Claim 3.1.8. Let $a, x \in \mathbb{R}$ with $x > 0$. Then the following holds:

$$\lim_{x\to\infty}\frac{\Gamma(x+a)}{\Gamma(x)x^a} = \lim_{x\to\infty}\frac{\Gamma(x)x^a}{\Gamma(x+a)} = 1.$$

This result can be found in [4, Formula 8.328.2].

Proposition 3.1.9. *Let $n \in \mathbb{N}, n > 1$. The direct sum of the following subspaces:*

$$L_{n-1} + L_{[n]}$$

is not closed.

Proof. To prove this fact we compute the minimal angle between the closed subspaces L_{n-1} and $L_{[n]}$.

For any $k \in \mathbb{N}$, let $x_k \in L_{[1]} \subset L_{n-1}$ and $y_k \in L_{[n]}$, given by:

$$x_k := \sqrt{\frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)}}\, z^k$$

and,

$$y_k := \sqrt{\frac{\Gamma(k+2n+\alpha)}{(k+2n-2)!\Gamma(\alpha+2)}}\, \overline{z}^{\,n-1} z^{k+n-1}.$$

In particular, by lemma 3.1.7, we have that $\|x_k\| = \|y_k\| = 1, \forall k \in \mathbb{N}$.

We compute:

$$
\begin{aligned}
\lim_{k\to\infty}|\langle x_k, y_k\rangle| &= \lim_{k\to\infty}\sqrt{\frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)}\,\frac{\Gamma(k+2n+\alpha)}{(k+2n-2)!\Gamma(\alpha+2)}}\,|\langle z^k,\ \overline{z}^{\,n-1}z^{k+n-1}\rangle| \\
&= \lim_{k\to\infty}\sqrt{\frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)}\,\frac{\Gamma(k+2n+\alpha)}{(k+2n-2)!\Gamma(\alpha+2)}}\left|\int_{\mathbb{D}} z^k\,\overline{(\overline{z}^{\,n-1}z^{k+n-1})}\,d\mu_\alpha(z)\right| \\
&= \lim_{k\to\infty}\sqrt{\frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)}\,\frac{\Gamma(k+2n+\alpha)}{(k+2n-2)!\Gamma(\alpha+2)}}\left|\int_{\mathbb{D}} |z^{2(n+k-1)}|\,d\mu_\alpha(z)\right| \\
&= \lim_{k\to\infty}\sqrt{\frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)}\,\frac{\Gamma(k+2n+\alpha)}{(k+2n-2)!\Gamma(\alpha+2)}}\,\|z^{k+n-1}\|^2_{2,\alpha} \\
&= \lim_{k\to\infty}\sqrt{\frac{\Gamma(k+\alpha+2)}{k!\Gamma(\alpha+2)}\,\frac{\Gamma(k+2n+\alpha)}{(k+2n-2)!\Gamma(\alpha+2)}}\,\frac{(k+n-1)!\Gamma(\alpha+2)}{\Gamma(k+n+\alpha+1)} \\
&= \lim_{k\to\infty}\sqrt{\frac{\Gamma(k+\alpha+2)}{k!}}\sqrt{\frac{\Gamma(k+2n+\alpha)}{(k+2n-2)!}}\,\frac{(k+n-1)!}{\Gamma(k+n+\alpha+1)}
\end{aligned}
$$

To proceed, we rewrite every quotient from the expression above as follows: The first one:

$$
\frac{\Gamma(k+\alpha+2)}{k!} = \frac{\Gamma(k+\alpha+2)}{\Gamma(k+1)} = \frac{\Gamma(k+1+(\alpha+1))}{\Gamma(k+1)(k+1)^{\alpha+1}}\,(k+1)^{\alpha+1}.
$$

The second one:

$$
\frac{\Gamma(k+2n+\alpha)}{(k+2n-2)!} = \frac{\Gamma(k+\alpha+2)}{\Gamma(k+2n-1)} = \frac{\Gamma(k+2n-1+(\alpha+1))}{\Gamma(k+1)(k+2n-1)^{\alpha+1}}\,(k+2n-1)^{\alpha+1}.
$$

And, the third one:

$$
\frac{(k+n-1)!}{\Gamma(k+n+\alpha+1)} = \frac{\Gamma(k+n)}{\Gamma(k+n+\alpha+1)} = \frac{\Gamma(k+n)(k+n)^{\alpha+1}}{\Gamma(k+n+(\alpha+1))}\,\frac{1}{(k+n)^{\alpha+1}}.
$$

We substitute this, and get:

$$
\begin{aligned}
\lim_{k\to\infty}|\langle x_k, y_k\rangle| &= \lim_{k\to\infty}\sqrt{\left(\frac{\Gamma(k+1+(\alpha+1))}{\Gamma(k+1)(k+1)^{\alpha+1}}(k+1)^{\alpha+1}\right)\left(\frac{\Gamma(k+2n-1+(\alpha+1))}{\Gamma(k+1)(k+2n-1)^{\alpha+1}}(k+2n-1)^{\alpha+1}\right)}\,\frac{\Gamma(k+n)(k+n)^{\alpha+1}}{\Gamma(k+n+(\alpha+1))}\,\frac{1}{(k+n)^{\alpha+1}} \\
&= \lim_{k\to\infty}\sqrt{\frac{\Gamma(k+1+(\alpha+1))}{\Gamma(k+1)(k+1)^{\alpha+1}}\,\frac{\Gamma(k+2n-1+(\alpha+1))}{\Gamma(k+1)(k+2n-1)^{\alpha+1}}}\,\frac{\Gamma(k+n)(k+n)^{\alpha+1}}{\Gamma(k+n+(\alpha+1))}\left(\frac{\sqrt{(k+1)(k+2n-1)}}{k+n}\right)^{\alpha+1}.
\end{aligned}
$$

In virtue of the limits stated before in Claim 3.1.8, this implies:

$$
\lim_{k\to\infty}|\langle x_k, y_k\rangle| = 1.
$$

By the definition of the minimal angle between closed subspaces of a Hilbert space, we get that $\phi^{(m)}(L_{n-1}, L_{[n]}) = 0$. The result follows from Theorem 1.7.2.

□

3.2 Pure Isometries on the Weighted L^2 Space

In these last sections, we try a different approach for applying the extended Fock space formalism to the weighted L^2 space.

We define a pure isometry acting on this space, initally defined through their action upon basis elements. Then, we find a basis-independent expression for it and recover a similar expression for the ladder operators that interest us.

We introduce the following operators, that are of course a natural generalization to those presented by Dr. Vasilevski in his article [11, Section 4]. In this case, we work with isometries defined for the $L^2(\mathbb{D}, \mu_\alpha)$ space.

Definition 3.2.1. We define the unilateral shift isometries as follows. Let $V_\alpha, \widetilde{V}_\alpha : L^2(\mathbb{D}, \mu_\alpha) \to L^2(\mathbb{D}, \mu_\alpha)$, defined on the basis elements of $L^2(\mathbb{D}, \mu_\alpha)$:

$$V_\alpha(b^\alpha_{k,l}) = b^\alpha_{k,l+1} \quad \text{and} \quad \widetilde{V}_\alpha(b^\alpha_{k,l}) = b^\alpha_{k+1,l}, \quad \forall k, l \in \mathbb{Z}_+.$$

These operators act on a very specific manner on the basis elements. Perhaps now it may not be quite clear what purpose they will serve, but the following results will help in understanding what they do and why they're defined like that.

In short, they raise the order of the Poly-Bergman spaces, so to speak. The last corollary of this section will illustrate this fact very well.

To begin, we prove that these operators are in fact isometries.

Proposition 3.2.2. *The operators $V_\alpha, \widetilde{V}_\alpha : L^2(\mathbb{D}, \mu_\alpha) \to L^2(\mathbb{D}, \mu_\alpha)$ are isometries.*

Proof. We will show that these unilateral shifts preserve the norm.

Let $f \in L^2(\mathbb{D}, \mu_\alpha)$. Since the disk polynomials are an orthonormal basis for this space, we get that there exists a sequence $\{\lambda_{k,l}\}_{k,l\in\mathbb{Z}_+}$ such that:

$$f = \sum_{k,l\in\mathbb{Z}_+} \lambda_{k,l} b^\alpha_{k,l}.$$

By Parseval's Identity, the norm of this function satisfies:

$$\|f\|^2_{\alpha,2} = \sum_{k,l\in\mathbb{Z}_+} |\lambda_{k,l}|^2.$$

Then, if we apply the operator V_α to this function f:

$$V_\alpha(f) = V_\alpha\left(\sum_{k,l\in\mathbb{Z}_+} \lambda_{k,l} b^\alpha_{k,l}\right) = \sum_{k,l\in\mathbb{Z}_+} \lambda_{k,l} V_\alpha(b^\alpha_{k,l}) = \sum_{k,l\in\mathbb{Z}_+} \lambda_{k,l} b^\alpha_{k,l+1}$$

Now, if we define a new sequence of complex numbers $\{\widetilde{\lambda}_{k,l}\}_{k,l\in\mathbb{Z}_+}$ as follows:

$$\widetilde{\lambda}_{k,0} = 0,$$

$$\widetilde{\lambda}_{k,l} = \lambda_{k,l-1}$$

for all $k, l \in \mathbb{Z}_+$, with $l > 0$.
Then, we get that:

$$V_\alpha(f) = \sum_{k,l\in\mathbb{Z}_+} \widetilde{\lambda}_{k,l} b^\alpha_{k,l}.$$

So, this implies that:

$$\|V_\alpha(f)\|^2_{\alpha,2} = \sum_{k,l\in\mathbb{Z}_+} |\widetilde{\lambda}_{k,l}|^2.$$

But, given the way the new sequence of scalars was defined, we have:

$$\sum_{k,l\in\mathbb{Z}_+} |\widetilde{\lambda}_{k,l}|^2 = \sum_{k,l\in\mathbb{Z}_+} |\lambda_{k,l}|^2.$$

Therefore, $\|V_\alpha(f)\|_{\alpha,2} = \|f\|_{\alpha,2}$, as we wanted. As f was arbitrary, the result holds for all elements of $L^2(\mathbb{D}, \mu_\alpha)$, so V_α preserves the norm.

The proof for $\widetilde{V}_\alpha$ is similar. □

Proposition 3.2.3. *The following characterization holds for the adjoint of these operators, $V^*_\alpha, \widetilde{V}^*_\alpha : L^2(\mathbb{D}, \mu_\alpha) \to L^2(\mathbb{D}, \mu_\alpha)$:*

$$V^*_\alpha(b^\alpha_{k,l}) = b^\alpha_{k,l-1} \quad \textit{and} \quad \widetilde{V}^*_\alpha(b^\alpha_{k,l}) = b^\alpha_{k-1,l}, \quad \forall k, l \in \mathbb{N}$$

with $V^*_\alpha(b^\alpha_{k,0}) = \widetilde{V}^*_\alpha(b^\alpha_{0,l}) = 0$, for all $k.l \in \mathbb{Z}_+$.

Proof. We appeal to the uniqueness of the adjoint operator. So, we show that these functions as defined above satisfy the defining property of the adjoint operator.

Furthermore, since the inner product is a sesquilinear continuous function, it suffices to verify this property for the basis elements, since their span forms a dense subset of $L^2(\mathbb{D}, \mu_\alpha)$.

For V_α, let $k, l, m, n \in \mathbb{Z}_+$.
If $n > 0$:

$$\begin{aligned}\langle V_\alpha(b^\alpha_{k,l}), b^\alpha_{m,n}\rangle_\alpha &= \langle b^\alpha_{k,l+1}, b^\alpha_{m,n}\rangle_\alpha \\ &= \delta_{k,m}\delta_{l+1,n} = \delta_{k,m}\delta_{l,n-1} \\ &= \langle b^\alpha_{k,l}, b^\alpha_{m,n-1}\rangle_\alpha = \langle b^\alpha_{k,l}, V^*_\alpha(b^\alpha_{m,n})\rangle_\alpha.\end{aligned}$$

If $n = 0$

$$\begin{aligned}\langle V_\alpha(b^\alpha_{k,l}), b^\alpha_{m,0}\rangle_\alpha &= \langle b^\alpha_{k,l+1}, b^\alpha_{m,0}\rangle_\alpha \\ &= \delta_{k,m}\delta_{l+1,0} = 0 \\ &= \langle b^\alpha_{k,l}, 0\rangle_\alpha = \langle b^\alpha_{k,l}, V^*_\alpha(b^\alpha_{m,0})\rangle_\alpha.\end{aligned}$$

So, in any case, we get that:

$$\langle V_\alpha(b^\alpha_{k,l}), b^\alpha_{m,n}\rangle_\alpha = \langle b^\alpha_{k,l}, V^*_\alpha(b^\alpha_{m,n})\rangle_\alpha.$$

An almost identical argument shows the equality for $\widetilde{V}^*_\alpha$. □

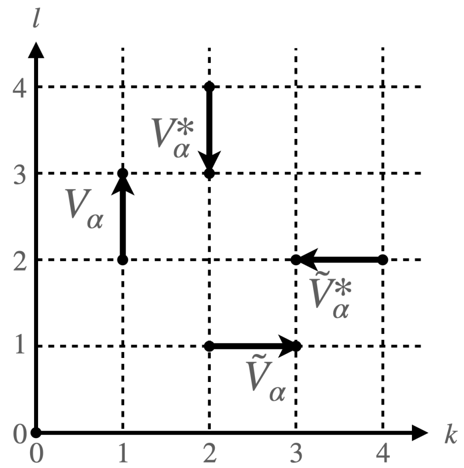

Figure 3.1: Graph of the product $\mathbb{Z}_+^2$
and the action of the operators V_α, $\widetilde{V}_\alpha$, V_α^*, $\widetilde{V}_\alpha^*$.

Remark 3.2.4. Before proceeding further, it is illustrative to identify the disk polynomials $b_{k,l}^\alpha$ with the ordered pair (k, l) from their sub-index to visualize how the operators V_α, $\widetilde{V}_\alpha$ and their adjoints act in the space. This is shown in Figure 3.1 below.

We proceed to show the next lemma, which is needed to prove the theorem that follows after it.

Lemma 3.2.5. *Let $n \in \mathbb{N}$. Then, the operator $(V_\alpha^*)^n\big|_{(\mathcal{A}_n^{\alpha,2})^\perp} : (\mathcal{A}_n^{\alpha,2})^\perp \to L^2(\mathbb{D}, \mu_\alpha)$ is invertible and its inverse is $(V_\alpha)^n : L^2(\mathbb{D}, \mu_\alpha) \to (\mathcal{A}_n^{\alpha,2})^\perp$.*

Proof. First, we elaborate upon the definition of the so-called inverse of the operator we are dealing with.

Initially, we may define the operator $(V_\alpha)^n : L^2(\mathbb{D}, \mu_\alpha) \to L^2(\mathbb{D}, \mu_\alpha)$, but we can restrict its co-domain to its own image, which coincides with $(\mathcal{A}_n^{\alpha,2})^\perp$, since this set has the sequence $\{b_{k,l}^\alpha\}_{k,l\in\mathbb{Z}_+, l\geq n}$ as a basis.

Now, we only check the compositions:

Let $f \in L^2(\mathbb{D}, \mu_\alpha)$. Then, there exists a sequence of complex numbers $\{\lambda_{k,l}\}_{k,l\in\mathbb{Z}_+}$ such that:

$$f = \sum_{k,l\in\mathbb{Z}_+} \lambda_{k,l} b_{k,l}^\alpha.$$

Then, we have the following:

$$\begin{aligned}(V_\alpha^*)^n\big|_{(\mathcal{A}_n^{\alpha,2})^\perp}(V_\alpha)^n(f) &= (V_\alpha^*)^n\big|_{(\mathcal{A}_n^{\alpha,2})^\perp}(V_\alpha)^n\Bigg(\sum_{k,l\in\mathbb{Z}_+}\lambda_{k,l}b_{k,l}^\alpha\Bigg)\\ &= (V_\alpha^*)^n\big|_{(\mathcal{A}_n^{\alpha,2})^\perp}\Bigg(\sum_{k,l\in\mathbb{Z}_+}\lambda_{k,l}b_{k,l+n}^\alpha\Bigg)\\ &= (V_\alpha^*)^n\Bigg(\sum_{k,l\in\mathbb{Z}_+}\lambda_{k,l}b_{k,l+n}^\alpha\Bigg)\\ &= \sum_{k,l\in\mathbb{Z}_+}\lambda_{k,l}b_{k,l}^\alpha = f = id_{L^2(\mathbb{D},\mu_\alpha)}(f).\end{aligned}$$

We proceed similarly for the other composition. Let $g \in (\mathcal{A}_n^{\alpha,2})^\perp$. Then, we get a sequence of complex numbers $\{\gamma_{k,l}\}_{k,l\in\mathbb{Z}_+}$ such that:

$$g = \sum_{k,l\in\mathbb{Z}_+}\gamma_{k,l}b_{k,l+n}^\alpha.$$

Thus:

$$\begin{aligned}(V_\alpha)^n(V_\alpha^*)^n\big|_{(\mathcal{A}_n^{\alpha,2})^\perp}(g) &= (V_\alpha)^n(V_\alpha^*)^n\big|_{(\mathcal{A}_n^{\alpha,2})^\perp}\Bigg(\sum_{k,l\in\mathbb{Z}_+}\gamma_{k,l}b_{k,l+n}^\alpha\Bigg)\\ &= (V_\alpha)^n(V_\alpha^*)^n\Bigg(\sum_{k,l\in\mathbb{Z}_+}\gamma_{k,l}b_{k,l+n}^\alpha\Bigg)\\ &= (V_\alpha)^n\Bigg(\sum_{k,l\in\mathbb{Z}_+}\gamma_{k,l}b_{k,l}^\alpha\Bigg)\\ &= \sum_{k,l\in\mathbb{Z}_+}\gamma_{k,l}b_{k,l+n}^\alpha = g = id_{(\mathcal{A}_n^{\alpha,2})^\perp}(g).\end{aligned}$$

With these results, we can conclude that:

$$(V_\alpha^*)^n\big|_{(\mathcal{A}_n^{\alpha,2})^\perp}(V_\alpha)^n = id_{L^2(\mathbb{D},\mu_\alpha)}$$

and,

$$(V_\alpha)^n(V_\alpha^*)^n\big|_{(\mathcal{A}_n^{\alpha,2})^\perp} = id_{(\mathcal{A}_n^{\alpha,2})^\perp}$$

i.e. the result holds. □

Now we verify the following equalities:

Theorem 3.2.6. *Let $n \in \mathbb{N}$. We have:*

1) $\mathcal{A}_n^{\alpha,2}(\mathbb{D}) = \ker(V_\alpha^*)^n$.

2) $\mathcal{A}_{(n)}^{\alpha,2}(\mathbb{D}) = V_\alpha^{n-1}(\mathcal{A}^{\alpha,2}(\mathbb{D}))$.

Proof. To see the equalities we check the corresponding set inclusions.

For 1:

First, recall that the space $\mathcal{A}_n^{\alpha,2}(\mathbb{D})$ has the sequence $\{b_{k,l}^{\alpha}\}_{k\in\mathbb{Z}_+,l=0}^{l=n-1}$ as basis.

Then, if we pick an element $f \in \mathcal{A}_n^{\alpha,2}(\mathbb{D})$, we get a sequence $\{\lambda_{k,l}\}_{k\in\mathbb{Z}_+,l=0}^{l=n-1}$ such that:

$$f = \sum_{l=0}^{n-1} \sum_{k\in\mathbb{Z}_+} \lambda_{k,l} b_{k,l}^{\alpha}.$$

Now we compute:

$$\begin{aligned}(V_\alpha^*)^n(f) &= (V_\alpha^*)^n \left(\sum_{l=0}^{n-1} \sum_{k\in\mathbb{Z}_+} \lambda_{k,l} b_{k,l}^{\alpha} \right) \\ &= \sum_{l=0}^{n-1} \sum_{k\in\mathbb{Z}_+} \lambda_{k,l} (V_\alpha^*)^n (b_{k,l}^{\alpha}) = 0.\end{aligned}$$

Where the last equality is given by the fact that all the basis terms in the sum have a second index lower than n, so the operator V_α^* maps them to zero.

This implies that:

$$\mathcal{A}_n^{\alpha,2}(\mathbb{D}) \subset \ker(V_\alpha^*)^n. \tag{3.1}$$

For the other inclusion, first remember that the space $\mathcal{A}_n^{\alpha,2}(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, \mu_\alpha)$, so it satisfies that:

$$L^2(\mathbb{D}, \mu_\alpha) = \mathcal{A}_n^{\alpha,2}(\mathbb{D}) \oplus (\mathcal{A}_n^{\alpha,2}(\mathbb{D}))^{\perp}.$$

So, if we take an element $f \in \ker(V_\alpha^*)^n$, it may be decomposed as a sum $f = g+h$, where g belongs to $\mathcal{A}_n^{\alpha,2}(\mathbb{D})$ and h belongs to its orthogonal complement.

Since $f \in \ker(V_\alpha^*)^n$, we have that $(V_\alpha^*)^n(f) = 0$.

In addition to this, $(V_\alpha^*)^n(f) = (V_\alpha^*)^n(g+h) = (V_\alpha^*)^n(g) + (V_\alpha^*)^n(h)$.

By the inclusion 3.1, we get that $(V_\alpha^*)^n(g) = 0$.

This implies that $(V_\alpha^*)^n(h) = 0$, so $h \in \ker(V_\alpha^*)^n$.

But by Lemma 3.2.5, the operator $(V_\alpha^*)^n$ is invertible when restricted to $(\mathcal{A}_n^{\alpha,2}(\mathbb{D}))^{\perp}$, and in particular, it is injective, so its kernel is trivial. Therefore, $h = 0$.

This in turn implies that $f = g$, and since g was already an element of $\mathcal{A}_n^{\alpha,2}(\mathbb{D})$, f is too.

Hence, we get the other inclusion:

$$\ker(V_\alpha^*)^n \subset \mathcal{A}_n^{\alpha,2}(\mathbb{D}).$$

That is,

$$\ker(V_\alpha^*)^n = \mathcal{A}_n^{\alpha,2}(\mathbb{D}).$$

For 2:

Recall that the sequence $\{b_{k,n-1}^{\alpha}\}_{k\in\mathbb{Z}_+}$ is an orthonormal basis for the space $\mathcal{A}_{(n)}^{\alpha,2}(\mathbb{D})$.

So, if we take an element $f \in \mathcal{A}^{\alpha,2}_{(n)}(\mathbb{D})$, we also get a sequence $\{\lambda_k\}_{k\in\mathbb{Z}_+}$ of complex numbers such that:

$$f = \sum_{k\in\mathbb{Z}_+} \lambda_k b^\alpha_{k,n-1}.$$

Now, let us define a new function g using the same sequence $\{\lambda_k\}_{k\in\mathbb{Z}_+}$ as follows:

$$g = \sum_{k\in\mathbb{Z}_+} \lambda_k b^\alpha_{k,0}.$$

Then, this function g is an element of $\mathcal{A}^{\alpha,2}(\mathbb{D})$ that satisfies:

$$\begin{aligned}(V_\alpha)^{n-1}(g) = (V_\alpha)^{n-1}\left(\sum_{k\in\mathbb{Z}_+} \lambda_k b^\alpha_{k,0}\right) \\ = \sum_{k\in\mathbb{Z}_+} \lambda_k (V_\alpha)^{n-1}(b^\alpha_{k,0}) \\ = \sum_{k\in\mathbb{Z}_+} \lambda_k b^\alpha_{k,n-1} = f.\end{aligned}$$

As such, $f \in (V_\alpha)^n(\mathcal{A}^{\alpha,2})$.
This gives the inclusion:

$$\mathcal{A}^{\alpha,2}_{(n)}(\mathbb{D}) \subset (V_\alpha)^n(\mathcal{A}^{\alpha,2}).$$

Now, consider an element $g \in (V_\alpha)^n(\mathcal{A}^{\alpha,2})$
That means there exists a function $h \in \mathcal{A}^{\alpha,2}(\mathbb{D})$ such that $g = (V_\alpha)^n(h)$.
Since $h \in \mathcal{A}^{\alpha,2}(\mathbb{D})$, we get a sequence of complex numbers $\{\lambda_k\}_{k\in\mathbb{Z}_+}$ such that:

$$h = \sum_{k\in\mathbb{Z}_+} \lambda_k b^\alpha_{k,0}.$$

Then, we get an expression for g:

$$\begin{aligned}g = (V_\alpha)^{n-1}(h) = (V_\alpha)^{n-1}\left(\sum_{k\in\mathbb{Z}_+} \lambda_k b^\alpha_{k,0}\right) \\ = \sum_{k\in\mathbb{Z}_+} \lambda_k (V_\alpha)^{n-1}(b^\alpha_{k,0}) \\ = \sum_{k\in\mathbb{Z}_+} \lambda_k b^\alpha_{k,n-1}.\end{aligned}$$

That is, $g \in \mathcal{A}^{\alpha,2}_{(n)}(\mathbb{D})$.
This gives the other inclusion:

$$(V_\alpha)^n(\mathcal{A}^{\alpha,2}) \subset \mathcal{A}^{\alpha,2}_{(n)}(\mathbb{D}).$$

Therefore,

$$(V_\alpha)^n(\mathcal{A}^{\alpha,2}) = \mathcal{A}^{\alpha,2}_{(n)}(\mathbb{D}).$$

□

And as a consequence, we get a couple of results:

Corollary 3.2.7. *Let $n, k \in \mathbb{N}$. The functions below are isometric isomorphisms:*

- $V_\alpha^k : \mathcal{A}_{(n)}^{\alpha,2}(\mathbb{D}) \to \mathcal{A}_{(n+k)}^{\alpha,2}(\mathbb{D})$
- $(V_\alpha^*)^k : \mathcal{A}_{(n+k)}^{\alpha,2}(\mathbb{D}) \to \mathcal{A}_{(n)}^{\alpha,2}(\mathbb{D})$

They are also inverse to one another.

Corollary 3.2.8. *The isometry V_α is in fact a pure isometry.*

Proof. Given the first equation proved in 3.2.6, we see that the weighted Bergman space $\mathcal{A}^{\alpha,2}(\mathbb{D})$ is actually a wandering subspace for the unilateral shift isometry V_α.

This, coupled with the fact that per Remark 2.3.5, the weighted L^2 space can be actually decomposed as the orthogonal sum of the true Poly-Bergman spaces, that is:

$$L(\mathbb{D}, \mu_\alpha) = \bigoplus_{n\in\mathbb{N}} \mathcal{A}_{(n)}^{\alpha,2}(\mathbb{D}) = \bigoplus_{n\in\mathbb{Z}_+} V_\alpha^n(\mathcal{A}^{\alpha,2}(\mathbb{D}))$$

implies that V_α is a pure isometry. □

With this, according to the theorem 1.3.11, we have an alternative extended Fock Space structure for $L^2(\mathbb{D}, \mu_\alpha)$ to the one presented before, when proposing the operators $\mathfrak{a}$ and $\mathfrak{b}$.

As that theorem specifies, from the existence of the pure isometry V_α in $L^2(\mathbb{D}, \mu_\alpha)$, we may recover the mutually adjoint operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$. However, the action of V_α is tied to the orthonormal basis of normalized disk polynomials. So, if we were to reconstruct the operators in this manner, using the isometry as is, it would also depend on this specific basis.

This is why, in the next section we aim to describe these isometries using differential operators, in order to obtain a basis-independent expression for their action.

Remark 3.2.9. Here we only make the quick comment regarding the fact that one can state and prove similar propositions to Theorem 3.2.6 and Corollaries 3.2.7 and 3.2.8 but considering the spaces of *anti-polyanalytic functions* and in turn using the other unilateral shift, $\widetilde{V}_\alpha$.

Since we concern ourselves only with polyanalytic functions, any more in-depth mention of this result would be out of place.

3.3 Operators on the Weighted L^2 Space

We define a new class of functions in terms of the disk polynomials and a set of different operators that act upon them. We describe some of their properties and show a few equalities regarding them.

Only for this section, we introduce a special subspace of the $L^2(\mathbb{D}, \mu_\alpha)$

Definition 3.3.1. We define the space of *anti-analytic* functions in $L^2(\mathbb{D}, \mu_\alpha)$ as:

$$\widetilde{\mathcal{A}}^{\alpha,2} = \text{clos}(\text{span}(\{b^\alpha_{0,l}\}_{l\in\mathbb{Z}_+}))$$

Remark 3.3.2. We mention that this notion of "anti-analytic" functions is in fact the same as the more commonly used, where one defines them as complex functions that are analytic when composed with the operation of complex conjugation. That is, they are essentially power series of $\overline{z}$. Of course, this also coincides with the class of continually differentiable functions that belong to the kernel of the other Wirtinger derivative $\frac{\partial}{\partial z}$, which is also a popular characterization of these types of functions.

We did not opt to make a more formal definition, or use any of these well-known descriptions of these functions because, as it was stated before, we avoid dealing with anything other than poly-analytic functions explicitly. And in particular, since the set from the previous definition will only serve an ancillary role at most.

Definition 3.3.3. We introduce the *disk function operator* $\Phi : L^2(\mathbb{D}, \mu_\alpha) \to L^2(\mathbb{D}, \mu)$, defined densely on $\mathcal{D}$ and describe their action upon their elements:

$$\Phi(b^\alpha_{k,l}(z)) = (1 - |z|^2)^{\alpha/2}\, b^\alpha_{k,l}(z)\,, \quad \forall z \in \mathbb{D}, k, l \in \mathbb{Z}_+$$

We refer to the images $\Phi(b^\alpha_{k,l})$ as the corresponding *disk function* associated to $b^\alpha_{k,l}$.

The disk function operator is a linear surjective isometry between the spaces $L^2(\mathbb{D}, \mu_\alpha)$ and $L^2(\mathbb{D}, \mu)$. Their inverse is defined and satisfies:

$$\Phi^{-1}(\Phi(b^\alpha_{k,l}(z))) = b^\alpha_{k,l}(z)\,, \quad \forall z \in \mathbb{D}, k, l \in \mathbb{Z}_+$$

That is, the inverse of any disk function, is its own associated disk polynomial.

Corollary 3.3.4. *The sequence $\{\Phi(b^\alpha_{k,l})\}_{k,l\in\mathbb{Z}_+}$ of disk functions is an orthonormal basis for the space $L^2(\mathbb{D}, \mu)$.*

We denote the linear span of this basis of disk functions as $\Phi(\mathcal{D})$, where $\mathcal{D}$ represents the linear span of the disk polynomials. Then, this previous corollary implies that this set is dense in $L^2(\mathbb{D}, \mu)$.

Now, in order to proceed, we need to define some operators. These are borrowed from Dr. Wünsche's article [12, Section 4].

Definition 3.3.5. Let $G^\alpha : \Phi(\mathcal{D}) \to L^2(\mathbb{D}, \mu)$ be densely defined on $\Phi(\mathcal{D})$ as:

$$G^\alpha(f)(z) = \left(-4\,\frac{\partial^2}{\partial z \partial \overline{z}} + 2\left(z\frac{\partial}{\partial z}\,\overline{z}\,\frac{\partial}{\partial \overline{z}} + \frac{\partial}{\partial z} z \frac{\partial}{\partial \overline{z}}\,\overline{z} \right) - 1 + \frac{\alpha^2}{1 - |z|^2} \right) f(z)\,, \quad \forall z \in \mathbb{D}$$

Theorem 3.3.6. *The operator G^α is Hermitian and satisfies the following eigenvalue equation:*

$$G^\alpha(\Phi(b^\alpha_{m,n}))(z) = (2m + \alpha + 1)(2n + \alpha + 1)\Phi(b^\alpha_{m,n})(z)\,, \quad \forall z \in \mathbb{D}, m, n \in \mathbb{Z}_+$$

for all $m, n \in \mathbb{Z}_+$.

Proof. First we prove the equation above. Let $m, n \in \mathbb{Z}_+$.

We start from the differential equation of the Jacobi polynomials.

$$\Big((1-x^2)\frac{\partial^2}{\partial x^2} + (\beta - \alpha - (\alpha+\beta+2)x)\frac{\partial}{\partial x} + n(n+\alpha+\beta+1)\Big)y = 0.$$

We substitute $\beta = m - n$ and get:

$$\Big((1-x^2)\frac{\partial^2}{\partial x^2} + (m-n-\alpha-(\alpha+m-n+2)x)\frac{\partial}{\partial x} + n(\alpha+m+1)\Big)P_n^{(\alpha,m-n)}(x) = 0.$$

Using the change of variables $x = 2r^2 - 1$ yields the following differential equation in polar coordinates:

$$\Big((1-r^2)\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\Big) - 2(1+\alpha)r\frac{\partial}{\partial r} + 4mn + 2(1+\alpha)(m+n)\Big)b^\alpha_{m,n}(re^{i\varphi}) = 0.$$

Then changing to the complex coordinates z and $\overline{z}$, we get:

$$\Big(2\Big((1-z\overline{z})\frac{\partial^2}{\partial z\partial\overline{z}} + \frac{\partial^2}{\partial z\partial\overline{z}}(1-z\overline{z})\Big) - \alpha\Big(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial\overline{z}} + \frac{\partial}{\partial z}z + \frac{\partial}{\partial\overline{z}}\overline{z}\Big) + 4mn + 2(1+\alpha)(m+n+1)\Big)b^\alpha_{m,n}(z) = 0.$$

We rearrange this last equation as follows:

$$\begin{aligned}&2\Big((1-z\overline{z})\frac{\partial^2}{\partial z\partial\overline{z}} + \frac{\partial^2}{\partial z\partial\overline{z}}(1-z\overline{z})\Big)b^\alpha_{m,n}(z)\\ &= \alpha\Big(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial\overline{z}} + \frac{\partial}{\partial z}z + \frac{\partial}{\partial\overline{z}}\overline{z}\Big) + 4mn + 2(1+\alpha)(m+n+1)\Big)b^\alpha_{m,n}(z).\end{aligned} \tag{3.2}$$

Using the definition of the operator G^α, we get:

$$\begin{aligned}&G^\alpha(\Phi(b^\alpha_{m,n}))(z) =\\ &\Big(-4\frac{\partial^2}{\partial z\partial\overline{z}} + 2\Big(z\frac{\partial}{\partial z}\overline{z}\frac{\partial}{\partial\overline{z}} + \frac{\partial}{\partial z}z\frac{\partial}{\partial\overline{z}}\overline{z}\Big) - 1 + \frac{\alpha^2}{1-|z|^2}\Big)\Phi(b^\alpha_{m,n})(z).\end{aligned} \tag{3.3}$$

An arduous yet straight-forward computation shows that this is in fact equal to:

$$\begin{aligned}G^\alpha(\Phi(b^\alpha_{m,n}))(z) &= (1+\alpha)^2\Phi(b^\alpha_{m,n})(z)\\ &+ 2(1-z\overline{z})^{\alpha/2}\Big(-\Big((1-z\overline{z})\frac{\partial^2}{\partial z\partial\overline{z}} + \frac{\partial^2}{\partial z\partial\overline{z}}(1-z\overline{z})\Big) - 1\\ &+ \alpha\Big(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial\overline{z}}\Big)\Big)\Phi(b^\alpha_{m,n})(z).\end{aligned} \tag{3.4}$$

So we substitute the expression from equation 3.2 into our last expression:

$$\begin{aligned}G^\alpha(\Phi(b^\alpha_{m,n}))(z) &= (1+\alpha)^2\Phi(b^\alpha_{m,n})(z)\\ &+ 2(1-z\overline{z})^{\alpha/2}\Big(-\frac{\alpha}{2}\Big(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial\overline{z}} + \frac{\partial}{\partial z}z + \frac{\partial}{\partial\overline{z}}\overline{z}\Big)\\ &+ 2mn + (1+\alpha)(m+n+1) - 1 + \alpha\Big(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial\overline{z}}\Big)\Big)\Phi(b^\alpha_{m,n})(z).\end{aligned} \tag{3.5}$$

Operating further, the terms in equation 3.5 can be reduced as follows:

$$\begin{aligned} &G^\alpha(\Phi(b^\alpha_{m,n}))(z) = (1+\alpha)^2\Phi(b^\alpha_{m,n})(z) \\ &+ \big(4mn + 2(1+\alpha)(m+n+1) - 2(\alpha+1)\big)\Phi(b^\alpha_{m,n}(z) \\ &= \big(1 + 2\alpha + \alpha^2 + 4mn + 2(1+\alpha)(m+n)\big)\Phi(b^\alpha_{m,n}(z) \\ &= (2m+\alpha+1)(2n+\alpha+1)\Phi(b^\alpha_{m,n}(z). \end{aligned} \tag{3.6}$$

So we get the desired equality.

Now we prove the hermiticity of the operator. We denote with $\langle , \rangle$ the standard inner product in $L^2(\mathbb{D}, \mu)$ throughout the next argument.

We verify the Hermitic property only for the basis elements, since the sesquilinearity of the inner product guarantees that it holds true for the whole set $\Phi(\mathcal{D})$.

Then, let $\Phi(b^\alpha_{m,n}), \Phi(b^\alpha_{k,l})$.

$$\begin{aligned} \langle G^\alpha(\Phi(b^\alpha_{m,n})), \Phi(b^\alpha_{k,l})\rangle &= \langle (2m+\alpha+1)(2n+\alpha+1)\Phi(b^\alpha_{m,n}), \Phi(b^\alpha_{k,l})\rangle \\ &= (2m+\alpha+1)(2n+\alpha+1)\langle \Phi(b^\alpha_{m,n}), \Phi(b^\alpha_{k,l})\rangle \\ &= (2m+\alpha+1)(2n+\alpha+1)\,\delta_{m,k}\delta_{n,l} \\ &= (2k+\alpha+1)(2l+\alpha+1)\,\delta_{m,k}\delta_{n,l} \\ &= (2k+\alpha+1)(2l+\alpha+1)\langle \Phi(b^\alpha_{m,n}), \Phi(b^\alpha_{k,l})\rangle \\ &= \langle \Phi(b^\alpha_{m,n}), (2k+\alpha+1)(2l+\alpha+1)\Phi(b^\alpha_{k,l})\rangle \\ &= \langle \Phi(b^\alpha_{m,n}), G^\alpha(\Phi(b^\alpha_{k,l}))\rangle \end{aligned}$$

□

Definition 3.3.7. We define the operators $L, \widetilde{G}^\alpha : \mathcal{D} \to L^2(\mathbb{D}, \mu_\alpha)$ densely defined on $\mathcal{D}$ as:

$$L(f)(z) = \left(z\frac{\partial}{\partial z} - \overline{z}\,\frac{\partial}{\partial\,\overline{z}}\right) f(z)\,, \quad \forall z \in \mathbb{D}$$

and,

$$\widetilde{G}^\alpha(f)(z) = \Phi^{-1}\,G^\alpha\,\Phi(f)(z)\,, \quad \forall z \in \mathbb{D}$$

Proposition 3.3.8. *The operators L and $\widetilde{G}^\alpha$ are Hermitian and satisfy the following eigenvalue equations:*

$$L(b^\alpha_{k,l}) = (k-l)\,b^\alpha_{k,l}$$

and

$$\widetilde{G}^\alpha(b^\alpha_{k,l}) = (\alpha+2k+1)(\alpha+2l+1)\,b^\alpha_{k,l}$$

for all $k, l \in \mathbb{Z}_+$.

Proof. For any Hermitian operator, its conjugation with a surjective linear isometry yields another Hermitian operator, so $\widetilde{G}^\alpha$ is Hermitian.

For L, a change of variables to polar coordinates yields another expression for the operator:

$$L = \frac{1}{i}\frac{\partial}{\partial\theta} = -i\frac{\partial}{\partial\theta}$$

Now, we need to show that L satisfies the inner product equality:

$$\langle L(f), g\rangle_\alpha = \langle f, L(g)\rangle_\alpha\,, \quad \forall f, g \in \mathcal{D}$$

We verify this fact for functions $f, g \in \mathcal{D}$.

In order to use the alternative expression of the operator L, we will make a change of variables to polar coordinates in the integrals of the inner product.

$$\begin{aligned}
\langle L(f), g\rangle_\alpha &= \int_{\mathbb{D}} L(f)\,\overline{g}\,d\mu_\alpha = \int_{\mathbb{D}} L(f)\,\overline{g}\,w_\alpha d\mu \\
&= \int_0^1\int_0^{2\pi} L(f)(re^{i\theta})\,\overline{g}\,(re^{i\theta})w_\alpha(re^{i\theta})rd\theta dr \\
&= \int_0^1 w_\alpha(re^{i\theta})r\int_0^{2\pi} -i\frac{\partial}{\partial\theta}f(re^{i\theta})\,\overline{g}\,(re^{i\theta})d\theta dr. \qquad (3.7)
\end{aligned}$$

The term corresponding to the weight function exits the innermost integral since it does not depend on the angular variable θ.

We will now focus in finding an equivalent representation for the innermost integral using integration by parts.

$$\begin{aligned}
\int_0^{2\pi} -i\frac{\partial}{\partial\theta}f(re^{i\theta})\,\overline{g}\,(re^{i\theta})d\theta &= -i\int_0^{2\pi}\frac{\partial}{\partial\theta}f(re^{i\theta})\,\overline{g}\,(re^{i\theta})d\theta \\
&= -i\left(f\,\overline{g}\,(re^{i\theta})\Big|_{\theta=0}^{\theta=2\pi} - \int_0^{2\pi} f(re^{i\theta})\frac{\partial}{\partial\theta}\,\overline{g}\,(re^{i\theta})d\theta\right) \\
&= i\int_0^{2\pi} f(re^{i\theta})\frac{\partial}{\partial\theta}\,\overline{g}\,(re^{i\theta})d\theta \\
&= \int_0^{2\pi} f(re^{i\theta})\,\overline{\left(-i\frac{\partial}{\partial\theta}g(re^{i\theta})\right)}\,d\theta
\end{aligned}$$

Now we return, and substitute it in equation 3.7:

$$\begin{aligned}
\langle L(f), g\rangle_\alpha &= \int_0^1 w_\alpha(re^{i\theta})r\int_0^{2\pi} f(re^{i\theta})\,\overline{\left(-i\frac{\partial}{\partial\theta}g(re^{i\theta})\right)}\,d\theta dr \\
&= \int_0^1\int_0^{2\pi} f(re^{i\theta})\,\overline{\left(-i\frac{\partial}{\partial\theta}g(re^{i\theta})\right)}\,w_\alpha(re^{i\theta})rd\theta dr \\
&= \int_0^1\int_0^{2\pi} f(re^{i\theta})\,\overline{L(g)}\,(re^{i\theta})w_\alpha(re^{i\theta})rd\theta dr \\
&= \int_{\mathbb{D}} f\,\overline{L}\,(g)w_\alpha d\mu = \int_{\mathbb{D}} f\,\overline{L}\,(g)d\mu_\alpha = \langle f, L(g)\rangle_\alpha
\end{aligned}$$

We also use the polar coordinates expression of L to check the equation. So, we express everything in polar coordinates:

$$\begin{aligned}
L(b^{\alpha}_{k,l})(re^{i\theta}) &= \frac{1}{i}\frac{\partial}{\partial\theta}(b^{\alpha}_{k,l})(re^{i\theta}) \\
&= \frac{1}{i}\frac{\partial}{\partial\theta}\big(e^{i(k-l)\theta}\, b^{\alpha}_{k,l}(r)\big) \\
&= \frac{1}{i}\, b^{\alpha}_{k,l}(r)\, \frac{\partial}{\partial\theta}\big(e^{i(k-l)\theta}\big) \\
&= \frac{1}{i}\, b^{\alpha}_{k,l}(r)\, \big(i(k-l)\big)e^{i(k-l)\theta} \\
&= (k-l)\, e^{i(k-l)\theta} b^{\alpha}_{k,l}(r) \\
&= (k-l)\, b^{\alpha}_{k,l}(re^{i\theta}) = (k-l) b^{\alpha}_{k,l}(re^{i\theta})
\end{aligned}$$

For the operator $\widetilde{G}^{\alpha}$ we have:

$$\begin{aligned}
G^{\alpha}(b^{\alpha}_{k,l}) &= \widetilde{G}^{\alpha}(f)(z) = \Phi^{-1}\, G^{\alpha}\, \Phi(b^{\alpha}_{k,l}) \\
&= \Phi^{-1}(G^{\alpha}(\Phi(b^{\alpha}_{k,l}))) \\
&= \Phi^{-1}((2k+\alpha+1)(2l+\alpha+1)\Phi(b^{\alpha}_{k,l})) \\
&= (2k+\alpha+1)(2l+\alpha+1)\Phi^{-1}(\Phi(b^{\alpha}_{k,l})) \\
&= (2k+\alpha+1)(2l+\alpha+1)\, b^{\alpha}_{k,l}
\end{aligned}$$

□

Definition 3.3.9. We define densely a new operator in terms of L and $\widetilde{G}^{\alpha}$, $H^{\alpha} : \mathcal{D} \to L^2(\mathbb{D}, \mu_{\alpha})$ as:

$$H^{\alpha} = L^2 + \widetilde{G}^{\alpha}$$

Right away we get that:

Lemma 3.3.10. *The operator H^{α} satisfies the next equation:*

$$H^{\alpha}(b^{\alpha}_{k,l}) = (\alpha + k + l + 1)^2 b^{\alpha}_{k,l}\,, \quad \forall k, l \in \mathbb{Z}_+$$

That is, the disk polynomials are eigenfunctions for this operator with the corresponding eigenvalues.

Proof. We only need a simple computation:

$$\begin{aligned}
H^{\alpha}(b^{\alpha}_{k,l}) &= L^2(b^{\alpha}_{k,l}) + \widetilde{G}^{\alpha}(b^{\alpha}_{k,l}) = (k-l)^2 b^{\alpha}_{k,l} + (2k+\alpha+1)(2l+\alpha+1) b^{\alpha}_{k,l} \\
&= (k^2 - 2kl + l^2 + 4kl + 2k\alpha + 2k + 2l\alpha + \alpha^2 + \alpha + 2l + \alpha + 1) b^{\alpha}_{k,l} \\
&= (\alpha^2 + k^2 + l^2 + 1 + 2k\alpha + 2l\alpha + 2\alpha + 2kl + 2k + 2l) b^{\alpha}_{k,l} \\
&= (\alpha + k + l + 1)^2 b^{\alpha}_{k,l}
\end{aligned}$$

□

Proposition 3.3.11. *The square and fourth root of the operator H^α, $(H^\alpha)^{1/2}, (H^\alpha)^{1/4} : \mathcal{D} \to L^2(\mathbb{D}, \mu_\alpha)$ are well-defined, Hermitian operators, and satisfy the following eigenvalue equations:*

$$(H^\alpha)^{1/2}(b^\alpha_{k,l}) = (\alpha + k + l + 1) b^\alpha_{k,l}\,, \quad \forall k, l \in \mathbb{Z}_+$$

$$(H^\alpha)^{1/4}(b^\alpha_{k,l}) = \sqrt{\alpha + k + l + 1}\, b^\alpha_{k,l}\,, \quad \forall k, l \in \mathbb{Z}_+$$

Proof. Since all eigenvalues of the operator H^α are positive, all of its powers are well-defined, and as powers of a Hermitian operator, they're also Hermitian.

The equations are a direct result of the previous lemma. □

Next we take these operators and combine them.

Definition 3.3.12. Let $L^\alpha_{(1)}, L^\alpha_{(2)} : \mathcal{D} \to L^2(\mathbb{D}, \mu_\alpha)$ be defined densely as:

$$L^\alpha_{(1)}(f)(z) = \frac{1}{2}\Big((H^\alpha)^{1/2} + L - (\alpha + 1)\Big) f(z)\,, \forall z \in \mathbb{D}$$

and,

$$L^\alpha_{(2)}(f)(z) = \frac{1}{2}\Big((H^\alpha)^{1/2} - L - (\alpha + 1)\Big) f(z)\,, \forall z \in \mathbb{D}.$$

Closely related to them, we define $\widetilde{L}^\alpha_{(1)}, \widetilde{L}^\alpha_{(2)} : L^2(\mathbb{D}, \mu_\alpha) \to L^2(\mathbb{D}, \mu_\alpha)$ densely on $\mathcal{D}$ as:

$$\widetilde{L}^\alpha_{(1)}(f)(z) = \frac{1}{2}\Big((H^\alpha)^{1/2} + L + \alpha - 1\Big) f(z)\,, \forall z \in \mathbb{D}.$$

and,

$$\widetilde{L}^\alpha_{(2)}(f)(z) = \frac{1}{2}\Big((H^\alpha)^{1/2} - L + \alpha - 1\Big) f(z)\,, \forall z \in \mathbb{D}$$

These operators satisfy:

$$\widetilde{L}^\alpha_{(1)} = L^\alpha_{(1)} + \alpha I$$

and,

$$\widetilde{L}^\alpha_{(2)} = L^\alpha_{(2)} + \alpha I.$$

Proposition 3.3.13. *The operators $L^\alpha_{(1)}$ and $L^\alpha_{(2)}$ are Hermitian and satisfy the following eigenvalue equations:*

$$L^\alpha_{(1)}(b^\alpha_{k,l}) = k b^\alpha_{k,l}$$

and,

$$L^\alpha_{(2)}(b^\alpha_{k,l}) = l b^\alpha_{k,l},$$

for all $k, l \in \mathbb{Z}_+$.
Similarly, the operators $\widetilde{L}^{\alpha}_{(1)}$ and $\widetilde{L}^{\alpha}_{(2)}$ are also Hermitian and satisfy the equations:

$$\widetilde{L}^{\alpha}_{(1)}(b^{\alpha}_{k,l}) = (k+\alpha)b^{\alpha}_{k,l}$$

and,

$$\widetilde{L}^{\alpha}_{(2)}(b^{\alpha}_{k,l}) = (l+\alpha)b^{\alpha}_{k,l}.$$

Proof. As before, these operators are Hermitian as they are linear combination of Hermitian operators.

For the equations, we compute:

$$\begin{aligned} L\alpha_{(1)}(b^{\alpha}_{k,l}) &= \frac{1}{2}\Big((H^{\alpha})^{1/2}(b^{\alpha}_{k,l}) + L(b^{\alpha}_{k,l}) - (\alpha+1)b^{\alpha}_{k,l}\Big) \\ &= \frac{1}{2}\Big((\alpha+k+l+1)b^{\alpha}_{k,l} + (k-l)b^{\alpha}_{k,l} - (\alpha+1)\Big) \\ &= \frac{1}{2}(2k)b^{\alpha}_{k,l} = kb^{\alpha}_{k,l} \end{aligned}$$

And:

$$\begin{aligned} L\alpha_{(2)}(b^{\alpha}_{k,l}) &= \frac{1}{2}\Big((H^{\alpha})^{1/2}(b^{\alpha}_{k,l}) - L(b^{\alpha}_{k,l}) - (\alpha+1)b^{\alpha}_{k,l}\Big) \\ &= \frac{1}{2}\Big((\alpha+k+l+1)b^{\alpha}_{k,l} - (k-l)b^{\alpha}_{k,l} - (\alpha+1)\Big) \\ &= \frac{1}{2}(2l)b^{\alpha}_{k,l} = lb^{\alpha}_{k,l} \end{aligned}$$

□

Proposition 3.3.14. *The operators $L^{\alpha}_{(1)}\big|_{(\widetilde{\mathcal{A}}^{\alpha,2})^{\perp}}, \widetilde{L}^{\alpha}_{(1)}\big|_{(\widetilde{\mathcal{A}}^{\alpha,2})^{\perp}} : (\widetilde{\mathcal{A}}^{\alpha,2})^{\perp} \to (\widetilde{\mathcal{A}}^{\alpha,2})^{\perp}$ and $L^{\alpha}_{(2)}\big|_{(\mathcal{A}^{\alpha,2})^{\perp}}, \widetilde{L}^{\alpha}_{(2)}\big|_{(\mathcal{A}^{\alpha,2})^{\perp}} : (\mathcal{A}^{\alpha,2})^{\perp} \to (\mathcal{A}^{\alpha,2})^{\perp}$ are diagonal with respect to their corresponding domains. They have all their powers defined.*

Proof. The first pair of operators are defined on the set $(\widetilde{\mathcal{A}}^{\alpha,2})^{\perp}$, which is spanned by the sequence $\{b^{\alpha}_{k,l}\}_{k,l\in\mathbb{Z}_+,k>0}$. On this set, all elements of the sequence satisfy an eigenvalue equation as per Proposition 3.3.13.

Similarly for the second pair of operators, considering the set $(\mathcal{A}^{\alpha,2})^{\perp}$ spanned by the sequence $\{b^{\alpha}_{k,l}\}_{k,l\in\mathbb{Z}_+,l>0}$.

As these diagonal operators have strictly positive eigenvalues, all their powers are well defined. So, in particular, their inverses are, too. □

Lemma 3.3.15. *The operators $L^{\alpha}_{(1)}$ and $L^{\alpha}_{(2)}$ satisfy the following:*

$$\ker(L^{\alpha}_{(1)}) = \widetilde{\mathcal{A}}^{\alpha,2}$$
$$\ker(L^{\alpha}_{(2)}) = \mathcal{A}^{\alpha,2}$$

Proof. Notice the fact that $\widetilde{\mathcal{A}}^{\alpha,2}$ is spanned by the sequence $\{b^{\alpha}_{0,l}\}_{l\in\mathbb{Z}_+}$, so by Proposition 3.3.13, we get:

$$\widetilde{\mathcal{A}}^{\alpha,2} \subset \ker(L^{\alpha}_{(1)}). \tag{3.8}$$

Now, since $\widetilde{\mathcal{A}}^{\alpha,2}$ is a closed subspace of $L^2(\mathbb{D},\mu_\alpha)$, a Hilbert space, it holds that:

$$L^2(\mathbb{D},\mu_\alpha) = \widetilde{\mathcal{A}}^{\alpha,2} \oplus (\widetilde{\mathcal{A}}^{\alpha,2})^{\perp}$$

That is, the space $L^2(\mathbb{D},\mu_\alpha)$ is a direct sum of $\widetilde{\mathcal{A}}^{\alpha,2}$ and its orthogonal complement.

So, if we take an element $f \in \ker(L^{\alpha}_{(1)})$, it may be decomposed as a sum $f = g+h$, where g belongs to $\widetilde{\mathcal{A}}^{\alpha,2}$ and h to its orthogonal complement.

As $f \in \ker(L^{\alpha}_{(1)})$, $L^{\alpha}_{(1)}(f) = 0$. This in turn implies that:

$$L^{\alpha}_{(1)}(f) = L^{\alpha}_{(1)}(g+h) = L^{\alpha}_{(1)}(g) + L^{\alpha}_{(1)}(h) = L^{\alpha}_{(1)}(h) = 0$$

where the third equality holds because of the set inclusion 3.8.

As $h \in (\widetilde{\mathcal{A}}^{\alpha,2})^{\perp}$, we may use the fact that this operator is invertible in this domain, so we get that $h = 0$.

That is, $f = g$, and since g belonged to $\widetilde{\mathcal{A}}^{\alpha,2}$, f does too.

Granted that this element $f \in \ker(L^{\alpha}_{(1)})$, was arbitrary, we get that:

$$\ker(L^{\alpha}_{(1)}) \subset \widetilde{\mathcal{A}}^{\alpha,2}$$

thereby concluding that

$$\ker(L^{\alpha}_{(1)}) = \widetilde{\mathcal{A}}^{\alpha,2}$$

An almost identical argument shows the other equality, using the same representation of elements and the invertibility of the corresponding operator.

□

We define another set of operators.

Definition 3.3.16. Let $K^{\alpha}_{(1^+)}, K^{\alpha}_{(1^-)}, K^{\alpha}_{(2^+)}, K^{\alpha}_{(2^-)} : \mathcal{D} \to L^2(\mathbb{D},\mu_\alpha)$ be defined densely as:

$$K^{\alpha}_{(1^+)}(f)(z) := \left(z(\widetilde{L}^{\alpha}_1 + 1) - (1 - z\,\overline{z})\frac{\partial}{\partial\,\overline{z}} \right) f(z)\,, \quad \forall z \in \mathbb{D}.$$

$$K^{\alpha}_{(1^-)}(f)(z) := \left(\overline{z}\,L^{\alpha}_1 + (1 - z\,\overline{z})\frac{\partial}{\partial z} \right) f(z)\,, \quad \forall z \in \mathbb{D}.$$

$$K^{\alpha}_{(2^+)}(f)(z) := \left(\overline{z}\,(\widetilde{L}^{\alpha}_2 + 1) - (1 - z\,\overline{z})\frac{\partial}{\partial z} \right) f(z)\,, \quad \forall z \in \mathbb{D}.$$

$$K^{\alpha}_{(2^-)}(f)(z) := \left(zL^{\alpha}_2 + (1 - z\,\overline{z})\frac{\partial}{\partial\,\overline{z}} \right) f(z)\,, \quad \forall z \in \mathbb{D}.$$

Proposition 3.3.17. *The operators $K^{\alpha}_{(1^+)}, K^{\alpha}_{(1^-)}, K^{\alpha}_{(2^+)}, K^{\alpha}_{(2^-)}$ satisfy the following equations:*

$$K^{\alpha}_{(1^+)}(P^{\alpha}_{k,l}) = (k+1+\alpha)P^{\alpha}_{k+1,l},$$

$$K^{\alpha}_{(1^-)}(P^{\alpha}_{k,l}) = kP^{\alpha}_{k-1,l},$$

$$K^{\alpha}_{(2^+)}(P^{\alpha}_{k,l}) = (l+1+\alpha)P^{\alpha}_{k,l+1},$$

$$K^{\alpha}_{(2^-)}(P^{\alpha}_{k,l}) = lP^{\alpha}_{k,l-1},$$

for all $k, l \in \mathbb{Z}_+$.

Proof. This time around we will be using the non-normalized disk polynomials. We make use of the recurrence relations for the disk polynomials from Theorem 2.1.10:

$$\begin{aligned}
K^{\alpha}_{(1^+)}(P^{\alpha}_{k,l})(z,\overline{z}) &= \left(z(\widetilde{L}^{\alpha}_1+1) - (1-z\,\overline{z})\frac{\partial}{\partial\,\overline{z}}\right)P^{\alpha}_{k,l}(z,\overline{z}) \\
&= z(\widetilde{L}^{\alpha}_1+1)(P^{\alpha}_{k,l})(z,\overline{z}) - (1-z\,\overline{z})\frac{\partial}{\partial\,\overline{z}}(P^{\alpha}_{k,l})(z,\overline{z}) \\
&= z(k+\alpha+1)(P^{\alpha}_{k,l})(z,\overline{z}) - (1-z\,\overline{z})\frac{\partial}{\partial\,\overline{z}}(P^{\alpha}_{k,l})(z,\overline{z}) \\
&= \big((k+\alpha+1)P^{\alpha}_{k+1,l}(z,\overline{z}) + lP^{\alpha}_{k,l-1}(z,\overline{z}) - lzP^{\alpha}_{k,l}(z,\overline{z})\big) \\
&+ \frac{(k+\alpha+1)l}{k+l+\alpha+1}\left(P^{\alpha}_{k+1,l}(z,\overline{z}) - P^{\alpha}_{k,l-1}(z,\overline{z})\right) \\
&= (k+\alpha+1)P^{\alpha}_{k+1,l}(z,\overline{z}) \\
&+ \left(lP^{\alpha}_{k,l-1}(z,\overline{z}) - lzP^{\alpha}_{k,l}(z,\overline{z}) + \frac{(k+\alpha+1)l}{k+l+\alpha+1}\left(P^{\alpha}_{k+1,l}(z,\overline{z}) - P^{\alpha}_{k,l-1}(z,\overline{z})\right)\right).
\end{aligned}$$

Next, we focus on the term in brackets, and verify that it is equal to zero:

$$\begin{aligned}
&lP^{\alpha}_{k,l-1}(z,\overline{z}) - lzP^{\alpha}_{k,l}(z,\overline{z}) + \frac{(k+\alpha+1)l}{k+l+\alpha+1}\left(P^{\alpha}_{k+1,l}(z,\overline{z}) - P^{\alpha}_{k,l-1}(z,\overline{z})\right) = \\
&\frac{(k+\alpha+1)l}{k+l+\alpha+1}P^{\alpha}_{k+1,l}(z,\overline{z}) + lP^{\alpha}_{k,l-1}(z,\overline{z})\left(1 - \frac{k+\alpha+1}{k+l+\alpha+1}\right) - lzP^{\alpha}_{k,l}(z,\overline{z}) = \\
&lzP^{\alpha}_{k,l}(z,\overline{z}) - \frac{l^2}{k+l+\alpha+1}P^{\alpha}_{k,l}(z,\overline{z}) + \frac{l^2}{k+l+\alpha+1}P^{\alpha}_{k,l}(z,\overline{z}) - lzP^{\alpha}_{k,l}(z,\overline{z}) = 0.
\end{aligned}$$

So, as was stated:

$$K^{\alpha}_{(1^+)}(P^{\alpha}_{k,l}(z,\overline{z})) = (k+1+\alpha)P^{\alpha}_{k+1,l}(z,\overline{z}), \quad \forall z \in \mathbb{D}.$$

For the second equality:

$$\begin{aligned}
K^{\alpha}_{(1^-)}(P^{\alpha}_{k,l})(z,\overline{z}) &= \left(\overline{z}\,L^{\alpha}_1 + (1-z\overline{z})\frac{\partial}{\partial z}\right)P^{\alpha}_{k,l}(z,\overline{z}) \\
&= \overline{z}\,L^{\alpha}_1(P^{\alpha}_{k,l})(z,\overline{z}) + (1-z\overline{z})\frac{\partial}{\partial z}(P^{\alpha}_{k,l})(z,\overline{z}) \\
&= \overline{z}\,k(P^{\alpha}_{k,l})(z,\overline{z}) + (1-z\overline{z})\frac{\partial}{\partial z}(P^{\alpha}_{k,l})(z,\overline{z}) \\
&= \big((l+\alpha+1)P^{\alpha}_{k,l+1}(z,\overline{z}) + kP^{\alpha}_{k-1,l}(z,\overline{z}) - (l+\alpha+1)\,\overline{z}\,P^{\alpha}_{k,l}(z,\overline{z})\big) \\
&+ \frac{(l+\alpha+1)k}{k+l+\alpha+1}\,(P^{\alpha}_{k-1,l}(z,\overline{z}) - P^{\alpha}_{k,l+1}(z,\overline{z})) \\
&= kb^{\alpha}_{k-1,l}(z) \\
&+ \left((l+\alpha+1)P^{\alpha}_{k,l+1}(z,\overline{z}) - (l+\alpha+1)\,\overline{z}\,P^{\alpha}_{k,l}(z,\overline{z}) + \frac{(l+\alpha+1)k}{k+l+\alpha+1}\,(P^{\alpha}_{k-1,l}(z,\overline{z}) - P^{\alpha}_{k,l+1}(z,\overline{z}))\right).
\end{aligned}$$

Just as before, we show that the term in brackets equals zero:

$$\begin{gathered}
(l+\alpha+1)P^{\alpha}_{k,l+1}(z,\overline{z}) - (l+\alpha+1)\,\overline{z}\,P^{\alpha}_{k,l}(z,\overline{z}) + \frac{(l+\alpha+1)k}{k+l+\alpha+1}\,(P^{\alpha}_{k-1,l}(z,\overline{z}) - P^{\alpha}_{k,l+1}(z,\overline{z})) = \\
\frac{(l+\alpha+1)k}{k+l+\alpha+1}\,P^{\alpha}_{k-1,l}(z,\overline{z}) + (l+\alpha+1)P^{\alpha}_{k,l+1}(z,\overline{z})\left(1-\frac{k}{k+l+\alpha+1}\right) - (l+\alpha+1)\,\overline{z}\,P^{\alpha}_{k,l}(z,\overline{z}) = \\
(l+\alpha+1)\,\overline{z}\,P^{\alpha}_{k,l}(z,\overline{z}) - \frac{(l+\alpha+1)^2}{k+l+\alpha+1}P^{\alpha}_{k,l+1}(z,\overline{z}) + \frac{(l+\alpha+1)^2}{k+l+\alpha+1}P^{\alpha}_{k,l+1}(z,\overline{z}) - (l+\alpha+1)\,\overline{z}\,P^{\alpha}_{k,l}(z,\overline{z}) = 0.
\end{gathered}$$

And thus:

$$K^{\alpha}_{(1^-)}(P^{\alpha}_{k,l}(z,\overline{z})) = kP^{\alpha}_{k-1,l}(z,\overline{z}), \quad \forall z \in \mathbb{D}.$$

Proceeding almost identically, using the corresponding recurrence relations for the disk polynomials yields the other two equations. □

Then, finally, to shift from the non-normalized to the normalized disk polynomials, we define another set of operators, closely related to the previous ones.

Definition 3.3.18. Let $\widetilde{K}^{\alpha}_{(1^+)}, \widetilde{K}^{\alpha}_{(1^-)}, \widetilde{K}^{\alpha}_{(2^+)}, \widetilde{K}^{\alpha}_{(2^-)} : \mathcal{D} \to L^2(\mathbb{D}, \mu_\alpha)$ be defined densely as:

$$\widetilde{K}^{\alpha}_{(1^+)}(f)(z) := H^{\alpha/4}K^{\alpha}_{(1^+)}H^{-\alpha/4}(f)(z), \quad \forall z \in \mathbb{D}.$$

$$\widetilde{K}^{\alpha}_{(1^-)}(f)(z) := H^{\alpha/4}K^{\alpha}_{(1^-)}H^{-\alpha/4}(f)(z), \quad \forall z \in \mathbb{D}.$$

$$\widetilde{K}^{\alpha}_{(2^+)}(f)(z) := H^{\alpha/4}K^{\alpha}_{(2^+)}H^{-\alpha/4}(f)(z), \quad \forall z \in \mathbb{D}.$$

$$\widetilde{K}^{\alpha}_{(2^-)}(f)(z) := H^{\alpha/4}K^{\alpha}_{(2^-)}H^{-\alpha/4}(f)(z), \quad \forall z \in \mathbb{D}.$$

Proposition 3.3.19. *The operators $\widetilde{K}^{\alpha}_{(1^+)}, \widetilde{K}^{\alpha}_{(1^-)}, \widetilde{K}^{\alpha}_{(2^+)}, \widetilde{K}^{\alpha}_{(2^-)}$ satisfy the following equations:*

$$\widetilde{K}^{\alpha}_{(1^+)}(b^{\alpha}_{k,l}) = \sqrt{(k+1)(k+1+\alpha)}\,b^{\alpha}_{k+1,l},$$

$$\widetilde{K}^{\alpha}_{(1^-)}(b^{\alpha}_{k,l}) = \sqrt{k(k+\alpha)}\, b^{\alpha}_{k-1,l},$$

$$\widetilde{K}^{\alpha}_{(2^+)}(b^{\alpha}_{k,l}) = \sqrt{(l+1)(l+1+\alpha)}\, b^{\alpha}_{k,l+1},$$

$$\widetilde{K}^{\alpha}_{(2^-)}(b^{\alpha}_{k,l}) = \sqrt{l(l+\alpha)}\, b^{\alpha}_{k,l-1},$$

for all $k, l \in \mathbb{Z}_+$.

Proof. To prove these formulas we recall the expresion for the norm of the disk polynomials from Definition 2.2.3:

$$\|P^{\alpha}_{k,l}\|_{2,\alpha} = \sqrt{\frac{(\alpha+1)k!l!\Gamma(\alpha+1)^2}{(k+l+\alpha+1)\Gamma(k+\alpha+1)\Gamma(l+\alpha+1)}}.$$

Let $k, l \in \mathbb{Z}_+$. We verify the first equality:

$$\begin{aligned}
&\widetilde{K}^{\alpha}_{(1^+)}(b^{\alpha}_{k,l}) = H^{\alpha/4}K^{\alpha}_{(1^+)}H^{-\alpha/4}(b^{\alpha}_{k,l})\\
&= H^{\alpha/4}K^{\alpha}_{(1^+)}\left(\frac{1}{\sqrt{k+l+1+\alpha}}\, b^{\alpha}_{k,l}\right)\\
&= H^{\alpha/4}K^{\alpha}_{(1^+)}\left(\frac{1}{\sqrt{k+l+1+\alpha}}\,\frac{1}{\|P^{\alpha}_{k,l}\|_{2,\alpha}}\, P^{\alpha}_{k,l}\right)\\
&= H^{\alpha/4}\left(\frac{1}{\sqrt{k+l+1+\alpha}}\,\frac{1}{\|P^{\alpha}_{k,l}\|_{2,\alpha}}\, K^{\alpha}_{(1^+)}(P^{\alpha}_{k,l})\right)\\
&= H^{\alpha/4}\left(\frac{1}{\sqrt{k+l+1+\alpha}}\,\frac{1}{\|P^{\alpha}_{k,l}\|_{2,\alpha}}\, (k+1+\alpha)\, P^{\alpha}_{k+1,l}\right)\\
&= \frac{k+1+\alpha}{\sqrt{k+l+1+\alpha}}\,\frac{\|P^{\alpha}_{k+1,l}\|_{2,\alpha}}{\|P^{\alpha}_{k,l}\|_{2,\alpha}} H^{\alpha/4}(b^{\alpha}_{k+1,l})\\
&= \sqrt{\frac{(k+1)(k+1+\alpha)}{(k+l+2+\alpha)}}\,\sqrt{k+l+2+\alpha}\, b^{\alpha}_{k+1,l}\\
&= \sqrt{(k+1)(k+1+\alpha)}\, b^{\alpha}_{k+1,l}.
\end{aligned}$$

For the second one:

$$\begin{aligned}
\widetilde{K}^{\alpha}_{(1^-)}(b^{\alpha}_{k,l}) &= H^{\alpha/4} K^{\alpha}_{(1^-)} H^{-\alpha/4}(b^{\alpha}_{k,l}) \\
&= H^{\alpha/4} K^{\alpha}_{(1^-)}\left(\frac{1}{\sqrt{k+l+1+\alpha}}\, b^{\alpha}_{k,l}\right) \\
&= H^{\alpha/4} K^{\alpha}_{(1^-)}\left(\frac{1}{\sqrt{k+l+1+\alpha}}\, \frac{1}{\|P^{\alpha}_{k,l}\|_{2,\alpha}}\, P^{\alpha}_{k,l}\right) \\
&= H^{\alpha/4}\left(\frac{1}{\sqrt{k+l+1+\alpha}}\, \frac{1}{\|P^{\alpha}_{k,l}\|_{2,\alpha}}\, K^{\alpha}_{(1^-)}(P^{\alpha}_{k,l})\right) \\
&= H^{\alpha/4}\left(\frac{1}{\sqrt{k+l+1+\alpha}}\, \frac{1}{\|P^{\alpha}_{k,l}\|_{2,\alpha}}\, k\, P^{\alpha}_{k-1,l}\right) \\
&= \frac{k}{\sqrt{k+l+1+\alpha}}\, \frac{\|P^{\alpha}_{k-1,l}\|_{2,\alpha}}{\|P^{\alpha}_{k,l}\|_{2,\alpha}} H^{\alpha/4}(b^{\alpha}_{k-1,l}) \\
&= \sqrt{\frac{k(k+\alpha)}{(k+l+\alpha)}}\, \sqrt{k+l+\alpha}\, b^{\alpha}_{k-1,l} \\
&= \sqrt{k(k+\alpha)}\, b^{\alpha}_{k-1,l}.
\end{aligned}$$

The other two equalities can be shown using very similar arguments. □

Theorem 3.3.20. *The following representations for the shift isometries hold*

$$V = (L^{\alpha}_{(2)})^{-1/2}\, (\widetilde{L}^{\alpha}_{(2)})^{-1/2}\, \widetilde{K}^{\alpha}_{(2^+)}$$

$$V^* = \begin{cases} \widetilde{K}^{\alpha}_{2^-}\, (\widetilde{L}^{\alpha}_{(2)})^{-1/2}\, (L^{\alpha}_{(2)})^{-1/2} & \text{on } (\mathcal{A}^{\alpha,2}_0)^{\perp} \\ 0 & \text{on } \mathcal{A}^{\alpha,2}_0 \end{cases}$$

$$\widetilde{V} = (L^{\alpha}_{(1)})^{-1/2}\, (\widetilde{L}^{\alpha}_{(1)})^{-1/2}\, \widetilde{K}^{\alpha}_{(1^+)}$$

$$(\widetilde{V})^* = \begin{cases} \widetilde{K}^{\alpha}_{1^-}\, (\widetilde{L}^{\alpha}_{(1)})^{-1/2}\, (L^{\alpha}_{(1)})^{-1/2} & \text{on } (\widetilde{\mathcal{A}}^{\alpha,2}_0)^{\perp} \\ 0 & \text{on } \widetilde{\mathcal{A}}^{\alpha,2}_0 \end{cases}$$

on their respective domains.

Proof. We only need to check the equality on their action upon the basis elements. Thus, the density of the basis guarantees equality everywhere.

For the first isometry:

$$\begin{aligned}
(L^{\alpha}_{(2)})^{-1/2}\, (\widetilde{L}^{\alpha}_{(2)})^{-1/2}\, \widetilde{K}^{\alpha}_{(2^+)}(b^{\alpha}_{k,l}) &= (L^{\alpha}_{(2)})^{-1/2}\, (\widetilde{L}^{\alpha}_{(2)})^{-1/2}\, (\sqrt{(l+1)(l+1+\alpha)}\, b^{\alpha}_{k,l+1}) \\
&= (L^{\alpha}_{(2)})^{-1/2} (\sqrt{l+1}\, b^{\alpha}_{k,l+1}) \\
&= b^{\alpha}_{k,l+1} = V_{\alpha}(b^{\alpha}_{k,l}).
\end{aligned}$$

And for the third:

$$\begin{aligned}(L_{(1)}^{\alpha})^{-1/2}\,(\widetilde{L}_{(1)}^{\alpha})^{-1/2}\,\widetilde{K}_{(1+)}^{\alpha}(b_{k,l}^{\alpha}) &= (L_{(1)}^{\alpha})^{-1/2}\,(\widetilde{L}_{(1)}^{\alpha})^{-1/2}\,(\sqrt{(k+1)(k+1+\alpha)}\,b_{k+1,l}^{\alpha})\\ &= (L_{(1)}^{\alpha})^{-1/2}(\sqrt{k+1}\,b_{k+1,l}^{\alpha})\\ &= b_{k+1,l}^{\alpha} = \widetilde{V}_{\alpha}(b_{k,l}^{\alpha}).\end{aligned}$$

For the others, we check the cases. Let $b_{k,l}^{\alpha}$, with $l > 0$.

$$\widetilde{K}_{2-}^{\alpha}\,(\widetilde{L}_{(2)}^{\alpha})^{-1/2}(L_{(2)}^{\alpha})^{-1/2}(b_{k,l}^{\alpha}) = \widetilde{K}_{2-}^{\alpha}\left(\frac{1}{\sqrt{l(l+\alpha)}}b_{k,l}^{\alpha}\right) = b_{k,l-1}^{\alpha}. \tag{3.9}$$

Recall that $\ker(V_{\alpha}^{*}) = \mathcal{A}^{\alpha,2}(\mathbb{D})$.

This implies that V_{α}^{*} only acts upon the elements of $(\mathcal{A}^{\alpha,2}(\mathbb{D}))^{\perp}$, where the equality holds.

Similarly, for a basis element $b_{k,l}^{\alpha}$, with $k > 0$:

$$\widetilde{K}_{1-}^{\alpha}\,(\widetilde{L}_{(1)}^{\alpha})^{-1/2}(L_{(1)}^{\alpha})^{-1/2}(b_{k,l}^{\alpha}) = \widetilde{K}_{1-}^{\alpha}\left(\frac{1}{\sqrt{k(k+\alpha)}}b_{k,l}^{\alpha}\right) = b_{k-1,l}^{\alpha}. \tag{3.10}$$

Since, $\ker(\widetilde{V}_{\alpha}^{*}) = \widetilde{\mathcal{A}}^{\alpha,2}(\mathbb{D})$, just as before, we get the desired result. □

Now, having found the basis-independent representations for the pure isometry V_{α}, we can recover the mutually adjoint operators $\mathfrak{a}$ and $\mathfrak{a}^{\dagger}$.

Following the abstract theory of the extended Fock space formalism, one can derive such ladder operators from a pure isometry per Theorem 1.3.11.

This next definition applies this to our particular case.

Definition 3.3.21. Let $\mathfrak{a}, \mathfrak{a}^{\dagger} : L^2(\mathbb{D}, d\mu_{\alpha}) \to L^2(\mathbb{D}, d\mu_{\alpha})$ be two operators densely defined through their actions on the basis elements of $L^2(\mathbb{D}, d\mu_{\alpha})$ as follows:

$$\mathfrak{a}(b_{p,q}^{\alpha}) = \begin{cases}\sqrt{q}\,b_{p,q-1}^{\alpha} & \text{if } p \in \mathbb{Z}_+, q > 0\\ 0 & \text{if } p \in \mathbb{Z}_+, q = 0\end{cases}$$

and,

$$\mathfrak{a}^{\dagger}(b_{p,q}^{\alpha}) = \sqrt{q+1}\,b_{p,q+1}^{\alpha}\,, \quad \forall p, q \in \mathbb{Z}_+$$

Next we verify that these operators are indeed the lowering and raising operators from Theorem 2.15.

Proposition 3.3.22. *The operators $\mathfrak{a}$ and $\mathfrak{a}^{\dagger}$ satisfy on $\mathcal{D}$ the commutation relation:*

$$[\mathfrak{a}, \mathfrak{a}^{\dagger}] = I$$

Proof. Granted the linearity of the operators $\mathfrak{a}$ and $\mathfrak{a}^{\dagger}$ we only need to verify this equality for the basis elements.

Let $b^{\alpha}_{p,q}$ be a normalized disk polynomial. If $q > 0$:

$$\begin{aligned}[\mathfrak{a}, \mathfrak{a}^{\dagger}](b^{\alpha}_{p,q}) &= (\mathfrak{a}\mathfrak{a}^{\dagger} - \mathfrak{a}^{\dagger}\mathfrak{a})(b^{\alpha}_{p,q}) \\ &= \mathfrak{a}(\sqrt{q+1}b^{\alpha}_{p,q}) - \mathfrak{a}(\sqrt{q}b^{\alpha}_{p,q}) \\ &= \sqrt{q+1}\mathfrak{a}(b^{\alpha}_{p,q}) - \sqrt{q}\mathfrak{a}(b^{\alpha}_{p,q}) \\ &= (q+1)b^{\alpha}_{p,q} - qb^{\alpha}_{p,q} = b^{\alpha}_{p,q}.\end{aligned}$$

If $q = 0$:

$$\begin{aligned}[\mathfrak{a}, \mathfrak{a}^{\dagger}](b^{\alpha}_{p,0}) &= (\mathfrak{a}\mathfrak{a}^{\dagger} - \mathfrak{a}^{\dagger}\mathfrak{a})(b^{\alpha}_{p,0}) \\ &= \mathfrak{a}(b^{\alpha}_{p,1}) = \mathfrak{a}(b^{\alpha}_{p,1}) \\ &= b^{\alpha}_{p,0}.\end{aligned}$$

In any case, the desired property holds. □

In fact, as it was mentioned before, these operators may be extended to the following common domain that is dense in $L^2(\mathbb{D}, \mu_\alpha)$:

$$\mathcal{D}_{ext} := \left\{ f = \sum_{j=1}^{\infty} f_j b^{\alpha}_{m_j,n_j} \in L^2(\mathbb{D}, \mu_\alpha) \;\middle|\; \sum_{j=1}^{\infty} n_j |f_j|^2 < \infty \right\}$$

and acting as:

$$\mathfrak{a}(f) = \sum_{j=1}^{\infty} \sqrt{n_j} f_j b^{\alpha}_{m_j,n_j-1}$$

(considering $n_j - 1 = 0$ if $n_j = 0$)

and,

$$\mathfrak{a}(f) = \sum_{j=1}^{\infty} \sqrt{n_j+1} f_j b^{\alpha}_{m_j,n_j+1}$$

for all elements $f \in \mathcal{D}_{ext}$.

In this domain, we have that the operators $\mathfrak{a}$ and $\mathfrak{a}^{\dagger}$ are mutually adjoint, and the linear combinations of elements of the spaces $(\mathfrak{a}^{\dagger})^{n-1} \ker \mathfrak{a}$ is dense in $L^2(\mathbb{D}, \mu_\alpha)$.

Finally, Theorem 3.3.23 gives a basis-independent representation for these operators.

Theorem 3.3.23. *The following representations for $\mathfrak{a}$ and $\mathfrak{a}^{\dagger}$ hold*

$$\mathfrak{a} = \begin{cases} \widetilde{K}^{\alpha}_{2^-} \, (\widetilde{L}^{\alpha}_{(2)})^{-1/2} & \text{on } (\mathcal{A}^{\alpha,2})^{\perp} \cap \mathcal{D}_{ext} \\ 0 & \text{on } \mathcal{A}^{\alpha,2}_0 \cap \mathcal{D}_{ext} \end{cases}$$

$$\mathfrak{a}^{\dagger} = (\widetilde{L}^{\alpha}_{(2)})^{-1/2} \, \widetilde{K}^{\alpha}_{(2+)}, \quad \text{on } \mathcal{D}_{ext}.$$

Proof. As was done in Theorem 3.3.20, we verify these equalities only on the basis elements.

For the operator $\mathfrak{a}$:

Let $b^{\alpha}_{k,l}$ be a basis element. If $l > 0$:

$$\widetilde{K}^{\alpha}_{2-}\,(\widetilde{L}^{\alpha}_{(2)})^{-1/2}(b^{\alpha}_{k,l}) = \widetilde{K}^{\alpha}_{2-}\left(\frac{1}{\sqrt{l+\alpha}}\,b^{\alpha}_{k,l}\right) = \frac{1}{\sqrt{l+\alpha}}\,\widetilde{K}^{\alpha}_{2-}\,(b^{\alpha}_{k,l}) = \sqrt{l}\,b^{\alpha}_{k,l-1}.$$

If $l = 0$, then the basis element $b^{\alpha}_{k,l}$ belongs to $(\mathcal{A}^{\alpha,2})^{\perp}$, so the equality holds.

Similarly, for $\mathfrak{a}^{\dagger}$:

Let $b^{\alpha}_{k,l}$ be a basis element. Then:

$$(\widetilde{L}^{\alpha}_{(2)})^{-1/2}\,\widetilde{K}^{\alpha}_{(2^{+})}(b^{\alpha}_{k,l}) = (\widetilde{L}^{\alpha}_{(2)})^{1/2}\big(\sqrt{(l+1)(l+1+\alpha)}\,b^{\alpha}_{k,l+1}\big) = \sqrt{l+1}\,b^{\alpha}_{k,l+1}.$$

□

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