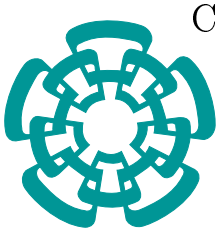

Centro de Investigación y de Estudios Avanzados
del Instituto Politécnico Nacional

Unidad Zacatenco

Departamento de Matemáticas

Cálculos combinatorios de resoluciones libres minimales de ideales de aristas de bosques

TESIS

Que presenta

David Camilo Molano Valbuena

Para obtener el grado de

Doctor en Ciencias

en la especialidad de Matemáticas

Director de la Tesis: Dr. Carlos Enrique Valencia Oleta

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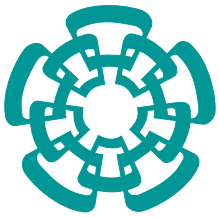

Center for Research and Advanced Studies
of The National Polytechnic Institute

Campus Zacatenco

Departament of Mathematics

Combinatorial Computations of Minimal Free Resolutions of Forests

THESIS

Presented by

David Camilo Molano Valbuena

to obtain the degree of

Doctor of Science

in the speciality of Mathematics

Thesis Advisor: Ph.D. Carlos Enrique Valencia Oleta

Mexico City. September, 2024.

Dedication and Aknowledgments

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- The people who are and will keep working with me on these topics since I've learned a lot from them.
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Last, but not least, I thank my family. All their support during my life made me the person who I can be proud to say I am today.

Thank you all.

Abstract

Minimal free resolutions (m.f.r.) are fundamental because many algebraic invariants can be derived from them. Even though m.f.r. of a finitely generated module over a polynomial ring are finite, their computation is complicated. Although much of the work done has resulted in algorithms like cellular resolutions, and despite recent advances, finding a m.f.r. remains difficult.

In this work, we develop the molecular cover (m.c.) algorithm, a technique for computing m.f.r. of forest edge ideals. This new algorithm allows us to cover a tree with stars using a particular procedure. Then, we relate these covers to its Betti numbers and, ultimately, compute minimal multi-graded free resolutions of their edge ideals, at least in some families.

In chapters 1 and 2, we introduce preliminary homological and combinatorial concepts.

These basic tools allow us to develop the m.c. algorithm in subsequent chapters. Then, we will discuss the versions of Hochster's formula.

In Chapter 3, we develop more advanced techniques that are the basis of the m.c. algorithm, which we will use in Chapter 4. This immediately allows us to prove some results using only combinatorial means.

In Chapter 4, we will first present a criterion for proving that a given chain complex is actually a resolution. Then, we'll formally introduce the concept of a *molecular cover* of a forest, along with the algorithm to compute them. Then, using combinatorial means and this algorithm, we will compute candidates of m.f.r. of some families of edge ideals.

In the Appendix, we attach a list of molecular trees and one with molecular trees with at most 18 vertices that remain molecular after removing all their leaves.

Resumen

Las resoluciones libres minimales (r.l.m.) son fundamentales porque muchos invariantes se pueden derivar de ellas. A pesar de que las r.l.m. de un módulo finitamente generado sobre un anillo de polinomios son finitas, sus cálculos son complicados. Aunque mucho del trabajo realizado ha resultado en algoritmos como resoluciones celulares, y a pesar del progreso reciente, encontrar una r.l.m. sigue siendo complicado.

En este trabajo desarrollamos el algoritmo de cubiertas moleculares (c.m.), para calcular r.l.m. de ideales de aristas de bosques. Este nuevo algoritmo nos permite cubrir un árbol con estrellas usando un procedimiento particular. Entonces relacionamos estas cubiertas con sus números de Betti con el objetivo último de calcular r.l.m. multigraduadas de sus ideales de aristas, al menos en algunas familias.

En los capítulos 1 y 2 calculamos conceptos preliminares homológicos y combinatorios.

Estas herramientas básicas nos permiten desarrollar el algoritmo de las c.m. en los capítulos siguientes. Entonces hablaremos de las versiones de la fórmula de Hochster.

En el capítulo 3 desarrollamos técnicas más avanzadas que son la base del algoritmo de la c.m. que usaremos en el capítulo 4. Esto inmediatamente nos permite demostrar algunos resultados vía medios combinatorios.

En el capítulo 4 primero presentaremos un criterio para demostrar que un complejo de cadenas dado es de hecho una resolución. Entonces introduciremos formalmente el concepto de una *cubierta molecular* de un bosque, junto con el algoritmo para calcularlas. Entonces, usando medios combinatorios y este algoritmo, calcularemos candidatos a r.l.m. de algunas familias de ideales de aristas.

En el apéndice, añadimos una lista de árboles moleculares, y de árboles moleculares que permanecen moleculares al remover las hojas.

Contents

Introduction

The minimal free resolution of a finitely generated module over a polynomial ring $S = k[\mathbf{x}]$, with variables $\mathbf{x} = \{x_1, \ldots, x_n\}$ and k a field, have been extensively studied. Nowadays, there is software to compute minimal free resolutions. However, it extensively uses iterative algorithms based on the Gröbner basis, which makes the computations very expensive, as the complexity to compute a Gröbner basis can be doubly exponential, as shown in [?]. Therefore, there is interest in finding other ways to compute minimal free resolutions or Betti numbers without having to compute them, skipping the iterative process. Some of this work can be seen in [?].

Some general methods have been developed to compute non-minimal free resolutions or complexes of a module, like the Taylor Resolution or the Scarf Complex. These resolutions and complexes can give us part of the information we'd get from a minimal free resolution, but not all. Some work has also been done on computing minimal free resolutions on families of ideals based on cellular complexes, but these computations are still highly complex.

Several techniques have been developed to compute Betti numbers, making this a manageable problem in a lot of cases. Some work that is related to ours can be found in [?], [?], and [?].

However, an explicit description of the differentials in the resolution is not often given, as this turns out to be a more difficult problem.

In this work, we develop some combinatorial tools that allow us to make computations of the differentials of minimal multigraded free resolutions of families of ideals, especially edge ideals of trees. Some work has been done to compute free resolutions of trees or their Betti numbers, as in [?] and [?]. The main concept developed in this work is called a *molecular cover*.

Objectives

The Objectives of this work are:

- To construct an algorithm that calculates the Betti numbers of the edge ideal of a given tree.
- To give a characterization on the family of all trees by using such algorithm.
- To use such characterization to calculate minimal free resolutions of the edge ideals of some families of trees.

Chapter 1

Preliminaries

1.1 Chain Complexes

Let R be a commutative ring.

Definition 1.1.1. For a (possibly finite) sequence (C, ϕ) of $R-$modules C_i and homomorphisms ϕ_i:

$$C : \cdots \overset{\phi_{-1}}{\leftarrow} C_{-1} \overset{\phi_0}{\leftarrow} C_0 \overset{\phi_1}{\leftarrow} C_1 \overset{\phi_2}{\leftarrow} \cdots$$

we say that C is a chain complex if $\phi_i \circ \phi_{i-1} = 0$ for all i. If, in fact, $\ker \phi_i = \text{im}\phi_{i+1}$ for some i, then C is said to be exact at the homological degree i. If it is exact at all the homological degrees, then we say it is an exact sequence. An exact sequence

$$0 \to L \to M \to N \to 0$$

is said to be a short exact sequence. In this case, this is equivalent to the fact that $L \to M$ is a monomorphism, $M \to N$ is an epimorphism and $\ker(M \to N) = \text{im}(L \to M)$. Sequences $0 \to L \to M \to 0$ and $0 \to L \to 0$ are also called short exact sequences.

We will use the notations $C_\bullet$, (C, ϕ), or simply C for a chain complex, depending on the need to differentiate two of them or state the maps ϕ_i.

1.2 Simplicial Homology and Cohomology

Here, we will cover a topological invariant for simplicial complexes: Their homology. Some of the results here can be generalized, adding some small technical hypotheses.

Definition 1.2.1. An $n-$simplex is just a topological space homeomorphic to

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} : \sum_{i=0}^{n} t_i = 1\}.$$

We define a face $\sigma \subseteq \Delta^n$ by taking a subset $A_\sigma \subseteq \{0, \dots, n\}$ and defining

$$\sigma = \{(t_0, \dots, t_n) \in \Delta^n \mid (\forall i \in A_\sigma), t_i = 0\}.$$

We then say that σ is a $n - |A_\sigma|-$face of Δ^n. In this case, it is also clear that σ is a $n - |A_\sigma|-$simplex for $|A_\sigma| \leq n + 1$.

We define $\Delta^{-1} = \varnothing$.

Definition 1.2.2 (Simplicial Complex)**.** A simplicial complex Δ is a topological space, with a family of simplices $\mathcal{S} \subseteq \{\sigma \subseteq \Delta\}$ such that

$$\Delta = \bigcup \mathcal{S}$$

and also for every $\sigma, \tau \in \mathcal{S}$ we have that $\sigma \cap \tau$ is a face of both σ and τ. We say that ρ is a face of Δ if it is a face of some simplex of Δ.

Remark. By the previous definition, we have that every simplex $\sigma \in \mathcal{S}$ is determined by its $0-$faces, i.e., vertices. Otherwise, if there were $\sigma \neq \tau$ both with the same vertices, this would mean $\sigma \cap \tau$ is a face of σ (resp. τ), having all its vertices, which can only be σ (since it would correspond to the face of the simplex Δ^n corresponding to the set $A = \varnothing$, which is Δ^n), thus $\sigma = \tau \cap \sigma = \tau$.

Definition 1.2.3. For a $n-$simplex σ, we define its dimension by $\dim \sigma = n$. For a simplicial complex Δ we define its dimension by $\dim \Delta = \max\{\dim \sigma : \sigma \in \mathcal{S}\}$.

Definition 1.2.4. For a simplicial complex Δ, we can define its face poset $F(\Delta)$ as the set of all its faces ordered by inclusion. It has a *graded* structure, by taking

$$F(\Delta) = \Delta_{-1} \sqcup \cdots \sqcup \Delta_{\dim \Delta}.$$

where $\Delta_i = \{\sigma \in F(\Delta) : \dim \sigma = i\}$.

Remark. A simplicial complex Δ remains unchanged if we replace $\mathcal{S}$ with $F(\Delta)$. So we can add an extra condition for $\mathcal{S}$: That every $\tau \subseteq \sigma$ is also in $\mathcal{S}$ for any $\sigma \in \mathcal{S}$. This means that a all the faces of a simplicial complex are determined by its maximal faces, its facets. This also gives us a recipe for the next definition.

Definition 1.2.5 (Abstract Simplicial Complex)**.** An abstract simplicial complex Δ on the vertex set $V = \Delta_0$ is a collection of subsets of V such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. The $i-$faces of Δ are the elements of Δ with $i+1$ elements. This way, we define the dimension of both the faces of Δ and Δ as above. We define Δ_i as the set of all the $i-$faces of Δ.

Remark. For a simplicial complex Δ, the face poset $F(\Delta)$ is an abstract simplicial complex on the vertex set Δ_0. Conversely, if $\Delta \neq \varnothing$ is instead an abstract simplicial complex, we get a simplicial complex $T(\Delta)$ by replacing all $i-$faces of Δ by $i-$simplices, in a way they intersect according to their intersections in Δ.

This way, the empty space $\varnothing$ corresponds to the *irrelevant* (abstract) simplicial complex $\{\varnothing\}$. The empty abstract simplicial complex $\varnothing$ does not correspond to a non-abstract simplicial complex.

Given an $i-$face $\sigma = \{v_{j_0}, \ldots, v_{j_i}\} \in \Delta$ such that $j_l \leq j_m$ for $l \leq m$, we shall denote it as $\sigma = [v_{j_0}, \ldots, v_{j_i}]$.

Definition 1.2.6 (Skeleton)**.** Let Δ be a simplicial complex. The simplicial complex given by all the faces of dimension $\leq i \in \mathbb{N}$ of Δ is called the $i-$skeleton of Δ.

Definition 1.2.7 (Euler Characteristic)**.** For a simplicial complex Δ, define the Euler characteristic of $\chi(\Delta)$ as $\chi(\Delta) = \rho - \iota - 1$ where ρ is the number of faces of even dimension and ι is the number of faces of odd dimension (counting the empty $-1-$face).

Definition 1.2.8 (Chain complex of a simplicial complex and its homology)**.** For an abstract simplicial complex Δ on a vertex set V(it is analogous for a simplicial complex using its face poset) and a commutative ring with unity R, we define, for every $-1 \leq i \leq \dim \Delta$, $C_i(\Delta; R) = R^{\Delta_i}$ formally, i.e. the free $R-$module generated by Δ_i, or more explicitly, the $R-$module of all the formal sums of the form $\sum_{\sigma \in \Delta_i} \alpha_\sigma \sigma$. We also define boundary maps

$$\partial_i : C_i(\Delta; R) \to C_{i-1}(\Delta; R)$$

by

$$\partial_i([v_{j_0},\ldots,v_{j_i}]) = \sum_{l=0}^{i}(-1)^l([v_{j_0},\ldots,\widehat{v_{j_l}},\ldots,v_{j_i}])$$

where $\widehat{v_{j_l}}$ means that component is removed. If we have some matrix

$$(a_{\sigma\sigma'})_{\sigma\in\Delta_i,\sigma'\in\Delta_{i-1}}$$

representing ∂, this means that

$$\partial(\sigma) = \sum_{\sigma'\in\Delta_{i-1},\sigma'\subseteq\sigma} a_{\sigma\sigma'}\sigma'.$$

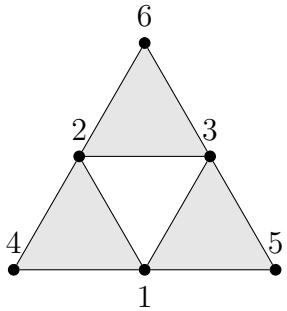
Since $\sigma-\sigma'$ consists in a single vertex k, we will define $\mathrm{sgn}(k,\sigma)$ as $a_{\sigma\sigma'}$. This also means that for $\sigma = [v_{j_0},\ldots,v_{j_i}]$ and $k = v_{j_l} \in \sigma$ we have $\mathrm{sgn}(k,\sigma) = (-1)^l$. We define the simplicial homology[(i)] of Δ with coefficients in R as $H_i(\Delta;R) = \ker\partial_i/\mathrm{im}\partial_{i+1}$. The elements of $Z_i(\Delta;R) = \ker\partial_i$ are called cycles, and the elements of $B_i(\Delta;R) = \mathrm{im}\partial_{i+1}$ are called boundaries.

When there is no risk of ambiguity, we will omit R in the notation, so $C_\bullet(\Delta;R)$ will be $C_\bullet(\Delta)$ and every module is an $R-$module.

Example 1.2.1. A simplicial complex (abstract or not) can be represented by a drawing. For example, consider the abstract simplicial complex Δ defined on the vertex set $\Delta_0 = \{1,\ldots,6\}$ determined by the facets

$$\{1,2,4\},\{1,3,5\},\{2,3,6\}.$$

This simplicial complex has the drawing:



[(i)] This is the definition of the reduced homology, usually denoted by $\tilde{H}$. Every homology calculation in this work will be reduced.

Computing its homology is not hard, but the calculations are a bit cumbersome. We have to compute the chain modules:

$$\begin{aligned}
C_2(\Delta;R) &= R[1,2,4] \oplus R[1,3,5] \oplus R[2,3,6],\\
C_1(\Delta;R) &= R[1,2] \oplus R[1,3] \oplus R[2,3] \oplus R[1,4] \oplus R[2,4] \oplus R[1,5] \oplus R[3,5]\\
&\quad \oplus R[2,6] \oplus R[3,6],\\
C_0(\Delta;R) &= \bigoplus_{i=1}^{6} R[i],\\
C_{-1}(\Delta;R) &= R\varnothing.
\end{aligned}$$

The boundary maps are also given by the following matrices (with respect to the given bases and considering the vectors of C_i as columns):

$$\partial_2 = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\\ 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\\ 0 & 0 & 1 \end{bmatrix}, \partial_1 = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0\\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1\\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and $\partial_0 = \mathbf{1}$ where $\mathbf{1}$ is the 1×6 vector with $1s$ as its entries. Now, we have to compute the kernel and image of the three maps. We will then have that $H_2(\Delta;R) \cong \ker\partial_2$, $H_1(\Delta;R) \cong \ker\partial_1/\mathrm{im}\partial_2$ and $H_0(\Delta;R) \cong \ker\partial_0/\mathrm{im}\partial_1$. To solve the system

$$\partial_2 \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}$$

we observe that, in particular, by computing the products with the first three rows, that $x = y = z = 0$, so $H_2(\Delta;R) \cong \ker\partial_2 = 0$. To compute $\mathrm{im}\partial_2$ we

see that for $x, y, z \in R$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ -x \\ x \\ -y \\ y \\ -z \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

so $\mathrm{im}\partial_2$ is generated by those three vectors, which are clearly a basis for it.

Now we solve the system

$$\begin{bmatrix} -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which in particular, implies that $x_8 = -x_9, x_6 = -x_7$ and $x_4 = -x_5$. We also get the relations

$$x_2 + x_3 - x_7 - x_9 = x_1 - x_3 - x_5 - x_8 = -(x_1 + x_2 + x_4 + x_6) = 0$$

which translate into

$$x_2 + x_3 + x_6 - x_9 = x_1 - x_3 + x_4 + x_9 = x_1 + x_2 + x_4 + x_6 = 0,$$

which are associated to the kernel of the 3×6 matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & 1 \end{bmatrix}$$

which can be reduced further by (carefully made) elementary row operations:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = x_3 - x_4 - x_9, x_2 = -x_3 - x_6 + x_9$. Therefore, any vector u in the kernel must have the form

$$u = \begin{bmatrix} x_3 - x_4 - x_9 \\ -x_3 - x_6 + x_9 \\ x_3 \\ x_4 \\ -x_4 \\ x_6 \\ -x_6 \\ -x_9 \\ x_9 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_9 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

These four vectors, u_1, u_2, u_3, u_4 are a basis, so this submodule of R^9 is isomorphic to R^4. We also see that u_2, u_3 and $u_4 + u_1$ are in $\mathrm{im}\partial_2$, which means that $u_2 + \mathrm{im}\partial_2 = u_3 + \mathrm{im}\partial_2 = 0$ and $u_1 + \mathrm{im}\partial_2 = -u_4 + \mathrm{im}\partial_2$. So $H_1(\Delta; R)$ is generated by $u_1 + \mathrm{im}\partial_2$. Since no multiple of u_1 is in $\mathrm{im}\partial_2$, we get $H_1(\Delta; R) \cong R$.

Lastly, $\ker \partial_0 : R^6 \to R$ consists on all the vectors

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

such that $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$, which means that

$$\begin{aligned} v &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ -x_1 - x_2 - x_3 - x_4 - x_5 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \end{aligned}$$

so $\ker \partial_0$ is generated by those five vectors v_1, v_2, v_3, v_4, v_5. The image of ∂_1 is just the submodule of all vectors v of the form

$$v = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix}$$

$$= \begin{bmatrix} -x_1 - x_2 - x_4 - x_6 \\ x_1 - x_3 - x_5 - x_8 \\ x_2 + x_3 - x_7 - x_9 \\ x_4 + x_5 \\ x_6 + x_7 \\ x_8 + x_9 \end{bmatrix}$$

Each of the v_i's is in there, it can be checked by making the following matrix products for each vector, so that we can see that $H_0(\Delta; R) = 0$:

$$\partial_1 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \partial_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \partial_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \partial_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

and lastly

$$\partial_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

The calculations above are very cumbersome, even when R is a field (in which case they're still simplified a lot because of theorems like rank-nullity and other properties of the dimension of finite dimensional vector spaces). But there are other tools to help us make these calculations in an easier way; two of them are the long exact sequence of homology and the Mayer-Vietoris sequence.

Proposition 1.2.2. The sequence

$$C_\bullet(\Delta; R) : 0 \leftarrow C_{-1}(\Delta; R) \overset{\partial_0}{\leftarrow} C_0(\Delta; R) \overset{\partial_1}{\leftarrow} \cdots$$

is indeed a chain complex.

Proof. it is very straightforward. Let $\sigma = [v_0, \ldots, v_n] \in C_n(\Delta)$. Then

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i (\sigma - v_i)$$

and

$$\begin{aligned} \partial_{n-1}\partial_n(\sigma) &= \sum_{j<i} (-1)^j (-1)^i (\sigma - v_i - v_j) + \sum_{j>i} (-1)^{j-1} (-1)^i (\sigma - v_i - v_j) \\ &= \sum_{j<i} (-1)^j (-1)^i (\sigma - v_i - v_j) + \sum_{j<i} (-1)^{i-1} (-1)^j (\sigma - v_i - v_j) \\ &= \sum_{j<i} (-1)^j (-1)^i (\sigma - v_i - v_j) - \sum_{j<i} (-1)^i (-1)^j (\sigma - v_i - v_j) \\ &= 0 \end{aligned}$$

■

This can be generalized a bit.

Definition 1.2.9 (Homology of chain complex)**.** For a chain complex

$$C : \cdots \overset{\partial_{i-1}}{\leftarrow} C_{i-1} \overset{\partial_i}{\leftarrow} C_i \overset{\partial_{i+1}}{\leftarrow} C_{i+1} \overset{\partial_{i+2}}{\leftarrow} \cdots$$

we define the homology of C, denoted $H_i(C)$ as $\ker\partial_i/\mathrm{im}\partial_{i+1}$. When the chain complex comes from a simplicial complex, we will denote $H_i(\Delta; R)$, if we take $C_{-1}(\Delta; R) = 0$ or $H_i(\Delta; R)$ if we consider $C_{-1}(\Delta; R) = R$. In the second case, the notation C will be replaced by $\tilde{C}$, and H will be called reduced homology. The elements of the set $Z_i(C) = \ker\partial_i$ will be called cycles and the elements of the set $B_i(C) = \mathrm{im}\partial_{i+1}$ will be called boundaries.

Definition 1.2.10. A homomorphism $\phi : C \to D$ of two chain complexes $(C, \partial), (D, \delta)$ is just a sequence of maps $\phi = (\phi_i : C_i \to D_i)_i$ such that the following diagram commutes (i.e. the rectangle in the diagram commutes for every i):

$$\begin{array}{ccccccc}
\cdots & \xleftarrow{\partial_{i-1}} & C_{i-1} & \xleftarrow{\partial_i} & C_i & \xleftarrow{\partial_{i+1}} & \cdots \\
 & & \downarrow{\scriptstyle \phi_{i-1}} & & \downarrow{\scriptstyle \phi_i} & & \\
\cdots & \xleftarrow{\delta_{i-1}} & D_{i-1} & \xleftarrow{\delta_i} & D_i & \xleftarrow{\delta_{i+1}} & \cdots
\end{array}$$

Definition 1.2.11. A chain complex (resp. exact sequence) of chain complexes is just a sequence $(A_i, \phi_i)_{i\in\mathbb{Z}}$:

$$\cdots \overset{\phi_{i-1}}{\leftarrow} A_{i-1} \overset{\phi_i}{\leftarrow} A_i \overset{\phi_{i+1}}{\leftarrow} A_{i+1} \overset{\phi_{i+2}}{\leftarrow} \cdots$$

such that for every $i \in \mathbb{Z}$, A_i is a chain complex, ϕ_i is a homomorphism of chain complexes and the sequence:

$$\cdots \overset{\phi_{i-1,j}}{\leftarrow} A_{i-1,j} \overset{\phi_{i,j}}{\leftarrow} A_{i,j} \overset{\phi_{i+1,j}}{\leftarrow} A_{i+1,j} \overset{\phi_{i+2,j}}{\leftarrow} \cdots$$

is a chain complex (resp. exact) for every j. In particular,

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of chain complexes if

$$0 \to A_j \to B_j \to C_j \to 0$$

is a short exact sequence of modules for every j.

Theorem 1.2.3 (Long exact sequence of homology)**.** If there is a short exact sequence of chain complexes:

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

for chain complexes $(A, \gamma), (B, \delta), (C, \epsilon)$then there is a long exact sequence:

$$\cdots \xrightarrow{\beta^*_{i+1}} H_{i+1}(C) \xrightarrow{\partial_{i+1}} H_i(A) \xrightarrow{\alpha^*_i} H_i(B) \xrightarrow{\beta^*_i} H_i(C) \xrightarrow{\partial_i} H_{i-1}(A) \xrightarrow{\alpha^*_{i-1}} \cdots$$

where the homomorphisms $\alpha^*_i, \beta^*_i, \partial_i$ are to be defined in the proof.

Proof. We have the following commutative mesh with exact rows:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{i+1} & \xrightarrow{\alpha_{i+1}} & B_{i+1} & \xrightarrow{\beta_{i+1}} & C_{i+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_{i+1}} & & \downarrow{\scriptstyle \delta_{i+1}} & & \downarrow{\scriptstyle \epsilon_{i+1}} & & \\
0 & \longrightarrow & A_i & \xrightarrow{\alpha_i} & B_i & \xrightarrow{\beta_i} & C_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \delta_i} & & \downarrow{\scriptstyle \epsilon_i} & & \\
0 & \longrightarrow & A_{i-1} & \xrightarrow{\alpha_{i-1}} & B_{i-1} & \xrightarrow{\beta_{i-1}} & C_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

First, we want to define the maps $\alpha^*, \beta^*, \partial$ of the sequence. To define α^*_i consider $\alpha_i : A_i \to B_i$. Since α is a chain complex homomorphism then $\alpha_i : A_i \to B_i$ maps $\ker \gamma_i$ into $\ker \delta_i$ and $\mathrm{im}\gamma_{i+1}$ into $\mathrm{im}\delta_{i+1}$. So first we can take $\alpha'_i = \alpha_i|_{\ker \gamma_i} : \ker \gamma_i \to \ker \delta_i$. Thus, there is a unique map $\alpha^*_i : H_i(A) \to H_i(B)$ such that the following diagram commutes:

$$\begin{array}{ccc}
\ker \gamma_i & \xrightarrow{\alpha'_i} & \ker \delta_i \\
\downarrow{\scriptstyle \pi_{A_i}} & & \downarrow{\scriptstyle \pi_{B_i}} \\
H_i(A) & \xrightarrow[\alpha^*_i]{} & H_i(B)
\end{array}$$

where π_{A_i}, π_{B_i} are the canonical maps. So that is how α^* and β^* are defined. About how ∂ is defined, we will focus on the following part of the mesh:

$$\begin{array}{ccccccc} & & & & B_i & \xrightarrow{\beta_i} & C_i \longrightarrow 0 \\ & & & & \downarrow{\scriptstyle \delta_i} & & \downarrow{\scriptstyle \epsilon_i} \\ 0 \longrightarrow & A_{i-1} & \xrightarrow{\alpha_{i-1}} & B_{i-1} & \xrightarrow{\beta_{i-1}} & C_{i-1} & \longrightarrow 0 \end{array}$$

So, take $[z] \in H_i(C)$ for $z \in \ker \epsilon_i$ (a cycle) so that $\epsilon_i(z) = 0$. Since β_i is surjective, then there exists some $b \in B_i$ such that $z = \beta_i(b)$. But since $\epsilon_i \beta_i = \beta_{i-1}\delta_i$, then $\delta_i(b) \in \ker \beta_{i-1} = \text{im}\alpha_{i-1}$. So there is $a \in A_{i-1}$ such that $\alpha_{i-1}(a) = \delta_i(b)$. But also $\alpha_{i-2}\gamma_{i-1} = \delta_{i-1}\alpha_{i-1}$, so since $\delta_{i-1}\alpha_{i-1}(a) = \delta_{i-1}\delta_i(b) = 0$, we get $\alpha_{i-2}\gamma_{i-1}(a) = 0$, and since α_{i-1} is injective this means $\gamma_{i-1}(a) = 0$ so $a \in \ker \gamma_{i-1}$ and $[a] \in H_{i-1}(A)$. So just define $\partial_i([z]) = [a]$. So we have just gone backward through the mesh

$$\begin{array}{ccc} B_i & \xrightarrow{\beta_i} & C_i \\ \downarrow{\scriptstyle \delta_i} & & b \longmapsto z \\ B_{i-1} & & \downarrow \\ A_{i-1} & \xrightarrow{\alpha_{i-1}} & B_{i-1} \\ a & \longmapsto & \delta_i(b) \end{array}$$

just to find a. This choice of a might a priori be dependent on the choice of the representative z of $[z]$ and on the choice of b, so we must prove these choices do not affect the homology class of a, by showing that ∂_i is a well defined map.

To do that, suppose $[z] = [z']$ so that $z - z' \in \text{im}\epsilon_{i+1}$, and we will do the process of finding a, a' for both of the representatives z, z'. Take $b, b' \in B$ such that $\beta_i(b) = z, \beta_i(b') = z'$, so $\beta_i(b - b') = z - z'$. Also, since $z - z' \in \text{im}\epsilon_{i+1}$, take c such that $\epsilon_{i+1}(c) = z - z'$. Since β_{i+1} is surjective there exists $b'' \in B_{i+1}$ such that $\beta_{i+1}(b'') = c$, so, since the diagram

$$\begin{array}{ccc} B_{i+1} & \xrightarrow{\beta_{i+1}} & C_{i+1} \\ {\scriptstyle \delta_{i+1}}\downarrow & & \downarrow{\scriptstyle \epsilon_{i+1}} \\ B_i & \xrightarrow[\beta_i]{} & C_i \end{array}$$

commutes, we have $\epsilon_{i+1}\beta_{i+1}(b'') = \beta_i\delta_{i+1}(b'') = z - z'$. Thus

$$\beta_i(b - b' - \delta_{i+1}(b'')) = 0.$$

This means that there is $a'' \in A_i$ such that $\alpha_i(a'') = b - b' - \delta_{i+1}(b'')$. Now,

$$\delta_i(b - b' - \delta_{i+1}(b'')) = \delta_i(b) - \delta_i(b') = \delta_i\delta_{i+1}(b'') = \delta(b) - \delta(b').$$

If we take a, a' such that $\alpha_{i-1}(a) = \delta_i(b), \alpha_{i-1}(a') = \delta_i(b')$, then

$$\alpha_{i-1}(a - a') = \delta_i(b) - \delta_i(b') = \delta_i(b - b' - \delta_{i+1}(b'')) = \delta_i\alpha_i(a'').$$

So, since the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\alpha_i} & B_i \\ {\scriptstyle \gamma_i}\downarrow & & \downarrow{\scriptstyle \delta_i} \\ A_{i-1} & \xrightarrow[\alpha_{i-1}]{} & B_{i-1} \end{array}$$

commutes, we have

$$\alpha_{i-1}\gamma_i(a'') = \delta_i\alpha_i(a'') = \alpha_{i-1}(a - a'),$$

so since α_{i-1} is a monomorphism, we get get $\gamma_i(a'') = a - a'$. So $a - a'$ is a boundary (lies in the image of γ_i) and $[a - a'] = [a] - [a'] = 0$, thus $[a] = [a']$, therefore $\partial_i([z]) = \partial_i([w])$, so ∂_i is indeed a well defined function.

To check it is a homomorphism, note that if you take cycles $z, w \in C_i$, and b, b' such that $\beta_i(b) = z, \beta_i(b') = w$, and then a, a' such that $\alpha_{i-1}(a) = \delta_i(b), \alpha_{i-1}(a') = \delta_i(b')$ then $\alpha_{i-1}(a + a') = \delta_i(b + b')$ and $\beta_i(b + b') = z + w$. So $\partial([z + w]) = [a + a'] = [a] + [a'] = \partial([z]) + \partial([w])$.

Now, we only have to prove that the new sequence is exact. Start with a cycle a such that $\alpha_i^*([a]) = 0$ so that $\alpha_i(a)$ is a boundary in B_i. So $\beta_i\alpha_i(a)$ is a

boundary in C_i, therefore $[\beta_i\alpha_i(a)] = \beta_i^*\alpha_i^*([a]) = 0$. Conversely suppose that $[b]$ is such that $\beta_i^*([b]) = 0$, so that $c = \beta_i(b)$ is a boundary in C_i. So there is some $c' \in C_{i+1}$ such that $\epsilon_{i+1}(c') = c$. Also since β_{i+1} is surjective there is $b'' \in B_{i+1}$ such that $\beta_{i+1}(b'') = c'$. Set $b' = \delta_{i+1}(b'')$. Then $\beta_i(b') = \beta_i(b) = c$. Thus $\beta_i(b - b') = 0$, so there is some $a \in A_i$ such that $\alpha_i(a) = b - b'$. But b is a cycle and b' is a boundary, so since the diagram above commutes, we have

$$\alpha_{i-1}\gamma_i(a) = \delta_i\alpha_i(a) = \delta_i(b - b') = 0,$$

but since α_{i-1} is injective, this means that $\gamma_i(a) = 0$, and a is a cycle. So $\alpha_i^*([a]) = [b - b'] = [b]$, therefore $\ker\beta_i^* = \mathrm{im}\alpha_i^*$.

Now, we need to prove that $\ker\partial_i = \mathrm{im}\beta_i^*$. Take a cycle z of C_i such that $\partial_i([z]) = 0$. This means that, for $b \in B_i$ such that $\beta_i(b) = z$ we have $0 = \delta_i(b)$, so b is a cycle in B_i and $\beta_i(b) = z$, which means that $\beta_i^*([b]) = [z]$. Conversely if we take a cycle b then $\partial_i(\beta_i^*([b])) = 0$ since $\delta_i(b) = 0$ and α_{i-1} is injective.

Lastly, to prove that $\ker\alpha_i^* = \mathrm{im}\partial_{i+1}$, suppose first that we take a cycle a such that $\alpha_i^*([a]) = 0$. Then $\alpha_i(a)$ is a boundary in B_i and there is $b \in B_{i+1}$ such that $\delta_{i+1}(b) = \alpha_i(a)$. Let $z = \beta_{i+1}(b)$. Then z is a cycle in C_{i+1} since $\epsilon_{i+1}(z) = \beta_i(\alpha_i(a)) = 0$. Also, by the construction of ∂ we have $\partial_{i+1}([z]) = [a]$. Conversely, if we take cycles $z \in C_{i+1}$ and $a \in A_{i+1}$ such that $\partial_{i+1}([z]) = [a]$ then $\alpha_i(a)$ is a boundary, thus $\alpha_i^*\partial_{i+1}([z]) = 0$. Therefore, the sequence is exact, which is the desired result. ■

We can learn two important things from this long and convoluted, but easy proof. First, we will talk about relative homology.

Definition 1.2.12 (Relative homology)**.** For a simplicial complex Δ and a subcomplex $A \subseteq \Delta$ (A subspace of Δ which is a simplicial complex with all its simplices being faces of Δ), define $C_i(\Delta, A) = C_i(\Delta)/C_i(A)$ and $H_i(\Delta, A) = H_i(C(\Delta, A))$.

This states implicitly that $C_i(\Delta, A)$ is a chain complex. We must wonder how are the boundary maps defined, but this is not a problem:

$$\delta_i : C_i(\Delta, A) \to C_{i-1}(\Delta, A)$$

is the only map making the following diagram commute

$$\begin{array}{ccc} C_{i-1}(\Delta) & \xleftarrow{\partial_i} & C_i(\Delta) \\ \Big\downarrow{\scriptstyle \pi_{i-1}} & & \Big\downarrow{\scriptstyle \pi_i} \\ C_{i-1}(\Delta, A) & \xleftarrow[\delta_i]{} & C_i(\Delta, A) \end{array}$$

This map is well defined since ∂_i sends $C_i(A)$ into $C_{i-1}(A)$. This can easily be made explicit: A basis for $C_i(\Delta, A)$ is given by all the faces $\sigma = [v_0, \ldots, v_i] \in \Delta_i - A_i$, so

$$\delta_i(\sigma) = \sum_{j=0}^{i} u_{\sigma,j}(-1)^j(\sigma - \{v_j\}).$$

where $u_{\sigma,j} = 1$ if and only if $\sigma - \{v_j\} \notin A$, otherwise $u_{\sigma,j} = 0$. This can be written as follows:

$$\delta_i(\sigma) = \sum_{\substack{v \in \sigma \\ \sigma - \{v\} \notin A}} \operatorname{sgn}(v, \sigma)(\sigma - \{v\}).$$

Proposition 1.2.4. There is a long exact sequence:

$$\cdots \overset{\beta_{i+1}^*}{\to} H_{i+1}(\Delta, A) \overset{\partial_{i+1}}{\to} H_i(A) \overset{\alpha_i^*}{\to} H_i(\Delta) \overset{\beta_i^*}{\to} H_i(\Delta, A) \overset{\partial_i}{\to} H_{i-1}(A) \overset{\alpha_{i-1}^*}{\to} \cdots$$

Proof. There is a short exact sequence

$$0 \to C(A) \overset{\alpha}{\to} C(\Delta) \overset{\beta}{\to} C(\Delta, A) \to 0,$$

where α is induced by the direct image of the inclusion $A \hookrightarrow \Delta$ on its faces (So it becomes the inclusion $C(A) \hookrightarrow C(\Delta)$), and β is the canonical projection. ■

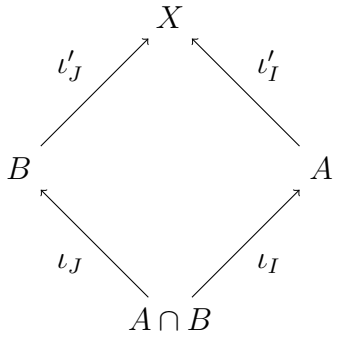
Proposition 1.2.5 (Mayer-Vietoris sequence)**.** For a simplicial complex X, and A, B subcomplexes such that $X = A \cup B$, there is a long exact sequence:

$$\cdots \overset{\partial_{i+1}}{\to} H_i(A \cap B) \overset{\alpha_i^*}{\to} H_i(A) \oplus H_i(B) \overset{\beta_i^*}{\to} H_i(X) \overset{\partial_i}{\to} H_{i-1}(A \cap B) \overset{\alpha_{i-1}^*}{\to} \cdots$$

Proof. There is a natural short exact sequence:

$$0 \to C(A \cap B) \overset{\alpha}{\to} C(A) \oplus C(B) \overset{\beta}{\to} C(X) \to 0,$$

for $\alpha = (\iota_A, -\iota_B), \beta = \iota'_A + \iota'_B$ where $\iota_A, \iota_B, \iota'_A, \iota'_B$ are induced by the inclusions given by the following diagram:



■

For the Mayer-Vietoris sequence, it is really easy to make the homomorphism $\partial : H_i(X) \to H_i(A \cap B)$ explicit. Take a cycle

$$z = \sum_{\sigma \in A_i} r_\sigma \sigma + \sum_{\sigma \in B_i - A_i} r_\sigma \sigma \in Z_i(X).$$

Since it is a cycle, its boundary is 0, and if we take

$$x = \sum_{\sigma \in A_i} r_\sigma \sigma \in C_i(A), y = \sum_{\sigma \in B_i - A_i} r_\sigma \sigma \in C_i(B),$$

we get

$$\partial(x) = -\partial(y),$$

so $\partial(x) \in C_{i-1}(A \cap B)$. So, just define $\partial([z]) = [\partial(x)]$, since $\alpha(\partial(x)) = (\partial(x), -\partial(x)) = (\partial(x), \partial(y))$. [(ii)]

Proposition 1.2.6 (Euler Characteristic)**.** For a simplicial complex of dimension n,

$$\chi(\Delta) = \sum_{i=0}^{n} \text{rank} H_i(\Delta, \mathbb{Z}).$$

By the fundamental theorem for finitely generated abelian groups, every abelian group Q can be decomposed as

$$Q = \mathbb{Z}^n \oplus T,$$

[(ii)] Here we are abusing the notation by letting lots of ∂s have the same name, making it tacit that when we say $\partial([z])$ we are talking about $\partial_i : H_i(X) \to H_i(A \cap B)$, where the i is determined by the location of $[z]$, and when we say $\partial(x)$ we are talking about $\partial_i : C_i(A) \to C_{i-1}(A)$ where A, i are determined by the location of x.

where T is a torsion subgroup. As used in the previous proposition, the rank of the abelian group Q will be defined as n. Similarly, for a free $R-$module R^n over a ring R, its rank will be n. A similar definition can be given more generally over principal ideal domains.

Proof of 1.2.6. Consider a chain complex (C, ϕ) denote $H_i(C)$ by H_i and by Z_i, B_i the cycles and boundaries of C respectively.

Then, for each i, we have an exact sequence

$$0 \to B_i \to Z_i \to H_i \to 0,$$

and one

$$0 \to Z_i \to C_i \to B_{i-1} \to 0.$$

Then it is easy to check that

$$\text{rank}Z_i = \text{rank}B_i + \text{rank}H_i$$

and

$$\text{rank}C_i = \text{rank}Z_i + \text{rank}B_{i-1}.$$

So, $\text{rank}C_i = \text{rank}B_i + \text{rank}H_i + \text{rank}B_{i-1}$, which means that

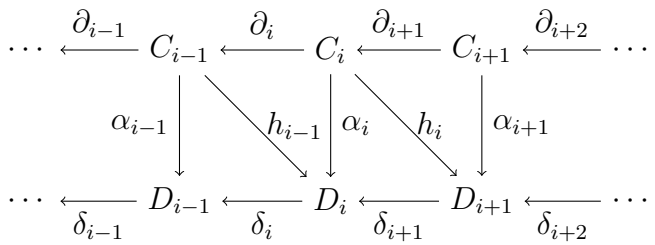
$$\sum_i (-1)^i \text{rank}C_i = \sum_i (-1)^i (\text{rank}B_i + \text{rank}B_{i-1}) + \sum_i (-1)^i \text{rank}H_i.$$

Since $\sum_i (-1)^i (\text{rank}B_i + \text{rank}B_{i-1}) = 0$, we get

$$\sum_i (-1)^i \text{rank}C_i = \sum_i (-1)^i \text{rank}H_i.$$

Applying it to the chain complex $C = C(\Delta; \mathbb{Z})$, and since $\text{rank}C_i = |\Delta_i|$ and $(-1)^i = 1$ if and only if i is even and -1 otherwise, we have the result. ■

Definition 1.2.13 (Chain homotopy)**.** We say a map $\alpha : C \to D$ are chain-null-homotopic if there is a sequence of maps $h_i : C_i \to D_{i+1}$ such that given the following diagram:



we have $h_{i-1}\partial_i + \delta_{i+1}h_i = \alpha_i$ for every i. We say two maps $\alpha, \beta : C \to D$ are chain-homotopic if $\alpha - \beta$ is chain-null-homotopic. The map h is called a chain homotopy, and we say $\alpha \simeq \beta$ or $\alpha \simeq_h \beta$ when the map h is relevant.

Proposition 1.2.7. If a map $\alpha : C \to D$ is chain-null-homotopic, then the induced homomorphisms in homology are 0.

Proof. As in the proof of the long exact sequence of homology, α maps cycles into cycles and boundaries into boundaries so it induces maps $\alpha_i^* : H_i(C) \to H_i(D)$ by $\alpha_i^*([z]) = [\alpha_i(z)]$ for $z \in Z_i(C)$. Since α is chain-null-homotopic we have

$$\alpha_i(z) = h_{i-1}\partial_i(z) + \delta_{i+1}h_i(z) = \delta_{i+1}(h_i(z))$$

which is clearly a boundary of D so its homology class is 0. Therefore $\alpha_i^* = 0$ for every i. ■

Corollary 1.2.8. If $\alpha, \beta : C \to D$ are chain-homotopic, then they induce the same homomorphisms in homology.

Definition 1.2.14. Given two chain complexes C, D, then we say they are chain-homotopically equivalent if there are maps $\alpha : C \to D$ and $\beta : D \to C$ such that $\beta\alpha \simeq 1_C$ and $\alpha\beta \simeq 1_D$.

Proposition 1.2.9. If two chain complexes C, D are chain-homotopically equivalent then they have the same homology.

Proof. Since there are maps $\alpha : C \to D$ and $\beta : D \to C$ such that $\beta\alpha \simeq 1_C$ and $\alpha\beta \simeq 1_D$ then $(\beta\alpha)_i^* = \beta_i^*\alpha_i^* : H_i(C) \to H_i(C)$ and $(\alpha\beta)_i^* = \alpha_i^*\beta_i^* : H_i(D) \to H_i(D)$ are identity maps, therefore α_i^*, β_i^* are isomorphisms for every i. ■

We will use these results to provide convoluted proof of an elementary fact: that a simplex has trivial homology. This can be proven using the fact that a simplex is contractible and that homology remains invariant under deformations. We haven't introduced deformations, but we can use the argument behind their validity to make it work here.

Example 1.2.10. Let Δ be the full simplex on $V = \{1, \ldots, n\}$, i.e. $\Delta = \wp(V)$. We will show that the complexes $C(\Delta)$ and $C(1)$ are homotopically equivalent. The candidate map $c : C(1) \to C(\Delta)$ is just the induced map $1 \mapsto k$ for a fixed $k \in V$. This one induces an inclusion $C_0(1) \to C_0(\Delta)$, the

identity map $C_{-1}(1) \to C_{-1}(\Delta)$ and the zero map everywhere else. For its homotopical inverse, we will take the map induced by sending every $x \in V$ into 1. This map induces the $C_{-1}(1) \leftarrow C_{-1}(\Delta)$, a kind of projection $C_0(1) \leftarrow C_0(\Delta)$ and the zero map everywhere else. By a composition of these two maps and a difference with the identity, we have a new map $f : C(\Delta) \to C(\Delta)$, which fits into the following diagram:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & C_{-1}(\Delta) & \xleftarrow{\partial_0} & C_0(\Delta) & \xleftarrow{\partial_1} & C_1(\Delta) & \xleftarrow{\partial_2} & \cdots \\
 & & \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\
0 & \longleftarrow & C_{-1}(\Delta) & \xleftarrow[\partial_0]{} & C_0(\Delta) & \xleftarrow[\partial_1]{} & C_1(\Delta) & \xleftarrow[\partial_2]{} & \cdots
\end{array}$$

There, f_0 is just the map induced by $v \mapsto v - 1$ (This difference is formal) for every $v \in V$, $f_{-1} = 0$ and $f_i = 1_{C_i(\Delta)}$ for $i > 0$. We want to prove that f is chain-null-homotopic. A chain-null-homotopy h is given by

$$h_i(\sigma) = \sigma \cup \{1\}$$

if $1 \notin \sigma$ and 0 if $1 \in \sigma$, for $i \geq 0$ and $\sigma \in \Delta_i$. For $i = -1$ just define $h_{-1} = 0$. Now we prove that, for every i,

$$\partial_{i+1} h_i + h_{i-1} \partial_i = f_i.$$

First, suppose that $i = -1$. Then since $h_{-1}(\varnothing) = 0$ we get that $\partial_0 h_{-1} = 0 = f_{-1}$. For $i = 0$, and a vertex $v \in V$ we have $h_0(v) = [1, v]$ and $\partial_1 h_0(v) = v - 1 = f_0(v)$. Since $h_{-1} = 0$, then the result holds. Now, for $i > 0$, consider $\sigma \in \Delta_i$. If $1 \in \sigma$ then $h_i(\sigma) = 0$, so that $\partial_{i+1} h_i(\sigma) = 0$. Furthermore, $\partial_i(\sigma)$ is a linear combination of faces σ' such that exactly one of them does not contain 1. The coefficient in such σ' is $(-1)^0 = 1$, so $h_{i-1}(\partial_i(\sigma)) = h_{i-1}(\sigma') = \sigma' \cup \{1\} = \sigma$. If, otherwise, $1 \notin \sigma = \{v_0, \cdots, v_i\}$, then not a single such face σ' contains 1, so

$$h_{i-1}(\partial_i(\sigma)) = \sum_{l=0}^{i} (-1)^l ((\sigma - \{v_l\}) \cup \{1\}).$$

Also $h_i(\sigma) = \sigma \cup \{1\}$ and

$$\begin{aligned}\partial_{i+1}(h_i(\sigma)) &= (-1)^0\sigma + \sum_{l=1}^{i+1}(-1)^l((\sigma \cup \{1\}) - \{v_{l-1}\}) \\ &= \sigma + \sum_{l=1}^{i+1}(-1)^l((\sigma - \{v_{l-1}\}) \cup \{1\}) \\ &= \sigma + \sum_{l=0}^{i}(-1)^{l+1}((\sigma - \{v_l\}) \cup \{1\}) \\ &= \sigma - \sum_{l=0}^{i}(-1)^l((\sigma - \{v_l\}) \cup \{1\}) \\ &= \sigma - h_{i-1}(\partial_i(\sigma)).\end{aligned}$$

Therefore, as expected, we have

$$\partial_{i+1}h_i + h_{i-1}\partial_i = f_i,$$

for every i. The composition $g : C(1) \to C(1)$ in the remaining order is just the identity map, so there is nothing to prove here. This means the chain complex of a $n-$simplex is chain-homotopically-equivalent to the chain complex of a $0-$simplex, which has trivial homology. we are thus done.

Definition 1.2.15. We say a simplicial complex Δ is connected if and only if for any two vertices x, y, there is a $x-y-$path, i.e. a path in the $1-$skeleton of Δ connecting x and y.

This means that Δ is connected if and only if its $1-$skeleton is connected.

Proposition 1.2.11. If Δ is a connected simplicial complex over the vertex set $V = \{v_1, \cdots v_n\}$ then $H_0(\Delta) = 0$.

Proof. We need to prove that $\ker\partial_0 = \text{im}\partial_1$. We already have that $\text{im}\partial_1 \subseteq \ker\partial_0$. So let

$$s = \sum_{i=1}^{n}\alpha_i v_i \in \ker\partial_0$$

so that

$$\sum_{i=1}^{n}\alpha_i = 0$$

Fix a vertex $v \in V$ and take a $v - v_i-$path $\tau_i = \{e_{1i}, \dots, e_{r_i i}\}$ for each i. Consider τ_i as the $1-$chain

$$\tau_i = \sum_{j=1}^{r_i} e_{ji}$$

so that by a telescopic property, $\partial_1(\tau_i) = v_i - v$. Let

$$t = \sum_{i=1}^{n} \alpha_i \tau_i.$$

Then

$$\partial_1(t) = \sum_{i=1}^{n} \alpha_i v_i - \sum_{i=1}^{n} \alpha_i v = \sum_{i=1}^{n} \alpha_i v_i = s$$

since

$$\sum_{i=1}^{n} \alpha_i v = v \sum_{i=1}^{n} \alpha_i = 0.$$

Therefore, $\ker \partial_0 = \mathrm{im} \partial_1$ and $H_0(\Delta) = 0$. ■

With all this, we can compute lots of homologies!

Proposition 1.2.12. Let $X = A \sqcup B$ for simplicial complexes A, B. Then, for all $i \geq 1$,

$$H_i(X) \cong H_i(A) \oplus H_i(B).$$

For $i = 0$ we have $H_i(X) \cong H_i(A) \oplus H_i(B) \oplus R$.

Proof. In the Mayer-Vietoris sequence for $X = A \cup B$, we have $A \cap B = \varnothing$, so for $i \geq 0$, $H_i(A \cap B) = 0$. But $H_{-1}(A \cap B) = R$. Therefore, the sequence divides into the short exact sequences:

$$0 \to H_i(A) \oplus H_i(B) \to H_i(X) \to 0$$

for each $i \geq 1$ and

$$0 \to H_0(A) \oplus H_0(B) \to H_0(X) \to R \to 0.$$

But a short exact sequence of this kind always splits: since $H_0(X) \to R$ is surjective, finding a section is just finding a preimage of $1 \in R$. Therefore $H_0(X) \cong H_0(A) \oplus h_0(B) \oplus R$. ■

Proposition 1.2.13. Let $X = A \vee B$, for simplicial complexes A, B. This is, $A \cap B$ consists of a single vertex. Then $H_i(X) \cong H_i(A) \oplus H_i(B)$ for **all** i.

Proof. In the Mayer-Vietoris sequence for $A \cup B = X$, $H_i(A \cap B) = 0$ for all i. ■

The space $A \vee B$ is called the wedge sum of A, B.

For a simplicial complex Δ and its associated chain complex $C_\bullet(\Delta; R)$ we can define its dual complex $C^\bullet$; its homology will be called the cohomology of Δ with coefficients in R.

Definition 1.2.16. For a simplicial complex Δ define $C^\bullet(\Delta; R)$ as

$$C_\bullet(\Delta; R)^* = \mathrm{Hom}(C_\bullet(\Delta; R), R).$$

The homology $\ker \partial^*_{i+1}/\mathrm{im}\partial^*_i$ of $C^\bullet(\Delta; R)$ will be called the cohomology of Δ with coefficients in R and denoted by $H^i(\Delta; R)$. The elements of the set $Z^i(\Delta; R) = \ker \partial^*_{i+1}$ will be called cocycles, and the elements of the set $B^i(\Delta, R) = \mathrm{im}\partial_i$ will be called coboundaries.

The boundary maps of the cochain complex $C^\bullet$, called coboundary maps can easily be made explicit: If we take $\phi \in C^j(\Delta; R)$ then, for

$$\tau = [v_0, \cdots, v_{n+1}] \in C_{j+1}(\Delta; R)$$

we have

$$\partial^*_{j+1}(\phi)(\tau) = \phi \circ \partial_{j+1}(\tau) = \sum_{i=0}^{j+1} (-1)^i \phi(\tau - \{v_i\}).$$

In particular, for the basis Δ_j of $C_j(\Delta; R)$ and $\sigma \in \Delta_j$ if we take the maps $f_\sigma : C_j(\Delta; R) \to R$ (These maps form a basis for $C^j(\Delta)$) determined by

$$f_\sigma(\sigma') = \delta_{\sigma\sigma'}$$

where δ is the Kronecker delta, then this means that

$$\partial^*_{j+1}(f_\sigma)(\tau) = \sum_{i=0}^{j+1} (-1)^i f_\sigma(\tau - \{v_i\})$$

so that

$$\partial^*_{j+1}(f_\sigma) = \sum_{\substack{v \notin \sigma \\ \sigma \cup v \in \Delta}} \mathrm{sgn}(v, \sigma \cup \{v\}) f_{\sigma \cup \{v\}}$$

since $f_\sigma(\tau - \{v\}) \neq 0$ if and only if $\tau - \{v\} = \sigma$, i.e. when $\tau = \sigma \cup \{v\}$, in which case $f_\sigma(\tau - \{v\}) = 1$; and since v_i is the $i-$th element of τ.

We will state the following particular version of the Universal Coefficient Theorem for Cohomology. It can be proven by using linear algebra, but it follows from the general version when the base ring is a field (so that Tor modules are 0).

Theorem 1.2.14 (Universal Coefficient Theorem for Cohomology)**.** Let k be a field and Δ a simplicial complex. Then there exists an isomorphism:

$$h : H^i(\Delta; k) \to \mathrm{Hom}_k(H_i(\Delta; k), k).$$

1.3 Duality

Here, we will talk about some dual objects of simplicial complexes that inherit the properties of the original objects. The homology of the dual of a simplicial complex will be, up to some permutation of the indexes, the same as the homology of the original complex.

Definition 1.3.1. Let Δ be the simplex with vertex set $\{1, \ldots, n\}$. Then we can define a duality map $\iota : \Delta \to \Delta$ given by $\sigma \mapsto \sigma^c$ for every $\sigma \in \Delta$. This map is just set complementation and is, of course, an involution, i.e. $\iota^2 = 1_\Delta$.

So we can define the *Alexander dual* of a simplicial complex:

Definition 1.3.2. For a simplicial complex X with vertex set $\{1, \ldots, n\}$ we define the Alexander dual $X^\vee$ of X as:

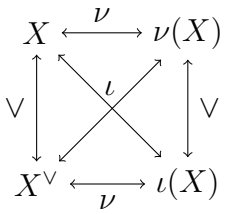
$$X^\vee = \{\iota(\sigma) : \sigma \notin X\} = \{\sigma : \iota(\sigma) \notin X\}.$$

Since the set of all the non-faces of X has a dual property to the one of the simplicial complexes (Every set containing a non-face is a non-face) and complementation reverses inclusions this makes clear that $X^\vee$ is a simplicial complex. We also have the next fact:

Proposition 1.3.1. For a simplicial complex X we have $X^{\vee\vee} = X$.

Proof. A face σ is in $X^{\vee\vee}$ if and only if $\iota(\sigma) \notin X^\vee$, which happens if and only if $\iota^2(\sigma) = \sigma \in X$. ■

We, then, have the following four related structures:



where $\nu(X)$ is the simplicial *cocomplex* (A fancy name for an upper set under the inclusion relation) given by $\iota(X^\vee)$; the set of all the nonfaces of X.

The following proof of the Simplicial Alexander duality theorem is due to Anders Björner and Martin Tancer in [**?**]. For every face σ of a simplicial complex X with vertex set $V = \{1, \ldots, n\}$, define

$$p(\sigma) = \prod_{i \in \sigma} (-1)^{i-1}.$$

Lemma 1.3.2 (Lemma 2.1,[**?**])**.** Let $k \in \sigma \subset \{1, \ldots, n\}$. Then

$$\operatorname{sgn}(k, \sigma) p(\sigma - \{k\}) = \operatorname{sgn}(k, \iota(\sigma) \cup \{k\}) p(\sigma).$$

Proof. We have that $\operatorname{sgn}(k, \sigma) = (-1)^j$ where k is the $j-$th element of σ, i.e.

$$\operatorname{sgn}(k, \sigma) = \prod_{\substack{i \in \sigma \\ i < k}} (-1).$$

Also

$$\operatorname{sgn}(k, \iota(\sigma) \cup \{k\}) = \prod_{\substack{i \notin \sigma \\ i < k}} (-1)$$

and

$$p(\sigma) = \prod_{i \in \sigma} (-1)^{i-1}.$$

Therefore, we have

$$p(\sigma) p(\sigma - \{k\}) = (-1)^{k-1} \prod_{i \in \sigma - \{k\}} (-1)^{i-1} \prod_{i \in \sigma - \{k\}} (-1)^{i-1} = (-1)^{k-1}$$

and

$$\mathrm{sgn}(k, \iota(\sigma) \cup \{k\})\mathrm{sgn}(k, \sigma) = \prod_{\substack{i \notin \sigma \\ i<k}} (-1) \prod_{\substack{i \in \sigma \\ i<k}} (-1) = \prod_{i<k} (-1) = (-1)^{k-1}.$$

Therefore

$$p(\sigma)p(\sigma - \{k\}) = \mathrm{sgn}(k, \iota(\sigma) \cup \{k\})\mathrm{sgn}(k, \sigma)$$

Since all the factors in the expressions above are in $\{-1, 1\}$, they are involutive in R, therefore multiplying at both sides by $p(\sigma)\mathrm{sgn}(k, \sigma)$ we have

$$\mathrm{sgn}(k, \sigma)p(\sigma - \{k\}) = \mathrm{sgn}(k, \iota(\sigma) \cup \{k\})p(\sigma)$$

which is the desired result. ■

Theorem 1.3.3 (Alexander Duality)**.** Let X be a simplicial complex over the vertex set $V = \{1, \ldots, n\}$. Then

$$H_i(X) \cong H^{n-i-3}(X^\vee).$$

Proof. First, if we take the simplex Δ on V, by the long exact sequence of homology, we have an exact sequence

$$\cdots \to H_{i+1}(\Delta) \to H_{i+1}(\Delta, X) \to H_i(X) \to H_i(\Delta) \to H_i(\Delta, X) \to \cdots,$$

but since a simplex has no homology, this sequence breaks into short sequences of the form

$$0 \to H_{i+1}(\Delta, X) \to H_i(X) \to 0$$

for every i, which means the maps

$$H_{i+1}(\Delta, X) \to H_i(X)$$

are isomorphisms.

Now, we must establish an isomorphism

$$H_{i+1}(\Delta, X) \to H^{n-i-3}(X^\vee).$$

We will do that by establishing an appropriate chain complex isomorphism between the complexes $C_\bullet(\Delta, X)$ and $C^\bullet(X^\vee)$. Define $\phi_i : C_i(\Delta, X) \to C^{n-i-2}(X^\vee)$ by

$$\phi_i(\sigma) = p(\sigma) f_{\iota(\sigma)}$$

for $\sigma \in \Delta_i - X_i$. This is a well defined map, since if $\sigma \in \Delta_i - X_i$ then $\iota(\sigma) \in X^\vee_{n-i-2}$, thus $f_{\iota(\sigma)} \in C^{n-i-2}(X^\vee)$. This also sends the standard basis of $C_i(\Delta, X)$ into the (up to signs) standard basis of $C^{n-i-2}(X^\vee)$, so it is an isomorphism at each dimension. The only thing left to prove is that it is a chain complex isomorphism so that it makes the following diagram commute for each i:

$$\begin{array}{ccc} C_{i-1}(\Delta, X) & \xleftarrow{\delta_i} & C_i(\Delta, X) \\ \Big\downarrow{\scriptstyle \phi_{i-1}} & & \Big\downarrow{\scriptstyle \phi_i} \\ C_{n-i-1}(\Delta, X) & \xleftarrow[\partial^*_{n-i-1}]{} & C^{n-i-2}(\Delta, X) \end{array}$$

But this is straightforward, for $\sigma \in \Delta_i - X_i$:

$$\begin{aligned} \phi_{i-1}\delta_i(\sigma) &= \phi_{j-1}\left(\sum_{\substack{v\in\sigma \\ \sigma-\{v\}\notin X}} \mathrm{sgn}(k,\sigma)(\sigma - \{v\})\right) \\ &= \sum_{\substack{v\in\sigma \\ \sigma-\{v\}\notin X}} \mathrm{sgn}(k,\sigma)p(\sigma - \{v\})f_{\iota(\sigma-\{v\})} \\ &= \sum_{\substack{v\in\sigma \\ \sigma-\{v\}\notin X}} p(\sigma)\mathrm{sgn}(k,\iota(\sigma)\cup\{v\})f_{\iota(\sigma-\{v\})} \\ &= \sum_{\substack{v\in\sigma \\ \sigma-\{v\}\notin X}} p(\sigma)\mathrm{sgn}(k,\iota(\sigma)\cup\{v\})f_{\iota(\sigma)\cup\{v\}} \\ &= \partial^*_{n-i-1}(p(\sigma)f_{\iota(\sigma)}) \\ &= \partial^*_{n-i-1}\phi_i(\sigma) \end{aligned}$$

which is the desired result. Therefore, since the chain complexes above are isomorphic, they have the same homology. Therefore, for each i,

$$H_i(X) \cong H_{i+1}(\Delta, X) \cong H^{n-i-3}(X^\vee).$$

■

Corollary 1.3.4 (Alexander Duality)**.** Let X be a simplicial complex over the vertex set $V = \{1, \ldots, n\}$ and k a field. Then

$$\dim_k H_i(X; k) = \dim_k H_{n-i-3}(X^\vee; k).$$

Proof. It follows from the universal coefficient theorem and the fact that X and $X^\vee$ have a finite number of faces of each dimension: Since in each (topological) dimension, the cohomology is the dual of the homology, and the homology is a finite dimensional $k-$vector space, they have the same dimension. ■

1.4 Hypergraphs and Their Ideals

The ideals described here will be the ones we will be mostly working on. There is a lot of theory about these ideals given in [?] or [?], but most of it won't be needed for our work here.

Definition 1.4.1. A hypergraph G is a pair (V, E) where $V = V(G)$ is a set (called vertex set) and $E = E(G)$ is a set of nonempty subsets of V (called edge set).

Note that when every $e \in E$ satisfies $|e| = 2$, then G is a graph. When for any $e \in E$ and $f \subseteq e$ we have $f \in E$, then G is a simplicial complex.

Definition 1.4.2. Let k be a field and G be a hypergraph on a vertex set $V = \{v_1, \ldots, v_n\}$. For a set of indeterminates (Which may as well be V) $\{x_1, \ldots, x_n\}$ define $x(v_i) = x_i$ for all i. The ideal

$$I = I_G = \left(\prod_{v \in e} x(v) \mid e \in E \right) \subseteq k[x_1, \ldots, x_n]$$

is called the edge ideal of G.

If G is a hypergraph with vertex set $\{1, \ldots, n\}$ this can be rewritten as

$$I_G = \left(\prod_{i \in e} x_i \mid e \in E \right).$$

Furthermore, if G is a graph, this can be rewritten further as

$$I_G = (x_i x_j \mid \{i, j\} \in E).$$

For a graph, we will use the notation vw for the edge $\{v, w\}$. A subgraph H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We

say that H is an induced subgraph if $E(H) = \{e \in E(G) \mid e \subseteq V(H)\}$; this definition can be generalized to define induced subhypergraph or induced simplicial subcomplex.

We say a cycle C in a graph G is a subgraph such that if $V(C) = \{v_1, \ldots, v_r\}$ then $E(C) = \{v_1v_2, v_2v_3, \ldots, v_{r-1}v_r, v_1v_r\}$. We also define the cycle and path C_n, P_n: C_n is the graph with n vertices, which is an induced cycle in itself, and P_n is the graph resulting from C_n after removing an edge. A path in a graph G is said to be a subgraph isomorphic to P_n for some n. A graph G is said to be connected if between any two vertices of G, there is a path containing both. A graph without cycles is called a forest. A connected graph without cycles is called a tree.

Chapter 2

Hochster's Formula

Here, we will talk about Hochster's formula. This formula shows that the Betti numbers of a monomial ideal can be computed by computing the homology of a family of simplicial complexes instead. Although its proof is very convoluted, it is fair enough because it allows us to avoid giving even more convoluted proofs of many other results. It is fundamental for this work.

2.1 Minimal Free Resolutions

Here we will talk about the fundamentals of homological algebra needed to understand Hochster's formula. Surprisingly, once we do that, we won't need many of the results given here. Hochster's formula and Mayer-Vietoris sequences are enough to prove many things in a more arithmetical and combinatorial way.

A chain complex

$$\mathcal{M}_\bullet : 0 \leftarrow M_0 \xleftarrow{\phi_1} M_1 \xleftarrow{\phi_2} M_2 \leftarrow \cdots$$

of $R-$modules is said to be a free resolution of an $R-$module M if $\mathcal{M}$ is exact everywhere except in the homological degree 0, and $M \cong \text{coker}\phi_1 = M_0/\text{im}\phi_1$. Analogously, $\mathcal{M}$ is a free resolution if the sequence

$$\mathcal{M}'_\bullet : 0 \leftarrow M \xleftarrow{\phi_0} M_0 \xleftarrow{\phi_1} M_1 \xleftarrow{\phi_2} M_2 \leftarrow \cdots$$

is exact everywhere, where ϕ_0 is the projection (since $M \cong \text{coker}\phi_1$). We will name both $\mathcal{M}$ and $\mathcal{M}'$ as $\mathcal{M}$ and use one or the other according to the context.

The following proposition can be generalized to projective resolutions, but since we will be working over a polynomial ring, we will have enough with free ones.

Proposition 2.1.1. Any two free resolutions $(\mathcal{M}, \phi), (\mathcal{N}, \psi)$ of a $R-$module M are chain-homotopically equivalent.

Our first ingredient for computing the Betti numbers of an ideal is a minimal free resolution of it.

Definition 2.1.1 (Minimal free resolution)**.** Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables,

$$\mathfrak{m} = (x_1, \ldots, x_n),$$

M a $R-$module, and $\mathcal{M} = (M_i, \phi_i)$ a free resolution of M. We say that $\mathcal{M}$ is minimal if $\mathrm{im}\phi_i \subseteq \mathfrak{m}M_{i-1}$ for all $i \geq 1$. The same definition will do when $(R, \mathfrak{m})$ is instead a local ring.

Remark. Equivalently, $\mathcal{M}$ is minimal if and only if $\mathcal{M} \otimes_R k$ is a complex with null morphisms, where $k = R/\mathfrak{m}$.

Lemma 2.1.2. Let M be a $R-$module and $\mathfrak{m}$ as before. Then, there is an isomorphism

$$\psi : M \otimes_R k \to M/\mathfrak{m}M.$$

Proof. Consider the exact sequence:

$$0 \to \mathfrak{m} \to R \to k \to 0,$$

where the two nonzero morphisms are, from left to right, the inclusion and canonical projection. By the exactness of the tensor product, the induced sequence:

$$\mathfrak{m} \otimes_R M \to R \otimes_R M \to k \otimes_R M \to 0$$

is exact. Furthermore, $R \otimes M \cong M$ by the isomorphism $m \mapsto 1 \otimes m$. By this isomorphism, the submodule $\mathfrak{m}M$ is given by $\mathfrak{m} \otimes_R M$, therefore

$$M/\mathfrak{m}M \cong R \otimes_R M/\mathfrak{m}(R \otimes_R M) \cong k \otimes_R M.$$

■

Proof of the Remark. By the lemma, for every i,

$$M_i/\mathfrak{m}M_i \cong k \otimes_R M_i$$

by the map $m+\mathfrak{m}M_i \mapsto 1\otimes m$, with inverse $r\otimes m \mapsto rm$ for $r \in R$. So, $\mathcal{M}$ is minimal if for every i, $\phi_i \otimes 1 = 0$ if and only if, for every i, m, $\phi_i(m)\otimes 1 = 0$, if and only if for every i, m, $\phi_i(m) + \mathfrak{m}M_{i-1} = 0$, if and only if for every i, m, $\phi_i(m) \in \mathfrak{m}M_{i-1}$, if and only if for every i, $\mathrm{im}\phi_i \subseteq \mathfrak{m}M_{i-1}$. ■

This means that if all the free modules of the free resolution are finitely generated (as in Hilbert's Syzygy Theorem), any matrix representing some ϕ_i has all its entries in M.

The following about free resolutions can generalized to projective resolutions over any commutative ring with unity.

Now, suppose that everything is $\mathbb{N}^n-$graded, so that a free graded module M of finite rank has a direct sum decomposition

$$M = R(-\mathbf{a}_1) \oplus \cdots \oplus R(-\mathbf{a}_r)$$

for some $\mathbf{a}_i \in \mathbb{N}^n$, with grading given by $R(-\mathbf{a}_i)_\mathbf{s} \cong R_{\mathbf{s}-\mathbf{a}_i}$ for $\mathbf{s} \in \mathbb{N}^n$ such that $\mathbf{s}-\mathbf{a}_i \in \mathbb{N}^n$. Also, a graded free resolution $\mathcal{M}$ will be a free resolution given by free graded modules with boundary maps of degree 0, i.e. $\phi(M_\mathbf{s}) = \phi(M)_\mathbf{s}$ for a boundary map ϕ of $\mathcal{M}$ and a module M in the resolution.

Definition 2.1.2. A monomial matrix is a matrix $M \in M_{n\times n}(k)$ with labels $\mathbf{a}_p, \mathbf{a}_q$ in its pth column and qth row, and entries λ_{qp} such that $\lambda_{qp} = 0$ unless $\mathbf{a}_p - \mathbf{a}_q \in \mathbb{N}^n$.

The previously stated condition is equivalent to $\frac{\mathbf{x}^{\mathbf{a}_p}}{\mathbf{x}^{\mathbf{a}_q}}$ being a well-defined polynomial or $\mathbf{x}^{\mathbf{a}_q} \mid \mathbf{x}^{\mathbf{a}_p}$.

Remark. The $R-$homomorphism

$$\bigoplus_p R(-\mathbf{a}_p) \to \bigoplus_q R(-\mathbf{a}_q)$$

determined by the monomial matrix $(\lambda_{qp})_{q,p}$ is just given by matricial product with the matrix $(\lambda_{qp}\mathbf{x}^{\mathbf{a}_p-\mathbf{a}_q})_{q,p}$.

Definition 2.1.3. A monomial matrix is minimal if $\lambda_{qp} = 0$ for $\mathbf{a}_p = \mathbf{a}_q$.

Remark. Given a free resolution of M given by minimal monomial matrices, then the free resolution is, in fact, minimal because all the entries of the matrices are in $\mathfrak{m}$. Conversely, if a minimal free resolution has a representation of its morphisms by monomial matrices, they must be minimal because all of its entries must lie in $\mathfrak{m}$.

In a free graded module $\bigoplus_p R(-\mathbf{a}_p)$, each $\mathbf{a}_p$ may appear more than once, in such case, we can instead use the notation

$$\bigoplus_{\mathbf{a}\in\mathbb{N}^n} R(-\mathbf{a})^{\beta_{\mathbf{a}}}.$$

In a minimal free resolution (M_i, ϕ_i) of a module M, with

$$M_i = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} R(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$$

each $\beta_{i,\mathbf{a}}$ is called the i^{th} Betti number of M in degree $\mathbf{a}$.

We can also define the Betti numbers by computing the derived functors $\operatorname{Tor}^R(-,k)$ of $-\otimes_R k$.[(i)] If we do so, it'll follow that the Betti numbers can, in theory, be computed without having a *minimal* free resolution, so that we can be content with any free resolution.

Remark. In the graded case, we have an isomorphism

$$\bigoplus_{\mathbf{a}\in\mathbb{N}^n} R(-\mathbf{a})^{\beta_{\mathbf{a}}} \otimes k \cong \bigoplus_{\mathbf{a}\in\mathbb{N}^n} k(-\mathbf{a})^{\beta_{\mathbf{a}}}.$$

Lemma 2.1.3. The i-th Betti number of an $\mathbb{N}^n-$graded module M in degree $\mathbf{a}$ equals the vector space dimension $\dim_k \operatorname{Tor}_i^R(k,M)_{\mathbf{a}}$.

Proof. Take a minimal free resolution

$$\mathcal{M} : 0 \leftarrow \cdots \leftarrow \bigoplus_{\mathbf{a}\in\mathbb{N}^n} R(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \leftarrow \bigoplus_{\mathbf{a}\in\mathbb{N}^n} R(-\mathbf{a})^{\beta_{i+1,\mathbf{a}}} \leftarrow \cdots \leftarrow 0.$$

When tensoring with k, we get the complex

$$\mathcal{M}\otimes k : 0 \leftarrow \cdots \leftarrow \bigoplus_{\mathbf{a}\in\mathbb{N}^n} k(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \leftarrow \bigoplus_{\mathbf{a}\in\mathbb{N}^n} k(-\mathbf{a})^{\beta_{i+1,\mathbf{a}}} \leftarrow \cdots \leftarrow 0,$$

[(i)] The functors Tor are computed by taking any projective resolution, in particular, free ones, tensoring with k and computing homology.

with all maps zero. Thus,

$$H_i(\mathcal{M} \otimes k) \cong \bigoplus_{\mathbf{a} \in \mathbb{N}^n} k(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$$

and

$$\beta_{i,\mathbf{a}} = \dim_k \operatorname{Tor}_i^R(k, M)_{\mathbf{a}}.$$

■

2.2 Hochster's Formula

Here, every ideal is supposed to be monomial, i.e., generated by monomials.

Definition 2.2.1 (Koszul Complex)**.** Consider the simplex Δ^{n-1} consisting on all the subsets of $\{x_1, \ldots, x_n\}$. In every matrix $(a_{ij})_{i,j}$ of the exact sequence $\tilde{C}_\bullet(\Delta^n, R)$, consider it as a monomial matrix, with labels in rows and columns given by χ_σ for faces σ generating the free summands of $R^{\binom{n}{i}}$ in homological degree $i+1$. Then, renumber the homological degrees so that the empty face $\varnothing$ is in homological degree 0. The resulting complex $K_\bullet$ is called the Koszul complex.

This is just done by taking the reduced chain complex of Δ^n and adjusting the morphisms so that they become graded.

Proposition 2.2.1. The Koszul complex is a minimal free resolution of k.

Proof. Since the boundary maps of the Koszul complex are given by considering the matrices of the boundary maps of $\tilde{C}_\bullet(\Delta^n, R)$ as monomial matrices, then every one of the maps in the Koszul complex are monomial. Also, since every entry of every one of these matrices is zero unless the face associated with the row is strictly contained in the face associated with the column, the maps of the Koszul complex are, in fact, minimal.

To prove the Koszul complex is a resolution, we must prove it is exact everywhere except in homological degree 0. So, restricting the Koszul complex to some degree $\mathbf{d} \in \mathbb{N}^n$, we get a subcomplex of $k-$vector spaces, which we want to prove is (isomorphic to) the reduced complex $\tilde{C}(\Delta^{\dim \mathbf{d}}, k)$ where $\dim \mathbf{d}$ is just the number of variables of $\mathbf{d}$ minus one. For degree $\mathbf{d} = 0$, we get the complex $0 \to k \to 0$, which is not exact. Otherwise, since a summand of each homological degree of the Koszul complex is $\neq 0$ in degree

$\mathbf{d}$ if and only if it is has the form $R(-\mathbf{d}') = R\mathbf{x}^{\mathbf{d}'}$ for $\mathbf{d}' \leq \mathbf{d}$, we get the complex $0 \leftarrow k \leftarrow k^s \leftarrow k^{\binom{s}{2}} \leftarrow \cdots \leftarrow k^{\binom{s}{s-2}} \leftarrow k^s \leftarrow k \leftarrow 0$, where, in each homological degree i, $k^{\binom{s}{i}}$ is considered as the direct sum

$$\bigoplus_{\mathbf{d}' \leq \mathbf{d}, \dim \mathbf{d}' = i-1} k\mathbf{x}^{\mathbf{d}}$$

and $s = \dim \mathbf{d} + 1$. And given that,

$$\partial(\mathbf{x}^{\mathbf{d}} e_{\mathbf{d}'}) = \sum_{\mathbf{d}'' \leq \mathbf{d}', \dim \mathbf{d}'' = i-2} \alpha_{\mathbf{d}'\mathbf{d}''} \mathbf{x}^{\mathbf{d}} e_{\mathbf{d}''}$$

where $\alpha_{\mathbf{d}'\mathbf{d}''}$ is the $\mathbf{d}'\mathbf{d}''$ entry of the matrix of the map ∂ in $\tilde{C}(\Delta^{\dim \mathbf{d}}, k)$. This means that when we restrict to degree $\mathbf{d}$ in $K_\bullet$, the resulting map is just the boundary map of the reduced chain complex of $\Delta^{\dim \mathbf{d}}$ with coefficients in k, which is what we expected.

Then, since every non-irrelevant simplex is contractible, this complex has null homology for $\mathbf{d} \neq 0$, from which it follows that $K_\bullet$ also has null homology except in the homology degree 0, on which the homology is k. It follows that $K_\bullet$ is a free resolution of k. ■

Definition 2.2.2 (Upper Koszul Complex)**.** For a monomial ideal I and a degree $\mathbf{b} \in \mathbb{N}^n$, define

$$K^{\mathbf{b}}(I) = \{\tau \in \{0,1\}^n : \mathbf{x}^{\mathbf{b}-\tau} \in I\}$$

to be the (upper) Koszul simplicial complex of I in degree $\mathbf{b}$.

If $\mathbf{b}$ is the degree associated with the least common multiple of all the generators of I, then $K^{\mathbf{b}}(I)$ will be denoted by $K(I)$. If I is the edge ideal of a hypergraph G then $K^{\mathbf{1}}(I)$ will be denoted by $K(G)$, where $\mathbf{1}$ is the characteristic vector of $V(G)$.

Definition 2.2.3 (Lower Koszul complex)**.** Let $\mathbf{b} \in \mathbb{N}^n$ and I a monomial ideal of the polynomial ring R in n variables. Define the lower Koszul complex of I as

$$K_{\mathbf{b}}(I) = \{\tau \in \{0,1\}^n \mid \tau \leq \mathbf{b}, \mathbf{x}^{\mathbf{b}-\mathbf{1}+\tau} \notin I\}$$

where $\mathbf{1} = \text{Supp}\mathbf{b}$.

it is easy to prove that $K_{\mathbf{b}}(I) = (K^{\mathbf{b}}(I))^\vee$, so we can denote $K_{\mathbf{b}}(I)$ as $K^\vee(I)$ or $K^\vee(G)$ in the respective cases.

We will now state Hochster's formula.

Theorem 2.2.2 (Hochster's Formula)**.** If $\beta_{i,\mathbf{b}}(I)$ is the i-th Betti number of a monomial ideal I in degree $\mathbf{b} \in \mathbb{N}^n$, then

$$\beta_{i,\mathbf{b}}(I) = \begin{cases} \dim_k H_{i-1}(K^{\mathbf{b}}(I); k), \\ \dim_k H^{n-i-2}(K_{\mathbf{b}}(I); k), \\ \dim_k H^{i-1}(K^{\mathbf{b}}(I); k), \\ \dim_k H_{n-i-2}(K_{\mathbf{b}}(I); k). \end{cases}$$

The first and second formulas are the well known versions of Hochster's formula, and the third and fourth follow from the Universal Coefficient Theorem.

We can also see that when I is the edge ideal of a graph, $K(G)$ and $K^\vee(G)$ are the complexes of non-edge covers and independent sets of G, respectively. So, in particular, we have the following:

Corollary 2.2.3 (Hochster's Formula)**.** If G is a graph, $I = I_G$ is the edge ideal of G, $K = K(G)$ is the simplicial complex of non-vertex covers of G and $K^\vee$ is the complex of independence of G,

$$\beta_{i,\mathbf{1}}(I) = \begin{cases} \dim_k H_{i-1}(K; k), \\ \dim_k H^{n-i-2}(K^\vee; k), \\ \dim_k H^{i-1}(K; k), \\ \dim_k H_{n-i-2}(K^\vee; k). \end{cases}$$

From now on, when we talk about the homology of a simplicial complex X, it will be the homology over k, unless stated otherwise.

2.3 Properties of $K^{\mathbf{b}}(I)$.

The operator $K^{\mathbf{b}}$ has some nice properties with respect to the order structure of the monomial ideals of R. it is known that the monomial ideals of R form a lattice, ordered by inclusion, with $+$ and $\cap$ being the lattice operations. It will be proven that $K^{\mathbf{b}}$ preserves this structure.

Here $I = (m_1, \ldots, m_r), J = (m'_1, \ldots, m'_s)$ are monomial ideals of R.

Proposition 2.3.1. $K^{\mathbf{b}}(I+J) = K^{\mathbf{b}}(I) \cup K^{\mathbf{b}}(J)$.

Proof. We have

$$\begin{aligned} K^{\mathbf{b}}(I+J) &= \{\tau \in \{0,1\}^n : \mathbf{x}^{b-\tau} \in I+J\} \\ &= \{\tau \in \{0,1\}^n : (\exists i \in [r]); m_i \mid \mathbf{x}^{b-\tau} \text{ o } (\exists j \in [s]); m'_j \mid \mathbf{x}^{b-\tau}\} \\ &= \{\tau \in \{0,1\}^n : (\exists i \in [r]); m_i \mid \mathbf{x}^{b-\tau}\} \\ &\cup \{\tau \in \{0,1\}^n : (\exists j \in [s]); m'_j \mid \mathbf{x}^{b-\tau}\} \\ &= K^{\mathbf{b}}(I) \cup K^{\mathbf{b}}(J). \end{aligned}$$

■

Proposition 2.3.2. $K^{\mathbf{b}}(I \cap J) = K^{\mathbf{b}}(I) \cap K^{\mathbf{b}}(J)$.

Proof. Using the previous proposition and the fact that

$$\begin{aligned} K^{\mathbf{b}}(m_i) \cap K^{\mathbf{b}}(m'_j) &= \{\tau \in \{0,1\}^n : m_i \mid \mathbf{x}^{b-\tau}\} \cap \{\tau \in \{0,1\}^n : m'_j \mid \mathbf{x}^{b-\tau}\} \\ &= \{\tau \in \{0,1\}^n : m_i \mid \mathbf{x}^{b-\tau} \text{ y } m'_j \mid \mathbf{x}^{b-\tau}\} \\ &= \{\tau \in \{0,1\}^n : \operatorname{lcm}(m_i, m'_j) \mid \mathbf{x}^{b-\tau}\} \\ &= K^{\mathbf{b}}(\operatorname{lcm}(m_i, m'_j)), \end{aligned}$$

we have

$$\begin{aligned} K^{\mathbf{b}}(I) \cap K^{\mathbf{b}}(J) &= \left(\bigcup_{i\in[r]} K^{\mathbf{b}}(m_i)\right) \cap \left(\bigcup_{i\in[r]} K^{\mathbf{b}}(m_i)\right) \\ &= \bigcup_{(i,j)\in[r]\times[s]} K^{\mathbf{b}}(m_i) \cap K^{\mathbf{b}}(m'_j) \\ &= \bigcup_{(i,j)\in[r]\times[s]} K^{\mathbf{b}}(\operatorname{lcm}(m_i, m'_j)) \\ &= K^{\mathbf{b}}((\operatorname{lcm}(m_i, m'_j) : (i,j) \in [r] \times [s])) \\ &= K^{\mathbf{b}}(I \cap J). \end{aligned}$$

■

There are also the following:

Proposition 2.3.3. Let G be a hypergraph with $E(G) = \{e_1, \ldots, e_r\}$. Then $K(G)$ is the simplicial complex generated by the complements of the edges.

Proof. For each edge e, let $\mathbf{b}_e$ be the characteristic vector of e and $\mathbf{c}_e$ the characteristic vector of e^c. Then $\mathbf{1} - \mathbf{c}_e = \mathbf{b}_e$. Therefore $\mathbf{x}^{\mathbf{1}-\mathbf{c}_e} \in I_G$.

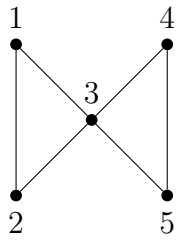
Now, if $\tau \in K(G)$, then $\mathbf{x}^{\mathbf{1}-\tau} \in I$, which means $e \subseteq e_{\mathbf{1}-\tau}$ for some $e \in E(G)$, where $e_{\mathbf{1}-\tau}$ is the subset of $V(G)$ associated to the vector $\mathbf{1} - \tau$. That means $e_\tau = e^c_{\mathbf{1}-\tau}$ does not contain any vertex in e, which means that $e_\tau \subseteq e^c$. ■

This means that to compute $K(G)$, we only need to compute the complements of the edges of G.

Chapter 3

Using Hochster's Formula Recursively

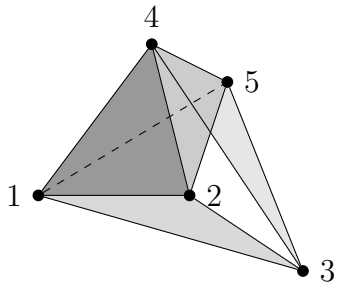
Hochster's formula allows us to compute Betti numbers of ideals. For example, consider the edge ideal I_G of the graph



i.e. the ideal $I_G = (x_1x_2, x_1x_3, x_2x_3, x_3x_5, x_3x_4, x_4x_5)$. We will first compute $K(G) = K^{\mathbf{1}}(I_G)$. The facets of $K(G)$ will be

$$145, 245, 345, 123, 124, 125.$$

The resulting complex is as follows.

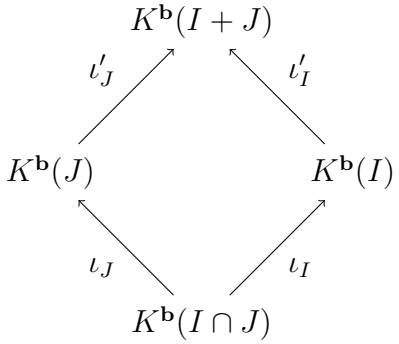


Considering $K(G)$ as a $CW-$complex, and using the proposition 0.17 from [**?**], we get, by computing $X_i = X_{i-1}/A_{i-1}$ sequentially, first for $X_0 = K^{\mathbf{1}}(_G)$ and $A_0 = x_1x_2x_3$, then $A_1 = x_4x_5$, and then $A_2 = x_{123}x_{45}$ (The strip resulting from contracting $x_1x_2x_3$ and then x_4x_5 to points x_{123}, x_{45}). So, $X_3 \simeq K^{\mathbf{1}}(I_G)$ but $X_3 \cong S^2/\{a, -a\}$ for some point $a \in S^2$. This is a sphere with two points identified, and using the long exact sequence for homology, we get $H_2(X_3) \cong H_2(S^2) \cong k$ and $H_1(X_3) \cong H_0(S^0) = k$ (where S^0 is the $0-$dimensional sphere, a two-point space). Being a bit more careful, we can check that $K^{\mathbf{1}}(I_G)$ has the homotopy lower neighbor of $S^2 \vee S^1$, which gives the same result.

Of course, these tools used are strongly topological. We are interested in finding tools to compute Betti numbers that do not rely so much on the topology of $K^{\mathbf{b}}$ but on the algebra and combinatorics behind it. The Mayer-Vietoris sequences that arise here are the first examples of such a tool.

3.1 Mayer-Vietoris Sequences

Since $K^{\mathbf{b}}$ preserves intersections and transforms $+$ into $\cup$, it would be a shame not to make use of this to build Mayer-Vietoris sequences. See [**?**] for a lot of work on this. We have the inclusions:



so, if $\mathbf{b} \in \mathbb{N}^n$ is such that $\mathbf{x}^{\mathbf{b}} \in K^{\mathbf{b}}(I \cap J)$, then there is a long exact sequence:

$$\cdots \xrightarrow{\partial} H_i(K^{\mathbf{b}}_{I\cap J}) \xrightarrow{\alpha_*} H_i(K^{\mathbf{b}}_I) \oplus H_i(K^{\mathbf{b}}_J) \xrightarrow{\beta_*} H_i(K^{\mathbf{b}}_{I+J}) \xrightarrow{\partial} H_{i-1}(K^{\mathbf{b}}_{I\cap J}) \xrightarrow{\alpha_*} H_{i-1}(K^{\mathbf{b}}_I) \oplus H_{i-1}(K^{\mathbf{b}}_J) \xrightarrow{\beta_*} H_{i-1}(K^{\mathbf{b}}_{I+J}) \xrightarrow{\partial} \cdots$$

where $K^{\mathbf{b}}_* = K^{\mathbf{b}}(*)$, $\alpha = (\iota_I, -\iota_J)$, $\beta = \iota'_I + \iota'_J$ and the maps α_*, β_* are the induced maps. In particular, if we take $I' = (m_1, \ldots, m_{r-1})$ and $I'' = (\operatorname{lcm}(m_1, m_r), \ldots, \operatorname{lcm}(m_{r-1}, m_r))$, then for $\mathbf{b}$ such that $\mathbf{x}^{\mathbf{b}} \in I''$ there is a long exact sequence:

$$\cdots \xrightarrow{\partial} H_i(K^{\mathbf{b}}_{I''}) \longrightarrow H_i(K^{\mathbf{b}}_{I'}) \oplus H_i(K^{\mathbf{b}}_{m_r}) \longrightarrow H_i(K^{\mathbf{b}}_I) \xrightarrow{\partial} H_{i-1}(K^{\mathbf{b}}_{I''}) \longrightarrow H_{i-1}(K^{\mathbf{b}}_{I'}) \oplus H_{i-1}(K^{\mathbf{b}}_{m_r}) \longrightarrow H_{i-1}(K^{\mathbf{b}}_I) \xrightarrow{\partial} \cdots$$

so if we can compute $H_i(K^{\mathbf{b}}(I'')), H_i(K^{\mathbf{b}}(I')), H_i(K^{\mathbf{b}}(m_r))$ then we will have a lot of information of $H_i(K^{\mathbf{b}}(I))$. In particular, maybe we will have in this information the numbers $\dim_k H_i(K^{\mathbf{b}}(I))$, from which we'd be able to compute the Betti numbers of I recursively. This is a reason this will be called a recursive Mayer-Vietoris sequence of I (A sequence gotten from decomposing the ideal into the sum of a principal ideal and of another ideal). These sequences are used extensively in [**?**]. By an application of a recursive Mayer-Vietoris sequence, we have the following result:

Proposition 3.1.1. Let $I = (m_1, \ldots, m_r)$ and $m = \operatorname{lcm}(m_1, \ldots, m_r) = \mathbf{x}^a$. Then $H_i(K^{\mathbf{b}}(I)) = 0$ for all i and $\mathbf{a} < \mathbf{b}$.

Proof. By induction on r. For such b, we can take the Mayer-Vietoris sequence:

$$\cdots \xrightarrow{\partial} H_i(K^{\mathbf{b}}_{I''}) \longrightarrow H_i(K^{\mathbf{b}}_{I'}) \oplus H_i(K^{\mathbf{b}}_{m_r}) \longrightarrow H_i(K^{\mathbf{b}}_I) \xrightarrow{\partial} H_{i-1}(K^{\mathbf{b}}_{I''}) \longrightarrow H_{i-1}(K^{\mathbf{b}}_{I'}) \oplus H_{i-1}(K^{\mathbf{b}}_{m_r}) \longrightarrow H_{i-1}(K^{\mathbf{b}}_I) \xrightarrow{\partial} \cdots$$

By induction hypothesis, and the base case in which $K^{\mathbf{b}}(m_r)$ will be a simplex, $H_i(K^{\mathbf{b}}(I')), H_i(K^{\mathbf{b}}(m_r)), H_{i-1}(K^{\mathbf{b}}(I''))$ are all 0, so $H_i(K^{\mathbf{b}}(I))$ is 0. The proof will only be complete if we prove the base case, where I is principal.

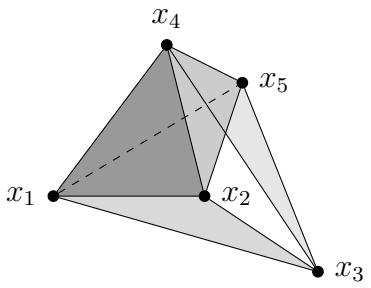
If $I = (\mathbf{x}^{\mathbf{a}})$ is principal then I is a free $R-$module, so the minimal free resolution of I is just $0 \to R\mathbf{x}^{\mathbf{a}} \to 0$, so the only Betti number is $\beta_{0,\mathbf{a}} = 1 = \dim_k H_{-1}(K^{\mathbf{a}}(\mathbf{x}^{\mathbf{a}}))$. Therefore, for $\mathbf{b} > \mathbf{a}$, $H_i(K^{\mathbf{b}}(\mathbf{x}^{\mathbf{a}})) = 0$. ∎

Using recursive Mayer-Vietoris sequences is a natural way to do induction proofs on the number of generating monomials since both the ideals I'' and I' have fewer generating monomials than I. However, the underlying decomposition of I is not unique, and it is not always the most useful we can use. An example of this is in order.

Example 3.1.2. Remember the ideal

$$I = (x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_3x_5, x_4x_5).$$

We wanted to compute $H_i(K^{\mathbf{1}}(I))$. Also remember that $K^{\mathbf{1}}(B)$ has a topological representation as:



Another method to compute its homology is to use (by a topological reasoning) a Mayer-Vietoris sequence for $X = K^{\mathbf{1}}(I) = A \cup B$ where

$$A \cong \partial\Delta^3 \cong S^2 = \{x_1x_2x_4, x_1x_2x_5, x_1x_4x_5, x_2x_4x_5\}$$

and

$$B \simeq \{x_3\} = \{x_1x_2x_3, x_3x_4x_5\}.$$

Also note that $A \cap B \simeq \{x_1, x_4\} = \{x_1x_2, x_4x_5\}$, in the Mayer-Vietoris sequence, we have the same sequences we got before

$$0 \to H_1(X) \to H_0(A \cap B) \to 0$$

and

$$0 \to H_2(A) \to H_2(X) \to 0$$

which mean that $H_2(X) \cong H_2(\partial\Delta^3) \cong k$ and $H_1(X) \cong H_0(A \cap B) \cong k$. The rest of the (reduced) homologies of X are zero. But we can do this without using so many *homotopical equivalence* arguments.

First, note that $K^{\mathbf{1}}(x_1x_2) = \{x_3x_4x_5\}$ and $K^{\mathbf{1}}(x_4x_5) = \{x_1x_2x_3\}$, so $K^{\mathbf{1}}(x_1x_2, x_4x_5) = B$. Also, we know that

$$H_i(K^{\mathbf{1}}(x_1x_2, x_4x_5)) = 0$$

for every i since $\operatorname{lcm}(x_1x_2, x_4x_5) = x_1x_2x_4x_5 \neq x_1x_2x_3x_4x_5$. Also

$$K^{\mathbf{1}}(x_1x_3, x_2x_3, x_3x_4, x_3x_5) = A,$$

and

$$A \cap B = K^{\mathbf{1}}(x_1x_2x_3, x_3x_4x_5),$$

so $H_0(A \cap B) = k$ and 0 elsewhere. Now, $A = K^{\mathbf{1}}(x_1, x_2, x_4, x_5) \cap K_1(x_3)$.

So we can now refer to the Mayer-Vietoris sequence of the ideal $\mathfrak{m}$. In this sequence, everything is 0 except $h_3(K^{\mathbf{1}}(\mathfrak{m})) \cong k$ since a minimal free resolution of $\mathfrak{m}$ is just the Koszul complex snipping the first R. So

$$H_2(A) \cong H_3(K^{\mathbf{1}}(\mathfrak{m})) \cong k.$$

While this decomposition works, there are some more general decompositions of I which we are interested in. The following lemma can be proven by other means, but we will use Mayer-Vietoris sequences to prove it.

Lemma 3.1.3. Let $I = (m_1, \ldots, m_r) \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal and $I' = (m_1, \ldots, m_r) \subseteq k[x_1, \ldots, x_n, y]$. Then, for a degree $\mathbf{b} \in \{0,1\}^{n+1}$ such that $y \mid \mathbf{x}^{\mathbf{b}}$ we have $\beta_{i,\mathbf{b}}(I') = 0$ for all i. Additionally, for $\mathbf{b} \in \{0,1\}^{n+1}$ such that $y \nmid \mathbf{x}^{\mathbf{b}}$, $\beta_{i,\mathbf{b}}(I') = \beta_{i,\mathbf{b}'}(I)$, where $\mathbf{b}' = (\mathbf{b}_i)_{i=1}^n$.

Proof. We will start with the last part. We know that

$$\beta_{i,\mathbf{b}}(I') = \dim_k H_{i-1}(K^{\mathbf{b}}(I'))$$

and

$$\beta_{i,\mathbf{b}'}(I) = \dim_k H_{i-1}(K^{\mathbf{b}'}(I)).$$

But for $\mathbf{b}$ such that $y \nmid \mathbf{x}^{\mathbf{b}}$ and $\tau \in \{0,1\}^{n+1}$, we have $\mathbf{x}^{\mathbf{b}-\tau} \in I'$ if and only if $\mathbf{x}^{\mathbf{b}'-\tau'} \in I$. Thus, the inclusion $\tau' \mapsto \tau$ induces an isomorphism $K^{\mathbf{b}}(I') \cong K^{\mathbf{b}'}(I)$ of simplicial complexes (They are just the same except for a last coordinate in each face which is 0). For $\mathbf{b}$ such that $y \mid \mathbf{x}^{\mathbf{b}}$ we will prove $\beta_{i,\mathbf{b}}(I') = 0$ by induction on r. If $\mathbf{x}^{\mathbf{b}} \notin I$, then there is nothing to prove. So we suppose $\mathbf{x}^{\mathbf{b}} \in I$.

By an induction argument, suppose $r = 1$, then since $m_1 \mid \mathbf{x}^{\mathbf{b}}$ but $m_1 \neq \mathbf{x}^{\mathbf{b}}$ we have that $K^{\mathbf{b}}(I')$ is a nontrivial simplex and therefore has trivial reduced homology. Now suppose $r > 1$ and that for every ideal I'' generated by less than r monomials in the variables $x_1, \ldots, x_n$ and $\mathbf{b}$ such that $y \mid \mathbf{x}^{\mathbf{b}}$, we have $\beta_{i,\mathbf{b}}(I'') = 0$.

If I' is such that there is only one generating monomial, say, m_r such that $m_r \mid \mathbf{x}^{\mathbf{b}}$ then $K^{\mathbf{b}}(I) = K^{\mathbf{b}}(m_r)$ which is already known to have trivial reduced homology. So we can suppose there are at least two monomials m_r, m_{r-1} such that $m_r, m_{r-1} \mid \mathbf{x}^{\mathbf{b}}$. So, for $I'' = (m_1, \ldots, m_{r-1})$ and $I''' = (m_1 m_r, \ldots, m_{r-1} m_r)$ there is a Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to H_i(K^{\mathbf{b}}(I''')) \to H_i(K^{\mathbf{b}}(I'')) \oplus H_i(K^{\mathbf{b}}(m_r)) \\ \to H_i(K^{\mathbf{b}}(I')) \to H_{i-1}(K^{\mathbf{b}}(I''')) \to \cdots \end{aligned}$$

By the induction hypothesis, all the homologies in the exact sequence, except for $H_i(K^{\mathbf{b}})(I')$ are zero, so by exactness $H_i(K^{\mathbf{b}})(I')$ is also zero for every i. We are, thus, done. ■

From this, it follows that if we have the edge ideal I_G of a graph G, then for a squarefree degree $\mathbf{b} \neq \mathbf{1}$ and every i, $H_i(K^{\mathbf{b}}(I_G)) \cong H_i(K^{\mathbf{1}'}(I_{G'}))$ where G' is the subgraph induced by the vertices which characteristic vector is $\mathbf{b}$ and $\mathbf{1}'$ is the vector filled with 1s in $\{0,1\}^{|G'|}$ and $I_{G'} \subseteq k[V(G')]$. It also follows that this number does not depend on the number of the variables (as long as there are as many as ones in $\mathbf{b}$).

Proposition 3.1.4. Let $I = (m_1, \ldots, m_r)$ be a monomial ideal of the ring $k[x_1, \ldots, x_n, y]$, and suppose that for all the monomials m_i such that $y \mid m_i$, there is some variable $x_j \in \mathrm{Supp}(m_1, \ldots, m_r)$ such that $x_j \nmid m_i$, and that $I' = (m_i : y \mid m_i) \neq I$. Then, for $I'' = (m_i : y \nmid m_i)$, $I''' = I' \cap I''$, and any $\mathbf{b} \in \mathbb{N}^{n+1}$ such that $\mathbf{x}^{\mathbf{b}} \in I'''$ there is an isomorphism:

$$H_i(K^{\mathbf{b}}(I)) \to H_{i-1}(K^{\mathbf{b}}(I''')).$$

For $\mathbf{b}$ such that $\mathbf{x}^{\mathbf{b}} \in I' - I''$ there is an isomorphism

$$H_i(K^{\mathbf{b}}(I')) \to H_i(K^{\mathbf{b}}(I)).$$

For $\mathbf{b}$ such that $\mathbf{x}^{\mathbf{b}} \in I'' - I'$ there is an isomorphism

$$H_i(K^{\mathbf{b}}(I'')) \to H_i(K^{\mathbf{b}}(I)).$$

Proof. For $\mathbf{b}$ such that $\mathbf{x}^{\mathbf{b}} \notin I'''$ we have $K^{\mathbf{b}}(I) = K^{\mathbf{b}}(m_i : m_i \mid \mathbf{x}^{\mathbf{b}})$. If $\mathbf{x}^{\mathbf{b}} \in I' - I''$ this means that the monomials m_i such that $m_i \mid \mathbf{x}^{\mathbf{b}}$ are in the generating set of I', so

$$K^{\mathbf{b}}(m_i : m_i \mid \mathbf{x}^{\mathbf{b}}) = K^{\mathbf{b}}(I').$$

If, otherwise $\mathbf{x}^{\mathbf{b}} \in I'' - I'$ then similarly $K^{\mathbf{b}}(m_i : m_i \mid \mathbf{x}^{\mathbf{b}}) = K^{\mathbf{b}}(I'')$. So, suppose that $\mathbf{x}^{\mathbf{b}} \in I'''$. This means that, in particular, $\mathbf{x}^{\mathbf{b}}$ is divisible by two generating monomials $m_i, m_{i'}$ of I, such that $y \mid m_i, y \nmid m_{i'}$, which means that $y \mid \mathbf{x}^{\mathbf{b}}$. Also there is some other variable x_j such that $x_j \mid \mathbf{x}^{\mathbf{b}}$ and $x_j \notin \mathrm{Supp} I'$. Therefore for all i, $H_i(K^{\mathbf{b}}(I')) \cong H_i(K^{\mathbf{b}}(I'')) \cong 0$ and in the Mayer-Vietoris sequence associated with this decomposition we will find the desired isomorphism. ■

The conditions for the proposition 3.1.4 seem to be harsh, but are not so much. This Mayer-Vietoris sequence will be called an *incomplete star* Mayer-Vietoris sequence. This is because when we have an edge ideal of a graph, the condition is equivalent to the graph having a vertex that is not adjacent to every other vertex of G.

3.2 Stars, Complete Bipartite Graphs, Cones

Here, we will study some families of ideals for which the Betti numbers can be computed rather easily. The first one of them is the family of the edge ideals of the *complete bipartite graphs*. Let $G = (V, E) = K_{n,m}$ be the complete bipartite graph with bipartition $H = \{1, \ldots, n\}, K = \{1, \ldots, m\}$ and

$$I = I_G \subseteq k[x_1, \ldots, x_n, y_1, \ldots, y_m]$$

its edge ideal. For a fixed vertex, which we can, by a permutation of A, B or $y_1, \ldots, y_m$ declare to be y_m, we can say $I = I_m + I'_m$ where $I_m = (e \in E : y_m \in e)$ and $I'_m = (e \in E : y_m \notin e)$. Given this, for $m = 1$, we can compute the Betti numbers of I in a harder than usual way.

Proposition 3.2.1. Let $G = K_{n,1}$ be a *star* with vertex set $\{1, \ldots, n, 0\}$, centered in 0, i.e. 0 is adjacent to all the vertices $1, \ldots, n$ which are not adjacent to each other. If $\mathbf{b} \in \{0,1\}^{n+1}$ is such that $\mathbf{x}^{\mathbf{b}} \in I_G$, we have

$$\beta_{i,\mathbf{b}}(I_G) = \dim_k H_{i-1}(K^{\mathbf{b}}(I_G)) = \dim_k H_i(K^{\mathbf{b}}(\mathfrak{m})) = \beta_{i+1,\mathbf{b}}(\mathfrak{m}),$$

for $\mathfrak{m} = (x_1, \ldots, x_n, y_1)$.

Proof. Take $y = y_0$, the variable associated with the vertex 0, and $I = I_G$. Since $x^{\mathbf{b}} \in I$ then we can assume by a permutation $\sigma \in S_n$ that $x_n y \mid x^{\mathbf{b}}$. So, since $I = (x_1, \ldots, x_n) \cap (y)$, we have, for $I' = (x_1, \ldots, x_n)$, a Mayer-Vietoris sequence:

$$\begin{aligned} \cdots \to H_i(K^{\mathbf{b}}(I)) &\to H_i(K^{\mathbf{b}}(I')) \oplus H_i(K^{\mathbf{b}}(y)) \\ &\to H_i(K^{\mathbf{b}}(\mathfrak{m})) \to H_{i-1}(K^{\mathbf{b}}(I)) \to \cdots \end{aligned}$$

Because of the lemma 3.1.3 we have that $H_i(K^{\mathbf{b}}(I')), H_i(K^{\mathbf{b}}(y))$ are both 0 for all i, therefore $H_i(K^{\mathbf{b}}(\mathfrak{m})) \cong H_{i-1}(K^{\mathbf{b}}(I))$ for all i. The result follows. ■

From the previous proposition, we can compute the Betti numbers of the star $K_{n,1}$. We know that the Koszul complex $K_\bullet$ is a minimal free resolution of $k = R/\mathfrak{m}$, so when we remove the R corresponding to the empty face from it, it becomes a minimal free resolution $K'_\bullet$ of $\mathfrak{m}$. Suppose that $\mathbf{b}$ is squarefree (otherwise its Betti number would be 0) and take $r = |\mathbf{b}|$. Then $\mathbf{x}^{\mathbf{b}}$ appears exactly once as a generator in homological degree $r-1$ and does not appear elsewhere in $K'_\bullet$. So

$$\beta_{i,\mathbf{b}}(I_G) = \beta_{i+1,\mathbf{b}}(\mathfrak{m}) = \begin{cases} 1 & \text{if } i = r-2 \\ 0 & \text{otherwise.} \end{cases}$$

The easier way is figuring out that $K^{\mathbf{b}}(I)$ is the simplicial sphere on the vertices $1, \ldots, n$.

it is no surprise if we can then compute the Betti numbers of the ideal I_G of the complete bipartite graph $G = K_{n,m}$ in a similar way.

Proposition 3.2.2. Let $G = K_{n,m}$ and $I = I_G \subseteq k[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Then, for $\mathbf{b}$ such that $\mathbf{x}^{\mathbf{b}} \in I$,

$$\beta_{\mathbf{b},i}(I) = \dim_k H_{i-1}(K^{\mathbf{b}}(I)) = \begin{cases} 1 & \text{if } i = n_{\mathbf{b}} + m_{\mathbf{b}} - 2 \\ 0 & \text{otherwise.} \end{cases}$$

where $n_{\mathbf{b}} = |\{x_i : x_i \mid \mathbf{x}^{\mathbf{b}}\}|, m_{\mathbf{b}} = |\{y_j : y_j \mid \mathbf{x}^{\mathbf{b}}\}|$.

Proof. For $\mathbf{b} < \mathbf{1}$ we have $K^{\mathbf{b}}(I) = K^{\mathbf{b}}(I_{K_{n_{\mathbf{b}},m_{\mathbf{b}}}})$ so we only have to prove the result for $\mathbf{b} = \mathbf{1}$. If m or n equals 1, then we are done: our graph is a star. So suppose that $m, n > 1$. Since no vertex is adjacent to every other vertex, the incomplete star Mayer-Vietoris sequence with respect to the vertex y_m gives an isomorphism:

$$H_i(K(I)) \cong H_{i-1}(K(I_1))$$

for

$$I_1 = (x_i y_j y_m : 1 \leq i \leq n, 1 \leq j \leq m) = I_{G-y_n} \cap I_{\Sigma_m},$$

where Σ_m is the star centered on m. Since not every generating monomial is divisible by y_{m-1}, we can repeat the process. So, this way we find a decreasing sequence $I_s, s \in \mathbb{N}$ of ideals such that $I_0 = I, I_{n+m-2} = (x_1 \dots x_n y_1 \dots y_m)$, and

$$H_i(K(I_s)) \cong H_{i-1}(K(I_{s+1})).$$

Therefore, $H_{n+m-3}(K(I)) \cong H_{-1}(K(I_{n+m-2})) = k$. ■

For these cases, Alexander's duality is especially powerful. The Lower Koszul Complex $K_{\mathbf{1}}(I_G)$ is just the disjoint union of two simplexes, one corresponding to the stable set of the variables x, and the other corresponding to the stable set of the variables y. Its reduced homology is clearly 0 everywhere besides at dimension 0, on which we have $H_0(K_{\mathbf{1}}(I_G)) = k$. This is exactly the same result. One can go even further. For a graph G which is the join of two graphs H, K (The graph resulting from taking the disjoint union of H and K, and adding all the possible $H-K-$edges), the Lower Koszul Complex $K_{\mathbf{1}}(I_G)$ is the disjoint union $K_{\mathbf{1'}}(I_H) \sqcup K_{\mathbf{1''}}(I_K)$, where $\mathbf{1'} = \chi(V(H)), \mathbf{1''} = \chi(V(K))$. Thus, for every dimension but 0, the homology of $K_{\mathbf{1}}(G)$ is the direct sum of the homologies of $K_{\mathbf{1'}}(H)$ and $K_{\mathbf{1''}}(K)$. In dimension 0, it is the direct sum of $K_{\mathbf{1'}}(H), K_{\mathbf{1''}}(K)$ and k.

Now, we are interested in generalizing some of the ideas in the proof above. One of them is, given a monomial ideal

$$I = (m_1, \dots, m_r) \subseteq k[x_1, \dots, x_n, y]$$

such that for all $i \in \{1, \dots, r\}$, $y \nmid m_i$, computing the Betti numbers of

$$I' = (m_1, \dots, m_r, y).$$

Using a recursive Mayer-Vietoris sequence, this is equivalent to computing the Betti numbers of $I'' = (m_1 y, \dots, m_r y)$. Ideals like I'' appear often when using incomplete star Mayer-Vietoris sequences, so it would be useful if we could compute these Betti numbers in terms of the Betti numbers of I. These ideals appear in the sequence of ideals in the proof above, for example.

Other ideas are to compute the Betti numbers of the cone of a graph, i.e., a graph with a vertex adjacent to every other one, and to compute the Betti numbers of the graph resulting from duplicating a vertex of a given graph.

Proposition 3.2.3. Let $I = (m_1, \ldots, m_r)$ be an ideal of $k[x_1, \ldots, x_n, y]$ not using the variable y. Then, for $I' = I + (y)$ and $\mathbf{b} \in \mathbb{N}^{n+1}$ such that $y \mid \mathbf{x}^{\mathbf{b}} \in I$ we have

$$\beta_{i+1,\mathbf{b}}(I') = \beta_{i,\mathbf{b}'}(I).$$

Proof. For any other $\mathbf{b}$, we have

$$K^{\mathbf{b}}(I') = K^{\mathbf{b}}(I) \cup K^{\mathbf{b}}(y).$$

We can characterize $K^{\mathbf{b}}(y)$ as the simplex of all the faces $\tau \leq \mathbf{b}$ such that $y \nmid \mathbf{x}^{\tau}$. Also since I does not use the variable y, we see that

$$\begin{aligned} K^{\mathbf{b}}(I) =& \{\tau = (t_1, \ldots, t_n, q) \in \{0,1\}^{n+1} : \mathbf{x}^{\mathbf{b}-\tau} \in I\} \\ =& \{\tau = (t_1, \ldots, t_n, 0) \in \{0,1\}^{n+1} : \mathbf{x}^{\mathbf{b}-\tau} \in I\} \\ & \cup \{\tau = (t_1, \ldots, t_n, 1) \in \{0,1\}^{n+1} : \mathbf{x}^{\mathbf{b}-\tau} \in I\} \end{aligned}$$

so $K^{\mathbf{b}}(I)$ is just the simplicial complex of all the subfaces of the faces of

$$T_1 = \{\tau = (t_0, \ldots, t_n, 1) \in \{0,1\}^{n+1} : \mathbf{x}^{\mathbf{b}-\tau} \in I\}.$$

This means that

$$\begin{aligned} K^{\mathbf{b}}(I) \cap K^{\mathbf{b}}(y) &= \{\tau = (t_1, \ldots, t_n, q) \in K^{\mathbf{b}}(I) : q = 0\} \\ &= K^{(\mathbf{b}',0)}(I) \end{aligned}$$

where $\mathbf{b}' \in \mathbb{N}^n$ is such that $\mathbf{b} = (\mathbf{b}', 1)$. Therefore, by a recursive Mayer-Vietoris sequence,

$$H_i(K^{\mathbf{b}}(I')) \cong H_{i-1}(K^{\mathbf{b}}(I \cap (y))) \cong H_{i-1}(K^{(\mathbf{b}',0)}(I))$$

as desired. ∎

We can also compute all the remaining Betti numbers of the ideal in the previous proposition:

For $\mathbf{b}$ such that $y \nmid \mathbf{x}^{\mathbf{b}}$ we have $K^{\mathbf{b}}(I') = K^{\mathbf{b}}(I)$ so $\beta_{i,\mathbf{b}}(I') = \beta_{i,\mathbf{b}}(I)$. Also if $\mathbf{x}^{\mathbf{b}} \notin I$ we have $K^{\mathbf{b}}(I') = K^{\mathbf{b}}(y)$ so $\beta_i(K^{\mathbf{b}}(I')) = \delta_{i0}$ where δ_{ij} is the Kronecker delta.

We can also compute the Betti numbers of the cone of a graph:

Proposition 3.2.4. Let G be a graph with vertex set $\{x_1, \ldots, x_n\}$, and $\kappa = C(G, y)$ the cone of G with y being the new vertex. Then

$$\beta_{n-1,\mathbf{1}}(I_\kappa) = \beta_{n-2,\mathbf{1}'}(I_G) + 1,$$

where $\mathbf{1} = (\mathbf{1}', 0)$. Furthermore $\beta_{i,\mathbf{1}}(I_\kappa) = \beta_{i,\mathbf{1}'}(I_G)$ for every $i \neq n-1$.

Proof. Here we take the Mayer-Vietoris sequence relative to the *complete* star $\Sigma(y)$ of y. We know that $H_i(K^{\mathbf{1}}(I_{\Sigma(y)})) \cong k$ if and only if $i = n-2$. Also $H_i(K^{\mathbf{1}}(I_G)) = 0$ for every i since G does not use the variable y and $H_{n-2}(K^{\mathbf{1}}(I')) = 0$ where $I' = I_G \cap I_{\Sigma(y)}$. So, we've got the following short exact sequence:

$$0 \to H_{n-2}(K^{\mathbf{1}}(I_{\Sigma(y)})) \to H_{n-2}(I_\kappa) \to H_{n-3}(K^{\mathbf{1}}(I')) \to 0$$

and isomorphisms $H_i(K^{\mathbf{1}}(I)) \to H_{i-1}(I')$ for every $i \neq n-2$. Therefore

$$\begin{aligned}\dim H_{n-2}(K^{\mathbf{1}}(I_\kappa)) &= \dim H_{n-2}(K^{\mathbf{1}}(I_{\Sigma(y)})) + \dim H_{n-3}(K^{\mathbf{1}}(I')) \\ &= 1 + \dim H_{n-3}(K^{\mathbf{1}}(I')).\end{aligned}$$

The trick lies, then, in disclosing the identity of $K^{\mathbf{1}}(I')$. Since $I' \neq 1$ then $\mathbf{1} \notin K^{\mathbf{1}}(I')$. Also $K^{\mathbf{1}}(I_{\Sigma(y)})$ is the $n-2-$skeleton of the simplex with vertices $x_1, \ldots, x_n$: The set of all the faces with $n-1$ vertices. So, with the same reasoning as the one in the proposition 3.2.3, the only thing we are doing when intersecting is removing the faces of $K^{\mathbf{1}}(I_G)$ containing y. As in the analysis of the proposition 3.2.3, we end up with $K^{(\mathbf{1}',0)}(I)$. So, by replacing it in the equation above and replacing it all with Betti, we get

$$\beta_{n-1,\mathbf{1}}(I_\kappa) = 1 + \beta_{n-2,\mathbf{1}'}(I_G),$$

which is what we wanted to prove. ■

Of course, we are also able to compute the remaining Betti numbers of κ, given. For $y \mid \mathbf{x}^{\mathbf{b}}$ we have the Betti numbers of the cone of an induced subgraph of G, from which we can apply the previous proposition, and for $y \nmid \mathbf{x}^{\mathbf{b}}$ we have the same Betti numbers as G. This, in particular, gives us another way to compute all the Betti numbers of any complete graph because it is the consecutive application of the cone operator to an edge.

The following proposition is clearly related to proposition 3.2.3; it is a sort of generalization of it, and it can be generalized further, but for the sake of simplicity, we will state it as follows.

Proposition 3.2.5. Let $I = (m_1, \ldots, m_{r-1})$ be a monomial ideal of $R = k[x_1, \ldots, x_n]$ and suppose $m_r = x_a x_b, a \neq b \in \{1, \ldots, n\}$ such that

$$\operatorname{Supp}(m_r) \cap \operatorname{Supp}(m_1, \ldots, m_{r-1}) = \varnothing^{(\mathrm{i})}$$

Let $I' = I + (m_r)$. Then

$$\beta_{i+1,\mathbf{b}}(I') = \beta_{i,\mathbf{b}'}(I),$$

where $\mathbf{b}'_i = (1 - \delta_{ia})(1 - \delta_{ib})\mathbf{b}_i$, i.e. $\mathbf{b}'$ is the result of letting the entries of $\mathbf{b}$ corresponding to x_a, x_b be zero.

Proof. By a permutation of the variables, suppose $m_r = x_{n-1}x_n$. As before, if $\mathbf{b}$ is such that $\mathbf{x}^{\mathbf{b}}$ is not divisible by m_r we have $\beta_{i,\mathbf{b}}(I') = \beta_{i,\mathbf{b}}(I)$. If $\mathbf{x}^{\mathbf{b}}$ is divisible only by m_r then $\beta_{i,\mathbf{b}}(I) = \beta_{i,\mathbf{b}}(m_r)$. So, suppose $\mathbf{x}^{\mathbf{b}}$ is divisible by both m_r and some other m_i. Then

$$\begin{aligned} K^{\mathbf{b}}(I) =& \{\tau \in \{0,1\}^n : x^{\mathbf{b}-\tau} \in I\} \\ =& \{\tau = (t_1, \ldots, t_{n-2}, 0, 0) \in \{0,1\}^n : x^{\mathbf{b}-\tau} \in I\} \\ & \cup \{\tau = (t_1, \ldots, t_{n-2}, t_{n-1}, t_n) \in \{0,1\}^n : x^{\mathbf{b}-\tau} \in I, (t_{n-1}, t_n) \neq 0\}. \end{aligned}$$

Also $K^{\mathbf{b}}(m_r)$ is the simplex on the vertex set $x_1, \ldots, x_{n-2}$. Therefore

$$K^{\mathbf{b}}(I \cap (m_r)) = \{\tau = (t_1, \ldots, t_{n-2}, 0, 0) \in \{0,1\}^n : x^{\mathbf{b}-\tau} \in I\} = K^{\mathbf{b}'}(I).$$

We get the result by using the isomorphism arising from the recursive Mayer-Vietoris sequence of I' with respect to m_r. ■

3.3 Forests, Paths, and Cycles

With the results of the previous section, we are able to compute the Betti numbers of a few more families of ideals. For example, edge ideals of *forests*. This has already been done in [**?**], but the approach used there is different, and the results are given to compute their graded Betti numbers. In counterpart, our result gets a simpler formula and algorithm to compute the multigraded Betti numbers. It also uses less terminology, so it has a simpler proof.

[(i)] If $I = I_G$, this means that G is disconnected being the edge $m_r = x_a x_b$ one of the connected components.

Lemma 3.3.1. Let $T = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ be a forest without isolated vertices. Suppose v_n is a leaf of T with $v_{n-1}v_n \in E$. Then

$$H_i(K(T)) \cong H_{i-d_T(v_{n-1})}(K(T - N_T[v_{n-1}])).$$

Proof. Let $I_T \subseteq k[x_1, \ldots, x_n]$ the edge ideal of T, with the variable x_i associated to v_i for each i. Let's assume by rearranging the vertices of V that v_n is a leaf, and that $v_n v_{n-1} \in E(T)$, so that $I_T = (m_1, \ldots, m_u, x_n x_{n-1})$. Then, since v_n is only adjacent to v_{n-1}, there is an isomorphism

$$H_i(K(T)) \to H_{i-1}(K(I'))$$

where $I' = I_{T-v_n} \cap (x_n x_{n-1})$.

We can split $E(T - v_n)$ in two sets E_1, E_2, where E_1 consists in all the edges in $T - v_n$ incident to v_{n-1} or its neighbors, and E_2 consists in the remaining edges. By a permutation of the m_i, suppose that there is some $k \in \{0, \ldots, u\}$ such that $E_2 = \{m_1, \ldots, m_k\}$ and $E_1 = \{m_{k+1}, \ldots, m_u\}$. We can deal with the edges in E_1 as follows:

$$I_{E_1} = (\operatorname{lcm}(m_i, x_{n-1}x_n) : x_{n-1} \mid m_i)$$

since every other edge in E_1 has the form $v_r v_s$ where v_s is adjacent to v_{n-1}, so $\operatorname{lcm}(x_r x_s, x_{n-1}x_n) = x_r x_s x_{n-1} x_n$ which is divisible by $x_s x_{n-1} x_n = \operatorname{lcm}(x_s x_{n-1}, x_{n-1}x_n)$. We can then reorder the edges in E_1 in such a way that there is some $l \in \{k+1, u\}$ such that $x_{n-1} \mid m_i$ for $k+1 \leq i \leq l$ and $x_{n-1} \nmid m_i$ for $l < i \leq u$. So, with this,

$$I' = (m_1 x_{n-1} x_n, \ldots, m_k x_{n-1} x_n, m_{k+1} x_n, \ldots, m_l x_n)$$

After this, we can go back and see that

$$H_i(K(T)) \cong H_i(K(I'_T)),$$

where $I'_T = (m_1, \ldots, m_k, x_{m_{k+1}}, \ldots, x_{m_l}, x_{n-1}x_n)$ for $x_{m_i} = \frac{m_i}{x_{n-1}}$. Now, since no generating monomial of I'_T besides $x_{n-1}x_n$ is divisible by x_{n-1} or x_n, we can use the proposition 3.2.5:

$$H_i(K^{\mathbf{1}}(I'_T)) \cong H_{i-1}(K(I'')),$$

where $I'' = (m_1, \ldots, m_k, x_{m_{k+1}}, \ldots, x_{m_l})$. Now, there is no generating monomial of I'' such that there is some variable in $\{x_{m_{k+1}}, \ldots, x_{m_l}\}$ dividing it,

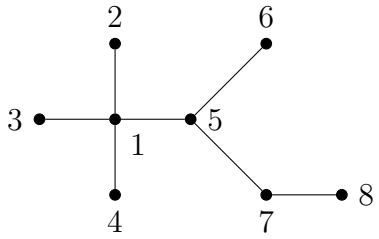
because E_2 consists in the edges of T not adjacent to x_{n-1} or any of its neighbors (The variables $x_{m_{k+1}}, \ldots, x_{m_l}$ are the neighbors of x_{n-1}). So, for all $i \in \{1, \ldots, k\}$, m_i is not divisible by any of the variables $x_{m_{k+1}}, \ldots, x_{m_l}$. Then, by an iterative application of the proposition 3.2.3, we have

$$H_{i-1}(K(I'')) \cong H_{i-d_T(v_{n-1})}(K(\tilde{I}))$$

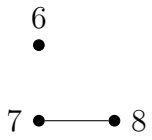
where $\tilde{I} = (m_1, \ldots, m_k)$. But $\tilde{I}$ is the edge ideal of a subforest T' of T, with fewer vertices than T. The identity of the subforest T' is clear: Its edges are just the edges not incident to v_{n-1} or any of its neighbors, i.e. it is $T - N_T[v_{n-1}]$. The result follows. ■

This lemma can be roughly stated as *The full Betti number of T is the same as the full Betti number of the forest resulting from removing a star of T centered in the neighbor of a leaf of T, in a dimension reduced by its degree.* By grouping all these stars together, we are constructing what we'll later call a molecular cover of T.

Example 3.3.2. Let T be the following graph:



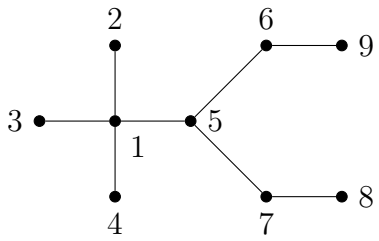
Then by the previous lemma $H_i(K(T)) \cong H_{i-4}(K(T'))$, where T' is as follows:



An isolated vertex means the ideal $I_{T'}$ does not use the variable associated to 6, which means $H_i(K^{\mathbf{1}'}(I_{T'})) = 0$ for all i. Therefore $H_i(K^{\mathbf{1}}(I_T))$ is also 0 for all i.

The following example has nonzero homology:

Example 3.3.3. Let T be the following graph:



With the same process as before, $H_i(K(T)) \cong H_{i-4}(K(T'))$ where T' is the following graph:

Then, using the proposition 3.2.5, we get that $H_i(K(T)) \cong H_{i-5}(K^{\mathbf{1}}(T''))$ where T'' is just an edge, so $H_i(K(T'')) \cong k^{\delta_{i,-1}}$, i.e. it has nonzero homology only at dimension -1, and that homology is k. Therefore $K(T)$ has nonzero homology only at dimension 4.

Chapter 4

Some Potential Resolutions

In the previous chapters, we developed some tools to compute Betti numbers of some families of edge ideals. In this one we will compute free resolutions of some subfamilies.

4.1 Unbalancing Diamonds

Let I be a monomial ideal of $R = k[x_1, \ldots, x_n]$ such that for any $\mathbf{b} \in \mathbb{N}^n$ and $i \in \mathbb{N}$, $\beta_{i,\mathbf{b}}(I) \in \{0, 1\}$ and $\beta_{i,\mathbf{b}}(I) = 1$ for at most one i. **Any monomial ideal in this section will satisfy these properties.**

Consider a chain of multigraded morphisms:

$$\mathcal{M} : 0 \leftarrow R/I \leftarrow R \xleftarrow{\sigma_1} F_1 \xleftarrow{\sigma_2} F_2 \leftarrow \cdots \leftarrow F_p$$

where

1. $p = \mathrm{pd}(R/I)$.
2. $F_i = \bigoplus_{\mathbf{b}\in\mathbb{N}^n} R(-\mathbf{b})^{\beta_{i,\mathbf{b}}}$.
3. The morphisms σ_i are all minimal multigraded of degree 0.

Any chain of morphisms in this section will satisfy these properties.

In particular, $\mathcal{M}$ can be a minimal multigraded free resolution of R/I. We can give a representation of this chain of morphisms as a graded poset $B = B(\mathcal{M})$ with tags in the edges, that we will call the *decorated poset* of

$\mathcal{M}$. In case $\mathcal{M}$ is a minimal free resolution, we will say B is a minimal poset of I.

In particular

$$B = V(B(\mathcal{M})) = \{\mathbf{b} \in \mathbb{N}^n : (\exists i \in \mathbb{N})\beta_{i,\mathbf{b}}(I) \neq 0\}$$

with

$$B_i = \{\mathbf{b} \in \mathbb{N}^n : \beta_{i,\mathbf{b}}(I) \neq 0\}$$

and for a map

$$\sigma_i : \bigoplus_{\mathbf{b}\in B_i} R(-\mathbf{b}) \to \bigoplus_{\mathbf{c}\in B_{i-1}} R(-\mathbf{c})$$

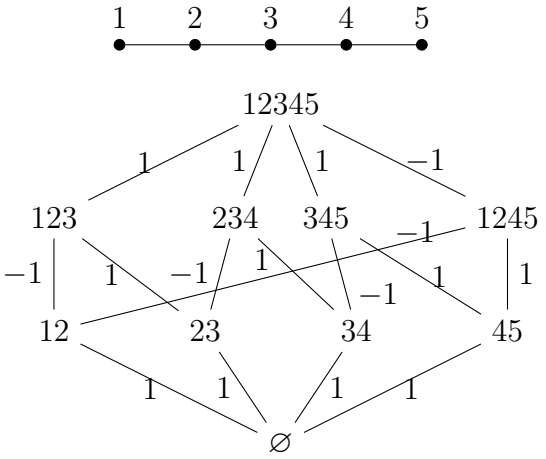
with matrix representation

$$(\sigma_{\mathbf{c},\mathbf{b}})_{\mathbf{c}\in B_{i-1},\mathbf{b}\in B_I},$$

we can add the edges $(\mathbf{c}, \mathbf{b})$ and tag them with the monomial $\sigma_{\mathbf{c},\mathbf{b}}$, or to make it simpler, with the coefficient of that monomial.

Since the multidegrees of I have squarefree coordinates, we can just describe each multidegree by the location of the nonzero coordinates, as in the following example:

Example 4.1.1. Here T is the path with 5 vertices.



If $\mathcal{M}$ is a minimal multigraded free resolution, all the coefficients on these maps are either $-1, 0$ or 1 (it holds for any graph such that the resolutions do not depend on the characteristic of the base field).

Definition 4.1.1 (Lower and Upper Neighborhoods)**.** For each i and each vertex $\mathbf{b}$ of B_i, we define the *upper and lower neighborhods* of $\mathbf{b}$ as:

$$N_{\uparrow}(\mathbf{b}) = \{\mathbf{a} \in B_{i+1} : \mathbf{b} \leq \mathbf{a}\}$$

and

$$N_{\downarrow}(\mathbf{b}) = \{\mathbf{a} \in B_{i-1} : \mathbf{a} \leq \mathbf{b} \in E(B)\}.$$

All the families of graphs we have studied up until this point satisfy the following property.

Definition 4.1.2 (Diamond Property)**.** Let B be a decorated poset. If, for any vertices $\mathbf{c}, \mathbf{b} \in B$,

$$|N_{\uparrow}(\mathbf{b}) \cap N_{\downarrow}(\mathbf{b})| \in \{0, 2\}$$

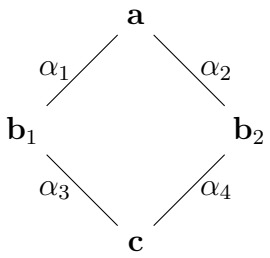
we say that B satisfies the *diamond property* .

Proposition 4.1.2. Let I be a monomial ideal and B a decorated poset corresponding to a chain of morphisms $\mathcal{M}$. If $B(\mathcal{M})$ satisfies the diamond property, then for any chain of morphisms $\mathcal{N}$, $B(\mathcal{N})$ also satisfies the diamond property.

Proof. The diamond property only depends on the Betti numbers of I, and all the chains of morphisms of this section have the same Betti numbers. ■

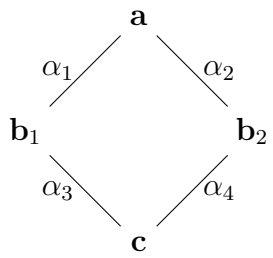
Definition 4.1.3. Let I be a monomial ideal, we will say I satisfies the diamond condition if some (therefore all) decorated poset of I satisfies the diamond condition.

If $\mathcal{M}$ satisfies the diamond property, as in the example 4.1.1, we can consider all the diamonds:



The tags of these diamonds can tell us if $\mathcal{M}$ is a chain complex. The condition that will determine that will be called *unbalanced*.

Definition 4.1.4. A diamond



is said to be unbalanced if $\alpha_1\alpha_2 = -\alpha_3\alpha_4$.

Proposition 4.1.3 (Proposition 12, [?])**.** A chain of morphisms satisfying the diamond property is a chain complex if and only if all its diamonds are unbalanced.

4.2 The Criterion

While the previous section allows us to give candidates to minimal multigraded free resolutions for a sufficiently nice monomial ideal, this section allows us to prove they are actually resolutions. The main purpose of this section is to give and explain a criterion to check when a multigraded free complex of a R-module is exact. The criterion includes two results. Before that, we will present the concept of *irredundancy*.

Definition 4.2.1. A set of vectors $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ in an R-module is called irredundant whenever

$$\gamma_i \notin \langle \gamma_1, \ldots, \hat{\gamma_i}, \ldots, \gamma_s \rangle \text{ for all } 1 \leq i \leq s.$$

The irredundancy is what is considered one of the equivalent definitions of *linear independence* in linear algebra. That said, in module theory, these definitions are not equivalent.

Lemma 4.2.1. Let N be a multigraded finitely generated R-module. If Γ is a minimal generating set of N and Λ is an irredundant subset of N with $|\Gamma_{\mathbf{c}}| = |\Lambda_{\mathbf{c}}|$ for all $\mathbf{c} \in \mathbb{N}^n$ and where $\Lambda_{\mathbf{c}} = \{u \in \Lambda : \deg(u) = \mathbf{c}\}$ and

$\Gamma_{\mathbf{c}} = \{u \in \Gamma : \deg(u) = \mathbf{c}\}$, then there exists an automorphism φ of N such that

$$\varphi(\Lambda_{\mathbf{c}}) = \Gamma_{\mathbf{c}}$$

and whose restriction on $\Lambda_{\mathbf{c}}$ is a k-linear map for all $\mathbf{c} \in \mathbb{N}^n$. Moreover, if M is a matrix representation of φ where Λ and Γ are ordered by their multidegrees in a non-decreasing way, then it is an upper triangular block matrix.

Theorem 4.2.2 (Theorem 2, [?]). If M is a finitely generated positively multigraded R-module,

$$\mathbf{F}_\bullet : 0 \leftarrow M \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \leftarrow \ldots \xleftarrow{d_p} F_p \leftarrow 0$$

is a multigraded minimal free resolution of M and

$$\mathbf{C}_\bullet : 0 \leftarrow M \xleftarrow{\delta_0} C_0 \xleftarrow{\delta_1} C_1 \leftarrow \ldots \xleftarrow{\delta_p} C_p \leftarrow 0$$

is a multigraded free complex of M such that

$$F_i = \bigoplus_{\mathbf{a} \in A_i \subset \mathbb{N}^n} R(-\mathbf{a}) = C_i$$

as free multigraded R-modules and the column sets, $C(D_i)$ of the matrix representations D_i of the differentials δ_i are irredundant for all $0 \leqslant i \leqslant p$, then $\mathbf{C}_\bullet$ is isomorphic to $\mathbf{F}_\bullet$.

4.3 Molecular Covers of Forests

Given a tree T, or more generally a forest, let $L(T)$ be its set of leaves. A star is a tree S with at least one edge and at most one non-leaf vertex. If it has a non-leaf vertex, then this vertex is called its center. Otherwise, any vertex can be its center when all its vertices are leaves. Note that a tree has only leaves if and only if it consists of only one edge.

If L is the set of leaves of a star S and w is its center, then S will be denoted by S_w^L, except when w and L are irrelevant. In particular, when S consists of only one edge uv, S will be denoted by either $S_u^{\{u,v\}}$ or $S_v^{\{u,v\}}$.

Definition 4.3.1. A sequence of stars $\mathbf{S} = (S_{w_1}^{L_1}, \ldots, S_{w_k}^{L_k})$ is a molecular cover of a forest F whenever the stars in $\mathbf{S}$ are induced subgraphs of F covering its vertices, and

$$L_i \cap L(F - (\cup_{j=1}^{i-1} S_{w_j}^{L_j})) \neq \emptyset \text{ for all } 1 \leq i \leq k.$$

We say that F has a molecular cover if there is a family of stars that is a molecular cover of F. If a forest F has a molecular cover, then F is called a *molecular forest*.

Theorem 4.3.1. Let T be a forest. If $\mathbf{S}$ is a molecular cover of T, $r = |S|$ and $K = K(T)$,

$$H_i(K) \cong \begin{cases} k & \text{if } i = n - r - 2 \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise $H_i(K) = 0$ for all i.

This means that r is uniquely defined no matter the choice of $\mathbf{S}$. In other words, every molecular cover of T has the same number of elements. Henceforth, when we state we *start a cover* of T, we mean we choose the star of the neighbor of a leaf in T; the first element of some molecular cover of T.

Proof. First, assume T is not molecular. Then, if we repeatedly use the lemma 3.3.1 until reaching a forest with no edges, this forest will still have some vertices (Otherwise, the neighborhoods calculated in each step will make a molecular cover). The homology of the complex associated with a nonempty forest with no edges is 0 everywhere.

Now assume T does have a molecular cover. If T is a star, then we already know that the formula holds. Now let's assume T is not a star and let $\mathbf{S}$ be a molecular cover of T such that $S = S_w$ is the first star in it. Then

$$H_i(K(T)) = H_{i-d_T(w)}(K(T - S_w)).$$

Assume that the formula holds for $T - S_w$, that is,

$$\begin{aligned} H_i(K(T - S_w)) &\cong \begin{cases} k & \text{if } i = (n - d_T(w) - 1) - (r - 1) - 2 \\ 0 & \text{otherwise.} \end{cases} \\ &\cong \begin{cases} k & \text{if } i = n - d_T(w) - r - 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then

$$\begin{aligned} H_i(K(T)) &\cong H_{i-d_T(w)}(K(T-S_w)) \\ &\cong \begin{cases} k & \text{if } i = n - d_T(w) - r - 2 + d_T(w) \\ 0 & \text{otherwise.} \end{cases} \\ &\cong \begin{cases} k & \text{if } i = n - r - 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

■

To formalize this, we will define the molecular cover number of a forest, and then give an algorithm to compute it (as well as a molecular cover).

Definition 4.3.2 (Molecular Cover Number)**.** Let $\mathbf{S}$ be a molecular cover of a forest T. We will define the molecular cover number of T as $r = |\mathbf{S}|$. If T is a forest that does not have a molecular cover, its molecular cover number will be defined as 0.

Proposition 4.3.2. If T is a forest with molecular cover number r, then $\mathrm{pd}(T) = |V(T)| - r$.

Proof. It follows from the fact that $\beta_{|V(T)|-r,|V(T)|} \neq 0$ and $\beta_{i,|V(T)|} = 0$ for all $i \neq |V(T)| - r$. ■

Corollary 4.3.3. The nonzero Betti numbers of the edge ideal of a forest are in the multidegrees corresponding to the subforests with molecular covers.

The following variation of the algorithm can compute the molecular cover number of T:

input: A forest T
output: The molecular cover number of T.
set $r = 0$;
while $E(T) \neq \varnothing$ **do**
    $r \leftarrow r + 1$;
    choose $v \in L(T)$;
    choose $w \in N_T(v)$;
    set $S_w = T[N_T[w]]$;
    $T \leftarrow T - S_w$;
end
if $V(T) = 0$ **then**
    return r
else
    return 0
end

And the following one can compute a molecular cover.

input: A forest T
output: A molecular cover of T.
set $\mathbf{S} = \varnothing$;
while $E(T) \neq \varnothing$ **do**
    choose $v \in L(T)$;
    choose $w \in N_T(v)$;
    set $S_w = T[N_T[w]]$;
    $T \leftarrow T - S_w$;
    $\mathbf{S} \leftarrow \mathbf{S} \cup \{S_w\}$;
end
return S.

The algorithms work due to Theorem 4.3.1, from which it also follows that a forest T has a molecular cover if and only if T has a molecular cover starting from any leaf in each step of the algorithm. Since a molecular cover is created recursively, it also means that every choice of stars in the algorithm gives us a new molecular cover.

Definition 4.3.3. Let T be a forest with a molecular cover such that $\mathrm{pd}(R/I_P) = i$. The lower neighborhood of T is defined as the induced subforests of T with a molecular cover such that their projective dimension is $i - 1$.

We have some implications on paths.

Corollary 4.3.4 (Paths)**.** Let P be a path with n vertices.

- The projective dimension of the edge ideal I_P is $\mathrm{pd}_S(R/I_P) = 2k + i$ where $n = 3k + i$ for $i \in \{-1, 0, 1\}$. For example, for $n = 8$, $\mathrm{pd}_S(R/I_P) = 5$.
- The subforests of a path with a molecular cover are the disjoint unions of subpaths such that none of them have a number of vertices congruent with 1 modulo 3. Thus, the multigraded component associated with a path with $3k$ or $3k - 1$ vertices appears in homological degree $2k$ or $2k - 1$ respectively. Similarly, the multigraded component associated with a disjoint union of paths $P_1, \ldots, P_s$ with projective dimensions $p_1, \ldots, p_s$ appears in homological degree

$$k = \sum_{i=1}^{n} p_i.$$

- For a given subforest T in homological degree i that has a molecular cover[(i)], its lower neighbors are calculated as follows.

  1. Take all the subforests resulting from removing a vertex from any connected component with $3k$ vertices such that the two resulting components have a number of vertices multiple of 3 and congruent with 2 modulo 3 each.
  2. Take all the subforests resulting from removing two vertices from any connected component with $3k' + 2$ vertices in such a way that every connected component has a number of vertices multiple of 3, and the subforests resulting from removing one vertex such that every component has a number of vertices congruent with 2 modulo 3.

Corollary 4.3.5. Let P be a path with projective dimension i and a molecular cover.

- If $|P| \equiv 0 \pmod 3$, and $V(P) = \{v_1, \ldots, v_{3k}\}$, its lower neighbors are computed by removing a single vertex v_i with $i \equiv 0 \pmod 3$ or $i \equiv 1 \mod 3$.

[(i)] This means that $\beta_{i,\mathbf{c}} \neq 0$ where $\mathbf{c}$ is the characteristic vector of the vertices of T

- If $|P| \equiv 2 \pmod 3$ and $V(P) = \{v_1, \ldots, v_{3k+2}\}$ its lower neighbors are computed by removing a single vertex v_i with $i \equiv 0 \pmod 3$ or by removing two vertices v_i, v_j with $i < j$ and $i \equiv 1 \pmod 3, j \equiv 2 \pmod 3$.

4.4 The lower Neighbors of a Forest

To compute the lower neighbors of a path, we had to consider the case of removing a single vertex and then the case of removing two vertices. Now, for any forest T and $U \subseteq V(T)$, $T - U$ is a lower neighbor if and only if $T - U$ has a molecular cover with $r - |U| + 1$ stars, where T has one with r stars. Now, we will consider the following results.

Proposition 4.4.1. Let T be a tree with a molecular cover with r stars and $u \in V(T)$. Then $T - u$ has a molecular cover with r stars if and only if u does not belong to any star with two vertices in any molecular cover of T and u is a leaf of a star with three or more vertices in a molecular cover of T.

Proof. Let $\mathbf{S}$ be a molecular cover of T. If $S \in \mathbf{S}$ is a star with two vertices and $u \in S$, then $T - S$ has a molecular cover, so $T - u$ does not have a molecular cover. Now, suppose $u \in S'$ where S' has at least 3 vertices. If u is not a leaf in S', then there is a molecular cover of $T - S'$, which means that $T - u$ does not have a molecular cover. Therefore, for $T - u$ to have a molecular cover, it needs to be a leaf of S'. Now, if u is a leaf of S', then $(\mathbf{S} - \{S'\}) \cup (S' - u)$ is a cover for $T - u$. ■

Lemma 4.4.2. Let T be a tree with a molecular cover with r stars, with a subpath P, such that every star S of T centered on the neighbor of a leaf intersects P. Suppose u and v are the leaves of P. Then $T - \{u, v\}$ has a molecular cover with $r - 1$ stars if and only if $T = P$ and $|V(P)| \equiv 2 \pmod 3$.

Proof. If T has two leaves, then $T = P$ is a path. T has a molecular cover if and only if $|V(T)| \equiv 1, 2 \pmod 3$, and $T - \{u, v\}$ has a molecular cover with one less star if and only if $|V(T)| \equiv 2 \pmod 3$. So we are done here.

Now we assume T is a tree with at least two leaves, that $T - \{u, v\}$ has a molecular cover with $r - 1$ stars. Suppose there is a star S centered on the

neighbor of a leaf different than u or v that intersects P. We will show this is impossible.

First suppose S is such that $u \in V(S) \cap V(P)$, and suppose that $|V(S) \cap V(P)| > 1$. Then u is the neighbor of a leaf of T not in P, $T-u$ has an isolated vertex and $T - \{u, v\}$ does not have a molecular cover; a contradiction. Now suppose that $|V(S) \cap V(P)| = \{u\}$. Then u is not a leaf of T, which means that $|S| \geq 3$. Now, $T-S$ has a molecular cover with $r-1$ stars, and $T-u$ has a molecular cover with r stars. There are two possibilities. First one, v is a leaf of T, and second one, there is another star S' such that $V(S') \cap V(P) = \{v\}$. In case there is such S', then $T - \{u, v\}$ has a molecular cover with r stars. In case v is a leaf, consider a molecular cover of T, say, $\mathbf{S}$, such that the star containing v, S', is chosen last, and S is chosen first. Let w be the neighbor of v. We have two cases.

1. $|S'| = 2$.
2. $|S'| \geq 3$.

In case 1, $\mathbf{S}$ restricts to a molecular cover of $T-u$ by removing u from S, and by removing S', it becomes a cover for $T - \{u, v, w\}$. This means $T - \{u, v\}$ does not have a molecular cover. In case 2, $\mathbf{S}$ restricts to a molecular cover of $T - \{u, v\}$ by removing u from S and v from S'. In this case, $T - \{u, v\}$ has a molecular cover with r stars. Therefore, S cannot intersect P on u or v.

Now, assume S meets P elsewhere, and consider a molecular cover $\mathbf{S}$ such that the stars containing u and v are chosen last. In a similar way as before, if both stars contain three or more vertices, then $T - \{u, v\}$ has a molecular cover with r stars, and otherwise, $T-\{u, v\}$ does not have a molecular cover.

Therefore, $P = T$ and $|V(P)| \equiv 2 \pmod 3$. The converse was done above. ■

Theorem 4.4.3. Let T be a forest with a molecular cover with r stars. Let $u, v \in V(T)$ and P be the path connecting them.

1. If $T - \{u, v\}$ has a molecular cover with $r - 1$ stars then $T - P$ has a molecular cover.
2. $T - \{u, v\}$ has a molecular cover with $r - 1$ stars if and only if there is a molecular cover $\mathbf{S}$ of T which can be described as

$$\mathbf{S} = \mathbf{T} \sqcup \mathbf{U},$$

where $\mathbf{T}$ is a molecular cover of $T - P$, $\mathbf{U}$ is a molecular cover of P, and $|P| \equiv 2 \pmod 3$ and the stars in $\mathbf{T}$ are chosen before[(ii)] the stars in $\mathbf{U}$.

Proof. We will prove (2) and then (1) will follow. First, suppose that there is a molecular cover $\mathbf{S}$ of T which can be described as

$$\mathbf{S} = \mathbf{T} \sqcup \mathbf{U}$$

where $\mathbf{T}$ is a molecular cover of $T - P$, $\mathbf{U}$ is a molecular cover of P, $|P| \equiv 2 \pmod 3$ and the stars in $\mathbf{T}$ are chosen before the stars in $\mathbf{U}$. To cover $T - \{u, v\}$ we replace $\mathbf{U}$ by any molecular cover $\mathbf{U}'$ of $P - \{u, v\}$ and set $\mathbf{S}' = \mathbf{T} \sqcup \mathbf{U}'$, which is a molecular cover of $T - \{u, v\}$.

Now, we will prove the converse by induction on the number k of leaves not in P. If $k = 0$, then $T = P$ and the result follows. Suppose now that $k > 0$ and that for every forest T' that has a molecular cover with $r(T')$ stars such that $T' - \{u', v'\}$ has a molecular cover with $r(T') - 1$ stars there is such a molecular cover, given that it has less than k leaves not in the path between u' and v'. Let T be a forest that has a molecular cover with r starts, with vertices $u, v \in V(T)$ such that there are k leaves not in the path P between u and v, and suppose $T - \{u, v\}$ has a molecular cover with $r - 1$ stars. Let w be the neighbor of one such leaves. Then the star S centered in w does not meet P. Let $T' = T - S$. Then T' has a molecular cover with $r - 1$ stars and has at most $k - 1$ leaves not in P. Since $T' - \{u, v\}$ also has a molecular cover with $r - 2$ stars then by induction hypothesis there is a molecular cover $\mathbf{S}'$ of T' which can be described as

$$\mathbf{S}' = \mathbf{T}' \sqcup \mathbf{U},$$

where $\mathbf{T}'$ is a molecular cover of $T' - P$, $\mathbf{U}$ is a molecular cover of P, $|P| \equiv 2 \pmod 3$ and the stars in $\mathbf{T}'$ are chosen before the stars in $\mathbf{U}$. Now, we build the cover $\mathbf{S}$ for T by setting $\mathbf{T} = \mathbf{T}' \cup \{S\}$ and $\mathbf{S} = \mathbf{T} \sqcup \mathbf{U}$, and since S is chosen first, the stars in $\mathbf{T}$ are chosen before the stars in $\mathbf{U}$.

Furthermore, both P and $P - \{u, v\}$ itself have molecular covers with the respective number of stars, so $|P| \equiv 2 \pmod 3$. So we are done. ■

In the case of paths, a lower neighbor of a subforest with a molecular cover differs from it by at most two vertices.

In the general situation that's not the case.

[(ii)] This means that if $\mathbf{S} = (S_1, \ldots, S_r)$ then $i < j$ whenever $S_i \in \mathbf{T}$ and $S_j \in \mathbf{U}$.

Example 4.4.4. Consider the tree T resulting from a subdivision of each of the edges of a star S with n leaves. Then T itself has n leaves, and has a molecular cover with n stars. Moreover, if L is the set of all its leaves, $T - L \cong S$ has a molecular cover with $1 = n - n + 1$ star.

The previous example shows an infinite family of trees with molecular covers with lower neighbors that can differ in an arbitrarily large number of vertices. We can generalize it further.

Let T be a graph resulting from subdividing each of the edges of a star with n leaves multiple times. Let l be a leaf of T, let P be the path from l to the center q of T and let x be the remainder of $|V(P)|$ when divided by 3. We have the following possibilities.

1. $x = 0$. In this case, we take a molecular cover of T starting with l until we get to q. This cover restricts to a cover of P. Thus, T has a molecular cover if each of the components of $T - P$ (which are paths) has a molecular cover. This means that each component C must have a number of vertices $x_C \equiv 0, 2 \pmod 3$. This means that the path from each leaf l' to q must have a number of vertices $x_{l'} \equiv 0, 1 \pmod 3$. Therefore, these trees can be represented by the following set of vectors in $\mathbb{Z}_3^3$:
$$\mathcal{T}_0^n = \left\{ \sum_{i=2}^{n} r_i e_i : r_i \in \{0, 1\} \right\} / S_n.$$
In the particular case of $n = 3$, we get
$$\mathcal{T}_0^3 = \{000, 001, 011\}.$$

2. $x = 2$. In this case, we get something similar: We take a molecular cover of T starting with l until we get to q. This cover does not restrict to a cover of P: The last star intersecting P is the entire star S_q centered in q. This means that the path from each leaf l' to q must have a number of vertices $x_{l'} \equiv 1, 2 \pmod 3$. In this case, we get
$$\mathcal{T}_2^n = \left\{ 2e_1 + \sum_{i=2}^{n} r_i e_i : r_i \in \{1, 2\} \right\} / S_n.$$
In the case of $n = 3$, we get
$$\mathcal{T}_2^3 = \{211, 221, 222\}.$$

3. $x = 1$. This case can only contain, if possible, the $\mathbf{1}$ vector, besides the cases mentioned before. In this case, we can't fully cover P and are stopped one vertex away from q. Then T has a molecular cover if and only if $T - (P - q)$ has one. $T - (P - q)$ is a tree resulting from subdividing each edge of a star with $n - 1$ leaves multiple times, so by recursion, we end up with a path with $\equiv 1 \pmod 3$ vertices, which does not have a molecular cover.

   Therefore, the only possible trees of this type with molecular covers are the ones in $\mathcal{T}_0^n$ and in $\mathcal{T}_2^n$.

Theorem 4.4.5. Let T be a forest with a molecular cover with r stars. Let $U \in V(T)$ and P be the smallest subtree containing U.

1. If $T - U$ has a molecular cover with $r - |U| + 1$ stars then $T - P$ has a molecular cover.

2. $T - U$ has a molecular cover with $r - |U| + 1$ stars if and only if there is a molecular cover $\mathbf{S}$ of T which can be described as

$$\mathbf{S} = \mathbf{T} \sqcup \mathbf{U},$$

   where $\mathbf{T}$ is a molecular cover of $T - P$, $\mathbf{U}$ is a molecular cover of P, and $P - U$ has a molecular cover with $|\mathbf{U}| - |U| + 1$ stars.

Proof. We use similar arguments to those used in the case of two vertices. First, suppose that there is a molecular cover $\mathbf{S}$ of T which can be described as

$$\mathbf{S} = \mathbf{T} \sqcup \mathbf{U}$$

where $\mathbf{T}$ is a molecular cover of $T - P$, $\mathbf{U}$ is a molecular cover of P and $P - U$ has a molecular cover $\mathbf{U}'$ with $|\mathbf{U}| - |U| + 1$ stars. Then $\mathbf{T} \sqcup \mathbb{U}'$ is a molecular cover of $T - U$ with $|\mathbf{S}| - |\mathbf{U}| + |\mathbf{U}| - |U| + 1 = r - |U| + 1$ stars. So, we are done in this case.

Now, we will prove the converse by induction on the number k of stars not meeting P. If $k = 0$ then $T = P$. Otherwise, if w is a leaf not in P, and S is a star starting a molecular cover, then $P - P \cap S$ is disconnected. Furthermore, every component of $P - P \cap S$ contains vertices of U; otherwise, P wouldn't be the smallest subtree containing U. Let U' be the set of vertices of U in a component of $P - P \cap S$. Then $P - U'$ has a molecular cover and therefore $P - U$ cannot have a molecular cover with $r - |U| + 1$ stars since $U' \subsetneq U$.

Suppose now that $k > 0$ and that for every forest T' that has a molecular cover with $r(T')$ stars such that $T' - U'$ has a molecular cover with $r(T') - |U'| + 1$ stars for $U' \subseteq V(T')$there is such a molecular cover, given that it has less than k stars not meeting the smallest subtree containing U.

Let T be a forest with a molecular cover with r stars and vertices $U \subseteq V(T)$ such that there are k stars not meeting the smallest subtree P containing U, and suppose $T - U$ has a molecular cover with $r - |U| + 1$ stars. Start a cover with a star S not meeting P. Let $T' = T - S$. Then T' has a molecular cover with $r - 1$ stars with at most $k - 1$ stars that do not meet P. Since $T' - U$ also has a molecular cover with $r - |U|$ stars then by induction hypothesis there is a molecular cover $\mathbf{S}'$ of T' which can be described as

$$\mathbf{S}' = \mathbf{T}' \sqcup \mathbf{U},$$

where $\mathbf{T}'$ is a molecular cover of $T' - P$, $\mathbf{U}$ is a molecular cover of P, and $P - U$ has a molecular cover with $|\mathbf{U}| - |U| + 1$ stars. Now, we build the cover $\mathbf{S}$ for T by setting $\mathbf{T} = \mathbf{T}' \cup \{S\}$ and $\mathbf{S} = \mathbf{T} \sqcup \mathbf{U}$. we are then done. ■

Proposition 4.4.6. Let T be a tree with a molecular cover with r stars, $U \subseteq V(T)$ such that $T - U$ has a molecular cover with $r - |U| + 1$ stars and assume there is no proper subtree of T containing U. Then $U = L(T)$.

Proof. First, $L(T) \subseteq U$; otherwise there would be $l \in L(T)$ such that $l \notin U$, which means $U \subseteq V(T - l)$. To prove the converse, assume otherwise that there is $u \in U$ such that $\deg u > 1$. Then $T - u$ is disconnected, with all the components of $T - u$ containing elements of U. Assume T has a molecular cover $\mathbf{S}$ and $S \in \mathbf{S}$ such that $|S| = 2$ and $u \in S$ Let $v \in S - \{u\}$, and let C be the component of $T - u$ such that $v \in C$. Let $U' = U \cap C$. Then we state that $(T - U') - u$ has a molecular cover. Indeed, start with a molecular cover of $T - U$, which induces a molecular cover $\mathbf{C}$ of $C - U'$. Now, take $\mathbf{S}' = \{S \in \mathbf{S} : V(S) \subseteq (V(T) - V(C)) - \{u\}\}$. Then $\mathbf{C} \cup \mathbf{S}'$ is a molecular cover of $T - U'$. Now assume there is not such a cover and instead that there is a cover $\mathbf{S}$ and $S \in \mathbf{S}$ with $|S| > 2$ and $S = S_u$. Let $C_1, \ldots, C_s$ be the connected components of $T - u$ and $U_i = U \cap C_i$ for each i. Then for each i, $C_i - U_i$ has a molecular cover $\mathbf{C}_i$, which creates a molecular cover

$$\mathbf{C} = \bigcup_{i=1}^{s} \mathbf{C}_i.$$

Consider C_i as a subgraph of $D_i = T[V(C_i) \cup V(S)]$. Then D_i has a molecular cover, say, with r_i stars. In this way,

$$r = 1 + \sum_{i=1}^{s} (r_i - 1).$$

Also, for each i, $|\mathbf{C}_i| \geq r_i - |U_i| + 1$. Therefore

$$|\mathbf{C}| \geq \sum_{i=1}^{s} (r_i - |U_i| + 1) = \sum_{i=1}^{s} (r_i - 1) - \sum_{i=1}^{s} |U_i| + 2s = r - |U| + 2s - 1.$$

If $s \geq 2$, which is the case here,

$$|\mathbf{C}| \geq r - |U| + 3,$$

which contradicts the hypothesis. ■

There are more complex "minimal" trees whose lower neighbors differ in multiple vertices.

Definition 4.4.1. Let T be a forest and L its sets of leaves. For $u, v \in V(T)$ that are connected, denote $d(u, v)$ as the number of vertices in the path connecting u, v. We say T is of type 0 if and only if

1. For all $u, v \in L$, $d(u, v) \equiv 2 \pmod 3$,
2. For all u and v vertices of degree ≥ 3, $d(u, v) \equiv 1 \pmod 3$,
3. For all $u \in L$ and $\deg(v) \geq 3$, $d(u, v) \equiv 0 \pmod 3$.

We say T is of type 2 if and only if

1. For all $u, v \in L$, $d(u, v) \equiv 2 \pmod 3$,
2. For all u and v vertices of degree ≥ 3, $d(u, v) \equiv 1 \pmod 3$,
3. For all $u \in L$ and $\deg(v) \geq 3$, $d(u, v) \equiv 2 \pmod 3$.

Theorem 4.4.7. Let T be a tree with a set of leaves L and $|L| = l$. Assume:

1. For all $u, v \in L$, $d(u, v) \equiv 2 \pmod 3$,
2. For all u and v vertices of degree ≥ 3, $d(u, v) \equiv 1 \pmod 3$,

3. For all $u \in L$ and $\deg(v) \geq 3$, $d(u, v) \equiv 0 \pmod 3$.

Then

1. Then both T and $T - L$ have molecular covers.
2. If T has a molecular cover with r stars, then $T - L$ has a molecular cover with $r - |L| + 1$ stars.

Theorem 4.4.8. Let T be a tree with set of leaves L such that

1. Both T and $T - L$ have molecular covers.
2. If T have molecular covers with r stars, then $T - L$ have molecular covers with $r - |L| + 1$ stars.

Then

1. If $u, v \in L$ then $d(u, v) \equiv 2 \pmod 3$.
2. If u and v are vertices of degree ≥ 3 then $d(u, v) \equiv 1 \pmod 3$.
3. If $u \in L$ and $\deg(v) \geq 3$ then $d(u, v) \equiv 0 \pmod 3$.

Alternatively, the Theorem 4.4.7 and Theorem 4.4.8 can be stated as

For a tree T with set of leaves L, $T - L$ is its lower neighbor if and only if T is of type 0.

The following lemmas will be crucial to prove theorem 4.4.7

Lemma 4.4.9. Let T be a tree and L its set of leaves. Let e be an edge of T. Let T' be the tree resulting from subdividing e three times and L' its set of leaves. Then T has a molecular cover with r stars if and only if T' has one with $s = r + 1$ stars and $T - L$ has a molecular cover $r - |L| + 1$ stars if and only if $T' - L'$ has a molecular cover with $s - |L| + 1$ stars.

Proof. Assume $e = uv$, and let $P = uw_1w_2w_3v$ be the path in T' resulting from subdividing e three times.

First, assume that T has a molecular cover. Let $\mathbf{S}$ be such a cover. Then we have two possibilities.

1. For all $S \in \mathbf{S}$, $e \notin E(S)$.

2. For exactly one $S \in \mathbf{S}$, $e \in E(S)$.

In case 1, $T' - \mathbf{S}$ has two leaves, w_1, w_3 and w_1, w_2, w_3 induce a star S_{w_2} in T' centered in w_2. In this case, $\mathbf{S} \cup \{S_{w_2}\}$ is a molecular cover for T' with $r+1$ stars.

In case 2, assume u is the center of the star S with $e \in E(S)$. Define S' as the star of T' resulting by replacing v by w_1, and S_{w_3} the star induced by the vertices w_2, w_3 and v. Then $(\mathbf{S} - \{S\}) \cup \{S_{w_3}, S'\}$ is a molecular cover for T' with $r+1$ stars.

Conversely, assume that T' has a molecular cover and let $\mathbf{S}'$ be such a cover. Then we have two possibilities.

1. There are three stars in $\mathbf{S}'$ having vertices of P.

2. There are two stars in $\mathbf{S}'$ having vertices of P.

In case 1, w_1, w_2, w_3 induce a star $S \in \mathbf{S}'$ and $\mathbf{S}' - \{S\}$ is a molecular cover of T. In case 2, let $S_1, S_2 \in \mathbf{S}'$ be the stars having vertices of P. Assume S_1 has only two vertices of P and S_2 has three vertices of P. Furthermore, assume that $V(S_2) = \{w_2, w_3, v\}$ and $V(S_1) = \{u, w_1\} \cup U$ where $U \cap V(P) = \varnothing$. Let S be the star of T induced by U, u and v. Then $(\mathbf{S}' - \{S_1, S_2\}) \cup \{S\}$ is a molecular cover of T with $|\mathbf{S}'| - 1$ stars.

We proved that T has a molecular cover with r stars if and only if T' has a molecular cover with $r-1$ stars. Since $T' - L$ is still the result of subdividing an edge of $T - L$ three times, the result follows. ■

Lemma 4.4.10. Let T be a tree and L its set of leaves. Let $v \in V(T)$ be a vertex of degree ≥ 4. Let T' be a tree with a path $v_1 w_1 w_2 v_2$ such that

$$V(T) = (V(T') - V(P)) \cup \{v\}$$

and

$$E(T) = E(T' - P) \cup \{wv : \text{ if there exists } u \in P \text{ such that } wu \in E(T')\}.$$

Then T has a molecular cover with r stars if and only if T' has a molecular cover with $s = r+1$ stars and $T' - L$ has a molecular cover with $s - |L| + 1$ stars if and only if $T - L$ has one with $r - |L| + 1$ stars.

Proof. Let $\mathbf{S}$ be a molecular cover of T with r stars. Then we have two possibilities.

1. $v \in L(S)$ for one $S \in \mathbf{S}$.

2. $S = S_v$ for one $S \in \mathbf{S}$.

In case 1, assume the vertices of S are all adjacent to v_1 in T' (They must either be all adjacent to v_1 or all to v_2 because they must all be on the same component of $T - v$). Take S' as the star of T' resulting by replacing v by v_1 and let S_{w_2} be the star centered on w_2, with vertices w_1, w_2, v_2. Then $(\mathbf{S} - \{S\}) \cup \{S', S_{w_2}\}$ is a molecular cover of T' with $r + 1$ stars.

In case 2, take $U_1 = (V(S) \cap N_{T'}(v_1)) \cup \{v\}$ and $U_2 = (V(S) \cap N_{T'}(v_2)) \cup \{v\}$. Let S_1 be the star of T' resulting from removing all the vertices of U_2 from S and adding w_1, and let S_2 be the star of T' resulting from removing all the vertices of U_1 from S and adding w_2. Then $(\mathbf{S} - \{S\}) \cup \{S_1, S_2\}$ is a molecular cover of T' with $r + 1$ stars.

Conversely, let $\mathbf{S}'$ be a molecular cover of T with s stars. We have two possibilities, up to a relabeling of v_1, v_2.

1. $v_1 \in L(S)$ for one $S \in \mathbf{S}'$.

2. $S = S_{v_1}$ for one $S \in \mathbf{S}'$.

In the first case, if S' is the star resulting from replacing v_1 by v in S and S_2 is the star of $\mathbf{S}$ containing v_2, then $(\mathbf{S}' - \{S, S_2\}) \cup \{S'\}$ is a molecular cover of T with $s - 1$ stars.

In the second case v_2 is also the center of a star in $\mathbf{S}$, and $V(S_{v_1}) \cup V(S_{v_2}) - \{w_1, w_2\}$ induce a star S in T, making $(\mathbf{S} - \{S_{v_1}, S_{v_2}\}) \cup \{S\}$ a molecular cover of T with $s - 1$ stars.

Since $T' - L$ is still the result of "subdividing" a vertex of $T - L$ and then subdividing the resulting edge twice, the result follows. ■

The lemmas above are important. It means that for us to prove the conjecture, we only need to prove it for the case where the minimal distances being considered are exactly $1, 2$ or 3 instead of congruent with $1, 2$ or $0 \pmod 3$, respectively. In the case of the theorem 4.4.7, we only need to prove it for the tree resulting from subdividing each edge of a star exactly once. But we already did that! So we have the following:

Theorem 4.4.11. Let T be a tree of type 0. Then T has a molecular cover. Furthermore, $T - L$ has a molecular cover with $r - |L| + 1$ stars.

Proof. Follows from the Example 4.4.4, and Lemmas 4.4.9 and 4.4.10. ■

The following will allow us to prove the theorem:

Definition 4.4.2. Let T be a tree and $v, w \in V(T)$ we say that v, w are almost adjacent in T if every other vertex of the only path connecting v and w has degree 2.

Definition 4.4.3. Let T be a tree. The reduced tree of T, $\mathcal{R}(T)$ is defined by setting

$$V(\mathcal{R}(T)) = \{v \in V(T) : \deg(v) \geq 3\}$$

and

$$E(\mathcal{R}(T)) = \{vw : v, w \text{ are almost adjacent in } T\}.$$

Lemma 4.4.12. Let T be a tree that has a molecular cover. Let $v \in V(T)$ be a leaf of $\mathcal{R}(T)$. Since $\deg(v) \geq 3$, then there are at least two leaves l_1, l_2 of T that are almost adjacent to v. Assume that

$$d(l_1, v) = d(l_2, v) = 3.$$

Let P be the path connecting l_1 and v. Then the following statements are equivalent:

- T has a molecular cover with r stars and $T - L(T)$ has a molecular cover with $r - |L(T)| + 1$ stars.
- $T' = T - (P - v)$ has a molecular cover with $s = r - 1$ stars and $T' - L(T')$ has a molecular cover with $s - |L(T')| + 1$ stars.

Furthermore T is of type 0 if and only if T' is.

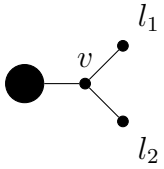
Proof. Let w_1, w_2 be the neighbors of l_1, l_2 respectively. Let $\mathbf{S}$ be a molecular cover of T such that the star S with vertices v, w_2, l_2 is in $\mathbf{S}$. Then the star S' with exactly two vertices l_1, w_1 is in $\mathbf{S}$. The set $\mathbf{S}' = \mathbf{S} - \{S'\}$ is a molecular cover of T' with $s = r - 1$ stars. Conversely, if we take a molecular cover $\mathbf{S}'$ of T' with s stars so that $S \in \mathbf{S}'$, then $\mathbf{S}' \cup \{S'\}$ is a molecular cover of T by $r = s + 1$ stars. Now assume further that $T - L(T)$ has a molecular cover with $r - |L(T)| + 1$ stars, and let $\mathbf{S}$ be such a cover so that the star S centered in v and containing both w_1 and w_2 is in $\mathbf{S}$. Then $(\mathbf{S} - \{S\}) \cup \{S - w_1\}$ is a molecular cover for $T' - L(T)$ with $r - |L(T)| + 1$ stars. But $|L(T')| = |L(T)| - 1$ and $s = r - 1$ so the cover has $s + 1 - |L(T')| - 1 + 1 = s - |L(T')| + 1$ stars. Conversely, if $T' - L(T')$ has a molecular cover with $s - |L(T')| + 1$ stars and

$\mathbf{S}'$ is such a cover so that the star S' centered in v and containing w_2 is in $\mathbf{S}'$, then $(\mathbf{S}' - \{S'\}) \cup \{S\}$ is a molecular cover for $T - L(T)$ with $r - |L(T)| + 1$ stars, where S is the star induced by S' and w_1. Now, assume that T is of type 0. If v has degree ≥ 3 in T' we are done, otherwise let w be the neighbor of v in $\mathcal{R}(T)$. Then $d(v, w) \equiv 1 \pmod 3$, and $d(w, l_2) = d(v, w) + d(v, l_2) - 1 \equiv 0 \pmod 3$. Therefore, T' is of type 0. Conversely, if T' is of class 0 and v has degree ≥ 3 in T' we are done, otherwise $d(v, l_2) = d(w, l_2) - d(v, w) + 1 \equiv 0 \pmod 3$. ■

Theorem 4.4.13. For a tree T that has a molecular cover, with its set of leaves L, $T - L$ is its lower neighbor if and only if T is of type 0.

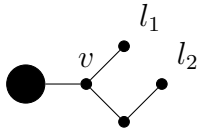
Proof. That for a tree T of type 0, $T - L$ is its lower neighbor is Theorem 4.4.11. To prove the converse, assume T has a molecular cover with r stars, $T - L$ has a molecular cover with $r - |L| + 1$ stars, and that T is not of type 0. Furthermore, assume that T has the minimum number of vertices of degree ≥ 3 possible. Let v be a leaf of $\mathcal{R}(T)$. Then there are at least two leaves of T almost adjacent to v, say, l_1, l_2. Let $d_1 = d(v, l_1)$ and $d_2 = d(v, l_2)$. First, $d_1, d_2 \geq 2$ (otherwise l_1, l_2 do not exist), so assume $d_1, d_2 \in \{2, 3, 4\}$ (if $d_1 > 4$ or $d_2 > 4$ then the same argument used for the correspondent of its class of congruence in the set $\{2, 3, 4\}$). Let P be the path connecting l_1, v, l_2. We have six cases:

1. $d_1 = d_2 = 2$:



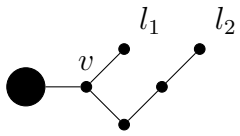
   In this case, in every molecular cover of T, v is the center of the one containing it (otherwise, l_1, l_2, or both wouldn't be covered). This means that $T - l_1$ is a lower neighbor of T; therefore, $T - L$ is not a lower neighbor of T.

2. $d_1 = 2, d_2 = 3$:



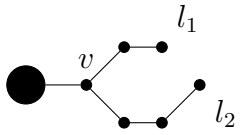
In this case, T does not have a molecular cover. By starting a cover from the leaf l_2, we isolate l_1.

3. $d_1 = 2, d_2 = 4$:



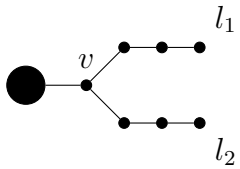
In this case, there is a molecular cover in which l_1 belongs to a star with two elements, l_1 and v. That means that there is a cover of T that can be separated in a cover of the path P between l_1 and l_2 and a cover of $T - P$, and P has 5 vertices. This means that $T - \{l_1, l_2\}$ is a lower neighbor of T, therefore $T - L$ is not a lower neighbor of T.

4. $d_1 = 3, d_2 = 4$:



In this case, l_2 must belong to a star with two vertices in some molecular cover of T. But that means in such a cover, v is the center of a star in the cover, which isolates l_1, making T unable to have a molecular cover.

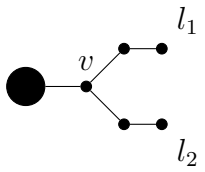
5. $d_1 = 4, d_2 = 4$:



Let $n = |V(T)|$, and $T'' = T - (P - v)$. Start a cover of T by choosing l_1 and l_2 as leaves. This induces a molecular cover of T'' with $r - 2$ stars. So, $\mathrm{pd}(T'') = (n - 6) - (r - 2) = n - r - 4$. Let $Q = T - L$ and

$Q' = Q - P$. By hypothesis, Q has a molecular cover with $r - |L| + 1$ stars. Start a cover of Q by choosing the former neighbors of l_1 and l_2 as leaves. Then Q' has a molecular cover with $r - |L| - 1$ stars. Therefore $\mathrm{pd}(Q') = (n - |L| - 5) - (r - |L| - 1) = n - r - 4 = \mathrm{T}''$, which is impossible, because $Q' = T'' - |L| - v$.

6. $d_1 = d_2 = 3$:



In this case, we repeatedly apply Lemma 4.4.12 until v has degree 2, then get a contradiction.

■

Proposition 4.4.14. Let T be a tree that has a molecular cover with r stars and $U, W \subseteq V(T)$ such that $T - U$ and $T - W$ have a molecular cover with $r - |U| + 1$ stars. Then $U \not\subseteq W$ and $W \not\subseteq U$. Therefore, the set of all such Us is an antichain.

4.5 The Scalar Function on Paths

With the results of the previous section, we have the following.

Definition 4.5.1. Let P be a subforest of a path that can be covered by stars, with n vertices $v_1, \ldots, v_n$ and projective dimension i. Let $\mathcal{S} = \mathcal{S}(P)$ be the set of all its lower neighbors. For each $T \in \mathcal{S}$ define $j_T = \min(j : v_j \in P - T)$ and $l_T = \min(l : v_l \in P - \{l_T\} - T)$. We order $\mathcal{S}$ lexicographically as follows:

For $T, U \in \mathcal{S}$, $T < U$ whenever one of the following holds:

1. $j_T < j_U$,
2. $j_T = j_U$ and $l_T < l_U$.

We also weakly order $\mathcal{S}$ as $T \prec U$ whenever $j_T < j_U$. We define $\iota : \mathcal{S} \to \{0, \ldots, |\mathcal{S}| - 1\}$ to be the enumeration of $\prec$.

Proposition 4.5.1. The relation $\leq$ defines a total order on $\mathcal{S}$ and the relation $\preceq$ defines a partial order on $\mathcal{S}$.

Definition 4.5.2. Let P be a subforest of a path with a molecular cover, and $\mathcal{S} = \mathcal{S}(P)$. Then for $T \in \mathcal{S}$ we define the scalar function:

$$\sigma(P, T) = (-1)^{\iota(T)}$$

where $\iota : \mathcal{S} \to \mathbb{N}$ is the enumeration of $\prec$.

This definition has a few particularities when P is a path.

Proposition 4.5.2. Let P be a subforest of a path with a molecular cover. The following hold:

1. If $T \in \mathcal{S}$,
$$\sigma(P, T) = \begin{cases} 1 & j_T \equiv 1 \pmod 3 \\ -1 & j_T \equiv 0 \pmod 3 \end{cases}.$$

2. If P is a path and $|P| \equiv 2 \pmod 3$ and $T \in \mathcal{S}$, then
$$\sigma(P, T) = \begin{cases} 1 & |P - T| = 2 \\ -1 & |P - T| = 1 \end{cases},$$

3. If $T = S \sqcup Q$ where Q is a path with a molecular cover and R is a subforest of Q also with a molecular cover, then
$$\sigma(T, R) = \begin{cases} \sigma(Q, R) & |S| \equiv 0 \pmod 3 \\ -\sigma(Q, R) & |S| \equiv 2 \pmod 3 \end{cases},$$

4. If $T = S \sqcup Q$ where S is a path with a molecular cover and R is a subforest of P with a molecular cover, then
$$\sigma(T, R) = \sigma(S, R).$$

Definition 4.5.3. For a forest T, the set $\mathcal{B}(T)$ of all the subforests of T with molecular covers, with the subset relation, will be called the *undecorated poset* of T.

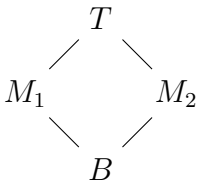
Definition 4.5.4. For a forest T, we define the following digraph $\mathcal{B}(P) = (V, E)$:

1. V is the set of all the subforests of P with molecular covers.
2. E consists of all the pairs (T, Q) where $T \in \mathcal{S}(Q)$.

This digraph will be called the *undecorated digraph* of P.

Now we just turn $\mathcal{B}(T)$ into a decorated poset $B(T)$ by assigning σ to it.

Proposition 4.5.3. Let P be a path with n vertices and consider a diamond D in $\mathcal{B}(P)$:



Then D is unbalanced in $B(P)$, that is, one of the following equivalent properties holds:

1. $\sigma(T, M_1)\sigma(T, M_2) \neq \sigma(M_1, B)\sigma(M_2, B)$.
2. $\sigma(T, M_1) = \sigma(T, M_2)$ if and only if $\sigma(M_1, B) \neq \sigma(M_2, B)$ and $\sigma(T, M_1) \neq \sigma(T, M_2)$ if and only if $\sigma(M_1, B) = \sigma(M_2, B)$.

Proof. First, suppose T is a path. We have that $|T - M_1|, |T - M_2| \in \{1, 2\}$ so we can consider four cases:

i) $|T - M_1| = |T - M_2| = 1$

ii) $|T - M_1| = |T - M_2| = 2$

iii) $|T - M_1| = 1, |T - M_2| = 2$

iv) $|T - M_1| = 2, |T - M_2| = 1$

Also, suppose that $M_1 \leq M_2$. Then, in each case, we prove that the products of the signs at the top and at the bottom differ.

i) We have some cases:

- First, assume that

$$j^T_{M_1} \equiv j^T_{M_2} \equiv 0 \pmod 3.$$

Then by proposition 4.5.2, (1),

$$\sigma(T, M_1) = \sigma(T, M_2) = -1.$$

Now, $M_1 = P_1 \sqcup Q_1$ and $M_2 = P_2 \sqcup Q_2$ where $|P_1| \equiv |P_2| \equiv 2 \pmod 3$. Then, since

$$j^{Q_1}_B \equiv j^{P_2}_B \equiv 0 \pmod 3,$$

it follows by proposition 4.5.2 (1), (3), and (4) that

$$\sigma(M_1, B) = -1 \text{ and } \sigma(M_2, B) = 1.$$

- Now, we assume that

$$j^T_{M_1} \equiv j^T_{M_2} \equiv 1 \pmod 3.$$

Then, by proposition 4.5.2, (1),

$$\sigma(T, M_1) = \sigma(T, M_2) = 1.$$

Now, $M_1 = P_1 \sqcup Q_1$ and $M_2 = P_2 \sqcup Q_2$ where $|P_1| \equiv |P_2| \equiv 0 \pmod 3$. In this case, though,

$$j^{Q_1}_B \equiv 0 \pmod 3 \text{ while } j^{P_2}_B \equiv 1 \pmod 3,$$

so it follows from proposition 4.5.2 (1), (3) and (4) that

$$\sigma(M_1, B) = -1 \text{ and } \sigma(M_2, B) = 1.$$

- Then, we assume that

$$j^T_{M_1} \equiv 0 \pmod 3, j^T_{M_2} \equiv 1 \pmod 3.$$

Then, by proposition 4.5.2, (1),

$$\sigma(T, M_1) = -1, \sigma(T, M_2) = 1.$$

Now, $M_1 = P_1 \sqcup Q_1$ and $M_2 = P_2 \sqcup Q_2$ where $|P_1| \equiv 2 \pmod 3, |P_2| \equiv 0 \pmod 3$. In this case,

$$j^{Q_1}_B \equiv 1 \pmod 3 \text{ and } j^{P_2}_B \equiv 0 \pmod 3$$

which means, by proposition 4.5.2 (1), (3) and (4), that $\sigma(M_1, B) = \sigma(M_2, B) = -1$.

- Last, we assume that

$$j^T_{M_1} \equiv 1 \pmod 3, j^T_{M_2} \equiv 0 \pmod 3.$$

Then, by proposition 4.5.2, (1),

$$\sigma(T, M_1) = 1, \sigma(T, M_2) = -1.$$

Now, $M_1 = P_1 \sqcup Q_1$ and $M_2 = P_2 \sqcup Q_2$ where $|P_1| \equiv 0 \pmod 3, |P_2| \equiv 2 \pmod 3$. In this case,

$$|M_1 - B| = |M_2 - B| = 2.$$

Otherwise B would have a subpath, between $v_{j^T_{M_1}}$ and $v_{j^T_{M_2}}$ with

$$j^T_{M_2} - j^T_{M_1} - 1 \equiv 0 - 1 - 1 \equiv 1 \pmod 3$$

vertices, which would make B unable to have a molecular cover. Therefore, since $|M_1 - B| = |M_2 - B| = 2$ and $|P_1| \equiv 0 \pmod 3$, it follows that

$$\sigma(M_1, B) = \sigma(M_2, B) = 1.$$

So, in this case, the diamond is unbalanced.

ii) In this case, first, we have that

$$\sigma(T, M_1) = \sigma(T, M_2) = 1.$$

Now, $|T| \equiv 2 \pmod 3$ and $|M_1 - B| = |M_2 - B| = 1$, which means that

$$|(T - M_1) \cap (T - M_2)| = 1.$$

We also have that

$$j^T_{M_1} \equiv j^T_{M_2} \equiv 1 \pmod 3, l^T_{M_1} \equiv l^T_{M_2} \equiv 2 \pmod 3.$$

Last, all the components of M_1 and M_2 have a number of vertices $\equiv 0 \pmod 3$. We have two subcases:

- First, assume that
$$j^T_{M_1} = j^T_{M_2}.$$
In this case, let Q_1 be the component of M_1 containing $v_{l^T_{M_2}}$ and P_2 the component of M_2 containing $v_{l^T_{M_1}}$. Then
$$j_B^{Q_1} \equiv 0 \pmod 3 \text{ and } j_B^{P_2} \equiv 1 \pmod 3.$$
Since there are no components of M_1, M_2 with a number of vertices $\equiv 2 \pmod 3$, because of proposition 4.5.2 (1) and (4), we have that
$$\sigma(M_2, B) = \sigma(P_2, B) = 1 \text{ and } \sigma(M_1, B) = \sigma(Q_1, B) = -1.$$

- Now assume that
$$l^T_{M_1} = l^T_{M_2}.$$
In this case, let Q_1 be the component of M_1 containing $v_{j^T_{M_2}}$ and P_2 the component of M_2 containing $v_{j^T_{M_1}}$. Then
$$j_B^{Q_1} \equiv 0 \pmod 3 \text{ and } j_B^{P_2} \equiv 1 \pmod 3.$$
Since there are no components of M_1, M_2 with a number of vertices $\equiv 2 \pmod 3$, because of proposition 4.5.2 (1) and (4), we have that
$$\sigma(M_2, B) = \sigma(P_2, B) = 1 \text{ and } \sigma(M_1, B) = \sigma(Q_1, B) = -1.$$
Therefore, this diamond is unbalanced.

iii) As in case ii), $|T| \equiv 2 \pmod 3$ and
$$\sigma(T, M_1) = -1, \sigma(T, M_2) = 1.$$
Also $|M_1 - B| = 2, |M_2 - B| = 1$. Let P_2 be the component of M_2 containing $v_{j_B^{M_2}}$. Then $j_B^{P_2} \equiv 0 \pmod 3$ so
$$\sigma(M_2, B) = -1.$$
Now, $M_1 = P_1 \sqcup Q_1$ where $|P_1| \equiv 2 \pmod 3$, so
$$\sigma(M_1, B) = -\sigma(Q_1, B) = -1.$$
Therefore, this diamond is unbalanced.

iv) As in cases ii) and iii), $|T| \equiv 2 \pmod 3$ and

$$\sigma(T, M_1) = 1, \sigma(T, M_2) = -1.$$

Also $|M_1 - B| = 1, |M_2 - B| = 2$. Let Q_1 be the component of M_1 containing $v_{j_B^{M_1}}$. Then $j_B^{Q_1} \equiv 1 \pmod 3$ so

$$\sigma(M_1, B) = 1.$$

Now, $M_2 = P_2 \sqcup Q_2$ and $j_B^{M_2} \equiv 0 \pmod 3$ so

$$\sigma(M_2, B) = \sigma(P_2, B) = -1.$$

Therefore, this diamond is unbalanced.

■

Definition 4.5.5. Let P be the path with vertex set $\{1, \ldots, n\}$ and edge ideal I. For each $i \in \{1, \ldots, \mathrm{pd}(P)\}$ set

$$\mathcal{B}_i = \{T \in \mathcal{B}(P) : \mathrm{pd}(T) = i\}.$$

For each such k, let

$$F_i = \begin{cases} R/I & \text{if } i = -1 \\ R & \text{if } i = 0 \\ \bigoplus_{Q \in \mathcal{B}_i} S(-\mathbf{x}^Q) & \text{if } 1 \leq i \leq \mathrm{pd}(P) \end{cases},$$

where $-\mathbf{x}^Q = \prod_{a \in V(Q)} \mathbf{x}^a$. Now define $d_i : F_i \to F_{i-1}$ by its matrix representation:

$$(d_i)_{P,Q} = \sigma(P, Q)\mathbf{x}^{P-Q}.$$

This gives us a complete candidate for a free resolution of a path P.

4.6 The Scalar Function of a Triple Subdivision

Let T be a forest and $e = v_1 w_1 \in E(T)$ an edge containing a leaf w. Let $\Sigma(T)$ be the tree resulting from subdividing e three times. Formally, $V(\Sigma(T)) = V(T) \cup \{t, v_2, w_2\}$ and $E(\Sigma(T)) = E(T) \cup \{wt, tv_2, v_2w_2\}$. If P is the path given by tv_2, v_2w_2 we will denote $\Sigma(T) = T + P$.

There are two kinds of lower neighbors for $\Sigma(T)$.

1. The ones emerging from a lower neighbor of T. In particular, for each lower neighbor C of T containing e, there is a lower neighbor of $\Sigma(T)$ given by $C + P$. For each lower neighbor of T resulting from removing w_1 (as well as possibly other vertices of T), there is an extra lower neighbor of $\Sigma(T)$ resulting from removing w_2 instead. And for each lower neighbor of T resulting from removing v_1 and w_1 there are two extra lower neighbors of $\Sigma(T)$ resulting from removing v_1 and w_2 or v_2 and w_2 instead.

2. The lower neighbor resulting from removing the vertex t.

We will define the following scalar function.

Definition 4.6.1. Let T be a molecular forest, and assume we have a scalar $\sigma(F, C)$ for each lower neighbor C of a molecular subforest F of T. For each lower neighbor C' of $\Sigma(T)$ we define $\sigma(\Sigma(T), C')$ as follows:

- If C' comes from a lower neighbor C of T, $\sigma(\Sigma(T), C') = \sigma(T, C)$.
- IF C' results from removing t, $\sigma(\Sigma(T), C') = (-1)^{n-r}$ where r is the molecular cover number of $\Sigma(T)$.

By using this, we can compute the scalar function for any other case. All the elements in the poset of $\Sigma(T)$ either have the forms described above or the form $Q \sqcup f$ where f is an edge and Q is in the poset of T. Then, we define the sign

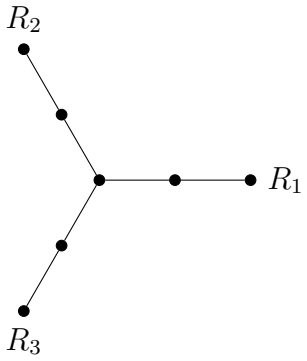
$$\sigma(Q \sqcup f, Q) = (-1)^{|Q \sqcup f| - r(Q \sqcup f) - 1}$$

and for a lower neighbor R of Q,

$$\sigma(Q \sqcup f, R \sqcup f) = \sigma(Q, R).$$

4.7 A Subdivision of a Star

Let $T = S_n^{(3,\ldots,3)}$ be the graph resulting from subdividing each edge of a star once.



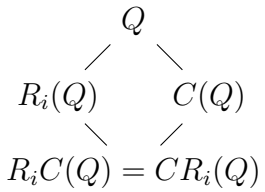
Each *branch*, the path between each leaf l_i and the center 0, can be denoted by R_i. The operations to compute the lower neighbors of T can be classified as follows:

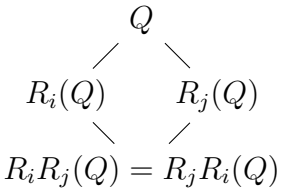
1. $R_i(T)$: Remove the vertices in the branch R_i at distance 1 and 2 of the center.
2. $C(T)$: remove the center 0.
3. $L(T)$: remove all the leaves of T.

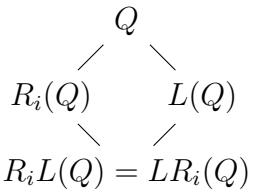
Furthermore:

- The tree resulting from the operation R_i is $S_{n-1}^{(3,\dots,3)}$. To compute the lower neighbors of this one, you can apply the same operations.
- The forest resulting from the operation C_i is a disjoint union of edges. For this one, the only possible operation is R_i.
- The tree resulting from the operation L is S_n. For this one, the only possible operation is R_i.

We can furthermore say that $R_iC(T) = CR_i(T)$, $R_iL(T) = LR_i(T)$, $R_iR_j(T) = R_jR_i(T)$ but LC and CL are not valid operations. Therefore, all the diamonds in the Poset of T can be described as the following three:







for a molecular subforest Q of T.

We will define the scalar function as follows:

- $\sigma(Q, R_i(Q)) = (-1)^i$
- $\sigma(Q, C(Q)) = (-1)^k$
- $\sigma(Q, L(Q)) = (-1)^k$ where k is the number of branches in Q.

Chapter 5

Future Work and Discussion

As can be seen, despite the fact we have been able to calculate candidates to minimal free resolutions of edge ideals, to prove they are actually resolutions is a different problem. The idea for a path to get to that is to prove the columns of the matrices are actually irredundant, and then, with the diamond property, that would prove they are resolutions, as seen in [**?**].

Next, the algorithms developed should be able to be generalized to other families of graphs. The fist hint is in the complete bipartite graphs: Their Betti numbers follow the same pattern as in trees.

The molecular cover number of a forest is a combinatorial invariant, so there can be some combinatorial results on that. The type 0 forests are also related to the concept of *degeneracy* in graph theory; see [**?**] for instance.

The upper Koszul Complex has some applications in logic and topology, as seen in [**?**]. There could be some applications to those topics.

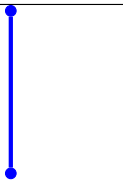
Chapter 6

Appendix

6.1 Molecular Trees

Here we attach a list with all the molecular trees with at most ten vertices and one with all the molecular trees with at most 18 vertices that remain molecular after removing all their leaves.

Table 6.1: List of all nonisomorphic trees of type 0 with more than two vertices and less than 18 along with a molecular cover and their projective dimension.

#	n	G	$\mathbf{S}$	pd
1	2			1
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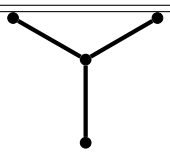
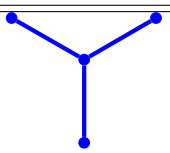
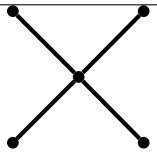
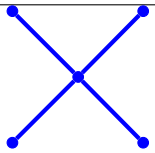
3	4			3
4	5			3

5				4
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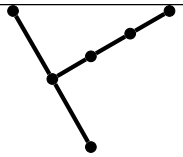
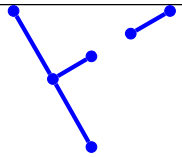
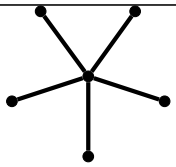
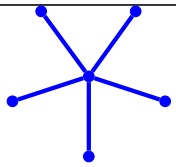
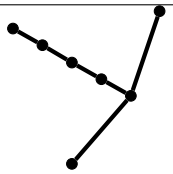
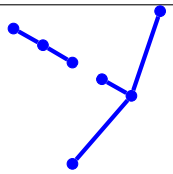
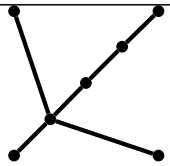
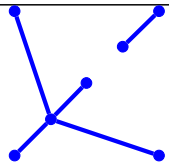
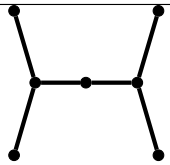
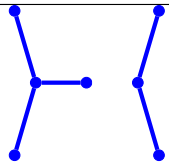
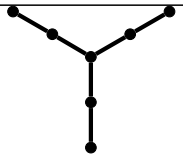
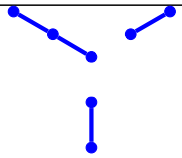
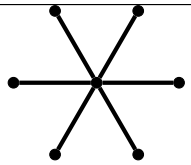
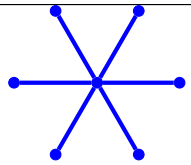
7				4
8				5
9	7			5
10				5
11				5
12				4
13				6

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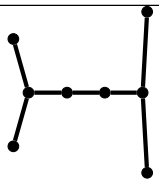
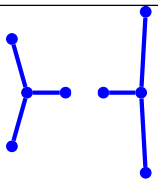
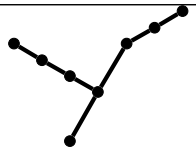
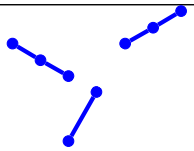
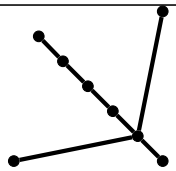
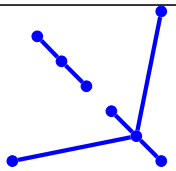
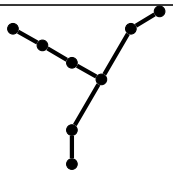
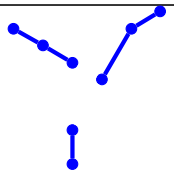
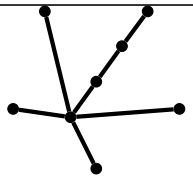
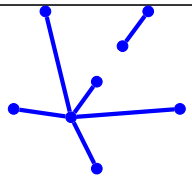
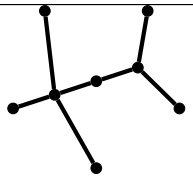
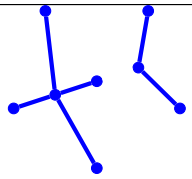
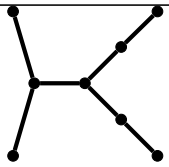
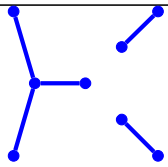
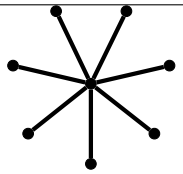
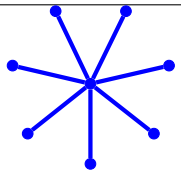
15				6
16				5
17				6
18				5
19				6
20				6
21				5

22				7
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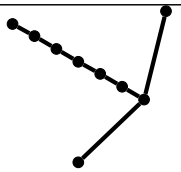
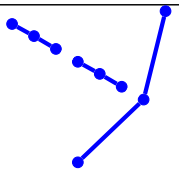
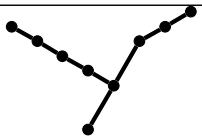
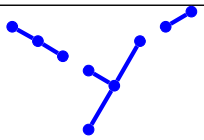
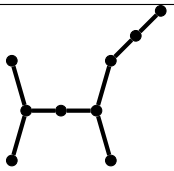
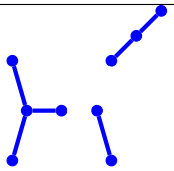
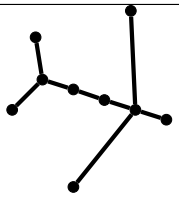
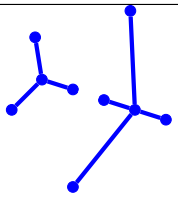
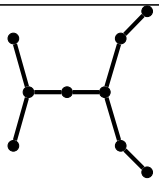
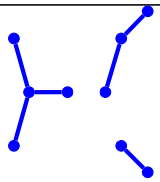
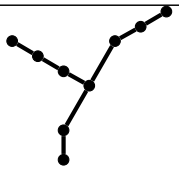
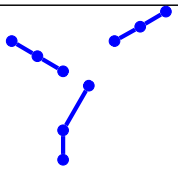
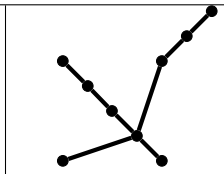
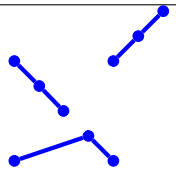
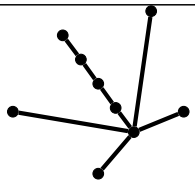
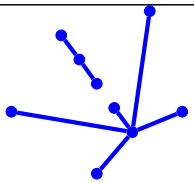
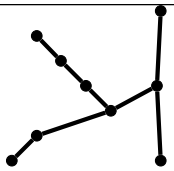
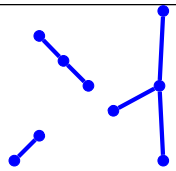
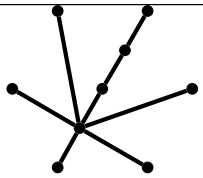
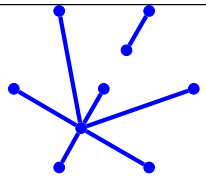
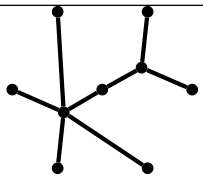
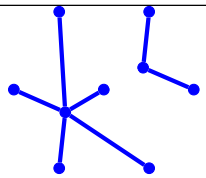
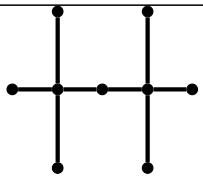
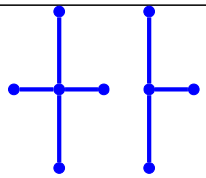
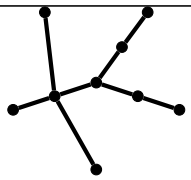
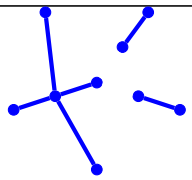
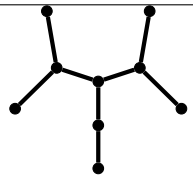
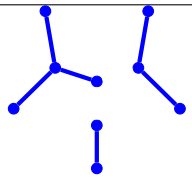
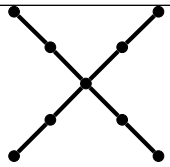
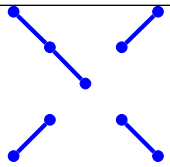
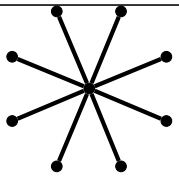
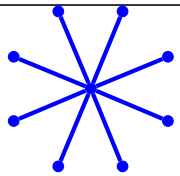
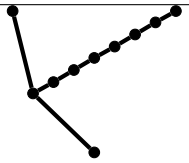
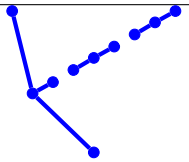
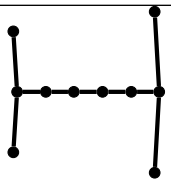
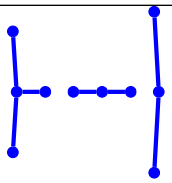
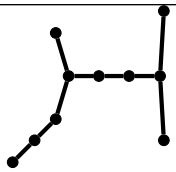
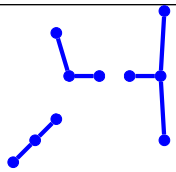
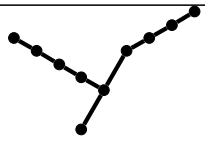
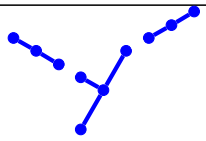
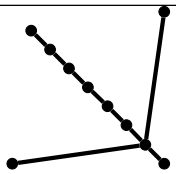
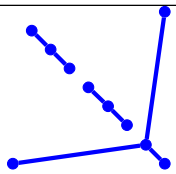
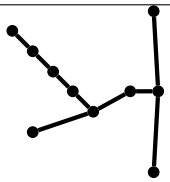
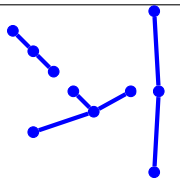
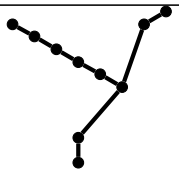
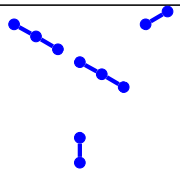
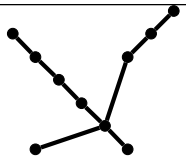
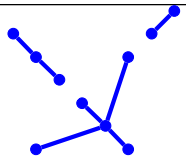
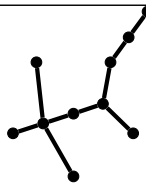
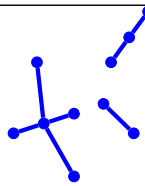
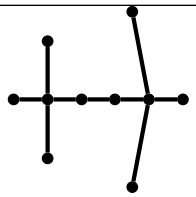
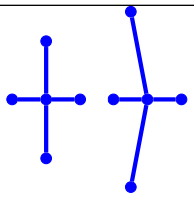
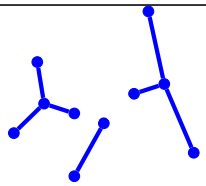
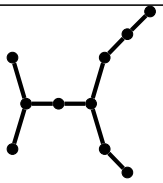
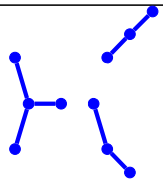
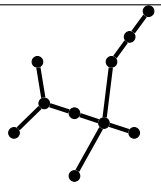
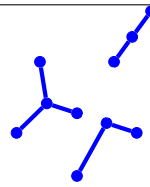
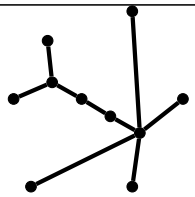
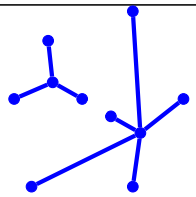
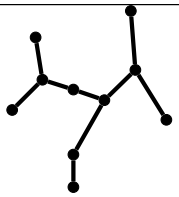
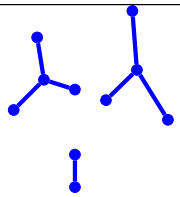
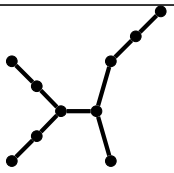
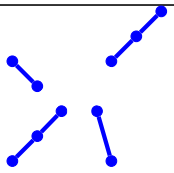
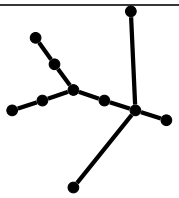
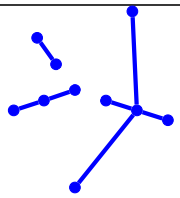
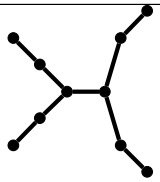
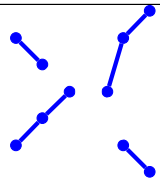
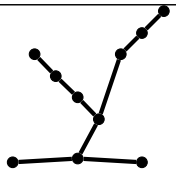
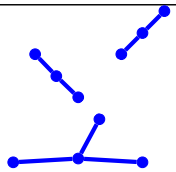
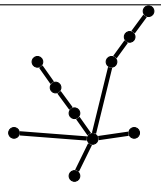
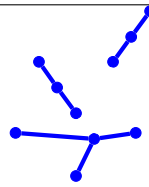
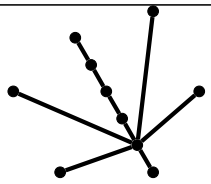
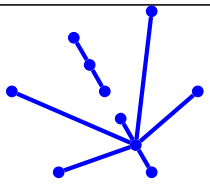
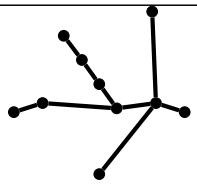
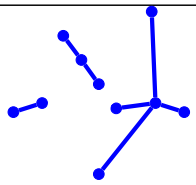
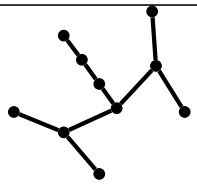
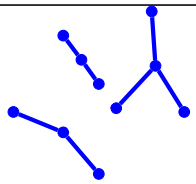
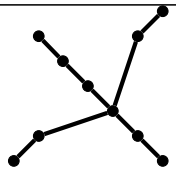
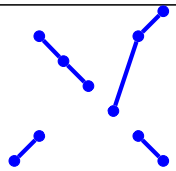
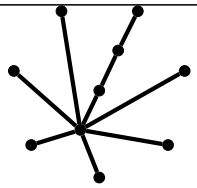
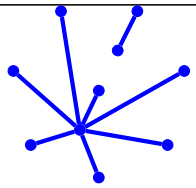
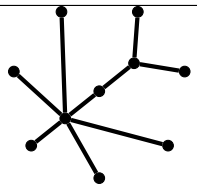
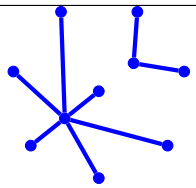
24				6
25				6
26				6
27				7
28				6
29				6

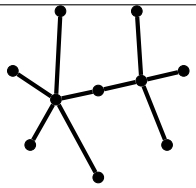
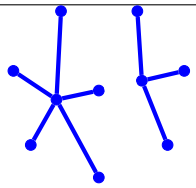
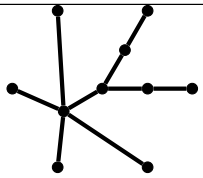
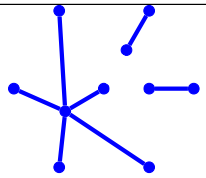
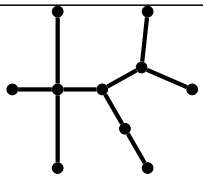
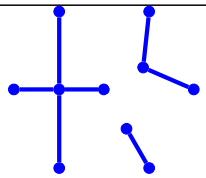
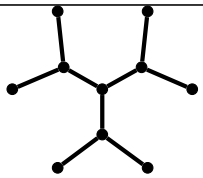
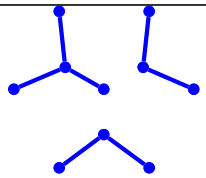
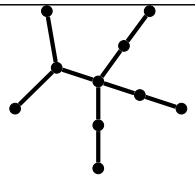
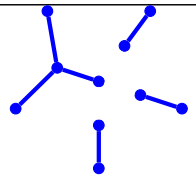
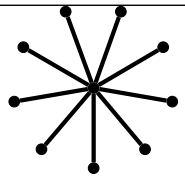
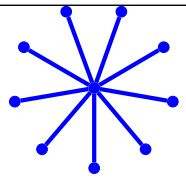
30				6
31				7
32				6
33				7
34				7
35				7
36				6

37				6
38				5
39				8
40	10			7
41				7
42				7
43				7
44				7

45				7
46				6
47				7
48				7
49				8
50				7
51				7

52				7
53				8
54				7
55				6
56				7
57				6
58				7

59				7
60				8
61				7
62				7
63				6
64				8
65				8

66				8
67				7
68				7
69				7
70				6
71				9

6.2 Type 0 Trees

Table 6.2: List of all nonisomorphic trees of type 0 with more than two vertices and less than 19 along with a molecular cover and their projective dimension.

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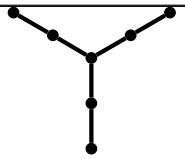
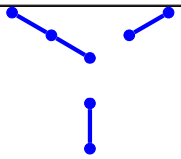
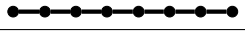
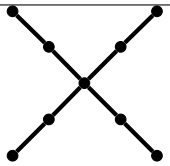
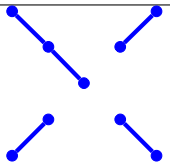
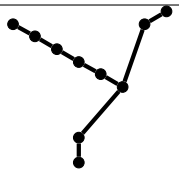
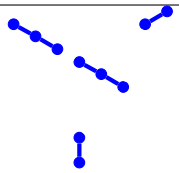
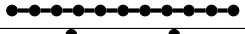
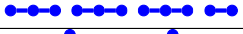
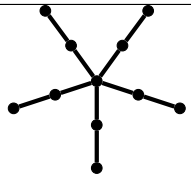
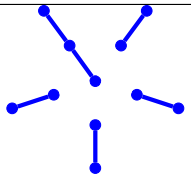
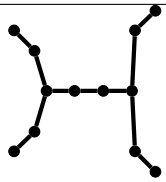
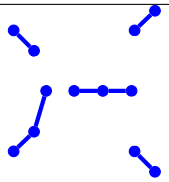
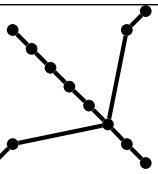
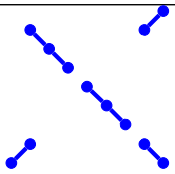
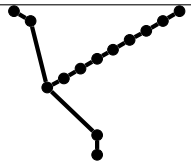
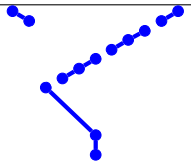
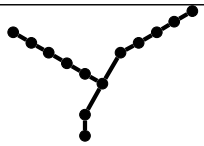
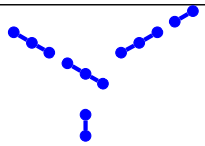
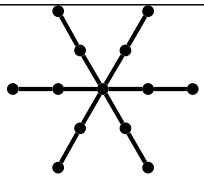
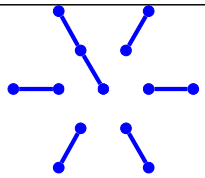
2	7			4
3	8			5
4	9			5
5	10			6
6	11			7
7				6
8	12			7
9				7

10	13			8
11				8
12				7
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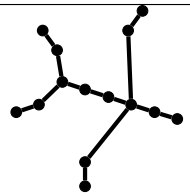
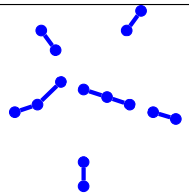
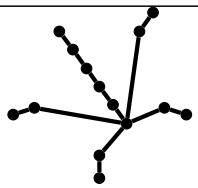
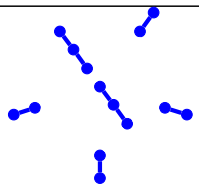
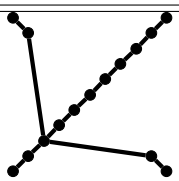
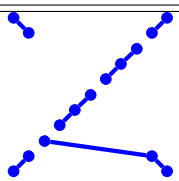
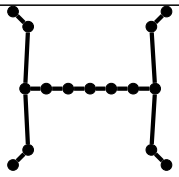
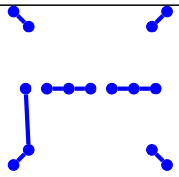
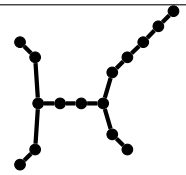
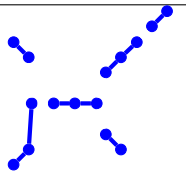
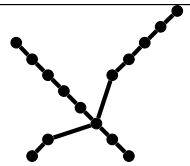
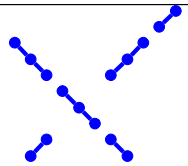
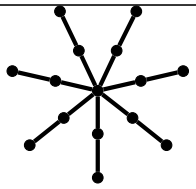
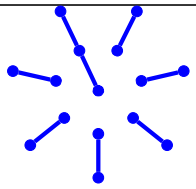
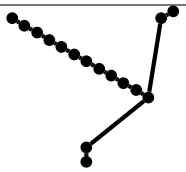
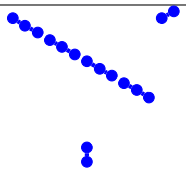
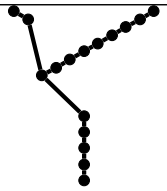
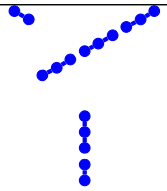
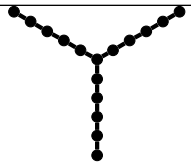
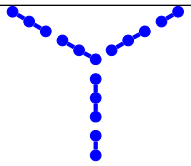
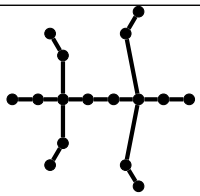
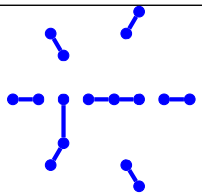
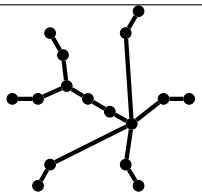
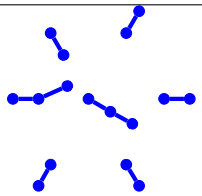
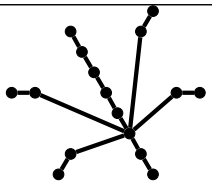
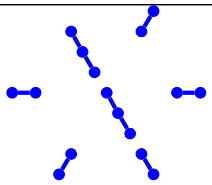
14				8
15				8
16	15			9
17				9

18				9
19				9
20				8
21	16			10
22				10
23				10
24				9

25				9
26				9
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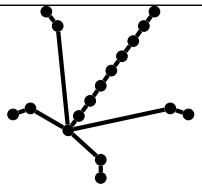
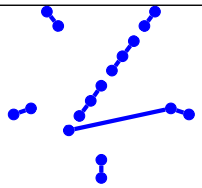
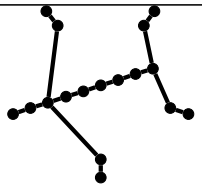
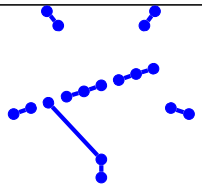
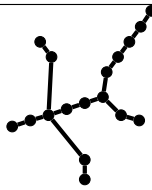
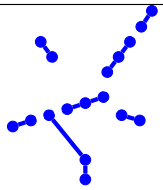
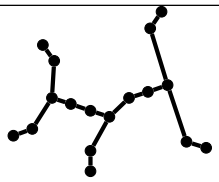
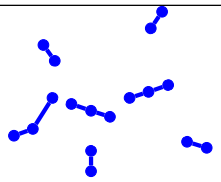
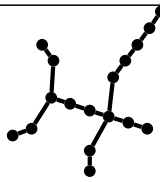
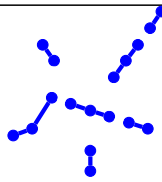
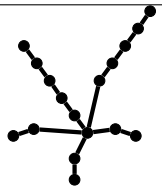
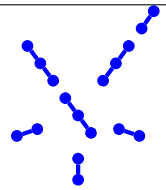
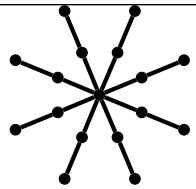
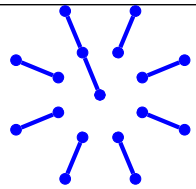
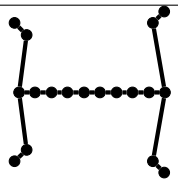
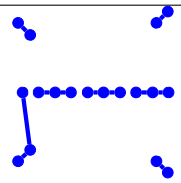
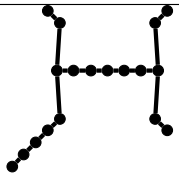
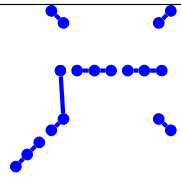
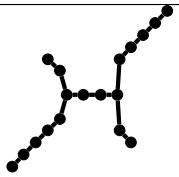
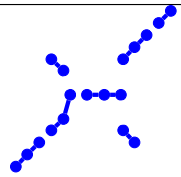
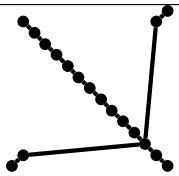
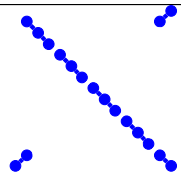
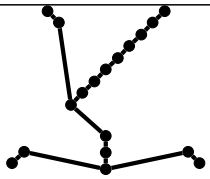
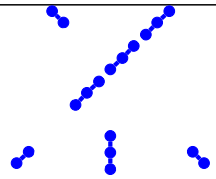
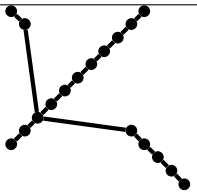
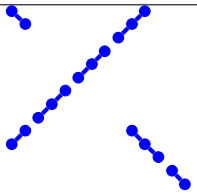
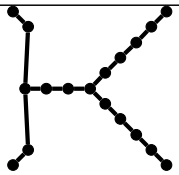
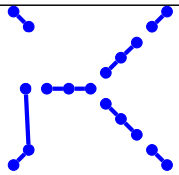
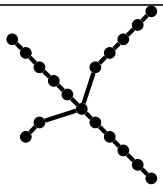
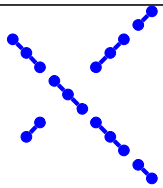
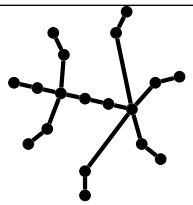
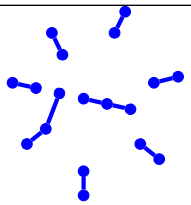
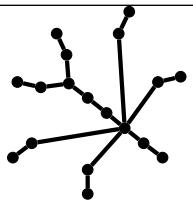
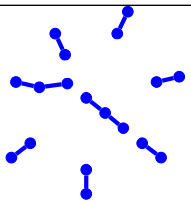
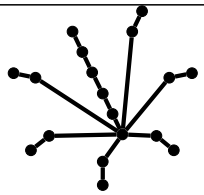
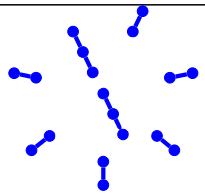
28				10
29				10
30				10
31				10
32				10

33				10
34				9
35	18			11
36				11
37				11
38				11
39				11

40				11
41				11
42				11
43				10
44				10
45				10