Centro de Investigación y de Estudios Avanzados
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Condiciones necesarias y suficientes de estabilidad para sistemas lineales de tipo neutral con retardo

T E S I S

Que presenta

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Necessary and sufficient stability conditions for neutral linear time delay systems

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Contents

Notation

$\mathbb{N}$ Natural numbers set.

$\mathbb{R}$ Real numbers set.

$\mathbb{R}^n$ Space of n-dimensional vector space with entries in $\mathbb{R}$.

$\mathbb{R}^{n\times n}$ Space of $n \times n$ matrices with entries in $\mathbb{R}$.

$\Re(s)$ Real part of an imaginary number s.

i Imaginary unit, $i^2 = -1$.

$\mathcal{C}^l\big([-h,0],\mathbb{R}^n\big)$ Space of $\mathbb{R}^n$-valued l times continuously differentiable functions on $[-h,0]$, $\forall l \in \mathbb{N}$

$\mathcal{C}^2_{[a,b]}$ Space of twice continuously differentiable $n \times n$ matrices defined on $[a,b]$.

$\mathcal{PC}^1\big([-h,0],\mathbb{R}^n\big)$ Space of $\mathbb{R}^n$-valued piecewise continuously differentiable on $[-h,0]$.

I_n Identity matrix of dimension $n \times n$.

$\mathbb{O}_n$ Zero matrix of dimension $n \times n$.

0_n Zero vector in $\mathbb{R}^n$.

A^T Transpose of matrix A.

$A > 0$ Symmetric positive definite matrix A.

$A \geq 0$ Symmetric non-negative definite matrix A.

$\lambda_{\min}(A)$ Minimum eigenvalue of the matrix A.

$\lambda_{\max}(A)$ Maximum eigenvalue of the matrix A.

$\{A_{ij}\}_{i,j=1}^r$ Square block matrix with i-th row block and j-th column block matrix $A_{ij} \in \mathbb{R}^{n\times n}$.

$\mathrm{He}(M)$ Means $M + M^T$.

$k = \overline{a,b}$ Means that k is an integer between a and b, included.

$\lceil r \rceil$ Ceil function, which maps r to the least integer greater or equal to r.

$\|\cdot\|$ Euclidean norm for vectors and matrices.

$\| \varphi \|_h$ Supremum norm of the function φ, defined as $\|\varphi\|_h = \sup_{\theta \in [-h,0]} \|\varphi(\theta)\|$.

$A \otimes B$ The Kronecker product given by:

$$A \otimes B \stackrel{\text{def}}{=} \begin{pmatrix} b_{11}A & b_{21}A & \cdots & b_{n1}A \\ b_{12}A & b_{22}A & \cdots & b_{n2}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n}A & b_{2n}A & \cdots & b_{nn}A \end{pmatrix}, \quad B = \{b_{ij}\}_{i,j=1}^{n}.$$

Resumen

Los sistemas lineales con retardo de tipo neutral permiten modelar procesos que dependen no solo de la derivada del estado actual sino también por derivadas pasadas del estado. Estos sistemas también pueden representar aquellos descritos por ecuaciones diferenciales parciales hiperbólicas, que pueden ser transformadas en esta clase. El efecto del retardo en las derivadas de estados pasados eleva la complejidad del análisis de estabilidad de esta clase de sistemas, debido a los comportamientos dinámicos intrincados resultantes. La teoria de Lyapunov-Krasovskii ha sido una herramienta esencial para abordar el problema de estabilidad, así como también para tratar temas de robustez y síntesis de control, entre otras. Por lo tanto, la presentación de criterios de estabilidad para esta clase de sistemas es un problema de interés.

En este trabajo de tesis, se presentan criterios de estabilidad, es decir, condiciones de estabilidad necesarias y suficientes, para sistemas lineales de retardo de tipo neutral. Este trabajo combina las funcionales de Lyapunov con derivada en el tiempo prescrita, las cuales son expresadas en términos de la matriz de Lyapunov de retardo, con una aproximación ya sea del argumento funcional o de los núcleos funcionales. Como resultado, la funcional de Lyapunov se aproxima a través de una forma cuadrática cuya matriz de bloques internos involucra la matriz de Lyapunov de retardo evaluada en puntos discretos o integrales de la matriz de Lyapunov de retardo y su primera y segunda derivada, dependiendo del enfoque de aproximación. La característica principal de estos criterios es que se verifican empleando un número moderado de operaciones, lo cual se logra estimando el error de aproximación funcional sobre un conjunto especial de funciones. Algunos ejemplos se presentan y se discuten para validar y comparar los resultados obtenidos.

Abstract

Neutral type time delay systems allow modeling processes that depend not only on the derivative of the current state but also on the past derivatives of the state. These systems can also represent those described by hyperbolic partial differential equations, which can be transformed into this class. The intricate dynamics behaviors of systems with delays in past state derivatives increase the complexity of the stability analysis. The Lyapunov-Krasovskii framework has been an essential tool for addressing this stability problem, as well as for dealing with issues of robustness and control synthesis, among others. Thus, the presentation of stability criteria for this class of systems is a problem of interest.

In this thesis work, we present stability criteria, namely necessary and sufficient stability conditions, for neutral type linear time-delay systems. We combine the Lyapunov functionals with a negative prescribed time derivative expressed in terms of the delay Lyapunov matrix with an approximation of either the functional argument or the functional kernels. As a result, the Lyapunov functional is approximated via a quadratic form whose inner block matrix involves the delay Lyapunov matrix evaluated at discrete points or integrals of the delay Lyapunov matrix and its first and second derivatives, depending on the approximation approach. The main feature of these criteria is that they are verified in a moderate number of operations, which is achieved by estimating the functional approximation error on a special set of functions. Some examples are presented and discussed to validate and compare the obtained results.

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Chapter 1

Introduction

To illustrate the background and difficulty motivating this thesis work, we present general comments on delay systems, particularly neutral type systems, to expose their relevance of the stability analysis. We also comprehensively summarize some complex properties needed to deal with this class of delay systems. Finally, we comment on the stability analysis techniques for time delay systems and some recent stability results for neutral-type systems.

General comments on delay systems

Time delay systems are dynamic systems characterized by a delay between the cause and its effect within the system feedback or control loops. This delay indicates the time required for information, signals, or material to cross the system or actions to manifest observable effects. These delays may stem from various factors, including interconnected systems (Bajodek, 2022), communication delays (Peet, 2021), processing times, or inherent physical properties. The presence of delays significantly impacts the stability, performance, and behavior of these systems.

In the classes of time delay systems, neutral differential-difference equations, which are considered as a generalization of those of retarded type (Hale & Lunel, 1993), are characterized by the presence of the derivative of the state at time t but also on past derivatives of the state. Some examples are:

- Drilling in the perforation process for oil and gas (Saldivar & Mondié, 2013; Saldivar, Mondié, Loiseau, & Rasvan, 2013):

  $$\dot{z}(t)-\alpha\dot{z}(t-2h)=-\psi z(t)-\alpha\psi z(t-2h)-\frac{1}{\beta}T(z(t))+\frac{1}{\beta}\alpha T(z(t-2h))+\pi\Omega(t-h)+w.$$

  Here, $z(t)$ is the angular velocity at the bottom of the extremity of the drill string, π, α and ψ are suitable constants depending on the inertia, shear modulus, and the geometrical moment of inertia, $T(\cdot)$ is a torque function, and $\Omega(t)$ is the angular velocity coming from the rotor;

- Lossless transmission lines (Brayton, 1968):

  $$\frac{\mathrm{d}}{\mathrm{d}t}\Big(u(t)-\hat{K}u(t-h)\Big)=\hat{\alpha}-\frac{1}{z}u(t)-\frac{\hat{K}}{z}u(t-h)-g(u(t))+\hat{K}g(u(t-h)),$$

where $u(t)$ is the voltage at the end of the transmission line, $g(\cdot)$ is a nonlinear function, constants $\hat{\alpha}$, z and $\hat{K}$ are given in terms of the system parameters.

- Vibrating masses attached to an elastic bar (Rubanik, 1969):

$$\ddot{x}(t) - \gamma_1 \ddot{y}(t-h) = \epsilon f_1(x(t), \dot{x}(t), y(t), \dot{y}(t)) - \omega_1^2 x(t),$$
$$\ddot{y}(t) - \gamma_2 \ddot{x}(t-h) = \epsilon f_2(x(t), \dot{x}(t), y(t), \dot{y}(t)) - \omega_2^2 x(t).$$

As shown in Brayton (1968), and Saldivar and Mondié (2013), physical phenomena modeled by hyperbolic partial differential equations also reduce, after some transformations, to this class of systems. This has motivated the study of several problems for this class of time delay systems, such as stability (Niculescu, 2001), robustness (Alexandrova, 2018), computation of the $\mathcal{H}_2$ norm (Jarlebring, Vanbiervliet, & Michiels, 2011), estimation of the critical parameters (G. Ochoa, Kharitonov, & Mondié, 2013), and controller design (Palmor, 1980; Yamanaka & Shimemura, 1987), among others.

The delayed state derivative leads to unique and complex dynamic behaviors, which differ considerably from those of retarded systems. To illustrate and clarify some of them, let us consider the scalar neutral type system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) - dx(t-h)\right) &= ax(t) + bx(t-h), \quad t \geqslant 0, \\ x(\theta) &= \varphi(\theta), \quad \theta \in [-h, 0], \end{aligned} \tag{1.1}$$

with real constant coefficients, constant delay $h > 0$ and the initial function $\varphi \in \mathcal{PC}^1\big([-h,0], \mathbb{R}^n\big)$, whose characteristic equation is given by:

$$H(s) = s(1 - de^{-sh}) - a - be^{-sh} = 0, \tag{1.2}$$

equivalently, for $s \neq 0$,

$$H(s) = se^{sh}\left(1 - \frac{a}{s}\right) - s\left(d + \frac{b}{s}\right) = 0 \tag{1.3}$$

Next, we detail some system properties:

- *System solution:* Consider system (1.1) in its integral form

$$x(t) = dx(t-h) + (\varphi(0) - d\varphi(-h)) + \int_0^t ax(s) + bx(s-h)\mathrm{d}s, \quad t \geqslant 0. \tag{1.4}$$

  Notice that if $\theta_1 \in [-h, 0]$ is a discontinuity point[1] of φ, then, at $t_1 = \theta_1 + h$, the solution (1.4) of system (1.1) has a jump whose size is

$$\lim_{t \to t_1 + 0} x(t) - \lim_{t \to t_1 - 0} x(t) = \Delta x(t_1) = d\Delta\varphi(\theta_1),$$

  where $\Delta x(t_1) = x(t_1 + 0) - x(t_1 - 0)$ and $\Delta\varphi(\theta_1) = \varphi(\theta_1 + 0) - \varphi(\theta_1 - 0)$. Since solution (1.4) is defined for $t \geqslant 0$, it suffers a jump discontinuity at all points

[1]At this point, the function may have a sudden jump, break, or gap in its value.

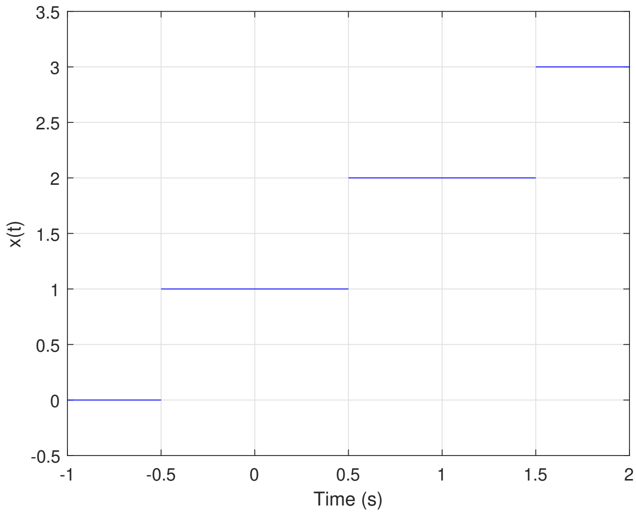


Figure 1.1: Solution of system (1.1) for $h = 1,\ d = 1$ and $a = b = 0$.

$t_k = \theta_1 + kh,\ k \in \mathbb{N}$. In addition, it follows from the previous analysis that:

$$\lim_{t\to t_1+0} \frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) - dx(t-h)\right) - \lim_{t\to t_1-0} \frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) - dx(t-h)\right) = a\Delta x(t_1) + b\Delta\varphi(\theta_1).$$

Thus, the discontinuity of the solutions implies the discontinuity of the left-hand side of system (1.1), concluding that, for piecewise continuously differentiable initial conditions, *the solutions only satisfy the system almost everywhere*. An example showing this property is depicted in Figure 1.1 for the solution of system (1.1) with parameters $h = 1,\ d = 1$ and $a = b = 0$ and initial function

$$\varphi(\theta) = \begin{cases} 0, & \theta \in [-1, -0.5), \\ 1, & \theta \in [-0.5, 0]. \end{cases}$$

This behavior contrasts with retarded type systems, which are well-known to smooth the solution for discontinuous initial functions.

- *Difference equation:* Observe that $\frac{a}{s}$ and $\frac{b}{s}$ approach zero as $|s| \to \infty$ in (1.3). Since $H(s)$ possesses roots with arbitrarily large moduli, we can conclude that the roots of (1.3) tend to be the roots of the difference equation $1 - de^{-sh} = 0$ at infinity, indicating that *the stability of the difference equation* $1 - de^{-sh} = 0$ *needs to be studied*. Notice that the roots of the characteristic equation to the difference equation $1 - de^{-sh} = 0$ are given by:

$$s = \begin{cases} \dfrac{\ln d}{h} + \dfrac{2k\pi i}{h}, & k = 0, \pm 1, \pm 2, \ldots, \text{ if } d > 0, \\ \dfrac{\ln |d|}{h} + \dfrac{(2k+1)\pi i}{h}, & k = 0, \pm 1, \pm 2, \ldots, \text{ if } d < 0. \end{cases}$$

  This means, in particular, that the roots appear in the right-half plane when $|d| > 1$, in the left-hand plane when $|d| < 1$, and on the imaginary axis when $|d| = 1$.

- *Eigenvalues in a vertical strip:* It follows from (1.3) that if $\Re(s) \to \infty$, then the left-hand side of (1.3) approaches 1 and not 0, thus equality (1.3) does not hold. In the contrary case, when $\Re(s) \to -\infty$, then equality (1.3) approaches to $d = 0$. It

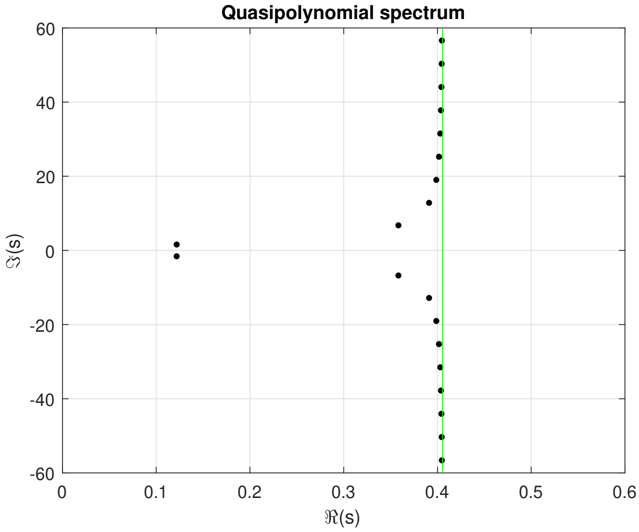


Figure 1.2: Spectrum of system (1.1) for $h = 1,\ d = 1.5$ and $a = b = -2$.

appears that if $d \neq 0$, then $\beta < \Re(s) < \alpha$ for some α and β. The previous analysis and the fact that the roots of (1.3) tend to the roots of the difference equation at infinity reveal that *a chain of eigenvalues may converge to a vertical strip in the complex plane*. For the system parameters $h = 1,\ d = 1.5$ and $a = b = -2$, the system spectrum is presented in Figure 1.2, showing that the characteristic roots (markers $\cdot$) indeed lie in a strip bounded by vertical blue lines. Notice that in the case of retarded type systems, roots at infinity go deep in the left-hand plane.

It is worth mentioning that these and a few other properties are discussed in the monograph of Hale and Lunel (1993) in greater detail.

Stability analysis of delay systems

Roughly speaking, stability in time delay systems refers to the ability of a dynamical system to converge to an equilibrium under small perturbations or changes in the initial state in the presence of delays. Thus, understanding the stability of time delay systems is essential in designing robust controllers, optimizing system performance, and mitigating the risk of failures resulting from potential instability.

In light of the above properties in the previous section and the concept of stability, addressing the case of neutral type systems complicates the stability analysis compared significantly with the retarded type case. Nevertheless, stability analysis can be tackled through the same approaches as in the retarded case: frequency domain and time domain analyses. The first one, frequency domain analysis, is based on finding stability regions by using the characteristic equation of the time-delay system and verifying the system stability with the help of the root tendency. It includes the D-subdivision methods and their variations (Gu, Niculescu, & Chen, 2005; Michiels & Niculescu, 2007; Vyhlídal & Zítek, 2003), and the CTCR (cluster treatment of the characteristic roots) paradigm (Sipahi & Olgac, 2006). For instance, Figure 1.3 depicts the stability/instability boundaries for system (1.1) with $h = 1$ and $d = -0.3$ using the D-subdivision method. This approach is well-suited for linear time-invariant systems as it provides valuable insights into system performance metrics (Fridman, 2014). However, notice that it is

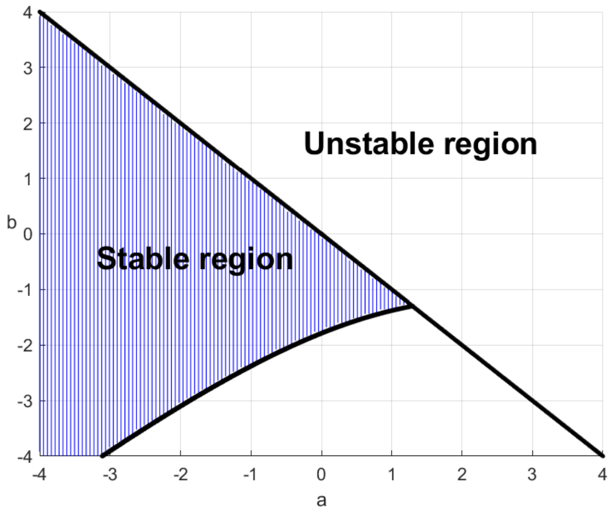


Figure 1.3: D-subdivision in the space of parameters (a, b) of system (1.1) for $h = 1,\ d = -0.3$.

restricted to the study of two, or at most three, parameters, and it becomes complex when the system dimension or the number of parameters increases. Moreover, it is restricted to the analysis of unknown but constant uncertainties.

The second approach for studying the stability of neutral time delay systems relies on the Lyapunov − Krasovskii and the Lyapunov-Razumikhin frameworks (Fridman, 2014), which allow a wide range of applications beyond stability analysis. A significant advantage of Lyapunov-type methods over the frequency domain approach is that it handles linear and non-linear systems without linearization, input-to-state analysis, time-varying uncertainties, multiple delays, estimates of the domain of attraction, estimates of solutions, among others, offering direct insights on behavior and stability (Fridman, 2014; Gu, Kharitonov, & Chen, 2003; Hale & Lunel, 1993). On the one hand, the Lyapunov-Krasovskii framework involves the construction of functionals that incorporate the entire history of the system state over the delay interval (Kharitonov, 2013). This approach provides a comprehensive assessment of the system stability by more explicitly capturing the effects of time delays. On the other hand, the Lyapunov-Razumikhin framework simplifies the stability analysis by using a Lyapunov function for the delay-free system and then applying Razumikhin conditions to account for the delay (Kolmanovskii & Myshkis, 1999). The Razumikhin condition states that if the Lyapunov function evaluated at the current time is greater than a scaled version of its value at the delayed time, and the derivative of the Lyapunov function is negative, then the system is stable. This method is generally more straightforward to implement as it avoids the construction of complex functionals, but since the delay is not directly integrated into the function (Gu et al., 2003), it can be conservative and may not fully capture the effects of delays as precisely as the Lyapunov-Krasovskii approach.

Despite the advantages of the time-domain approach, it may require substantial computational resources (Peet, Papachristodoulou, & Lall, 2009) and the challenging construction of Lyapunov functionals. However, we use the Lyapunov-Krasovskii approach in this thesis work because of its versatility, the power of analysis of systems with complex delay structures, and less conservative results compared to the Lyapunov-Razumikhin approach. It is worth mentioning that N.N. Krasovskii introduced the stability theorems for time delay systems, which are collected in Temple (1965).

The construction of Lyapunov-Krasovskii functionals, which are positive definite and whose time derivatives along the solutions of the time-delay system satisfy a negativity condition, leads to an outstanding number of LMI-based sufficient stability conditions. Initial schemes of these functionals were proposed in Kolmanovskii and Myshkis (1999) and also discussed in Niculescu (2001); Seuret, Gouaisbaut, and Baudouin (2016), conducting to the following two basic ideas:

1. To construct quadratic Lyapunov candidate functionals such as

   - delay-independent Lyapunov-Krasovskii functionals:

   $$v(\varphi) = \varphi^T(0)\Omega_1\varphi(0) + \int_{-h}^{0} \varphi^T(s)\Omega_2\varphi(s)\mathrm{d}s,$$

   and,

   - delay-dependent Lyapunov-Krasovskii functionals:

   $$v(\varphi) = \varphi^T(0)\Omega_1\varphi(0) + \int_{-h}^{0} \varphi^T(s)\Omega_2\varphi(s)\mathrm{d}s + h\int_{-h}^{0}\int_{s_1}^{0} \dot{\varphi}^T(s_2)\Omega_3\dot{\varphi}(s_2)\mathrm{d}s_2\mathrm{d}s_1,$$

   where *the functional argument* φ is the initial condition of the system and *the functional kernels* Ω_1, Ω_2 and Ω_3 are symmetric matrices, achieving delay-independent and delay-dependent sufficient stability conditions, respectively (see Fridman (2014) for different classes of time delay systems).

2. To construct complete Lyapunov-Krasovskii functionals

   $$v(\varphi) = \varphi^T(0)\widehat{\Omega}_1\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} \widehat{\Omega}_2(s)\varphi(s)\mathrm{d}s + \int_{-h}^{0}\int_{-h}^{0} \varphi^T(s_1)\widehat{\Omega}_3(s_1,s_2)\varphi(s_2)\mathrm{d}s_2\mathrm{d}s_1. \tag{1.5}$$

   Here, *the functional kernel* $\widehat{\Omega}_1$ is a constant symmetric matrix, and *functional kernels* $\widehat{\Omega}_2(s)$ and $\widehat{\Omega}_3(s_1,s_2) = \widehat{\Omega}_3^T(s_2,s_1)$ are continuous matrices. This functional was introduced in Repin (1965) and proved to give necessary and sufficient stability conditions in Infante and Castelan (1978) and Huang (1989) for the retarded case and Castelan and Infante (1979) for the neutral case. Unfortunately, the verification of the positive definiteness of the Lyapunov-Krasovskii functionals for all *functional argument* φ is impossible for practical problems (Niculescu, 2001).

For the reduction of conservatism of sufficient conditions, we find in the literature two main strategies. The first one is based on an appropriate conjunction of the above Lyapunov-Krasovskii functionals and functional argument or functional kernels approximation. In particular, the Legendre polynomials addressed in Seuret and Gouaisbaut (2015) for the functional argument φ, and the sum of the square Peet (2018); Peet and Bliman (2011) and a discretized Lyapunov functional method in Gu (1997) for the functional kernels arrived at an asymptotic non-conservative set of matrix inequalities. The second strategy relies on the descriptor model transformation and the descriptor type Lyapunov-Krasovskii functionals, resulting in less conservative conditions, even for uncertain systems (Fridman, 2014).

The so-called complete type functionals with a negative definite prescribed derivative are a major progress in the Lyapunov-Krasovskii framework, which give necessary conditions. For linear neutral time-delay systems, they are constructed in the monograph by Kharitonov (Kharitonov, 2013), thus reducing the stability analysis to verify the functional positive definiteness uniquely. They are characterized by:

- *functional kernels* in terms of the delay Lyapunov matrix, whose concept for neutral type systems first appeared implicitly in Castelan and Infante (1979) and then developed in Rodriguez, Kharitonov, Dion, and Dugard (2004) and Kharitonov (2005); it is determined by three properties: dynamic, symmetry, and algebraic, and it is unique, if and only if the Lyapunov condition is satisfied (Kharitonov, 2010);

- the construction of these functionals is motivated by the converse Lyapunov approach for the delay-free case, where the system $\dot{x} = Ax$ is exponentially stable if and only if there exists a quadratic positive definite Lyapunov function $V(x) = x^T Px$ that satisfies the prescribed time derivative along the solutions $\frac{\mathrm{d}V(x)}{\mathrm{d}t} = -x^T Wx$ for a positive definite matrix W. The latter is equivalent to verifying the following stability criterion: the delay-free system is exponentially stable if and only if the Lyapunov matrix $P = P^T$, solution of the algebraic equation $A^T P + PA = -W$, is positive definite. Here, the Lyapunov matrix P encapsulates the system information as the delay Lyapunov matrix does for time delay systems.

The above antecedents raise the following two questions:

- is it possible to obtain a stability criterion exclusively in terms of the delay Lyapunov matrix analogous to the one for the delay-free case?

- as time delay systems are infinite-dimensional ones, must the stability criterion also be infinite-dimensional?

These two questions have been addressed in recent research where the authors pursue a stability criterion stemming from functionals with prescribed derivatives verified in a finite number of operations. These criteria have been mainly addressed via a functional argument approximation (Alexandrova, 2023; Bajodek, Gouaisbaut, & Seuret, 2023; Egorov, 2016; Gomez, Egorov, & Mondié, 2019) and a functional kernels approximation (Alexandrova & Belov, 2024; Belov & Alexandrova, 2022). Considering a functional argument approximation, a stability criterion expressed in terms of discrete evaluations of the delay Lyapunov matrix was presented for retarded (Egorov, 2016; Gomez, Egorov, & Mondié, 2019) and neutral (Gomez et al., 2021) type systems. The necessity of this criterion follows from substituting a special function depending on the system fundamental matrix in the functional argument. To prove the sufficiency of the result, the positivity of the functional must be verified for all functional arguments φ, which is an intractable task in practice. Here, one has to resort to new instability results and to arguments establishing that arbitrary functional arguments φ can be approximated by the above-mentioned class of initial functions depending on fundamental matrices. The approximation order for which sufficiency is achieved turns out to be very large. To address this issue, for the retarded case, some authors introduced other function approximation classes, such as piecewise linear (Alexandrova, 2023), and Legendre polynomials (Bajodek et al., 2023). These proposals substantially reduced the approximation at the cost of time-consuming integral computations and the loss of the nice matrix form for the criterion.

Considering a functional kernel approximation, for the retarded type case, the discretization method of Gu (1997); Gu et al. (2003) was applied to the functionals with prescribed derivative (Alexandrova & Belov, 2024; Belov & Alexandrova, 2022), combining the ele gant structure of the delay Lyapunov matrix-based stability conditions of Gomez, Egorov, and Mondié (2019) with approximation orders comparable to those in Alexandrova (2023); Bajodek et al. (2023). It is worth mentioning that the discretized Lyapunov functional method was extended to neutral type systems in Han, Yu, and Gu (2004) and Han (2005), arriving at LMI-based stability conditions. For retarded type systems, some earlier combinations of the discretized Lyapunov functional method and the delay Lyapunov matrix approach are available in Mondié and Kharitonov (2004); B. M. Ochoa and Mondié (2006, 2007) and Gu (2013).

1.1 Problem Statement

For neutral type time delay systems, the only Lyapunov matrix-based stability test available in the literature, based on the functional argument approximation via the system fundamental matrix, is the one presented by Gomez et al. (2021). A major drawback of this test is that it leads to large, however finite, approximation orders. As a result, the sufficiency of this stability criterion can only be verified in some cases due to the computational burden.

The obtained improvement concerning approximation orders for sufficiency in the retarded type case allows formulating the following questions:

- is it possible to achieve a stability criterion that can be tested in a moderate number of mathematical operations for neutral type systems similar to the one obtained for the retarded type case by overcoming the complex properties of this class of delay systems?

- knowing that a reduction of the approximation orders was obtained at the cost of a stability test in terms of integrals of the delay Lyapunov matrix for retarded type systems, where a recursive method was implemented, is a recursive method feasible for the computation of integrals of the delay Lyapunov matrix for neutral type systems taking into account that, for neutral type systems, functionals with prescribed derivative depend on derivatives of the delay Lyapunov matrix or the derivative of the functional argument?

- furthermore, in view of the results for retarded systems, is it possible to find for neutral time delay systems a criterion that merges the elegant form in terms of the delay Lyapunov matrix and a test in a tractable number of operations?

Motivated by the substantially reduced approximation orders provided by the stability tests of Bajodek et al. (2023), Alexandrova (2023), and Alexandrova and Belov (2024) for retarded type systems, the aim of this thesis work is to answer these questions by extending the above works to the case of neutral type systems. Overcoming the complexity of neutral type system properties in each approach and developing computationally verifiable stability tests are our most challenging tasks.

1.2 Thesis Objectives

The main purpose of this work is *to provide stability tests, necessary and sufficient stability conditions, for neutral type time-delay systems in a tractable number of operations.*

More precisely, our specific objectives are:

- To approximate Lyapunov functionals with prescribed derivative through either a functional argument approximation or a functional kernel approximation to arrive at a discretized functional expressed explicitly as a quadratic form.

- To determine and estimate the functional approximation error on a special class of functions, delivering an expression for the functional approximation error by calculating appropriately the approximation order.

- To provide necessary and sufficient stability conditions for neutral type time delay systems, verified in a tractable number of mathematical operations.

- To present a recursive method to deal with computational issues involved in the approximation scheme of the tests.

1.3 Methodology

Our work is developed in the Lyapunov-Krasovskii framework using functionals with prescribed derivative expressed in terms of the delay Lyapunov matrix. Fundamental concepts and results in the approximation theory, particularly considering piecewise linear and Legendre polynomial approximation, as well as an explicit bound for the functional approximation error approximating the functional argument on a special set of functions, are also crucial to reach sufficiency. In the case of functional argument approximation, the necessity follows from the fact that the exponential stability of the system implies the non-negativity of the functional for any functional argument. On the contrary, for functional kernels approximation, the necessity of the approximated functional is validated by showing its connection with the presented necessary stability conditions in terms of discrete evaluations of the delay Lyapunov matrix in recent research.

1.4 Structure of the manuscript

This manuscript is organized as follows:

Chapter 2 starts with a review of the theoretical preliminaries on neutral type time delay systems. It is followed by a review of concepts and results on the Lyapunov-Krasovskii functionals and the delay Lyapunov matrix. Thereafter, we revisit the discretized Lyapunov functional method presented in Gu (1997). Based on that method, we introduce a sufficient stability condition for neutral type systems. Finally, a recent stability criterion expressed in terms of the delay Lyapunov matrix is also reminded.

Chapter 3 presents a general outline of proofs based on either functional argument or functional kernels approximation. The stability criterion of Gomez et al. (2021) is verified following this outline.

In Chapter 4, a stability test in a moderate[2] number of operations based on a piecewise linear approximation for the functional argument is presented. The result follows from the general outline of proofs based on approximations developed in Chapter 2 and an instability result on a special set of functions.

Chapter 5 is devoted to approximating the functional argument via Legendre polynomials relying on their convergence rate properties. As a result, a stability test with tractable orders of approximation is given. Following the general outline of proofs of Chapter 2, we exploit the super-geometric convergence property of the Legendre polynomial to provide an estimate for the functional argument error.

In Chapter 6, the discretized Lyapunov functional method introduced in Chapter 2 is extended to Lyapunov functionals with prescribed derivative for neutral type systems. The inner block matrix of the resulting approximated functional is proved to be related to the elegant stability conditions presented in (Gomez et al., 2021). Thus, stability conditions of the same form as those of Gomez et al. (2021) are obtained but with a tractable number of operations.

In Chapter 7, the computation of the integral of the delay Lyapunov matrix and its derivatives are solved by using a recursive method based on the properties of the delay Lyapunov matrix.

In Chapter 8, some illustrative examples are presented and discussed. The first example validates the moderate approximation orders provided by the stability criterion based on piecewise linear approximations, which are compared with the large orders of the sole stability criterion available in the literature Gomez et al. (2021). The stability criterion based on Legendre polynomials approximation is tested and compared in the second example with our main result in Chapter 4 and the result in Gomez et al. (2021), delivering similar or tighter approximations orders than those using piecewise linear approximations. It also corroborates the advantage of using a better basis of approximation. Ultimately, the third and fourth examples validate the obtained result using a discretized Lyapunov functional method, which outperforms those of the previous chapters. Finally, Chapter 9 is devoted to concluding remarks and future research directions.

The specific contributions of this thesis work are outlined below:

1. Theorem 4: a stability criterion based on a piecewise linear approximation for the functional argument in a moderate number of mathematical operations.

2. Theorem 5: a stability criterion based on the Legendre polynomial approximation for the functional argument in a reduced number of mathematical operations.

3. Theorem 6: a stability criterion based on a discretized Lyapunov functional method on the functional kernels in a low number of mathematical operations, while it keeps the elegant form of the criterion in terms of the delay Lyapunov matrix of Gomez et al. (2021).

4. A recursive method for the computation of integrals of the delay Lyapunov matrix required for testing Theorem 4 and 5.

[2]In the sequel, moderate means substantially reduced size with respect to previous results.

1.5 Publications

The main results of our work on the analysis of time-delay systems are currently published and submitted to international conferences and journals:

Journal papers related to this thesis work:

- Portilla, G., Alexandrova, I. V., & Mondié, S. (2024). Stability tests for neutral-type delay systems: A delay Lyapunov matrix and piecewise linear approximation of the functional argument approach. International Journal of Robust and Nonlinear Control, 34(9), 5664-5685.
- Portilla, G., Bajodek, M., & Mondié, S. Necessary and sufficient condition for neutral-type delay systems: Polynomial approximations, Submitted to International Journal of Control.
- Portilla, G., Alexandrova, I. V., & Mondié, S. An Elegant Lyapunov Stability Test for Neutral Type Delay Systems: a Discretized Functional Approach. Submitted to IEEE Transactions on Automatic Control.

Other journal papers:

- Portilla, G., Seuret, A., & Mondié, S. Robust data-driven control for linear discrete-time systems with unknown delay. Submitted to International Journal of Control.
- Portilla, G., Albea, C., & Seuret, A. Predictive control design for switched affine systems subject to an input delay in the switching signal. Submitted to IEEE Control Systems Letters.

International Conference papers:

- Portilla, G., Alexandrova, I. V., & Mondié, S. (2023). Stability test for neutral type delay systems: a piecewise linear approximation scheme. IFAC-PapersOnLine, 56(2), 186-191.
- Portilla, G., Castaño, A., Bajodek, M., & Mondié, S. (2024, June). Stability test for some classes of linear time-delay systems: A Legendre polynomial approximation-based approach. In 2024 European Control Conference (ECC) (pp. 1045-1050). IEEE.

National Conference papers:

- Castaño, A., Santos-Estudillo, O., Portilla, G. & Mondié, S. (2022). Necessary and sufficient stability conditions for time-delay systems: a comparison. Memorias del Congreso Nacional de Control Automático, pp. 50-55.
- Portilla, G., Castaño, A. & Mondié, S. (2023). Necessary and Sufficient Stability Conditions: Traffic Systems. Memorias del Congreso Nacional de Control Automático, pp. 278-283.

Chapter 2

Theoretical preliminaries

In this chapter, we present theoretical preliminaries concerning neutral type time delay systems. We recall the concept of the delay Lyapunov matrix, its properties, and the semi-analytic method for its computation. The Lyapunov functionals with prescribed derivatives, which are given in terms of the delay Lyapunov matrix, are also introduced. Furthermore, we remind some stability results in the Lyapunov-Krasovskii framework for neutral type systems. The discretized Lyapunov functional method of Gu (1997) is introduced, as well as the sufficient stability conditions achieved through this approach. Finally, a stability criterion in terms of the delay Lyapunov matrix for neutral type systems is reminded.

2.1 Basic concepts on neutral type time delay systems

Consider a neutral type time delay system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(x(t) - Dx(t-h)\Big) = A_0 x(t) + A_1 x(t-h), \ \ a.e., \ \ t \geqslant 0, \tag{2.1}$$

with initial function

$$x(\theta) = \varphi(\theta), \quad \varphi \in \mathcal{PC}^1\big([-h,0],\mathbb{R}^n\big),$$

where $h \geqslant 0$, and $A_0,\ A_1,\ D \in \mathbb{R}^{n\times n}$. Considering $\varphi \in \mathcal{PC}^1\big([-h,0],\mathbb{R}^n\big)$, the solution $x(t) = x(t,\varphi)$ is a piecewise continuous function such that $x(\theta) = \varphi(\theta)$, $\theta \in [-h,0]$, which satisfies system (2.1) almost everywhere ($a.e.$) for $t \geqslant 0$. The difference $x(t) - Dx(t-h)$ is assumed continuous and differentiable for $t \geqslant 0$, except for possibly a countable number of points. The restriction of the solution to the interval $[t-h,t],\ t \geqslant 0$, is denoted by

$$x_t : \ \theta \mapsto x(t+\theta), \quad \theta \in [-h,0].$$

Definition 1 *(Hale & Lunel, 1993) The fundamental matrix $Y(t)$ of system (2.1) satisfies the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(Y(t) - DY(t-h)\Big) = A_0 Y(t) + A_1 Y(t-h), \ \ \text{a.e.}, \ \ t \geqslant 0, \tag{2.2}$$

with the initial condition $Y(t) = I_n$ *for* $t = 0$ *and* $Y(t) = \mathbb{O}_n$ *for* $t < 0$.

Remark 1 *It is worth mentioning that*

$$Y(t) = e^{A_0 t}, \quad t \in [0, h),$$

which allows computing the Lipschitz constant L *of the system fundamental matrix* $Y(t)$ *on* $t \in (0, h)$, *i.e., it is such that* $\|Y'(t)\| = \|A_0 e^{A_0 t}\| \leqslant L$, $t \in (0, h)$.

Definition 2 *System* (2.1) *is said to be exponentially stable if there exist* $\gamma \geqslant 1$ *and* $\sigma > 0$ *such that for any initial function* $\varphi \in \mathcal{PC}^1\big([-h, 0], \mathbb{R}^n\big)$,

$$\|x(t, \varphi)\| \leqslant \gamma e^{-\sigma t} \|\varphi\|_h, \quad t \geqslant 0.$$

We also introduce the following assumption on matrix D, which is a necessary condition for the exponential stability of system (2.1).

Assumption 1 *The matrix* D *is Schur stable, i.e., with all eigenvalues inside the open unit circle.*

Lemma 1 *(Kharitonov, Mondie, & Collado, 2005) If the matrix* D *is Schur stable then there exist* $\rho \in (0, 1)$ *and* $d \geqslant 1$ *such that*

$$\|D^k\| \leqslant d\rho^k, \quad k = 0, 1, \dots$$

Assumption 2 *System* (2.1) *satisfies the Lyapunov condition (Kharitonov, 2013), i.e., there exists* $\varepsilon > 0$ *such that* $|s_1 + s_2| > \varepsilon$ *for any* $s_1, s_2 \in \Lambda$, *where*

$$\Lambda = \Big\{ s \in \mathbb{C} \;\Big|\; \det\Big(sI - se^{-sh} D - A_0 - e^{-sh} A_1\Big) = 0 \Big\}.$$

2.2 Delay Lyapunov matrix

Here, we remind the definition of the delay Lyapunov matrix for neutral type time delay systems.

Definition 3 *(Kharitonov, 2013) Given a symmetric matrix* $W \in \mathbb{R}^{n\times n}$, *a continuous matrix function* $U : [-h, h] \to \mathbb{R}^{n\times n}$ *is called delay Lyapunov matrix if it satisfies the following properties:*

1. *dynamic property*

$$U'(s) - U'(s - h)D = U(s)A_0 + U(s - h)A_1, \; s \in (0, h), \tag{2.3}$$

2. *symmetry property*

$$U^T(s) = U(-s), \; s \in [-h, h], \tag{2.4}$$

3. *algebraic property*

$$\Delta U'(0) - D^T \Delta U'(0) D = -W, \tag{2.5}$$

   where $\Delta U'(0) = \lim\limits_{s\to+0} \frac{\mathrm{d}U(s)}{\mathrm{d}s} - \lim\limits_{s\to-0} \frac{\mathrm{d}U(s)}{\mathrm{d}s}$.

Observe that, for negative values of the argument, the dynamic property of the delay Lyapunov matrix also satisfies

$$U'(s) - D^T U'(s+h) = -A_0^T U(s) - A_1^T U(s+h), \; s \in (-h, 0). \tag{2.6}$$

The existence and uniqueness of the delay Lyapunov matrix, proved in Theorem 6.6 and Theorem 6.8 in Kharitonov (2013), are merged in the following lemma:

Lemma 2 *The following statements are equivalent:*

(a) *System* (2.1) *admits a unique Lyapunov matrix associated with any given symmetric matrix W;*

(b) *System* (2.1) *satisfies the Lyapunov condition;*

(c) $\det\left(\mathcal{M} + \mathcal{N} e^{\mathcal{L}h}\right) \neq 0$, *where*

$$\mathcal{M} = \begin{pmatrix} I_{n^2} & \mathbb{O}_{n^2} \\ I_n \otimes A_0 - A_1^T \otimes D & I_n \otimes A_1 - A_0^T \otimes D \end{pmatrix},$$
$$\mathcal{N} = \begin{pmatrix} \mathbb{O}_{n^2} & -I_{n^2} \\ A_1^T \otimes I_n - D^T \otimes A_0 & A_0^T \otimes I_n - D^T \otimes A_1 \end{pmatrix},$$
$$\mathcal{L} = \begin{pmatrix} I_{n^2} & -I_n \otimes D \\ -D^T \otimes I_n & I_{n^2} \end{pmatrix}^{-1} \begin{pmatrix} I_n \otimes A_0 & I_n \otimes A_1 \\ -A_1^T \otimes I_n & -A_0^T \otimes I_n \end{pmatrix}.$$

Notice that Lemma 2 explains the sense of Assumption 2 and shows how to verify it. Now, under Assumption 2, the delay Lyapunov matrix may be computed by solving a delay-free boundary value problem resulting from properties (2.3), (2.4) and (2.5). Its vectorized form (see Appendix A for this property), given in Kharitonov (2013), is:

$$\mathcal{U}(s) = \begin{pmatrix} I_{n^2} & \mathbb{O}_{n^2} \end{pmatrix} e^{\mathcal{L}s} \left(\mathcal{M} + \mathcal{N} e^{\mathcal{L}h}\right)^{-1} \begin{pmatrix} 0_{n^2} \\ -\mathcal{W} \end{pmatrix}, \; s \in [0, h], \tag{2.7}$$

with $\mathcal{U}(s) = \operatorname{vec}(U(s))$, $\mathcal{W} = \operatorname{vec}(W)$. For negative arguments, $\mathcal{U}(s), s \in [-h, 0)$, is obtained via the symmetry property (2.4).

Remark 2 $U \in \mathcal{C}^2_{[0,h]}$ *and* $U \in \mathcal{C}^2_{[-h,0]}$ *considering the corresponding one-sided derivatives at* 0 *and* $\pm h$*. Its derivatives can be computed directly by differentiating expression* (2.7)*.*

2.3 Lyapunov–Krasovskii functional

For a given positive definite matrix W, we introduce the delay Lyapunov matrix-based functional $v_0(\varphi)$, $\varphi \in \mathcal{PC}^1\left([-h, 0], \mathbb{R}^n\right)$, whose time derivative along the solutions of system (2.1) is equal to:

$$\frac{\mathrm{d}v_0(x_t)}{\mathrm{d}t} = -x^T(t) W x(t), \quad t \geqslant 0. \tag{2.8}$$

As shown in Kharitonov (2013), its expression is:

$$v_0(\varphi) = \varphi^T(0) P \varphi(0) + 2\varphi^T(0) \int_{-h}^{0} Q(s)\gamma(s)\mathrm{d}s + \int_{-h}^{0}\int_{-h}^{0} \gamma^T(s_1) R(s_1, s_2) \gamma(s_2) \mathrm{d}s_2 \mathrm{d}s_1, \tag{2.9}$$

where

$$\begin{aligned} P &= U(0) - D^T U(h) - U(-h)D + D^T U(0) D, \\ Q(s) &= U^T(h+s) - D^T U(-s), \\ R(s_1, s_2) &= U(s_1 - s_2), \\ \gamma(s) &= D\varphi'(s) + A_1\varphi(s). \end{aligned}$$

An alternative representation of the functional, which involves the first and second derivatives of U instead of the derivative of φ, is also available in Kharitonov (2013), as follows:

$$v_0(\varphi) = (\varphi(0) - D\varphi(-h))^T U(0)(\varphi(0) - D\varphi(-h)) + \sum_{j=2}^{4} I_j, \tag{2.10}$$

where,

$$I_2 = 2(\varphi(0) - D\varphi(-h))^T \int_{-h}^{0} \Phi(h+\theta)\varphi(\theta)\mathrm{d}\theta, \quad I_3 = \int_{-h}^{0}\int_{-h}^{0} \varphi^T(\theta_1)\Psi(\theta_1 - \theta_2)\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1,$$

$$I_4 = -\int_{-h}^{0}\int_{-h}^{0} \varphi^T(\theta_1) D^T U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1.$$

Here,

$$\Phi(s) = U^T(s)A_1 - U'^T(s)D, \quad \Psi(s) = A_1^T U(s)A_1 - D^T U'(s)A_1 + A_1^T U'(s)D, \quad s \in (0, h),$$

and $\Psi(-s) = \Psi^T(s)$. Due to the fact that $U'(s)$ admits a jump at zero as evidenced by (2.5), the term I_4 should be calculated as:

$$\begin{aligned} I_4 = &-\int_{-h}^{0}\int_{-h}^{\theta_1 - 0} \varphi^T(\theta_1) D^T U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 - \int_{-h}^{0}\int_{\theta_1 + 0}^{0} \varphi^T(\theta_1) D^T U''(\theta_1 - \theta_2) D\varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &- \int_{-h}^{0} \varphi^T(\theta) D^T \Delta U'(0) D\varphi(\theta)\mathrm{d}\theta. \end{aligned}$$

Next, a fifth term was added to functional $v_0(\varphi)$ in Egorov and Mondié (2014) and Gomez, Egorov, and Mondié (2017a):

$$v_1(\varphi) = v_0(\varphi) + \int_{-h}^{0} \varphi^T(\theta) W \varphi(\theta)\mathrm{d}\theta, \tag{2.11}$$

yielding the functional whose time derivative along the solutions of system (2.1) is

$$\frac{\mathrm{d}v_1(x_t)}{\mathrm{d}t} = -x^T(t-h)Wx(t-h), \quad t \geqslant 0.$$

The main advantage of functional $v_1(\varphi)$ with respect to $v_0(\varphi)$ is that it allows obtaining a quadratic lower bound on the set of functions $\mathcal{PC}^1\big([-h, 0], \mathbb{R}^n\big)$. We remind here that, in contrast, functional $v_0(\varphi)$ (2.10) does not admit such bound (Kharitonov, 2013, Example 2.1, p. 58) and satisfies only a local cubic one (Huang, 1989) even for a class of retarded type systems.

Remark 3 *It is worth mentioning that the main results of this manuscript are based on the approximation of either argument or kernels of Lyapunov functional, considering successively each summand of Lyapunov functional. Despite both functionals allow concluding on the system stability, the additional term in functional $v_1(\varphi)$ leads to a stability test based on verifying the positive definiteness of a matrix. On the contrary, functional $v_0(\varphi)$ arrives at stability tests based on verifying the non-negative definiteness of a matrix. However, a matrix with eigenvalues located close to zero may lead to inaccurate results.*

2.4 Stability theorems in the Lyapunov-Krasovskii framework

Next, we remind the fundamental stability result for neutral type time-delay systems in the Lyapunov-Krasovskii framework.

Theorem 1 *(Lyapunov–Krasovskii Theorem)(Hale & Lunel, 1993; Kharitonov, 2013). Suppose that D is Schur stable. If there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$ and a differentiable along the solution of system (2.1) functional $v(\varphi)$, such that*

$$\alpha_1 \|\varphi(0) - D\varphi(-h)\|^2 \leqslant v(\varphi) \leqslant \alpha_2 \|\varphi\|_h^2, \tag{2.12}$$

$$\frac{dv(x_t)}{dt} \leqslant -\alpha_3 \|\varphi(0) - D\varphi(-h)\|^2, \tag{2.13}$$

then system (2.1) is exponentially stable.

As Theorem 1 requires a quadratic lower bound for the candidate functional, which is not satisfied for functional v_0 on the set of an arbitrary functional argument, the special compact set of functions satisfying a Razumikhin-type condition

$$\mathcal{S} = \Big\{ \varphi \in \mathcal{C}^l\big([-h,0], \mathbb{R}^n\big) \,\Big|\, \|\varphi\|_h = \|\varphi(0)\| = 1, \|\varphi^{(k)}\|_h \leqslant K^k, \ \forall k \in \mathbb{N} \Big\}, \tag{2.14}$$

where $K = \dfrac{d}{1-\rho}(\|A_0\| + \|A_1\|)$ with d and ρ defined in Lemma 1, was introduced in Alexandrova and Zhabko (2019), proving that the verification of the positivity of functional v_0 can be checked on this set of functions. In addition, it also allowed presenting the following instability result:

Lemma 3 *(Alexandrova & Zhabko, 2019) If system (2.1) is unstable, then there exists a function $\varphi \in \mathcal{S}$ such that*

$$v_0(\varphi) < -a_0 \stackrel{\text{def}}{=} -\frac{\lambda_{\min}(W)}{4\alpha}.$$

Here, α is such that $\Re(s) \leqslant \alpha$ for any eigenvalue s with $\Re(s) > 0$.

Remark 4 *The meaning of the constant K in $\mathcal{S}$, first appeared in this form for neutral type systems in Gomez, Egorov, and Mondié (2018), is that any $s \in \Lambda$ with $\Re(s) > 0$ satisfies $|s| \leqslant K$ Alexandrova and Zhabko (2019). As outlined in Remark 4 in Alexandrova (2023), it can be replaced by any upper bound for the modulus of an "unstable" eigenvalue.*

The value α in Lemma 3 can be computed either as $\alpha = K$ following Remark 4, which provides an overly conservative bound, or based on the following lemma inspired by the ideas of Gomez, Egorov, and Mondié (2019); Tissir and Hmamed (1996).

Lemma 4 *Any eigenvalue $s \in \Lambda$ with $\Re(s) > 0$ satisfies $\Re(s) < \alpha$, where $\alpha > 0$ is such that there exist matrices $\bar{X} > 0$ and $\bar{Y} > 0$, solution of the LMI:*

$$\Omega_\alpha(\bar{X}, \bar{Y}) = \begin{pmatrix} -\widehat{A}_0^T \bar{X} - \bar{X}\widehat{A}_0 - \bar{Y} & \widehat{A}_0^T \bar{X}\widehat{D} - \bar{X}\widehat{A}_1 \\ \widehat{D}^T \bar{X}\widehat{A}_0 - \widehat{A}_1^T \bar{X} & \widehat{D}^T \bar{X}\widehat{A}_1 + \widehat{A}_1^T \bar{X}\widehat{D} + \bar{Y} \end{pmatrix} > 0 \tag{2.15}$$

with

$$\widehat{A}_0 = A_0 - \alpha I, \quad \widehat{A}_1 = e^{-\alpha h}(A_1 + \alpha D), \quad \widehat{D} = e^{-\alpha h} D.$$

Moreover, the LMI (2.15) is feasible for a sufficiently large α.

Proof : Following the ideas of Gomez, Egorov, and Mondié (2019); Tissir and Hmamed (1996), we apply the change of variable $z = s - \alpha$ for some $\alpha > 0$, which shifts the spectrum to the left, to the characteristic equation of system (2.1). Then, $s \in \Lambda$ if and only if $z \in \widehat{\Lambda}$, where

$$\widehat{\Lambda} = \left\{ z \in \mathbb{C} \;\middle|\; \det\left(zI - ze^{-zh}\widehat{D} - \widehat{A}_0 - e^{-zh}\widehat{A}_1 \right) = 0 \right\}.$$

Let us prove that $\Re(z) < 0$ for any $z \in \widehat{\Lambda}$. To do this, first notice that Schur stability of D implies that of $\widehat{D}$ when $\alpha > 0$. Denote a solution of system (2.1) with matrices $\widehat{A}_0$, $\widehat{A}_1$, $\widehat{D}$ by $\widehat{x}(t)$. Next, introduce the functional

$$v(\varphi) = [\varphi(0) - \widehat{D}\varphi(-h)]^T \bar{X} [\varphi(0) - \widehat{D}\varphi(-h)] + \int_{-h}^{0} \varphi^T(\theta)\bar{Y}\varphi(\theta) \mathrm{d}\theta.$$

Condition (2.15) yields negative-definiteness of its derivative along the solutions $\widehat{x}(t)$:

$$\frac{\mathrm{d}v(\widehat{x}_t)}{\mathrm{d}t} = -\begin{pmatrix} \widehat{x}^T(t) & \widehat{x}^T(t-h) \end{pmatrix} \Omega_\alpha(X, Y) \begin{pmatrix} \widehat{x}(t) \\ \widehat{x}(t-h) \end{pmatrix}.$$

Hence, $\Re(z) < 0$, and consequently, $\Re(s) < \alpha$. Based on the ideas of Kudryakov and Alexandrova (2023), we now prove the feasibility of LMI (2.15) for a sufficiently large α. Choose $\bar{Y} > 0$ and define an increasing sequence $\{\alpha_n\}_{n=1}^{+\infty}$, $\alpha_n \to +\infty$, $\alpha_n > \|A_0\| \geqslant |\lambda_{\max}(A_0)|$. Since for any n the matrix $\widehat{A}_0 = A_0 - \alpha_n I$ is Hurwitz, there exists the corresponding $\bar{X}_n > 0$, solution to the Lyapunov matrix equation

$$(A_0 - \alpha_n I_n)^T \bar{X}_n + \bar{X}_n (A_0 - \alpha_n I_n) = -2\bar{Y}.$$

It is not difficult to see that

$$\begin{aligned} &2\alpha_n \bar{X}_n = A_0^T \bar{X}_n + \bar{X}_n A_0 + 2\bar{Y}, \\ &\alpha_n \|\bar{X}_n\| \leqslant \|A_0\| \|\bar{X}_n\| + \|\bar{Y}\|, \\ &\|\bar{X}_n\| \leqslant \frac{\|\bar{Y}\|}{\alpha_n - \|A_0\|} \xrightarrow[n \to +\infty]{} 0, \\ &\|\widehat{A}_1\| \leqslant e^{-\alpha_n h}\|A_1\| + \alpha_n e^{-\alpha_n h}\|D\| \xrightarrow[n \to +\infty]{} 0, \end{aligned}$$

$$\|\widehat{A}_0^T \bar{X}_n \widehat{D}\| \leqslant (2\|A_0\|\|\bar{X}_n\| + \|\bar{Y}\|)\|\widehat{D}\| \xrightarrow[n\to+\infty]{} 0,$$

since $\|\widehat{D}\| = e^{-\alpha_n h}\|D\| \xrightarrow[n\to+\infty]{} 0$. Hence,

$$\Omega_{\alpha_n}(\bar{X}_n, \bar{Y}) \xrightarrow[n\to+\infty]{} \begin{pmatrix} \bar{Y} & \mathbb{O} \\ \mathbb{O} & \bar{Y} \end{pmatrix} > 0,$$

that is, it is positive definite for a sufficiently large n. □

Remark 5 *In practice, a minimum value of α delivering a solution to LMI* (2.15) *is computed by a binary search procedure on* $(0, K]$. *It is worth noting that α is computable for any, not necessarily unstable, system.*

2.5 Discretized Lyapunov functional method

Next, we remind the discretized functional approach of Han (2005); Han et al. (2004) for neutral type systems to explore the functional kernels approximation approach to obtain a stability criterion. It is worth mentioning that this approach was initially presented to retarded type systems in Gu (1997); Gu et al. (2003). To do so, consider the Lyapunov-Krasovskii functional

$$\begin{aligned} v(\varphi) &= (\varphi(0) - D\varphi(-h))^T \mathcal{R}(\varphi(0) - D\varphi(-h)) + 2(\varphi(0) - D\varphi(-h))^T \int_{-h}^{0} \mathcal{X}(s)\varphi(s)\mathrm{d}s \\ &+ \int_{-h}^{0}\int_{-h}^{0} \varphi^T(s_1)\mathcal{Y}(s_1, s_2)\varphi(s_2)\mathrm{d}s_2\mathrm{d}s_1 + \int_{-h}^{0} \varphi^T(s)\mathcal{Z}(s)\varphi(s)\mathrm{d}s, \end{aligned} \tag{2.16}$$

where

$$\mathcal{R} = \mathcal{R}^T \in \mathbb{R}^{n\times n},$$

and, for all $s, s_1, s_2 \in [-h, 0]$,

$$\mathcal{X}(s) \in \mathbb{R}^{n\times n},$$
$$\mathcal{Y}(s_1, s_2) = \mathcal{Y}^T(s_2, s_1) \in \mathbb{R}^{n\times n},$$
$$\mathcal{Z}(s) = \mathcal{Z}^T(s) \in \mathbb{R}^{n\times n},$$

and $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are continuous *functional kernels*. Its time derivative along the solutions of system (2.1) is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}v(x_t) &= 2(x(t) - Dx(t-h))^T \mathcal{R}(A_0 x(t) + A_1 x(t-h)) \\ &+ 2(A_0 x(t) + A_1 x(t-h))^T \int_{t-h}^{t} \mathcal{X}(s-t)x(s)\mathrm{d}s + 2(x(t) - Dx(t-h))^T \mathcal{X}(0)x(t) \end{aligned}$$

$$- 2(x(t) - Dx(t-h))^T \mathcal{X}(-h)x(t-h) - 2(x(t) - Dx(t-h))^T \int_{t-h}^{t} \left.\frac{\mathrm{d}\mathcal{X}(u)}{\mathrm{d}u}\right|_{u=s-t} x(s)\mathrm{d}s$$

$$+ 2x^T(t) \int_{t-h}^{t} \mathcal{Y}(0, s-t)x(s)\mathrm{d}s - 2x^T(t-h) \int_{t-h}^{t} \mathcal{Y}(-h, s-t)x(s)\mathrm{d}s$$

$$- \int_{t-h}^{t}\int_{t-h}^{t} x^T(s_1)\left[\left.\frac{\partial\mathcal{Y}(u_1,u_2)}{\partial u_1}\right|_{\substack{u_1=s_1-t\\u_2=s_2-t}} + \left.\frac{\partial\mathcal{Y}(u_1,u_2)}{\partial u_2}\right|_{\substack{u_1=s_1-t\\u_2=s_2-t}}\right] x(s_2)\mathrm{d}s_2\mathrm{d}s_1$$

$$+ x^T(t)\mathcal{Z}(0)x(t) - x^T(t-h)\mathcal{Z}(-h)x(t-h) - \int_{t-h}^{t} x^T(s)\left.\frac{\mathrm{d}\mathcal{Z}(u)}{\mathrm{d}u}\right|_{u=s-t} x(s)\mathrm{d}s.$$

Now, let us discretize the interval $[-h, 0]$ by equidistant points $\theta_i = -i\tau$, $i = \overline{0, N}$ and $\tau = h/N$, and choose $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ to be continuous piecewise linear functions. The expressions for the piecewise linear functions are given by

$$\begin{aligned}
\mathcal{X}^N(s+\theta_i) &= \mathcal{X}_i + \left(\mathcal{X}_{i-1} - \mathcal{X}_i\right)\frac{s}{\tau}, \\
\mathcal{Z}^N(s+\theta_i) &= \mathcal{Z}_i + \left(\mathcal{Z}_{i-1} - \mathcal{Z}_i\right)\frac{s}{\tau}, \quad s \in [0, \tau], \\
\mathcal{Y}^N(s_1+\theta_i, s_2+\theta_j) &= \left(1 - \frac{s_2}{\tau}\right)\mathcal{Y}_{ij} + \frac{s_1}{\tau}\mathcal{Y}_{i-1,j-1} + \frac{s_2 - s_1}{\tau}\mathcal{Y}_{i,j-1}, \quad 0 \leqslant s_1 \leqslant s_2 \leqslant \tau, \\
\mathcal{Y}^N(s_1+\theta_i, s_2+\theta_j) &= \left(1 - \frac{s_1}{\tau}\right)\mathcal{Y}_{ij} + \frac{s_2}{\tau}\mathcal{Y}_{i-1,j-1} + \frac{s_1 - s_2}{\tau}\mathcal{Y}_{i-1,j}, \quad 0 \leqslant s_2 \leqslant s_1 \leqslant \tau,
\end{aligned} \tag{2.17}$$

$i, j = \overline{1, N}$, where

$$\mathcal{X}_i = \mathcal{X}(\theta_i), \quad \mathcal{Z}_i = \mathcal{Z}(\theta_i), \quad \mathcal{Y}_{ij} = \mathcal{Y}(\theta_i, \theta_j).$$

Dividing the integration in (2.16) into segments $[\theta_i, \theta_{i-1}]$ and substituting their kernels $\mathcal{X}(s)$ and $\mathcal{Y}(s_1, s_2)$ by (2.17), functional (2.16) can be rewritten as follows

$$\begin{aligned}
v^{(N)}(\varphi) = {} & (\varphi(0) - D\varphi(-h))^T\mathcal{R}(\varphi(0) - D\varphi(-h)) \\
& + 2(\varphi(0) - D\varphi(-h))^T \sum_{i=1}^{N} \int_0^{\tau} \mathcal{X}^N(s+\theta_i)\varphi(s+\theta_i)\mathrm{d}s \\
& + \sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^{\tau}\int_0^{\tau} \varphi^T(s_1+\theta_i)\mathcal{Y}^N(s_1+\theta_i, s_2+\theta_j)\varphi(s_2+\theta_j)\mathrm{d}s_2\mathrm{d}s_1 \\
& + \sum_{i=1}^{N} \int_0^{\tau} \varphi^T(s+\theta_i)\mathcal{Z}^N(s+\theta_i)\varphi(s+\theta_i)\mathrm{d}s.
\end{aligned} \tag{2.18}$$

Applying the integration by parts to the second summand in (2.18), we obtain

$$\int_0^{\tau} \mathcal{X}^N(s+\theta_i)\varphi(s+\theta_i)\mathrm{d}s = \mathcal{X}_{i-1}\int_0^{\tau} \varphi(\xi+\theta_i)\mathrm{d}\xi - \int_0^{\tau}\int_0^{s} \frac{\mathrm{d}\mathcal{X}^N(s+\theta_i)}{\mathrm{d}s}\varphi(\xi+\theta_i)\mathrm{d}\xi\mathrm{d}s$$

$$= \mathcal{X}_{i-1}\int_0^\tau \varphi(\xi+\theta_i)\mathrm{d}\xi - \int_0^\tau\int_0^s \frac{1}{\tau}\left(\mathcal{X}_{i-1}-\mathcal{X}_i\right)\varphi(\xi+\theta_i)\mathrm{d}\xi\mathrm{d}s$$

$$= \mathcal{X}_{i-1}\int_0^\tau \frac{1}{\tau}\left(\int_0^s \varphi(\xi+\theta_i)\mathrm{d}\xi + \int_s^\tau \varphi(\xi+\theta_i)\mathrm{d}\xi\right)\mathrm{d}s - \int_0^\tau\int_0^s \frac{1}{\tau}(\mathcal{X}_{i-1}-\mathcal{X}_i)\varphi(\xi+\theta_i)\mathrm{d}\xi\mathrm{d}s$$

$$= \frac{1}{\tau}\int_0^\tau \left[\mathcal{X}_{i-1}\left(\int_s^\tau \varphi(\xi+\theta_i)\mathrm{d}\xi\right) + \mathcal{X}_i\left(\int_0^s \varphi(\xi+\theta_i)\mathrm{d}\xi\right)\right]\mathrm{d}s.$$

Similarly, we apply the integration by part to the third summand in (2.18) with respect to s_2, yielding

$$\int_0^\tau\int_0^\tau \varphi^T(s_1+\theta_i)\mathcal{Y}^N(s_1+\theta_i, s_2+\theta_j)\varphi(s_2+\theta_j)\mathrm{d}s_2\mathrm{d}s_1$$

$$= \int_0^\tau \varphi^T(s_1+\theta_i)\left[\mathcal{Y}^N(s_1+\theta_i,\theta_{j-1})\int_0^\tau \varphi(\xi+\theta_j)\mathrm{d}\xi\right.$$

$$\left. - \int_0^\tau\int_0^{s_2} \frac{\partial \mathcal{Y}^N(s_1+\theta_i, s_2+\theta_j)}{\partial s_2}\varphi(\xi+\theta_j)\mathrm{d}\xi\mathrm{d}s_2\right]\mathrm{d}s_1$$

$$= \int_0^\tau \varphi^T(s_1+\theta_i)\left[\left(\frac{s_1}{\tau}\mathcal{Y}_{i-1,j-1} + \left(1-\frac{s_1}{\tau}\right)\mathcal{Y}_{i,j-1}\right)\int_0^\tau \varphi(\xi+\theta_j)\mathrm{d}\xi\right.$$

$$\left. - \int_0^{s_1}\int_0^{s_2} \frac{1}{\tau}(\mathcal{Y}_{i-1,j-1}-\mathcal{Y}_{i-1,j})\varphi(\xi+\theta_j)\mathrm{d}\xi\mathrm{d}s_2 - \int_{s_1}^\tau\int_0^{s_2} \frac{1}{\tau}(\mathcal{Y}_{i,j-1}-\mathcal{Y}_{i,j})\varphi(\xi+\theta_j)\mathrm{d}\xi\mathrm{d}s_2\right]\mathrm{d}s_1.$$

Further, the integration by parts with respect to s_1 of the previous expression allows arriving at

$$\int_0^\tau\int_0^\tau \varphi^T(s_1+\theta_i)\mathcal{Y}^N(s_1+\theta_i, s_2+\theta_j)\varphi(s_2+\theta_j)\mathrm{d}s_2\mathrm{d}s_1$$

$$= \int_0^\tau \varphi^T(\xi+\theta_i)\mathrm{d}\xi\left[\mathcal{Y}_{i-1,j-1}\int_0^\tau \varphi(\xi+\theta_j)\mathrm{d}\xi - \frac{1}{\tau}\int_0^\tau\int_0^s (\mathcal{Y}_{i-1,j-1}-\mathcal{Y}_{i-1,j})\varphi(\xi+\theta_j)\mathrm{d}\xi\mathrm{d}s\right]$$

$$- \int_0^\tau\int_0^s \varphi^T(\xi+\theta_i)\mathrm{d}\xi\left[\frac{1}{\tau}(\mathcal{Y}_{i-1,j-1}-\mathcal{Y}_{i,j-1})\int_0^\tau \varphi(\xi+\theta_j)\mathrm{d}\xi - \frac{1}{\tau}(\mathcal{Y}_{i-1,j-1}-\mathcal{Y}_{i-1,j})\int_0^s \varphi(\xi+\theta_j)\mathrm{d}\xi\right.$$

$$\left. + \frac{1}{\tau}(\mathcal{Y}_{i,j-1}-\mathcal{Y}_{i,j})\int_0^s \varphi(\xi+\theta_j)\mathrm{d}\xi\right]\mathrm{d}s$$

Taking into account that, for $\theta \in [-h, 0]$,

$$\int_0^\tau \varphi(\gamma+\theta)\mathrm{d}\gamma = \frac{1}{\tau}\int_0^\tau\int_0^\tau \varphi(\gamma+\theta)\mathrm{d}\gamma\mathrm{d}s = \frac{1}{\tau}\int_0^\tau\left(\int_0^s \varphi(\gamma+\theta)\mathrm{d}\gamma + \int_s^\tau \varphi(\gamma+\theta)\mathrm{d}\gamma\right)\mathrm{d}s,$$

equivalently,

$$\frac{1}{\tau}\int_0^\tau\int_s^\tau \varphi(\gamma+\theta)\mathrm{d}\gamma\mathrm{d}s = \int_0^\tau \varphi(\gamma+\theta)\mathrm{d}\gamma - \frac{1}{\tau}\int_0^\tau\int_0^s \varphi(\gamma+\theta)\mathrm{d}\gamma\mathrm{d}s,$$

the previous integral reduces to

$$\begin{aligned}
&\int_0^\tau\int_0^\tau \varphi^T(s_1+\theta_i)\mathcal{Y}^N(s_1+\theta_i,s_2+\theta_j)\varphi(s_2+\theta_j)\mathrm{d}s_2\mathrm{d}s_1\\
&=\int_0^\tau \varphi^T(\xi+\theta_i)\mathrm{d}\xi\left[\mathcal{Y}_{i-1,j-1}\int_0^\tau \varphi(\xi+\theta_j)\mathrm{d}\xi - \int_0^\tau\int_0^s \frac{1}{\tau}(\mathcal{Y}_{i-1,j-1}-\mathcal{Y}_{i-1,j})\varphi(\xi+\theta_j)\mathrm{d}\xi\mathrm{d}s\right]\\
&-\int_0^\tau\int_0^s \varphi^T(\xi+\theta_i)\mathrm{d}\xi\left[\frac{1}{\tau}(\mathcal{Y}_{i-1,j-1}-\mathcal{Y}_{i,j-1})\int_0^\tau \varphi(\xi+\theta_j)\mathrm{d}\xi - \frac{1}{\tau}(\mathcal{Y}_{i-1,j-1}-\mathcal{Y}_{i-1,j})\int_0^s \varphi(\xi+\theta_j)\mathrm{d}\xi\right.\\
&\left.+\frac{1}{\tau}(\mathcal{Y}_{i,j-1}-\mathcal{Y}_{i,j})\int_0^s \varphi(\xi+\theta_j)\mathrm{d}\xi\right]\mathrm{d}s\\
&=\frac{1}{\tau}\int_0^\tau\Bigg[\left(\int_s^\tau \varphi^T(\xi+\theta_i)\mathrm{d}\xi\right)\mathcal{Y}_{i-1,j-1}\left(\int_s^\tau \varphi(\xi+\theta_j)\mathrm{d}\xi\right)\\
&+\left(\int_s^\tau \varphi^T(\xi+\theta_i)\mathrm{d}\xi\right)\mathcal{Y}_{i-1,j}\left(\int_0^s \varphi(\xi+\theta_j)\mathrm{d}\xi\right)+\left(\int_0^s \varphi^T(\xi+\theta_i)\mathrm{d}\xi\right)\mathcal{Y}_{i,j-1}\left(\int_s^\tau \varphi(\xi+\theta_j)\mathrm{d}\xi\right)\\
&+\left(\int_0^s \varphi^T(\xi+\theta_i)\mathrm{d}\xi\right)\mathcal{Y}_{i,j}\left(\int_0^s \varphi(\xi+\theta_j)\mathrm{d}\xi\right)\Bigg]\mathrm{d}s.
\end{aligned}$$

Using the Jensen inequality, it was proven in Gu et al. (2003) that the last summand in functional (2.18) satisfies the following inequality

$$\begin{aligned}
&\sum_{i=1}^N\int_0^\tau \varphi^T(s+\theta_i)\mathcal{Z}^N(s+\theta_i)\varphi(s+\theta_i)\mathrm{d}s \geqslant\\
&\int_0^\tau\Bigg[\left(\int_s^\tau \varphi^T(\xi+\theta_1)\mathrm{d}\xi\right)\mathcal{Z}_0\left(\int_s^\tau \varphi(\xi+\theta_1)\mathrm{d}\xi\right)+\left(\int_0^s \varphi^T(\xi+\theta_N)\mathrm{d}\xi\right)\mathcal{Z}_N\left(\int_0^s \varphi(\xi+\theta_N)\mathrm{d}\xi\right)\\
&+\sum_{i=1}^{N-1}\left(\int_s^\tau \varphi^T(\xi+\theta_{i+1})\mathrm{d}\xi+\int_0^s \varphi^T(\xi+\theta_i)\mathrm{d}\xi\right)\mathcal{Z}_i\left(\int_s^\tau \varphi(\xi+\theta_{i+1})\mathrm{d}\xi+\int_0^s \varphi(\xi+\theta_i)\mathrm{d}\xi\right).
\end{aligned}$$

Combining the integration by parts of all summands, it is possible to get the following lower bound for the discretized functional:

$$v^{(N)}(\varphi) \geqslant \frac{1}{\tau}\int_0^{\tau} \left((\varphi(0)-D\varphi(-h))^T \quad \Psi^T(s)\right) \begin{pmatrix} \mathcal{R} & \widetilde{\mathcal{X}}_N \\ \widetilde{\mathcal{X}}_N^T & \widetilde{\mathcal{Y}}_N + \tau\widetilde{\mathcal{Z}}_N \end{pmatrix} \begin{pmatrix} \varphi(0)-D\varphi(-h) \\ \Psi(s) \end{pmatrix} \mathrm{d}s, \tag{2.19}$$

where

$$\widetilde{\mathcal{X}}_N = \Big(\mathcal{X}_0, \mathcal{X}_1, \cdots, \mathcal{X}_N\Big), \quad \widetilde{\mathcal{Y}}_N = \Big\{\mathcal{Y}_{ij}\Big\}_{i,j=0}^{N}, \quad \widetilde{\mathcal{Z}}_N = \mathrm{diag}\Big(\mathcal{Z}_0, \mathcal{Z}_1, \cdots, \mathcal{Z}_N\Big),$$

$$\Psi(s) = \Big(\psi_0^T(s), \psi_1^T(s), \ldots, \psi_N^T(s)\Big)^T,$$

$$\psi_0(s) = \int_{\theta_1+s}^{0} \varphi(\xi)\mathrm{d}\xi, \quad \psi_N(s) = \int_{\theta_N}^{\theta_N+s} \varphi(\xi)\mathrm{d}\xi,$$

$$\psi_i(s) = \int_{\theta_{i+1}+s}^{\theta_i+s} \varphi(\xi)\mathrm{d}\xi, \quad i = \overline{1, N-1}.$$

The discretized time derivative of functional (2.16), which is addressed in Han (2005) for neutral type systems, admits an upper bound as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}v^{(N)}(x_t) \leqslant -\zeta^T(x_t)\Theta_N(\mathcal{R}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})\zeta(x_t), \quad t \geqslant 0,$$

where $\zeta(x_t)$ is a vector in terms of integrals of the system state and Θ_N is a constant matrix whose explicit expressions are omitted for their irrelevancy in the following developments.

Based on the previous discretized Lyapunov functional method, where a piecewise linear approximation for the functional kernels is considered, LMI-based sufficient stability conditions were derived in Han (2005). This result is recalled in the following theorem.

Theorem 2 *(Han, 2005) If there exists matrices $\mathcal{R} = \mathcal{R}^T$, $\mathcal{X}_i$, $\mathcal{Z}_i = \mathcal{Z}_i^T$ and $\mathcal{Y}_{ij} = \mathcal{Y}_{ji}^T$, $i, j = \overline{0, N}$, such that*

$$\begin{pmatrix} \mathcal{R} & \widetilde{\mathcal{X}}_N \\ \widetilde{\mathcal{X}}_N^T & \widetilde{\mathcal{Y}}_N + \tau\widetilde{\mathcal{Z}}_N \end{pmatrix} > 0, \quad \Theta_N(\mathcal{R}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}) > 0, \tag{2.20}$$

then system (2.1) *is exponentially stable.*

2.6 Stability test in terms of the delay Lyapunov matrix

Here, a stability test in terms of discretized evaluations of the delay Lyapunov matrix is reminded. This result is based on functional $v_1(\varphi)$ and a functional argument approximation considering the system fundamental matrix. The sufficiency conditions are achieved thanks to an instability result regarding the special set of functions $\mathcal{S}$. It is worth highlighting that this is the sole Lyapunov matrix-based stability criterion available in the literature to check the stability of neutral type systems. Some details of this result are presented in more detail in the next chapter.

The necessary conditions of the above-mentioned stability test were first presented in Gomez, Egorov, and Mondié (2017b) for neutral type systems. Next, we remind this result, whose elegant structure is given in terms of the delay Lyapunov matrix.

Lemma 5 *(Gomez et al., 2017b) If system* (2.1) *is exponentially stable, then matrices* $\mathcal{K}_0 = U(0) > 0$, *and*

$$\mathcal{K}_N = \Big\{U((j-i)\tau)\Big\}_{i,j=0}^{N} > 0 \quad \forall\, N \in \mathbb{N},$$

where $\tau = h/N$.

The following instability result played a significant role in obtaining the sufficient part of the stability test:

Lemma 6 *(Gomez et al., 2021) If system* (2.1) *is unstable, then there exists a function* $\varphi \in \mathcal{S}$ *such that*

$$v_1(\varphi) \leqslant -a_1 \overset{\text{def}}{=} -\frac{\lambda_{\min}(W)e^{-2\alpha h}}{4\alpha}\cos^2(b),$$

where $b \in \left(0, \frac{\pi}{2}\right)$ *is a solution of the equation*

$$\sin^4(b)((\alpha h)^2 + b^2) = (\alpha h)^2.$$

Here, α *is such that* $\Re(s) \leqslant \alpha$ *for any eigenvalue* s *with* $\Re(s) > 0$.

Now, let us introduce the following constants:

$$\beta = (1 + \|D\|)^2\|U(0)\| + 2h(1 + \|D\|)\widehat{F}_1 + h^2\widehat{F}_2 + h\|\Delta U'(0)\|,$$

$$\widehat{F}_1 = \sup_{\theta\in(0,h)} \|\Phi(\theta)\|, \quad \widehat{F}_2 = \sup_{\theta\in(0,h)} \|\Psi(s) - D^T U''(s)D\|,$$

where Φ and Ψ are given in (2.10).

Next, we remind a Lyapunov matrix-based stability test for neutral type systems via functional $v_1(\varphi)$ and the system fundamental matrix $Y(t)$.

Theorem 3 *(Gomez et al., 2021) Assume that matrix* D *is Schur stable. System* (2.1) *is exponentially stable if and only if the Lyapunov condition holds and*

$$\mathcal{K}_{\hat{N}} = \Big\{U((j-i)\tau)\Big\}_{i,j=0}^{\hat{N}} > 0,$$

where

$$\hat{N} = \left\lceil e^{Lh}h(K+L)\left(\beta^\star + \sqrt{\beta^\star(\beta^\star+1)}\right) - Lh\right\rceil$$

with $\tau = h/\hat{N}$, $\beta^\star = \frac{\beta}{a_1}$. *Here,* a_1 *and* L *are determined by Lemma 6 and Remark 1, respectively.*

2.7 Conclusions

This chapter provides some basic concepts on neutral type systems and the explicit expression of functional $v_0(\varphi)$ expressed in terms of the delay Lyapunov matrix $U(s)$.

Furthermore, a discretized Lyapunov functional method, essential to achieve the main result of Chapter 6, is reminded. This chapter also mentions some stability/instability conditions for neutral type systems.

Chapter 3

General outline of proofs based on approximations

In recent research, considering the Lyapunov functionals with prescribed derivative introduced in Section 2.3, the functional argument approximations (Alexandrova, 2023; Bajodek et al., 2023; Gomez, Egorov, & Mondié, 2019) (retarded type systems), (Gomez et al., 2021) (neutral type systems), and functional kernel approximations (Alexandrova & Belov, 2024; Belov & Alexandrova, 2022; Gu, 1997)(retarded type systems), (Han, 2005; Han et al., 2004)(neutral type systems), have been essential in achieving necessary and sufficient stability conditions in a moderate number of operations for time delay systems. A quadratic form, characterized by an inner block matrix, is derived in both cases. It is worth mentioning that the above contributions, based on functional argument approximations or functional kernel approximations, are obtained under common steps such as obtaining an approximated function, estimating the functional approximation error, and computing an approximation order for sufficiency. Next, those common steps are formulated as a general outline of proofs based on approximations, which we shall use in the main results in the next chapters.

To do so, let us express the Lyapunov functional $v_0(\varphi)$ as the sum of two terms, namely

$$v_0(\varphi) = v_0^{\text{approx}}(\varphi) + \Upsilon_N, \quad \varphi \in \mathcal{S}. \tag{3.1}$$

The first one corresponds to the functional $v_0(\varphi)$ evaluated at the approximation of the functional argument or the functional kernels, resulting in an approximated functional. The second one is related to the functional approximation error.

3.1 General outline of the proofs based on functional approximations

Here, the general outline of the proofs is summed up in five steps. In the sequel, a functional argument approximation is considered for each step. As these steps can be applied to the case of functional kernel approximations, some comments are separately given in Section 3.4, indicating slight variations.

Step 1. The functional argument φ is presented in the form

$$\varphi(\theta) = \varphi_N(\theta) + \tilde{\varphi}_N(\theta), \quad \theta \in [-h, 0], \tag{3.2}$$

where $\varphi_N(\theta)$ denote the approximation of φ, given by any class of approximation (piecewise linear approximations, polynomial approximations, among others), and $\tilde{\varphi}_N(\theta)$ stands for the corresponding approximation error.

Step 2. Here, φ_N is substituted into the functional $v_0(\varphi_N)$. It leads to an quadratic expression

$$v_0^{\text{approx}}(\varphi) = v_0(\varphi_N) = \xi_N^T \mathbf{P}_N \xi_N, \tag{3.3}$$

where $\mathbf{P}_N$ is a symmetric inner block matrix and ξ_N is a vector whose entries are determined by the type of approximation. The entries of matrix $\mathbf{P}_N$ might depend on the delay Lyapunov matrix, its derivatives, and its integrals. As a rule $v_0^{\text{approx}}(\varphi) = v_0(\varphi_N)$.

Step 3. The error $\tilde{\varphi}_N(\theta)$ defined in (3.2) is bounded on the set $\mathcal{S}$ in such a way that

$$\|\tilde{\varphi}_N\|_h = \|\varphi - \varphi_N\|_h \leqslant \varepsilon_N, \quad \forall \varphi \in \mathcal{S},$$

and that $\varepsilon_N \to 0$ as $N \to \infty$. Here, the set $\mathcal{S}$ is defined by (2.14).

Step 4. The approximation error on the functional argument φ is conveyed to functional $v_0(\varphi)$, leading to the following expression for the functional approximation error

$$\Upsilon_N = v_0(\varphi) - v_0^{\text{approx}}(\varphi), \quad \varphi \in \mathcal{S}. \tag{3.4}$$

The value Υ_N quantifies the error between $v_0^{\text{approx}}(\varphi)$ and $v_0(\varphi)$ on the set of functions $\varphi \in \mathcal{S}$. Thus, this step aims to obtain an upper bound of the functional approximation error in terms of the approximation order N with the help of *Step 3* as follows:

$$|\Upsilon_N| = |v_0(\varphi) - v_0^{\text{approx}}(\varphi)| \leqslant \delta_N, \quad \forall \varphi \in \mathcal{S}. \tag{3.5}$$

Here, δ_N is determined by the class of approximation under consideration and its approximation error of φ, in particular, ε_N defined by *Step 3*.

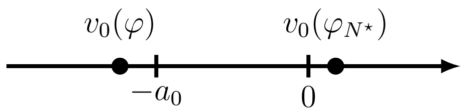
Step 5. The stability criterion for neutral type systems is then derived by calculating the approximation order $N^\star$ such that

$$|\Upsilon_{N^\star}| \leqslant \delta_{N^\star} \leqslant a_0, \quad \varphi \in \mathcal{S},$$

with a_0 determined in Lemma 3, and verifying the non-negative definiteness of $\mathbf{P}_{N^\star}$.

On the one hand, for a better understanding of this step, notice that if system (2.1) is unstable, then it follows from Lemma 3 that there exists a function $\varphi \in \mathcal{S}$ such that $v_0(\varphi) < -a_0$. On the other hand, non-negative definiteness of $\mathbf{P}_{N^\star}$ implies $v_0(\varphi_{N^\star}) = \xi_{N^\star}^T \mathbf{P}_{N^\star} \xi_{N^\star} \geqslant 0$ for any $\varphi \in \mathcal{S}$. Clearly, if $N^\star$ is chosen in such a way that $\delta_{N^\star} \leqslant a_0$, equation (3.5) immediately leads us to a contradiction, as it is illustrated by Figure 3.1.

Figure 3.1: Tolerance to conclude the system stability



Remark 6 *It is worth mentioning that the steps of the general outline of the proofs can be used with functional $v_1(\varphi)$ instead of $v_0(\varphi)$. Unlike functional $v_0(\varphi)$, we need to approximate an additional term in Step 3. Also, the expression of the bilinear functional is extended to functional $v_1(\varphi)$; hence, the bounding of the additional term can be carried out as in Step 4.*

3.2 Functional approximation error estimate

Here, we proceed to compute and estimate the functional approximation error

$$\Upsilon_N = v_0(\varphi) - v_0^{\text{approx}}(\varphi),$$

on the set of functions $\varphi \in \mathcal{S}$ in order to transfer the convergence properties of the class of approximation to the error Υ_N. To do so, considering that if the Lyapunov condition holds, then $U(s)$ is bounded, define the values

$$M_1 = \sup_{\theta\in(0,h)} \|\Phi(\theta)\|, \quad M_2 = \sup_{\theta\in(0,h)} \|\widetilde{\Psi}(s)\|, \quad M_3 = |\lambda_{\max}(D^T \Delta U'(0)D)|, \tag{3.6}$$

with $\widetilde{\Psi}(s) = \Psi(s) - D^T U''(s)D$ and $\Phi(\theta)$ and $\Psi(s)$ defined in (2.10).

Here, we use the way of bounding through the bilinear functional introduced in Gomez et al. (2021) to determine an explicit expression for the functional approximation error Υ_N:

$$\begin{aligned} z(\varphi_1,\varphi_2) &= (\varphi_1(0) - D\varphi_1(-h))^T U(0)(\varphi_2(0) - D\varphi_2(-h)) \\ &+ (\varphi_1(0) - D\varphi_1(-h))^T \int_{-h}^{0} \Phi(h+s)\varphi_2(s)\mathrm{d}s + (\varphi_2(0) - D\varphi_2(-h))^T \int_{-h}^{0} \Phi(h+s)\varphi_1(s)\mathrm{d}s \\ &+ \int_{-h}^{0}\int_{-h}^{0} \varphi_1^T(s_1)\widetilde{\Psi}(s_1-s_2)\varphi_2(s_2)\mathrm{d}s_2\mathrm{d}s_1, \quad \varphi_1,\varphi_2 \in \mathcal{PC}^1\big([-h,0],\mathbb{R}^n\big). \end{aligned}$$

Notice that $v_0(\varphi) = z(\varphi,\varphi)$ and $z(\varphi_1,\varphi_2) = z(\varphi_2,\varphi_1)$ for any φ_1,φ_2. Then, considering a function $\varphi \in \mathcal{S}$ and $\varphi_N = \varphi - \tilde{\varphi}_N$, we get

$$v_0(\varphi_N) = z(\varphi - \tilde{\varphi}_N, \varphi - \tilde{\varphi}_N) = v_0(\varphi) - 2z(\varphi,\tilde{\varphi}_N) + v_0(\tilde{\varphi}_N).$$

Observe that the term Υ_N can be presented as

$$\Upsilon_N = v_0(\varphi) - v_0(\varphi_N) = 2z(\varphi,\tilde{\varphi}_N) - v_0(\tilde{\varphi}_N) = z(\varphi,\tilde{\varphi}_N) + z(\varphi_N,\tilde{\varphi}_N). \tag{3.7}$$

Based on the previous expression, we present the following lemma, which provides a suitable estimate for the functional approximation error Υ_N on the special set of functions $\mathcal{S}$.

Lemma 7 *Let $\varphi \in \mathcal{S}$. If its approximation φ_N is such that $\varphi_N(0) = \varphi(0)$, $\varphi_N(-h) = \varphi(-h)$ then*

1. $|\Upsilon_N| \leqslant 2\varkappa \int_{-h}^{0} \|\tilde{\varphi}_N(\theta)\|\mathrm{d}\theta + (M_2 h + M_3) \int_{-h}^{0} \|\tilde{\varphi}_N(\theta)\|^2\mathrm{d}\theta,$

2. *if, in addition,* $\|\varphi_N\|_h \leqslant 1$ *then* $|\Upsilon_N| \leqslant 2\varkappa \int\limits_{-h}^{0} \|\tilde{\varphi}_N(\theta)\| \mathrm{d}\theta$.

Here, $\varkappa = (1 + \|D\|)M_1 + M_2 h + M_3$.

Proof : To prove the first statement, we first compute $z(\psi, \tilde{\varphi}_N)$ for an arbitrary ψ and in particular, $z(\tilde{\varphi}_N, \tilde{\varphi}_N)$ as follows

$$z(\psi, \tilde{\varphi}_N) = (\psi(0) - D\psi(-h))^T U(0)(\tilde{\varphi}_N(0) - D\tilde{\varphi}_N(-h))$$
$$+ (\psi(0) - D\psi(-h))^T \int\limits_{-h}^{0} \Phi(h+s)\tilde{\varphi}_N(s)\mathrm{d}s + (\tilde{\varphi}_N(0) - D\tilde{\varphi}_N(-h))^T \int\limits_{-h}^{0} \Phi(h+s)\psi(s)\mathrm{d}s$$
$$+ \int\limits_{-h}^{0}\int\limits_{-h}^{0} \psi^T(s_1)\widetilde{\Psi}(s_1 - s_2)\tilde{\varphi}_N(s_2)\mathrm{d}s_2\mathrm{d}s_1,$$

$$z(\tilde{\varphi}_N, \tilde{\varphi}_N) = (\tilde{\varphi}_N(0) - D\tilde{\varphi}_N(-h))^T U(0)(\tilde{\varphi}_N(0) - D\tilde{\varphi}_N(-h))$$
$$+ 2(\tilde{\varphi}_N(0) - D\tilde{\varphi}_N(-h))^T \int\limits_{-h}^{0} \Phi(h+s)\tilde{\varphi}_N(s)\mathrm{d}s + \int\limits_{-h}^{0}\int\limits_{-h}^{0} \tilde{\varphi}_N^T(s_1)\widetilde{\Psi}(s_1 - s_2)\tilde{\varphi}_N(s_2)\mathrm{d}s_2\mathrm{d}s_1.$$

Under conditions of the lemma, observe that $\tilde{\varphi}_N(-h) = \varphi(-h) - \varphi_N(-h) = 0$ and $\tilde{\varphi}_N(0) = \varphi(0) - \varphi_N(0) = 0$, then the previous equations reduce to

$$z(\psi, \tilde{\varphi}_N) = (\psi(0) - D\psi(-h))^T \int\limits_{-h}^{0} \Phi(h+s)\tilde{\varphi}_N(s)\mathrm{d}s + \int\limits_{-h}^{0}\int\limits_{-h}^{0} \psi^T(s_1)\widetilde{\Psi}(s_1 - s_2)\tilde{\varphi}_N(s_2)\mathrm{d}s_2\mathrm{d}s_1, \tag{3.8}$$

$$z(\tilde{\varphi}_N, \tilde{\varphi}_N) = \int\limits_{-h}^{0}\int\limits_{-h}^{0} \tilde{\varphi}_N^T(s_1)\widetilde{\Psi}(s_1 - s_2)\tilde{\varphi}_N(s_2)\mathrm{d}s_2\mathrm{d}s_1. \tag{3.9}$$

Now, noticing that $v_0(\tilde{\varphi}_N) = z(\tilde{\varphi}_N, \tilde{\varphi}_N)$ and considering equations (3.7), (3.8) and (3.9), the functional approximation error is bounded as

$$|\Upsilon_N| = |2z(\varphi, \tilde{\varphi}_N) - v_0(\tilde{\varphi}_N)| \leqslant 2\|\varphi(0) - D\varphi(-h)\| M_1 \int\limits_{-h}^{0} \|\tilde{\varphi}_N(s)\|\mathrm{d}s$$
$$+ 2\left|\int\limits_{-h}^{0}\int\limits_{-h}^{0} \varphi^T(s_1)\widetilde{\Psi}(s_1 - s_2)\tilde{\varphi}_N(s_2)\mathrm{d}s_2\mathrm{d}s_1\right| + \left|\int\limits_{-h}^{0}\int\limits_{-h}^{0} \tilde{\varphi}_N^T(s_1)\widetilde{\Psi}(s_1 - s_2)\tilde{\varphi}_N(s_2)\mathrm{d}s_2\mathrm{d}s_1\right|. \tag{3.10}$$

Next, we bound the absolute value of the first double integral in (3.10), taking into account the jump of U' at zero,

$$\left| \int_{-h}^{0}\int_{-h}^{\theta_1 - 0} \varphi^T(\theta_1)\widetilde{\Psi}(\theta_1 - \theta_2)\tilde{\varphi}_N(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 + \int_{-h}^{0}\int_{\theta_1 + 0}^{0} \varphi^T(\theta_1)\widetilde{\Psi}(\theta_1 - \theta_2)\tilde{\varphi}_N(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 \right.$$
$$\left. - \int_{-h}^{0} \varphi^T(\theta) D^T \Delta U'(0) D\tilde{\varphi}_N(\theta)\mathrm{d}\theta \right| \leqslant (M_2 h + M_3) \int_{-h}^{0} \|\tilde{\varphi}_N(\theta)\|\mathrm{d}\theta.$$

Dealing similarly with the second double integral in the in (3.10), we have

$$\Bigg| \int_{-h}^{0}\int_{-h}^{\theta_1-0} \tilde{\varphi}_N^T(\theta_1)\widetilde{\Psi}(\theta_1-\theta_2)\tilde{\varphi}_N(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 + \int_{-h}^{0}\int_{\theta_1+0}^{0} \tilde{\varphi}_N^T(\theta_1)\widetilde{\Psi}(\theta_1-\theta_2)\varphi_N(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1 - \int_{-h}^{0} \tilde{\varphi}_N^T(\theta)D^T\Delta U'(0)D\varphi_N(\theta)\mathrm{d}\theta \Bigg| \leqslant (M_2h+M_3)\int_{-h}^{0}\|\tilde{\varphi}_N(\theta)\|^2\mathrm{d}\theta.$$

Gathering the above inequalities, we prove that the functional approximation error for $\varphi \in \mathcal{S}$ admits the estimate

$$|\Upsilon_N| \leqslant 2\varkappa \int_{-h}^{0} \|\tilde{\varphi}_N(s)\|\mathrm{d}s + (M_2h+M_3)\int_{-h}^{0}\|\tilde{\varphi}_N(s)\|^2\mathrm{d}s,$$

with $\varkappa = (1+\|D\|)M_1 + M_2h + M_3$.

To prove the second statement, we consider the functional approximation error in the form

$$\Upsilon_N = v_0(\varphi) - v_0(\varphi_N) = z(\varphi,\tilde{\varphi}_N) + z(\varphi_N,\tilde{\varphi}_N)$$

Taking into account equation (3.8), the functional approximation error Υ_N then admits the following expression

$$\Upsilon_N = (\varphi(0)-D\varphi(-h))^T\int_{-h}^{0}\Phi(h+\theta)\tilde{\varphi}_N(s)\mathrm{d}s + \int_{-h}^{0}\int_{-h}^{0}\varphi^T(s_1)\widetilde{\Psi}(s_1-s_2)\tilde{\varphi}_N(s_2)\mathrm{d}s_2\mathrm{d}s_1$$

$$+(\varphi_N(0)-D\varphi_N(-h))^T\int_{-h}^{0}\Phi(h+\theta)\tilde{\varphi}_N(s)\mathrm{d}s + \int_{-h}^{0}\int_{-h}^{0}\varphi_N^T(s_1)\widetilde{\Psi}(s_1-s_2)\tilde{\varphi}_N(s_2)\mathrm{d}s_2\mathrm{d}s_1,$$

Observe that if $\|\varphi_N\|_h \leqslant 1$ then an upper bound of $|\Upsilon_N|$ is given by

$$|\Upsilon_N| \leqslant 2((1+\|D\|)M_1 + M_2h + M_3)\int_{-h}^{0}\|\tilde{\varphi}_N(s)\|\mathrm{d}s,$$

concluding the proof. □

Remark 7 *Notice that if there exists ε_N in Step 3 such that $\varepsilon_N \to 0$ as $N \to \infty$, then $\Upsilon_N \to 0$ as $N \to \infty$, which implies that the functional approximation approaches the value of the exact functional $v_0(\varphi)$ when N tends to infinity.*

3.3 Illustration of the outline using the approach of Gomez et al. (2021)

To validate the previous general outline of proofs based on approximations, we briefly address the steps described in Section 3.1 to obtain the stability criterion of Theorem 3 of Gomez et al. (2021).

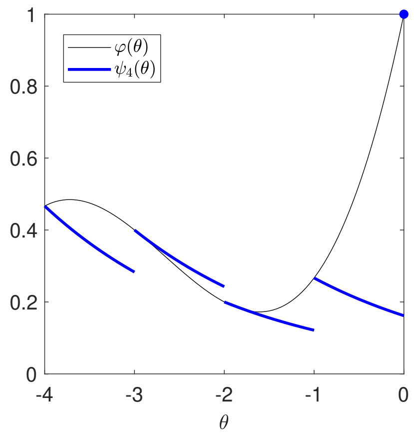


Figure 3.2: Function ψ_N in a scalar example.

Let us consider the function

$$\psi_N(\theta) = \sum_{j=0}^{N} Y(j\tau + \theta)\gamma_j, \quad \theta \in [-h, 0],$$

where $Y(s)$ is the system fundamental matrix, $\tau = \frac{h}{N}$ and $\gamma_j \in \mathbb{R}^n$, $j = \overline{0, N}$, are constant arbitrary vectors. An example of this class of function is depicted in Figure 3.2 for the scalar case $y(t) = e^{-0.5t}$, $t \in [0, h]$, with $h = 4$, $N = 4$, $\gamma_0 = 0.83$, $\gamma_1 = 0.14$, $\gamma_2 = -0.04$, $\gamma_3 = 0.11$ and $\gamma_4 = 0.46$.

Following *Step 1* of the general outline of proofs, a functional argument approximation $\varphi_N = \psi_N$ is considered, leading to the functional argument representation

$$\varphi(\theta) = \psi_N(\theta) + \tilde{\varphi}_N(\theta), \quad \theta \in [-h, 0],$$

Taking into account functional $v_1(\varphi)$, address *Step 2* by substituting $\varphi_N = \psi_N$ into the functional argument, resulting in a quadratic form structure with the help of some properties concerning the delay Lyapunov matrix and the system fundamental matrix. The approximated functional is given by the following expression

$$v_1^{\text{approx}}(\varphi) = v_1(\psi_N) = \gamma^T \mathcal{K}_N \gamma \tag{3.11}$$

where $\gamma = (\gamma_0^T \ \cdots \ \gamma_N^T)^T$ and

$$\mathcal{K}_N = \Big\{ U((j-i)\tau) \Big\}_{i,j=0}^{N}.$$

Now, focus on the approximation error $\tilde{\varphi}_N$ in order to validate *Step 3*. An estimate of $\tilde{\varphi}_N$ considering the function $\varphi_N = \psi_N$ is provided by the next lemma.

Lemma 8 *(Gomez et al., 2021) For every $\varphi \in \mathcal{S}$*

$$\|\tilde{\varphi}_N\|_h \leqslant \varepsilon_N = \frac{(K + L)e^{Lh}}{N/h + L},$$

where K is the constant defined in the set $\mathcal{S}$ and L is determined in Remark 1.

According to *Step 4* of the outline of proofs, an estimate for the functional approximation error Υ_N defined in (3.7) must be computed. In Gomez et al. (2021), an estimate of $|\Upsilon_N|$ is supplied for $\varphi \in \mathcal{S}$, which is given by the following expression

$$|\Upsilon_N| \leqslant 2\beta\|\tilde{\varphi}_N\|_h + \beta\|\tilde{\varphi}_N\|_h^2 \leqslant \delta_N = 2\beta\varepsilon_N + \beta\varepsilon_N^2, \quad \varphi \in \mathcal{S}, \tag{3.12}$$

$$\beta = (1 + \|D\|)^2\|U(0)\| + 2h(1 + \|D\|)M_1 + h^2 M_2 + h\|\Delta U'(0)\|,$$

with M_1 and M_2 defined in (3.6). Notice that if we choose γ_j so that $\psi_N(\theta_j) = \varphi(\theta_j)$, then Lemma 7 can be applied, yielding a less conservative bound than (3.12). However, we keep the estimate (3.12) of the functional approximation error because it leads to Theorem 3 introduced in Section 2.6.

Having in mind *Step 5* of the outline of proofs, the value a_1 in Lemma 6 and δ_N in (3.12) allow us to compute an approximation order $\hat{N}$ such that $\delta_{\hat{N}} = a_1$ and $\varphi \in \mathcal{S}$. The approximation order $\hat{N}$ for sufficiency is then given by

$$\hat{N} = \left\lceil e^{Lh} h(K + L)\left(\beta^\star + \sqrt{\beta^\star(\beta^\star + 1)}\right) - Lh \right\rceil,$$

delivering, under the assumption that matrix D is Schur stable, the stability criterion

$$\mathcal{K}_{\hat{N}} = \Big\{U((j - i)\tau)\Big\}_{i,j=0}^{\hat{N}}.$$

3.4 Comments on the approach based on functional kernels approximation

In this section, we state a slight variation of the approach for the functional kernels approximation case.

- Considering *Step 1*, we regard the following representation for the functional kernels:

  $$\begin{aligned} Q(\theta) &= Q^N(\theta) + \tilde{Q}^N(\theta), \quad \theta \in [-h, 0], \\ R(\theta_1, \theta_2) &= R^N(\theta_1, \theta_2) + \tilde{R}^N(\theta_1, \theta_2), \quad \theta_1 \in [-h, 0], \quad \theta_2 \in [-h, 0] \end{aligned}$$

  where Q^N and R^N denote the approximation of order N of Q and R, respectively, and $\tilde{Q}^N$ and $\tilde{R}^N$ stand for their corresponding approximation errors.

- In *Step 2*, we similarly substitute Q^N and R^N into the functional $v_0(\varphi)$ in the form of (2.9) to obtain an approximated functional $v_0^{\text{approx}}(\varphi) = v_0^{(N)}(\varphi)$:

  $$v_0^{(N)}(\varphi) = \varphi^T(0)P\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} Q^N(s)\gamma(s)\mathrm{d}s + \int_{-h}^{0}\int_{-h}^{0} \gamma^T(s_1)R^N(s_1, s_2)\gamma(s_2)\mathrm{d}s_2\mathrm{d}s_1, \tag{3.13}$$

  which is characterized by a quadratic form. In particular, notice that the discretized Lyapunov functional method of Section 2.5 allows us to achieve a quadratic form determined by an inner block matrix in terms of discrete evaluations of the kernels.

- Unlike the functional argument approximation case, *Step 3* concerns the error estimation of $\tilde{Q}^N$ and $\tilde{R}^N$ instead of $\tilde{\varphi}_N$. Thus, we must compute an estimate in such a way that

$$\|\tilde{Q}^N\|_h = \|Q - Q^N\|_h \leqslant \zeta_N,$$
$$\|\tilde{R}^N\|_h = \|R - R^N\|_h \leqslant \eta_N,$$

and that $\zeta_N \to 0$ as $N \to \infty$ and $\eta_N \to 0$ as $N \to \infty$.

- Considering functional $v_0(\varphi)$ in the form of (2.9) and a function $\varphi \in \mathcal{S}$, the functional approximation error in *Step 4* is tackled in the following lemma:

Lemma 9 *Given $\varphi \in \mathcal{S}$, the functional approximation error admits the following bound:*

$$\left|\Upsilon_N\right| = \left|v_0(\varphi) - v_0^{(N)}(\varphi)\right| \leqslant \delta_N = 2\mu \int_{-h}^{0} \left\|\tilde{Q}^N(s)\right\| \mathrm{d}s + \mu^2 \int_{-h}^{0}\int_{-h}^{0} \|\tilde{R}^N(s_1, s_2)\| \mathrm{d}s_2 \mathrm{d}s_1.$$

Here, $\mu = \|D\|K + \|A_1\|$.

Proof : Considering functional $v_0(\varphi)$ in the form of (2.9) and the expression (3.4) for the functional approximation error, we get

$$\Upsilon_N = v_0(\varphi) - v_0^{\text{approx}}(\varphi) = v_0(\varphi) - v_0^{(N)}(\varphi)$$
$$= 2\varphi^T(0) \int_{-h}^{0} (Q(s) - Q^N(s))\gamma(s) \mathrm{d}s + \int_{-h}^{0}\int_{-h}^{0} \gamma^T(s_1)(R(s_1, s_2) - R^N(s_1, s_2))\gamma(s_2) \mathrm{d}s_2 \mathrm{d}s_1.$$

Finally, bounding each summand of the previous expression and taking into account that $\varphi \in \mathcal{S}$, we arrive at the following upper bound for the approximation error

$$|\Upsilon_N| \leqslant 2(\|D\|K + \|A_1\|) \int_{-h}^{0} \left\|\tilde{Q}^N(s)\right\| \mathrm{d}s + (\|D\|K + \|A_1\|)^2 \int_{-h}^{0}\int_{-h}^{0} \|\tilde{R}^N(s_1, s_2)\| \mathrm{d}s_2 \mathrm{d}s_1.$$

□

- Finally, *Step 5* is addressed in the same manner. However, we calculate the approximation order $N^\star$ such that

$$|\Upsilon_{N^\star}| \leqslant \delta_{N^\star} \leqslant a_0, \quad \varphi \in \mathcal{S},$$

with $|\Upsilon_N|$ determined in Lemma 9.

3.5 Conclusions

This chapter presents a general outline of proofs based on approximations deployed in five steps for either functional argument or functional kernel approximation. It also gives an overview of the approximation machinery for the latter chapters to obtain our main results. The presented outline of proofs is verified for Theorem 3 introduced in Section 2.6.

Chapter 4

Stability tests for neutral-type delay systems: Piecewise linear approximation of the functional argument approach

The obtention of stability conditions for neutral type systems using piecewise linear approximations were first attempted in Alexandrova and Zhabko (2019). This class of approximation on a special set of functional arguments was used as a discretization scheme to develop constructive sufficient stability conditions for *scalar neutral type systems*. Nevertheless, this approach allowed obtaining a stability criterion for the retarded type case in Alexandrova (2023), notably reducing the approximation order for sufficiency in comparison with those provided by the stability criterion based on discretized values of the delay Lyapunov matrix in Gomez, Egorov, and Mondié (2019). In this chapter, motivated by Alexandrova and Zhabko (2019) and Alexandrova (2023), we use the general outline of proofs described in Chapter 3 with a piecewise linear approximation of the functional argument to find necessary and sufficient stability conditions via a finite number of operations for multivariable neutral type systems.

This chapter is organized as follows. In Section 4.1, an approximated functional is obtained by substituting a piecewise linear approximation into the functional argument. In Section 4.2, starting from the representation (3.1) of Chapter 3, the functional approximation error is given by noticing that if $\varphi \in \mathcal{S}$ then the piecewise linear approximation belongs to the set $\mathcal{S}$. In Section 4.3, the main result of this chapter is presented. There, the necessity part follows from the fact that if system (2.1) is exponentially stable, then we are able to find a non-negative lower bound and, in particular, for the approximated functional argument. The sufficiency condition is proved with the help of the instability result Lemma 3 introduced in Section 2.4. We prove that, if the system is unstable, the approximated functional remains negative for a sufficiently large approximation order, and this order is tractable compared to the one in Gomez et al. (2021). Finally, we conclude with some remarks in Section 4.4.

4.1 A piecewise linear approximation scheme

In this section, for the case of piecewise linear approximation of the functional argument, we focus on *Step 1* and *2* of the general outline of Chapter 3, in other words, introduce the piecewise linear approximation l_N and calculate the approximated functional $v_0^{\text{approx}}(\varphi) = v_0(\varphi_N)$. We show that functional $v(\varphi_N)$ can be expressed as a quadratic form.

Let us discretize the interval $[-h, 0]$ at points $\theta_j = -j\tau$, $j = \overline{0, N}$, $\tau = h/N$, and consider the piecewise linear approximation of the function φ:

$$l_N(s+\theta_j) = \varphi(\theta_j)\left(1+\frac{s}{\tau}\right) - \varphi(\theta_{j+1})\frac{s}{\tau}, \quad s \in [-\tau, 0], \quad j = \overline{0, N-1}. \tag{4.1}$$

The functional argument φ can be presented as

$$\varphi(s) = \varphi_N(s) + \tilde{\varphi}_N(s), \quad s \in [-h, 0], \tag{4.2}$$

where, $\varphi_N(s) = l_N(s)$. Let us introduce the following vector:

$$\widehat{\varphi} = \begin{pmatrix} \varphi(0) \\ \varphi(\theta_1) \\ \vdots \\ \varphi(\theta_N) \end{pmatrix},$$

and denote its constituents by $\widehat{\varphi}_j = \varphi(\theta_j)$, $j = \overline{0, N}$. A sketch of the approximation scheme is depicted in Figure 4.1 for a given scalar function. The scalar function φ and its piecewise linear approximation l_N are depicted in red and blue lines, respectively.

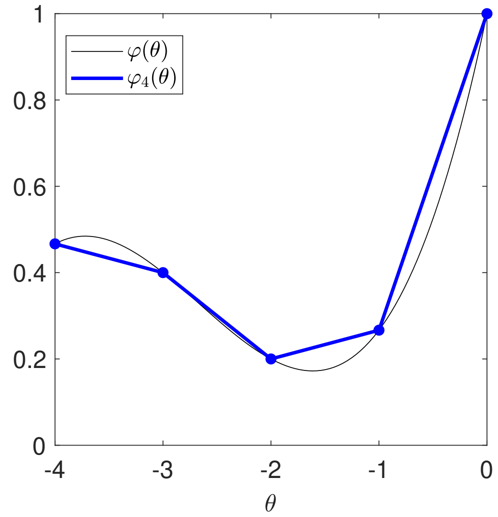


Figure 4.1: Function l_N in a scalar example.

Next, we evaluate the approximated functional $v_0^{\text{approx}}(\varphi) = v_0(\varphi_N)$. We start evaluating each summand successively. The first one is equal to

$$I_1(\varphi_N) = (\varphi(0) - D\varphi(-h))^T U(0)(\varphi(0) - D\varphi(-h)) = (\widehat{\varphi}_0 - D\widehat{\varphi}_N)^T U(0)(\widehat{\varphi}_0 - D\widehat{\varphi}_N).$$

The summand $I_2(\varphi)$ is rewritten as follows:

$$I_2(\varphi) = 2(\varphi(0) - D\varphi(-h))^T \int_{-h}^{0} \Phi(h+\theta)\varphi(\theta)\mathrm{d}\theta$$

$$=2(\varphi(0)-D\varphi(-h))^T\sum_{k=1}^{N}\int\limits_{-(N-k+1)\tau}^{-(N-k)\tau}\Phi(h+\theta)\varphi(\theta)\mathrm{d}\theta$$
$$=2(\varphi(0)-D\varphi(-h))^T\sum_{k=1}^{N}\int\limits_{-\tau}^{0}\Phi(s+k\tau)\varphi(s+\theta_{N-k})\mathrm{d}s,$$

with the change of variable $s=\theta+(N-k)\tau$. Substituting the piecewise linear approximation (4.1) instead of $\varphi(s+\theta_{N-k})$ into the previous expression, the approximation of $I_2(\varphi)$ is given by:

$$I_2(\varphi_N)=2(\varphi(0)-D\varphi(-h))^T\sum_{k=1}^{N}\int\limits_{-\tau}^{0}\Phi(s+k\tau)\left(\varphi(\theta_{N-k})\left(1+\frac{s}{\tau}\right)-\varphi(\theta_{N-k+1})\frac{s}{\tau}\right)\mathrm{d}s$$
$$=2(\widehat{\varphi}_0-D\widehat{\varphi}_N)^T\sum_{k=1}^{N}\left(\mathcal{M}_k\widehat{\varphi}_{N-k}+\mathcal{N}_k\widehat{\varphi}_{N-k+1}\right),$$

where

$$\mathcal{M}_k=\int_{-\tau}^{0}\Phi(s+k\tau)\left(1+\frac{s}{\tau}\right)\mathrm{d}s,\quad \mathcal{N}_k=\int_{-\tau}^{0}\Phi(s+k\tau)\left(-\frac{s}{\tau}\right)\mathrm{d}s.$$

Similarly, consider the third and fourth summands of the functional:

$$I_3(\varphi)+I_4(\varphi)=\int_{-h}^{0}\int_{-h}^{0}\varphi^T(\xi_1)\widetilde{\Psi}(\xi_1-\xi_2)\varphi(\xi_2)\mathrm{d}\xi_2\mathrm{d}\xi_1$$
$$=\sum_{k=1}^{N}\sum_{j=1}^{N}\int\limits_{-(N-k+1)\tau}^{-(N-k)\tau}\int\limits_{-(N-j+1)\tau}^{-(N-j)\tau}\varphi^T(\xi_1)\widetilde{\Psi}(\xi_1-\xi_2)\varphi(\xi_2)\mathrm{d}\xi_2\mathrm{d}\xi_1$$
$$=\sum_{k=1}^{N}\sum_{j=1}^{N}\int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0}\varphi^T(s_1+\theta_{N-k})\widetilde{\Psi}(s_1-s_2+(k-j)\tau)\varphi(s_2+\theta_{N-j})\mathrm{d}s_2\mathrm{d}s_1,$$

where $\widetilde{\Psi}(s)=\Psi(s)-D^TU''(s)D$. Substituting approximation (4.1) instead of $\varphi(s_1+\theta_{N-k})$ and $\varphi(s_2+\theta_{N-j})$, we get

$$I_3(\varphi_N)+I_4(\varphi_N)=\sum_{k=1}^{N}\sum_{j=1}^{N}\int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0}\left(\varphi(\theta_{N-k})\left(1+\frac{s_1}{\tau}\right)-\varphi(\theta_{N-k+1})\frac{s_1}{\tau}\right)^T$$
$$\times\widetilde{\Psi}(s_1-s_2+(k-j)\tau)\left(\varphi(\theta_{N-j})\left(1+\frac{s_2}{\tau}\right)-\varphi(\theta_{N-j+1})\frac{s_2}{\tau}\right)\mathrm{d}s_2\mathrm{d}s_1$$

$$=\sum_{k=1}^{N}\sum_{j=1}^{N}\left(\widehat{\varphi}_{N-k}^T\mathcal{P}_{k-j}\widehat{\varphi}_{N-j}+\widehat{\varphi}_{N-k}^T\mathcal{Q}_{k-j}\widehat{\varphi}_{N-j+1}+\widehat{\varphi}_{N-k+1}^T\mathcal{Q}_{j-k}^T\widehat{\varphi}_{N-j}+\widehat{\varphi}_{N-k+1}^T\mathcal{P}_{k-j}\widehat{\varphi}_{N-j+1}\right).$$

Here,

$$\mathcal{P}_l=\int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0}\widetilde{\Psi}(s_1-s_2+l\tau)\frac{s_1s_2}{\tau^2}\mathrm{d}s_2\mathrm{d}s_1=\int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0}\widetilde{\Psi}(s_1-s_2+l\tau)\left(1+\frac{s_1}{\tau}\right)\left(1+\frac{s_2}{\tau}\right)\mathrm{d}s_2\mathrm{d}s_1,$$

$$\mathcal{Q}_l = \int_{-\tau}^{0}\int_{-\tau}^{0} \widetilde{\Psi}(s_1 - s_2 + l\tau)\left(1 + \frac{s_1}{\tau}\right)\left(-\frac{s_2}{\tau}\right)\mathrm{d}s_2\mathrm{d}s_1,$$

for $l = \overline{-(N-1), N-1}$. It is useful to note that $\mathcal{P}_l = \mathcal{P}_{-l}^T$ for any l. When $l = 0$, we stress out that while calculating the integrals involving $U''(s)$ one has to take into account that the first derivative of the Lyapunov matrix is discontinuous at zero. Hence, those integrals should be interpreted in the same way as in formula (2.11), namely,

$$\begin{aligned}
\mathcal{P}_0 = &\int_{-\tau}^{0}\int_{-\tau}^{0} \Psi(s_1 - s_2)\frac{s_1 s_2}{\tau^2}\mathrm{d}s_2\mathrm{d}s_1 - \int_{-\tau}^{0} \frac{s_1}{\tau^2}\left(\int_{-\tau}^{s_1-0} s_2 D^T U''(s_1 - s_2)D\mathrm{d}s_2\right.\\
&\left. + \int_{s_1+0}^{0} s_2 D^T U''(s_1 - s_2)D\mathrm{d}s_2\right)\mathrm{d}s_1 - \frac{\tau}{3}D^T\Delta U'(0)D,\\
\mathcal{Q}_0 = &\int_{-\tau}^{0}\int_{-\tau}^{0} \Psi(s_1 - s_2)\left(1+\frac{s_1}{\tau}\right)\left(-\frac{s_2}{\tau}\right)\mathrm{d}s_2\mathrm{d}s_1 + \int_{-\tau}^{0}\left(1+\frac{s_1}{\tau}\right)\left(\int_{-\tau}^{s_1-0} \frac{s_2}{\tau}D^T U''(s_1 - s_2)D\mathrm{d}s_2\right.\\
&\left. + \int_{s_1+0}^{0} \frac{s_2}{\tau}D^T U''(s_1 - s_2)D\mathrm{d}s_2\right)\mathrm{d}s_1 - \frac{\tau}{6}D^T\Delta U'(0)D.
\end{aligned} \tag{4.3}$$

Here, the integrals appeared due to the jump of U' at zero are explicitly calculated:

$$\int_{-\tau}^{0} \frac{s^2}{\tau^2}\mathrm{d}s = 2\int_{-\tau}^{0}\left(1 + \frac{s}{\tau}\right)\left(-\frac{s}{\tau}\right)\mathrm{d}s = \frac{\tau}{3}. \tag{4.4}$$

Adding the four summands leads to the approximation of the functional as a quadratic form,

$$v_0^{\text{approx}}(\varphi) = v_0(\varphi_N) \stackrel{\text{def}}{=} \widehat{\varphi}^T \Lambda_N \widehat{\varphi}. \tag{4.5}$$

Here, the blocks of $\Lambda_N = \{\Lambda^{ij}\}_{i,j=0}^{N}$ are given by:

$$\Lambda^{00} = U(0) + \mathcal{M}_N + \mathcal{M}_N^T + \mathcal{P}_0, \quad \Lambda^{NN} = D^T U(0)D - D^T\mathcal{N}_1 - \mathcal{N}_1^T D + \mathcal{P}_0,$$

$$\Lambda^{0N} = \mathcal{N}_1 - (U(0) + \mathcal{M}_N^T)D + \mathcal{Q}_{N-1}, \quad \Lambda^{k,k+l} = \Omega_l + \Omega_{-l}^T,\ l = \overline{0, N-k-1},$$

$$\Lambda^{0k} = \mathcal{M}_{N-k} + \mathcal{N}_{N-k+1} + \Omega_k, \quad \Lambda^{kN} = \Omega_{N-k} - (\mathcal{M}_{N-k}^T + \mathcal{N}_{N-k+1}^T)D, \quad k = \overline{1, N-1},$$

where $\Omega_j = \mathcal{P}_j + \mathcal{Q}_{j-1}$, $j = \overline{-(N-2), N-1}$, and $\Lambda^{jk} = \Lambda^{kj^T}$ for other indices.

In order to delve into the quadratic form of $v_0^{\text{approx}}(\varphi)$, we express the matrix Λ_N as the sum of the following parts:

$$\Lambda_N = \sum_{j=1}^{3} \Lambda^{(j)},$$

where,

$$\Lambda^{(1)} = \begin{pmatrix} U(0)+\mathcal{M}_N+\mathcal{M}_N^T & \mathcal{M}_{N-1}+\mathcal{N}_N & \cdots & \mathcal{M}_1+\mathcal{N}_2 & \mathcal{N}_1-U(0)D-\mathcal{M}_N^T D \\ * & \mathbb{O}_n & \cdots & \mathbb{O}_n & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & -D^T(\mathcal{M}_{N-1}+\mathcal{N}_N) & \cdots & -D^T(\mathcal{M}_1+\mathcal{N}_2) & D^T U(0)D-D^T\mathcal{N}_1-\mathcal{N}_1^T D \end{pmatrix},$$

$$\Lambda^{(2)} = \begin{pmatrix} \mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \cdots & \mathcal{P}_{N-1} & \mathbb{O}_n \\ * & 2\mathcal{P}_0 & \mathcal{P}_1 & \cdots & \mathcal{P}_{N-2} & \mathcal{P}_{N-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ * & * & * & 2\mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 \\ * & * & * & * & 2\mathcal{P}_0 & \mathcal{P}_1 \\ * & * & * & \cdots & * & \mathcal{P}_0 \end{pmatrix},$$

$$\Lambda^{(3)} = \begin{pmatrix} \mathbb{O}_n & \mathcal{Q}_0 & \mathcal{Q}_1 & \cdots & \mathcal{Q}_{N-2} & \mathcal{Q}_{N-1} \\ * & \mathcal{Q}_{-1}+\mathcal{Q}_{-1}^T & \mathcal{Q}_0+\mathcal{Q}_{-2}^T & \cdots & \mathcal{Q}_{N-3}+\mathcal{Q}_{1-N}^T & \mathcal{Q}_{N-2} \\ * & * & \mathcal{Q}_{-1}+\mathcal{Q}_{-1}^T & \cdots & \mathcal{Q}_{N-4}+\mathcal{Q}_{2-N}^T & \mathcal{Q}_{N-3} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ * & * & * & \cdots & \mathcal{Q}_{-1}+\mathcal{Q}_{-1}^T & \mathcal{Q}_0 \\ * & * & * & \cdots & * & \mathbb{O}_n \end{pmatrix}.$$

Notice that all matrices here are of dimension $n(N+1) \times n(N+1)$, and each matrix is symmetric.

Remark 8 *Compared to retarded-type systems, the integral terms $\mathcal{M}_k$, $\mathcal{N}_k$, $\mathcal{P}_l$, $\mathcal{Q}_l$ involved in the matrix Λ_N depend not only on the Lyapunov matrix itself as in the retarded-type case, but also on its first and second derivatives and the monomials s_1, s_2 and combinations of them. In practice, the computation of those terms turns out to be more complex, even if a recursive method is implemented, as the one presented in Chapter 7.*

4.2 Estimate of the functional approximation error using a piecewise linear approximation

Here, following *Step 3* and *4* of the general outline of Chapter 3, we provide an estimate of the functional argument error and the functional approximation error. The last one is partially addressed in Lemma 7, which is given in terms of the approximation error $\tilde{\varphi}_N$. In this way, it only remains to provide an expression for $\tilde{\varphi}_N$. To do so, we introduce the following lemma, which was presented in recent work. It allows us to get an estimate for the error $\tilde{\varphi}_N$ of the piecewise linear approximation of φ defined in (4.1).

Lemma 10 *(Alexandrova, 2023) The piecewise linear approximation error admits a bound*

$$\|\tilde{\varphi}_N(s+\theta_j)\| \leqslant \frac{1}{2}K^2(-s)(s+\tau), \;\; s \in [-\tau, 0],$$

for all $j = \overline{0, N-1}$ and any function $\varphi \in \mathcal{S}$.

Proof : Using the Taylor formula with the remainder in the integral form (see Appendix A), we get for $s \in [-\tau, 0]$:

$$\varphi(s+\theta_j) = \varphi(\theta_j) + s\varphi'(\theta_j) + \int_s^0 (t-s)\varphi''(t+\theta_j)\mathrm{d}t,$$

$$\varphi(\theta_{j+1}) = \varphi(\theta_j) - \tau\varphi'(\theta_j) + \int_{-\tau}^0 (t+\tau)\varphi''(t+\theta_j)\mathrm{d}t.$$

Substituting this in the expression for $\tilde{\varphi}_N(s+\theta_j)$ obtained from (4.1) we arrive at

$$\begin{aligned}\|\tilde{\varphi}_N(s+\theta_j)\| &= \left\| \int_s^0 (t-s)\varphi''(t+\theta_j)\mathrm{d}t + \frac{s}{\tau}\int_{-\tau}^0 (t+\tau)\varphi''(t+\theta_j)\mathrm{d}t \right\| \\ &= \left\| \left(1+\frac{s}{\tau}\right)\int_s^0 t\varphi''(t+\theta_j)\mathrm{d}t + s\int_{-\tau}^s \left(1+\frac{t}{\tau}\right)\varphi''(t+\theta_j)\mathrm{d}t \right\|.\end{aligned}$$

Since $\varphi \in \mathcal{S}$, we have

$$\|\tilde{\varphi}_N(s+\theta_j)\| \leqslant K^2\left(\left(1+\frac{s}{\tau}\right)\int_s^0(-t)\mathrm{d}t + (-s)\int_{-\tau}^s\left(1+\frac{t}{\tau}\right)\mathrm{d}t\right) = K^2\frac{(-s)(s+\tau)}{2},\ s\in[-\tau,0].$$

□

Following Alexandrova (2023), we notice that if $\varphi \in \mathcal{S}$ then $\|\varphi(\theta)\| \leqslant \|\widehat{\varphi}_0\| = 1$, $\|\varphi_N(\theta)\| \leqslant \|\widehat{\varphi}_0\| = 1$, $\theta \in [-h, 0]$, and

$$\int_{-h}^0 \|\tilde{\varphi}_N(\theta)\|\mathrm{d}\theta = \sum_{k=1}^N \int_{-\tau}^0 \|\tilde{\varphi}_N(s+\theta_{N-k})\|\mathrm{d}s \leqslant \frac{1}{2}K^2 N\int_{-\tau}^0 (-s)\,(s+\tau)\,\mathrm{d}s = \frac{1}{12}K^2\frac{h^3}{N^2}, \tag{4.6}$$

By construction of the piecewise linear approximation, we also have that $\tilde{\varphi}_N(-h) = \varphi(-h) - \varphi_N(-h) = 0$ and $\tilde{\varphi}_N(0) = \varphi(0) - \varphi_N(0) = 0$. Then, It follows from the statement 2 of Lemma 7 and inequality (4.6) that the functional approximation error admits the following estimate:

$$|\Upsilon_N| \leqslant 2\varkappa\int_{-h}^0 \|\tilde{\varphi}_N(\theta)\|\mathrm{d}\theta \leqslant \varkappa\frac{1}{6}K^2\frac{h^3}{N^2},$$

equivalently,

$$|\Upsilon_N| \leqslant \frac{c_0}{N^2} = \delta_N, \quad c_0 = \frac{1}{6}K^2h^3\varkappa, \tag{4.7}$$

where $\varkappa = (1+\|D\|)M_1 + M_2h + M_3$, and constants M_1, M_2 and M_3 are defined in *Step 4* of the general outline of Chapter 3.

Remark 9 *Notice that, for any* $\varepsilon > 0$, *if*

$$N \geqslant \sqrt{\frac{c_0}{\varepsilon}},$$

then $|\Upsilon_N| \leqslant \varepsilon$.

4.3 Stability criterion based on a piecewise linear approximation

In this section, we present our main result reflecting a comprehensive treatment of a necessary and sufficient stability test in a moderate number of operations, which may be given based on the general outline of Chapter 3, particularly in Sections 4.1 and 4.2 for a piecewise linear approximation. As discussed in *Step 5* of the general outline, the stability criterion is derived by calculating an approximation order N such that the estimate of the functional approximation error $\delta_N = a_0$. We emphasize that the criterion expressed in terms of the non-negative semi-definiteness of matrix Λ_N is mathematically tractable, i.e., it requires the verification of just one condition evaluated at a fixed, explicitly giving a value of N.

Theorem 4 *System* (2.1) *is exponentially stable, if and only if the Lyapunov condition holds and the matrix*

$$\Lambda_{N_0} \geq 0,$$

where $N_0 = \left\lceil \sqrt{\frac{c_0}{a_0}} \right\rceil$, *with* c_0 *and* a_0 *determined in* (4.7) *and Lemma 3, respectively.*

Proof : *Necessity.* It follows from the fact that

$$v_0(\varphi) = \int_0^{+\infty} x^T(t,\varphi) W x(t,\varphi) \mathrm{d}t \geqslant 0$$

for any $\varphi \in \mathcal{PC}^1\big([-h,0],\mathbb{R}^n\big)$ by construction of the functional, if system (2.1) is exponentially stable. In particular, for $\varphi_N = l_N \in \mathcal{PC}^1\big([-h,0],\mathbb{R}^n\big)$ we have

$$v_0(\varphi_N) = \widehat{\varphi}^T \Lambda_N \widehat{\varphi} \geqslant 0,$$

for any $\widehat{\varphi}$ and N which implies $\Lambda_N \geq 0$.

Sufficiency. We remind that the functional approximation error can be expressed as:

$$\Upsilon_N = v_0(\varphi) - v_0^{\text{approx}}(\varphi) = v_0(\varphi) - \widehat{\varphi}^T \Lambda_N \widehat{\varphi}, \quad \varphi \in \mathcal{S}.$$

Formulae (4.5) and (4.7) imply that for any N and any $\varphi \in \mathcal{S}$,

$$\widehat{\varphi}^T \Lambda_N \widehat{\varphi} = v_0(\varphi) - \Upsilon_N \leqslant v_0(\varphi) + \delta_N. \tag{4.8}$$

By contradiction, assume that system (2.1) is unstable. By Lemma 3, there exists $\widetilde{\varphi} \in \mathcal{S}$ such that $v_0(\widetilde{\varphi}) < -a_0$. Moreover, the choice of N_0 implies $\delta_{N_0} = c_0/N_0^2 \leqslant a_0$. Hence, taking (4.8) with $N = N_0$ and $\varphi = \widetilde{\varphi}$ we get

$$\widehat{\varphi}^T \Lambda_{N_0} \widehat{\varphi} \leqslant v_0(\widetilde{\varphi}) + \delta_{N_0} < -a_0 + a_0 = 0,$$

which contradicts $\Lambda_{N_0} \geq 0$. □

4.4 Conclusions

Thanks to the piecewise linear approximation, which better approximates arbitrary functional arguments than fundamental matrices combinations, and thanks to the instability result on the set $\mathcal{S}$, which allows bounding the approximation error, the obtained stability criterion relies on verifying the non-negativity of a matrix with substantially reduced approximation orders.

Notice, however, that this criterion requires the computation of integrals of the delay Lyapunov matrix and its first and second derivative, which imply a substantial computational burden. A recursive method that overcomes this issue is introduced in Chapter 7.

Chapter 5

Stability test for neutral-type delay systems: Legendre polynomial approximation of the functional argument

As evidenced in the previous chapter, a better approximation of the functional argument provides tractable approximation orders for evaluating the system stability. It is well-known that piecewise linear approximation is not the best; however, it is also shown that this approximation allows a considerable reduction of the approximation order. Thus, a natural idea is to explore the approximation theory for better approximations. In the approximation theory, orthogonal polynomials used as a base of approximation are notable for their fast convergence towards a given function. In particular, the Legendre polynomials are considered in this chapter to approximate the functional argument. They are chosen because of their rapid convergence for smooth arguments, orthogonality with respect to the Lebesgue measure, and second-order recurrence satisfied by the coefficients (Wang & Xiang, 2012). Moreover, it is worth highlighting that, for retarded type systems, this approach allowed obtaining similar or tighter approximation orders in Bajodek et al. (2023) than those using piecewise linear approximations in Alexandrova (2023). Thus, this chapter aims to approximate the functional argument via the Legendre polynomial approximation in order to achieve a stability criterion for neutral type systems. The result is obtained following the general outline of proofs of Chapter 3.

This chapter is organized as follows. Section 5.1 introduces some preliminaries on Legendre polynomials, such as definition and properties. In Section 5.2, a Legendre polynomial approximation scheme is presented, relying on substituting the Legendre polynomial approximation instead of φ into the functional. This scheme derives a quadratic form, whose inner block matrix is characterized by integrals of the delay Lyapunov matrix multiplied by the Legendre polynomials. The approximation error of the functional argument and the functional approximation error are estimated and bounded in Section 5.3. The main result of this chapter, a stability criterion in a tractable number of operations, is proven in Section 5.4. Finally, some conclusions are given in Section 5.5.

5.1 Preliminaries on Legendre polynomials

In this section, we briefly remind the definition and some properties of the Legendre polynomials.

The Legendre polynomials p_k on the segment $[-h, 0]$ are defined as

$$p_k(s) = (-1)^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{k+i}{i} \left(\frac{s+h}{h}\right)^i, \quad \forall k \in \mathbb{N},$$

where $\binom{k}{i}$ stands for the binomial coefficient. Notice that the first polynomials are equal to

$$p_0(s) = 1, \quad p_1(s) = \frac{2s}{h} + 1, \quad p_2(s) = \frac{6s}{h} + \frac{6s^2}{h^2} + 1.$$

The set of Legendre polynomials $\{p_k\}_{k\in\mathbb{N}}$ satisfy the following properties (Gautschi, 2006; Gottlieb & Orszag, 1977):

1. *Orthogonality:* $\forall i, j \in \mathbb{N}$,

$$\int_{-h}^{0} p_i(s)p_j(s)\mathrm{d}s = \begin{cases} 0, & i \neq j, \\ \dfrac{h}{2i+1}, & i = j. \end{cases}$$

2. *Boundary conditions:*

$$\forall k \in \mathbb{N}, \quad p_k(0) = 1, \quad p_k(-h) = (-1)^k.$$

3. *Differentiation:* The following rule is satisfied

$$\forall k \geqslant 2, \quad p_k'(s) - p_{k-2}'(s) = \frac{2(2k-1)}{h} p_{k-1}(s).$$

The first derivative of the Legendre polynomials satisfies

$$p_k'(s) = \sum_{i=0}^{k-1} \frac{(2i+1)}{h} (1 - (-1)^{k+i}) p_i(s), \quad k \geqslant 1.$$

5.2 A Legendre polynomial approximation scheme

This section introduces a Legendre polynomial approximation for the functional argument φ of functional (2.10). It relies on the general outline of proofs described in Chapter 3, whose approximation leads to a quadratic form determined by the Legendre polynomials and the delay Lyapunov matrix.

It follows from *Step 1* in the general outline of Chapter 3 that any function $\varphi \in \mathcal{PC}^1\big([-h, 0], \mathbb{R}^n\big)$ can be written as

$$\varphi(s) = \varphi_N(s) + \tilde{\varphi}_N(s), \quad s \in [-h, 0],$$

where φ_N is the functional argument approximation and $\tilde{\varphi}_N$ is its approximation error. A sketch of the approximation scheme using Legendre polynomials is depicted in Figure 5.1 for a given scalar function and low orders of approximations.

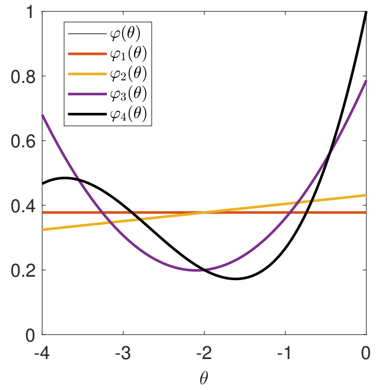


Figure 5.1: Function φ approximated by Legendre polynomials in a scalar example.

Let us consider a N-order Legendre polynomial approximation of this function, given by

$$\varphi_N(\theta) = \begin{cases} \varphi(-h), & \theta = -h \\ \mathbf{p}_N^T(\theta)\Phi_N, & \theta \in (-h, 0), \quad N \in \mathbb{N}, \\ \varphi(0), & \theta = 0. \end{cases} \tag{5.1}$$

Here,

$$\Phi_N = \left(\int_{-h}^{0} \mathbf{p}_N(s)\mathbf{p}_N^T(s)\mathrm{d}s \right)^{-1} \int_{-h}^{0} \mathbf{p}_N(s)\varphi(s)\mathrm{d}s \ \in \mathbb{R}^{Nn},$$

$$\mathbf{p}_N(\theta) = [p_0(\theta)I_n \ p_1(\theta)I_n \ \cdots \ p_{N-1}(\theta)I_n]^T, \ \theta \in [-h, 0],$$

where $\{p_k\}_{k\in\overline{0,N-1}}$ is the set of Legendre polynomials of order lower than N and, vector $\Phi_N \in \mathbb{R}^{nN}$ collects the projection coefficients associated to the Legendre polynomials, which is guaranteed thanks to the orthogonal property of the Legendre polynomials. Following *Step 2* of the general outline, we substitute the Legendre polynomial approximation (5.1) instead of φ into the functional (2.10), resulting in the approximated functional $v_0(\varphi_N)$, which is the Legendre approximation of functional (2.10). Observe that I_2 reduces to

$$I_2(\varphi_N) = 2(\varphi(0) - D\varphi(-h))^T \int_{-h}^{0} \Phi(h+s)\mathbf{p}_N^T(s)\Phi_N\mathrm{d}s = 2(\varphi(0) - D\varphi(-h))^T \mathcal{J}_{1N}\Phi_N, \tag{5.2}$$

where

$$\mathcal{J}_{1N} = \int_{-h}^{0} \Phi(h+s)\mathbf{p}_N^T(s)\mathrm{d}s.$$

Evaluating the summand I_3 and I_4 of functional (2.10), we get

$$\begin{aligned} I_3(\varphi_N) + I_4(\varphi_N) &= \int_{-h}^{0}\int_{-h}^{0} \Phi_N^T\mathbf{p}_N(s_1)\Psi(s_1 - s_2)\mathbf{p}_N^T(s_2)\Phi_N\mathrm{d}s_2\mathrm{d}s_1 \\ &\quad - \int_{-h}^{0}\int_{-h}^{s_1} \Phi_N^T\mathbf{p}_N(s_1)D^TU''(s_1 - s_2)D\mathbf{p}_N^T(s_2)\Phi_N\mathrm{d}s_2\mathrm{d}s_1 \\ &\quad - \int_{-h}^{0}\int_{s_1}^{0} \Phi_N^T\mathbf{p}_N(s_1)D^TU''(s_1 - s_2)D\mathbf{p}_N^T(s_2)\Phi_N\mathrm{d}s_2\mathrm{d}s_1 \\ &\quad - \int_{-h}^{0} \Phi_N^T\mathbf{p}_N(s)D^T\Delta U'(0)D\mathbf{p}_N^T(s)\Phi_N\mathrm{d}s \\ &= \Phi_N^T(\mathcal{J}_{2N} - \mathcal{J}_{3N} - \mathcal{J}_{4N})\Phi_N. \end{aligned} \tag{5.3}$$

Here,

$$\mathcal{J}_{2N} = \int_{-h}^{0}\int_{-h}^{0} \mathbf{p}_N(s_1)\Psi(s_1-s_2)\mathbf{p}_N^T(s_2)\mathrm{d}s_2\mathrm{d}s_1,$$

$$\mathcal{J}_{3N} = \int_{-h}^{0}\left(\int_{-h}^{s_1-0} \mathbf{p}_N(s_1)D^TU''(s_1-s_2)D\mathbf{p}_N^T(s_2)\mathrm{d}s_2 + \int_{s_1+0}^{0} \mathbf{p}_N(s_1)D^TU''(s_1-s_2)D\mathbf{p}_N^T(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1,$$

$$\mathcal{J}_{4N} = \int_{-h}^{0} \mathbf{p}_N(s)D^T\Delta U'(0)D\mathbf{p}_N^T(s)\mathrm{d}s.$$

Note that, for any $\Theta(s) = \Theta^T(-s),\ s > 0$, we have

$$\int_{-h}^{0}\int_{s_1}^{0} p_i(s_1)p_j(s_2)\Theta(s_1-s_2)\mathrm{d}s_2\mathrm{d}s_1 = \int_{-h}^{0}\int_{-h}^{s_1} p_i(s_2)p_j(s_1)\Theta^T(s_1-s_2)\mathrm{d}s_2\mathrm{d}s_1.$$

Hence, $\mathcal{J}_{2N}$ can be rewritten as

$$\mathcal{J}_{2N} = \mathrm{He}\left(\int_{-h}^{0}\int_{-h}^{s_1} \mathbf{p}_N(s_1)\Psi(s_1-s_2)\mathbf{p}_N^T(s_2)\mathrm{d}s_2\mathrm{d}s_1\right),$$

In view of (5.2) and (5.3), we rewrite the approximation of the functional corresponding to the Legendre polynomial approximation (2.10) as:

$$v_0^{\text{approx}}(\varphi) = v_0(\varphi_N) \overset{\text{def}}{=} \Xi^T\Pi_N\Xi, \tag{5.4}$$

where

$$\Pi_N = \begin{bmatrix} U(0) & \mathcal{J}_{1N} \\ \star & \mathcal{J}_{2N} - \mathcal{J}_{3N} - \mathcal{J}_{4N} \end{bmatrix}, \quad \Xi = [(\varphi(0) - D\varphi(-h))^T \quad \Phi_N^T]^T.$$

Remark 10 *Unlike the integral terms $\mathcal{M}_k$, $\mathcal{N}_k$, $\mathcal{P}_l$, $\mathcal{Q}_l$ involved in the matrix Λ_N in Chapter 4, where these integrals are expressed in terms of monomials of degree equal to one, the integral terms $\mathcal{J}_{1N}$, $\mathcal{J}_{2N}$, $\mathcal{J}_{3N}$ and $\mathcal{J}_{4N}$ are now given in terms of monomials s^k, $k \in \overline{0, N-1}$, whose maximum degree is determined by the approximation order of the Legendre polynomial approximation. The recursive method of integral computations in Section 7 also allows computing the integrals $\mathcal{J}_{1N}$, $\mathcal{J}_{2N}$, $\mathcal{J}_{3N}$ and $\mathcal{J}_{4N}$.*

5.3 Estimate of the functional approximation error using the Legendre polynomial approximation

Considering *Step 3* and *4* of the general outline of proofs of Chapter 3, this section aims to provide an upper bound estimation for the approximation error $\tilde{\varphi}_N$ and the functional approximation error Υ_N.

As formulated in *Step 3*, we require the error quantification of the approximation error $\tilde{\varphi}_N$ on the set $\mathcal{S}$ in terms of the approximation order. Next, a crucial lemma on the Legendre convergence rate error is presented, which is slightly modified to the one in (Bajodek, 2022, Lemma 2.1).

Lemma 11 *The Legendre polynomial approximation error admits a bound*

$$\|\tilde{\varphi}_N\|_h \leqslant \frac{7\mu^N}{N!}, \quad \mu = \max\left(1, \tfrac{2hK}{3}\right),$$

for any function $\varphi \in \mathcal{S}$.

Proof : For $N = \overline{1,2}$, we have roughly

$$\begin{aligned} \|\tilde{\varphi}_1\|_h &= \left\| \varphi(\theta) - \tfrac{1}{h}\int_{-h}^{0} \varphi(s)\mathrm{d}s \right\|_h \leqslant 2\|\varphi\|_h = 2, \\ \|\tilde{\varphi}_2\|_h &= \left\| \varphi(\theta) - \tfrac{1}{h}\int_{-h}^{0} \varphi(s)\mathrm{d}s - \tfrac{2\theta+h}{h}\tfrac{3}{h}\int_{-h}^{0} \tfrac{2s+h}{h}\varphi(s)\mathrm{d}s \right\|_h \leqslant 3.5\|\varphi\|_h = 3.5. \end{aligned} \tag{5.5}$$

According to Wang and Xiang (2012) and Bajodek (2022, Lemma 2.1), for any $N \geqslant 3$ and $2 \leqslant k \leqslant N-1$, an upper bound of the Legendre approximation error is given by

$$\|\tilde{\varphi}_N\|_h \leqslant \frac{(hK)^{k+1}}{2^k(k-1)(N-\frac{3}{2})\dots(N-k+\frac{1}{2})}, \tag{5.6}$$

with k and K defined in the set $\mathcal{S}$. Taking $k = N-1$ and multiplying and dividing by $N!$, we have that

$$\begin{aligned} \|\tilde{\varphi}_N\|_h &\leqslant \frac{2(\frac{hK}{2})^N}{N!}\frac{N(N-1)(N-2)\dots 2}{(N-2)(N-\frac{3}{2})\dots(\frac{3}{2})} = \frac{2(\frac{hK}{2})^N}{N!}\frac{N(N-1)(N-2)}{(N-2)(N-\frac{3}{2})(N-\frac{5}{2})}\prod_{j=2}^{N-3}\frac{j}{j-\frac{1}{2}} \\ &\leqslant \frac{2(\frac{hK}{2})^N}{N!}\frac{N(N-1)}{(N-\frac{3}{2})(N-\frac{5}{2})}\prod_{j=2}^{N-3}\frac{4}{3} = \frac{2(\frac{hK}{2})^N}{N!}\frac{N(N-1)}{(N-\frac{3}{2})(N-\frac{5}{2})}\left(\frac{4}{3}\right)^{N-4} \\ &\leqslant \frac{2(\frac{hK}{2})^N}{N!}\frac{N}{(N-\frac{5}{2})}\left(\frac{4}{3}\right)^{N-3} \leqslant \frac{12(\frac{hK}{2})^N}{N!}\left(\frac{4}{3}\right)^{N-3} = \frac{\frac{81}{16}(\frac{2hK}{3})^N}{N!}. \end{aligned} \tag{5.7}$$

Finally, considering the maximum between (5.5) and (5.7), we arrive at

$$\|\tilde{\varphi}_N\|_h \leqslant \frac{7\mu^N}{N!}.$$

□

Now, we focus on bounding the functional approximation error. It follows from the functional argument approximation (5.1) and statement 1 of Lemma 7 that the functional approximation error Υ_N is bounded as

$$|\Upsilon_N| \leqslant 2\varkappa \int_{-h}^{0} \|\tilde{\varphi}_N(\theta)\|\mathrm{d}\theta + (M_2h + M_3)\int_{-h}^{0} \|\tilde{\varphi}_N(\theta)\|^2\mathrm{d}\theta,$$

where $\varkappa = (1 + \|D\|)M_1 + M_2h + M_3$. Using Lemma 11, the functional approximation error Υ_N using the Legendre polynomial approximation admits the following upper bound:

$$\begin{aligned} |\Upsilon_N| \leqslant \delta_N &= 2c_1\|\tilde{\varphi}_N\|_h + c_2\|\tilde{\varphi}_N\|_h^2 \\ &\leqslant 2c_1\frac{7\mu^N}{N!} + c_2\left(\frac{7\mu^N}{N!}\right)^2, \end{aligned} \tag{5.8}$$

$$c_1 = h\varkappa, \quad c_2 = h(hM_2 + M_3),$$

with μ defined in Lemma 11.

The following lemma plays a key role in achieving our stability criterion for neutral type systems based on Legendre approximation. It relates the approximation order N with a given upper bound of the approximation error $\tilde{\varphi}_N$.

Lemma 12 *Consider $\varphi \in \mathcal{S}$. The Legendre approximation error $\tilde{\varphi}_N$ satisfies the following inequality*

$$\|\tilde{\varphi}_N\|_h \leqslant \varepsilon, \tag{5.9}$$

$$\forall N \geqslant \mu \exp\left[1 + \mathcal{W}\left(-\frac{1}{\exp(1)\mu}\ln\left(\frac{\exp(1)\varepsilon}{7}\right)\right)\right], \quad \mu = \max\left(1, \tfrac{2hK}{3}\right),$$

where $\mathcal{W}(z)$ is the Lambert function given by $\mathcal{W} : z \to y,\ z \in \mathbb{R}_+,\ y \in \mathbb{R}_+$, which is uniquely defined by the relation $ye^y = z$. Here, $\varepsilon > 0$ is assumed small enough so that the function $\mathcal{W}$ is well defined.

Proof : It follows from Lemma 11 that the approximation error of φ considering Legendre approximations is given by

$$\|\tilde{\varphi}_N\|_h \leqslant \frac{7\mu^N}{N!}.$$

Applying the logarithm to the previous inequality, we have

$$\ln(\|\tilde{\varphi}_N\|_h) \leqslant \ln(7) + N\ln(\mu) - \sum_{j=1}^{N}\ln(j) = \ln(7) + N\ln(\mu) - \sum_{j=2}^{N}\ln(j). \tag{5.10}$$

Since the $\ln(\cdot)$ function is monotonically increasing, we obtain

$$\ln(j) \geqslant \int_{j-1}^{j}\ln(s)\mathrm{d}s, \quad j = \overline{2,N}.$$

Thus, we are able to bound inequality (5.10) as follows

$$\ln(\|\tilde{\varphi}_N\|_h) \leqslant \ln(7) + N\ln(\mu) - \int_{1}^{N}\ln(s)\mathrm{d}s = \ln(7) + N\ln(\mu) - N(\ln(N) - 1) + (\ln(1) - 1)$$
$$= \ln\left(\frac{7}{\exp(1)}\right) - N\ln\left(\frac{N}{\exp(1)\mu}\right),$$

equivalently,

$$\frac{1}{\exp(1)\mu}\ln\left(\frac{\exp(1)\|\tilde{\varphi}_N\|_h}{7}\right) \leqslant -\frac{N}{\exp(1)\mu}\ln\left(\frac{N}{\exp(1)\mu}\right).$$

Now, if N is chosen in such a way that

$$-\frac{N}{\exp(1)\mu}\ln\left(\frac{N}{\exp(1)\mu}\right) \leqslant \frac{1}{\exp(1)\mu}\ln\left(\frac{\exp(1)\varepsilon}{7}\right),$$

then this implies that $\|\tilde{\varphi}_N\|_h \leqslant \varepsilon$. Denoting $y = \ln\left(\frac{N}{\exp(1)\mu}\right)$, the previous relation can be written as

$$-ye^y \leqslant \frac{1}{\exp(1)\mu}\ln\left(\frac{\exp(1)\varepsilon}{7}\right),$$

which is solved by using the Lambert function given by

$$y = \ln\left(\frac{N}{\exp(1)\mu}\right) \geqslant \mathcal{W}\left(-\frac{1}{\exp(1)\mu}\ln\left(\frac{\exp(1)\varepsilon}{7}\right)\right).$$

Here, we assume that ε is small enough so that the argument of $\mathcal{W}$ is positive. Hence, if

$$N \geqslant \mu \exp\left[1 + \mathcal{W}\left(-\frac{1}{\exp(1)\mu}\ln\left(\frac{\exp(1)\varepsilon}{7}\right)\right)\right],$$

then $\|\tilde{\varphi}_N\|_h \leqslant \varepsilon$. □

Remark 11 *Notice that Lemma 12 is particularly satisfied with* $\varepsilon \leqslant \frac{7}{\exp(1)}$*, ensuring that the argument of function* $\mathcal{W}$ *is non-negative definite. Also, observe that if* $N \geqslant \mu\exp(1)$*, then* $\|\tilde{\varphi}_h\| \leqslant \varepsilon = \frac{7}{\exp(1)}$*.*

5.4 Stability criterion based on the Legendre polynomials approximation

Summarizing, we have proved that the substitution of the Legendre polynomial approximation of φ into functional (2.10) leads to a quadratic form (5.4) characterized by a constant block matrix and that the functional approximation error Υ_N admits the upper bound (5.8).

Next, the approximated functional (5.4) and the instability result in Lemma 3 allow us to present the main result of this chapter, a stability criterion for neutral type systems based on the Legendre polynomials approximation. This criterion relies on verifying the non-negativity of matrix Π_N for a given tractable value of N.

Theorem 5 *System* (2.1) *is exponentially stable, if and only if the Lyapunov condition holds and the matrix*

$$\Pi_{N_1} \geq 0,$$

where,

$$N_1 = \left\lceil \mu \exp\left[1 + \mathcal{W}\left(-\frac{1}{\exp(1)\mu}\ln\left(\frac{\exp(1)\hat{\mu}}{7}\right)\right)\right]\right\rceil,$$
$$\mathcal{E}(a_0) = -\frac{c_1}{c_2} + \sqrt{\left(\frac{c_1}{c_2}\right)^2 + \frac{a_0}{c_2}}, \quad \hat{\mu} = \min\left(\mathcal{E}(a_0), \frac{7}{\exp(1)}\right). \tag{5.11}$$

with c_1*,* c_2 *and* μ *defined in Lemma 12 and* a_0 *determined in Lemma 3.*

Proof : *Necessity.* Since system (2.1) is exponentially stable, it is also stable, thus by the construction of the functional, we have

$$v_0(\varphi) = \int_0^{+\infty} x^T(t,\varphi)Wx(t,\varphi)\mathrm{d}t \geqslant 0,$$

for any $\varphi \in \mathcal{PC}^1\big([-h,0],\mathbb{R}^n\big)$. In particular, for the Legendre polynomial approximation $\varphi_N : [-h,0] \to \mathbb{R}^n$ given by (5.1), it follows from (5.4) that:

$$v_0(\varphi_N) = \Xi^\top \Pi_N \Xi \geqslant 0,$$

for any vector Ξ and any approximation order N, which implies that $\Pi_N \geq 0$.

Sufficiency. By contradiction, assume that system (2.1) is unstable but that Π_{N_1} is non-negative definite. Moreover, recall that $v_0(\varphi)$ can be expressed as:

$$v_0(\varphi) = v_0(\varphi_{N_1}) + \Upsilon_{N_1}.$$

By Lemma 3, there exists $\widetilde{\varphi} \in \mathcal{S}$ such that $v_0(\widetilde{\varphi}) < -a_0$, which implies that

$$v_0(\widetilde{\varphi}_{N_1}) = v_0(\widetilde{\varphi}) - \Upsilon_{N_1} < -a_0 + |\Upsilon_{N_1}|.$$

Now, consider the equation

$$0 = a_0 - 2c_1\|\tilde{\varphi}_N\|_h - c_2\|\tilde{\varphi}_N\|_h^2.$$

It is straightforward to see that its unique positive root is given by:

$$\|\tilde{\varphi}_N\|_h = \mathcal{E}(a_0) = -\frac{c_1}{c_2} + \sqrt{\left(\frac{c_1}{c_2}\right)^2 + \frac{a_0}{c_2}}.$$

Then, if we take $N = N_1$ and consider $\hat{\mu} = \min\left(\mathcal{E}(a_0), \frac{7}{\exp(1)}\right)$ so that the Lambert function $\mathcal{W}$ is well-defined, it implies that $\|\tilde{\varphi}_{N_1}\|_h \leqslant \hat{\mu}$ and consequently, $|\Upsilon_{N_1}| \leqslant a_0$. Hence,

$$v_0(\tilde{\varphi}_{N_1}) < -a_0 + a_0 = 0,$$

which contradicts $\Pi_{N_1} \geq 0$. □

5.5 Conclusions

This chapter is dedicated to approximating the functional argument using Legendre polynomials projections, extending the ideas introduced in Bajodek et al. (2023) for retarded type systems. A stability criterion for neutral-type linear is formulated. This result benefits from the super-geometric convergence property of Legendre polynomials approximation, which allows the reduction of the numerical complexity of the criterion. It is due to the presence of logarithms in the computation of the approximation order for sufficiency.

For Legendre approximation of the functional argument, the numerical computation of integrals $\mathcal{J}_{1N}$, $\mathcal{J}_{2N}$, $\mathcal{J}_{3N}$ and $\mathcal{J}_{4N}$ is more demanding than for the piecewise case. Notice that matrix Λ_N in the piecewise linear approximation approach incorporates integrals of the delay Lyapunov matrix multiplied by monomials up to the third degree, contrary to the case of Π_N in the Legendre polynomial approach, where the involved integrals depend on the delay Lyapunov matrix multiplied by Legendre polynomials of N-order. A recursive approach for this case is presented in Chapter 7.

Chapter 6

Stability test for neutral-type delay systems: A discretized Lyapunov functional method

In this chapter, the functional kernels approximation scheme of Gu (1997) recalled in Section 2.5 is applied to functional with prescribed derivative for neutral type systems. An interesting feature of this approach is that it achieves sufficient stability conditions depending on discrete values of the functional kernels (see Theorem 2). It is worth noticing that, for Lyapunov functionals with prescribed derivative, the functional kernels take the form of the delay Lyapunov matrix that can be computed through the semi-analytic method (2.7). Then, the computation of discretized values of the delay Lyapunov matrix is possible. This suggests that a similar criterion to Theorem 3, given in terms of the delay Lyapunov matrix, is feasible without the need for integrals of the delay Lyapunov matrix as in Theorem 4 and Theorem 5.

The above observations are indeed developed in Belov and Alexandrova (2022) and Alexandrova and Belov (2024) for retarded type systems in the Lyapunov functional with prescribed derivative framework. As a result, the obtained stability criterion combines the elegant structure of the delay Lyapunov matrix-based stability conditions of Lemma 5 with approximation orders comparable to those in Alexandrova (2023); Bajodek et al. (2023). In this chapter, we extend the result of Alexandrova and Belov (2024) to the case of neutral type systems.

This chapter is organized as follows. In Section 6.1, functional (2.9) is discretized considering a piecewise linear approximation of its kernels. We also present a key lemma, relating the quadratic form of the discretized functional to the necessary conditions of Theorem 3. The quantification of the functional approximation error over the special set $\mathcal{S}$ of bounded functions is developed in Section 6.2. The main result of this chapter, a stability criterion expressed through the positive definiteness of a block matrix based on discrete evaluations of the delay Lyapunov matrix, is addressed in Section 6.3. Finally, we conclude with some remarks and conclusions in Section 6.4.

6.1 Functional kernels approximation: functional $v_0(\varphi)$

Here, the discretized Lyapunov functional method proposed in Gu (1997) and reminded for the neutral case in Section 2.5 is applied to functional (2.9), which has a structure similar to the retarded case structure and admits a quadratic lower bound on a special set of functions as proved for the retarded case in Medvedeva and Zhabko (2015). It is worth emphasizing that neither the discretization of the functional derivative nor an additional positive term like those of the complete type functional (Kharitonov & Zhabko, 2003) is required in this approach.

To this aim, discretize the interval $[-h, 0]$ by equidistant points $\theta_i = -i\tau,\ i = \overline{0, N}$, where $\tau = h/N$, and introduce

$$Q_i = Q(\theta_i) = U^T((N-i)\tau) - D^T U(i\tau),$$
$$R_{ij} = R(\theta_i, \theta_j) = U((j-i)\tau), \quad i, j = \overline{0, N}.$$

Following *Step 1* of the general outline of Chapter 3, we consider a representation for the functional kernels as follows:

$$Q(\theta) = Q^N(\theta) + \tilde{Q}^N(\theta), \quad \theta \in [-h, 0],$$
$$R(\theta_1, \theta_2) = R^N(\theta_1, \theta_2) + \tilde{R}^N(\theta_1, \theta_2), \quad \theta_1 \in [-h, 0], \quad \theta_2 \in [-h, 0]$$

where $Q^N(s)$ and $R^N(s_1, s_2)$ denote the approximation of order N of $Q(s)$ and $R(s_1, s_2)$, respectively, and $\tilde{Q}^N$ and $\tilde{R}^N$ stand for their corresponding approximation errors. As shown in Section 2.5, the method relies on approximating the functional kernels $Q(s)$ and $R(s_1, s_2)$ by piecewise linear matrix functions Q^N and R^N given by the following formulae:

$$\begin{aligned} Q^N(s+\theta_i) &= Q_i + \big(Q_{i-1} - Q_i\big)\frac{s}{\tau}, \quad s \in [0, \tau], \\ R^N(s_1+\theta_i, s_2+\theta_j) &= R_{ij} + (R_{i,j-1} - R_{ij})\frac{s_2 - s_1}{\tau}, \quad 0 \leqslant s_1 \leqslant s_2 \leqslant \tau, \\ R^N(s_1+\theta_i, s_2+\theta_j) &= R_{ij} + (R_{i-1,j} - R_{ij})\frac{s_1 - s_2}{\tau}, \quad 0 \leqslant s_2 \leqslant s_1 \leqslant \tau, \end{aligned} \tag{6.1}$$

$i, j = \overline{1, N}$. Here, the expressions are simplified thanks to $R_{ij} = R_{i-1,j-1}$ in comparison with more general expressions in (2.17). Considering *Step 2* of the general outline of Chapter 3, we replace the kernels in functional (2.9) by their piecewise linear approximations, arriving at:

$$v_0^{(N)}(\varphi) = \varphi^T(0)P\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} Q^N(s)\gamma(s)\mathrm{d}s + \int_{-h}^{0}\int_{-h}^{0} \gamma^T(s_1)R^N(s_1, s_2)\gamma(s_2)\mathrm{d}s_2\mathrm{d}s_1, \tag{6.2}$$

where $\gamma(s) = D\varphi'(s) + A_1\varphi(s)$. Furthermore, following the same steps as in Section 2.5, we conclude that for any $\varphi \in \mathcal{C}^1$, the approximated functional (6.2) admits a representation (Gu, 1997; Han, 2005)

$$v_0^{\text{approx}}(\varphi) = v_0^{(N)}(\varphi) = \frac{1}{\tau}\int_0^{\tau} \begin{pmatrix}\varphi^T(0) & \Psi^T(s)\end{pmatrix} \mathcal{A}_N \begin{pmatrix}\varphi(0) \\ \Psi(s)\end{pmatrix} \mathrm{d}s. \tag{6.3}$$

Here,

$$\begin{aligned} \mathcal{A}_N &= \begin{pmatrix} P & \mathcal{Q}_N \\ \mathcal{Q}_N^T & \mathcal{K}_N \end{pmatrix}, \\ \mathcal{Q}_N &= \Big(Q_0, Q_1, \cdots, Q_N\Big) \\ &= \Big(U^T(N\tau) - D^T U(0), \cdots, U^T(\tau) - D^T U((N-1)\tau), U(0) - D^T U(N\tau)\Big), \\ \mathcal{K}_N &= \Big\{R_{ij}\Big\}_{i,j=0}^{N} = \Big\{U((j-i)\tau)\Big\}_{i,j=0}^{N}, \\ \Psi(s) &= \Big(\psi_0^T(s), \psi_1^T(s), \ldots, \psi_N^T(s)\Big)^T, \end{aligned}$$

$$\psi_0(s) = \int\limits_{\theta_1+s}^{0} \gamma(\xi)\mathrm{d}\xi, \quad \psi_N(s) = \int\limits_{\theta_N}^{\theta_N+s} \gamma(\xi)\mathrm{d}\xi, \quad \psi_i(s) = \int\limits_{\theta_{i+1}+s}^{\theta_i+s} \gamma(\xi)\mathrm{d}\xi, \quad i = \overline{1, N-1}.$$

Remark 12 *We would like to highlight that the quadratic form* (6.3) *represents an exact equality, as opposed to the lower bound obtained in* (2.19)*, which arose from majorizations. This precise form is derived from the exact functional* (2.9)*. An important implication of this is that there is no need to discretize the functional derivative* (2.8).

Noticing that matrix $\mathcal{K}_N$ in Lemma 5 is a block of matrix $\mathcal{A}_N$, the next lemma aims to prove the direct connection between matrix $\mathcal{A}_N$ and matrix $\mathcal{K}_N$. The result is achieved using the Schur complement.

Lemma 13 *Given* $N \in \mathbb{N}$, $\mathcal{K}_N > 0$ *implies* $\mathcal{A}_N \geq 0$.

Proof : Let us decompose matrices $\mathcal{K}_N$ and $\mathcal{A}_N$ as follows:

$$\mathcal{K}_N = \begin{pmatrix} \mathcal{K}_{N-1} & \mathcal{B}_N \\ \mathcal{B}_N^T & U(0) \end{pmatrix},$$

$$\mathcal{A}_N = \begin{pmatrix} P & \mathcal{Q}_N \\ \mathcal{Q}_N^T & \mathcal{K}_N \end{pmatrix} = \begin{pmatrix} P & \widetilde{\mathcal{Q}}_N & \mathcal{C}_N \\ \widetilde{\mathcal{Q}}_N^T & \mathcal{K}_{N-1} & \mathcal{B}_N \\ \mathcal{C}_N^T & \mathcal{B}_N^T & U(0) \end{pmatrix},$$

where

$$\mathcal{K}_{N-1} = \Big\{U((j-i)\tau)\Big\}_{i,j=0}^{N-1}, \; \mathcal{B}_N = \Big(U^T(N\tau), \cdots, U^T(\tau)\Big)^T,$$
$$\widetilde{\mathcal{Q}}_N = \Big(Q_0, \cdots, Q_{N-1}\Big)$$
$$= \Big(U^T(N\tau) - D^T U(0), \cdots, U^T(\tau) - D^T U((N-1)\tau)\Big),$$
$$\mathcal{C}_N = U(0) - D^T U(N\tau).$$

First, $\mathcal{K}_N > 0$ implies $U(0) > 0$ and, by the Schur complement,

$$\mathcal{T} = \mathcal{K}_{N-1} - \mathcal{B}_N U^{-1}(0) \mathcal{B}_N^T > 0.$$

Here, $\mathcal{T} = \Big\{\mathcal{T}_{ij}\Big\}_{i,j=0}^{N-1}$ with

$$\mathcal{T}_{ij} = U((j-i)\tau) - U((N-i)\tau)U^{-1}(0)U^T((N-j)\tau).$$

Now, consider the matrix $\mathcal{A}_N$ and construct the Schur complement $\mathcal{P}$ of the block $U(0) > 0$ in (6.4):

$$\mathcal{P} = \begin{pmatrix} P & \widetilde{\mathcal{Q}}_N \\ \widetilde{\mathcal{Q}}_N^T & \mathcal{K}_{N-1} \end{pmatrix} - \begin{pmatrix} \mathcal{C}_N \\ \mathcal{B}_N \end{pmatrix} U^{-1}(0) \begin{pmatrix} \mathcal{C}_N^T & \mathcal{B}_N^T \end{pmatrix} = \begin{pmatrix} \mathcal{D}^T \mathcal{T} \mathcal{D} & -\mathcal{D}^T \mathcal{T} \\ -\mathcal{T}\mathcal{D} & \mathcal{T} \end{pmatrix},$$
$$\mathcal{D} = \Big(D^T \; \mathbb{O}_n \; \cdots \; \mathbb{O}_n\Big)^T.$$

Next, the Schur complement of block $\mathcal{T} > 0$ of matrix $\mathcal{P}$ is

$$\mathcal{D}^T \mathcal{T} \mathcal{D} - \mathcal{D}^T \mathcal{T} \mathcal{T}^{-1} \mathcal{T} \mathcal{D} = \mathbb{O}_{nN}.$$

This implies $\mathcal{P} \geq 0$ and consequently, $\mathcal{A}_N \geq 0$. □

Remark 13 *It follows from Lemmas 5 and 13 that if system* (2.1) *is exponentially stable, then* $\mathcal{K}_N > 0$ *and consequently, the discretized functional* (6.3) *is nonnegative for any* $N \in \mathbb{N}$.

6.2 Estimate of the functional approximation error Υ_N

In this instance, a quadratic form for functional (2.9) is obtained via piecewise linear approximation for its kernels functional. Now, let us turn our attention to seeking an estimate for the functional approximation error

$$\Upsilon_N = v_0(\varphi) - v_0^{(N)}(\varphi), \quad \varphi \in \mathcal{S}.$$

The developments rely on the following technical lemma related to *Step 3* of the general outline, which estimates the error between a twice-differentiable matrix function and its linear approximation on the basis of Taylor formula with integral remainder:

Lemma 14 *(Alexandrova & Belov, 2024) Given a matrix function $G \in \mathcal{C}^2_{[0,\tau]}$ and the corresponding linear matrix function*

$$G^{\text{lin}}(s) = G(0) + \Big(G(\tau) - G(0)\Big)\frac{s}{\tau},$$

the following bound holds true:

$$\|G(s) - G^{\text{lin}}(s)\| \leqslant G''_{\max}\frac{s(\tau - s)}{2}, \quad s \in [0,\tau],$$

$$G''_{\max} = \max_{s\in[0,\tau]} \|G''(s)\|.$$

Following *Step 4* of the general outline of Chapter 3, it stems from Lemma 9 that the functional approximation error admits the following estimate:

$$|\Upsilon_N| \leqslant 2\|\varphi\|_h\|\gamma\|_h \int_{-h}^{0} \left\|Q(\xi) - Q^N(\xi)\right\| \mathrm{d}\xi + \|\gamma\|_h^2 \int_{-h}^{0}\int_{-h}^{0} \|R(\xi_1,\xi_2) - R^N(\xi_1,\xi_2)\|\mathrm{d}\xi_2\mathrm{d}\xi_1. \quad (6.4)$$

Now, let us bound the first integral in (6.4):

$$I_2 = \int_{-h}^{0} \left\|Q(\xi) - Q^N(\xi)\right\| \mathrm{d}\xi = \sum_{j=1}^{N} \int_{\theta_j}^{\theta_{j-1}} \left\|Q(\xi) - Q^N(\xi)\right\| \mathrm{d}\xi$$
$$= \sum_{j=1}^{N} \int_0^{\tau} \left\|Q(s+\theta_j) - Q^N(s+\theta_j)\right\| \mathrm{d}s.$$

Note that $s + (N-j)\tau \in [0,h]$, $j\tau - s \in [0,h]$ for any $j = \overline{1,N}$ and $s \in [0,\tau]$. Therefore, according to Remark 2 in Chapter 2,

$$Q(s+\theta_j) = U^T(s + (N-j)\tau) - D^T U(-s + j\tau) \in \mathcal{C}^2_{[0,\tau]},$$

considering the corresponding one-sided derivatives of U at 0 and h. Hence, it is possible to apply Lemma 14 with $G(s) = Q(s+\theta_j)$ and

$$G''_{\max} \leqslant \widetilde{M_1} \overset{\text{def}}{=} \max_{s\in[0,h]} \|U''^T(h-s) - D^T U''(s)\|$$

for any j. Finally,

$$I_2 \leqslant \widetilde{M}_1 \sum_{j=1}^{N} \int_0^{\tau} \frac{s(\tau - s)}{2} \mathrm{d}s = \frac{\widetilde{M}_1 h^3}{12N^2}.$$

Let us now move to the second integral in (6.4):

$$\begin{aligned} I_3 &= \int_{-h}^{0}\int_{-h}^{0} \|R(\xi_1,\xi_2) - R^N(\xi_1,\xi_2)\| \mathrm{d}\xi_2 \mathrm{d}\xi_1 \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^{\tau}\int_0^{\tau} \|R(s_1+\theta_i, s_2+\theta_j) - R^N(s_1+\theta_i, s_2+\theta_j)\| \mathrm{d}s_2 \mathrm{d}s_1. \end{aligned}$$

To bound this expression, introduce first matrix functions $G_k(s) = U(s + k\tau)$, and the corresponding linear functions

$$G_k^{\mathrm{lin}}(s) = U(k\tau) + \Big(U((k+1)\tau) - U(k\tau)\Big)\frac{s}{\tau},\ s \in [0,\tau],$$

for $k = \overline{-N, N-1}$. Notice that, by the symmetry property (2.4),

$$R(s_1+\theta_i, s_2+\theta_j) = U(s_1 - s_2 + (j-i)\tau) = \begin{cases} G_{j-i}(s_1 - s_2), & 0 \leqslant s_2 \leqslant s_1 \leqslant \tau \\ G_{i-j}^T(s_2 - s_1), & 0 \leqslant s_1 \leqslant s_2 \leqslant \tau. \end{cases}$$

Similarly, substituting expressions for R_{ij}, $R_{i-1,j}$, $R_{i,j-1}$ in (2.17), we get

$$R^N(s_1+\theta_i, s_2+\theta_j) = \begin{cases} G_{j-i}^{\mathrm{lin}}(s_1 - s_2), & 0 \leqslant s_2 \leqslant s_1 \leqslant \tau \\ [G_{i-j}^{\mathrm{lin}}(s_2 - s_1)]^T, & 0 \leqslant s_1 \leqslant s_2 \leqslant \tau. \end{cases}$$

This directly implies

$$\begin{aligned} I_3 &\leqslant \sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^{\tau}\int_0^{s_1} \|G_{j-i}(s_1 - s_2) - G_{j-i}^{\mathrm{lin}}(s_1 - s_2)\| \mathrm{d}s_2 \mathrm{d}s_1 \\ &\quad + \sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^{\tau}\int_{s_1}^{\tau} \|G_{i-j}^T(s_2 - s_1) - [G_{i-j}^{\mathrm{lin}}(s_2 - s_1)]^T\| \mathrm{d}s_2 \mathrm{d}s_1 \\ &= 2\sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^{\tau}\int_0^{s_1} \|G_{j-i}(s) - G_{j-i}^{\mathrm{lin}}(s)\| \mathrm{d}s \mathrm{d}s_1. \end{aligned}$$

Let us pay attention to the fact that $G_k \in \mathcal{C}^2_{[0,\tau]}$ for any fixed k from $-N$ to $N-1$ (Rodriguez et al., 2004), considering the corresponding right and left derivatives at the end-points. Indeed, the only discontinuity points of the second derivative of U on $[-h, h]$ are 0 and $\pm h$, and they can only be the end-points of $[k\tau, (k+1)\tau]$ for any k. Hence, Lemma 14 can be applied:

$$I_3 \leqslant 2\sum_{i=1}^{N}\sum_{j=1}^{N} \int_0^{\tau}\int_0^{s_1} \frac{M_2 s(\tau - s)}{2} \mathrm{d}s \mathrm{d}s_1 = \frac{\widetilde{M}_2 h^4}{12N^2}$$

with

$$\widetilde{M}_2 \stackrel{\mathrm{def}}{=} \max_{s\in[0,h]} \|U''(s)\| = \max_{s\in[-h,0]} \|U''(s)\|.$$

The last equality is due to the symmetry property (2.4). Collecting the bounds for I_2 and I_3, we finally arrive at

$$|\Upsilon_N| \leqslant \frac{h^3}{12N^2}\left(2\widetilde{M}_1\|\varphi\|_h\|\gamma\|_h + h\widetilde{M}_2\|\gamma\|_h^2\right). \tag{6.5}$$

The following lemma completes the developments of this section.

Lemma 15 *Given $\varphi \in \mathcal{S}$, the functional approximation error admits the following bound for any $N \in \mathbb{N}$:*

$$\left|\Upsilon_N\right| = \left|v_0(\varphi) - v_0^{(N)}(\varphi)\right| \leqslant \delta_N = \frac{c}{N^2}. \tag{6.6}$$

Here,

$$c = \frac{h^3}{12}\mu\left(2\widetilde{M}_1 + h\widetilde{M}_2\mu\right), \quad \mu = \|D\|K + \|A_1\|,$$
$$\widetilde{M}_1 = \max_{s\in[0,h]}\|U''(h-s) - U''^T(s)D\|,\ \widetilde{M}_2 = \max_{s\in[0,h]}\|U''(s)\|$$

considering the right-hand-side and left-hand-side derivatives of the delay Lyapunov matrix at 0 *and h, respectively.*

Proof : The result follows directly from bound (6.5) having in mind that $\varphi \in \mathcal{S}$ implies that $\|\varphi\|_h = 1$ and $\|\gamma\|_h \leqslant \mu$. □

6.3 Stability criterion based on a discretized functional approach

This section presents our main result, a finite delay Lyapunov matrix-based stability criterion for neutral type systems. It follows from *Step 5* of the general outline of Chapter 3 that the error tolerance δ_N equal to the value a_0 defined in Lemma 3 allows computing the approximation order such that the non-negativity of the approximated functional can be verified. This approximation order and the connection between matrix $\mathcal{A}_N$ in (6.3) and the matrix $\mathcal{K}_N$ of Lemma 5 allow delivering the same stability criterion of Theorem 3 with reduced approximation orders for sufficiency.

Theorem 6 *System* (2.1) *is exponentially stable, if and only if the Lyapunov condition holds, D is a Schur stable matrix, and*

$$\mathcal{K}_{N_2} = \left\{U((j-i)\tau)\right\}_{i,j=0}^{N_2} > 0,$$

where $N_2 = \left\lceil \sqrt{\dfrac{c}{a_0}}\right\rceil$, *with a_0 and c determined in Lemma 3 and 15, respectively.*

Proof : *Necessity.* The exponential stability of system (2.1) implies the Lyapunov condition (Kharitonov, 2013), the Schur stability of D, and, by Lemma 5, $\mathcal{K}_N > 0$ for any N, in particular, $N = N_2$.

Sufficiency. By contradiction, assume that system (2.1) is unstable but $\mathcal{K}_{N_2} > 0$. On the one hand, by definition of Υ_N,

$$v_0^{(N_2)}(\varphi) = v_0(\varphi) - \Upsilon_{N_2}.$$

By Lemma 3, there exists $\widetilde{\varphi} \in \mathcal{S} \subset \mathcal{C}^1$ such that $v_0(\widetilde{\varphi}) < -a_0$, thus Lemma 15 implies that

$$v_0^{(N_2)}(\widetilde{\varphi}) = v_0(\widetilde{\varphi}) - \Upsilon_{N_2} < -a_0 + \frac{c}{(N_2)^2}.$$

Observe that the value $N_2 = \left\lceil \sqrt{\frac{c}{a_0}} \right\rceil$ is precisely selected so that the last expression is non-positive, that is, $v_0^{(N_2)}(\widetilde{\varphi}) < 0$.

On the other hand, by Lemma 13, $\mathcal{K}_{N_2} > 0$ implies that $\mathcal{A}_{N_2} \geq 0$. Consequently, expression (6.3) gives $v_0^{(N_2)}(\varphi) \geqslant 0$ for any $\varphi \in \mathcal{C}^1$, which leads us to a contradiction. □

Remark 14 *Although the stability criterion of Theorem 6 has exactly the same matrix structure as in Theorem 3, the parameter N_2 is calculated by a completely different approach which yields a considerable reduction of the matrix dimension. In contrast to Theorem 4 and Theorem 5, where the dimension is reduced at the cost of complicating the matrix structure, we managed to keep a nice matrix structure of Theorem 3 which is made possible thanks to Lemma 13.*

6.4 Conclusions

This chapter delivers a stability criterion for neutral type time delay systems based on verifying the positive definiteness of a matrix in a tractable number of operations. It is shown that the discretization of Lyapunov functionals with prescribed derivative through the discretized Lyapunov functional method in Section 2.5 and ideas of Alexandrova and Belov (2024) arrives at the same elegant criterion form of Theorem 3 with a significantly reduced matrix dimension. The use of functional (2.9) is the key factor for achieving the dimension reduction. Notice that this last approach completely avoids the computations of integrals of the delay Lyapunov matrix and its derivatives.

Chapter 7

Recursive computation of integrals

This section is devoted to the development of effective tools for the computation of the integrals involved in the determination of matrices Λ_N and Π_N defined in (4.5) and (5.4), respectively, which are required to carry out the stability test presented in the previous section. The numerical burden linked to the computation of integrals is heavy and can be highly detrimental to the proper testing of the condition. Fortunately, if the matrix

$$\mathcal{L} = \begin{pmatrix} I_{n^2} & -I_n \otimes D \\ -D^T \otimes I_n & I_{n^2} \end{pmatrix}^{-1} \begin{pmatrix} I_n \otimes A_0 & I_n \otimes A_1 \\ -A_1^T \otimes I_n & -A_0^T \otimes I_n \end{pmatrix},$$

involved in (2.7) for the computation of the Lyapunov matrix is non-singular, it is possible to develop a recursive numerical scheme that makes this task much more efficient. To do so, the dynamic property (2.3) is discretized and integrated, resulting in a recursivity. A semi-analytic procedure similar to the one employed in the construction of the Lyapunov matrix is then developed.

Next, the involved integrals in Λ_N and Π_N are addressed separately to clarify the slight difference in each recursive method.

7.1 Recursive computation of the integrals in piecewise linear approach

Observe that the blocks of matrix Λ_N defined in (4.5) are determined by the following integrals:

$$J_l = \int_{-\tau}^{0} U(s+l\tau)\mathrm{d}s, \quad F_l = \int_{-\tau}^{0} (s+\tau)U(s+l\tau)\mathrm{d}s, \quad \bar{F}_l = \int_{-\tau}^{0} (-s)U(s+l\tau)\mathrm{d}s, \quad l = \overline{0,N},$$

$$Y_k^{(1)} = \int_{-\tau}^{0}\int_{-\tau}^{0} s_1 U(s_1 - s_2 + k\tau)\mathrm{d}s_2\mathrm{d}s_1, \quad Y_k^{(2)} = \int_{-\tau}^{0}\int_{-\tau}^{0} s_2 U(s_1 - s_2 + k\tau)\mathrm{d}s_2\mathrm{d}s_1,$$

$$Z_k = \int_{-\tau}^{0}\int_{-\tau}^{0} s_1 s_2 U(s_1 - s_2 + k\tau)\mathrm{d}s_2\mathrm{d}s_1 = \int_{-\tau}^{0}\int_{-\tau}^{0} (s_1+\tau)(s_2+\tau)U(s_1 - s_2 + k\tau)\mathrm{d}s_2\mathrm{d}s_1,$$

$k = \overline{0, N-1}$, and similar integrals involving the first and second derivatives of the Lyapunov matrix. Notice that $Y_k^{(1)} = Y_{-k}^{(2)T}$, and it is easy to verify equality of two representations of Z_k with the change of variables $s_1 + \tau = -s_2'$, $s_2 + \tau = -s_1'$.

Let us define the vectors

$$u_l = \mathrm{vec}(U(l\tau)),\ a_l = \mathrm{vec}(J_l),\ f_l = \mathrm{vec}(F_l),$$
$$\bar{f}_l = \mathrm{vec}(\bar{F}_l),\ y_k^{(j)} = \mathrm{vec}(Y_k^{(j)}),\ j = 1,2,\ z_k = \mathrm{vec}(Z_k),$$

and operation $\star$ which means vectorization of a transposed matrix, i.e. $u_l^\star = \mathrm{vec}(U^T(l\tau))$, $a_l^\star = \mathrm{vec}(J_l^T)$, etc.

The following propositions address the computation of the integrals.

Proposition 1 *If* $\det(\mathcal{L}) \neq 0$ *and the Lyapunov condition in Lemma 2 holds then*

i) a_k, f_k *and* $\bar{f}_k$ *satisfy*

$$\begin{pmatrix} a_k \\ a_{N-k+1}^\star \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} u_k - u_{k-1} \\ u_{N-k}^\star - u_{N-k+1}^\star \end{pmatrix},$$
$$\begin{pmatrix} f_k \\ \bar{f}_{N-k+1}^\star \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \tau u_k - a_k \\ \tau u_{N-k}^\star - a_{N-k+1}^\star \end{pmatrix}, \quad k = \overline{1, N},$$

ii) $y_k^{(1)}$, $y_k^{(2)}$, *and* z_k *satisfy*

$$\begin{pmatrix} y_k^{(1)} \\ y_{N-k}^{(2)\star} \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \bar{f}_k - \bar{f}_{k+1} \\ f_{N-k+1}^\star - f_{N-k}^\star. \end{pmatrix},$$
$$\begin{pmatrix} z_k \\ z_{N-k}^\star \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \tau \bar{f}_{k+1} + y_k^{(1)} \\ \tau f_{N-k}^\star + y_{N-k}^{(2)\star} \end{pmatrix}, \quad k = \overline{1, N-1}.$$

Proof : *i)* To compute a_k, $k = \overline{1, N}$, notice that integrating the dynamic property (2.3) with $\tau = s + k\tau$, $s \in [-\tau, 0]$, yields

$$\int_{-\tau}^{0} U'(s + k\tau)\mathrm{d}s - \int_{-\tau}^{0} U'(s + (k - N)\tau)\mathrm{d}sD = \int_{-\tau}^{0} U(s + k\tau)\mathrm{d}sA_0 + \int_{-\tau}^{0} U(s + (k - N)\tau)\mathrm{d}sA_1$$

for $k = \overline{1, N}$. Hence,

$$J_k A_0 + J_{N-k+1}^T A_1 = U(k\tau) - U((k-1)\tau) - U^T((N-k)\tau)D + U^T((N-k+1)\tau)D, \quad k = \overline{1, N}.$$

Notice that this equation can be transformed to

$$-A_1^T J_k - A_0^T J_{N-k+1}^T = U^T((N-k)\tau) - U^T((N-k+1)\tau) - D^T U(k\tau) + D^T U((k-1)\tau), \quad k = \overline{1, N}.$$

By combining the last two equations and using vectorization, we get

$$\begin{pmatrix} I_n \otimes A_0 & I_n \otimes A_1 \\ -A_1^T \otimes I_n & -A_0^T \otimes I_n \end{pmatrix} \begin{pmatrix} a_k \\ a_{N-k+1}^\star \end{pmatrix} = \begin{pmatrix} I_{n^2} & -I_n \otimes D \\ -D^T \otimes I_n & I_{n^2} \end{pmatrix} \begin{pmatrix} u_k - u_{k-1} \\ u_{N-k}^\star - u_{N-k+1}^\star \end{pmatrix}, \; k = \overline{1, N},$$

as required. To tackle the computation of f_k and g_k we first multiply the same dynamic property by $(s + \tau)$. Then, integrating, we obtain

$$\tau U(k\tau) - J_k - \tau U^T((N-k)\tau)D + J_{N-k+1}^T D = F_k A_0 + \bar{F}_{N-k+1}^T A_1. \tag{7.1}$$

Similarly, changing k by $N - k + 1$ in the dynamic property, and multiplying by $(-s)$, implies

$$\tau U^T((N-k)\tau) - J_{N-k+1}^T - \tau D^T U(k\tau) + D^T J_k = -A_0^T \bar{F}_{N-k+1}^T - A_1^T F_k. \tag{7.2}$$

Now, vectorizing the system (7.1)–(7.2), gives the desired expression for $(f_k^T, \bar{f}_{N-k+1}^{\star T})^T$, since $\det(\mathcal{L}) \neq 0$.

ii) Let us prove the second equation. To this end, write the dynamic property for $\tau =$

$s_1 - s_2 + k\tau \geqslant 0,\ s_1, s_2 \in [-\tau, 0],\ k = \overline{1, N-1}$, in the form

$$s_1 s_2 \frac{\partial U(s_1 - s_2 + k\tau)}{\partial s_2} - s_1 s_2 \frac{\partial U(s_1 - s_2 + (k-N)\tau)}{\partial s_2} D$$
$$= -s_1 s_2 U(s_1 - s_2 + k\tau) A_0 - s_1 s_2 U(s_1 - s_2 + (k-N)\tau) A_1. \quad (7.3)$$

Integrating it successively with respect to s_2 and s_1, we get

$$\tau \bar{F}_{k+1} + Y_k^{(1)} - \tau F_{N-k}^T D - Y_{N-k}^{(2)T} D = Z_k A_0 + Z_{N-k}^T A_1.$$

Similarly, multiplying by $(s_1 + \tau)(s_2 + \tau)$ instead of $s_1 s_2$ in (7.3), replacing k with $N - k$ and then integrating we arrive at the equation

$$\tau F_{N-k}^T + Y_{N-k}^{(2)T} - \tau D^T \bar{F}_{k+1} - D^T Y_k^{(1)} = -A_1^T Z_k - A_0^T Z_{N-k}^T.$$

Combining the two equations for Z_k and Z_{N-k}^T and using vectorization, we get the result. The proof of the first equation is carried out following similar steps. □

For the next proposition, we introduce the auxiliary terms

$$X^l = \int_{-\tau}^{0} (s+\tau)^l U(s+\tau) \mathrm{d}s, \quad \bar{X}^l = \int_{-\tau}^{0} s^l U(s + N\tau) \mathrm{d}s,$$

where $l = 2, 3$, and denote $x^l = \mathrm{vec}(X^l)$ and $\bar{x}^l = \mathrm{vec}(\bar{X}^l)$.

Proposition 2 *If the Lyapunov condition in Lemma 2 is satisfied, then*

i) it holds that $J_0 = J_1^T$, $\bar{F}_0 = F_1^T$ and $F_0 = \bar{F}_1^T$,

ii) furthermore, $Y_0^{(1)}$ and Z_0 may be found as

$$Y_0^{(1)} = -\frac{\tau^2}{2}\left(J_0 + J_0^T\right) + \tau \bar{F}_0^T + \frac{1}{2}\left(X^{2T} - X^2\right),$$
$$Z_0 = \frac{\tau^3}{3}\left(J_0 + J_0^T\right) - \frac{\tau^2}{2}\left(\bar{F}_0 + \bar{F}_0^T\right) + \frac{1}{6}\left(X^3 + X^{3T}\right),$$

and $Y_0^{(2)} = Y_0^{(1)T}$. If $\det(\mathcal{L}) \neq 0$ then the terms X^2 and X^3 can be computed by devectorization of

$$\begin{pmatrix} x^2 \\ \bar{x}^{2\star} \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \tau^2 u_1 - 2\bar{f}_0^\star \\ \tau^2 u_{N-1}^\star - 2\bar{f}_N^\star \end{pmatrix},$$
$$\begin{pmatrix} x^3 \\ -\bar{x}^{3\star} \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \tau^3 u_1 - 3x^2 \\ \tau^3 u_{N-1}^\star - 3\bar{x}^{2\star} \end{pmatrix}.$$

Proof : Relations in *i)* are obtained directly using the symmetry property of the Lyapunov matrix. Furthermore, changing the variable s_2 to $\tau = s_1 - s_2$ in $Y_0^{(1)}$ we get the following representation

$$Y_0^{(1)} = \frac{1}{2} \int_{-\tau}^{0} \int_{s_1}^{s_1+\tau} U(\tau) \mathrm{d}\tau \mathrm{d}(s_1^2).$$

Integrating by parts and observing that

$$X^2 = \int_{-\tau}^{0} s^2 U(s+\tau) \mathrm{d}s - 2\tau \bar{F}_1 + \tau^2 J_1,$$

it is straightforward to get the desired result. The formula for Z_0 may be found in Alexandrova (2023), notice that the proof does not depend on the type of system thus it holds in particular for a neutral type system of the form (2.1). The vectorized relations for X^2 and X^3 are obtained in the same vein as in the proof of Proposition 1. □

Remark 15 *The following relation is useful to reduce the computational burden:*

$$Y_k^{(1)} + Y_k^{(2)} = -\tau(F_k + \bar{F}_{k+1}), \quad k = \overline{0, N-1}.$$

Indeed, comparing two representations of Z_k *it is easy to see that*

$$Y_k^{(1)} + Y_k^{(2)} = -\tau \int_{-\tau}^{0} \int_{-\tau}^{0} U(s_1 - s_2)\mathrm{d}s_2\mathrm{d}s_1 = -\tau \int_{-\tau}^{0} \int_{s_1}^{s_1+\tau} U(\tau)\mathrm{d}\tau\mathrm{d}s_1.$$

One proceeds integrating by parts. It is also easy to see that $F_l + \bar{F}_l = \tau J_l$ *for any* l.

Now, the matrix terms $J_l,\ F_l,\ \bar{F}_l,\ l = \overline{0, N}$, and $Y_k^{(j)},\ Z_k,\ k = \overline{0, N-1}$ are obtained by devectorization of $a_l,\ f_l,\ \bar{f}_l,\ y_k^{(j)}$, and z_k, respectively. The computation of Λ_N involves similar integral terms based on the first and the second derivatives of the Lyapunov matrix as well. However, they reduce to already calculated terms by using integration by parts. For example,

$$\int_{-\tau}^{0} \int_{-\tau}^{0} s_1 s_2 U'(s_1 - s_2 + k\tau)\mathrm{d}s_2\mathrm{d}s_1 = \tau \bar{F}_{k+1} + Y_k^{(1)},$$

$$\int_{-\tau}^{0} \int_{-\tau}^{0} s_1 s_2 U''(s_1 - s_2 + l\tau)\mathrm{d}s_2\mathrm{d}s_1 = -\tau^2 U(l\tau) + F_{l+1} + \bar{F}_l, \quad k = \overline{0, N-1}, \quad l = \overline{1, N-1},$$

$$\int_{-\tau}^{0} \left(\int_{-\tau}^{s_1 - 0} s_2 U''(s_1 - s_2)\mathrm{d}s_2 + \int_{s_1+0}^{0} s_2 U''(s_1 - s_2)\mathrm{d}s_2 \right) s_1 \mathrm{d}s_1 = -\tau^2 U(0) + \bar{F}_0 + \bar{F}_0^T - \frac{\tau^3}{3}\Delta U'(0),$$

and so on.

7.2 Recursive computation of the integrals in Legendre polynomials approach

Next, we present a recursive method for computing the integrals involved in determining matrix $\mathbf{P}_N$ defined in (5.4), which are given in terms of the following integrals for $k, l = \overline{0, N-1}$:

$$G_k = \int_{-h}^{0} p_k(s)U(h+s)\mathrm{d}s, \quad \bar{G}_k = \int_{-h}^{0} p_k(s)U(-s)\mathrm{d}s,$$

$$H_{kl} = \int_{-h}^{0} \int_{-h}^{s_1} p_k(s_1)p_l(s_2)U(s_1 - s_2)\mathrm{d}s_2\mathrm{d}s_1, \quad \bar{H}_{kl} = \int_{-h}^{0} \int_{-h}^{s_1} p_k(s_1)p_l(s_2)U(h - s_1 + s_2)\mathrm{d}s_2\mathrm{d}s_1,$$

and similar integrals involving the first and second derivatives of the Lyapunov matrix.

To do so, let us consider the notation of the previous subsection for $N = 1$, that is

$$J_1 = \int_{-h}^{0} U(s+h)\mathrm{d}s, \quad \bar{F}_0 = \int_{-h}^{0} (-s)U(s)\mathrm{d}s, \ \bar{F}_1 = \int_{-h}^{0} (-s)U(s+h)\mathrm{d}s,$$

$$X^l = \int_{-h}^{0} (s+h)^l U(s+h)\mathrm{d}s, \quad \bar{X}^l = \int_{-h}^{0} s^l U(s+h)\mathrm{d}s, \quad l = 2, 3,$$

and define the vectors

$$u_0 = \mathrm{vec}(U(0)),\ u_1 = \mathrm{vec}(U(h)),\ a_1 = \mathrm{vec}(J_1),\ \bar{f}_0 = \mathrm{vec}(\bar{F}_0),\ \bar{f}_1 = \mathrm{vec}(\bar{F}_1),$$
$$g_k = \mathrm{vec}(G_k),\ \bar{g}_k = \mathrm{vec}(\bar{G}_k),\ h_{kl} = \mathrm{vec}(H_{kl}),\ \bar{h}_{kl} = \mathrm{vec}(\bar{H}_{kl})$$

and operation * which means vectorization of a transposed matrix, i.e. $u_1^\star = \mathrm{vec}(U^T(h))$, $g_k^\star = \mathrm{vec}(G_k^T)$, etc. Next, we introduce two propositions allowing the recursive computation of the above integral.

Proposition 3 *If* $\det(\mathcal{L}) \neq 0$ *and the Lyapunov condition in Lemma 2 holds then*

i) g_k *and* $\bar{g}_k$ *satisfy*

$$\begin{pmatrix} g_k \\ \bar{g}_k^\star \end{pmatrix} = \begin{pmatrix} g_{k-2} \\ \bar{g}_{k-2}^\star \end{pmatrix} - \frac{2(2k-1)}{h}\mathcal{L}^{-1}\begin{pmatrix} g_{k-1} \\ \bar{g}_{k-1}^\star \end{pmatrix}, \quad k \geqslant 2. \tag{7.4}$$

ii) h_{jk} *and* $\bar{h}_{jk}$ *satisfy*

$$\begin{pmatrix} H_{jk} \\ \bar{H}_{jk} \end{pmatrix} = (-1)^{j+k}\begin{pmatrix} H_{kj} \\ \bar{H}_{kj} \end{pmatrix}, \quad \forall k < j, \tag{7.5}$$

$$\begin{pmatrix} h_{jk} \\ \bar{h}_{jk}^\star \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} -\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})u_0 + \frac{2(2k-1)}{h}h_{j,k-1} \\ -\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})u_1^\star + \frac{2(2k-1)}{h}\bar{h}_{j,k-1}^\star \end{pmatrix} + \begin{pmatrix} h_{j,k-2} \\ \bar{h}_{j,k-2}^\star \end{pmatrix}, \ \forall k \geqslant \max(2,j). \tag{7.6}$$

Here, δ_{jk} *denotes the Kronecker delta such that* $\delta_{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$

Proof : *i)* To compute $G_k,\ \bar{G}_k,\ \forall k \geqslant 2$, the dynamic property (2.3) is multiplied by the difference of Legendre polynomials $p_k(s) - p_{k-2}(s)$ and then integrated, yielding

$$\int_{-h}^{0} U'(h+s)(p_k(s) - p_{k-2}(s))\mathrm{d}s - \int_{-h}^{0} U'(s)(p_k(s) - p_{k-2}(s))\mathrm{d}sD$$
$$= \int_{-h}^{0} U(h+s)(p_k(s) - p_{k-2}(s))\mathrm{d}sA_0 + \int_{-h}^{0} U(s)(p_k(s) - p_{k-2}(s))\mathrm{d}sA_1.$$

Applying integration by parts and taking into account the derivation property of the Legendre polynomials in Section 5.1, we get

$$U(h)(p_k(0) - p_{k-2}(0)) - U(0)(p_k(-h) - p_{k-2}(-h)) - \frac{2(2k-1)}{h}\int_{-h}^{0} U(h+s)p_{k-1}(s)\mathrm{d}s$$
$$- U(0)(p_k(0) - p_{k-2}(0))D + U^T(h)(p_k(-h) - p_{k-2}(-h))D + \frac{2(2k-1)}{h}\int_{-h}^{0} U(s)p_{k-1}(s)\mathrm{d}sD$$
$$= \int_{-h}^{0} U(h+s)(p_k(s) - p_{k-2}(s))\mathrm{d}sA_0 + \int_{-h}^{0} U(s)(p_k(s) - p_{k-2}(s))\mathrm{d}sA_1.$$

Knowing that $p_k(-h) = p_{k-2}(-h) = (-1)^k$ and $p_k(0) = p_{k-2}(0) = 1$, it leads to

$$(G_k - G_{k-2})A_0 + (\bar{G}_k^T - \bar{G}_{k-2}^T)A_1 = -\frac{2(2k-1)}{h}(G_{k-1} - \bar{G}_{k-1}^T D). \tag{7.7}$$

Now, multiply by $p_k(s) - p_{k-2}(s)$ and integrate from $-h$ to 0 the dynamic property for negative values of the argument (2.6):

$$\int_{-h}^{0} U'(s)(p_k(s) - p_{k-2}(s))\mathrm{d}s - D^T \int_{-h}^{0} U'(s+h)(p_k(s) - p_{k-2}(s))\mathrm{d}s$$
$$= -A_0^T \int_{-h}^{0} U(s)(p_k(s) - p_{k-2}(s))\mathrm{d}s - A_1^T \int_{-h}^{0} U(s+h)(p_k(s) - p_{k-2}(s))\mathrm{d}s.$$

Similarly, applying integration by parts and knowing that $p_k(-h) = p_{k-2}(-h) = (-1)^k$ and $p_k(0) = p_{k-2}(0) = 1$, we arrive at

$$-\frac{2(2k-1)}{h} \int_{-h}^{0} U(s)p_{k-1}(s)\mathrm{d}s + \frac{2(2k-1)}{h} D^T \int_{-h}^{0} U(s+h)p_{k-1}(s)\mathrm{d}s$$
$$= -A_0^T \int_{-h}^{0} U(s)(p_k(s) - p_{k-2}(s))\mathrm{d}s - A_1^T \int_{-h}^{0} U(s+h)(p_k(s) - p_{k-2}(s))\mathrm{d}s,$$

consequently,

$$-A_0^T(\bar{G}_k^T - \bar{G}_{k-2}^T) - A_1^T(G_k - G_{k-2}) = -\frac{2(2k-1)}{h}(\bar{G}_{k-1}^T - D^T G_{k-1}). \tag{7.8}$$

Finally, the algebraic equations (7.7) and (7.8) are rewritten in vectorized form as follows

$$\begin{pmatrix} I_n \otimes A_0 & I_n \otimes A_1 \\ -A_1^T \otimes I_n & -A_0^T \otimes I_n \end{pmatrix} \left[\begin{pmatrix} g_k \\ \bar{g}_k^\star \end{pmatrix} - \begin{pmatrix} g_{k-2} \\ \bar{g}_{k-2}^\star \end{pmatrix} \right] = -\frac{2(2k-1)}{h} \begin{pmatrix} I_{n^2} & -I_n \otimes D \\ -D^T \otimes I_n & I_{n^2} \end{pmatrix} \begin{pmatrix} g_{k-1} \\ \bar{g}_{k-1}^\star \end{pmatrix},$$

or,

$$\mathcal{L} \left[\begin{pmatrix} g_k \\ \bar{g}_k^\star \end{pmatrix} - \begin{pmatrix} g_{k-2} \\ \bar{g}_{k-2}^\star \end{pmatrix} \right] = -\frac{2(2k-1)}{h} \begin{pmatrix} g_{k-1} \\ \bar{g}_{k-1}^\star \end{pmatrix}.$$

Since $\det(\mathcal{L}) \neq 0$, we arrive at a unique solution of this system given by (7.4).

ii) Firstly, using the symmetry of the Legendre polynomials $p_k(-s-h) = (-1)^k p_k(s)$, $s \in [-h, 0]$ and the changes of variable $s_1' = -s_1 - h$ and $s_2' = -s_2 - h$, we have

$$H_{jk} = \int_{-h}^{0} \int_{-h}^{s_1} p_j(s_1)p_k(s_2)U(s_1 - s_2)\mathrm{d}s_2\mathrm{d}s_1 = \int_{-h}^{0} \int_{s_1'}^{0} p_j(-s_1' - h)p_k(-s_2' - h)U(s_2' - s_1')\mathrm{d}s_2'\mathrm{d}s_1'$$
$$= \int_{-h}^{0} \int_{-h}^{s_2'} (-1)^{j+k} p_j(s_1')p_k(s_2')U(s_2' - s_1')\mathrm{d}s_1'\mathrm{d}s_2'.$$

Making the change of variable $s_1'' = s_2'$ and $s_2'' = s_1'$, we finally arrive at the following expression

$$H_{jk} = \int_{-h}^{0} \int_{-h}^{s_1''} (-1)^{j+k} p_j(s_2'') p_k(s_1'') U(s_1'' - s_2'') \mathrm{d}s_2'' \mathrm{d}s_1'' = (-1)^{j+k} H_{kj}.$$

To compute H_{jk}, $\forall k \geqslant \max(2, j)$, we repeat the steps used in the proof of (7.4). In this case, the dynamic property (2.3) for $s_1 - s_2 > 0$ gives

$$-\frac{\partial U(s_1 - s_2)}{\partial s_2} + \frac{\partial U(s_1 - s_2 - h)}{\partial s_2} D = U(s_1 - s_2)A_0 + U(s_1 - s_2 - h)A_1.$$

Now, multiply the discretized dynamic property (2.3) by $p_j(s_1)(p_k(s_2) - p_{k-2}(s_2))$:

$$\begin{aligned} &- p_j(s_1)(p_k(s_2) - p_{k-2}(s_2)) \frac{\partial U(s_1 - s_2)}{\partial s_2} + p_j(s_1)(p_k(s_2) - p_{k-2}(s_2)) \frac{\partial U(s_1 - s_2 - h)}{\partial s_2} D \\ &\quad = p_j(s_1)(p_k(s_2) - p_{k-2}(s_2)) U(s_1 - s_2) A_0 + p_j(s_1)(p_k(s_2) - p_{k-2}(s_2)) U(s_1 - s_2 - h) A_1. \end{aligned}$$

Then, we integrate with respect to s_2 the previous expression, from $-h$ up to s_1 in such a way that the difference $s_1 - s_2 > 0$. It implies that

$$\begin{aligned} &-\int_{-h}^{s_1} p_j(s_1)(p_k(s_2) - p_{k-2}(s_2)) \frac{\partial U(s_1 - s_2)}{\partial s_2} \mathrm{d}s_2 + \int_{-h}^{s_1} p_j(s_1)(p_k(s_2) - p_{k-2}(s_2)) \frac{\partial U(s_1 - s_2 - h)}{\partial s_2} \mathrm{d}s_2 D \\ &= \int_{-h}^{s_1} p_j(s_1)(p_k(s_2) - p_{k-2}(s_2)) U(s_1 - s_2) \mathrm{d}s_2 A_0 + \int_{-h}^{s_1} p_j(s_1)(p_k(s_2) - p_{k-2}(s_2)) U(s_1 - s_2 - h) \mathrm{d}s_2 A_1. \end{aligned} \tag{7.9}$$

Compute the integrals in the above expression:

$$\begin{aligned} &\int_{-h}^{s_1} p_j(s_1)(p_k(s_2) - p_{k-2}(s_2)) \frac{\partial U(s_1 - s_2)}{\partial s_2} \mathrm{d}s_2 \\ &\qquad = p_j(s_1) \left((p_k(s_1) - p_{k-2}(s_1)) U(0) - \frac{2(2k-1)}{h} \int_{-h}^{s_1} p_{k-1}(s_2) U(s_1 - s_2) \mathrm{d}s_2 \right), \end{aligned}$$

$$\begin{aligned} &\int_{-h}^{s_1} p_j(s_1)(p_k(s_2) - p_{k-2}(s_2)) \frac{\partial U(s_1 - s_2 - h)}{\partial s_2} \mathrm{d}s_2 \\ &\qquad = p_j(s_1) \left((p_k(s_1) - p_{k-2}(s_1)) U(-h) - \frac{2(2k-1)}{h} \int_{-h}^{s_1} p_{k-1}(s_2) U(s_1 - s_2 - h) \mathrm{d}s_2 \right). \end{aligned}$$

Thus, system (7.9) reduces to

$$\begin{aligned} &-p_j(s_1) \left((p_k(s_1) - p_{k-2}(s_1)) U(0) - \frac{2(2k-1)}{h} \int_{-h}^{s_1} p_{k-1}(s_2) U(s_1 - s_2) \mathrm{d}s_2 \right) \\ &+p_j(s_1) \left((p_k(s_1) - p_{k-2}(s_1)) U(-h) - \frac{2(2k-1)}{h} \int_{-h}^{s_1} p_{k-1}(s_2) U(s_1 - s_2 - h) \mathrm{d}s_2 \right) D \end{aligned}$$

$$= \int_{-h}^{s_1} p_j(s_1)(p_k(s_2)-p_{k-2}(s_2))U(s_1-s_2)\mathrm{d}s_2 A_0 + \int_{-h}^{s_1} p_j(s_1)(p_k(s_2)-p_{k-2}(s_2))U(s_1-s_2-h)\mathrm{d}s_2 A_1.$$

Further, integrate with respect to s_1

$$\int_{-h}^{0} -p_j(s_1)\left((p_k(s_1)-p_{k-2}(s_1))U(0) - \frac{2(2k-1)}{h}\int_{-h}^{s_1} p_{k-1}(s_2)U(s_1-s_2)\mathrm{d}s_2\right)\mathrm{d}s_1$$
$$+\int_{-h}^{0} p_j(s_1)\left((p_k(s_1)-p_{k-2}(s_1))U(-h) - \frac{2(2k-1)}{h}\int_{-h}^{s_1} p_{k-1}(s_2)U(s_1-s_2-h)\mathrm{d}s_2\right)\mathrm{d}s_1 D$$
$$= \int_{-h}^{0}\int_{-h}^{s_1} p_j(s_1)(p_k(s_2)-p_{k-2}(s_2))U(s_1-s_2)\mathrm{d}s_2\mathrm{d}s_1 A_0$$
$$+ \int_{-h}^{0}\int_{-h}^{s_1} p_j(s_1)(p_k(s_2)-p_{k-2}(s_2))U(s_1-s_2-h)\mathrm{d}s_2\mathrm{d}s_1 A_1.$$

Reminding the orthogonality property of the Legendre polynomials, we obtain

$$-\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})U(0)+\frac{2(2k-1)}{h}H_{j,k-1}+\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})U(-h)D-\frac{2(2k-1)}{h}\bar{H}^T_{j,k-1}D$$
$$= \left(H_{j,k}-H_{j,k-2}\right)A_0 + \left(\bar{H}^T_{j,k}-\bar{H}^T_{j,k-2}\right)A_1.$$

Using similar steps for the dynamic property (2.6), we arrive at the two following algebraic equations

$$-\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})U(0)+\frac{2(2k-1)}{h}H_{j,k-1}+\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})U^T(h)D-\frac{2(2k-1)}{h}\bar{H}_{j,k-1}D$$
$$= \left(H_{j,k}-H_{j,k-2}\right)A_0 + \left(\bar{H}^T_{j,k}-\bar{H}^T_{j,k-2}\right)A_1$$

$$-\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})U^T(h)+\frac{2(2k-1)}{h}\bar{H}^T_{j,k-1}+\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})D^TU(0)-\frac{2(2k-1)}{h}D^TH_{j,k-1}$$
$$= -A_0^T\left(\bar{H}^T_{j,k}-\bar{H}^T_{j,k-2}\right) - A_1^T\left(H_{j,k}-H_{j,k-2}\right)$$

By vectorization, the following system of linear algebraic equations is obtained

$$\begin{pmatrix} I_n\otimes A_0 & I_n\otimes A_1 \\ -A_1^T\otimes I_n & -A_0^T\otimes I_n \end{pmatrix}\begin{pmatrix} h_{jk} \\ \bar{h}^\star_{jk}\end{pmatrix} = \begin{pmatrix} I_{n^2} & -I_n\otimes D \\ -D^T\otimes I_n & I_{n^2}\end{pmatrix}$$
$$\times\begin{pmatrix} -\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})u_0 + \frac{2(2k-1)}{h}h_{j,k-1} \\ -\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})u_1^\star + \frac{2(2k-1)}{h}\bar{h}^\star_{j,k-1}\end{pmatrix} + \begin{pmatrix} I_n\otimes A_0 & I_n\otimes A_1 \\ -A_1^T\otimes I_n & -A_0^T\otimes I_n\end{pmatrix}\begin{pmatrix} h_{j,k-2} \\ \bar{h}^\star_{j,k-2}\end{pmatrix},$$

equivalently,

$$\mathcal{L}\begin{pmatrix} h_{jk} \\ \bar{h}^\star_{jk}\end{pmatrix} = \begin{pmatrix} -\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})u_0 + \frac{2(2k-1)}{h}h_{j,k-1} \\ -\frac{h}{2j+1}(\delta_{j,k}-\delta_{j,k-2})u_1^\star + \frac{2(2k-1)}{h}\bar{h}^\star_{j,k-1}\end{pmatrix} + \mathcal{L}\begin{pmatrix} h_{j,k-2} \\ \bar{h}^\star_{j,k-2}\end{pmatrix}.$$

When $\det(\mathcal{L}) \neq 0$, we conclude that the unique solution of the previous system of algebraic equations is given by (7.6). □

Proposition 4 *If the Lyapunov condition in Lemma 2 is satisfied, then*

i) it holds that

$$\begin{pmatrix} g_0 \\ \bar{g}_0^\star \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} u_1 - u_0 \\ u_0^\star - u_1^\star \end{pmatrix},$$
$$\begin{pmatrix} g_1 \\ \bar{g}_1^\star \end{pmatrix} = \mathcal{L}^{-1}\left[\begin{pmatrix} u_1 + u_0 \\ u_0^\star + u_1^\star \end{pmatrix} - \frac{2}{h}\begin{pmatrix} g_0 \\ \bar{g}_0 \end{pmatrix}\right],$$

ii) furthermore, H_{00}, $\bar{H}_{00}$, H_{01}, $\bar{H}_{01}$, H_{11} *and* $\bar{H}_{11}$ *may be found as*

$$H_{00} = \bar{F}_1, \quad \bar{H}_{00} = \bar{F}_0^T, \quad H_{10} = \bar{F}_1 - \frac{1}{h}\bar{X}^2, \quad \bar{H}_{10} = \bar{F}_0^T - \frac{1}{h}X^2,$$
$$H_{11} = -\bar{F}_1 + \frac{2}{3h^2}\bar{X}^3 + \frac{2}{h}\bar{X}^2, \quad \bar{H}_{11} = -\bar{F}_0^T - \frac{2}{3h^2}X^3 + \frac{2}{h}X^2.$$

Proof : *i)* Considering condition *i)* of Proposition 2 with $N = 1$, $\tau = h$, notice that the vectorized form of matrices G_0, $\bar{G}_0$, G_1 and $\bar{G}_1$ are given by the following expressions:

$$\begin{pmatrix} g_0 \\ \bar{g}_0^\star \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1^\star \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} u_1 - u_0 \\ u_0^\star - u_1^\star \end{pmatrix},$$

$$\begin{pmatrix} g_1 \\ \bar{g}_1^\star \end{pmatrix} = \mathcal{L}^{-1}\left[\frac{2}{h}\begin{pmatrix} hu_1 - a_1 \\ hu_0^\star - a_1^\star \end{pmatrix} - \begin{pmatrix} u_1 - u_0 \\ u_0^\star - u_1^\star \end{pmatrix}\right] = \mathcal{L}^{-1}\left[\begin{pmatrix} u_1 + u_0 \\ u_0^\star + u_1^\star \end{pmatrix} - \frac{2}{h}\begin{pmatrix} g_0 \\ \bar{g}_0^\star \end{pmatrix}\right].$$

ii) Observe that

$$H_{00} = \int_{-h}^{0}\int_{-h}^{s_1} U(s_1 - s_2)\mathrm{d}s_2\mathrm{d}s_1 = \int_{-h}^{0}\int_{0}^{s_1+h} U(s)\mathrm{d}s\mathrm{d}s_1,$$

with the change of variable $s = s_1 - s_2$. Using integration by parts, then we obtain

$$H_{00} = s_1\int_{0}^{s_1+h} U(s)\mathrm{d}s\bigg|_{-h}^{0} - \int_{-h}^{0} s_1 U(s_1 + h)\mathrm{d}s_1 = \bar{F}_1.$$

Similarly, $\bar{H}_{00}$, H_{10} and $\bar{H}_{10}$ are computed by using integration by parts as follows:

$$\bar{H}_{00}^T = \int_{-h}^{0}\int_{-h}^{s_1} U(s_1 - s_2 - h)\mathrm{d}s_2\mathrm{d}s_1 = \int_{-h}^{0}\int_{-h}^{s_1} U(s)\mathrm{d}s\mathrm{d}s_1 = s_1\int_{-h}^{s_1} U(s)\mathrm{d}s\bigg|_{-h}^{0} - \int_{-h}^{0} s_1 U(s_1)\mathrm{d}s_1 = \bar{F}_0,$$

$$H_{10} = \int_{-h}^{0}\int_{-h}^{s_1}\left(\frac{2s_1}{h} + 1\right)U(s_1 - s_2)\mathrm{d}s_2\mathrm{d}s_1 = H_{00} + \frac{2}{h}\int_{-h}^{0}\int_{-h}^{s_1} s_1 U(s_1 - s_2)\mathrm{d}s_2\mathrm{d}s_1$$
$$= H_{00} + \frac{2}{h}\int_{-h}^{0}\int_{0}^{s_1+h} U(s)\mathrm{d}s\mathrm{d}\left(\frac{s_1^2}{2}\right) = H_{00} - \frac{2}{h}\int_{-h}^{0}\frac{s_1^2}{2}U(s_1 + h)\mathrm{d}s_1 = \bar{F}_1 - \frac{1}{h}\bar{X}^2,$$

$$\bar{H}_{10}^T = \int_{-h}^{0}\int_{-h}^{s_1}\left(\frac{2s_1}{h} + 1\right)U(s_1 - s_2 - h)\mathrm{d}s_2\mathrm{d}s_1 = \bar{H}_{00}^T + \frac{2}{h}\int_{-h}^{0}\int_{-h}^{s_1} s_1 U(s_1 - s_2 - h)\mathrm{d}s_2\mathrm{d}s_1$$

$$= \bar{H}_{00}^{T} + \frac{2}{h}\int_{-h}^{0}\int_{-h}^{s_1} U(s)\mathrm{d}s\mathrm{d}\left(\frac{s_1^2}{2}\right) = \bar{H}_{00}^{T} - \frac{2}{h}\int_{-h}^{0}\frac{s_1^2}{2}U(s_1)\mathrm{d}s_1$$

$$= \bar{H}_{00}^{T} - \frac{1}{h}\int_{-h}^{0}(s+h)^2U^T(s+h)\mathrm{d}s = \bar{F}_0 - \frac{1}{h}X^{2T}.$$

Now, we calculate H_{11} and $\bar{H}_{11}$, which admit the following expressions:

$$H_{11} = \int_{-h}^{0}\int_{-h}^{s_1}\left(\frac{2s_1}{h}+1\right)\left(\frac{2s_2}{h}+1\right)U(s_1-s_2)\mathrm{d}s_2\mathrm{d}s_1 = H_{00} - \frac{1}{h}\bar{X}^2 + \frac{4}{h^2}\Sigma_1 + \frac{2}{h}\Sigma_2,$$

$$\bar{H}_{11}^{T} = \int_{-h}^{0}\int_{-h}^{s_1}\left(\frac{2s_1}{h}+1\right)\left(\frac{2s_2}{h}+1\right)U(s_1-s_2-h)\mathrm{d}s_2\mathrm{d}s_1 = \bar{H}_{00}^{T} - \frac{1}{h}X^{2T} + \frac{4}{h^2}\Sigma_3 + \frac{2}{h}\Sigma_4,$$

where,

$$\Sigma_1 = \int_{-h}^{0}\int_{-h}^{s_1} s_1s_2U(s_1-s_2)\mathrm{d}s_2\mathrm{d}s_1, \quad \Sigma_2 = \int_{-h}^{0}\int_{-h}^{s_1} s_2U(s_1-s_2)\mathrm{d}s_2\mathrm{d}s_1,$$

$$\Sigma_3 = \int_{-h}^{0}\int_{-h}^{s_1} s_1s_2U(s_1-s_2-h)\mathrm{d}s_2\mathrm{d}s_1, \quad \Sigma_4 = \int_{-h}^{0}\int_{-h}^{s_1} s_2U(s_1-s_2-h)\mathrm{d}s_2\mathrm{d}s_1.$$

Let us calculate Σ_i , $i = \overline{1,4}$, using integration by parts:

$$\Sigma_1 = \int_{-h}^{0}\int_{-h}^{s_1} s_1s_2U(s_1-s_2)\mathrm{d}s_2\mathrm{d}s_1 = \int_{-h}^{0}\int_{-h-s1}^{0} s_1(s_1+s)U^T(s)\mathrm{d}s\mathrm{d}s_1$$

$$= \int_{-h}^{0}\int_{\xi_1}^{0}(-h-\xi_1)(-h-\xi_1+s)U^T(s)\mathrm{d}s\mathrm{d}\xi_1 = \int_{-h}^{0}\int_{\xi_1}^{0}U^T(s)\mathrm{d}s\mathrm{d}\left(\frac{(h+\xi_1)^3}{3}\right)$$

$$- \int_{-h}^{0}\int_{\xi_1}^{0}sU^T(s)\mathrm{d}s\mathrm{d}\left(\frac{(h+\xi_1)^2}{2}\right) = \frac{1}{3}\int_{-h}^{0}(h+\xi_1)^3U^T(\xi_1)\mathrm{d}\xi_1 - \frac{1}{2}\int_{-h}^{0}(h+\xi_1)^2\xi_1U^T(\xi_1)\mathrm{d}\xi_1$$

$$= \frac{1}{3}\int_{-h}^{0}(-s)^3U(s+h)\mathrm{d}s - \frac{1}{2}\int_{-h}^{0}(-s)^2(-s-h)U(s+h)\mathrm{d}s$$

$$= -\frac{1}{3}\int_{-h}^{0}s^3U(s+h)\mathrm{d}s + \frac{1}{2}\int_{-h}^{0}s^2(s+h)U(s+h)\mathrm{d}s = \frac{1}{6}\bar{X}^3 + \frac{h}{2}\bar{X}^2,$$

$$\Sigma_2 = \int_{-h}^{0}\int_{-h}^{s_1} s_2U(s_1-s_2)\mathrm{d}s_2\mathrm{d}s_1 = \int_{-h}^{0}\int_{-h-s1}^{0}(s_1+s)U^T(s)\mathrm{d}s\mathrm{d}s_1$$

$$= \int_{-h}^{0}\int_{\xi_1}^{0}(-h-\xi_1+s)U^T(s)\mathrm{d}s\mathrm{d}\xi_1 = -\int_{-h}^{0}\int_{\xi_1}^{0}U^T(s)\mathrm{d}s\mathrm{d}\left(\frac{(h+\xi_1)^2}{2}\right) + \int_{-h}^{0}\int_{\xi_1}^{0}sU^T(s)\mathrm{d}s\mathrm{d}\xi$$

$$= -\frac{1}{2}\int_{-h}^{0}(h+\xi_1)^2U^T(\xi_1)\mathrm{d}\xi_1 + h\int_{-h}^{0} sU^T(s)\mathrm{d}s + \int_{-h}^{0}\xi_1^2U^T(\xi_1)\mathrm{d}\xi_1 = -\frac{1}{2}\int_{-h}^{0}(-s)^2U^T(-s-h)\mathrm{d}s$$

$$+ h\int_{-h}^{0}(-s-h)U^T(-s-h)\mathrm{d}s + \int_{-h}^{0}(s+h)^2U^T(-s-h)\mathrm{d}s = -\frac{1}{2}\int_{-h}^{0}s^2U(s+h)\mathrm{d}s$$

$$- h\int_{-h}^{0}(s+h)U(s+h)\mathrm{d}s + \int_{-h}^{0}(s+h)^2U(s+h)\mathrm{d}s = \frac{1}{2}\bar{X}^2 - h\bar{F}_1,$$

$$\Sigma_3 = \int_{-h}^{0}\int_{-h}^{s_1} s_1s_2U(s_1-s_2-h)\mathrm{d}s_2\mathrm{d}s_1 = \int_{-h}^{0}\int_{-h}^{s_1} s_1(s_1-s-h)U(s)\mathrm{d}s\mathrm{d}s_1$$

$$= \int_{-h}^{0}\int_{-h}^{-h-\xi_1}(-h-\xi_1)(-h-\xi_1-s-h)U(s)\mathrm{d}s\mathrm{d}\xi_1 = \int_{-h}^{0}\int_{-h}^{-h-\xi_1}U(s)\mathrm{d}s\mathrm{d}\left(\frac{(h+\xi_1)^3}{3}\right)$$

$$+ \int_{-h}^{0}\int_{-h}^{-h-\xi_1}(s+h)U(s)\mathrm{d}s\mathrm{d}\left(\frac{(h+\xi_1)^2}{2}\right) = \frac{1}{3}\int_{-h}^{0}(h+\xi_1)^3U^T(h+\xi_1)\mathrm{d}\xi_1$$

$$+ \frac{1}{2}\int_{-h}^{0}(-h-\xi_1+h)(h+\xi_1)^2U^T(h+\xi_1)\mathrm{d}\xi_1 = -\frac{1}{6}X^{3T} + \frac{h}{2}X^{2T}$$

$$\Sigma_4 = \int_{-h}^{0}\int_{-h}^{s_1} s_2U(s_1-s_2-h)\mathrm{d}s_2\mathrm{d}s_1 = \int_{-h}^{0}\int_{-h}^{s_1}(s_1-s-h)U(s)\mathrm{d}s\mathrm{d}s_1$$

$$= \int_{-h}^{0}\int_{-h}^{-h-\xi_1}(-h-\xi_1-s-h)U(s)\mathrm{d}s\mathrm{d}\xi_1 = -\int_{-h}^{0}\int_{-h}^{-h-\xi_1}U(s)\mathrm{d}s\mathrm{d}\left(\frac{(h+\xi_1)^2}{2}\right) - \int_{-h}^{0}\int_{-h}^{-h-\xi_1}(s+h)U(s)\mathrm{d}s\mathrm{d}\xi_1$$

$$= -\frac{1}{2}\int_{-h}^{0}(h+\xi_1)^2U^T(h+\xi_1)\mathrm{d}\xi_1 - h\int_{-h}^{0}(s+h)U(s)\mathrm{d}s + \int_{-h}^{0}(s+h)^2U(s)\mathrm{d}s$$

$$= -\frac{1}{2}X^{2T} + h\bar{F}_0 + X^{2T} - 2h\bar{F}_0 = \frac{1}{2}X^{2T} - h\bar{F}_0.$$

Finally, we obtain the desired expression for H_{11} and $\bar{H}_{11}$, which are given by

$$H_{11} = H_{00} - \frac{1}{h}\bar{X}^2 + \frac{4}{h^2}\left(\frac{1}{6}\bar{X}^3 + \frac{h}{2}\bar{X}^2\right) + \frac{2}{h}\left(\frac{1}{2}\bar{X}^2 - h\bar{F}_1\right) = -\bar{F}_1 + \frac{2}{3h^2}\bar{X}^3 + \frac{2}{h}\bar{X}^2,$$

$$\bar{H}_{11}^T = \bar{H}_{00}^T - \frac{1}{h}X^{2T} + \frac{4}{h^2}\left(-\frac{1}{6}X^{3T} + \frac{h}{2}X^{2T}\right) + \frac{2}{h}\left(\frac{1}{2}X^{2T} - h\bar{F}_0\right) = -\bar{F}_0 - \frac{2}{3h^2}X^{3T} + \frac{2}{h}X^{2T}.$$

□

The matrix terms G_k, $\bar{G}_k$, H_{jk} and $\bar{H}_{jk}$ are finally obtained by devectorization of $g_k, \bar{g}_k, h_{jk}$ and $\bar{h}_{jk}$, respectively. With the help of Proposition 3 and 4, we can give explicit expressions for the integral of matrix $\mathbf{P}_N$ involving the delay Lyapunov matrix and its derivatives. For example,

$$\int_{-h}^{0}\int_{-h}^{s_1} p_k(s_1)p_l(s_2)U'(s_1-s_2)\mathrm{d}s_2\mathrm{d}s_1 = -\delta_{kl}U(0) + (-1)^lG_k + \sum_{i=0}^{l-1}\left(\frac{(2i+1)}{h}(1-(-1)^{l+i})\right)H_{ik},$$

$$\int_{-h}^{0}\int_{-h}^{s_1-0} p_k(s_1)p_l(s_2)U''(s_1-s_2)\mathrm{d}s_2\mathrm{d}s_1$$
$$= -\delta_{kl}U'(+0) + (-1)^l \left(U(h) - (-1)^k U(0) - \sum_{i=0}^{k-1} \frac{(2i+1)}{h}(1-(-1)^{k+i})G_i \right)$$
$$+ \sum_{i=0}^{l-1} \left(\frac{(2i+1)}{h}(1-(-1)^{l+i}) \right) \left(-\delta_{ki}U(0) + (-1)^i G_k + \sum_{j=0}^{i-1} \left(\frac{(2j+1)}{h}(1-(-1)^{i+j}) \right) H_{jk} \right),$$

$k = \overline{0, N-1}, \quad l = \overline{1, N-1}$, and so on.

Remark 16 *It is worth mentioning that if the condition* $\det(\mathcal{L}) \neq 0$ *does not hold, then the previous propositions are not still true. As an alternative to this problem, the above integrals may be either computed using a numerical method or calculating an auxiliary exponential matrix of higher dimension as in Aliseyko (2019).*

Numerical algorithm: The general procedure for computing matrices Λ_N and Π_N is presented, when $\det(\mathcal{L}) \neq 0$. It is summarised by the following steps:

1. Compute the delay Lyapunov matrix $U(\tau),\ \tau \in [0, h]$, associated with a positive definite matrix W. It is solved by using the semi-analytic method (2.7) introduced in Chapter 2.
2. Set an arbitrary integer number N. For matrix Λ_N, also compute $\tau = h/N$.
3. Compute either integrals $J_l,\ F_l,\ \bar{F}_l,\ Y_k^{(1)},\ Y_k^{(2)}$ and Z_k by using Proposition 1 and 2, or integrals $G_k,\ \bar{G}_k,\ H_{kl}$ and $\bar{H}_{kl}$ by using Proposition 3 and 4.
4. Represent either $\mathcal{M}_k,\ \mathcal{N}_k,\ \mathcal{P}_l$ and $\mathcal{Q}_l$, or $\mathcal{J}_{1N},\ \mathcal{J}_{2N},\ \mathcal{J}_{3N}$ and $\mathcal{J}_{4N}$, in terms of the integrals of step 3.
5. Construct the corresponding matrix Λ_N or Π_N as required.

Chapter 8

Illustrative examples

In this section, we validate the stability criterion presented in Theorem 6. We also compare it with results based on other functional argument approximations: system fundamental matrix (Theorem 3), piecewise linear functions (Theorem 4) and Legendre polynomials (Theorem 5). In each example, the delay Lyapunov matrix is computed by (2.7) in Chapter 2 with $W = I_n$, the elapsed time is measured in seconds (s), and the matrix dimension of Theorem 3, Theorem 4, Theorem 5 and Theorem 6 is computed as $n(\hat{N}+1)$, $n(N_0+1)$, $n(N_1+1)$ and $n(N_2+1)$, respectively. In the figures of Example 1, 2 and 3, the space of parameters is presented. The stability/instability candidate boundaries, obtained by the D-subdivision technique (Neimark, 1949), are depicted by a solid line. The isolated dots represent points in the space of parameters where the outcome of the necessary conditions in the stability criterion is affirmative. The non-negativity of Λ_N and Π_N and the positivity of $\mathcal{K}_N$ is verified by using the function *"cholcov"* in Matlab. The implementation is done on a desktop computer with an Intel Core i7-8700, 3.20 GHz, 6 cores, and 32 GB RAM.

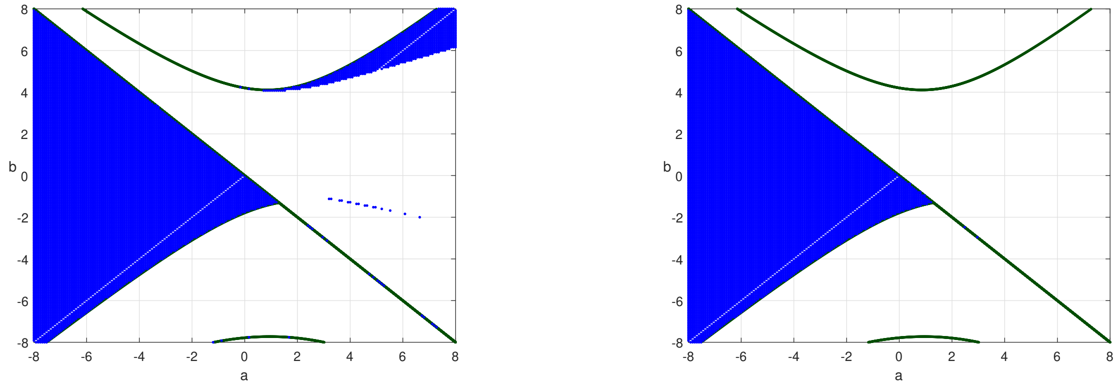
Example 1: Consider a scalar neutral type equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) - dx(t-h)) = ax(t) + bx(t-h).$$

For the parameter values $h = 1$ and $d = -0.3$, the stability boundaries are depicted by a solid line in Figure 8.1. The constants a_0 and a_1 defined in Lemma 3 and Lemma 5, respectively, are computed with $\alpha = K$ with the constant K defined in the set $\mathcal{S}$. The isolated dots represent points in the space of parameters where the non-negativity test of Λ_N holds. The involved integrals of Λ_N are computed by using the recursive method in Chapter 7. In this example, the stability criterion of Theorem 4 is also tested and compared with the one of Theorem 3.

In order to show the effectiveness of the necessary conditions of Theorem 4, the non-negativity of Λ_1 and Λ_3 is verified. The line $a = b$ is omitted since $\det(\mathcal{L}) = \frac{b^2-a^2}{1-d^2} = 0$. Figure 8.1 exhibits that the exact stability region is visually achieved already for small values of N.

In Table 8.1, for $a = 1$, $b = -2$, $d = -0.3$, and different values of h, we compare the matrix dimension provided by Theorem 3 and Theorem 4. The symbol (—) indicates that the numerical burden of the test surpasses the computer RAM. We remind that the stability in Theorem 3 is tested via discretization of the delay Lyapunov matrix. The matrix dimension of Theorem 3 and Theorem 4 is computed as $n\hat{r}$ and $n(N_0+1)$, respectively. Notice that the sufficiency matrix dimension of Theorem 4 is substantially smaller than that of Theorem 3. It also evidences the impact of using piecewise linear approximations for the functional argument, where a better and faster convergence toward any function φ is obtained instead of using the system fundamental matrix.



(a) Test $\Lambda_1 \geq 0$ (b) Test $\Lambda_3 \geq 0$

Figure 8.1: Necessary conditions of Theorem 4, *Example 1*

Table 8.1: Matrix dimension given by Theorem 3 and Theorem 4 with $a = 1,\ b = -2,\ d = -0.3$, and different values of h, *Example 1*

h	Theorem 3	Outcome	Theorem 4	Outcome
0.3	13981	—	4	Stable
0.5	1029069	—	8	Stable
0.73	2.9×10^9	—	97	Stable
0.74	2.7×10^9	—	87	Unstable
1	6.3×10^9	—	23	Unstable
2	4.8×10^{19}	—	42	Unstable

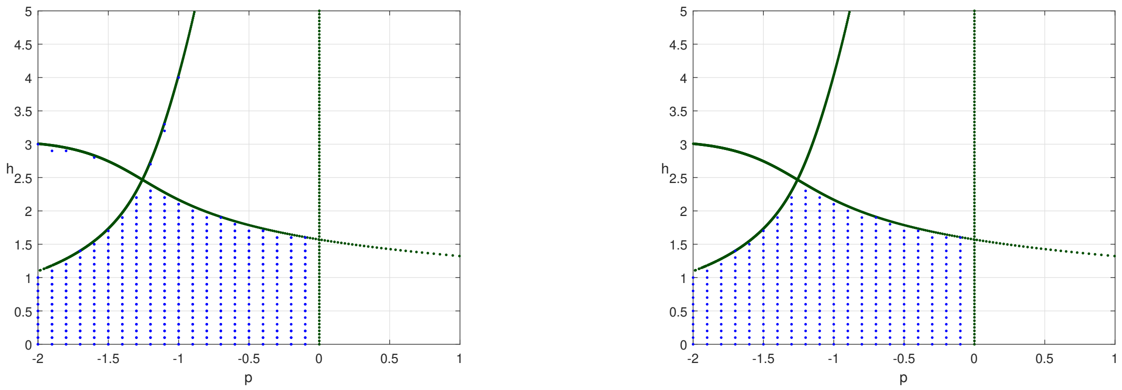
Example 2: Consider system (2.1) and define the matrices

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0.25 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0.2 \\ p & 0 \end{pmatrix},$$

where $p \in \mathrm{R}$ and $h > 0$ are free parameters. Here, the constants a_0 and a_1 defined in Lemma 3 and Lemma 5, respectively, are computed with $\alpha = K$. For the space of parameter (p, h), the stability boundaries are depicted by a solid line in Figure 8.2. In this example, the necessary conditions of Theorem 5 are verified to Π_1 and Π_3. The isolated dots represent points where the non-negativity of the criterion holds. Notice that the necessary conditions of Theorem 5 reveal the exact stability region for small values of N similar to the necessary conditions of Theorem 4 (piecewise linear approximations).

Next, considering the vicinity of the point $(p, h) = (-1.2, 2.395)$ at the stability/instability boundary in Figure 8.2, the stability of system (2.1) is tested with the stability criteria presented in Theorem 3 (system fundamental matrix approximation), Theorem 4 (piecewise linear approximation) and Theorem 5 (Legendre polynomial approximation), for $p = -1.2$ and different values of h.

Table 8.2 shows that Theorem 5 based on Legendre polynomials approximation also provides a matrix dimension substantially reduced compared with those provided by Theorem 3. In comparison with Theorem 4, which is based on piecewise linear approximations, the matrix dimension of Theorem 5 is notably smaller when approaching the stability boundary of the delay h. It is due to the tendency towards infinity of the delay Lyapunov matrix at the stability/instability boundary. However, the value N_2 attenuates these large values thanks to



(a) Test $\Pi_1 \geq 0$ (b) Test $\Pi_3 \geq 0$

Figure 8.2: Necessary conditions of Theorem 5, *Example 2*

Table 8.2: Matrix dimension given by Theorem 3, Theorem 4 and Theorem 5 with $p = -1.2$, *Example 2*

h	Theorem 3	Outcome	Theorem 4	Outcome	Theorem 5	Outcome
1.5	10.8×10^7	—	58	Stable	12	Stable
2	6×10^9	—	150	Stable	15	Stable
2.39	19×10^{11}	—	1240	Stable	19	Stable
2.4	3×10^{12}	—	1576	Unstable	19	Unstable
3	4×10^{12}	—	444	Unstable	20	Unstable
3.5	2.2×10^{13}	—	344	Unstable	22	Unstable

the involved logarithm in its computation as evidenced in the proof of Lemma 12.

Example 3: The σ-stability analysis of the proportional-integral control of a passive linear system leads to studying a quasipolynomial of neutral type (Castaños, Estrada, Mondié, & Ramírez, 2018). Its time domain representation is of the form (2.1), with matrices $D = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\alpha_2}{\alpha_1} \end{pmatrix}$,

$$A_0 = \frac{1}{\alpha_1} \begin{pmatrix} 0 & \alpha_1 \\ -\sigma^2\alpha_1 + \sigma\beta_1 - \gamma_1 & -\beta_1 + 2\sigma\alpha_1 \end{pmatrix},$$

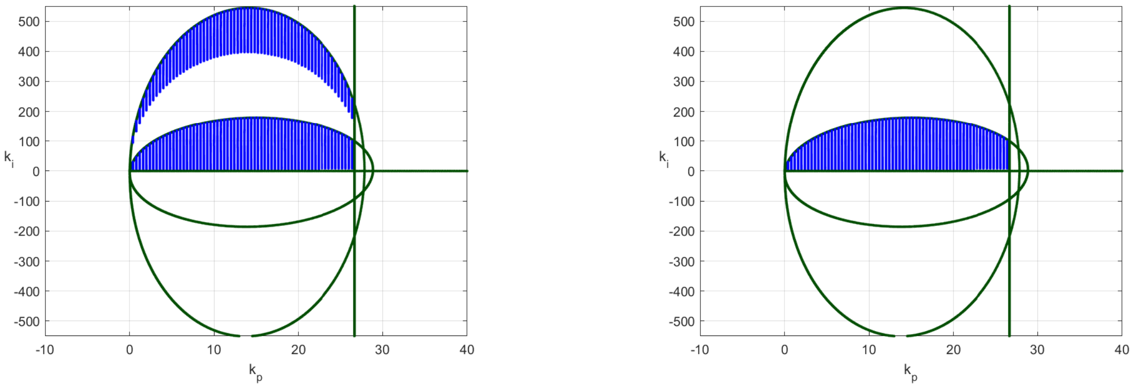
$$A_1 = \frac{1}{\alpha_1} \begin{pmatrix} 0 & 0 \\ -\sigma^2\alpha_2 + \sigma\beta_2 - \gamma_2 & -\beta_2 + 2\sigma\alpha_2 \end{pmatrix},$$

where

$$\alpha_1 = d + k_p, \quad \gamma_1 = bk_i d + ak_i,$$
$$\alpha_2 = (d - k_p)e^{\sigma h}, \quad \gamma_2 = (bk_i d - ak_i)e^{\sigma h},$$
$$\beta_1 = (bk_p + a)d + bd^2 + ak_p + k_i,$$
$$\beta_2 = ((bk_p + a)d - bd^2 - ak_p - k_i)e^{\sigma h}.$$

In this example, we assess the effect of using α computed as in Lemma 4 instead of $\alpha = K$. Then, we proceed with the comparison with other stability tests presented in the previous chapters.

For the parameters, $a = 0.4$, $b = 50$, $h = 0.2$, $d = 0.8$, $\sigma = 0.3$, the stability boundaries are depicted by a solid line in Figure 8.3. The stability of the difference operator imposes in the D-subdivision map the additional condition $|k_p| < 26.67$. In Figure 8.3, the positivity of $\mathcal{K}_1$



(a) Test $\mathcal{K}_1 > 0$ (b) Test $\mathcal{K}_3 > 0$

Figure 8.3: Necessary conditions of Theorem 6, *Example 3*

and $\mathcal{K}_3$ of the necessity conditions of Theorem 6 is tested for the space of parameter (k_p, k_i). The isolated points indicate that the positivity of $\mathcal{K}_N$ holds. It is worth mentioning that each of these conditions achieves visually the whole stability region for $N = 3$.

In Table 8.3, for the stable pair $(k_p, k_i) = (1, 1)$ and unstable pair $(k_p, k_i) = (1, -1)$, we compare the matrix dimension of the stability test in Theorem 6 when $\alpha = K$ and when α is computed using Lemma 4. In this case, a binary search on the interval $(0, K]$ with a precision 10^{-6} is carried out. Clearly, a better estimate of α in the instability result in Lemma 3, reduces substantially the dimension of the stability criterion.

Table 8.3: Matrix dimension given by Theorem 6 with different α, *Example 3*

(k_p, k_i)	$\alpha = K$	Theorem 6	α as in Lem. 4	Theorem 6
$(1, 1)$	72.74	68	0.2893	8
$(1, -1)$	88.03	110	1.2103	16

Next, we compare the matrix dimension and computational time of the stability test of Theorem 4 with other proposals in the previous chapters. For a fair comparison, we use the improved bound α of Lemma 4 instead of $\alpha = K$ in all cases. The results are presented in Table 8.4 and Table 8.5 for $(k_p, k_i) = (1, 1)$ and $(k_p, k_i) = (1, -1)$, respectively. The symbol (—) indicates that the numerical burden of the test surpasses the computer RAM.

Table 8.4: Matrix dimension and elapsed time given by Theorem 3, Theorem 4, Theorem 5 and Theorem 6 with $(k_p, k_i) = (1, 1)$, *Example 3*

Stability criterion	Matrix dim.	Outcome	Elapsed time (s)
Theorem 3	5157685	—	—
Theorem 4	10	Stable	0.551 s
Theorem 5	33	Stable	20.56 s
Theorem 6	8	Stable	0.1154 s

Table 8.4 and Table 8.5 show that the matrix dimension for sufficiency in Theorem 3, even with our improved bound for α, exceeds the RAM capacity. It appears that Theorem 4, and Theorem 5 provide tractable matrix dimensions. It is worth noticing that, among functional

Table 8.5: Matrix dimension and elapsed time given by Theorem 3, Theorem 4, Theorem 5 and Theorem 6 with $(k_p, k_i) = (1, -1)$, *Example 3*

Stability criterion	Matrix dim.	Outcome	Elapsed time (s)
Theorem 3	42647927	—	—
Theorem 4	24	Unstable	0.683 s
Theorem 5	38	Unstable	35.41 s
Theorem 6	16	Unstable	0.1241 s

argument approximation approaches, the piecewise linear one (Theorem 4) gives a matrix dimension close to Theorem 6. This is due to a similar convergence rate for the error quantification of the functional argument or functional kernel, which is determined by the factor N^{-2}.

Theorem 6 has notable advantages over the other criteria. First, it substantially reduces the matrix dimension for sufficiency. Second, its computational burden outperforms Theorem 4 and Theorem 5. The reason is that the numerical test only depends on the computation of the Lyapunov matrix and its evaluation at discrete values, followed by a positivity test. On the contrary, the conditions of Theorem 4 and Theorem 5 are semi-definite positivity tests that also require the computation of time-consuming integrals depending on the Lyapunov matrix and its derivatives.

Example 4: The Proportional-Derivative control $u(s) = k_p + k_d s$ of the system described by the transfer functions

$$H(s) = \frac{15s^2 + 3s - 20}{125s^3 + 70s^2 + 10s + 8} e^{-hs},$$

introduced in Méndez (2011), is described by system (2.1) with $h > 0$ and matrices

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{15}{125}k_d \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{8}{125} & -\frac{10}{125} & -\frac{70}{125} \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{20}{125}k_p & -\frac{1}{125}(3k_p - 20k_d) & -\frac{8}{125}(15k_p + 3k_d) \end{pmatrix},$$

where parameters k_p and k_d are the proportional and derivative gains, respectively.

In Tables 8.6 and 8.7, for the stable pair $(k_p, k_d) = (0.25, 0.25)$ and different values of the delay h, the matrix dimension and elapsed calculation time of the test of Theorem 6 are compared to those obtained in Theorem 3, Theorem 4 and Theorem 5. In all cases, the less conservative estimates of α of Lemma 4 are employed. It is worthy of mention that the matrix dimension of Theorem 5 notably outperforms the one of Theorem 6 when the delay is very close to the stability boundary. This is due to the fact that the Lyapunov matrix and its second derivative increase when the Lyapunov condition is violated, which directly affects the dimension, see Lemma 15. In the case of Theorem 5, these expressions are attenuated through logarithms, resulting in less sensitive matrix dimensions. It confirms the presented results in *Example 2* and *3*.

Table 8.6 and Table 8.7 indeed corroborate the previous example, demonstrating a significant reduction in the matrix dimension and computational burden of the stability test of Theorem 6 compared to the other approaches, except for parameter values extremely close to stability boundaries. The poor quality of the approximation by fundamental matrices explains why the computer's memory capacity is exceeded when verifying the conditions of Theorem 3. A larger computational elapsed time of testing the conditions of Theorem 5 and Theorem 4 is due to the evaluation of integrals of the delay Lyapunov matrix and its derivatives implemented by recursive methods.

Table 8.6: Matrix dimension given by Theorem 3, Theorem 4, Theorem 5 and Theorem 6 with $(k_p, k_d) = (0.25, 0.25)$, *Example 4*

h	α	Theorem 3	Theorem 4	Theorem 5	Theorem 6
1.5	0.13	1×10^5	33	9	9
2.5	0.12	1×10^6	54	12	12
3.5	0.11	12×10^6	84	15	18
4.96	0.10	6×10^9	756	22	246
4.97	0.10	6×10^{10}	2478	24	807
5.5	0.10	6×10^8	162	21	42
6.5	0.09	27×10^8	171	24	36
7.5	0.09	15×10^9	195	27	39

Table 8.7: Elapsed time given by Theorem 3, Theorem 4, Theorem 5 and Theorem 6 with $(k_p, k_d) = (0.25, 0.25)$, *Example 4*

h	α	Theorem 3	Theorem 4	Theorem 5	Theorem 6
1.5	0.13	—	0.61 s	9.84 s	0.15 s
2.5	0.12	—	0.61 s	10.07 s	0.13 s
3.5	0.11	—	0.61 s	12.65 s	0.13 s
4.96	0.10	—	1.32 s	15.47 s	0.18 s
4.97	0.10	—	7.4 s	22.01 s	0.38 s
5.5	0.10	—	0.64 s	18.68 s	0.15 s
6.5	0.09	—	0.64 s	21.32 s	0.13 s
7.5	0.09	—	0.66 s	23.71 s	0.14 s

Chapter 9

Conclusions and Future Work

9.1 Conclusions

This thesis presents three tractable stability tests for neutral linear time delay systems. They are based on approximating either the functional argument or the functional kernels in the framework of Lyapunov-Krasovskii functionals with prescribed derivative. The results are made possible by the instability results on the set of bounded functions $\mathcal{S}$, which allows determining both an instability constant and a bound on the approximation error. Our results are necessary and sufficient, unlike results presented in the literature based on quadratic functionals of prescribed form stated as LMI-type sufficient stability conditions (for example, see Theorem 2). Furthermore, they are verified in a moderate number of mathematical operations compared with the large dimensions provided by the stability criterion of Theorem 3 (Gomez et al., 2021).

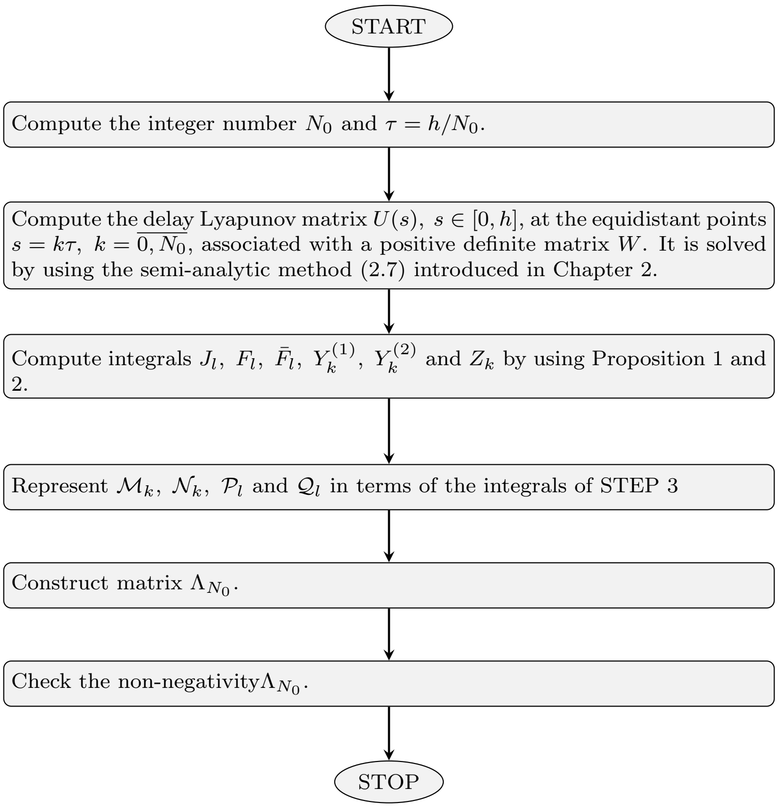
In order to clarify and remind the construction and verification of our main results, we present a flowchart of each stability criterion given by Theorem 4, Theorem 5 and Theorem 6 in Figure 9.1, 9.2 and 9.3, respectively. These flowcharts briefly allow illustrating the algorithm to implement our stability criteria. In addition, Table 9.1 compares our main results and Theorem 3 considering two main aspects: approximation order and simplicity of the criterion. This comparison is carried out with the help of the outcomes obtained from the examples in previous chapters.

From a control design perspective, the use of the delay Lyapunov matrix and the proposed stability criteria is an ongoing challenge. The existing approaches, available for the retarded case only, employ the delay Lyapunov matrix framework in predictor-based feedback design for systems with both input and state delays (Juárez, Mondié, & Kharitonov, 2020), as well as in optimizing the pseudo-spectral abscissa and the $\mathcal{H}_2$-norm (Gomez & Michiels, 2019) along with the quadratic cost function (Gomez, Michiels, & Mondié, 2019). Moreover, in predictor-based feedback design, the delay Lyapunov matrix of the target system defines a Lyapunov functional

	Simple	**Complex**
High approximation order	Theorem 3	—
Low approximation order	Theorem 6	Theorem 4, Theorem 5

Table 9.1: Simplicity of the criterion vs. Approximation order

Figure 9.1: Flowchart of the construction and verification of the stability criteria of Theorem 4



that can be used for robustness analysis (Juárez et al., 2020), ISS analysis, etc. Thus, our stability criteria allow the extension of these approaches to the neutral type case, among other uses.

For the case of functional argument approximation, we explore the use of piecewise linear and Legendre polynomial functions as the bases for approximation, yielding efficient criteria involving the verification of the non-negativity of a matrix in both cases. Next, we give some conclusions based on these results:

- Table 9.1 shows that the Legendre and piecewise linear approximation of φ give tractable approximation orders than those using the system fundamental matrix (Theorem 3), but at the cost of losing the simplicity, particularly the elegant form of discrete values of the delay Lyapunov matrix of Theorem 3.

- Regarding Legendre and piecewise approximations, the obtained orders of approximation are similar in spite of the super-geometric property of Legendre, except in the vicinity of critical delays. The main reason is that the approximation orders using Legendre polynomials depend on logarithms, which attenuate the increased values of the norm of the delay Lyapunov matrix at the stability/instability boundaries.

- Notice that the implementation of the criterion of Legendre polynomials is more time-consuming than the one of the criterion of the piecewise linear approximations due to the computations of integrals of the delay Lyapunov matrix multiplied by N-order Legendre polynomials, compared to the third-order monomials required in piecewise linear approximation of φ as shown in the recursive method presented in Chapter 7.

The functional kernels approximation criterion, characterized by the positivity of the matrix, stands out among the presented results:

Figure 9.2: Flowchart of the construction and verification of the stability criteria of Theorem 5

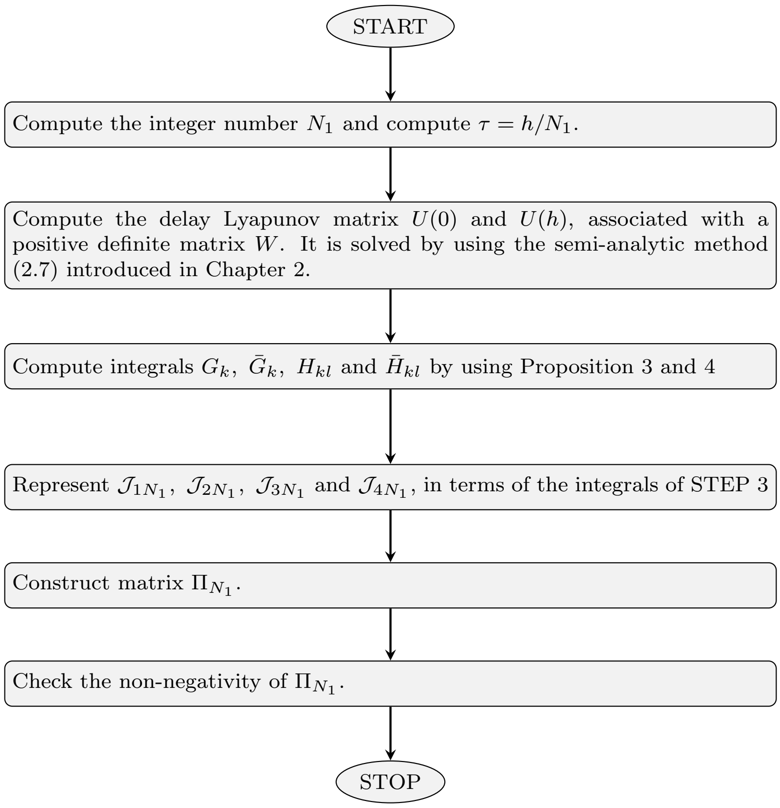
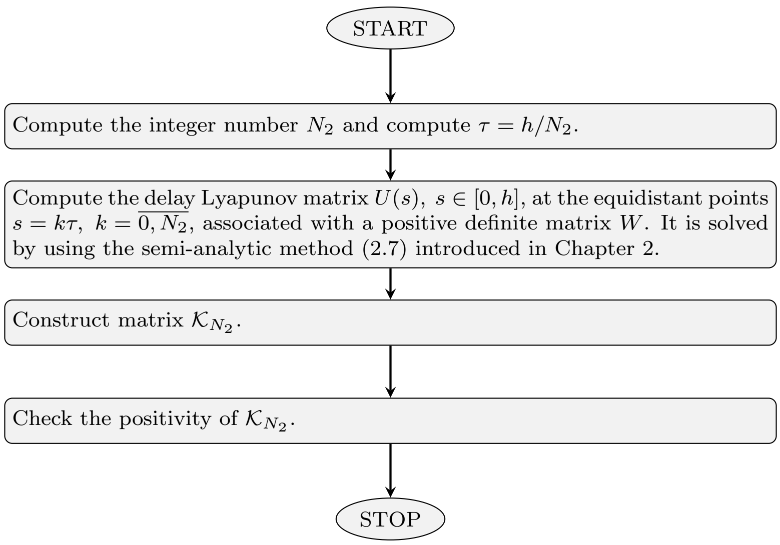


Figure 9.3: Flowchart of the construction and verification of the stability criteria of Theorem 6



- It has the elegant form in terms of discrete evaluations of an array of Lyapunov matrices as the criterion of Theorem 3.

- Compared with the functional argument approximation, the approximation order of this criterion outperforms the previous methods with substantially reduced time execution. Unfortunately, when approaching the stability/instability boundaries of the delay, this approach also shows large orders of approximation compared to those using Legendre polynomials, which is a good choice for these special issues.

- Finally, Table 9.1 indicates that the stability criterion of Theorem 6 is more appropriate since it trades off between approximation orders and the simplicity of the criterion.

9.2 Future Work

This thesis work so far has implemented approximations for the functional argument or functional kernels to obtain stability tests in a moderate number of computations for neutral type systems. As a result, a stability criterion given in terms of discrete evaluations of the delay Lyapunov matrix gave the best outcomes. Based on the results, the following is considered as future work:

- Improve the bound for the instability result of Lemma 3, as the bound computed introduces a conservatism in the assessment of the stability conditions presented in this work.
- From a control design perspective, use the delay Lyapunov matrix and the proposed stability criteria for predictor-based feedback design for neutral type systems with both input and state delays, as well as in optimizing the pseudo-spectral abscissa and the $\mathcal{H}_2$-norm along with the quadratic cost function.
- Extend the stability criterion of Theorem 6 to the multiple commensurate or incommensurate delays (incommensurate even for retarded type systems) delay case for neutral type systems.
- Extend the discretized Lyapunov method of Section 6.1 to the case of distributed delay systems. There, the system complexity is increased due to the presence of the integral of a distributed kernel multiplied by past states, whose discretization may be carried out only for special cases of the distributed kernel.

Appendix A

Auxiliary results

In this appendix, we remind the Schur complement, the theorem of Taylor and the concept of the Kronecker product, which are used to achieve the results presented in this thesis work.

Lemma 16 *(Schur Complement) Let*

$$X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}.$$

The following statements are true:
(a). Let $A > 0$. Then, $X \geqslant 0$, if and only if

$$C - B^T A^{-1} B \geqslant 0.$$

(b). Matrix $X > 0$, if and only if $C > 0$ and

$$A - BC^{-1}B^T > 0.$$

Theorem 7 *(Theorem of Taylor) Let $k \geqslant 1$ be an integer and let the function $f : \mathbb{R} \to \mathbb{R}$ be k times differentiable at the point $a \in \mathbb{R}$. Then,*

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k \mathrm{d}t.$$

The Kronecker product: Using the non-standard definition introduced in Kharitonov (2013), the Kronecker product is defined by

$$A \otimes B = \begin{pmatrix} b_{11}A & b_{21}A & \cdots & b_{n1}A \\ b_{12}A & b_{22}A & \cdots & b_{n2}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n}A & b_{2n}A & \cdots & b_{nn}A \end{pmatrix}.$$

The next property holds:

$$\mathrm{vec}(AQB) = (A \otimes B)\mathrm{vec}(Q),$$

where $Q \in \mathbb{R}^{n \times n}$ and $\mathrm{vec}(Q)$ stands for the vectorization of matrix Q, which is obtained from Q by stacking up its columns.

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