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Arithmetical structures on paths and cycles

A thesis presented by:

Miguel Alonso Izquierdo

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Thesis Advisor: Ph.D. Carlos Enrique Valencia Oleta

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Miguel Alonso Izquierdo

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Dr. Carlos Enrique Valencia Oleta

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Resumen

El objetivo de esta tesis es el de estudiar las estructuras aritméticas de gráficas. En particular nos centraremos en enumerar las estructuras aritméticas del camino $\mathcal{P}_n$ y el ciclo C_n con n vértices. Para esto, primero desarrollaremos algunas propiedades generales de cierta clase de matrices no negativas y nos adentraremos en la combinatoria de Catalán.

La tesis está organizada de la siguiente manera: En el Capítulo 1, cubriremos los preliminares de álgebra lineal y combinatoria que necesitaremos en el resto de la tesis. En el Capitulo 2, exploraremos algunas propiedades de las matrices no negativas, donde destaca el clásico Teorema de Perron-Frobenius, del cual daremos una demostración autocontenida. El Teorema de Perron-Frobenius se usará para demostrar varios resultados en el estudio de M-matrices, las cuales, como observaremos tienen una profunda relación con las estructuras aritméticas de gráficas.

Finalmente, en el Capítulo 3, estudiaremos las estructuras aritméticas tanto del camino como del ciclo. A cada estructura aritmética le asociaremos cierto objeto combinatorio presentado en los capítulos anteriores, lo cual nos permitirá enumerar las estructuras aritméticas en el camino, cuyo número resulta ser los famosos números de Catalán. Y en el caso del ciclo con n vértices, su número de estructuras aritméticas es el coeficiente binomial $\binom{2n-1}{n-1}$.

Abstract

The goal of this thesis is to study the arithmetical structures of graphs. In particular, we will focus on enumerating the arithmetical structures of the path $\mathcal{P}_n$ and the cycle C_n with n vertices. To do this, we will first develop some general properties of a particular class of non-negative matrices, and we will go into Catalan combinatorics.

The thesis is organized as follows: In Chapter 1, we will cover the preliminaries of linear algebra and combinatorics that we will need in the rest of the thesis. In Chapter 2, we will explore some properties of non-negative matrices, including the classic Perron-Frobenius Theorem, for which we will give a self-contained proof. The Perron-Frobenius Theorem will be used to prove several results in the study of M-matrices, which, as we will observe, have a deep relationship with the arithmetical structures of graphs.

Finally, in Chapter 3, we will study the arithmetical structures of the path and the cycle. We will associate each arithmetic structure with lattice paths, which will be presented in the preliminaries. This will allow us to enumerate the arithmetic structures in the path whose numbers turn out to be the famous Catalan numbers. And, in the case of the cycle with n vertices, its number of arithmetic structures is the binomial coefficient $\binom{2n-1}{n-1}$.

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Chapter 1

Introduction

The concept of an arithmetical graph was introduced in 1989 by Dino Lorenzini. It came from Arithmetic Geometry, more precisely appearing in the study of algebraic curves that degenerate into different components and how they intersect.

One way to codify how two curves intersect with each other is by using a matrix L whose entries are the number of intersections between the curves. More precisely, let C be a curve that degenerates in n components $C_1, \ldots, C_n$, and let L be the matrix defined by setting its off-diagonal entries equal to

$$L_{i,j} = -|C_i \cap C_j| \text{ for all } 1 \leq i \neq j < n$$

and the number of self-intersections on the diagonal. When this matrix has a nontrivial kernel, we have, in essence, what we call an arithmetical structure or arithmetical graph when L is associated with a graph.

This serves as a historical motivation for the study of arithmetical structures, but for the purposes of our work, we will not be inclined to study this angle. Instead, we will start with the following definition.

Definition 1.0.1. *An arithmetical graph is a triplet* $(G, \mathbf{d}, \mathbf{r})$*, composed by a finite connected graph* G*, and positive integer vectors* $\mathbf{d}$ *and* $\mathbf{r}$*, with* $\mathbf{r}$ *primitive, such that*

$$(\operatorname{diag}(\mathbf{d}) - A_G)\mathbf{r} = \mathbf{0}.$$

Here, primitive is understood as having all entries in the vector be setwise coprime; that is, the largest positive integer that divides all entries of $\mathbf{r}$ is 1.

We should note that this condition is essential, as it will guarantee the uniqueness of the pair $(\mathbf{d}, \mathbf{r})$; that is, there is only one $\mathbf{r}$ associated with each $\mathbf{d}$.

The pair $(\mathbf{d}, \mathbf{r})$ in definition 1.0.1 is said to be an *arithmetical structure*, and its components $\mathbf{d}, \mathbf{r}$ can be referred to as an arithmetical d-structure and arithmetical r-structure respectively. The matrix $\operatorname{diag}(\mathbf{d}) - A_G$ is what we will calle the *pseudo-Laplacian* of G and we denote it by $L(G, \mathbf{d})$. Further along, we will show that this indeed gives us a unique pair of vectors, as the kernel of the pseudo-Laplacian will have rank 1.

Once we have formally defined an arithmetical graph, we proceed to question ourselves about its most basic and essential properties and which of them are of most interest to us.

It is important to recall that any arithmetical structure will give us a solution to the Diophantine equation

$$\det(A - \operatorname{diag}(x_1, \ldots, x_n)) = 0.$$

Our interest will be centered around the path $\mathcal{P}_n$ and the cycle C_n. We will be especially interested in counting the number of arithmetical structures on these two families of graphs.

Chapter 2

Preliminaries

For the purposes of this work, it will be useful to review some concepts and results in graph theory, linear algebra, and combinatorics.

First, we will start with a review of basic concepts in graph theory; then, we will introduce some results in linear algebra, which will prove vital once we begin our study of nonnegative matrices, especially their eigenvalues. Finally, we will review Catalan combinatorics, particularly their relation to Lattice paths.

2.1 Graph theory

In this section, we will first briefly review basic concepts in graph theory and some essential results regarding the study of critical groups in arithmetical structures, which will be needed for the following chapters.

Definition 2.1.1. *A graph G is a pair (V, E), where V is a set and E is a set of non-ordered pairs in V.*

The elements in V are called the vertices of G, and the elements of E are called its edges, and we consider that $\{a, b\} = \{b, a\}$ for any $a, b \in V$. Usually, we will denote by V and E the vertex and edge set of a graph G, respectively.

We say the graph G is finite whenever the set V is finite. A loop is an edge that goes from a vertex to itself, that is, an edge of the form $\{v, v\}$ for some $v \in V$.

There are two alterations that we can make to our definition of a graph to obtain different types of it. First, we can consider edges as ordered pairs of vertices, in which case we would call G a *directed graph*, or digraph for short. We can also allow E to be a multiset, so there may be multiple edges between vertices; in this case, we say that G is a multigraph.

If a graph is not a multigraph and does not have loops, it is called simple. In this work, we will consider all of our graphs simple unless specified otherwise.

Definition 2.1.2. *If $G = (V, E)$ and $G' = (V', E')$ are graphs with $V' \subset V$ and $E' \subset E$, then we say that G' is a subgraph of G, denoted by $G' \subset G$.*

Besides, if $G' \subset G$ and $V' = V$, we say that G' is a *spanning* subgraph of G.

Example 2.1.3. *Some common examples of graphs are:*

1. *The null graph with its vertex and edge set equal to $\emptyset$.*
2. *The complete graph K_n on n vertices defined by*

$$V(K_n) = \{v_1, \dots, v_n\} \text{ and } E(K_n) = \{\{v_i, v_j\} : 1 \le i \ne j \le n\}.$$

3. *The path with n vertices $\mathcal{P}_n$ is the graph on the n vertices $\{v_1, \dots, v_n\}$ and the $n-1$ edges*

$$E(\mathcal{P}_n) = \{\{v_i, v_{i+1}\} : 1 \le i \le n-1\}.$$

4. *The cycle with n vertices C_n is the graph obtained by adding the edge $\{v_1, v_n\}$ to $\mathcal{P}_n$.*

These last two graphs will be of importance to us in later chapters. We say that a graph G has a *cycle* whenever there is some $C_n \subset G$ for some n. We call a graph acyclic or a *forest* whenever it has no cycles.

A graph G is connected whenever, for each pair of vertices u, v of G, there is a path that starts at $u = v_1$ and ends at $v = v_n$. A connected forest is called a *tree*. Spanning trees will play an important part in the study of arithmetical structures of a graph.

We will denote the edge $\{u, v\}$ by uv to simplify the notation. A vertex u is adjacent to a vertex v whenever $uv \in E$. This relation can be codified in what we call the adjacency matrix.

Definition 2.1.4. *The adjacency matrix of a graph G is the matrix A_G whose rows and columns are indexed by the vertices of G, and its entries are given by*

$$(A_G)_{u,v} = \begin{cases} 1 & \text{if } uv \in E, \\ 0 & \text{in other case.} \end{cases}$$

Note that the adjacency matrix of a graph is symmetric because $uv \in E$ if and only if $vu \in E$.

Example 2.1.5. *We describe the adjacency matrix of the complete graph and the path.*

1. *The adjacency matrix of the complete graph in n vertices is the $n \times n$ matrix with all entries equal to* 1 *except the diagonal, which are equal to* 0*.*

$$\begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}$$

2. *The adjacency matrix of the path in n vertices $\mathcal{P}_n$ is the tridiagonal matrix with diagonal entries equal to* 0 *and entries equal to* 1 *on the secondary diagonals adjacent to the main diagonal.*

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

We call an edge $e = uv$ incident to the vertices u and v. The degree of a vertex v of graph G, denoted by $\deg_G(v)$, is the number of edges incident to it. The degree vector of G is the vector $\deg_G$ whose entries are indexed by the vertices of G, and the entry indexed by v is the degree of v in G.

Example 2.1.6.

1. *The degree vector of the complete graph $\mathcal{K}_n$ is*

$$(n-1, n-1, \dots, n-1, n-1).$$

2. *The degree vector of the path $\mathcal{P}_n$ is*

$$(1, 2, \dots, 2, 1).$$

3. *The degree vetor of the cycle C_n is*

$$(2, 2, \dots, 2, 2).$$

With these definitions, we can give our first result on arithmetical structures.

Theorem 2.1.7. *For a connected graph G, the pair*

$$\mathbf{d} := \deg(G) \text{ and } \mathbf{r} := \mathbf{1} = (1, \dots, 1),$$

define an arithmetical structure on G, which will be called the Laplacian arithmetical structure of G.

2.2 Linear algebra

In the following chapter, we study the nonnegative matrices; we will be particularly interested in the behavior of their spectrum. Thus, it will be useful to review some results in linear algebra.

We introduce some notation that will be useful for the rest of this section. Given $n \in \mathbb{N}$, let $[n]$ be denote the set $\{1, 2, \dots, n\}$. Now, given $I, J \subseteq [n]$, let $M[I, J]$ be the submatrix of M conformed of the rows indexed by I and columns indexed by J. When $I = J$, the submatrix $N = M[I, J]$ of M it is called a principal submatrix of M. Moreover, if $I = J = \{1, 2, \dots, s\}$ for some $1 \le s \le n$, then N it is called a leading principal submatrix of M.

The determinant of a principal submatrix is called a principal minor, and a leading principal minor is the determinant of a leading principal submatrix.

Similarly, $M[I^c, J^c]$ is the submatrix of M obtained by erasing the rows and columns of M indexed by I and J, respectively. In this section, the space of square $n \times n$ matrices over a field $\mathbb{F}$ will be denoted by $\mathbb{F}^{n\times n}$.

Theorem 2.2.1 (Gershgorin's circle theorem)**.** *If M is a square complex matrix of size n and λ is one of its eigenvalues, then*

$$\lambda \in \bigcup_{i=1}^{n} D(M_{ii}, t_i),$$

where $t_i = \sum_{j\neq i} |M_{ij}|$ and $D(c, t)$ is the disk with center at c and radius t.

Proof. Let x be an eigenvector of the eigenvalue λ and $1 \le i \le n$ such that $|x_i| \ge |x_j|$ for all j. Since $Mx = \lambda x$, then $\sum_{j=1}^{n} M_{ij} x_j = \lambda x_i$, thas is, $\sum_{j\neq i} M_{ij} x_j = (\lambda - M_{ii}) x_i$.

Thus

$$|\lambda - M_{i,i}| = \left| \frac{\sum_{j\neq i} M_{i,j} x_j}{x_i} \right| \le \sum_{j\neq i} |M_{i,j}| \frac{|x_j|}{|x_i|} \le \sum_{j\neq i} |M_{i,j}| = t_i$$

and therefore $\lambda \in D(M_{i,i}, r_i)$. □

In the following part of this section, we will be working with matrices called *positive definite* or *positive semi-definite*. This last class of matrices will also contain the pseudo-Laplacian matrix of an arithmetical structure.

We start by defining them.

Definition 2.2.2. *A real symmetric matrix M is called positive definite whenever the number*

$$z^T M z$$

is positive for each $z \in \mathbb{R}^n \setminus 0$. Besides, M is called positive semi-definite whenever $z^T M z \geq 0$ for all $z \in \mathbb{R}^n \setminus 0$. Here z^T denotes the transpose of z.

In the complex case, a matrix is called positive definite whenever

$$z^H M z \text{ is a positive real number for any } z \in \mathbb{C}^n \setminus 0,$$

where $z^H = \overline{z}^H$ denotes the conjugate transpose of z. We define a positive semi-definite matrix similarly in the complex case.

Note that $z^H M z$ being a real number for all $z \in \mathbb{C}^n$ implies that M is equal to its conjugate transpose. More precisely, if $z^H M z$ is always a real number, then

$$z^H M z = (z^H M z)^H = z^H M^H z \text{ for all } z \in \mathbb{C}^n \setminus 0$$

and therefore $M = M^H$, so M is Hermitian whenever it is positive definite or semi-definite. For the rest of this section, M will be a Hermitian matrix of size n, which also implies symmetry in the real case.

Then, an important theorem for this section is the Spectral theorem for Hermitian matrices, so it will be useful to recall it. Given a square matrix A, the spectrum of A, denoted by Λ_A, is the vector with its eigenvalues.

Theorem 2.2.3 (Spectral theorem for Hermitian matrices). *If A is a Hermitian matrix in $\mathbb{C}^{n \times n}$, then*

1. *all of its eigenvalues are real,*
2. *eigenvectors corresponding to different eigenvalues are orthonormal,*
3. *there exist an orthogonal basis of $\mathbb{C}^n$ consisting of eigenvectors of A.*

Moreover, there exists an unitary matrix Q such that $Q^{-1} A Q = diag(\Lambda_A)$.

Taking z to be the vector $x = (x_1, \ldots, x_n)$ formed by a set of variables, the product

$$x^T Mx = \sum_{1 \leq i \leq n} M_{ii} x_i^2 + \sum_{1 \leq i < j \leq n} 2M_{ij} x_i x_j,$$

is the quadratic form associated with M.

We note that that if Q is invertible, then for all $w \in \mathbb{C}^n \setminus 0$ there exists $z \in \mathbb{C}^n \setminus 0$ such that $w = Qz$. That is, Q defines a bijection in $\mathbb{C}^n$.

Lemma 2.2.4. *If P is an invertible matrix and M is a Hermitian matrix, then M is positive definite if and only if $P^H MP$ is positive definite.*

Proof. It follows from the relation $z^H (P^H MP)z = (Pz)^H M(Pz)$ and the fact that for all $w \in \mathbb{C}^n$ there exists some $z \in \mathbb{C}^n$ such that $Pz = w$, by the invertibility of P. □

If M is a Hermitian matrix, then by the Spectral theorem for Hermitian matrices 2.2.3, its eigenvalues are real, and there exists a unitary matrix Q such that $M = Q^H \mathrm{diag}(\Lambda_M)Q$ where $\Lambda_M = (\lambda_1, \ldots, \lambda_n)$ is the vector whose entries are the eigenvalues of M. Moreover, without loss of generality, we can assume $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Theorem 2.2.5. *A matrix M is positive (semi-)definite if and only if its eigenvalues are (nonnegative) positive.*

Proof. We shall only prove the positive definite case, as the arguments for the semi-definite case are similar. ($\Rightarrow$) Let λ be an eigenvalue of M with eigenvector x. That is, $Mx = \lambda x$ with $x \in \mathbb{C}^n \setminus 0$. Multiplying both sides of $Mx = \lambda x$ by x^H we get

$$\lambda x^H x = x^H \lambda x = x^H Mx.$$

Since $x^H x = ||x||^2$, the square of its euclidean norm, is positive and $x^H Mx$ is positive for all $x \in \mathbb{C}^n \setminus 0$, it follows that λ is positive.

($\Leftarrow$) Using that M is Hermitian, by the Spectral theorem for Hermitian matrices 2.2.3, there exists a unitary matrix Q such that $M = Q^H \mathrm{diag}(\Lambda)Q$.

Then

$$x^H Mx = x^H Q^H \mathrm{diag}(\Lambda)Qx = (Qx)^H \mathrm{diag}(\Lambda)(Qx) = y^H \mathrm{diag}(\Lambda)y = \sum_{i=1}^{n} \lambda_i |y_i|^2$$

is positive. Finally, the result follows from Lemma 2.2.4. □

Theorem 2.2.6. *A square matrix M of size n is positive definite if and only if all its leading principal minors are positive.*

Proof. ($\Rightarrow$) Let $I \subseteq [n]$ and $M_I = M[I, I]$ (the submatrix obtained by erasing from M the rows and columns not indexed by I). Let $x \in \mathbb{C}^n \setminus 0$ with $x_i = 0$ for all $i \notin I$ and $y \in \mathbb{C}^I \setminus 0$ be the vector obtained from x by erasing the entries not indexed by I. As $x^H M x = y^H M_I y$ for all $x \in \mathbb{C}^n \setminus 0$ with $x_i = 0$ for all $i \notin I$, then is clear that any principal submatrix of M is also positive definite.

As M_I is positive definite, by Theorem 2.2.5, all of its eigenvalues are positive. The determinant of a matrix is the product of its eigenvalues, thus it follows that the determinant is also positive.

($\Leftarrow$) We now assume that every leading principal minor of M is positive. We proceed by induction on the size of the matrix n. The result is clear for $n = 1$. Now, we will assume it is valid for all matrices of size less than or equal to $n - 1$. Let N be the matrix obtained from M by erasing its last row and column. As the leading principal submatrices of N are also the leading principal submatrices of M, their determinants are also positive according to our hypothesis. By our induction hypotesis N is positive definite so its eigenvalues $\Lambda_N = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1})$ are positive. Let $\Lambda_M = (\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n)$ be the spectrum of M. By the Cauchy interlacing Theorem 2.2.9 on M and N (which will be proven in the next section), we get

$$\mu_1 \leq \lambda_1 \leq \mu_2 \leq \lambda_2 \leq \cdots \lambda_{n-1} \leq \mu_n.$$

As $\lambda_1 > 0$, then $\mu_i > 0$ for all $2 \leq i \leq n$ and it only remais to prove that $\mu_1 > 0$.

As M is also a leading principal submatrix, by hypothesis, its determinant is positive. Finally, since the determinant of M is the product of its eigenvalues, μ_1 must also be positive. □

Note that by Lemma 2.2.4, and because permutation matrices are invertible, the previous Theorem 2.2.6 implies that any principal minor must also be positive.

Theorem 2.2.7. *A matrix M is positive semi-definite if and only if its principal minors are nonnegative.*

Proof. ($\Rightarrow$) Using similar arguments to those used in the proof of 2.2.6, we obtain that any principal submatrix M_I is positive semi-definite. Now, by Theorem 2.2.5, all of its eigenvalues are nonnegative, and so its determinant is nonnegative.

($\Leftarrow$) Let K be a principal matrix of M of size s and $K_t = K + tI_s$ with $t \in \mathbb{R}_+$. A known formula for the characteristic polynomial of a matrix A of size s is

$$\det(tI - A) = \sum_{j=0}^{s} (-1)^j \sigma_j(K) t^{s-j}, \tag{2.1}$$

where $\sigma_j(K)$ is the sum over the principal minors of size j of A, and $\sigma_0 = 1$ (the proof of this formula can be consulted in [12], p.52). Then, by substituting $A = -K$ in the formula, we get

$$\det(K_t) = \sum_{j=0}^{s} \sigma_j(K) t^{s-j}.$$

Since the principal minors of K are principal minors of M and these are nonnegative, then

$$\det(K_t) = \sum_{j=0}^{s} \sigma_j(K) t^{s-j} \geq t^s > 0 \text{ for all } t \in \mathbb{R}_+.$$

That is, all the principal minors $M_t = M + tI_n$ are positive. By Theorem 2.2.6, M_t is positive definite for all $t \in \mathbb{R}_+$. Finally, because $M = \lim_{t\to 0^+} M_t$ and $z^H M z = \lim_{t\to 0^+} z^H M_t z \geq 0$ for all $z \in \mathbb{C}^k \setminus 0$, then M is positive semi-definite. □

2.2.1 Min-max theorem and Cauchy's interlacing theorem.

Given a Hermitian matrix M, the *Rayleigh–Ritz quotient* $R_M : \mathbb{C}^n \setminus 0 \to \mathbb{R}$ is given by

$$R_M(x) = \frac{x^H M x}{x^H x}.$$

Theorem 2.2.8 (Min-max theorem)**.** *If M is a Hermitian matrix with spectrum $\Lambda_M = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n)$, then*

$$\begin{aligned}\lambda_k &= \min_U\{\max_x\{R_M(x) : x \in U \setminus 0\} : \dim(U) = k\} \\ &= \max_U\{\min_x\{R_M(x) : x \in U \setminus 0\} : \dim(U) = n - k + 1\}.\end{aligned}$$

Proof. By the Spectral theorem for Hermitian matrices 2.2.3, there is an orthonormal basis of eigenvectors of M.

Let $Q = \{q_1, \ldots, q_n\}$ be the set of orthonormal eigenvectors of M, W the subspace of $\mathbb{C}^n$ generated by $\{q_k, \ldots, q_n\}$, and U a subspace of $\mathbb{C}^n$ of dimension k.

As the dimensions of U and W are k and $n-k+1$, respectively, there is some $u \in W \cap U$, $u \neq 0$, so that $u = \sum_{i=k}^n c_i q_i \in U$. Then

$$R_M(u) = \frac{u^T M u}{u^T u} = \frac{\sum_{i=k}^n \lambda_i |c_i|^2}{\sum_{i=k}^n |c_i|^2} \geq \frac{\sum_{i=k}^n \lambda_k |c_i|^2}{\sum_{i=k}^n |c_i|^2} = \lambda_k$$

and so $\max_x \{R_M(x) : x \in U \setminus 0\} \geq \lambda_k$. As this is valid for any subspace U of $\mathbb{C}^n$ of dimension k, then

$$\lambda_k \leq \min_U \{\max_x \{R_M(x) : x \in U \setminus 0\} : \dim(U) = k\}.$$

On the other hand, if V is the subspace $\mathbb{C}^n$ generated by $\{q_1, \ldots, q_k\}$, then

$$R_M(v) = \frac{v^T M v}{v^T v} = \frac{\sum_{i=i}^k \lambda_i |c_i|^2}{\sum_{i=1}^k |c_i|^2} \leq \lambda_k \text{ for all } v = \sum_{i=1}^k c_i q_i \in V \setminus 0.$$

Thus $\lambda_k \geq \max_x \{R_M(x) : x \in V \setminus 0\}$ and

$$\min_U \{\max_x \{R_M(x) : x \in U \setminus 0\} : \dim(U) = k\} \leq \max_x \{R_M(x) : x \in V \setminus 0\} \leq \lambda_k,$$

from which the result follows.

The second identity follow from similar arguments in the matrix $-M$ noting that its spectrum is equal to $\Lambda_{-M} = (-\lambda_n \leq -\lambda_{n-1} \leq \cdots \leq -\lambda_1)$. □

Theorem 2.2.9 (Cauchy interlacing theorem or Poincare separation theorem)**.** *Let M be a Hermitian matrix and N the submatrix of M obtained by erasing its last row and column. If $\Lambda_M = (\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n)$ and $\Lambda_N = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1})$ are the spectrum of M and N respectively, then*

$$\mu_1 \leq \lambda_1 \leq \mu_2 \leq \lambda_2 \leq \cdots \lambda_{n-1} \leq \mu_n.$$

Proof. By the Spectral theorem for Hermitian matrices 2.2.3, there is an orthonormal basis of $\mathbb{C}^n$ composed by eigenvectors of M, and another orthonormal basis of $\mathbb{C}^{n-1}$ composed by eigenvectors of N, as it is also Hermitian.

First we prove that $\mu_k \leq \lambda_k$ for all $1 \leq k \leq n-1$. Let $k \in \{1, \cdots, n-1\}$, and $Q = \{q_1, \ldots, q_n\}$ be a set of orthonormal eigenvectors of M in $\mathbb{C}^n$, $\mathcal{P} = \{p_1, \ldots, p_{n-1}\}$ a set of orthonormal eigenvectors of N in $\mathbb{C}^{n-1}$, U the $n-k+1$-dimensional subspace $\mathbb{C}^n$ generated by $\{q_k, \ldots, q_n\}$ and W the k-dimensional subspace of $\mathbb{C}^{n-1}$ generated by $\{p_1, \ldots, p_k\}$.

If $u \in U$, then $u = \sum_{i=k}^{n} c_i q_i$ and

$$R_M(u) = \frac{u^H M u}{u^H u} = \frac{\sum_{i=k}^{n} \mu_i |c_i|^2}{\sum_{i=k}^{n} |c_i|^2} \geq \frac{\sum_{i=k}^{n} \mu_k |c_i|^2}{\sum_{i=k}^{n} |c_i|^2} = \mu_k$$

therefore $\mu_k = \min\{R_M(u) : u \in U \setminus 0\}$.

Similarly, if $w \in W$, then $w = \sum_{i=1}^{k} d_i p_i$ and

$$R_N(w) = \frac{w^H N w}{w^H w} = \frac{\sum_{i=1}^{k} \lambda_i |d_i|^2}{\sum_{i=1}^{k} |d_i|^2} \leq \frac{\sum_{i=1}^{k} \lambda_k |d_i|^2}{\sum_{i=1}^{k} |d_i|^2} = \lambda_k$$

and so, $\lambda_k = \max\{R_N(w) : w \in W \setminus 0\}$.

On the other hand, let W' be the set of vectors obtained by adding a 0 at the end of each vector in W. As $\dim(U) = n - k + 1$ and $\dim(W) = k = \dim(W')$, then there exists $0 \neq v \in U \cap W'$. In particular, the last entry of v is equal to zero. Then, it is clear that

$$R_M(v) = \frac{v^H M v}{v^H v} = \frac{v'^H N v'}{v'^H v'} = R_N(v'),$$

where v' is the vector obtained by v from removing its last entry. As $v \in U$ and $v' \in W$, then

$$\mu_k = \min\{R_M(u) : u \in U \setminus 0\} \leq \frac{v^H M v}{v^H v} = \frac{v'^H N v'}{v'^H v'} \leq \max\{R_N(w) : w \in W \setminus 0\} = \lambda_k.$$

We show that $\lambda_k \leq \mu_{k+1}$ for all $1 \leq k \leq n-1$. Following the notation and similar arguments to those used for the previous inequality, let U be the subspace of $\mathbb{C}^n$ generated by $\{q_1, \ldots, q_{k+1}\}$ and W the subspace of $\mathbb{C}^{n-1}$ generated by $\{p_k, \ldots, p_{n-1}\}$. Then

$$\lambda_k = \min\{R_N(w) : w \in W \setminus 0\} \leq \frac{v^H M v}{v^H v} = \frac{v'^H N v'}{v'^H v'} \leq \max\{R_M(u) : u \in U \setminus 0\} = \mu_{k+1}$$

for some $v \in U \cap W'$. □

2.2.2 Continuity of the spectrum of a matrix

In this subsection, we talk about the continuity of the spectrum and spectral radius of a matrix. For that, we will use the following norm on the vector space $\mathbb{C}^{n \times n}$:

$$||A||_\infty = \max |a_{ij}|.$$

We begin by recalling the continuity of the zeroes of a polynomial; the proof of this result can be consulted, for example, in [11].

Theorem 2.2.10. *If $a_1(t), \ldots, a_n(t)$ are continuous complex valued functions defined in an interval $I \subseteq \mathbb{R}$, then the zeroes $\alpha_1(t), \ldots, \alpha_n(t)$ of the polynomial*

$$z^n - z^{n-1}a_1(t) + \cdots + (-1)^n a_n(t)$$

on z, are continuous functions on t.

Let $\{a_{ij}(t)\}_{1\leq i,j\leq n}$ be continuous functions on an interval $I \subset \mathbb{R}$ and

$$A(t) : I \longrightarrow \mathbb{C}^{n\times n},$$

a function from the interval I to the matrices in $\mathbb{C}^{n\times n}$. Moreover, let $\Lambda_A(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ be the functions on I to $\mathbb{C}^n$ given by the ordered eigenvalues of $A(t)$. Also, let $\rho : \mathbb{C}^{n\times n} \longrightarrow \mathbb{R}$ given by

$$\rho(A(t)) = \max(|\Lambda_A(t)|),$$

be the spectral radius function of the matrix $A(t)$, where $|\Lambda_A(t)|$ is the set composed by $|\lambda_1(t)|, \ldots, |\lambda_n(t)|$.

The following result establishes the continuity of the spectrum and spectral radius of $A(t)$ as a function of t.

Theorem 2.2.11. *If $A(t)$ is a continuous function defined in an interval $I \subseteq \mathbb{R}$ into the space $\mathbb{C}^{n\times n}$, then the spectrum $\Lambda_A(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ and spectral radious $\rho(A(t))$ of $A(t)$ are continuous functions on t.*

Proof. Using the Laplace expansion of $\det(zI - A(t))$, the coefficients of the characteristic polynomials will be continuous functions of the $a_{ij}(t)$. By applying Theorem 2.2.10, we obtain the result.

Finally, since the maximum of the set of continuous functions $|\lambda_1(t)|, \ldots, |\lambda_n(t)|$ is a continuous function, we get that $\rho(A(t))$ is continuous. □

2.3 Catalan combinatorics

In the fourth chapter, we will use a connection between the number of lattice paths and the number of arithmetical structures on paths and cycles. Therefore, it is helpful to learn how to count them. For this, we need to study Catalan numbers and their combinatorics.

Before introducing Catalan's triangle, let's recall the well-known Pascal's triangle.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}$$

In which we put one's in both sides of the triangle, and every entry in a row is the resulting sum of the two numbers above it.

2.3.1 Catalan numbers

The Catalan's triangle is very similar to Pascal's triangle: , it looks like this

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 2 & & \\
 & 1 & & 3 & & 5 & & 5 & \\
1 & & 4 & & 9 & & 14 & & 14 \\
 & & & & \vdots & & & &
\end{array}$$

To understand how to construct it, it is easier if we revisualize it in the following way:

$$\begin{array}{ccccc}
1 & & & & \\
1 & 1 & & & \\
1 & 2 & 2 & & \\
1 & 3 & 5 & 5 & \\
1 & 4 & 9 & 14 & 14 \\
 & & \vdots & &
\end{array}$$

Figure 2.1: Catalan's triangle

Here, we can see that we start with a column of numbers one, and every number at the right of this column is the sum of the number directly to their left and the one directly above it. Note that when no number is directly above, we may treat it as if there is a 0.

An interesting observation is that the sum of numbers at each row will be equal to the number at the end of the next row. The number at the end of the n-th row is called the n-th Catalan number and will be represented as C_n.

We shall show that a general formula for the n-th Catalan number is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

2.3.2 Lattice paths

Definition 2.3.1 (Lattice path). *A lattice path is a finite sequence of pairs $v_k = (a_k, b_k)$ in $\mathbb{Z}^2$, such that*

1. $v_0 = (0, 0)$,

2. $a_k, b_k \geq 0$, *and*

3. *if $v_k = (a_k, b_k)$, then either $v_{k+1} = (a_k + 1, b_k)$ or $v_{k+1} = (a_k, b_k + 1)$.*

Last condition is equivalent to saying that either $v_{k+1} = v_k + (1, 0)$ or $v_{k+1} = v_k + (0, 1)$. In the first case, we say that our path took a horizontal step, and in the second case, we say that our path took a vertical step. Then, a path $(v_1, \ldots, v_n)$ can also be represented as a string of H and V's, where we put an H or V in position i depending if our path took a horizontal or vertical step at the point v_i. A lattice path $(v_1, \ldots, v_n)$ is said to have length n. We recall that a *string* over a non empty set Σ is a finite sequence of elements

$$a_1 a_2 \ldots a_n,$$

where $a_1, \ldots, a_n \in \Sigma$. An *integer string* will be a string over $\mathbb{Z}$.

We can visualize the number of lattice paths that end at the point (k, l) while not going above the diagonal $x = y$ in the following figure.

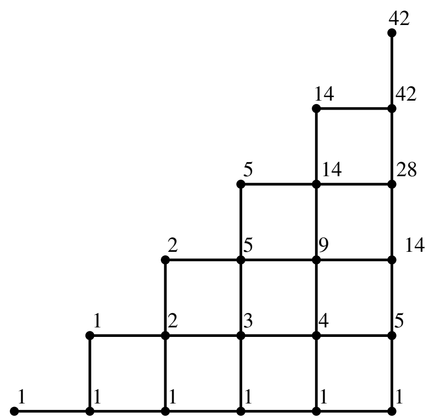


Figure 2.2: Number of lattice paths below the diagonal to each point.

In this figure, the number next to each point (k, l) equals the number of different lattice paths that reach this point while not going above the diagonal. In this case, we consider the empty path the unique lattice path that begins and finishes in $(0, 0)$.

Here, we can see some similarities with the Catalan triangle, mainly that it follows a similar set of rules to its own:

1. There is a row of numbers one at the bottom because there is only one lattice path to any of the points $(a, 0)$.

2. The number of paths to a point (a, b) is equal to the sum of the number of paths to the points $(a - 1, b)$ and $(a, b - 1)$, this is the sum of the numbers directly below and to the left.

By rotating 90 degrees clockwise the Figure 2.2, one can see that we are left with the visualization of Catalan's triangle given in Figure 2.1.

The following result counts the number of lattice paths that end at the point (m, n) that does not cross above the diagonal $x = y$.

Theorem 2.3.2. *If $m > n$ are non-negative integers, then the number of lattice paths that end in (m, n) that does not cross above the diagonal is equal to*

$$\frac{m - n + 1}{m + 1}\binom{m + n}{n}.$$

Proof. We may asssume that $m > n$, as there are the same number of paths to (n,n) as to $(n, n-1)$. It is a known result that the number of paths from $(0,0)$ to the point (m,n) is $\binom{m+n}{n}$. Then, we could count the number of this path that stays below the diagonal $x = y$.

It is not difficult to check that there are no paths that do not touch or cross the diagonal when $m < n$, so we may assume that $m > n$. We also observe that the first step of the path will be to the right, as the paths do not touch the diagonal. This is $v_1 = (1,0)$, so it is the same as the number of paths from $(1,0)$ to (m,n) that don't touch or cross the diagonal.

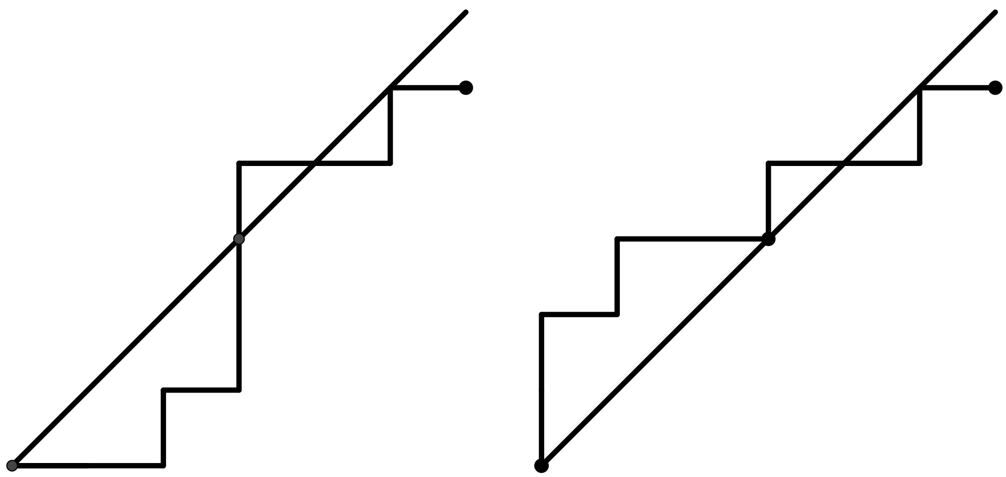


Figure 2.3: A path to $(6,5)$ that crosses the diagonal at $(3,3)$, and the resulting path after reflecting the points before it.

For any path that touches the diagonal, let v_a be the first point at which the path does so. If we reflect all the points before it around the diagonal $x = y$, we will obtain a path from $(0,1)$ to (m,n). An example of this procedure is shown in Figure 2.3. We note that any path from $(0,1)$ to (m,n) will touch the diagonal at some point because $m > n$. The number of these paths is $\binom{m+n-1}{m}$.

As the reflection is a bijection, we may conclude that the number of paths from $(1,0)$ to (m,n) that touch the diagonal is $\binom{m+n-1}{m}$. So the number of paths from $(1,0)$ to (m,n) that do not touch the diagonal is

$$\binom{m+n-1}{m-1} - \binom{m+n-1}{m} = \frac{m-n}{m+n}\binom{m+n}{m}.$$

So far, we have only counted paths that stay strictly below the diagonal $x = y$. In our theorem, we allow paths to touch the diagonal but not go above it.

To count this type of paths, we note that by taking any path that stays strictly below the diagonal $x = y$, from $(1, 0)$ to $(m + 1, n)$, with $m > n$, and translating it one step to the left, we get a path from $(0, 0)$ to (m, n) that may touch the diagonal, but won't go above it.

Putting these values in our previous formula, the number of paths from $(0, 0)$ to (m, n) that may touch the diagonal is

$$\frac{m-n+1}{m+n+1}\binom{m+n+1}{m+1} = \frac{m-n+1}{m+1}\binom{m+n}{m}.$$

□

Remark 2.3.3. *The numbers*

$$\frac{m-n+1}{m+1}\binom{m+n}{m},$$

are called Ballot numbers and they are represented by $B(m, n)$*, and they count the numbers of paths from* $(0, 0)$ *to* (m, n) *that do not cross above the diagonal.*

Thus, the number of paths from $(0, 0)$ to (n, n) is equal to

$$\frac{n-n+1}{n+1}\binom{n+n}{n} = \frac{1}{n+1}\binom{2n}{n} = C_n.$$

We also get the following result for lattice paths, which stay strictly below the diagonal.

Corollary 2.3.4. *The number of lattice paths from* $(0, 0)$ *to* (m, n) *for* $n < m$ *that stay strictly below the diagonal* $x = y$ *is equal to*

$$B(m-1, n).$$

Moreover, the number of lattice paths from $(0, 0)$ *to* (m, m) *that stay strictly below the diagonal* $x = y$ *until the end is equal to* C_{m-1}*.*

In particular, we have another identity for the Ballot numbers: any path from $(0, 0)$ to (n, k) (with $k < n$) can be subdivided in a path from $(0, 0)$ to (i, i), where (i, i) is the last point before (n, k) in which the path touches the diagonal and the subsequent path (which won't touch the diagonal). This gives us

$$B(n, k) = \sum_{i=0}^{k} B(i, i)B(n-i-1, k-i). \tag{2.2}$$

A similar reasoning gives us the following identity for Catalan numbers.

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1} = \sum_{i=1}^{n} C_{i-1} C_{n-i}. \tag{2.3}$$

This identity is crucial because it implies that any integer sequence a_n satisfying $a_0 = 1, a_n = \sum_{i=1}^{n} a_{i-1} a_{n-i}$ will, in fact, be the Catalan numbers.

2.3.3 Other appearances of Catalan numbers

Catalan numbers appear in many combinatorial problems; see, for instance [16] for 214 different kinds of objects which are counted using Catalan numbers. In this section, we will focus on some that we will find useful in the following chapters.

We begin by introducing the so-called Up-down walks.

Up-down walks

Definition 2.3.5 (Up-down walk). *An up-down path is a finite sequence of pairs $v_k = (a_k, b_k)$ in $\mathbb{Z}^2$, such that*

1. $a_0, b_0 = 0$*, and*
2. *if* $v_k = (a_k, b_k)$*, then* $v_{k+1} = (a_k + 1, b_k + 1)$ *or* $v_{k+1} = (a_k + 1, b_k - 1)$.

The last condition implies that $v_{k+1} = v_k + (1, 1)$ or $v_{k+1} = v_k + (1, -1)$. In the first case, we say that it took one step upwards, and in the second, we say that it took one step downwards to position $k + 1$.

We may associate any up-down walk to a string of U's and D's, representing at each position a step upwards and downwards, respectively.

Theorem 2.3.6. *The number of nonnegative up-down walks, that is, up-down walks that do not go below the x-axis, that end on $(2n, 0)$, is equal to the Catalan number C_n.*

Proof. At any position i, there will be at least as many steps upwards as steps downwards, otherwise the walk would't be positive. We also note that the walk ends at $(2n, 0)$, then there must be n steps upwards and n steps downwards.

We will use a bijection between these walks and paths that don't cross above the diagonal $x = y$. For a positive walk, create the following path:

- For every upward step in our walk, do a horizontal step in the path in the same position.
- for every downward step, do a vertical step in our path in the same position.

Based on our observations above, this association will result in a path from $(0, 0)$ to (n, n). For example, the positive walk $UUDUDDUD$ (see Figure 2.4)

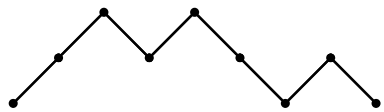


Figure 2.4: The positive walk $UUDUDDUD$

has associate the path $HHVHVVHV$ (see Figure 2.5).

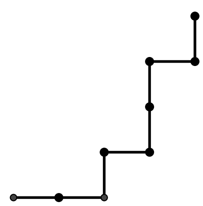


Figure 2.5: The path $HHVHVVHV$

As every path to (n, n) will have at most as many vertical steps as it does horizontal steps at each position i, the previous procedure can be used to get a nonnegative walk to $(2n, 0)$. It is not difficult to check that this is a bijection, and, therefore, we get the result. □

Another combinatorial interpretation of the Catalan number is the triangulations of the n-gon, which will be particularly useful when we count the arithmetical structures of the path with n vertices.

Triangulations of a n-gon

The following discussion will be useful once we count arithmetical structures on the path.

A triangulation of a n-gon is a collection of $n - 3$ diagonals of the n-gon, which do not cross in their interiors. This set of diagonals will partition the polygon into $n - 2$ triangles.

Theorem 2.3.7. *The number of triangulations of a $n + 2$-gon, with vertices $u_0, \ldots, u_{n+1}$ equals the Catalan number C_n.*

Proof. We proceed by mathematical induction on n. For $n = 1$, there is only $C_1 = 1$ triangulation of the triangle or 3-gon.

Suppose that for any $k \leq n$, the number of triangulations of a $k+2$-gon equals C_k. Then, for the $n+3$-gon, the edge $u_{n+1}u_{n+2}$ must be the side of a triangle. Thus, if u_j, $j \neq n+1, n+2$ is the other vertex of this triangle, by removing this triangle $u_{n+1}u_{n+2}u_j$, and gluing the ends of the remaining paths by adding an edge, we obtain an $j+2$-gon and a $n-j+3$-gon. Then, as the choice of the third vertex is free, the total number of triangulations is

$$\sum_{j=0}^{n} C_j C_{n+1-j} = C_{n+1}.$$

and we get the result. □

Moreover, we can obtain the following result, highlighting the connection between Ballot numbers and Catalan numbers.

Theorem 2.3.8. *The number of triangulations of a $n+3$-gon, with vertices $\{v_1, \ldots, v_{n+3}\}$, such that $n-k+1$ triangles are incident to a distinguished vertex i is equal to the Ballot number B(n,k).*

Proof. First, we make the observation that, by symmetry, it is enough to prove the case where the distinguished vertex is the first.

We proceed by induction. When $n = 1$, it is straightforward to check that the Ballot numbers coincide with the number of triangulations with 1 and 2 triangles incident to the first vertex. Now, suppose then that the result is valid for all $m \leq n$.

For the cases where all $n+2$ triangles of the triangulation are incident to v_1, the result is true because there is only one such triangulation. Then, we can suppose that there is at least one triangle not incident to v_1.

For a triangulation of $n+4$-gon with $(n+1)-k+1$ triangles incident to v_1, note that v_1v_2 is the edge of a triangle incident to v_1.

Then, if v_i is the other vertex of this triangle, all of the triangles that are not incident to v_1 have vertices in the set $v_2, \ldots, v_i$.

Then, any triangulation of the $n+4$-gon with $n+1-k+1$ vertices incident to v_1 can be decomposed in a triangulation of the $n-i+6$-gon with vertices $v_i, \ldots, v_{n+4}, v_1$ with $n-k+1$ vertices incident to v_1, a triangle $v_1v_2v_i$ and a triangulation of the $i-1$-gon with vertices $v_2, \ldots, v_i$. Noting that different choices of v_i give strictly different triangulations of these two polygons and using our induction hypothesis and the identity 2.2, we get the total number of

$$\sum_{i=3}^{k+3} C_{i-3}B(n-i+3, k-i+3) = \sum_{i=0}^{k} C_i B(n-i, k-i) = B(n+1, k),$$

triangulations of the $n+4$-gon with $n+1-k+1$-triangles incident to v_1. □

Binary trees

The next appearance of Catalan's numbers that we will explore is related to the number of *binary trees*. We start by reviewing some concepts related to trees in graph theory.

First, we recall that a *directed graph* is a graph where the edge set is composed of ordered pairs of vertices. This means that an edge uv will not be equal to vu. In the case of a directed graph, for an edge uv, we will say that u is the *parent* of v or that v is the *child* of u. We define the *descendants* of a vertex u recursively as any child of it or a descendant of a child of it.

We will say that a vertex v is a *root* of a tree if there is a path from v to any other vertex u in the tree. A *rooted tree* is a tree in which we specify a root vertex v. Finally, an *ordered tree* is a directed tree in which an order is specified for the set of children of each vertex.

Now, we are ready to define the concept of a *binary tree*.

Definition 2.3.9 (Binary tree). *A binary tree is an rooted, ordered tree in which no vertex has more than two children, who will be labeled (ordered) as "left child" and "right child".*

Some examples of binary trees are the following.

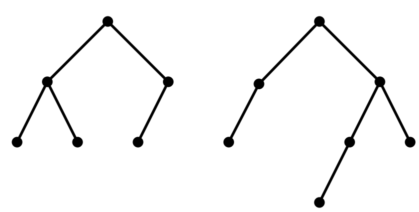


Figure 2.6: Binary trees on 5 and 7 vertices.

As we see in the example, we draw binary trees by putting the root vertex at the top. Now, we proceed to count the number of this type of graphs.

Theorem 2.3.10. *The number of binary trees with n vertices is equal to the Catalan number* C_n.

Proof. Let B_n denote the number of binary trees in n vertices. By taking out the root vertex, we are left with two binary trees having as root vertex the left and right child of the original root vertex. Each of these will have at most $n-1$ vertices, and the sum of their vertices will be $n-1$.

From this, it follows that by taking any two binary trees that fulfill this condition, one can obtain any binary tree on n vertices by adding a root vertex that has the root vertices of these two binary trees as its children. Then B_n is equal to

$$B_n = \sum_{i=0}^{n-1} B_i B_{n-i-1}.$$

Since $B_0 = 1$ and B_n follows the Catalan recurrence 2.3, then $B_n = C_n$. □

A *leaf* in a tree is a vertex of degree 1. It is a well-known result of *generating functions* that the number of binary trees on n vertices with k leafs equals to

$$\binom{n-1}{2k-2} 2^{n+1-2k} C_{k-1}. \tag{2.4}$$

This formula will be useful for our next appearance of Catalan numbers.

Admissible sequences.

Our next appearance will be related to what we will call *admissible sequences*; these sequences will be of great importance, as they follow a similar behavior to arithmetical structures on the path and cycle.

Definition 2.3.11. *Let $a_1, \ldots, a_n$ be a sequence of integers with $a_i > 1$, and set $a_0 = a_{n+1} = 1$. We say that $a_1, \ldots, a_n$ is an admissible sequence if $a_i | a_{i-1} + a_{i+1}$ for $i = 1, \ldots, n$*

Here, n is the length of the sequence. We simplify our notation by denoting the sequence $a_1, \ldots, a_n$ with the integer string

$$a_1 a_2 \ldots a_n.$$

Theorem 2.3.12. *The number of admissible sequences of length n equals C_n.*

First, we will discuss some useful properties of admissible sequences.

If $a_1, \ldots, a_n$ is an admissible sequence of length n, then

1. $(a_i, a_{i+1}) = 1$, where (a, b) denotes the GCD of a and b.

2. $a_1 \dots a_i(a_i + a_{i+1})a_{i+1} \dots a_n$ is an admissible sequence of length $n + 1$.

3. $a_1 \dots a_{i-1}a_{i+1} \dots a_n$ is an admissible sequence of length $n - 1$ whenever there exists some i such that $a_i = a_{i-1} + a_{i+1}$. Moreover, some i of that type always exists, and such a_i is called a *local maximum*.

We shall only prove the last claim.

Let $a_i = \max\{a_1, \dots, a_n\}$, then $a_i > a_{i-1}$ and $a_i > a_{i+1}$, also by the divisibility condition, $a_{i-1} + a_{i+1} = da_i$ for some d. Then

$$2a_i > a_{i+1} + a_{i-1} = da_i.$$

From this, we conclude that $d = 1$. Notice that this also implies that any local maximum, that is, an a_i such that $a_i > a_{i-1}$ and $a_i > a_{i+1}$, must necessarily have $a_i = a_{i-1} + a_{i+1}$. In this case, we say that a_i is a local maximum.

We define the *in-order* listing of vertices in a binary tree recursively; the left descendants of a vertex v are listed before it, and the right descendants are listed after it.

As an example of this listing, in the following binary tree, the vertices would appear in this listing in the order of their label.

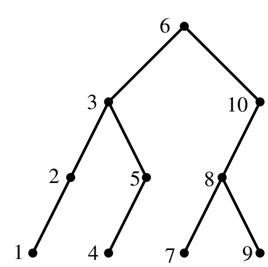
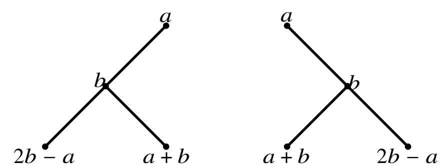


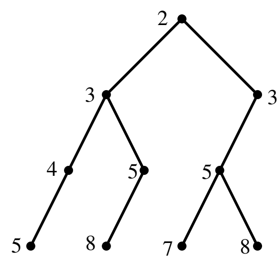
Figure 2.7: In-order listing of the vertices.

We will prove Theorem 2.3.12 by defining a bijection between binary trees with n vertices and admissible sequences of length n.

For this, for a binary tree B, label the vertices as follows: label the root vertex with a 2 and its children with a 3. Recursively label the vertices as follows: let v be a vertex with u as its father and that they are labeled with b and a respectively, then label the children of v as it is shown in the following figure:



Thus, the label of the children of v depends on whether they were left or right children of u. Using Figure 2.7, we would obtain the following labeling:



Using the in-order listing, the labels of the vertices would appear in the order 5438527583, which is an admissible sequence. This example shows how we will construct the bijection.

Before we start the proof, we observe the following diagram, which will tell us how to label a sequence of left or right children coming from some vertex.

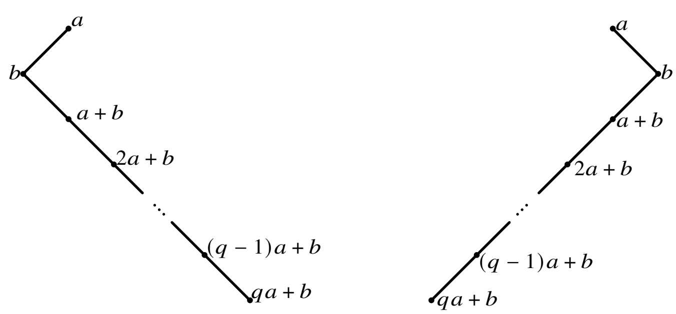


Figure 2.8

This figure will be useful to illustrate what happens in the bijection.

Proof. Define a function

$$T : \text{Binary trees with } n \text{ vertices} \to \text{admissible sequences of length } n$$

by labeling the vertices as shown above and listing them in the in-order listing.

We show that this map is well defined and a bijection. First, we show that it indeed gives us an admissible sequence. We proceed by induction. For $n \leq 2$, it is clear; one can check individually that this process will give us all admissible sequences of length 2. Suppose that that the result is valid for all $k \leq n$. Let B be a binary tree on $n + 1$ vertices labeled as described above. Let a_i be the label of a leaf v.

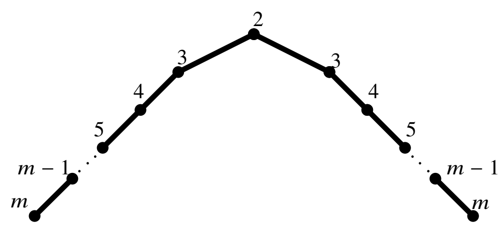


Figure 2.9: Labeling of a sequence of sequences of left and right children from the root vertex.

If $i = 1$ or $i = n + 1$, then by being a leaf, and in consequence of the in-order listing, there must be a path composed exclusively of left or right children starting from the root vertex to v, as shown in figure 2.9. From this, we have that $a_1 = a_2 + 1 = a_2 + a_0$ or $a_{n+1} = a_n + 1 = a_n + a_{n+2}$.

Suppose then that $1 < i < n + 1$, let u be the in-order predecessor of v, with label a_{i-1}, and w it successor, with label a_{i+1}. Then we have one of the situations in figure 2.10

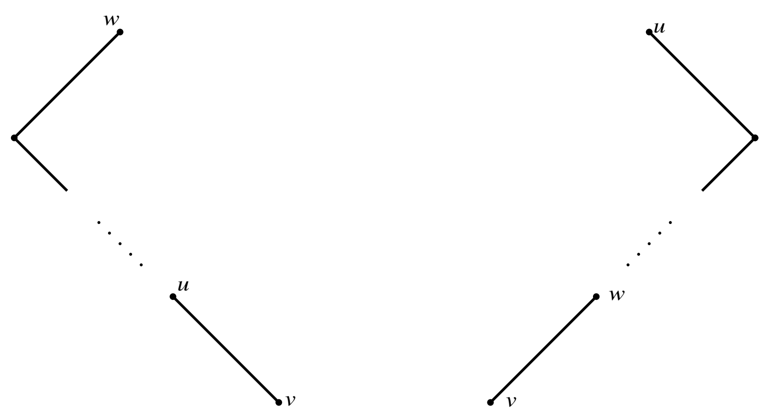


Figure 2.10

Then, with help of figure 2.8, one can see that $a_i = a_{i+1} + a_{i-1}$ in either case. Deleting this leaf from the tree, we obtain a binary tree in n vertices with the same in-order listing, which is mapped to the sequence $a_1 \dots a_{i-1}a_{i+1} \dots a_{n+1}$ which is admissible by our induction hypothesis.

The first of our properties in admissible sequences tells us that the original sequence $a_1 \dots a_{i-1}a_ia_{i+1} \dots a_{n+1}$ is, in fact, admissible. With this, we have proven that the map is well defined and injective.

Now, we prove the surjectivity by induction. Clearly, For $n \leq 2$, one can find a binary tree on n vertices that gives us any admissible sequence of this length. This can be illustrated with the following binary tree on 3 vertices.

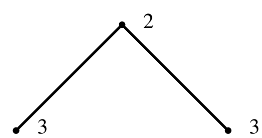


Figure 2.11

Here, in Figure 2.11, we can find all admissible sequences of length less than 2, which are 2, 23 and 32.

Suppose, then, that the result is valid for all admissible sequences of length $k \leq n$. Let $a_1 \dots a_i \dots a_{n+1}$ be an admissible sequence of length $n + 1$ such that $a_i = a_{i-1} + a_{i+1}$ (existence of such i is guaranteed by the second part of property 3). Then the sequence $a_1 \dots a_{i-1}a_{i+1} \dots a_{n+1}$ is an admissible sequence and, as such, is associated with a binary tree on n vertices B.

If $i = 1$ or $i = n + 1$, then the vertex with this label will have at most one right or left child, respectively. In the first case, we add a vertex with the label a_1 as a left child, and in the second case, we add a vertex with the label a_{n+1} as a right child. In either case, we will be left with a new binary tree such that the labels in the in-order listing will be $a_1 \dots a_{n+1}$.

Suppose then that $1 < i < n + 1$. Using figure 2.10, one can see that we can add a vertex with label a_i as a leaf to the tree B in such a way that the labels of the vertices in the in-order listing will appear as $a_1 \dots a_{i-1}a_ia_{i+1} \dots a_{n+1}$. This concludes the proof of the theorem. □

An important observation is that an element of an admissible sequence will be a local maximum if and only if it appears as a leaf in the binary tree associated with the sequence. This, combined with the identity 2.4, gives us the following.

Corollary 2.3.13. *The number of admissible sequences of length n with k local maxima is equal to*

$$\binom{n-1}{2k-2} 2^{n+1-2k} C_{k-1}.$$

This result will be helpful when we count the number of arithmetical structures on the path.

Chapter 3

Non-negative matrices

The adjacency matrix of a graph will play an essential role in the study of its arithmetical structures. This matrix is closely related to the set of what is called non-negative matrices. For this reason, it will be helpful for us to know some of its most important properties, the first of which is the Perron-Frobenius theorem.

Moreover, the Pseudo-Laplacian is another type of matrix called M-matrix. So, it will also be useful for us to explore some results of this type of matrices.

Studying this set of matrices will play an important role in answering the question of how many arithmetical structures have a graph.

3.1 Irreducible matrices

A square matrix A over a field $\mathbb{F}$ is called reducible whenever there exists a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} E & F \\ 0 & G \end{pmatrix},$$

where E, F, G are square matrices of positive dimension. We say that a matrix is irreducible when it is not reducible.

Given $A \in \mathbb{C}^{n\times m}$, let $|A| \in \mathbb{R}^{n\times m}$ be the matrix whose entries are given by $|A|_{ij} = |A_{ij}|$. Besides, given $A, B \in \mathbb{R}^{n\times m}$,

$$A > B,$$

means that $A_{ij} > B_{ij}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. In a similar way, $A \geq B$ means that $A_{ij} \geq B_{ij}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. A matrix A is called positive whenever $A > 0$

and nonnegative whenever $A \geq 0$. We shall use the same notation for vectors in $\mathbb{R}^n$. That is, for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we shall say that $x > y$ whenever $x_i > y_i$ (and $x \geq y$ whenever $x_i \geq y_i$ for all $1 \leq i \leq n$).

The inequality

$$A \gneqq B,$$

for vectors or matrices A and B means that $A \geq B$, but $A \neq B$.

Proposition 3.1.1. *A matrix A is positive if and only if $Ax > 0$ for all $x \gneqq 0$.*

Proof. Let $x \gneqq 0$ and A positive, that is, $A > 0$. Since $x \gneqq 0$, then there exists i such that $x_i > 0$. Thus

$$(Ax)_j = A_{j1}x_1 + \ldots + A_{ji}x_i + \ldots + A_{jn}x_n \geq A_{ji}x_i > 0 \text{ for all } j$$

and therefore $Ax > 0$.

In the opposite direction, we choose the vector $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the i-th position and 0 otherwise, multiplying it by A we get

$$(Ae_i)_j = A_{ji} > 0.$$

Thus, it follows that all entries of A are positive. □

Lemma 3.1.2. *If A is a nonnegative irreducible matrix of size n, then*

$$(A + I)^{n-1} > 0.$$

Proof. Let $x = (x_1, \ldots, x_n) \gneqq 0$ be a vector. If $x_i > 0$, then $((A + I)x)_i = (Ax)_i + x_i > 0$. This tells us that by multiplying the matrix $A + I$ by some vector, the resulting vector will have at least the same positive entries as the original vector. We proceed by showing that the resulting vector will have more positive entries unless the original vector is positive.

Suppose that the positive entries of $(A + I)x$ are the same as in the vector x. That is, $((A + I)x)_i > 0$ if and only if $x_i > 0$. Then, there is a permutation matrix P such that $Px = \begin{pmatrix} v \\ 0 \end{pmatrix}$, where v is a positive vector. As we are assuming that the positive entries of $(A + I)x$ are equal to the positive entries of x, then $P(A + I)x = \begin{pmatrix} u \\ 0 \end{pmatrix}$, where u is a positive vector of the same size as v. Since $P^T = P^{-1}$, we have that

$$P(A + I)P^T \begin{pmatrix} v \\ 0 \end{pmatrix} = P(A + I)P^T Px = P(A + I)x = \begin{pmatrix} u \\ 0 \end{pmatrix}.$$

Writing the matrix $A + I$ in the block form $P(A + I)P^T = \begin{pmatrix} A_{11} + I & A_{12} \\ A_{21} & A_{22} + I \end{pmatrix}$, with A_{11} of the same size as v. We get $\begin{pmatrix} A_{11} + I & A_{12} \\ A_{21} & A_{22} + I \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}$. Hence, the only possibility is that $A_{21} = 0$, implying that A is reducible, which is a contradiction. So $(A + I)x$ has at least one more positive entry than the original vector. Applying this result recursively, we obtain that $(A + I)^{n-1}x$ must be positive. □

3.2 The Perron-Frobenius Theorem

In this section, we will provide self-contained proof of the Perron-Frobenius Theorem, which will be crucial in studying M-matrices. This theorem is concerned with studying the spectrum of irreducible positive matrices.

We recall that the spectral radius of a matrix $N \in \mathbb{C}^{n\times n}$, denoted by $\rho(N)$, is the maximum of the magnitude of its eigenvalues. That is,

$$\rho(N) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } N\}.$$

Theorem 3.2.1 (Perron-Frobenius). *If $A \geq 0$ is irreducible, then*

1. *the spectral radious $\rho(A)$ is an eigenvalue of A,*
2. *the spectral radious $\rho(A)$ is a simple eigenvalue,*
3. *the spectral radious $\rho(A)$ has a positive eigenvector,*
4. *the eigenvalues of A of magnitude $\rho(A)$ have both algebraic and geometric multiplicity equal to one. Moreover, if there are h of them, then they are the solutions to the equation $\lambda^h = \rho(A)^h$.*
5. *the spectrum of A as a multiset is mapped to itself by the rotation by $\frac{2\pi}{h}$ of the complex plane,*
6. *if $h > 1$, then there exists a permutation matrix π such that*

$$\pi^{-1}A\pi = \begin{pmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_{h-1} \\ B_h & 0 & 0 & \dots & 0 \end{pmatrix},$$

where all blocks on the diagonal are square.

Proof. Consider the following function on the orthant $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0, x \neq 0\}$:

$$r(x) = \min_{\substack{j=1,\dots,n \\ x_j \neq 0}} \frac{(Ax)_j}{x_j}.$$

Then $r(x)x \leq Ax$. Indeed, for $x_j \neq 0$, $(Ax)_j = \frac{(Ax)_j}{x_j} x_j \geq r(x)x_j$, from which the inequality follows. Even more,

$$r(x) = \max\{\rho : \rho x \leq Ax\}, \tag{3.1}$$

and $r(x)$ is a continuous function on all $x > 0$; nevertheless, it can be discontinuous at the boundary because some entries of x can be 0 at the boundary of $\mathbb{R}^n_+$.

We proceed to prove the existence of a vector $z \in \mathbb{R}^n_+$ such that

$$r(z) = \sup\{r(x) : x \in \mathbb{R}^n_+\}. \tag{3.2}$$

For this purpose, we shall consider the following set:

$$Y = \{(A+I)^{n-1}x : ||x|| = 1\}. \tag{3.3}$$

This set fulfills all of the conditions required in the Bolzano-Weierstrass Theorem; it is compact, and the function r is continuous on it as $(A+I)^{n-1}x > 0$ by Proposition 3.1.2. Thus there exists $z \in Y$ such that

$$r(z) = \sup\{r(x) : x \in Y\}. \tag{3.4}$$

To prove that this z is indeed the vector that we want, we observe that the function r satisfies $r(\alpha x) = r(x)$ for any scalar $\alpha > 0$.

Indeed, from the definition, we get

$$r(\alpha x) = \min_{\substack{j=1,\dots,n \\ \alpha x_j \neq 0}} \frac{\alpha (Ax)_j}{\alpha x_j} = \min_{\substack{j=1,\dots,n \\ x_j \neq 0}} \frac{(Ax)_j}{x_j} = r(x).$$

This tells us that for $x \in \mathbb{R}^n_+$, with $x \neq 0$, $r(x) = r(\frac{x}{||x||})$.

Now, we shall prove that $r(z)$ as defined in equation 3.4 is the maximum over $\mathbb{R}^n_+$. We do this by showing that $r(x) \leq r(y)$ whenever $y = (A+I)^{n-1}x$. Let $y = (A+I)^{n-1}x$ for some x, and ρ be any real number such that $\rho x \leq Ax$. Multiplying by $(A+I)^{n-1}$ on both sides of the inequality, we get $\rho y \leq Ay$ so $\rho \leq r(y)$. By 3.1, this implies that $r(x) \leq r(y)$.

From this, we get the following:

$$\sup\{r(x) : x \in \mathbb{R}^n_+\} = \sup\{r(x) : ||x|| = 1\} \leq \sup\{r(y) : y \in Y\} = \max\{r(y) : y \in Y\} = r(z).$$

We will prove that z is the desired vector; that is, it is an eigenvector of A and has eigenvalue $\rho(A)$.

Let $r' = r(z)$, then, for any $u \in \mathbb{R}^n_+$ that satisfies $r(u) = r'$, set $y = (A + I)^{n-1}u$. If we had the inequality in $r'u \leq Au$, multiplying by $(A + I)^{n-1}$ we obtain $(A + I)^{n-1}(Au - r'u) > 0$ by the Lemma 3.1.2. This implies that $Ay > r'y$, which in turn implies the existence of some $\varepsilon > 0$ such that $Ay > (r' + \varepsilon)y$, which contradicts the maximality of r'. Then, $Au = r'u$, which means that u is an eigenvector of A with eigenvalue r'.

As $A > 0$, if $x > 0$, $Ax > 0$ and so $r(x) > 0$. It follows that $r' > 0$ and any eigenvector u with eigenvalue r' must be positive.

Finally, we will show that $r' = \rho(A)$. For this, we only need to show that $r' \geq ||\lambda||$ for all λ eigenvalue of A. Let λ be an eigenvalue of A with eigenvector y. From the equality $Ay = \lambda y$, we get

$$|Ay|_j = |\lambda y_j| = |\lambda||y_j| = (|\lambda||y|)_j,$$

$$|Ay|_j = |A_{j1}y_1 + \ldots + A_{jn}y_n| \leq A_{j1}|y_1| + \ldots + A_{jn}|y_n| = (A|y|)_j \quad \text{(by the triangle inequality)}.$$

Combining these expressions, we get

$$|\lambda||y| \leq A|y|, \tag{3.5}$$

which in turn implies that $|\lambda| \leq r(y) \leq r'$. So $r' = \rho(A)$

Before continuing with the proof of the Perron-Froebenius Theorem, we need the following lemmas.

Lemma 3.2.2. *Let $A \geq 0$ be an irreducible matrix and B with $|B| \leq A$. If β is an eigenvalue of B, then $|\beta| \leq \rho(A)$. Moreover, $\beta = \rho(A)e^{i\theta}$ for some θ if and only if $|B| = A$ and*

$$B = e^{i\theta}DAD^{-1},$$

where D is a diagonal complex matrix with $|D| = I$.

Proof. Let y be an eigenvector of B with eigenvalue β, then using the inequality 3.5 and the fact that $|B| < A$ we get the inequality

$$|\beta||y| \leq |B||y| \leq A|y|.$$

So $|\beta| \leq r(y) \leq \rho(A)$.

If $|\beta| = \rho(A)$, then $\beta = \rho(A)e^{i\phi}$. Moreover, if $|y|$ is a vector with eigenvalue $\rho(A)$, then $|\beta||y| = A|y|$. Using the previous inequality we get that $|B||y| = A|y|$, as $A \geq |B|$ it must happen that $|B| = A$.

We can write $y = (e^{i\phi_1}|y_1|, \ldots, e^{i\phi_n}|y_n|)$.

Define D as the diagonal matrix $\mathrm{diag}\{e^{i\phi_1}, \ldots, e^{i\phi_n}\}$, then it is clear that $|D| = I$ and $D|y| = y$. It follows that

$$By = BD|y| = \rho(A)e^{i\phi}D|y| = \rho(A)e^{i\phi}y.$$

$$\Longrightarrow e^{-i\phi}D^{-1}BD = \rho(A)|y|.$$

If $C = e^{-i\phi}D^{-1}BD$, then $C|y| = |B||y| = |A||y|$. It is clear that $|C| = |B| = A$, from which $|C||y| = C|y|$, as $|y| > 0$, it follows that $C = |C| = A$, from which the desired result follows. The other direction of the theorem is clear. □

The next lemma follows directly from *Jacobi's formula*

$$\frac{d}{dt}\det(A(t)) = \operatorname{tr}\left(\operatorname{adj}(A(t))\frac{dA(t)}{dt}\right)$$

in matrix calculus.

Lemma 3.2.3. *If A is a square matrix, then*

$$(\det(tI - A))' = \sum_{i=1}^{n} \det(tI - A_i),$$

where A_i are the principal submatrices obtained by removing the i-th row and column.

Finally, we will make use of the following lemma.

Lemma 3.2.4. *If $A \geq 0$ is an irreducible $n \times n$ matrix and B is a square principal submatrix of A, then $\rho(B) < \rho(A)$.*

Proof. As B is a square principal submatrix of A, there is a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} B & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Applying Lemma 3.2.2 to $C = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ and we get that $|C| \leq A$ and $|C| \neq A$. So for any β eigenvalue of C, $\beta \neq \rho(A)e^{i\phi}$ for all $\phi \in [0, 2\pi]$, combining this with the fact that $|\beta| \leq \rho(A)$, we get that $\rho(B) = \rho(C) < \rho(A)$. □

Now, we will prove that $\rho(A)$ is a simple eigenvalue of A. Let $\phi(t) = \det(tI - A)$ be the characteristic polynomial of A, then by Lemma 3.2.3

$$\phi'(t) = \sum_{i=1}^{n} \det(tI - A_i).$$

We know that $\rho(A)$ is a zero of multiplicity bigger than 1 if and only if it is a zero of the derivative of the characteristic polynomial. By Lemma 3.2.4, $\rho(A_i) < \rho(A) = r'$. Thus $\rho(A)$ is not an weigenvalue of any A_i, which means that $\det(r'I - A_i) \neq 0$ for all i.

We are left to prove that $\det(\rho(A)I - A_i) > 0$ for all i. If $\det(\rho(A)I - A_i) < 0$, then for a large enough value of t, $\det(tI - A_i) > 0$ by the intermediate value theorem, which would imply the existence of some a between $\rho(A)$ and t such that $\det(aI - A_i) = 0$. This is impossible because it would imply that

$$\rho(A) < a \leq \rho(A_i) < \rho(A).$$

Thus $\det(\rho(A)I - A) > 0$ for all i and $\phi'(\rho(A)) > 0$. This means that $\rho(A)$ is a zero of algebraic multiplicity 1, and as we have shown the existence of an eigenvector, it also has geometric multiplicity 1.

Now, we will proceed to prove (4). Suppose $\lambda_1, \ldots, \lambda_{h-1}$ are all of the other eigenvalues with $|\lambda_i| = \rho(A)$, that is, $\lambda_i = e^{i\theta_i}\rho(A)$, ordered in such way that $0 < \theta_1 < \theta_2 < \cdots < \theta_{h-1}$. We note that λ_j is a simple eigenvector of A, as it is a multiple of $\rho(A)$.

Applying Lemma 3.2.2 with $B = A$ and $\beta = \lambda_i = \rho(A)e^{i\theta_i}$. The equality holds and so there exists a diagonal matrix D_i with $|D_i| = I$ for all i such that

$$A = e^{i\theta_i} D_i A D_i^{-1}.$$

Let $y_j = D_j z$, $j = 1, \ldots, h-1$, where z is a positive eigenvector of $\rho(A)$. Then

$$Ay_j = e^{i\theta_j} D_j A D_j^{-1} D_j z = e^{i\theta_j} D_j A z = e^{i\theta_j} D_j \rho(A) z = \rho(A) e^{i\theta_j} D_j z = \lambda_j y_j.$$

Thus, y_j is an eigenvector of λ_j. This implies that y_j and, by the way it was defined, D_j are defined uniquely up to scalar multiplication.

Choose D_i so its first entry is equal to 1. It follows that

$$AD_jD_kz = e^{i\theta_j} D_j A D_j^{-1} D_j D_k z = e^{i\theta_j} A y_k = \rho(A) e^{i(\theta_j+\theta_k)} D_j y_k = \rho(A) e^{i(\theta_j+\theta_k)} D_j D_k z$$
$$AD_jD_k^{-1}z = e^{-i\theta_k} D_k^{-1} A D_k D_j D_k^{-1} z = e^{-i\theta_k} D_K^{-1} A y_j = \rho(A) e^{i(\theta_j-\theta_k)} D_k^{-1} y_j = D_j D_k^{-1} z$$

Then $\rho(A)e^{i(\theta_i \pm \theta_k)}$ is an eigenvalue with eigenvector $D_j D_k^{\pm 1} z$, as our list of eigenvalues with magnitude $\rho(A)$ is exhaustive, we must have that for every i, k, there is some j such that $\theta_i \pm \theta_k = \theta_j \mod 2\pi$. We prove that the θ_i make up a cyclic group of order h, generated by θ_1.

We have that $\theta_2 - \theta_1 > 0$, and it is equal to some θ_k that must lie between 0 and θ_2, this can only be θ_1, and with that, we have shown that $\theta_2 = 2\theta_1$. Then, we can suppose that $\theta_n = n\theta_1$, for $n + 1 \leq h - 1$, $0 < \theta_{n+1} - \theta_1$ is some θ_k between 0 and θ_{n+1}. But $\theta_k = k\theta_1$ by hypothesis so the only choice possible is $k = n$, that is, $\theta_{n+1} = \theta_n + \theta_1 = (n+1)\theta_1$.

Thus, the θ_i make up a cyclic group of order h by taking them modulo 2π, which means that $e^{i\theta_i}$ is a root of unity for all i, and $\theta_1 = 2\pi/h$. We note that there is an isomorphism between D_i and $e^{i\theta_i}$, from which it follows that $D_i^h = I$ for all i.

(5) it follows from the fact that $e^{i2\pi/h} = D_1^{-1} A D_1$, and the spectrum is conserved under similarity.

For (6), let $D = D_1$, as $D^h = I$, the entries of D are roots of unity, and so, there exists some permutation matrix P such that

$$PDP^{-1} = \begin{pmatrix} I_0 e^{i\delta_0} & 0 & \dots & 0 \\ 0 & I_1 e^{i\delta_1} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & I_{s-1} e^{i\delta_{s-1}} \end{pmatrix}.$$

Where $\delta_j = (2\pi/h)n_j$ with $0 = n_0 < n_1 < \dots < n_{s-1} < h$.

Consider the same permutation for A

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ A_{21} & A_{22} & \dots & A_{2s} \\ \vdots & \vdots & & \vdots \\ A_{s1} & A_{s2} & \dots & A_{ss} \end{pmatrix}.$$

With A_{jj} of the same size as I_{j-1}. With this, we write $A = e^{i2\pi/h} DAD^{-1}$ and so $PAP^{-1} = e^{i2\pi/h}(PDP^{-1})(PAP^{-1})(PD^{-1}P^{-1})$, which looks like

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ A_{21} & A_{22} & \dots & A_{2s} \\ \vdots & \vdots & & \vdots \\ A_{s1} & A_{s2} & \dots & A_{ss} \end{pmatrix} = e^{i2\pi/h} \begin{pmatrix} A_{11} & e^{i(\delta 0 - \delta_1)} A_{12} & \dots & e^{i(\delta_0 - \delta s - 1)} A_{1s} \\ e^{i(\delta_1 - \delta_0)} A_{21} & A_{22} & \dots & e^{i(\delta_1 - \delta_{s-1})} A_{2s} \\ \vdots & \vdots & & \vdots \\ e^{i(\delta_{s-1} - \delta_0)} A_{s1} & e^{i(\delta_{s-1}) - \delta 1} A_{s2} & \dots & A_{ss} \end{pmatrix}.$$

From this, we obtain the linear system of s^2 equations

$$A_{pq} = e^{i2\pi/h} e^{i(\delta_{p-1}-\delta_{q-1})} A_{pq} = e^{i2\pi(n_{p-1}-n_{q-1}+1)/h} A_{pq}.$$

So $A_{pq} \neq 0$ if and only if $n_q = n_p + 1 \mod h$. By the irreducibility of A, for all p, there is some q such that $A_{pq} \neq 0$.

We show that $s = h$. By the previous observation, there is some n_q such that $n_q = n_{s-1} + 1$ mod h, which implies that $n_{s-1} - n_q = mh - 1$ for some m. But $0 < n_{s-1} - n_q < h$, which implies $m = 1$ and so $n_{s-1} - n_q = h - 1$, which is only possible when $s = h$ and $n_{s-1} = h - 1$. This in turn implies that $n_i = i$ and $A_{pq} \neq 0$ only if $q = p + 1 \mod h$. Then

$$PAP^{-1} = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{h-1h} \\ A_{h1} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

This concludes our proof of the Perron-Frobenius theorem. □

We mention some immediate consequences from the Perron Frobenius Theorem 3.2.1. First, we note that any positive matrix is irreducible, as for a positive matrix A, PAP^{-1} is positive, for any permutation matrix P.

Then, for any nonnegative matrix $M \in \mathbb{R}^{n\times n}$, if $N = (1_{ij})$ is the $n \times n$ matrix with all of its entries equal to 1, then $M + tN$ is a positive matrix for all $t > 0$. By continuity of eigenvalues 2.2.11, and applying the Perron-Frobenius Theorem to $M + tN$, we also get that M has $\rho(M)$ as one of its eigenvalues. This is the Perron-Frobenius Theorem for nonnegative matrices.

Theorem 3.2.5 (Perron-Frobenius for nonnegative matrices). *If $M \geq 0$, then $\rho(M)$ is an egenvalue of M with nonnegative eigenvector.*

3.3 M-matrices

We conclude our study of nonnegative matrices by studying a closely related set of matrices, the so-called M-matrices.

As we have mentioned, the study of M-matrices is vital because the pseudo-Laplacian of an arithmetical structure is a M-matrix. In particular, it will be helpful when discussing the finiteness of path and cycle arithmetic structures.

First, a square matrix M is called a Z-matrix whenever its entries satisfy that $M_{ij} \leq 0$ for all $i \neq j$.

Definition 3.3.1. *A Z-matrix M is a M-matrix whenever there is a nonnegative matrix N such that*

$$M = rI_n - N$$

for some $r \geq \rho(N)$, where $\rho(N)$ is the maximum of all the absolute values of eigenvalues of N.

Note that in the previous definition, by the Perron-Frobenius Theorem for nonnegative matrices 3.2.5, an M-matrix is nonsingular if and only if $r > \rho(N)$. Indeed, because N is nonnegative, $\rho(N)$ is an eigenvalue of N. Then, if $r = \rho(N)$, then $\det(rI - N) = 0$. On the other hand, if $r > \rho(N)$, then $\det(rI - N) > 0$.

It is not difficult to check that the spectrum of N and $rI - N$ are directly related. More precisely, if λ is an eigenvalue of N with eigenvector v, then

$$Nv = \lambda v \leftrightarrow (rI - N)v = (r - \lambda)v.$$

Let $p_A(t) = \det(tI - A)$ be the characteristic polynomial of a matrix A. Since $p_{-N}(t) = (-1)^n p_N(-t)$ and $p_{rI-N}(t) = (-1)^n p_N(r - t)$, one can conclude that $\Lambda_{rI_n - N} = (r - \lambda_n \leq r - \lambda_{n-1} \leq \cdots \leq r - \lambda_1)$ whenever $\Lambda_N = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n)$.

Remark 3.3.2. *A matrix N of size n has spectral radius zero if and only if all of its eigenvalues are zero. This only happens if and only if the characteristic polynomial is t^n, which implies that N is nilpotent.*

Corollary 3.3.3. *If M is an M-matrix, then its real eigenvalues are nonnegative. Moreover, if it is nonsingular, then its eigenvalues are positive.*

Proof. Let $A = rI - N$ be an M-matrix. If λ is an eigenvalue of A with eigenvector v, then

$$\lambda v = Av = (rI - N)v.$$

Thus $(r - \lambda)v = Nv$, that is, $r - \lambda$ is an eigenvalue of N. Since $r \geq \rho(N)$ we get that $\lambda \geq 0$. Besides, if it is nonsingular then $r > \rho(N)$, then it is clear that $\lambda > 0$. □

As we shall see, many times, it will be easier to show a result for non-singular $M-$matrices, so it will be useful to show a result that relates non-singular and singular M-matrices. In particular, the following result will tell us that any M-matrix can be seen as a 'limit' of non-singular M-matrices.

Theorem 3.3.4. *A Z-matrix A is an M-matrix if and only if $A+\varepsilon I$ is a nonsingular M-matrix for all $\varepsilon > 0$.*

Proof. ($\Leftarrow$) Let $A + \varepsilon I$ be an non singular M-matrix for all $\varepsilon > 0$. Since A is a Z-matrix, $A = rI - N$ for some $N \leq 0$, and by hypothesis, $(r + \varepsilon) - N$ is a non singular M-matrix, so $r + \varepsilon > \rho(N)$ for all $\varepsilon > 0$. Then $r \geq \rho(N)$, which implies that A is an M-matrix.

($\Rightarrow$) If A is a M-matrix, then $A = rI - N$ for $r \geq \rho(N)$, So $A + \varepsilon I = (r + \varepsilon)I - N$, with $r + \varepsilon > \rho(N)$, therefore $A + \varepsilon I$ is a non-singular M-matrix. □

The following lemma will be useful for the rest of this section.

Lemma 3.3.5. *A nonnegative matrix $T \in \mathbb{R}^{n\times n}$ is convergent, that is, $\rho(T) < 1$ if and only if $(I - T)^{-1}$ exists and*

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k \geq 0.$$

Proof. ($\Rightarrow$) If T is convergent, then the identity

$$(I - T)(T^{k-1} + \cdots + I) = I - T^k$$

gives us one implication of the theorem by letting n tend to infinity, as we get that $(I - T)(\sum_{k=0}^{\infty} T^k) = I$.

($\Leftarrow$) If $T = 0$, then the result is clear. On the other hand, as a consequence of the Perron-Frobenius Theorem for nonnegative matrices 3.2.5, if $T \neq 0$, there exists some $x \geq 0, x \neq 0$ such that $Tx = \rho(T)x$. Then, by hypothesis $(I - T)^{-1}$ exists, so $\rho(T) \neq 1$. Then

$$(I - T)x = (1 - \rho(T))x.$$

This, in turn, implies that

$$(I - T)^{-1}x = \frac{1}{1 - \rho(T)}x.$$

As $I - T$ is nonnegative and $x \gneqq 0$, it must happen that $1 - \rho(T) > 0$. That is $\rho(T) < 1$. □

Theorem 3.3.6. *If M is a M-matrix, then*

- *its diagonal entries are nonnegative,*
- *if M is nonsingular, then its diagonal entries are positive,*
- *its principal minors are nonnegative,*

- *if M is nonsingular, then its principal minors are positive.*

Proof. We will show the results for nonsingular M-matrices. Let $M = rI - N$, with $r > \rho(N)$. Then $\rho(\frac{1}{r}N) < 1$. By the previous Lemma 3.3.5

$$M^{-1} = \frac{1}{r}(I - \frac{1}{r}N)^{-1} \geq 0.$$

If $M^{-1} = (m'_{ij})$, then $m'_{ij} \geq 0$ and

$$1 = m_{i1}m'_{1i} + \cdots + m_{ii}m'_{ii} + \cdots + m_{in}m'_{ni}$$

for all i because $MM^{-1} = I$. As $m_{ij} \leq 0$ for $i \neq j$, $m_{ij}m'_{ji} \leq 0$, therefore $m_{ii} > 0$ for the previous equation to happen.

Let $x = M^{-1}e \geq 0$, with $e = (1, \ldots, 1)$. If $D = \text{diag}(x)$, then $MDe = Ax = e$ implies that the sum of the rows of MD are positive. This implies that MD is strictly diagonally dominant, as the off-diagonal entries are nonpositive.

Suppose that some $x_i = 0$. This would imply that

$$1 = m_{i1}x_1 + \cdots + m_{ii-1}x_{i-1} + m_{ii+1}x_{i+1} + \cdots + m_{in}x_n.$$

All of the numbers on the right side of this equation are nonpositive. This is impossible, and therefore $x > 0$.

As $\text{diag}(x) > 0$, the entries of MD are of the same sign as M and the sign of the principal minors is conserved.

Because MD is strictly diagonally dominant, any principal submatrix will also be strictly diagonally dominant. By the Gershgorin circle Theorem 2.2.1, any eigenvalue will have positive real part.

Then, as the complex eigenvalues come in conjugate pairs, for a principal submatrix M'

$$\det(M') = \prod_{\lambda \in \Lambda(M')} \lambda = (\prod_{\text{Im}(\lambda)=0} \lambda)(\prod_{\text{Im}(\lambda \neq 0)} \lambda\overline{\lambda}) = (\prod_{\text{Im}(\lambda)=0} \lambda)(\prod_{\text{Im}(\lambda \neq 0)} |\lambda|^2).$$

So, the determinant is positive, and the principal minors of M are positive. Finally, for the singular case, we may use Theorem 3.3.4 for an M-matrix M, the matrix $M + \varepsilon I$ is nonsingular for all $\varepsilon > 0$. Letting $\varepsilon \longrightarrow 0$, we get the desired results for singular M-matrices. □

The following theorems will show us the necessary and sufficient conditions for a Z-matrix to be a M-matrix.

Theorem 3.3.7. *[2, Theorem 6.4.6 (A_1), page 149] A Z-matrix M is an M-matrix if and only if all of its principal minors are nonnegative.*

Proof. A M-matrix having nonnegative principal minors is shown in Theorem 3.3.6.

Then, we only need to show that if the principal minors are nonnegative, then M is an M-matrix. By formula 2.1, matrices with nonnegative principal minors will have nonnegative real eigenvalues.

Suppose that M is not an M-matrix, then $M = rI - B$ with $B \geq 0$ and $r < \rho(B)$. By Perron-Frobenius, there would be an eigenvector $x \geq 0$ with eigenvalue $\rho(B)$, so $Mx = (rI - B)x = (r - \rho(B))x$. This would imply that $r - \rho(B) < 0$ is an eigenvalue of M, which is a contradiction. □

Theorem 3.3.8. *[2, Theorem 6.4.16, page 156] If M is an irreducible singular M-matrix, then any proper principal submatrix of M is a nonsingular M-matrix.*

Proof. As $M = rI - B$, is irreducbible, B is irreducible. So, if B' is a proper principal submatrix of B, by Lemma 3.2.4 $\rho(B') < \rho(B)$. Then $rI - B'$ is a non singular M-matrix. □

Almost non-singular M-matrices

There is a subset of singular M-matrices that will be pretty important to us, as our arithmetical structures will give rise to singular M-matrices that will have rank $n - 1$, as we shall see. In particular, they will be part of a subset of singular M-matrices that are known as *almost non-singular M-matrices.*

Definition 3.3.9. *A real matrix $A = (a_{i,j})$ is an almost non-singular M-matrix whenever it is a Z-matrix with positive proper principal minors and nonnegative determinant.*

In a sense, the following theorem shows us that almost non-singular M-matrices come very close to being non-singular. Recall that we use the notation $M[I, I]$ to refer to the submatrix of $M \in \mathbb{F}^{n\times n}$ obtained by erasing rows and columns not indexed by $I \subset \{1, \dots, n\}$.

Lemma 3.3.10. *Given $1 \leq s \leq n$, let $E_s = (e_{ij})$ be the matrix with e_{ij} equal to 1 if i and j are equal to s and 0 otherwise. If M is an almost non-singular M-matrix, then $M' = M + d \cdot E_s$ is a non singular M-matrix for all $d > 0$.*

Proof. It is enough to show that all principal minors of M' are positive. Let $\emptyset \neq I \subseteq [n]$. If $s \notin I$, then $M'[I; I] = M[I; I] > 0$. The last inequality follows, as M is an almost nonsingular M-matrix and $I \neq [n]$. On the other hand, if $s \in I$, then

$$\det(M'[I, I]) = \det(M[I, I]) + d \cdot \det(M[I \setminus s, I \setminus s]) \overset{(I\setminus s\neq[n],\, d>0)}{>} \det(M[I, I]) \geq 0.$$

□

Using Lemma 3.3.10, we get the following equivalencies for a M-matrix to be almost nonsingular.

Theorem 3.3.11. *If M is a real Z-matrix, then the following conditions are equivalent:*

- *M is an almost non-singular M-matrix.*
- *$M + D$ is a non-singular M-matrix for any diagonal matrix $D \gneqq 0$.*
- $\det(M) \geq 0$ *and* $\det(M+D) > \det(M+D') > 0$ *for any diagonal matrices* $D \gneqq D' \gneqq 0$.

Proof. (1)⇒ (2) Since any diagonal matrix D is equal to $\sum_{i=1}^{n} d_i \cdot E_i$ for some $d_i \in \mathbb{R}_+$, then the result follows by using Lemma 3.3.10 several times.

(2)⇒ (3) We first prove that $\det(M) \geq 0$. If we take $D_m = (1/m)I_n$ ($m \in \mathbb{N}_+$), then $\det(M + D_m) > 0$ and so $\lim_{m\to\infty} \det(M + D_m) \geq 0$. Since $\lim_{m\to\infty} D_m = 0$ and the determinant is a continuous function with respect to the Hilbert–Schmidt norm, $\det(M) \geq 0$.

Now, let $D \gneqq D' \gneqq 0$ be diagonal matrices. By hypothesis, $M + D'$ is a non-singular M-matrix and, in particular, an almost non-singular M-matrix. By Lemma 3.3.10 we get that $M + D' + E_s$ is an almost non-singular M-matrix for any $1 \leq s \leq n$ and $\det(M + D' + E_s) > \det(M + D')$. Similarly, it is not difficult to prove that $\det(M + D' + F) > \det(M + D')$ for any diagonal matrix $F > 0$. Now, clearly, the result follows by taking $F = D - D'$.

On the other hand, let $F = D - D' \gneqq 0$, let f_{ii} be the first non-zero diagonal entry of F, let $C' = M + D'$, and let $C = C' + F$. Then, since C' is a non-singular M-matrix and $f_{ii} > 0$, $\det(C) = \det(C') + f_{ii} \cdot \det(C\big[[n] \setminus i, [n] \setminus i\big]) > \det(C')$.

(3)⇒(1) Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x_1, \dots, x_n) = \det(M + \operatorname{diag}(x_1, \dots, x_n))$. By hypothesis f is a nonnegative and strictly increasing function on $(\mathbb{R}_+ \cup \{0\})^n$. Also, it is not difficult to see that

$$f(x_1, \dots, x_n) = \sum_{I \subseteq [n]} \det(M[I; I]) x_{I^c},$$

where $x_J = \prod_{j\in J} x_j$ for all $J \subseteq [n]$.

First, we prove that M is an M-matrix. By Theorem 3.3.7, we only need to prove that $\det(M[I; I]) \geq 0$ for each $I \subseteq [n]$. Let $I \subseteq [n]$. If $I = [n]$, then $M[I; I] = M$ and thus $\det(M[I; I]) = f(0, \dots, 0) \geq 0$. If $I = [n] \setminus j$ for some $j \in [n]$, then $M[I; I] = \partial f/\partial x_j(0, \dots, 0) > 0$ since $\partial f/\partial x_j$ is positive on $(\mathbb{R}_+ \cup \{0\})^n$.

If $J \subsetneq [n] \setminus j$ for some $j \in [n]$, then let $a_i = x$ for $i \notin J$ and $a_i = 0$ for $i \in J$. Thus, if $\det(M[J; J]) < 0$, then the leading coefficient of $\partial f/\partial x(a_1, \dots, a_n)$ will be $\det(M[J; J])$, which is a contradiction since $\partial f/\partial x_i$ is positive on $(\mathbb{R}_+ \cup \{0\})^n$. Thus, $\det(M[J; J]) \geq 0$.

Since we already proved that $\det(M) \geq 0$ and that $\det(M[J;J]) > 0$ if $J \subseteq [n]$ with $|J| = n-1$, then, in order to prove (1), we need to show that $\det(M[J;J]) > 0$ for each $J \subseteq [n]$ with $|J| < n-1$. Let $J \subseteq [n]$ with $|J| < n-1$. Since $|J| < n-1$, there exists $j \in [n]$ such that $J \subsetneq [n] \setminus j$. Let $I = [n] \setminus j$. Since M is an M-matrix, it follows that $M[I;I]$ is also an M-matrix. But $\det(M[I;I]) > 0$ since $|I| = n-1$. This means that $M[I;I]$ is a non-singular M-matrix. By [2, Theorem 6.2.3], all the principal minors of $M[I;I]$ are positive. In particular, $\det(M[J;J]) > 0$. □

3.3.1 Finiteness of M-matrices of fixed determinant

Given $\alpha \geq 0$ and a nonnegative integer $n \times n$ matrix M with all the diagonal entries equal to zero, let

$$\mathcal{A}_{\geq\alpha}(M) = \{\mathbf{d} \in \mathbb{N}_+^n \mid A = \operatorname{diag}(\mathbf{d}) - M \text{ is an } M\text{-matrix and } \det(A) \geq \alpha\}.$$

Also, let $\mathcal{A}_{\alpha}(M) = \{\mathbf{d} \in \mathcal{A}_{\geq\alpha}(M) \mid \det(\operatorname{diag}(\mathbf{d}) - M) = \alpha\}$. The last set is closely related to the arithmetical structures on a graph; see Chapter 4. More precisely, when M is equal to the adjacency matrix of a graph and $\alpha = 0$. However, to recover the main properties of the arithmetical structures on a graph, we must add some extra conditions to obtain the correct definition.

By Dickson's Lemma, $\mathcal{A}_{\geq\alpha}(M)$ has a finite number of minimal elements with respect to the partial order $\leq$ in $\mathbb{N}^n$. Let

$$\operatorname{Minimal}(\mathcal{A}_{\geq\alpha}(M)) = \{\mathbf{d} \in \mathcal{A}_{\geq\alpha}(M) \mid \text{ if } \mathbf{d}' \leq \mathbf{d} \text{ for some } \mathbf{d}' \in \mathcal{A}_{\geq\alpha}(M), \text{ then } \mathbf{d}' = \mathbf{d}\}$$

be the set of minimal elements of $\mathcal{A}_{\geq\alpha}(M)$. Moreover, if M is an almost non-singular M-matrix, then by Theorem 3.3.11 we have that

$$\mathcal{A}_{\geq\alpha}(M) = \operatorname{Minimal}(\mathcal{A}_{\geq\alpha}(M)) + (\mathbb{N}_+ \cup \{0\})^n.$$

That is, $\mathcal{A}_{\geq\alpha}(M)$ is an infinite monoid. However, as the following theorem shows, the set $\mathcal{A}_{\alpha}(M)$ is finite when M is a nonnegative integer matrix and $\alpha > 0$.

Theorem 3.3.12. *If M is a nonnegative integer matrix, then $\mathcal{A}_{\alpha}(M)$ is finite for any $\alpha > 0$.*

Proof. We claim that $\mathcal{A}_{\alpha}(M) \subseteq \operatorname{Minimal}(\mathcal{A}_{\geq\alpha}(M))$. We prove this by contradiction. Let $\mathbf{d} \in \mathcal{A}_{\alpha}(M)$ and assume that $\mathbf{d} \notin \operatorname{Minimal}(\mathcal{A}_{\geq\alpha}(M))$. This means that there exists an

$\mathbf{e} \in \mathcal{A}_{\geq\alpha}(M)$ such that $\mathbf{e} \lneq \mathbf{d}$. Since $\det(\operatorname{diag}(\mathbf{e}) - M) \geq \alpha > 0$, $\operatorname{diag}(\mathbf{e}) - M$ is a non-singular M-matrix. By Theorem 3.3.11, $\det(\operatorname{diag}(\mathbf{d}) - M) > \det(\operatorname{diag}(\mathbf{e}) - M) \geq \alpha$, which is a contradiction since $\det(\operatorname{diag}(\mathbf{d}) - M) = \alpha$.

Finally, since $\mathcal{A}_{\geq\alpha}(M) \subseteq \mathbb{N}_+^n$, by Dickson's Lemma $\text{Minimal}(\mathcal{A}_{\geq\alpha}(M))$ is finite and thus $\mathcal{A}_\alpha(M)$ is also finite. □

The inclusion $\mathcal{A}_\alpha(M) \subseteq \text{Minimal}(\mathcal{A}_{\geq\alpha}(M))$ is in general not an equality, as the following example shows.

Example 3.3.13. *If*

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then $\mathcal{A}_6(M) = \{(3,2,2)^t, (2,2,3)^t\}$ *and* $\min \mathcal{A}_{\geq 6}(M) = \{(3,2,2)^t, (2,3,2)^t, (2,2,3)^t\}$.

The particular case of $\mathcal{A}_\alpha(M)$ when α is equal to zero is more challenging to deal with. For instance, if M is reducible, then $\mathcal{A}_0(M)$ can be infinite, as the following example shows.

Example 3.3.14. *Let*

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

which is reducible because

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

On the other hand, it is not difficult to check that $\left\{(1,x,1,y)^t \,|\, x, y \in \mathbb{N}_+\right\} \subsetneq \mathcal{A}_0(M)$ *and therefore infinite.*

Chapter 4

Arithmetical structures of graphs

In this chapter, we start working with arithmetical structures. First, we will show some results that are true for connected graphs in general, and once we have them, we will proceed to do a more detailed analysis of arithmetical structures of the path $\mathcal{P}_n$ and the cycle C_n with n vertices. Finally, we will count the number of arithmetical structures on these two.

First, we recall that an arithmetical structure on a connected graph G is a pair of positive integer vectors $(\mathbf{d}, \mathbf{r})$, with $\mathbf{r}$ primitive such that

$$(\text{diag}(\mathbf{d}) - A_G)\mathbf{r} = 0,$$

where A_G is the adjacency matrix of G. Also, we recall that the matrix $\text{diag}(\mathbf{d}) - A_G$ is called the pseudo-Laplacian.

We will start by showing that the proper principal minors of a pseudo-Laplacian matrix of an arithmetical structure of a connected graph are positive.

Theorem 4.0.1. *If $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure on a connected graph G, the proper principal minors of its pseudo-Laplacian will be positive.*

Proof. We begin by showing that the principal minors of $L(G, \mathbf{d})$ are non-negative. Since $\mathbf{r} > \mathbf{0}$, the sign of the determinant of any submatrix of $K = L(G, \mathbf{d})\text{diag}(\mathbf{r})$ will be the same as of the corresponding submatrix of $L(G, \mathbf{d})$. Let $K' = K[I, I]$ for some $I \subsetneq [n]$. If λ is an eigenvalue of of K', then by Gershgorin's circle theorem

$$\lambda \in \bigcup_{i \in I} D(K'_{ii}, t_i),$$

where $D(K'_{ii}, t_i)$ is the disk centered at K'_{ii} with radius t, and $t_i = \sum_{j\in I\setminus i} |K'_{ij}|$. Since $K\mathbf{1} = \mathbf{0}$, for $\mathbf{1} = (1, \dots, 1)$, and $K_{ij} \leq 0$ for all $i \neq j$, it follows that

$$t_i = \sum_{j\in I\setminus i} -K_{ij} = K_{ii} + \sum_{j\notin I} K_{ij} \leq K_{ii}.$$

So $t_i \leq K_{ii}$, therefore the eigenvalues of K' are nonnegative. Since the determinant of K' is equal to the product of its eigenvalues, our desired result follows.

Now, we show that the minors are indeed positive. We proceed by contradiction, that is, suppose that $\det(K') = 0$. This would imply that there exists some $t \in \mathbb{Z}^n \setminus 0$, such that

$$K't = 0.$$

Let i be such that t_i is maximized. Then, we can rescale our vector so that $t_i = 1$ and $t_j \leq 1$ for all $j \neq i \in I$. Then

$$-K_{ii} = \sum_{j\in I\setminus i} K_{ij} t_j.$$

On the other hand, $K\mathbf{1} = \mathbf{0}$ so that $K_{ii} = -\sum_{j\neq i} K_{ij}$ and

$$\sum_{j\notin I} K_{ij} = \sum_{j\in I\setminus i} (1 - t_j) K_{ij}.$$

The left hand of this equation is nonnegative, and the right hand is non-positive, so they must be equal to 0. Then $K_{ij} = 0$ for all $j \notin I$, and $K_{ij} = 0$ for all $j \in I \setminus i$ such that $t_j \neq 1$.

Thus, if $J = \{j \in I | t_j = 1\}$, then $K_{ij} = 0$ for all $i \in J$ and $j \notin J$. Since $i \in J \neq \emptyset$ and $J \subset I \neq [n]$, the vertices in J are separate from those in its complement $J^c \neq \emptyset$, which would contradict the fact that G is connected. □

Using this, it is not difficult to prove that the pseudo-Laplacian of an arithmetical structure $(\mathbf{d}, \mathbf{r})$ on a connected graph G has rank $n - 1$.

Corollary 4.0.2. *If* $(\mathbf{d}, \mathbf{r})$ *is an arithmetical structure on a connected graph* G*, then its pseudo-Laplacian*

$$L(G, \mathbf{d}) = \operatorname{diag}(\mathbf{d}) - A_G,$$

has rank $n - 1$.

Proof. It follows from the fact that any principal submatrix of size $n - 1$ has positive determinant and, therefore, is nonsingular. □

Moreover, we also have that $L(G, \mathbf{d})$ is positive semidefinite as it is a symmetric matrix.

Corollary 4.0.3. *If* $(\mathbf{d}, \mathbf{r})$ *is an arithmetical structure on a connected graph* G*, then its pseudo-Laplacian is positive semidefinite.*

As the pseudo-Laplacian is a Z-matrix, it also follows that the pseudo-Laplacian of an arithmetical structure is an M-matrix. Moreover, it is an almost non-singular M-matrix.

We will use the notation $\mathrm{Arith}(G)$ to denote the set of all arithmetical structures $(\mathbf{d}, \mathbf{r})$ in the graph G.

For our next important concept in arithmetical structures, we have to define the *cokernel* of a pseudo-Laplacian.

Definition 4.0.4 (Cokernel)**.** *Let* G *be a connected graph and* $(\mathbf{d}, \mathbf{r})$ *be an arithmetical structure on* G*. Seeing the pseudo-Laplacian* $L(G, \mathbf{d})$ *as an homomorphism*

$$L(G, \mathbf{d}) : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n.$$

Then, we define the cokernel of $L(G, \mathbf{d})$ *to be the the quotient group*

$$\mathbb{Z}^n \big/ \mathrm{Im}(L(G, \mathbf{d})),$$

and we use the notation $\mathrm{cok}(L(G, \mathbf{d}))$.

Corollary 4.0.2 tells us that the image of $L(G, \mathbf{d})$ in Definition 4.0.4 above has dimension $n - 1$. This implies that

$$\mathrm{cok}(L(G, \mathbf{d})) \cong \mathbb{Z} \oplus K.$$

Where K is a finite abelian group, that is, K is the *torsion* part of $\mathrm{cok}(L(G, \mathbf{d}))$. This leads us to the following definition.

Definition 4.0.5. *The critical group associated with an arithmetical structure* $(\mathbf{d}, \mathbf{r})$ *on a graph* G *is the torsion part of the cokernel of the pseudo-Laplacian*

$$L(G, d) = \mathrm{diag}(\mathbf{d}) - A_G.$$

The following is a significant result that comes from the study of *Chip firing games*; it is proof that can be consulted in [9, Corrolary 3, page 119].

Theorem 4.0.6. *For a connected graph* G*, the order of the critical group of the Laplacian arithmetical structure is equal to the number of spanning trees in* G*.*

4.1 Arithmetical structures on the path

In this subsection, we will study the arithmetical structures on the path.

Lemma 4.1.1. *If* $\mathbf{r} = (r_1, \dots, r_n)$ *is an arithmetical r-structure on the path with* $n \geq 2$ *vertices* $\mathcal{P}_n$*, then*

$$r_1 = r_n = 1.$$

Moreover, if $r_i = 1$ *for some* $1 < j < n$*, then* $\mathbf{r}' = (r_1, \dots, r_i)$ *and* $\mathbf{r}'' = (r_i, \dots, r_n)$ *are arithmetical r-structures on the paths* $\mathcal{P}_i$ *and* $\mathcal{P}_{n-i+1}$*.*

Proof. Let $(\mathbf{d}, \mathbf{r})$ be an arithmetical structure on the path $\mathcal{P}_n$. Then its pseudo-Laplacian satisfies

$$L(\mathcal{P}_n, \mathbf{d})\mathbf{r} = \begin{pmatrix} d_1 & -1 & 0 & \dots & 0 & 0 \\ -1 & d_2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & d_n \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, we get the following set of equations

$$\begin{aligned} r_1 d_1 &= r_2, \\ r_i d_i &= r_{i-1} + r_{i+1} \quad \text{for } 1 < i < n, \\ r_n d_n &= r_{n-1}. \end{aligned} \tag{4.1}$$

From these equations, we get the following sequence of implications

$$r_1 | r_2 \Longrightarrow r_1 | r_3 \Longrightarrow \cdots \Longrightarrow r_1 | r_n.$$

By hypothesis, $\mathbf{r}$ is a primitive vector, that is, $\gcd(\mathbf{r}) = 1$, so $r_1 = 1$. Using a similar argument, just going backward from r_n to r_1, one gets

$$r_n | r_{n-1} \Longrightarrow r_n | r_{n-1} \Longrightarrow \cdots \Longrightarrow r_n | r_1$$

and therefore, also $r_n = 1$.

Now, suppose that $r_i = 1$ for some $1 < i < n$. We define the vectors $\mathbf{d}', \mathbf{r}' \in \mathbb{Z}^i$ as follows,

$$d'_j = \begin{cases} r_{i-1} & \text{if } j = i, \\ d_j & \text{if } j < I, \end{cases} \text{ and } r'_j = r_j \text{ for all } j \leq i.$$

Then, we shall verify that these vectors satisfy the set of equations 4.1 in order to see that they indeed define an arithmetical structure on $\mathcal{P}_i$. Remembering that by the previous part and by hypothesis $r_i = 1 = r_1$, we have that

$$\begin{aligned} d'_1 r'_1 &= d_1 r_1 = r'_2, \\ r'_j d'_j = r_j d_j &= r_{j-1} + r_{j+1} = r'_{j-1} + r_{j+1} \text{ for } 1 < j < i, \\ r'_i d'_i &= r_i r_{i-1} = r_{i-1} = r'_{i-1} \text{ for } i > 1. \end{aligned}$$

So the equations 4.1 are satisfied. Defining another pair $\mathbf{d}'', \mathbf{r}'' \in \mathbb{Z}^{n-i+1}$ similarly, we obtain the other arithmetical structure. □

As a consequence of Lemma 4.1.1, equivalent conditions for an integer vector $\mathbf{r}$ to be an r-strucutre on the path $\mathcal{P}_n$ are that

$$r_1 = r_n = 1 \text{ and } r_i | (r_{i-1} + r_{i+1}) \text{ for } 1 < i < n. \tag{4.2}$$

Given an arithmetical r-structure $\mathbf{r}$, let

$$\mathbf{r}(1) = |\{i : r_i = 1\}|,$$

be the number of entries of $\mathbf{r}$ that are equal to one.

We are now ready to count the arithmetical structures of the path.

Theorem 4.1.2. *The number of arithmetical structures on the path $\mathcal{P}_n$ with with $\mathbf{r}(1) = 2$ is the Catalan number C_{n-2}. Moreover, the number of arithmetical structures on $\mathcal{P}_n$ is the Catalan number C_{n-1}.*

Proof. As seen in Theorem 2.3.12, an interpretation of the Catalan number C_{n-2} is integer strings of length n of the form $1a_1a_2 \ldots a_{n-2}1$ such that $a_i > 1$ and $a_i | (a_{i-1} + a_{i+1})$ for all i. Thus, these strings coincide with the set of arithmetical r-structures in P_n with $\mathbf{r}(1) = 2$, and therefore we get the first claim.

Now, we will prove the second claim by induction. For $n = 2$, $C_1 = 1$ and there is only arithmetical structure on the path $\mathcal{P}_2$, the Laplacian arithmetical structure, and therefore our first case follows.

Assume, then, that the result is true for any $k < n$. Then, by Lemma 4.1.1 we may decompose any r-structure $(r) = (r_1, \ldots, r_n)$ on $\mathcal{P}_n$ into two r-structures $\mathbf{r}' = (r_1, \ldots, r_j)$ and $\mathbf{r}'' = (r_j, \ldots, r_n)$ on $\mathcal{P}_j$ and $\mathcal{P}_{n-j+1}$ respectively, where j is the smallest index greater than 1 such that $r_i = 1$. We note that $\mathbf{r}'$ will always have $\mathbf{r}'(1) = 2$, by definition of j.

Then, combining the first statement of our theorem and our induction hypothesis, the number of r-structures on $\mathcal{P}_n$ such that j is the minimum index greater than 1 such that $r_i = 1$ is $C_{j-2}C_{n-j}$.

Then then total number of r-structures on $\mathcal{P}_n$ is equal to the sum over all j's

$$\sum_{j=2}^{n} C_{j-2}C_{n-j} = \sum_{i=0}^{n-2} C_j C_{n-j-2}.$$

The result follows from the standard Catalan recurrence 2.3

$$C_{n-1} = \sum_{j=0}^{n-2} C_j C_{n-j-2}.$$

□

The following result gives us information about the arithmetical d-structures of a connected graph.

Lemma 4.1.3. *Let G be a connected graph with at least three vertices, $A = (a_{ij})$ be its adjacency matrix, and $v_1v_2 \in E(G)$. If $\mathbf{d}$ is an arithmetical d-structure on G with $d_1 = 1$, then $d_2 > 1$.*

Proof. Let $\mathbf{r} = (r_1, \dots, r_n)$ be the corresponding r-structure. Since the pseudo-Laplacian satisfies $L(G, \mathbf{d})\mathbf{r} = 0$,

$$d_1 r_1 = \sum_{i=1}^{n} a_{1i} r_i \quad \text{and} \quad d_2 r_2 = \sum_{i=1, i\neq 2}^{n} a_{2i} r_i.$$

Since G is connected, then either v_1 or v_2 must be neighbors of another vertex. Without loss of generality, we may assume that v_2 and v_3 are neighbors, so $a_{23} > 0$. If $d_1 = d_2 = 1$, then

$$\begin{aligned} r_1 r_2 = d_1 r_1 d_2 r_2 &= (\sum_{i=1}^{n} a_{1i} r_i)(\sum_{i=1}^{n} a_{2i} r_i) \\ &= a_{12}^2 r_1 r_2 + a_{12} a_{23} r_2 r_3 + \sum_{(i,j)\neq(1,2),(2,3)} a_{1i} a_{2j} r_i r_j \end{aligned}$$

and therefore

$$(a_{12}^2 - 1) r_1 r_2 + a_{12} a_{23} r_2 r_3 + \sum_{(i,j)\neq(1,2),(2,3)} a_{1i} a_{2j} r_i r_j = 0.$$

Which is impossible, as $a_{12}^2 - 1, a_{1i}a_{2j} \geq 0$ and $a_{23} r_2 r_3 > 0$. □

The following propositions will tell us the procedures for constructing arithmetical structures from one another.

Proposition 4.1.4. *Let $n \geq 2$ and $(d', r') \in Arith(P_n)$. Given $2 \leq i \leq n$, define integer vectors $\mathbf{d}$ and $\mathbf{r}$ of length $n+1$ as follows*

$$d_j = \begin{cases} d'_j & \text{if } j < i-1, \\ d'_{i-1} + 1 & \text{if } j = i-1, \\ 1 & \text{if } j = i, \\ d'_i + 1 & \text{if } j = i+1, \\ d'_{j-1} & \text{if } j > i+1, \end{cases} \quad \text{and } r_j = \begin{cases} r'_j & \text{if } j < i, \\ r'_{i-1} + r'_i & \text{if } j = i, \\ r'_{j-1} & \text{if } j > i, \end{cases}$$

for $1 \leq j \leq n+1$. Then $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure on the path $\mathcal{P}_{n+1}$. Moreover, the cokernel of the resulting arithmetical structure will be isomorphic to the cokernel of the original arithmetic structure.

Proof. We directly verify that $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure on the path $\mathcal{P}_{n+1}$. That is, we show that

$$(\operatorname{diag}(\mathbf{d}) - A_{\mathcal{P}_{n+1}})\mathbf{r} = 0.$$

If $j < i-1$ or $j > i+1$, then the entries of the new vectors are the same as the original, so the equations follow. That is

$$r_j d_j - r_{j+1} - r_{j-1} = r'_j d'_j - r'_{j+1} - r'_{j-1} = 0.$$

If $j = i-1$, then

$$r_{i-1} d_{i-1} - r_i - r_{i-2} = r'_{i-1}(d'_{i-1} + 1) - r'_{i-1} - r'_i - r'_{i-2} = r'_{i-1} d'_{i-1} - r'_{i-2} - r'_i = 0.$$

If $j = i+1$, then we obtain an equation similar to the previous case. In a similar way, if $j = i$, then

$$r_i d_i - r_{i-1} - r_{i+1} = (r'_{i-1} + r'_i) - r'_{i-1} - r'_i = 0.$$

Now, we proceed to the second part of the proposition. If $L(\mathcal{P}_n, \mathbf{d}') = M'$ and $L(\mathcal{P}_{n+1}, \mathbf{d}) = M$, by letting Q be the vector of size $n+1$ with $Q_{i+1} = Q_{i-1} = 1$ and zeroes otherwise. Then, it can be shown that the matrices

$$\begin{pmatrix} M' + Q^t Q & -Q^t \\ -Q & 1 \end{pmatrix} \quad \begin{pmatrix} M' & 0 \\ 0 & 1 \end{pmatrix}.$$

Are $\mathbb{Z}$-equivalent, and finally, that $\begin{pmatrix} M' + Q^tQ & -Q^t \\ -Q & 1 \end{pmatrix}$ is $\mathbb{Z}$-equivalent to M. Thus, the cokernels of M' and M are isomorphic. □

Proposition 4.1.5. *Let $n \geq 3$ and $(\mathbf{d}, \mathbf{r}) \in Arith(P_n)$ such that $d_i = 1$ for some $1 < i < n$. Define integer vectors $\mathbf{d}'$ and $\mathbf{r}'$ of length n as follows*

$$d'_j = \begin{cases} d_j & \text{if } j < i-1, \\ d_{i-1} - 1 & \text{if } j = i-1, \\ d_{i+1} - 1 & \text{if } j = i, \\ d_{j+1} & \text{if } n-1 \geq j > I, \end{cases} \quad \text{and } r'_j = \begin{cases} r_j & \text{if } j < i, \\ r_{j+1} & \text{if } j \geq i. \end{cases}$$

Then, $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on $\mathcal{P}_{n-1}$. Moreover, the cokernel of the resulting arithmetical structure will be isomorphic to the cokernel of the original arithmetic structure.

Proof. Firstly, we observe that the operation makes sense, as $d_{i-1} > 1$ and $d_{i+1} > 1$ by Lemma 4.1.3, so we indeed obtain a positive integer vector.

We directly verify that $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on the path $\mathcal{P}_n$. That is, we show that

$$(\text{diag}(\mathbf{d}') - A_{\mathcal{P}_{n+1}})\mathbf{r}' = 0.$$

If $j < i$ or $n - 1 \geq j > i$, then, once again, the entries of the new vectors are the same as the original, so the equations follow. That is, $r'_j d'_j - r'_{j+1} - r'_{j-1} = r_j d_j - r_{j+1} - r_{j-1} = 0$. If $j = i - 1$, then

$$\begin{aligned} r'_{i-1}d'_{i-1} - r'_i - r_{i-2} = r_{i-1}(d_{i-1} - 1) - r_{i+1} - r_{i-2} &= r_{i-1}(d_{i-1} - 1) - (r_i - r_{i-1}) - r_{i-2} \\ &= r_{i-1}d_{i-1} - r_{i-2} - r_i = 0 \end{aligned}$$

because $d_i = 1$, $r_i = r_{i-1} + r_{i+1}$ and $r_{i+1} = r_i - r_{i-1}$. If $j = i$, then

$$\begin{aligned} r'_i d'_i - r'_{i-1} - r'_{i+1} &= r_{i+1}(d_i - 1) - r_{i-1} - r_{i+1} = r_{i+1}(d_i - 1) - (r_i - r_{i+1}) - r_{i+2} \\ &= r_{i+1}d_{i+1} - r_i - r_{i+2} = 0 \end{aligned}$$

because $r_{i-1} = r_i - r_{i+1}$.

The second part of the proposition follows from the second part of Proposition 4.1.4, as the original structure on $\mathcal{P}_n$ is the subdivision of $(\mathbf{d}', \mathbf{r}')$. □

The resulting arithmetical structures in Propositions 4.1.4 and 4.1.5 are called the subdivisions and smoothing of the original arithmetical structure.

Theorem 4.1.6. *The only arithmetical structure* $(\mathbf{d}, \mathbf{r})$ *on* $\mathcal{P}_n$ *with* $d_i \geq 2$ *for all* $1 < i < n$ *is the Laplacian arithmetical structure.*

Proof. If $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure with $\mathbf{d}' \neq \mathbf{d} = (1, 2 \dots, 2, 1)$, then it can not happen that $\mathbf{d}' > \mathbf{d}$, $L(\mathcal{P}_n, d)$ is an M-matrix, and by Theorem 3.3.10, $\mathbf{d}' \gneq \mathbf{d}$ would imply that $\det(L(\mathcal{P}_n), \mathbf{d}') > 0$, in contradiction to the fact that it is an arithmetical structure. So for some $1 < i < n$, $d'_i = 1$. □

Theorem 4.1.7. *If* $(\mathbf{d}, \mathbf{r}) \in Arith(\mathcal{P}_n)$ *is an arithmetical structure on the path, then the associated critical group* $\mathcal{K}$ *of the arithmetical structure* $(\mathbf{d}, \mathbf{r})$ *on* $\mathcal{P}_n$ *is trivial. Moreover,*

$$\mathbf{r}(1) = 3n - 2 - \sum_{i=1}^{n} d_i.$$

Proof. We proceed by induction on n. Our base case is $n = 2$. Here the only arithmetical structure on $\mathcal{P}_2$ is the Laplacian arithmetical structure $\mathbf{d} = \mathbf{r} = (1, 1)$. Then, the critical group can be calculated directly. It is the torsion part of the cokernel

$$\mathbb{Z}^2 \Big/ Im \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Applying reductions by rows and columns, we get

$$\mathbb{Z}^2 \Big/ Im \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \cong \mathbb{Z}^2 \Big/ Im \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cong \mathbb{Z}^2 / \mathbb{Z} \cong \mathbb{Z}.$$

Clearly, the second statement of the theorem follows.

Now, suppose the result is true for all $2 \leq k \leq n$. If $(\mathbf{d}, \mathbf{r})$ is the Laplacian arithmetical structure, then it is clear that

$$3(n+1) - 2 - \sum_{i=1}^{n+1} d_i = 3n + 1 - 2(n-1) - 2 = n = \mathbf{r}(1).$$

By Theorem 4.0.6, the critical subgroup's order is equal to the number of spanning trees of the path $\mathcal{P}_{n+1}$, which is equal to one.

If $(\mathbf{d}, \mathbf{r})$ is different from the Laplacian arithmetical structure, then $d_i = 1$ for some $1 < i < n$ by Theorem 4.1.6, and by Proposition 4.1.4 it can be obtained by subdividing an arithmetical structure $(\mathbf{d}', \mathbf{r}')$ in $\mathcal{P}_n$.

We also have that $r_i = r_i d_i = r_{i-1} + r_{i+1} \geq 2$, so $r_i \neq 1$, and this implies that $\mathbf{r}(1) = \mathbf{r}'(1)$. So, by induction hypothesis

$$\begin{aligned}\mathbf{r}'(1) &= 3n - 2 - \sum_{j=1}^{n} d'_j \\ &= 3n - 2 - (\sum_{j=1}^{i-1} d_j + \sum_{j=i+1}^{n+1} d_j - 2) + d_i - d_i \\ &= 3n + 1 - \sum_{j=1}^{n+1} d_j \\ &= 3(n+1) - 2 - \sum_{j=1}^{n+1} d_j.\end{aligned}$$

Combined with the fact that $\mathbf{r}(1) = \mathbf{r}'(1)$, this gives us the second part of the result.

Moreover, the cokernels of $\mathcal{L}(\mathcal{P}_n, \mathbf{d}')$ and $\mathcal{L}(\mathcal{P}_{n+1}, \mathbf{d})$ will be isomorphic by the second part of Proposition 4.1.4. Thus, the resulting critical subgroups will also be isomorphic and, according to the induction hypothesis, trivial. □

Remark 4.1.8. *During the proof of Theorem 4.1.7 we also showed that subdividing an arithmetical r-structure $\mathbf{r}$ will result in an arithmetical structure $\mathbf{r}'$ such that $\mathbf{r}(1) = \mathbf{r}'(1)$.*

Using Propositions 4.1.4 and 4.1.5, we will describe the process to obtain arithmetical structures on $\mathcal{P}_n$ from the Laplacian arithmetical structure on $\mathcal{P}_m$, when $m < n$ by applying the subdivision operation on the paths $\mathcal{P}_m, \mathcal{P}_{m+1}, \ldots, \mathcal{P}_{m+(n-m)} = \mathcal{P}_n$, on selected $n - m$ vertices.

If $2 \leq m \leq n$, then let b be an interger vector of length $n-m$ such that $1 \leq b_i \leq m+i-2$. Then, define $(\mathbf{d_0}, \mathbf{r_0})$ as the Laplacian arithmetical structure in $\mathcal{P}_m$. Then, for $i \geq 1$ define $(\mathbf{d_i}, \mathbf{r_i})$ to be the arithmetical structure on $\mathcal{P}_{m+i}$ obtained from the arithmetical structure $(\mathbf{d_{i-1}}, \mathbf{r_{i-1}})$ by subdividing the edge b_i in $\mathcal{P}_{m+i-1}$. We will denote the arithmetical structure on the path $\mathcal{P}_n$ obtained at the end of this procedure by $\mathbf{A}_n(b) := (\mathbf{d}_{n-m}, \mathbf{r}_{n-m})$.

We proceed to show an example of this procedure.

Example 4.1.9. *Let $n = 5$, $m = 2$. Then, we must construct an integer vector b satisfying $1 \leq b_i \leq 2 + i - 2 = i$. The vector $b = (1, 2, 3)$ suffices.*

Then, we define $\mathbf{d_0} = \mathbf{r_0} = (1, 1)$, the Laplacian arithmetical structure on $\mathcal{P}_2$. By applying the procedure, we get:

- $\mathbf{d_1} = (2,1,2), \quad \mathbf{r_1} = (1,2,1)$
- $\mathbf{d_2} = (2,2,1,3), \quad \mathbf{r_2} = (1,2,3,1)$
- $\mathbf{d_3} = (2,2,2,1,4), \quad \mathbf{r_3} = (1,2,3,5,1)$

From which we obtain the arithmetical structure $\mathbf{d} = (1,3,2,1,3)$, $\mathbf{r} = (1,2,3,5,1)$ *on* $\mathcal{P}_5$.

We can represent our process in the following sequence:

$$\frac{(1,1)}{(1,1)} \to \frac{(2,1,2)}{(1,2,1)} \to \frac{(2,2,1,3)}{(1,2,3,1)} \to \frac{(2,2,2,1,4)}{(1,2,3,5,1)},$$

where the upper part are the entries d_i'*s and the bottom part are the entries* r_i'*s*

Remark 4.1.10. *A problem with our method of generating arithmetical structures when it comes to counting them is that it allows for two different integer vectors,* b *and* b'*, to give the same resulting arithmetical structures.*

For example, if $m = 2$*,* $n = 5$*, then both* $b = (1,2,1)$ *and* $b' = (1,1,3)$ *give us the following sequence of arithmetical structures:*

$$b : \frac{(1,1)}{(1,1)} \to \frac{(2,1,2)}{(1,2,1)} \to \frac{(3,1,2,2)}{(1,3,2,1)} \to \frac{(3,1,3,1,3)}{(1,3,2,3,1)}$$

$$b' : \frac{(1,1)}{(1,1)} \to \frac{(2,1,2)}{(1,2,1)} \to \frac{(2,2,1,3)}{(1,2,3,1)} \to \frac{(3,1,3,1,3)}{(1,3,2,3,1)}.$$

For our purpose of counting arithmetical structures, we must find a way to associate an arithmetical strucure to exactly one vector.

The following lemma will allow us to associate an unique vector to an arithmetical structure, even more, this vectors will have a certain structure that will allow us to count them with what we know.

Lemma 4.1.11. *Let* $n \geq m \geq 2$ *and* $b = (b_1, \dots, b_{n-m})$ *be an integer vector such that* $1 \leq b_i \leq m + i - 2$*, and such that* $b_i > b_{i+1}$ *for some* $1 \leq i \leq n - m$*. Define the vector* b' *as follows:*

$$b_j = \begin{cases} b_{i+1} & \text{if } j = i, \\ b_i + 1 & \text{if } j = i+1, \\ b_j & \text{otherwise}. \end{cases}$$

Then the arithmetical structures $A_n(b)$ *and* $A_n(b')$ *arising from* b *and* b' *will be equal.*

Proof. We shall show what happens in our process once we reach the i-th step. Let $\mathbf{d} = (d_1, \ldots, d_{m+i-1}), \mathbf{r} = (r_1, \ldots, r_{m+i-1})$ be the arithmetical structure obtained at the $(i-1)$-th step in our procedure.

Since our vector b' only differs from b in the i and $i+1$ index, the arithmetical structures attained from both will only differ in the i and $i+1$-th steps. We show this in the diagram

$$
\begin{aligned}
b :&(\ldots, d_{b_{i+1}-1}, d_{b_{i+1}, d_{b_{i+1}+1}}, \ldots, d_{b_{i-1}}, d_{b_i}, d_{b_{i+1}}, \ldots)\\
&\to (\ldots, d_{b_{i+1}-1}, d_{b_{i+1}}, d_{b_{i+1}+1}, \ldots, d_{b_{i-1}}+1, 1, d_{b_i}+1, d_{b_i+1}, \ldots)\\
&\to (\ldots, d_{b_{i+1}-1}+1, 1, d_{b_{i+1}}+1, d_{b_{i+1}+1}, \ldots, d_{b_{i-1}}+1, 1, d_{b_i}+1, d_{b_i+1}, \ldots)\\
b' :&(\ldots, d_{b_{i+1}-1}, d_{b_{i+1}, d_{b_{i+1}+1}}, \ldots, d_{b_{i-1}}, d_{b_i}, d_{b_{i+1}}, \ldots)\\
&\to (\ldots, d_{b_{i+1}-1}+1, 1, d_{b_{i+1}}+1, d_{b_{i+1}+1}, \ldots, d_{b_{i-1}}, d_{b_i}, d_{b_i+1}, \ldots)\\
&\to (\ldots, d_{b_{i+1}-1}+1, 1, d_{b_{i+1}}+1, d_{b_{i+1}+1}, \ldots, d_{b_{i-1}}+1, 1, d_{b_i}+1, d_{b_i+1}, \ldots)
\end{aligned}
$$

From this, it is clear that the resulting arithmetical structures at the end of the procedure will be equal. We notice that in b', the fact that we subdivided the edge $b_{i+1} < b_i$ makes it so that the edge b_i in $\mathcal{P}_{m+i+1}$ becomes the edge $b_i + 1$ in the standard enumeration of edges in $\mathcal{P}_{m+i+1}$. □

Repeatedly applying this lemma to a vector b satisfying our conditions, we can obtain a vector b' with $b'_i \le b'_{i+1}$ for all i, that will give us the same arithmetical structure as b. This will lead us to the following result.

Lemma 4.1.12. *Every arithmetical structure on $\mathcal{P}_n$ is equal to $\mathbf{A}_n(b)$ for a unique vector $b = (b_1, \ldots, b_{n-m})$ satisfying $1 \le b_i \le i + m - 2$ and $b_i \le b_{i+1}$ for all i.*

Proof. Applying Proposition 4.1.5 repeatedly, we will get to a Laplacian arithmetical structure on $\mathcal{P}_m$ for some $m \le n$. The order in which we subdivide the edges of this resulting arithmetical structure will give us our vector b. From this, any arithmetical structure will be equal to $\mathbf{A}_n(b)$ for some b.

Finally, if $b \ne b'$, then our subdivision process will yield different arithmetical structures, and we get the result. □

Remark 4.1.13. *By Remark 4.1.8, we note that in Lemma 4.1.12 the resulting arithmetical structure will have $\mathbf{r}(1) = m$.*

With this, we proceed to count the number of arithmetical structures on the path $\mathcal{P}_n$. By Lemma 4.1.12, we know this will be the same as counting the number of integer sequences $b = (b_1, \ldots, n - m)$ with $2 \le m \le n$ such that $1 \le b_i \le m + i - 2$ and $b_i \le b_{i+1}$.

Adding $m-2$ entries equal to 1 to this vector, we get a vector of the form

$$(1,\dots,1,b_{m-1},\dots,b_{n-2}),$$

such that $1 \le b_i \le i$ and $b_i \le b_{i+1}$. Sequences of this type are called *ballot sequences*.

We recall that the ballot number $B(n,m)$ counts the number of lattice paths from $(0,0)$ to (n,m) that does not cross above the diagonal $x=y$.

Any path from $(1,1)$ to $(n-1,n-1)$ that does not cross above the diagonal $x=y$ will give rise to a sequence of this type by associating to b_i the y-coordinate of the path at $x=i$. Note that $b_i \le i$ because the path does not cross the diagonal $x=y$. Note that this association would give us a $b_{n-1}=n-1$, so we can omit it to be left with a ballot sequence $b_1,\dots,b_{n-2}$.

If we require our path to start with $m-2$ horizontal steps, then we are left with a path that can be divided into a path from $(1,1)$ to $(m-1,1)$ and a path from $(m-1,1)$ to $(n-1,n-1)$. Reversing the order of the tail, that is, starting doing the step taken at the end of the original path and changing each vertical step to a horizontal one, and likewise, for horizontal steps, we get a path from $(1,1)$ to $(n-1,n-1)$ that ends in $m-2$ vertical steps. These paths are clearly in bijection to paths from $(0,0)$ to $(n-2,n-k)$. An example of this is shown in the following figure.

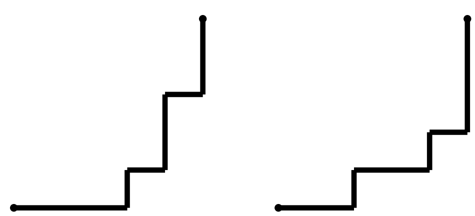


Figure 4.1: The path HHHVHVVHVV is converted into the path HHVHHVHVVV

Thus, the number of these paths is $B(n-2,n-m)$. This discussion, along with Remark 4.1.13 lead us to the next main result of this section.

Theorem 4.1.14. *Let $n \ge 2$ and $1 \le k \le n$, and let $A(n,k)$ be the number of arithmetical structures $(\mathbf{d},\mathbf{r})$ on $\mathcal{P}_n$ such that $\mathbf{r}(1)=k$. Then*

$$A(n,k)=B(n-2,n-k).$$

We could have also proven this result by mathematical induction on $\mathbf{r}(1)=k$, with n fixed and $k \le n$. For $k=1$, we know that $\mathbf{r}(1) \ge 2$, so the number of arithmetic structures on $\mathcal{P}_n$ with $\mathbf{r}(1) = 1$ is $0 = B(n-1, n-2)$. In Theorem 4.1.2 we saw that the number of arithmetical structures on $\mathcal{P}_n$ with $\mathbf{r}(1) = 2$ is the Catalan number $C_{n-2} = B(n-2, n-2)$.

Assume the result is true for all $2 \le j \le k$. Then, if $\mathbf{r}$ is an arithmetical structure with $\mathbf{r}(1) = k+1$, then let m be the first entry after the first of $\mathbf{r}$ such that $r_i = 1$. Then by Lemma 4.1.1, $\mathbf{r}' = (r_1, \ldots, r_m)$ and $\mathbf{r}'' = (r_m, \ldots, r_n)$ will be arithmetical r-structures on $\mathcal{P}_m$ and $\mathcal{P}_{n-m+1}$ respectively, with $\mathbf{r}'(1) = 2, \mathbf{r}''(1) = k$. Every pair r', r'' of this kind will define an arithmetical r-structure $\mathbf{r}$ on $\mathcal{P}_n$, with $\mathbf{r}(1) = k+1$.

With this reasoning, the number of arithmetical r- structures on $\mathcal{P}_n$ with $\mathbf{r}(1) = k+1$ is

$$A(n, k+1) = \sum_{m=2}^{n-k+1} A(m, 2)A(n-m+1, k) \tag{4.3}$$

$$= \sum_{m=2}^{n-k+1} B(m-2, m-2)B(n-m-1, n-m-k+1) \tag{4.4}$$

$$= B(n-2, n-k-1) \tag{4.5}$$

Which follows from the identity 2.2. This also proves Theorem 4.1.14.

We obtain the following corollary.

Corollary 4.1.15. *For $n \ge 2$ and any k, we have*

$$|\{(\mathbf{d}, \mathbf{r}) \in Arith(\mathcal{P}_n) : \sum_{j=1}^{n} d_j = k\}| = B(n-2, k-2n+2).$$

In particular, there are no arithmetical structures with $\sum_{j=1}^{n} d_j = k$ unless $2n-2 \le k \le 3n-4$.

Proof. By Theorem 4.1.7, if $(\mathbf{d}, \mathbf{r})$ is an arithmetical structure, then $\mathbf{r}(1) = 3n-2-\sum_{j=1}^{n} d_j$. Thus, if $\sum_{j=1}^{n} d_j = k$, $\mathbf{r}(1) = 3n-2-k$.

Then, by Theorem 4.1.14,

$$\begin{aligned} |\{(\mathbf{d}, \mathbf{r}) \in \text{Arith}(\mathcal{P}_n) : \sum_{j=1}^{n} d_j = k\}| &= A(n, 3n-2-k) = B(n-2, n-3n+2+k) \\ &= B(n-2, k-2n+2). \end{aligned}$$

□

In what follows, it will be useful to adopt a new way to generate arithmetical structures. For a $(n+1)$-gon, label its vertices as $0, \dots, n$ in clockwise order.

Given a triangulation T of the $(n+1)$-gon, let $D(T) = (D_0, \dots, D_n)$ be the integer vector such that D_i is the number of triangles adjacent to the vertex i.

Example 4.1.16. *For the following triangulation of the heptagon, we have* $D(T) = (2, 1, 4, 1, 3, 1, 3)$.

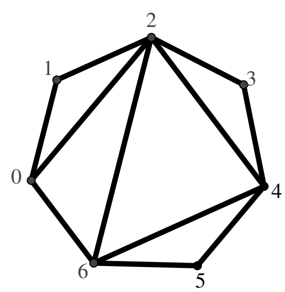


Figure 4.2

We notice that $(1, 4, 1, 3, 1, 3)$ defines an arithmetical d-structure on $\mathcal{P}_6$. This motivates the following result.

Theorem 4.1.17. *If T is a triangulation of the $n+1$-gon, then $(D_1, \dots, D_n)$ is an arithmetical d-structure on $\mathcal{P}_n$.*

Proof. Thinking D as a map from the set of triangulations of the $n+1$-gon to the set of arithmetical d-structures on $\mathcal{P}_n$.

We prove the theorem by induction on n. For the base case $n = 2$, the only triangulation of the 3-gon (triangle) is trivial and it is given by $D(T) = (1, 1, 1)$, and the only arithmetical d-structure on the path $\mathcal{P}_2$ is $(1, 1)$.

For $n \geq 3$, we can obtain any triangulation of an $n+1$-gon by gluing a triangle to the exterior of a triangulation of an n-gon. Let T' be the triangulation of the n-gon, $D(T') = (d'_0, d'_1, \dots, d'_{n-1})$, T the triangulation obtained, $D(T) = (d_0, d_1, \dots, d_n)$.

If the triangle was glued on the edge between vertices $i, i+1$, then the entries of $D(T)$

are defined by

$$d_j = \begin{cases} d'_j & \text{if } j < I, \\ d'_{i-1} + 1 & \text{if } j = i - 1, \\ 1 & \text{if } j = i, \\ d'_i + 1 & \text{if } j = i + 1, \\ d'_{j-1} & \text{if } j < i + 1. \end{cases}$$

This agrees with our process of subdividing an edge as described in Proposition 4.1.4, by induction hypothesis $D(T')$ was an arithmetic d-structure, and so, $D(T)$ is an arithmetic d-structure.

By our previous discussion of the subdivision process, the map D is surjective. Two triangulations associated with the same arithmetic structure will be the same. Therefore, D is bijective. □

Example 4.1.18. *We shall use the previous example to illustrate this procedure.*

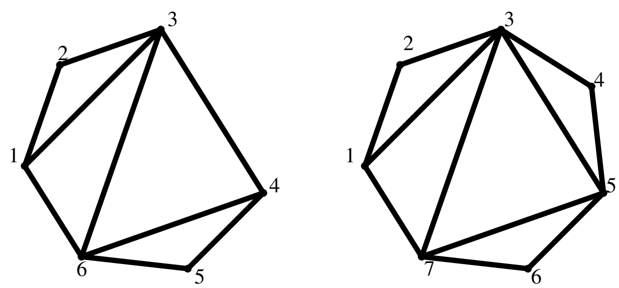


Figure 4.3

For the first polygon $D(T') = (2, 1, 3, 2, 1, 3)$. *By gluing the triangle to obtain the triangulation of the* 7*-gon, we obtain* $D(T) = (2, 1, 4, 1, 3, 1, 3)$.

This allows us to prove the following theorem.

Theorem 4.1.19. *For each* $1 \leq i \leq n$ *and* $0 \leq k \leq n - 2$, *the number of arithmetical* d*-structures* $(d_1, \ldots, d_n)$ *on* $\mathcal{P}_n$ *with* $d_i = n - k - 1$ *is equal to* $B(n - 2, k)$.

Proof. Define $d_0 = 3n - 3 - \sum_{j=1}^{n} d_j$, and then, by Corollary 4.1.15, the result extends to the case $i = 0$, as $d_0 = \mathbf{r}(1) - 1$.

For the rest of the cases, from Theorem 2.3.8, the number of triangulations of a $n + 1$-gon with $d_i = n - k - 1$ is $B(n - 2, k)$. This, combined with the above results, gives us our desired conclusion. □

Let $\mathbf{d}(1)$ defined similarly to $\mathbf{r}(1)$.

Proposition 4.1.20. *If n is a positive integer, then the number of arithmetical structures $(\mathbf{d}, \mathbf{r})$ on $\mathcal{P}_{n+2}$ with $\mathbf{r}(1) = 2$ and $\mathbf{d}(1) = k$ is equal to*

$$\binom{n-1}{2k-2} 2^{n+1-2k} C_{k-1}.$$

Proof. From Corollary 2.3.13, we know that the number of admissible sequences, sequences $a_0, \ldots, a_{n+1}$ such that: $a_0 = a_{n+1} = 1$ and $a_i | (a_{i-1} + a_{i+1})$ for $1 \leq i \leq n$, and which attain k local maxima (that is, $a_i = a_{i-1} + a_{i+1}$), is

$$\binom{n-1}{2k-2} 2^{n+1-2k} C_{k-1}.$$

We shall show then that these sequences are in correspondence to arithmetical structures in $\mathcal{P}_{n+2}$ that fulfill the conditions of the theorem.

Let $\mathbf{d} = (d_1, \ldots, d_{n+2})$, $\mathbf{r} = (r_1, \ldots, r_{n+2})$ be arithmetical structures on $\mathcal{P}_{n+2}$. As $\mathbf{r}(1) = 2$, then $r_2 \ldots r_n$ is an admissible sequence satisfying 4.1. Then, for $1 < i < n+2$, $r_i = r_{i-1} + r_{i+1}$ if and only if $d_i = 1$, that is, if the sequence r_j attains a maxima at i. If $d_1 = 1$ or $d_{n+2} = 1$, then $1 = r_1 = r_2$ or $r_{n+1} = r_{n+2} = 1$ respectively. This can not happen as $\mathbf{r}(1) = 2$.

Thus, the arithmetical structures satisfying the theorem conditions are in correspondence to admissible sequences with k local maxima, and we get the result. □

4.2 Arithmetical structures on the cycle

In this section, we will conduct a similar analysis of arithmetical structures on the cycle C_n as we did with the path $\mathcal{P}_n$.

The cycle C_2 on two vertices is the multigraph with vertex set $\{1, 2\}$ and edge set $\{(1, 2), (2, 1)\}$. We begin by describing the complete set of arithmetical structures on C_2.

Theorem 4.2.1. *The 2-cycle C_2 has only three arithmetical structures, namely* $(1, 4), (2, 2), (4, 1)$.

Proof. The adjacency matrix of the 2-cycle is equal to

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

So for $L(C_2, \mathrm{diag}(x_1, x_2))$ to have non trivial kernel we must have

$$x_1x_2 - 4 = 0.$$

The solutions of this equation over the positive integers are $(1,4)$, $(4,1)$, $(2,2)$. These are all the arithmetical d-structures.

Solving for the possible r-structures, we obtain the following arithmetical structures for C_2

$$\frac{(1,4)}{(1,2)}, \frac{(2,2)}{(1,1)}, \text{ and } \frac{(4,1)}{(2,1)}.$$

□

As with the path, the condition

$$(\mathrm{diag}(\mathbf{d}) - A_{C_n})\mathbf{r} = 0,$$

will produce a similar set of conditions for an integer vector $\mathbf{r}$ to be an r-strucure in C_n.

Proposition 4.2.2. *If* $\mathbf{r}$ *is an arithmetical* r*-structure on the cycle* C_n*, then, by taking the indices modulo n, we have*

$$r_i | (r_{i-1} + r_{i+1}) \quad \textit{for all } 1 \le i \le n.$$

Proof. For $n > 2$, the adjacency matrix for the n-cycle is equal to

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

If $(\mathbf{d}, \mathbf{r}) \in \mathrm{Arith}(C_n)$, then $L(A_{C_n}, \mathbf{d})\mathbf{r} = 0$. This implies the following system of equations

$$\begin{aligned} d_1r_1 - r_2 - r_n &= 0, \\ d_2r_3 - r_1 - r_3 &= 0, \\ &\vdots \\ d_nr_n - r_1 - r_{n-1} &= 0. \end{aligned} \tag{4.6}$$

This system of equations directly implies the result. □

For an arithmetical structure $(\mathbf{d}, \mathbf{r}) \in \mathrm{Arith}(C_n)$, we adopt the convention $d_0 = d_n$. The next result proves that the subdivision operation on an arithmetical structure of the cycle C_n is again an arithmetical structure of the cycle C_{n+1}.

Theorem 4.2.3. *If $(\mathbf{d}', \mathbf{r}') \in Arith(C_n)$ and $1 \leq i \leq n+1$, then the vectors $\mathbf{d}, \mathbf{r}$ whose entries are given by*

$$d_j = \begin{cases} d'_j & \text{if } j < i-1, \\ d'_{i-1} + 1 & \text{if } j = i-1, \\ 1 & \text{if } j = i, \\ d'_i + 1 & \text{if } j = i+1, \\ d'_{j-1} & \text{if } j > i+1, \end{cases} \quad \text{and } r_j = \begin{cases} r'_j & \text{if } j < i, \\ r'_{j-1} + r'_j & \text{if } j = i, \\ r'_{j-1} & \text{if } j > i, \end{cases}$$

form an arithmetical structure on C_{n+1}. Moreover, the cokernels of $L(C_n, \mathbf{d}')$ and $L(C_{n+1}, \mathbf{d})$ are isomorphic.

Proof. We shall show that the system of equations 4.6 is satisfied by the newly defined vectors.

For $j < i-1$ and $j > i+1$ the equation is clearly fulfilled, as we will get the same equations as in the original vectors. So, we shall only check the equations in which the vectors $\mathbf{d}, \mathbf{r}$ differ from $\mathbf{d}', \mathbf{r}'$.

For $j = i-1$, we have

$$\begin{aligned} r_j d_j &= r'_j (d'_j + 1) \\ &= r'_j d'_j + r'_j = r'_{j-1} + r'_{j+1} + r'_j \\ &= r_{j-1} + r_{j+1}. \end{aligned}$$

For $j = i$, we have that $r_j d_j = (r'_{j-1} + r'_j)(1) = r_{j-1} + r_{j+1}$. Finally, for $j = i+1$, we have

$$\begin{aligned} r_j d_j &= r'_{j-1}(d'_{j-1} + 1) = r'_{j-1} d'_{j-1} + r'_{j-1} \\ &= r'_{j-2} + r'_j + r'_{j-1} = r'_{j-2} + r'_{j-1} + r'_j \\ &= r_{j-1} + r_{j+1}. \end{aligned}$$

By a similar reasoning to Proposition 4.1.4, one shows the pseudo-Laplacian matrices are $\mathbb{Z}$-equivalent, so the cokernels are isomorphic. □

The next result proves something similar for the smoothing operation, which is the inverse of the subdivision operation.

Theorem 4.2.4. *Let $n \geq 3$ and $(d, r) \in Arith(C_{n+1})$ such that $d_{i-1} > d_i = 1 < d_{i+1}$ for some $1 \leq i \leq n$. Define integer vectors $\mathbf{d}'$ and $\mathbf{r}'$ of length n as follows*

$$d'_j = \begin{cases} d_j & \text{if } j < i-1, \\ d_{i-1} - 1 & \text{if } j = i-1, \\ d_{i+1} - 1 & \text{if } j = i, \\ d_{j+1} & \text{if } n-1 > j > I, \end{cases} \qquad \text{and } r'_j = \begin{cases} r_j & \text{if } j < i, \\ r_{j+1} & \text{if } j \geq i. \end{cases}$$

Then $(\mathbf{d}', \mathbf{r}')$ is an arithmetical structure on C_{n-1}. Moreover, the cokernels of $L(C_{n-1}, \mathbf{d}')$ and $L(C_{n+1}, \mathbf{d})$ are isomorphic.

Proof. We will use similar arguments to those given in the previous proposition. More precisely, for $j < i-1$ and $n-1 > j > i$, the Equations 4.6 follow, as the vectors have the same entries as the original pair. Then, we only need to check for the entries in which they will differ.

For $j = i-1$, as $d_i = 1, r_i = r_{i-1} + r_{i+1}$

$$\begin{aligned} r'_{i-1}d'_{i-1} &= r_{i-1}(d_{i-1} - 1), \\ &= r_{i-2} + r_i - r_{i-1} = r_{i-2} + r_{i-1} + r_{i+1} - r_{i-1}, \\ &= r_{i-2} + r_{i+1} = r_{i-2} + r_i. \end{aligned}$$

And for $j = i$,

$$\begin{aligned} r'_i d_i &= r_{i+1}(d_{i+1} - 1) = r_i + r_{i+2} - r_{i+1}, \\ &= r_{i-1} + r_{i+1} + r_{i+2} - r_{i+1} = r_{i-1} + r_{i+2}, \\ &= r'_{i-1} + r'_{i+1} \end{aligned}$$

By a similar reasoning to Proposition 4.1.3. one shows the pseudo-Laplacian matrices are $\mathbb{Z}$-equivalent, so the cokernels are isomorphic. □

One can see that in the process of subdividing an arithmetical structure, we have $\mathbf{r}'(1) = \mathbf{r}(1)$.

Proposition 4.2.5. *There is only one arithmetical structure on C_n such that $d_i \geq 2$ for all i, namely* $\mathbf{d} = (2, \ldots, 2) := \mathbf{2}, \mathbf{r} = (1, \ldots, 1) := \mathbf{1}$.

Proof. The result follows similarly to Theorem 4.1.6. □

Corollary 4.2.6. *If* $\mathbf{r}$ *is an arithmetical r-structure on* C_n*, then* $\mathbf{r}(1) > 0$.

Proof. We prove the result by induction on n. By Theorem 4.2.1, the result is clear for $n = 2$.

Suppose, then, that the result is true for all $2 \le k \le n$. For $n + 1$, if $\mathbf{d} = \mathbf{2}, \mathbf{r} = \mathbf{1}$ and the result is clear.

If $(\mathbf{d}, \mathbf{r}) \neq (\mathbf{2}, \mathbf{1})$, then by Proposition 4.2.5, we have $d_i = 1$ for some i. By Lemma 4.1.3, we have $d_{i-1} > d_i < d_{i+1}$, this implies that the arithmetical structure $(\mathbf{d}, \mathbf{r})$ can be obtained by subdividing an arithmetical structure on C_n (the subdivision of its smoothing).

Since $\mathbf{r}'(1) = \mathbf{r}(1)$, the result follows by our inductive hypothesis. □

Theorem 4.2.7. *If* $(\mathbf{d}, \mathbf{r}) \in Arith(C_n)$ *is an arithmetical structure of the cycle* C_n*, then*

$$\mathbf{r}(1) = 3n - \sum_{j=1}^{n} d_j$$

and $K(C_n, \mathbf{d}, \mathbf{r}) = \mathbb{Z}_{\mathbf{r}(1)}$.

Proof. We proceed by induction on n. First, for $n = 2$, one can directly calculate by checking the structures on Theorem 4.2.1.

For $n \ge 3$. If $(\mathbf{d}, \mathbf{r}) = (\mathbf{2}, \mathbf{1})$ is the Laplacian arithmetical structure, then $3n - \sum_{j=1}^{n} d_j = 3n - 2n = n$, and $K(C_n, \mathbf{2}, \mathbf{1}) = \mathbb{Z}_n$, this result can be consulted on [9], p.121.

If $\mathbf{d} \neq \mathbf{2}$, then $(\mathbf{d}, \mathbf{r})$ is obtained by subdividing an edge on an arithmetical structure $(\mathbf{d}', \mathbf{r}')$ in C_{n-1}. Then

$$\sum_{j=1}^{n} d_j = (\sum_{j=1}^{n-1} d'_j) + 3 = 3(n-1) - \mathbf{r}'(1) + 3 = 3n - \mathbf{r}'(1) = 3n - \mathbf{r}(1).$$

From which the first part follows. Because the cokernels of $L(C_n, \mathbf{d})$ and $L(C_{n-1}, \mathbf{d}')$ are isomorphic, the critical groups are isomorphic. As $\mathbf{r}(1) = \mathbf{r}'(1)$, the las part of the theorem follows. □

Before we start counting the number of arithmetical structures on C_n, we introduce the set

$$MSet_i(n) = \{S \text{ is a multiset of } [n] \text{ of cardinality } i\},$$

and we define similarly the set $MSet_{\le i}(n)$. A known result is that

$$|MSet_k(n)| \overset{\text{def}}{=} \left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

We now start counting the number of arithmetical structures on C_n with the following theorem.

Theorem 4.2.8. *If* $1 \leq k \leq n$*, then*

$$|\{(\mathbf{d}, \mathbf{r}) \in Arith(C_n) : k = \mathbf{r}(1)\}| = \left(\!\!\binom{n}{n-k}\!\!\right) = \binom{2n-k-1}{n-k}.$$

Moreover, the total number of arithmetical structures on C_n *is*

$$\sum_{k=1}^{n} \left(\!\!\binom{n}{n-k}\!\!\right) = \left(\!\!\binom{n+1}{n-1}\!\!\right) = \binom{2n-1}{n-1}.$$

Before beginning with the proof, we introduce two actions of $\mathbb{Z}_n$, the first one over $\text{Arith}(C_n)$ and the second one over $MSet_l(n)$. More precisely, given $c \in \mathbb{Z}_n$, let

$$\rho_c(r_1, \ldots, r_n) = (r_{1+c}, \ldots, r_n, r_1, \ldots, r_c),$$

the action that rotates the positions on the vector $\mathbf{r} = (r_1, \ldots, r_n)$ and

$$\phi_c([a_1, \ldots, a_l]) = [b_1, \ldots, b_l],$$

where $b_i = (a_i + c) \bmod n$ for $1 \leq i \leq l$. That is, this action rotates the values of $[a_1, \ldots, a_l]$ by taking it modulo n.

Example 4.2.9. *The* ρ*-orbit of the arithmetical* r*-structure* $\mathbf{r} = (2, 3, 4, 1, 3)$ *in* C_5 *is given by*

$$\{(2,3,4,1,3), (3,4,1,3,2), (4,1,3,2,3), (1,3,2,3,4), (3,2,3,4,1)\}.$$

The ϕ*-orbit of the multiset* $m = [2, 2, 3, 5]$ *in* $MSet_4(5)$ *is given by*

$$\{[2,2,3,5], [3,3,4,1], [4,4,5,2], [5,5,1,3], [1,1,2,4]\}.$$

With this, we are ready to prove Theorem 4.2.8.

Proof. We will give an explicit bijection

$$\Omega : MSet_{\leq n-1}(n) \to \text{Arith}(C_n).$$

Even though the output of this function is an arithmetical structure $(\mathbf{d}, \mathbf{r})$, for our convenience, we will only refer to the r-structure; that is, we will treat it as if $\Omega(S) = \mathbf{r}$. This makes sense, as the vector $\mathbf{d}$ is uniquely defined by $\mathbf{r}$.

The function Ω will have the following properties:

- The number of ones of $\Omega(S)$ is equal to $\mathbf{r}(1) = n - |S|$ for all S.
- It is equivariant respect to the actions ϕ and ρ, that is, $\Omega(\phi_c(S)) = \rho_c(\Omega(S))$.
- Given a nonempty multiset S, let $\tilde{S} = \phi_c(S)$ be the element of the ϕ-orbit of S that is first in reverse-lex order. Then $\Omega(\tilde{S}) = \tilde{\mathbf{r}}$ is first in reverse-lex order in its ρ-orbit. In particular, $\tilde{r}_n = 1$.

First, $\Omega(\emptyset)$ is defined as $\mathbf{1}$. Now, given a nonempty multiset S, let $\tilde{S} = [s_1, \ldots, s_l]$ with $s_i \leq s_{i+1}$ be the first element in the ϕ-orbit of S in reverse lex order. Then, as $l = |S| < n$, some element in the ϕ-orbit of S includes no instance of n, in particular, $s_l < n$. Also, $s_1 = 1$; otherwise, we could subtract 1 from every entry and obtain another element in the orbit that precedes it.

We describe an algorithm to generate arithmetic r-structures starting from the Laplacian arithmetical structure on C_1.

Algorithm A. Let $\tilde{S}$ be a multiset $[s_1, \ldots, s_l]$ as before.

1. Set $\tilde{r}_0 = 1$ and $n_0 = 1$ on C_1.
2. For $1 \leq i \leq l$ we construct an r-structure on C_{n_i} by following the steps

   a) If $n_{i-1} < s_i$ add vertices with respective $\tilde{r}_j = 1$ until there are s_i vertices, then add the vertex with respective $\tilde{r}_{s_i} = 2$ and set $n_i = s_i + 1$.

   b) If $n_{i-1} = s_i$, add a vertex with respective $\tilde{r}_{s_i} = \tilde{r}_{s_{i-1}} + 1$ and set $n_i = s_i + 1$.

   c) If $n_i > s_i$ insert a vertex in position s_i, so that it will shift all vertex with their respective labels forward one position. Once this is done, set the entry at position s_i equal to $\tilde{r}_{s_i} + \tilde{r}_{s_i - 1}$. Then set $n_i = n_{i-1} + 1$.
3. If $n_l < n$ add $n - n_l$ vertices with respective $r_i = 1$.
4. The resulting arithmetical r-structure is $\tilde{\mathbf{r}} = (\tilde{r}_1, \ldots, \tilde{r}_n) = \Omega(\tilde{S})$ (recall that $\tilde{r}_0 = \tilde{r}_n$). Set $\mathbf{r} = \rho_{-c}(\tilde{\mathbf{r}}) = \Omega(S)$.

We show that at every step of this process, we obtain an arithmetical r-structure.

- We know that $s_1 = 1$, so at the first iteration we obtain $(1, 2) \in \text{Arith}(C_2)$.
- If the algorithm proceeds to step 2a, we obtain an arithmetical r-structure with $\tilde{r}_0 = 1$. Furthermore, the added tail of $1'$s ending with a 2 preserves the divisibility properties required so that the resulting vector is an arithmetical r-structure.

- If it proceeds to step 2b, then $\tilde{r}_{n_{i-1}} = \tilde{r}_{n_{i-1}-1} + 1 = \tilde{r}_{n_{i-1}-1} + \tilde{r}_0$, so the divisibility properties are once again met.

- If it proceeds to step 2c, it is clear that the divisibility properties are once again met.

- If it proceeds to step 3, the tail of 1's keeps the divisibility properties required to be an r-strucure.

- By applying a rotation to the values of an arithmetical r-structure on the cycle, we once again obtain an arithmetical r-structure.

Example 4.2.10. *Let $n = 7$, $k = 3$, and $S = [1, 3, 3, 4]$. It is clear that S is the first element of its ϕ-orbit.*

Applying our algorithm, we obtain the following sequence of arithmetical r-structures: First, we set $n_0 = r_0 = 1$.

1. *For $i = 1$, $n_0 = 1 = s_1$, so we proceed to step 2b, we add an entry $r_1 = r_0 + 1 = 2$, we set $n_1 = s_1 + 1 = 2$ and the resulting arithmetical r-structure at this step is $(1, 2)$.*

2. *For $i = 2$, $n_1 = 2 < 3 = s_2$, so we proceed to step 2a, we add an entry $r_2 = 1$, and a final vertex $r_3 = 2$, then we set $n_2 = s_2 + 1 = 4$. The resulting arithmetical r-structure after this step is $(1, 2, 1, 2)$.*

3. *For $i = 3$, $n_2 = 4 > 3 = s_3$, so we proceed to step 2c, we add an entry in position 3 equal to $2 + 1 = 3$, then we set $n_3 = n_2 + 1 = 5$. The resulting arithmetical r-structure after this step is $(1, 2, 1, 3, 2)$.*

4. *For $i = 4$, $n_3 = 5 > 4 = s_4$, so we proceed to step 2c, we add an entry in position 4 equal to $2 + 3$, then we set $n_4 = n_3 + 1 = 6$. The resulting arithmetical r-structure after this step is $(1, 2, 1, 3, 5, 2)$.*

As $n_4 = 6 < 7$,we do step 3 and we add an entry $r_6 = 1$ and the resulting arithmetical r-structure is then $(1, 2, 1, 3, 5, 2, 1)$.

Now, we shall show that the function Ω satisfies our desired properties. First, every iteration of step 2 of our algorithm adds exactly one vertex with $\tilde{r}_i > 1$, this step is repeated $|S|$ times. This is also the only step in which we add entries different from 1. The number of 1's is unaffected by the rotation in step 4, so the number of 1's in the resulting vector is equal to $\Omega(S)(1) = n - |S|$. This proves the first property.

For the second property, set $T = \phi_t(S)$ for $t \in \mathbb{Z}_n$. Let c be such that $\tilde{S} = \phi_c(S)$ is the first element in reverse lex order as required. Then, $\tilde{S} = \phi_c(S) = \phi_c(\phi_{-t}(T)) = \phi_{c-t}(T)$. Then $\Omega(T) = \rho_{t-c}(\tilde{\mathbf{r}}) = \rho_t(\rho_{-c}(\tilde{\mathbf{r}})) = \rho_t(\Omega(S))$, as desired.

For the third property, suppose that $S = [s_1, \ldots, s_n]$ is such that they are the first element of its orbit in reverse lex order, the resulting $\tilde{\mathbf{r}}$ has $\tilde{r}_j = 1$ for $s_l < j \leq n$ with $r_{s_l} > 1$. For $\tilde{\mathbf{r}}$ to contain a longer sequence of entries equal to 1 than this one, it would require that for some i, that the gap $s_{i+1} - s_i - 1$ to be greater than the gap $n - s_l$. Then, by adding $n - s_{i+1} + 1$ to every entry of S (taken modulo n), we would obtain a new multiset with maximal entry $n - s_{i+1} + s_i + 1 < s_l$. This would contradict the fact that S was the first element in reverse lex order.

We have proven the properties of Ω, now we prove that it is indeed a bijection.

It will be useful to consider Ω as the union of the maps:

$$\Omega_l : MSet_l(n) \longrightarrow \{\text{arithmetical } r\text{-structures on } C_n \text{ with } \mathbf{r}(1) = n - l\}.$$

We show that every Ω_l is a bijection by induction on l. For $l = 0$, then $S = \emptyset$ and the only arithmetical structure with $\mathbf{r}(1) = n - 0 = n$ is the Laplacian arithmetical structure, so the result is clear. For $l = 1$ if $S \in MSet_1(n)$, then $S = [a]$ and its firs element in its ϕ orbit is $\tilde{S} = \phi_{n-a+1}(S) = [1]$. Step 2 of the algorithm is executed once and we obtain the arithmetical r-strucure $\tilde{\mathbf{r}} = (2, 1, \ldots, 1)$ and $\mathbf{r} = \rho_{n-a+1}(\tilde{\mathbf{r}})$, is the vector with $r_{n-a+1} = 2$ and every other entry equal to 1. It is not difficult to check that Ω_1 is a bijection.

Suppose $l \geq 2$. At every iteration of our algorithm the vector $\tilde{\mathbf{r}}$ has a local maximum at position s_i, in the sense that $\tilde{r}_{s_i - 1} < \tilde{r}_{s_i}$ and, either $\tilde{r}_{s_{i+1}} < \tilde{r}_{s_i}$ or $\tilde{r}_{s_{i+1}}$ does not exist. Also,at this iteration claim that if $m > s_i$, then $\tilde{r}_m$ is not a local maximum. This is clear whenever steps 2a and 2b of our algorithm occur. If step 2c is applied multiple times, then at each step, we are adding an entry that is greater than the one to its right, and each of these insertions occurs to the right of previous insertions, so r_m cannot be a local maximum.

We proceed to show an algorithm that will recover $\tilde{S}$ from $\mathbf{r}$. Let $\tilde{\mathbf{r}}$ be the first element in the ρ-orbit of $\mathbf{r}$ in reverse lex order. Label the vertices of the cycle graph C_n with $\tilde{\mathbf{r}}$ and proceed with the following algorithm.

Algorithm B Let $\tilde{\mathbf{r}}$ be as above.

1. Let $\tilde{S}$ be the empty multiset.
2. Let j be the greatest integer so that $\tilde{r}_j$ is a local maximum and add j to $\tilde{S}$.
3. Delete entry j form $\tilde{\mathbf{r}}$, this will give us an arithmetical structure on C_{n-1} (by the smoothing operation).

4. Repeat the previous steps until we are left with the arithmetical r-structure $\mathbf{1}$ on C_{n-l}. The multiset $\tilde{S}$ will now be a multiset with $l = n - \mathbf{r}(1)$ elements. And will be first in reverse lex order in its ϕ-orbit.

Once the algorithm is completed, we have recovered $\tilde{S}$, and we set $S = \phi_{-c}(\tilde{S})$.

Steps 2 and 3 of the algorithm will be executed exactly l times, as they remove one entry greater than 1 in every iteration.

With this, we have concluded the proof of our theorem. □

We can do another less constructive proof of this theorem by exploring ways in which arithmetical r-structures on the cycle give rise to arithmetical r-structures for paths.

Lemma 4.2.11. *Let $n \geq 2$.*

a) Suppose that $\mathbf{r} = (r_1, \ldots, r_n)$ is an arithmetical r-structure on C_n such that $r_j = 1$ for some j, then $\mathbf{r}' = (r_j, \ldots, r_n, r_1, \ldots, r_j)$ is an arithmetical structure on $\mathcal{P}_{n+1}$.

b) Suppose that $\mathbf{r} = (r_1, \ldots, r_n)$ in an arithmetical r-structure in C_n such that $r_\alpha = r_\beta = 1$ for some $1 \leq \alpha < \beta$ then $\mathbf{r}' = (r_\alpha, r_{\alpha+1}, \ldots, r_\beta)$ and $\mathbf{r}'' = (r_\beta, r_{\beta+1}, \ldots, r_n, r_1, \ldots, r_\alpha)$ are arithmetical structures on $\mathcal{P}_{\beta-\alpha+1}$ and $\mathcal{P}_{n-(\beta-\alpha)+1}$, respectively.

Proof. In both cases, it is not difficult to check that

$$r_1 = r_n = 1 \text{ and } r_i | (r_{i-1} + r_{i+1}) \text{ for all } 1 < i < n.$$

That is, satisfy conditions (4.2) for an integer vector to be an r-structure on the path $\mathcal{P}_n$. □

Second proof of Theorem 4.2.8. We will proceed by induction on k. For the base case $k = 1$, one can see by Lemma 4.2.11 that for each $j \in [n]$ there is a bijection between arithmetical r-structures on the cycle C_n with exactly one entry equal to 1 in position j, and arithmetical structures on the path $\mathcal{P}_{n+1}$ with $\mathbf{r}(1) = 2$ given by

$$(r_1, r_2, \ldots, r_{j-1}, 1, r_{j+1}, \ldots, r_n) \longrightarrow (1, r_{j+1}, \ldots, r_n, r_1, \ldots, 1).$$

Then, by Theorem 4.1.14 we conclude that

$$\begin{aligned} |\{\mathbf{r} \in \mathrm{Arith}(C_n) : \mathbf{r}(1) = 1\}| &= n \cdot |\{\mathbf{r} \in \mathrm{Arith}(\mathcal{P}_{n+1}) : \mathbf{r}(1) = 2\}| \\ &= n \cdot A(n+1, 2) = \binom{2n-2}{n-2} = \left(\!\!\binom{n}{n-1}\!\!\right). \end{aligned}$$

Then, we assume that $k > 1$.

The number of lattice paths (without restriction) from $(0,0)$ to $(n-1, n-k)$ is $\binom{2n-k-1}{n-k}$. Each of these paths can be decomposed into a path from $(0,0)$ to (z,z) and a path from (z,z) to $(n-1,n-k)$, where z is the greatest number where the path crosses the line $x = y$. Then, the second path will start by doing an east step from (z,z) to $(z+1,z)$.

There are

$$\binom{2z}{z} = (z+1)C_z,$$

choices for the first path. And, for the second path, there is a bijection from choices of this kind of path and paths from $(0,0)$ to $(n-z-2, n-k-z)$ that do not go above the line $x = y$, of which there are $B(n-z-2, n-k-z) = A(n-z,k)$ paths of this kind.

This gives us the expression

$$\binom{2n-k-1}{n-k} = \sum_{z=0}^{n-k}(z+1)C_z A(n-z,k).$$

We shall show that the right-hand expression counts the number of arithmetical r-structures on C_n with $\mathbf{r}(1) = k$. Label the vertices of C_n as $v_0, \ldots, v_{n-1}$. Let $\mathbf{r} = (r_0, \ldots, r_{n-1})$ be an arithmetical r-structure with $\mathbf{r}(1) = k$ and

$$\alpha = \min\{i : r_i = 1\}, \quad \beta = \max\{i : r_i = 1\}.$$

Then $0 \le \alpha \le \beta \le n-1$.

By Lemma 4.2.11, the vector $\mathbf{r}' = (r_\alpha, \ldots, r_\beta)$ is an arithmetical r-structure on the path $\mathcal{P}_{\beta-\alpha+1}$ with $\mathbf{r}'(1) = k$. We note that $\beta - \alpha + 1 \ge k$. By Theorem 4.1.14, the number of possibilities for these arithmetical structures equals $A(\beta-\alpha+1, k)$.

Again, by Lemma (4.2.11), the vector

$$\mathbf{r}'' = (r_\beta, \ldots, r_{n-1}, r_0, \ldots, r_\alpha)$$

is an arithmetical r-structure on $\mathcal{P}_{n-(\beta-\alpha)+1}$ with $\mathbf{r}''(1) = 2$. And there is $A(n-(\beta-\alpha)+1, 2) = C_{n-(\beta-\alpha)-1}$ of this arithmetical structures. Let $z = n-(\beta-\alpha)-1 \le n-k$: Each choice of β, α satisfying the inequality $\beta - \alpha + 1 \ge k$ gives a total of $C_z \cdot A(n-z,k)$ posible arrithemtical r-structures.

Each value of z arises from one of the $z+1$ pairs $(0, n-z-1), (1, n-z), \ldots, (z-1, n)$. Putting all this together, we get

$$\sum_{z=0}^{n-k}(z+1)C_z \cdot A(n-z,k) = \binom{2n-k-1}{n-k}$$

arithmetical structures. □

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