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Una extensión de la Fórmula de Cambio de Variables para integrales

Tesis que presenta

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An extension of the Change of Variables integral formula

A THESIS PRESENTED BY

Luis Contreras Moreno

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Para Mary Jose.

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Resumen

En esta tesis trabajamos con una extensión de la llamada Fórmula de Cambio de Variable para integrales; esta extensión es conocida como la Fórmula del Área. Lo hacemos para funciones Lipschitz y la integración con respecto a la llamada medida de Hausdorff. Para ello presentamos una exposición casi autocontenida acerca de las propiedades de las funciones Lipschitz y la medida de Hausdorff. A continuación, utilizando estas propiedades, probaremos la Fórmula del Área y daremos algunas de sus aplicaciones. En el Capítulo 1 estudiamos a las funciones Lipschitz y la medida de Hausdorff en espacios Euclidianos. En el Capítulo 2 demostraremos el célebre Teorema de Rademacher. En el Capítulo 3 probamos la Fórmula del Área en Espacios Euclidianos y en Conjuntos Rectificables, así como también presentamos ejemplos y aplicaciones. Finalmente en el Cápitulo 4 extendemos lo anterior a espacios métricos.

Abstract

In this thesis we work with an extension of the so-called Change of Variables integral formula, this extension is known as the Area formula. We do it for Lipschitz functions and the integration with respect to he so-called Hausdorff measure. For this purpose we present an almost self-contained exposition of the properties of Lipschitz functions and Hausdorff measure. Thereafter, using these properties we will prove the Area formula and give some of its applications. In Chapter 1 we study Lipschitz functions and the Hausdorff measure in Euclidean spaces. In Chapter 2 we will prove the famous Rademacher's Theorem. In Chapter 3 we prove the Area formula in Euclidean spaces and Rectifiable sets, we also present examples and applications. Finally, in Chapter 4 we extend the above to metric spaces.

Contents

Introduction

The goal of this work is to present an extension of the so-called Change of Variables integral formula. This extension is called the Area formula and we do it for Lipschitz functions and the integration with respect to he so-called Hausdorff measure, for this reason we give an extensive almost self contained and detailed exposition of the principal properties of Lipschitz functions and Hausdorff measure, which will later be used to prove the mentioned extension, and provide some applications. In the first three chapters we will work on Euclidean spaces endowed with the usual norm, while in the last chapter we will work on general metric spaces.

The concepts of Lebesgue measure and integral play an important role in mathematics due to their good properties and multiple applications in many areas, such as probability and real analysis. However, a disadvantage of the Lebesgue measure lies in the fact that it does not distinguish between sets of dimension less than n. For example, in $\mathbb{R}^3$, for λ^3 there is no distinction between a line segment and a plane figure, since for both sets λ^3 will be equal to 0. To solve this problem, in 1918, in the article *Dimension and Outer Measure*, Felix Hausdorff introduced the concept of $s-$*dimensional measure*, denoted by $\mathcal{H}^s$, where $s \in [0, \infty)$. This measure, later known as the *Hausdorff measure*, apart from having similar properties to those of the Lebesgue measure (they coincide when $n = s$), allows us to measure lengths, areas, volumes, hypervolumes and even fractal-sets of lower dimensional in $\mathbb{R}^n$. Naturally, since the Hausdorff measure has similar properties to the Lebesgue measure, in the mathematicians quest for generalization, attempts were made to extend the known results of Lebesgue measure and integration to this new measure. This generalization effort was proved particularly fruitful in the case of one of the most useful results of Lebesgue measure and integration, the Change of Variables Theorem, as in 1969 Herbert Federer proved the Area Formula, which generalizes the Change of Variables Theorem.
Recall that the change of variables theorem for Lebesgue measure and integration states that:

Let $V \subset \mathbb{R}^n$ be an open set and $\varphi : V \to \mathbb{R}^n$ be an injective function of class C^1. Then

$$\int_{\varphi(V)} f(x) d\lambda^n(x) = \int_V (f \circ \varphi)(y) |det(d_y\varphi)| d\lambda^n(y), \quad \forall f \in C_C(\varphi(V)),$$

where $d_y\varphi$ is the differential of φ at y.

In this context, the Area Formula generalizes the previous theorem, as firstly, we are no limited to the particular case of a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, since we will use with functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $n \leq m$. Furthermore, we are no longer restricted to the use of C^1 functions, because we will use Lipschitz functions. The latter class of functions are themselves an interesting object of study, since they possess operational and extension properties that make them useful in various fields of mathematics.
Lipschitz functions, although not necessarily differentiable over their entire domain, share certain similarities with functions of class C^1, which makes them in a sense a good replacement for

continuously differentiable functions. In this sense, Rademacher's Theorem is relevant, since it states that a Lipschitz function is differentiable almost everywhere with respect to the Lebesgue measure. This implies that, although Lipschitz functions may present problems on a set of zero measure, they still retain some degree of smoothness.

As we can notice we have highlighted two important tools which are the Hausdorff measure and Lipschitz functions, this is not causality because these two will be useful to develop and prove the main topic of this work, for this purpose we will follow the following order:

In Chapter 1, we first give a brief reminder about some general concepts of measure theory. Subsequently, we define and prove the main properties of the Hausdorff measure and Lipschitz functions, highlighting in the case of Lipschitz functions the extension properties. Then, we show some relations between the Hausdorff measure and Lipschitz functions. We end this chapter with the introduction of two useful tools: the Steiner symmetrization (that provides a way to tranform a object in a symmetric object with the samen Lebesgue measure) and the Isodiametric inequality (wich states that among all sets of fixed diameter, balls have maximum volume), we use these tools combining by some results proved in this chapter to show that

$$\mathcal{H}^n = \lambda^n \text{ on } \mathbb{R}^n.$$

In Chapter 2, we state and prove Rademacher's Theorem. To do so, we will begin by recalling the concept of differentiability and stating some lemmas that will be useful for proving the main theorem of this chapter. This theorem will be of great importance as it will allow us to introduce, in the following chapter, the concept of Jacobian for a Lipschitz function (wich represents a generalization of the classical Jacobian and by Rademacher's Theorem can be defined in all $\mathbb{R}^n$), and likewise, many of the subsequent results will depend on it.

Chapter 3 will be the centerpiece of this work, as we will present the Area formula. In simple terms, the Area formula states that given a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ with $n \leq m$, the Hausdorff measure of every Lebesgue measurable set E of $\mathbb{R}^n$ can be computed as a integral, i.e.,

$$\mathcal{H}^n(f(E)) = \int_E Jf d\lambda^n,$$

provided f is injective, where Jf denotes the Jacobian of f. In the case that f is not injective, we introduce the multiplicity function (which counts the cardinality of the inverse image of every point in the codomain), and we can obtain a generalization of the last equation, wich represents the general version of the Area formula
The first part of this chapter will consist of the proof of this theorem in Euclidean spaces, which we will do constructively: we will start by proving the case when we have a linear map, then we will add the injectivity hypothesis (in this part we present as corollary the Change of Variables Theorem), subsequently we introduce the multiplicity function and use these two last versions of the Area formula to prove the general case where we only require the function in question to be Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $n \leq m$. Throughout this part of the chapter, we will introduce three important lemmas, which by themselves are already interesting results. These lemmas refer to one characterization of $\mathcal{H}^n$-measurable sets on $\mathbb{R}^m$, the linearization of Lipschitz functions, and the $\mathcal{H}^n-$measure of the set of critical values of Lipschitz functions.
Having established the above results, in the second part of this chapter, we introduce the

concept of Rectifiable sets, which are sets that looks like a countable union of images under some Lipschitz functions of $\mathbb{R}^k$ for some $k \in \mathbb{N}$, i.e., $M \subset \mathbb{R}^n$ is Rectifiable if there exist countably Lipschitz functions $f_h : \mathbb{R}^k \to \mathbb{R}^n$, $h \in \mathbb{N}$ such that

$$\mathcal{H}^k \left(M \setminus \bigcup_{h \in \mathbb{N}} f_h(\mathbb{R}^k) \right) = 0.$$

These sets have interesting geometric properties and can be considered as a generalization of the $k-$dimensional embedded surfaces in $\mathbb{R}^n$. Rectifiable sets allow us to extend the concept of surface or curve defined by differentiable functions to sets that may be more general and still admit a well defined notion of length or area. This is especially useful in contexts where structures may be more irregular, but one still seeks to have tools to measure and analyze their geometric properties.
We presented some properties and examples of rectifiables sets and, extend some concepts introduced at the beginning of the chapter, as well as a kind of Rademacher's Theorem, which, together with the results obtained in Euclidean spaces, we will use to prove the Area formula in Rectifiable sets.
We conclude this chapter by presenting some applications of the Area formula, which, along with the examples showed throughout the entire chapter, we hope will serve as a good complement to showcase some uses of the Area formula.

Finally in Chapter 4 we generalize the concepts and some results seen in Chapter 1 to more general metric spaces, as well as introduce new concepts and results, and conclude by giving a brief look at the generalization of the Area formula to general metric spaces.

Chapter 1

Hausdorff Measure and Lipschitz functions on $\mathbb{R}^n$

1.1 Outer Measures on $\mathbb{R}^n$

We will review some concepts from measure theory that will recur throughout this work. For this reason, we have limited to state only the respective definitions and results, however we refer to [8], [11], [14], and [15], for a more exhaustive and complete development of this theory.

Definition 1.1. *An **outer measure** on $\mathbb{R}^n$ is a set function, $\mu : \mathscr{P}(\mathbb{R}^n) \to [0, \infty]$, with $\mu(\emptyset) = 0$ and*

$$E \subset \bigcup_{k\in\mathbb{N}} E_k \quad \Longrightarrow \quad \mu(E) \leq \sum_{k\in\mathbb{N}} \mu(E_k).$$

Example 1.1. *Let $E \subset \mathbb{R}^n$, and define*

$$\lambda_n^*(E) := \inf_{\{Q_k\}_{k\in\mathbb{N}}\in\mathcal{F}} \left\{ \sum_{k\in\mathbb{N}} l(Q)^n \right\},$$

where $\mathcal{F}$ is the family of countable covers $\{Q_k\}_{k\in\mathbb{N}}$ of E by cubes with sides parallel to the coordinate axes, and $l(Q_k)$ denotes the side length of Q_k (the cubes Q_k are not assumed to be open, nor closed).
Then λ_n^* is an outer measure on $\mathbb{R}^n$ (see [14] or [15]), that is called the Lebesgue outer measure on $\mathbb{R}^n$. ◀

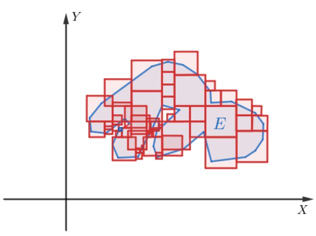

Figure 1.1: Covering of E by squares.

The next theorem (first proved by Constantin Carathéodory in 1914) is a fundamental result in measure theory.

Theorem 1.1. *(**Carathéodory's Theorem**) If μ is an outer measure on $\mathbb{R}^n$, and $\mathcal{M}(\mu)$ is the family of those $E \subset \mathbb{R}^n$ such that*

$$\mu(F) = \mu(F \cap E) + \mu(F \cap E^c), \quad \forall F \subset \mathbb{R}^n,$$

then $\mathcal{M}(\mu)$ is a $\sigma-$algebra, and μ is a measure on $\mathcal{M}(\mu)$.

Remark 1.1. *We recall that $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^n)$ is a $\sigma-$**algebra** on $\mathbb{R}^n$ if $E \in \mathcal{M}$ implies $E^c \in \mathcal{M}$, $\{E_k\}_{k=1}^{\infty} \subset \mathcal{M}$ implies $\cup_{k=1}^{\infty} E_k \in \mathcal{M}$, and $\mathbb{R}^n \in \mathcal{M}$.*
*If $\mathcal{M}$ is a $\sigma-$algebra, then a set function $\mu : \mathcal{M} \to [0, \infty]$ is a **measure** on $\mathcal{M}$ if $\mu(\emptyset) = 0$ and μ is $\sigma-$additive on $\mathcal{M}$ i.e., if $\{E_k\}_{k \in \mathbb{N}} \subset \mathcal{M}$, is such that $E_i \cap E_j = \emptyset$ for $i \neq j$, then*

$$\mu\left(\bigcup_{k \in \mathbb{N}} E_k\right) = \sum_{k \in \mathbb{N}} \mu(E_k).$$

◁

Note that, by Theorem 1.1 we can pass from an outer measure to a measure whenever we restrict ourselves to a suitable family of subsets, i.e., any outer measure μ, provides a natural domain of $\sigma-$additivity for μ, so every outer measure on $\mathbb{R}^n$ can be seen as a measure on a σ-algebra on $\mathbb{R}^n$. In this way, various classical results from Measure Theory are immediately recovered in the context of outer measures.

Example 1.2. *Applying Theorem 1.1 to the Lebesgue outer measure, we obtain a $\sigma-$algebra $\mathcal{M}(\lambda_n^*)$, this $\sigma-$algebra is called the Lebesgue $\sigma-$algebra on $\mathbb{R}^n$, which we denote by $\mathscr{L}(\mathbb{R}^n)$, and the restriction of λ_n^* on $\mathscr{L}(\mathbb{R}^n)$ is called the Lebesgue measure on $\mathbb{R}^n$ and we denote by λ^n.* ◀

Definition 1.2. *A **Borel measure** on $\mathbb{R}^n$ is an outer measure μ on $\mathbb{R}^n$ such that $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{M}(\mu)$, where $\mathcal{B}(\mathbb{R}^n)$ is the Borel $\sigma-$algebra on $\mathbb{R}^n$.*

Theorem 1.2. *(**Carathéodory's criterion**) If μ is an outer measure on $\mathbb{R}^n$, then μ is a Borel measure on $\mathbb{R}^n$ if and only if*

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2),$$

for every $E_1, E_2 \subset \mathbb{R}^n$ such that $\operatorname{dist}(E_1, E_2) > 0$.

Definition 1.3. *We say that a Borel measure μ on $\mathbb{R}^n$ is **regular** if for every $F \subset \mathbb{R}^n$ there exists a Borel set E such that*

$$F \subset E \quad \text{and} \quad \mu(E) = \mu(F).$$

Thus, a regular Borel measure is completely determined by its values on Borel sets.

Definition 1.4. *An outer measure μ is **locally finite** if $\mu(K) < \infty$ for every compact set $K \subset \mathbb{R}^n$.*

Definition 1.5. *An outer measure μ is a **Radon measure** on $\mathbb{R}^n$ if it is a locally finite, Borel regular measure on $\mathbb{R}^n$.*

Definition 1.6. *Given an outer measure μ on $\mathbb{R}^n$, and $E \subset \mathbb{R}^n$, the **restriction of** μ **to** E is the set function $\mu \llcorner E$ defined as*

$$\mu \llcorner E(F) := \mu(E \cap F), \quad \forall F \subset \mathbb{R}^n.$$

It is easy to see that $\mu \llcorner E$ is an outer measure on $\mathbb{R}^n$ and $\mathcal{M}(\mu) \subset \mathcal{M}(\mu \llcorner E)$.

Proposition 1.1. *If μ is a Borel regular measure on $\mathbb{R}^n$, and $E \in \mathcal{M}(\mu)$ is such that $\mu \llcorner E$ is locally finite, then $\mu \llcorner E$ is a Radon measure on $\mathbb{R}^n$.*

Theorem 1.3. *The Lebesgue outer measure is a Radon measure on $\mathbb{R}^n$.*

Proposition 1.2. *If μ is a Radon measure on $\mathbb{R}^n$, then:*
1.- $\mu(E) = \inf\{\mu(A) : E \subset A, A \text{ is open}\}$, for every $E \subset \mathbb{R}^n$.
2.- $\mu(E) = \inf\{\mu(K) : K \subset E, K \text{ is compact}\}$, for every $E \in \mathcal{M}(\mu)$.

Definition 1.7. *Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$ and an outer measure μ on $\mathbb{R}^n$, the **push-forward of** μ **through** f is the outer measure $f_\# \mu$ defined by the formula*

$$f_\# \mu(E) := \mu(f^{-1}(E)), \quad \forall E \subset \mathbb{R}^m.$$

Proposition 1.3. *If μ is a Radon measure on $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and proper, then $f_\# \mu$ is a Radon measure on $\mathbb{R}^m$.*

1.2 Hausdorff Measure on $\mathbb{R}^n$

The concept of Hausdorff measure generalizes counting, length, area and volume like the Lebesgue measure; the only difference is that the Hausdorff measure can measure the length, area and volume of objects that live in a higher dimensional space. In this section we will state and prove the most important properties of this measure.

Definition 1.8. *Let $n \in \mathbb{N}$, $s \in [0, \infty[$, and $\delta > 0$. For $E \subset \mathbb{R}^n$, define*

$$\mathcal{H}^s_\delta(E) := \inf_{\{F_k\}_{k \in \mathbb{N}} \in \mathcal{F}} \left\{ \frac{\omega_s}{2^s} \sum_{k \in \mathbb{N}} (\mathrm{diam}(F_k))^s \right\},$$

where the infimum is taken over the family $\mathcal{F}$ of all countable covers of E consisting of sets $F \subset \mathbb{R}^n$ such that $\mathrm{diam}(F) < \delta$ (which we will call δ-covers), and ω_s is given by:

$$\omega_s := \frac{\pi^{\frac{s}{2}}}{\Gamma\left(1 + \frac{s}{2}\right)},$$

where $\Gamma :]0, \infty[\to [1, \infty[$ is the Euler Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \ \ s > 0.$$

We also define:

$$\mathcal{H}^s(E) := \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

Remark 1.2. *Let $E \subset \mathbb{R}^n$, $s \in [0,\infty[$ and $\delta_1, \delta_2 > 0$, with $\delta_1 < \delta_2$. We claim that, the family*

$$\mathcal{F}_{\delta_2} = \left\{ \{F_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n : E \subset \bigcup_{k=1}^{\infty} F_k,\ \text{diam}(F_k) < \delta_2 \right\},$$

contains the family

$$\mathcal{F}_{\delta_1} = \left\{ \{F_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n : E \subset \bigcup_{k=1}^{\infty} F_k,\ \text{diam}(F_k) < \delta_1 \right\}.$$

Indeed, if $\{F_k\}_{k\in\mathbb{N}} \in \mathcal{F}_{\delta_1}$ is arbitrary, then $\text{diam}(F_k) < \delta_1 < \delta_2$, $\forall k \in \mathbb{N}$, so $\{F_k\}_{k\in\mathbb{N}} \in \mathcal{F}_{\delta_2}$. Hence,

$$\mathcal{H}^s_{\delta_2}(E) := \inf_{\{F_k\}_{k\in\mathbb{N}} \in \mathcal{F}_{\delta_2}} \left\{ \frac{\omega_s}{2^s} \sum_{k\in\mathbb{N}} (\text{diam}(F_k))^s \right\} \leq \inf_{\{F_k\}_{k\in\mathbb{N}} \in \mathcal{F}_{\delta_1}} \left\{ \frac{\omega_s}{2^s} \sum_{k\in\mathbb{N}} (\text{diam}(F_k))^s \right\} = \mathcal{H}^s_{\delta_1}(E).$$

We have proved that, given $\delta_1 < \delta_2$, then $H^s_{\delta_1}(E) \geq H^s_{\delta_2}(E)$, so $H^s_\delta(E)$ is decreasing in δ. Therefore, we can write:

$$\mathcal{H}^s(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(E).$$

◁

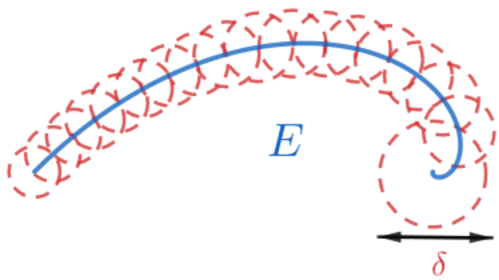

Figure 1.2: $\delta-$Covering of E.

Remark 1.3. *Notice that the sets F in the definition of $\mathcal{H}^s_\delta$ are arbitrary subsets of $\mathbb{R}^n$. However, the same results would be obtained if one considers the subsets F to be either closed convex sets intersecting E or open sets intersecting E.* ◁

Proposition 1.4. *For every $s \in [0,\infty[$ and every $\delta > 0$, $\mathcal{H}^s_\delta$ and $\mathcal{H}^s$ are outer measures on $\mathbb{R}^n$. We will call $\mathcal{H}^s_\delta$ the s-dimensional Hausdorff outer measure of step δ and $\mathcal{H}^s$ the s-dimensional Hausdorff measure.*

Proof. Let $\varepsilon \in]0,1[$ and $E \subset \{F_j\}_{j\in\mathbb{N}}$, for every j let $\{F_{j,k}\}_{k\in\mathbb{N}}$ be a cover of F_j such that $\text{diam}(F_{j,k}) < \delta$ and

$$\frac{\omega_s}{2^s} \sum_{k\in\mathbb{N}} (\text{diam}(F_{j,k}))^s < \mathcal{H}^s_\delta(F_j) + \varepsilon^j.$$

Then $\cup_{k\in\mathbb{N}}\{F_{j,k}\}_{k\in\mathbb{N}}$ is a cover of E, thus

$$\begin{aligned}\mathcal{H}^s_\delta(E) &\leq \frac{\omega_s}{2^s}\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}(\operatorname{diam}(F_{j,k}))^s \\ &< \sum_{j\in\mathbb{N}}[\mathcal{H}^s_\delta(F_j)+\varepsilon^j] \\ &= \sum_{j\in\mathbb{N}}\mathcal{H}^s_\delta(F_j)+\frac{\varepsilon}{1-\varepsilon}\end{aligned}$$

Since ε was arbitrary, it follows that $\mathcal{H}^s_\delta$ is subadditive. Noting that $\mathcal{H}^s_\delta(\emptyset)=0$, we obtain that $\mathcal{H}^s_\delta$ is an outer measure. To finally see that $\mathcal{H}^s$ is an outer measure, we can use the fact that the supremum of any arbitrary family of outer measures is indeed an outer measure. ■

As a comment, the construction of the Hausdorff measures is a special case of Carathéodory's construction in measure theory.

In this work, we will use $||\cdot||$ to always refer to the Euclidean norm in $\mathbb{R}^n$, unless otherwise specified.

Remark 1.4. *Recall that given $A, B \subset \mathbb{R}^n$. The distance between A and B is defined as*

$$\operatorname{dist}(A,B) := \inf\{||x-y|| : x\in A, y\in B\}.$$

This concept can be extended to arbitrary metric spaces (as will be used in Chapter 4). ◁

Proposition 1.5. *For every $s\in[0,\infty[$, $\mathcal{H}^s$ is a Borel measure on $\mathbb{R}^n$.*

Proof. Let A and B be subsets of $\mathbb{R}^n$ such that $\operatorname{dist}(A,B)>0$. Given $\delta>0$ with $\operatorname{dist}(A,B)>\delta$ and $\varepsilon>0$ arbitrary, there exist a $\delta-$cover $\{E_k\}_{k\in\mathbb{N}}$ of $A\cup B$ such that

$$\frac{\omega_s}{2^s}\sum_{k\in\mathbb{N}}(\operatorname{diam}(E_k))^s \leq \mathcal{H}^s_\delta(A\cup B)+\varepsilon. \tag{1.1}$$

We define

$$I_A=\{i\in\mathbb{N} : E_i\cap A\neq\emptyset\} \text{ and } I_B=\{i\in\mathbb{N} : E_i\cap B\neq\emptyset\}.$$

Note that $I_A\cap I_B=\emptyset$, because if $I_A\cap I_B\neq\emptyset$ then there would exist $i_0\in\mathbb{N}$, $a\in E_{i_0}\cap A$ and $b\in E_{i_0}\cap B$, where $||a-b||<\delta$ (since $a,b\in E_{i_0}$ and $\operatorname{diam}(E_{i_0})<\delta$), therefore $\operatorname{dist}(A,B)<\delta$ (because $\operatorname{dist}(A,B)\leq||a-b||$). However, this contradicts our choice of δ.
By the definitions of I_A and I_B, it is easy to see that $\{E_i\}_{i\in I_A}$ forms a δ-cover of A, and $\{E_i\}_{i\in I_B}$ forms a δ-cover of B. Therefore, by the definition of $\mathcal{H}^s_\delta(A)$ and $\mathcal{H}^s_\delta(B)$, we have

$$\begin{aligned}\mathcal{H}^s_\delta(A)+\mathcal{H}^s_\delta(B) &\leq \frac{\omega_s}{2^s}\sum_{i\in I_A}(\operatorname{diam}(E_i))^s+\frac{\omega_s}{2^s}\sum_{i\in I_B}(\operatorname{diam}(E_i))^s \\ &= \frac{\omega_s}{2^s}\sum_{i}(\operatorname{diam}(E_i))^s.\end{aligned}$$

Thus, from (1.1), it follows that

$$\mathcal{H}^s_\delta(A)+\mathcal{H}^s_\delta(B)\leq\mathcal{H}^s_\delta(A\cup B)+\varepsilon.$$

Taking $\delta \to 0$ we obtain $\mathcal{H}^s(A) + \mathcal{H}^s(B) \leq \mathcal{H}^s(A \cup B) + \varepsilon$. Since ε was arbitrary, we obtain

$$\mathcal{H}^s(A) + \mathcal{H}^s(B) \leq \mathcal{H}^s(A \cup B). \tag{1.2}$$

Now, from the subadditivity of $\mathcal{H}^s$, it follows that

$$\mathcal{H}^s(A \cup B) \leq \mathcal{H}^s(A) + \mathcal{H}^s(B). \tag{1.3}$$

Therefore, from (1.2) and (1.3)

$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Thus, from the latter, using Carathéodory's criterion we obtain that $\mathcal{H}^s$ is a Borel measure on $\mathbb{R}^n$. ■

It can be proven that $\mathcal{H}^s_\delta$ is not a Borel measure when $s < n$. To see this, consider the case when $n = 2$ and $s = 1$. Given any arbitrary $\varepsilon > 0$, let $A := [0, 1] \times \{0\}$ and $B_\varepsilon := [0, 1] \times \{\varepsilon\}$, then

$$\mathcal{H}^1_\delta(A) = 1 = \mathcal{H}^1_\delta(B_\varepsilon).$$

Thus,

$$\mathcal{H}^1_\delta(A) + \mathcal{H}^1_\delta(B_\varepsilon) = 2.$$

However, we can observe that $\operatorname{diam}(A \cup B) = \sqrt{1 + \varepsilon^2}$, choosing ε sufficiently small, will have to $\operatorname{diam}(A \cup B) = \sqrt{1 + \varepsilon^2} < 2$. Then, by choosing $\delta(\varepsilon) > 0$ sufficiently large such that $\operatorname{diam}(A \cup B) < \delta(\varepsilon)$ and using the fact that $A \cup B$ is a $\delta(\varepsilon)-$cover of itself, we will have for the above that

$$\mathcal{H}^1_\delta(A \cup B_\varepsilon) < 2.$$

Therefore

$$\mathcal{H}^1_\delta(A \cup B_\varepsilon) \neq \mathcal{H}^1_\delta(A) + \mathcal{H}^1_\delta(B_\varepsilon),$$

and as $\operatorname{dist}(A, B_\varepsilon) = \varepsilon > 0$ from Carathéodory's criterion we will have that $\mathcal{H}^1_\delta$ is not Borel. Another way to prove the above is as follows: Given U an open ball on $\mathbb{R}^n$ $(n \geq 2)$, such that $\operatorname{diam}(U) = \delta$, it can be show, that for $0 \leq s \leq 1$

$$\mathcal{H}^s_\delta(U) = \mathcal{H}^s_\delta\left(\overline{U}\right) = \mathcal{H}^s_\delta(\partial U).$$

So $\mathcal{H}^s_\delta$ cannot be Borel, since it is not additive on $U = \overline{U} \cup \partial U$.

Proposition 1.6. *For every $s \in [0, \infty[$, $\mathcal{H}^s$ is Borel regular on $\mathbb{R}^n$.*

Proof. Let $E \subset \mathbb{R}^n$, for each $k \in \mathbb{N}$, we can find a cover $C_k := \cup_{j\in\mathbb{N}} E^k_j \supset E$, where every E^k_j is a closed set such that $\operatorname{diam}(E^k_j) < \frac{1}{k}$ and

$$\frac{\omega_s}{2^s} \sum_{j\in\mathbb{N}} (\operatorname{diam}(E^k_j))^s < \mathcal{H}^s_{\frac{1}{k}}(E) + \frac{1}{k},$$

if we put $F := \cap_{k\in\mathbb{N}} C_k$, then F is a Borel set, $E \subset F$ and

$$\mathcal{H}^s_{\frac{1}{k}}(F) \leq \mathcal{H}^s_{\frac{1}{k}}(C_k) \leq \frac{\omega_s}{2^s} \sum_{j\in\mathbb{N}} (\operatorname{diam}(E^k_j))^s < \mathcal{H}^s_{\frac{1}{k}}(E) + \frac{1}{k},$$

taking $k \to \infty$, we have $\mathcal{H}^s(F) \leq \mathcal{H}^s(E)$. Since $E \subset F$, we obtain the opposite inequality. ■

The proof of the next proposition follows from the fact that the diameter is invariant under translations and rotations.

Proposition 1.7. *For every $s \in [0, \infty)$, $\mathcal{H}^s$ is invariant under translations and rotations.*

Proposition 1.8. *Let $E \subset \mathbb{R}^n$ and $\alpha > 0$. Then*

$$\mathcal{H}^s(\alpha E) = \alpha^s \mathcal{H}^s(E).$$

Proof. If $\{E_k\}_{k\in\mathbb{N}}$ is an arbitrary $\delta-$cover of E, then $\{\alpha E_k\}_{k\in\mathbb{N}}$ is an $\alpha\delta-$cover of αE. Hence

$$\mathcal{H}^s_{\alpha\delta}(E) \leq \frac{\omega_s}{2^s} \sum_{k\in\mathbb{N}} (\mathrm{diam}(\alpha E_k))^s = \alpha^s \left(\frac{\omega_s}{2^s} \sum_{k\in\mathbb{N}} (\mathrm{diam}(E_k))^s \right).$$

Therefore

$$\mathcal{H}^s_{\alpha\delta}(\alpha E) \leq \alpha^s \mathcal{H}^s_\delta(E).$$

Taking $\delta \to 0$, we have

$$\mathcal{H}^s(\alpha E) \leq \alpha^s \mathcal{H}^s(E).$$

Replacing α by $\frac{1}{\alpha}$ and E by αE yields the oposite inequality. ■

We now introduce a measure-theoretic notion of dimension.

Definition 1.9. *Let $E \subset \mathbb{R}^n$, the* ***Hausdorff dimension of*** *E is defined as*

$$\mathcal{H}\text{-}\dim(E) := \inf\{s \in [0, \infty[: \mathcal{H}^s(E) = 0\}.$$

Its use as a notion of dimension is justified by Propositions 1.9, 1.10, 1.11, 1.13, and Theorems 1.6 and 1.8.

Proposition 1.9. *If $E \subset \mathbb{R}^n$ and $s > n$, then $\mathcal{H}^s(E) = 0$.*

Proof. Let $Q = (0, 1)^n$, since $\alpha^s(\mathcal{H}^s(Q)) = \mathcal{H}^s(\alpha Q) \to \mathcal{H}^s(\mathbb{R}^n)$ as $\alpha \to \infty$, it suffices to prove that $\mathcal{H}^s(Q) = 0$, to do this, consider a partition of Q into k^n cubes of diameter $\frac{\sqrt{n}}{k}$, thus

$$\mathcal{H}^s_{\frac{\sqrt{n}}{k}}(Q) \leq \omega_s k^n \left(\frac{\sqrt{n}}{2k}\right)^s = \frac{\omega_s n^{\frac{s}{2}}}{2^s} k^{n-s},$$

taking $k \to \infty$ we can conclude. ■

Proposition 1.10. *If $E \subset \mathbb{R}^n$, then $\mathcal{H}\text{-}\dim(E) \in [0, n]$ and $\mathcal{H}^s(E) = \infty$ for each $s < \mathcal{H}\text{-}\dim(E)$.*

Proof. From Proposition 1.9, we know that $\mathcal{H}\text{-}\dim(E) \in [0, n]$. We will now prove that if $\mathcal{H}^s(E) < \infty$ for some $s \in [0, n)$, then $\mathcal{H}^t(E) = 0$ for every $t > s$. Indeed, if $\{E_k\}_{k\in\mathbb{N}}$ is a $\delta-$cover of E, then

$$\mathcal{H}^t_\delta(E) \leq \frac{\omega_t}{2^t} \sum_{k\in\mathbb{N}} (\mathrm{diam}(E_k))^t \leq \left(\frac{\delta}{2}\right)^{t-s} \frac{\omega_t}{\omega_s} \left(\frac{\omega_s}{2^s} \sum_{k\in\mathbb{N}} (\mathrm{diam}(E_k))^s \right).$$

Hence $\mathcal{H}^t_\delta(E) \leq C(t, s)\delta^{t-s}\mathcal{H}^s(E)$, taking $\delta \to 0$ we find that $\mathcal{H}^t(E) = 0$. ■

Proposition 1.11. *$\mathcal{H}^0$ is the counting measure.*

Proof. If $x \in \mathbb{R}^n$ y $\delta > 0$, then $\mathcal{H}^0_\delta(\{x\}) = \omega_0 = 1$, therefore $\mathcal{H}^0 = 1$. Since $\mathcal{H}^0$ is a Borel measure, if E is finite or countable, then

$$\mathcal{H}^0(E) = \sum_{x \in E} \mathcal{H}^0(\{x\}) = \#(E).$$

If E is uncountable, then there exist a countable subset $F \subset E$, so

$$\infty = \mathcal{H}^0(F) \leq \mathcal{H}^0(E).$$

■

Proposition 1.12. *If $E \subset \mathbb{R}^n$ with $\mathcal{H}^s_\infty(E) = 0$, then $\mathcal{H}^s = 0$.*

Proof. Let $s > 0$, since $\mathcal{H}^s_\infty(E) = 0$ for $\varepsilon > 0$ arbitrary, there exist a cover $\{E_k\}_{k \in \mathbb{N}}$ of E such that

$$\frac{\omega_s}{2^s} \sum_{k \in \mathbb{N}} (\operatorname{diam}(E_k))^s < \varepsilon,$$

thus

$$\sup_{k \in \mathbb{N}} \operatorname{diam}(E_k) < 2 \left(\frac{\varepsilon}{\omega_s} \right)^{\frac{1}{s}} = \delta(\varepsilon).$$

Hence $\mathcal{H}^s_{\delta(\varepsilon)(E)}(E) < \varepsilon$ with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. ■

Proposition 1.13. *The Hausdorff dimension satisfies the following properties:*
1.- If $E \subset F$, then $\mathcal{H}\text{-}\dim(E) \leq \mathcal{H}\text{-}\dim(F)$.
2.- If $\{E_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$, then

$$\mathcal{H}\text{-}\dim \left(\bigcup_{k \in \mathbb{N}} E_k \right) = \sup_{k \in \mathbb{N}} \{\mathcal{H}\text{-}\dim(E_k)\}, \text{ (Countable Stability).}$$

3.- If $E \subset \mathbb{R}^n$ is countable, then $\mathcal{H}\text{-}\dim(E) = 0$.
4.- If $A \subset \mathbb{R}^n$ is open, then $\mathcal{H}\text{-}\dim(A) = n$.

The proof of this proposition can be found in [1].

If $s \in [0, n)$, then $\mathcal{H}^s$ is not a Radon measure, this is because from Propositions 1.9 and 1.13, we deduce that $\mathcal{H}^s(A) = \infty$, for every open set A on $\mathbb{R}^n$, which implies that $\mathcal{H}^s$ is not locally finite. However, we have the following:

Proposition 1.14. *If $E \subset \mathbb{R}^n$ is a Borel set such that $\mathcal{H}^s(E) < \infty$, then $\mathcal{H}^s \llcorner E$ is a Radon measure on $\mathbb{R}^n$.*

Proof. Let $K \subset \mathbb{R}^n$ be a compact set, then

$$\mathcal{H}^s \llcorner E(K) = \mathcal{H}^s(E \cap K) \leq \mathcal{H}^s(E) < \infty$$

from which we obtain that $\mathcal{H}^s \llcorner E$ is locally finite, thus we can use Proposition 1.1 for conclude. ■

1.3 Lipschitz functions on $\mathbb{R}^n$

In this section we study the basic properties of Lipschitz functions, for this purpose we provides some operational properties and extension results about these functions, which will also be useful in later sections and chapters.

Definition 1.10. *The **Lipschitz constant*** $\text{Lip}(f; E)$ *of a function* $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ *is defined as the infimum over all non negative constants* L *(it exist) such that*

$$||f(x) - f(y)|| \leq L||x - y||, \quad \forall x, y \in E.$$

If $\text{Lip}(f; E) < \infty$, *we say that* f *is a **Lipschitz function on*** E, *also if* $\text{Lip}(f; E) \leq L$ *we say that* f *is* $L-$***Lipschitz function****, if there is no confusion, we simply say that* f *is Lipschitz. In the case where* $E = \mathbb{R}^n$, *we write* $\text{Lip}(f) := \text{Lip}(f; \mathbb{R}^n)$.

Remark 1.5. *Since the norms on* $\mathbb{R}^d$ *are equivalent, the above definition does not depend on the choice of norms on* $\mathbb{R}^n$ *and* $\mathbb{R}^m$. *That is, if we change the norms of the domain and the codomain, then* f *remains Lipschitz, although possibly with a different Lipschitz constant.* $\triangleleft$

It is immediate from the definition that a Lipschitz function is uniformly continuous. In fact, these class of functions has interesting properties, which we will discuss in later chapters. If $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, we can prove that

$$\text{Graph}(f) = \{(x, y) \in \mathbb{R}^2 : y = f(x)\} \subset \bigcap_{x_0 \in \mathbb{R}} C(x_0; \text{Lip}(f)),$$

where $C(x_0; \text{Lip}(f)) := \{(x, y) \in \mathbb{R}^2 : |y - f(x_0)| \leq \text{Lip}(f)|x - x_0|\}$ is a double cone with vertex at $(x_0, f(x_0))$. This gives us a geometric interpretation of a Lipschitz function in the case $n = m = 1$.

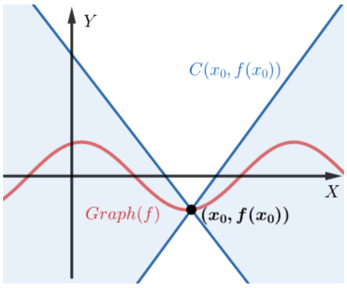

Figure 1.3: Given $(x_0, f(x_0)) \in \text{Graph}(f)$ we can draw a double cone $C(x_0, \text{Lip}(f))$ such that its vertex is at $(x_0, f(x_0))$ and contains $\text{Graph}(f)$.

The following proposition gives us some closure properties regarding Lipschitz functions.

Proposition 1.15. *1.- Let* $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$, $F \subset f(E)$ *and* $g : F \to \mathbb{R}^l$. *If* f *and* g *are Lipschitz functions, then* $g \circ f$ *is Lipschitz and*

$$\text{Lip}(g \circ f; E) \leq \text{Lip}(g; F)\text{Lip}(f; E).$$

2.- If $f, g : A \subset \mathbb{R}^n \to \mathbb{R}^m$ *are Lipschitz functions, then* $f + g$ *and* αf *are Lipschitz functions* $\forall \alpha \in \mathbb{R}$.

3.- If $f, g : A \subset \mathbb{R}^n \to \mathbb{R}$ are Lipschitz and bounded functions, then fg is a Lipschitz function.
4.- If $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of functions from $A \subset \mathbb{R}^n$ to $\mathbb{R}^m$, such that f_k is L_k-Lipschitz, and there exist $L > 0$ such that $L_k \leq L$, $\forall k \in \mathbb{N}$ and $f_k \to f$ uniformly, then f is a $L-$Lipschitz function.
5.- Let $\mathcal{F}$ be a family of $L-$Lipschitz functions from an $A \subset \mathbb{R}^n$ to $\mathbb{R}$. Then,

$$g_*(x) := \inf_{f\in\mathcal{F}}\{f(x)\} \quad \text{and} \quad g^*(x) := \sup_{f\in\mathcal{F}}\{f(x)\}$$

are $L-$Lipschitz functions.

Proof. For (1), let $x, y \in E$. Then

$$||g \circ f(x) - g \circ f(y)|| \leq \text{Lip}(g; F)||f(x) - f(y)|| \leq \text{Lip}(g; F)\text{Lip}(f; E)||x - y||.$$

The proof of (2) is an immediate consequence of the triangle inequality (for addition) and the definition of Lipschitz function (scalar multiplication). To prove (3), given $x, y \in A$ note that if $||f(v)|| \leq M_1$ and $||g(v)|| \leq M_2$ $\forall v \in A$, then

$$\begin{aligned} ||(fg)(x) - (fg)(y)|| &\leq ||f(x)g(x) - f(x)g(y)|| + ||f(x)g(y) - f(y)g(y)|| \\ &\leq M_1||g(x) - g(y)|| + M_2||f(x) - f(y)|| \\ &\leq (M_1\text{Lip}(g) + M_2\text{Lip}(f))||x - y||. \end{aligned}$$

For (4), first note that from the hypotheses,

$$||f_k(x) - f_k(y)| \leq L||x - y||, \quad \forall x, y \in A, \forall k \in \mathbb{N}.$$

Thus,

$$||f(x) - f(y)|| = \lim_{k\to\infty} ||f_k(x) - f_k(y)|| \leq L||x - y||.$$

To prove (5), let $x, y \in A$. Then, given $\varepsilon > 0$ arbitrary, there exist $f \in \mathcal{F}$ such that

$$g^*(x) - \varepsilon \leq f(x).$$

Then,

$$\begin{aligned} g^*(x) - g^*(y) &\leq f(x) - g^*(y) + \epsilon \\ &\leq f(x) - f(y) + \epsilon \\ &\leq |f(x) - f(y)| + \epsilon \\ &\leq L||x - y|| + \epsilon, \end{aligned}$$

where the second inequality follows from the definition of g^*, taking $\varepsilon \to 0$ we have

$$g^*(x) - g^*(y) \leq L||x - y||. \tag{1.4}$$

Exchanging the roles of x and y, we obtain

$$g^*(y) - g^*(x) \leq L||x - y||. \tag{1.5}$$

Thus, from (1.4) and (1.5), we deduce

$$|g^*(x) - g^*(y)| \leq L||x - y||.$$

From which we have the desired result for g^*. Now, considering g_*, given $\varepsilon > 0$ arbitrary, there exist $h \in \mathcal{F}$ such that

$$h(x) \leq g_*(x) + \varepsilon.$$

Thus,

$$\begin{aligned} g_*(y) - g_*(x) &\leq g_*(y) - h(x) + \varepsilon \\ &\leq h(y) - h(x) + \varepsilon \\ &\leq |h(x) - h(y)| + \varepsilon \\ &\leq L||x - y|| + \varepsilon. \end{aligned}$$

Then, following a similar reasoning as already done for g^*, we can conclude. ■

A natural question that may arise is: What happens for the product of two Lipschitz functions? In this case, the result may not be favorable, as shown by the following example.

Example 1.3. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$. This is a Lipschitz function. However $x^2 = f(x)f(x)$ is not a Lipschitz function, because if x^2 were Lipschitz, there would exist $L > 0$ such that*

$$|x^2 - y^2| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}. \tag{1.6}$$

Then,

$$|k^2 - 0^2| \leq L|k - 0| \Rightarrow k \leq L, \quad \forall k \in \mathbb{N},$$

which would imply that the set of natural numbers is bounded, but this is a contradiction. Therefore, x^2 is not Lipschitz.

◀

Proposition 1.16. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and denote $f = (f_1, \ldots, f_m)$. Then f is Lipschitz if and only if f_i is Lipschitz for each $i = 1, \ldots, m$.*

Proof. Suppose that f is Lipschitz, then for given $x, y \in \mathbb{R}^n$ with $i = 1, \ldots, m$, we have

$$|f_i(x) - f_i(y)| \leq ||f(x) - f(y)|| \leq \mathrm{Lip}(f)||x - y||,$$

thus the claim follows. Now suppose that f_i is Lipschitz for each $i = 1, \ldots, m$. Then, for any $x, y \in \mathbb{R}^n$

$$||f(x) - f(y)|| \leq \sum_{i=1}^{m} |f_i(x) - f_i(y)| \leq \left(\sum_{i=1}^{m} \mathrm{Lip}(f_i) \right) ||x - y||.$$

■

Proposition 1.17. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then L is a Lipschitz function and $\mathcal{N}(L) = \mathrm{Lip}(L)$ (here $\mathcal{N}(\cdot)$ denotes the norm of L as an operator).*

Proof. The first assertion follows from the characterization of the continuity of L as a linear operator, while the equality $\mathcal{N}(T) = \mathrm{Lip}(T)$ follows from the definition of $\mathcal{N}(\cdot)$. ■

1.3.1 Extension of Lipschitz functions on $\mathbb{R}^n$

Lemma 1.1. *(**McShane's extension lemma**) Let $f : E \subset \mathbb{R}^n \to \mathbb{R}$ be an $L-$Lipschitz function. Then there exist an $L-$Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F|_E = f$.*

Proof. For each $y \in \mathbb{R}^n$, we define

$$F(y) := \inf\{f(x) + L||x - y|| \ : \ x \in E\}. \tag{1.7}$$

Note that if $y \in E$, then

$$f(y) = f(y) + L||y - y|| \geq F(y).$$

Thus, on E we have

$$f \geq F. \tag{1.8}$$

Also, given $x, y \in E$, with x arbitrary and y fixed, from the Lipschitz condition

$$f(y) - f(x) \leq L||x - y||,$$

hence

$$f(y) \leq f(x) + L||x - y||.$$

Then, minimizing the right hand side of the last inequality over all x in E, it follows

$$f \leq F. \tag{1.9}$$

And therefore, from (1.8) and (1.9) we obtain that, indeed, (1.7) is an extension of f. It remains to show that F is an $L-$Lipschitz function. For this, it suffices to note that given $s, t \in \mathbb{R}^n$ and $z \in E$, then

$$\begin{aligned} |f(z) + L||z - s|| - f(z) - L||z - t||\,| &= L|\,||z - s|| - ||z - t||\,| \\ &\leq L||s - t||. \end{aligned}$$

So for each $z \in E$, we have that $f_z(s) := f(z) + L||z - s||, \forall s \in \mathbb{R}^n$, is a Lipschitz function. Hence, using Proposition 1.15 we can conclude. ■

Observe that the previous extension has a geometric interpretation because the graph of F is obtained by taking the lower envelope over all sets

$$C_x = \{(y, z) \in \mathbb{R}^n \times \mathbb{R} : z = f(x) + L||y - x||\}, \quad x \in E,$$

where each C_x represents a half-cone with slope L, with its vertex on the graph of the original function f.

Corollary 1.1. *Let $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ be an $L-$Lipschitz function. Then there exist a $\sqrt{m}L-$Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$ such that $F|_E = f$.*

Proof. Put $f = (f_1, \ldots, f_m)$, then for any $s, t \in E$, we have

$$|f_i(s) - f_i(t)| \leq ||f(s) - f(t)|| \leq L||s - t||$$

for each $i \in \{1, \ldots, m\}$. Thus each coordinate function is an $L-$Lipschitz function from E to $\mathbb{R}^n$. Therefore, by McShane's lemma, for each $i \in \{1, \ldots, m\}$ there exist an extension $F_i : \mathbb{R}^n \to \mathbb{R}$, $L-$Lipschitz of f_i. Defining $F := (F_1, \ldots, F_m)$, note that $F|_E = f$. Furthermore, for $x, y \in \mathbb{R}^n$ we have

$$||F(x) - F(y)|| \leq \sqrt{m}||F(x) - F(y)||_\infty \leq \sqrt{m}L||x - y||,$$

which implies the result. ■

In the case where we work with the infinity norm on $\mathbb{R}^m$, the factor $\sqrt{m}$ in the previous corollary is irrelevant. We should note that in the previous corollary, we are guaranteed that the found extension will have a Lipschitz constant greater than or equal to that of the original function. However, the following result tells us that we can find an extension with the same Lipschitz constant as the original function.

Theorem 1.4. *(**Kirszbraun Theorem**) Consider $\mathbb{R}^n$ and $\mathbb{R}^m$ both of them equipped with the Euclidean norm. If $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ is a Lipschitz function. Then there exist a function $G : \mathbb{R}^n \to \mathbb{R}^m$ such that $G|_E = f$ and $\mathrm{Lip}(G) = \mathrm{Lip}(f; E)$.*

The proof of the previous theorem can be consulted in [11]. In a more general context, it can be shown that Kirszbraun's Theorem remains valid whenever $\mathbb{R}^n$ and $\mathbb{R}^m$ are replaced by Hilbert spaces H_1, H_2, although it can be seen (Example 4.1) that the result may fail if we consider finite dimensional Banach spaces.

1.4 $\mathcal{H}^1$ and the classical notion of length

We first present the following result, which provides an initial relationship between Hausdorff measure and Lipschitz functions. It is worth mentioning that this result will be useful throughout this work.

Theorem 1.5. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, then*

$$\mathcal{H}^s(f(E)) \leq \mathrm{Lip}(f)^s \mathcal{H}^s(E) \tag{1.10}$$

for each $s \in [0, \infty[$ and $E \subset \mathbb{R}^n$. In particular $\mathcal{H}$-dim$(f(E)) \leq \mathcal{H}$-dim(E).

Proof. Let $\{E_k\}_{k\in\mathbb{N}}$ be a $\delta-$cover of E. Then $\{f(E_k)\}_{k\in\mathbb{N}}$ is a $\mathrm{Lip}(f)\delta-$cover of $f(E)$, since

$$\mathrm{diam}(f(E_k)) \leq \mathrm{Lip}(f)\mathrm{diam}(E_k) < \mathrm{Lip}(f)\delta, \quad \forall k \in \mathbb{N}.$$

Moreover, note that

$$\mathcal{H}^s_{\mathrm{Lip}(f)\delta}(f(E)) \leq \frac{\omega_s}{2^s} \sum_{k\in\mathbb{N}} (\mathrm{diam}(f(E_k)))^s \leq \mathrm{Lip}(f)^s \left(\frac{\omega_s}{2^s} \sum_{k\in\mathbb{N}} (\mathrm{diam}(E_k))^s \right).$$

Then, as the cover is arbitrary,

$$\mathcal{H}^s_{\mathrm{Lip}(f)\delta}(f(E)) \leq \mathrm{Lip}(f)\mathcal{H}^s_\delta(E).$$

Taking $\delta \to 0$, we can deduce the conclusion (1.10). ■

In terms of the Hausdorff dimension, the above theorem tells us that Lipschitz functions behave in a regular manner with respect to it.

Definition 1.11. *A set $\Gamma \subset \mathbb{R}^n$ is a curve if there exist $a > 0$ and a continuous and injective function $\gamma : [0, a] \to \mathbb{R}^n$ such that $\Gamma = \gamma([0, a])$. The function γ is called a **parametrization** of Γ.*

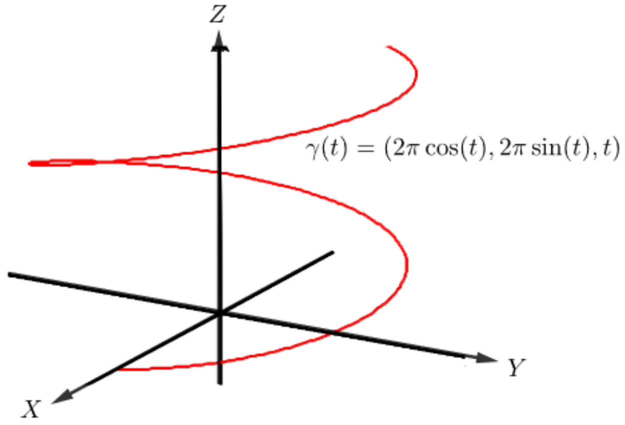

Figure 1.4: Curve given by the parametrization $\gamma(t) = (2\pi\cos(t), 2\pi\sin(t), t)$.

Given a parametrization $\gamma : [0, a] \to \mathbb{R}^n$ and a subinterval $[b, c]$ of $[0, a]$, the **length** of γ over $[b, c]$ is defined as:

$$\ell(\gamma; [b, c]) := \sup \left\{ \sum_{k=1}^{K} |\gamma(t_k) - \gamma(t_{k-1})| b = t_0 < t_{k-1} < t_k < t_K = c, K \in \mathbb{N} \right\}.$$

It can be proven that $\ell(\gamma; [0, a])$ is independent of the parametrization γ of Γ. This allows us to define the **length** of Γ as

$$length(\Gamma) := \ell(\gamma; [0, a]).$$

The following are well known properties of ℓ:

$$\ell(\gamma; [b, c]) \geq |\gamma(b) - \gamma(c)|, \text{ whenever } 0 \leq b \leq c \leq a \tag{1.11}$$

$$\ell(\gamma; [b, c]) = \ell(\gamma; [b, d]) + \ell(\gamma; [d, c]), \text{ whenever } 0 \leq b \leq d \leq c \leq a \tag{1.12}$$

Lemma 1.2. *Let $H \subset \mathbb{R}^n$ be an affine subspace, and let $p : \mathbb{R}^n \to \mathbb{R}^n$ be the projection from $\mathbb{R}^n$ onto H. Then p is a 1-Lipschitz function.*

Proof. Firstly, as H is an affine subspace, we have

$$H = V + a$$

where V is a linear subspace of $\mathbb{R}^n$ and $a \in \mathbb{R}^n$. Therefore

$$p(x) = a + f_V(x - a),$$

where f_v is the projection from $\mathbb{R}^n$ onto V. Since $\mathbb{R}^n = V \oplus V^\perp$, for $x, y \in \mathbb{R}^n$, there exist $v_1, v_2 \in V$ and $w_1, w_2 \in V^\perp$ such that

$$x = v_1 + w_1 \quad \text{and} \quad y = v_2 + w_2.$$

As

$$||f_V(x) - f_V(y)|| = ||v_1 - v_2|| \tag{1.13}$$

and

$$||x-y|| = ||(v_1-v_2)+(w_1-w_2)|| = \sqrt{||v_1-v_2||^2+||w_1-w_2||^2}. \tag{1.14}$$

From (1.13) and (1.14), we have

$$||f_V(x)-f_V(y)|| \leq ||x-y||. \tag{1.15}$$

Thus

$$||p(x)-p(y)|| = ||f_V(x-a)-f_V(y-a)|| \leq ||x-y||,$$

which implies the result. ■

Definition 1.12. *Given $x, y \in \mathbb{R}^n$, we define the* ***line through*** x ***in the direction*** y *(or* ***the line defined by*** x ***and*** y*), as the set*

$$[x,y] := \{x+ty : t \in \mathbb{R}^n\}.$$

Theorem 1.6. *If Γ is a curve on $\mathbb{R}^n$, then*

$$\mathcal{H}^1(\Gamma) = length(\Gamma).$$

Proof. Let $\gamma : [0,a] \to \mathbb{R}^n$ be a parametrization of Γ and put $\ell = length(\Gamma) = \ell(\gamma;[0,a])$. First, prove that

$$\mathcal{H}^1(\Gamma) \geq |\gamma(a)-\gamma(0)|. \tag{1.16}$$

Indeed, as the projection $p : \mathbb{R}^n \to \mathbb{R}^n$ onto the affine subspace given by the line defined by $\gamma(0)$ and $\gamma(a)$, satisfies $\text{Lip}(p) \leq 1$, from Theorem 1.5 we obtain that

$$\mathcal{H}^1(p(\Gamma)) \leq \mathcal{H}^1(\Gamma). \tag{1.17}$$

Also, note that $[\gamma(0),\gamma(a)] \subset p(\Gamma)$, since otherwise $\Gamma = \gamma([0,a])$ would be disconnected, which would contradict the continuity of γ. Then, by the monotonicity of $\mathcal{H}^1$ we have

$$\mathcal{H}^1(p(\Gamma)) \geq \mathcal{H}^1([\gamma(0),\gamma(a)]) = |\gamma(a)-\gamma(0)|. \tag{1.18}$$

Thus, from (1.17) and (1.18), we obtain (1.16). Now, let $\{t_k\}_{k=0}^K$ be a partition of $[a,b]$, and define $\Gamma_k := \gamma([t_{k-1},t_k])$. Then $\Gamma = \cup_{k=1}^K \Gamma_k$ and, by the injectivity of γ

$$\mathcal{H}^1(\Gamma_k \cap \Gamma_{k+1}) = \mathcal{H}^1(\{t_k\}) = 0.$$

Therefore, from this and (1.17), we have

$$\mathcal{H}^1(\Gamma) = \sum_{k=1}^K \mathcal{H}^1(\Gamma_k) \geq \sum_{k=1}^K |\gamma(t_k)-\gamma(t_{k-1})|.$$

This implies

$$\mathcal{H}^1(\Gamma) \geq \ell. \tag{1.19}$$

To prove the reverse inequality, define $v : [0,a] \to [0,\ell]$ by $v(t) := \ell(\gamma;[0,t])$, $\forall t \in [0,a]$. Note that

$$v(0) = 0 \quad \text{and} \quad v(a) = \ell$$

and also that if $s < t$, then $v(s) < v(t)$. Thus v is strictly increasing and hence injective. Moreover we can observe that v is continuous, then by the Intermediate Value Theorem, v will be surjective, and thus v is invertible, with inverse $w : [0, \ell] \to [0, a]$, which is strictly increasing and continuous. Now define : $\gamma^* : [0, \ell] \to \mathbb{R}^n$, by $\gamma^*(s) := \gamma(w(s))$, $\forall s \in [0, \ell]$. Then γ^* is injective and continuous. Furthermore, from (1.11) and (1.12), if $[s_1, s_2] \subset [0, \ell]$, then

$$\begin{aligned} |\gamma^*(s_2) - \gamma^*(s_1)| &\leq \ell(\gamma; [s_1, s_2]) \\ &= \ell(\gamma^*; [0, s_2]) - \ell(\gamma^*; [0, s_1]) \\ &= s_2 - s_1. \end{aligned} \tag{1.20}$$

From all the above, we have that γ^* is a $1-$Lipschitz parametrization of Γ. Thus, by Theorem 1.5, we obtain

$$\mathcal{H}^1(\Gamma) = \mathcal{H}^1(\gamma^*([0, \ell])) \leq \mathcal{H}^1([0, \ell]) = \ell. \tag{1.21}$$

Finally, from (1.19) and (1.21) we can conclude. ■

If $\gamma : [0, a] \to \mathbb{R}^n$ is a C^1 parametrization of Γ, we know that

$$length(\Gamma) = \int_0^a ||\gamma\prime(t)||dt.$$

In particular, from Theorem 1.6, we obtain

$$\mathcal{H}^1(\Gamma) = \int_0^a ||\gamma\prime(t)||dt.$$

This represents a particular $1-$dimensional case of the Area formula, which we will discuss in Chapters 3 and 4.

1.5 Relationship between Hausdorff and Lebesgue measures

In order to prove Theorem 1.7 we introduce a first tool, which is called Steiner Symmetrization (named after the mathematician Jakob Steiner). This is a geometric technique used to transform a geometric object into another with greater symmetry, through reflection, rotation and translation operations, we will study a particular case, however, we refer to [5] for a more general case of this symmetrization.
In this work we denote by $\langle \cdot, \cdot \rangle$ the usual dot product on $\mathbb{R}^n$.

Definition 1.13. *Let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $q : \mathbb{R}^n \to \mathbb{R}$ be the projections onto the first $(n-1)-$coordinates and the last coordinate, respectively. Given $E \subset \mathbb{R}^n$, we define the vertical sections of E, as:*

$$E_z := \{t \in \mathbb{R} : (z, t) \in E\} \subset \mathbb{R}, \quad z \in \mathbb{R}^{n-1}$$

and we define the ***Steiner symmetrization of E with respect to*** *$\langle e_n \rangle^\perp := \{x \in \mathbb{R}^n : \langle x, e_n \rangle = 0\}$ (here $e_n = (0, \ldots, 0, 1)$), as:*

$$\begin{aligned} E^s = E^{s_n} &= \left\{(z, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : E_z \neq \emptyset, \ |t| \leq \frac{1}{2}\mathcal{H}^1(E_z)\right\} \\ &= \left\{x \in \mathbb{R}^n : E_{p(x)} \neq \emptyset, \ |q(x)| \leq \frac{1}{2}\mathcal{H}^1(E_{p(x)})\right\}. \end{aligned}$$

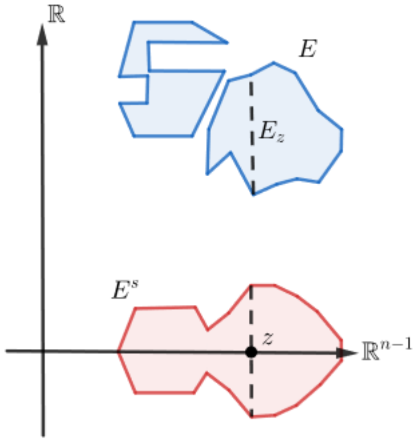

Figure 1.5: Steiner symmetrization of E.

We can think in the Steiner symmetrization with respect to $\langle e_j\rangle^\perp$ as a set function $s : \mathscr{P}(\mathbb{R}^n) \to \mathscr{P}(\mathbb{R}^n)$ such that $s(E) = E^s$, $\forall E \subset \mathbb{R}^n$.

Proposition 1.18. *(**Steiner symmetrization properties**) Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. Then:*
1.- E_z is Lebesgue measurable, the function $z \mapsto \lambda^1(E_z)$, $\forall z \in \mathbb{R}^{n-1}$, is Lebesgue measurable and

$$\lambda^n(E) = \lambda^n(E^s). \tag{1.22}$$

2.-

$$\operatorname{diam}(E^s) \leq \operatorname{diam}(E). \tag{1.23}$$

Proof. 1) From Fubini's theorem, we can deduce the first two statements. Furthermore, from the same theorem and the translation invariance of λ^1, we obtain

$$\lambda^n(E) = \int_{\mathbb{R}^{n-1}} \lambda^1(E_z)dz = \int_{\mathbb{R}^{n-1}} \lambda^1(E_z^s)dz = \lambda^n(E^s),$$

from which (1.22) follows.
2) Suppose $\operatorname{diam}(E) < \infty$, otherwise the result is immediate. Let $x, y \in E^s$, we claim that

$$||x - y|| \leq \max\{||M(x) - m(y)||, ||M(y) - m(x)||\}, \tag{1.24}$$

where $M(x), m(x), M(y), m(y) \in \overline{E}$ are defined as follows (see Figure 1.6) :

$$m(x) = (p(x), r),\ M(x) = (p(x), s)$$

$$m(y) = (p(y), u),\ M(y) = (p(y), v)$$

with:

$$r = \inf\{E_{p(x)}\},\ s = \sup\{E_{p(x)}\}$$
$$u = \inf\{E_{p(y)}\},\ v = \sup\{E_p(y)\}.$$

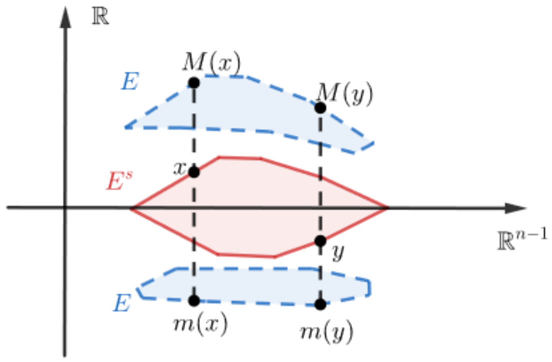

Figure 1.6:

Without loss of generality, suppose that

$$v - r \geq s - u.$$

Then,

$$\begin{aligned} v - r &\geq \frac{1}{2}(v-r) + \frac{1}{2}(s-u) \\ &= \frac{1}{2}(s-r) + \frac{1}{2}(v-u) \\ &\geq \lambda^1(E_{p(x)}) + \lambda^1(E_{p(y)}), \end{aligned}$$

wich implies that

$$v - r \geq |q(x)| + |q(y)| \geq |q(x) - q(y)|,$$

where the second inequality follows from the definitions of r, s, u, v. Hence

$$||x-y|| = \sqrt{||p(x-y)||^2 + |q(x-y)|^2} \leq \sqrt{||p(x-y)||^2 + |v-r|^2}.$$

And as

$$\sqrt{||p(x-y)||^2 + |v-r|^2} = \sqrt{||p(x-y)||^2 + |q(M(y)-m(x))|^2} = ||M(y) - m(x)||.$$

We will have:

$$||x-y|| \leq ||M(y) - m(x)||,$$

and therefore we obtain (1.24).
Since $x, y \in E^s$ were arbitrary and $M(x), m(x), M(y), m(y) \in \overline{E}$, then

$$\operatorname{diam}(E^s) \leq \operatorname{diam}(\overline{E}) = \operatorname{diam}(E).$$

■

If $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$, in a similarly way that we defined the Steinner symmetrization with respect to $\langle e_n \rangle^\perp$, we can define the Steiner symmetrization with respect to $\langle e_j \rangle^\perp$ for $j = 1, \ldots, n-1$. We do it as follows:

Definition 1.14. *We define the* ***Steiner symmetrization of*** $E \subset \mathbb{R}^n$ ***with respect to*** $\langle e_j \rangle^\perp = \{x \in \mathbb{R}^n : \langle x, e_j \rangle = 0\}$*, as*

$$E^{s_j} := \bigcup_{\substack{b \in \langle e_j \rangle^\perp \\ E \cap [b, e_j] \neq \emptyset}} \left\{ b + te_j : |t| \leq \frac{1}{2}\mathcal{H}^1(A \cap [b, e_j]) \right\}$$

It is possible to prove that this definition coincides with the particular case showed in the Definition 1.13, indeed these new symmetrizations satisfies analogue properties to those mentioned in the Proposition 1.18, and also, we can think as a set functions $s_{e_j} : \mathscr{P}(\mathbb{R}^n) \to \mathscr{P}(\mathbb{R}^n)$, such that $s_{e_j}(E) = E^{s_j}$, $\forall E \subset \mathbb{R}^n$.
We now introduce a second tool to prove Theorem 1.7, namely, the isodiametric inequality which states that among all sets of fixed diameter, balls have maximum volume.

Theorem 1.7. ***(Isodiametric inequality)*** *Let* $E \subset \mathbb{R}^n$*, then*

$$\lambda_n^*(E) \leq \frac{\omega_n}{2^n}(\text{diam}(E))^n.$$

Proof. Let $\tilde{s} = s_{e_1} \circ \cdots \circ s_{e_n}$ be the composition of Steiner's symmetrizations s_{e_i} with respect to $\langle e_i \rangle^\perp$ (thinking of them as a set functions) . We claim that

$$\lambda^n(\overline{E}^{\tilde{s}}) = \lambda_n^*(\overline{E}^{\tilde{s}}) \leq \omega_n \left(\frac{\text{diam}(\overline{E}^{\tilde{s}})}{2} \right)^n. \tag{1.25}$$

Indeed, it suffices to prove that

$$\overline{E}^{\tilde{s}} \subset \overline{B}\left(0, \frac{\text{diam}(\overline{E}^{\tilde{s}})}{2}\right),$$

which follows from the fact that:

$$y \in \overline{E}^{s_{e_i}} \Rightarrow (y_1, \ldots, -y_i, \ldots, y_n) \in \overline{E}^{s_{e_i}}, \quad \forall i = 1, \ldots, n.$$

Then $x \in \overline{E}^{\tilde{s}} \Rightarrow -x \in \overline{E}^{\tilde{s}}$ and as

$$||x - (-x)|| \leq \text{diam}(\overline{E}^{\tilde{s}}).$$

We obtain:

$$x \in \overline{B}\left(0, \frac{\text{diam}(\overline{E}^{\tilde{s}})}{2}\right).$$

And thus we can deduce (1.25). Finally, using Propositions 1.18 and 1.25, we obtain:

$$\lambda_n^*(E) \leq \lambda_n^*(\overline{E}) = \lambda^n(\overline{E}) = \lambda^n(\overline{E}^{\tilde{s}}) \leq \omega_n \left(\frac{\text{diam}(\overline{E}^{\tilde{s}})}{2}\right)^n \leq \omega_n \left(\frac{\text{diam}(\overline{E})}{2}\right)^n = \omega_n \left(\frac{\text{diam}(E)}{2}\right)^n.$$

■

We conclude this chapter showing the equivalence of the Lebesgue outer measure and the $n-$dimensional Hausdorff measure $\mathcal{H}^n$ on $\mathbb{R}^n$.

Theorem 1.8. *If $E \subset \mathbb{R}^n$ and $\delta > 0$, then*

$$\lambda_n^*(E) = \mathcal{H}^n(E) = \mathcal{H}_\delta^n(E).$$

Proof. Let $\{E_k\}_{k\in\mathbb{N}}$ be a cover of E in the sense of the definition of Lebesgue outer measure (see Example 1.1). Then by Pythagorean theorem, we have that $\operatorname{diam}(E_k) = \sqrt{n}l(E_k)$, where $l(E_k)$ denotes the side length of the cube E_k, $\forall k \in \mathbb{N}$. Hence

$$\mathcal{H}_\infty^n(E) \le \frac{\omega_n}{2^n}\sum_{k\in\mathbb{N}}(\operatorname{diam}(E_k))^n = \omega_n\left(\frac{\sqrt{n}}{2}\right)^n\sum_{k\in\mathbb{N}} l(E_k)^n.$$

Thus, we have

$$\mathcal{H}_\infty^n(E) \le \omega_n\left(\frac{\sqrt{n}}{2}\right)^n \lambda_n^*(E). \tag{1.26}$$

Now, suppose without loss of generality that $\lambda_n^*(E) < \infty$. Let $\varepsilon > 0$ and $\delta > 0$, by the regularity of Lebesgue outer measure, there exist an open set $A \subset \mathbb{R}^n$ such that $E \subset A$ and

$$\lambda_n^*(A) \le \lambda_n^*(E) + \varepsilon.$$

By Vitali's Covering Theorem (see *Appendix* A), there exist a countable disjoint family of closed balls $\mathcal{F}$ contained in A, with diameters strictly less than δ, such that

$$\lambda_n^*\left(A \setminus \bigcup\{\overline{B} : \overline{B} \in \mathcal{F}\}\right) = 0.$$

If $F = \bigcup_{\overline{B}\in\mathcal{F}} \overline{B}$, then

$$\lambda_n^*(E) + \varepsilon \ge \lambda_n^*(A) = \lambda_n^*(F) = \sum_{\overline{B}\in\mathcal{F}} \lambda_n^*(\overline{B}) = \frac{\omega_n}{2^n}\sum_{\overline{B}\in\mathcal{F}}(\operatorname{diam}(\overline{B}))^n \ge \mathcal{H}_\delta^n(F). \tag{1.27}$$

Also, from (1.26) applied to $A\backslash F$, we have $\mathcal{H}_\infty^n(A\backslash F) = 0$, and by monotonicity, $\mathcal{H}_\infty^n(E\backslash F) = 0$. Then by Proposition 1.12, $\mathcal{H}^n(E \setminus F) = 0$ and from the definition of $\mathcal{H}^n$, we obtain that $\mathcal{H}_\delta^n(E \setminus F) = 0$. Thus, from (1.27),

$$\mathcal{H}_\delta^n(E) \le \mathcal{H}_\delta^n(E \cap F) + \mathcal{H}_\delta^n(E \setminus F) \le \mathcal{H}_\delta^n(F) \le \lambda_n^*(E) + \varepsilon.$$

Taking $\delta \to 0$ and $\varepsilon \to 0$

$$\mathcal{H}^n(E) \le \lambda_n^*(E). \tag{1.28}$$

To prove the reverse inequality, let $\delta \in (0, \infty]$ and $\{F_k\}_{k\in\mathbb{N}}$ be a $\delta-$cover of E, from the isodiametric inequality,

$$\begin{aligned}\frac{\omega_n}{2^n}\sum_{k\in\mathbb{N}}(\operatorname{diam}(F_k))^n &\ge \sum_{k\in\mathbb{N}}\lambda_n^*(F_k)\\ &\ge \lambda_n^*\left(\bigcup_{k\in\mathbb{N}} F_k\right)\\ &\ge \lambda_n^*(E).\end{aligned}$$

Hence

$$\mathcal{H}^n(E) \ge \mathcal{H}_\delta^n(E) \ge \lambda_n^*(E). \tag{1.29}$$

Thus, from (1.28) and (1.29) the result holds. ■

Corollary 1.2. *If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then*

$$\lambda^n(E) = \mathcal{H}^n(E) = \mathcal{H}_\delta^n(E).$$

Chapter 2

Rademacher's Theorem

In this chapter, we will study Rademacher's Theorem (named after Hans Rademacher, who first proved it in 1930). Which is an important result for this work, because it establishes the differentiability properties of Lipschitz functions, providing a relation between the regularity of the functions and their geometric properties, specifically, the theorem establishes that Lipschitz functions are almost everywhere differentiable in the sense of Lebesgue measure.

Definition 2.1. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **differentiable** at $x \in \mathbb{R}^n$ if there exist a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{y \to x} \frac{||f(y) - f(x) - L(y - x)||}{||y - x||} = 0. \tag{2.1}$$

In such a case, we will say that L is the differential of f at x, and denote it by $d_x f$.

We begin with the following lemma regarding the differentiability of a Lipschitz function in the case where $f : \mathbb{R} \to \mathbb{R}$, which we can indeed think of as a particular case of Rademacher's Theorem and will be useful in the proof of the latter.

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The following statements are equivalent:*
1.- f is a Lipschitz function.
2.- f is differentiable a.e. with respect to the Lebesgue measure, $f' \in L^\infty(\mathbb{R})$ and

$$f(t) = f(0) + \int_0^t f'(s)ds, \quad \forall t \in \mathbb{R}.$$

3.- There exist $g \in L^\infty(\mathbb{R})$ such that

$$f(t) = f(0) + \int_0^t g(s)ds, \quad \forall t \in \mathbb{R}.$$

Moreover, it holds that

$$\text{Lip}(f) = ||f'||_\infty = ||g||_\infty.$$

Corollary 2.1. *Let $S \subset \mathbb{R}$ such that $\lambda^1(S) > 0$ and let $f : S \to \mathbb{R}$ be a Lipschitz function. Then f is differentiable a.e. in S and $||f'||_\infty \leq \text{Lip}(f)$. If S is an interval, then $||f'||_\infty = \text{Lip}(f)$ and*

$$f(t) = f(t_0) + \int_{t_0}^t f'(s)ds, \quad \forall t, t_0 \in S,$$

while if S is disconnnected, then $||f'||_\infty < \text{Lip}(f)$.

The proof of the two previous results requires tools from real analysis and will be omitted, but we refer to [17] for further details.

Definition 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $v \in \mathbb{S}^{n-1}$. We define:*

$$D_v f(x) := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}, \quad \forall x \in \mathbb{R}^n, \tag{2.2}$$

*whenever the limit exist, and $D_v f(x)$ will be called the **directional derivative of f at x in the direction of the vector** v.*

Proposition 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function, $v \in \mathbb{S}^{n-1}$ and $x \in \mathbb{R}^n$. If $D_v f(x)$ exist, then:*

$$|D_v f(x)| \leq \text{Lip}(f).$$

Proof. From the continuity of the absolute value and the Lipschitz condition:

$$\begin{aligned} |D_v f(x)| = \left| \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} \right| &= \lim_{t \to 0} \left| \frac{f(x+tv) - f(x)}{t} \right| \\ &\leq \lim_{t \to 0} \text{Lip}(f) \frac{||x + tv - x||}{|t|} = \text{Lip}(f). \end{aligned}$$

■

In the following, we denote by $C_C^\infty(\mathbb{R}^n, \mathbb{R})$ to the set of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$ with compact support. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is smooth provided that all mixed partial derivatives of f of all orders exist and are continuous at every point of $\mathbb{R}^n$.
The importance of the following lemma in the proof of Rademacher's Theorem lies mainly in its corollary, which is an immediate consequence.

Lemma 2.2. ***(Fundamental Lemma of Variational Calculus)*** *Let $f \in L^1_{loc}(\mathbb{R}^n, \mathbb{R})$. If*

$$\int_{\mathbb{R}^n} f(x)\varphi(x) d\lambda^n(x) = 0, \quad \forall \varphi \in C_C^\infty(\mathbb{R}^n, \mathbb{R}),$$

then $f = 0$ $\lambda^n - a.e.$

Corollary 2.2. *Let $f, g \in L^1_{loc}(\mathbb{R}^n, \mathbb{R})$. If*

$$\int_{\mathbb{R}^n} f(x)\varphi(x) d\lambda^n(x) = \int_{\mathbb{R}^n} g(x)\varphi(x) d\lambda^n(x), \quad \forall \varphi \in C_C^\infty(\mathbb{R}^n, \mathbb{R}),$$

then $f = g$ $\lambda^n - a.e.$

Now, we have the necessary tools to state and prove Rademacher's Theorem.

Theorem 2.1. ***(Rademacher's Theorem)*** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $L-$Lipschitz function. Then f is differentiable a.e. with respect to the Lebesgue measure on $\mathbb{R}^n$.*

Proof. We will divide the proof into three steps, as follows:
Step 1) We claim that, for $v \in \mathcal{S}^{n-1}$ fixed, $D_v f(x)$ exist for λ^n−a.e. x in $\mathbb{R}^n$.
From the continuity of f (due to the Lipschitz property), the functions

$$\overline{D_v} f(x) := \limsup_{t\to 0} \frac{f(x+tv)-f(x)}{t} = \lim_{k\to\infty} \sup_{0<|t|<\frac{1}{k}, t\in\mathbb{Q}} \frac{f(x+tv)-f(x)}{t}$$

and

$$\underline{D_v} f(x) := \liminf_{t\to 0} \frac{f(x+tv)-f(x)}{t} = \lim_{k\to\infty} \inf_{0<|t|<\frac{1}{k}, t\in\mathbb{Q}} \frac{f(x+tv)-f(x)}{t},$$

are Borel measurable, thus

$$A_v := \{x \in \mathbb{R}^n : D_v f(x) \text{ does not exist } \} = \{x \in \mathbb{R}^n : \underline{D_v} f(x) < \overline{D_v} f(x)\}$$

is a Borel set and hence is Lebesgue measurable. Define now $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = f(x+tv), \quad \forall t \in \mathbb{R}.$$

Note that

$$\begin{aligned} |\phi(l)-\phi(c)| &= |f(x+lv)-f(x+cv)| \\ &\leq \mathrm{Lip}(f)||x+lv-x-cv|| = \mathrm{Lip}(f)|l-c|. \end{aligned}$$

Thus φ is Lipschitz, then by Lemma 2.1 φ is differentiable λ^1−a.e. Therefore,

$$\mathcal{H}^1(A_v \cap L) = 0,$$

for each parallel line L to v (see figure 2.1). Then by Fubini's Theorem and the invariance under rotations and translations of the Lebesgue measure:

$$\begin{aligned} \lambda^n(A_v) &= \int_{\mathbb{R}^n} \chi_{A_v} d\lambda^n \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_{A_v \cap L} dx d\lambda^{n-1} \\ &= \int_{\mathbb{R}^{n-1}} \lambda^1(A_v \cap L) d\lambda^{n-1} \\ &= 0, \end{aligned}$$

hence we have the claim, in particular we will have that

$$\nabla f(x) = (D_{e_1} f(x), \ldots, D_{e_n} f(x))$$

exist λ^n−a.e. on $\mathbb{R}^n$.

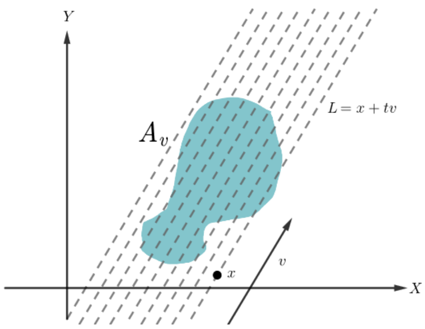

Figure 2.1:

Step 2) We claim that, for $v \in \mathbb{S}^{n-1}$ fixed,

$$D_v f(x) = \langle v, \nabla f(x) \rangle$$

λ^n-a.e. on $\mathbb{R}^n$

Let $\varphi \in C_C^\infty(\mathbb{R}^n, \mathbb{R})$ be arbitrary. Using the classical change of variables theorem for the Lebesgue integral, we have:

$$\int_{\mathbb{R}^n} f(x)\varphi(x - tv) d\lambda^n(x) = \int_{\mathbb{R}^n} f(x + tv)\varphi(x) d\lambda^n(x).$$

Then:

$$\int_{\mathbb{R}^n} \frac{f(x + tv) - f(x)}{t} \varphi(x) d\lambda^n(x) = \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(x - tv)}{t} f(x) d\lambda^n(x)$$
$$\Rightarrow \int_{\mathbb{R}^n} \frac{f(x + tv) - f(x)}{t} \varphi(x) d\lambda^n(x) = -\int_{\mathbb{R}^n} \frac{\varphi(x - tv) - \varphi(x)}{t} f(x) d\lambda^n(x).$$

Taking $t \to 0$ and using the Dominated Convergence Theorem, we obtain:

$$\int_{\mathbb{R}^n} D_v f(x)\varphi(x) d\lambda^n(x) = -\int_{\mathbb{R}^n} D_{-v}\varphi(x) d\lambda^n(x). \tag{2.3}$$

Now notice that $D_{-v}\varphi$ corresponds precisely to the directional derivative of φ in the direction of $-v$. Using the fact that $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ we can deduce that

$$D_{-v}\varphi(x) = \langle \nabla\varphi(x), -v \rangle = -\langle \nabla\varphi(x), v \rangle = -D_v\varphi(x), \quad \forall x \in \mathbb{R}^n.$$

Thus, from this and (2.3)

$$\int_{\mathbb{R}^n} D_v f(x)\varphi(x) d\lambda^n(x) = \int_{\mathbb{R}^n} D_v\varphi(x) d\lambda^n(x). \tag{2.4}$$

In particular, if $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} \frac{\partial f(x)}{\partial x_i} \varphi(x) d\lambda^n(x) = \int_{\mathbb{R}^n} \frac{\partial \varphi(x)}{\partial x_i} f(x) d\lambda^n(x).$$

Now, again using the fact that $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$, we obtain

$$D_v\varphi(x) = \langle \nabla\varphi(x), v \rangle = \sum_{i=1}^n v_i \frac{\partial \varphi(x)}{\partial x_i}, \quad \forall x \in \mathbb{R}^n.$$

Then, from this and (2.4)

$$\int_{\mathbb{R}^n} D_v f(x)\varphi(x) d\lambda^n(x) = \int_{\mathbb{R}^n} \sum_{i=1}^n v_i \frac{\partial \varphi(x)}{\partial x_i} f(x) d\lambda^n(x)$$
$$= \sum_{i=1}^n v_i \int_{\mathbb{R}^n} \frac{\partial \varphi(x)}{\partial x_i} f(x) d\lambda^n(x) = \sum_{i=1}^n v_i \int_{\mathbb{R}^n} \frac{\partial f(x)}{\partial x_i} \varphi(x) d\lambda^n(x)$$
$$= \int_{\mathbb{R}^n} \sum_{i=1}^n v_i \frac{\partial f(x)}{\partial x_i} \varphi(x) d\lambda^n(x) = \int_{\mathbb{R}^n} \langle v, \nabla f(x) \rangle \varphi(x) d\lambda^n(x).$$

Thus,

$$\int_{\mathbb{R}^n} D_v f(x)\varphi(x)d\lambda^n(x) = \int_{\mathbb{R}^n} \langle v, \nabla f(x)\rangle \varphi(x)d\lambda^n(x).$$

Since $D_v f$, $\langle v, \nabla f(x)\rangle \in L^1_{loc}(\mathbb{R}^n, \mathbb{R})$ (this follows from Proposition 2.1 and the fact that λ^n is a Radon measure), from (2.1) and Corollary 2.2

$$D_v f(x) = \langle v, \nabla f(x)\rangle \quad \lambda^n - a.e.,$$

as desired.

Step 3) Let $\{v_k\}_{k\in\mathbb{N}}$ be a countable dense subset of $\mathbb{S}^{n-1}$ and define:

$$A_k := \{x \in \mathbb{R}^n : D_{v_k} f(x) \text{ exist and } D_{v_k} f(x) = \langle v_k, \nabla f(x)\rangle\}, \quad \forall k \in \mathbb{N}.$$

Put $A := \bigcap_{k\in\mathbb{N}} A_k$, then by step 2 $\lambda^n(\mathbb{R}^n \setminus A_k) = 0$, $\forall k \in \mathbb{N}$, thus

$$\begin{aligned} \lambda^n(\mathbb{R}^n \setminus A) &= \lambda^n\left(\bigcup_{k\in\mathbb{N}} (\mathbb{R}^n \setminus A_k)\right) \\ &\le \sum_{k=1}^{\infty} \lambda^n(\mathbb{R}^n \setminus A_k) \\ &= 0, \end{aligned}$$

hence

$$\lambda^n(\mathbb{R}^n \setminus A) = 0.$$

We claim that f is differentiable on A. Indeed, let $x \in A$ fixed, for $v \in \mathbb{S}^{n-1}$ and $t \in \mathbb{R}$ with $t \neq 0$ define

$$Q(x, v, t) = \frac{f(x + tv) - f(x)}{t} - \langle v, \nabla f(x)\rangle.$$

Given $u \in \mathbb{S}^{n-1}$ with $u \neq v$ note that

$$\begin{aligned} |Q(x, v, t) - Q(x, u, t)| &= \left| \frac{f(x + tv) - f(x)}{t} - \langle v, \nabla f(x)\rangle - \frac{f(x + tu) - f(x)}{t} + \langle u, \nabla f(x)\rangle \right| \\ &\le \left| \frac{f(x + tv) - f(x + tu)}{t} \right| + |\langle v - u, \nabla f(x)\rangle|. \end{aligned}$$

Using the Lipschitz condition and the Cauchy Schwarz inequality, we obtain:

$$|Q(x, v, t) - Q(x, u, t)| \le \text{Lip}(f)||v - u|| + ||\nabla f(x)||||v - u||. \tag{2.5}$$

Moreover, since we are assuming that $x \in A$, we can use Proposition 2.1, hence

$$||\nabla f(x)|| \le \sum_{i=1}^{n} \left| \frac{\partial f(x)}{\partial x_i} \right| \le n\text{Lip}(f),$$

and replacing into (2.5)

$$|Q(x, v, t) - Q(x, u, t)| \le (1 + n)\text{Lip}(f)||v - u||. \tag{2.6}$$

Given $\varepsilon > 0$ arbitrary, from the compactness of $\mathbb{S}^{n-1}$ and the density of $\{v_k\}_{k\in\mathbb{N}}$ there exist $v_1, \dots v_N$ such that

$$\mathbb{S}^{n-1} \subset \bigcup_{i=1}^{N} B(v_i, \varepsilon).$$

Thus, for $v \in \mathbb{S}^{n-1}$, there exist v_j such that

$$||v - v_j|| < \frac{\varepsilon}{2(1+n)\mathrm{Lip}(f)},$$

taking $u = v_j$ in (2.6) and combining it with the latter

$$|Q(x, v, t) - Q(x, v_j, t)| < \frac{\varepsilon}{2}.$$

Also, due to how we chose x

$$\lim_{t\to 0} Q(x, v_j, t) = 0,$$

hence for ε there exist $\delta > 0$ such that $|t| < 0$ implies $|Q(x, v_j, t)| \leq \frac{\varepsilon}{2}$. Combining all of the above

$$|Q(x, v, t)| \leq |Q(x, v, t) - Q(x, v_j, t)| + |Q(x, v_j, t)| < \varepsilon. \tag{2.7}$$

Finally taking ε and δ as above, if $y \in \mathbb{R}^n$ satisfies $y \neq x$ and $||x - y|| < \delta$ puting $v = \frac{x-y}{||x-y||}$ and $t = ||x - y|| < \delta$, and replacing this in (2.7), we get

$$\frac{|f(y) - f(x) - \langle y - x, \nabla f(x)\rangle|}{||y - x||} < \varepsilon,$$

and thus

$$\lim_{t\to 0} \frac{|f(y) - f(x) - \langle y - x, \nabla f(x)\rangle|}{||y - x||} = 0.$$

Therefore f is differentiable on A (and hence $\lambda^n - a.e.$ on $\mathbb{R}^n$) with $d_x f = \langle \cdot, \nabla f(x)\rangle$. ■

Corollary 2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an $L-$Lipschitz function. Then f is differentiable a.e. with respect to the Lebesgue measure on $\mathbb{R}^n$.*

Proof. Since f is $L-$Lipschitz, every coordinate function of f is $L-$Lipschitz, thus by Rademacher's Theorem every coordinate function is differentiable λ^n-a.e. on $\mathbb{R}^n$ and this implies the claim. ■

The previous proof of Rademacher's Theorem represents the most common approach, as this theorem can also be proven using other tools such as using the notion of weak gradient of a function and its properties, for an approach like this, we refer to [11]. It is worth mentioning that with this approach the Lemma 2.2 and its corollary play a more significant role.

Chapter 3

Area formula

3.1 Area formula on Euclidean Spaces

Given $E \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ $(1 \leq n \leq m)$ a Lipschitz function, Theorem 1.5 and the properties of Hausdorff measure imply that $f(E)$ is at most $n-$dimensional on $\mathbb{R}^m$, so it makes sense to ask about the measure $\mathcal{H}^n$ of $f(E)$. In this chapter we will study the Area formula, which gives us a way to express $\mathcal{H}^n(f(E))$ in terms of an integral over E, whenever E is a Lebesgue measurable set and f is a Lipschitz function.
The approach we will take for the above will be in a similar way to the classical proof of the change of variables theorem. First, we will prove it for the special case when f is a linear map, then when f is injective, and finally, we will provide the proof of the general case. We will now give some definitions and lemmas which will be useful in the following sections of this chapter. In the following we will denote by $(d_x f)^*$ to the adjoint of $d_x f$ (see Apendix B).

Definition 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function, with $1 \leq n \leq m$. We define the **Jacobian of** f as the function $Jf : \mathbb{R}^n \to [0, \infty]$ given by*

$$Jf(x) = \begin{cases} [det((d_x f)^* \circ d_x f)]^{1/2}, & \text{if } f \text{ is differentiable at } x. \\ \infty, & \text{if } f \text{ is not differentiable at } x. \end{cases}$$

Note that Jf is a Borelian function (it is defined in terms of Borelian functions) and furthermore we can notice that the set $\{x \in \mathbb{R}^n : Jf(x) < \infty\}$, coincides precisely with the set of points where f is differentiable, which by Rademacher's Theorem has full Lebesgue measure on $\mathbb{R}^n$.

The following lemma establishes a condition for a set to be $\mathcal{H}^n-$measurable on $\mathbb{R}^m$, this is important as firstly we would like to compute $\mathcal{H}^n(f(E))$ as integration of its characteristic function with respect to $\mathcal{H}^n$. This will enable us to develop some results.

Lemma 3.1. *If E is a Lebesgue measurable set on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ $(1 \leq n \leq m)$ is a Lipschitz function, then $f(E)$ is $\mathcal{H}^n-$measurable on $\mathbb{R}^m$.*

Proof. First, suppose that E is bounded, so $\lambda^n(E) < \infty$. Given $\varepsilon > 0$, since E is Lebesgue measurable, we can find a sequence of compact sets $\{K_i\}_{i\in\mathbb{N}}$ such that $K_i \subset E$ and

$$\lambda^n(E \setminus K_i) < \frac{\varepsilon}{2^i},$$

for each $i \in \mathbb{N}$. Using the continuity of f we obtain that $f(K_i)$ is compact for each i, this implies that $\bigcup_{i\in\mathbb{N}} f(K_i)$ is a Borel set on $\mathbb{R}^m$, and hence $\mathcal{H}^n-$mesurable on $\mathbb{R}^m$ because $\mathcal{H}^n$ is a Borel measure.
Putting $N = f(E) \setminus \bigcup_{i\in\mathbb{N}} f(K_i)$ notice that

$$\begin{aligned}
\mathcal{H}^n(N) &= \mathcal{H}^n\left(f(E) \setminus \bigcup_{i\in\mathbb{N}} f(K_i)\right) \\
&\leq \mathcal{H}^n\left(f\left(E \setminus \bigcup_{i\in\mathbb{N}} K_i\right)\right) \\
&\leq (\mathrm{Lip}(f))^n \mathcal{H}^n\left(E \setminus \bigcup_{i\in\mathbb{N}} K_i\right) \\
&= (\mathrm{Lip}(f))^n \lambda^n\left(E \setminus \bigcup_{i\in\mathbb{N}} K_i\right) \\
&= (\mathrm{Lip}(f))^n \lambda^n\left(\bigcup_{i\in\mathbb{N}} (E \setminus K_i)\right) \\
&\leq (\mathrm{Lip}(f))^n \sum_{i\in\mathbb{N}} \lambda^n(E \setminus K_i) \\
&< (\mathrm{Lip}(f))^n \sum_{i\in\mathbb{N}} \frac{\varepsilon}{2^i} \\
&= \varepsilon(\mathrm{Lip}(f))^n.
\end{aligned}$$

Taking $\varepsilon \to 0$, we obtain $\mathcal{H}^n(N) = 0$, and this implies that N is $\mathcal{H}^n-$measurable on $\mathbb{R}^m$. Since

$$f(E) = N \cup \left(\bigcup_{i\in\mathbb{N}} f(K_i)\right),$$

we obtain the claim. In the case where E is unbounded, as $\mathbb{R}^n = \cup_{i\in\mathbb{N}} B(0,i)$, we only need to set $C_i = E \cap B(0,i)$, to obtain $E = \bigcup_{i\in\mathbb{N}} C_i$. Since it has already been proven that $f(C_i)$ is $\mathcal{H}^n-$mesurable for each $i \in \mathbb{N}$, then as $f(E) = \bigcup_{i\in\mathbb{N}} f(C_i)$ we can conclude. ■

For the proof of the following theorem, we introduce the notation:
1.- Given $k \in \mathbb{N}$ y $r > 0$, B_r^k denotes the open ball of radius r centered at the origin of $\mathbb{R}^k$, i.e.,

$$B_r^k = \{x \in \mathbb{R}^k : ||x|| < r\}.$$

2.- Given $F \subset \mathbb{R}^m$ and $\varepsilon > 0$, $I_\varepsilon(F)$ denotes the epsilon neighborhood of F on $\mathbb{R}^m$, i.e.,

$$I_\varepsilon(F) = \{x \in \mathbb{R}^m : \mathrm{dist}(x, F) < \varepsilon\}.$$

3.- D_s denotes a $k-$dimensional disk of radius $s > 0$ on $\mathbb{R}^m$, i.e.,

$$D_s = \{(z, y) \in \mathbb{R}^k \times \mathbb{R}^{m-k} : ||z|| < s \text{ and } y = 0\}.$$

Theorem 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ $(1 \leq n \leq m)$ be a Lipschitz function, if $E = \{x \in \mathbb{R}^n : Jf(x) = 0\}$. Then*

$$\mathcal{H}^n(f(E)) = 0.$$

Proof. We divide the proof into three steps:
Step 1) We first prove that, if $1 \leq k \leq n-1$, then

$$\mathcal{H}^n_\infty(I_\delta D_s)) \leq C(n,s)\delta, \quad \forall \delta \in (0,1). \tag{3.1}$$

For this purpose, let

$$K = \{(z,y) \in \mathbb{R}^k \times \mathbb{R}^{m-k} : ||z|| < \delta s,\, ||y|| < \delta\} = B^k_{\delta s} \times B^{m-k}_\delta.$$

In order to prove (3.1) we prove that $I_\delta(D_s)$ can be covered by at most $C(n,s)\delta^{-k}$ sets of the form K, then first notice that

$$I_\delta(D_s) \subset \{(z,y) \in \mathbb{R}^k \times \mathbb{R}^{m-k} : ||z|| < s+\delta,\, ||y|| < \delta\} = B^k_{s+\delta} \times B^{m-k}_\delta,$$

and by Fubini's Theorem,

$$\begin{aligned} \mathcal{H}^m(K) &= \mathcal{H}^m(B^k_{\delta s} \times B^{m-k}_\delta) \\ &= \mathcal{H}^k(B^k_{\delta s})\mathcal{H}^{m-k}(B^{m-k}_\delta) \\ &= \omega_k \omega_{m-k} s^k \delta^m. \end{aligned} \tag{3.2}$$

Thus

$$I_\delta(D_s) \subset \{(z,y) \in \mathbb{R}^k \times \mathbb{R}^{m-k} : ||z|| < s+\delta,\, ||y|| < \delta\} = B^k_{s+\delta} \times B^{m-k}_\delta,$$

and

$$\mathcal{H}^m(I_\delta(D_s)) \leq \omega_k \omega_{m-k}(s+\delta)^k \delta^{m-k}. \tag{3.3}$$

Now, let $\{K_i\}_{i=1}^N$ be a maximal disjoint collection of sets obtained by translations of the set K such that $K_i \subset I_\delta(D_s)$, $i = 1, \ldots, N$ (We can assume this because by how K is defined, we have $K \subset I_\delta(D_s)$). Since the F_i are translations of K, then $\mathcal{H}^m(K) = \mathcal{H}^m(F_i)$, for $i = 1, \ldots, N$. Using (3.2), (3.3) and the previous considerations, we make the following Bishop-Gromov type estimate:

$$\begin{aligned} N\mathcal{H}^m(K) &= \sum_{i=1}^N \mathcal{H}^m(F_i) \\ &= \mathcal{H}^m\left(\bigcup_{i=1}^N F_i\right) \\ &\leq \mathcal{H}^m(I_\delta(D_s)) \\ &\leq \mathcal{H}^m(B^k_{s+\delta} \times B^{m-k}_\delta). \end{aligned}$$

Hence

$$N \leq \frac{\mathcal{H}^m(B^k_{s+\delta} \times B^{m-k}_\delta)}{\mathcal{H}^m(K)} = \frac{(s+\delta)^k}{s^k\delta^k},$$

since $\delta \in (0,1)$

$$N \leq \left(1 + \frac{1}{s}\right)^k \delta^{-k}.$$

Using that $1 + \frac{1}{s} \geq 1$ and $k \leq n-1$

$$N \leq \left(1 + \frac{1}{s}\right)^n \delta^{-k}.$$

Now, we can find a family $\mathcal{F}$ of disjoint sets obtained by translations of K that cover $I_\delta(D_s)$. From the previous estimate, we have an upper bound for its cardinality. Also, for $F \in \mathcal{F}$, note that

$$\text{diam}(F)^2 = \text{diam}(K)^2 = \text{diam}(B^k_{\delta s})^2 + \text{diam}(B^{m-k}_\delta)^2 = 4\delta^2(1+s).$$

Thus

$$\begin{aligned}\mathcal{H}^n_\infty(I_\delta(D_s) &\leq \frac{\omega_n}{2^n}\sum_{F\in\mathcal{F}}(\text{diam}(F))^n \\ &\leq \frac{\omega_n}{2^n}\left(1+\frac{1}{s}\right)^n \delta^{-k}(1+s)^{n/2}2^n\delta^n \\ &= C(n,s)\delta.\end{aligned}$$

From which (3.1) follows.

Step 2) Now, let $x \in E$, then $V_x := d_xf(\mathbb{R}^n)$ is a vector subspace of $\mathbb{R}^m$ with

$$k = \dim(V_x) \leq n-1 < m.$$

We distinguish two cases:

Case 1) If $k \geq 1$, by linearity of d_xf and Proposition 2.1 it follows that $d_xf(B^n_r)$ is contained in a $k-$dimensional disk of radius $\text{Lip}(f)r$ on $\mathbb{R}^m$, i.e.,

$$d_xf(B^n_r) \subset B^m_{\text{Lip}(f)r} \cap V_x, \quad \forall r > 0.$$

Thus (3.1), implies that for each $\varepsilon \in]0,1[$ and $r > 0$

$$\begin{aligned}\mathcal{H}^n_\infty(I_{\varepsilon r}(d_xf(B^n_r))) &\leq \mathcal{H}^n_\infty(I_{\varepsilon r}(B^m_{\text{Lip}(f)r} \cap V_x)) \\ &= r^n\mathcal{H}^n_\infty(I_\varepsilon(B^m_{\text{Lip}(f)} \cap V_x)) \leq C(n,\text{Lip}(f))r^n\varepsilon.\end{aligned} \tag{3.4}$$

Case 2) If $k = 0$, then $V_x = \{0\}$, thus for each $\varepsilon \in]0,1[$ and $r > 0$

$$\begin{aligned}\mathcal{H}^n_\infty(I_{\varepsilon r}(V_x)) &= \mathcal{H}^n_\infty(I_{\varepsilon r}(\{0\})) \\ &= \mathcal{H}^n_\infty(B^m_{\varepsilon r}) = \omega_n\varepsilon^n r^n \leq \omega_n\varepsilon r^n.\end{aligned} \tag{3.5}$$

Step 3) Let $x \in E$ and $\varepsilon \in \;]0,1[$, since f is differentiable at x, there exist $r(\varepsilon,x) \in (0,1)$ such that

$$||f(x+v) - f(x) - d_xf(v)|| \leq \varepsilon||v||,$$

whenever $||v|| < r(\varepsilon,x)$. In particular, for each $r < r(\varepsilon,x)$

$$f(B^n(x,r)) \subset f(x) + I_{\varepsilon r}(d_xf(B^n_r)).$$

Combining this with (3.4),(3.5), and the properties of $\mathcal{H}^n_\infty$ we have

$$\mathcal{H}^n_\infty(f(B^n(x,r)) \leq C(n,\text{Lip}(f))\varepsilon r^n. \tag{3.6}$$

Now, given $R > 0$ define

$$\mathcal{F} := \{B^n(x,r) : x \in E \cap B^n_R \quad \text{and} \quad 0 < r < r(\varepsilon,x)\}.$$

From the construction of $\mathcal{F}$ we can notice that the family of its centers is bounded, therefore by Besicovitch's Covering Theorem (A.2) there exist a collection of subfamilies $\{\mathcal{F}_i\}_{i=1}^{\xi(n)}$ such that $E \cap B^n_R \subset \bigcup_{i=1}^{\xi(n)} \mathcal{F}_i$, where each $\mathcal{F}_i$ is countable and disjoint. Thus from (3.6) (putting $C := C(\varepsilon, x)$ for simplicity)

$$\begin{aligned}
\mathcal{H}^n_\infty(f(E \cap B^n_R)) &\leq \sum_{i=1}^{\xi(n)} \sum_{B^n(x,r)\in\mathcal{F}_i} \mathcal{H}^n_\infty(f(B^n(x,r))) \\
&\leq C\varepsilon \sum_{i=1}^{\xi(n)} \sum_{B^n(x,r)\in\mathcal{F}_i} r^n \\
&= \frac{C\varepsilon}{\omega_n} \sum_{i=1}^{\xi(n)} \sum_{B^n(x,r)\in\mathcal{F}_i} \lambda^n(B(x,r)) \\
&= \frac{C\varepsilon}{\omega_n} \sum_{i=1}^{\xi(n)} \lambda^n \left(\bigcup_{B(x,r)\in\mathcal{F}_i} B(x,r) \right) \\
&\leq \frac{C\varepsilon\xi(n)}{\omega_n} \lambda^n(I_1(E \cap B^n_R)).
\end{aligned}$$

In the last inequality, we used the fact that $r(\varepsilon, x) \in]0, 1[$. Thus, we obtain

$$\mathcal{H}^n_\infty(f(E \cap B^n_R)) \leq \frac{C\varepsilon\xi(n)}{\omega_n} \lambda^n(I_1(E \cap B^n_R)),$$

taking $\varepsilon \to 0$ we find that $\mathcal{H}^n_\infty(f(E \cap B^n_R)) = 0$, and using Proposition 1.12 we get that $\mathcal{H}^n(f(E \cap B^n_R)) = 0$ from which taking $R \to \infty$ we can conclude. ■

As a comment, we can note that the above theorem is in a way a "Sard-type" theorem for Lipschitz functions, since it tells us that the measure of the set of critical values of f is an $\mathcal{H}^n-$null set.

3.1.1 Area formula for linear maps

Next we will prove the particular case of the Area formula when the function in question is linear.
In the following we will denote by dim to the dimension in the sense of a vector space.

Theorem 3.2. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$, $(1 \leq n \leq m)$ be a linear map. Then, for every $A \subset \mathbb{R}^n$*

$$\mathcal{H}^n(L(A)) = JL\lambda^*_n(A). \tag{3.7}$$

In particular, if A is Lebesgue measurable

$$\mathcal{H}^n(L(A)) = JL\lambda^n(A).$$

Proof. From Polar Decomposition Theorem B.1, we can write $L = O \circ S$, where O is an orthogonal linear map and S is a symmetric linear map. Then $JL = |\det(S)|$, We can distinguish two cases:
Case 1: If $JL = 0$, then $\det(S) = 0$. Thus S cannot be injective, by the rank nullity theorem, we obtain that $\text{Ker}(S) \neq \{0\}$, which implies that $\dim(\text{Ker}(S)) \geq 1$ and hence

$\dim(\text{Im}(S)) \leq n-1$, consequently $dim(L(\mathbb{R}^n)) \leq n-1$. Therefore $\mathcal{H}^n(L(\mathbb{R}^n)) = 0$, concluding this case.
Case 2: If $JL > 0$, note that given $x \in \mathbb{R}^n$ and $r > 0$

$$\begin{aligned}\frac{\mathcal{H}^n(L(B(x,r))}{\lambda^n(B(x,r))} &= \frac{\lambda^n(O^* \circ L(B(x,r)))}{\lambda^n(B(x,r))} = \frac{\lambda^n(S(B(x,r)))}{\lambda^n(B(x,r))} \\ &= \frac{\lambda^n(S(x)+S(B(0,r)))}{\lambda^n(B(x,r))} = \frac{\lambda^n(S(B(0,r))}{\lambda^n(B(x,r))} = \frac{\lambda^n(S(B(0,1)))}{\omega_n}. \end{aligned} \tag{3.8}$$

Using the Change of Variables Theorem for the Lebesgue integral

$$\frac{\lambda^n(S(B(0,1)))}{\omega_n} = |detS| = JL. \tag{3.9}$$

Substituting (3.9) into (3.8)

$$\frac{\mathcal{H}^n(L(B(x,r)))}{\lambda^n(B(x,r))} = JL. \tag{3.10}$$

Now, define $\nu(A) = \mathcal{H}^n(L(A))$ for each $A \subset \mathbb{R}^n$. It is easy to see that ν is an outer measure on $\mathbb{R}^n$, and furthermore, it is a Radon measure. To see the latter, first note that for any $K \subset \mathbb{R}^n$

$$\nu(K) = \mathcal{H}^n(L(K)) \leq \text{Lip}(L)\mathcal{H}^n(K) = \text{Lip}(L)\lambda^n(K) < \infty,$$

which implies that ν is locally finite, to see that ν is Borel regular, let $A \subset \mathbb{R}^n$, then $L(A) \subset L(\mathbb{R}^n)$ and since $\mathcal{H}^n$ is Borel regular, there exists a Borel set $C \subset \mathbb{R}^m$ such that $L(A) \subset C$, and

$$\mathcal{H}^n(L(A)) = \mathcal{H}^n(C). \tag{3.11}$$

Using the same argument for A on $\mathbb{R}^n$, there exists a Borel set $B \subset \mathbb{R}^n$ such that $A \subset B$ and

$$\mathcal{H}^n(A) = \mathcal{H}^n(B),$$

furthermore, we have that $L(A) \subset L(B) \subset L(\mathbb{R}^n)$. Define $D := C \cap L(B) \subset L(B) \subset L(\mathbb{R}^n)$ and note that D is a Borel set, because it is the intersection of two Borel sets ($L(B)$ is Borel since it is the image of B under an injective and continuous function). Also, since we are assuming that L is injective and $D \subset L(\mathbb{R}^n)$, there exists $E \subset \mathbb{R}^n$ such that $L(E) = D$ or equivalently $E = L^{-1}(D)$. Again, using a previously used argument, we conclude that E is a Borel set, thus we have:

$$C \cap L(B) \leq C \quad \text{and} \quad L(A) \subset C \cap L(B).$$

Thus, from this and (3.11)

$$\begin{aligned}\mathcal{H}^n(C \cap L(B)) &\leq \mathcal{H}^n(C) \\ &= \mathcal{H}^n(L(A)) \leq \mathcal{H}^n(C \cap L(B)).\end{aligned}$$

This implies that

$$\nu(A) = \nu(E),$$

and therefore, ν is Borel regular. Also, as we had already proven that ν is locally finite, we will have that ν is Radon measure on $\mathbb{R}^n$.
We claim that $\nu \ll \lambda^n$, indeed, if $A \subset \mathbb{R}^n$ is such that $\lambda^n(A) = 0$, then

$$\mathcal{H}^n(L(A)) \leq (\text{Lip}(L))^n\mathcal{H}^n(A) = (\text{Lip}(L))^n\lambda^n(A) = 0 \Rightarrow \nu(A) = 0,$$

hence $\nu \ll \lambda^n$.
Now, note that from (3.10)

$$D_{\lambda^n}\nu(x) = \lim_{r\to 0}\frac{\nu(B(x,r))}{\lambda^n(B(x,r))} = \lim_{r\to 0}\frac{\mathcal{H}^n(L(B(x,r)))}{\lambda^n(B(x,r))} = JL.$$

Then, by Radon-Nikodym Theorem, for any Borel set $B \subset \mathbb{R}^n$

$$\nu(B) = \mathcal{H}^n(L(B)) = \int_B D_{\lambda^n}\nu(x)d\lambda^n(x) = \int_B JLd\lambda^n(x) = JL\lambda^n(B). \tag{3.12}$$

So, we have obtained (3.7) for all Borel sets. Now, we need to prove it for any arbitrary set A on $\mathbb{R}^n$. To prove this, since ν is a Borel measure, there exist a Borel set B_1 such that $A \subset B_1$ and $\nu(A) = \nu(B_1)$. Also, since λ^n is a Borel measure, there exist a Borel set B_2 such that $A \subset B_2$ and $\lambda^n(A) = \lambda^n(B_2)$. Putting $B := B_1 \cap B_2$, note that $A \subset B \subset B_1$ and $A \subset B \subset B_2$, thus

$$\nu(A) \leq \nu(B) \leq \nu(B_1) = \nu(A) \quad \Rightarrow \quad \nu(A) = \nu(B) \tag{3.13}$$

and

$$\lambda^n(A) \leq \lambda^n(B) \leq \lambda^n(B_2) = \lambda^n(A) \quad \Rightarrow \quad \lambda^n(A) = \lambda^n(B). \tag{3.14}$$

By (3.12), (3.13) and (3.14)

$$\nu(A) = \nu(B) = JL\lambda^n(B) = JL\lambda^n(A),$$

which implies (3.7). ■

Example 3.1. *Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ given by*

$$L(x,y) = (y, x+2y, x+y).$$

Then

$$L^*(x,y,z) = (y+z, x+2y+z).$$

Since L and L^* are linear maps, they are differentiable at $\mathbb{R}^n$ with $d_xL = L$ and $(d_xL)^* = L^* = d_xL^*$, thus

$$(d_xL)^* \circ d_xL = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix},$$

hence

$$JL = [det((d_xL)^* \circ d_xL)]^{1/2} = \sqrt{3}.$$

Thus, for example, if we consider the triangle E with vertices at $(0,0)$, $(0,\sqrt{3})$ and $(1,0)$, then

$$\mathcal{H}^2(L(E)) = JL\lambda^2(E) = \sqrt{3}\frac{\sqrt{3}}{2} = \frac{3}{2}.$$

◀

Remark 3.1. *In what follows, $GL(k)$ will denote the set of invertible linear maps of $\mathbb{R}^k$ into $\mathbb{R}^k$ and $\mathcal{N}(\cdot)$ denotes the operator norm on $GL(k)$.*
1.- Let $L \in GL(n)$, then by linearity of L, both L and L^{-1} are differentiable at $\mathbb{R}^n$. Therefore, by the chain rule

$$(JL)(JL^{-1}) = J(L \circ L^{-1}) = J(I) = 1,$$

which implies that $JL > 0$ *and* $(JL)^{-1} = JL^{-1}$.

2.- We claim that

$$\mathcal{N}(L^{-1})^{-n} \leq JL \leq \mathcal{N}(L)^n, \quad \forall L \in GL(n). \tag{3.15}$$

Indeed, by Theorem 1.5 and Proposition 1.17, given $B \subset \mathbb{R}^n$ *with* $0 < \mathcal{H}^n(B) < \infty$, *note that*

$$\mathcal{H}^n(L(B)) \leq \mathcal{N}(L)^n \mathcal{H}^n(B),$$

and

$$\mathcal{H}^n(B) = \mathcal{H}^n(L^{-1} \circ L(B)) \leq \mathcal{N}(L^{-1})^n \mathcal{H}^n(B),$$

using Theorem 3.2

$$\mathcal{H}^n(B)\mathcal{N}(L^{-1})^{-n} \leq JL\mathcal{H}^n(B) \leq \mathcal{N}(L)^n \mathcal{H}^n(B),$$

so (3.15) follows from here.

3.- A similar argument to the one used in the previous point item that if $n \leq m$, $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ *is linear transformation and* $L_2 \in GL(m)$, *then*

$$\mathcal{N}(L_2^{-1})^{-m} JL_1 \leq J(L_2 \circ L_1) \leq \mathcal{N}(L_2)^n JL_1.$$

4.- Let $L, S \in GL(n)$ *such that* $\mathcal{N}(L - S) \leq \delta$, *then*

$$\mathcal{N}(L \circ S^{-1}) \leq 1 + \delta\mathcal{N}(S^{-1}) \quad \text{and} \quad \mathcal{N}(S \circ L^{-1}) \leq 1 + \delta\mathcal{N}(L^{-1}).$$

$\triangleleft$

We will now prove an important theorem regarding Lipschitz functions (under some conditions). This theorem will be utilized initially in the proof of Theorem 3.4 and will also play a crucial role in the theory of Rectifiable sets. A fundamental technique from elementary calculus involves extending properties of linear maps to C^1 functions by leveraging the continuity of gradients to deduce they are locally almost constant, unfortunately this approach is not applicable to Lipschitz functions, however a brilliant result introduced by Federer allows us to reformulate approximation via linear maps in this framework too.

Theorem 3.3. *(**Lipschitz linearization**) Let* $f : \mathbb{R}^n \to \mathbb{R}^m$, $(1 \leq n \leq m)$ *be a Lipschitz function and*

$$F := \{x \in \mathbb{R}^n : 0 < Jf(x) < \infty\}.$$

Then, there exist a partition of F, $\{F_h\}_{h\in\mathbb{N}}$ *into Borel sets such that:*
1.- $f|_{F_h}$ *is injective.*
2.- $\forall t > 1$, $\forall h \in \mathbb{N}$ *there exist* $S_h \in GL(n)$ *such that:*
(i)

$$t^{-1}||S_h(x) - S_h(y)|| \leq ||f(x) - f(y)|| \leq t||S_h(x) - S_h(y)||, \quad \forall x, y \in F_h.$$

Thus

$$\text{Lip}\left(f \circ S^{-1}_{h|_{S_h(F_h)}}\right) \leq t,$$

and

$$\text{Lip}(S_h \circ (f|_{F_h})^{-1}) \leq t.$$

(ii)

$$t^{-1}||S_h(v)|| \le ||d_x f(v)|| \le t||S_h(v)||, \quad \forall x \in F_h, v \in \mathbb{R}^n.$$

(iii)

$$t^{-n} JS_h \le Jf(x) \le t^n JS_h, \quad \forall x \in F_h.$$

Proof. Given $t > 1$, fix $\varepsilon > 0$ such that

$$t^{-1} + \varepsilon < 1 < t - \varepsilon,$$

and let $\mathcal{C}$ be a countable dense set F, $\mathcal{G}$ be a countable dense set of $GL(n)$. For each $c \in \mathcal{C}$, $S \in \mathcal{G}$ and $h \in \mathbb{N}$ we define

$$F(c, S, h)$$

as the set of those $x \in F \cap B(c, 1/h)$ such that

$$(t^{-1} + \varepsilon)||S(v)|| \le ||d_x f(v)|| \le (t - \varepsilon)||S(v)||, \quad \forall v \in \mathbb{R}^n \tag{3.16}$$

and

$$||f(y) - f(x) - d_x f(y - x)|| \le \varepsilon||S(y - x)||, \quad \forall y \in B(x, 2/h). \tag{3.17}$$

Claim 1: All sets $F(c, S, h)$ satisfy property (i).
By definition of $F(c, S, h)$ and the triangle inequality, for $x, y \in F(c, S, h)$ note that $y \in B(x, 2/h)$, thus

$$\begin{aligned} ||f(y) - f(x)|| &\le ||d_x f(y - x)|| + \varepsilon||S(y - x)|| \\ &\le (t - \varepsilon)||S(y - x)|| + \varepsilon||S(y - x)|| = t||S(y - x)||. \end{aligned}$$

Similarly,

$$\begin{aligned} ||f(y) - f(x)|| &\ge ||d_x f(y - x)|| - \varepsilon||S(y - x)|| \\ &\ge (t^{-1} + \varepsilon)||S(y - x)|| + \varepsilon||S(y - x)|| = t^{-1}||S(y - x)||. \end{aligned}$$

And the claim holds.
For condition (ii), by the way we defined the sets $F(c, S, h)$, (ii) is be trivially satisfied in each of them.
Claim 2: All sets $F(c, S, h)$ satisfy property (iii).
Given $x \in F(c, S, h)$, it suffices to prove that:

$$(t^{-1} + \varepsilon)^n JS \le Jf(x) \le (t - \varepsilon)^n JS. \tag{3.18}$$

By (3.16) and taking $v \in S^{-1}(B(0, 1))$, we can deduce that:

$$B(0, t^{-1} + \varepsilon) \subset d_x f \circ S^{-1}(B(0, 1)) \subset B(0, t - \varepsilon)$$

$$\Rightarrow \quad \mathcal{H}^n(B(0, t^{-1} + \varepsilon)) \le \mathcal{H}^n(d_x f \circ S^{-1}(B(0, 1))) \le \mathcal{H}^n(B(0, t - \varepsilon)).$$

From here, using Theorem 3.2, Remark 3.1, and the fact that $J(d_x f \circ S^{-1}) = (Jd_x f)(JS^{-1})$

$$(t^{-1} + \varepsilon)^n \mathcal{H}^n(B(0, 1)) \le J(d_x f) JS^{-1} \mathcal{H}^n(B(0, 1)) \le (t - \varepsilon)^n \mathcal{H}^n(B(0, 1)),$$

which implies (3.18).
Claim 3: $F \subset \cup F(c, S, h)$.
Given $x \in F$, write (using the polar decomposition)

$$d_x f = P_x \circ S_x.$$

Since $Jf(x) > 0$, we have that S_x is invertible, then using Remark 3.1, we can find $S \in \mathcal{G}$ (see Remark 3.2) such that

$$\mathcal{N}(S_x \circ S^{-1}) \leq t - \varepsilon \quad \text{and} \quad \mathcal{N}(S \circ S_x^{-1}) \leq (t^{-1} + \varepsilon)^{-1}.$$

Therefore

$$||S_x(v)|| \leq (t - \varepsilon)||S(v)|| \quad \text{and} \quad ||S(v)|| \leq (t^{-1} + \varepsilon)^{-1}||S_x(v)||, \quad \forall v \in \mathbb{R}^n,$$

since $d_x f = P_x \circ S_x$ and P_x preserves norm, we obtain

$$||d_x f(v)|| = ||S_x \circ P_x(v)|| \leq (t - \varepsilon)||S(v)||, \quad \forall v \in \mathbb{R}^n$$

and

$$||S(v)|| \leq (t^{-1} + \varepsilon)^{-1}||S_x(v)|| = (t^{-1} + \varepsilon)^{-1}||d_x f(v)||, \quad \forall v \in \mathbb{R}^n.$$

So we obtain (3.16), for (3.17), since $S \in GL(n)$, then

$$||x' - y'|| \leq \text{Lip}(S^{-1})||S(x') - S(y')||, \quad \forall x', y' \in \mathbb{R}^n.$$

From the definition of $d_x f$, there exist $\delta(x, \varepsilon, S)$, such that $||y - x|| < \delta$ implies

$$||f(y) - f(x) - d_x f(y - x)|| \leq \frac{\varepsilon}{\text{Lip}(S^{-1})}||y - x|| \leq \varepsilon||S(x) - S(y)||.$$

Choosing $h = h(x, \varepsilon, S) \in \mathbb{N}$ such that $\frac{2}{h} < \delta$,

$$||f(y) - f(x) - d_x f(y - x)|| \leq \varepsilon||S(x) - S(y)||, \quad \forall y \in B(x, 2/h).$$

Finally, taking $c \in C$ such that $||x - c|| < \frac{1}{h}$ we have the claim.
Claim 4: f is injective in each $F(c, S, h)$.
Given $x, y \in F(c, S, h)$ by Claim 1

$$t^{-1}||S(x) - S(y)|| \leq ||f(x) - f(y)|| \leq t||S(x) - S(y)||,$$

this together with injectivity of S implies that

$$x = y \quad \Longleftrightarrow \quad f(x) = f(y)$$

To conclude the proof, using Claim 3 and renumbering $\{F(c, S, h)\}_{c,S,h}$ as $\{\overline{F}_j\}_{j \in \mathbb{N}}$, we can replace each $\overline{F}_j$ by $F_1 := \overline{F}_1$ and $F_j := \overline{F}_j \setminus \left(\bigcup_{k=1}^{j-1} F_k\right)$ for each $j \geq 2$, to obtain the desired partition. This completes the proof. ■

Remark 3.2. *Let ε, t, S_x as in the last theorem, using Remark 3.1 we need find $S \in \mathcal{G}$ and $\delta > 0$ such that*

$$1 + \delta \mathcal{N}(S) \leq t - \varepsilon, \tag{3.19}$$

and

$$1 + \mathcal{N}(S_x^{-1}) = (t^{-1} + \varepsilon)^{-1}. \tag{3.20}$$

By solving (3.20), we find that

$$\delta = \frac{t - 1 - \varepsilon t}{\mathcal{N}(S_x^{-1})(1 + \varepsilon t)} > 0,$$

which implies that

$$\mathcal{N}(S^{-1}) \leq \frac{\mathcal{N}(S_x^{-1})(1 + \varepsilon t)(t - \varepsilon - 1)}{t - 1 - \varepsilon t}.$$

Using the inequality

$$\frac{1}{\mathcal{N}(S^{-1})} \leq \mathcal{N}(S),$$

and combining with the above, we have the following condition for S,

$$\frac{t - 1 - \varepsilon t}{\mathcal{N}(S_x^{-1})(1 + \varepsilon t)(t - \varepsilon - 1)} \leq \mathcal{N}(S). \tag{3.21}$$

Since $GL(n)$ is dense in the set of bounded linear operators of $\mathbb{R}^n$ into $\mathbb{R}^n$, then as $\mathcal{G}$ is dense in $GL(n)$, without loss of generality we can assume that G is dense in the set of bounded linear operators of $\mathbb{R}^n$ into $\mathbb{R}^n$, using this consideration it is easy to see that we can find $S \in \mathcal{G}$ such that it satisfies (3.21), in geometric terms S needs to be out of the open ball with center at 0 and radius $\frac{t-1-\varepsilon t}{\mathcal{N}(S_x^{-1})(1+\varepsilon t)(t-\varepsilon-1)}$. Additionally to this, in order to use Remark 3.1 for concluding, we also need that S satisfies

$$\mathcal{N}(S - S_x) < \delta = \frac{t - 1 - \varepsilon t}{\mathcal{N}(S_x^{-1})(1 + \varepsilon t)}. \tag{3.22}$$

Using the density of $\mathcal{G}$, if

$$\frac{t - 1 - \varepsilon t}{\mathcal{N}(S_x^{-1})(1 + \varepsilon t)(t - \varepsilon - 1)} \leq \mathcal{N}(S_x),$$

then it is possible find $S \in \mathcal{G}$ satisfying (3.21) and (3.22), therefore for this S it holds (3.19) and (3.20). Now if

$$\mathcal{N}(S_x) < \frac{t - 1 - \varepsilon t}{\mathcal{N}(S_x^{-1})(1 + \varepsilon t)(t - \varepsilon - 1)},$$

since

$$\frac{1}{\mathcal{N}(S_x^{-1})} \leq \mathcal{N}(S_x),$$

then

$$\frac{1}{\mathcal{N}(S_x^{-1})} < \frac{t - 1 - \varepsilon t}{\mathcal{N}(S_x^{-1})(1 + \varepsilon t)(t - \varepsilon - 1)},$$

which implies that

$$t < \varepsilon + t^{-1},$$

but this is a contradiction because we assume (in the last theorem) that $t^{-1}+\varepsilon<t$, then neccesarily

$$\frac{t-1-\varepsilon t}{\mathcal{N}(S_x^{-1})(1+\varepsilon t)(t-\varepsilon-1)}\leq \mathcal{N}(S_x),$$

and we can conclude. ◁

3.1.2 Area formula for injective functions

Before stating the main result of this section, we make the following remark, which will be useful later on.

Remark 3.3. *Let $E\subset\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}^m$, $(1\leq n\leq m)$ be a Lipschitz function, if*

$$F:=\{x\in\mathbb{R}^n : 0<Jf(x)<\infty\}=\{0<Jf<\infty\}.$$

Then, from Rademacher's Theorem and Theorem 3.1, we can assume that $E\subset F$. This is because from the aforementioned results, we can deduce that $\mathcal{H}^n(f(F^c))=0$, and as

$$\mathcal{H}^n(f(E))=\mathcal{H}^n(f(E\cap F))+\mathcal{H}^n(f(E\cap F^c)).$$

Then

$$\mathcal{H}^n(f(E))=\mathcal{H}^n(f(E\cap F)).$$

Therefore, we can make the aforementioned consideration. ◁

Theorem 3.4. *(**Area formula for injective functions**) Let $f:\mathbb{R}^n\to\mathbb{R}^m$, $(1\leq n\leq m)$ be a injective Lipschitz function and $E\subset\mathbb{R}^n$ be a Lebesgue measurable set. Then*

$$\mathcal{H}^n(f(E))=\int_E Jf(x)d\lambda^n(x).$$

If additionally f is proper, then $\mathcal{H}^n\llcorner f(\mathbb{R}^n)$ is a Radon measure on $\mathbb{R}^m$.

Proof. By remark 3.3 we can assume that $E\subset F:=\{0<Jf<\infty\}$. Given $t>1$ fixed, and consider the partition $\mathcal{F}=\{F_k\}_{k\in\mathbb{N}}$ of F given by Theorem 3.3, we have that $\mathcal{F}$ induces a partition on E, namely $\{E\cap F_k\}_{k\in\mathbb{N}}$. Thus, from the global injectivity of f, we can express $f(E)$ as a disjoint union of the $\mathcal{H}^n-$measurable sets $\{f(E\cap F_k)\}_{k\in\mathbb{N}}$, combining this with Theorems 1.5, 3.2 and 3.3

$$\begin{aligned}
\mathcal{H}^n(f(E)) &= \sum_{k\in\mathbb{N}}\mathcal{H}^n(f(E\cap F_k))\\
&= \sum_{k\in\mathbb{N}}\mathcal{H}^n((f|_{F_k}\circ S_k^{-1})(S_k(E\cap F_k)))\\
&\leq \sum_{k\in\mathbb{N}}(\mathrm{Lip}(f|_{F_k}\circ S_k^{-1}))^n\lambda^n(S_k(E\cap F_k))\\
&\leq t^n\sum_{k\in\mathbb{N}}JS_k\lambda^n(E\cap F_k)\\
&\leq t^{2n}\sum_{k\in\mathbb{N}}\int_{E\cap F_k}Jf(x)d\lambda^n(x)\\
&= t^{2n}\int_E Jf(x)d\lambda^n(x).
\end{aligned}\tag{3.23}$$

Similarly, we obtain

$$\begin{aligned}\int_E Jf(x)d\lambda^n(x) &= \sum_{k\in\mathbb{N}} \int_{E\cap F_k} Jf(x)d\lambda^n(x) \\ &\le t^n \sum_{k\in\mathbb{N}} JS_k \lambda^n(E\cap F_k) \\ &= t^n \sum_{k\in\mathbb{N}} \lambda^n([S_k \circ (f|_{F_k})^{-1}](f(E\cap F_k))) \\ &\le t^{2n} \sum_{k\in\mathbb{N}} \mathcal{H}^n(f(E\cap F_k)) \\ &= t^{2n}\mathcal{H}^n(f(E)).\end{aligned} \tag{3.24}$$

Thus, taking $t \to 1$ in (3.23) and (3.24), we can deduce the desired equality. To show that $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is a Radon measure, we use Proposition 1.1, because $\mathcal{H}^n$ is a Borel measure on $\mathbb{R}^m$, $f(\mathbb{R}^n)$ is $\mathcal{H}^n-$measurable set and $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is locally finite, to see the latter, given $K \subset \mathbb{R}^m$ compact , since f is proper, then $f^{-1}(K)$ is compact on $\mathbb{R}^n$, hence

$$\begin{aligned}\mathcal{H}^n \llcorner f(\mathbb{R}^n)(K) &= \mathcal{H}^n(f(\mathbb{R}^n)\cap K) \\ &\le \text{Lip}(f)^n \mathcal{H}^n(\mathbb{R}^n \cap f^{-1}(K)) \\ &\le \text{Lip}(f)^n \mathcal{H}^n(f^{-1}(K)) < \infty.\end{aligned}$$

■

Corollary 3.1. ***(Change of Variables)*** *Let f be as in the previous theorem, and $g : \mathbb{R}^m \to [-\infty, \infty]$ be a Borel measurable function such that $g \ge 0$ or $g \in L^1(\mathbb{R}^m, \mathcal{H}^n \llcorner f(\mathbb{R}^n))$. Then $g \circ f$ is Borel measurable on $\mathbb{R}^n$ and*

$$\int_{f(\mathbb{R}^n)} g(x)d\mathcal{H}^n(x) = \int_{\mathbb{R}^n} g(f(x))Jf(x)d\lambda^n(x).$$

Proof. Suppose firstly that g is a non negative simple function, then

$$g = \sum_{h\in\mathbb{N}} c_h \chi_{F_h},$$

with $c_h \ge 0$ and F_h Borel sets on $\mathbb{R}^m$, for each $h \in \mathbb{N}$. If we define $E_h := f^{-1}(F_h)$, then

$$g \circ f = \sum_{h\in\mathbb{N}} c_h \chi_{E_h}.$$

Using Theorem 3.4

$$\begin{aligned}\int_{\mathbb{R}^m} g d\mathcal{H}^n(x) &= \sum_{h\in\mathbb{N}} c_h \mathcal{H}^n(F_h) \\ &= \sum_{h\in\mathbb{N}} c_h \int_{E_h} Jf(x)d\lambda^n(x) \\ &= \int_{\mathbb{R}^n} g(f(x))Jf(x)d\lambda^n(x).\end{aligned}$$

If we now assume that g is a non negative Borel measurable function, from what has been proven above and from the approximation of g by non negative simple functions, we obtain the desired result. Finally, if $g \in L^1(\mathbb{R}^m, \mathcal{H}^n \llcorner f(\mathbb{R}^n))$ it suffices to write $g = g^+ - g^-$. ■

Corollary 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ $(1 \leq n \leq m)$ be an isometry. Then, for every Lebesgue measurable set E on $\mathbb{R}^n$*

$$\mathcal{H}^n(f(E)) = \mathcal{H}^n(E).$$

Proof. Since f is an isometry, it follows that f is injective and Lipschitz and there exist an orthogonal linear map $O : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$f(x) = f(0) + O(x), \quad \forall x \in \mathbb{R}^n.$$

Then, by linearity we can observe that $d_x f = O$, therefore $Jf = 1$. Thus given a Lebesgue measurable set E on $\mathbb{R}^n$, using Theorem 3.4

$$\begin{aligned}\mathcal{H}^n(f(E)) &= \int_E Jf d\lambda^n(x) \\ &= \int_E d\lambda^n(x) \\ &= \lambda^n(E) = \mathcal{H}^n(E).\end{aligned}$$

■

Example 3.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ define by*

$$f((x, y)) = (x, y, |x| - |y|).$$

It is immediately to see that f is an injective function and furthermore, it is Lipschitz (since each component is Lipschitz). Note that if $(x, y) \in \mathbb{R}^2$ is such that $x = 0$ or $y = 0$, then f is not differentiable at (x, y), thus for such points

$$Jf((x, y)) = +\infty. \tag{3.25}$$

Let now $(x, y) \in \mathbb{R}^2$ such that $x, y \neq 0$, then

$$d_{(x,y)} f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{x}{|x|} & -\frac{y}{|y|} \end{bmatrix},$$

hence

$$(d_{(x,y)} f)^* = \begin{bmatrix} 1 & 0 & \frac{x}{|x|} \\ 0 & 1 & -\frac{y}{|y|} \end{bmatrix}.$$

thus

$$(d_{(x,y)} f)^* \circ d_{(x,y)} f = \begin{bmatrix} 2 & -\frac{xy}{|xy|} \\ -\frac{xy}{|xy|} & 2 \end{bmatrix},$$

which implies that

$$Jf((x, y)) = \sqrt{3}. \tag{3.26}$$

Using (3.25) and (3.26), we have

$$Jf((x, y)) = \begin{cases} \sqrt{3} & \text{if } x, y \neq 0. \\ +\infty & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

If we consider, for example $E = \overline{B}(0,2)$, *from Theorem* 3.4

$$\mathcal{H}^2(f(E)) = \int_{\overline{B}(0,2)} Jf(x)d\lambda^2(x,y) = \int_0^{2\pi}\int_0^2 \sqrt{3}rdrd\theta = 4\sqrt{3}\pi,$$

where in the second equality we have used change of variables to polar coordinates. ◀

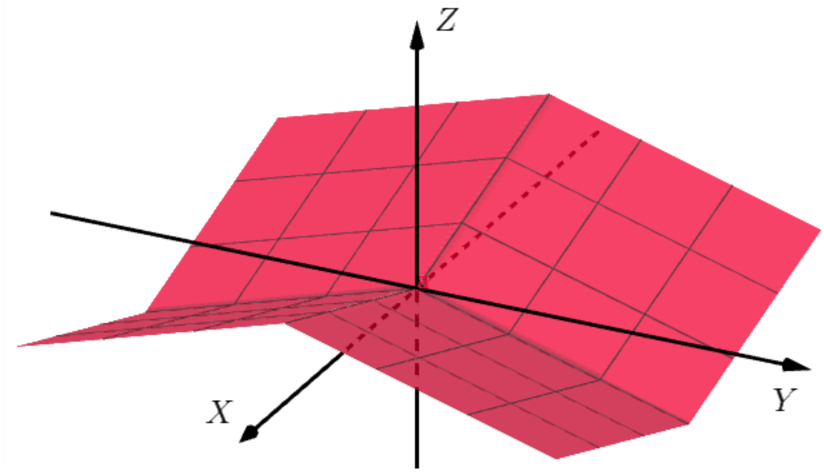

Figure 3.1: Image of f.

In the case when $n = m$, Corollary 3.1 represents a generalization of the Change of Variables integral formula, in the case when $n < m$ we have the next application.

Example 3.3. *Let* $g : \mathbb{R}^3 \to \mathbb{R}$ *given by* $f(u,v,w) = e^{u+v-w}$, *compute*

$$\int_T e^{u+v-w} d\mathcal{H}^2(u,v,w).$$

Where T *is the triangle defined by the vertices* $(-1,0,-1)$, $(1,0,1)$ *and* $(1,1,2)$.
We use Corollary 3.1 *for change this problem of integration with respect of the measure* $\mathcal{H}^2$ *on* $\mathbb{R}^3$, *for a problem of integration on* $\mathbb{R}^2$ *with respect to* λ^2. *Indeed, define* $f : \mathbb{R}^2 \to \mathbb{R}^3$ *given by* $f(x,y) = (x,y,x+y)$, *then* f *is Lispchitz and injective. Furthermore, we can see that* $T = f(C)$, *where* C *is the triangle defined by the vertices* $(-1,0)$, $(1,0)$ *and* $(1,1)$. *By using the Change of Variables integral formula (Corollary* 3.1*), we obtain that*

$$\begin{aligned}\int_{T=f(C)} e^{u+v-w} d\mathcal{H}^2(u,v,w) &= \int_C e^{f(x,y)} Jf(x,y)d\lambda^2(x,y)\\ &= \int_C e^{x+y-x-y}\sqrt{3}d\lambda^2(x,y)\\ &= \int_C \sqrt{3}d\lambda^2(x,y)\\ &= \sqrt{3}\lambda^2(C)\\ &= \sqrt{3},\end{aligned}$$

where we used the fact that $Jf(x,y) = \sqrt{3}$, $\forall (x,y) \in \mathbb{R}^2$. *Thus, we can conclude that*

$$\int_T e^{u+v-w} d\mathcal{H}^2(u,v,w) = \sqrt{3}.$$

◀

3.1.3 Area formula with multiplicities

In the previous section, we proved the Area formula for the particular case when f is injective and Lipschitz. However, imposing the condition of injectivity limits us. In this section, we present the general result of the Area formula in Euclidean spaces, where we only need f to be a Lipschitz function. First, we present the following:

Example 3.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$f((x,y)) = \begin{cases} (x,y) & \text{if } x > 0. \\ (-x,y) & \text{if } x \leq 0. \end{cases}$$

It is straightforward to verify that f is a Lipschitz function and is not injective. Furthermore, we can also observe that $Jf(x) = 1$. So, for example, if T is the triangle with vertices at $(1,0)$, $(-1,0)$, and $(0,1)$, then integrating Jf over T will precisely yield its Lebesgue measure, which is $\lambda^2(E) = 1$. Similarly, we can see that $f(T)$ is the triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. Thus, $\mathcal{H}^2(f(T)) = \frac{1}{2}$. Therefore,

$$\mathcal{H}^2(f(T)) \neq \lambda^2(T) = \int_T Jf(x)d\lambda^2(x).$$

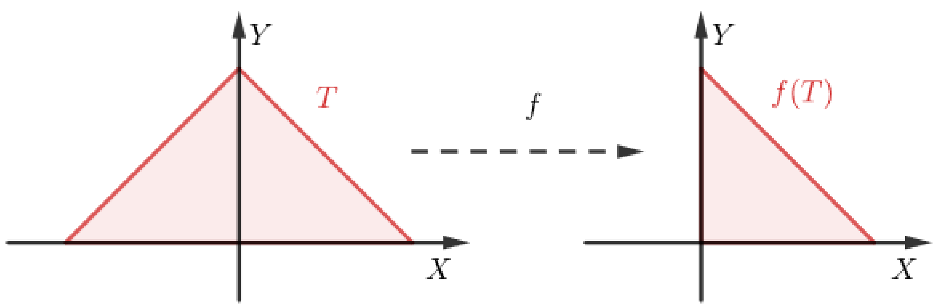

Figure 3.2: "Overlap effect" in the image of f.

◀

Definition 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function and $E \subset \mathbb{R}^n$. We define $m_E^f : \mathbb{R}^m \to \mathbb{N} \cup \{+\infty\}$ as*

$$m_E^f(y) := \mathcal{H}^0(E \cap \{f = y\}), \quad \forall y \in \mathbb{R}^m,$$

where $\{f = y\} = \{x \in \mathbb{R}^n : f(x) = y\}$. We will call m_E^f the ***multiplicity function of f on E****.*

The multiplicity function will play a significant role as it helps us compensate the "overlap effects" in the image of f. We can illustrate this again using Example 3.4, where we saw that

$$\mathcal{H}^2(f(E)) \neq \int_T Jf(x)d\lambda^2(x).$$

However, noting that $m_T^f(y) = 2$ for λ^2-a.e. y on $\mathbb{R}^2$, by integrating m_T^f over $f(T)$, we obtain

$$\int_{f(T)} m_T^f(y) d\mathcal{H}^2(y) = 2 \int_{f(T)} d\mathcal{H}^2(y) = 1,$$

hence

$$\int_{f(T)} m_T^f(y) d\mathcal{H}^2(y) = \int_T Jf(x) d\lambda^2(x).$$

This equality precisely corresponds to the Area formula.

Theorem 3.5. *(**Area formula with multiplicities**) Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $(1 \leq n \leq m)$ be a Lipschitz function and $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. Then m_E^f is an $\mathcal{H}^n$ measurable function and*

$$\int_{\mathbb{R}^m} m_E^f(y) d\mathcal{H}^n(y) = \int_E Jf(x) d\lambda^n(x). \tag{3.27}$$

Proof. We divide the proof into three steps:

Step 1) Let us prove that m_E is $\mathcal{H}^n-$ measurable, to do this, let $\mathcal{Q}_k$ be the standard dyadic partition of $\mathbb{R}^n$ by semi-open cubes of side length 2^{-k}, and define $m_k : \mathbb{R}^m \to \mathbb{N} \cup \{+\infty\}$ as

$$m_k(y) := \sum_{Q \in \mathcal{Q}_k} \chi_{f(E \cap Q)}(y), \quad \forall y \in \mathbb{R}^m.$$

Lemma 3.1 implies that for each $Q \in \mathcal{Q}_k$, $f(E \cap Q)$ is $\mathcal{H}^n-$measurable on $\mathbb{R}^m$, thus $\chi_{f(E \cap Q)}$ is $\mathcal{H}^n-$measurable and therefore m_k is measurable for each $k \in \mathbb{N}$. We can also observe that

$$m_k(y) = \text{ number of cubes in } \mathcal{Q}_k \text{ such that } f^{-1}(y) \cap (E \cap Q) \neq \emptyset.$$

Then, as $k \to \infty$, we can deduce that

$$m_k \to m_E^f,$$

and therefore m_E^f is $\mathcal{H}^n-$measurable.

Step 2) We will now prove that for every Lebesgue measurable set $E \subset \mathbb{R}^n$ the following inequality holds:

$$\int_{\mathbb{R}^m} \mathcal{H}^0(E \cap \{f = y\}) d\mathcal{H}^n(y) \leq \text{Lip}(f)^n \lambda^n(E). \tag{3.28}$$

To do this, observe that from the construction of the sequence $\{m_k\}_{k \in \mathbb{N}}$, we have $m_k \leq m_{k+1}$. Also, note that

$$\lambda^n(E) = \sum_{Q \in \mathcal{Q}_k} \lambda^n(E \cap Q).$$

Thus, from the above and the Monotone Convergence Theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^m} m_E^f(y) d\mathcal{H}^n(y) &= \lim_{k \to \infty} \int_{\mathbb{R}^m} m_k(y) d\mathcal{H}^n(y) \\
&= \lim_{k \to \infty} \int_{\mathbb{R}^m} \sum_{Q \in \mathcal{Q}_k} \chi_{f(E \cap Q)}(y) d\mathcal{H}^n(y) \\
&= \lim_{k \to \infty} \sum_{Q \in \mathcal{Q}_k} \int_{\mathbb{R}^m} \chi_{f(E \cap Q)}(y) d\mathcal{H}^n(y) \\
&= \lim_{k \to \infty} \sum_{Q \in \mathcal{Q}_k} \mathcal{H}^n(f(E \cap Q)) \\
&\leq \text{Lip}(f)^n \lim_{k \to \infty} \sum_{Q \in \mathcal{Q}_k} \lambda^n(E \cap Q),
\end{aligned} \tag{3.29}$$

and as

$$\sum_{Q\in\mathcal{Q}_k} \lambda^n(E\cap Q) = \lambda^n(E), \quad \forall k\in\mathbb{N},$$

combining this with (3.29), yields (3.28).

Step 3) To prove (3.27), from Step 2 and using Remark 3.3, we can assume without loss of generality that $E\subset F := \{0 < Jf < \infty\}$, Now, consider the partition $\mathcal{F} = \{F_k\}_{k\in\mathbb{N}}$ of F given by Theorem 3.3. This partition induces a partition on E, namely $\{E\cap F_k\}_{k\in\mathbb{N}}$, where f is injective on each $E\cap F_k$. Thus, by Theorem 3.4, we have

$$\begin{aligned}
\int_{\mathbb{R}^m} m_E^f(y)d\mathcal{H}^n(y) &= \int_{\mathbb{R}^m}\sum_{k\in\mathbb{N}} m^f_{E\cap F_k}(y)d\mathcal{H}^n(y)\\
&= \sum_{k\in\mathbb{N}} \mathcal{H}^n(f(E\cap F_k))\\
&= \sum_{k\in\mathbb{N}} \int_{E\cap F_k} Jf(x)d\lambda^n(x)\\
&= \int_E Jf(x)d\lambda^n(x).
\end{aligned}$$

■

The following corollary is a generalization of Corollary 3.1.

Corollary 3.3. *Let $f:\mathbb{R}^n\to\mathbb{R}^m$, $(n\le m)$ be a Lipschitz function and $g:\mathbb{R}^n\to[-\infty,\infty]$ be a Borel function such that $g\ge 0$ or $g\in L^1(\mathbb{R}^n;\lambda^n)$. Then*

$$\int_{\mathbb{R}^m}\left(\int_{\{f=y\}} g d\mathcal{H}^0\right)d\mathcal{H}^n(y) = \int_{\mathbb{R}^n} g(x)Jf(x)d\lambda^n(x). \tag{3.30}$$

Proof. Firstly, assume that $g\ge 0$. Then, there exist two sequences, one of Borel sets $\{E_h\}_{h\in\mathbb{N}}$ and $\{c_h\}_{h\in\mathbb{N}}\subset]0,\infty[$ such that

$$g = \sum_{h\in\mathbb{N}} c_h\chi_{E_h}.$$

Now, using Theorem 3.5 and the Dominated Convergence Theorem

$$\begin{aligned}
\int_{\mathbb{R}^n} g(x)Jf(x)d\lambda^n(x) &= \int_{\mathbb{R}^n}\sum_{h\in\mathbb{N}} c_h\chi_{E_h}Jf(x)d\lambda^n(x)\\
&= \sum_{h\in\mathbb{N}} c_h\int_{E_h} Jf(x)d\lambda^n(x)\\
&= \sum_{h\in\mathbb{N}} c_h\int_{\mathbb{R}^m} \mathcal{H}^0(E_h\cap\{f=y\})d\mathcal{H}^n(y)\\
&= \int_{\mathbb{R}^m}\sum_{h\in\mathbb{N}}\left(\int_{\{f=y\}} c_h\chi_{E_h}d\mathcal{H}^0\right)d\mathcal{H}^n(y)\\
&= \int_{\mathbb{R}^m}\left(\int_{\{f=y\}}\left(\sum_{h\in\mathbb{N}} c_h\chi_{E_h}\right)d\mathcal{H}^0\right)d\mathcal{H}^n(y)\\
&= \int_{\mathbb{R}^m}\left(\int_{\{f=y\}} g d\mathcal{H}^0\right)d\mathcal{H}^n(y).
\end{aligned}$$

■

Proposition 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $(1 \leq n \leq m)$ be a Lipschitz function, $E \subset \mathbb{R}^n$ a Lebesgue measurable set, and $F \subset \mathbb{R}^m$ a Borel set. Then*

$$\begin{aligned} f_{\#}(\lambda^n \llcorner E)(F) &= \lambda^n(f^{-1}(F) \cap E \cap \{Jf = 0\}) \\ &\quad + \int_{F \cap f(E \cap \{Jf > 0\})} \left(\int_{\{f=y\}} \frac{d\mathcal{H}^0}{Jf(x)} \right) d\mathcal{H}^n(y). \end{aligned} \tag{3.31}$$

In particular, if f is injective, proper, $Jf = 1$ a.e. on E and $\lambda^n(E) < \infty$, then

$$f_{\#}(\lambda^n \llcorner E)(C) = \mathcal{H}^n \llcorner f(E)(C), \quad \forall C \subset \mathbb{R}^m. \tag{3.32}$$

Proof. First, note that for any Borel set F in $\mathbb{R}^m$

$$\begin{aligned} f_{\#}(\lambda^n \llcorner E)(F) &= \lambda^n(E \cap f^{-1}(F)) \\ &= \lambda^n(f^{-1}(F) \cap E \cap \{Jf = 0\}) + \lambda^n(f^{-1}(F) \cap E \cap \{Jf > 0\}) \\ &= \lambda^n(f^{-1}(F) \cap E \cap \{Jf = 0\}) + \int_{\mathbb{R}^n} \chi_{\{f^{-1}(F) \cap E \cap \{Jf > 0\}}(x) d\lambda^n(x). \end{aligned}$$

Define

$$g(x) = \begin{cases} \frac{1}{Jf(x)} & \text{if } x \in f^{-1}(F) \cap E \cap \{Jf > 0\}. \\ 0 & \text{if } x \notin f^{-1}(F) \cap E \cap \{Jf > 0\}. \end{cases}$$

Then we can observe that g is well-defined, Borel, and non-negative. Moreover

$$\int_{\mathbb{R}^n} \chi_{\{f^{-1}(F) \cap E \cap \{Jf > 0\}}(x) d\lambda^n(x) = \int_{\mathbb{R}^n} g(x) Jf(x) d\lambda^n(x).$$

Thus, from Corollary 3.3, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \chi_{\{f^{-1}(F) \cap E \cap \{Jf > 0\}}(x) d\lambda^n(x) &= \int_{\mathbb{R}^n} g(x) Jf(x) d\lambda^n(x) \\ &= \int_{\mathbb{R}^m} \left(\int_{\{f=y\}} g d\mathcal{H}^0 \right) d\mathcal{H}^n(y) \\ &= \int_{(F \cap f(E \cap \{Jf > 0\}))^c} \left(\int_{\{f=y\}} g d\mathcal{H}^0 \right) d\mathcal{H}^n(y) \\ &\quad + \int_{F \cap f(E \cap \{Jf > 0\})} \left(\int_{\{f=y\}} g d\mathcal{H}^0 \right) d\mathcal{H}^n(y). \end{aligned}$$

Then, using the definition of g, we can verify that

$$\int_{(F \cap f(E \cap \{Jf > 0\}))^c} \left(\int_{\{f=y\}} g d\mathcal{H}^0 \right) d\mathcal{H}^n(y) = 0$$

and

$$\int_{F \cap f(E \cap \{Jf > 0\})} \left(\int_{\{f=y\}} g d\mathcal{H}^0 \right) d\mathcal{H}^n(y) = \int_{F \cap f(E \cap \{Jf > 0\})} \left(\int_{\{f=y\}} \frac{d\mathcal{H}^0}{Jf(x)} \right) d\mathcal{H}^n(y).$$

Putting all of the above together, we prove that (3.31) holds.
Now assume that f is injective, proper and $Jf = 1$ a.e. on E, then from these new assumptions,

$$0 \leq \lambda^n(f^{-1}(F) \cap E \cap \{Jf = 0\}) \leq \lambda^n(E \cap \{Jf \neq 1\}) = 0,$$

which implies

$$\lambda^n(f^{-1}(F) \cap E \cap \{Jf = 0\}) = 0.$$

Thus, from the identity above and (3.31), it follows that

$$\begin{aligned} f_{\#}(\lambda^n \llcorner E)(F) &= \int_{F\cap f(E\cap\{Jf>0\})} \left(\int_{\{f=y\}} \frac{d\mathcal{H}^0}{Jf(x)} \right) d\mathcal{H}^n(y) \\ &= \int_{F\cap f(E\cap\{Jf=1\})} \left(\int_{\{f=y\}} \frac{d\mathcal{H}^0}{Jf(x)} \right) d\mathcal{H}^n(y) \\ &+ \int_{F\cap f(E\cap\{Jf>1\})} \left(\int_{\{f=y\}} \frac{d\mathcal{H}^0}{Jf(x)} \right) d\mathcal{H}^n(y). \end{aligned}$$

Now note that

$$\mathcal{H}^n(F \cap f(E \cap \{Jf > 1\})) \leq \mathcal{H}^n(f(E \cap \{Jf > 1\})) \leq (\mathrm{Lip}(F))^n \mathcal{H}^n(E \cap \{Jf > 1\}),$$

and again using the hypothesis about Jf on E and the fact that in $\mathbb{R}^n$, $\mathcal{H}^n(E \cap \{Jf > 1\}) = \lambda^n(\mathcal{H}^n(E \cap \{Jf > 1\})) = 0$, we obtain

$$\mathcal{H}^n(F \cap f(E \cap \{Jf > 1\})) = 0.$$

Then, when integrating over a set of measure 0 in (3.1), we get

$$f_{\#}(\lambda^n \llcorner E)(F) = \int_{F\cap f(E\cap\{Jf>1\})} \left(\int_{\{f=y\}} \frac{d\mathcal{H}^0}{Jf(x)} \right) d\mathcal{H}^n(y).$$

From this last equality, using the injectivity of f, and the fact that in the integration set $Jf(x) = 1$ for $x \in f^{-1}(\{y\})$

$$\begin{aligned} f_{\#}(\lambda^n \llcorner E)(F) &= \int_{F\cap f(E\cap\{Jf=1\})} \left(\int_{\{f=y\}} \frac{d\mathcal{H}^0}{Jf(x)} \right) d\mathcal{H}^n(y) \\ &= \int_{F\cap f(E\cap\{Jf=1\})} \mathcal{H}^n(y) \\ &= \mathcal{H}^n(F \cap f(E \cap \{Jf = 1\})). \end{aligned} \tag{3.33}$$

Now, since Jf is a Borel function, $\{Jf \neq 1\}$ and $\{Jf = 1\}$ are Lebesgue measurables. Moreover, since E is Lebesgue measurable, then $E\cap\{Jf \neq 1\}$ and $E\cap\{Jf = 1\}$ are Lebesgue measurables. Moreover, since f is Lipschitz and F is a Borel set on $\mathbb{R}^m$, $F \cap f(E\cap\{Jf \neq 1\})$ and $F \cap f(E\cap \{Jf = 1\})$ are $\mathcal{H}^n-$measurables and disjoint sets, because by the injectivity of f

$$\begin{aligned} F \cap f(E \cap \{Jf \neq 1\}) \cap F \cap f(E \cap \{Jf = 1\}) &= F \cap f(E) \cap f(\{Jf \neq 1\}) \cap f(\{Jf = 1\}) \\ &= F \cap f(E) \cap f(\{Jf \neq 1\} \cap \{Jf = 1\}) \\ &= F \cap f(E) \cap f(\emptyset) \\ &= \emptyset. \end{aligned}$$

Also, by a reasoning analogous to one already made, we can deduce that

$$\mathcal{H}^n(F \cap f(E) \cap \{Jf \neq 1\}) = 0.$$

Combining this with the sigma-additivity of $\mathcal{H}^n$

$$\begin{aligned}\mathcal{H}^n(F \cap f(E)) &= \mathcal{H}^n(F \cap f(E) \cap \{Jf = 1\}) + \mathcal{H}^n(F \cap f(E) \cap \{Jf \neq 1\}) \\ &= \mathcal{H}(F \cap f(E) \cap \{Jf = 1\}),\end{aligned}$$

hence from this and (3.33),

$$f_{\#}(\lambda^n \llcorner E)(F) = \mathcal{H}^n(F \cap E) = \mathcal{H}^n \llcorner f(E)(F),$$

for each Borel set $F \subset \mathbb{R}^m$, and thus by Proposition 1.3, Proposition 1.14 and Theorem 1.5 we obtain (3.32). ■

Example 3.5. *Let $f : \mathbb{R} \to \mathbb{R}^2$ given by*

$$f(t) = (r\cos(|t|), r\sin(|t|)),$$

where $r > 0$. We can note that f is Lipschitz because each coordinate function is Lipschitz (composition of Lipschitz functions). Let $E =]-2n\pi, 2n\pi[$ with $n \in \mathbb{N}$, note that

$$m_E^f((x,y)) = \begin{cases} 0 & \text{if } (x,y) \notin f(E). \\ 2n & \text{if } (x,y) \in f(E). \end{cases}$$

Since

$$f(E) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r\} := S_r,$$

we will have

$$\int_{\mathbb{R}^2} m_E^f((x,y)) d\mathcal{H}^1(x,y) = \int_{S_r} 2n d\mathcal{H}^1(x,y) = 2n\mathcal{H}^1(S_r). \tag{3.34}$$

Also, we can see that

$$Jf(t) = \begin{cases} r & \text{if } t \neq 0. \\ +\infty & \text{if } t = 0. \end{cases} \tag{3.35}$$

This is because f is not differentiable at $t = 0$ (since $\sin(|t|)$ is not differentiable at 0). Thus, from (3.34), (3.35) and Theorem 3.5

$$2n\mathcal{H}^1(S_r) = \int_E Jf(t) d\lambda^1(t) = 4n\pi r.$$

Therefore,

$$\mathcal{H}^1(S_r) = 2\pi r.$$

◀

3.2 Tangential differentiability and the Area formula

3.2.1 Rectifiable sets

We will now introduce the concept of a Rectifiable set, which extends the idea of a surface and plays a crucial role in geometric measure theory. We start by fixing some concepts.

Definition 3.3. *Given $k \in \mathbb{N}$, $1 \le k \le n-1$, $h \ge 1$, we shall say that $M \subset \mathbb{R}^n$ is a $k-$**dimensional (embedded) surface of class** C^h **in** $\mathbb{R}^n$ if for each $x \in M$, there exist an open neighborhood A of x, an open set $E \subset \mathbb{R}^k$ and a bijection $f : E \to A \cap M$ with $f \in C^h(E)$ and $Jf > 0$ on E. Each map f is called a **coordinate mapping of** M.*

If M is a $k-$dimensional surface of class C^h in $\mathbb{R}^n$ then by definition, M is a relatively open set on $\mathbb{R}^n$, furthermore it is easy to show that M can be covered by countably set of images $f(E)$, with f and E as in the previous definition. Below, we present some well-known examples.

Example 3.6. *1.- $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : ||x|| = 1\}$ is a $n-1$ dimensional surface of class C^∞ in $\mathbb{R}^n$.*

2.- $M = [0, l] \times [0, l] \subset \mathbb{R}^2$ is not a surface of $\mathbb{R}^2$.

3.- $V = \{(x, y, z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 = 1; z = xy\}$ is a $1-$dimensional surface of class C^∞ in $\mathbb{R}^3$.

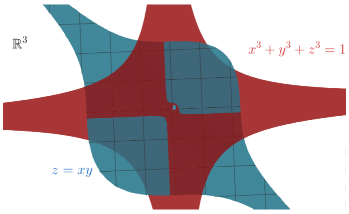

Figure 3.3: V is obtained from the intersection of the sets $\{(x, y, z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 = 1\}$ and $\{(x, y, z) \in \mathbb{R}^3 : z = xy\}$.

4.- $C := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$ is not a surface of $\mathbb{R}^3$.

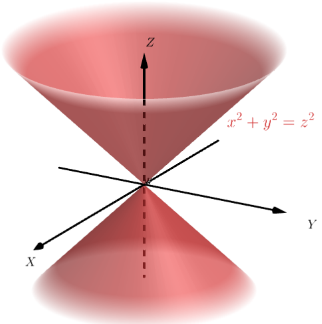

Figure 3.4: C turns out not to be a surface of $\mathbb{R}^3$ due to the impossibility of finding any suitable neighborhood for the point $(0, 0, 0)$.

◀

Of course, the comprehensive study of surfaces in $\mathbb{R}^n$ goes beyond the scope of this work, but an interested reader can refer to [4] or [7].
As we mentioned at the beginning of this section, we will be interested in studying more general objects than surfaces. More precisely, we have the following:

Definition 3.4. *Let $k \in \mathbb{N}$ $(1 \le k \le n)$ and $M \subset \mathbb{R}^n$. We say that M is* ***countably $\mathcal{H}^k-$ Rectifiable*** *if there exist countably Lipschitz functions $f_h : \mathbb{R}^k \to \mathbb{R}^n$, $h \in \mathbb{N}$ such that*

$$\mathcal{H}^k\left(M \setminus \bigcup_{h\in\mathbb{N}} f_h(\mathbb{R}^k)\right) = 0.$$

We say that M is ***locally $\mathcal{H}^k$-rectifiable*** *if additionally $\mathcal{H}^k \llcorner M(K) < \infty$ for every compact set $K \subset \mathbb{R}^n$, and M is simply called* ***$\mathcal{H}^k$-rectifiable*** *if additionally $\mathcal{H}^k \llcorner M(\mathbb{R}^n) < \infty$.*

This concept was first introduced by Besicovitch for 1−dimensional sets in the plane, then his work was extended by Federer to $m-$subsets of $\mathbb{R}^n$, and finally generalized by Marstrand to fractal sets in the plane whose Hausdorff dimension is any positive real number. As we have already mentioned above, Rectifiable sets in Euclidean space can be considered theoretical generalizations of C^1-surfaces, in fact we can think of a Rectifiable set as our best definition of a generalized k-dimensional surface on $\mathbb{R}^n$, possibly with infinite singularities and infinite topological type (see Example 3.7), but structured enough to do differential geometry.

Example 3.7. *Adding countably many handles of finite total area to the sphere can produce a Rectifiable set S with infinite topological type.*

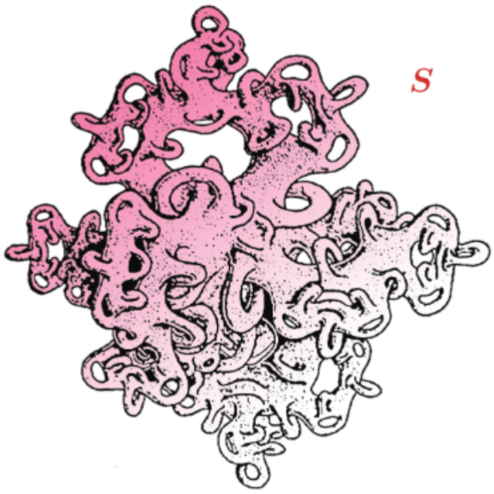

Figure 3.5: Rectifiable set S.

◀

Example 3.8. *Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function, define the graph $\overline{f} : \mathbb{R}^{n-1} \to \mathbb{R}^n$ by $\overline{f}(x) = (x, f(x))$, then $\Gamma := \overline{f}(\mathbb{R}^{n-1})$ is locally $\mathcal{H}^{n-1}-$Rectifiable.* ◀

The following remark gives us a characterization of countably Rectifiable sets, which will be useful.

Remark 3.4. *From McShane's Lemma and the regularity properties of Radon measures, we will have that $M \subset \mathbb{R}^n$ is a countably $\mathcal{H}^k$-Rectifiable set if and only if there exist a Borel set*

$M_0 \subset \mathbb{R}^n$ with $\mathcal{H}^k(M_0) = 0$, countably many Lipschitz functions $f_h : \mathbb{R}^k \to \mathbb{R}^n$ and Borel sets $E_h \subset \mathbb{R}^k$ such that

$$M = M_0 \cup \left(\bigcup_{h \in \mathbb{N}} f_h(E_h) \right).$$

$\triangleleft$

In the following example, we will make use of the previous remark to show that a set can be $\mathcal{H}^2-$countably rectifiable.

Example 3.9. *Let the square $M = [0,1] \times [0,1] \times \{0\} \subset \mathbb{R}^3$. Then M is a countably $\mathcal{H}^2$-Rectifiable set. Indeed, first note that*

$$M = L \cup M_1, \tag{3.36}$$

where

$$L = ([0,1] \times \{0\} \times \{0\}) \cup ([0,1] \times \{1\} \times \{0\}) \cup (\{0\} \times [0,1] \times \{0\}) \cup (\{1\} \times [0,1] \times \{0\})$$

and

$$M_1 = (0,1) \times (0,1) \times \{0\},$$

consider the function $i : \mathbb{R}^2 \to \mathbb{R}^3$ given by :

$$i((x,y)) = (x,y,0),$$

which is Lipschitz and satisfies $i((0,1) \times (0,1)) = M_1$ with $(0,1) \times (0,1)$ being a Borel set. Additionally, we can observe that $\mathcal{H}^2(L) = 0$, as each member of the union is an $\mathcal{H}^2-$null set in $\mathbb{R}^3$. Therefore, combining all of the above and (3.36), we conclude that M is countably $\mathcal{H}^2$-rectifiable. $\blacktriangleleft$

We can observe that any $M \subset \mathbb{R}^n$, will be countably $\mathcal{H}^n-$rectifiable, since by taking the identity function $Id : \mathbb{R}^n \to \mathbb{R}^n$ (which is Lipschitz), we can note that

$$\mathcal{H}^n(M \setminus I_d(\mathbb{R}^n)) = \mathcal{H}^n(M \setminus \mathbb{R}^n) = \mathcal{H}^n(\emptyset) = 0. \tag{3.37}$$

Note that the identity above is valid for all subsets of $\mathbb{R}^n$, this happens, because recall that in the strictly sense $\mathcal{H}^n$, is an outer measure on $\mathbb{R}^n$. Then if M is not a Lebesgue measurable set of $\mathbb{R}^n$, using Theorem 1.8, we can write (3.37) as

$$\mathcal{H}^n(M \setminus I_d(\mathbb{R}^n)) = \lambda_n^*(M \setminus I_d(\mathbb{R}^n)) = \lambda_n^*(M \setminus \mathbb{R}^n) = \lambda_n^*(\emptyset) = 0.$$

And if M is a Lebesgue measurable set of $\mathbb{R}^n$, using Corollary 1.2, we can write (3.37) as

$$\mathcal{H}^n(M \setminus I_d(\mathbb{R}^n)) = \lambda^n(M \setminus I_d(\mathbb{R}^n)) = \lambda^n(M \setminus \mathbb{R}^n) = \lambda^n(\emptyset) = 0.$$

Thus, in both cases we have obtained (3.37).
Using well-known properties of the Hausdorff measure, we will have that for $m > n$, any set $M \subset \mathbb{R}^n$ will be countably $\mathcal{H}^m-$rectifiable.
The following lemma shows elementary properties of Rectifiable sets.

Lemma 3.2. *The following properties hold:*
1.- Every subset of an $\mathcal{H}^k$-Rectifiable set is $\mathcal{H}^k$-rectifiable.
2.- The countable union of $\mathcal{H}^k$-Rectifiable sets is $\mathcal{H}^k$-rectifiable.

Below, we present two interesting examples of Rectifiable sets, among which we highlight Example 3.11 for the properties that the set exhibits.

Example 3.10. *For $q \in \mathbb{Q}$, define*

$$M_q := \{(x, y) \in \mathbb{R}^2 : y = qx\},$$

then, note that if $f_q : \mathbb{R} \to \mathbb{R}^2$ is the function

$$f(t) := (t, qt),$$

this is a Lipschitz function, since for any $t, r \in \mathbb{R}$

$$\begin{aligned} ||f(t) - f(r)|| &= ||(t, qt) - (r, qr)|| \\ &= ||(t - r, q(t - r)|| \\ &\leq |t - r| + q|t - r| = (1 + q)|t - r|, \end{aligned}$$

and since $f_q(\mathbb{R}) = M_q$

$$\mathcal{H}^1(M_q \setminus f(\mathbb{R})) = 0.$$

Thus, M_q will be a countably $\mathcal{H}^1$-Rectifiable set for each $q \in \mathbb{Q}$. Then, by Lemma 3.2,

$$M_q := \bigcup_{q \in \mathbb{Q}} M_q$$

is a countably $\mathcal{H}^1$-Rectifiable. ◀

Example 3.11. *There exist a dense subset E of $\mathbb{R}^2$ such that E is $\mathcal{H}^1$-measurable, $\mathcal{H}^1$-Rectifiable and*

$$\mathcal{H}^1(E) < \infty.$$

Indeed, consider $\mathbb{Q}^2 \subset \mathbb{R}^2$, without loss of generality, enumerate all the elements of $\mathbb{Q}^2$ as follows $\mathbb{Q}^2 := \{\vec{q}_k\}_{k \in \mathbb{N}}$ and let

$$E := \bigcup_{k \in \mathbb{N}} \partial(B(\vec{q}_k, 2^{-k})),$$

where $B(\vec{q}_k, 2^{-k})$ is the open ball with center at $\vec{q}_k$ and radius 2^{-k}.

For each $k \in \mathbb{N}$, we define $f_k : [0, 2\pi[\to \mathbb{R}^2$ by

$$f_k(t) := (2^{-k} \cos(t), 2^{-k} \sin(t)) + \vec{q}_k,$$

and note that f_k is Lipschitz, since for any $s, t \in [0, 2\pi[$

$$\begin{aligned} ||f_k(s) - f_k(t)|| &= ||(2^{-k} \cos(t), 2^{-k} \sin(t)) + \vec{q}_k - (2^{-k} \cos(s), 2^{-k} \sin(s)) - \vec{q}_k|| \\ &= ||2^{-k}(\cos(t) - \cos(s), \sin(t) - \sin(s))|| \\ &\leq 2^{-k}(|\cos(t) - \cos(s)| + |\sin(t) - \sin(s)|) \\ &\leq 2^{-k+1}|t - s|. \end{aligned}$$

Then, since $[0, 2\pi[$ *is Lebesgue measurable (as a Borel set), for each* $k \in \mathbb{N}$,

$$f_k([0, 2\pi[) = \partial(B(\vec{q}_k, 2^{-k}))$$

is $\mathcal{H}^1-$*measurable. Moreover, it follows from Remark 3.3 that* $\partial(B(\vec{q}_k, 2^{-k}))$ *is countably* $\mathcal{H}^1-$*rectifiable. Therefore, from the definition of* E, *Lemma 3.2, and the above, we have that* E *is an* $\mathcal{H}^1$*-measurable and* $\mathcal{H}^1$*-countably Rectifiable set.*
Now, from the subadditivity of $\mathcal{H}^1$

$$\mathcal{H}^1(E) \leq \sum_{k\in\mathbb{N}} \mathcal{H}^1(\partial(B(\vec{q}_k, 2^{-k}))) = \sum_{k\in\mathbb{N}} 2\pi 2^{-k} = 2\pi,$$

and thus $\mathcal{H}^1(E) < \infty$.
To show that E *is dense in* $\mathbb{R}^2$, *let* $x \in \mathbb{R}^2$. *Then there exist a nontrivial subsequence* $\{\vec{q}_{k_j}\}_{j\in\mathbb{N}}$ *(i.e.,* $\mathcal{H}^0(\{\vec{q}_{k_j}\}) = \infty$*) such that*

$$\vec{q}_{k_j} \xrightarrow[j\to\infty]{} x.$$

Taking $y_{k_j} \in \partial(B(\vec{q}_k, 2^{-k})) \subset E$, *we form a nontrivial subsequence (which exist because* $\{\vec{q}_{k_j}\}_{j\in\mathbb{N}}$ *is nontrivial, implying* $\#(2^{-k_j}) = \infty$*). Thus, by the triangle inequality and the way we chose* y_{k_j} *and* $\vec{q}_{k_j}$,

$$d(x, y_{k_j}) \leq d(x, \vec{q}_{k_j}) + d(\vec{q}_{k_j}, y_{k_j}) \to 0.$$

This implies that E *is dense in* $\mathbb{R}^2$. ◀

The following results and concepts will be used in an auxiliary way for the proof of the Area formula in Rectifiable sets, for this reason we will omit their respective proofs, but we refer to [11] for further details.

Definition 3.5. *Given a Lipschitz function* $f : \mathbb{R}^k \to \mathbb{R}^n$ *and a bounded Borel set* $E \subset \mathbb{R}^k$, *we say that* (f, E) *defines a* ***Lipschitz regular image*** $f(E)$ *if:*
1.- f *is injective and differentiable on* E, *with* $Jf(x) > 0$ *for every* $x \in E$.
2.- Every $x \in E$ *is a point of density* 1 *for* E.
3.- Every $x \in E$ *is a Lebesgue point of* ∇f.

Theorem 3.6. *(**Decomposition of Rectifiable sets**) If* $M \subset \mathbb{R}^n$ *is countably* $\mathcal{H}^k-$ *rectifiable and* $t > 1$, *then there exist:*
1.- A Borel set $M_0 \subset \mathbb{R}^n$, *such that* $\mathcal{H}^k(M_0) = 0$.
2.- Countably many Lipschitz functions $f_h : \mathbb{R}^k \to \mathbb{R}^n$.
3.- Bounded Borel sets $E_h \subset \mathbb{R}^k$.
Such that:
(i) (f_h, E_h) *define a Lipchitz regular image.*
(ii)

$$M = M_0 \cup \left(\bigcup_{h\in\mathbb{N}} f_h(M_h) \right)$$

Definition 3.6. *Let* $\Phi_{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ *given by*

$$\Phi_{x,r}(y) := \frac{y - x}{r}, \quad \forall y \in \mathbb{R}^n$$

and μ *be a Radon measure on* $\mathbb{R}^n$. *The* ***blow-up of*** μ ***of dimension*** k ***centered at*** x ***of size*** r *is defined as*

$$\mu_{x,r} := \frac{(\Phi_{x,r})_{\#}\mu}{r^k}.$$

Note that $\Phi_{x,r}$ is a continuous and proper function with

$$\Phi_{x,r}^{-1}(y) = x + ry.$$

Therefore, from Proposition 1.3, it follows that $\mu_{x,r}$ is a Radon measure.

Theorem 3.6 provides a means to establish the existence (in a measure theoretic sense) of tangent spaces to Rectifiable sets, more precisely:

Theorem 3.7. ***(Existence of approximate tangent spaces)*** *If $M \subset \mathbb{R}^n$ is a locally $\mathcal{H}^k-$Rectifiable set, then for $\mathcal{H}^k-$a.e. $x \in M$ there exist a unique $k-$ dimensional vector subspace Π_x of $\mathbb{R}^n$ such that, as $r \to 0^+$*

$$\frac{(\Phi_{x,r})_{\#}(\mathcal{H}^k \llcorner M)}{r^k} \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner \Pi_x, \tag{3.38}$$

where $\overset{*}{\rightharpoonup}$ denotes weak convergence.

Definition 3.7. *If Π_x is as in the previous theorem, we write*

$$T_x M := \Pi_x,$$

and name it the ***approximate tangent space to M at x****.*

Thus, for example despite its peculiar appearance, the Rectifiable set S defined in Example 3.7 has an approximate tangent plane at almost every point.
The set of points of M such that (3.38) holds true depends only on the Radon measure $\mu = \mathcal{H}^k \llcorner M$, this set it is a locally $\mathcal{H}^k-$Rectifiable set in $\mathbb{R}^n$, which is left unchanged if we modify M on and by $\mathcal{H}^k$-null sets. The following results concern properties of $T_x M$.

Lemma 3.3. *If $M = f(E)$ is a $k-$dimensional regular Lipschitz image in $\mathbb{R}^n$ and $z \in E$, then*

$$T_x M = d_z f(\mathbb{R}^n), \quad x = f(z).$$

Proposition 3.2. ***(Locality of approximate tangent spaces)*** *If M_1 and M_2 are locally $\mathcal{H}^k-$Rectifiable sets in $\mathbb{R}^n$, then for $\mathcal{H}^k-$a.e. $x \in M_1 \cap M_2$,*

$$T_x M_1 = T_x M_2.$$

3.2.2 Area formula on Rectifiable sets

Given a locally $\mathcal{H}^k-$Rectifiable set M in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ a Lipschitz function, similar to the previous section, we would like to express $\mathcal{H}^k(f(M))$ in terms of some integral over M, i.e.,

$$\mathcal{H}^k(f(M)) = \int_M (\dots).$$

To achieve this goal, we first need to introduce some concepts and results (which are indeed analogous to the introduced in the last chapter and section).

Definition 3.8. *Let M be a locally $\mathcal{H}^k-$Rectifiable set in $\mathbb{R}^n$ and $x \in M$ such that T_xM exist. We say that $f : \mathbb{R}^n \to \mathbb{R}^m$ is* ***tangentially differentiable (t.d.) with respect to*** *M* ***at*** *x if there exist a linear map $d_x^M f : T_xM \to \mathbb{R}^m$, such that uniformly on $\{v \in T_xM : ||v|| = 1\}$,*

$$\lim_{h\to 0} \frac{f(x+hv) - f(x)}{h} = d_x^M f v.$$

The above definition remains unchanged if we consider M as a $k-$dimensional C^1-surface in $\mathbb{R}^n$.

Remark 3.5. *1.- $f : \mathbb{R}^n \to \mathbb{R}^m$ may not be differentiable at every $x \in M$, despite being t.d. at every $x \in M$. For example, consider $M = \{x_n = 0\} \subset \mathbb{R}^n$, $\varphi \in C^1(\mathbb{R}^{n-1}, \mathbb{R}^m)$, and define*

$$f(x) = \varphi(x') + |x_n|v,$$

where $v \in \mathbb{R}^m$ is fixed, and $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$.

2.- If $f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, M is a $k-$dimensional C^1-surface and $x \in M$, then f is t.d. at x and $d_x^M f = d_x f|_{T_xM}$. ◁

Lemma 3.4. *If $M = g(E)$ is a $k-$dimensional regular Lipschitz image in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, and $f \circ g$ is differentiable at $z \in E$, then f is t.d. with respect to M at $x = g(z)$, with*

$$d_x^M f = d_z(f \circ g)(d_zg)^{-1} \quad \text{in} \quad T_xM = d_zg(\mathbb{R}^k). \tag{3.39}$$

Here, we have denoted by $(d_zg)^{-1}$ the inverse of d_zg seen as an isomorphism between $\mathbb{R}^k$ and $T_xM = d_zg(\mathbb{R}^k)$.

Proof. By Lemma 3.3, M admits the approximate tangent space $T_xM = d_zg(\mathbb{R}^k)$ at $x = g(z)$. Let $v \in T_xM$ with $||v|| = 1$, then there exists $w \in \mathbb{R}^k$ such that $v = d_zg(w)$, thus

$$\begin{aligned}\lim_{t\to 0} \frac{f(x+tv) - f(x)}{t} &= \lim_{t\to 0} \frac{f(g(z) + td_zg(w)) - f(g(z))}{t} \\ &= \lim_{t\to 0} \frac{f(g(z+tw)) - f(g(z))}{t} \\ &= \lim_{t\to 0} \frac{f \circ g(z+tw) - f \circ g(z)}{t}.\end{aligned}$$

In the third equality, we use the fact that

$$|g(z+tw) - g(z) - td_zg(w)| = o(t)$$

and that f is Lipschitz. Also, since $f \circ g$ is differentiable at z

$$\lim_{t\to 0} \frac{f \circ g(z+tw) - f \circ g(z)}{t} = d_z(f \circ g)(w) = d_z(f \circ g)(d_zg)^{-1}(v).$$

As d_zg is a linear isomorphism between $\mathbb{R}^k$ and T_xM we will have that $v \mapsto d_z(f \circ g)(d_zg)^{-1}(v)$ is a linear map. By combining all the above, we can conlude. ■

We now prove a "Rademacher type" theorem concerning tangential differentiability on locally $\mathcal{H}^k-$Rectifiable sets, which will allow us to define the notion of Jacobian in this context. The proof of the next theorem follows directly from Proposition 3.2, Theorem 3.6 and Lemma 3.4.

Theorem 3.8. *If M is locally $\mathcal{H}^k-$Rectifiable, and $f:\mathbb{R}^n\to\mathbb{R}^m$ is a Lipschitz function, then f is t.d. $\mathcal{H}^k-$a.e. on M.*

Definition 3.9. *The* ***tangential jacobian of f with respect to M at $x\in M$,*** *is defined by*

$$J^M f(x) := \begin{cases} [det((d_x^M f)^* \circ d_x^M f)]^{1/2} & \text{if } f \text{ is t.d. at } x. \\ \infty & \text{if } f \text{ is not t.d. at } x. \end{cases}$$

We can note that Theorem 3.8 implies that $\mathcal{H}^k(\{J^M f=\infty\})=0$.

Lemma 3.5. *If V is a $k-$dimensional subspace of $\mathbb{R}^n$, $T_1:\mathbb{R}^k\to\mathbb{R}^n$ is a linear map such that $T_1(\mathbb{R}^k)=V$ and $T_2:V\to\mathbb{R}^m$ is a linear map, then*

$$J(T_2\circ T_1)=JT_2JT_1. \tag{3.40}$$

Proof. Indeed, let us consider the polar decompositions $T_1=P_1\circ S_1$ and $T_2=P_2\circ S_2$, we obtain

$$(T_2\circ T_1)^*\circ(T_2\circ T_1)=T_1^*\circ T_2^*\circ T_2\circ T_1=S_1\circ P_1^*\circ S_2^2\circ P_1\circ S_1=S_1\circ U\circ S_1,$$

where $U:=P_1^*\circ S_2^2\circ P_1$ is a linear map from $\mathbb{R}^k$ on $\mathbb{R}^k$, thus

$$J(T_2\circ T_1)=J(S_1)\sqrt{det(U)}=J(T_1)\sqrt{det(U)}=J(T_1)J(T_2).$$

■

We are now ready to prove the Area formula for Rectifiable sets. An attentive reader may notice that in this case, we state the general form directly, without the need to go through the linear and injective cases as in Euclidean spaces, we do this because this new result rests on the work already done.

Theorem 3.9. *(**Area formula on locally Rectifiable sets**) Let M be a locally $\mathcal{H}^k-$ Rectifiable set in $\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}^m$ $(1\le k\le m)$ a Lipschitz function, then*

$$\int_{\mathbb{R}^m}\mathcal{H}^0(M\cap\{f=y\})d\mathcal{H}^k(y)=\int_M J^M f d\mathcal{H}^k.$$

In particular, if f is injective in M, then

$$\mathcal{H}^k(f(M))=\int_M J^M f d\mathcal{H}^k.$$

Proof. Without loss of generality, suppose that $M=g(E)$ is a Lipschitz regular image. Thus, by Lemma 3.3 and Lemma 3.4

$$T_xM=d_zg(\mathbb{R}^k),\quad d_x^M f(v)=d_z(f\circ g)(d_zg)^{-1}(v)$$

whenever $f\circ g$ is differentiable at $z\in E$. In particular,

$$d_z(f\circ g)=d_{g(z)}^M f\circ d_zg.$$

Now, using Lemma 3.5 and Rademacher's Theorem, we obtain

$$J(f \circ g)(z) = J^M f(g(z))Jg(z) \quad \text{for } \mathcal{H}^k - \text{a.e. } z \in E.$$

Applying the Area formula to $f \circ g : \mathbb{R}^k \to \mathbb{R}^m$, we have

$$\int_{\mathbb{R}^m} \mathcal{H}^0(E \cap \{f \circ g = y\}) d\mathcal{H}^k(y) = \int_E J(f \circ g)(z) dz.$$

Now, from the injectivity of $g|_E$,

$$\begin{aligned} g(E \cap \{f \circ g = y\}) &= g(E \cap (f \circ g)^{-1}\{y\}) \\ &= g(E) \cap g(g^{-1}((f^{-1}\{y\})) \\ &= M \cap \{f = y\}, \end{aligned}$$

which implies that

$$\#(E \cap \{f \circ g = y\}) = \#(M \cap \{f = y\}).$$

So,

$$\begin{aligned} \int_{\mathbb{R}^m} \mathcal{H}^0(M \cap \{f = y\}) d\mathcal{H}^k(y) &= \int_{\mathbb{R}^m} \mathcal{H}^0(E \cap \{f \circ g = y\}) d\mathcal{H}^k(y) \\ &= \int_E J(f \circ g)(z) dz \\ &= \int_E J^M f(g(z)) Jg(z) dz \\ &= \int_M J^M f d\mathcal{H}^k. \end{aligned}$$

■

Reasoning as in the proof of the previous theorem allow us to prove the **Area formula for k-dimensional C^1-surfaces**, which states: *If $M \subset \mathbb{R}^n$ is a $k-$dimensional C^1-surface and $f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $(m \geq k)$ is injective, then*

$$\mathcal{H}^k(f(M)) = \int_M J^M f d\mathcal{H}^k.$$

The following corollary, represents the analogue of Corollary 3.3.

Corollary 3.4. *If S is a locally $\mathcal{H}^{n-2}-$Rectifiable set in $\mathbb{R}^{n-1}$, $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, $\Gamma = \{(z, u(z)) \in \mathbb{R}^n; z \in S\}$, $g : \mathbb{R}^n \to [-\infty, \infty]$ is a Borel function, and $g \geq 0$ or $g \in L^1(\mathbb{R}^n, \mathcal{H}^{n-2} \llcorner \Gamma)$, then*

$$\int_\Gamma g d\mathcal{H}^{n-2} = \int_S \overline{g} \sqrt{1 + |d^S u|^2} d\mathcal{H}^{n-2},$$

where $\overline{g}(z) = g(z, u(z))$, $z \in \mathbb{R}^{n-1}$.

We conclude this section by presenting two illustrative examples of the use of Theorem 3.9.

Example 3.12. *Let*

$$M := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \;\; 0 < y \leq 2\pi\},$$

and $f : \mathbb{R}^2 \to \mathbb{R}^3$ *given by*

$$f(x, y) := (x\cos(y), x\sin(y), x).$$

Then, it can be shown that

$$f(M) = \{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{x^2 + y^2}, \quad x^2 + y^2 \leq 1\}.$$

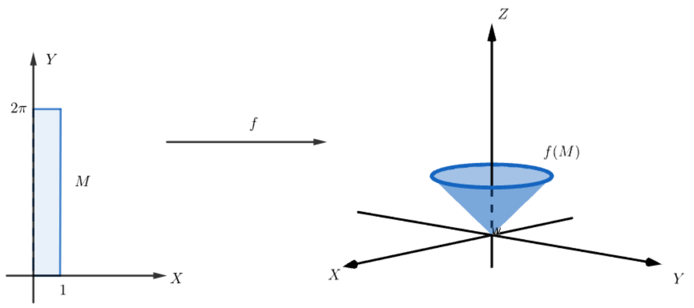

Figure 3.6: The semi-closed rectangle M is mapped by f to the cone $\{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{x^2 + y^2}, x^2 + y^2 \leq 1\}$.

Note that in the usual sense

$$d_{(x,y)}f = \begin{bmatrix} \cos(y) & -x\sin(y) \\ \sin(y) & x\cos(y) \\ 1 & 0 \end{bmatrix}.$$

Using the fact that $f \in C^1(\mathbb{R}^2, \mathbb{R}^3)$, *we have*

$$d^M_{(x,y)}f = d_{(x,y)}f \llcorner T_{(x,y)}M = d_{(x,y)}f \llcorner \mathbb{R}^2.$$

Therefore, $d^M_{(x,y)}f = d_{(x,y)}f$ *i.e.,*

$$d^M_{(x,y)}f = \begin{bmatrix} \cos(y) & -x\sin(y) \\ \sin(y) & x\cos(y) \\ 1 & 0 \end{bmatrix}.$$

Hence

$$(d^M_{(x,y)}f)^* = \begin{bmatrix} \cos(y) & \sin(y) & 1 \\ -x\sin(y) & x\cos(y) & 0 \end{bmatrix},$$

from which

$$(d^M_{(x,y)}f)^* \circ d^M_{(x,y)}f = \begin{bmatrix} 2 & 0 \\ 0 & x^2 \end{bmatrix}.$$

Therefore,

$$J^M f((x,y)) = \sqrt{2}x,$$

so, using Theorem 3.9

$$\mathcal{H}^2(f(M)) = \int_M \sqrt{2}x d\mathcal{H}^2(x,y) = \int_0^1 \int_0^{2\pi} \sqrt{2}x dy dx = \sqrt{2}\pi,$$

where the third equality follows from the fact that $\mathcal{H}^2 = \lambda^2$ in $\mathbb{R}^2$ and $M = [0,1] \times [0, 2\pi[$. ◀

For showing the use of the Area Formula on rectifiable sets, we provided the solution of the next exercise, which appears as an exercise for the reader in [13].

Example 3.13. *Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by*

$$f((x,y)) := (\sin(x)\cos(y), \sin(x)\sin(y), \cos(x)).$$

Thus, if $M := [0,\pi] \times]0, 2\pi]$, then $f(M) = \mathbb{S}^2$.

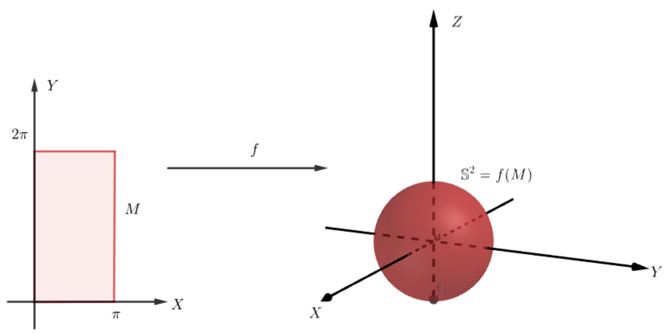

Figure 3.7: The semi-closed rectangle M is mapped by f to $\mathbb{S}^2$.

Furthermore, proceeding analogously to the previous example, we can see that

$$d^M_{(x,y)} f = \begin{bmatrix} \cos(y)\cos(x) & -\sin(x)\sin(y) \\ \cos(x)\sin(y) & \sin(x)\cos(y) \\ -\sin(x) & 0 \end{bmatrix},$$

therefore

$$(d^M_{(x,y)} f)^* = \begin{bmatrix} \cos(y)\cos(x) & \cos(x)\sin(y) & -\sin(x) \\ -\sin(x)\sin(y) & \sin(x)\cos(y) & 0 \end{bmatrix}.$$

Thus

$$(d^M_{(x,y)} f)^* d^M_{(x,y)} f = \begin{bmatrix} \cos^2(y)\cos^2(x) + \cos^2(x)\sin^2(y) + \sin^2(x) & 0 \\ 0 & \sin^2(x)\sin^2(y) + \sin^2(x)\cos^2(y) \end{bmatrix},$$

hence

$$(d^M_{(x,y)} f)^* \circ d^M_{(x,y)} f = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(x) \end{bmatrix},$$

which implies that

$$J^M f((x,y)) = \sin(x),$$

using Theorem 3.9

$$\mathcal{H}^2(\mathbb{S}^2) = \int_M \sin(x) d\mathcal{H}^2(x,y) = \int_0^{2\pi} \int_0^{\pi} \sin(x) dx dy = 4\pi.$$

◀

3.3 Applications

From now on, we will use the well-known Cauchy-Binet formula, which will be useful for us in calculating the corresponding Jacobians. A reader interested in this formula can refer to [5].

Length of a curve

Let $f : \mathbb{R} \to \mathbb{R}^m$ $(1 \leq m)$ be an injective Lipschitz function, and consider the curve $\Gamma := f([a,b])$, where $-\infty < a < b < \infty$. Using the Area formula for injective functions, we can note that

$$\int_a^b Jf(x) d\lambda^1(x) = \mathcal{H}^1(f([a,b]) = \mathcal{H}^1(\Gamma).$$

In this case, it is immediately verified that

$$Jf(x) = ||f'(x)||, \quad \text{whenever } f \text{ is differentiable at } x.$$

Thus, by Rademacher's Theorem

$$\int_a^b ||f'(x)|| d\lambda^1(x) = \mathcal{H}^1(\Gamma).$$

And by Theorem 1.6

$$\int_a^b ||f'(x)|| d\lambda^1(x) = length(\Gamma).$$

Surface area of a graph

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function, and for $U \subset \mathbb{R}^n$ open, the graph of f over U is defined as the set

$$G_U(f) := \{(x, f(x)) : x \in U\} \subset \mathbb{R}^{n+1}.$$

Define $g : \mathbb{R}^n \to \mathbb{R}^{n+1}$ by $g(x) = (x, f(x))$, then we can notice that g is an injective Lipschitz function (since each coordinate function is), furthermore, given $x \in \mathbb{R}^n$ such that f is differentiable at x, we will also have that g is differentiable at x, and

$$d_x g = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \\ 0 & \dots & \dots & 1 \\ \frac{\partial f(x)}{\partial x_1} & \dots & \dots & \frac{\partial f(x)}{\partial x_n} \end{bmatrix}.$$

Using the Cauchy-Binet formula,

$$Jg(x)^2 = 1 + \left(\frac{\partial f(x)}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial f(x)}{\partial x_n}\right)^2,$$

hence

$$Jg(x) = \sqrt{1 + \left(\frac{\partial f(x)}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial f(x)}{\partial x_n}\right)^2}.$$

Thus, by Rademacher's Theorem

$$\int_U Jg(x)d\lambda^n(x) = \int_U \sqrt{1 + \left(\frac{\partial f(x)}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial f(x)}{\partial x_n}\right)^2} d\lambda^n(x).$$

Also, from the Area Formula for injective functions,

$$\int_U Jg(x)d\lambda^n(x) = \mathcal{H}^n(f(U)) = \mathcal{H}^n(G_U(f)),$$

and therefore

$$\mathcal{H}^n(G_U(f)) = \int_U \sqrt{1 + \left(\frac{\partial f(x)}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial f(x)}{\partial x_n}\right)^2} d\lambda^n(x).$$

We can observe that this coincides with the classical case.

Sard's Theorem

Sard's theorem is an important result in differential geometry and analysis, since it helps to understand the behavior of the critical points of smooth functions. It highlights a fundamental property: even in high-dimensional spaces, most points in the target space are regular values, allowing for a deeper understanding of the geometry of smooth mappings. This theorem has applications in some fields, including optimization, geometric analysis, and topology.
Next, we provide a proof of Sard's Theorem for the case where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function and $n \leq m$, for this purpose, the following lemmas will be useful.

Lemma 3.6. *Let $A \subset \mathbb{R}^n$ be a non empty open set and $f \in C^\infty(A, \mathbb{R}^m)$. If $K \subset A$ is a non empty compact set, then $f|_K$ is Lipschitz.*

We could think of the last lemma as a kind of converse to Rademacher's theorem, i.e., it establishes under what conditions a differentiable function is Lipschitz.

Lemma 3.7. *Let $A \subset \mathbb{R}^n$ be an open set, and define*

$$K_j := \left\{x \in A : \text{dist}(x, \partial A) \geq \frac{1}{j}\right\} \cap \overline{B}(0, j).$$

Then K_j is compact for each $j \in \mathbb{N}$ and

$$A = \bigcup_{j=1}^{\infty} K_j.$$

Theorem 3.10. ***(Sard's Theorem)*** *Let $A \subset \mathbb{R}^n$ be a non empty open set and $f \in C^\infty(A, \mathbb{R}^m)$. If $E := \{x \in U : \operatorname{rank}(d_x f) < n\}$, then*

$$\mathcal{H}^n(f(E)) = 0.$$

Proof. We can see that if $x \in E$, then as $\operatorname{rank}((d_x f)^*) = \operatorname{rank}(d_x f) < n$, and since

$$\operatorname{rank}((d_x f)^* d_x f) \leq \min\{\operatorname{rank}(d_x f)^*, \operatorname{rank}(d_x f)\} < n,$$

one automatically have that $det((d_x f)^* d_x f) = 0$, thus $Jf(x) = 0$. Using Lemma 3.7, yields

$$E = \bigcup_{j=1}^{\infty} (E \cap K_j).$$

By Lemma 3.6, as $E \cap K_j \subset K_j$, then $f|_{E \cap K_j}$ is Lipschitz. These considerations above will help us prove the desired result. Put $f_j := f|_{E \cap K_j}$ and fix $0 < \varepsilon \leq 1$, then if $g_j : (E \cap K_j) \to \mathbb{R}^m \times \mathbb{R}^n$ is the Lipschitz function given by

$$g(x) := (f_j(x), \varepsilon x)$$

and $p : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is the projection (which will also be Lipschitz)

$$p(y, z) = y,$$

we can factor $f_j = p \circ g_j$. We claim that there exists a constant C such that

$$0 < Jg_j(x) \leq C\varepsilon, \quad \text{for } x \in E \cap K_j.$$

Write $g_j = (f_j^1, \ldots, f_j^m, \varepsilon x_1, \ldots, \varepsilon x_n)$, then

$$d_x g_j = \begin{bmatrix} d_x f \\ \varepsilon I \end{bmatrix}_{(n+m) \times n}.$$

Since $Jg_j(x)^2$ equals the sum of the squares of the $(n \times n)-$subdeterminants of $d_x g_j$, according to the Cauchy-Binet formula, we have

$$Jg_j(x)^2 \geq \varepsilon^{2n} > 0. \tag{3.41}$$

Furthermore, since $||d_x f_j|| \leq \operatorname{Lip}(f_j) < \infty$, we may also employ the Cauchy-Binet formula to compute

$$Jg_j(x)^2 = Jf_j(x)^2 + \alpha \leq C\varepsilon^2, \quad \text{for each } x \in E \cap K_j, \tag{3.42}$$

where $\alpha :=$ sum of squares of terms, each involving at least one ε. Thus (3.41) and (3.42) implies the claim.

Since p is a projection we can compute, using Theorem 3.4 and Theorem 1.5

$$\begin{aligned} \mathcal{H}^n(f_j(E \cap K_j)) &\leq \mathcal{H}^n(g_j(E \cap K_j)) \\ &\leq \int_{\mathbb{R}^{n+m}} m^{g_j}_{E \cap K_j}(y, z) d\mathcal{H}^n(y, z) \\ &= \int_{E \cap K_j} Jg_j(x) d\lambda^n(x) \\ &\leq \varepsilon C \lambda^n(E \cap K_j). \end{aligned}$$

Let $\varepsilon \to 0$ to conclude that $\mathcal{H}^n(f_j(E \cap K_j)) = 0$,and thus

$$\begin{aligned}\mathcal{H}^n(f(E)) &= \mathcal{H}^n\left(f\left(\bigcup_{j=1}^{\infty}(E \cap K_j)\right)\right) \\ &= \mathcal{H}^n\left(\bigcup_{j=1}^{\infty} f(E \cap K_j)\right) \\ &= \mathcal{H}^n\left(\bigcup_{j=1}^{\infty} f_j(E \cap K_j)\right) \\ &\leq \sum_{j=1}^{\infty} \mathcal{H}^n(f_j(E \cap K_j)) \\ &= 0.\end{aligned}$$

■

The sharp version of Sard's theorem, the Morse–Sard–Federer theorem, can be found in [6].

Chapter 4

Hausdorff measure and Lipschitz functions on metric spaces

4.1 Hausdorff measure on metric spaces

Definition 4.1. *Let $X = (X, d)$ be a metric space, $n \in \mathbb{N}$, $s \in [0, \infty[$, and $\delta > 0$. For $E \subset X$, define*

$$\mathcal{H}^s_\delta(E) := \inf_{\mathcal{F}} \frac{\omega_s}{2^s} \sum_{F \in \mathcal{F}} (\mathrm{diam}(F))^s,$$

where the infimum is taken over all countable covers $\mathcal{F}$ of E consisting of sets $F \subset X$ such that $\mathrm{diam}(F) < \delta$ *(which we will call δ-covers), and ω_s is given by:*

$$\omega_s := \frac{\pi^{\frac{s}{2}}}{\Gamma\left(1 + \frac{s}{2}\right)},$$

where $\Gamma :]0, \infty[\to [1, \infty[$ is the Euler Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \;\; s > 0.$$

We also define

$$\mathcal{H}^s(E) := \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

The above definition is an exact copy of Definition 1.8, however, it is important to notice that each $\mathcal{H}^s$ depends on the underlying metric of the space and that this dependence is not visible in our notation.
We will have analogous properties to those stated in Propositions 1.4, 1.5, 1.6, 1.11, 1.12 and 1.14. While Propositions 1.7 and 1.8 will be valid whenever X has a vector space structure.
We can also extend the Definition 1.9 and we will obtain analogues to Propositions 1.9, 1.10 by replacing in both n to $\mathcal{H}\text{-}\dim(E)$. In Proposition 1.13 we can replace n to $\mathcal{H}\text{-}\dim(E)$ for obtain an analogue to the items (1-3), since (4) is not necessarily valid in arbitrary metric spaces (take for example $\mathbb{R}$ with the discrete metric).

4.2 Lipschitz functions on metric spaces

We can extend Definition 1.10 and its corresponding terminology to functions $f : X \to Y$, where X and Y are metric spaces In such a case, we will have analogous properties to those stated in Proposition 1.15 (in the case of 1.15 (2), provided the image is $\mathbb{R}^n$, and in 1.15 (3) and (5), provided the image is $\mathbb{R}$). We also have the analogous of Theorem 1.5.
As with the Haudorff measure, the property of being Lipschitz will depend on the underlying metric of the space.

4.2.1 Extensions of Lipschitz functions on metric spaces

To start this subsection, we present the following result and its subsequent corollary, which we already know for functions from $\mathbb{R}^n$ to $\mathbb{R}$, and whose proofs are similar to those already carried out.

Lemma 4.1. *(**McShane's extension lemma**) Let $X = (X, d)$ be a metric space and $f : E \subset X \to \mathbb{R}$ a $L-$Lipschitz function. Then there exist an $L-$Lipschitz function $F : X \to \mathbb{R}$ such that $F|_E = f$.*

Corollary 4.1. *Let $X = (X, d)$ be a metric space and $f : E \subset X \to \mathbb{R}^m$ be a $L-$Lipschitz function. Then there exist an $\sqrt{m}L-$Lipschitz function $F : X \to \mathbb{R}^m$ such that $F|_E = f$.*

At the moment we can notice that everything is turning out to be a complete analog to the study already done for Euclidean spaces, so one would expect that the next result to be stated would be Kirszbraun's Theorem, however as we have already anticipated previously in metric spaces it is not always possible to find the extension given by Kirzbraun's Theorem, as the following example shows.

Example 4.1. *Kirszbraun theorem does not hold in general for Lipschitz functions defined between Banach spaces of dimension greather than or equal to 2. Indeed, let $X = (\mathbb{R}^2, ||\cdot||_\infty)$ and $Y = (\mathbb{R}^2, ||\cdot||)$. Consider $A \subset X$ given by $A := \{(-1,1), (1,-1), (1,1)\}$, and let $f : A \to Y$ defined by*

$$f((-1,1)) = (-1,0),\ f((1,-1)) = (1,0),\ f((1,1)) = (0,\sqrt{3}).$$

Then, f cannot be extended to $A \cup \{(0,0)\}$, to see this, first note that f is $1-$Lipschitz, in fact $\mathrm{Lip}(f) = 1$ (to see this, it suffices to calculate the respective norms of the elements of A and their respective images). Then, it can be shown that if f had a $1-$Lipschitz extension to $A \cup \{(0,0)\}$, this would lead to a contradiction. Thus, in particular, f does not admit a $1-$Lipschitz extension in X, and therefore we have what was claimed. ◀

The following results provide extensions of Lipschitz functions when the codomains are more general sets.

Proposition 4.1. *Let $X = (X, d)$ be a metric space, $f : A \subset X \to L^\infty(Y)$ an $L-$Lipschitz function, where Y is any set. Then there exist an $L-$Lipschitz function $F : X \to L^\infty(Y)$ such that $F|_A = f$.*

Proof. For each $a \in A$, we denote by f_a to the image of a under f, i.e., $f_a = f(a)$ is a function $f_a : Y \to \mathbb{R}$ such that $f_a \in L^\infty(Y)$. Given $x \in X$, we define the function $F_x : Y \to \mathbb{R}$, as

$$F_x(y) := \inf\{f_a(y) + Ld(x,a) : a \in A\}, \quad \forall y \in Y.$$

In a similar way to McShane's Lemma (Lemma 1.1), we can verify that $F|_A = f$. We claim that $F_x \in L^\infty(Y)$ $\forall x \in X$, to see this choose $a_0 \in A$ fix, and notice that for each $a \in A$ and $y \in Y$, it holds

$$\begin{aligned} f_a(y) + Ld(x,a) &\geq (f_a(y) + Ld(a,a_0)) - Ld(x,a_0) \\ &\geq F_{a_0}(y) - Ld(x,a_0) \\ &= f_{a_0}(y) - Ld(x,a_0). \end{aligned} \tag{4.1}$$

Here we have used the fact that $F|_A = f$, then as $f_{a_0} \in L^\infty(Y)$ from (4.1) by taking the infimum on the left hand side

$$F_x(y) \geq -||f_{a_0}||_{L^\infty(Y)} - Ld(x,a_0), \quad \forall y \in Y. \tag{4.2}$$

Also, from the definition of F

$$F_x(y) \leq ||f_{a_0}||_{L^\infty(Y)} + Ld(x,a_0), \quad \forall y \in Y.$$

So from this and (4.2) we obtain

$$|F_x(y)| \geq ||f_{a_0}||_{L^\infty(Y)} + Ld(x,a_0), \quad \forall y \in Y.$$

Therefore $F_x \in L^\infty(Y)$, $\forall x \in X$. Finally, noticing that F_x is the infimum of a family of $L-$Lipschitz functions, it follows that F_x is $L-$Lipschitz. ■

Theorem 4.1. *(**Kuratowski embedding theorem**) Every metric space $Y = (Y,d)$ embeds isometrically into the Banach space $L^\infty(Y)$.*

Proof. Fix a point $y_0 \in Y$. For each $x \in Y$ define

$$f_x(y) := d(y,x) - d(y,y_0), \forall y \in Y.$$

From the triangle inequality, we have

$$|f_x(y)| \leq d(x,y_0), \forall y \in Y.$$

Therefore $f_x \in L^\infty(Y)$, $\forall x \in Y$. Moreover, notice that for any $x, z \in Y$

$$|f_x(y) - f_z(y)| = |d(x,y) - d(y,z)| \leq d(x,z), \forall y \in Y.$$

So, by taking $y = x$

$$||f_x - f_z||_{L^\infty(Y)} = d(x,z).$$

This implies that we have an isometry. ■

The target space for the embedding in Kuratowski theorem depends on the space itself. For separable metric spaces, we can use a universal target.

Theorem 4.2. *(**Fréchet embedding theorem**) Every separable metric space $Y = (Y,d)$ embeds isometrically into the Banach space ℓ^∞.*

Proof. Let $\{y_k\}_{k\in\mathbb{N}}$ be a countable dense set in Y. We define $\varphi : Y \to \ell^\infty$ as

$$\varphi(y) = \{\varphi_h(y)\}_{h\in\mathbb{N}} \quad \text{where} \quad \varphi_k(y) := d(y, y_k) - d(y_k, y_0), \quad \forall y \in Y.$$

Notice that φ is well-defined because

$$|\varphi_k(y)| = |d(y, y_k) - d(y_k, y_0)| \leq d(y, y_0), \quad \forall k \geq 1.$$

Hence

$$\varphi(y) \in \ell^\infty, \quad \forall y \in Y.$$

Given $x, y \in Y$, note that

$$|\varphi_k(x) - \varphi_k(y)| = |d(x, y_k) - d(y, y_k)| \leq d(x, y), \quad \forall k \geq 1. \tag{4.3}$$

Using the density of $\{y_k\}_{k\in\mathbb{N}}$, we can find a subsequence $\{y_{k_j}\}_{j\in\mathbb{N}}$ such that $y_{k_j} \to x$. Combining this with (4.3), we have

$$||\varphi(x) - \varphi(y)||_{\ell^\infty} = d(x, y),$$

which implies that φ is an isometry. ■

We can notice that the inclusion in the previous results is not a canonical inclusion because it depends on the choice of the element $y_0 \in Y$ (Kuratowski's theorem) and the dense set $\{y_h\}_{h\in\mathbb{N}}$ (Fréchet's theorem).

Doubling spaces

A doubling space, in the area of metric geometry, refers to a space equipped with a metric that satisfies certain doubling conditions. These conditions essentially quantify how quickly the volume of balls in the space can grow as the radius increases.
Doubling spaces are of great importance in analysis and geometry, particularly in the study of geometric measure theory, harmonic analysis, and PDE's. They provide a framework for understanding the distribution of mass and energy in various contexts, and their properties often lead to deep results.

Definition 4.2. *Let $X = (X, d)$ be a metric space and let $\varepsilon > 0$. A subset $A \subset X$ is called a $\varepsilon-$**separated set in** X, if for every $x, y \in A$ with $x \neq y$, it holds that*

$$d(x, y) \geq \varepsilon.$$

Example 4.2. *If $A = \{x_1, x_2, x_3, x_4\} \subset \mathbb{R}^3$ are the vertices of a regular tetrahedron with side length $\varepsilon > 0$, then A is an $\varepsilon-$separated set.* ◀

Definition 4.3. *A metric space $X = (X, d)$ is said to be **doubling with constant** N, where $N \in \mathbb{N}$, if every $\frac{r}{2}-$separated subset of $B(x, r)$ contains at most N points $\forall x \in X$ and $\forall r > 0$. Sometimes we simply say that X is **doubling** if there is no need to mention the constant N.*

It is immediate to verify that any subset of a doubling metric space is also doubling with the same constant.

Definition 4.4. *A **metric measure space** is a triple (X, d, μ), where (X, d) is a separable metric space and μ is a non trivial Radon measure on X.*

Definition 4.5. *Let μ be a regular Borel measure on a metric space (X, d). We say that μ is a **doubling measure**, if every ball in X has positive and finite measure, and there exist a constant $C \geq 1$ such that for any $x \in X$ and $r > 0$,*

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)). \tag{4.4}$$

*The smallest constant satisfying (4.4) is called the **doubling constant of** μ and is denoted by C_μ.*
*If (X, d, μ) is a metric measure space and μ is a doubling measure, we call (X, d, μ) a **doubling metric measure space (DMMS).***

The following result shows the relationship between DMMS and doubling spaces.

Theorem 4.3. *If (X, d, μ) is a DMMS, then X is doubling.*

Proof. Let $x \in X$ and $r > 0$ be arbitrary, and let $A = \{x_1, x_2, \dots, x_n\}$ be an $\frac{r}{2}$−separated subset of $B(x, r)$. Note that $B\left(x_i, \frac{r}{4}\right) \subset B(x, 2r)$, because if $y \in B\left(x_i, \frac{r}{4}\right)$, then

$$d(x, y) \leq d(x, x_i) + d(x_i, y) < r + \frac{r}{4} < 2r.$$

Without loss of generality, suppose that

$$\mu\left(B\left(x_1, \frac{r}{4}\right)\right) \leq \mu\left(B\left(x_i, \frac{r}{4}\right)\right), \quad i = 2, \dots, n.$$

Thus,

$$\begin{aligned} n\mu\left(B\left(x_1, \frac{r}{4}\right)\right) &\leq \sum_{i=1}^{n} \mu\left(B\left(x_i, \frac{r}{4}\right)\right) \\ &= \mu\left(\bigcup_{i=1}^{n} B\left(x_1, \frac{r}{4}\right)\right) \\ &\leq \mu(B(x, 2r)), \end{aligned} \tag{4.5}$$

where the equality follows because the balls are disjoint. It is now we claim that $B(x, 2r) \subset B(x_1, 4r)$, indeed, if $y \in B(x, 2r)$, then

$$d(y, x_1) \leq d(x, y) + d(x, x_1) < 2r + r < 4r.$$

From the latter and (4.5) we obtain

$$n\mu\left(B\left(x_1, \frac{r}{4}\right)\right) \leq \mu(B(x_1, 4r)),$$

and by the doubling property of μ we have that

$$\begin{aligned} \mu(B(x_1, 4r)) &\leq C_\mu \mu(B(x_1, 2r)) \\ &\leq C_\mu^2 \mu(B(x_1, r)) \\ &\leq C_\mu^3 \mu\left(B\left(x_1, \frac{r}{2}\right)\right) \\ &\leq C_\mu^4 \mu\left(B\left(x_1, \frac{r}{4}\right)\right). \end{aligned}$$

From which it follows that

$$n\mu\left(B\left(x_1,\frac{r}{4}\right)\right)\leq C_\mu^4\mu\left(B\left(x_1,\frac{r}{4}\right)\right).$$

Combining all of the above,

$$n\leq C_\mu^4.$$

■

It can be proved that the reciprocal of the above theorem holds whenever X is a complete metric space [10].

Proposition 4.2. $(\mathbb{R}^n,||.||,\lambda^n)$ *is a DMMS.*

Proof. It is immediate to verify that the triple is a metric measure space. We need to show that λ^n is a doubling measure, for this, note that

$$\lambda^n(B(x,2r))=\omega_n(2r)^n=2^n\omega_n r^n=2^n\lambda^n(B(x,r)).$$

Furthermore, since $\omega_n>0$, the measure of every ball is positive. ■

From the previous proposition and Theorem 4.3, we conclude that $\mathbb{R}^n$ is a doubling space. In fact, it can be shown that the doubling constant depends on the dimension of the space. Below, we will give some properties of doubling spaces that are often mentioned or left as exercises in the literature. We have provided corresponding proofs in the hope that they may facilitate future consultations and references. I thank Professor Andres Sabino Diaz Castro of ESFM-IPN for his suggestions in some details of those proofs.

Lemma 4.2. *Let X be a doubling metric space with constant N. Then, for any $x\in X$ and $r>0$, we can cover $B(x,r)$ with at most N open balls of radius $\frac{r}{2}$, whose centers lie in $B(x,r)$. Conversely, if X is a metric space in which, for any $x\in X$ and $r>0$, the ball $B(x,r)$ can be covered by at most M open balls of radius $\frac{r}{2}$ with centers in $B(x,r)$, then X is doubling with constan M^2.*

Proof. Suppose X is doubling. Given arbitrary $x\in X$ and $r>0$, consider $B(x,r)$ and let $A=\{x_1,\ldots,x_k\}\subset B(x,r)$ be an $r/2-$ separated subset of $B(x,r)$. Without loss of generality, using the doubling condition we can that assume $k<N$. Indeed if $k=N$ then necessarily $B(x,r)\subset\cup_{i=1}^N B(x_i,r/2)$, because otherwise there would exists $x_{N+1}\in B(x,r)$ such that $A'=\{x_1,\ldots,x_N,x_{N+1}\}$ would be an $r/2-$separated subset of $B(x,r)$ which would be a contradiction (because X is doubling with constant N). Then we can assume that $k<N$ and we have two cases

1.- If $B(x,r)\subset\cup_{i=1}^k B(x_i,r/2)$, we are done.

2.- If $B(x,r)\not\subset\cup_{i=1}^k B(x_i,r/2)$, then there exist $x_{k+1}\in B(x,r)$ such that $x_{k+1}\notin B(x_i,r/2)$ for each $i=1,\ldots,k$. Thus $A'=\{x_1,\ldots,x_k,x_{k+1}\}$ is an $r/2-$separated subset of $B(x,r)$. Note the following:

If $k+1=N$, then necessarily $B(x,r)\subset\cup_{i=1}^{k+1}B(x_i,r/2)$, because otherwise there would exists $x_{N+1}\in B(x,r)$ such that $A''=\{x_1,\ldots,x_N,x_{N+1}\}$ would be an $r/2-$separated subset of $B(x,r)$ which would be a contradiction (because X is doubling with constant N).

If $k+1<N$, then we can proceed as in the first part of the proof, i.e., by continuing this

process with the same reasoning, which will eventually terminate since $N < \infty$.
Now, suppose every ball can be covered by at most M open balls of radius $r/2$. Given arbitrary $x \in X$ and $r > 0$, consider $B(x, r)$. From the hypothesis, we have

$$B(x,r) \subset \bigcup_{i=1}^{M} B(y_i, r/2),$$

for some $y_1, \dots y_M \in X$. Let $A = \{x_1, \dots, x_k\}$ be an $r/2-$separated subset of $B(x, r)$. Note that $\#(A \cap B(y_i, r/2)) \leq M$ for each $i = 1, \dots, M$. Since, if there exist $j \in \{1, \dots, M\}$ such that $\#(A \cap B(y_j, r/2)) > M$, by taking $r' = r/2$ and $s, t \in A \cap B(y_j, r')$, we would have

$$d(s,t) \geq \frac{r}{2} > \frac{r}{4} = \frac{r'}{2},$$

and thus, we could cover $B(y_j, r')$ with more than M open balls of radius $r'/2$ which would be a contradiction. Hence, $\#(A \cap B(y_i, r/2)) \leq M$ for each $i = 1, \dots, M$, consequently

$$\begin{aligned} \#(A) &\leq \# \left(\bigcup_{i=1}^{M} (A \cap B(y_i, r/2)) \right) \\ &\leq \sum_{i=1}^{M} \#(A \cap B(y_i, r/2)) \\ &\leq \sum_{i=1}^{M} M = M \sum_{i=1}^{M} 1 \\ &= M^2, \end{aligned}$$

which implies

$$\#(A) \leq M^2.$$

■

The proof of the following lemma is completely analogous (under an iteration argument) to the one in the previous result.

Lemma 4.3. *Let X be a doubling metric space with constant N, and let $k \in \mathbb{N}$. Then, any $\frac{r}{2^k}$-separated set in any ball $B(x, r)$ in X has at most N^k points.*

Lemma 4.4. *Every doubling metric space is separable.*

Proof. Let $x_0 \in X$ be fixed. Then,

$$X = \bigcup_{n \in \mathbb{N}} B(x_0, n).$$

It suffices to show that for each $n \in \mathbb{N}$, $B(x_0, n)$ is separable. To prove this result, note that for $n/2$, there exist $x_1^1, \dots, x_N^1$ elements of $B(x_0, n)$ such that

$$B(x_0, n) \subset \bigcup_{i=1}^{N} B(x_i^1, n/2).$$

Then, given $x \in B(x_0, n)$, there exist $i \in \{1, \ldots, N\}$ such that $x \in B(x_i, n/2)$, implying $d(x, x_i) < n/2$. By repeating this process for each $k \in \mathbb{N}$, we find $x_1^k, \ldots, x_{N^k}^k$ elements of $B(x_0, n)$ such that

$$B(x_0, n) \subset \bigcup_{i=1}^{N^k} B(x_i^k, n/2^k).$$

Thus, we can find x_i^k such that $d(x, x_i^k) < n/2^k$. Following this process for each $x \in B(x_0, n)$, we can find a sequence $\{y_j\}_{j\in\mathbb{N}}$ formed by the centers of some of the balls from the previous covers, with the property that

$$d(x, y_j) < \frac{n}{2^j},$$

thus

$$\lim_{j\to\infty} y_j = x.$$

From the construction above, the set $D_n := \{\{x_i^h\}_{i=1,\ldots,N^h}\}_{h\in\mathbb{N}}$ is countable and dense in $B(x_0, n)$ for each $n \in \mathbb{N}$. Thus, by taking $D = \cup_{n\in\mathbb{N}} D_n$, we have that D is a countable dense subset of X. ■

Definition 4.6. *A metric space is called **proper** if every closed ball in it is compact.*

Lemma 4.5. *If X is a doubling metric space, then its metric completion will also be doubling with the same constant. Furthermore, every complete and doubling metric space is proper.*

Proof. Let $\overline{X}$ be the completion of X as a metric space. Suppose there exist $x_0 \in \overline{X}$ and $r_0 > 0$ such that $B(x_0, r_0)$ has a subset $r_0/2-$separated with cardinality strictly greater than N, where N is the doubling constant of X. Let A be such a subset, and without loss of generality, assume that $\#(A) = N + 1$. Then we can write $A = \{x_1, \ldots, x_{N+1}\}$, since $\varphi(X)$ is dense in $\overline{X}$, we can find sequences $\{x_n^0\}_{n\in\mathbb{N}}, \{x_n^1\}_{n\in\mathbb{N}}, \ldots, \{x_n^{N+1}\}_{n\in\mathbb{N}}$ in X such that

$$\varphi(x_n^0) \xrightarrow[n\to\infty]{} x_0$$

$$\varphi(x_n^1) \xrightarrow[n\to\infty]{} x_1$$

$$\vdots$$

$$\varphi(x_n^{N+1}) \xrightarrow[n\to\infty]{} x_{N+1},$$

where we can find $n_0, n_1, \ldots, n_{N+1} \in \mathbb{N}$ sufficiently large such that

$$d(\varphi(x_{n_i}^i), \varphi(x_{n_j}^j)) \geq \frac{r_0}{2}, \quad i \neq j \text{ in the set } \{1, \ldots, N+1\}, \tag{4.6}$$

and

$$d(\varphi(x_{n_0}^0), \varphi(x_{n_i}^i)) < r_0, \quad \forall i \in \{1, \ldots, N+1\}. \tag{4.7}$$

Using the fact that φ is an isometry, from (4.6) and (4.7), we have that $A' = \{x_{n_1}^1, \ldots, x_{n_{N+1}}^{N+1}\}$ is an $r_0/2-$separated subset of $B(x_{n_0}^0, r_0)$ with cardinality $N+1 > N$, which is a contradiction. Thus, $\overline{X}$ is doubling with constant N.
For the second assertion, if X is complete and doubling. By Lemmas 4.2 and 4.3, it follows that $B(x, r)$ is totally bounded for each $x \in X$ and $r > 0$. Then $\overline{B}(x, r)$ is totally bounded, and using the fact that in every complete metric space, the property of being totally bounded implies compactness, we have that X is proper. ■

The previous lemmas will be applied in the following result, which tells us that open subsets of doubling spaces can be covered by balls that constitute a covering akin to the classical Whitney decomposition of open subsets of $\mathbb{R}^n$.

Theorem 4.4. ***(Whitney decomposition)*** *Let $X = (X, d)$ be a doubling metric space with constant N, and let A be an open subset of X such that $X \setminus A \neq \emptyset$. Then there exist a countable collection $\mathcal{W}_A = \{B(x_i, r_i)\}_{i\in\mathbb{N}}$ of open balls in A such that*

$$A = \bigcup_i B(x_i, r_i), \tag{4.8}$$

and

$$\sum_i \chi_{B(x_i,2r_i)} \leq 2N^5, \tag{4.9}$$

where

$$r_i = \frac{1}{8}\operatorname{dist}(x_i, X \setminus A). \tag{4.10}$$

Proof. For $x \in A$, we define $d(x) := \operatorname{dist}(x, X \setminus A)$. Also, for each $k \in \mathbb{N}$ we set

$$\mathcal{F}_k := \left\{ B\left(x, \frac{1}{40}d(x)\right) \ : \ x \in A \quad \text{and} \quad 2^{k-1} < d(x) \leq 2^k \right\}.$$

By the $5B$ Covering Theorem (see Theorem A.3 of Appendix A), we can find a countable and disjoint subfamily $\mathcal{G}_k$ of $\mathcal{F}_k$ such that

$$\bigcup_{B\in\mathcal{F}_k} B \subset \bigcup_{B\in\mathcal{G}_k} 5B.$$

We claim that

$$\mathcal{W}_A = \bigcup_{k=1}^{\infty}\{5B : B \in \mathcal{G}_k\},$$

satisfies (4.8) - (4.10). Indeed, from the construction, (4.8) and (4.10) are satisfied. To prove (4.9), take $x \in A$ and suppose that x is in M balls of the form $2B$, where $B \in \mathcal{W}_A$. Without loss of generality, we write these balls as $B\left(x_1, \frac{1}{4}d(x_1)\right)$, $B\left(x_2, \frac{1}{4}d(x_2)\right)$, ..., $B\left(x_M, \frac{1}{4}d(x_M)\right)$, with $d(x_1) \geq d(x_i)$ for each $i = 1, \ldots, M$. Then, for each $i = 1, \ldots, M$, we have

$$d(x_i) \geq \frac{3}{5}d(x_1) \tag{4.11}$$

and

$$B\left(x_i, \frac{1}{4}d(x_i)\right) \subset B\left(x_1, \frac{3}{4}d(x_1)\right). \tag{4.12}$$

To prove (4.11), note that since $x \in B\left(x_1, \frac{1}{4}d(x_1)\right) \cap B\left(x_i, \frac{1}{4}d(x_i)\right)$, then

$$d(x_1, x_i) \leq d(x_1, x) + d(x, x_i) < \frac{1}{4}d(x_1) + \frac{1}{4}d(x_i). \tag{4.13}$$

Thus, from this and using the definition of $d(x_1)$, for any $y \in X \setminus A$, we have

$$\begin{aligned} d(x_1) &\leq d(x_1, y) \\ &\leq d(x_1, x_i) + d(x_i, y) \\ &< \frac{1}{4}d(x_1) + \frac{1}{4}d(x_i) + d(x_i, y). \end{aligned}$$

Hence

$$\frac{3}{4}d(x_1) - \frac{1}{4}d(x_i) < d(x_i, y).$$

Taking the infimum on the right-hand side with respect to $y \in X \setminus A$, we obtain

$$\frac{3}{4}d(x_1) - \frac{1}{4}d(x_i) \leq d(x_i).$$

From this result, (4.11) follows. To verify (4.12), note that if $y \in B\left(x_i, \frac{1}{4}d(x_i)\right)$, then

$$\begin{aligned} d(x_1, y) &\leq d(x_1, x_i) + d(x_i, y) \\ &< \frac{1}{4}d(x_1) + \frac{1}{4}d(x_i) + \frac{1}{4}d(x_i) \\ &\leq \frac{3}{4}d(x_1). \end{aligned}$$

Here, we have used (4.13) and the assumption made about $d(x_1)$ and $d(x_i)$. Also note that if x_i and x_j are centers of balls in the same family $\mathcal{F}_k$, then

$$d(x_i, x_j) \geq \frac{1}{20}\min\{d(x_i), d(x_j)\} \geq \frac{1}{40}d(x_1),$$

whenever $i \neq j$. From this and (4.12), we conclude that there exist a ball with radius $\frac{3}{4}d(x_1)$ that contains a $\frac{1}{40}d(x_1)$-separated set of M elements. Then, from Lemma 4.3, at most N^5 balls can have their centers in $\mathcal{F}_k$, for a fixed k. Now, if $x_1 \in \mathcal{F}_{k_1}$, then

$$2^{k_1-1} < d(x_1) \leq 2^{k_1}. \tag{4.14}$$

Also, from (4.11) and the given assumptions and hypothesis,

$$\frac{3}{5}d(x_1) \leq d(x_i) \leq d(x_1). \tag{4.15}$$

Thus, from (4.14) and (4.15), it follows that

$$2^{k_1-2} < d(x_i) \leq 2^{k_1}.$$

Therefore $x_i \in \mathcal{F}_{k_1} \cup \mathcal{F}_{k_1-1}$. Since in these sets there can be at most N^5 balls, we can deduce the conclusion (4.9). ■

In the previous result, it is important to consider $\operatorname{dist}(x, X \setminus A)$ instead of $\operatorname{dist}(x, \partial A)$, because it may happen that $B(x, d(x, \partial A))$ intersects $X \setminus A$.

In the following, we will use the hypotheses and notation introduced in Theorem 4.4. Given $B(x_i, r_i) \in \mathcal{W}_A$, we define for $x \in A$

$$\psi_i(x) := \min\left\{\frac{1}{r_i}\operatorname{dist}(x, X \setminus B(x_i, 2r_i)), 1\right\}.$$

Note that ψ_i is a $\frac{1}{r_i}$-Lipschitz function (since it is the minimum of two $\frac{1}{r_i}$-Lipschitz functions). Also, from Theorem 4.4, we have that

$$1 \leq \sum_i \psi_i \leq 2N^5.$$

Define

$$\varphi_i(x) := \frac{\psi_i(x)}{\sum_k \psi_k(x)}.$$

We can note that the functions φ_i satisfy the following properties:

1.- $\varphi_i(x) = 0$ if $x \notin B(x_i, 2r_i)$. Moreover, $x \in A$, $\varphi_i(x) \neq 0$, for all $x \in A$ and at most $2N^5$ indices $i \in \mathbb{N}$.

2.- $0 \leq \varphi_i \leq 1$ and $\varphi_i|_{B(x_i,2r_i)} \geq \frac{1}{2N^5}$, for each $i \in \mathbb{N}$.

3.- φ_i is $\frac{5N^5}{r_i}-$Lipschitz, for each $i \in \mathbb{N}$.

4.- $\sum_i \varphi_i(x) = 1$, for each $x \in A$ and $i \in \mathbb{N}$.

A collection $\{\varphi_i\}$ that satisfies the above conditions is called a **Lipschitz partition of unity** of the open set A.
We present the following theorem on the extension of Lipschitz functions when the codomain is a Banach space.

Theorem 4.5. *Let $X = (X, d)$ be a doubling metric space and $f : E \subset X \to V$ be an L-Lipschitz function from E into the Banach space $(V, |\cdot|_V)$. Then there exist a CL-Lipschitz function $F : X \to V$ such that $F|_E = f$, where C is a constant depending only on the doubling constant of X.*

Proof. Without loss of generality, assume that X is complete (since we can isometrically embed X into its metric completion). Then, since X is complete, by Lemma 4.5, we have that X is proper. Also, since we can uniquely extend f to the closure of E, we can assume that E is closed. With all this said, let $\{\varphi_i\}$ be a Lipschitz partition of unity of the open set $A := X \setminus E$. For each index i, since X is proper and A is closed, we can find $y_i \in E$ such that

$$8r_i = \text{dist}(x_i, A) = d(x_i, y_i).$$

Define

$$F(x) = \begin{cases} \sum_i \varphi_i(x) f(y_i) & \text{if } x \in A. \\ f(x) & \text{if } x \in E. \end{cases}$$

Note that F is well-defined, as if $x \in A$,

$$F(x) = \sum_i \varphi_i(x) f(y_i) \leq \sum_{i=1}^{2N^5} f(y_i) < \infty.$$

Here, N is the doubling constant of X. Also, as F is defined, it is immediately observed that F is indeed an extension of f.

In the following we will denote by $|\cdot|_V$ the metric on the Banach space V. Now, we need to verify that F is a $CL-$Lipschitz function, where C is a constant (depending only of the doubling constant N), for which we have the following cases:

Case 1) Suppose that $x \in A$ and $y \in E$, then

$$\begin{aligned}|F(x) - F(y)|_V &= \left|\sum_i \varphi_i(x) f(y_i) - f(y)\right|_V \\ &= \left|\sum_i \varphi_i(x) f(y_i) - \sum_i \varphi_i(x) f(y)\right|_V \\ &= \left|\sum_i (f(y_i) - f(y))\varphi_i(x)\right|_V \\ &\leq \sum_i \varphi_i(x)|f(y_i) - f(y)|_V \\ &\leq 2N^5 \max_{x \in B(x_i, r_i)} |f(y_i) - f(y)|_V \\ &\leq 2N^5 L \max_{x \in B(x_i, r_i)} d(y, y_i) \\ &\leq \frac{16}{3} N^5 L d(x, y),\end{aligned}$$

where the last inequality follows from

$$\begin{aligned}d(y, y_i) &\leq d(y, x) + d(x, x_i) + d(x_i, y_i) \\ &\leq d(x, y) + \frac{5}{4} d(x_i, y_i) \\ &\leq \frac{8}{3} d(x, y),\end{aligned} \tag{4.16}$$

since

$$\begin{aligned}d(x_i, y_i) &\leq d(x_i, y) \\ &\leq d(x_i, x) + d(x, y) \\ &< \frac{1}{4} d(x_i, y_i) + d(x, y) \\ &\leq \frac{1}{4}\left(\frac{1}{4} d(x_i, y_i) + d(x, y)\right) + d(x, y) \\ &= \frac{1}{4^2} d(x_i, y_i) + \frac{1}{4^0} d(x, y) + \frac{1}{4} d(x, y) \\ &\leq \ldots \leq \frac{1}{4^n} d(x_i, y_i) + d(x, y) \sum_{k=0}^{n} \frac{1}{4^k},\end{aligned}$$

thus

$$\begin{aligned}d(x_i, y_i) &\leq \lim_{n \to \infty} \frac{1}{4^n} d(x, y) + d(x, y) \sum_{n=0}^{\infty} \frac{1}{4^n} \\ &= \frac{4}{3} d(x, y),\end{aligned}$$

and therefore we have (4.16).
Then taking $C_1(N) = \frac{16}{3} N^5$, the above results imply that

$$|F(x) - F(y)|_V \leq C_1 L d(x, y). \tag{4.17}$$

Case 2) Suppose that a and b lie in the intersection $A \cap B\left(x_j, \frac{1}{4}r_j\right)$ for some j. Then

$$\begin{aligned}
|F(a)-F(b)|_V &= \left|\sum_i (\varphi_i(a)-\varphi_i(b)) f(y_i)\right|_V \\
&= \left|\sum_i (\varphi_i(a)-\varphi_i(b))(f(y_i)-f(y_j))\right|_V \\
&\leq \sum_i |\varphi_i(a)-\varphi_i(b)||f(y_i)-f(y_j)|_V \\
&\leq \sum_i Ld(y_i,y_j)|\varphi_i(a)-\varphi_i(b)|.
\end{aligned}$$

Since y_j is fixed, we can replicate what was done in (4.16) to obtain

$$d(y_i,y_j) \leq \frac{8}{3}d(x_j,y_j) = \frac{8}{3}r_j.$$

Hence,

$$|F(a)-F(b)|_V \leq L\frac{8}{3}\sum_i r_j|\varphi_i(a)-\varphi_i(b)|.$$

Then, since each φ_i is $\frac{5N^5}{r_i}-$Lipschitz, we have

$$|F(a)-F(b)|_V \leq L\frac{8}{3}\sum_i \frac{5N^5 r_j}{r_i}d(a,b). \tag{4.18}$$

Following a similar reasoning as that done in (4.11), if $B(x_i,2r_i)\cap B(x_j,2r_j) \neq \emptyset$, then $r_i \geq \frac{3}{5}r_j$, so

$$\begin{aligned}
L\frac{8}{3}\sum_i \frac{5N^5 r_j}{r_i}d(a,b) &\leq L\frac{40}{9}5N^5 4N^{10} d(a,b) \\
&= \frac{800}{9}N^{15}Ld(a,b).
\end{aligned} \tag{4.19}$$

Setting $C_2(N) = \frac{800}{9}N^{15}$, from (4.18) and (4.19) we have

$$|F(a)-F(b)|_V \leq C_2 Ld(a,b). \tag{4.20}$$

Case 3) Suppose that $a, b \in A$ and $b \notin B(x_k, 2r_k)$, for any k, such that $a \in B(x_k, r_k)$. Let $y_a, y_b \in E$ be such that $d(a,E) = d(a,y_a)$ and $d(b,E) = d(b,y_b)$. Then, from the estimates

$$d(a,y_a) < 4d(a,b), \quad d(b,y_b) < d(a,b) \quad \text{and} \quad d(y_a,y_b) \leq d(a,y_a) + d(a,b) + d(b,y_b),$$

and using case 1, we obtain

$$\begin{aligned}
|F(a)-F(b)|_V &\leq |F(a)-F(y_a)|_V + |F(y_a)-F(y_b)|_V + |F(y_b)-F(b)|_V \\
&\leq C_1(N)Ld(a,y_a) + Ld(y_a,y_b) + C_1(N)Ld(b,y_b) \\
&\leq 4C_1(N)Ld(a,b) + L(d(a,y_a)+d(a,b)+d(b,y_b)) + 4C_1(N)Ld(a,b) \\
&\leq C_1(N)L4d(a,b) + 9Ld(a,b) + C_1(N)L4d(a,b) \\
&= (8C_1(N)+9)Ld(a,b).
\end{aligned}$$

Thus, taking $C_3(N) = 8C_1(N) + 9 = \frac{128}{3}N^5 + 9$, we have

$$|F(a) - F(b)|_V \leq C_3(N)Ld(a,b). \tag{4.21}$$

Then, if $C := \max\{C_1(N), C_2(N), C_3(N)\}$, from (4.17), (4.20) and (4.21) we conclude that for any $x, y \in X$

$$|F(a) - F(b)|_V \leq CLd(a,b),$$

with C depending only on the doubling constant N. ■

4.2.2 Lipschitz functions in L^p spaces

Lipschitz functions are of significant importance in L^p spaces due to their stability under integration, usefulness in approximation tasks, smoothness and regularity properties, favorable convergence behavior, and applications in functional analysis.
In this section some results concerning the classical theory of integration and the Bochner integral will be useful, which can be consulted in [8], [14] and [15].

Definition 4.7. *Let $X = (X, d)$ be a metric space. A function $f : X \to]-\infty, \infty]$ is said to be* ***lower semicontinuous at*** *$x \in X$, if for every $\varepsilon > 0$ there exist $\delta > 0$ such that*

$$f(x) - \varepsilon < f(y), \quad \forall y \in B(x, \delta).$$

We say that f is ***lower semicontinuous on*** *X, if it is lower semicontinuous at every point in X.*

The following proposition gives us a useful characterization of the lower semi-continuity condition.

Proposition 4.3. *The following statements are equivalent for a function $f : X \to]-\infty, \infty]$ on a metric space $X = (X, d)$:*
1.- f is lower semicontinuous on X
2.- For each $a \in \mathbb{R}$, the set $\{x \in X : f(x) > a\}$ is open.
3.- For any $x \in X$ and any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$, we have

$$\liminf_{n\to\infty} f(x_n) \geq f(x).$$

Example 4.3. *Let A be an open subset of X and $a \in \mathbb{R}$. Then,*

$$\chi_A^{-1}(a, \infty] = \begin{cases} X & \textit{if } a < 0. \\ A & \textit{if } 0 \leq a < 1. \\ \emptyset & \textit{if } t \geq 1. \end{cases}$$

Thus, we can observe that χ_A is lower semicontinuous if and only if A is open. ◀

Example 4.4. *Let (X, μ) be a measure space and $L^+(X, \mu)$ be the set of non-negative measurable functions equipped with the topology induced by convergence in measure μ. By Fatou's Lemma, the operator $\int_X (\cdot) d\mu : L^+(X, \mu) \to]-\infty, \infty]$ is lower semicontinuous.* ◀

The following two results provide operational properties of lower semicontinuous functions.

Proposition 4.4. *The class of lower semicontinuous functions on a metric space X forms a positive cone. Moreover, the pointwise supremum of any arbitrary family of lower semicontinuous functions is also lower semicontinuous.*

Corollary 4.2. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of lower semicontinuous functions of a metric space X on $[0,\infty]$. Then $\sum_{n=1}^{\infty} f_n$ is lower semicontinuous.*

Proof. Let $u \in [0,\infty)$. We want to show that

$$U = \left\{ x : \sum_{n\geq 1} f_n(x) > u \right\},$$

is an open set. Note, that if $x \in U$, since $f_n \geq 0$ for each $n \in \mathbb{N}$, we have that

$$\sum_{n\geq 1} f_n(x) = \sup_{N\geq 1} \sum_{n=1}^{N} f_n(x) > u,$$

so there exist $N_x \geq 1$ such that $\sum_{n=1}^{N_x} f_n(x) > u$. Therefore, $x \in U_{N_x} \subset U$, where $U_{N_x} := \left\{ y : \sum_{n=1}^{N_x} f_n(y) > u \right\}$ is open (because the finite sum of lower semicontinuous functions is lower semicontinuous). ■

Proposition 4.5. *Let $X = (X,d)$ be a metric space, $c \in \mathbb{R}$ and $f : X \to [c,\infty]$ be a lower semicontinuous function. Then there exist a sequence $\{f_k\}_{k\in\mathbb{N}}$ of Lipschitz functions of X on $\mathbb{R}$, such that*

$$c \leq f_k(x) \leq f_{k+1}(x) \leq f(x)$$

and

$$\lim_{k\to\infty} f_k(x) = f(x), \quad \forall x \in X.$$

Proof. For each $k \in \mathbb{N}$, define

$$f_k(x) = \inf\{f(y) + kd(x,y) : y \in X\}.$$

Then f_k is a $k-$Lipschitz function for each $k \in \mathbb{N}$ (since it is the infimum of a family of k-Lipschitz functions). Also, note that for any $y \in X$, $f(y) + kd(x,y) \leq f(y) + (k+1)d(x,y)$. Then, by taking the infimum on both sides over $y \in X$, we obtain

$$f_k \leq f_{k+1}. \tag{4.22}$$

Moreover, from the definition of f_k, we obtain

$$f_k(x) \leq f(x) + kd(x,x) = f(x), \ \ \forall k \in \mathbb{N}. \tag{4.23}$$

Considering the fact that $f : X \to [c,\infty]$ and (4.22), we get

$$c \leq c + d(x,y) \leq f(x) + d(x,y) \Rightarrow c \leq f_1(x) \Rightarrow c \leq f_k(x), \quad \forall k \in \mathbb{N}. \tag{4.24}$$

Then, from equations (4.22), (4.23), and (4.24), we have that

$$c \leq f_k(x) \leq f_{k+1}(x) \leq f(x), \quad \forall x \in X. \tag{4.25}$$

To verify pointwise convergence, suppose initially that $f(x) = \infty$. For any $M > 0$, choose $\varepsilon > 0$ such that $f > M$ in $B(x, \varepsilon)$. Thus $f_k(x)$ is at least the minimum between M and $c + k\varepsilon$. For each sufficiently large k such that $c + k\varepsilon > M$, we have $f_k(x) \geq M$. Therefore

$$\lim_{k\to\infty} f_k(x) = \infty = f(x). \tag{4.26}$$

Suppose now that $f(x) < \infty$. For any $\varepsilon > 0$ there exist $x_k \in X$ such that

$$f_k(x) \geq f(x_k) + kd(x, x_k) - \varepsilon. \tag{4.27}$$

From (4.25)

$$\begin{aligned} f(x) &\geq f_k(x) \\ &\geq f(x_k) + k(d(x, x_k) - \varepsilon \\ &\geq c + kd(x, x_k) - \varepsilon. \end{aligned} \tag{4.28}$$

Thus,

$$\frac{f(x) - c + \varepsilon}{k} \geq d(x, x_k) \geq 0,$$

hence

$$d(x, x_k) \to 0,$$

and consequently $x_k \to x$. Using the fact that f is lower semicontinuous and (4.27), we obtain

$$\begin{aligned} \liminf_{k\to\infty} f_k(x) &\geq \liminf_{k\to\infty} [f(x_k) + kd(x, x_k) - \varepsilon] \\ &\geq \liminf_{k\to\infty} [f(x_k)] - \varepsilon \\ &\geq f(x) - \varepsilon. \end{aligned} \tag{4.29}$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\liminf_{k\to\infty} f_k(x) \geq f(x). \tag{4.30}$$

Also, from construction of f_k

$$\limsup_{k\to\infty} f_k(x) \leq f(x). \tag{4.31}$$

From (4.30) and (4.31), it follows that

$$\lim_{k\to\infty} f_k(x) = f(x). \tag{4.32}$$

Thus, from (4.26) and (4.32), we can deduce the conclusion. ■

Corollary 4.3. *Let $X = (X, d, \mu)$ be a metric space, $1 \leq p < \infty$ and $f : X \to [0, \infty]$ be a lower semicontinuous $p-$integrable function. Then there exist a sequence $\{f_k\}_{k\in\mathbb{N}}$ of Lipschitz functions of X on $\mathbb{R}$, such that*

$$0 \leq f_k \leq f_{k+1} \leq f,$$

and $f_k \to f$ both pointwise and in $L^p(X, \mu)$ as $k \to \infty$.

Proof. It follows from the previous proposition and the Dominated Convergence Theorem. ■

In every metric measure space nonnegative p-integrable functions can be approximated in L^p by a pointwise decreasing sequence of lower semicontinuous functions. This is the so called Vitali Carathéodory theorem, has turned out to be handy in the geometric theory of Sobolev spaces.

Theorem 4.6. ***(Vitali Caratheodory theorem)*** *Let $X = (X, d, \mu)$ be a metric measure space, $1 \leq p < \infty$ and $f : X \to [0, \infty]$ be a $p-$integrable function. There exist a pointwise decreasing sequence $\{g_k\}_{k\in\mathbb{N}}$ of lower semicontinuous functions on X, such that*

$$f \leq g_{k+1} \leq g_k$$

and $g_k \to f$ in $L^p(X, \mu)$.

Proof. Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be an increasing sequence of non negative simple functions converging to f, then we can express f as

$$f = \varphi_1 + \sum_{k=2}^{\infty}(\varphi_k - \varphi_{k-1}).$$

Moreover, using the representation of each φ_k as a simple function, we have that

$$f = \sum_{j=0}^{\infty} a_j \chi_{E_j},$$

with $a_0 = \infty$, $a_j \in]0, \infty[$ for $j \geq 1$, and $E_j \subset X$ is $\mu-$measurable for all $j = 0, 1, \ldots$. Note that since f is $p-$integrable, we have $\mu(E_0) = 0$. Now given $\varepsilon > 0$, for each $j \geq 1$ we can find an open set A_j, such that $E_j \subset A_j$ and

$$\mu(A_j) \leq \mu(E_j) + \frac{\varepsilon^p}{2^{jp}a_j^p}.$$

Furthermore, we can find a sequence of open sets O_j such that $E_0 \subset O_j$ and

$$\mu(O_j) \leq \frac{\varepsilon^p}{2^{jp}},$$

for each $j \geq 1$. Now define

$$g := \sum_{j=1}^{\infty} a_j \chi_{A_j} + \sum_{j=1}^{\infty} \chi_{O_j}.$$

From Corollary 4.2, we know that g is lower semicontinuous. Furthermore, from the construction of g, it follows that $f \leq g$ on X and

$$\begin{aligned}||g - f||_p &= \left|\left|\sum_{j=1}^{\infty} a_j \chi_{A_j} + \sum_{j=1}^{\infty} \chi_{O_j} - \sum_{j=0}^{\infty} a_j \chi_{E_j}\right|\right|_p \\ &\leq \sum_{j=1}^{\infty} ||a_j(\chi_{A_j} - \chi_{E_j}||_p + \sum_{j=1}^{\infty} ||\chi_{O_j}||_p \\ &= \sum_{j=1}^{\infty} a_j ||\chi_{A_j\setminus E_j}||_p + \sum_{j=1}^{\infty} ||\chi_{O_j}||_p\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} a_j \mu(A_j \setminus E_j)^{\frac{1}{p}} + \sum_{j=1}^{\infty} \mu(O_j)^{\frac{1}{p}} \\
&\leq \sum_{j=1}^{\infty} a_j \left(\frac{\varepsilon^p}{2^{jp} a_j^p} \right)^{\frac{1}{p}} + \sum_{j=1}^{\infty} \left(\frac{\varepsilon^p}{2^{jp}} \right)^{\frac{1}{p}} \\
&= \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} + \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} \\
&= 2\varepsilon.
\end{aligned}
$$

From this, we obtain that

$$||g - f||_p \leq 2\varepsilon.$$

Finally, to construct the desired sequence, first, for $\varepsilon = 1$, there exist a lower semicontinuous function h_1 such that $||h_1 - f||_p \leq 1$. Also, for $\varepsilon = \frac{1}{2}$, there exist a lower semicontinuous function h_2 such that $||h_2 - f||_p < \frac{1}{2}$. Following this process for each $k \in \mathbb{N}$, there exist a lower semicontinuous function h_k such that $||h_k - f||_p \leq \frac{1}{k}$. Defining $g_k := \min\{h_1, \ldots, h_k\}$, we have that $\{g_k\}_{k\in\mathbb{N}}$ is a sequence of lower semicontinuous functions (since it is the minimum of lower semicontinuous functions) and, by construction, $g_{k+1} \leq g_k$. Moreover, by the previous argument and the made construction, $f \leq g_k$ and $\lim_{k\to\infty} g_k(x) = f(x)$ in $L^p(X, \mu)$ because

$$\lim_{k\to\infty} ||g_k - f||_p \leq \lim_{k\to\infty} \frac{1}{k}.$$

■

Theorem 4.7. *Let (X, d, μ) be a metric measure space, $1 \leq p < \infty$ and V be a Banach space. Then Lipschitz functions are dense in $L^p(X, V, \mu)$. If in addition (X, d) is locally compact then Lipschitz functions with compact support are dense in $L^p(X, V, \mu)$.*

4.3 Area formula on metric spaces

The idea of including this section (and in general this chapter) is to show that, using analogous ideas to the those developed in Chapters 1, 2 and 3, we can extend the Area formula to metric spaces. This generalization of the Area formula first appears in the article *Rectifiable metric spaces: Local estructure and regularity of the Hausdorff measure* published by Bernd Kirchhein in 1994, later this work was retomated by Kirchhein and Ambrosio for a more general approach. The ideas and concepts used for the proofs that appear in these works are more specialized and by themselves would need a very careful development, for this reason we will omit them, because as we have mentioned at this moment we are only interested in showing the scope of this theory, however for an interested reader, we refer to [9] and [2] for the consultation of these proofs.

In order to deduce this generalization of the area formula, we can only work with Lipschitz functions from $\mathbb{R}^n$ to V, with $(V, |\cdot|_V)$ be an Banach space. We only need to work in Banach spaces over $\mathbb{R}$, because by Fréchet's and Kuratowski theorems (4.2 and 4.1), every metric space admits an isometric embedding in some Banach space. This is useful because in Banach spaces additionally to the good properties in the metric sense, we have a vector space structure.

In Euclidean spaces, we can think of an the area formula as not so straightforward application of Rademacher's theorem. If we can try to extended the results of the previous chapter, firstly we need to extend the notions of Lipschitz function and Hausdorff measure to metric spaces, which we have already done in the previous sections of this chapter, also we like to have extension theorems for Lipschitz function, because we would like to consider functions with domain $\mathbb{R}^n$ instead of have domain only in a subset of $\mathbb{R}^n$. For this point Theorem 4.5 will be useful. Indeed given $E \subset \mathbb{R}^n$ if V is a Banach space then as $\mathbb{R}^n$ is a doubling space using the mentioned theorem we can find a extension $F : \mathbb{R}^n \to V$ Lipschitz, then the assumption that domain of f is $\mathbb{R}^n$ it is not an restriction.

Finally, given $E \subset \mathbb{R}^n$ be a Lebesgue measurable subset on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to V$ Lipschitz and injective, we would like have a expression with the form

$$\mathcal{H}^n(f(E)) = \int_E (\dots) d\lambda^n,$$

such as Theorem 3.2 and in a similar way a expression to Theorem 3.5. In the last chapter the integrand is a function that depends to the differential of f, but in the case of this chapter we work in metric spaces, so we need introduce a notion of differentiability. For this purpose firstly we have the next results and definition.

Theorem 4.8. *Let $f : \mathbb{R}^n \to (V, |\cdot|_V)$ be a Lispchitz function, where $(V, |\cdot|_V)$ is a Banach space. If $u \in \mathbb{S}^{n-1}$ is arbitrary, then:*
1) λ^n-a.e. $x \in \mathbb{R}^n$ the following limit exists

$$\lim_{h\to 0^+} \frac{|f(x+hu) - f(x)|_V}{h}. \tag{4.33}$$

2) Given $x \in \mathbb{R}^n$, if (4.33) exists, then $y \mapsto \lim_{h\to 0^+} \frac{|f(x+hy)-f(x)|_V}{h}$ is a seminorm on $\mathbb{R}^n$, and

$$|f(z) - f(y)|_V - \lim_{h\to 0^+} \frac{|f(x+h(z-y)) - f(x)|_V}{h} = o(||z-x|| + ||y-x||), \quad \forall y, z \in \mathbb{R}^n. \tag{4.34}$$

Definition 4.8. *Let $f : \mathbb{R}^n \to (V, |\cdot|_V)$ be a Lipschitz function, where $(V, |\cdot|_V)$ is a Banach space, and $x \in \mathbb{R}^n$. In case that there is a seminorm s_x on $\mathbb{R}^n$ satisfying*

$$|f(z) - f(y)|_V - s_x(z-y) = o(||z-x|| + ||y-x||), \quad \forall\ y, z \in \mathbb{R}^n.$$

*We say that f is **metrically differentiable at** x, and we call s_x the **metric differential of** f **at** x.*

Suppose that s_x is the metric differential of f at x, then using (4.34), by uniqueness we can see that

$$s_x(y) = \lim_{h\to 0^+} \frac{|f(x+hy) - f(x)|_V}{h}, \quad \forall y \in \mathbb{R}^n.$$

We denoted the metric differential of f at x as $md_x f$, so

$$md_x f(y) = \lim_{h\to 0^+} \frac{|f(x+hy) - f(x)|_V}{h}, \quad \forall y \in \mathbb{R}^n.$$

Note that, Theorem 4.8 represents an analogue to Rademacher's Theorem for the metric differential. Using the new terminology we can state Theorem 4.8 as:

Let $f : \mathbb{R}^n \to V$ be a Lipschitz function, then f is metrically differentiable λ^n-a.e. $x \in \mathbb{R}^n$.

Remark 4.1. *Let $f : \mathbb{R}^n \to V$ be Lipschitz and $x \in \mathbb{R}^n$, such that $md_x f$ there exists. Given $\lambda \in \mathbb{R}$, with $\lambda \geq 0$, using the change $\lambda h \mapsto t$, note that*

$$\begin{aligned} md_x f(\lambda y) &= \lim_{h\to 0^+} \frac{|f(x+h(\lambda y)) - f(x)|_V}{h} \\ &= \lambda \lim_{h\to 0^+} \frac{|f(x+(\lambda h)y) - f(x)|_V}{\lambda h} \\ &= \lambda \lim_{t\to 0^+} \frac{|f(x+ty) - f(x)|_V}{t} \\ &= \lambda md_x f(y). \end{aligned}$$

Hence for any $\lambda \geq 0$

$$md_x f(\lambda y) = \lambda md_x f(y), \quad \forall y \in \mathbb{R}^n.$$

◁

The above does not represent the classical generalization of differentiability in Banach spaces, since this role is played by the well-known differentiability in the Fréchet sense. In the case where the Fréchet derivative $Df(x)$ exists at $x \in \mathbb{R}^n$, it is possible to prove that

$$md_x f(y) = |Df(x)(y)|_V, \quad \forall y \in \mathbb{R}^n.$$

In [2] additionally to the definition of metric differentiability, they use the notion of w^*- differentiability to show an "Rademacher's-type" theorem for the w^*- differentiability of Lipschitz functions. The notion of w^*- differentiability that they use is:

Let $W = V^*$, with V a Banach space and let $f : \mathbb{R}^n \to W$ be a function. We say that f is w^*- differentiable at x if there exists a linear map $L : \mathbb{R}^n \to W$ satisfying

$$w^*\text{-}\lim_{y\to x} \frac{f(y) - f(x) - L(y-x)}{||y-x||} = 0.$$

This map L will be said to be the w^*- differential of f at x and it will be denoted by $wd_x f$.

Let $W = V^*$, with V a Banach space, if $\mathcal{N}$ denotes the operator norm on $W = V^*$, we can see that using the w^*- lower semicontinuity of the norm, the metric differential and the w^*- differential are related by

$$\mathcal{N}(wd_x f(y)) \leq md_x f(y), \quad \forall y \in \mathbb{R}^n$$

And in the particular case of Lipschitz functions, we have the following:

Theorem 4.9. *Let $W = V^*$, with V a separable Banach space. Any Lipschitz function $f : \mathbb{R}^n \to W$ is w^*- differentiable and*

$$\mathcal{N}(wd_x f(y)) = md_x f(y), \quad \forall y \in \mathbb{R}^n,$$

for λ^n-a.e. $x \in \mathbb{R}^n$.

Now, at this moment, we have a notion of differentiability, naturally the next step is define a Jacobian, for this reason we introduce the next definition.

Definition 4.9. *Let s be a seminorm on $\mathbb{R}^n$. We define the* ***Jacobian of*** *s by*

$$J(s) := \frac{\omega_n}{\mathcal{H}^n(\{x \in \mathbb{R}^n : s(x) \leq 1\})},$$

with ω_n given by

$$\omega_n := \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1 + \frac{n}{2}\right)}.$$

For a Lipschitz function $f : \mathbb{R}^n \to V$, we denote by $\mathcal{MD}(f)$ the set of all $x \in \mathbb{R}^n$ where the metric differential exists and by $\mathcal{MD}_r(f)$ the subset of $\mathcal{MD}(f)$ where it is a norm on $\mathbb{R}^n$. We can see that both $\mathcal{MD}(f)$ and $\mathcal{MD}_r(f)$ are Borel sets.
We can think at the following lemma as an analogue of Theorem 3.3 (Lipschitz linearization).

Lemma 4.6. *Let $f : \mathbb{R}^n \to V$ be a Lipschitz function and $t > 1$. Then there are Borel subsets $\{E_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}^n$ and norms $\{n(\cdot)_k\}_{k\in\mathbb{N}}$ on $\mathbb{R}^n$ such that*
1.-

$$\bigcup_{k\in\mathbb{N}} E_k = \mathcal{MD}_r(f).$$

2.-

$$t^{-1}n(x-y)_k \leq |f(x) - f(y)|_V \leq tn(x-y)_k, \quad \forall x, y \in E_k.$$

Lemma 4.7. *Let $n(\cdot)$ be a norm on $\mathbb{R}^n$. Then*

1.- $\mathcal{H}^n_{n(\cdot)}(B_{n(\cdot)}(0,1)) = \omega_n$.

2.- $\mathcal{H}^n_{n(\cdot)}(A) = J(n(\cdot))\lambda^n(A), \quad \forall A \subset \mathbb{R}^n$.

In a similar way that we define the multiplicity function for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ on an subset A of $\mathbb{R}^n$, we can define a multiplicty function for functions from $\mathbb{R}^n$ to V, more precisely we have the next definition.

Definition 4.10. *Let $f : \mathbb{R}^n \to V$ be a function, with $V = (V, |\cdot|_V)$ a Banach space, and $E \subset \mathbb{R}^n$. We define $m^f_E : V \to \mathbb{N} \cup \{+\infty\}$ as*

$$m^f_E(y) := \mathcal{H}^0(E \cap \{f = y\}), \quad \forall y \in V,$$

where $\{f = y\} = \{x \in \mathbb{R}^n : f(x) = y\}$. We will call m^f_E the ***multiplicity function of*** *f* ***on*** *E.*

Now we can state the Area formula on metric spaces, and his subsequent corollaries.

Theorem 4.10. ***(Area formula on metric spaces)*** *Let $f : \mathbb{R}^n \to V$ be a Lipschitz function, and let $E \subset \mathbb{R}^n$ an Lebesgue measurable subset. Then*

$$\int_E J(md_x f) d\lambda^n(x) = \int_V m^f_E(y) d\mathcal{H}^n_{|\cdot|_V}(y),$$

in particular, if f is injective

$$\int_E J(md_x f) d\lambda^n(x) = \mathcal{H}^n_{|\cdot|_V}(f(E)).$$

The following corollaries are the generalization of the Corollaries 3.1 and 3.3 respectively.

Corollary 4.4. *Let $f : \mathbb{R}^n \to V$ be a Lipschitz function. If $g : V \to [-\infty, \infty]$ is $\mathcal{H}^n_{|\cdot|_V}-$measurable, and $E \subset \mathbb{R}^n$ is Lebesgue measurable, then*

$$\int_E g(f(x))J(md_x f)d\lambda^n(x) = \int_V g(y)m^f_E(y)d\mathcal{H}^n_{|\cdot|_V}(y),$$

provided at least one of the integrals exists. In particular if f is injective

$$\int_E g(f(x))J(md_x f)d\lambda^n(x) = \int_{f(E)} g(y)d\mathcal{H}^n_{|\cdot|_V}(y).$$

Note that in the last corollary we implicitly have the assumption that $f(E)$ is $\mathcal{H}^n_{|\cdot|_V}-$measurable, this is not a problem, because we can prove (in a similar form that the proof of Lemma 3.1) that in fact $f(E)$ is $\mathcal{H}^n_{|\cdot|_V}-$measurable, whenever f is a Lipschitz function and E be an Lebesgue measurable subset.

Corollary 4.5. *Let $f : \mathbb{R}^n \to V$ be a Lipschitz function. If $g : \mathbb{R}^n \to [-\infty, \infty]$ is Lebesgue integrable, then*

$$\int_V \left(\int_{\{f=y\}} g d\mathcal{H}^0_{|\cdot|_V} \right) d\mathcal{H}^n_{|\cdot|_V}(y) = \int_{\mathbb{R}^n} g(x)J(md_x f)d\lambda^n(x).$$

Note that in the previous generalization we have considered that the metric spaces have a vector space structure, so a natural question would be: Can we have these results without the need to have a vector space structure?
This question was addressed by Magnani in [12], for this purpose he considered metric measurable spaces which do not necessarily have an vector space structure, and introduced concepts like Jacobian, but in terms of the measures of the spaces.

Appendix A

Covering theorems

This appendix presents the statments of the three covering theorems that we use in this work: Vitali's Covering Theorem, Besicovitch's Covering Theorem, and the $5B$ Covering Theorem. These theorems, as we have already noted, have important applications in some areas of mathematics, in particular in analysis and measure theory.
Vitali's Covering Theorem provides conditions under which a collection of balls can be covered by a countable and disjoint subfamily, ensuring precise control over the measure of the uncovered set. Besicovitch's Covering Theorem extends this concept, offering a more general framework for covering collections of balls in Euclidean spaces. Additionally, it establishes the existence of countable and disjoint subfamilies that effectively cover the original set.
The $5B$ Covering Theorem, on the other hand, focuses on covering collections of balls in separable metric spaces. It guarantees the existence of a countable and disjoint subfamily whose union encompasses the original set, along with specific intersection properties.

Theorem A.1. ***(Vitali's Covering Theorem)*** *Let μ be a Radon measure on $\mathbb{R}^n$, $\mathcal{F}$ a family of non degenerate closed balls such that the set C of their centers is bounded, $\mu-$measurable, and for each $x \in C$*

$$\inf\{\operatorname{diam}(\overline{B}) : \overline{B} \in \mathcal{F} \text{ and } \overline{B} \text{ has its center at } x\} = 0.$$

Then there exist a countable and disjoint subfamily $\mathcal{G}$ of $\mathcal{F}$, such that

$$\mu\left(C \setminus \bigcup\{\overline{B} : \overline{B} \in \mathcal{G}\}\right) = 0.$$

Theorem A.2. ***(Besicovitch's Covering Theorem)*** *If $n \geq 1$, then there exist a positive constant $\xi(n)$ with the following property:*
If $\mathcal{F}$ is a family of non-degenerate open (or closed) balls in $\mathbb{R}^n$ and the set C of the centers of the balls in $\mathcal{F}$ is bounded or

$$\sup\{\operatorname{diam}(\overline{B}) : \overline{B} \in \mathcal{F}\} < \infty.$$

Then there exist $\mathcal{F}_1, \ldots, \mathcal{F}_{\xi(n)}$ subfamilies (possibly empty) of $\mathcal{F}$ such that:
1.- Each subfamily $\mathcal{F}_i$ is at most countable and disjoint.
2.- $C \subset \cup_{i=1}^{\xi(n)} \cup_{\overline{B}\in\mathcal{F}_i} \overline{B}$.

Theorem A.3. ***($5B$ Covering Theorem)*** *Let $\mathcal{F}$ be an arbitrary collection of balls in a separable metric space, such that*

$$\sup\{\operatorname{diam}(B) : B \in \mathcal{F}\} < \infty,$$

where rad(B) *denotes the radius of the ball* B*. Then there exist a countable and disjoint subfamily* $\mathcal{G}$ *of* $\mathcal{F}$ *such that*

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{C\in\mathcal{G}} 5C.$$

Moreover, each $B \in \mathcal{F}$ *intersects some* $C \in \mathcal{G}$ *with* $B \subset 5C$*.*

In the case where the family in the previous theorem is finite, we can replace the covering of the form $5C$ with one of the form $3C$.

Appendix B

Linear Algebra

In Chapter 3 we work with some tools of linear algebra on several occasions, in this appendix we summarize these concepts and results.

Definition B.1. *(i) A linear map $O : \mathbb{R}^n \to \mathbb{R}^m$ is **orthogonal** if*

$$\langle O(x), O(y)\rangle = \langle x, y\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

*(ii) A linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ is **symmetric** if*

$$\langle x, S(y)\rangle = \langle S(x), y\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

*(iii) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The **adjoint** of L is the linear map $L^* : \mathbb{R}^m \to \mathbb{R}^n$ defined by*

$$\langle x, L^*(y)\rangle = \langle L(x), y\rangle, \quad x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

Proposition B.1. *The following properties hold:*
(i) $L^{**} = L$.
(ii) $(L \circ T)^* = T^* \circ L^*$.
(iii) $O^* = O^{-1}$ if $O : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal.
(iv) $S^* = S$ if $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric.
(v) If $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric, then there exist an orthogonal linear map $O : \mathbb{R}^n \to \mathbb{R}^n$ and a diagonal transformation $D : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S = O \circ D \circ O^{-1}.$$

(vi) If $O : \mathbb{R}^n \to \mathbb{R}^m$ is orthogonal, then $n \leq m$ and

$$O^* \circ O = I, \ \text{in } \mathbb{R}^n,$$

$$O \circ O^* = I, \ \text{in } \mathbb{R}^m.$$

Theorem B.1. ***(Polar Decomposition Theorem)*** *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. If $n \leq m$, then there exist a symmetric linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ and an orthogonal linear map $O : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$L = O \circ S.$$

Using the terminology introduced in Chapter 3, we have the following result:

Proposition B.2. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and $L = O \circ S$ its polar decomposition, then:*

$$JL = |\det(S)|.$$

List of Figures

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