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Dimensión homotópica de espacios de configuraciones de cuadrados duros vía teoría de Morse discreta

Tesis que presenta:
Mary Carmen Pérez Morales

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Director de Tesis:
Dr. Jesús Gonzáles Espino Barros

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Homotopy dimension of configuration spaces of hard squares via discrete Morse theory

A dissertation presented by
Mary Carmen Pérez Morales

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Dr. Jesús González Espino Barros

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Resumen

En este trabajo se estudia la dimensión homotópica del espacio de configuraciones $C(n;p,q)$ de n cuadrados duros en un rectángulo $p\times q$. En [1], H. Alpert y sus colaboradores, describen un algoritmo para construir un campo vectorial gradiente sobre el complejo cúbico $X(n;p,q)$, el cual es un retracto por deformaciòn de $C(n;p,q)$, y conjeturan cotas superiores sobre j para las cuales $H_j[X(n;p,q)] \neq 0$. Aquí se propone usar un algoritmo distinto para construir un campo vectorial gradiente V sobre $X(n;p,q)$, el cual es diferente al propuesto en [1]. Luego, usando el algoritmo propuesto, se obtiene la dimensión homotópica de $X(n;p,q)$ para $1 \leq q \leq 2$; mientras que para $n = 3$ las cotas para el valor de j son verificadas para valores arbitrarios de p y q. Finalmente, mediante una implementación computacional de este algoritmo sobre $X(4;p,q)$, las cotas para j son verificadas también para valores relativamente pequeños de p y q.

Abstract

In this work, the homotopy dimension of configuration spaces $C(n;p,q)$ of n hard squares in a rectangle $p \times q$ are studied. In [1], H. Alpert and her collaborators, give a procedure for constructing a gradient vector field over the cubical complex $X(n;p,q)$, which is a deformation retract of $C(n;p,q)$, and conjecture upper bounds over j when $H_j[X(n;p,q)] \neq 0$. This work proposes to use a different algorithm to construct a gradient vector field V over $X(n;p,q)$. Then, using the algorithm proposed, for grids of height $q \in \{1,2\}$, the homotopic dimension of $X(n;p,q)$ is obtained. Meanwhile for $n=3$ the conjectured bounds for j are verified for arbitrary values of p and q. Finally, via a computational implementation of the same algorithm used for the previous cases, for $n=4$ and relatively small values of p and q, the bounds are again verified too.

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"De quien olvidé sus generales
pero recuerdo sus particulares."

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1 Introduction

For a topological space X, the classical configuration space $F(n, X)$ of n labelled non-collisioning particles in X consists of the n-tuples $(x_1, \ldots, x_n)$ in the topological product X^n satisfying $x_i \neq x_j$ for $i \neq j$. When X is in fact a metric space and $\epsilon > 0$, we can consider in addition the subspace $F_\epsilon(n, X) \subset F(n, X)$ of n labelled non-overlapping *ϵ-thick* particles in X. To fix ideas, assume X is the closure of a convex bounded open region $\mathrm{Int}(X)$ of an Euclidean space endowed with a specific metric d, and let ∂X stand for the boundary of X. Then $F_\epsilon(n, X)$ consists of the n-tuples $(x_1, \ldots x_n) \in F(n, \mathrm{Int}(X))$ for which the corresponding closed ϵ-balls $B_1, \ldots B_n$ in X centered at the points $x_1, \ldots, x_n$ satisfy $B_i \cap B_j = \partial B_i \cap \partial B_j$ for $i \neq j$, as well as $B_i \cap \partial X = \partial B_i \cap \partial X$ for all i.

In this work we focus on the case where X is a bounded rectangle in the plane with sides parallel to the axes and of integer sides, say p and q. Additionally we consider the max metric and $\epsilon = 1/2$. The resulting space, also denoted as $C(n; p, q)$, thus consists of configurations of n labelled non-overlapping squares of unit side that are confined to a $p \times q$ grid. This object was initially studied by Alpert et al in [1]. In that work, inspired by the classical moves in the well-known 15 puzzle which holds with $(n, p, q) = (15, 4, 4)$, the authors constructed a natural cubical complex $X(n; p, q)$ sitting inside $C(n; p, q)$ as a deformation retract. In particular, computational techniques can be used to assess the topological properties of $C(n; p, q)$ via its homotopy equivalent combinatorial model $X(n; p, q)$. Indeed, Alpert and her collaborators constructed a discrete gradient field on the cubical complex $X(n; p, q)$ which lead them to prove that the j-dimensional homology group $H_j(C(n; p, q))$ is non-zero only for

$$j \leq \min\left\{pq - n, n, \frac{pq}{3}\right\}.$$

In fact, a closer analisis of the methods reveals that their argument gives an explicit upper bound for the homotopy dimension of $C(n; p, q)$, namely

$$\mathrm{hdim}(C(n; p, q)) \leq \min\left\{pq - n, n, \frac{pq}{3}\right\}. \tag{1}$$

Here, by *homotopy dimension* of a space Z we mean the smallest dimension of a cell complex having the homotopy type of Z. As showed by Alpert and her collaborators, the assessment of homotopy dimension in (1) is surprisingly close to being optimal on a *large scale* perspective. Yet, based on extensive computer calculations, Alpert and her collaborators realized that, in particular instances, the homotopy dimension of $C(n; p, q)$ would seem to be smaller than the bound indicated by (1).

In this work we address the problem of determining the homotopy dimension of $C(n; p, q)$ for small values of n or q. With this in mind we use a gradient field on $X(n; p, q)$, which is valid for arbitrary values of n, p and q, but completely different from the one developed by Alpert et al. The gradient field constructed and used in the present work seems to read off the homotopy dimension of $C(n; p, q)$ with a precision that is apparently better —almost optimal— than the one provided by the discrete gradient field originally used by Alpert and her collaborators. Such a situation might be a result, in part, of the fact that, unlike Alpert's et al field, our discrete gradient field is not equivariant with respect to the free action of the symmetric group on n letters and, consequently, has the ability of picking up topological aspects inherent to the ordered, rather than the unordered, configuration space of hard squares.

More specifically, in some of the cases we study in detail here, all critical cells in our gradient field lie in dimensions that are actually smaller than the upper bound in (1), thus improving on the homotopy dimension bound. In some other instances , by comparing with the homology computations by Alpert et al, we verify that our homotopy dimension upper bounds are in fact optimal.

A point that must be spelled out arises from the fact that our gradient field is extremely involved in the general situation. Actually, this is the reason why we are forced to refrain from analyzing the field in situations with large values of n and $min\{p,q\}$. On our view, this situation is an unavoidable consequence of the fact that the topology of $C(n;p,q)$ (and therefore of $X(n;p,q)$) appears to be highly complex. Having said so, it should be remarked that as p and q increase, the topology of $C(n;p,q)$ stabilizes into the much better understood topology of the classical configuration space of n labelled (infinitesimal) points in the plane $\mathbb{R}^2$, i.e., the Eilenberg-MacLane space corresponding to Artin's classical braid group with n strands.

Together with Alpert's and collaborators ideas, our techniques can be considered as a discrete counterpart of those used by Plachta in [7] to study configuration spaces of labelled non-overlapping hard squares moving on rectangular environments whose size do not have to be integer numbers. In such a more general situation, affine Morse-Bott theory and smooth flows are used to study the resulting spaces. Yet, it should be noted that the connectivity properties in these slightly more general configuration spaces are deduced by Pachta from their interplay with the configuration spaces $C(n;p,q)$ based on integer-sized rectangles.

After reviewing the needed essentials on combinatorial complexes and Forman's discrete Morse theory, we introduce our discrete gradient field, and apply it into the study of the topology of $C(n;p,q)$ for small values of n or q. We stress on the homotopy-dimension optimality of our gradient field in the case of a few families of triples (n,p,q), as well as for some additional examples based on computer calculations. We remark that the full determination of the homotopy dimension of $C(n;p,q)$ as an explicit function of n, p and q is a currently open problem with important insights into phase transition problems for hard-spheres systems.

We close this introduction with a statement that summarizes the goals achieved in this work, together with an analysis of their implications within the context of previous results in the area.

Theorem 1.1. *The gradient field constructed in this work for the homotopy model $X(n;p,q)$ of the configuration space of n non-overlapping squares in a $p \times q$ rectangle has the following properties:*

a) When $min\{p,q\} = 1$ (and $max\{p,q\} \geq n$, so that $C(n;p,1) \neq \emptyset$), there are $n!$ critical cells, all of dimension 0. In particular $C(n;p,1)$ has $n!$ components, each of which is contractible, so the homotopy dimension of $C(n;p,q)$ is 0.

Assuming in addition $min\{p,q\} \geq 2$:

b) *When* $n = 2$ *there are only 2 critical cells, one of dimension 0 and the other one of dimension 1. In particular* $C(2; p, q)$ *has homotopy dimension 1 with the homotopy type of a circle.*

c) *When* $n = 3$ *and* $p = q = 2$*, the gradient field has 8 critical cells, 4 of which have dimension 0, and the other 4 have dimension 1. In particular the homotopy dimension of* $C(3; 2, 2)$ *is 1. In fact, has 2 components, each homotopy equivalent to a circle.*

d) *When* $n = 3$*,* $min\{p, q\} = 2$ *and* $max\{p, q\} \geq 3$*, the gradient field has no critical cells in dimension greater than 1, so that the homotopy dimension of* $C(3; p, q)$ *is 1. In fact,* $C(3; p, q)$ *is connected and has homotopy type of a wedge of 7 circles.*

e) *When* $n = 3$ *and* $min\{p, q\} \geq 3$*, the gradient field has no critical cells of dimension greater than 2, and only 9 critical cells of dimension 2. In fact, the homotopy dimension of* $C(3; p, q)$ *is 2.*

In order to contextualize the above results, it is convenient to take a slightly closer look at the work of Alpert et al in [1], where it is proved that any non-trivial homology group $H_j[C(n; p, q)]$ must hold with

$$j \leq min\Big\{pq - n, n, \frac{pq}{3}\Big\}. \tag{2}$$

Furthermore, on the basis of extensive homological computations, the authors conjectured that the estimated in (2) could be sharpened to

$$j \leq min\Big\{pq - n, n - \frac{8n^2}{9pq}, \frac{pq}{4}\Big\}. \tag{3}$$

In this line of thinking, the theorem above allows us to see that the conjecture improvement in (3) holds true in a number of cases, sometimes being sharp and some other times admitting further improvements.

For instance, a simple initial situation where (3) holds true in a sharp way arises with $min\{p, q\} \geq 2 = n$, in view of item (b) of the theorem above.

Concerning cases where (3) can be further improved, take $min\{p, q\} = 1$ with $max\{p, q\} \geq n$. Then item (a) of the theorem above implies that the actual homotopy dimension of $C(n; p, 1)$ is zero, while the right-hand side in (3) can be arbitrarily large (take for instance $q = 1$ and $p = n^2$). A more interesting and substantial example of this improvement situation arises with $n = 3$, $min\{p, q\} = 2$ and $max\{p, q\} \geq 4$, say $p \geq 4$ and $q = 2$. In such a case, the integral part of the right-hand side in (3) satisfies

$$\left\lfloor min\Big\{pq - n, n - \frac{8n^2}{9pq}, \frac{pq}{4}\Big\} \right\rfloor = \left\lfloor n - \frac{8n^2}{9pq} \right\rfloor = 2, \tag{4}$$

however item (d) of the theorem above implies that, in fact, $C(n; p, q) {=} C(3; p, 2)$ has homotopy dimension 1, so that the upper stimate in (3) can be improved by one unit and so that the conclusion applies in homotopical (rather than just homological) terms.

2 Preliminaries

In this section, we introduce standard definitions and useful results for the purposes of this work. The results of subsections 2.1 to 2.5 can be found on [5], [4], [6], [2] and [3] respectively.

2.1 Simplicial complexes

Definition 2.1. *Let $n \geq 0$ be an integer and $[v_n]$:= $\{v_0, v_1, ..., v_n\}$ a collection of n+1 symbols. An (abstract) simplicial complex K on $[v_n]$ or simply a complex is a collection of subsets of $[v_n]$, excluding $\emptyset$, such that*

1. *If $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$*
2. *$\{v_i\}\in K$ for every $v_i \in [v_n]$*

The set $[v_n]$ is called the vertex set of K and the elements $\{v_i\}$ are called vertices or 0-simplices and the set with all of them is denoted by V(K).

As an easy example, with $[v_3]$ ={A, B, C, D} then defining K :={{A}, {B}, {C}, {D}, {A, B}, {A, C}, {A, D}, {B, C}, {B, D}, {C, D}, {A, D, C}, {A, B, C}, {A, D, B}, {D, B, C}} it is easy to check K satisfies the definition of a simplicial complex. This simplicial complex may be viewed as in the figure below.

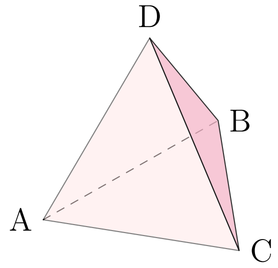


Figure 1: Simplicial complex associated to K

Definition 2.2. *Given a simplicial complex K; a set $\sigma \in K$ of cardinality $i+1$ is called an **i-dimensional simplex** or **i-simplex**. The **dimension** of K, denoted by $dim(K)$ is the maximum of the dimensions of all its simplices.*

In the last example, $dim(K) = 2$ since the maximum of the cardinalities of all the simplices of K is 3.

Definition 2.3. *Let $L \subset K$, L is said to be a **subcomplex** of K if L is a simplicial complex by itself.*
For $\sigma \in K$ the subcomplex generated by σ (denoted by $\overline{\sigma}$) is defined as

$$\overline{\sigma} := \{\tau \in K \mid \tau \subset \sigma\}$$

*Besides if $\sigma, \tau \in K$ and $\tau \subset \sigma$, then τ is said to be a **face of** σ and σ is said to be a **coface** of τ, and is common to use the notation $\tau < \sigma$ for this. And finally, the **i-skeleton** of K is given by $K^i = \{\sigma \in K \mid dim(\sigma) \leq i\}$.*

In the last example, by the definition, the complex generated by a simplex, it is easy to see

$$\{\overline{D, B, C}\} := \{ \{B\}, \{C\}, \{D\}, \{B, C\}, \{B, D\}, \{C, D\}, \{D, B, C\}\}$$

where all of its elements are faces of {D, B, C}.

Let K be a simplicial complex and $\tau, \sigma \in K$ The notation σ^i is used to denote the dimension of σ and $\tau < \sigma^i$ is used for any face of σ of dimension strictly less than i.
For $\tau < \sigma$, the number $dim(\sigma) - dim(\tau) > 0$ is called the codimension of τ with respect to σ.

Definition 2.4. *Let K be a simplicial complex, for any simplex $\sigma \in K$, the boundary of σ in K is defined by*

$$\partial_K(\sigma) := \partial(\sigma) := \{\tau \in K \mid \tau \textit{ is a codimension 1-face of } \sigma\}.$$

*Also, if σ is not properly contained in any other simplex of K, then σ is called to be a **facet** of K.*

Definition 2.5.

1. *A geometric $n-$simplex σ is the convex hull of a set A of $n+1$ affine independent points in $\mathbb{R}^N$, for some $N \geq n$. The convex hulls of subsets of A are called subsimplices of σ.*

2. *The standard $n-$simplex is the convex hull of the set of endpoints of the standard unit basis $\{(1, 0, ..., 0), (0, 1, 0, ..., 0), ...(0, ..., 0, 1)\}$ in $\mathbb{R}^{n+1}$.*

More generally, given a finite set A, taking the vector space $\mathbb{R}^A$, whose coordinates are indexed by the elements of A; and correspondingly, for any $B \subset A$, then a standard $B-$simplex in $\mathbb{R}^A$ as the one that is spanned by the endpoints of the part of the standard unit basis indexed by elements in B in that vector space.

Definition 2.6. *Given a finite abstract simplicial complex K, its **standard geometric realization** is defined to be the topological space whose underlying space is obtained by taking the union of standard $\sigma-$simplices in $\mathbb{R}^{V(K)}$, for all $\sigma \in K$.*

Given a nonempty abstract simplicial complex K and thinking of K^σ as the standard $\sigma-$simplex of $\sigma \in K$ the constructive definition of the geometric realization of K goes as follows:

- Start with an arbitrary vertex of K, and then add new simplices one by one, in any order, with the only condition being that all the proper subsimplices of the simplex that is being added have already been glued on at this point.

- Assume that we are at the situation in which we would like to glue the simplex $\sigma \in K$ onto the part of the realization of X that we have so far. The new space is $X \cup_{\partial K^\sigma} K^\sigma$, which is obtained by identifying the boundary of K^σ with the subspace of X, which is the result of gluing the simplices corresponding to the proper subsimplices of σ. In shortly, the simplex K^σ is glued onto X along its boundary in the usual way.

As an illustrative example, given K ={{A},{B},{C},{D},{A,B},{B,C},{A,C},{A,B,C},{C,D}}, by the first item, first all the vertices are needed, and then one by one each of the edges are glued, and finally is glued the 2-simplex {A,B,C}, as shown in the figure below.

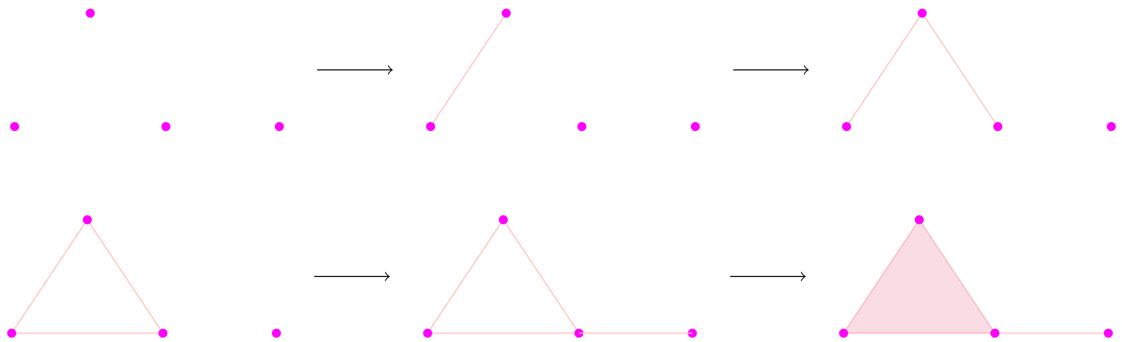


Figure 2: Caption

One way to think of this gluing process is the following. Given a collection of simplices $\{K^\sigma\}_{\sigma\in K}$, together with inclusion maps $i_{\sigma,\tau} : K^\sigma \hookrightarrow K^\tau$ whenever $\sigma < \tau$. The geometric realization $|K|$ of K is obtained from the disjoint union of the simplices by one extra condition: two points are identified whenever one of them maps to the other one by one of these inclusion maps.

Definition 2.7. *A geometric simplicial complex $\mathcal{K}$ in $\mathbb{R}^n$ is a collection of simplices in $\mathbb{R}^n$ such that every subsimplex of a simplex of $\mathcal{K}$ is a simplex of $\mathcal{K}$ and the intersection of any two simplices of $\mathcal{K}$ is a subsimplex of each of them.*

As for abstract simplicial complexes, the concepts of face, subcomplex and skeleton can be defined for geometric simplicial complexes.

Definition 2.8. *For a geometric simplicial complex $\mathcal{K}$, let $|\mathcal{K}|$ denote the union of all simplices of $\mathcal{K}$. The topology on $|\mathcal{K}|$ is defined as follows: every simplex σ of $\mathcal{K}$ has the induced topology, and in general $A \subset |\mathcal{K}|$ is open if and only if $A \cap \sigma$ is open in σ, for all simplices $\sigma \in K$ (equivalently, the word "open" could be replaced with the word "closed").*

It is importan to note that

- If $\mathcal{K}$ has finitely many simplices, then the intrinsic topology of $|\mathcal{K}|$ is the same as the topology induced from the encompassing space $\mathbb{R}^n$.
- In general, a geometric realization of an abstract simplicial complex is a geometric simplicial complex.

Remark: From here, we will refer to geometric simplicial complexes as **simplicial complexes**.

Definition 2.9. *Let K be a simplicial complex and $\alpha, \beta \in K$ with $\alpha < \beta$ then α is said to be a **free face** of β if α is a face of β but not a face of any other simplex.*

Definition 2.10. *Let K be a simplicial complex and $\alpha, \beta \in K$ such that $\alpha < \beta$ with α a free face of β, $dim(\alpha + 1) = dim(\beta)$ and β a facet of K, the pair (α, β) is said to be a free pair and the act of removing a free pair of K is called an **elementary simplicial collapse**.*

For example, on the simplicial complex K_1 on the right of figure 3, the face $\{B, C\}$ of $\{A, B, C\}$ is free and $\{A, B, C\}$ is a facet of K_1, and $K_2 = K_1 - \{\{B, C\}, \{A, B, C\}\}$ on the left is the result of the the elementary collapse.

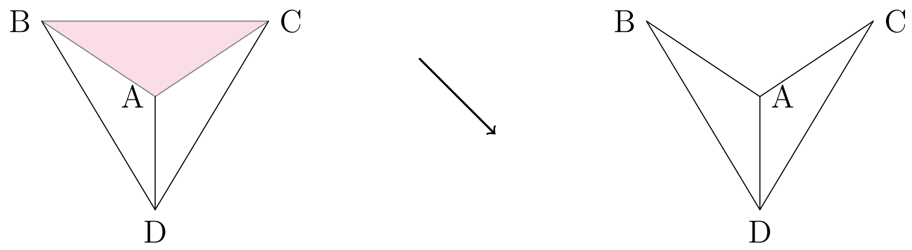


Figure 3: Example of an elementary simplicial collapse

Since for a free pair neither of the elements are faces of other elements, it follows that the result of an elementary collapse is another simplicial complex and a finite sequence of elementary collapses is a simplicial complex too. As usual, if K_2 is obtained from K_1 by elementary simplicial collapses then it is said that K_1 collapses to K_2 and this is indicated by $K_1 \searrow K_2$. Moreover, as can be seen in figure 3, for the geometric realization of K_1, a collapse can be seen as pushing down the edge $\{B, C\}$ through the triangle $\{A, B, C\}$ and stopping at K_2, i.e. K_1 is a deformation retract onto K_2 so both of them are homotopy equivalent.

Definition 2.11. *For K and L simplicial complexes, they are said to be simple homotopy equivalent , denoted by $K \sim L$, if they are related by a sequence $K = K_0, K_1, ..., K_k = L$ where for each $0 \leq i < k$ there is an elementary collapse from either K_i to K_{i+1} or vice versa.*

Definition 2.12. *A discrete vector field V on a simplical complex K is a collection of pairs $\{\alpha^p < \beta^{p+1}\}$ of simplices of K such that each simplex is in at most one pair of V. i.e.,*

$$V := \{(\alpha^p, \beta^{p+1}) \mid \alpha, \beta \in K, \alpha < \beta \text{ and each simplex is in at most one pair}\}$$

Let K be a simplicial complex and V a discrete vector field on it, a V-path is a sequence of simplices of K

$$\alpha_0^p, \beta_0^{p+1}, \alpha_1^p, \beta_1^{p+1}, ..., \alpha_l^p, \beta_l^{p+1}, \alpha_{l+1}^p$$

such that for each $i = 0, ..., l$, $\alpha_i < \beta_i$, $(\alpha, \beta) \in V$ and $\beta_i > \alpha_{i+1} \neq \alpha_i$, such a path is a non-trivial closed path if $l \geq 0$ and $\alpha_0 = \alpha_{l+1}$. A discrete gradient field $V$with no non-trivial closed paths is said to be a **gradient vector field**.

2.2 Cubical complexes

Definition 2.13. *An elementary interval is a closed interval $I \in \mathbb{R}$ of the form*

$$I = [l, l+1] \text{ or } I = [l, l]$$

for some $l \in \mathbb{Z}$. Elementary intervals that consist of a single point are known as degenerate, while those of length 1 as nondegenerate.

To simplify the notation, it is written

$$[l]{=}[l, l]$$

Definition 2.14. *An elementary cube Q is a finite product of elementary intervals, i.e.*

$$Q = I_1 \times I_2 \times ... \times I_n \subseteq \mathbb{R}^n$$

where each I_i is an elementary interval. $\mathcal{K}^n$ is used to denote the set of all elementary cubes in $\mathbb{R}^n$ and $\mathcal{K}$ for the set of all elementary cubes, namely

$$\mathcal{K}^n := \{Q \subseteq \mathbb{R}^n \mid Q \text{ is elementary cube}\}$$

and

$$\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}^n$$

respectively.

Figure 1 shows examples of the elementary cubes $I_1 = [1] \times [1]$, $I_2 = [2] \times [2, 3]$, $I_3 = [3, 4] \times [1]$ and $I_4 = [5, 6] \times [2, 3]$, all in $\mathbb{R}^2$

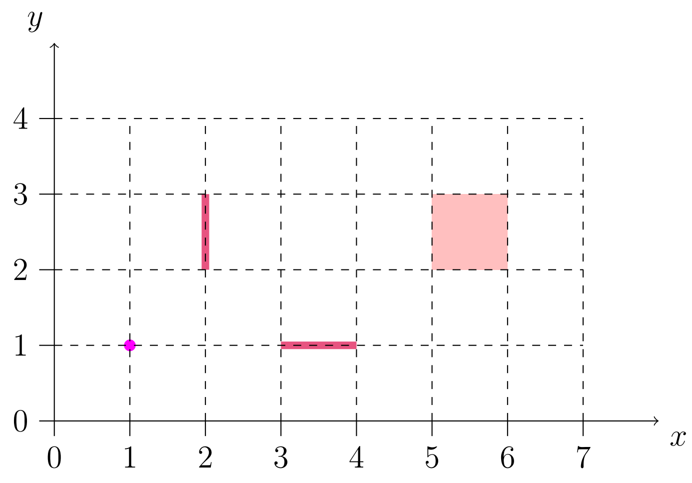


Figure 4: Examples of elementary cubes in $\mathbb{R}^2$.

Definition 2.15. *Let $Q = I_1 \times I_2 \times ... \times I_n \subseteq \mathbb{R}^n$ be an elementary cube. The dimension of Q, denoted as $dim(Q)$, is defined to be the number of nondegenerate components in Q. The interval I_i is referred to as the ith component of Q and is written as $I_i(Q)$. The embedding number of Q, denoted by $emb(Q)$, is defined as n since $Q \subseteq \mathbb{R}^n$.*

Since $emb(Q) = n$, then $Q \in \mathcal{K}^n$

Definition 2.16. *$\mathcal{K}_k$ is defined as*

$$\mathcal{K}_k := \{Q \in \mathcal{K} \mid dim(Q) = k\}$$

and

$$\mathcal{K}_k^n := \mathcal{K}_k \cap \mathcal{K}^n$$

By definition the elements of $\mathcal{K}_k^n$ are elementary cubes Q with k nondegenerate components and such that they are embedded in $\mathbb{R}^n$. Is easy to see that for any Q the inequality

$$0 \leq dim(Q) \leq emb(Q)$$

holds.

Proposition 2.17. *Let $Q \in \mathcal{K}_k^n$ and $Q \in \mathcal{K}_{k'}^{n'}$. Then*

$$Q \times P \in \mathcal{K}_{k+k'}^{n+n'}$$

Proof. By definition, Q and P can be written as $Q = I_1 \times I_2 \times ... \times I_n$ and $P = J_1 \times J_2 \times ... \times J_{n'}$ respectively, with I_i and $J_{i'}$ elementary intervals. So

$$Q \times P = I_1 \times I_2 \times ... \times I_n \times J_1 \times J_2 \times ... \times J_{n'}$$

is the product of $n + n'$ elementary intervals, also is easy to see the nondegenerate intervals in $Q \times P$ are exactly the same that those in Q or P, it is $dim(Q) \times dim(P) = dim(Q) + dim(P)$ □

Definition 2.18. *Let $Q,P \in \mathcal{K}$. If $Q \subseteq P$, then Q is a face of P. This is denoted by $Q \preceq P$ and if Q is a face of P and $Q \neq P$ then Q is said to be a proper face of P, which is written as $Q \prec P$*

For Q and P in $\mathcal{K}$, with $Q \preceq P$ it is clear $dim(Q) \leq dim(P)$ with the equality only when Q=P, also if $Q \prec P$ then $dim(Q) \leq dim(P) - 1$

Definition 2.19. *A set $X \subseteq \mathbb{R}^n$ is* **cubical** *if X a finite union of elementary cubes.*

As for elementary cubes, if $X \subseteq \mathbb{R}^n$ is a cubical set, then the following notation is adopted

$$\mathcal{K}(X) := \{\ Q \in \mathcal{K} \mid Q \subset X\ \}$$

and

$$\mathcal{K}_k(X) := \{\ Q \in \mathcal{K}(X) \mid dim(Q) = k\ \}$$

Since for $Q \subset X$ with $X \subset \mathbb{R}^n$ and $Q \in \mathcal{K}$ it follows $emb(Q) = n$ hence $X \subset \mathcal{K}^n$, then for X the notation $\mathcal{K}^d(X)$ is somewhat redundant. It is important to note that, for $X \subseteq \mathbb{R}^n$ cubical $X = \bigcup_{l=0}^{n} \mathcal{K}_l(X)$, since $emb(X) \geq dim(Q)$ for every $Q \in K(X)$.

Example 2.1. *In figure 4, the elementary intervals* $I_1 = [1] \times [1]$, $I_2 = [2] \times [2,3]$, $I_3 = [3,4] \times [1]$ *and* $I_4 = [5,6] \times [2,3]$ *in* $\mathbb{R}^2$ *are shown. Now if* $X = I_1 \cup I_2 \cup I_3 \cup I_4$ *then* $X \subset \mathbb{R}^2$ *is a cubical set with*

$\mathcal{K}_0(X) = \{[1]\times[1], [2]\times[2], [2]\times[3], [3]\times[1], [4]\times[1], [5]\times[2], [5]\times[3], [6]\times[2], [6]\times[3]\}$
$\mathcal{K}_1(X) = \{[2]\times[2,3], [3,4]\times[1], [5,6]\times[2], [5,6]\times[3], [5]\times[2,3], [6]\times[2,3]\}$
$\mathcal{K}_2(X) = \{[5,6]\times[2,3]\}$
Figure 5 shows $\mathcal{K}_0(X)$ *in red,* $\mathcal{K}_1(X)$ *in pink, and* $\mathcal{K}_2(X)$ *in a darker pink for* X

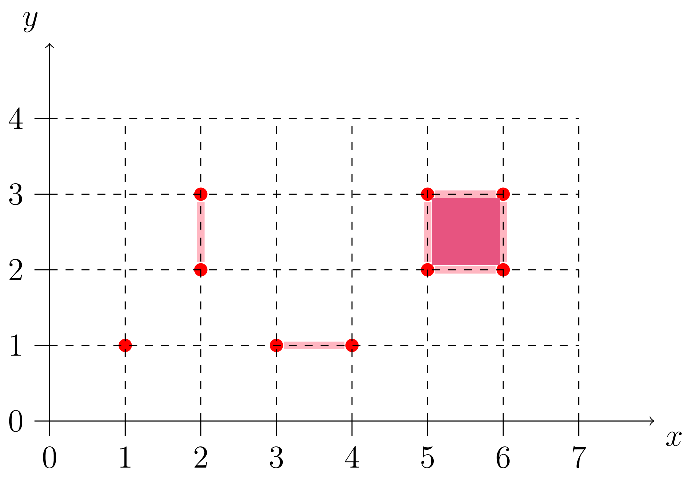


Figure 5: $\mathcal{K}_0(X)$, $\mathcal{K}_1(X)$ and $\mathcal{K}_2(X)$ for X.

Proposition 2.20. *If* $X \subset \mathbb{R}^d$ *is cubical, then* X *is closed and bounded.*

Proof. Since by definition a cubical set is the finite union of elementary cubes, and elementary cubes are finite products of closed intervals, then X is closed. Let $Q \in K(X)$, $Q = I_1 \times I_2 \times ... \times I_n$ where I_i are elementary interval Letting $\rho(Q) = max\{|l_i| + 1 \mid i \in i, ..., d\}$, since $K(X) < \infty$ then $R = max\{\rho(Q) \mid Q \in K(X)\}$ satisfies $X \subset B_0(0, R)$. □

As an important part and in a similar way to that for simplicial complexes, cells in the context of cubes are defined.

Definition 2.21. *Let* I *be an elementary interval. The associated* ***elementary cell*** *is*

$$\mathring{I} = \begin{cases} (l, l+1) & if I = [l, l+1], \\ [l] & if I = [l, l]. \end{cases}$$

This definition is extended to a general elementary cube in the easiest way.

Definition 2.22. *Given an elementary cube* $Q = I_1 \times I_2 \times ... \times I_n \subset \mathbb{R}^n$, *the associated* ***elementary cubical cell*** *or* ***elementary*** $n-$***cubical cell*** *is defined as*

$$Q = \mathring{I}_1 \times \mathring{I}_2 \times ... \times \mathring{I}_n$$

By its utility for the next section, some properties of elementary cells are summarized in the following proposition.

Proposition 2.23. *Elementary cells have the following properties:*

1. $A \subset \mathbb{R}^d$ *bounded implies that* $card\{Q \in K^d \mid \mathring{Q} \cap A \neq \emptyset\} < \infty$
2. *For every* $Q \in K$, $cl(\mathring{Q}) = Q$
3. *If* $P, Q \in K^d$, *then* $\mathring{P} \cap \mathring{Q} = \emptyset$ *or* $P = Q$.
4. *If* X *is a cubical set and* $\mathring{Q} \cap X \neq \emptyset$ *for some elementary cube* Q *then* $Q \subset X$.

2.3 Cell complexes

Definition 2.24. *A closed Euclidean* ***n-cell*** E^n *is a homeomorphic image of the Euclidean n-elementary cube* I^n*, i.e. the cartesian product of n copies of the closed unit interval* $I = \{t \in \mathbb{R} \mid 0 \leq t \leq 1\}$

Other n-cells besides the n-cube are

- The unit n-disc $D^n = \{(t_1, t_2, ..., t_n) \in \mathbb{R}^n \mid \sum_{i=1}^n t_i^2 \leq 1\}$
- The standard n-simplex Δ^n=$\{(t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \text{ for } 0 \leq i \leq n, \sum_{i=1}^n t_i = 1\}$

Definition 2.25. *The* ***boundary*** $\partial(E^n)$ *of a closed Euclidean n-cell embedded as a convex set in* $\mathbb{R}^n$ *is its topological frontier and is homeomorphic to the standard* $(n-1)$*-sphere*

$$S^{n-1} = \{t_1, t_2, ..., t_n \in \mathbb{R}^n \mid \sum_{i=1}^n t_i^2 = 1\}$$

An ***open*** *n-cell is a homeomorphic image of* $E^n - \partial(E^n)$.

All these spaces except the open n-cell are compact (closed and bounded for being in $\mathbb{R}^n$), and the open n-cell is locally compact. Next is stated the definition of a cell structure within a set, it is important to note X is not necessarily a topological set for the following definition.

Definition 2.26. *Let X be a set. A* ***cell structure*** *on X is a pair* (X, Φ)*, where* Φ *is a collection of maps of closed Euclidean cells into X satysfying the following conditions*

1. *If* $\phi \in \Phi$ *and* $\phi : E^n \to X$ *then* ϕ *is injective on* $E^n - \partial(E^n)$.
2. *The images* $\{\phi(E^n - \partial E^n) \mid \phi \in \Phi\}$ *partition X, i.e., they are disjoint and have union X.*
3. *If* $\phi \in \Phi$ *has domain* E^n*, then* $\phi(\partial E^n) \subset \bigcup$ $\{\psi(E^k - \partial E^k) \mid \psi \in \Phi,\ \psi : E^k \to X, \text{ for } k \leq n-1\}$

If $\phi \in \Phi$, $\phi : E^n \to X$, then $\phi(E^n) = \sigma^n$ is called an $\mathbf{n-cell}$ or *closed n-cell* of (X, Φ), as well ϕ is called the *characteristic map* of σ^n, so Φ is a set of characteristic maps for the cells of (X, Φ). Similarly $\phi(\partial E^n) = \partial\sigma^n$ and $\phi(E^n - \partial E^n)$ are called the **boundary** and the **interior** of the cell σ^n respectively, besides for $n > 0$, $\phi(E^n - \partial E^n)$ is called an **open n-cell**. All of the terminology is useful, but since X is just a set (and not a topological one necessarily) these terms do not mean the cells are neither open nor closed topologically, and even with a topology on X open cells need not be topological open sets and closed cells need not be topological closed sets.

Definition 2.27. *Let X be a set with a cell structure, the* $(n-1)$ *skeleton* X^{n-1} *is defined as*

$$X^{n-1} = \bigcup\{\psi(E^k - \partial E^k) \mid \psi \in \Phi \text{ has domain } E^k \text{ and } k \leq n-1\}$$

Lemma 2.28. *Let* (X, Φ) *be a cell structure. Then*

1. If σ^n is a cell with charateristic map ϕ, then $\phi(E^n - \partial E^n) = \sigma^n - \partial\sigma^n$ is the interior of σ^n

2. each n-cell is a subset of X^n,

3. for each n $X^n = \bigcup\{\sigma^k \mid \sigma^k$ is a k-cell of (X, Φ) and $k \leq n.\}$

Proof. 1. By 3. in the definition, $\partial\sigma^n = \phi(\partial E^n)$ is in X^{n-1} and by 2. $\partial\sigma^n$ is disjoint from $\phi(E^n - \partial E^n)$. Hence $\sigma^n = \phi((E^n - \partial E^n) \cup \partial E^n) = \phi(E^n - \partial E^n) \cup \phi(\partial E^n)$ is a disjoint union, hence $\sigma^n - \partial\sigma^n$ is the interior.

2. By the first statement, $\sigma^n - \partial\sigma^n \subset X^n$ and by 3. in the definition, $\partial\sigma^n$ is in $X^{n-1} \subset X^n$. Therefore $\sigma^n = (\sigma^n - \partial\sigma^n) \cup \partial\sigma^n \subset X^n$.

3. It is always true that $\sigma^n - \partial\sigma^n \subset \sigma^n$ so $X^n \subset \bigcup \{\sigma^k \mid \sigma^k$ is a k-cell of (X, Φ) and $k \leq n\}$, and by the statement before, the reverse inclusion is true too.

□

By definition of X^n, it is easy to see that

$$X^0 \subset X^1 \subset ... \subset X^n \subset ... \subset X$$

Definition 2.29. *Let (X, Φ) and (X, Φ') be two cell structures, it is said that they are strictly equivalent if there is a one-to-one correspondence between Φ and Φ' such that a characteristic function with domain E^n, and its corresponding function differ only by a reparametrization of their domain. That is, if ϕ and ϕ' are corresponding functions of Φ and Φ' respectively, then $\phi' = \phi \circ h$, where $h : (E^n, \partial E^n) \to (E^n, \partial E^n)$ is a homeomorphism of pairs.*

Given that the functions h are homeomorphism of pairs, then *strictly equivalence* is actually an equivalence relation on the collection of cell structures. If (X, Φ) is a cell structure, let $\mathcal{C}_\Phi = \{(\sigma^n, [\phi]) \mid \sigma^n = \phi(E^n)$, $[\phi]$ the strict equivalence class of ϕ $\}$. In an abuse of notation such a pair $(\sigma^n, [\phi])$ is denoted just by σ^n.

Definition 2.30. *A **cell complex** on a set X or a **cellular decomposition** on a set X is an equivalence class of cell structures (X, Φ) under the equivalence relation of strict equivalence. A cell complex on X will be denoted by a pair $(X, \mathcal{C})$, where $\mathcal{C} = \mathcal{C}_\Phi$ for some representative cell structure (X, Φ). The set $\mathcal{C}$ is called the set of (closed) cells of $(X, \mathcal{C})$*

Definition 2.31. *A **subcomplex** $(A, \mathcal{F})$ of a cell complex $(X, \mathcal{C})$, denoted as $(A, \mathcal{F}) \subset (X, \mathcal{C})$, is a cell complex (by its own) such that $A \subset X$ and $\mathcal{F} \subset \mathcal{C}$.*

Letting $(X, \mathcal{C})$ be a cell complex and taking $\mathcal{C}^n = \{\sigma^p \in \mathcal{C} \mid p \leq n\}$, then it is clear that $(X^n, \mathcal{C}^n)$ is a subcomplex of $(X, \mathcal{C})$ and it is called the n-skeleton of $(X, \mathcal{C})$. Note that, for n-skeletons

$$(X^0, \mathcal{C}^0) \subset (X^1, \mathcal{C}^1) \subset ... \subset (X, \mathcal{C}).$$

Proposition 2.32. *Let $(A, \mathcal{F})$ be a subcomplex of the cell complex $(X, \mathcal{C})$, and let σ be a cell in $\mathcal{C}$. Then σ is a cell of $\mathcal{F}$ if and only if $(\sigma - \partial\sigma) \cap A \neq \emptyset$.*

Proof. The right implication is clear. Conversely, if $(\sigma - \partial\sigma) \cap A \neq \emptyset$, choosing $\mathcal{F} = \mathcal{F}_\Psi \mathcal{C} = \mathcal{C}_\Phi$ and $\phi \in \Phi$ such that $\sigma = \phi(E^n) = \phi_\sigma$, then $\phi_\sigma(E^n - \partial E^n) \cap A \neq \sigma$ and since the open cells partition X, in order for this intersection being non empty then $\phi_\sigma \in \mathcal{F}_\Psi$. □

Definition 2.33. *For a cell complex* $(X, \mathcal{C})$:

- *It is finite or countable if* $\mathcal{C}$ *is a finite or countable set.*
- *It is locally finite or locally countable if each closed cell meets only a finite or countable number of cells.*
- *It is closure finite if each n-cell meets only a finite number of open cells* $\sigma^p - \partial\sigma^p$ *with* $p < n$*. Also, the cell complex* $(X, \mathcal{C})$ *has dimension* n *if it has no cells* σ^p *of dimension greater than* n *and at least one cell of dimension* n*.*

The following definitions are just a useful remainder

Definition 2.34. *A topological space* X *is said to be a Hausdorff space if any two distinct points* $x, y \in X$ *posses non intersecting neighborhoods.*

Definition 2.35. *Let* $\{\phi_\sigma \mid \sigma \in \mathcal{C}\}$ *a set of characteristic functions for cells of* $\mathcal{C}$*. Then the weak topology on* X *with respect to* $\mathcal{C}$ *is obtained by:*

- *Giving each cell* $\sigma \in \mathcal{C}$ *the quotient topology with respect to its characteristic function.*
- *Giving* X *the weak topology with respect to the subsets* $\sigma \in \mathcal{C}$*, i.e., a set* $F \subset X$ *is closed if and only if* $F \cap \sigma$ *is closed in* σ *for each* $\sigma \in \mathcal{C}$*.*

Definition 2.36. *Let* X *be a Hausdorff topological space,* X *is said to be a CW complex with respect of a family of sets* $\mathcal{C}$ *provided:*

1. *The pair* $(X, \mathcal{C})$ *is a cell complex such that each cell* $\sigma \in \mathcal{C}$ *has a continuous characteristic function.*
2. *The space* X *has the weak topology with respect to* $\mathcal{C}$*.*
3. *the cell complex* $(X, \mathcal{C})$ *is closure finite.*

If $(X, \mathcal{C})$ is a cell complex such that satisfies 1., 2., and 3. with respect to $\mathcal{C}$, then is said X is a CW-complex with cells in $\mathcal{C}$.
CW-complexes are constructed by gluing together cells, i.e. topological cells homeomorphic to standard disks. For example, the torus T^2 can be obtained by gluing together of opposite sides of a square, so T^2 can be presented as the union of one 2-dimensional cell, two 1-dimensional cells and one 0-dimensional cell.

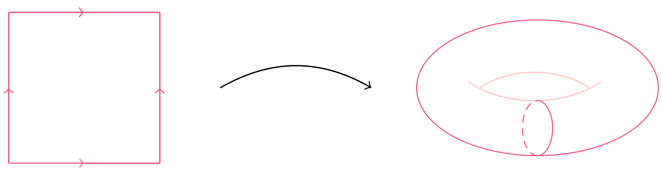


Figure 6: T^2 gluing cells

The first point in the definition of a CW-complex relates $\sigma \in \mathcal{C}$ to the topology of the more familiar space E_σ, and the following can be proved.

Lemma 2.37. *Let X be a CW complex with cells $\mathcal{C}$. Then*

1. *Each cell $\sigma \in \mathcal{C}$ is a closed subset of X.*

2. *For each cell $\sigma \in \mathcal{C}$, the restriction of the characteristic map ϕ_σ to $E_\sigma - \partial E_\sigma$ is a homeomorphism onto $\sigma - \partial\sigma$.*

Proof.

1. Given any closed set $K \subset E_\sigma$ then K is compact because E_σ is, and by the continuity of ϕ_σ it follows $\phi_\sigma(K)$ is compact in X, with X a Hausdorff space, so $\phi_\sigma(K)$ is closed in X.

2. By ϕ_σ is a closed function and for the subspace $E_\sigma - \partial E_\sigma$ the restricted function $\phi_\sigma \mid (E_\sigma - \partial E_\sigma)$ is continuous and injective by definition and by lemma 2.16 it is bijective so $\phi_\sigma \mid (E_\sigma - \partial E_\sigma)$ is a homeomorphism onto $\phi - \partial\phi$.

□

Definition 2.38. *Let X and Y be topological spaces, let $A \subset X$ closed and let $f : A \to Y$ be a continuous map. The adjunction space $Y \cup_f X$ is the quotient of the disjoint union of Y and X by the small equivalence relation which identifies each $a \in A$ with its image $f(a) \in Y$.*

If $\Phi : Y \cup X \to Y \cup_f X$ is the quotient map, then Φ is continuous and it is easy to see that

- $\Phi|_Y$ is a homeomorphism of Y onto the closed subset $\Phi(Y)$.
- $\Phi_{(X-A)}$ is a homeomorphism of $X - A$ onto the open subset $\Phi(X - A)$.
- $Y \cup_f X = \Phi(Y) \cup \Phi(X - A)$ is a partition of $Y \cup_f X$

Now, supposing that X is a topological space and for each $\lambda \in \Lambda$ there is a map $f_\lambda : \partial E_\lambda \to X$ of the boundary of the Euclidean cell E_λ into X. Letting $\xi = \cup\{E_\lambda \mid \lambda \in \Lambda\}$ and $\partial\xi = \{\partial E_\lambda \mid \lambda \in \Lambda\}$ be the disjoint unions, and let $F = \cup_\lambda f_\lambda : \partial\xi \to X$ be the union map. The adjunction space $X \cup_F \xi$ is said to be obtained by *attaching cells* E_λ to X. The maps $f_\lambda : \partial E_\lambda \to X$ are called *attaching maps* for the cells E_λ of $X \cup_F \xi$. The adjunction of a family of $0-$cells is just the disjoint union. The following proposition is stated as an important result and its proof can be found in [4].

Proposition 2.39. *Let X be a CW complex, and for each $\lambda \in \Lambda$, let $f_\lambda : \partial E_\lambda \to X^{n-1}$ be a map attaching the cell E^n_λ to the $(n-1)-$skeleton of X. Define $\xi^n = \cup\{E^n_\lambda \mid \lambda \in \Lambda\}$, $\partial\xi = \cup\{\partial E_\lambda \mid \lambda \in \Lambda\}$, and let $F = \cup : \partial\xi^n \to X^{n-1}$ be the union map. Then $X \cup_F \xi^n$ is a CW complex*

To end this subsection, the concept of regular CW complex is introduced.

Definition 2.40. *A CW complex X is called* ***regular*** *if for each cell α of X, the restriction of the characteristic map $f_\alpha : \partial E_\alpha \to f_\alpha(\partial E_\alpha)$ is a homeomorphism.*

By definitions 2.7 and 2.8 it follows that simplicial complexes are regular CW complexes.

2.4 Discrete Morse theory

Let M be simplicial complex, K the set of simplices of M, and K_p the set of simplices of dimension p, then the definition of a discrete Morse function is as follows.

Definition 2.41. *A function*

$$f : K \to \mathbb{R}$$

is a discrete Morse function if for every $\alpha^p \in K$

$$\text{(i)}\ \left|\{\beta^{p+1} > \alpha \mid f(\beta) \leq f(\alpha)\}\right| \leq 1$$

and

$$\text{(ii)}\ \left|\{\gamma^{p-1} < \alpha \mid f(\gamma) \geq f(\alpha)\}\right| \leq 1.$$

In the figure below are shown two different functions for the same simplicial complex. For the function on the right, it is easy to see it is not a Morse function since the edge with value zero has both endpoints with higher value. Instead, for the function on the left, all the elements fulfill the requirements of a discrete Morse function.

Figure 7: In the left a discrete Morse function and in the right a discrete Morse function.

Definition 2.42. *A simplex α^p is* ***critical*** *if*

$$\left|\{\beta^{p+1} > \alpha \mid f(\beta) \leq f(\alpha)\}\right| = 0$$

and

$$\left|\{\gamma^{p-1} < \alpha \mid f(\gamma) \geq f(\alpha)\}\right| = 0.$$

Any simplex that is not critical is called ***regular*** *and the total number of critical simplices of dimension p is denoted by m_p.*

In this definition, taking $\alpha^p \in K$ a non-critical simplex, then either of the following conditions holds

1. There exists a unique $\beta^{p+1} \in K$ such that $\beta > \alpha$ and $f(\beta) \leq f(\alpha)$.
2. There exists a unique $\sigma^{p-1} \in K$ such that $\sigma < \alpha$ and $f(\sigma) \geq f(\alpha)$.

An important thing to note about this, is that just one of the conditions can hold. This is proved in the following lemma, sometimes referred as the exclusion lemma.

Lemma 2.43. *Let $f : K \to \mathbb{R}$ be a discrete Morse function of the simplicial complex and $\sigma \in K$ a regular simplex. Then the conditions i) and ii) cannot be simultaneously true. Hence, exactly one of the conditions holds whenever σ is a regular simplex.*

Proof. Suppose by contradiction, for $\alpha = \{a_0, a_1, ..., a_p\} = a_0, a_1, ..., a_p$ there exist $\beta = a_0, a_1, ..., a_p, a_{p+1}$ and $\sigma = a_0, a_1..., a_{p-1}$ such that $\sigma < \alpha < \beta$ and $f(\beta) \leq f(\alpha) \leq f(\sigma)$, taking $\widetilde{\sigma} = a_0, a_1, ...a_{p-1}, a_{p+1}$ satisfies $\sigma < \widetilde{\sigma} < \beta$ then it follows that $f(\sigma) < f(\widetilde{\sigma}) < f(\beta)$ since we already have $f(\sigma) \geq f(\alpha)$ and $f(\beta) \leq f(\alpha)$ and our function is a discrete Morse function. Hence

$$f(\beta) \leq f(\alpha) \leq f(\sigma) < f(\widetilde{\sigma}) < f(\beta)$$

which is a contradiction. □

For example, in the last figure, all the edges labeled with four, are critical, but they are not the only ones, also the 2-simplex and the right edge are critical.

By the exclusion lemma, for $0 \leq p < dim(K)$ taking $\alpha^p \in K$ a regular simplex then exists a unique 1-coface $\beta^{p+1} \in K$ of α such that $f(\beta) \leq f(\alpha)$, and taking all of these pairs (α, β) a discrete vector field can be constructed. So, for any Morse function over K one can obtain a discrete vector field that is, in fact, a discrete vector gradient field, as is proved in the next proposition.

Proposition 2.44. *If V is the discrete vector field on a simplicial complex K, V is the gradient vector field obtained from a Morse function $f : K \to \mathbb{R}$ if and only if there are no non-trivial closed $V-$paths. It is said that V is* **acyclic**

Proof. For $\alpha_0^p, \beta_0^{p+1}, \alpha_1^p, \beta_1^{p+1}, ..., \alpha_l^p, \beta_l^{p+1}, \alpha_{l+1}^p$ a non-trivial V-path, then $f(\alpha_i) \geq f(\beta_i)$ and $f(\beta_i) > f(\alpha_{i+1})$ for all $0 \leq i < l$. So the chain of inequalities

$$f(\alpha_0) \geq f(\beta_0) > f(\alpha_1) \geq f(\beta_1) > ... > f(\alpha_{l+1})$$

guarantees $\alpha_0 \neq \alpha_{l+1}$. So no non-trivial V-path is closed. □

The converse of this proposition is going to be proved after giving some other necessary concepts. Consider a simplicial complex K, it can be "drawn" by representing each simplex as a point, and drawing an arrow from β^{p+1} to α^p if $\alpha^p \subset \beta^{p+1}$. This is called the Hasse diagram of K.
As an example of a Hasse diagram, suposse the simplicial complex is an abstract one $K = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, then the Hasse diagram of K is like

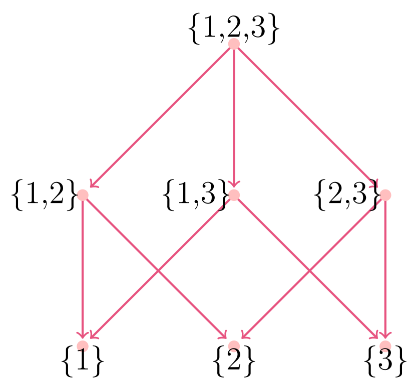


Figure 8: Example of the Hasse diagram of a simplicial complex K

Now, supposing V is a discrete vector field on K, the Hasse diagram can be modified by reversing the direction of the arrow from β to α whenever $(\alpha, \beta) \in V$.

In the above example, if $V = \{(\{2,3\},\{1,2,3\}),(\{1\},\{1,2\}),(\{3\},\{1,3\})\}$, then modified Hasse diadram results in the directed graph shown below

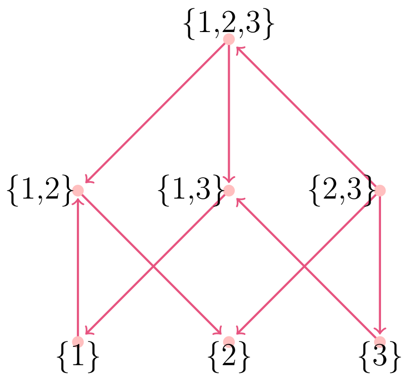


Figure 9: Example of the modified Hasse diagram of a simplicial complex K

As can be seen in the example, in general, for every simplicial complex with a discrete vector field V, the new Hasse diagram H_v, i.e. that resulting from modifying the original one by inverting the arrows whose head and tail are in V, is a *directed graph.*

Proposition 2.45. *If V is a discrete vector field on a simplicial complex K and there are no closed $V-$paths then the corresponding directed graph H_v has no cycles.*

Proof. Suposse there is a cycle in the directed graph, since the dimension of adjacent simplices differs by 1, the cycle must have an even number of elements, let call them $\alpha_0, \beta_0, ..., \alpha_r, \beta_r$. Without loss of generality $\alpha_0^p < \beta_0^{p+1}$. For some simplex α_i in the cycle, if β_i is such that the arrow connecting them is pointing up $\alpha_i \nearrow \beta_i$ then the following must be an arrow pointing down $\beta_i \searrow \alpha_{i+1}$ by the definition of V, $dim(\alpha_i) = dim(\alpha_{i+1})$ and the same can be deduced for $\beta's$. So the considered cycle is actually a closed $V-$path □

Theorem 2.46. *If G is a finite directed graph, there exist f from the vertices of G to $\mathbb{R}$ such that $f(a) > f(b)$ whenever there is an arrow from a to b if and only if G has no cycles.*

Proof. Is clear that if such function exists, G cannot have a cycle since the strict inequality would yield a contradiction. Now, since F is finite and acyclic, calling a node to each point that is not the tail of any arrow, then for a path starting on any vertex, its must eventually hit a node. Then, for a vertex α, let $f(\alpha)$ be the greatest number of vertices in a path from a to a node (which is finite). Then if there is an arrows from α to β, any path from α to a node can be appended to the arrow from α to β to make that path one vertex longer, so necessarily $f(\alpha) > f(\beta)$. □

The converse of proposition 2.44 is proven directly using the function obtained from the Hasse diagram associated to the gradient vector field V A very relevant result about discrete Morse function is the following.

Theorem 2.47. *Let M be a regular CW complex, then M is homotopy equivalent to a CW complex with $m_p(f)$ cells of dimension p.*

A proof of this theorem can be found in [2].

2.5 The algorithm

Proposition 2.44 says that if there is a gradient vector field V over a simplicial complex K then it is associated to a discrete Morse function. So, constructing a gradient vector field over K is sufficient to be able to use theorem 2.47. In such setting, it is clear that the critical cells correspond to vertices on the modified Hasse diagram such that they are not in V i.e. they do not have a partner. On reference [3], an algorithm for constructing a gradient vector field over subcomplexes of powers of simplicial complexes is described. Here we will adapt the idea in order to construct a suitable gradient field on an arbitrary subcomplex of more general cartesian products. Details follow.

Let $K_1, ..., K_n$ be simplicial complexes and X a subcomplex of the ordered product $K_1 \times ... \times K_n$, where for every $c \in \{1, ..., n\}$, the sets $V_c = V(K_c)$ have a linear order, such that every $d-$cell $\sigma \in K_c$, has a unique expression like a word $\sigma = v_0...v_d$. In such a context, it is said that, the vertex v_i appears in the position i of the cell σ. In particular, the $d-$cells of K_c are ordered linearly in a lexicographic form, and establishing that any cell of dimension p is greater than any other of dimension q providing $p < q$, then all the cells of K_c are linearly ordered too. In such terms, the $d'-$cells $(c_1, ..., c_n)$ of $K_1 \times ... \times K_n$ are ordered lexicographically:

$$(c_1, ..., c_n) < (d_1, ..., d_n)$$

if $c_i < d_i$ for some $i \in \{1, ..., n\}$, and $c_j = d_j$ for every $j < i$.
The gradient field on X is built through an iterative process (described in the following paragraph), in which $d'-$cells $\alpha = (e_1, ..., e_n)$ of X are paired with $(d+1)'-$cells of X such that $\beta = (e_1, ..., e_{c-1}, e_c +_p v, e_{c+1}, ..., e_n)$, where the notation $e_c +_p v$ refers to the potential cell of K_c that would be obtained by inserting the vertex $v \in K_c$ in the coordinate e_c of α and such that v is in the position p of $e_c +_p v$ $(0 \leq p \leq d+1)$. At each moment of the algorithm, for all $0 \leq m \leq dim(X)$ there is a list of all available $m-$cells, that is, those that have not been paired yet. Each time a pair is identified, both of these cells are deleted of their corresponding lists. The identification and formation of pairs is organized inside a process $\mathcal{P}$. Such process consist of other sub-processes described below:

- $\mathcal{P}(D)$ is a sub-process of $\mathcal{P}$ executed at dimension D, where D starts at $D = dim(X) - 1$ and ends at $D = 0$.

- $\mathcal{P}(D, c)$ is a sub-process of $\mathcal{P}(D)$ executed over the coordinates c, where c starts at $c = n$ and ends at $c = 1$.

- $\mathcal{P}(D, c, p)$ is a sub-process of $\mathcal{P}(D, c)$ executed by position, where p starts at $p = d + 1$ and ends at $p = 0$.

- $\mathcal{P}(D, c, p, v)$ is a sub-process of $\mathcal{P}(D, c, p)$ executed over each vertex v of K_c, starting from the greatest to the lowest

  According to the last two items, the position refers to that which will have v inside the coordinate $e_c +_p v$.

- $\mathcal{P}(D,c,p,v,\alpha)$ is a sub-process of $\mathcal{P}(D,c,p,v)$ executed over all the $d'-$cells which are available at the moment when $\mathcal{P}(D,c,p,v)$ starts. Concretely, for every $d'-$cell, say $\alpha=(e_1,...,e_n)$ which is available, $\mathcal{P}(D,c,p,r,\alpha)$ asks whether $\beta=(e_1,...,e_{c-1},e_c+_p v,e_{c+1},...,e_n)$ is a $(d+1)-$cell of X and if it is available. If the answer is positive, then the pair $\alpha\nearrow\beta$ is added to the field, and α,β are deleted of their respective lists, i.e. they are not available anymore.

Remark 1. *In the last part of the algorithm, i.e. on $\mathcal{P}(D,c,p,r,\alpha)$, it is not relevant the order of how the $d'-$cells α are taken. In fact, if $A^{d'}$ is the set of all $d'-$cells which are available when $\mathcal{P}(d,c,p,v)$ starts, it is not possible that two different $d'-$cells, say α,α' in $A^{d'}$, to be paired with the same $(d+1)'-$cell β in $A^{(d+1)'}$, this is because all the parameters for inserting v are fixed, so then β determines its pair α according to where the parameters indicate that vertex v is being inserted. Thus the task of $\mathcal{P}(D,c,p,v)$ is to make all the possible pairings $\alpha\nearrow\beta$ at the moment when the vertex $v\in K_c$ is being inserted in the position p of the coordinate c. Still, for the purpose of demonstrating acyclicity of the resulting field, it is considered that the instructions $P(D,c,p,r,\alpha)$ are executed in order, from the lowest to the greatest of all the available $d'-$cells α at the moment when $\mathcal{P}(D,c,p,v)$ starts.*

This concludes the definition of the algorithm that builds the vector field that concerns us. The demonstration of the acyclicity of this field is based on the following version of the main lemma in the proof of acyclicity on [3]:

Lemma 2.48. *Assume $\alpha=(e_1,...,e_n)\nearrow\beta=(e_1,...,e_{c-1},e_c+_p v,e_{c+1},e_n)$ is a pair produced during the stage $\mathcal{P}(l,c,p,v,\alpha)$ of the algorithm. Let $\gamma\in X$ a $l-$cell obtained from β by removing the vertex w that is at the position q of the coordinate d of β, such that $\gamma\neq\alpha$ (particularly $(c,p)\neq(d,q)$) and suppose one and only one of the following conditions holds:*

a) $c<d$.

b) $c=d$ and $p<q$.

Then γ is part of a pair $\gamma\nearrow\delta$ constructed by the algorithm during the sub-process $\mathcal{P}(l,e,r,u,\gamma)$, for parameters l,e,r,u,γ that satisfies one, and necessarily just one, of the following conditions

a') $d<e$.

b') $d=e$ and $q<r$.

c') $d=e$ and $q=r$ and $w<u$.

Particularly, the pair $\gamma\nearrow\delta$ is done before the pair $\alpha\nearrow\beta$

Proof. Given $e_c=(e_{c_0},...,e_{c_p},...,e_{c_s})$ then

$$\alpha=(e_1,...,(e_{c_0},...,e_{c_{p-1}},e_{c_p},...,e_{c_s}),...,e_n)$$

and

$$\beta=(e_1,...,(e_{c_0},...,e_{c_{p-1}},v,\underbrace{e_{c_p}...,e_{c_s}),...,e_n)}.$$

By the hypotheses a) and b), γ is obtained from β by removing the vertex w from any of the places indicated by the horizontal key. Since the algorithm is executed from the largest to the smallest of the values of both coordinates and positions, then $\mathcal{P}(l,c,p,v)$ is posterior to $\mathcal{P}(l,d,q,w)$, an so β is available at the moment when $\mathcal{P}(l,d,q,w)$ is executing, but then γ could not be available in such a moment since, if that were the case, the pair $\gamma \nearrow \beta$ could have been realized first, but that is not possible. So, γ had to be paired before the sub-process $\mathcal{P}(l,d,q,w)$ starts. □

In the case that concerns us, all the complexes K_i are of dimension 1, so the acyclicity of the field is proved under such conditions.

Proposition 2.49. *The discrete field V obtained by the algorithm is gradient when each K_i, $1 \leq i \leq n$ is of dimension 1.*

Proof. Assume, for a contradiction, the algorithm produced a cycle

$$\alpha_0 \nearrow \beta_0 \searrow \alpha_1 \nearrow \beta_1 \searrow ...\alpha_n \nearrow \beta_n \searrow \alpha_{n+1} = \alpha_0$$

. Without loss of generality, it can be supposed $\alpha_0 \nearrow \beta_0$ is the first pair produced by the algorithm. Say, β_i is obtained from α_i by inserting the vertex v_i on the position p_i of the coordinate c_i of β_i, i.e. $\alpha_i \nearrow \beta_i = \alpha_i + c_i, p_i v_i$, then it is produced during the sub-process $\mathcal{P}(l, c_i, p_i, v_i, \alpha_i)$, where $l = dim(\alpha_i)$, for every i. By hypothesis, and in view of remark 1, for every $i = 1, ...n$, one and only one of the following conditions holds:

- $c_i < c_0$.
- $c_i = c_0$ and $p_i < p_0$.
- $c_i = c_0$ and $p_i = p_0$ and $v_i < v_0$.
- $c_i < c_0$ and $p_i = p_0$ and $v_i = v_0$ and $\alpha_0 < \alpha_i$.

Case 1: $p_0 = 0$. Given that $dim(K_{c_0}) = 1$, α_0 and β_0 are of the form $\alpha_0 = (..., \alpha_{00}, ...)$ and $\beta = (..., v_0\alpha_{00}, ...)$, where the vertex α_{00} of K_{c_0} and the $1-$simplex $v_0\alpha_{00}$ appears on the coordinate c_0 of α_0 and β_0 respectively. By the lemma and given that $\alpha_1 \neq \alpha_0$, α_1 is obtained from β_0 by removing a vertex from a simplex at a coordinate less than c_0 of β_0, while the pair $\alpha_1 \nearrow \beta_1$ necessarily happens with $c_1 < c_0$, as $dim(K_{c_0}) = 1$. In particular, both α_1 and β_1 are of the form $(..., v_0\alpha_{00}, ...)$, i.e. with the $1-$simplex $v_0\alpha_0$ on the coordinate c_0. inductively, it follows that $\alpha_i \nearrow \beta_i$ happens with $c_i < c_0$ and with both cells of the form $(..., v_0\alpha_{00}, ...)$, where the $1-$simplex $v_0\alpha_{00}$ is at the coordinate c_0. By the lemma, α_{i+1} is obtained from β_i by removing a vertex from a simplex on a coordinate less than c_0 of β_i, while the pair $\alpha_{i+1} \nearrow \beta i + 1$ happens with $c_{i+1} < c_0$, so all the α_j, $j \geq 1$, have the $1-$simplex $v_0\alpha_{00}$ at coordinate c_0, but this impossible since $\alpha_{n+1} \neq \alpha_0$.
Case 2: $p_0 = 1$. This time $\alpha = (..., \alpha_0, ...)$ and $\beta_0 = (..., \alpha_{00}v_0, ...)$, with α_{00} and $\alpha_{00}v_0$ at the coordinate c_0. Say v_0 remains as vertex of the coordinate c_0 for all the cells of the curve $\beta_0 \searrow ... \searrow \alpha_r \nearrow \beta_r$, $r \geq 1$, but α_{r+1} is obtained from β_r by removing v_0 of the coordinate c_0, this situation necessarily happens since $\alpha_1 \neq \alpha_0$ and v_0 is not a vertex of the coordinate c_0 of $\alpha_{n+1} = \alpha_0$. Then, there are two different options for β_r, either $\beta_r = (..., wv_0, ...)$ or $\beta = (..., v_0w, ...)$, with wv_0 and v_0w at the coordinate c_0. The second is not possible as w had to have been inserted in a pairing previous

to $\alpha_0 \nearrow \beta_0$. Hence $\beta_r = (..., wv_0, ...)$, and by the election of $\alpha_0 \nearrow \beta_0$, it follows $\alpha_r \nearrow \beta_r$ arises with $c_r \leq c_0$. If $c_r < c_0$ it follows is not possible for the curve to be a cycle. If $c_r = c_0$, then $v_r = w < v_0$, and in that case, removing v_0 from the simplex wv_0 to form the face $\beta_r \searrow \alpha_{r+1}$, the lemma says that $\alpha_{r+1} \nearrow \beta_{r+1}$ happens before $\alpha_o \nearrow \beta_0$, which is a contradiction. $\square$

3 Configuration spaces of hard squares

Configuration spaces of non-overlapping hard squares in a $p \times q$ rectangle were studied by Alpert et al in [1]. In this short chapter we review relevant definitions and basic results. Readers whishing to check proof details are referred to [1].

As its name says, taking $n, p, q \in \mathbb{N}$, the configuration space $C(n; p, q)$ of n unit squares in a $p \times q$ rectangle consists of all the possible forms to have n non-intersecting —they can intersect each other at the boundary or at the rectangle's boundary, but nothing more — unit squares inside a bounded rectangle of size $p \times q$. By keeping track of the coordinates of the centers of the squares, as can be seen in the figure below, and taking for convenience, the $p \times q$ rectangle as $[\frac{1}{2}, p+\frac{1}{2}] \times [\frac{1}{2}, q+\frac{1}{2}] \subset \mathbb{R}^2$, if (x_i, y_i) is the coordinate of the center of the $i-th$ square, then $(x_1, y_1, x_2, x_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$ completely characterizes a typical element of $C(n; p, q)$ and thus considered as a subspace of $\mathbb{R}^{2n}$

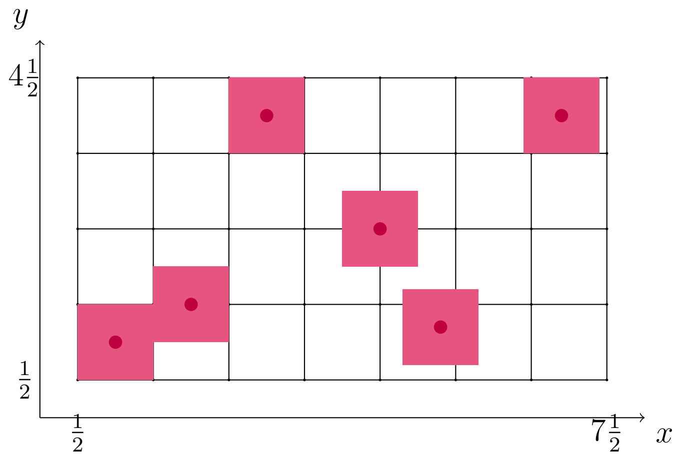


Figure 10: Example of an element in $C(6; 7, 4)$

Definition 3.1. *The configuration space $C(n; p, q)$ of n unit squares in a $p \times q$ rectangle is defined to be the set of all points*

$$(x_1, y_1, x_2, x_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$$

such that

- $1 \leq x_k \leq p$ *and* $1 \leq y_k \leq q$ *for all* $1 \leq k \leq n$ *and*
- $|x_k - x_l| \geq 1$ *or* $|y_k - y_l| \geq 1$ *for all* $1 \leq k < l \leq n$

There are two different ways to draw a grid on the rectangle that will be used. For the one, since the rectangle used is $[\frac{1}{2}, p + \frac{1}{2}] \times [\frac{1}{2}, q + \frac{1}{2}]$, the set of all possible positions for the centers of squares is $[1, p] \times [1, q]$ which can be thought of as having **vertices** at the points where both coordinates are integers, **edges** between vertices have distance one, there are $(p-1)(q-1)$ **square** 2-cells, this grid is denoted as G_1. These vertices, edges and squares are referred as *coordinate grid vertices, coordinate grid edges* and *coordinate grid squares* respectively, together are called *coordinate grid cells*. The second grid (G_2) is that of the $[\frac{1}{2}, p+\frac{1}{2}] \times [\frac{1}{2}, q+\frac{1}{2}]$ itself. An example of grids is shown in the figure below.

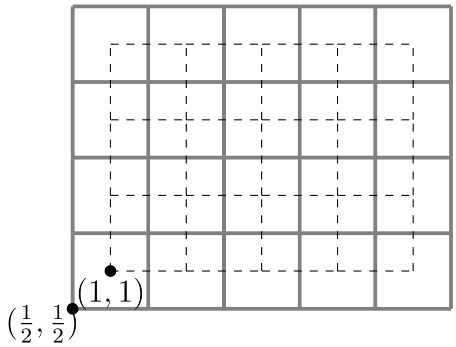


Figure 11: Grids G_1 and G_2, continuous gray lines are for G_2 and the dashed ones for G_1

Taking a coordinate grid vertex v, and considering of the rectangle $[\frac{1}{2}, p+\frac{1}{2}] \times [\frac{1}{2}, q+\frac{1}{2}]$ as a $p \times q$ chessboard, the unit square centered at v is called a *board square*. Also the other coordinate grid cells have a rectangle of board squares given by taking the union of all unit squares for which the center lies on that coordinate grid cell, so for a coordinate grid edge there is a pair of adjacent board squares, and for a coordinate grid square there is a 2×2 rectangle of board squares. An example of these rectangles is shown in the next figure

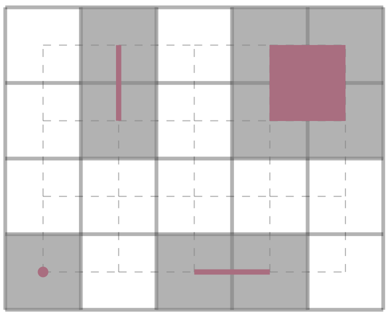


Figure 12: Rectangles of board squares for different coordinate grid cells

Let $G(n; p, q) := ([1, p] \times [1, q])^n \subset \mathbb{R}^{2n}$, then $G(1; p, q) = [1, p] \times [1, q])$ so $G(n; p, q) = G(1; p, q)^n$, where G stands for *grid*. We take $G(n; p, q)$ with its natural cubical complex structure, i.e. as $G(n; p, q) = \bigcup_{l=1}^{2n} \mathcal{K}_l(G(n; p, q))$. Now, taking an element Q of $\mathcal{K}(G(n; p, q))$ then $Q = I_1 \times ... \times I_n$ with I_i the product of two elementary, but possibly degenerate) intervals in $p \times q$, setting I_i inside the rectangle according to its own components and doing this for every elementary interval of Q, then $G(n; p, q)$ can be seen as the set of configurations of labeled squares (from here on the intervals will be referred to as squares) in the rectangle where the squares are allowed to overlap. As a cubical complex, each cell of $G(n; p, q)$ corresponds to a n-tuple in which each entry is a coordinate grid cell, so a cell of $G(n; p, q)$ can be drawn as n rectangles of board squares on $p \times q$. Any list of n rectangles of board squares of sizes 1×1, 1×2, 2×1, 2×2 is the rectangle arrangement of some cell in $G(n; p, q)$.

Definition 3.2. *The subcomplex $X(n; p, q) \subset G(n; p, q)$ consists of all the cells of $G(n; p, q)$ that are fully contained in $C(n; p, q)$.*

Proposition 3.3 below is one of the central results in [1].

Proposition 3.3. *X(n;p,q) consists of all the cells of $G(n;p,q)$ for which their corresponding rectangle arrangement has none of its pieces overlapping. Futhermore, the inclusion $X(n;p,q) \hookrightarrow C(n;p,q)$ is a homotopy equivalence. Indeed, $X(n;p,q)$ sits inside $C(n;p,q)$ as a strong deformation retract.*

4 Deeply inside particular cases

From the deformation retract $X(n;p,q)$ of $C(n;p,q)$, the configuration space of hard squares can be seen as the collection of at most $2n-$dimensional cells whose components are unit squares, unit horizontal or vertical segments, or points over the grid G_1 such that their board squares over the grid G_2 do not intersect each other (maybe just on their boundary). On the other hand, by proposition 2.49, it follows that a vector gradient field can be constructed for the cubical complex $X(n;p,q)$ with the algorithm described, since $X(n;p,q)$ is actually a subcomplex of the Cartesian product $[1,p]\times[1,q]\times[1,p]\times[1,q]\times ... \times[1,p]\times[1,q]$. Then, by proposition 2.44 it follows V is related to a discrete Morse function, so theorem 2.47 can be used to obtain information about $X(n;p,q)$.

Before starting with calculations, it is useful to remark some facts:

- Given any cell $\alpha \in X(n;p,q)$, $\alpha=(\alpha_1,...,\alpha_n)$, there are four possible options for α_i, $i=1,...n$, namely
  1. α_i is a point, in such case $\alpha_i=(a_{i_{1,1}},a_{i_{1,2}})$.
  2. α_i is a horizontal line, in such case $\alpha_i=(a_{i_{1,1}}a_{i_{1,2}},a_{i_{2,1}})$, with $a_{i_{1,2}}=a_{i_{1,1}}+1$.
  3. α_i is a vertical line, in such case $\alpha_i=(a_{i_{1,1}},a_{i_{2,1}}a_{i_{2,2}})$, with $a_{i_{2,2}}=a_{i_{2,1}}+1$.
  4. α_i is a square, in such case $\alpha_i=(a_{i_{1,1}}a_{i_{1,2}},a_{i_{2,1}}a_{i_{2,2}})$, with $a_{i_{1,2}}=a_{i_{1,1}}+1$ and $a_{i_{2,2}}=a_{i_{2,1}}+1$.
- In particular, we think of $X(n;p,q)$ on a cubical subcomplex of a product of $1-$dimensional simplicial complexes, namely, as a cubical subcomplex of

  $$[1,p]\times[1,q]\times[1,p]\times[1,q]\times ... \times[1,p]\times[1,q],$$

  where there are n factors $[1,p]\times[1,q]$. Thus, for the purposes of the algorithm of section 2.5, we have $2n$ coordinates which are packed as $((1,2),(3,4),...,(2n-1,2n))$ to form the original n coordinates of $X(n;p,q)$. Note that even coordinates are "vertical" and correspond to factors $[1,q]$, while odd coordinates are "horizontal" and correspond to factors $[1,p]$
- At a fixed dimension d, the algorithm consists of the $2n$ processes

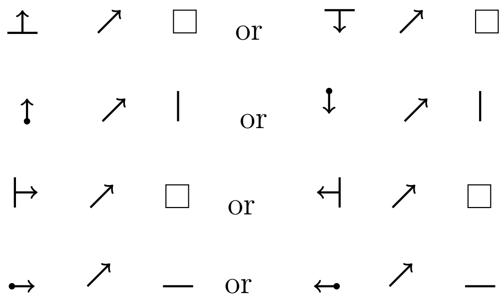
  $$\mathcal{P}(d,2n),\mathcal{P}(d,2n-1),...,\mathcal{P}(d,2),\mathcal{P}(d,1),$$

  executed in that order (from left to right). Since $dim[1,p]=dim[1,q]=1$, each of these processes $\mathcal{P}(d,c)$ $(1\leq c\leq 2n)$ is limited to two sub-processes:

  $$\mathcal{P}(d,c,1) \text{ and } \mathcal{P}(d,c,0),$$

  which are run in that order. When $c=2i$, with $1\leq i\leq n$, i.e. for a vertical coordinate, the identification of pairs by $\mathcal{P}(d,c,1)$ (respectively by $\mathcal{P}(d,c,0)$), i.e. by insertion of vertices in position 1 (respectively in position 0) of coordinate c, is based on checking whether the $i-th$ coordinate of available grid cells of dimension d of $X(n;p,q)$ can be "*stretched upwards*" (respectively "*stretched downwards*") yielding an available grid cell of dimension $d+1$ of $X(n;p,q)$.

Likewise, when $c = 2i - 1$ with $1 \leq i \leq n$, i.e. for a horizontal coordinate, the identification of pairs by $\mathcal{P}(d, c, 1)$ (respectively by $\mathcal{P}(d, c, 0)$), i.e. by insertion of vertices in position 1 (respectively in position 0) of coordinate c, is based on checking whether the $i - th$ coordinate of available grid cells of dimension d of $X(n; p, q)$ can be *"stretched to the right"* (respectively *"stretched to the left"*) yielding an available grid cell of dimension $d + 1$ of $X(n; p, q)$.
In particular, the way in which each of the final instructions $\mathcal{P}(d, c, r, v, \alpha)$ can assemble a pair $\alpha \nearrow \beta$ is forced to obey the "stretching" descriptions above. For instance, when $\alpha = (\alpha_1, ..., \alpha_n)$, with $\alpha_i = (\alpha_{i_{1,1}}, \alpha_{i_{2,1}} \alpha_{i_{2,2}})$, $\mathcal{P}(d, 2i, r, v, \alpha)$ cannot produce a pair $\alpha \nearrow \beta$, while $\mathcal{P}(d, 2i - 1, r, v, \alpha)$ can produce a pair $\alpha \nearrow \beta$ only when $v = \alpha_{i_{1,1}} + 1$ and $r = 1$ (in which case a "stretching-to-the-right pairing" arises), or when $v = \alpha_{i_{1,1}} - 1$ and $r = 0$ (in which case a "stretching-to-the-left pairing" arises). In each case, the potential partner β is well defined and the pairing $\alpha \nearrow \beta$ is assembled as β is a cell of $X(n; p, q)$ which is available at that moment of the algorithm. All possible stretching patterns of coordinates of cells are shown below.



- Since the processes $\mathcal{P}(d, 2i, 1), \mathcal{P}(d, 2i, 0), \mathcal{P}(d, 2i - 1, 1)$ and $\mathcal{P}(d, 2i - 1, 0)$ run in that order, for the viewpoint of a given cell $\alpha = (\alpha_1, ..., \alpha_n)$ of dimension d, the algorithm amounts to running i from n down to 1 and, in each such iteration, the component α_i of an available cell, α is first attempted to be stretched upwards, then stretched downwards, then stretched to the right and, finally, stretched to the left. As soon as the first one of these stretching trials matches an available cell of $X(n; p, q)$, the process stops for α and the two paired cells are marked as no longer available. If neither of these processes do not find a partner for α then the process repeats one dimension below and α is then the subject to be the partner of a $(d - 1)-$dimensional cell via the corresponding sequence of stretching trials.

- An important feature in the application of the algorithm is that all the cells are ordered tuples if grid cells which are unequivocally determined by their components (□, —, | or •) and their position within the cells. Then for a fixed component, there is just one way at a time for pairing it, so in this sense the order is not relevant, thus recovering the observations in Remark 1.

All of these observations are considered and used for the following sections. Moreover, arrows $\uparrow$ and $\rightarrow$ are used to denote whether cells are being matched by adding a vertex at a coordinate $2i$ or $2i - 1$, respectively. This does not mean that cells are being matched by stretching the respective component only up or to the right, in other words, in each case we take into account both positions 0 or 1.
The calculations that will be made in the following sections cover the cases:

- $q = 1$ with $p, n \in \mathbb{N}$ arbitrary.
- $n = 2$ with $p, q \in \mathbb{N}$ arbitrary.
- $n = 3$ with $p, q \in \mathbb{N}$ arbitrary.

4.1 q=1

In order to illustrate and prepare the argument for values of n and p general, we start with some particular cases n, q. The first explored case is $n = 2$ and $q = 1$, i.e. the grid is of height one with arbitrary length, so there are no cells with squares or vertical lines as components. When $p = 2$ there are just two cells of dimension zero. If $p = 3$ then there are cells of dimension one, and by the size of the grid, there are four cells of this dimension, and six cells of dimension zero and it is easy to deduce all 1-cells can be paired except two critical cells, both of dimension zero, so that $X(2; 3, 1)$ has two components each of which is contractible. Our first goal is to show that this phenomenon holds for all $X(2; p, 1)$, which therefore, recovers the fact that $C(2; p, 1)$ has zero homotopy dimension.

For $p \geq 4$ there are cells of dimension 2 and all the types of cells are listed in the next table

Dimension 0	$(\bullet, \ \ \bullet)$
Dimension 1	$(\bullet, \ \ \text{—})$ $(\text{—} \ \ \bullet)$
Dimension 2	$(\text{—}, \ \ \text{—})$

Table 4.1: Types of cells for $n = 2$ and $p \geq 4$

The algorithm starts matching cells of dimension 2 with cells of dimension 1, more precisely

$$(\text{—}, \ \ \bullet\!\!\rightarrow) \nearrow (\text{—}, \ \ \text{—})\,.$$

It is easy to see that any 2-cell of this form with an empty space at the right side of the second coordinate gets paired to a 2−cell by stretching the second component to the right. This procedure deletes all 2−cells, while the only 1−cells remaining have either the form $(\bullet, \ \ \text{—})$ or $(\text{—}, \ \ \bullet\times)$, where the blue cross mark indicates that the spot at the right of the vertex is taken (either by the horizontal cell or by the end of the board).

With all the 2-cells matched, then the process continues with the pairings from dimension 0 to dimension 1, starting with those of the form

$$(\bullet, \ \ \bullet\!\!\rightarrow) \nearrow (\bullet, \ \ \text{—})\,.$$

With the same argument as before, but for cells of dimension 1, it is deduced at all these 1−cells are deleted and the 0−cells that remain look like $(\bullet, \ \ \bullet\times)$. Finally, the last pairing has the form

$$(\bullet\!\!\rightarrow \ \ \bullet) \nearrow (\text{—}, \bullet)\,.$$

By the matching $1 \nearrow 2$, these 1−cells look like $(\relbar, \ \bullet\!\times)$, and the 0-cells have a blockage on the right side of their second coordinate too, so for each of these 1−cells, there is a 0−cell partner. Then all the 1−cells are paired and the critical 0−cells are those of the form

$$(\bullet\!\times, \ \bullet\!\times)\,.$$

It is easy to deduce, that there are exactly 2 of these cells, namely

where the figures illustrate $(p-1, p)$ and $(p, p-1)$. Thus, for $p \geq 2$, by the theorem 2.47, it follows $X(2, p, 1)$ is homotopic equivalent to S^0.
We next consider the case of $X(3; p, 1)$, for which $p \geq 3$ is needed in order to have non-empty spaces. When $p = 3$ there are only 6 cells of dimension zero so no pairing can be performed, and we get $X(3; 3, 1) = \vee_5 S^0$.
For $p = 4$ there are no cells of dimension greater than 1, whereas there are three types of cells of dimension 1, namely

$$(\relbar, \ \bullet, \ \bullet) \qquad (\bullet, \ \relbar, \ \bullet) \qquad (\bullet \ \bullet, \ \relbar)$$

and according to the algorithm, the first step is

$$(\bullet, \ \bullet, \ \mapsto) \nearrow (\bullet \ \bullet, \ \relbar)\,.$$

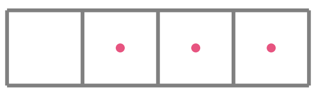
It is clear that every one of these 1−cells is paired with one of these 0−cells, so that all of these 1−cells are deleted, and since the 1-cells are paired with the 0-cells related to take the left end point of the horizontal line and this corresponds to all the 0-cells whose third coordinate is able to stretch to the right, then the remaining 0-cells of this step are those in $(\bullet, \ \bullet, \ \bullet\!\times)$ [**a**]. The next step is

$$(\bullet, \ \mapsto, \ \bullet) \nearrow (\bullet, \ \relbar, \ \bullet)\,.$$

Since all the grid is filled (recall $p = 4$), then considering the 0-cells related to take the left end point of the horizontal line corresponds with the 0-cells whose second coordinate is able to grow to the right and whose third coordinate is blocked to grow to the right, so all of these 1-cells are deleted and remain those 0-cells in $(\bullet, \ \bullet\!\times, \ \bullet\!\times)$.
The last step of the algorithm is

$$(\mapsto, \ \bullet, \ \bullet) \nearrow (\relbar, \ \bullet, \ \bullet)$$

and with the same reasoning as before, it is concluded that there is no cell of dimension one remaining and the critical cells of dimension 0 are those in $(\bullet\!\times, \ \bullet\!\times, \ \bullet\!\times)$, and it is easy to see that this corresponds to all the components together and next to the right border, i.e.,



Clearly, all permutations of this configuration need to be considered, so again we get that $X(3; 4, 1)$ is homotopy equivalent to $\vee_5 S^0$.
Before addressing the general case of $X(3; p, 1)$, let us consider the situation for

$p = 5$, as here we encounter for the first time cells of dimension two. In fact, all the types of cells for this case are shown in the table below

Dimension 0	$(\bullet,\ \bullet,\ \bullet)$
Dimension 1	$(\bullet,\ \bullet,\ \underline{\ \ })\ (\bullet,\ \underline{\ \ },\ \bullet)\ (\underline{\ \ },\ \bullet,\ \bullet)$
Dimension 2	$(\bullet,\ \underline{\ \ },\ \underline{\ \ })\ (\underline{\ \ },\ \bullet,\ \underline{\ \ })\ (\underline{\ \ },\ \underline{\ \ },\ \bullet)$

Table 4.2: Types of cells for n=3, p=5 and q=1

For these cells, the first pairings are

$$(\bullet,\ \underline{\ \ },\ \bullet\!\!\rightarrow) \nearrow (\bullet,\ \underline{\ \ },\ \underline{\ \ })$$

$$(\underline{\ \ },\ \bullet,\ \bullet\!\!\rightarrow) \nearrow (\underline{\ \ },\ \bullet,\ \underline{\ \ })\,.$$

Considering the 0-cells related to take the left end point of the third coordinate of the $2-$cell, then it is clear all $2-$cells are deleted and the remaining $1-$cells are modified as $(\bullet,\ \underline{\ \ },\ \bullet\!\times)$ and $(\underline{\ \ },\ \bullet,\ \bullet\!\times)$.
As can be seen in the next pairing

$$(\underline{\ \ },\ \bullet\!\!\rightarrow,\ \bullet) \nearrow (\underline{\ \ },\ \underline{\ \ },\ \bullet)$$

all these $2-$cells are deleted and the modified $1-$cells are $(\underline{\ \ },\ \bullet\!\times,\ \bullet\!\times)$. So all $2-$cells have been paired and the process continues with the pairing

$$(\bullet,\ \bullet,\ \bullet\!\!\rightarrow) \nearrow (\bullet\ \ \bullet,\ \underline{\ \ })\,.$$

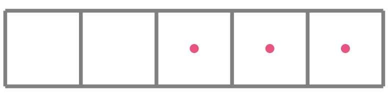
As in [a], these 1-cells are all deleted and the 0-cells are modified as $(\bullet,\ \bullet,\ \bullet\!\times)$. The next step is given by

$$(\bullet,\ \bullet\!\!\rightarrow,\ \bullet) \nearrow (\bullet,\ \underline{\ \ },\ \bullet\!\times)\,.$$

Since for both of these cells, their third coordinate is blocked like $\bullet\!\times$, taking the 0-cell related to the left end point of the vertical line gives rise to a pair, and the remaining 0-cells are those whose second coordinate is unable to grow to the right, i.e. those in $(\bullet,\ \bullet\!\times,\ \bullet\!\times)$. The final pairing is

$$(\bullet\!\!\rightarrow,\ \bullet,\ \bullet) \nearrow (\underline{\ \ },\ \bullet\!\times,\ \bullet\!\times)$$

and since a similar reasoning as before can be applied, it is deduced that all the 1-cells are deleted, and as in the case $p = 4$ the remaining cells are those in $(\bullet\!\times,\ \bullet\!\times,\ \bullet\!\times)$, which again correspond to all the permutations of the points that are all together, next to the right border, i.e. like,



So, once again $X(3; 5, 1)$ is homotopy equivalent to $\vee_5 S^0$.

Is it true that, for every $p \geq 4$ our gradient field leads to a homotopic equivalence of $X(3; p, 1)$ and $\vee_5 S^0$? We next answer this question on the positive. For $p \geq 6$, all the types of cells are shown on the table below

Dimension 0	$(\bullet,\ \bullet,\ \bullet)$
Dimension 1	$(\bullet,\ \bullet,\ \text{—})$ $(\bullet,\ \text{—},\ \bullet)$ $(\text{—},\ \bullet,\ \bullet)$
Dimension 2	$(\bullet,\ \text{—},\ \text{—})$ $(\text{—},\ \bullet,\ \text{—})$ $(\text{—},\ \text{—},\ \bullet)$
Dimension 3	$(\text{—},\ \text{—},\ \text{—})$

Table 4.3: Types of cells for n=3, $p \geq 6$ and q=1

Starting with the process, the first pairing is

$$(\text{—},\ \text{—},\ \mapsto) \nearrow (\text{—},\ \text{—},\ \text{—})\,.$$

All the 3−cells can be paired with a 2−cell related to take the left end point of the third coordinate, while all the paired 2−cells are those whose third coordinate is allowed to grow to the right [**b**] so the remaining 2−cells are $(\text{—},\ \text{—},\ \bullet\!\times)$.
The following pairings start to match 1−cells with 2−cells as

$$(\bullet,\ \text{—},\ \mapsto) \nearrow (\bullet,\ \text{—},\ \text{—})$$

$$(\text{—},\ \bullet,\ \mapsto) \nearrow (\text{—},\ \bullet,\ \text{—})\,.$$

With a reasoning as in [b] but this time for 2−cells and not for 3−cells, it is clear all the 2−cells are deleted and the modified 1−cells are $(\bullet,\ \text{—},\ \bullet\!\times)$ and $(\text{—},\ \bullet,\ \bullet\!\times)$. Following with the algorithm, the last pairing from dimension 2 to dimension 1 is

$$(\text{—},\ \mapsto,\ \bullet) \nearrow (\text{—},\ \text{—},\ \bullet)\,.$$

For both types of cells, the third coordinate is blocked on the right, and for a 1−cell whose second coordinate is able to grow to the right, its related 2−cell is an available 2−cell, and since for a 2−cell with its third coordinate blocked on the right, taking the 1−cell related to take the left end point of the second coordinate is one of these cells [**c**], then all the 2−cells are deleted and the remaining 1−cells are $(\text{—},\ \bullet\!\times,\ \bullet\!\times)$. Finally, the process considers pairings between dimensions zero and one, starting with

$$(\bullet,\ \bullet,\ \mapsto) \nearrow (\bullet\ \ \bullet,\ \text{—})\,.$$

As in [b] but for 1−cells, it is concluded all these 1cells are deleted and the remaining 1−cells are $(\bullet,\ \bullet,\ \bullet\!\times)$. The following pairing is given by

$$(\bullet,\ \mapsto,\ \bullet) \nearrow (\bullet,\ \text{—},\ \bullet\!\times)\,.$$

Using reasonings as those on [b] and [c], it follows all of these 1−cells are deleted and the remaining 0−cells are modified as $(\bullet,\ \bullet\!\times,\ \bullet\!\times)$. The last pairing is

$$(\mapsto,\ \bullet,\ \bullet) \nearrow (\text{—},\ \bullet\!\times,\ \bullet\!\times)$$

and again by [b] and [c] the 1−cells are all deleted and the critical 0−cells are those in $(\bullet\!\times,\ \bullet\!\times,\ \bullet\!\times)$. Then, is concluded that, indeed, $X(3;p,1)$ is homotopic equivalent to $\vee_5 S^0$.

From the above calculations, it can be expected that by applying the same algorithm, taking values $q = 1$ and $p > n \geq 4$, then an analogue conclusion can be drawn, but instead of $\vee_5 S^0$, these cases gives rise to an homotopy equivalence between $X(n,p,1)$ and $\vee_{n!-1} S^0$. To give the details of such an assertion, we first note that there are three different cases:

1. If $p \geq 2n$ then the process starts pairing cells from dimension n to dimension $n-1$.
2. If $p = 2n-1$ then the process starts pairing cells from dimension $n-1$ to dimension $n-2$.
3. If $p < 2n-1$ then the maximum dimension of the cells is at most $n-2$.

Then, for $p \geq 2n$ the process stars with

$$\left(\text{—}, \quad ..., \quad \text{—}, \quad \bullet\!\!\rightarrow\right) \nearrow \left(\text{—}, \quad ..., \quad \text{—}, \quad \text{—}\right).$$

Considering the $(n-1)-$cell related to take the left end point of the $n-$coordinate gives rise to a match, and then all the $n-$cells are deleted **[c]** while the remaining $(n-1)$-cells are like $\left(\text{—}, \quad ..., \quad \text{—}, \quad \bullet\!\times\right)$.

Next it, is time to match cells of dimension $n-2$ with cells of dimension $n-1$, for this the algorithm can start or continue, depending on the value of p. Indeed, it is important to note that, for $p = 2n-1$ the $(n-1)-$cells are of the form

$$\left(\text{—}, \quad ..., \quad \text{—}, \quad \bullet_i, \quad \text{—}, \quad ..., \quad \text{—}, \quad ...\right)$$

where the point can be in any of the n entries of the cell, i.e. $i = 1, ..., n$ and given the size of the grid being considered, the points are actually like

$$\left(\text{—}, \quad ..., \quad \text{—}, \quad \bullet\!\times_i, \quad \text{—}, \quad ..., \quad \text{—}, \quad ...\right).$$

In particular the description of the remaining cells of dimension $n-1$ in the previous instance are kept, which justifies our assertion that, for $p \geq 2n-1$, the process starts or continues as stated here. This situation applies in all further instances of our analysis of the gradient field. Analogously, the $(n-2)-$cells are of the form

$$\left(\text{—}, \quad ..., \quad \text{—}, \quad \bullet_i, \quad ..., \quad \text{—}, \quad \bullet_j, \quad ...\right)$$

for $i, j = 1, ..., n,\ i \neq j$. So, for $p \geq 2n-1$ the process starts/continue with the pairings from dimension $n-2$ to dimension $n-1$ of the form

$$\left(\text{—}, \quad ..., \quad \text{—}, \quad \bullet_i, \quad ..., \quad \bullet\!\!\rightarrow_n\right) \nearrow \left(\text{—}, \quad ..., \quad \text{—}, \quad \bullet_i, \quad ..., \quad \text{—}_n\right)$$

for $i > n$. If $p = 2n-1$, with an argument analogue to [c] it is concluded that all the $(n-1)-$-cells in this pairing are deleted, and the remaining $(n-2)-$cells are modified like $\left(\text{—}, \quad ..., \quad \text{—}, \quad \bullet_i, \quad ..., \quad \bullet\!\times_n\right)$. If $p > 2n-1$, using [c] too, it turns out again that all the $(n-1)-$cells are deleted and the modified $(n-2)-$cells are modified like $\left(\text{—}, \quad ..., \quad \text{—}, \quad \bullet_i, \quad ..., \quad \bullet\!\times_n\right)$.
As can be expected, each time a pairing is done, [c] can be used as the case may be, and this gives rise to the same deletions and modifications. An $(n-1)-$cell is unequivocally determined by the coordinate of its point coordinate, of which there are n possible positions, so given that $i > n$, the only type of $(n-1)-$cell left is the one that has point on its $n-$coordinate, so, the following pairing is

$$\left(\text{—}, \quad ..., \text{—} \quad \bullet\!\!\rightarrow_{n-1}, \bullet_n\right) \nearrow \left(\text{—}, \quad ..., \text{—} \quad \text{—}_{n-1}, \bullet_n\right).$$

If $p = 2n-1$, making use of [c], it is clear that all of these $(n-1)-$cells are deleted and the remaining $(n-2)-$cells are modified as $\big(\_\_,\ \ ...,\_\_,\ \ \bullet\!\times_{n-1}, \bullet_n\big)$, but since the size of the grid is exactly $p = 2n-1$, these cells can be seen like $\big(\_\_,\ \ ...,\_\_,\ \ \bullet\!\times_{n-1}, \bullet\!\times_n\big)$. On the other hand, if $p > 2n-1$, the modified $(n-1)-$cells of the match $(n-1) \nearrow n$ are paired here, and given that the $(n-2)-$cells which are paired with those $(n-1)-$cells are the cells $\big(\_\_,\ \ ...,\ \ \_\_,\ \ \bullet_{n-1},\ \ \bullet\!\times_n\big)$, then both of them are blocked on the right of the $n-$coordinate, and taking one of these $(n-1)-$cells, the corresponding $(n-2)-$cell related to take the left end point of the $(n-1)-$coordinate is a match for this cell, and the right end point of $(n-1)-$coordinate corresponds to the left end point of another similar cell but whose $(n-1)-$coordinate moves one step, so while the $(n-1)-$coordinate of a $(n-2)-$cell has an empty space at its right side, then this cells is paired **[c.2]**. So in this case, the modified $(n-2)-$cells are $\big(\_\_,\ \ ...,\ \ \_\_,\ \ \bullet\!\times_{n-1},\ \ \bullet\!\times_n\big)$. Then, for the pairing $(n-2) \nearrow (n-1)$ there are 2 different kind of modified cells, which are those on the table below.

Modified $(n-2)-$cells
$\big(\_\_,\ \ ...,\ \ \_\_,\ \ \bullet_i,\ \ ...,\ \ \bullet\!\times_n\big)$
$\big(\_\_,\ \ ...,\ \ \_\_,\ \ \bullet\!\times_{n-1},\ \ \bullet\!\times_n\big)$

Table 4.4: Modified $(n-2)-$cells remaining for the $(n-2) \nearrow (n-1)$ pairing.

Following with the procedure, the next pairing is for cells of dimension $n-3$ to cells of dimension $n-2$. In this case, for $p < 2n-2$ the process cannot start, meanwhile for $p \geq 2n-2$ the process stars (if $p = 2n-2$) or continue if $p > 2n-2$ with the pairing

$$\big(\_\_,\ \ ...,\ \ \bullet_i,\ \ ...,\ \ \bullet_j,\ \ ...\ \ \bullet\!\!\rightarrow_n\big) \nearrow \big(\_\_,\ \ ...,\ \ \bullet_i,\ \ ...,\ \ \bullet_j,\ \ ...\ \ \_\_{}_n\big)$$

with $i, j > n$ and $i \neq j$. For this pairing, all the $(n-2)-$cells have not been modified by any previous process because their last coordinate is an horizontal line. Making use of [c], it is clear all of the $(n-2)-$cells are deleted, meanwhile the $(n-3)-$cells are modified as $\big(\_\_,\ \ ...,\ \ \bullet_i,\ \ ...,\ \ \bullet_j,\ \ ...\ \ \bullet\!\times_n\big)$. In the next step, all the previous modified $(n-2)-$cells are paired, with only one exception, and given that all the $(n-2)-$cells which have an horizontal line on its last coordinate were paired already, then the pairings are specifically

$$\big(\_\_,\ \ ...,\ \ \bullet_i,\ \ ...,\ \ \bullet\!\!\rightarrow_{n-1},\ \ \bullet_n\big) \nearrow \big(\_\_,\ \ ...,\ \ \bullet_i,\ \ ...,\ \ \_\_{}_{n-1},\ \ \bullet_n\big)$$

for $i > n-1$. For $p = 2n-2$ all the $(n-2)-$cells occupied all the spaces available on the grid, so they have in particular the same form as the remaining cells of the last pairing. Then, with $p \geq 2n-2$, using the reasoning on [c.2], it can be seen that all of these $(n-2)-$cells are paired and the modified $(n-3)-$cells are like $\big(\_\_,\ \ ...,\ \ \bullet_i,\ \ ...,\ \ \bullet\!\times_{n-1},\ \ \bullet\!\times_n\big)$. Finally, the last pairing is

$$\big(\_\_,\ \ ...,\ \ \_\_,\ \ ...,\ \ \bullet\!\!\rightarrow_{n-2},\ \ \bullet_{n-1}\ \ \bullet_n\big) \nearrow \big(\_\_,\ \ ...,\ \ \_\_,\ \ ...,\ \ \_\_{}_{,n-2},\ \ \bullet_{n-1}\ \ \bullet_n\big).$$

Since for both cells on the pairing, their two final points are blocked on its right side, so using the reasoning used in [c.2], it follows that the $(n-2)-$cells are all deleted and the remaining $(n-3)-$cells are modified like $\big(\_\_,\ \ ...,\ \ \_\_,\ \ ...,\ \ \bullet\!\times_{n-2},\ \ \bullet\!\times_{n-1}\ \ \bullet\!\times_n\big)$. Summarizing, the three different kind of modified $(n-3)-$cells are indicated in the following table, where $i < n-1$.

Modified $(n-3)-$cells
$\big(\text{—}, \;\; \ldots, \;\; \bullet_i, \;\; \ldots, \;\; \text{—}_{n-1}, \;\; \bullet\!\!\times_n\big)$
$\big(\text{—}, \;\; \ldots, \;\; \bullet_i, \;\; \ldots, \;\; \bullet\!\!\times_{n-1}, \;\; \bullet\!\!\times_n\big)$
$\big(\text{—}, \;\; \ldots, \;\; \text{—}, \;\; \ldots, \;\; \bullet\!\!\times_{n-2}, \;\; \bullet\!\!\times_{n-1} \;\; \bullet\!\!\times_n\big)$

Table 4.5: Modified $(n-3)-cells$ remaining for the $(n-3) \nearrow (n-2)$ pairing.

Let $0 \leq l < n$, then for $p \geq 2n-l-1$, the pairing from cells of dimension $n-l-1$ to cells of dimension $n-l$ starts or continues (as the case may be) with

$$\big(\text{—}, \;\; \ldots, \;\; \bullet_{i_1}, \;\; \ldots, \;\; \bullet_{i_{l-1}}, \;\; \cdots \;\; \bullet\!\!\rightarrow_n\big) \nearrow \big(\text{—}, \;\; \ldots, \;\; \bullet_{i_1}, \;\; \ldots, \;\; \bullet_{i_{l-1}}, \;\; \cdots \;\; \text{—}_n\big).$$

As in [c], all the $(n-l)-$cells whose last coordinate is an horizontal line are deleted, these cells are in fact those that have not been used by any step before, meanwhile the remaining $(n-l-1)-$cells are modified as $\big(\text{—}, \;\; \ldots, \;\; \bullet_{i_1}, \;\; \ldots, \;\; \bullet_{i_{l-1}}, \;\; \cdots \;\; \bullet\!\!\times_n\big)$. With all the $(n-l)-$cells with an horizontal line in its last coordinate paired, then the following pairing is given precisely by

$$\big(\text{—}, \;\; \ldots, \;\; \bullet_{i_1}, \;\; \ldots, \;\; \bullet_{i_{l-2}}, \;\; \cdots \;\; \bullet\!\!\rightarrow_{n-1}, \;\; \bullet_n\big) \nearrow$$
$$\big(\text{—}, \;\; \ldots, \;\; \bullet_{i_1}, \;\; \ldots, \;\; \bullet_{i_{l-2}}, \;\; \cdots \;\; \text{—}_{n-1}, \;\; \bullet_n\big).$$

It is important to note that, since the components of a cell are limited to be a horizontal line or a point and its position within the components of the cell determines it unequivocally, then for any $(n-l)-$cell with a point on its last coordinate, this cell has been modified by the matching from $(n-l)$ to $(n-l+1)$, and then these kind of cells have a block on the right side of the $n-$coordinate, and since for the last pairing, the $(n-l-1)-$cells have a block on the right of its last coordinate too, then applying [c] it is clear the $(n-l)-$cells are deleted, and the remaining $(n-l-1)-$cells are modified as $\big(\text{—}, \;\; \ldots, \;\; \bullet_{i_1}, \;\; \ldots, \;\; \bullet_{i_{l-2}}, \;\; \cdots \;\; \bullet\!\!\times_{n-1}, \;\; \bullet\!\!\times_n\big)$. Following with the process, and given that all the $(n-l)-$cells whose $(n-1)-$coordinate is an horizontal line were paired already, then the next pairings are specifically

$$\big(\text{—}, \;\; \ldots, \;\; \bullet_{i_{l-3}}, \;\; \cdots \;\; \bullet\!\!\rightarrow_{n-2}, \;\; \bullet_{n-1}, \;\; \bullet_n\big) \nearrow \big(\text{—}, \;\; \ldots, \;\; \bullet_{i_{l-3}}, \;\; \cdots \;\; \text{—}_{n-2}, \;\; \bullet_{n-1}, \bullet_n\big),$$

for $1 \leq i < n-2$. Again, since these $(n-l-1)-$cells have a point on their last two coordinates, then by the pairings made before both of the coordinates must be blocked on their right side, and given that the $(n-l)-$cells have a point on their last two coordinates too, these cells were modified by a matching from $(n-l)$ to $(n-l+1)$. This is clear because the $(n-l+1)-$cells with a point on its last coordinate can be paired just with this kind of $(n-l)-$cells, and these in turn were matched before with $(n-l+1)-$cells with an horizontal line on its last coordinate first and after with those cells with an horizontal line on the $(n-1)-$coordinate and a point on the last, so both cells have blockages on the right side of their last two coordinates [c.4]. Using a reasoning as [c], it follows these $(n-l)-$cells are deleted and the $(n-l-1)-$cells are modified as $\big(\text{—}, \;\; \ldots, \;\; \bullet_{i_{l-3}}, \;\; \cdots \;\; \bullet\!\!\times_{n-2}, \;\; \bullet\!\!\times_{n-1}, \;\; \bullet\!\!\times_n\big)$ for $1 \leq i < n-3$.
As can be expected for the last two pairings, as the procedure progresses the matches on the $jth-$step, $j < l$, of $(n-l-1) \nearrow (n-l)$ can be identified by how many of their final coordinates have a point as a component (at most $l+1$ for the $(n-l-1)-$cells since they must have $n-l-1$ horizontal lines), since all of those $(n-l)-$cells with a horizontal line at some $kth-$coordinate, $n-j < k \leq n$, must have been paired before.

On the case of the jth matching of the $(n-l-1) \nearrow (n-l)$, the $(n-l-1)-$cells must have a point on their last j coordinates, while the $(n-l)-$cells must have a point on their last $(j-1)$ coordinates and a horizontal line on the $j-$coordinate. i.e.,

$$\left(—,\ \ldots,\ \bullet_{i_{l+1-j}},\ \cdots\ \bullet\!\!\rightarrow_j,\ \ldots,\ \bullet_n\right) \nearrow \left(—,\ \ldots,\ \bullet_{i_{l+1-j}},\ \cdots\ —_j,\ \ldots, \bullet_n\right).$$

Now, using a reasoning similar to that in [c.4], it is clear the last $j-1$ coordinates of the cells are blocked on their right side, and by [c], it follows the $(n-l)-$cells are all deleted, while the remaining $(n-l-1)-$cells are modified like $\left(—,\ \ldots,\ \bullet_{i_{l+1-j}},\ \cdots\ \bullet\!\times_j,\ \ldots,\ \bullet\!\times_n\right)$.
Then, for $(n-l-1) \nearrow (n-l)$, $l > n$, the $(n-l)-$cells are all deleted, while the modified remaining cells are those in the table below.

Modified $(n-l-1)-$**cells**
$\left(—,\ \ldots,\ \bullet_{i_1},\ \ldots,\ \bullet_{i_{l-1}},\ \cdots\ —\ \bullet\!\times_n\right)$
$\left(—,\ \ldots,\ \bullet_{i_1},\ \ldots,\ \bullet_{i_{l-2}},\ \cdots\ \bullet\!\times_{n-1},\ \bullet\!\times_n\right)$
$\left(—,\ \ldots,\ \bullet_{i_1},\ \ldots,\ \bullet_{i_{l-2}},\ \cdots\ —\ \bullet\!\times_{n-1},\ \bullet\!\times_n\right)$
$\left(—,\ \ldots,\ \bullet_{i_{l-3}},\ \cdots\ \bullet\!\times_{n-2},\ \bullet\!\times_{n-1},\ \bullet\!\times_n\right)$
$\cdots$
$\left(—,\ \ldots,\ —,\ \ldots,\ —_{n-l},\ \ldots,\ \bullet\!\times_n\right)$
$\left(—,\ \ldots,\ —,\ \bullet\!\times_{n-l},\ \ldots,\ \bullet\!\times_n\right)$

Table 4.6: Modified $(n-l-1)-$*cells* remaining for the $(n-l-1) \nearrow (n-l)$ pairing.

Since all of this was done for $0 \leq l < n$ and $p \geq 2n-l-1$, then for $l = n-1$ and $p \geq n$, all the $1-$cells are deleted and the critical $0-$cells are exactly those in

$$\left(\bullet\!\times_1,\ \bullet\!\times_2,\ \ldots\ \bullet\!\times_n\right).$$

Then, for this kind of cells, the only possible option is to have all the points along the right border of the grid next to each other. Taking into account all their $n!$ possible permutations, then it follows that $X(n;p,1)$ is homotopy equivalent to $\vee_{n!-1}S^0$.

As we will see in the rest of the thesis, for $n = 2$ and p, q arbitrary, the pairing process turns out to be reasonably short and there are just a few types of cells, while in the case $n = 3$ and p, q arbitrary, the total number of pairings increases significantly, but it is still feasible to handle the analysis of the whole algorithm. However for $n \geq 4$ the total number of pairings increases a lot compared to the two previous cases, mainly due to the fact that the type of cells grows considerably because of the combinatorial nature of their components. Nonetheless, final part of the thesis we describe the critical cells reported by a computational implementation of the gradient field for $X(4;p,q)$ when p and q are reasonably small. In particular this will suggest that, for general values of n, p and q, the gradient field is surprisingly close to detecting the homotopy dimension of $X(n;p,q)$.

4.2 n=2

Dimension	Cell types for n=2
Dimension 0	$(\bullet, \ \bullet)$
Dimension 1	$(\bullet, \ \vert)$, $(\bullet, \ —)$ $(\vert, \ \bullet)$, $(—, \ \bullet)$
Dimension 2	$(\square, \ \bullet)$, $(\bullet, \ \square)$ $(—, \ \vert)$, $(\vert, \ —)$ $(\vert, \ \vert)$, $(—, \ —)$
Dimension 3	$(\square, \ \vert)$, $(\square, \ —)$ $(—, \ \square)$, $(\vert, \ \square)$
Dimension 4	$(\square, \ \square)$

Table 4.7: Cell types of $X(2; p, q)$, for $p, q > 1$

In the last subsection, the vector gradient field of $X(2; p, 1)$ has been described using the algorithm of subsection 2.5, as particularized to the case of $X(n; p, q)$ in Chapter 3. Here $n = 2$ is taken too, but now p and q are strictly greater than 1. In this case, the components of a cell can be points or horizonal lines but, unlike the previous, also squares or vertical lines. All the types of cells are listed on table 4.7.

The procedure begins with the process $\mathcal{P}(3, 4, i)$, for $i = 0, 1$, so the first match is

$$(\square, \ \underline{\uparrow}) \nearrow (\square, \ \square) \, .$$

All the 4−cells have an available 3−face, namely the 3−cell related to take the bottom edge of the second component (figure 18).

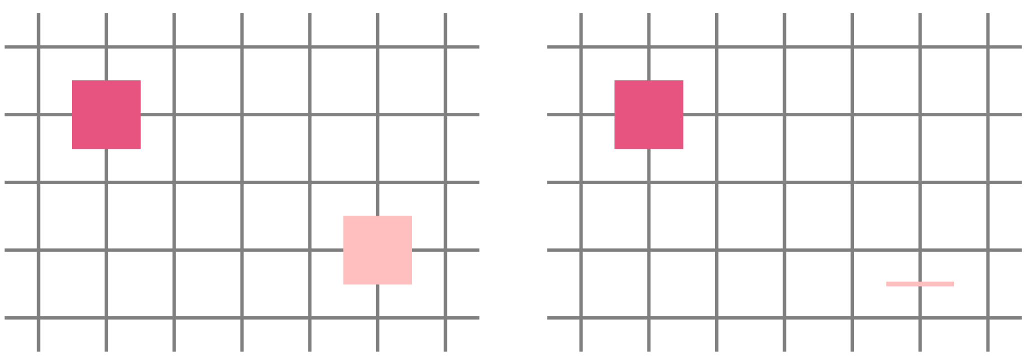


Figure 13: Example of a 4−cell (on the left) and its 3−face related to take the bottom side of the second component (on the right)

Besides, for a 3−cell for which its horizontal line is unblocked above, then growing up this component, the cell can be matched with a unique 4−cell. Moreover, if its horizontal line is unblocked below too, it is not possible for this same cell to be paired with a 4−cell by growing downwards its second component because the vertices are added from the largest to the smallest, so first the cell is matched by growing upwards its horizontal line and when the vertex $v-1$ is added, this cell is no longer available. So all 4−cells are deleted and the modified 3−cells are all characterized by having their second component blocked above in some way, so all of them can be denoted by $(\square, \ \underline{\asymp})$.

Remark 2. *For the rest of the calculations, when a process $\mathcal{P}$ is written, the letter i will always be $i = 0, 1$, since for a given $\mathcal{P}(d, c)$, the calculation is performed for stretching up and down or to the sides, as the case may be, simultaneously.*

Since all the 4−cells were deleted, the next pairing step is from cells of dimension two to cells of dimensions three. The pairings of $\mathcal{P}(2, 4, i)$ are

$$\left(|, \ \underline{\uparrow}\right) \nearrow \left(| \ \square\right)$$

$$(\_, \ \underline{\uparrow}) \nearrow (\_ \ \square)$$

$$(\square, \ \dot{\updownarrow}) \nearrow \left(\square, \ |\right).$$

Again, all the 3−cells have the 2−cell related to take the bottom side of its second component available, and all the 2−cells for which its second component is unblocked above can be paired with a 3−cell by growing up this component. Moreover if it is unblocked below too, since the vertices are added from greater to lower, there is no confusion about how this cell match. Then, the 3−cells in $\mathcal{P}(2, 4, i)$ are deleted, while the modified 2−cells are those in the table below.

Dimension 2	**Dimension 3**
$\left(|, \ \underline{\asymp}\right)$ $(\_, \ \underline{\asymp})$ $(\square, \ \overset{\times}{\bullet})$	$(\square, \ \underline{\asymp})$

Table 4.8: Modified cells of $\mathcal{P}(2, 4, i)$

The following step is $\mathcal{P}(2, 3, i)$, and the corresponding pair is

$$(\square, \ \bullet\!\!\rightarrow) \nearrow (\square, \ \_).$$

From table 4.8, it is known the remaining cells have their second coordinate blocked above, so their horizontal line can be blocked like

$$\underline{\times\checkmark} \qquad\qquad \underline{\checkmark\ \times} \qquad\qquad \underline{\times\,\times}$$

As noted at the beginning of this section, given that the algorithm adds vertices from greater to lower, a component tends to grow to the right first, if it can do so. For an available 2−cell for which its second component does not have a block on its right side, then by growing to the right this component, it can be matched with a 3−cell; and since the component is moving to the right, it is clear the 3−cell must have its second component blocked like the first or third case. Furthermore, all 3−cells with their second component blocked above in the left or in both sides have an available 2−cell, namely, that related to take the left end point of the line. So all of these cells get paired. It is important to note that there are cases for 2−cells like the one presented below

$$\square_{\bullet\times}$$

These kind of cells just can be paired by growing their point to the left, but the resulting 3−cells have already been eliminated. So they remain unpaired. For the second case, there are two different possibilities

$$\overset{\checkmark\ \times}{\text{---}}{}_{\times} \qquad \text{or} \qquad \overset{\checkmark\ \times}{\text{---}}{}_{\checkmark}$$

If the line is blocked on its right side too, the 2−cell related to take its left final point has not been paired yet, so a 3−cell with its second component like this can be matched. But, if the line has the blocks like in the second option, the 3−cell does not have a pair, since the only 2−cell it can match with is the one relative to taking its right endpoint, but this cell has already been eliminated. On the other hand, it is clear every every 2−cell with its second component blocked on the right and unblocked on its left side get a pair. Then, the unmated 2−cells are those with their point blocked like

$$\overset{\times}{\times\bullet\times} \qquad \text{or} \qquad \overset{\times\times}{\ \bullet\times}$$

To simplify notation, both types of locks are denoted by

$$\chi\overset{\times}{\bullet}\times$$

And taking this into account, the modified cells are shown in the following table

Dimension 2	**Dimension 3**
$(\vert,\ \overset{\times}{=})$ $(\text{—},\ \overset{\times}{=})$ $(\square,\ \chi\overset{\times}{\bullet}\times)$	$\left(\square,\ \overset{\checkmark\ \times}{\text{---}}{}_{\checkmark}\right)$

Table 4.9: Modified cells of $\mathcal{P}(2, 3, i)$

The last step for delete all 3−cells is $\mathcal{P}(2, 2, i)$, i.e.

$$(\perp,\ \text{—}) \nearrow (\square,\ \text{—})\,.$$

From table 4.9, it is clear that all the 3−cells are paired, since the blockages such as $\overset{\checkmark\ \times}{\text{---}}{}_{\checkmark}$ are all on $\overset{\times}{=}$ and given that the 2−cells have no restrictions on their first component, it is enough that it is unlocked from above to be able to be paired. The modified cells from $\mathcal{P}(2, 2, i)$ are listed on table 4.10

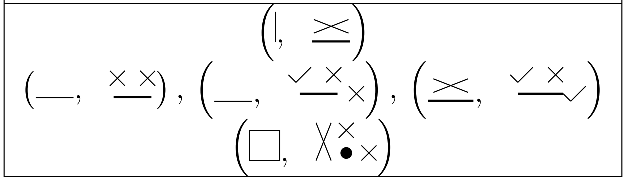
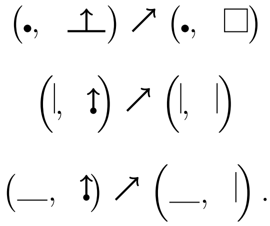
Dimension 2


Table 4.10: Modified cells of $\mathcal{P}(2, 2, i)$

Remark 3. *From here, the notation* $\underline{\times\ \times}$ *used in the tables, denotes that both end-points of the line are blocked, but also denotes the block* $\underline{\times\ }$*.*

With all 3−cells removed, the next step is $\mathcal{P}(1, 4, i)$.



For these pairings, the same reasoning in $\mathcal{P}(2, 4, i)$ or $\mathcal{P}(3, 4, i)$ can be done here, so the remaining cells are listed right in the table below.

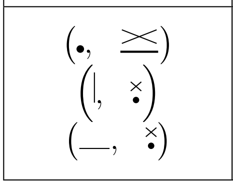
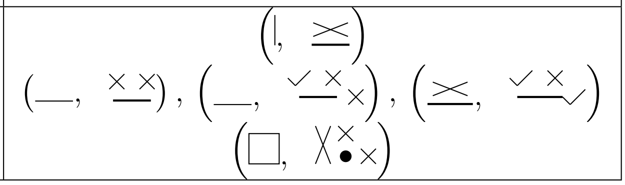
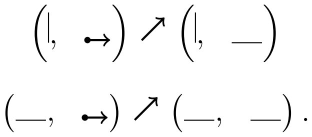
Dimension 1	Dimension 2
	

Table 4.11: Modified cells of $\mathcal{P}(1, 4, i)$

The following step is $\mathcal{P}(1, 3, i)$ and the pairings are



For these pairings, a reasoning analogous to that in $\mathcal{P}(2, 3, i)$ can be done, so the remaining cells are straightforward obtained.

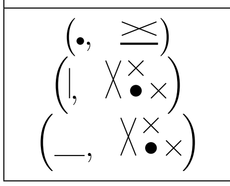
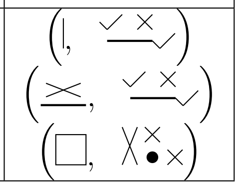
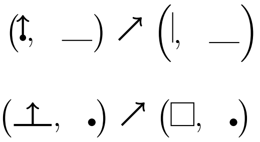
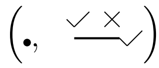
Dimension 1	Dimension 2
	

Table 4.12: Modified cells of $\mathcal{P}(1, 3, i$

For $P(1, 2, i)$ the pairings are

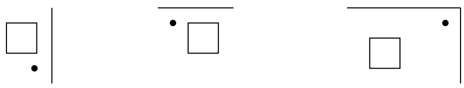


For the first pair, it is easy to see that the only kind of available 2−cell is of the form

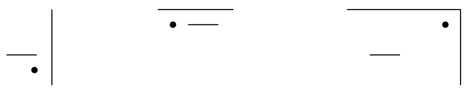
And since the only way to have 1−cells like  is

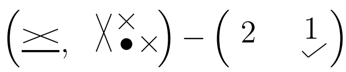
then, it is clear all the 2−cells can be matched with a 1−cell like the one above, for which its point is not blocked from going up, so all of these cells are deleted.

For the second pair, it is straightforward to verify that there are only three different 2−cell shapes, namely



while for the 1−cells the possibilities are



So, it is clear that the 2−cells of the options one and three are paired with those 1−cells related to take the bottom edge of the square, while the second with those related to take its top edge. Also, it is clear that the remaining 1-cells are those whose horizontal line is not blocked above, except for those for which the first component is just to the right of the second and the line is able to go down. These cells are described with the notation . The modified cells are listed on table 4.13.

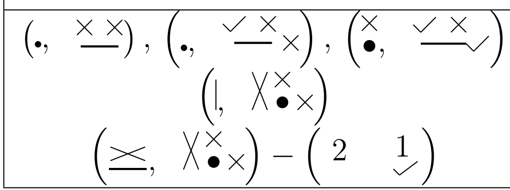
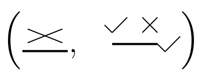
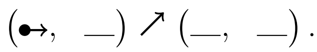
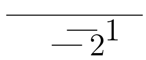
Dimension 1	**Dimension 2**
	

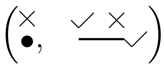
Table 4.13: Modified cells of $\mathcal{P}(1,2,i)$

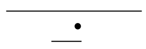
The last step for these dimensions is $\mathcal{P}(1,1,i)$, i.e.



The only available shape of 2−cells is



while for 1−cells like  there is just one possibility too, namely



So, as long as the point can grow to the right, all the 2−cells are paired. Furthermore, since the right side of the line must be free, it follows that all 1−cells are also deleted. The modified 1−cells are described on the following table.

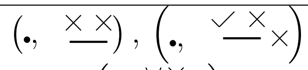
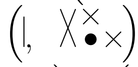
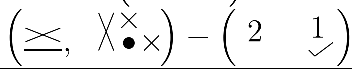
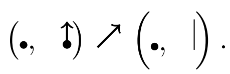
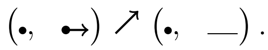
Dimension 1




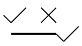
Table 4.14: Modified cells of $\mathcal{P}(1,1,i)$

Continuing with the process, the next step is $\mathcal{P}(0,4,i)$, with pairing rule



As in $\mathcal{P}(3,4,i)$ or $\mathcal{P}(2,4,i)$, it follows all of these 1−cells are deleted and the 0−cells are modified like . The next step is $\mathcal{P}(0,3,i)$, i.e.



Since all the 1−cells with its line blocked like  were all deleted already and using the reasoning in $\mathcal{P}(2,3,i)$ it follows all of these 1−cells are all deleted. The modified cells are those on table 4.15.

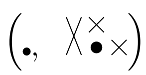
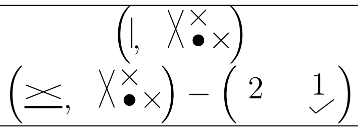
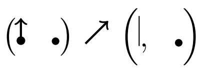
Dimension 0	**Dimension 1**
	

Table 4.15: Modified cells of $\mathcal{P}(0,3,i)$

For $\mathcal{P}(0,2,i)$ the pairing rule is



For 1−cells there are just two possible shapes

Meanwhile, for 0−cells, the possible shapes are

For the first 1−cell, the 0−cell related to take the top point of the line is available, so a match can be done. For the second 2−cell, the 0−cell related to take the bottom point is available, so it can be matched too.

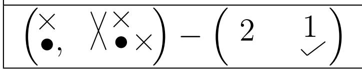
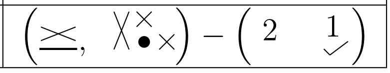
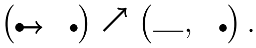
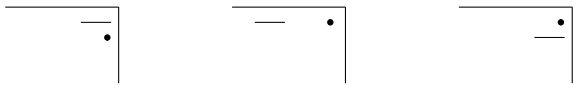
Dimension 0	Dimension 1
	

Table 4.16: Modified cells of $\mathcal{P}(0, 2, i)$

The last pairing is given by $\mathcal{P}(0, 1, i)$, i.e.



An straightforward verification shows there are three different shapes for 1−cells, namely



Also for 0−cells there are two possibilities

For the second options of 1−cells, the 0−cell related to take the left end point of the line is available, so it is matched. For the third options, the 0−cells related to take the right final point of the line is available too, so it is matched. In the first case, neither of the final points of the line are related to an available 0−cell, so this is not deleted. Summarizing, we have shown that $X(2; p, q)$ has only two critical cells, one in dimension zero, and the other in dimension one, with explicit forms described on Table 4.17.

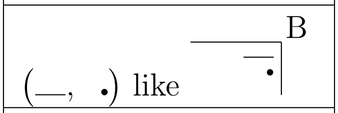
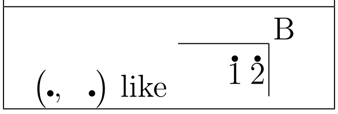
Dimension 1

Dimension 0


Table 4.17: Critical cells of $X(2, p, q)$

So, for any $p, q \geq 2$, $X(2, p, q)$ is homotopy equivalent to S^1. In particular we recover the homotopy dimension of $X(2; p, q)$.

Using the algorithm described in [1] for $X(2, 2, 2)$, the authors obtained four critical cells of dimensions one and another four in dimension zero, so for this case, the algorithm proposed in this work is quite powerful. Even more, using our gradient field the total number of critical cells of dimension 1 and 0 for $X(2; p, q)$ is optimal and the resulting homotopy type corresponds with that of 2 non-collisioning particles on $\mathbb{R}^2$.

4.3 n=3

Dimension	Cells Types for n=3
Dimension 0	$(\bullet,\ \bullet,\ \bullet)$
Dimension 1	$(\bullet,\ \bullet,\ \vert),\ (\bullet,\ \vert,\ \bullet),\ (\vert,\ \bullet,\ \bullet),\ (\bullet,\ \bullet,\ \_),\ (\bullet,\ \_,\ \bullet),\ (\_,\ \bullet,\ \bullet)$
Dimension 2	$(\bullet,\ \bullet,\ \square),\ (\bullet,\ \square,\ \bullet),\ (\square,\ \bullet,\ \bullet),\ (\bullet,\ \vert,\ \vert),\ (\vert,\ \bullet,\ \vert),\ (\vert,\ \vert,\ \bullet)$ $(\bullet,\ \_,\ \_),\ (\_,\ \bullet,\ \_),\ (\_,\ \_,\ \bullet),\ (\bullet,\ \vert,\ \_),\ (\vert,\ \bullet,\ \_),\ (\vert,\ \_,\ \bullet)$ $(\bullet,\ \_,\ \vert),\ (\_,\ \bullet,\ \vert),\ (\_,\ \vert,\ \bullet)$
Dimension 3	$(\_,\ \_,\ \_),\ (\vert,\ \vert,\ \vert),\ (\vert,\ \vert,\ \_),\ (\vert,\ \_,\ \vert),\ (\_,\ \vert,\ \vert)$ $(\_,\ \_,\ \vert),\ (\_,\ \vert,\ \_),\ (\vert,\ \_,\ \_),\ (\square,\ \vert,\ \bullet),\ (\square,\ \bullet,\ \vert),\ (\bullet,\ \square,\ \vert)$ $(\square,\ \_,\ \bullet),\ (\square,\ \bullet,\ \_),\ (\bullet,\ \square,\ \_),\ (\_,\ \square,\ \bullet),\ (\_,\ \bullet\ \square,),\ (\bullet,\ \_,\ \square)$ $(\vert,\ \square,\ \bullet),\ (\vert,\ \bullet\ \square,),\ (\bullet,\ \vert,\ \square)$
Dimension 4	$(\square,\ \square,\ \bullet),\ (\square,\ \bullet\ \square,),\ (\bullet,\ \square,\ \square),\ (\square,\ \vert,\ \vert),\ (\vert,\ \square,\ \vert),\ (\vert,\ \vert,\ \square)$ $(\square,\ \vert,\ \_),\ (\vert,\ \square,\ \_),\ (\vert,\ \_,\ \square),\ (\square,\ \_,\ \_),\ (\_,\ \square,\ \_),\ (\_,\ \_,\ \square)$ $(\square,\ \_,\ \vert),\ (\_,\ \square,\ \vert),\ (\_,\ \vert,\ \square)$
Dimension 5	$(\square,\ \square,\ \vert),\ (\square,\ \vert\ \square),\ (\vert,\ \square,\ \square),\ (\square,\ \square,\ \_),\ (\square,\ \_\ \square),\ (\_,\ \square,\ \square)$
Dimension 6	$(\square,\ \square,\ \square)$

Table 4.18: Table with dimensions and cell types

In the subsection before, with a grid of height 1, was relatively straightforward to obtain the homotopy type of $X(n;p,1)$, and as mentioned before, it was easy because the total number of cell types does not have a very large variety despite being large in number, also the algorithm used makes the procedure simple. Now, for $n = 3$, the cells can be at most dimension six and can be classified by the shape of its components too, which could be a square $\square$, an horizontal line $\_$, a vertical line $\vert$ or a point $\bullet$. Table 4.18 shows all the types of cells that there are for $n = 3$.
It can be seen from table 4.18, the algorithm starts by verifying the possible pairs from the 5-cells $(\square,\ \square,\ \vert)$ to the 6-cells $(\square,\ \square,\ \square)$, and as usual it is denoted by

$$(\square,\ \square,\ \underline{\uparrow}) \nearrow (\square,\ \square,\ \square)\,.$$

Every $\left(\square,\ \square,\ \square\right)$ cell has a codimension-1 face which is available, so every sixth dimensional cell is paired. Moreover, for a cell in $\left(\square,\ \square,\ \_\right)$ for which its third component, the horizontal line, does not have a block above for any of the squares components or by the boundary of the grid, then it has a unique available 1-coface, and given the preference to grow upwards, then all of the 6−cells are paired, while the remaining 5−cells in this process are those for which its horizontal line is blocked for growing upwards [**A**]. Graphically the blockages can be like:

$$\underline{\times\ \ }\qquad\qquad \underline{\times\ \times}\qquad\qquad \underline{\ \ \times}\qquad\qquad \underline{\ B\ }$$

Here the position of × tells if the line is blocked by another component (left, right or both) or by the boundary (both sides blocked or the letter "B" above). All the kind of blocks are included in the notation $\underline{\asymp}$, so the remaining 5-cells are $\left(\square,\ \square,\ \underline{\asymp}\right)$ and all the others fifth dimensional cells.

Having finished all the 6-cells, then the process of pairing 4-cells with 5-cells starts with the matches

$$\left(|,\ \square,\ \underline{\uparrow}\right) \nearrow \left(|,\ \square,\ \square\right)$$

$$\left(\_,\ \square,\ \underline{\uparrow}\right) \nearrow \left(\_,\ \square,\ \square\right)$$

$$\left(\square,\ |,\ \underline{\uparrow}\right) \nearrow \left(\square,\ |\ \square\right)$$

$$\left(\square,\ \_,\ \underline{\uparrow}\right) \nearrow \left(\square,\ \_\ \square\right)$$

$$\left(\square,\ \square,\ \underset{\bullet}{\uparrow}\right) \nearrow \left(\square,\ \square,\ |\right).$$

Using the same argument as in [A], it is clear the modified cells after these pairings are those listed in table 4.19.

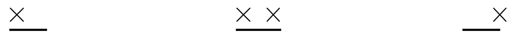
Dimension four	**Dimension five**
$\left(|,\ \square,\ \underline{\asymp}\right)$	
$\left(\_,\ \square,\underline{\asymp}\right)$	
$\left(\square,\ |,\ \underline{\asymp}\right)$	$\left(\square,\ \square,\ \underline{\asymp}\right)$
$\left(\square,\ \_,\ \underline{\asymp}\right)$	
$\left(\square,\ \square,\ \overset{\times}{\bullet}\right)$	

Table 4.19: Modified cells of $\mathcal{P}(4,6,i)$

For the remaining 5−dimensional cells, the pairing that follows is

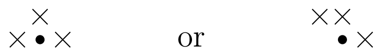
$$\left(\square,\ \square,\ \mapsto\right) \nearrow \left(\square,\ \square,\ \_\right).$$

For both dimensions, the cells have their third coordinate blocked above, for the 5−cells the blocks are like:

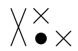


Given the preference for moving right first, as long as the point in the 4-cell is free to move to the right (i.e. ×•✓) then it can be paired with a 5−cell whose horizontal line has a block like the first or the second option (by the left or in both sides), while for —× there are two options

For the first option a match can be done by growing the point to the left, but not for the second because its 5-coface was already deleted. Thus, the remaining cells of dimension five are (□, □, ✓×—✓). Meanwhile for dimension four, the remaining cells are those for which the point satisfies [**B**]:



and again, in order to simplify the notation, it is agreed that both cases are contemplated in the notation:

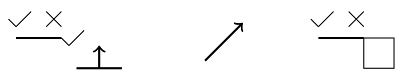


Then the remaining 4-cells are (□, □, ╳×•×).

Given that there are still cells of dimension five, the next step is $\mathcal{P}(4,4,i)$

(□, ↥, —) ↗ (□, □, —) .

It is clear all the 5−cells have an available face of dimension four. For the remaining fourth dimensional cells, as in [A], a 4-cell is paired as long as its second coordinate does not have a block above but also the condition ✓×—✓ must be hold. This means that the second coordinate can grow up and, when it does, the third coordinate is still unblocked on the right, but since the second is increasing, it may happen that



in which case the cell is not paired, this kind of unpaired cell is denoted with (□, ↥—;, ✓×—✓) [**C**]. Then, after this pairing, the modified cells are:

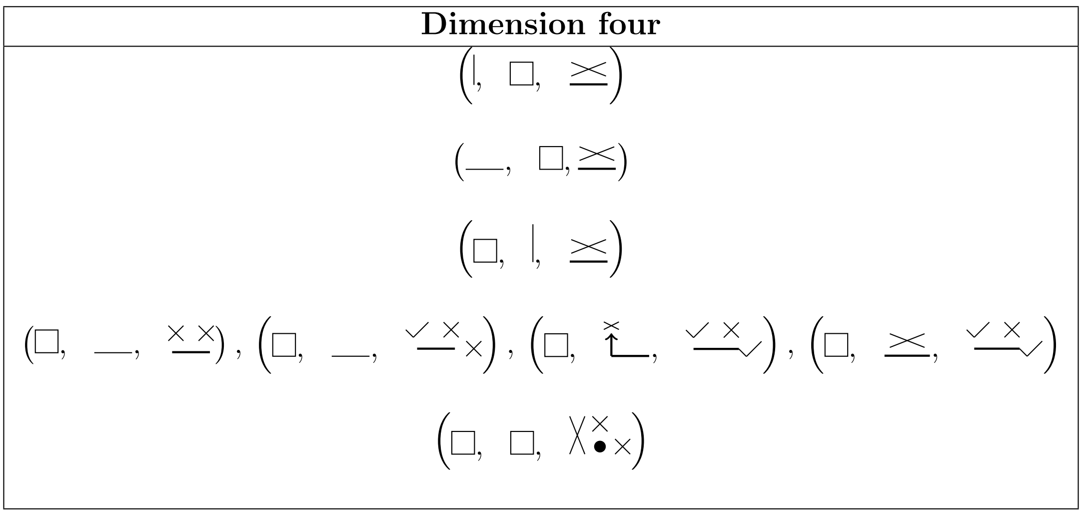


Table 4.20: Modified cells of $\mathcal{P}$(4,4,i)

Remark 4. *In this and the next tables, the notation* $\underline{\times\ \times}$ *is used for denote this specific block and also* $\underline{\times\ }$.

Next, the first pairings from dimension 3 to dimension 4 are the following ones:

$$(\bullet,\ \square,\ \underline{\uparrow})\nearrow(\bullet,\ \square,\ \square)$$

$$(|,\ |,\ \underline{\uparrow})\nearrow(|,\ |,\ \square)$$

$$(|,\ \_,\ \underline{\uparrow})\nearrow(|,\ \_,\ \square)$$

$$(|,\ \square,\ \updownarrow)\nearrow(|,\ \square,\ |)$$

$$(\_,\ |,\ \underline{\uparrow})\nearrow(\_,\ |,\ \square)$$

$$(\_,\ \_,\ \underline{\uparrow})\nearrow(\_,\ \_,\ \square)$$

$$(\_,\ \square,\ \updownarrow)\nearrow(\_,\ \square,\ |)$$

$$(\square,\ \bullet,\ \underline{\uparrow})\nearrow(\square,\ \bullet,\ \square)$$

$$(\square,\ |,\ \updownarrow)\nearrow(\square,\ |,\ |)$$

$$(\square,\ \_,\ \updownarrow)\nearrow(\square,\ \_,\ |).$$

Again by [A] and table 4.20, it is easy to verify that the modified cells are those in table 4.21.

Dimension three	**Dimension four**
$(\bullet,\ \square,\ \underline{\asymp})$	
$(|,\ |,\ \underline{\asymp})$	
$(|,\ \_,\ \underline{\asymp})$	$(|,\ \square,\ \underline{\asymp})$
$(|,\ \square,\ \overset{\times}{\bullet})$	$(\_,\ \square, \underline{\asymp})$
$(\_,\ |,\ \underline{\asymp})$	$(\square,\ |,\ \underline{\asymp})$
$(\_,\ \_,\ \underline{\asymp})$	$(\square,\ \_,\ \underline{\times\times})$, $(\square,\ \_,\ \underline{\checkmark\times}\times)$, $(\square,\ \overset{\times}{\llcorner},\ \underline{\checkmark\times}\checkmark)$, $(\square,\ \underline{\asymp},\ \underline{\checkmark\times}\checkmark)$
$(\_,\ \square,\ \overset{\times}{\bullet})$	$(\square,\ \square,\ \rangle\!\langle\overset{\times}{\bullet}\times)$
$(\square,\ \bullet,\ \underline{\asymp})$	
$(\square,\ |,\ \overset{\times}{\bullet})$	
$(\square,\ \_,\ \overset{\times}{\bullet})$	

Table 4.21: Modified cells of $\mathcal{P}(3,6,i)$

The next step is given by the third coordinate going left or right and according to table 4.21, the pairings are:

$$\left(|,\ \square,\ \bullet\!\!\rightarrow\right) \nearrow \left(|,\ \square,\ \_\right)$$

$$\left(\_,\ \square,\ \bullet\!\!\rightarrow\right) \nearrow \left(\_,\ \square,\ \_\right)$$

$$\left(\square,\ |,\ \bullet\!\!\rightarrow\right) \nearrow \left(\square,\ |,\ \_\right)$$

$$\left(\square,\ \_,\ \bullet\!\!\rightarrow\right) \nearrow \left(\square,\ \_,\ \_\right).$$

For all the pairings, the argument explained in [B] can be applied, but for the last, since some cells have the restriction $\overset{\checkmark\ \times}{\underline{\quad}}\checkmark$ they cannot be paired because the point in 3-cells first went to the right and got paired with the 4-cells whose third coordinate satisfy $\overset{\times\ \times}{\underline{\quad}}$. Moreover those with $\overset{\checkmark\ \times}{\underline{\quad}}\times$ are paired too since all of them have an available 3-face whose third coordinate fulfills the blocks and can grow to the left. The modified cells of this process are listed in table 4.22.

Dimension three	**Dimension four**
$\left(\bullet,\ \square,\ \underline{\asymp}\right)$	
$\left(|,\ |,\ \underline{\asymp}\right)$	
$\left(|,\ \_,\ \underline{\asymp}\right)$	$\left(|,\ \square,\ \overset{\checkmark\ \times}{\underline{\quad}}\checkmark\right)$
$\left(|,\ \square,\ \chi\overset{\times}{\bullet}\times\right)$	$\left(\_,\ \square,\ \overset{\checkmark\ \times}{\underline{\quad}}\checkmark\right)$
$\left(\_,\ |,\ \underline{\asymp}\right)$	$\left(\square,\ |,\ \overset{\checkmark\ \times}{\underline{\quad}}\checkmark\right)$
$\left(\_,\ \_,\ \underline{\asymp}\right)$	$\left(\square,\ \overset{\times}{\uparrow}\!\underline{\quad},\ \overset{\checkmark\ \times}{\underline{\quad}}\checkmark\right), \left(\square,\ \underline{\asymp},\ \overset{\checkmark\ \times}{\underline{\quad}}\checkmark\right)$
$\left(\_,\ \square,\ \chi\overset{\times}{\bullet}\times\right)$	$\left(\square,\ \square,\ \chi\overset{\times}{\bullet}\times\right)$
$\left(\square,\ \bullet,\ \underline{\asymp}\right)$	
$\left(\square,\ |,\ \chi\overset{\times}{\bullet}\times\right)$	
$\left(\square,\ \_,\ \chi\overset{\times}{\bullet}\times\right)$	

Table 4.22: Modified cells of $\mathcal{P}(3,5,i)$

The next step leads to:

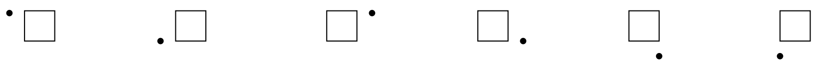
$$\left(|,\ \underline{\uparrow},\ \_\right) \nearrow \left(|,\ \square,\ \_\right)$$

$$(\_\!\_,\ \underline{\uparrow},\ \_\!\_) \nearrow (\_\!\_,\ \square,\ \_\!\_)$$

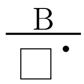
$$(\square,\ \updownarrow,\ \_\!\_) \nearrow \left(\square,\ |,\ \_\!\_\right)$$

$$(\square,\ \underline{\uparrow},\ \bullet) \nearrow (\square,\ \square,\ \bullet)\,.$$

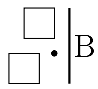
According to table 4.22, it is easy to see that the first three fourth dimensional kind of cells have available faces, so all of them are paired. By [A] and [C], the remaining 3−cells are those in table 4.23. The last pairing must be examined carefully, as it is necessary ensure that the blocks in the third coordinate are met and since what is moving is the second coordinate, it is convenient to take a look at how the square can be blocking the point:



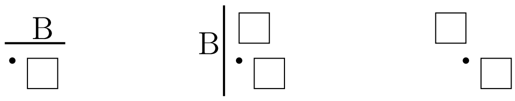
For the last two, as long as the other blocks holds, they can be paired with a three dimensional cell with a similar configuration and such that the third coordinate is not block for going up. The fourth option cannot happen and for the third it is only possible have the boundary or the square of the first coordinate blocking the point and the upper right side of the second coordinate, i.e.



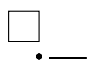
or



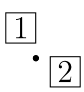
in both cases the cells have an available 3−face. The second option only could happen if the first coordinate is just above the point, in which case it has an available face for being paired. Meanwhile for the first if the upper blockage is due to the boundary, it is paired with a three dimensional cell but this time the second coordinate must go down, but if it is due to the square of the first coordinate then there are three possible options:



in the first two cases there exists an available 3-face taking the upper side of the square and the 3-cell with this same configuration, which can be paired by increasing its second coordinate downwards. However for the first case, the only face that can be taken is that corresponding to the upper side of the square of the second coordinate too, i.e.



but this type of cells has already been paired by going up in the second option. So cells like:



cannot be paired [**D**]. It is important to remember that a cell of dimension three is paired by going down when the second coordinate is at the right side of third and it is blocked above and completely free below, this kind of configuration is denoted by $\begin{pmatrix} 3 & \overset{\times}{\underset{\checkmark}{2}} \end{pmatrix}$ [**E**].

Keeping in mind these modifications to the remaining cells, the algorithm continues by doing pairings fixing the first coordinate of the second component with the rules

$$\left(\square,\ \mapsto,\ \bullet\right) \nearrow \left(\square,\ \square,\ \bullet\right)$$

$$\left(\square,\ \bullet\!\!\rightarrow,\ \_\right) \nearrow \left(\square,\ \_,\ \_\right).$$

As for the first pairing, the remaining cells are of the specific type in table 4.22. It is clear that taking the left side of the second square, the resulting three dimensional cell is available, so these fourth dimensional cells are deleted while the 3-cells like:

$$\begin{matrix} \boxed{1} & \\ & \bullet\!\vdash\!2 \end{matrix}$$

are also deleted, this type of cells are denoted by $\begin{pmatrix} 1 & & \checkmark \\ 3 & 2 & \checkmark \end{pmatrix}$.

For the last pairing, one part of the cells according to table 4.2 are like those described in [C]. Then the 3-cells whose second coordinate also fulfills that can grow to the right and are all paired, remainly only those with a block on their right side, like $\overset{\times}{\updownarrow}\!\bullet\times$ [**G**]. Finally for:

$$\left(\square,\ \overset{\times}{\bullet},\ \overset{\checkmark\ \times}{\text{---}}\!\checkmark\right) \nearrow \left(\square,\ \overset{\times}{\underline{\ \ }},\ \overset{\checkmark\ \times}{\text{---}}\!\checkmark\right).$$

Since both comply with $\overset{\checkmark\ \times}{\text{---}}\!\checkmark$ then by [B], the modified cells of dimension four and three are listed on table 4.23.

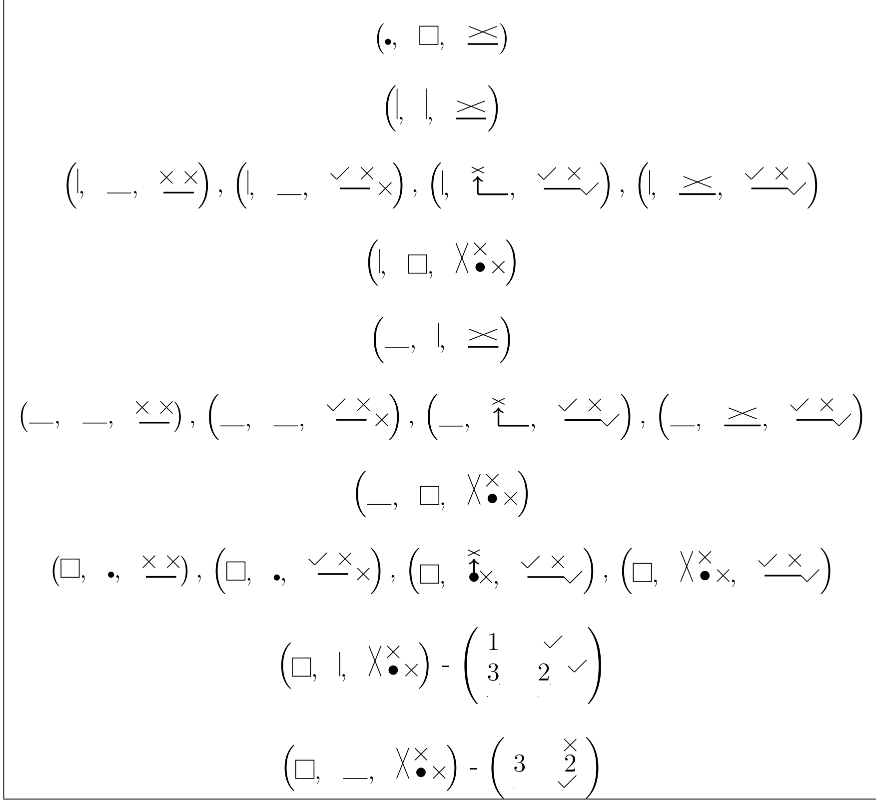
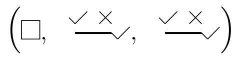
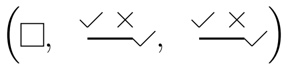
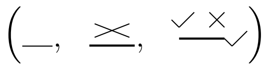
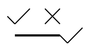
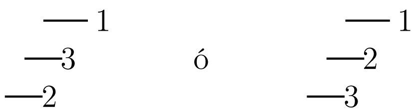
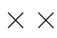
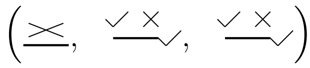
Dimension three	Dimension four
	

Table 4.23: Modified cells of $\mathcal{P}(3,3,i)$

To remove all the 4-cells, the last pairing is:

Evidently every cell in  has a free face in  and since there are just two different forms to have  in both second and third coordinate **[G.0]**, namely:



then the unpaired cells are those whose second coordinate is blocked like  or  or those whose first coordinate is blocked above, i.e. the cells in . So the 3-cells remaining from this process are in table 4.24

Dimension three

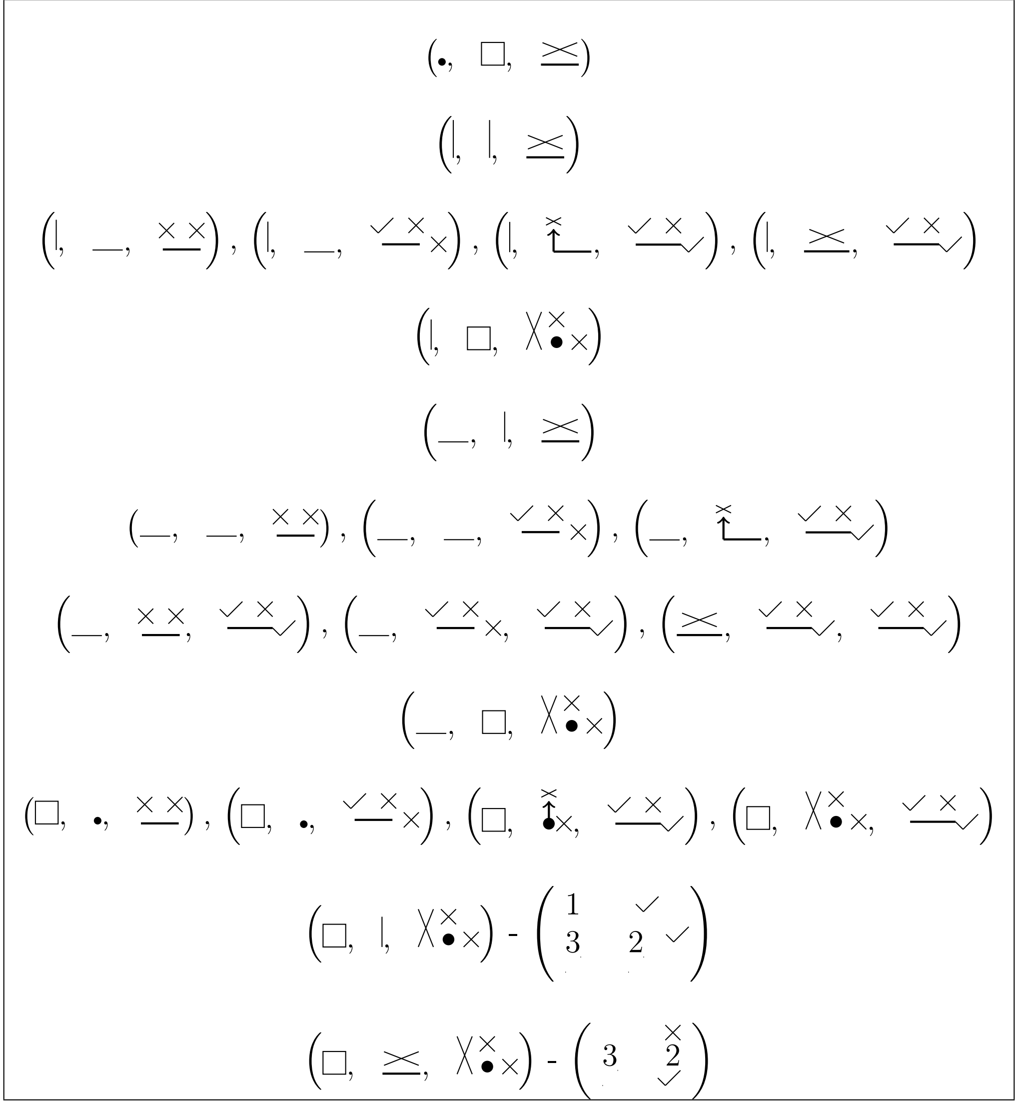
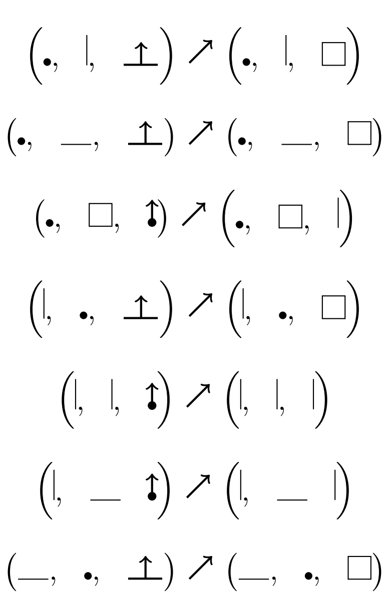
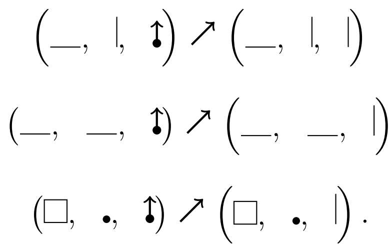


Table 4.24: Modified cells of $\mathcal{P}(3,2,i)$

Now it is time to start with the pairings from dimension two to dimension three:





As before, by [A] all these 3-cells are deleted. So the modified cells are in the table 4.25.

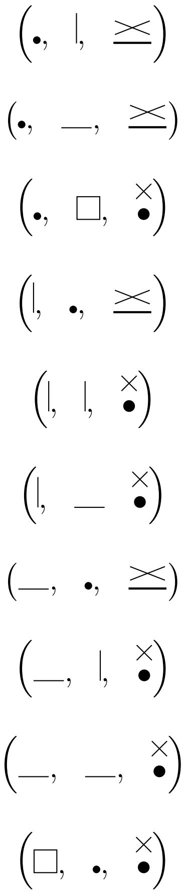
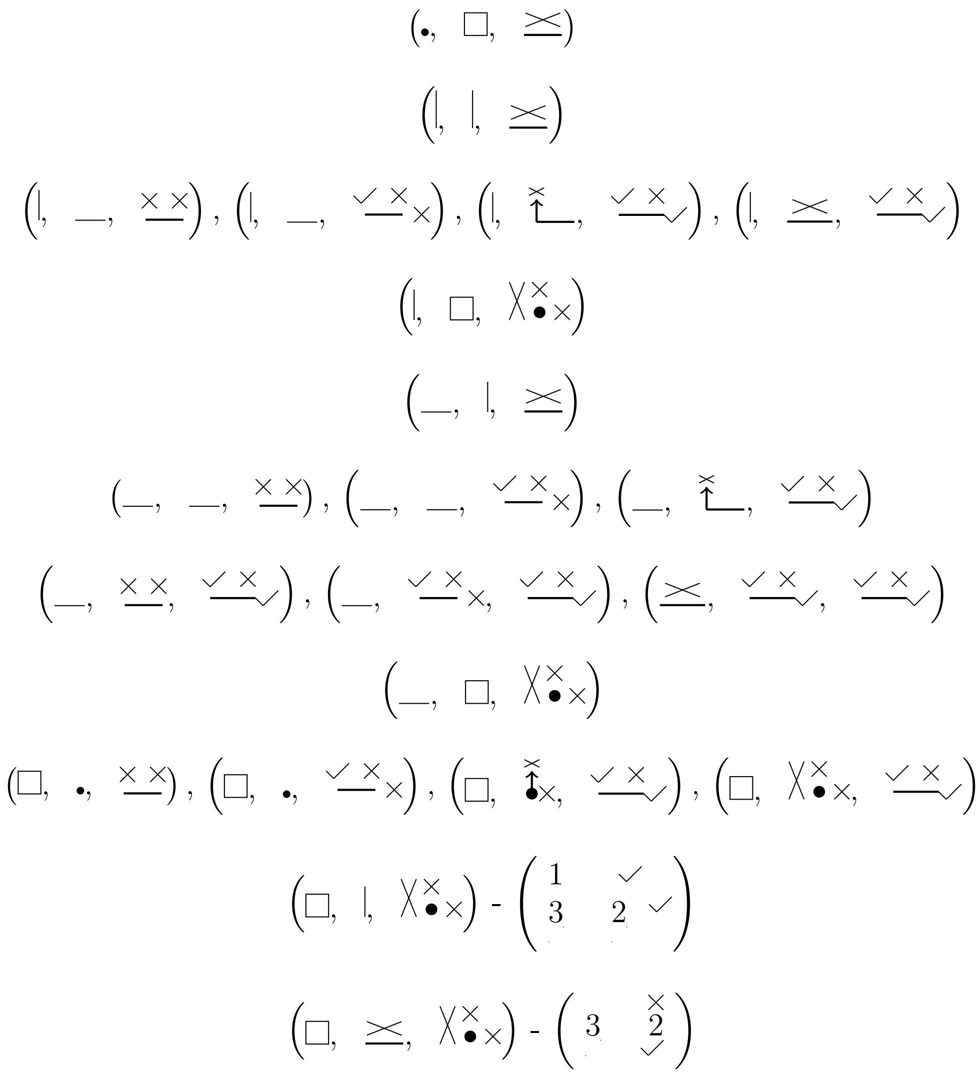
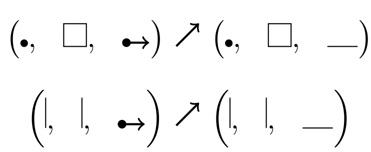
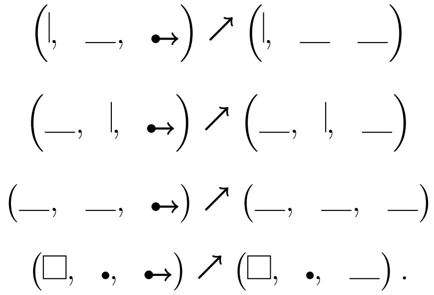
Dimension two	**Dimension three**
	

Table 4.25: Modified cells of $\mathcal{P}(2,6,i)$

The next step is given by fixing the third component, but now fixing its first coordinate, namely





Using [B] it is easy to see that the modified cells are those in table below (4.26).

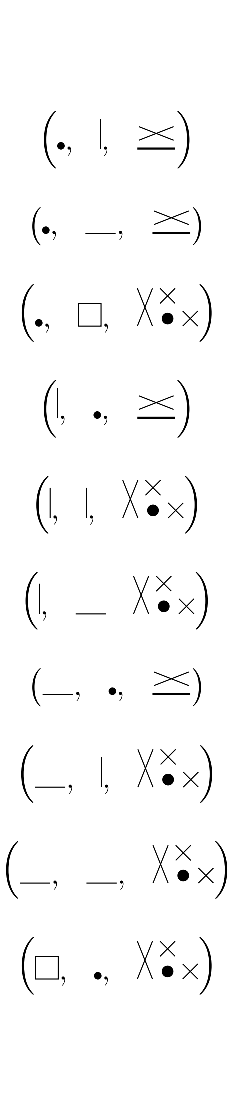
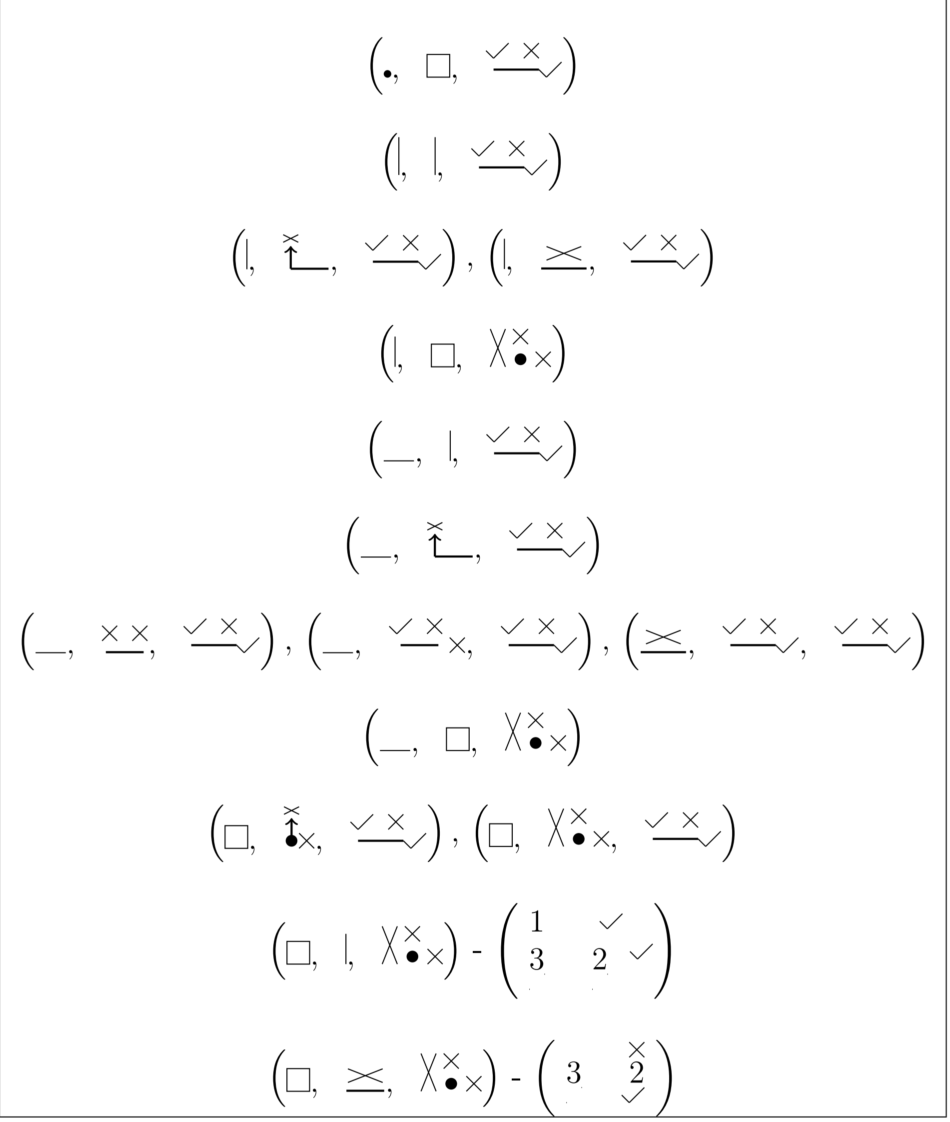
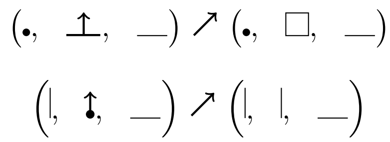
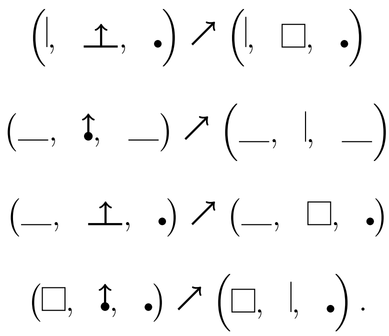
Dimension two	**Dimension three**
	

Table 4.26: Modified cells of $\mathcal{P}(2, 5, i)$

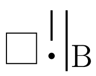
Now, fixing the second component, given that there are already multiple restrictions for the leftover cells, the pairings need to be examined carefully.





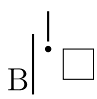
According to table 4.26, the pairings one, two and four are easy to do using the same idea as in [B]. For the fifth match, with the same reasoning as [D] and [E] the unpaired cells are easily deduced. For the third, an idea similar to [D] can be done but now taking into account the shape of the first coordinate. Namely, again the second coordinate can block the point as:

Now, while the other blocks holds, for the last two just taking the bottom side of the square and a 2-cell with a similar configuration and with its second coordinate unblocked from above then a pair is obtained. Unlike [D] in this case the fourth option is possible:

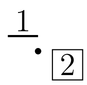


and again taking the bottom side of the square and a 2-face similar to it whose second coordinate is free to grow up, then it can be paired. For the third one, there are two possible options:

For the second, the 2-cell related to take the bottom side of the square is a pair of it, but for the first the only face this type of cell could have is the one due to the top side of the square but this has been paired already in the fourth option, so this remains unpaired. Finally as in [D], for the first and second options cells like:

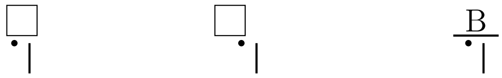


remain too **[D.1]** Taking into account [D] and noting that throughout the explanation what really influences is the bottom side of the square in the first coordinate, then the remaining 3-cells are those like:

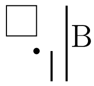


For the last pairing a similar procedure to that done in [D] is useful, so considering the second coordinate is moving:

For the first, while all the other blocks hold, it is clear the cell has a free 2-face such that it has the same configuration as the 3-cell when the lower point of the line is taken and it is able to go up, and the same for options two an four, this due to the preference to move upwards. The third one also gets a pair by a three dimensional cell going up because the block above the point can only be done for a square or the boundary and in both cases the upper side of the line needs to be covered too. In the last option it is clear that the pairing needs to be done going downwards, if is possible, since it is necessary to have $\bullet \times$. Then the upper block of the point is done by the boundary or by the square, i.e.



The former is paired by a 2-face with a similar configuration and taking the upper side of the line, and the same for the last. But the second case no longer has a pair because it was paired before by growing upwards in the fourth option. Now, taking into consideration that there are cells that were deleted before as can be seen in table 4.26 , then the unpaired cells are precisely like:



Having finished these pairings, the remaining cells are listed in table 4.27.

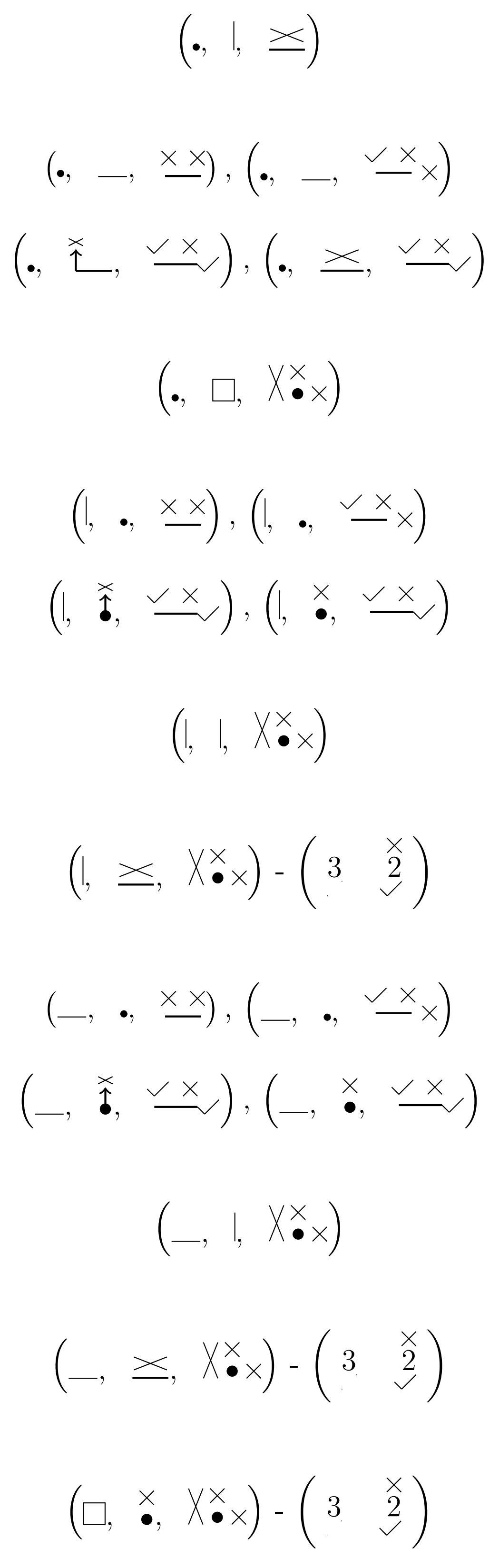
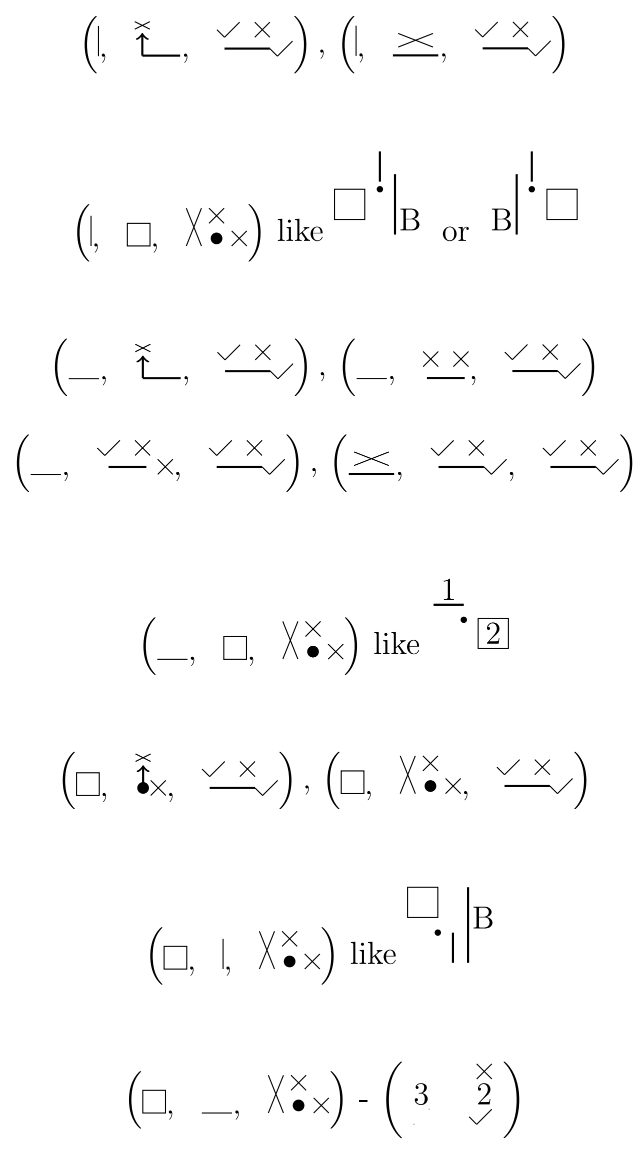
Dimension two	**Dimension three**
	

Table 4.27: Modified cells of $\mathcal{P}(2, 4, i)$

The following pairings are:

$$(|,\ \bullet\!\!\rightarrow,\ \_)\nearrow(|,\ \_,\ \_)$$

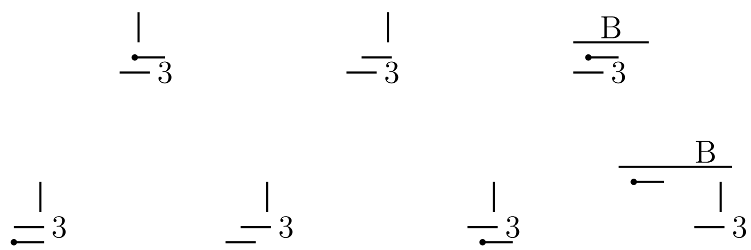
$$(|,\ \mapsto,\ \bullet)\nearrow(|,\ \square,\ \bullet)$$

$$(\_,\ \bullet\!\!\rightarrow,\ \_)\nearrow(\_,\ \_,\ \_)$$

$$(\_,\ \mapsto,\ \bullet)\nearrow(\_,\ \square,\ \bullet)$$

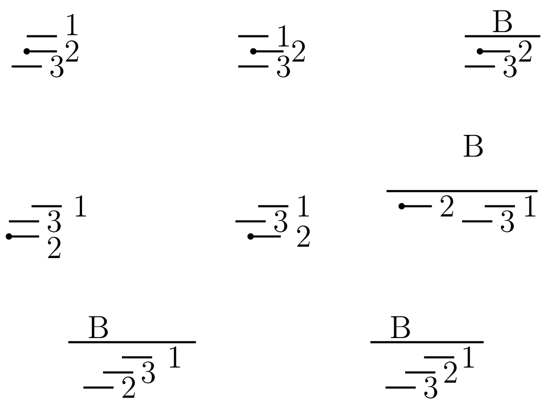
$$(\square,\ \bullet\!\!\rightarrow,\ \bullet)\nearrow(\square,\ \_,\ \bullet)\,.$$

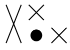
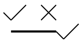
In the first pairing, by [F] the cells (|, ⤒_, ✓×_✓) are all deleted, meanwhile for (|, ×_, ✓×_✓) the possible ways for this to hold are:

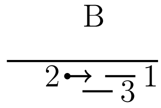


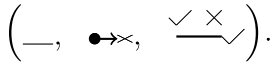
For those configurations with a point at the start point of the second coordinate it is clear that they have a free face and are paired, but for those without a point it is not possible to get a match. In the second case, taking any of the end points of the second coordinate, the 2-face related to it is no longer available, meanwhile in the fifth case, the 2-face related to take the right end point were matched before, precisely in the option number four. Furthermore, it is important to see that both cases are the only possible to have (|, ✓×_✓, ✓×_✓) with the exception that for the second case it is also possible that the second coordinate is blocked at the right end point by the boundary, such cells are denoted by (|, ✓×_B, ✓×_✓) **[G.1]**.

For the third pairing, using [F] it is easy to see the cells (_, ⤒_, ✓×_✓) are all deleted and for the other kind of cells the reasoning is very similar to that in the first pairing, only with slight changes. Explicitly, the 3-cells are:

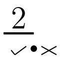


Those cells with a point in its second coordinate are clearly matched with a 2-face, and the 2-cells not paired are those with , with an exception, given that the condition  is necessary, seeing the option number six, it is easy to note that there is an specific kind of cells which is not matched [C.0]:

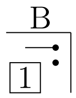


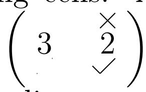
As in [C], these cells are denoted by .

The last pairing has different types of unpaired cells. First:

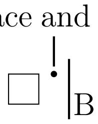
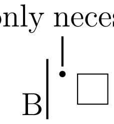
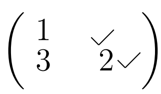
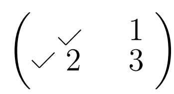
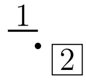
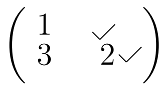


is not paired because taking neither of the ends points in the second coordinate and the 2-face related to it, it is not available. It is important to note that the point must be free at its left side, if there was a blockage there; then the cell is paired by taking the 2-cell related to the right end point of the second coordinate [**I**]. Specifically the cell:



is matched with the 2-cell related to take the right end point of the second coordinate, so this kind of cell needs to be deleted of the remaining cells. Another kind of unpaired cells are those due to the fact that the cells  were removed. As can be seen, the available 3-cells with the second coordinate at the right of the second coordinate are precisely:

For those with a point at the left end of the second coordinate, there is a partner which is the 2-cells related to take that point, but for the last the left end point does not comply with being locked down, and the right point does not block the left of the third coordinate. Then these cells cannot be paired. The last type of cell that does not have partner are those deduced by a reasoning as [B].

For the pairings two and four, it is clear that all the third dimensional cells have an available 2-face and it is only necessary to remove those 2-dimensional cells which were paired, so  and  are denoted by  and  respectively. Moreover the cells  are denoted by  too and there should be no confusion since, despite of being the same notation, the entries of the cells are not the same. Cells that were modified for these pairings are listed in table 4.28.

The next step in the process is given by:

$$\left(\updownarrow_{\bullet},\ \_,\ \_\right) \nearrow \left(|,\ \_,\ \_\right)$$

$$\left(\underline{\uparrow},\ \bullet,\ \_\right) \nearrow \left(\square,\ \bullet,\ \_\right)$$

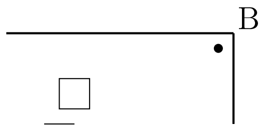
$$\left(\underline{\uparrow},\ |,\ \bullet\right) \nearrow \left(\square,\ |,\ \bullet\right)$$

$$\left(\underline{\uparrow},\ \_,\ \bullet\right) \nearrow \left(\square,\ \_,\ \bullet\right).$$

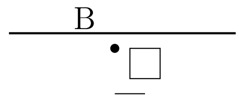
In the first pairing, it is clear that all the 3-cells have a pair, and as in [G.0] and [G.1], it is easy to see which cells are left, but in this case the cells with $\overset{\checkmark\ \times}{\rule{1em}{0.4pt}}\times$ or $\overset{\checkmark\ \times}{\rule{1em}{0.4pt}}\checkmark$ in the second and third coordinate are denoted by $\left(\overset{\times}{\bullet},\ \overset{\checkmark\ \times}{\rule{1em}{0.4pt}}B,\ \overset{\checkmark\ \times}{\rule{1em}{0.4pt}}\checkmark\right)$
For the second, doing:

$$\left(\_,\ \overset{\times}{\updownarrow_{\bullet}}\times,\ \overset{\checkmark\ \times}{\rule{1em}{0.4pt}}\checkmark\right) \nearrow \left(\square,\ \overset{\times}{\updownarrow_{\bullet}}\times,\ \overset{\checkmark\ \times}{\rule{1em}{0.4pt}}\checkmark\right),$$

all the 3-cells are deleted and remain the 2-cells with its first coordinate as $\underline{\asymp}$. For cells in $\left(\square,\ \overset{\times}{\bullet}\times,\ \overset{\checkmark\ \times}{\rule{1em}{0.4pt}}\checkmark\right)$ there exists only two different ways to meet the restrictions, namely [**H**]:



or



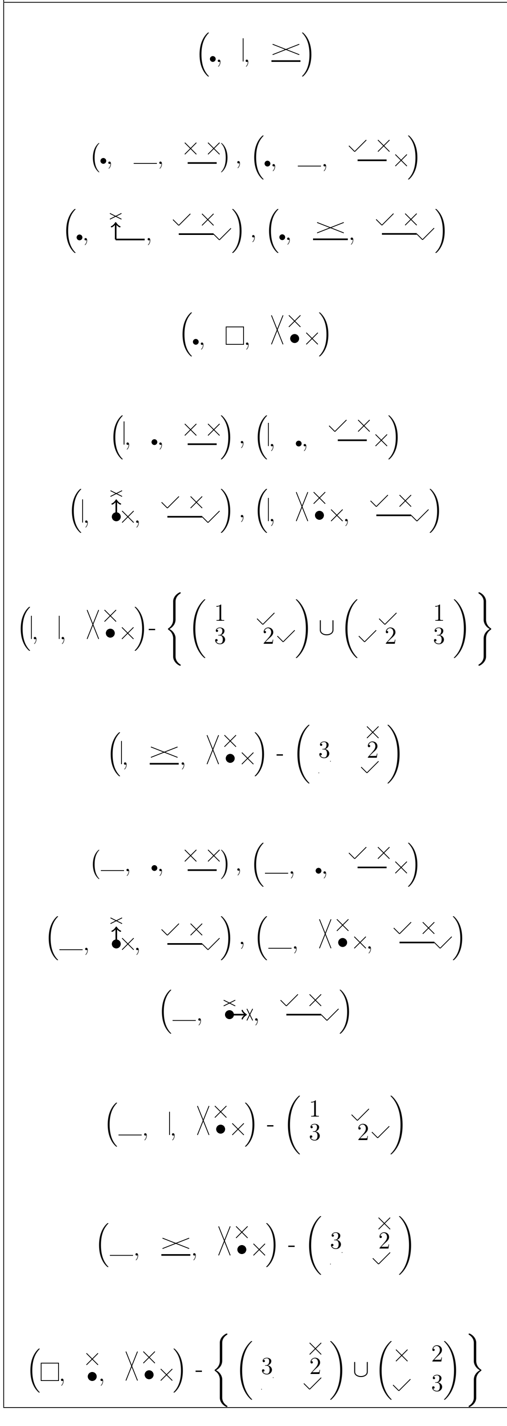
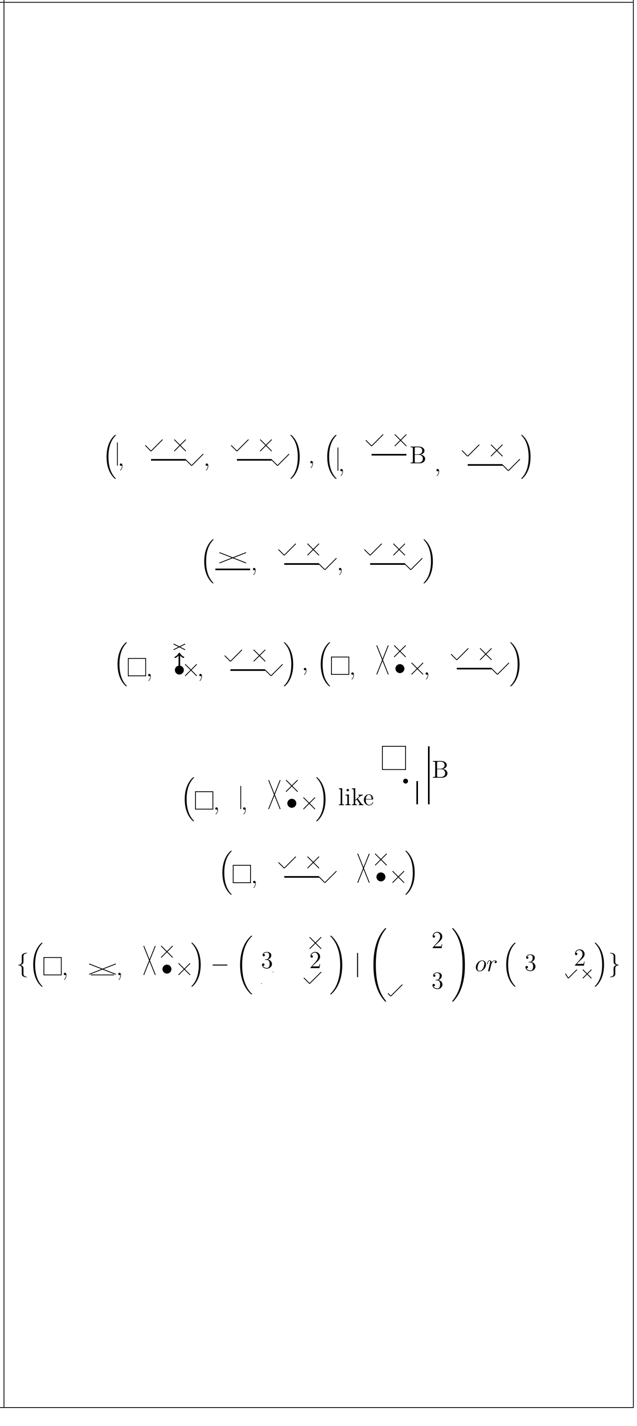
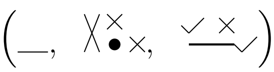
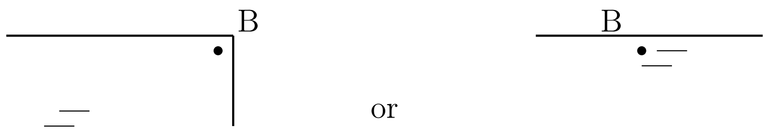
Dimension two	Dimension three
	

Table 4.28: Modified cells of $\mathcal{P}(2,3,i)$

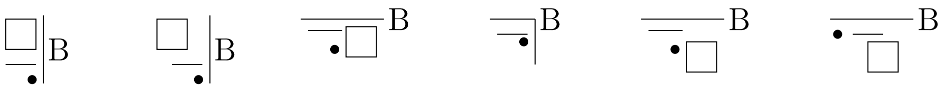
Meanwhile in  the options are:



The first options in both kind of cells, as long as the first coordinate is not like , then they are paired. But the 3-cells in the second option no longer have pairs.

The 2-cells in both options comply $\asymp$. Given that the available 3-cells in the third pairing are of the specific type in table 4.28 since its second coordinate is right next to the boundary then have a free 2-face to mate with and it is necessary to delete the paired 2-cells, which are denoted by $\begin{pmatrix} \check{1} & \checkmark \\ 3 & 2\times \end{pmatrix}$.

For the last one the same thing happens as in [H] for $\left(\square,\ \overset{\checkmark\times}{\_\_}\checkmark,\ \times\!\overset{\times}{\bullet}\!\times\right)$ so the paired 2-cells denoted by $\begin{pmatrix} \check{1} \\ 2 \end{pmatrix}$ must to be deleted from their respective kind of cells. While for the others, the specific 2-cells are:



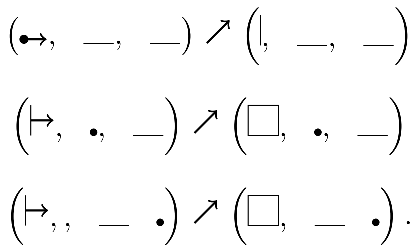
Except for the penultimate option, all 3-cells have a cell to pair with by considering the bottom side of the square and taking the 2-cell relative to this configuration. The first four and the last one are respectively on the sets

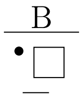
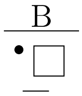
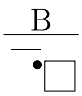
$$S_0 = \left\{ \begin{pmatrix} 2 \\ \checkmark 3 \end{pmatrix} \mid \text{its first coordinate can grow upwards and does not block the right of the point.} \right\}$$

$$S_1 = \left\{ \begin{pmatrix} 3 & \underset{\checkmark\times}{2} \end{pmatrix} \mid \text{its first coordinate is able to grow downwards} \right\}$$

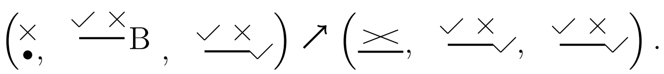
and both of them must be deleted. The modified cells are presented in table 4.29.

The final pairings from dimension two to dimension three are

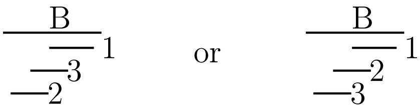


From table 4.29, for the last two pairings just taking the right side of the square and the 2-cell with a similar configuration to it, then the 3-cells can be paired, so it is just necessary to delete these cells. Cells like  in the penultimate case are denoted by $\begin{pmatrix} \check{2} \ 1\checkmark \\ 3 \end{pmatrix}$, meanwhile cells as  or  in the last case are denoted by $\begin{pmatrix} \check{3} \ 1\checkmark \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & \checkmark \\ 3 & 1\checkmark \end{pmatrix}$.

For the first, specifically the pairing is:



And the only available 3-cells are:



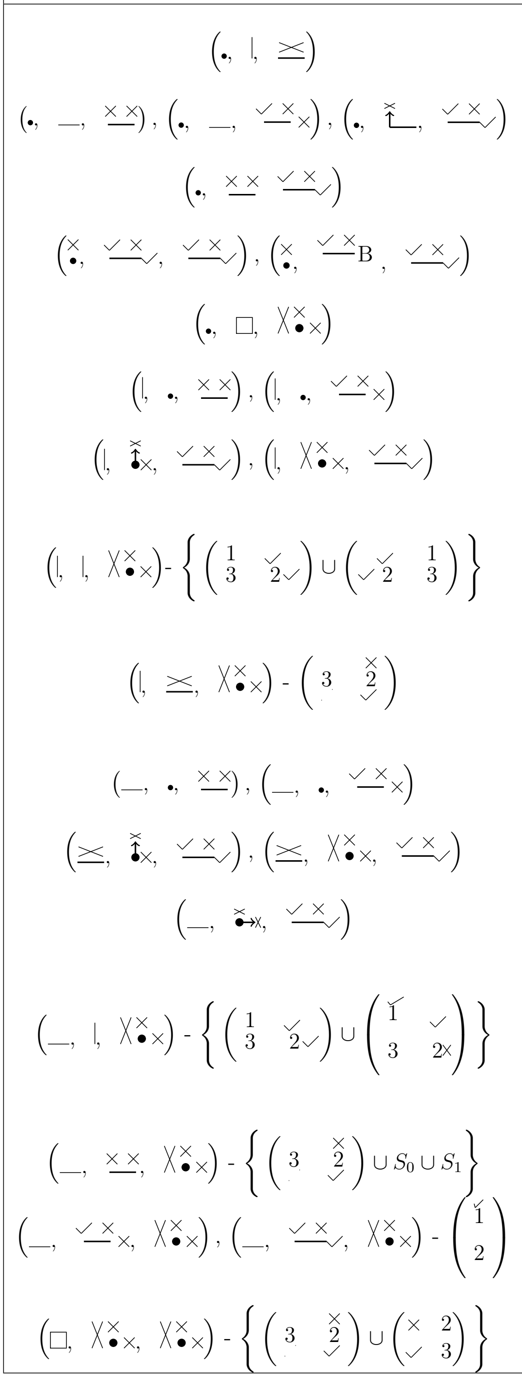
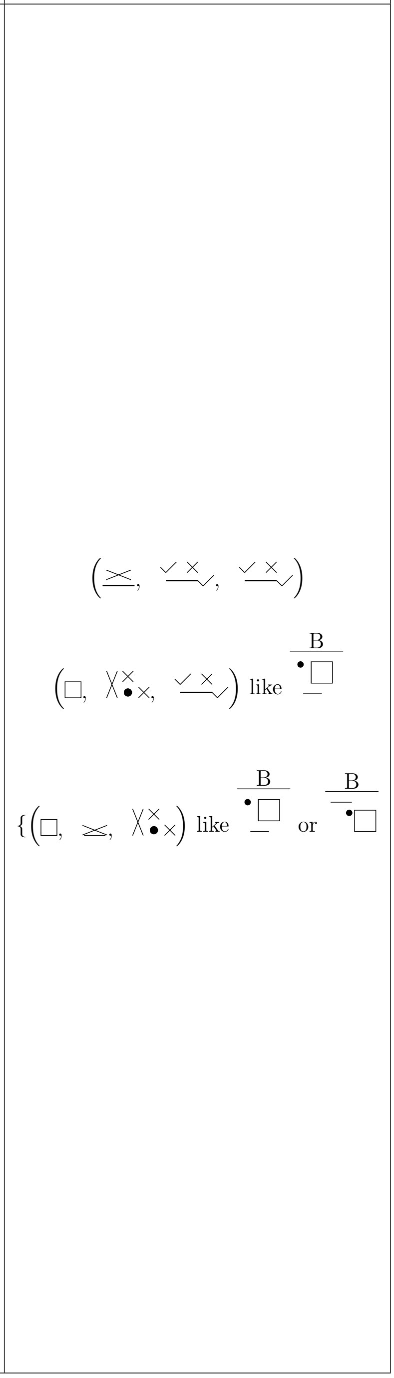
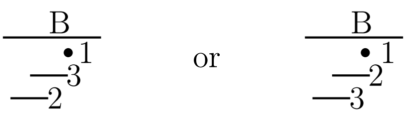
Dimension two	Dimension three
	

Table 4.29: Modified cells of $\mathcal{P}(2, 2, i)$

And similarly for the 2-cells:



so every 3-cells has a partner. All the modifications are listed on table 4.30.

Dimension two

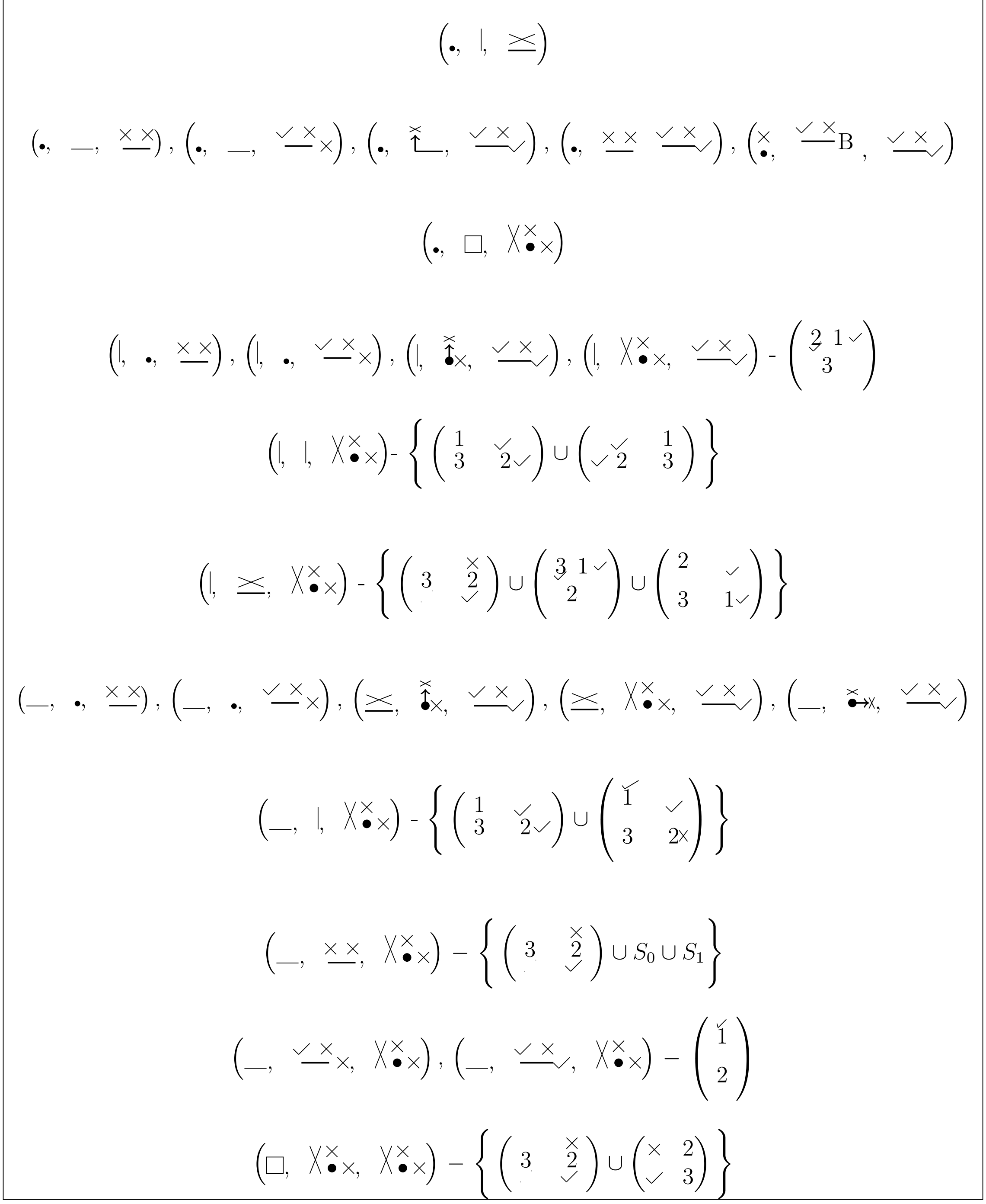


Table 4.30: Modified cells of $\mathcal{P}(2,1,i)$

The process of $dim(1) \nearrow \dim(2)$ starts with the pairing rules

$$(\bullet, \quad \bullet, \quad \underset{\perp}{\uparrow}) \nearrow (\bullet, \quad \bullet, \quad \square)$$

$$(\bullet, \quad |, \quad \updownarrow) \nearrow (\bullet, \quad |, \quad |)$$

$$(\bullet, \quad \_, \quad \updownarrow) \nearrow (\bullet, \quad \_, \quad |)$$

$$(|, \quad \bullet, \quad \updownarrow) \nearrow (|, \quad \bullet, \quad |)$$

$$(\_, \quad \bullet, \updownarrow) \nearrow (\_, \quad \bullet, \quad |)\,.$$

And the remaining cells of dimension one and two are described in table 4.30 and 4.31 respectively.

Dimension one
$(\bullet, \quad \bullet, \quad \underline{\times\!\!\!\times}),\ (\bullet, \quad |, \quad \overset{\times}{\bullet})$
$(\bullet, \quad \_, \quad \overset{\times}{\bullet}),\ (|, \quad \bullet, \quad \overset{\times}{\bullet})$
$(\_, \quad \bullet, \overset{\times}{\bullet})$

Table 4.31: Modified cells of dimension one of $\mathcal{P}(1, 6, i)$

The process continues in the third component and its first coordinate:

$$(\bullet, \quad |, \quad \mapsto) \nearrow (\bullet, \quad |, \quad \_)$$

$$(\bullet, \quad \_, \quad \mapsto) \nearrow (\bullet, \quad \_, \quad \_)$$

$$(|, \quad \bullet, \quad \mapsto,) \nearrow (|, \quad \bullet, \quad \_)$$

$$(\_, \quad \bullet, \mapsto) \nearrow (\_, \quad \bullet, \quad \_)\,.$$

For this step, by [B] it is easy to see that all the 2-cells such that its third component is blocked as $\overset{\checkmark\ \times}{\text{—}}\checkmark$ cannot be paired because those whose third component is blocked as $\underline{\times\ \times}$ or $\overset{\checkmark\ \times}{\text{—}}\times$ are matched first, so the resulting cells are in table 4.32.

For the following step, the second component and its second coordinate are fixed, then the corresponding matches are

$$(\bullet, \quad \updownarrow, \quad \_) \nearrow (\bullet, \quad |, \quad \_)$$

$$(\bullet, \quad \underset{\perp}{\uparrow}, \quad \bullet) \nearrow (\bullet, \quad \square, \quad \bullet)$$

$$(|, \quad \updownarrow, \quad \bullet) \nearrow (|, \quad |, \quad \bullet)$$

$$(\_, \quad \updownarrow \quad , \bullet) \nearrow (\_, \quad |, \quad \bullet)\,.$$

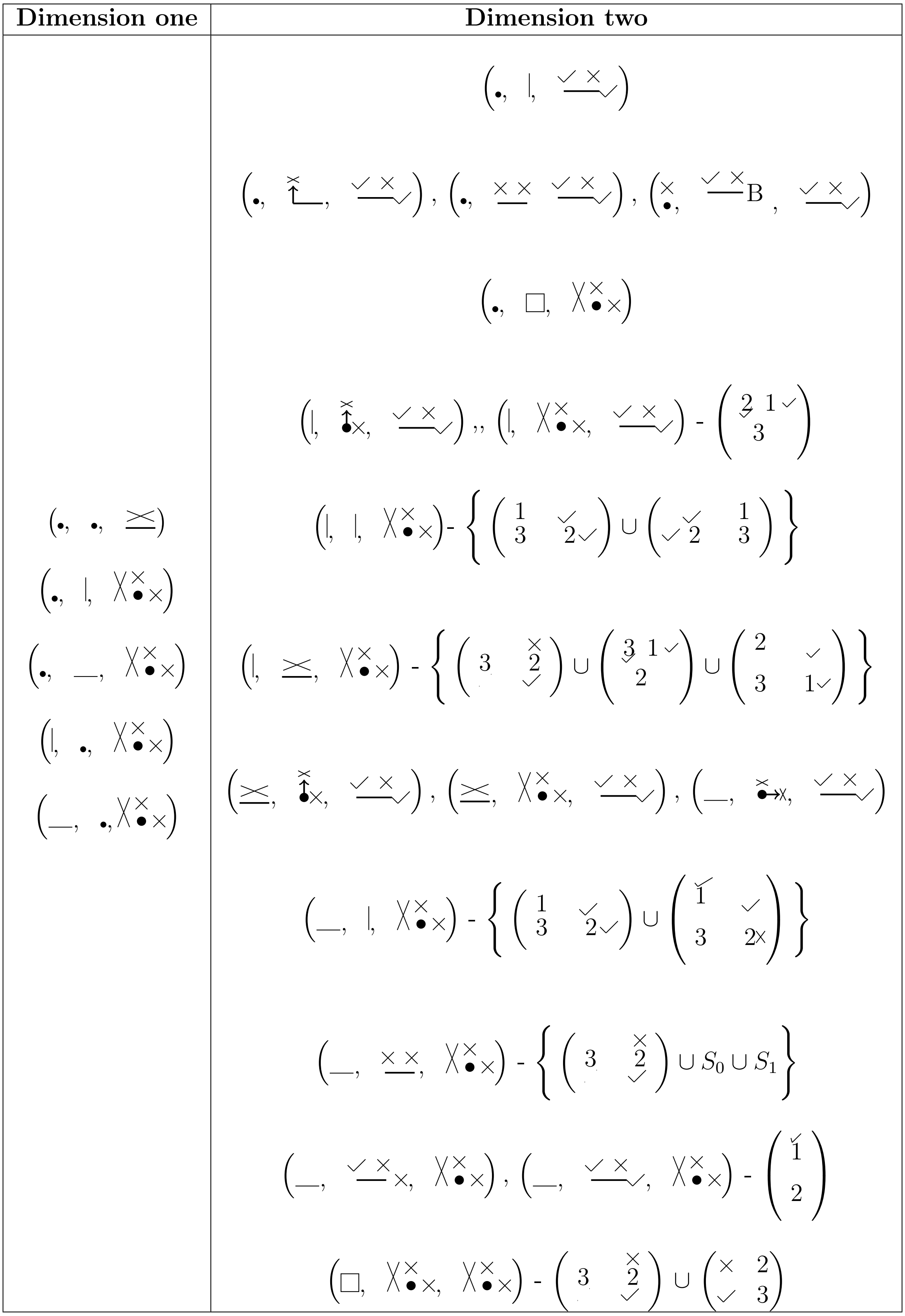


Table 4.32: Modified cells of $\mathcal{P}(1,5,i)$

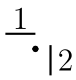
For the first, it is evident that all the 2-cells are deleted, and for the 1-cells, it is easy to see which are the unmatched cells. It is important to note there are cells like those in [C] which are not paired too. For the second, as explained in [D.1], there are still remaining cells of dimension two, namely

For the last two, if the second coordinate is above the third and all the other blocks hold, then while the second coordinate is able to grow up, then it is paired. In fact, as it can be seen in previous cases the problem arises when it is in the same side but in different positions, i.e.

and

For the third, while the correct blocks of the third component hold, the first 2-cells to be matched are the first ones of both cases, and for the others, there are just two possible options:

Both kinds of cells are not matched since their 1-cell available were matched before by growing up, but most of these cells were deleted already according to table 4.32. So the only remaining cells are those with their second coordinate blocked on the left or on the right respectively, a situation that only happens when $p = 2$. For the last pairing the previous cases also hold, but since the first coordinate of the cells is an horizontal line, using [D] and noticing the same reasoning can be used even if the second coordinate is a vertical line and not a square, it is easy to see that cells like



Since the remaining 2-cells for the last three pairings are of the specific types already show and all of the first were deleted, for this time just the modifications of the remaining 1-cells are listed on the table below (4.33).

Dimension one

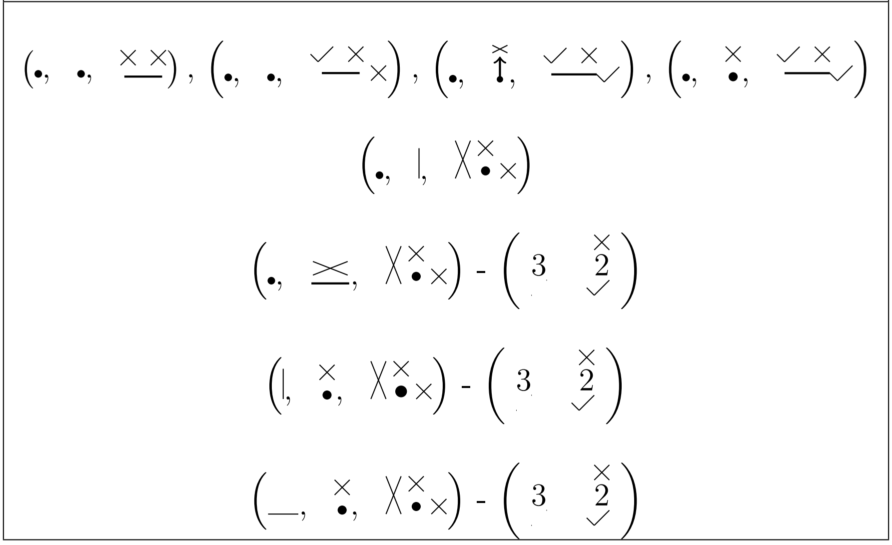
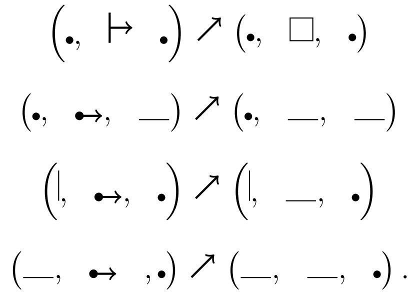
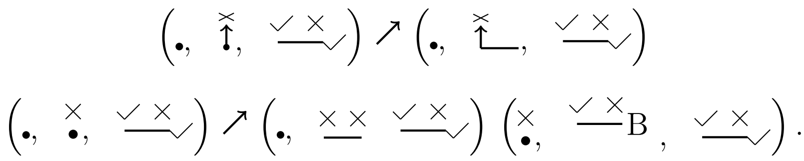


Table 4.33: Modified 1-cells of $\mathcal{P}(1,4,i)$

Still in the second component, the process continues by setting the first coordinate

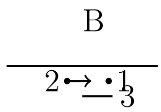


For the first, just taking the 1-cell related to the right or left side of the square as the case may be, makes all the 2-cells be deleted. On the second, the pairings are:

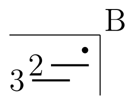


Given the specific shape of the cells in the first case, it becomes clear all the 2-cells are deleted. For the cells in (•, ×× ✓×✓), since ×× also denotes ×— and given the components of the cells, this blocks can be realized as

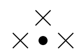
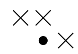
Then, while the block on the third component holds, it is clear that by the preference for moving to the right, all of these 2-cells are matched. For the unpaired 1-cells, as in [C.0] there is the special case



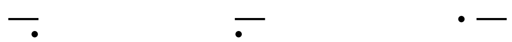
that fulfills that its second coordinate is able for move to the right but whose 2-cell is not available, these cells are not paired and are denoted as those in [C.0] too. Finally, for this pairing, it remains exactly one 2-cell, namely

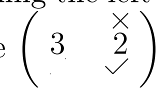


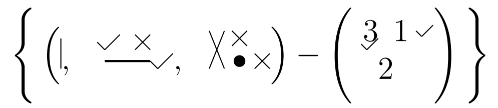
For the third, since the blockages of the third coordinate are like

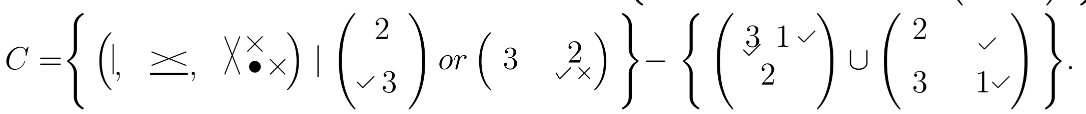
 or 

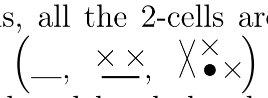
then it is easy to deduce that, given that it is the second coordinate which is under consideration, a deeper look must be taken about when it is making the upper or the right block. This is because when it makes the left block, since both coordinates must be blocked above, then they are right below the boundary and taking the 1-cell related to the point on the right side of the horizontal line, then the 2-cell is paired. Now, there are three options

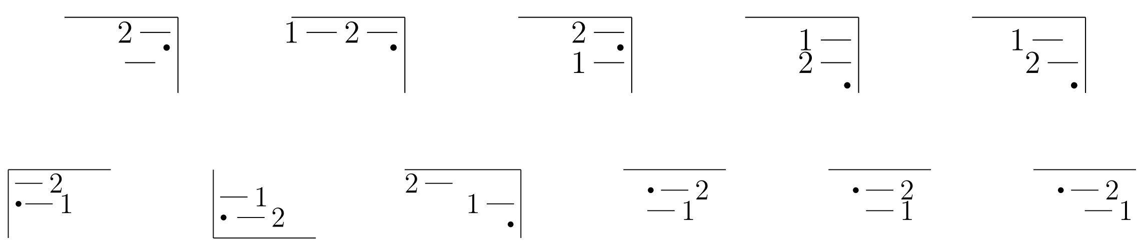


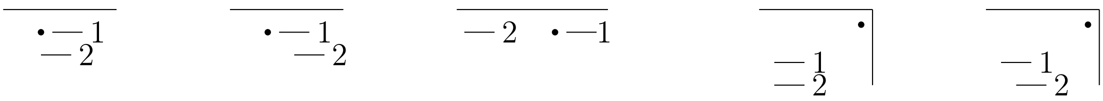
while the other corresponding blocks hold, for the first if there is a block on the left of the point, the cell is paired, otherwise the 2-cell remains unpaired. For the middle option, there must be blockages on the left and right, so taking the left point of the line, a match is achieved. For the last one, since the cells like  were deleted already, the cells actually considered are

The first case can be paired just taking the left side of the horizontal line, but not the second because neither of the ends of the line are related to available 1-cells. Finally, even if the second coordinate is not making any of the blockages of the point, there is still the case on [B], and this case must be considered on the remaining cells. So the remaining cells are those on  or



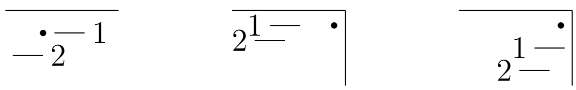
For the last one, given that there are several kinds of cells, all the 2-cells are illustrated below for the three cases. First, for the 2-cells in  and considering all the special types of 2-cells with these blocks but deleted already, all the cells are (1 to 16)





Meanwhile for the cells on $\left(\_\_, \ \overset{\checkmark\ \times}{\overline{\ \ }}\times, \ \chi\overset{\times}{\bullet}\times\right)$ there are just two different options, namely (17 and 18)

Finally, for those cells in $B = \left(\_\_, \ \overset{\checkmark\ \times}{\overline{\ \ }}\checkmark, \ \chi\overset{\times}{\bullet}\times\right) - \begin{pmatrix} \overset{\checkmark}{1} \\ 2 \end{pmatrix}$ there are two possible options too (19 to 21)



For these, since the possible 1-cell available is that related to take the right end point of the second coordinate, but this 1-cell was already paired on 12 and 15 respectively, then 19 and 20 are not paired, also 21 cannot be paired since neither of the end points of the first coordinate are related to available 1-cells. For 17 and 18, either are matched with the 1-cell related to take the right end point of the second coordinate. Now, for 1 to 5, since the point is not blocked on the left, neither of the end points of the second coordinate have an available related 1-cell. For 6 to 10 and 12 to 16, taking the 1-cell related to take the left end point of the second coordinate, then the 2-cell is paired, but not for 11 since neither of its end points are related to available 1-cells. So for the remaining cells, setting

$A_0 = \left\{ \begin{pmatrix} 2 \\ \checkmark 3 \end{pmatrix} \mid \text{ its first coordinate is blocked above } \right\}$,

$A_1 = \left\{ \begin{pmatrix} 2 \\ \checkmark 3 \end{pmatrix} \mid \text{ its first coordinate can block the left side of the point if it grows up } \right\}$

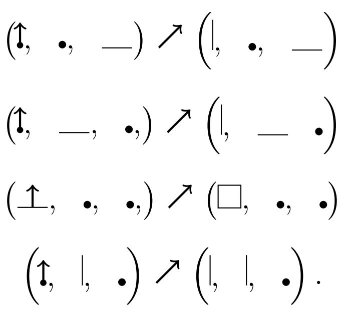
and

$A_2 = \left\{ \begin{pmatrix} 3 & 2 \end{pmatrix} \mid \text{ its second coordinate is blocked below at the right by the first coordinate } \right\}$

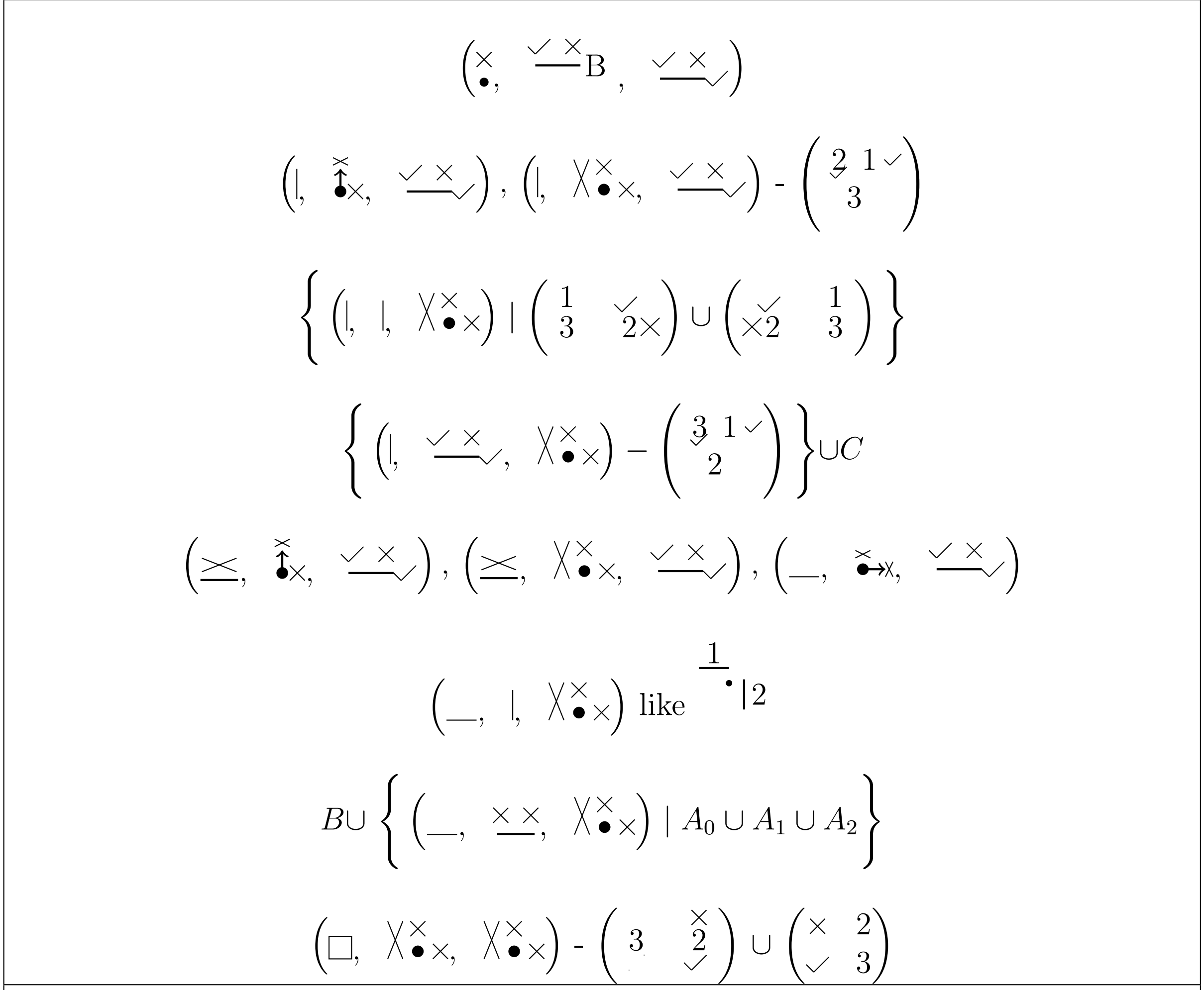
then the remaining 2-cells are those in $B \cup \left\{ \left(\_\_, \ \overset{\times\ \times}{\overline{\ \ }}, \ \chi\overset{\times}{\bullet}\times\right) \mid A_0 \cup A_1 \cup A_2 \right\}$.

All the modified cells are listed on table 4.34.

Now, almost finishing, the first component is fixed, and the pairings are those where the first coordinate can go up or down, i.e.,



Dimension two



Dimension one

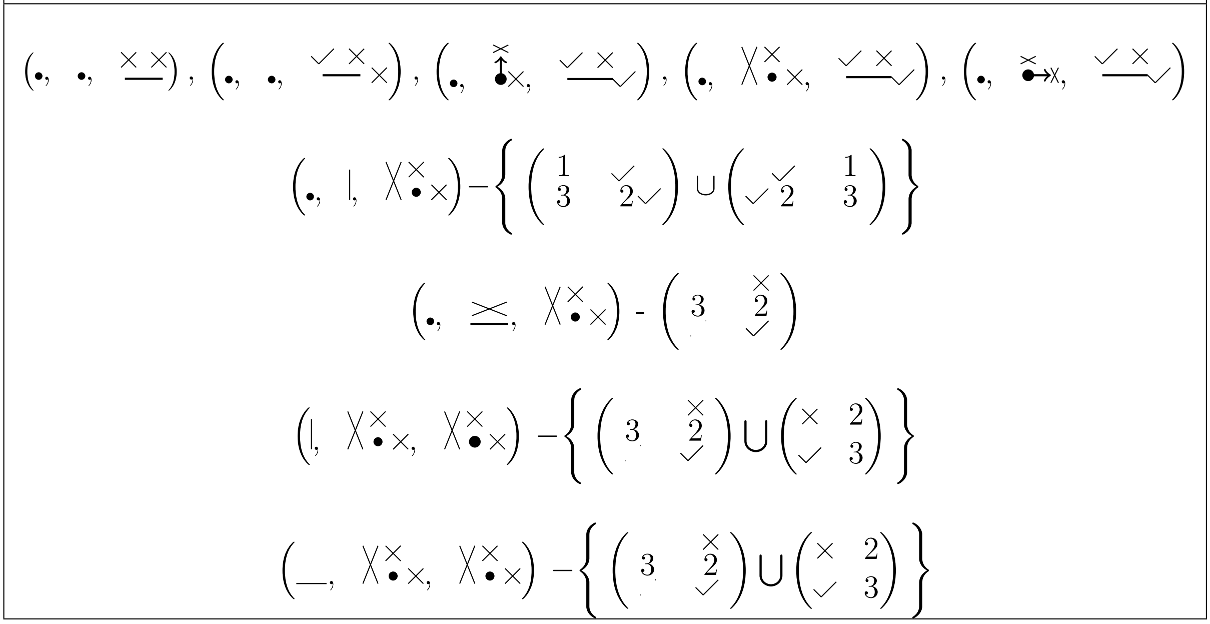
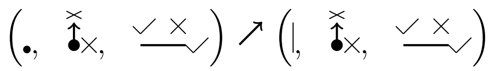
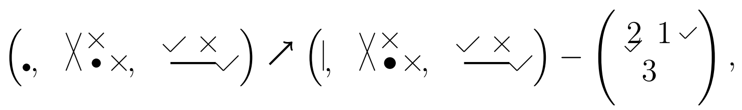


Table 4.34: Modified cells of $\mathcal{P}(1,3,i)$

First, for



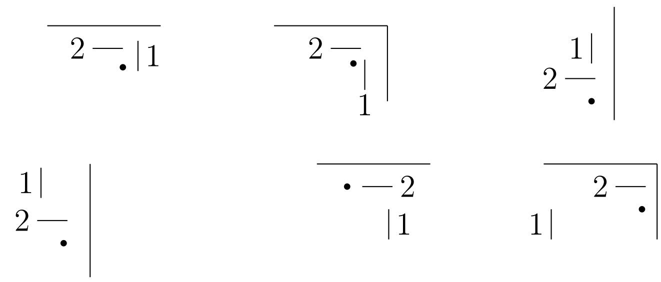
it is easy to see that all the 2-cells are paired and for the 1-cells there is just one unique remaining cell whose first coordinate is below the upper boundary. Concerning

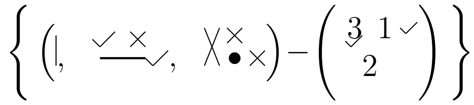


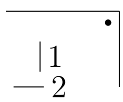
for the 1-cells, there are just two possible options with these blocks, namely

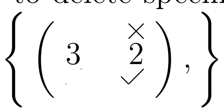
And for the 2-cells the options are

As can be verified on table 4.34, the last option was deleted already, and taking the 1-cell related to take the end point of the vertical line above the third coordinate gives a match for the first option.
For the second pairing, an exhaustive verification gives that all the possible options for C are (1 to 6)



Meanwhile for the cells on  there is just one option (7)



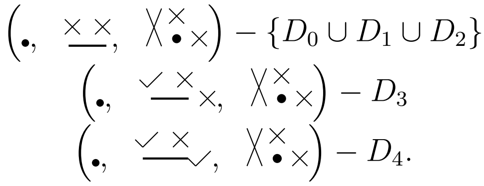
These last kinds of 2-cells are all paired since taking the 1-cells related to take the end point of the vertical line above the second coordinate gives a pair for it, because this 1-cell does not move to the right first since that 2-cell were deleted before. For options 1 and 2 to 6, taking the final end point of the vertical line and the 1-cell related to it gives a match for the 2-cell, but not for 2 since for it the starting point must be taken, but this 1-cell was paired with a 2-cell like 1. For the remaining 1-cells, it is just necessary to delete specific cells as the case may be. So, dividing the cells and taking $D_0 =$ 

$$D_1 = \left\{ \begin{pmatrix} 2 \\ \checkmark 3 \end{pmatrix} \mid \text{its first coordinate can grow up and does not block the left side of the point if it does.} \right\}$$

$$D_2 = \left\{ \begin{pmatrix} 3 & 2 \end{pmatrix} \mid \text{its second coordinate is blocked below at the right by the first and this can grow downwards} \right\}$$

$$D_3 = \left\{ \begin{pmatrix} 2 \\ \checkmark 3 \end{pmatrix} \mid \text{its first coordinate can grow up} \right\} \text{ and } D_4 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid \text{its first coordinate can grow up} \right\}$$

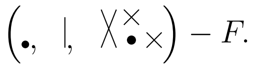
then the remaining 1-cells are those in



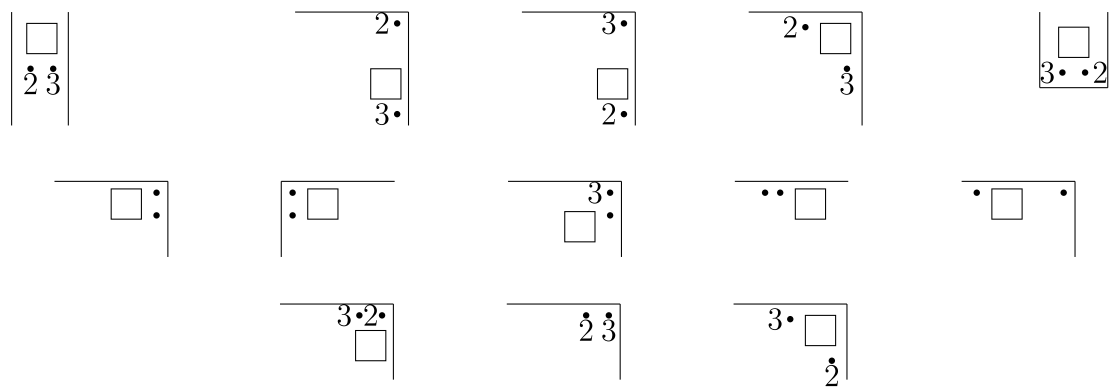
For the third pairing, since the remaining 2-cells are of the specific types in table 4.34, and taking the related 1-cell obtained by taking the end point of the first coordinate which is exactly the one that has not been eliminated yet, then all of these 2-cells are deleted and taking $F = F_0 \cup F_1$ with

$$F_0 = \left\{ \begin{pmatrix} 1 & \checkmark \\ 3 & 2\checkmark \end{pmatrix} \cup \begin{pmatrix} \checkmark & 1 \\ \checkmark 2 & 3 \end{pmatrix} \right\} \text{ and } F_1 = \left\{ \begin{pmatrix} 1 & \checkmark \\ 3 & 2\times \end{pmatrix} \cup \begin{pmatrix} \checkmark & 1 \\ \times 2 & 3 \end{pmatrix} \mid \text{its first coordinate can grow up} \right\}$$

then the remaining 1-cells are those in



Finally, for the fourth pairing, it is convenient to illustrate all possible 2-cells (1 to 13)



where all the cells without numbers of coordinates means that the point can take any of the values 2 or 3. For the options 1, 2, 3, 5, 6 and 12, the 2-cells are paired by taking the bottom side of the square and the 1-cell related to it (in 6, this only can be done if the third coordinate is above and the second is below). For the options 6, 9, 10 and 11 the 2-cells are paired by taking the upper side of the square and the 1-cell related to it (in 6, this is done just when the second coordinate is the one on the top and the second is the one below, this because some of the 1-cells were deleted already, but there is still a partner for it). Finally, for the cells like 4, 7, 8 and 13 neither of the upper or bottom sides of the square can be taken to form an available 1-cell, so these specific cells still remain. For the unpaired 1-cells, it is only necessary to delete those 1-cells related to the 2-cells paired before. So denoting the next sets

$$E_0 = \left\{ \begin{pmatrix} \times & 2 \\ \checkmark & 3 \end{pmatrix} \right\},\ E_1 = \left\{ \begin{pmatrix} 2 & & 1 \end{pmatrix} \mid \text{ the first is able to grow downwards } \right\}$$

$$E_2 = \left\{ \begin{pmatrix} 3 & & 1 \end{pmatrix} \mid \text{ the first coordinate is able to grow downwards } \right\},\ E_3 = \left\{ \begin{pmatrix} 1 & 2 \\ \checkmark & 3 \end{pmatrix} \right\}$$

and

$$E_4 = \left\{ \begin{pmatrix} 3 & 2 \\ & 1 \end{pmatrix} \mid \text{ the first coordinate is able to grow downwards } \right\}$$

then the remaining cells are those in

$$\left(\underline{\asymp},\ \ \chi\overset{\times}{\bullet}\times,\ \ \chi\overset{\times}{\bullet}\times\right) - \{D_0 \cup E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4\}.$$

Then, the modified cells are those presented on table 4.35

For the pairings from dimension one to dimension two, the last step is given by the following matches

$$(\bullet\!\!\rightarrow,\ \ \bullet,\ \ \text{—}) \nearrow (\text{—},\ \ \bullet,\ \ \text{—})$$

$$(\bullet\!\!\rightarrow,\ \ \text{—},\ \ \bullet,) \nearrow (\text{—},\ \ \text{—},\ \ \bullet)$$

$$\left(\mapsto,\ \ \bullet,\ \ \bullet\right) \nearrow (\square,\ \ \bullet,\ \ \bullet)$$

$$\left(\bullet\!\!\rightarrow\ \ |,\ \ \bullet\right) \nearrow \left(\text{—},\ \ |,\ \ \bullet\right).$$

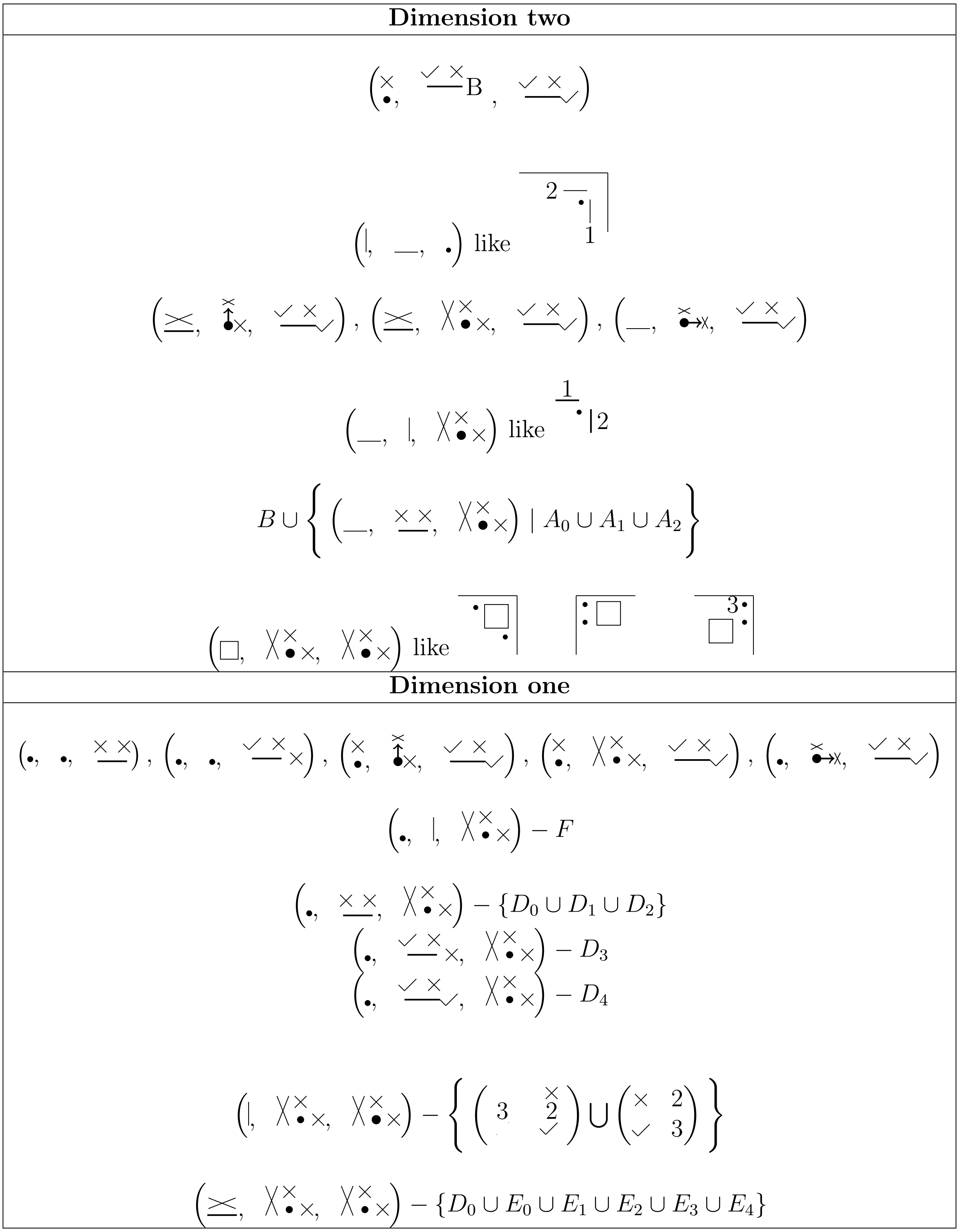
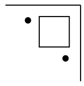


Table 4.35: Modified cells of $\mathcal{P}(1,2,i)$

Getting started with the third pairing that is the easiest, according to table 4.35, the cells like

are paired with the 1-cells related to take the left or the right side of the square respectively. But for the cells



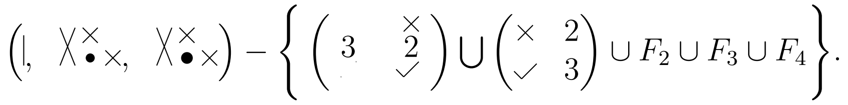
neither of its sides is related to an available 1-cell, so they are not deleted. And for the remaining 1-cells, we set

$$F_2 = \left\{ \begin{pmatrix} \begin{smallmatrix} 2 & \\ 3 & 1 \end{smallmatrix} \end{pmatrix} \mid \text{the first coordinate is unblocked on its right side} \right\},$$

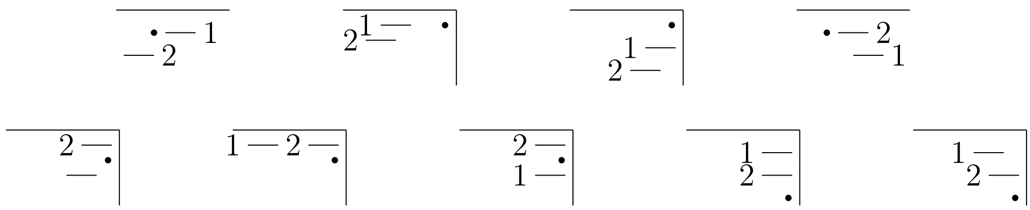
$$F_3 = \left\{ \begin{pmatrix} \begin{smallmatrix} 3 & \\ 2 & 1 \end{smallmatrix} \end{pmatrix} \mid \text{the first coordinate is unblocked on its right side} \right\} \text{ and}$$

$$F_4 = \left\{ \begin{pmatrix} \begin{smallmatrix} \checkmark 3 \\ 1\,2 \end{smallmatrix} \end{pmatrix} \mid \text{the first coordinate is unblocked on its right side} \right\}.$$

Then the remaining 1-cells are those in



As seen before, the remaining 2-cells for the pairing number two are (1 to 9)

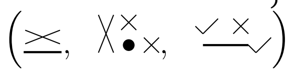


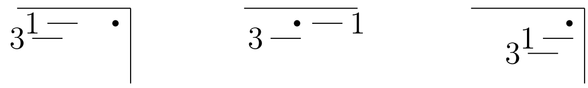
For the options 1, 2, 4, 6, 7 and 8, according to table 4.35, just considering the 1-cell obtained by taking the left end point of the first coordinate gives an available 1-cell, so these cells are paired. For 3, 5 and 9, neither of the end points of the first coordinate give rise to an available 1-cell. On table 4.36 are the remaining 1-cells, for which just were deleted the 1-cells deleted here, and for which

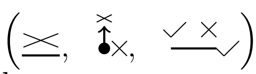
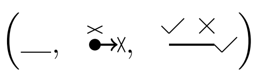
$$D_5 = \left\{ \begin{pmatrix} \begin{smallmatrix} 2 \\ \checkmark 3 \end{smallmatrix} \end{pmatrix} \mid \text{its first coordinate is blocked above and unblocked on its right side.} \right\}$$

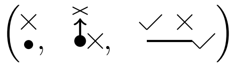
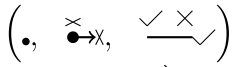
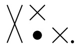
and

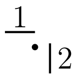
$$D_6 = \left\{ \begin{pmatrix} 3 \quad 2 \end{pmatrix} \mid \text{its second coordinate is blocked below at the right by the first and this is unblocked on its right side} \right\}.$$

For the first pairing, there are few remaining 2-cells, those in  are (1 to 3)



Meanwhile for those in  and  there is respectively one option, namely

For these two, the 1-cell related to take the left end point of the coordinate 1, gives rise to matches for these 2-cells. For options 1 and 2, the same can be done and then both kinds of cells are paired. For the third, neither of the end points of the first coordinates is related to an available 1-cell, so this cell is not deleted. For the remaining 1-cells, it is evident that all the cells on  and  are all paired here, and the only modified cells of the form  are those whose first coordinate is blocked like .

Finally, for the cell  since the third coordinate is not blocked on its left side and neither of the end points of the first coordinate can block the point properly, this cell is not deleted.
The modified remaining cells, are all listed on the table below (4.36).

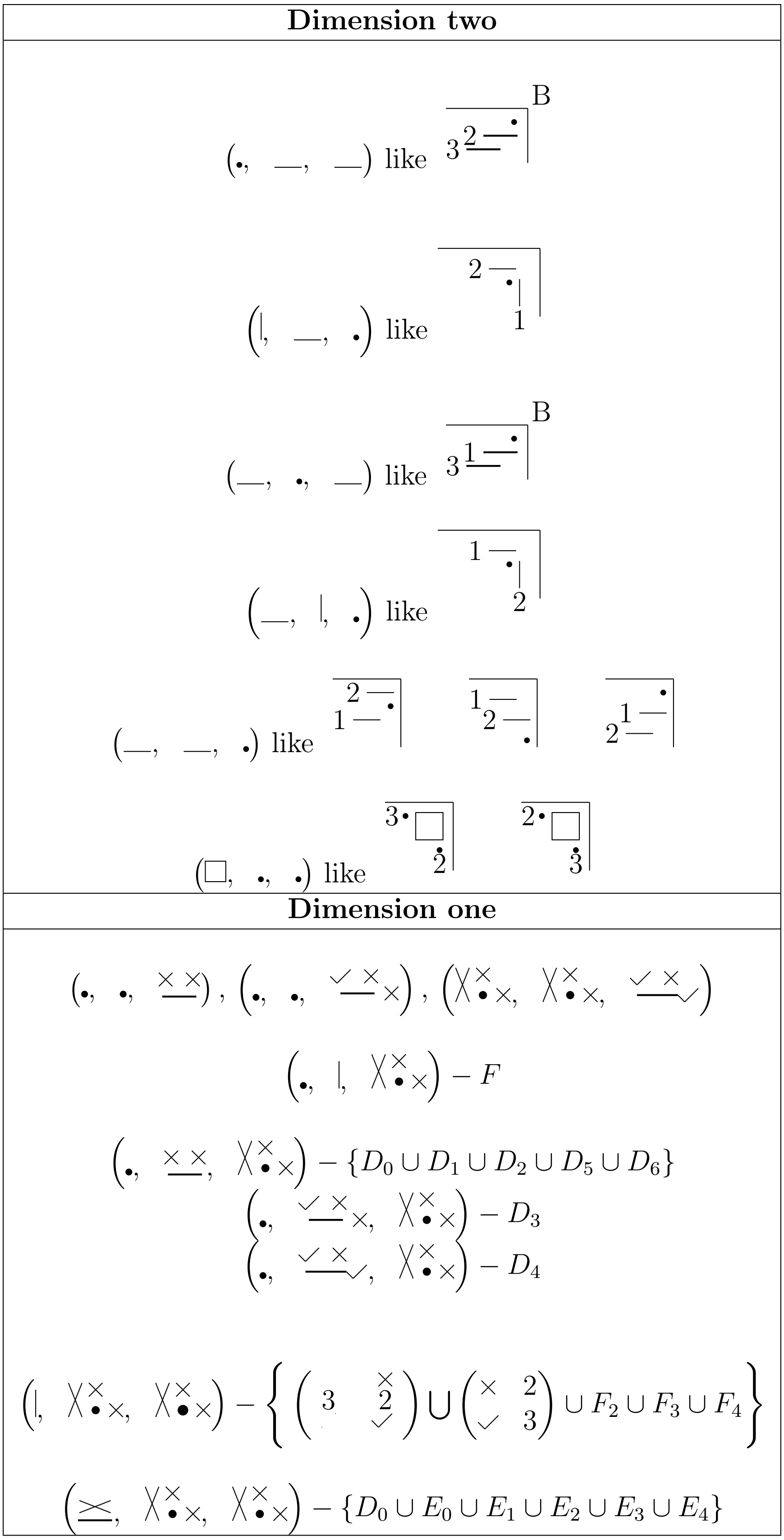


Table 4.36: Modified cells of $\mathcal{P}(1,1,i)$

We refrain to describe the pairing process between dimension 0 and 1 and, instead, we close this section with a few final remarks that complete a proof of the theorem on the introduction:

Proof of item (e). The analysis above of the gradient field gives the assertions about the dimension of critical cells, together with the inequality $hdim(C(3;p,q)) \leq 2$, where $hdim$ stands for "homotopy dimension of". The fact that the previous inequality is an equality follow from the homological calculations in [1] giving $H_2(C(3;3,3)) \neq 0$ (see table 1 in [1]) together with the fact (communicated by Matthew Kahle) that, when $min\{p,q\} \geq n$, the homotopy type of $C(n;p,q)$ depends solely on n (and agrees with that of the classical configuration space of n ordered points in the plane).

Proof of item (d). Under the present conditions there are no critical $2-$cells on table 4.36, so that $C(n;p,q)$ has the homotopy type of a graph which is connected in view of Proposition 21 in [7]. The asserted homotopy equivalence then follows from the homological calculations in [1] (again, see table 1 in [1]).
Lastly, we leave as a very illustrative exercise for the reader to verify the assertions in item (c) for $X(3;2,2)$. In any case, it is transparent from its definition that $C(3;2,2)$ is in fact homeomorphic to the disjoint union of two circles.

Items (a) and (b) are fully contained in the analysis of the gradient field in subsections (4.1) and (4.2).

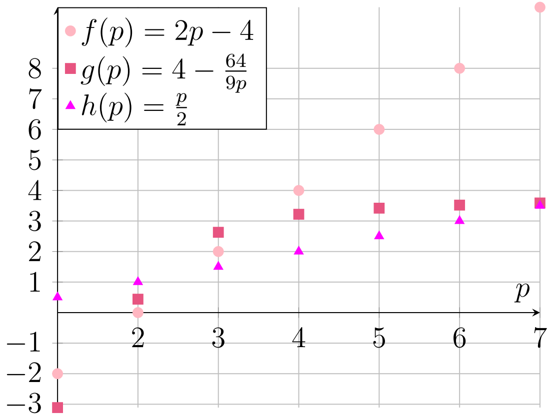
4.4 n=4

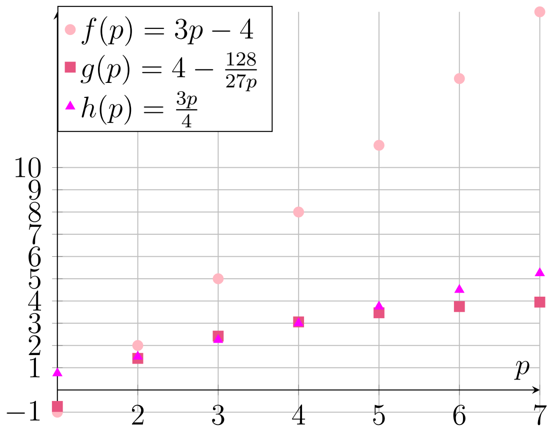
A computational implementation for the case $n = 4$ yields the information reported in the following table.

q \ p	2	3	4	5	6	7
2	0	1	2	2	2	2
3	1	2	3	3	3	

Table 4.37: Maximum value of j such that $m_j \neq 0$ depending of $p \times q$ for $n = 4$

This table indicates, for different sizes of the grid $p \times q$, the maximum value of j such that $m_j \neq 0$. The following tables are obtained considering the values $q = 2, 3$ respectively





From table 4.37 and these graphics, it is easy to see for $q = 2$ and $p = 2, 3, 4$, the bounds established by the conjecture are very well adjusted, meanwhile for $p > 4$ are more than necessary. Moreover, for $q = 3$ and again for $p = 2, 3, 4$, the bounds are met very well. It is of special interest investigate over the cases with $p < n$ or $q < n$ because as noted at the end of the previous subsection, if $p \geq n$ and $q \geq n$, then the configuration space $C(n; p, q)$ of n hard squares in a $p \times q$ rectangle is homotopy equivalent to the classical and well known configuration space of n points in the plane.

5 Conclusions

The homotopy type of $X(n, p, 1)$ was calculated in a very simple way, mainly because the combinatorial nature of the components of a cell is not too difficult to handle with the algorithm described, since there are just two different components, a horizontal line or a point, and even if the total number of cells can be very huge for very large values of n, the procedure followed allows us to deduce the homotopy dimension of $X(n, p, 1)$ for arbitrary n and p. Even more, for this case, is deduced that both upper bounds for j (those established in theorem 1.1 in [1] and those that are conjectured) are larger than necessary.

For $X(2, p, q)$, the gradient field described in [1] for $X(2; 2, 2)$ produced four critical cells of dimensions one and another four in dimension zero, while our gradient field have only one critical $0-$cell and one critical $1-$cell. So for this case, the algorithm proposed in this work is quite useful. Even more, the total number of critical cells of dimension 1 and 0 for $X(2, p, q)$ is optimal and the resulting homotopic type corresponds with that of 2 non-collisioning particles on $\mathbb{R}^2$.

On the other hand, for $n = 3$, the total cell types is not very large, but despite this, the calculations become much more laborious. Indeed, the modifications to the remaining cells become more restrictive even in similar pairings to those made for the case $n = 2$. Nevertheless, it could be deduced that for $p = 2$ the maximum dimension of critical cells is one, which makes it clear that the upper bounds for j in theorem 1.1 of [1] can be larger than necessary. In all other cases analyzed here we see that the conjectured upper bounds fit quite well. Finally, with $n = 4$ a similar conclusion can be drawn.

Unlike the gradient field in [1], our gradient field fails to be equivariant with respect of the natural free action of the symmetric group. Such a characteristic stands in sharp contrast with (and is perhaps responsible for) its apparent optimality (at least homotopy-dimension wise). The algorithm for producing our gradient field is very natural, easy to visualize, and simple to implement on a computer. Yet, given the combinatorial (and complex) nature of the cells in $X(n; p, q)$, the analysis of the algorithm, and description of resulting critical cells, becomes a hard major challenge, even for a computer dealing with relatively small values of the parameters n, p and q.

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