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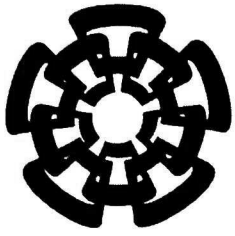

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Análisis de formas normales de alto orden en sistemas de potencia resonantes usando enfoques no recursivos

Tesis que presenta:
Ramón Octavio Jiménez Betancourt

para obtener el grado de:
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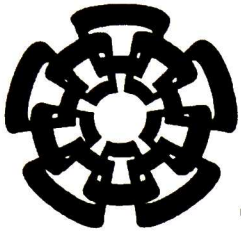
Directores de Tesis
Dr. Arturo Román Messina
Dr. Emilio Barocio Espejo

Guadalajara, Jalisco, Febrero de 2007.



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Higher Order Normal Forms Analysis of Resonant Power Systems Using Non-recursive Approaches

A thesis presented by:
Ramón Octavio Jiménez Betancourt

to obtain the degree of:
Doctor in Science

in the subject of:
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Thesis Advisors:
Dr. Arturo Román Messina
Dr. Emilio Barocio Espejo

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Tesis de Doctorado en Ciencias
Ingeniería Eléctrica

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Higher Order Normal Forms Analysis of Resonant Power Systems Using Non-recursive Approaches

Doctor of Science Thesis
In Electrical Engineering

By:
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Análisis de formas normales de alto orden en sistemas de potencia resonantes usando enfoques no recursivos

Resumen

En esta tesis, se presenta un marco de referencia sistemático para la determinación de la forma normal más simple de campos vectoriales resonantes. El desarrollo teórico presentado en este trabajo puede ser aplicado en forma general al modelado de sistemas no lineales con resultados más precisos que los obtenidos con los ya existentes. Adicionalmente, el marco de referencia matemático permite la consideración de sistemas dinámicos con variaciones paramétricas y puede aplicarse a sistemas no resonantes y resonantes.

Se desarrollan dos extensiones analíticas a la teoría de formas normales y se investiga la precisión de las soluciones obtenidas. Primero se deriva un modelo conceptual para determinar la representación de forma normal de sistemas no lineales de alta dimensión, basándose en una transformación de un solo paso entre el sistema original y la forma normal en la cual se evita la reducción de la variedad central. Esta transformación permite establecer una metodología selectiva para el tratamiento de efectos no lineales inherentes al cambio no lineal de coordenadas. Se obtienen criterios analíticos que describen la precisión de la representación de forma normal son obtenidos y se sugiere una interpretación física del modelo.

Se presenta también una extensión de esta formulación para el cálculo de la forma normal real de un campo vectorial que esta en la vecindad de una condición de resonancia fuerte. Utilizando una transformación especial de coordenadas real-valuadas en el espacio físico, se propone una técnica para el cálculo de la forma normal real. A diferencia de las representaciones existentes, los coeficientes de la transformación de forma normal y el campo vectorial simplificado son reales, lo cual permite el estudio de campos vectoriales resonantes directamente en el espacio físico original.

Los resultados presentados en este trabajo son los primeros en abordar el problema del análisis de la forma normal real de campos vectoriales de dimension grande que exhiben una condición de resonancia fuerte. Se presentan cálculos numéricos y simbólicos detallados con el objeto de ilustrar la efectividad y aplicabilidad de la técnica analítica propuesta.

Higher order normal forms analysis of resonant power systems using non-recursive approaches

Abstract

In this dissertation, a systematic framework for computing the simplest high-order normal form of resonant vector fields is presented. The theoretical development in this work may be applied rather generally to model the dynamics of general nonlinear systems and results in more accurate solutions than existing procedures. In addition, the mathematical framework permits consideration of general dynamic systems with general parametric variations and is applicable to both, resonant and non-resonant systems.

Two analytical extensions to existing normal form theory are developed, and the accuracy of the solutions is investigated. A conceptual model is first derived to construct the simplest normal form representation for high-dimensional nonlinear systems, based on a one step transformation between the original system and the normal form in which the use of center manifold reduction is avoided. Such a transformation allows the methodology to more selectively treat the occurrence of higher order nonlinear effects arising from the inherent nonlinear change of coordinates, and leads to the simplest normal form. Analytical criteria to describe the accuracy of the resulting normal form representation are derived and a physical interpretation of the system model is suggested.

An extension to this formulation is also presented for computing the real normal form of a vector field in the vicinity of a strong resonance condition. Making use of a special real-valued nonlinear transformation of coordinates in physical space, a technique for the computation of a real normal form is proposed. Unlike existing representations, the normal form transformation coefficients and the simplified vector fields are real, thereby enabling the study of resonant vector fields directly in the original, physical space.

The results presented here are the first to address the question of real normal form analysis and simplification of high-dimensional vector fields in the vicinity of a strong resonance condition. Detailed symbolic and numerical calculations are presented to illustrate the effectiveness and applicability of the

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Contents

Chapter 1

Introduction

Chapter 2

The method of normal forms for vector fields

Chapter 3

Computation of the simplest normal form

Chapter 4

High-order normal form analysis for diagonalizable systems

Chapter 5

Results

List of figures

List of tables

Chapter 1

Introduction

1.1 Background and motivation

Nonlinearity is a ubiquitous feature of physical systems which has profound implications for their dynamics. Recent analytical studies reported in the literature, have shown that the dynamic response of stressed power systems subjected to critical contingencies may result in a complex nonlinear dynamic phenomenon that is not accounted by conventional linear analysis techniques [1]-[3] A clear understanding of the underlying mechanisms leading to the observed nonlinear phenomena is only emerging and accurate analytical techniques are required to analyze and characterize the observed system behavior [4].

The mathematical analysis of system dynamic behavior begins with the derivation of a nonlinear system model obtained by approximating the center manifold of the system model at an equilibrium, by a truncated power series [5],[6]. This approach has the advantage that each term in the truncated series has an obvious physical interpretation. The use of these expansions, however, has been limited to low-order representations in which the non-essential part of the system dynamics is neglected.

The criteria for choosing the relevant dynamics seem to be based on numerical experimentation to simplify the system model. Errors introduced by neglected dynamics can be considerable, especially for operating conditions near critical equilibria. For reliable analysis of complex systems, the influence of the neglected

terms has to be inferred, and methods to improve the basic system formulation have to be developed.

In systems where nonlinearities are strong, or highly stressed, intermode coupling effects are shown to be significant and the higher degree terms can not be neglected [7]-[12]. Because of the complexity involved in the analysis, the prediction of this phenomenon is extremely difficult.

Current simulation-based analysis procedures fail to predict the ensuing oscillations or may provide an incomplete characterization of the underlying phenomena giving rise to the oscillatory phenomena. This has prompted the need for improved modeling and characterization of nonlinear oscillations.

Various approximations have been used in the literature to investigate nonlinear behavior [13]-[16]. Among them, the method of normal forms has been successfully applied to investigate various aspects of system dynamic behavior such as intermode coupling [17], harmonic generation and intermodulation [18], and dynamic bifurcations. This method is particularly attractive for the study of nonlinear effects arising from the series expansion of the original nonlinear power system representation, and can be used to analyze and design system controllers.

Analysis, however, has so far been limited to low-order (second order) approximations. Such an approximation may fail to identify and characterize the nonlinear process underlying the dynamical system or result in incomplete system characterization. Results from recent investigation suggests that higher dimensional representations may be needed to fully extract nonlinear power system behavior especially under heavy stress or heavy loading operating conditions.

Existing formulations present several limitations when applied to complex, higher order representations. Furthermore, current approaches are sensitive to the presence of various types of resonances and the use of physical units in the analysis [19]-[21].

As discussed in this document, new features must be devised because existing analytical approaches are no longer effective to represent the essential system dynamics.

1.2 Problem statement and research objective

Nonlinear power system analysis by means of perturbation theory has been the subject of considerable interest over the years. Power system behavior is governed by highly-complex nonlinear systems of equations, the modeling of which has been approached in a number of ways. A rich variety of nonlinear behavior has been found by direct simulation, some of which can be explained with normal form theory.

In the field of power system dynamics, few models exist that can be studied purely analytically. In current practice, the mathematical analysis of system nonlinear behavior begins with the derivation of a system model obtained by approximating the original nonlinear system model by a truncated power series (a low-order representation). This higher-order dimensional representation is implicitly assumed to contain more information about the system dynamics than the low-dimensional models. Extracting nonlinear characteristics from these models, however, is difficult due to the complex nature of dynamic coupling arising from the nonlinear interaction of the system modes of oscillation.

Of special physical interest is the case of nonlinear processes with dynamics close to a linear (or higher-order) resonance condition in which two or more eigenvalues interact. These conditions may arise from the natural variation of power system parameters or result from the mathematical linearization of the power system model.

Nonlinear models for the study of resonant vector fields have been the subject of extensive theoretical study. The fundamental idea behind these approaches is to employ successive, nonlinear coordinate transformations to systematically obtain a simpler or normal form of the original differential equations up to the de-

sired order of approximation that keeps the dynamical properties of the original system unchanged. Based on this analysis, the normal form coefficients of the reduced vector field on the center manifold in terms of the Taylor coefficients of the original differential equations can be computed.

The NF computation is carried out order by order in a recursive form using separate nonlinear transformations. This approach has several disadvantages:

- At each order of the normal form computation, higher order terms are generated which are not fully accounted for in successive computations. This may add to the complexity of the problem or result in additional errors.
- It is difficult to determine the exact order at which to truncate the normal form approximation.
- The explicit formulas of normal forms in terms of the original nonlinear system are difficult and computationally time-consuming.
- The analysis of nonlinear behavior in the neighborhood of a linear resonance conditions does not present reliable information.
- A final issue that naturally arises is that of determining the nonlinearities and the order at which normal form analysis is truncated.

This has hampered the application of normal form theory to practical problems. In addition, we can make the following observations based on the previous discussion:

- At each step of the NF transformation procedure, the initial conditions in the new coordinate system are determined by solving a set of nonlinear algebraic (constraint) equations. The results are then mapped back into physical space, enabling closed form solutions to be determined. Since, the normal form coefficients are complex, this formulation requires the solution of a complex set of nonlinear algebraic equations.

- The conventional normal form does not yield the simplest normal form. Such a further reduction is very useful in analyzing higher dimensional and/or higher order nonlinear dynamical systems.

In the light of the above discussion, this thesis discusses the use of alternate normal form representations to compute the simplest normal form representation of a nonlinear vector field. An efficient, explicit method is devised to compute the normal form of a high-dimensional system which avoids the use of center manifold reduction. This method enables the essential dynamic of the system to be captured, and retains the physically meaningful state variables and parameter structure of the full system. In this procedure, a novel, non-recursive algebraic procedure is derived that enables the normal form and associated nonlinear transformations for both resonant and non-resonant systems to be reliably computed.

For simplicity, attention is restricted to the study of third-dimensional representations, but the theory and analysis methods, can be easily generalized to other types of nonlinear representations. These results are then used along with the normal forms theory to predict the behavior of resonance and off-resonance conditions. Several nonlinear measures are evaluated as a by-product of these computations. In particular, we propose techniques for identifying the sources of error affecting normal form results and derive analytical criteria to determine nonlinear modal interactions in both, physical and modal coordinates.

In the following section we provide a brief and selective review of some aspects of nonlinear systems theory that are directly relevant to the analysis of resonant vector fields. Recent work on a conceptually new and computationally efficient technique for computing higher order normal form representations based on general framework described above is outlined. Further, areas in which our understanding is incomplete, and which deserve further study are discussed, in particular the effects of higher order resonances, fundamental limitations of existing approaches, and the development of structure-preserving models.

The methodology is not restricted to non-resonant systems, and we can envision employing the procedures presented here in a variety of other nonlinear processes.

1.3 Review of previous work

Higher-order perturbation analysis is becoming of increasing importance in many power system applications because of the need to more accurately approximate system behavior. The first applications of normal form theory in the field of power system dynamics date back to the 1990s. In these works, investigators at Iowa State University discussed the use of weak perturbation theory to analyze and characterize structural properties resulting in nonlinear system behavior [1],[7],[8],[10]. These early simulations of classical power system models suggested that low-dimensional models obtained from normal form theory can capture, and help explain, major aspects of system dynamic behavior such as mode interaction and the effects of structural characteristics on system behavior. More recent applications of these methodologies included the analysis of the significance of the higher order Taylor's representation in the behavior of stressed power systems and the derivation of nonlinear interaction indices [1]. A particularly interesting aspect of this study is the development of closed-form analytical solutions to system behavior.

In the bulk of these applications, normal form theory is used to investigate nonlinear effects arising from the perturbation model of the power system dynamic representation with a view to detect the occurrence of nonlinear interaction between the fundamental natural modes of oscillation of the system.

Common and notable features of these approaches are the reliance on the use of regular perturbation methods based on a low-dimensional Taylor's formula and on center manifold reduction to develop approximate models of system dynamic behavior. A nonlinear coordinate transformation coupled with an optimization problem is then used to map the information in normal form space to physical space. This results in closed-form solutions in which the effect of mode interaction is clearly separated.

This approach has been advocated by many researchers and forms the basis of most existing power system normal form analysis results [4],[9]. The need for improved modeling has led to the formation of the IEEE Task Force on Assessing the Need to Include Higher Order Terms for Small-Signal (Modal) Analysis. The work summarizes and discusses current advances in modeling and controlling inter-area oscillations using small-signal (modal) analysis [4].

The following sections review previously published literature pertaining to the mathematical methods that are used in this dissertation. We review several applications that explicitly acknowledge normal form theory to evaluate various aspects of system nonlinear behavior in order to place in perspective the relevance of this research.

1.3.1. *Generalization to high-dimensional nonlinear systems*

A key challenge for obtaining reliable reduced order dynamic models becomes accurate computation of normal forms for high-dimensional nonlinear systems. Despite the wider application of conventional methods, the generalization of these techniques to address higher-dimensional systems remains an open problem. Among the major impediments to a more extensive use of nonlinear methods are: (i) the problems involved in the construction and representation of high-dimensional nonlinear systems, and (ii) the computational difficulties encountered in subsequent oscillation analyses to determine the normal form and the associated transformation coefficients.

As pointed out in this research, current formulations may be adapted to more general system representations, but substantial difficulties arise in the solving of high-dimensional systems without the simplifying assumption of system reduction. The only theory to date with that attempts to solve the problem of dimensionality is the structure-preserving NF approach introduced by Martinez *et al.* [3],[14] In this work, a structure preserving model of the system is proposed based on the combined use of normal form theory and singular perturbation techniques.

The simplest NF of high-dimensional systems was introduced in the context of power system normal form analysis by Betancourt et al. [23]. More recently, Amano *et al.* proposed the development of nonlinear stability indexes of power oscillation modes using high-order normal form analysis [3].

These investigations are based on a recursive approach [14],[3] in which the normal form of a high-dimensional vector field is derived recursively, order by order. The procedure has been used in various disciplines, other than power systems, including fluid mechanics, process identification and control in chemical engineering and Neuroscience [5],[6].

There are several drawbacks to classical normal form analysis. The first is that, using recursive approaches, nonlinear interactions that need to be eliminated by successive transformations are nor properly modelled. Further problems with classical NF analysis arise in the computation of the simplest normal form representation for high-dimensional systems.

Chapter 3 discusses in detail this aspect of normal form analysis.

1.3.2. *Assessment of nonlinear mode interaction*

In the past decade studies on nonlinear mode interaction have received considerable attention [1],[2],[4] In particular a great deal of effort in the study of power system behavior has focused on the role of mode interaction on control performance and system nonlinearity. Results from this early work indicated that the inter-area mode phenomenon occurs as a result of nonlinear interaction between the dominant natural modes of oscillation of the power system [1]. Although evidence has emerged that this information can be used to provide estimates of nonlinearity, and the interactions of the fundamental modes of oscillation with other system modes, its effects remain largely unknown.

The role of nonlinear mode interaction was first described by Vittal *et al.* in his study of stressed power system behavior [1]. The results indicate that, nonlinear oscillations involving a large number of machines may occur as a result of

nonlinear interactions between the dominant natural modes of oscillation of the power system. To capture properly the influence of dynamic nonlinearities on modal interaction, analytical indices based on the closed-form system response have been developed.

Research on nonlinear mode interaction took a major step forward with the work of Yorino *et al.* [13] who characterized parametric resonance in simplified power system representations. Other examples of recent work on nonlinear mode interaction include the work of Messina *et al.* [24] and Xi *et al.* [25], among others. Parallel work Barocio and Messina explored the use of normal form theory to assess various aspects of nonlinear behavior and the location of system controllers [9-11], [23-24]. More recently, Liu *et al.* have extended power system normal form theory to locate and design power system controllers [12].

The key fact to us of importance here is that mode interaction may affect control performance and location.

1.3.3. *Near resonance conditions*

Evidence from analytical simulations has shown that linear and higher order resonances can affect normal form results and interpretation. Among the earliest studies of system nonlinear behavior were those of Yorino and other Japanese scholars who considered the possibility of auto-parametric resonance in power system behavior [13].

The concept of strong resonance was introduced in the concept of power system dynamic behavior by Dobson and Barocio [19], and Dobson [26]. Using symbolic examples, the authors showed that normal form coefficients and indices may become very large near a strong resonance in the system linearization. One of the interesting findings of this work is that the problem can be avoided if one chooses scaling appropriate to the physical phenomenon under study.

Other studies addressing the role of resonance conditions in system performance include the works of Barocio *et al.* on the analysis of factors affecting

normal form results and the study of strong resonance arising from linearization of a parameter-dependent state matrix [21],[27] and Liu *et al.* [12].

More recently, the author introduced a novel, efficient method to compute the system nonlinear representation of resonant vector fields that avoids the use of center manifold reduction and arbitrary scaling [16]. The work examines both the resonance and off-resonance behavior of complex systems using a non-recursive approach. The analysis approach provides significant advantages over existing formulations by providing a framework in which to treat linear and nonlinear resonances, and by incorporating real, nonlinear coordinate transformations. These works demonstrate and emphasize the need for higher-order nonlinear estimation of dynamics in physical space.

1.3.4. *Computation of the simplest normal form*

Recently, many researchers have paid attention to further reduction of conventional normal form [29]-[32]. In conventional normal form theory, the vector field is expanded into vector homogeneous polynomials up to the desired degree of approximation. Then, the normal form computation is carried out order by order. Conventional normal form (CNF) analysis approaches consider only simplified models and do no take account of higher-order effects of interest in the analysis of highly stressed systems.

For system in which the dynamic response can be approximated by a low-order system representation, classical normal form theory provides a good approach to system modeling. It is observed that the approximation of true system behavior by a low-order system model can introduce large inaccuracies into the solution process, depending upon the solution procedure.

For very large systems, determination of the simplest normal form has not been pursued, due to challenges in solving the constraint (network) equations. As pointed out by many researchers [4][23], calculation of high-dimensional normal form representation requires accurate modeling of neglected dynamics.

Efforts to improve normal form computations beyond the CNF analysis include step-by-step recursive approaches to normal form computation [14],and the computation of nonlinear stability indexes of power oscillation flow [3]. Unlike CNF analysis, the nonlinear terms arising in the nonlinear transformation are assumed to be small, but not negligible. While these approaches may be applied to medium size problems, the methods are very computationally intensive and their ability to produce highly stressed conditions has not been shown in the literature.

A noteworthy recent attempt to investigate the question of simplified normal form representations in the case of stressed power systems is that of Betancourt *et al.* [16] in which a real, one-step nonlinear transformation between the original system and the simplest NF is used. The method does not rely on center manifold reduction and is more direct and efficient that other existing formulations. The results are particularly interesting since information about resonances and bifurcations can be obtained directly in physical space.

This aspect of normal form computations has, to date, received little attention, and consequently the development of models which might be employed is at an early stage.

Various analytical issues remain to be resolved, however, especially concerning the need to preserve network structure [22]. As discussed in [23], general analytical results are not possible and even specific results are rare. A final issue arises in the extraction of the essential dynamics using indices or measures of mode interaction, and the assessment of network and load characteristics on system dynamics. Further, improved methods for nonlinear analysis of higher-order resonant systems are needed.

The work presented in this document is a step in this direction.

1.4 Objectives of this dissertation

This thesis is concerned with the development and generalization of existing normal form theory to compute the simplest normal form of resonant vector fields, which are described by nonlinear ordinary differential equations with the following objectives:

- To introduce a new analysis approach to compute the simplest normal form representation, which overcomes the limitations of existing analysis methods
- To explore the use of non-recursive technique to simplify the nonlinear structure of vector fields. A particular aspect of interest is the derivation of the simplest normal form for a given operating condition
- To assess of the effects of higher-order nonlinearities on system behavior and investigate errors arising from the various simplifying assumptions in simplified normal form representations.
- To develop a general framework for the study and characterization of system dynamic behavior in the vicinity of resonant conditions.

A further objective is the computation of real normal forms or nearly resonant power systems of interest in the study of voltage stability.

1.5 Contributions of this dissertation

The following aspects of his work constitute original contributions.

1. The development and evaluation of a new formulation, based on normal form theory, for the study and characterization of nonlinear oscillatory behavior in complex, high-dimensional systems.
2. The derivation of higher-order normal form representations for the analysis of stressed system performance.
3. The generalization of existing approaches to permit the study of both, resonant and non-resonant vector fields.

4. The establishment of a general framework for the derivation of real normal form representations of nonlinear systems

1.6 Outline of the dissertation

This document is organized as follow. Chapter 2 contains a brief introduction to the standard normal form theory for weakly nonlinear systems which are based on regular approaches. The review emphasizes the main limitations of present analysis approaches and motivates the need for improved system modeling. A novel technique based on the non-recursive approach, to compute the normal form and the associated nonlinear transformation is introduced. Attention is focused on the computation of residual terms generated by high-dimensional system representations. We close this chapter with a brief discussion of deterministic perturbations on the system.

Chapter 3 extends and generalizes the non-recursive approach to analyze resonant vector fields. Expressions for the determination of the normal form coefficients under non-resonant and resonant conditions are devised. Mechanisms responsible for the formation of linear resonances of parameter-dependent systems, and for the generation of nonlinear system characteristics are discussed. The procedure allows both, the nonlinear characterization of resonant vector fields as well as the identification of mechanisms responsible for nonlinear behavior.

Chapter 4 discusses the application of the proposed methods to the study of power system nonlinear behavior operating close to a resonant condition. In particular, a detailed bifurcation and stability analysis is given for the resonant case, which shows that conventional analysis techniques may fail to provide complete characterization of system behavior. We end the chapter with a critical interpretation of the results.

Chapter 5 provides a general discussion of the overall results and presents recommendations for future related areas of study. Finally, general conclusions in

Chapter 6 are followed by Appendix A, which provides some details about real normal form theory, and Appendix B, which provides data for the test systems.

1.7 Publications

1.7.1. *Papers in conference proceedings*

1. J. Arroyo, E. Barocio, **R.J. Betancourt**, A.R. Messina, "Quantifying nonlinearity in power systems using normal forms theory and higher-order statistics", *IEEE/PES General Meeting 2005*, St. Francisco, USA, June 22-25.

2. **R.J. Betancourt**, E. Barocio, J. Arroyo, A.R. Messina, "Higher-order normal forms analysis of stressed power systems: A non-recursive approach", *Power System Computations Conference 2005*, Liege, Belgium, August 16-20.

3. E. Barocio, **R.J. Betancourt**, J. Arroyo, A.R. Messina, "Assessing nonlinearity and non-stationary in power systems behaviour using higher order statistical analysis techniques", *Power System Computations Conference 2005*, Liege, Belgium, August 16-20.

4. J. Arroyo, E. Barocio, **R.J. Betancourt**, A.R Messina, "A Bilinear Method for Detection and Quantification of Nonlinear Modal Interactions in Power Systems" *IEEE/PES General Meeting 2006*, Montreal, Canada June 22-25.

5. **R.J. Betancourt**, E. Barocio, I. Martinez, A.R. Messina, "Use of Linear Normal Modes to Analyze Inter-Area Oscillations", *IEEE/PES Power Systems Conference and Exposition*, 2006, Georgia, Atlanta, USA, August 22-25.

1.7.2. *Refereed Journal Papers*

1. **R.J. Betancourt**, E. Barocio, A.R. Messina, "A Real Normal Form Approach to the Study of Resonant Power Systems", *IEEE Trans. on Power Systems*, vol. 21, no. 1, February 2006.

2. J. Arroyo, **R.J. Betancourt**, E. Barocio, A. R. Messina, "Development of bilinear power system representation for small-signal stability analysis", To appear in *Electric Power Systems Research*, 2007.

1.7.3. *Submitted papers*

1. **R.J. Betancourt**, E. Barocio, A.R. Messina, "Normal mode analysis of inter-area oscillations: A normal form approach" Submitted to Electric Power Systems Research, 2006.

2. **R.J. Betancourt**, E. Barocio, A.R. Messina, "A new approach to compute the simplest normal form of resonant vector fields " Submitted to IEEE Trans. on Circuits and Systems, 2007.

3. E. Barocio, A.R. Messina, **R. J. Betancourt** " Analysis of composite inter-area oscillations using the method of normal forms" Submitted to the Journal of the Franklin Institute, 2005.

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Chapter 2

The Method of Normal Forms for Vector Fields

The method of normal forms is one of the more useful and important techniques to analyze nonlinear dynamical systems near singularities. With this technique, it is possible to obtain the simplest form of a set of non-linear differential equations and, hence to study and identify the nature of local flows using standard linear analysis techniques.

The chapter contains two parts aimed at addressing each of these issues. The first part is a critical, focused survey, of the current state-of-the-arte approach in classical normal form theory and power system normal form computational capabilities. The review treats recursive algorithms, the role of diagonalization, the dynamical interpretation of normal form coefficients, and some relevant limitations.

The second part of this chapter introduces a new approach for normal form analysis of nonlinear systems described by differential equations, based on a modified Poincaré-Dulac normal form theory. A general nonlinear theory is presented for treating nonlinear vector fields and computing the normal forms of ordinary differential equations based on Kronecker algebra and perturbation theory, along with ideas of extension of the methodology to allow construction of the simplest possible normal form. Such an expansion simplifies the analysis of high dimensional nonlinear systems and can be used for studying internal resonances and dynamic bifurcations.

The present method is quite general and extends readily to high dimensional systems.

2.1 Introduction

Normal form theory is a powerful tool that allows the systematic simplification of nonlinear vector fields. The theory of normal forms plays an important role in the study of the dynamic behavior of nonlinear systems near equilibria because it greatly simplifies the analysis and formulations.

As outlined in the introductory Chapter, the underlying idea of normal form theory is to find an analytical nonlinear change of coordinates with the origin as a fixed point, such that the original vector field becomes simpler to study in terms of the new variables [1]; the simplified vector field is called the *normal form of the original system*. Several review articles describe and classify existing normal form approaches.

While the underlying idea is straightforward, the amount and complexity of the analyses required to determine the normal form increases rapidly with order. It recent years, a number of systematic approaches have been developed and applied to large-scale nonlinear models with limited success. However, a number of open issues remain with these methods, including the complexity of computations, the non-uniqueness of the normal form, and the computation of the simplest normal form.

In the following, we give a detailed overview of order-by-order normal form methodology with an emphasis on the development of higher order representations. Attention is also focused on identifying the limitations of exiting approaches in the context of practical system applications.

Drawing on the combined use of perturbation theory, and Kronecker forms, a systematic approach to compute high-dimensional normal forms is presented that circumvents some of the limitations of existing approaches. By extending the basic approach, we develop in this Chapter a more general approach for nonlinear systems whose nonlinearities are nor necessarily small and that can treat internal resonances.

2.2 Classical normal form theory

The method of NF enables to study the behavior of a vector field near a singularity by reducing it via a suitable change of coordinates to a simpler form. Existing approaches to higher normal form analysis are based on recursive computation of the NF coefficients and the associated nonlinear transformations. Conventional normal form (CNF) analysis uses the k-th order near identity transformation to remove the k-th order nonlinear terms. Given several simplifying assumptions, a straightforward technique can be used to estimate the normal form up to a desired order of accuracy.

The process usually includes three main steps [2],[3]:

(a). The system is expanded in power series about an equilibrium condition

(b). The resulting system is represented in the Jordan canonical form by a linear transformation of the original variables, and

(c). The system is represented in the normal form by a sequence of nonlinear coordinate transformations

Steps (a) through (c) are discussed in more detail below.

We begin with a short survey of present-day research.

2.2.1. *Description of the nonlinear model*

Consider a general, n-dimensional nonlinear dynamical system described by the nonlinear ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2.1}$$

where $\mathbf{x}$ is an n-dimensional vector of system states , the function $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ is a real smooth vector field satisfying $\mathbf{f}(0) = 0$.

Assuming without loss of generality that $\mathbf{x}(0) = \mathbf{x}^0$, represents an equilibrium point and expanding the vector field in a Taylor series about that point, we can write [3]

$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}) + \sum_{k=2}^{p} \mathbf{f}_k(\mathbf{x}) + O\left(\|\mathbf{x}\|^{p+1}\right) \tag{2.2}$$

where $\mathbf{f}_1(\mathbf{x}) = \mathbf{A}_1\mathbf{x}$ with $\mathbf{A}_1 = D\mathbf{f}(\mathbf{x})\big|_{\mathbf{x}^0}$ the Jacobian matrix of the system evaluated at the equilibrium condition, $\mathbf{f}_k(\mathbf{x})$, $k = 2,...,p$ contains the nonlinear part, p represents the desired order of approximation and $O\left(\|\mathbf{x}\|^{p+1}\right)$ are the terms in $\mathbf{x}$ of order $p+1$ and higher.

In (2.2), the $\mathbf{f}_k(\mathbf{x}) \in \mathbb{R}^n$, the real vector space of vector fields whose components are homogeneous polynomials of degree k given by

$$\mathbf{f}_k(\mathbf{x}) = \begin{bmatrix} f_k^1(\mathbf{x}) \\ f_k^2(\mathbf{x}) \\ \vdots \\ f_k^n(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \sum_{j1=1}^{n} \cdots \sum_{jk=1}^{n} F^1_{k_{j1}\cdots jk} x_{j1} \cdots x_{jk} \\ \sum_{j1=1}^{n} \cdots \sum_{jk=1}^{n} F^2_{k_{j1}\cdots jk} x_{j1} \cdots x_{jk} \\ \vdots \\ \sum_{j1=1}^{n} \cdots \sum_{jk=1}^{n} F^n_{k_{j1}\cdots jk} x_{j1} \cdots x_{jk} \end{bmatrix} \tag{2.3}$$

where

$$F^i_{2_{j1,j2}} = \left.\frac{\partial^2 f_2^i(\mathbf{x})}{\partial x_{j1}\partial x_{j2}}\right|_{\mathbf{x}^0}, \quad F^i_{3_{j1,j2,j3}} = \left.\frac{\partial^3 f_3^i(\mathbf{x})}{\partial x_{j1}\partial x_{j_2}\partial x_{j_3}}\right|_{\mathbf{x}^0} \cdots \text{ for } i = 1,2,\cdots,n.$$

In matrix form (2.2) may be rewritten as [5], [6]

$$\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \begin{bmatrix} \mathbf{x}^T\mathbf{H}_2^1\mathbf{x} \\ \mathbf{x}^T\mathbf{H}_2^2\mathbf{x} \\ \vdots \\ \mathbf{x}^T\mathbf{H}_2^n\mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{x}^T\mathbf{H}_3^1 \begin{bmatrix} \mathbf{x} & & \\ & & \mathbf{x} \end{bmatrix} \mathbf{x} \\ \mathbf{x}^T\mathbf{H}_3^2 \begin{bmatrix} \mathbf{x} & & \\ & & \mathbf{x} \end{bmatrix} \mathbf{x} \\ \vdots \\ \mathbf{x}^T\mathbf{H}_3^n \begin{bmatrix} \mathbf{x} & & \\ & & \mathbf{x} \end{bmatrix} \mathbf{x} \end{bmatrix} + \cdots \tag{2.4}$$

where $\mathbf{A}_1\mathbf{x} = \mathbf{f}_1(\mathbf{x})$ and the $\mathbf{H}_k^j$, $k = 1, \cdots, p$ are variational matrices associated with the series expansion (2.3). See Martinez et al. [4],[6], for more details of this approximation.

We remark that such an approach may be feasible for low-order systems but becomes prohibitive as the order of the approximations increases. Further, the special nature of this model makes it very difficult to find general analytical solutions.

Hence, a different kind of formulation is advisable as discussed in section 2.3.

2.2.2. *Transformation to the Jordan canonical form*

Consider the n dimensional vector field (2.2). Let $\mathbf{J}$ have an eigenvalue set $(\lambda_1\ \lambda_2 \cdots \lambda_n)$ with associated eigenvectors $\mathbf{U} = [\mathbf{u}_1\ \ \mathbf{u}_2\ \ \cdots\ \ \mathbf{u}_n]$, and reciprocal eigenvectors $\mathbf{V} = [\mathbf{v}_1\ \ \mathbf{v}_2\ \ \cdots\ \ \mathbf{v}_n]$. Then the linear change of coordinates $\mathbf{x} = \mathbf{U}\mathbf{y}$ transforms the system in (2.2) into its canonical form

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \sum_{k=2}^{p} \hat{\mathbf{f}}_k(\mathbf{y}) + O\left(\|\mathbf{y}\|^{p+1}\right) \tag{2.5}$$

where $\mathbf{y} \in \mathbb{C}^n$ the n-dimensional vector of Jordan coordinates, $\mathbf{J}$ is in the Jordan canonical form $\mathbf{J} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$, and the nonlinear terms can be represented in the form

$$\hat{\mathbf{f}}_k(\mathbf{y}) = \mathbf{U}^{-1}\mathbf{f}_k(\mathbf{U}\mathbf{y}) = \begin{bmatrix} \hat{f}_k^1(\mathbf{y}) \\ \hat{f}_k^2(\mathbf{y}) \\ \vdots \\ \hat{f}_k^n(\mathbf{y}) \end{bmatrix} = \begin{bmatrix} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \hat{F}^1_{k j_1 \cdots j_k} y_{j_1} \cdots y_{j_k} \\ \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \hat{F}^2_{k j_1 \cdots j_k} y_{j_1} \cdots y_{j_k} \\ \vdots \\ \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \hat{F}^n_{k j_1 \cdots j_k} y_{j_1} \cdots y_{j_k} \end{bmatrix}$$

The basic idea of normal forms theory is to performance a sequence of transformations of a given differential equation, each successive sequence simplifying the terms of the next degree, without affecting the simplified terms of lower degree so that the end result yields a transformed differential equation in a normal (simple) form which is easier to solve or analyze than the original.

2.2.3. *Recursive formulation*

The higher order normal form of a system of ordinary differential equations may be obtained by performing a sequence of nonlinear transformations, which eliminate or greatly simplify successively higher order terms in the Taylor expansion of the right hand side of the system, up to an arbitrarily high order. The idea is to simplify the higher order terms without changing the linear terms.

To make this statement precise we introduce the Poincaré-Dulac theorem.

Normal Form Theorem [1]. *For the system (2.5), given the sequence of decompositions*

$$\mathbf{z}_{m-1} = \varphi(\mathbf{z}_m) = \mathbf{z}_m + \hat{\mathbf{h}}_m(\mathbf{z}_m),\ \mathbf{z}_1 = \mathbf{y},\ m = 2,3,...,q \tag{2.6}$$

and any finite integer m, there exists a series of changes of coordinates and a neighborhood of the origin, such that for $\mathbf{z} \in \mathbb{C}^n$, the system is transformed to a system of the form

$$\dot{\mathbf{z}}_q = \mathbf{J}\mathbf{z}_q + \hat{\mathbf{g}}_2(\mathbf{z}_q) + \hat{\mathbf{g}}_3(\mathbf{z}_q) + ... + \hat{\mathbf{g}}_q(\mathbf{z}_q) + O(\|\mathbf{z}\|^{q+1}) \tag{2.7}$$

In the equations above, $\varphi(\mathbf{z}_m)$ is an invertible transformation, $\hat{\mathbf{h}}_m(\mathbf{z}_m) = [\hat{h}^1_m(\mathbf{z}_m) \quad \hat{h}^2_m(\mathbf{z}_m) \quad \cdots \quad \hat{h}^n_m(\mathbf{z}_m)]^T$ is a vector consisting of homogeneous mth-order polynomials whose coefficients are to be determined, and

$\hat{\mathbf{g}}_m(\mathbf{z}_m) = [\hat{g}^1_m(\mathbf{z}_m) \quad \hat{g}^2_m(\mathbf{z}_m) \quad \cdots \quad \hat{g}^n_m(\mathbf{z}_m)]^T$ is the vector of homogeneous mth-order polynomials whose coefficients are the resonant terms that can not be eliminated by the nonlinear transformation (2.6). Equation (2.7) is said to be in normal form. A proof of this theorem is given in [1], [3], and will not be repeated here.

Remark 1. *As long as the number of iterations of the change of coordinates is finite, the normal form (2.7) of (2.5) must be convergent on some neighborhood* $\mathbb{C}$ *of the origin. If the sequence of changes of coordinates is taken to infinity, the resulting sequence of transformations may not be convergent.*

In most cases, the value of the series (2.6) is not known, and the above simplification procedure is carried out order by order [1],[6].

Precise mathematical criteria for its computation are discussed below.

Differentiating (2.6) with respect to time, we obtain

$$\dot{\mathbf{y}} = D(\varphi(\mathbf{z}_m))\dot{\mathbf{z}}_m = \dot{\mathbf{z}}_m + \left(D\hat{\mathbf{h}}_m(\mathbf{z}_m)\right)\dot{\mathbf{z}}_m = \left[\mathbf{I} + D\hat{\mathbf{h}}_m(\mathbf{z}_m)\right]\dot{\mathbf{z}}_m \tag{2.8}$$

where $D\hat{\mathbf{h}}_m(\mathbf{z}_m)$ is the Jacobian of the nonlinear term $\hat{\mathbf{h}}_m(\mathbf{z}_m)$.

Substitution of (2.8) and (2.6) into (2.5) yields

$$\dot{\mathbf{z}}_m = [\mathbf{I} + D\hat{\mathbf{h}}_m(\mathbf{z}_m)]^{-1}\left[\mathbf{J}\left(\mathbf{z}_m + \hat{\mathbf{h}}_m(\mathbf{z}_m)\right) + \sum_{k=2}^{p}\hat{\mathbf{f}}_k\left(\mathbf{z}_m + \hat{\mathbf{h}}_m(\mathbf{z}_m)\right) + O\left(\left|\mathbf{z}_m + \hat{\mathbf{h}}_m(\mathbf{z}_m)\right|^{p+1}\right)\right] \tag{2.9}$$

where $\mathbf{I}$ denotes the nxn identity matrix. For $|\mathbf{z}_m|$ sufficiently small, $\left\|D\hat{\mathbf{h}}_m(\mathbf{z}_m)\right\|$ can be made arbitrarily small and thus $[\mathbf{I} + D\hat{\mathbf{h}}_m(\mathbf{z}_m)]^{-1}$ exists. More precisely, it then follows that the inverse of $D\varphi(\mathbf{z}_m)$ can be approximated by the binomial inverse theorem[7],

$$[\mathbf{I} + D\hat{\mathbf{h}}_m(\mathbf{z}_m)]^{-1} = \mathbf{I} - D\hat{\mathbf{h}}_m(\mathbf{z}_m) + O\left(\|\mathbf{z}_m\|^{m+1}\right) \quad \text{as} \quad |\mathbf{z}_m| \to 0 \tag{2.10}$$

Substitution of (2.10) into (2.9) results in

$$\dot{\mathbf{z}}_m = \mathbf{J}\mathbf{z}_m + \mathbf{J}\hat{\mathbf{h}}_m(\mathbf{z}_m) - \left(D\hat{\mathbf{h}}_m(\mathbf{z}_m)\right)\mathbf{J}\mathbf{z}_m + \tilde{\mathbf{f}}_m(\mathbf{z}_m) + O(\|\mathbf{z}_m\|^{m+1}) \tag{2.11}$$

Note that, for each $m \geq 2$, $\hat{\mathbf{h}}_m(\mathbf{z}_m)$ is arbitrary. Now we will choose a specific form for $\hat{\mathbf{h}}_m(\mathbf{z}_m)$ so as to simplify the m*th*-order terms as much as possible. This would mean choosing $\hat{\mathbf{h}}_m(\mathbf{z}_m)$ such that

$$\left(D\hat{\mathbf{h}}_m(\mathbf{z}_m)\right)\mathbf{J}\mathbf{z}_m - \mathbf{J}\hat{\mathbf{h}}_m(\mathbf{z}_m) = \tilde{\mathbf{f}}_m(\mathbf{z}_m) \tag{2.12}$$

This equation is the homological equation associated with the linear vector field $\mathbf{J}\mathbf{z}_m$. Equation (2.12) is a system of linear equation acting on a linear vector space [1]. The solution for $\hat{\mathbf{h}}_m$ is determined by defining the linear operator in an appropriate form.

In order to introduce these ideas, let us define the notion of the Lie bracket operation associated with (2.12).

Definition 2.1[3]. *Given two vector fields* $\mathbf{X}$ *and* $\mathbf{Y}$ *belonging to* $\mathbb{C}^n$*, the Lie-bracket operation on these two vectors is defined as*

$$[\mathbf{X},\mathbf{Y}] = (D_y\mathbf{Y})\mathbf{X} - \mathbf{Y}(D_x\mathbf{X}) \tag{2.13}$$

Using this definition, the homological equation (2.12) can be written as

$$L_J(\hat{\mathbf{h}}_m(\mathbf{z}_m)) = -\left[\hat{\mathbf{h}}_m(\mathbf{z}_m), \mathbf{J}\mathbf{z}_m\right] = \tilde{\mathbf{f}}_m(\mathbf{z}_m) \tag{2.14}$$

where $L_J(\cdot)$ is a linear operator acting over the linear vector space of homogeneous polynomials of m*th*-order [1].

Since L_J is a linear operator, then the solution to (2.14) can be determined in terms of the properties of a space complementary to L_J. Solving (2.14) is like solving the equation $\mathbf{B}\mathbf{w} = \mathbf{b}$ from linear algebra. More precisely, if $\tilde{\mathbf{f}}_m(\mathbf{z}_m)$ is in the range of $L_J(\cdot)$, then all the *m*th-order terms ($O(\|\mathbf{z}_m\|^m)$) can be eliminated from (2.11). In any case, we can choose $\hat{\mathbf{h}}_m(\mathbf{z}_m)$ so that only the *m*th-order terms that are

in a space complementary to $L_J(\cdot)$, remain. We denote these terms by

$$\hat{\mathbf{g}}_m(\mathbf{z}_m) \in \mathbf{Null}(L_J(\cdot))$$

Thus, (2.11) can be simplified to

$$\dot{\mathbf{z}}_m = \mathbf{J}\mathbf{z}_m + \hat{\mathbf{g}}_m(\mathbf{z}_m) + O(\|\mathbf{z}_m\|^{m+1}) \tag{2.15}$$

The demonstration of this assertion is discussed in detail in Arnold [1].

A case of special importance arises when the matrix $\mathbf{J}$ is in diagonal form, $\mathbf{J} = \mathbf{\Lambda} = diag[\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n]$. In this case, the Lie bracket equations (2.14) have a straightforward solution. It can be easily shown that these solutions are of the form [3]

$$\hat{h}^i_{m\,j_1,j_2,\cdots j_m} = \frac{\hat{F}^i_{m\,j_1,j_2,\cdots,j_m}}{\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_m} - \lambda_i}, \text{ for } (\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_m} - \lambda_i) \neq 0 \tag{2.16}$$

or

$$\begin{aligned} \hat{h}^i_{m\,j_1,j_2,\cdots j_m} &= 0 \\ \hat{g}^i_{m\,j_1,j_2,\cdots j_m} &= \hat{F}^i_{m\,j_1,j_2,\cdots j_m} \end{aligned} \quad \text{for } (\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_m} - \lambda_i) = 0 \tag{2.17}$$

In the non-resonant case, $(\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_m} - \lambda_i) \neq 0$, the computation of normal form coefficients is straightforward and is given by (2.16). This case corresponds to the simplest possible normal form (2.15). In the more general case in which matrix $\mathbf{J}$ is not diagonal, however, few results are known [9]. The study of such systems is postponed until Chapter 3.

The above approach results in time-consuming, recursive computations involving the application of a sequence of nonlinear transformations. A detailed analysis of the limitations of these approaches is given below.

2.2.4. *Ste-by-step nonlinear transformations*

Existing approaches to higher order normal form analysis are based on a recursive computation of the normal form coefficients and the associated normal form transformations. The derivation of higher order terms is described in [4],[8], and is summarized here for completeness.

In conventional normal form theory, mth-order nonlinear transformation terms are used to eliminate mth-order nonlinear terms of the system. We begin with second order $m = 2$.

More formally, let

$$\mathbf{y} = \varphi(\mathbf{z}_2) = \mathbf{z}_2 + \hat{\mathbf{h}}_2(\mathbf{z}_2) \tag{2.18}$$

denote a near identity coordinate change with $\hat{\mathbf{h}}_2(\mathbf{z}_2) = [\hat{h}_2^1(\mathbf{z}_2) \quad \hat{h}_2^2(\mathbf{z}_2) \quad \cdots \quad \hat{h}_2^n(\mathbf{z}_2)]^T$ consisting of homogeneous second order polynomials $\hat{h}_2^j(\mathbf{z}_2) = \sum_{j1=1}^{n} \sum_{j2=1}^{n} \hat{h}_{2\,j1,j2}^{j} z_{2\,j1} z_{2\,j2}$ whose coefficients $\hat{h}_{2\,j1,j2}^{j}$ are to be determined.

To the desired second order accuracy, the inverse transformation of $\varphi(\mathbf{z}_2)$ is [3]

$$\mathbf{z}_2 = \mathbf{y} - \hat{\mathbf{h}}_2(\mathbf{y}) + O\left(\|\mathbf{y}\|^3\right) \tag{2.19}$$

Then

$$D(\varphi(\mathbf{z}_2)) = \mathbf{I} + D(\hat{\mathbf{h}}_2(\mathbf{z}_2)) \tag{2.20}$$

where $D\hat{\mathbf{h}}_2(\mathbf{z}_2)$ is the Jacobian matrix of $\hat{\mathbf{h}}_2(\mathbf{z}_2)$.

For $|\mathbf{z}_2|$ sufficiently small, $\left\|D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right\|$ can be made arbitrarily small and thus $\left[\mathbf{I} + D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right]^{-1}$ exists. More generally, it then follows that the inverse of $D\varphi(\mathbf{z}_2)$ can

be approximated by

$$\left[\mathbf{I}+D\hat{\mathbf{h}}_m(\mathbf{z}_m)\right]^{-1}=\mathbf{I}-D\hat{\mathbf{h}}_m(\mathbf{z}_m)+\left(D\hat{\mathbf{h}}_m(\mathbf{z}_m)\right)^2-\left(D\hat{\mathbf{h}}_m(\mathbf{z}_m)\right)^3+\cdots+O\left(\|\mathbf{z}_m\|^{r+1}\right) \tag{2.21}$$

or

$$\left[\mathbf{I}+D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right]^{-1}=\mathbf{I}-D\hat{\mathbf{h}}_2(\mathbf{z}_2)+O\left(\|\mathbf{z}_2\|^{2m-2}\right)\quad\text{as}\quad|\mathbf{z}_2|\to 0 \tag{2.22}$$

where $O\left(\|\mathbf{z}_2\|^{2m-2}\right)$ denotes linear combinations of terms of degree $(2m-2)$ or higher, and r is the order of the approximation.[1]

Substitution of (2.18) and (2.22) into (2.9) results in

$$\begin{aligned}\dot{\mathbf{z}}_2=&\left(\mathbf{I}-D\hat{\mathbf{h}}_2(\mathbf{z}_2)+\left(D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)^2-\left(D\hat{\mathbf{h}}_k(\mathbf{z}_2)\right)^3+\cdots+O(|\mathbf{z}_2|^p)\right)\\&\left[\mathbf{J}\left(\mathbf{z}_2+\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)+\sum_{k=2}^{p}\hat{\mathbf{f}}_k\left(\mathbf{z}_2+\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)+O(\|\mathbf{z}_2+\hat{\mathbf{h}}_2(\mathbf{z}_2)\|^{p+1})\right]\end{aligned} \tag{2.23}$$

Consequently,

$$\dot{\mathbf{z}}_2=\mathbf{J}\mathbf{z}_2+\overbrace{\mathbf{J}\hat{\mathbf{h}}_2(\mathbf{z}_2)+\tilde{\mathbf{f}}_2(\mathbf{z}_2)-D\hat{\mathbf{h}}_2(\mathbf{z}_2)\mathbf{J}\mathbf{z}_2}^{Second\ \ Order\ \ Terms}+\tilde{\mathbf{f}}_3(\mathbf{z}_2)+\sum_{k=4}^{p}\tilde{\mathbf{f}}_k(\mathbf{z}_2)+O(\|\mathbf{z}_2\|^{p+1}) \tag{2.24}$$

where the $\tilde{\mathbf{f}}_k(\mathbf{z}_2)$ represent terms of order k which have been modified by the transformation (2.18) and the approximation (2.22). The first few terms are:

$$\begin{aligned}\tilde{\mathbf{f}}_3(\mathbf{z}_2)=&\hat{\mathbf{f}}_3(\mathbf{z}_2)-D\hat{\mathbf{h}}_2(\mathbf{z}_2)\mathbf{J}\hat{\mathbf{h}}_2(\mathbf{z}_2)+\left(D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)^2\mathbf{J}\mathbf{z}_2-D\hat{\mathbf{h}}_2(\mathbf{z}_2)\hat{\mathbf{f}}_2(\mathbf{z}_2)+\\&D\hat{\mathbf{f}}_2(\mathbf{z}_2)\hat{\mathbf{h}}_2(\mathbf{z}_2)\\\tilde{\mathbf{f}}_4(\mathbf{z}_2)=&\hat{\mathbf{f}}_4(\mathbf{z}_2)-D\hat{\mathbf{h}}_2(\mathbf{z})\hat{\mathbf{f}}_3(\mathbf{z}_2)-\left(D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)^3\mathbf{J}\mathbf{z}_2+\left(D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)^2\mathbf{J}\hat{\mathbf{h}}_2\mathbf{z}_2+\\&\left(D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)^2\hat{\mathbf{f}}_2(\mathbf{z}_2)-D\hat{\mathbf{h}}_2(\mathbf{z}_2)D\hat{\mathbf{f}}_2(\mathbf{z}_2)\hat{\mathbf{h}}_2(\mathbf{z}_2)\\&\vdots\end{aligned}$$

It then follows that if $\hat{\mathbf{h}}_2(\mathbf{z}_2)$ can be chosen so that the commutator or Lie

[1] Existing approaches are based on a first-order approximation ($r=1$) [2]

bracket $L_J(\hat{\mathbf{h}}_2(\mathbf{z}_2)) = \left[\hat{\mathbf{h}}_2(\mathbf{z}_2), \mathbf{J}\mathbf{z}_2\right] = \mathbf{J}\hat{\mathbf{h}}_2(\mathbf{z}_2) - \left(D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)\mathbf{J}\mathbf{z}_2$ cancels every term in $\hat{\mathbf{f}}_2(\mathbf{z}_2)$, all quadratic terms in (2.24) are transformed into cubic and higher order terms[2].

The system obtained from this procedure is called a *normal form.* In CNF theory a particular choice of the coordinate change $\hat{\mathbf{h}}_2(\mathbf{z}_2)$ can be made by setting resonant terms to zero ($\hat{\mathbf{g}}_2(\mathbf{z}_2) = 0$), such that

$$L_J(\hat{\mathbf{h}}_2(\mathbf{z}_2)) = \left[\hat{\mathbf{h}}_2(\mathbf{z}_2), \mathbf{J}\mathbf{z}_2\right] = \mathbf{J}\hat{\mathbf{h}}_2(\mathbf{z}_2) - D\hat{\mathbf{h}}_2(\mathbf{z}_2)\mathbf{J}\mathbf{z}_2 = -\hat{\mathbf{f}}_2(\mathbf{z}_2) \tag{2.25}$$

and thus remove all second order terms in (2.24). In CNF theory, the resonant terms $\hat{\mathbf{g}}_2(\mathbf{z}_2)$ are set to zero, which simplifies the normal form transformation. Under these circumstances, Eqn. (2.25) asserts that $\hat{\mathbf{f}}_2(z)$ satisfying $L_J\hat{\mathbf{h}}_2(\mathbf{z}_2) = -\hat{\mathbf{f}}_2(\mathbf{z}_2)$ can be found if the inverse operator L_J^{-1} exists. With this choice for $\hat{\mathbf{h}}_2(\mathbf{z}_2)$, all the second order terms of (2.24) will vanish and the approximation will be accurate to second order.

By accounting for the resonant terms (the undetermined coefficients), however, further reduction of the normal form is possible, leading to the simplest normal form. In other words, the undetermined coefficients can be used to further simplify the normal form. This aspect is discussed in detail in Chapter 3, where a general procedure is proposed to deal with both, the normal form coefficients and the resonant terms.

In this more general case in which resonant terms can not be eliminated, Equation (2.24) takes the form

$$\dot{\mathbf{z}}_2 = \mathbf{J}\mathbf{z}_2 + \hat{\mathbf{g}}_2(\mathbf{z}_2) + \tilde{\mathbf{f}}_3(\mathbf{z}_2) + O(\|\mathbf{z}_2\|^4) \tag{2.26}$$

---

[2] The quantity $\left[\hat{\mathbf{h}}_2(\mathbf{z}_2), \mathbf{J}\mathbf{z}_2\right] = (D(\mathbf{J}\mathbf{z}_2))\hat{\mathbf{h}}_2(\mathbf{z}_2) - (D(\hat{\mathbf{h}}_2(\mathbf{z}_2)))\mathbf{J}\mathbf{z}_2$ is known as the Lie or Poisson bracket of the vector fields $\mathbf{J}\mathbf{z}_2$ and $\hat{\mathbf{h}}_2(\mathbf{z}_2)$ [3].

where the $\hat{\mathbf{g}}_2(\mathbf{z}_2)$ are the terms that can not be eliminated by the second order nonlinear polynomial change of variables, and the terms $\tilde{\mathbf{f}}_3(\mathbf{z}_2)$ are third order terms.

Remark 2. *We note that the elimination of second order terms requires solution of the linear problem (2.25). The terms which cannot be eliminates, which are the resonant terms constitute the Poincaré-Dulac normal form and are related to the existence of some form of resonance between the eigenvalues of the linear system.*

Remark 3. *We note that the $\tilde{\mathbf{f}}_3(z)$ contain three components: the original terms in the Taylor series expansion in (2.5), the third order terms introduced by the nonlinear transformation (2.18), and the approximation of (2.22)* $(\hat{\mathbf{f}}_3(\mathbf{z}_2) - \left(D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)\mathbf{J}\hat{\mathbf{h}}_2(\mathbf{z}_2) + \left(D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)^2 \mathbf{J}\mathbf{z}_2 - \left(D\hat{\mathbf{h}}_2(\mathbf{z}_2)\right)\hat{\mathbf{f}}_2(\mathbf{z}_2) + \left(D\hat{\mathbf{f}}_2(\mathbf{z}_2)\right)\hat{\mathbf{h}}_2(\mathbf{z}_2))$. *In other words, the cubic terms depend on the quadratic part of the transformation $\hat{\mathbf{h}}_2(\mathbf{z}_2)$ as is typical in perturbation theory.*

We then proceed to the next order and seek to simplify the third order terms in (2.24) by introducing the near identity transformation $\mathbf{z}_2 = \mathbf{z}_3 + \hat{\mathbf{h}}_3(\mathbf{z}_3)$. Application of this transformation to (2.24) results in

$$\begin{aligned}\dot{\mathbf{z}}_3 = \Lambda\mathbf{z}_3 + \hat{\mathbf{g}}_2(\mathbf{z}_3 + \hat{\mathbf{h}}_3(\mathbf{z}_3)) + \overbrace{\mathbf{J}\left(\hat{\mathbf{h}}_3(\mathbf{z}_3)\right) + \tilde{\mathbf{f}}_3(\mathbf{z}_3) - \left(D\hat{\mathbf{h}}_3(\mathbf{z}_3)\right)\mathbf{J}\mathbf{z}_3}^{Third\ \ Order\ \ Terms} + \\ \tilde{\mathbf{f}}_4(\mathbf{z}_3 + \hat{\mathbf{h}}_3(\mathbf{z}_3)) + \sum_{k=5}^{p} \tilde{\mathbf{f}}_k(\mathbf{z}_3 + \hat{\mathbf{h}}_3(\mathbf{z}_3)) + O(\left\|\mathbf{z}_3 + \hat{\mathbf{h}}_3(\mathbf{z}_3)\right\|^{p+1})\end{aligned} \tag{2.27}$$

Clearly, we would now like to choose $\hat{\mathbf{h}}_3(\mathbf{z}_3)$ such that

$$\mathbf{J}\hat{\mathbf{h}}_3(\mathbf{z}_3) - \left(D\hat{\mathbf{h}}_3(\mathbf{z}_3)\right)\mathbf{J}\mathbf{z}_3 = -\tilde{\mathbf{f}}_3(\mathbf{z}_3) + \hat{\mathbf{g}}_3(\mathbf{z}_3) \tag{2.28}$$

The resulting system is

$$\dot{\mathbf{z}}_3 = \mathbf{J}\mathbf{z}_3 + \mathbf{g}_2(\mathbf{z}_3) + \mathbf{g}_3(\mathbf{z}_3) + \bar{\mathbf{f}}_4(\mathbf{z}_3) + \sum_{k=5}^{p} \bar{\mathbf{f}}_k(\mathbf{z}_3) + O(\left\|\mathbf{z}_3\right\|^{p+1}) \tag{2.29}$$

where

$$\bar{\mathbf{f}}_4(\mathbf{z}_3) = \tilde{\mathbf{f}}_4(\mathbf{z}_3) + \left(D\hat{\mathbf{h}}_3(\mathbf{z}_3)\right)^2 \mathbf{J}\mathbf{z}_3$$
$$\vdots$$

The key point to stress is that, at each order, a set of linear algebraic equations of the form

$$\left[\hat{\mathbf{h}}_k(\mathbf{z}_k), \mathbf{J}\mathbf{z}_k\right] = \mathbf{J}\hat{\mathbf{h}}_k(\mathbf{z}_k) - \left(D\hat{\mathbf{h}}_k(\mathbf{z}_k)\right)\mathbf{J}\mathbf{z}_k = -\hat{\mathbf{f}}_k(\mathbf{z}_k) \qquad k = 2,3,\cdots,q \tag{2.30}$$

needs to be solved for computing the normal form and the associated normal form coefficients. This set of equations will have a unique solution for the coefficient vectors $\hat{\mathbf{h}}_k$ in terms of the $\hat{\mathbf{f}}_k$, provided that the Lie bracket is invertible.

We must note that the procedure can continue up to the desired order p.

Remark 4. *At each application of a transformation of the form* $\mathbf{z}_k = \mathbf{z}_{k+1} + \hat{\mathbf{h}}_{k+1}(\mathbf{z}_{k+1})$, *the* $k+1$ *terms are changed but not the lower order terms.*

Figure 2.1 gives an overview of the solution procedure.

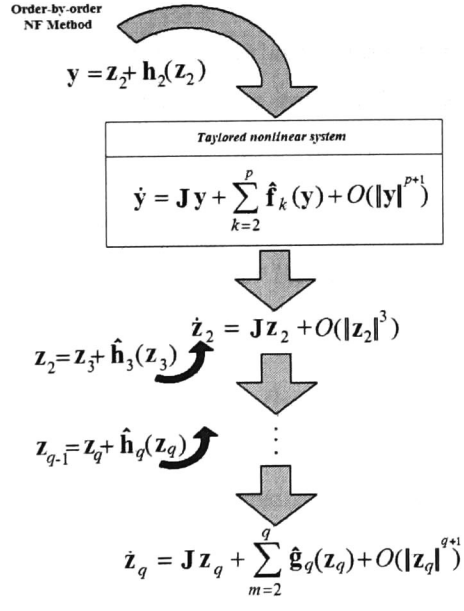

Figure 2.1. The order-by-order normal form computation

2.2.5. *Error considerations*

There are three main sources of computational error associated with CNF theory:

(a). *Errors due to truncation in the numbers of terms of the inverse matrix* (2.10). The size of this error is dependent on both, the order of the approximation r, and amplification errors due to the sequential process

(b). *Amplification of accumulative errors due to unaccounted terms in recursive solutions.* In the conventional approach discussed in section 2.2.4, these terms are not fully accounted for in successive computations

(c). *Residual errors arising from neglected resonant terms.*

These observations are substantially confirmed in our numerical results in Chapter 4.

We conclude this section with a few comments on CNF theory.

Comment 1. *The NF theorem determines how much a vector field can be simplified, depending uniquely on its linear part. As discussed in Chapter 3, however, taking into account the nonlinear terms, it is possible to obtain further simplifications in the classical normal form. This leads to the notion of the simplest possible normal form.*

Comment 2. *The coordinate transformations are nonlinear functions of the dependent variables, but these coordinate transformations are found by solving a sequence of linear problems.*

Comment 3. *It is difficult to determine the exact order at which to truncate (2.2) and (2.10). Further, finding the explicit formulas for the normal form coefficients in terms of the original nonlinear system is difficult and time consuming.*

Furthermore, classical normal form theory is not adequate when the linearized system has Jordan blocks of different sizes with the same eigenvalue. Also, the normal forms for complex high-dimensional systems and systems with inner resonances are very hard to obtain by conventional normal form theories. This provides motivation for more advanced formulations that address these issues.

In an effort to reduce or eliminate these limitations, several analytical methods have been proposed over the years to account for neglected terms in the above formulation [9],[11].

The next section introduces a unified framework for performing local analysis of nonlinear power systems based on Kronecker products of matrices. Complete expressions are derived for calculating the leading-order terms in the normal form, using the concept of Kronecker forms and perturbation theory. This approach allows the study of high-dimensional systems and can be used to study various aspects of nonlinear behavior such as internal resonances, bifurcations, and the simplest normal form.

2.3 General framework for high-dimensional normal forms

This section generalizes the theory of normal forms to higher dimensional systems. An innovative approach, based on Kronecker forms, to characterize high-dimensional system is proposed. The method extends existent approaches [5] to higher dimensions resulting in a systematic technique for the study of complex systems.

2.3.1. *Mathematical formulation*

Consider an n-dimensional nonlinear system (2.1). We consider a consistent approximation up to and including a given order p and write the vector field in a perturbation series in $\mathbf{x}$:

$$\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x}^{[1]} + \sum_{k=2}^{p} \mathbf{A}_k\mathbf{x}^{[k]} + O(\|\mathbf{x}\|^{[p+1]}) \tag{2.31}$$

where, in terms of Kronecker forms

$$\mathbf{x}^{[1]} = \mathbf{x}$$

$$\mathbf{x}^{[2]} = \mathbf{x} \otimes \mathbf{x} = \begin{bmatrix} x_1\mathbf{x} \\ x_2\mathbf{x} \\ \vdots \\ x_n\mathbf{x} \end{bmatrix} = \begin{bmatrix} x_1x_1 & \cdots & x_1x_n & x_2x_1 & \cdots & x_2x_n & \cdots & x_nx_1 & \cdots & x_nx_n \end{bmatrix}^T$$

$$\mathbf{x}^{[3]} = \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}$$
$$\vdots$$

in which the symbol $\otimes$ represents the Kronecker product of two matrices.

In the equations above, $\mathbf{A}_1 \in \mathbb{R}^{nxn}$ is the Jacobian of $\mathbf{f}$, the $\mathbf{A}_k \in \mathbb{R}^{nxn^k}$, $k = 1, \cdots, p$ are the kth-order derivative matrices of $\mathbf{f}$, and $\mathbf{x}^{[k]} = \underbrace{\mathbf{x} \otimes \cdots \otimes \mathbf{x}}_{k-times}$, and $\mathbf{A}_k \in \mathbb{R}^{nxn^k}$ represents a matrix whose rows are obtained stacking the columns of $\mathbf{H}_k^j$ for $j = 1,2,\cdots,n$. We point out that (2.31) is the generalization to p-th order of (2.4) which forms the basis of the proposed formulation.

2.3.2. *Vectorized operations and Kronecker forms*

Before presenting details of the proposed model, we review some basic principles concerning Kronecker forms that will be especially important in our use of normal form theory.

2.3.2.1. Approximation by a Kronecker sum

Definition 2.2. *Consider two square matrices $\mathbf{M}_{m*m}$ and $\mathbf{N}_{n*n}$, of dimensions mxm and nxn, respectively. Then the Kronecker sum between M and N denoted $\mathbf{M} \oplus \mathbf{N}$ is defined as [10]*

$$\mathbf{M} \oplus \mathbf{N} = \mathbf{M} \otimes \mathbf{I}_n + \mathbf{I}_m \otimes \mathbf{N} \tag{2.32}$$

where $\mathbf{I}_n$ and $\mathbf{I}_m$ are identity matrices of dimension n and m, respectively.

Definition 2.3. *The "vec" operation transforms a matrix* $\mathbf{M} = [M_1 \quad M_2 \quad \cdots \quad M_n]$ *into a column vector by stacking the columns of the matrix below each other. Thus, with the matrix* $\mathbf{M}_{m*n}$, *the* $(n*m)$ *vector is defined as*

$$vec(\mathbf{M}) = \mathbf{M}_S = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix} \tag{2.33}$$

where the subscript " s "denotes stacking the columns of the matrix.

Some useful properties that relate the *"vec"* operation and the Kronecker product are:

1) $vec(\mathbf{MNP}) = (\mathbf{P}^T \otimes \mathbf{M})\mathbf{N}_S$

2) $vec(\mathbf{MN}) = vec(\mathbf{MNI}) = (\mathbf{I} \otimes \mathbf{M})\mathbf{N}_S$

3) $\mathbf{x} \otimes \mathbf{y} = vec(\mathbf{y}\mathbf{x}^T)$

4) $vec(\mathbf{x}) = \mathbf{x}$

In terms of the notation above, it can be seen directly that the state representation (2.4) can be written in alternate form as

$$\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \begin{bmatrix} vec(\mathbf{x}^T\mathbf{H}_2^1\mathbf{x}) \\ vec(\mathbf{x}^T\mathbf{H}_2^2\mathbf{x}) \\ \vdots \\ vec(\mathbf{x}^T\mathbf{H}_2^n\mathbf{x}) \end{bmatrix} + \begin{bmatrix} vec\left(\mathbf{x}^T\mathbf{H}_3^1(\mathbf{I} \otimes \mathbf{x})\mathbf{x}\right) \\ vec\left(\mathbf{x}^T\mathbf{H}_3^2(\mathbf{I} \otimes \mathbf{x})\mathbf{x}\right) \\ \vdots \\ vec\left(\mathbf{x}^T\mathbf{H}_3^n(\mathbf{I} \otimes \mathbf{x})\mathbf{x}\right) \end{bmatrix} + \cdots \tag{2.34}$$

where $\mathbf{I} \otimes \mathbf{x} = \begin{bmatrix} \mathbf{x} & & \\ & & \\ & & \mathbf{x} \end{bmatrix}$.

Further, noting that

$$
\begin{aligned}
vec(\mathbf{x}^T \mathbf{H}_2^j \mathbf{x}) &= \left(\mathbf{x}^T \otimes \mathbf{x}^T\right)\mathbf{H}_{2S}^j \\
&= \left(\mathbf{H}_{2S}^j\right)^T (\mathbf{x} \otimes \mathbf{x})
\end{aligned}
$$

$$
\begin{aligned}
vec\left(\mathbf{x}^T \mathbf{H}_3^j (\mathbf{I} \otimes \mathbf{x})\mathbf{x}\right) &= ((\mathbf{I} \otimes \mathbf{x})\mathbf{x})^T \otimes \mathbf{x}^T \mathbf{H}_{3S}^j \\
&= \left(\mathbf{x}^T \otimes \mathbf{x}^T \otimes \mathbf{x}^T\right)\mathbf{H}_{3S}^j \\
&= \left(\mathbf{H}_{3S}^j\right)^T (\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x})
\end{aligned}
$$

$$\vdots$$

$$
vec\left(\mathbf{x}^T \mathbf{H}_k^j (\mathbf{I} \otimes \cdots \otimes \mathbf{x})\mathbf{x}\right) = \left(\mathbf{H}_{kS}^j\right)^T \left(\mathbf{x}^{[k]}\right)
$$

$$\vdots$$

the terms of the series expansion in (2.34) can be rewritten as

$$
\begin{aligned}
\mathbf{A}_2 \mathbf{x}^{[2]} &= \begin{bmatrix} \left(vec(\mathbf{H}_2^1)\right)^T \mathbf{x} \otimes \mathbf{x} \\ \vdots \\ \left(vec(\mathbf{H}_2^n)\right)^T \mathbf{x} \otimes \mathbf{x} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{H}_{2S}^1\right)^T \\ \vdots \\ \left(\mathbf{H}_{2S}^n\right)^T \end{bmatrix} \mathbf{x} \otimes \mathbf{x} \\
\mathbf{A}_3 \mathbf{x}^{[3]} &= \begin{bmatrix} \left(vec(\mathbf{H}_3^1)\right)^T \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} \\ \vdots \\ \left(vec(\mathbf{H}_3^n)\right)^T \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{H}_{3S}^1\right)^T \\ \vdots \\ \left(\mathbf{H}_{3S}^n\right)^T \end{bmatrix} \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} \\
&\vdots
\end{aligned}
\tag{2.35}
$$

2.3.2.2. The diagonal case

Similar to the previous developments in CNF theory, it can be shown that (2.31) may be transformed via the linear change of coordinates $\mathbf{x} = \mathbf{U}\mathbf{y}$ to

$$
\dot{\mathbf{y}} = \Lambda \mathbf{y} + \mathbf{V} \sum_{k=2}^{p} \mathbf{A}_k (\mathbf{U}\mathbf{y})^{[k]} + O(\|\mathbf{U}\mathbf{y}\|^{[p+1]}) \tag{2.36}
$$

Using the Kronecker property $[\mathbf{N} \otimes \mathbf{M}][\mathbf{P} \otimes \mathbf{Q}] = \mathbf{NP} \otimes \mathbf{MQ}$, the system (2.36) can be rewritten as

$$\dot{\mathbf{y}} = \mathbf{\Lambda}\mathbf{y} + \sum_{k=2}^{p} \mathbf{C}_k \mathbf{y}^{[k]} + O(\|\mathbf{y}\|^{[p+1]}) \tag{2.37}$$

where $\mathbf{C}_k = \mathbf{V}\mathbf{A}_k\mathbf{U}^{[k]} = \mathbf{V}\mathbf{A}_k \underbrace{\mathbf{U}\otimes\cdots\otimes\mathbf{U}}_{k-times} \in \mathbb{C}^{n\times n^k}$ $\mathbf{\Lambda} = \mathbf{U}^{-1}\mathbf{A}_1\mathbf{U}$ is the diagonal matrix of system eigenvalues; $O(\|\mathbf{y}\|^{[p+1]})$ are the terms in $\mathbf{y}$ of order higher to $p+1$.

Another problem associated with the use of modal decomposition is the computation of matrices $\mathbf{C}_k$, since the determination of $\underbrace{\mathbf{U}\otimes\cdots\otimes\mathbf{U}}_{k-times}$ demands a great computational effort. The matrices $\mathbf{C}_k$ are full and it is therefore required that the identical monomials be reduced.

2.3.3. *Simplifying the second order nonlinearities*

Following the procedure described in section 2.2, we use the m th nonlinear transformations to remove m th-order nonlinear terms. Starting with $m = 2$, consider a formal nonlinear near-identity coordinate change of the form

$$\mathbf{y} = \mathbf{z}_2 + \mathbf{h}_2(\mathbf{z}_2 \otimes \mathbf{z}_2) = \mathbf{z}_2 + \mathbf{h}_2\mathbf{z}_2^{[2]} \tag{2.38}$$

where $\mathbf{z}_2$ is the vector of second order normal form coordinates and $\mathbf{h}_2 \in \mathbb{C}^{n\times n^2}$ complex valued matrix with coefficients to be determined so that the system in (2.37) not have second order nonlinearities.

Hence, insertion of (2.38) into (2.37), yields

$$\left[\mathbf{I} + \mathbf{h}_2 D_z(\mathbf{z}_2^{[2]})\right]\dot{\mathbf{z}}_2 = \mathbf{\Lambda}\mathbf{z}_2 + \mathbf{\Lambda}\mathbf{h}_2\mathbf{z}_2^{[2]} + \sum_{k=2}^{p} \mathbf{C}_k(\mathbf{z}_2 + \mathbf{h}_2\mathbf{z}_2^{[2]})^{[k]} + O(\|\mathbf{z}_2 + \mathbf{h}_2\mathbf{z}_2^{[2]}\|^{[p+1]}) \tag{2.39}$$

Now, applying the Kronecker rule for differentiation of vector products, $\frac{\partial(\mathbf{z}\otimes\mathbf{z})}{\partial\mathbf{z}} = (\mathbf{I}\otimes\mathbf{z} + \mathbf{z}\otimes\mathbf{I})\dot{\mathbf{z}}$ [10], the normal form system can be recast in the form

$$\dot{\mathbf{z}}_2 = \left[\mathbf{I} + \mathbf{h}_2(\mathbf{z}_2 \oplus \mathbf{z}_2)\right]^{-1}\Big[\Lambda\mathbf{z}_2 + \Lambda\mathbf{h}_2\mathbf{z}_2^{[2]} + \sum_{k=2}^{p}\mathbf{C}_k(\mathbf{z}_2 + \mathbf{h}_2\mathbf{z}_2^{[2]})^{[k]} + O(\left\|\mathbf{z}_2 + \mathbf{h}_2\mathbf{z}_2^{[2]}\right\|^{[p+1]})\Big] \tag{2.40}$$

where $\mathbf{z}_2 \oplus \mathbf{z}_2$ denotes the Kronecker sum .

To allow comparison with CNF theory, we expand the inverse matrix $\left[\mathbf{I} + \mathbf{h}_2(\mathbf{z}_2 \oplus \mathbf{z}_2)\right]^{-1}$ in an infinite series as [7]

$$\left[\mathbf{I} + \mathbf{h}_2(\mathbf{z}_2 \oplus \mathbf{z}_2)\right]^{-1} \approx \mathbf{I} + \sum_{j=1}^{\infty}(-1)^j\left(\mathbf{h}_2(\mathbf{z}_2 \oplus \mathbf{z}_2)\right)^j \tag{2.41}$$

The use of more general formulations that avoid the use of simplifying assumptions is discussed in Chapter 3 of this document.

Introducing (2.41) into (2.40), we can write

$$\dot{\mathbf{z}}_2 = \left[\mathbf{I} + \sum_{j=1}^{\infty}(-1)^j\left(\mathbf{h}_2(\mathbf{z}_2 \oplus \mathbf{z}_2)\right)^j\right]\Big[\Lambda\mathbf{z}_2 + \Lambda\mathbf{h}_2\mathbf{z}_2^{[2]} + \sum_{k=2}^{p}\mathbf{C}_k(\mathbf{z}_2 + \mathbf{h}_2\mathbf{z}_2^{[2]})^{[k]} + O(\left\|\mathbf{z}_2 + \mathbf{h}_2\mathbf{z}_2^{[2]}\right\|^{[p+1]})\Big] \tag{2.42}$$

Carrying out the indicated operations, equating terms of like power, and retaining terms up to order p, it is obtained that

$$\begin{aligned}\dot{\mathbf{z}}_2 = {} & \Lambda\mathbf{z}_2 + \Lambda\mathbf{h}_2\mathbf{z}_2^{[2]} - \mathbf{h}_2(\mathbf{z}_2 \oplus \mathbf{z}_2)\Lambda\mathbf{z}_2 + \mathbf{C}_2\mathbf{z}_2^{[2]} + \\ & \sum_{k=3}^{p}\tilde{\mathbf{C}}_k\mathbf{z}_2^{[k]} + \sum_{j=2}^{p-1}(-1)^j\left(\mathbf{h}_2(\mathbf{z}_2 \oplus \mathbf{z}_2)\right)^j\Lambda\mathbf{z}_2 + \\ & \sum_{j=1}^{p-2}(-1)^j\left(\mathbf{h}_2(\mathbf{z}_2 \oplus \mathbf{z}_2)\right)^j\Lambda\mathbf{h}_2\mathbf{z}_2^{[2]} + \\ & \sum_{j=1}^{p-2}\sum_{k=2}^{p-j}(-1)^j\left(\mathbf{h}_2(\mathbf{z}_2 \oplus \mathbf{z}_2)\right)^j\tilde{\mathbf{C}}_k\mathbf{z}_2^{[k]} + O(\left\|\mathbf{z}_2\right\|^{[p+1]})\end{aligned} \tag{2.43}$$

In order to emphasize these limitations and to have a comparison basis, we consider the case in which these expressions are derived up to 3rd order.

It follows that the, above expression can be rewritten in synthesized form as

$$\dot{\mathbf{z}}_2 = \mathbf{\Lambda z}_2 + \overbrace{\mathbf{\Lambda h}_2\mathbf{z}_2^{[2]} - \mathbf{h}_2(\mathbf{z}_2 \oplus \mathbf{z}_2)\mathbf{\Lambda z}_2 + \mathbf{C}_2\mathbf{z}_2^{[2]}}^{second\ order\ terms} + \widetilde{\mathbf{C}}_3\mathbf{z}_2^{[3]} + \sum_{k=4}^{q} \widetilde{\mathbf{C}}_k\mathbf{z}_2^{[k]} + O(\|\mathbf{z}_2\|^{[p+1]}) \quad (2.44)$$

where, in view of the above transformations,

$$\widetilde{\mathbf{C}}_3 = \mathbf{C}_3 + \mathbf{C}_2(\mathbf{I}\otimes\mathbf{h}_2 + \mathbf{h}_2\otimes\mathbf{I}) + \mathbf{\Gamma}_3 + \mathbf{\Psi}_3 + \mathbf{\Omega}_3 \quad (2.45)$$

$$\vdots$$

with

$$\mathbf{\Gamma}_3 = \mathbf{h}_2\left[\mathbf{h}_2\otimes\mathbf{I}(\mathbf{\Lambda}\otimes\mathbf{I}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{\Lambda}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{h}_2(\mathbf{I}\otimes\mathbf{\Lambda}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{I}\otimes\mathbf{\Lambda})\right] \quad (2.46a)$$

$$\vdots$$

$$\mathbf{\Psi}_3 = -\mathbf{h}_2(\mathbf{\Lambda h}_2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{\Lambda h}_2) \quad (2.46b)$$

$$\vdots$$

$$\mathbf{\Omega}_3 = -\mathbf{h}_2(\mathbf{C}_2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{C}_2) \quad (2.46c)$$

$$\vdots$$

We emphasize that the above terms arise due to change of coordinates (2.38) and the use of (2.41).

2.3.4. *The homological equation*

The adopted approach provides a convenient form to interpret and solve the homological equation in terms of the eigenvalues of the linear system. From (2.44), it is found after a little algebra that all second order terms are annihilated when

$$\mathbf{\Lambda h}_2\mathbf{z}_2^{[2]} + \mathbf{C}_2\mathbf{z}_2^{[2]} - \mathbf{h}_2(\mathbf{I}\otimes\mathbf{z}_2 + \mathbf{z}_2\otimes\mathbf{I})\mathbf{\Lambda z}_2 = 0 \quad (2.47)$$

Moreover, by analogy with (2.25), we can write

$$-L_{\Lambda}(\mathbf{h}_2\mathbf{z}_2^{[2]}) = -\left[\mathbf{h}_2\mathbf{z}_2^{[2]}, \mathbf{\Lambda z}_2\right] = \mathbf{C}_2\mathbf{z}_2^{[2]} \quad (2.48)$$

where $[\,.\,]$ denotes the Lie bracket operator.

Referring to the Kronecker property

$$[\mathbf{N}\otimes\mathbf{M}][\mathbf{P}\otimes\mathbf{Q}]=\mathbf{NP}\otimes\mathbf{MQ}$$

one thus has that (2.48) is transformed into

$$\mathbf{h}_2[\mathbf{I}\otimes\boldsymbol{\Lambda}+\boldsymbol{\Lambda}\otimes\mathbf{I}]-\boldsymbol{\Lambda}\mathbf{h}_2=\mathbf{C}_2 \tag{2.49}$$

where the term $\mathbf{z}_2^{[2]}$ has been dropped for simplicity.

Equation (2.49) represents a set of linear algebraic equations which must be solved for $\mathbf{h}_2$. It should be emphasized that this equation has the structure of a Sylvester equation and may exhibit, under certain circumstances, multiple solutions, as discussed below.

2.3.4.1. Solution of the homological equation

In a variation to the present approach, the system (2.49) can be solved in closed form using the "*vec*" operation [12]. The corresponding expression for the homological equation is

$$vec(\mathbf{h}_2(\mathbf{I}\otimes\boldsymbol{\Lambda}+\boldsymbol{\Lambda}\otimes\mathbf{I})-\boldsymbol{\Lambda}\mathbf{h}_2)=vec(\mathbf{C}_2) \tag{2.50}$$

from which it follows immediately that

$$[\mathbf{I}\otimes(\mathbf{I}\otimes\boldsymbol{\Lambda}+\boldsymbol{\Lambda}\otimes\mathbf{I})-\boldsymbol{\Lambda}\otimes\mathbf{I}\otimes\mathbf{I}]\mathbf{h}_{2S}^T=\mathbf{C}_{2S}^T \tag{2.51}$$

where the subscript "s" is equivalent to the *vec* operation. Solution is straightforward, since for completely diagonal matrices, the solution is obtained by inverting each diagonal term. In order to obtain a general derivation for the elements of the transformation, we introduce the following definitions [12].

Definition 2.4. *The eigenvalues of the Kronecter product of two matrices,* $\mathbf{M}\otimes\mathbf{N}$, *are the* mn *numbers* $\lambda_k\lambda_l$, $k=1,2,\cdots,m$, $l=1,2,\cdots,n$.

Definition 2.5. *The eigenvalues of* $\mathbf{I}_m\otimes\mathbf{M}+\mathbf{N}\otimes\mathbf{I}_n$ *are the* mn *numbers* $\lambda_k+\lambda_l$, $k=1,2,\cdots,m$, $l=1,2,\cdots,n$

The linear system of (2.51) is block diagonal with n sub-blocks of dimension

$n^2 x n^2$ having the next structure

$$\mathbf{I}\otimes(\mathbf{I}\otimes\mathbf{\Lambda}+\mathbf{\Lambda}\otimes\mathbf{I})-\mathbf{\Lambda}\otimes\mathbf{I}\otimes\mathbf{I}=\begin{bmatrix}\mathbf{\Pi}_1 & & & \\ & \mathbf{\Pi}_2 & & \\ & & & \\ & & & \mathbf{\Pi}_n\end{bmatrix} \tag{2.52}$$

Concentrating on a diagonal term on the right hand side of (2.52), we observe that

$$\mathbf{\Pi}_j=(\mathbf{I}\otimes\mathbf{\Lambda}+\mathbf{\Lambda}\otimes\mathbf{I})-\lambda_j\mathbf{I}^{[2]}=\begin{bmatrix}\mathbf{\Lambda}+(\lambda_1-\lambda_j)\mathbf{I} & & & \\ & \mathbf{\Lambda}+(\lambda_2-\lambda_j)\mathbf{I} & & \\ & & \ddots & \\ & & & \mathbf{\Lambda}+(\lambda_n-\lambda_j)\mathbf{I}\end{bmatrix} \tag{2.53}$$

Therefore, closed form solution of the linear system can be determined by inverting (2.51) through the relation

$$\begin{bmatrix}\mathbf{h}_{2_1}{}^T\\ \mathbf{h}_{2_2}{}^T\\ \\ \mathbf{h}_{2_n}{}^T\end{bmatrix}=\begin{bmatrix}\mathbf{\Pi}_1 & & & \\ & \mathbf{\Pi}_2 & & \\ & & \ddots & \\ & & & \mathbf{\Pi}_n\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{C}_{2_1}{}^T\\ \mathbf{C}_{2_2}{}^T\\ \vdots\\ \mathbf{C}_{2_n}{}^T\end{bmatrix}=\begin{bmatrix}\mathbf{\Pi}_1^{-1}\mathbf{C}_{2_1}{}^T\\ \mathbf{\Pi}_2^{-1}\mathbf{C}_{2_2}{}^T\\ \vdots\\ \mathbf{\Pi}_n^{-1}\mathbf{C}_{2_n}{}^T\end{bmatrix} \tag{2.54}$$

where $\mathbf{h}_{2_j}{}^T$ and $\mathbf{C}_{2_j}{}^T$ denote the transposed of the jth row of $\mathbf{h}_2$ and $\mathbf{C}_2$ respectively. It is immediately seen from the relation above that

$$\mathbf{h}_{2_j}=\mathbf{C}_{2_j}\mathbf{\Pi}_j^{-1}=\mathbf{C}_{2_j}\begin{bmatrix}\left[\mathbf{\Lambda}+(\lambda_1-\lambda_j)\mathbf{I}\right]^{-1} & & & \\ & \left[\mathbf{\Lambda}+(\lambda_2-\lambda_j)\mathbf{I}\right]^{-1} & & \\ & & \ddots & \\ & & & \left[\mathbf{\Lambda}+(\lambda_n-\lambda_j)\mathbf{I}\right]^{-1}\end{bmatrix} \tag{2.55}$$

Equation (2.55) has a unique solution when the eigenvalues of matrix $\left[\mathbf{I}\otimes\mathbf{\Lambda}+\mathbf{\Lambda}\otimes\mathbf{I}\right]$ are not equal to the j-th eigenvalue of $\mathbf{\Lambda}$, i.e.

$$\lambda_k + \lambda_l \neq \lambda_j \qquad k = 1,2,...,n \qquad l = 1,2,...,n \qquad j = 1,2,...,n \tag{2.56}$$

Solving (2.56), for $\mathbf{h}_2$ yields

$$\mathbf{h}_2 = \begin{bmatrix} \frac{C_{2_{1,1}}}{\lambda_1+\lambda_1-\lambda_1} & \cdots & \frac{C_{2_{1,n}}}{\lambda_1+\lambda_n-\lambda_1} & \cdots & \frac{C_{2_{1,n^2-n+1}}}{\lambda_n+\lambda_1-\lambda_1} & \cdots & \frac{C_{2_{1,n^2}}}{\lambda_n+\lambda_n-\lambda_1} \\ \frac{C_{2_{2,1}}}{\lambda_1+\lambda_1-\lambda_2} & \cdots & \frac{C_{2_{2,n}}}{\lambda_1+\lambda_n-\lambda_2} & \cdots & \frac{C_{2_{2,n^2-n+1}}}{\lambda_n+\lambda_1-\lambda_2} & \cdots & \frac{C_{2_{2,n^2}}}{\lambda_n+\lambda_n-\lambda_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{C_{2_{n,1}}}{\lambda_1+\lambda_1-\lambda_n} & \cdots & \frac{C_{2_{n,n}}}{\lambda_1+\lambda_n-\lambda_n} & \cdots & \frac{C_{2_{n,n^2-n+1}}}{\lambda_n+\lambda_1-\lambda_n} & \cdots & \frac{C_{2_{n,n^2}}}{\lambda_n+\lambda_n-\lambda_n} \end{bmatrix} \tag{2.57}$$

As expected, this solution is agreement with that obtained using CNF analysis in section 2.2 [5].

Thus, the second order normal form is

$$\dot{\mathbf{z}}_2 = \Lambda \mathbf{z}_2 + \mathbf{g}_2 \mathbf{z}_2^{[2]} + \widetilde{\mathbf{C}}_3 \mathbf{z}_2^{[3]} + O(|\mathbf{z}_2|^{[4]}) \tag{2.58}$$

where the term $\mathbf{g}_2 \in \mathbb{C}^{nxn^2}$ represents the second order resonant terms which appear when (2.50) does not hold. Therefore,

$$\mathbf{g}_{2_{jk}} = \mathbf{C}_{2_{jk}} \tag{2.59}$$

2.3.4.2. Generalization to high-dimensional systems

The results of the preceding section can be readily extended to the case of high-dimensional systems. Referring to section (2.2.3), consider the nonlinear transformation

$$\mathbf{z}_2 = \mathbf{z}_3 + \mathbf{h}_3 \mathbf{z}_3^{[3]} \tag{2.60}$$

where $\mathbf{h}_3 \in \mathbb{C}^{nxn^3}$ is a complex valued matrix with coefficients to be determined so that the system in (2.58) not have third order nonlinearities.

Using the same idea presented for the second order case, the third order

terms are simplified and the process continues order-by-order just to the desired q order, choosing the components of the q-*th* transformation successively to eliminate or simplify as many terms as possible.

At q-*th* order the approximate NF representation for the system in (2.37) may be written as

$$\dot{\mathbf{z}}_q = \Lambda \mathbf{z}_q + \sum_{k=2}^{q} \mathbf{g}_k \mathbf{z}_q^{[k]} + O(|\mathbf{z}_2|^{[q+1]}) \tag{2.61}$$

where $\Lambda = diag[\lambda_1, \lambda_2, ..., \lambda_n]$, $\mathbf{g}_k$ are resonance terms that can not be eliminated by the *k-th* nonlinear transformations and $O(|\mathbf{z}_2|^{[q+1]})$ denotes an expression containing residual terms in $\mathbf{z}_q$ of order $q+1$ and higher.

The details for higher order representations are omitted but significant difference with respect to developments using polynomials representations are obvious.

2.4 Discussion

In this Chapter, a systematic analytical tool based on a modified normal form approach to assess the influence of higher order terms on system dynamic behavior has been proposed. Apart from its simplicity, the method is thought to have potentially important applications for dynamic analysis of general nonlinear systems. These include the computation of high-dimensional normal forms and the associated coefficients under resonant conditions and the study of various bifurcations.

For systems in which the dynamic response can be approximated by a low-order system representation, classical normal form theory provides a good approach to system modeling. It is observed that the approximation of true system behavior by a low-order system model can introduce large inaccuracies into the solution process, depending upon the degree of stress in the system and the characteristics of the system model.

The generalization of this approach to simplify the computation of the simplest normal form is discussed in the next Chapter.

2.5 References

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Chapter 3

Computation of the Simplest Normal Form

In the previous chapter, a general framework for modeling of nonlinear systems based on recursive, normal form computations was proposed. Some of the characteristics of recursive approaches, however, limit their usefulness for certain applications. These limitations include: numerical complexity of the algorithms, dependence of the recursive solutions on neglected terms, difficulty in interpreting the solutions when the order of the power series expansion is high, and complexity in the computation of initial conditions.

In this Chapter, assuming such a representation has been generated, a fundamentally new technique for determining the simplest possible normal form is presented. Making use of a special real-valued nonlinear transformation of coordinates in physical space, a general technique for the computation of the simplest, real normal form is proposed. The approach performs the computation in a single step using sequential formulations based on linear equations and does not require reduction of the center manifold. Expressions utilizing the proposed procedure are then derived to compute the normal form transformation coefficients using an efficient linear procedure.

This approach is suitably general for application to a wide variety of nonlinear system and can be used to study nonlinear behavior in the neighborhood of resonant vector fields and the study of system bifurcations. Analytical examples are provided to demonstrate the properties of the proposed formulation and improved performance realized by using high-dimensional approaches.

3.1 Introduction

The normal form method is an important tool for studying the behavior of dynamic systems described by differential equations near singularities. So far, most of the research effort has focused on developing normal form analysis techniques suitable for non-resonant, low-dimensional systems.

Existing analytical methods for the analysis of nonlinear systems have traditionally been restricted to low-dimensional system representations. Center manifold theory is used to determine formulae for evaluating the coefficients of the normal form coefficients. Whilst the center manifold approach is useful for obtaining reduced-order system representations, solution of the manifold equation becomes very complicated when higher order perturbation terms are considered. These early approaches have a limited range of applicability due primarily to the enormous algebraic complexity for systems of high order.

However, there are further results in the literature about the normal form that suggest that the conventional normal form is not unique [1] nor is the simplest form and can be simplified further by introducing a similar nonlinear change of coordinates [2]-[4].

One of the drawbacks of CNF analysis is that the calculation of the coefficients involved in the nonlinear problem is specific to each problem, i.e. reduction of Hopf and Saddle-node bifurcations to normal form, and is much more complicated than the calculation of the conventional normal form. Depending on the physical situation, certain non-algebraic conditions need to be satisfied in order to have a solution [4]. Moreover, these approaches rely on symbolic computations that make them unsuitable for real-world applications.

The increased complexity of such systems and the difficulties associated with the solution and analysis of the corresponding models has motivated the need for improved modeling techniques with the ability to analyze complex systems [5]. What is needed is a general approach for analyzing more general structures and that can deal with resonant and non-resonant vector fields.

The problem of finding the simplest possible normal form of a nonlinear differential equation has been recently addressed by some researchers [3]-[6]. However, a general method for obtaining the SNF for a general nonlinear system is lacking.

In this chapter, a general method for determining the simplest possible normal form is presented, which is characterized by high accuracy and low computational cost. Using a modified Poincaré-Birkhoff normal form theory for nonlinear systems, a new technique is introduced that computes the simplest possible real normal form. The method is general, requiring no preprocessing or special model scaling. In addition, the algorithm is extended to deal with both resonant and non-resonant vector fields and enables therefore the computation of resonant normal forms

The mathematical framework permits consideration of general dynamical systems which are not necessarily assumed in conventional normal form. We demonstrate that the proposed approach can successfully resolve complex dynamics and that it can capture the nonlinear behavior in cases where conventional normal form theory fails.

3.2 Extension to higher order systems

3.2.1 Theoretical background

Drawing on the general formulation in Chapter 2, consider a general n-dimensional nonlinear system described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \approx \mathbf{A}_1\mathbf{x} + \sum_{k=2}^{p} \mathbf{A}_k \mathbf{x}^{[k]} + O\left(\|\mathbf{x}\|^{[p+1]}\right) \tag{3.1}$$

where p is the order of the analytical approximation, and $\mathbf{A}_1$ denotes the linearization of $\mathbf{f}$

The method of normal forms enables us to analyze the behavior of the vector field (3.1) by reducing it via a suitable change of near-identity coordinates to a simpler form. In CNF theory, the coefficients of the qth-order nonlinear transformation are used to simplify or annihilate the qth-order nonlinear terms.

Standard theory indicates that non-resonant terms may always be eliminated from the normal form. As discussed in our introductory discussion in Chapter 2, this approach has several disadvantages: (i) At each step of the NF computation, higher order terms are generated which are not fully accounted for in successive computations, (ii) It is difficult to determine the exact order at which to truncate the normal form approximation.

Significant improvement in normal form computations can be gained by noting that that reduction to normal form can be combined in a single step [6],[7]. In power system applications involving a large number of states, this approach is often advantageous and may result in improved characterization of system nonlinear behavior [5].

The key idea is to find an appropriate formal near-identity transformation $\mathbf{x} = \varphi(\mathbf{z})$

$$\mathbf{x} = \varphi(\mathbf{z}) = \mathbf{z} + \sum_{m=2}^{q} \widetilde{\mathbf{h}}_m \mathbf{z}^{[m]} \tag{3.2}$$

such that the resulting system

$$\dot{\mathbf{z}} = \mathbf{A}_1 \mathbf{z} + \sum_{m=2}^{q} \widetilde{\mathbf{g}}_k \mathbf{z}^{[m]} + O\left(\|\mathbf{x}\|^{[q+1]}\right) \tag{3.3}$$

becomes as simple as possible. In the equations above, $\mathbf{z}$ is the new set of coordinates, $\varphi(\mathbf{z})$ represents a nonlinear function of $\mathbf{z}$, and $\widetilde{\mathbf{h}}_m$ and $\widetilde{\mathbf{g}}_k \in \mathbb{R}^{nxn^k}$ denote the m th (k th)-order matrix homogeneous. We remark that, in general $q \geq p$, which provides some degree of freedom to eliminate nonlinear terms. In other words, in the proposed formulation, the coefficients of the q th-order nonlinear transformation are used not only to annihilate the q th-order terms, but also to annihilate higher order terms.

Substitution of (3.2) into (3.1) yields

$$\left[D_z \varphi(\mathbf{z})\right]\dot{\mathbf{z}} = \mathbf{A}_1 \varphi(\mathbf{z}) + \sum_{k=2}^{q} \mathbf{A}_k (\varphi(\mathbf{z}))^{[k]} + O\left(\|\mathbf{z}\|^{[p+1]}\right) \tag{3.4}$$

It is clearly possible to choose the normal form transformation $\varphi(\mathbf{z})$ in various ways to simplify (3.3). More specifically, the analysis suggests that under appropriate conditions, a one-step nonlinear transformation may be used to annihilate nonlinear terms of order p. Motivated by this idea we next explore the use of a sequential procedure which avoids some of the limitations of existing approaches.

3.2.2 *One-step nonlinear transformation*

In the light of the previous discussion, we conjecture that a single nonlinear coordinate-transformation of order q between the original system and the normal form system can be used to annihilate terms of order p (with $q \geq p$). More specifically, we seek a near-identity change of coordinates of the form

$$\mathbf{x} = \varphi(\mathbf{z}) = \mathbf{z} + \tilde{\mathbf{h}}_2(\mathbf{z} \otimes \mathbf{z}) + \tilde{\mathbf{h}}_3(\mathbf{z} \otimes \mathbf{z} \otimes \mathbf{z}) + \cdots + \tilde{\mathbf{h}}_q(\mathbf{z} \otimes \cdots \otimes \mathbf{z}) = \mathbf{z} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \tag{3.5}$$

where the vector $\mathbf{z}$ denotes the new set of coordinates, $\mathbf{z}^{[m]} = \underbrace{\mathbf{z} \otimes \cdots \otimes \mathbf{z}}_{m-times}$.

Use of the nonlinear transformation (3.5) in the system model (3.4), yields

$$\frac{\partial \varphi(\mathbf{z})}{\partial \mathbf{z}} \dot{\mathbf{z}} = \left[\mathbf{I} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m D_z(\mathbf{z}^{[m]})\right] \dot{\mathbf{z}} = \mathbf{A}_1 \left[\mathbf{z} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]}\right] + \sum_{k=2}^{p} \mathbf{A}_k \left(\mathbf{z} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]}\right)^{[k]} + O\left(\left\|\mathbf{z} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]}\right\|^{[p+1]}\right) \tag{3.6}$$

Making use of the procedure outlined in the Chapter 2, the above normal form system can be solved assuming that the term $\left[\mathbf{I} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m D_z(\mathbf{z}^{[m]})\right]$ have inverse, then the above system is rewritten as

$$\dot{\mathbf{z}} = \left[\mathbf{I} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m D_z(\mathbf{z}^{[m]})\right]^{-1} \left\{ \begin{array}{l} \mathbf{A}_1 \left(\mathbf{z} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]}\right) + \sum_{k=2}^{p} \mathbf{A}_k \left(\mathbf{z} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]}\right)^{[k]} + \\ O\left(\left\|\mathbf{z} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]}\right\|^{[p+1]}\right) \end{array} \right\} \tag{3.7}$$

where the k - fold product is defined to be

$$\left[\mathbf{z}+\sum_{m=2}^{q}\widetilde{\mathbf{h}}_m\mathbf{z}^{[m]}\right]^{[k]}=\underbrace{\left[\mathbf{z}+\sum_{m=2}^{q}\widetilde{\mathbf{h}}_m\mathbf{z}^{[m]}\right]\otimes\cdots\otimes\left[\mathbf{z}+\sum_{m=2}^{q}\widetilde{\mathbf{h}}_m\mathbf{z}^{[m]}\right]}_{k-times}$$

In standard normal form analysis, the inverse $\left[\mathbf{I}+\sum_{m=2}^{q}\widetilde{\mathbf{h}}_m D_z(\mathbf{z}^{[m]})\right]^{-1}$ is approximated by

$$\left[\mathbf{I}+\sum_{m=2}^{q}\widetilde{\mathbf{h}}_m D_z(\mathbf{z}^{[m]})\right]^{-1}=\sum_{j=0}^{\infty}(-1)^j\left[\sum_{m=2}^{q}\widetilde{\mathbf{h}}_m D_z(\mathbf{z}^{[m]})\right]^{j}\qquad \text{as } |\mathbf{z}|\to 0 \qquad (3.8)$$

In a variation to the present approach the inverse expression in (3.8) is avoided. One convenient way to do this is to express the analytical model (3.6) in terms of the assumed normal form representation. Substitution of (3.3) in (3.6) gives

$$\left[\mathbf{I}+\sum_{m=2}^{q}\widetilde{\mathbf{h}}_m D_z(\mathbf{z}^{[m]})\right]\left[\mathbf{A}_1\mathbf{z}+\sum_{k=2}^{q}\widetilde{\mathbf{g}}_k\mathbf{z}^{[m]}+O(\|\mathbf{z}\|^{[q+1]})\right]=\mathbf{A}_1\left[\mathbf{z}+\sum_{m=2}^{q}\widetilde{\mathbf{h}}_m\mathbf{z}^{[m]}\right]+\sum_{k=2}^{p}\mathbf{A}_k\left[\mathbf{z}+\sum_{m=2}^{q}\widetilde{\mathbf{h}}_m\mathbf{z}^{[m]}\right]^{[k]}+O(\|\mathbf{z}\|^{[p+1]}) \qquad (3.9)$$

Equation (3.9) can be solved explicitly for $\widetilde{\mathbf{h}}_m$ in terms of the resonant terms $\widetilde{\mathbf{g}}_k$ in order to determine the q-th order normal form. Note that the only approximation in equation (3.7) is that we have neglected higher order terms $O\left(\|\mathbf{z}\|^{[k]}\right)$.

We emphasize that by avoiding the approximation (3.8), a significant number of terms that depend on powers of $[\sum_{m=2}^{q}\widetilde{\mathbf{h}}_m D_z(\mathbf{z}^{[m]})]^j$ vanish; thus reduces the complexity of the expressions involved in the compute of the nonlinear transformations.

A brief outline of this procedure is given below.

3.2.3 *Reduction to the simplest normal form*

In order to more clearly illustrate the nature of the proposed model, we analyze each term in (3.9) separately. It is readily apparent that that the left hand side term in (3.9) can be expressed as

$$\begin{aligned} &D_z(\mathbf{z}+\sum_{m=2}^{q}\tilde{\mathbf{h}}_m\mathbf{z}^{[m]}) = \\ &(\mathbf{I}+\tilde{\mathbf{h}}_2(\mathbf{I}\otimes\mathbf{z}+\mathbf{z}\otimes\mathbf{I})+\tilde{\mathbf{h}}_3(\mathbf{I}\otimes\mathbf{z}\otimes\mathbf{z}+\mathbf{z}\otimes\mathbf{I}\otimes\mathbf{z}+\mathbf{z}\otimes\mathbf{z}\otimes\mathbf{I})+\cdots)\dot{\mathbf{z}} \end{aligned} \tag{3.10}$$

or

$$D_z(\mathbf{z}+\sum_{m=2}^{q}\tilde{\mathbf{h}}_m\mathbf{z}^{[m]}) = \mathbf{I}+\tilde{\mathbf{h}}_2\mathbf{z}_{\mathbf{I}_2}+\mathbf{h}_3\tilde{\mathbf{h}}_{\mathbf{I}_3}+\cdots+\tilde{\mathbf{h}}_q\mathbf{z}_{\mathbf{I}_q} = (\mathbf{I}+\sum_{m=2}^{q}\tilde{\mathbf{h}}_m\mathbf{z}_{\mathbf{I}_m})\dot{\mathbf{z}} \tag{3.11}$$

where in the above use has been made of the property $\dfrac{\partial(\mathbf{z}\otimes\mathbf{z})}{\partial\mathbf{z}} = (\mathbf{I}\otimes\mathbf{z}+\mathbf{z}\otimes\mathbf{I})\dot{\mathbf{z}}$ of the Kronecker product [8], and

$$\begin{cases} \mathbf{z}_{\mathbf{I}_2} = \mathbf{z}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{z} \\ \mathbf{z}_{\mathbf{I}_3} = \mathbf{I}\otimes\mathbf{z}\otimes\mathbf{z}+\mathbf{z}\otimes\mathbf{I}\otimes\mathbf{z}+\mathbf{z}\otimes\mathbf{z}\otimes\mathbf{I} \\ \vdots \end{cases}$$

In a similar manner, the first term on the right hand side (rhs) of (3.9) can be arranged to give

$$\mathbf{A}_1\left[\mathbf{z}+\sum_{m=2}^{q}\tilde{\mathbf{h}}_m\mathbf{z}^{[m]}\right] = \mathbf{A}_1\mathbf{z}+\sum_{m=2}^{q}\mathbf{A}_1\tilde{\mathbf{h}}_m\mathbf{z}^{[m]} \tag{3.12}$$

in which no further simplification is possible.

The second term on the rhs of equation (3.9) is significantly more complicated since it involves the k-fold Kronecker product of terms of the form $\mathbf{z}+\sum_{m=2}^{q}\tilde{\mathbf{h}}_m\mathbf{z}^{[m]}$ This makes the analysis much more difficult; the reduction is not straightforward and a general expression is not possible.

To gain insight into the nature of the proposed model and to illustrate application of the procedure, assume that $p=q$ in (3.9). Further, in the interest of brevity and clarity, the following development is restricted to $q=3$. Under these assumptions, the second term on the rhs of (3.9), can be rewritten as

$$\sum_{k=2}^{3} \mathbf{A}_k \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[3]} \right]^{[k]} = \mathbf{A}_2 \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[3]} \right]^{[2]} + \mathbf{A}_3 \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[3]} \right]^{[3]} \tag{3.13}$$

Performing the above calculations, it can be proved that the terms of (3.13) are given by:

Second order summations

$$\begin{aligned}
&\mathbf{A}_2 \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right]^{[2]} = \mathbf{A}_2 \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right] \otimes \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right] \\
&= \mathbf{A}_2 \left[\mathbf{z}^{[2]} + \mathbf{z} \otimes \left(\sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right) + \left(\sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right) \otimes \mathbf{z} + \left(\sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right) \otimes \left(\sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right) \right] \\
&= \mathbf{A}_2 \left[\mathbf{z}^{[2]} + \sum_{m=2}^{3} (\mathbf{I} \otimes \tilde{\mathbf{h}}_m + \tilde{\mathbf{h}}_m \otimes \mathbf{I}) \mathbf{z}^{[m+1]} + \sum_{m=2}^{3} \sum_{l=2}^{3} (\tilde{\mathbf{h}}_m \otimes \tilde{\mathbf{h}}_l) \mathbf{z}^{[m+l]} \right] \\
&= \mathbf{A}_2 \mathbf{z}^{[2]} + \mathbf{A}_2 (\mathbf{I} \otimes \tilde{\mathbf{h}}_2 + \tilde{\mathbf{h}}_2 \otimes \mathbf{I}) \mathbf{z}^{[3]} + \mathbf{O}(\|\mathbf{z}\|^{[4]}) \\
&= \mathbf{A}_2 \mathbf{z}^{[2]} + \overline{\mathbf{A}}_3 \mathbf{z}^{[3]} + O(\|\mathbf{z}\|^{[4]})
\end{aligned} \tag{3.14a}$$

Third order summations

$$\begin{aligned}
&\mathbf{A}_3 \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right]^{[3]} = \mathbf{A}_3 \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right] \otimes \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right] \otimes \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right] \\
&= \mathbf{A}_3 \left[\mathbf{z} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right] \otimes \left[\mathbf{z}^{[2]} + \sum_{m=2}^{3} (\mathbf{I} \otimes \tilde{\mathbf{h}}_m + \tilde{\mathbf{h}}_m \otimes \mathbf{I}) \mathbf{z}^{[m+1]} + \sum_{m=2}^{3} \sum_{l=2}^{3} (\tilde{\mathbf{h}}_m \otimes \tilde{\mathbf{h}}_l) \mathbf{z}^{[m+l]} \right] \\
&= \mathbf{A}_3 \left[\mathbf{z}^{[3]} + \mathbf{z} \otimes \sum_{m=2}^{3} (\mathbf{I} \otimes \tilde{\mathbf{h}}_m + \tilde{\mathbf{h}}_m \otimes \mathbf{I}) \mathbf{z}^{[m+1]} + \mathbf{z} \otimes \sum_{m=2}^{3} \sum_{l=2}^{3} (\tilde{\mathbf{h}}_m \otimes \tilde{\mathbf{h}}_l) \mathbf{z}^{[m+l]} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \otimes \mathbf{z}^{[2]} \right] + \\
&\mathbf{A}_3 \left[\sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \otimes \sum_{m=2}^{3} (\mathbf{I} \otimes \tilde{\mathbf{h}}_m + \tilde{\mathbf{h}}_m \otimes \mathbf{I}) \mathbf{z}^{[m+1]} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \otimes \sum_{m=2}^{3} \sum_{l=2}^{3} (\tilde{\mathbf{h}}_m \otimes \tilde{\mathbf{h}}_l) \mathbf{z}^{[m+l]} \right] \\
&= \mathbf{A}_3 \mathbf{z}^{[3]} + \overline{\mathbf{A}}_4 \mathbf{z}^{[3]} + O(\|\mathbf{z}\|^{[5]})
\end{aligned} \tag{3.14b}$$

In general this terms, can be obtained for a determined degree of truncation of the series. More generally, these equations are of the form

$$\begin{aligned}
&\mathbf{A}_k \left[\mathbf{z} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m \mathbf{z}^{(m)} \right]^{[k]} = \mathbf{A}_k \underbrace{\left[\mathbf{z} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right] \otimes \cdots \otimes \left[\mathbf{z} + \sum_{m=2}^{q} \tilde{\mathbf{h}}_m \mathbf{z}^{[m]} \right]}_{k-times} \\
&= \mathbf{A}_k \mathbf{z}^{[k]} + \overline{\mathbf{A}}_{k+1} \mathbf{z}^{[k+1]} + \cdots + O(\|\mathbf{z}\|^{[k*q]})
\end{aligned} \tag{3.15}$$

where $\overline{\mathbf{A}}_{k+1}$ ($k = 2,3,...$) is a matrix generated by the nonlinear change of variable involving products between the matrices of the original vector field and the matrices of the nonlinear transformation. These terms are the so-called *residual terms*.

From the above expressions, we can make the following observations:

- The k-fold product in (3.14) results in terms of order $k+1$ being generated
- In particular, second order elements give rise to third order residual elements
- The formulae to compute the residual elements are explicit, and result in direct formulations.

We now illustrate the application of this approach for deriving the simplest normal form of a cubic system.

Following the same approach as that used in the first two sections of the Chapter, we derive a high order normal from representation. Substitution of (3.11), (3.12) and (3.14) into (3.9) results in

$$\left[\mathbf{I} + \sum_{m=2}^{3} \tilde{\mathbf{h}}_m \mathbf{z}_{\mathbf{I}_m}\right]\left[\mathbf{A}_1\mathbf{z} + \sum_{k=1}^{3} \tilde{\mathbf{g}}_k \mathbf{z}^{[k]} + O(\|\mathbf{z}\|^{[4]})\right] = \mathbf{A}_1\mathbf{z} + \left[\mathbf{A}_1\tilde{\mathbf{h}}_2 + \mathbf{A}_2\right]\mathbf{z}^{[2]} + \left[\mathbf{A}_1\tilde{\mathbf{h}}_3 + \overline{\mathbf{A}}_3 + \mathbf{A}_3\right]\mathbf{z}^{[3]} + O(\|\mathbf{z}\|^{[4]}) \tag{3.16}$$

where the coefficients $\tilde{\mathbf{h}}_m$ are chosen such that the system in (3.6) is taken to the normal form (3.3).

Carrying out the operations indicated in (3.16) and collecting terms at each order, we obtain the following sequence of linear equations in terms of the system parameters[1] with the unknowns $\tilde{\mathbf{h}}_k$ and $\tilde{\mathbf{g}}_k$ for each equation.

[1] This is referred to in this dissertation as the real normal form, since all the coefficients of the nonlinear transformation are purely real. Appendix B provides a more detailed discussion of this formulation.

$$
\begin{aligned}
\tilde{\mathbf{g}}_2\mathbf{z}^{[2]} + \tilde{\mathbf{h}}_2\mathbf{z}_{\mathbf{I}_2}\mathbf{A}_1\mathbf{z} &= [\mathbf{A}_1\tilde{\mathbf{h}}_2 + \mathbf{A}_2]\mathbf{z}^{[2]} \\
\tilde{\mathbf{g}}_3\mathbf{z}^{[3]} + \tilde{\mathbf{h}}_3\mathbf{z}_{\mathbf{I}_3}\mathbf{A}_1\mathbf{z} + \tilde{\mathbf{h}}_2\mathbf{z}_{\mathbf{I}_2}\tilde{\mathbf{g}}_2\mathbf{z}^{[2]} &= [\mathbf{A}_1\tilde{\mathbf{h}}_3 + \overline{\mathbf{A}}_2 + \mathbf{A}_3]\mathbf{z}^{[3]} \\
&\vdots \\
\tilde{\mathbf{g}}_3\mathbf{z}^{[k]} + \tilde{\mathbf{h}}_k\mathbf{z}_{\mathbf{I}_k}\mathbf{A}_1\mathbf{z} + \bar{\mathbf{g}}_k\mathbf{z}^{[k]} &= [\mathbf{A}_k\tilde{\mathbf{h}}_k + \overline{\mathbf{A}}_k + \mathbf{A}_k]\mathbf{z}^{[k]}
\end{aligned}
\tag{3.17}
$$

where $\bar{\mathbf{g}}_k$ are the terms obtained from the simplification of the product $\sum_{m=2}^{3}\tilde{\mathbf{h}}_m\mathbf{z}_{\mathbf{I}_m}\sum_{k=1}^{3}\tilde{\mathbf{g}}_k\mathbf{z}^{[k]}$ These terms are referred as the *residual resonant terms.*

And also in another useful form in terms of the Lie bracket operator, as

$$
\begin{aligned}
\tilde{\mathbf{g}}_2\mathbf{z}^{[2]} - L_{A_1}(\tilde{\mathbf{h}}_2\mathbf{z}^{[2]}) &= \mathbf{A}_2\mathbf{z}^{[2]} \\
\tilde{\mathbf{g}}_3\mathbf{z}^{[3]} - L_{A_1}(\tilde{\mathbf{h}}_3\mathbf{z}^{[3]}) + \tilde{\mathbf{h}}_2\mathbf{z}_{\mathbf{I}_2}\tilde{\mathbf{g}}_2\mathbf{z}^{[3]} &= [\overline{\mathbf{A}}_3 + \mathbf{A}_3]\mathbf{z}^{[3]} \\
&\vdots \\
\tilde{\mathbf{g}}_k\mathbf{z}^{[k]} - L_{A_1}(\tilde{\mathbf{h}}_k\mathbf{z}^{[k]}) + \bar{\mathbf{g}}_k\mathbf{z}^{[k]} &= [\overline{\mathbf{A}}_k + \mathbf{A}_k]\mathbf{z}^{[k]}
\end{aligned}
\tag{3.18}
$$

After further simplification, equations (3.18) finally reduces to

$$
\begin{aligned}
\mathbf{g}_2 + \tilde{\mathbf{h}}_2[\mathbf{A}_1\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{A}_1] - \mathbf{A}_1\tilde{\mathbf{h}}_2 &= \mathbf{A}_2 \\
\mathbf{g}_3 + \tilde{\mathbf{h}}_3[\mathbf{A}_1\otimes\mathbf{I}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{A}_1\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{I}\otimes\mathbf{A}_1] - \mathbf{A}_1\tilde{\mathbf{h}}_3 + & \\
\tilde{\mathbf{h}}_2[\tilde{\mathbf{g}}_2\otimes\mathbf{I} + \mathbf{I}\otimes\tilde{\mathbf{g}}_2] &= \overline{\mathbf{A}}_3 + \mathbf{A}_3 \\
&\vdots
\end{aligned}
\tag{3.19}
$$

More generally, for the k-th order, (3.19) can be written as

$$
\tilde{\mathbf{g}}_k + \tilde{\mathbf{h}}_k\sum_{j=1}^{k}\overbrace{\cdots\mathbf{I}\otimes\cdots\otimes\underbrace{\mathbf{A}_1}_{j-th\ \text{position}}\otimes\cdots\otimes\mathbf{I}\cdots}^{k\ \text{matrices}} - \mathbf{A}_1\tilde{\mathbf{h}}_k = \overline{\mathbf{A}}_k + \mathbf{A}_k - \bar{\mathbf{g}}_k \tag{3.20}
$$

A few remarks are in order concerning the interpretation of these results.

Remark 1. *We note that at each order k of the normal form computation, the algorithm generates a set of algebraic equations. The equations obtained are linear in the unknowns, but the terms involved in these linear systems contain nonlinear information about the structure of the original model.*

Remark 2. *Observe that, because the above expressions are linear in the unknowns, these expressions can be transformed into an equivalent linear system. By setting the normal form coefficients* $\widetilde{\mathbf{g}}_k$, *to zero a solution is possible that ensures the simplest normal form representation.*

These homological equations are in general non-diagonal, sparse, and require an iterative solution as discussed in the following.

3.2.4 *Solution of the homological equation*

Equations (3.19) represent a set of decoupled algebraic matrix equations which must be solved sequentially starting with the second order case, since the terms of higher order $\widetilde{\mathbf{h}}_k$, $k>2$ depend on the lower order terms. This equation is recognizable as a Sylvester-like equation which is present in other fields of engineering and physical sciences [9].

There are two general approaches that can be used to solve these equations: using iterative methods or direct linear methods. The second option is used here.

Now, following the general procedure outlined the Chapter 2, these matrix equation can setting in a linear systems by applying the "*vec*" operator to the (3.19)

$$\begin{aligned}
vec(\widetilde{\mathbf{g}}_2 + \widetilde{\mathbf{h}}_2[\mathbf{A}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}_1] - \mathbf{A}_1\widetilde{\mathbf{h}}_2) &= vec(\mathbf{A}_2) \\
vec(\widetilde{\mathbf{g}}_3 + \widetilde{\mathbf{h}}_3[\mathbf{A}_1 \otimes \mathbf{I} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{A}_1] - \mathbf{A}_1\widetilde{\mathbf{h}}_3) &= vec(\overline{\mathbf{A}}_3 + \mathbf{A}_3 - \overline{\mathbf{g}}_3) \\
&\vdots \\
vec(\widetilde{\mathbf{g}}_k + \widetilde{\mathbf{h}}_k \sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\mathbf{A}_1}_{j\text{-}th\ \text{position}} \otimes \cdots \otimes \mathbf{I} \cdots}^{k\ \text{matrices}} - \mathbf{A}_1\widetilde{\mathbf{h}}_k) &= vec(\overline{\mathbf{A}}_k + \mathbf{A}_k - \overline{\mathbf{g}}_k)
\end{aligned} \tag{3.21}$$

Equation (3.21) is a rather long and involved expression. It can be simplified by first recognizing that the *vec* rule enables the product of matrices to be rewritten as

$$vec(\mathbf{AB}) = (\mathbf{I} \otimes \mathbf{A})vec(\mathbf{B}) = (\mathbf{B}^T \otimes \mathbf{I})\mathbf{A}_S \tag{3.22}$$

With the aid of (3.22, (3.21) becomes

$$
\begin{aligned}
&\widetilde{\mathbf{g}}_{2S}^{T} + \left[\mathbf{I}\otimes(\mathbf{A}_1^{T}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{A}_1^{T})-\mathbf{A}_1\otimes\mathbf{I}\otimes\mathbf{I}\right]\widetilde{\mathbf{h}}_{2S}^{T} = \mathbf{A}_{2S}^{T} \\
&\widetilde{\mathbf{g}}_{3S}^{T} + \begin{bmatrix} \mathbf{I}\otimes(\mathbf{A}_1^{T}\otimes\mathbf{I}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{A}_1^{T}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{A}_1^{T})- \\ \mathbf{A}_1\otimes\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{I} \end{bmatrix}\widetilde{\mathbf{h}}_{3S}^{T} = \overline{\mathbf{A}}_{3S}^{T} + \mathbf{A}_{3S}^{T} - \overline{\mathbf{g}}_{3S}^{T} \\
&\qquad\qquad\vdots \\
&\widetilde{\mathbf{g}}_{kS}^{T} + \left[\begin{array}{l} \mathbf{I}\otimes\sum_{j=1}^{k}\overbrace{\cdots\mathbf{I}\otimes\cdots\otimes\underbrace{\mathbf{A}_1^{T}}_{j\text{-th position}}\otimes\cdots\otimes\mathbf{I}\cdots}^{k\text{ matrices}}- \\ \mathbf{A}_1\otimes\underbrace{\mathbf{I}\otimes\cdots\otimes\mathbf{I}}_{k\text{-times}} \end{array}\right]\widetilde{\mathbf{h}}_{kS}^{T} = \overline{\mathbf{A}}_{kS}^{T} + \mathbf{A}_{kS}^{T} - \overline{\mathbf{g}}_{kS}^{T}
\end{aligned}
\tag{3.23}
$$

Under quite general circumstances, these equations admit a set of real or complex solutions depending upon the adopted formulation. For the general case the solution is obtained by introducing the Schur transformation. Details about the use of Schur decomposition are given in Appendix B.

Note that the system in (3.23) is real and sparse; of order n^3 n^4 and n^{k+1} respectively. This is a linear system of the form $\mathbf{Bw}=\mathbf{b}$, where $\mathbf{w}$ is the vector solution, for which several solution techniques are available.

From linear systems theory, the solution for this type of equations, can be summarized in the following theorems [10].

Theorem 3.1. *The system* $\mathbf{Bw}=\mathbf{b}$ *is solvable if and only if*

$$\mathbf{rank}[\mathbf{B}\quad\mathbf{b}]=\mathbf{rank}[\mathbf{B}]$$

Theorem 3.2. *A matrix equation of the form* $\mathbf{MX}+\mathbf{XN}=\mathbf{P}$ *has a unique solution if and only if the matrices* $\mathbf{M}$ *and* $-\mathbf{N}$ *have no eigenvalues in common.*

The set of equations (3.23) has a unique solution when the following conditions hold [10]

$$
\begin{aligned}
&eig(\mathbf{A}_1\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{A}_1)\neq eig(\mathbf{A}_1) \\
&eig(\mathbf{A}_1\otimes\mathbf{I}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{A}_1\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{A}_1)\neq eig(\mathbf{A}_1) \\
&\qquad\vdots \\
&eig\Big(\sum_{j=1}^{k}\overbrace{\cdots\mathbf{I}\otimes\cdots\otimes\underbrace{\mathbf{A}_1}_{j\text{-th position}}\otimes\cdots\otimes\mathbf{I}\cdots}^{k\text{ matrices}}\Big)=eig(\mathbf{A}_1)
\end{aligned}
\tag{3.24}
$$

If any of these conditions holds, then the system is said to exhibit a high-

order resonant condition; the order of the resonance is given by the leading resonance condition in (3.24).

For this particular case, the solution for the linear system is infinite and the resonant terms are determined by

$$\widetilde{\mathbf{g}}_k = \mathbf{Null}(\mathbf{I} \otimes \sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\mathbf{A}_1^T}_{j-th\ \mathbf{position}} \otimes \cdots \otimes \mathbf{I} \cdots}^{k\ \mathbf{matrices}} - \mathbf{A}_1 \otimes \underbrace{\mathbf{I} \otimes \cdots \otimes \mathbf{I}}_{k-times})$$

At this point is important to emphasize that the modified homological equations obtained using this approach are general since they for the general structure of the linear part.

Having computed the normal form transformation coefficients, the system can be represented in the form

$$\dot{\mathbf{z}} = \mathbf{A}_1 \mathbf{z} + O(\|\mathbf{z}\|^{[q+1]}) \tag{3.25}$$

which reduces to $\dot{\mathbf{z}} = \mathbf{A}_1 \mathbf{z}$ if nonlinear terms $O(\|\mathbf{z}\|^{[q+1]})$ are neglected.

The proposed normal form methodology can be synthesized in the next proposition.

Proposition 3.1. *Given a nonlinear vector field of the form of (3.1), this can be transformed into a simplified linear system of the form*

$$\dot{\mathbf{z}} = \mathbf{A}_1 \mathbf{z} + \sum_{k=2}^{q} \widetilde{\mathbf{g}}_k \mathbf{z}^{[k]} + O\left(\|\mathbf{z}\|^{[q+1]}\right)$$

by introducing a one-step nonlinear change of coordinates $\mathbf{x} = \mathbf{z} + \sum_{m=2}^{q} \widetilde{\mathbf{h}}_m \mathbf{z}^{[m]}$ *and solving a sequence of linear systems of the form*

$$\widetilde{\mathbf{g}}_{k\,S}^T + \left[\begin{array}{l} \mathbf{I} \otimes \sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\mathbf{A}_1^T}_{j-th\ \mathbf{position}} \otimes \cdots \otimes \mathbf{I} \cdots}^{k\ \mathbf{matrices}} - \\ \mathbf{A}_1 \otimes \underbrace{\mathbf{I} \otimes \cdots \otimes \mathbf{I}}_{k-times} \end{array} \right] \widetilde{\mathbf{h}}_{k\,S}^T = \overline{\mathbf{A}}_{k\,S}^T + \mathbf{A}_{k\,S}^T - \overline{\mathbf{g}}_k^T \quad k = 2, \cdots, q.$$

3.3 Comparison with existing approaches

A major consequence of the presence of unaccounted terms in CNF is that the computation of the normal form transformation coefficients is more complicated. To illustrate this fact, consider a third order normal form representation. For the sake of clarity, the analysis is restricted to the non-resonant case. Table 3.1 compares the formulae for computing third order terms obtained using CNF theory and the proposed formulation.

A comparison of the expressions obtained using the SNF formulations with that of conventional formal form theory is enlightening. We first note that third order terms are dependent on second order computations[2]. Whereas the computation of third order terms in the proposed formulation is straightforward, the computation of third order terms in CNF theory is elaborated. In this case, a set of linear algebraic equations must be solved recursively.

Table 3.1: Nature of third order coefficients for CNF and the proposed approach

Normal form analysis Approach	Third order terms
CNF (refer to section 2.2)	$\widetilde{\mathbf{C}}_3 = \mathbf{C}_3 + \mathbf{C}_2[\mathbf{I}\otimes\mathbf{h}_2 + \mathbf{h}_2\otimes\mathbf{I}] + \boldsymbol{\Gamma}_3 + \boldsymbol{\Psi}_3 + \boldsymbol{\Omega}_3$ $\boldsymbol{\Gamma}_3 = \mathbf{h}_2[\mathbf{h}_2\otimes\mathbf{I}[\boldsymbol{\Lambda}\otimes\mathbf{I}\otimes\mathbf{I} + \mathbf{I}\otimes\boldsymbol{\Lambda}\otimes\mathbf{I}] + \mathbf{I}\otimes\mathbf{h}_2[\mathbf{I}\otimes\boldsymbol{\Lambda}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{I}\otimes\boldsymbol{\Lambda}]]$ $\boldsymbol{\Psi}_3 = -\mathbf{h}_2[\boldsymbol{\Lambda}\mathbf{h}_2\otimes\mathbf{I} + \mathbf{I}\otimes\boldsymbol{\Lambda}\mathbf{h}_2]$ $\boldsymbol{\Omega}_3 = -\mathbf{h}_2[\mathbf{C}_2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{C}_2]$
Proposed Approach	$\mathbf{A}_3 + \mathbf{A}_2[\mathbf{I}\otimes\hat{\mathbf{h}}_2 + \hat{\mathbf{h}}_2\otimes\mathbf{I}]$

Another important point is implicit in these expressions. As shown in Table 3.1, the order-by-order method requires three terms more that our approach to compute the normal form representation. As discussed in our application of the method in Chapter 5, this results in increased computer usage and a less efficient

[2] The second order terms are identical in both formulations and are therefore not shown.

computation of normal form coefficients. This effect becomes more pronounced as the order of the system increases.

A number of remarks are in order in the derivation of (3.25).

- The above formulation is general and allows for the rigorous treatment of both, resonant and non-resonant vector fields.
- The computation of normal form coefficients requires the solution of a set of real, linear algebraic equations. This is an important characteristic since complex solutions are avoided.
- Furthermore, the normal coefficients obtained by this procedure are generally fairly efficient relative to conventional approaches

We now develop expressions for some particular cases and discuss the nature of closed-form system solutions.

3.4 Illustrative example

In order to further illustrate the applicability and accuracy of the above approach and introduce the general ideas that follow, consider a second order nonlinear dynamical system represented by [11],[12],

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ \mu & -1 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & \mu & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ \varepsilon x_1^2 \\ 0 \\ 0 \end{bmatrix} \tag{3.26}$$

where μ and ε are real parameters, and μ controls the proximity to strong resonance; ε represent the degree of nonlinearity. The real-valued parameter μ introduces coupling between the two nonlinear systems. Such coupling has been shown to produce a close resonance condition when $\mu = 0$ [11],[12].

Equivalently, we can express the system model in equations (3.26) as

$$\dot{\mathbf{x}} = \mathbf{A}_1 \mathbf{x} + \mathbf{A}_2 \mathbf{x}^{[2]} \tag{3.27}$$

where, by definition $\mathbf{x}^{[2]} = \mathbf{x} \otimes \mathbf{x} = [x_1^2 \quad x_1 x_2 \cdots \quad x_2 x_1 \quad x_2^2 \quad \cdots \quad x_4^2]$, and

$$\mathbf{A}_1 = \mathbf{A}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In terms of this notation, the eigenvalues of the linear matrix $\mathbf{A}$ are given by

$$\lambda_1 = -1 + i + \sqrt{\mu} \qquad \lambda_2 = -1 + i - \sqrt{\mu}$$

and the right and left eigenvectors are

$$\mathbf{U} = \frac{1}{\sqrt{2}\sqrt{1+\mu}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ \sqrt{\mu} & -\sqrt{\mu} & \sqrt{\mu} & -\sqrt{\mu} \\ i & i & -i & -i \\ i\sqrt{\mu} & -i\sqrt{\mu} & -i\sqrt{\mu} & i\sqrt{\mu} \end{bmatrix}$$

$$\mathbf{V} = \frac{\sqrt{1+\mu}}{2\sqrt{2}} \begin{bmatrix} 1 & \frac{1}{\sqrt{\mu}} & -i & \frac{-1}{\sqrt{\mu}} \\ 1 & \frac{-1}{\sqrt{\mu}} & -i & \frac{1}{\sqrt{\mu}} \\ 1 & \frac{1}{\sqrt{\mu}} & i & \frac{1}{\sqrt{\mu}} \\ 1 & \frac{-1}{\sqrt{\mu}} & i & \frac{-1}{\sqrt{\mu}} \end{bmatrix}$$

where $\mathbf{V}^T\mathbf{U} = \mathbf{I}$.

In many practical cases, two situations are of particular interest: the non-resonance case ($\mu \neq 0$), and the case in which the two eigenvalues coincide exactly ($\mu = 0$). These two cases are examined separately.

3.4.1 The non-resonant case

As a special case, we first assume that the system (3.27) exhibits distinct eigenvalues. In this case, the linear transformation $\mathbf{x} = \mathbf{U}\mathbf{y}$ transforms the system in (3.27) into the Jordan canonical form

$$\dot{\mathbf{y}} = \mathbf{U}^{-1}\mathbf{A}_1\mathbf{U}\mathbf{y} + \mathbf{U}^{-1}\mathbf{A}_2\left((\mathbf{U}\otimes\mathbf{U})(\mathbf{y}\otimes\mathbf{y})\right) = \Lambda\mathbf{y} + \mathbf{C}_2\mathbf{y}^{[2]} \tag{3.28}$$

or

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\\ \dot{y}_4\end{pmatrix} = \begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda^*_1 & \\ & & & \lambda^*_2\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\\ y_4\end{pmatrix} + \frac{\varepsilon(y_1+y_2+y_3+y_4)^2}{8\sqrt{2}\sqrt{\mu}\sqrt{1+\mu}}\begin{pmatrix}1\\ -1\\ 1\\ -1\end{pmatrix} \tag{3.29}$$

where $\mathbf{C}_2 = \mathbf{U}^{-1}\mathbf{A}_2[(\mathbf{U}\otimes\mathbf{U})]$. Explicit computation of the second order coefficients C_{kl}, in terms of the original coefficients yields

$$C_{kl} = \frac{\varepsilon(-1)^{j+1}}{8\sqrt{2}\sqrt{\mu}\sqrt{1+\mu}}\ , k = 1,\cdots,4\ , l = 1,\cdots,16 \tag{3.30}$$

The above type of simplification can provide physical insight into the structural behavior, since the set of differential equations is now coupled through the non-linear term only. It can also offer substantial reduction in computational effort, particularly when the nonlinear term takes a simple form.

3.4.1.1 Transformation to the normal form

Consider the nonlinear transformation $\mathbf{y} = \varphi(\mathbf{z}) = \mathbf{z} + \mathbf{h}_2(\mathbf{z}\otimes\mathbf{z})$. With reference to (3.28) and applying the procedure outlined above, we conclude that

$$\begin{aligned}[\mathbf{I} + \mathbf{h}_2 D_z\mathbf{z}^{[2]}]\dot{\mathbf{z}} &= \Lambda\left[\mathbf{z} + \mathbf{h}_2(\mathbf{z}\otimes\mathbf{z})\right] + \mathbf{C}_2(\mathbf{z} + \mathbf{h}_2(\mathbf{z}\otimes\mathbf{z}))\otimes(\mathbf{z} + \mathbf{h}_2(\mathbf{z}\otimes\mathbf{z}))\\ &= \Lambda\mathbf{z} + \mathbf{C}_2\left[\mathbf{z}\otimes(\mathbf{z} + \mathbf{h}_2(\mathbf{z}\otimes\mathbf{z})) + (\mathbf{h}_2(\mathbf{z}\otimes\mathbf{z}))\otimes(\mathbf{z} + \mathbf{h}_2(\mathbf{z}\otimes\mathbf{z}))\right]\\ &= \Lambda\mathbf{z} + \mathbf{C}_2\left[(\mathbf{z}\otimes\mathbf{z}) + (\mathbf{z}\otimes\mathbf{h}_2(\mathbf{z})) + (\mathbf{h}_2(\mathbf{z})\otimes\mathbf{z}) + (\mathbf{h}_2(\mathbf{z})\otimes\mathbf{h}_2(\mathbf{z}))\right]\\ &= \Lambda\mathbf{z} + \Lambda\mathbf{h}_2(\mathbf{z}\otimes\mathbf{z}) + \mathbf{C}_2(\mathbf{z}\otimes\mathbf{z}) + O(\|\mathbf{z}^3\| + O(\|\mathbf{z}^3\|)\end{aligned} \tag{3.31}$$

or, in simplified form,

$$\left[\mathbf{I} + \mathbf{h}_2(\mathbf{I}\otimes\mathbf{z} + \mathbf{z}\otimes\mathbf{I})\right]\left\{\Lambda\mathbf{z} + \mathbf{g}_2(\mathbf{z}\otimes\mathbf{z}) + O\left(\|\mathbf{z}\|^3\right)\right\} = \Lambda\mathbf{z} + \Lambda\mathbf{h}_2(\mathbf{z}\otimes\mathbf{z}) + \mathbf{C}_2(\mathbf{z}\otimes\mathbf{z}) \tag{3.32}$$

which results from setting $\mathbf{g}_2(\mathbf{z}\otimes\mathbf{z}) = \mathbf{0}$, and neglecting terms $O\left(\|\mathbf{z}\|^3\right)$.

Equating the coefficients of second order results in

$$\mathbf{h}_2(\mathbf{I}\otimes\mathbf{z} + \mathbf{z}\otimes\mathbf{I})\Lambda\mathbf{z} - \Lambda\mathbf{h}_2(\mathbf{z}\otimes\mathbf{z}) = \mathbf{C}_2(\mathbf{z}\otimes\mathbf{z}) \tag{3.33}$$

Carrying out the operations indicated, and equating like powers in $\mathbf{z}\otimes\mathbf{z}$, one obtains

$$\mathbf{h}_2\left(\mathbf{I}\otimes\Lambda+\Lambda\otimes\mathbf{I}\right)-\Lambda\mathbf{h}_2=\mathbf{C}_2$$

Following the procedure described in the section 2.3.4.1 the system solution is then written as

$$\begin{bmatrix} h_{2_{1,1}} & h_{2_{1,2}} & \cdots & h_{2_{1,16}} \\ h_{2_{2,1}} & h_{2_{2,2}} & \cdots & h_{2_{2,16}} \\ h_{2_{3,1}} & h_{2_{3,2}} & \cdots & h_{2_{3,16}} \\ h_{2_{4,1}} & h_{2_{4,2}} & \cdots & h_{2_{4,16}} \end{bmatrix} = \begin{bmatrix} \frac{C_{2_{1,1}}}{\lambda_1+\lambda_1-\lambda_1} & \frac{C_{2_{1,2}}}{\lambda_1+\lambda_2-\lambda_1} & \cdots & \frac{C_{2_{1,16}}}{\lambda_n+\lambda_n-\lambda_1} \\ \frac{C_{2_{2,1}}}{\lambda_1+\lambda_1-\lambda_2} & \frac{C_{2_{2,2}}}{\lambda_1+\lambda_2-\lambda_2} & \cdots & \frac{C_{2_{2,16}}}{\lambda_n+\lambda_n-\lambda_2} \\ \frac{C_{2_{3,1}}}{\lambda_1+\lambda_1-\lambda_3} & \frac{C_{2_{3,2}}}{\lambda_1+\lambda_2-\lambda_3} & \cdots & \frac{C_{2_{3,16}}}{\lambda_n+\lambda_n-\lambda_3} \\ \frac{C_{2_{4,1}}}{\lambda_1+\lambda_1-\lambda_4} & \frac{C_{2_{4,2}}}{\lambda_1+\lambda_2-\lambda_4} & \cdots & \frac{C_{2_{4,16}}}{\lambda_n+\lambda_n-\lambda_4} \end{bmatrix} \tag{3.34}$$

Finally, solving (3.34) for the coefficients of the normal form transformation, $h_{2_{k,l}}$, as a function of the mode combinations $\lambda_k+\lambda_l-\lambda_j$, and system parameters, one obtains that

$$\mathbf{h}_2=\frac{\varepsilon}{8\sqrt{2}\sqrt{\mu}\sqrt{1+\mu}}\begin{bmatrix} \frac{1}{\lambda_1+\lambda_1-\lambda_1} & \frac{1}{\lambda_1+\lambda_2-\lambda_1} & \cdots & \frac{1}{\lambda_n+\lambda_n-\lambda_1} \\ \frac{-1}{\lambda_1+\lambda_1-\lambda_2} & \frac{-1}{\lambda_1+\lambda_2-\lambda_2} & \cdots & \frac{-1}{\lambda_n+\lambda_n-\lambda_2} \\ \frac{1}{\lambda_1+\lambda_1-\lambda_3} & \frac{1}{\lambda_1+\lambda_2-\lambda_3} & \cdots & \frac{1}{\lambda_n+\lambda_n-\lambda_3} \\ \frac{-1}{\lambda_1+\lambda_1-\lambda_4} & \frac{-1}{\lambda_1+\lambda_2-\lambda_4} & \cdots & \frac{-1}{\lambda_n+\lambda_n-\lambda_4} \end{bmatrix} \tag{3.35}$$

Equation (3.31) then becomes, upon ignoring the terms in $O(\|\mathbf{z}^{p+1}\|)$

$$\dot{\mathbf{z}}=\Lambda\mathbf{z}$$

where $\mathbf{z}=[z_1 \quad z_2 \quad z_3 \quad z_4]^T$

Remark 3. *It is readily apparent from (3.35) that even if the nonlinearity, ε is small, a smaller value of the parameter μ can make the coefficients C^j_{2kl} and h^j_{2kl} to become very large near a strong, non-diagonalizable resonance occurring in the system linearization. In other words, NF coefficients and the associated indices in Jordan coordinates can become very large despite very small amounts of nonlinearity in the original (physical) coordinates [11].*

Since the second order coefficients are also used to compute initial conditions in the NF space, numerical problems may also arise in the computation of closed form solutions even if an exact resonance is not found. See for instance Liu *et. al.* [13]. We postpone discussion of the numerical methods until the next Chapter.

3.4.2 The resonant case

The nature of system behavior is best understood by considering the operation of the system in the neighborhood of the resonance condition. At $\mu = 0$ the eigenvalues $\lambda_{1,2}, \lambda_{3,4}$ coincide in a strong resonance, and $\mathbf{A}_1$ is non diagonalizable. To circumvent this problem, we use the technique sketched in the section 3.2.4.

Using (3.23), and with the aid of Maple, the normal form coefficients, $\tilde{h}_{2\,k,l}$, can be expressed in terms of μ as

$$\tilde{h}_{2\,k,l} = [p_{k,l}(u)]\frac{\varepsilon}{p(\mu)}10^{-3} \qquad j = 1,2...,\ n\ ,\ k = 1,2,...,\ n^2 \tag{3.36}$$

where the $p_{k,l}$ are polynomial functions of μ, and

$$p(\mu) = 0.0065\mu^8 + 0.116\mu^7 + 0.877\mu^6 + 1.768\mu^5 + 4.256\mu^4 + 5.271\mu^3 + 3.448\mu^2 + 0.256\mu + 0.160$$

From (3.36) we see that the $\tilde{h}_{2\,j,k}$ coefficients are proportional to ε and finite when $p(\mu) \neq 0$; this condition is satisfied for all real values of μ since the roots of $p(\mu)$ are complex.

The graphs in Fig. 3.1 show the real and conventional normal form coefficients for $\varepsilon = -0.05$ as a function of μ. We remark that the $\tilde{h}_{2j,k}$ coefficients are always finite and real-valued. This is a unique characteristic of the suggested method that makes it useful for the study of resonant conditions.

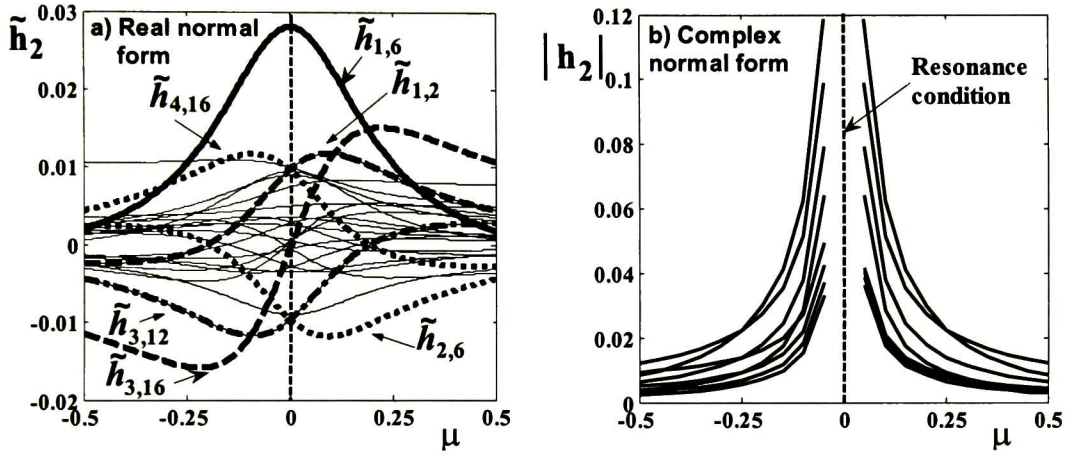

Figure 3.1. Behavior of normal form coefficients as a function of μ.

Both resonant and non-resonant conditions can be studied and analyzed using the proposed technique; the resonant condition is a particular case of the more general condition.

3.5 Discussion

In this chapter, a higher order normal form representation suitable for the analysis of nonlinear behavior in real space has been proposed. This works extends CNF theory to higher dimensional systems and allows the analysis of non-conventional formulations. A unique feature of the above procedure is its ability to avoid higher-order resonances arising from the interactions of multiple eigenvalues. The central idea behind this approach is a real, normal change of coordinates in physical space. This procedure allows both, the treatment of strong resonant and non-resonant power systems and the derivation of a general purpose higher order normal form method.

Compared to the conventional normal form approaches, perturbation theories based on this approach offer three substantial advantages: (i) they yield the transformation of state variables in real coordinates, (ii) these transformations are function of the original variables, and (iii) the inverse transformation can be built in the same way. The distinction between real and complex normal form formulations characteristics is clearly pointed out. This aspect is entirely new: it contains all previous formulations as special cases.

A symbolic example shows that conventional NF results and interpretation can be affected by both, linear and higher order resonances. The analysis of these factors in this research motivated the need for improved modeling and analysis techniques, particularly with respect to the analysis of complex power systems.

The information obtained form this analysis could be used to provide insight into the selection of appropriate initial approximation to system solutions, the range of physically meaningful parameters, and ultimately to check on the adequacy of the NF method used in the analysis.

3.6 References

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Chapter 4

Higher-Order Normal Form Analysis for Diagonalizable Systems

The application of normal form theory to nearly diagonalizable systems yields a variety of interesting information about nonlinear behavior that is of particular interest to many physical systems. This chapter discusses the important case in which the linear part of the system is diagonalizable and presents general criteria for analyzing resonant systems.

A general method of computing the simplest normal form for diagonalizable systems is presented, which is characterized by high accuracy and low computational cost. The method is based on an efficient, physically-motivated, linear algebraic procedure with desirable properties for the study of complex systems. Techniques from linear systems theory are then used to derive explicit equations which describe nonlinear oscillatory behavior and the treatment of higher order nonlinear resonances in terms of the inertial modes of oscillation of the system.

Exact mathematical criteria for this behavior are derived and procedures to analyze both, resonant and non-resonant power systems are given. The utility of the proposed approach in obtaining closed-form time domain solutions of both resonant and non-resonant systems is discussed, and technique are suggested for the computation of high-dimensional nonlinear transformations. Analytical expressions utilizing the normal form representation are developed for the various criteria.

A detailed analytical example is presented to illustrate the practical application of the algorithms.

The Chapter concludes with an outline of some directions for future work.

4.1 Introduction

Normal form analysis has been traditionally investigated using the approach outlined in section 2.2, where a canonical transformation is introduced to simplify the original vector field. A standing assumption in these approaches is that the linear part of the vector field can be taken to a diagonal form and that the linear eigenvalues do not interact to produce a modal resonance. In the simplest case, the Jacobian of the linear part of the vector field is the diagonal matrix of system eigenvalues. In the most general case, however, the Jacobian matrix will have a non-diagonal structure, which makes the analysis difficult.

The presence of internal resonances essentially complicates the construction of the normal form representation since the analysis and interpretation of normal form coefficients may be uninformative. Previous investigators have pointed out that in the neighborhood of internal resonances, the normal form coefficients become quite large, even if the actual nonlinearity is small [1],[2]. Further, existing approaches are unable to deal with higher-order resonances and bifurcations arising from the parameter-dependent linearization of the system [3].

In this Chapter, we address these specific issues by developing a new analytical technique for the study of nearly resonant power systems. This approach results in universal relations and model dependent properties of nonlinear systems which may yield information about the nonlinear dynamic behavior in terms of the fundamental natural modes of oscillation.

4.2 Higher order normal form analysis

4.2.1. *Derivation of the normal form*

A case of particular importance in the study of power system dynamic behavior arises in the study of diagonalizable systems. In this section we recapitulate the derivation of the model equations with emphasis in the analysis of diagonalizable systems and present some new results and extensions which will be useful in the context of dynamic models of resonant power systems.

In order to introduce the general ideas that follow, let us consider a nonlinear power system described by the set of differential equations (3.1). On the basis of our previous work, we assume that the nonlinear system can be taken to the q th- order normal form

$$\dot{\mathbf{z}} = \mathbf{A}_1\mathbf{z} + \sum_{m=2}^{q} \widetilde{\mathbf{g}}_m \mathbf{z}^{[m]} + O(\|\mathbf{x}\|^{[q+1]}) \tag{4.1}$$

under the nonlinear coordinate transformation

$$\mathbf{x} = \mathbf{z} + \sum_{m=2}^{q} \widetilde{\mathbf{h}}_m \mathbf{z}^{[k]} \tag{4.2}$$

Following the same procedure as that used in Chapter 3, the normal form and the associated nonlinear transformation coefficients are obtained from the solution of the homological equations

$$\widetilde{\mathbf{g}}_k + \widetilde{\mathbf{h}}_k \sum_{j=1}^{k} \cdots \overbrace{\mathbf{I} \otimes \cdots \otimes \underbrace{\mathbf{A}_1}_{j-th \text{ position}} \otimes \cdots \otimes \mathbf{I}}^{k \text{ matrices}} \cdots - \mathbf{A}_1 \widetilde{\mathbf{h}}_k = \overline{\mathbf{A}}_k + \mathbf{A}_k - \overline{\mathbf{g}}_k \qquad k = 2,3,\cdots,q \tag{4.3}$$

Our departure from a similar model in Chapter 3 at this stage is that we assume the matrix $\mathbf{A}_1$ to be diagonalizable. For this case, the homological equations (4.3) can be rewritten as[1]

$$\widetilde{\mathbf{g}}_k + \widetilde{\mathbf{h}}_k \sum_{j=1}^{k} \cdots \overbrace{\mathbf{I} \otimes \cdots \otimes \underbrace{\mathbf{U}\Lambda\mathbf{V}}_{j-th \text{ position}} \otimes \cdots \otimes \mathbf{I}}^{k \text{ matrices}} \cdots - \mathbf{U}\Lambda\mathbf{V}\widetilde{\mathbf{h}}_k = \overline{\mathbf{A}}_k + \mathbf{A}_k - \overline{\mathbf{g}}_k \tag{4.4}$$

where $\mathbf{A}_1 = \mathbf{U}\Lambda\mathbf{V}$ $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, and $\mathbf{U}$ and $\mathbf{V} = \mathbf{U}^{-1}$ are the matrices of right and left eigenvectors, respectively.

Generally, several possibilities exist for the solution of this equation. Systematic examination of the modal structure of the structure of (4.4) is used in succeeding sections to deduce the nature of physically meaningful solutions.

[1] The linear transformation can be applied previous to the introduction of the nonlinear change of coordinates. This procedure is avoided since the expressions are identical to those obtained here.

Introducing the identity $\mathbf{I} = \mathbf{VIU}$ on the rhs of (4.4), we obtain

$$\tilde{\mathbf{g}}_k + \tilde{\mathbf{h}}_k \sum_{j=1}^{k} \overbrace{\cdots \mathbf{VIU} \otimes \cdots \otimes \underbrace{\mathbf{U\Lambda V}}_{j-th\ position} \otimes \cdots \otimes \mathbf{VIU} \cdots}^{k-matrices} - \mathbf{U\Lambda V}\tilde{\mathbf{h}}_k = \bar{\mathbf{A}}_k + \mathbf{A}_k - \tilde{\mathbf{g}}_k \tag{4.5}$$

Noting that $[\mathbf{A} \otimes \mathbf{B}][\mathbf{C} \otimes \mathbf{D}] = [\mathbf{AC}] \otimes [\mathbf{BD}]$[4],[5], (4.5) reduces to

$$\tilde{\mathbf{g}}_k + \tilde{\mathbf{h}}_k \mathbf{U}^{[k]} \left[\sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\Lambda}_{j-th\ position} \otimes \cdots \otimes \mathbf{I} \cdots}^{k-matrices} \right] \mathbf{V}^{[k]} - \mathbf{U\Lambda V}\tilde{\mathbf{h}}_k = \bar{\mathbf{A}}_k + \mathbf{A}_k - \bar{\mathbf{g}}_k \tag{4.6}$$

Now post-multiplying by $\mathbf{U}^{[k]}$ $(k = 2,3,\cdots)$ yields

$$\tilde{\mathbf{g}}_k \mathbf{U}^{[k]} + \tilde{\mathbf{h}}_k \mathbf{U}^{[k]} \left[\sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\Lambda}_{j-th\ position} \otimes \cdots \otimes \mathbf{I} \cdots}^{k-matrices} \right] - \mathbf{U\Lambda V}\tilde{\mathbf{h}}_k \mathbf{U}^{[k]} = \bar{\mathbf{A}}_k \mathbf{U}^{[k]} + \mathbf{A}_k \mathbf{U}^{[k]} - \bar{\mathbf{g}}_k \mathbf{U}^{[k]} \tag{4.7}$$

Finally, pre-multiplying both sides of (4.7) by $\mathbf{V}$, we obtain

$$\mathbf{V}\tilde{\mathbf{g}}_k \mathbf{U}^{[k]} + \mathbf{V}\tilde{\mathbf{h}}_k \mathbf{U}^{[k]} \left[\sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\Lambda}_{j-th\ position} \otimes \cdots \otimes \mathbf{I} \cdots}^{k-matrices} \right] - \Lambda \mathbf{V}\tilde{\mathbf{h}}_k \mathbf{U}^{[k]} = \mathbf{V}\bar{\mathbf{A}}_k \mathbf{U}^{[k]} + \mathbf{V}\mathbf{A}_k \mathbf{U}^{[k]} - \mathbf{V}\bar{\mathbf{g}}_k \mathbf{U}^{[k]} \tag{4.8}$$

Consequently,

$$\mathbf{g}_k + \mathbf{h}_k \left[\sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\Lambda}_{j-th\ position} \otimes \cdots \otimes \mathbf{I} \cdots}^{k-matrices} \right] - \Lambda \mathbf{h}_k = \bar{\mathbf{C}}_k + \mathbf{C}_k - \bar{\bar{\mathbf{g}}}_k \tag{4.9}$$

where, at order k, the coefficients of above homological equation are related to the those obtained in real coordinates through the next relations

$$\mathbf{h}_k = \mathbf{V}\tilde{\mathbf{h}}_k \mathbf{U}^{[k]} \quad \bar{\mathbf{C}}_k = \mathbf{V}\bar{\mathbf{A}}_k \mathbf{U}^{[k]} \quad \mathbf{C}_k = \mathbf{V}\mathbf{A}_k \mathbf{U}^{[k]} \quad \mathbf{g}_k = \mathbf{V}\tilde{\mathbf{g}}_k \mathbf{U}^{[k]} \quad \bar{\bar{\mathbf{g}}}_k = \mathbf{V}\bar{\mathbf{g}}_k \mathbf{U}^{[k]}$$

Note from above that the quantities are identical with those obtained in the Chapter 2 for non-resonant conditions $\mathbf{g}_k = 0$. This approach readily generalizes to more complex representations, and to high-dimensional systems.

It is worth commenting on the remarkable simplicity of this result, especially considering the very complicated form of the conventional normal form approach (refer to Section 2.2.3). In the following, we discuss the generalization of this approach to the nonlinear resonant condition case and derive explicit formulae for computing the SNF and the associated nonlinear transformation coefficients.

4.2.2. *Closed-form solution of resonant homological equations*

A key part of the proposed analysis scheme is solving the homological equation (4.9). In an effort to find a simpler model containing the qualitative behavior of interest and to reduce computational complexity, the homological equations previously determined, are transformed into a linear system form, by applying the "vec" operator on both sides. It can be readily shown that

$$\mathbf{g}_{kS}^{T} + \left[\mathbf{I} \otimes \left(\sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\Lambda}_{j\text{-th position}} \otimes \cdots \otimes \mathbf{I} \cdots}^{k\text{-matrices}} \right) - \Lambda \otimes \mathbf{I}^{[k]} \right] \mathbf{h}_{kS}^{T} = \overline{\mathbf{C}}_{kS}^{T} + \mathbf{C}_{kS}^{T} - \overline{\overline{\mathbf{g}}}_{kS}^{T} \quad (4.10)$$

In a similar manner to the linear system of the section 2.3.4.1, (4.10) is equivalent to a block diagonal system of the form

$$\mathbf{g}_{kS}^{T} + \begin{bmatrix} \mathbf{\Pi}^{k} - \lambda_1 \otimes \mathbf{I}^{[k]} & & & \\ & \mathbf{\Pi}^{k} - \lambda_2 \otimes \mathbf{I}^{[k]} & & \\ & & & \\ & & & \mathbf{\Pi}^{k} - \lambda_n \otimes \mathbf{I}^{[k]} \end{bmatrix} \mathbf{h}_{kS}^{T} = \overline{\mathbf{C}}_{kS}^{T} + \mathbf{C}_{kS}^{T} - \overline{\overline{\mathbf{g}}}_{kS}^{T} \quad (4.11)$$

where

$$\mathbf{\Pi}^{k} = \sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\Lambda}_{j\text{-th position}} \otimes \cdots \otimes \mathbf{I} \cdots}^{k\text{-matrices}} \quad (4.12)$$

We see immediately from (4.11) that, the homological equation has a unique solution when the following constrain holds

$$eig\left[\sum_{j=1}^{k}\overbrace{\cdots\mathbf{I}\otimes\cdots\otimes\underbrace{\Lambda}_{j-th\ position}\otimes\cdots\otimes\mathbf{I}\cdots}^{k-matrices}\right]\neq eig(\Lambda)$$

If (4.11) is strictly diagonal, then the solution for the resonant terms are particularly simple:

$$g_{k_i,m}+(\Pi^k{}_{m,m}-\lambda_i)h_{k_i,m}=\overline{C}_{k_i,m}+C_{k_i,m}-\overline{\overline{g}}_{k_i,m}\qquad m=1,2,\cdots,n^k \tag{4.13}$$

where $\Pi^k_{m,m}$ is the mth diagonal element of matrix Π^k.

This linear system constitutes a set of n^k decoupled linear equations with unknowns $g_{k_i,m}$ and $h_{k_i,m}$ The results agree well with conventional analysis for the case of non-resonance conditions [7].

Depending on the physical situation, several solutions are possible. From the above expression, three possible cases for the solution of (4.11) are identified:

1. *Non-resonant case.* The modal spectrum of the system $(\sum_{j=1}^{k}\overbrace{\cdots\mathbf{I}\otimes\cdots\otimes\underbrace{\Lambda}_{j-th\ position}\otimes\cdots\otimes\mathbf{I}\cdots}^{k-matrices})$ is different from any eigenvalue λ_i, i.e.

$$eig(\sum_{j=1}^{k}\overbrace{\cdots\mathbf{I}\otimes\cdots\otimes\underbrace{\Lambda}_{j-th\ position}\otimes\cdots\otimes\mathbf{I}\cdots}^{k-matrices})\neq eig(\lambda_i\mathbf{I}^{[k]})$$

In this case, the system has an unique solution. Setting $\mathbf{g}_{k_j}$ equals to zero, we conclude that sufficient condition for the solution of the non-resonant system will be to find coefficients $\mathbf{h}^i_k$ such that

$$\mathbf{h}_{k_i}=\mathbf{C}_{k_i}\left[\sum_{j=1}^{k}\overbrace{\cdots\mathbf{I}\otimes\cdots\otimes\underbrace{\Lambda}_{j-th\ position}\otimes\cdots\otimes\mathbf{I}\cdots}^{k-matrices}-\lambda_i\mathbf{I}^{[k]}\right]^{-1}$$

2. *Resonant case.* There exists a set of eigenvalues of $(\sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\Lambda}_{j-th\ position} \otimes \cdots \otimes \mathbf{I} \cdots}^{k-matrices})$ equals to the eigenvalue λ_i. Use of this conditions in (4.11) leads to the following set of algebraic equations

$$\hat{\mathbf{g}}_{k_i} + \mathbf{0}\hat{\mathbf{h}}_{k_i} = \hat{\mathbf{C}}_{k_i}$$

$$\hat{\bar{\mathbf{g}}}_{k_i} + \overline{\mathbf{\Pi}}^{i}\hat{\bar{\mathbf{h}}}_{k_i} = \hat{\bar{\mathbf{C}}}_{k_i}$$

where the $\hat{\mathbf{h}}_{k_i}$ are the elements for which $(\mathbf{\Pi}^{k}{}_{m,m} - \lambda_i)$ vanishes, $\hat{\bar{\mathbf{h}}}_{k_i}$ are the elements for which $(\mathbf{\Pi}^{k}{}_{m,m} - \lambda_i) \neq 0$ and $\overline{\mathbf{\Pi}}^{i}$ is a diagonal matrix containing the elements for which this condition holds. Similar to the previous development, we remark that the first equation has trivial solution and is associated with the resonant elements of the transformation. As in the previous case, the second equation can be solved by setting $\hat{\bar{\mathbf{g}}}_{k_i} = 0$. This leads to a general expression for the analysis of resonant conditions.

3. In addition to the two types of resonance conditions considered above, a further resonance condition is possible when all the eigenvalues of $\left[\sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\Lambda}_{j-th\ position} \otimes \cdots \otimes \mathbf{I} \cdots}^{k-matrices}\right]$ are identical to the eigenvalue λ_i or

$$eig\left[\sum_{j=1}^{k} \overbrace{\cdots \mathbf{I} \otimes \cdots \otimes \underbrace{\Lambda}_{j-th\ position} \otimes \cdots \otimes \mathbf{I} \cdots}^{k-matrices}\right] = eig(\lambda_i \mathbf{I}^{[k]})$$

In the latter case it is easy to see that the system has unique solution given by

$$\mathbf{g}_{k_i} + \mathbf{0}\mathbf{h}_{k_i} = \mathbf{C}_{k_i}$$

Clearly, for this situation, the elements of the transformation are zero since the nonlinear coordinate transformation cannot annihilate the nonlinear terms. Closed-form expressions are given below in terms of the fundamental mode of oscillation.

4.3 Closed-form time-domain solutions

Once the system has been transformed into the simplest form, time domain solutions are customarily determined from (4.1) by assuming that the higher order terms $O(\|\mathbf{x}\|^{[q+1]})$ are negligible. More generally, the simplest normal form can be written

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + \sum_{k=2}^{q} \mathbf{g}_k \mathbf{z}^{[k]} \tag{4.14}$$

The goal is to find the time evolution of the normal form states. The solution for the above system is not straightforward and the time domain solution depends on the structure of resonant terms.

4.3.1. *Nonlinear resonant conditions*

From the resonance conditions in (4.13), we may distinguish two cases are of especial interest.

(a). The resonant states are uncoupled. The time domain solution for these conditions is given by

$$\mathbf{z}(t) = e^{\Lambda t}\mathbf{z}_0 + \sum_{m=2}^{q} t^{(m-1)} \mathbf{g}_m e^{(\Lambda \otimes \mathbf{I}^{[m-1]})t} \mathbf{z}_0^{[m]} \tag{4.15}$$

(b). The resonant states are nonlinearly coupled. In this case, the time domain solution for this case cannot be determined in analytic form and can be expressed as

$$\mathbf{z}(t) = \int [\Lambda\mathbf{z} + \sum_{m=2}^{q} \mathbf{g}_m \mathbf{z}^{[m]}] dt \tag{4.16}$$

In the equations above, $\mathbf{z}_0$ are the initial conditions in normal form coordinates and their values depend of the initial conditions in Jordan space. The determination of initial conditions in normal form space is described in detail in succeeding sections.

4.3.2. *Non-resonant conditions*

Setting $\sum_{k=2}^{q} \mathbf{g}_k \mathbf{z}^{[k]} = 0$, and solving the linear system yields

$$\mathbf{z}(t) = e^{\Lambda t}\mathbf{z}_0 \tag{4.17}$$

These solutions can then be transformed back to the physical coordinates using the inverse nonlinear transformation (4.2) [6],[7].

We next examine the properties of time domain representations.

4.3.3. *Time domain solution in Jordan coordinates*

To obtain a better understanding of the nonlinear mode coupling phenomenon, we first analyze the nature of system response for the cases of interest above. Substituting (4.15) into the nonlinear transformation $\mathbf{y} = \mathbf{z} + \sum_{m=2}^{q} \mathbf{h}_m \mathbf{z}^{[k]}$. we obtain the following relationships.

a) Resonant condition

$$\mathbf{y}(t) = e^{\Lambda t}\mathbf{z}_0 + \sum_{m=2}^{q} (\mathbf{h}_m \left[e^{\Lambda t}\right]^{[m]} + t^{(m-1)}\mathbf{g}_m e^{(\Lambda \otimes \mathbf{I}^{[m-1]})t})\mathbf{z}_0^{[m]} \tag{4.18}$$

b) Non-resonant condition

$$\mathbf{y}(t) = e^{\Lambda t}\mathbf{z}_0 + \sum_{m=2}^{q} \mathbf{h}_k \left[e^{\Lambda t}\right]^{[m]} \mathbf{z}_0^{[m]} \tag{4.19}$$

By accounting for the resonant terms, a more accurate description of system's dynamic behavior is provided as suggested in (4.19).

4.3.4. *Approximate time-domain solutions in physical coordinates*

Using the linear transformation $\mathbf{x} = \mathbf{U}\mathbf{y}$, (4.18) may be solved immediately to yield

$$\mathbf{x}(t) = \mathbf{U}\mathbf{y}(t) = \mathbf{U}e^{\Lambda t}\mathbf{z}_0 + \mathbf{U}\sum_{m=2}^{q} (\mathbf{h}_m \left[e^{\Lambda t}\right]^{[m]} + t^{(m-1)}\mathbf{g}_m e^{(\Lambda \otimes \mathbf{I}^{[m-1]})t})\mathbf{z}_0^{[m]} \tag{4.20}$$

in which the initial conditions are determined from a nonlinear optimization problem as discussed below.

This final result has interesting physical implications.

- Equation (4.20) is quite important because it allows us to approximate system behavior by a linear combination of the fundamental modes of oscillation of the system.

- At $t=0$, we may therefore interpret $\mathbf{U}\tilde{\mathbf{h}}_k \mathbf{z}_0^{[k]}$ as the relative contribution of the higher order mode combination to the j th state.

- The nonlinear coefficients are usually ordered according to some physically motivated criterion (decreasing magnitude).

4.3.5. *Initial conditions*

The points that satisfy (4.17) lie in a hyperplane in the n-dimensional space. For the particular case in which matrix $\mathbf{A}$ is diagonalizable, the solutions $\mathbf{z}_0$ are complex-valued.

Since $\mathbf{x}_0$ is known, a basic solution to this nonlinear problem can be obtained using a Newton-based optimization technique. A brief outline of this procedure is given below:

1. Given the initial conditions, $\mathbf{x}_o$ in physical space, we seek an estimate for $\mathbf{z}_o$ by requiring this value to minimize the expression

$$\min \left\{ \mathbf{V}\mathbf{x}_0 - \mathbf{z}_0 - \sum_{m=2}^{q} \mathbf{h}_k \mathbf{z}_0^{[m]} \right\} \tag{4.21}$$

2. Solve the nonlinear optimization problem (4.21) using an appropriate numerical method

The treatment of this problem can be addressed by efficient nonlinear optimization methods. In Barocio *et al.* [3] and Liu *et al* [7] are given several directions

in order to solve this problem.

Table 4.1 compares the nature of normal form solutions for the two cases of interest.

Table 4.1: Nature of system solutions

Solution characteristic	Non-diagonalizable system	Diagonalizable system
Initial conditions in physical space	Real	Real
Initial condition in Jordan space		Complex
Initial conditions in NF space	Real-valued	Complex-valued
Normal form transformation coefficients	Real-valued	Complex-valued

4.4 Quantification of nonlinear modal interaction

An attractive feature of normal form analysis is that it enables to characterize and quantify nonlinear mode interaction. For convenience, the equations describing the time evolution of the system stares are repeated below.

In the simple case where no resonant conditions hold, the time evolution of the modal components can be written as

$$\mathbf{y}(t) = e^{\Lambda t}\mathbf{z}_0 + \sum_{m=2}^{q} \mathbf{h}_k \left[e^{\Lambda t}\right]^{[m]} \mathbf{z}_0^{[m]} \tag{4.22}$$

At $t = 0$, (4.22) becomes

$$\mathbf{y}(0) = \mathbf{z}_0 + \sum_{m=2}^{q} \mathbf{h}_k \mathbf{z}_0^{[m]} \tag{4.23}$$

In analogy with [8], we generalize the definition of nonlinear interaction indexes for mode j, $I1_{2\lambda_j}$ to the higher order case:

$$
\begin{aligned}
I1_{2\lambda_j} &= \left|y_{0_j}\right| - \left|z_{0_j}\right| + \max\left(\left|\mathbf{h}_{2_j}\mathbf{z}_0{}^{[2]}\right|\right) \\
I1_{3\lambda_j} &= \left|y_{0_j}\right| - \left|z_{0_j}\right| + \max\left(\left|\mathbf{h}_{2_j}\mathbf{z}_0{}^{[2]}\right|\right) + \max\left(\left|\mathbf{h}_{3_j}\mathbf{z}_0{}^{[3]}\right|\right) \\
&\vdots \\
I1_{q\lambda_j} &= \left|y_{0_j}\right| - \left|z_{0_j}\right| + \max\left(\left|\mathbf{h}_{2_j}\mathbf{z}_0{}^{[2]}\right|\right) + \cdots + \max\left(\left|\mathbf{h}_{q_j}\mathbf{z}_0{}^{[q]}\right|\right)
\end{aligned}
\tag{4.24}
$$

These indices determine the magnitude of the maximum components of nonlinear part for two or more modes. The maximum entries determine the dominant mode combinations interacting nonlinearly.

Remark 1: Note that higher order interaction indices defined in (4.24) are obtained from a single-step high order optimization problem (refer to 4.21) whilst those existing in the literature are obtained form a series of truncated optimization problems as suggested in Chapter 2. As a result, the indices in (4.24) provide a more accurate characterization of system behavior.

Similar to the above case, we can define a normalized index

$$
\begin{aligned}
I2_{2\lambda_j} &= \frac{\max\left(\left|\mathbf{h}_{2_j}\mathbf{z}_0{}^{[2]}\right|\right)}{\mathbf{z}_{0_j}} \\
I2_{3\lambda_j} &= \frac{\max\left(\left|\mathbf{h}_{3_j}\mathbf{z}_0{}^{[3]}\right|\right)}{\mathbf{z}_{0_j}} \\
&\vdots \\
I2_{q\lambda_j} &= \frac{\max\left(\left|\mathbf{h}_{q_j}\mathbf{z}_0{}^{[q]}\right|\right)}{\mathbf{z}_{0_j}}
\end{aligned}
\tag{4.25}
$$

Again, these measures generalize existing indices in the literature to the high dimensional case [8].

In this case the nonlinear part is normalized with respect to the linear part in order to determine the proportion with respect of the components to respect to the Jordan variables. This measures the contribution of the linear part and nonlinear

part. These two indices are used in the application of the method to assess the extent and distribution of nonlinearity in the system.

To conclude this section, we note that it is possible to generalize these definitions to higher dimensional systems. The analysis of contribution factors in physical space follows the same lines and is therefore omitted.

We now illustrate the procedure by the following example in five dimensions.

4.5 Assessment of nonlinearity in physical coordinates

Following the approach in [7][8], the concept of nonlinear contribution factors is extended to the high dimensional case. In the power system model, of particular interest are the contribution factors associated to speeds of the machines. Making use of (4.20), the time evolution of the system states can be expressed

$$\mathbf{x}_i(t) = P_{1_i} e^{\Lambda t} + \sum_{m=2}^{q} P_{m_i} \left[e^{\Lambda t}\right]^{[m]} \qquad i = 1,2,\cdots,ng \tag{4.26}$$

where P_1 and P_m are row vectors of dimensions n and n^m respectively. The entries of these rows represent the linear and nonlinear contributions of order m respectively, of the higher order modes to the system states. These contribution factors are defined as

$$P_{1_i} = \mathbf{U}_i \cdot \mathbf{z}_0^T \tag{4.27}$$

$$P_{m_i} = (\mathbf{U}_i \mathbf{h}_m) \cdot (\mathbf{z}_0^{[m]})^T \tag{4.28}$$

4.6 Analysis of nonlinear systems with strong resonance conditions

To give an idea of how a general high-order resonant system is treated under the proposed approach, consider the five-dimensional first order nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -\eta & 1 & & & \\ -1 & -\eta & & & \\ & & -\sigma & 1 & \\ & & -1 & -\sigma & \\ & & & & -\mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \mathbf{A}_2 \mathbf{x}^{[2]} \tag{4.29}$$

where

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By adjusting the parameters η, μ, σ it is possible to obtain a variety of nonlinear behavior.

It then follows that the eigenvalues of the linear part are

$$\lambda_{1,2} = -\eta \pm j \quad \lambda_{3,4} = -\sigma \pm j \quad \lambda_5 = -\mu$$

with eigenvectors

$$\mathbf{U} = \begin{bmatrix} -j & j & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -j & j \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} j\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -j\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & j\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & -j\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

As pointed out before, the system exhibits a wealth of dynamic behavior. First, we note that for $\eta = \sigma = 0$, the system exhibits double Hopf bifurcation points characterized by two pairs of complex conjugate eigenvalues, $\lambda_{1,2} = \lambda_{3,4} = \pm j$ Depending on the selection of the physical parameters, other interesting behavior is obtained as discussed below.

Prior to the application of normal form theory, we introduce the linear transformation $\mathbf{x} = \mathbf{U}\mathbf{y}$. Applying the above change of coordinates, we write the system (4.29) in the form

$$\dot{\mathbf{y}} = \begin{bmatrix} -\eta + j & 0 & 0 & 0 & 0 \\ 0 & -\eta - j & 0 & 0 & 0 \\ 0 & 0 & -\sigma + j & 0 & 0 \\ 0 & 0 & 0 & -\sigma - j & 0 \\ 0 & 0 & 0 & 0 & -\mu \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} + \mathbf{C}_2 \mathbf{y}^{[2]} \tag{4.30}$$

where $\mathbf{C}_2 = \mathbf{V}\mathbf{A}_2\mathbf{U} \otimes \mathbf{U}$ is a nxn^2 matrix.

To transform the system (4.30) to a normal form we consider a one-step nonlinear transformation of the form $\mathbf{y} = \mathbf{z} + \sum_{m=2}^{3} \mathbf{h}_m \mathbf{z}^{[m]}$ On substituting this transformation into (4.30) we find the set of algebraic equations , ($q > p$)

$$\begin{aligned} \mathbf{g}_{2\,S}^{T} + \left[\mathbf{I} \otimes \left[\Lambda \otimes \mathbf{I} + \mathbf{I} \otimes \Lambda\right] - \Lambda \otimes \mathbf{I} \otimes \mathbf{I}\right] \mathbf{h}_{2\,S}^{T} &= \mathbf{C}_{2\,S}^{T} \\ \mathbf{g}_{3\,S}^{T} + \left[\mathbf{I} \otimes \left[\Lambda \otimes \mathbf{I} \otimes \mathbf{I} + \mathbf{I} \otimes \Lambda \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{I} \otimes \Lambda\right] - \Lambda \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}\right] \mathbf{h}_{3\,S}^{T} &= \overline{\mathbf{C}}_{3\,S}^{T} - \overline{\overline{\mathbf{g}}}_{3\,S}^{T} \end{aligned} \tag{4.31}$$

where $\mathbf{C}_3 = 0$, $\overline{\mathbf{C}}_3 = \mathbf{C}_2[\mathbf{I} \otimes \mathbf{h}_2 + \mathbf{h}_2 \otimes \mathbf{I}]$ and $\overline{\overline{\mathbf{g}}}_3 = \mathbf{h}_2\left[\mathbf{g}_2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{g}_2\right]$.

The normal form transformation coefficients are then solved for from this set of coupled, algebraic equations. These equations are solved sequentially, since third-order terms depend on second-order terms. The same procedure may be used to obtain higher order relationships.

For this system, the most interesting case of nonlinear resonance occurs when the sum of two eigenvalues equals to a third, namely

$$\eta + \mu = \sigma$$

Analytical solutions for the cases of interest are given below.

4.6.1. *Non resonant conditions*

A particular case of non-resonant condition occurs when the parameters of the linear part take the values

$$\eta = 2,\ \mu = 2,\ \sigma = 3$$

In accordance with the above discussion, the homological equations (4.31) become

$$\begin{aligned}\left[\mathbf{I}\otimes(\Lambda\otimes\mathbf{I}+\mathbf{I}\otimes\Lambda)-\Lambda\otimes\mathbf{I}\otimes\mathbf{I}\right]\mathbf{h}_{2S}^{T}&=\mathbf{C}_{2S}^{T}\\ \left[\mathbf{I}\otimes(\Lambda\otimes\mathbf{I}\otimes\mathbf{I}+\mathbf{I}\otimes\Lambda\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{I}\otimes\Lambda)-\Lambda\otimes\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{I}\right]\mathbf{h}_{3S}^{T}&=\overline{\mathbf{C}}_{3S}^{T}\end{aligned} \tag{4.32}$$

Figure 4.1 shows the magnitude of the corresponding transformation coefficients, $\mathbf{h}_2$ and $\mathbf{h}_3$, obtained from (4.32).

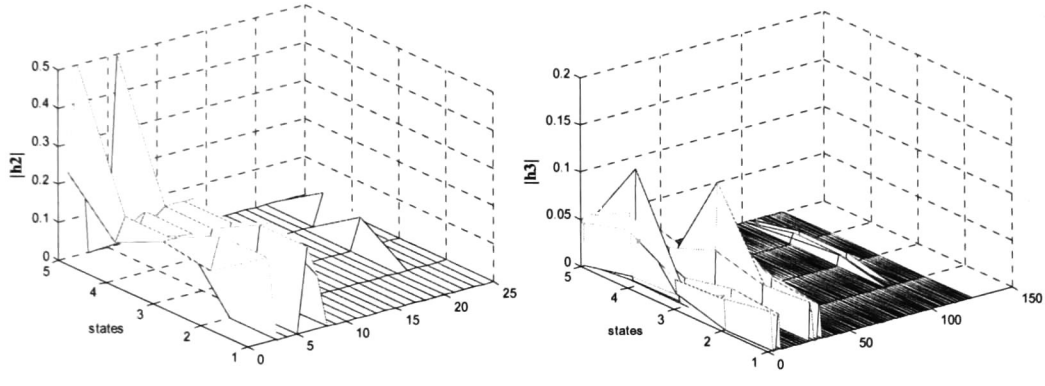

Figure 4.1. Coefficients of the third order nonlinear transformation, Non-resonant case.

Then the simplest normal form for this case is

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\\ \dot{z}_5\end{bmatrix}=\begin{bmatrix}(-2+j)z_1\\ (-2-j)z_2\\ (-2+j)z_3\\ (-2-j)z_4\\ -3z_5\end{bmatrix} \tag{4.33}$$

Solution for this case is straightforward and is therefore omitted.

4.6.2. *Second-order resonant condition*

One further special case occurs for the values

$$\eta=1,\ \mu=1,\ \sigma=2$$

It can be easily seen that this particular choice of parameters results in the sec-

second-order resonance conditions:

$$\lambda_1 + \lambda_5 = \lambda_3$$
$$\lambda_2 + \lambda_5 = \lambda_4$$

From these conditions, the homological to be solved acquire the structure of (4.31). For the second order coefficients two homological equations to be solved

$$\begin{aligned}
\hat{\mathbf{g}}_{23,5} + (\lambda_1 + \lambda_5 - \lambda_3)\hat{\mathbf{h}}_{23,5} &= 0.25 \\
\hat{\mathbf{g}}_{23,21} + (\lambda_5 + \lambda_1 - \lambda_3)\hat{\mathbf{h}}_{23,21} &= 0.25 \\
\hat{\mathbf{g}}_{24,10} + (\lambda_2 + \lambda_5 - \lambda_4)\hat{\mathbf{h}}_{24,10} &= 0.25 \\
\hat{\mathbf{g}}_{24,22} + (\lambda_5 + \lambda_2 - \lambda_4)\hat{\mathbf{h}}_{24,22} &= 0.25
\end{aligned} \tag{4.34a}$$

$$\hat{\hat{\mathbf{h}}}_{k_i} = \left[\overline{\overline{\mathbf{\Pi}}}^{i}\right]^{-1}\hat{\hat{\mathbf{C}}}_{k_i} \tag{4.34b}$$

Following the previous analysis we determined the structure of the nonlinear system representation. Figure 4.2 shows, schematically, the terms of the nonlinear part of the vector field associated with these singularities along with the location of resonances. Here, a circle indicates the elements of vector field that can be annihilated and the x's denote the resonant elements that are retained in the simplest normal form. These terms which can not be eliminated are precisely those nonlinear terms appearing in the simplified normal form below.

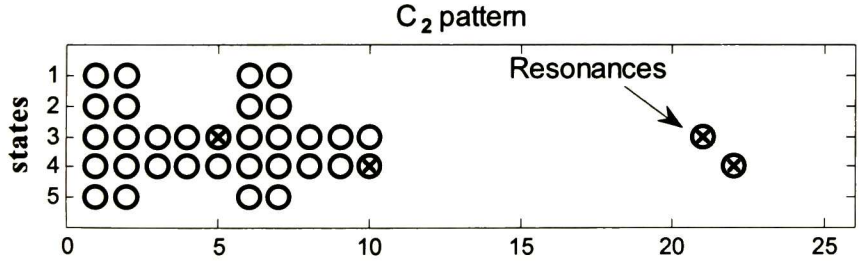

Figure 4.2. Pattern of C_2 indicating the location of resonant elements.

To complete the analysis we take the system to the SNF. In the case of two resonant modes, the SNF is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \end{bmatrix} = \begin{bmatrix} (-1+j)z_1 \\ (-1+j)z_2 \\ (-2+j)z_3 + 0.5z_1 z_5 \\ (-2-j)z_4 + 0.5z_2 z_5 \\ -z_5 \end{bmatrix} \tag{4.35}$$

which has a general solution of the form

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \\ z_5(t) \end{bmatrix} = \begin{bmatrix} e^{(-1+j)t} z_{01} \\ e^{(-1-j)t} z_{02} \\ e^{(-2+j)t}(z_{03} + 0.5tz_{01}z_{05}) \\ e^{(-2-j)t}(z_{04} + 0.5tz_{02}z_{05}) \\ e^{-t} z_{05} \end{bmatrix} \tag{4.36}$$

Comparing the SNF found by the proposed technique with the results of CNF theory, we observe some differences.

(a). For the non-resonant case, the results obtained above completely agree with those obtained using CNF theory. In this case, it is straightforward to show that $\mathbf{\Gamma}_3 + \mathbf{\Psi}_3 + \mathbf{\Omega}_3 = 0$ (refer to Table 3.1). The details are omitted.

(b). Observe that the above solution differs from those obtained for the non-resonant case since it contains a second component that depends on time.

(c). CNF analysis in general, can not treat resonant cases

Figure 4.3 shows schematically the magnitude of the third-order nonlinear transformation coefficients for the resonant case.

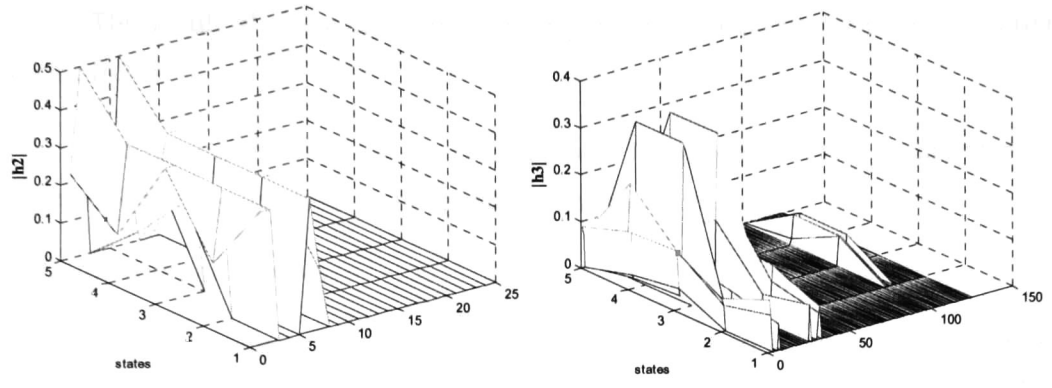

Figure 4.3. Coefficients of the third order nonlinear transformation. Resonant case.

Careful examination of these plots indicates that the second order and third order elements of the transformation exhibit similar order of magnitude. For the case of the third order elements $\mathbf{h}_3$, these are composed by the residual terms and the second order resonant residual terms $\overline{\overline{\mathbf{g}}}_3$. Figure 4.4 illustrates the magnitude of the elements associated with the residual resonant terms, $\overline{\overline{\mathbf{g}}}_3$.

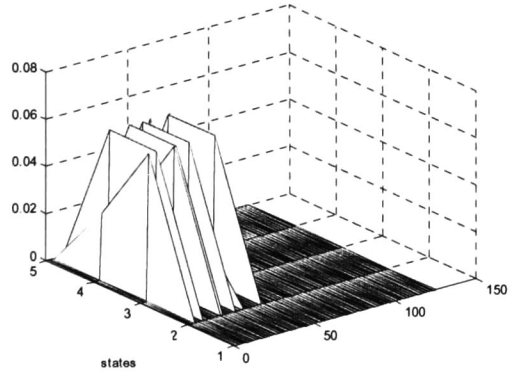

Figure 4.4. Magnitude of resonant residual terms $\overline{\overline{\mathbf{g}}}_3 = \mathbf{h}_2[\mathbf{I} \otimes \mathbf{g}_2 + \mathbf{g}_2 \otimes \mathbf{I}]$.

The analysis of residual terms in Fig. 4.4 indicates that while residual terms are small compared with the third order terms, their influence has to be accounted for to capture system's dynamics properly. It is important to emphasize that, unlike CNF theory in which these terms are disregarded, the proposed formulation assumes these terms to be small but not negligible.

The result above can be applied in general no resonant and non-resonant vector fields.

Further discussion of these concepts will be presented later in Chapter 5 in relation to the practical use of the proposed methodology.

4.7 Concluding remarks

In this chapter, a higher order normal form representation suitable for the analysis of nonlinear behavior has been proposed. A unique feature of the above procedure is its ability to avoid higher-order resonances arising from the interactions of multiple eigenvalues. The central idea behind this approach is a real, normal change of coordinates in physical space. This procedure allows both, the treatment of strong resonant and non-resonant power systems and the derivation a general purpose higher order normal form method.

Compared to the conventional normal form analysis approach, the analysis procedure described in this paper contains two additional steps.

Perturbation theories based on this approach offer three substantial advantages: (i) they yield the transformation of state variables in real coordinates, (ii) these transformations are function of the original variables, and (iii) the inverse transformation can be built in the same way. The distinction between real and complex normal form formulations characteristics is clearly pointed out. This aspect is entirely new: all previous formulations

Analysis of theoretical systems illustrates that conventional NF results and interpretation can be affected by both, linear and higher order resonances. The analysis of these factors in this research motivated the need for improved modeling and analysis techniques, particularly with respect to the analysis of complex power systems. The proposed approach allows routine incorporation of unaccounted nonlinear terms

The information obtained form this analysis could be used to provide in-

sight into the selection of appropriate initial approximation to system solutions, range of physically meaningful parameters, and ultimately to check on the adequacy of the NF method used in the analysis. Furthermore, dynamic models obtained using normal form theory can be quantitatively validated via comparisons with detailed system simulations.

4.8 References

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Chapter 5

Results

This chapter discusses the application of the developed procedures to the analysis of power system dynamic behavior. The utilized methodology is tested on two practical systems. The first system is a two-area, four-machine test system which has been used extensively in the literature to investigate the onset of nonlinear behavior. The second test system is a sixteen-machine reduced order model of a real power system for which extensive linear analysis results are available.

The analysis consists of two parts. In the first part, real normal form theory is used to study a resonant power system in which voltage control leads to a near-strong resonance condition involving two inertial modes of oscillation. Attention is focused on the analysis of nonlinear behavior in the neighborhood of resonant vector fields. The influence of near-resonant conditions on the normal form computation and interpretation is discussed.

In the second part, higher order normal form analysis is used to study the influence of nonlinear terms on system behavior. Based upon a third-order system representation, normal form analysis studies are performed to analyze the physical mechanisms underlying the dynamical interactions among modes. The study concentrates on the ability of the method to analyze various aspects of system non-linear behavior following large perturbations. The efficiency and accuracy of the method is demonstrated by comparisons to fully nonlinear simulations.

Finally, a review of the results and an outline of some directions for further work is presented.

5.1 Real normal form analysis of a near-resonant power system

In this section, normal form theory applied to investigate the ability of normal form analysis to analyze resonant vector fields. The study generalizes conventional normal form analysis to the study of resonant power systems and provides a more accurate characterization of nonlinearity in physical coordinates.

5.1.1. *The test system*

Figure 5.1 shows the system under investigation adapted from Refs. [1],[2]. The test system comprises two areas connected by a relatively weak tie, 4 generators, 11 buses and two loads. The generator and system parameters are given in Appendix A.

For the purposes of this study, all generators are represented by a fourth order $d-q$ axis model and equipped with a simple dc exciter; the loads are modeled by constant impedances and the network is reduced to the generator internal nodes. The case of operation under study corresponds to an interchange of energy between areas of 250 MW sending from Area 1 to Area 2. For this condition, all four generators have similar loadings

5.1.2. *Modeling considerations*

Modeling considerations are essentially those in Ref. [3]. For the purposes of this study, a second-order approximation is adopted to analyze system dynamic behavior.

The power or series expansion about the equilibrium is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{A}_1\mathbf{x} + \frac{1}{2}\mathbf{A}_2(\mathbf{x}\otimes\mathbf{x}) \tag{5.1}$$

with initial conditions $\mathbf{x}(0) = \mathbf{x}_o$, where $\mathbf{A}_1$ contains the real part of the original vector field, and $\mathbf{A}_2$ accounts for the second-order approximation to the system model.

5.1.3. *Modal analysis*

Modal analysis was conducted to determine both, the dominant natural modes of oscillation, and the machines with the largest participation in these modes. In addition, detailed linear system studies were conducted to determine operating conditions resulting in near strong resonance conditions.

Table 5.1 lists the eigenvalues of the system for the base case condition. The system exhibits three electromechanical modes of interest [2]:

- One inter-area mode at 0.50 Hz associated with the exchange of energy between the two areas, and
- Two electromechanical modes associated with the local behavior of Areas 1 and 2.

In what follows we investigate the potential for nonlinear behavior arising from interaction between these modes.

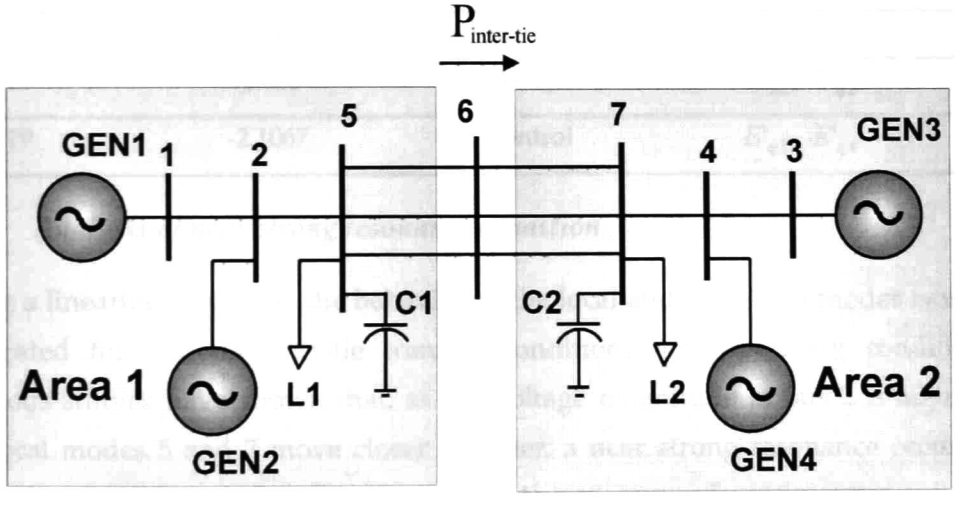


Figure 5.1. Two-area four-machine test system.

Table 5.1: System eigenvalues for the base case condition.

Mode #	Value	Mode Type	Source
1	-46.8675	Control	E_{fd1}, E_{fd2}
2	-46.9437	Control	E_{fd3}, E_{fd4}
3	-38.3565	Control	E_{fd3}, E_{fd4}
4	-36.1104	Control	E_{fd1}, E_{fd2}
5,6	-1.3518 ±j8.0783	Plant	ω_1, ω_2
7,8	-1.3397±j8.0964	Plant	ω_3, ω_4
9,10	-10.4572±j1.5811	Control	E'_{q1}, E'_{q2}
11,12	-9.8982±j1.9250	Control	E'_{q3}, E'_{q4}
13,14	-0.2474 ±j3.1625	Inter-area	ω_1, ω_2 vs. ω_3, ω_4
15	-6.4603	Flux	E'_{d1}, E'_{d2}
16	-7.5086	Flux	E'_{d4}
17	-0.4481	Reference	--
18	-2.053	Control	E'_{q1}, E'_{q2}
19	-2.1067	Control	E'_{q3}, E'_{q4}

5.1.4. *Analysis of near strong resonance condition*

Using a linearized analysis, the behavior of the local and inter-area modes was investigated for various inter-tie transfer conditions and operating conditions. Previous studies have shown that, as the voltage magnitude at bus 2 is adjusted the local modes 5 and 7 move closer together; a near strong resonance occurs at about $V_2 = 1.018$ p.u. as shown in Figure 5.2 [4].

To facilitate the analysis and interpretation of nonlinear behavior, two cases are considered: (i) Case 1 representing a non-resonant condition, and (ii) Case 2 representing a near strong resonance condition.

Case 1 illustrates the normal operating performance characterized by different eigenvalues. A standing assumption in this analysis is that all eigenvalues are different.

Case 2 is essentially the situation used by Betancourt *et al.* to illustrate voltage collapse [4]. At the resonant condition, two eigenvalues coincide in magnitude and damping leading to a complex dynamic behavior [5],[6].

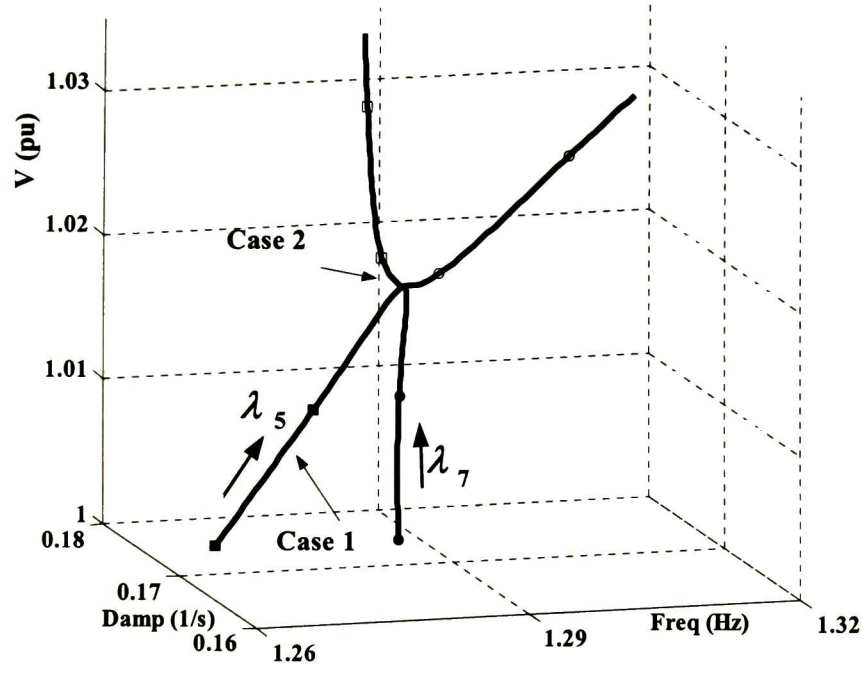


Figure 5.2. Root loci of local modes 5 and 7

5.1.5. *Behavior of second order normal form coefficients near a strong resonance*

In an effort to asses the effect of resonance on the performance of the method, the real normal forms from (3.23) and the complex normal forms coefficients from the analysis of the CNF in (2.57) were determined. Figure 5.3 shows the behavior of the normal form coefficients as V_2 is varied whilst Figure 5.4 depicts the behavior of the corresponding coefficients, obtained using the quadratic real normal form representation.

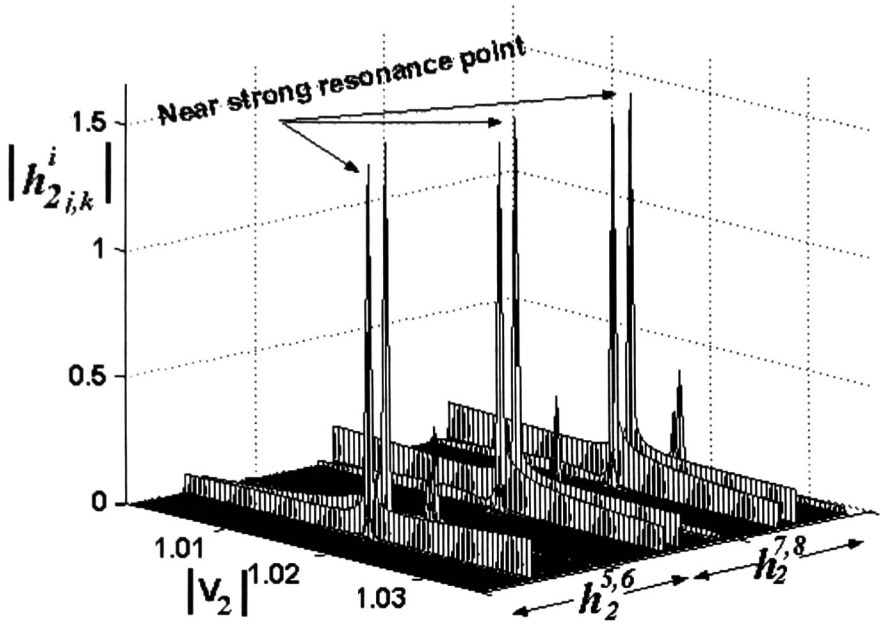


Figure 5.3. Behavior of complex normal form coefficients as V_2 is varied

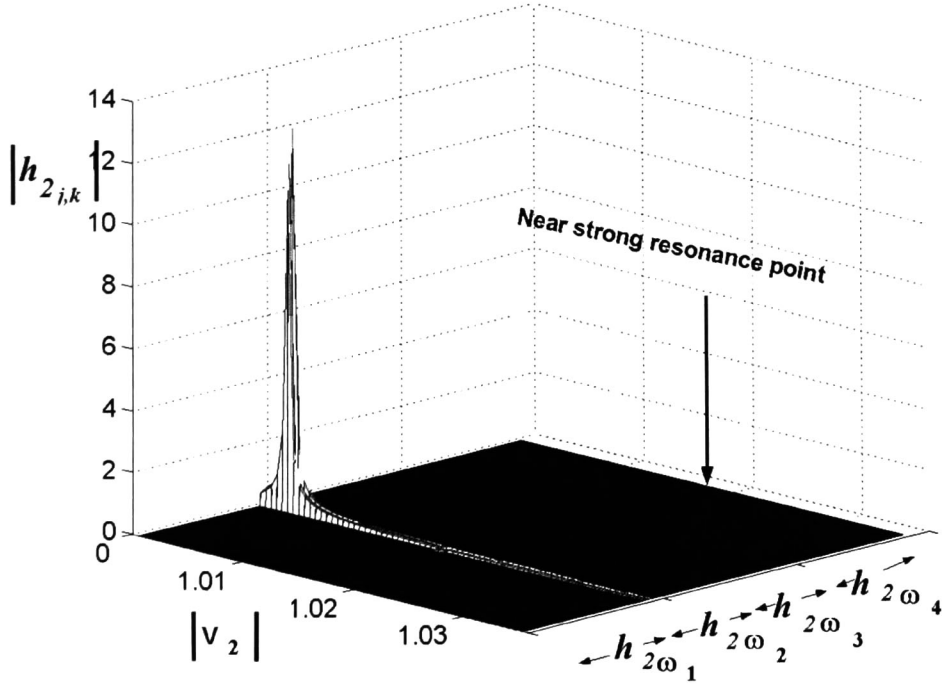


Figure 5.4. Behavior of the real normal form coefficients as V_2 is varied

From the above results, we can observe that the real normal form presents a smooth behavior close to the resonance condition. In contrast to this, the sensitivity of the conventional normal form coefficients to the change of the voltage at bus 2 is higher. This analysis confirms the results obtained for the symbolic case in Chapter 3. Also of interest, the peaks in Figure 5.3 are associated with the local modes 5-6 and 7-8, in agreement with the results obtained in previous section.

In order to determine the significance of the real normal form transformation under near resonant conditions, we computed second order contribution factors using the coefficients of the real normal form for Cases 1 and 2. To this end, a three phase stub fault at bus 5 cleared in 0.05 seconds was applied.

To provide a basis of comparison with the results obtained using linear analysis, the speed deviations of the system generators were expressed in the form (4.28) for $m = 2$

$$\mathbf{x}_{\omega_i}(t) = P_{1_i} e^{\Lambda t} + P_{2_i} e^{(\Lambda \otimes I + I \otimes \Lambda)t} \qquad i = 1,2,\cdots,4 \tag{5.1}$$

where P_{2_i} is a row vector describing the contribution of the mode combination $(\lambda_k + \lambda_l)$ to the speed deviation of the i th-machine. Using this representation, several interaction measures were computed using the procedures outlined in Chapter 4.

Figure 5.5 shows the largest second-order contribution factors to speed deviations, $max(P_{2_i})$ along with the interacting modes following a stub fault at bus 5.

The results are found to be very accurate. However, as previously noted, conventional interaction coefficients $h_2 z_0^{[2]}$ tend to infinity as the resonance is approached making the assessment of nonlinearity in physical space unreliable [5]. Thus, for instance, for mode 9 (E'_{q1}, E'_{q2}), $h_2 z_0^{[2]}$ increases from 160.04 for Case 1 to 186.01 for Case 2, and for mode 11 (E'_{q3}, E'_{q4}) from 114.37 for Case 1 to 179.1 for Case 2. Closer to resonance the coefficients h_2 become arbitrarily large, even if nonlinearity is small.

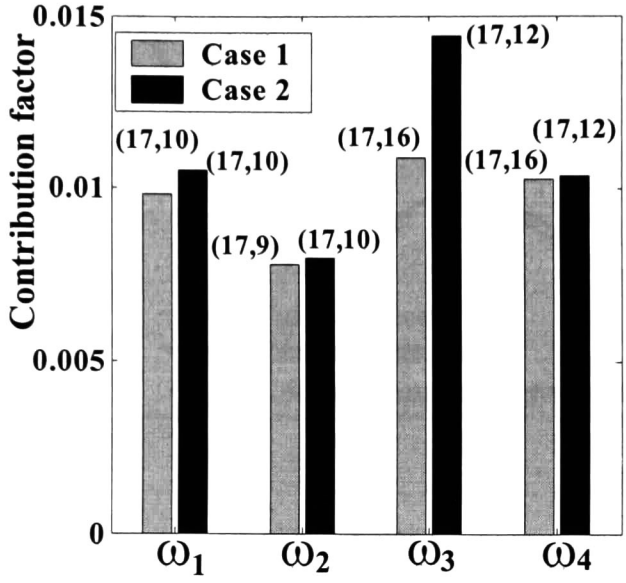


Figure 5.5. Nonlinear speed-based contribution factors.

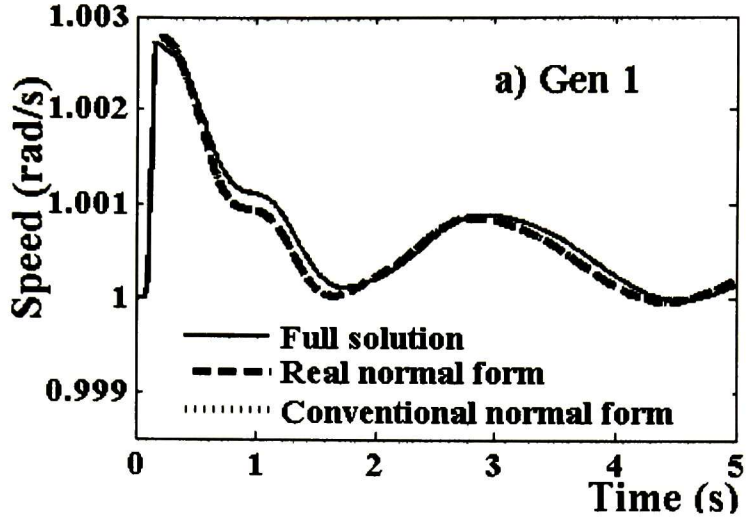
5.1.6. *Comparison with time domain solutions*

In order to verify our previous findings and determine the accuracy of the real normal form representations, time domain solutions were determined. Figure 5.6 provides a comparison of the speed deviations of generator 1 and 3 obtained from the full system solution $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with the solution from (4.20) for $q = 2$, and the approach in [2]. We use as initial condition, the values at clearing time determined using a transient stability program as explained below.

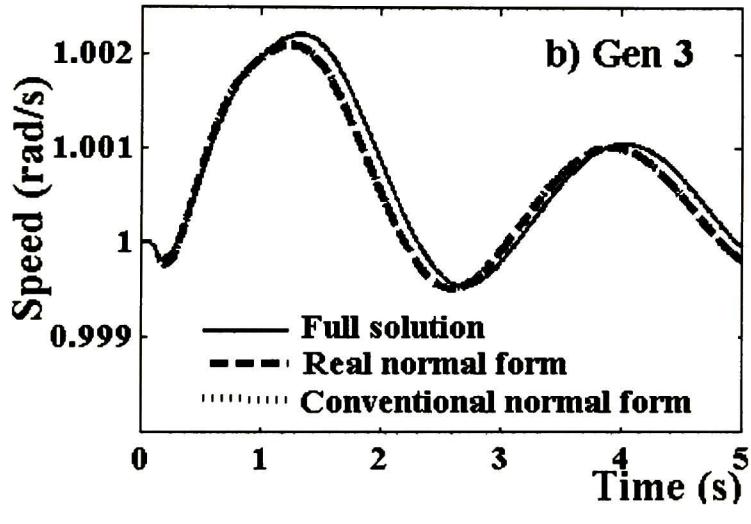
The analysis of initial conditions in physical space consists of two parts:

1). For a given disturbance, determine the conditions $\mathbf{x}_{cl}$, at the end of the disturbance using conventional time-domain simulation.

2). Move the origin to the post-disturbance SEP, such that $\mathbf{x}_o = \mathbf{x}_{cl} - \mathbf{x}_{SEP}$, and obtain the initial conditions in the Jordan and NF spaces using the procedure in sections 4.3.5.



a) Generator 1



b) Generator 3

Figure 5.6 Comparison of rotor speed deviations

The results of the simulations are undistinguishable showing the correctness of the results. Similar results are obtained for other generators and are not presented here.

5.2 Higher-order normal form analysis of stressed power systems

Highest order normal form computation is demonstrated on a reduced version of the NPCC system which contains 16 generators and 68 buses [7],[8]. Attention is restricted to the study of the influence of third-order effects in the series expansion of the power system representation on system behavior.

5.2.1. *The 16-machine NPCC system*

Figure 5.7 provides a schematic illustration of the 16-machine NPCC showing the location of coherent areas and major inter-ties [7],[8]. This system consists of five coherent areas designated as Area 1 (A1), Area 2 (A2), Area 3 (A3), Area 4(A4), and Area 5 (A5).

The coherent machine groups are [8]:

Table 5.2: Coherent groups for the NPCC system

AREA 1	AREA 2	AREA 3	AREA 4	AREA 5
G1-G9	G10-G13	G14	G15	G16

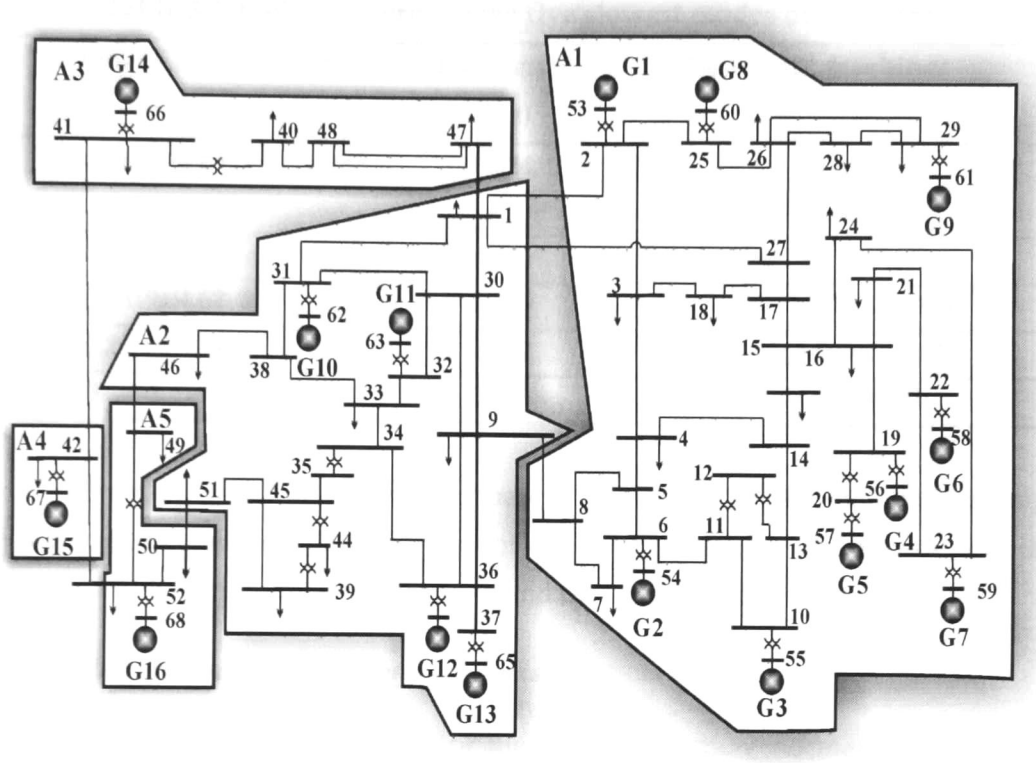


Figure 5.7. One line diagram of the five-area, 16-machine test system.

For the purpose of this analysis, all machines are represented by a fourth order model and equipped a simple gain dc exciter. In this representation, the loads are modeled as constant impedances and the network is reduced to the generator internal nodes. The system operating conditions and machine data are taken from [7]. The base case condition corresponds to a total demand of 18,234 MW.

5.2.2. *Small signal analysis*

Modal analysis was used to determine the dominant natural modes of oscillation and the machines having the largest participation in these modes. For the base case condition, the 16-machine NPCC system exhibits five critical lightly damped inter-area modes with damping ratios below 5% at 0.42, 0.55, 0.73, and 0.80 Hz. A higher frequency mode at about 1.05 Hz is also identified representing an oscillation local to Area 1. Table 5.2 shows the oscillation frequency and the oscillation patterns of these modes.

Table 5.3: Swing pattern for the five slowest modes of the system

Eigenvalue	Mode	Frequency (Hz)	Dominant machines
-0.0686±j2.66	48	0.424	9(**A1**) 13(**A2**) vs. 14(**A3**),15(**A4**),16(**A5**)
-0.1230±j3.47	46	0.553	14(**A3**) vs. 16(**A5**)
-0.0819±j4.62	44	0.735	9(**A1**) vs. 12(**A2**),13(**A2**)
-0.2360±j5.04	42	0.802	14(**A3**) 15(**A4**) vs. 16(**A5**)
-0.2436±j6.75	40	1.075	2(**A1**), 3(**A1**), 5(**A1**)

From Table 5.3, it is readily apparent that modes 48, 46 and 42 have a significant participation in the machines of areas A3, A4 and A5. The modes excited depend on the structure of the network and the location of the disturbance.

5.2.3. *Higher-order normal form analysis*

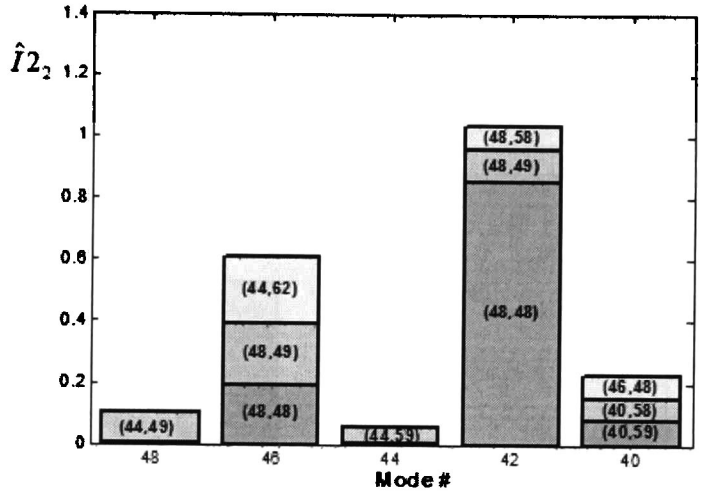
In the light of the linear analysis of the previous section, normal form analyses were conducted to investigate the potential for nonlinear behavior arising from the interaction of the critical inter-area modes as well as to assess the influence of higher order terms on system dynamic performance. The critical contingency is a three phase fault at bus 27 cleared by opening the line between buses 1 and 27. This fault triggers poorly damped oscillations of the dominant modes associated with various geographical regions.

Normal form coefficients were computed following the procedure described in section 4.2.2. In this analysis a third order normal form transformation ($q = 3$) was used to annihilate second order nonlinear terms ($p = 2$). The adopted analysis approach includes two main steps:

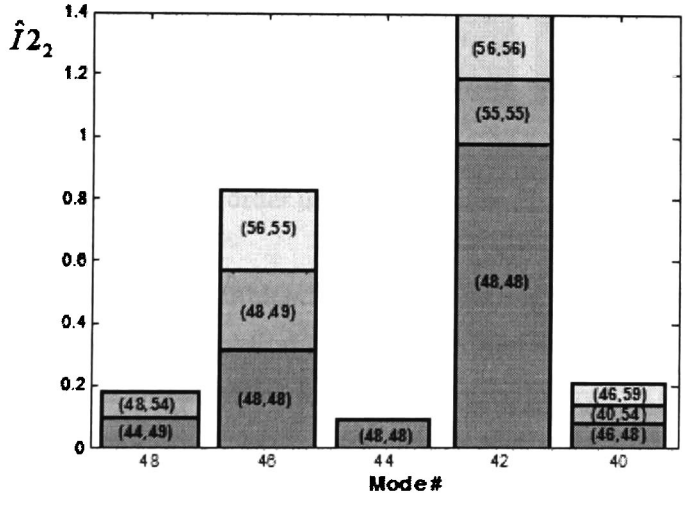
(i) The perturbed system (2.5) is represented in the Jordan canonical form by a linear transformation of the original variables, and

(ii) The system is represented by the simplest normal form by a one-step nonlinear coordinate transformation.

Figures 5.8 and 5.9 show the second and third-order interaction indices, $I2_{2_{\lambda_j}}$ $I2_{3_{\lambda_j}}$ obtained from the proposed procedure in section 4.4, for the five inter-area modes in Table 5.3. For comparison purposes, the interaction indices obtained from conventional normal form theory and the proposed approach ($\hat{I}2_{2_{\lambda_j}}$) are also shown in Figure 5.8a).



a) Conventional normal form theory



b) Proposed formulation

Figure 5.8 Comparison of second-order interaction indices

Of particular interest, a study of the second-order nonlinear interaction indices in Figures 5.8a) and 5.8b) reveals a strong interaction between modes 48 and 44 and mode 56. Careful inspection of the second-order indices obtained using the proposed procedure, however, discloses the presence of mode 55 associated with the controls of machine 15, and mode 57 involving the participation of the control states of machines 12 and 14.

The corresponding third-order interaction indices, ($I2_{3_{\lambda_j}}$), for the modes of interest, are shown in Fig. 5.9.

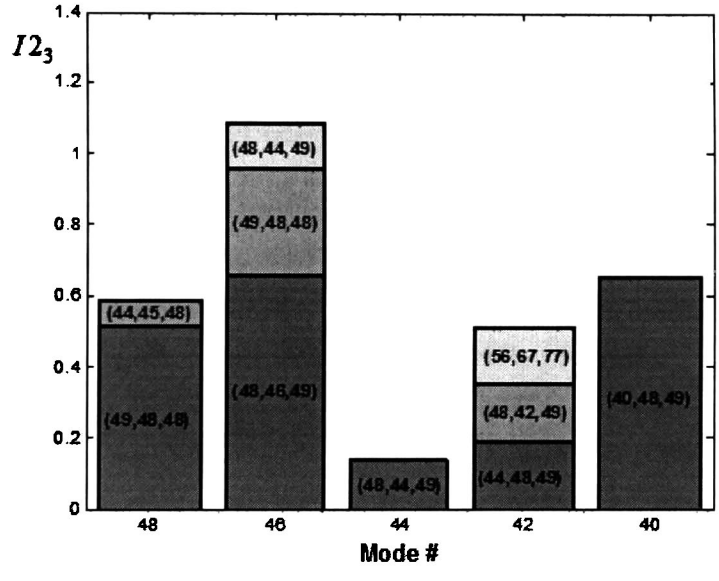


Figure 5.9. Third order interaction index $I2_{3_{\lambda_j}}$

The analysis of third-order interaction indices in Figure 5.9 confirms these findings. It is also interesting to note that the magnitude of third-order interactions is of the same order of magnitude that the second-order interaction showing the presence of higher-order nonlinear modal interaction and increased nonlinearity. This is an important finding since the analysis suggests that third order effects may become significant under high stress coupled with severe disturbances.

The result of primary importance to us is that third-order interaction modes

may exist that are not properly accounted for in conventional analysis.

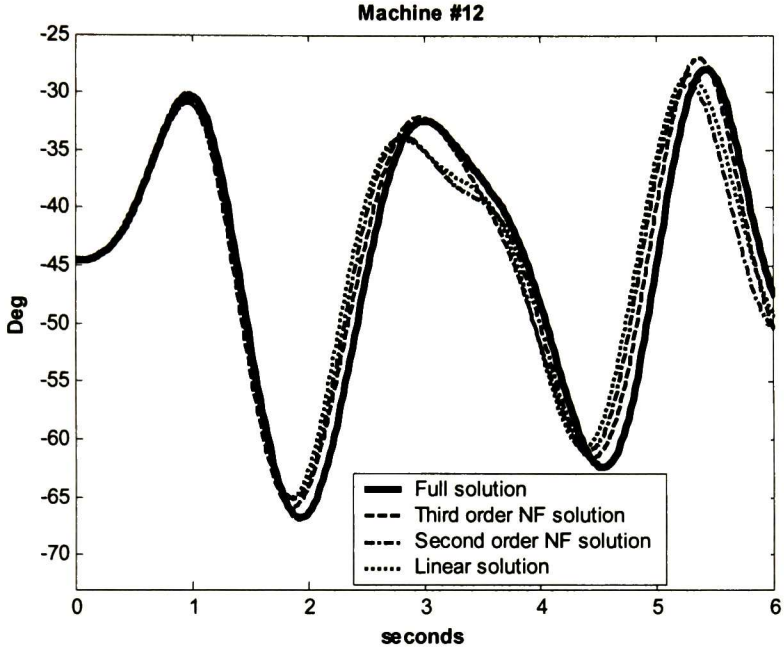
5.2.4. *Validation*

Detailed time-domain simulations were conducted to examine system performance following large perturbations. The same contingency case previously analyzed is used to test the accuracy of the procedure for determining approximate closed-form time-domain solutions.

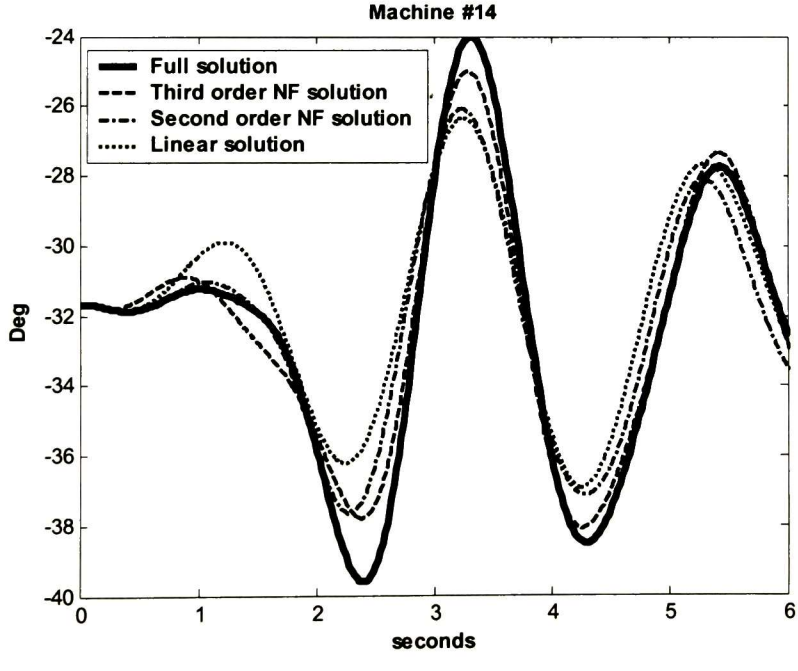
Based on the analysis of interaction indices, machines # 12, 14, and 15 were selected for study. Three system representations were tried in the study, namely: (1) a linear approximation $\dot{\mathbf{x}} = \mathbf{f}_1\mathbf{x}$ obtained by ignoring higher order terms in (2.2), (2) a second order system representation ($q = 2$), and (3) a third-order representation ($q = 3$) obtained from the proposed approach of this work.

Figure 5.10 compare the results of the proposed high-order NF solution with the linear and second-order representations for the machines of concern. Our results are compared to those obtained using detailed step-by-step simulation (full system solution) using a commercial transient stability program.

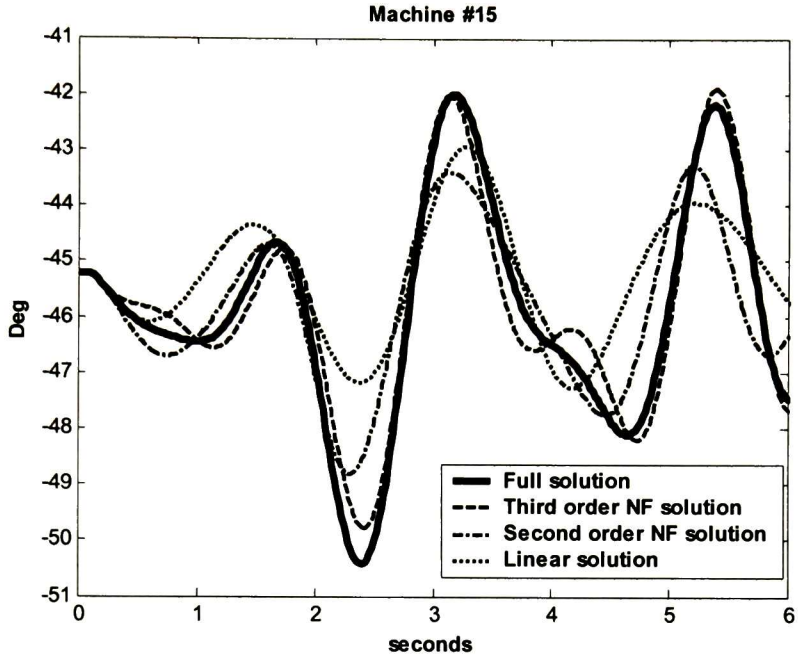
Similar results are also obtained for other machines.



a) Machine 12



b) Machine 14



c) Machine 15

Figure 5.10. Comparison of relative rotor angle swings

5.2.5. *Analysis of results*

For machine # 12, examination of the system results in Figure 5.10 shows that the response is given mainly by the linear terms. It can also be seen that all solutions remain in phase; the linear solution becomes less accurate as time increases. The analysis of machines #14 and 15, on the other hand, shows that higher-order NF solutions provide a more accurate approximation for the full system solution than the lower order approximations.

Overall, the agreement between the linear solution and the NF solutions is good over the entire study period although some discrepancies are noted. Of particular interest, simulations results show that third-order NF solutions are in close agreement with the full solutions for the entire study period thus showing the correctness of the proposed procedures.

In contrast to this, linear solutions provide a poor approximation to system

behavior both in magnitude and phase. Clearly nonlinearity and nonlinear modal interaction are not uniformly distributed and may exhibit complex characteristics.

The above results confirm the validity of the concepts and methodologies developed to analyze the extent and distribution of system nonlinear behavior and provide motivation to investigate nonlinear dynamic phenomena.

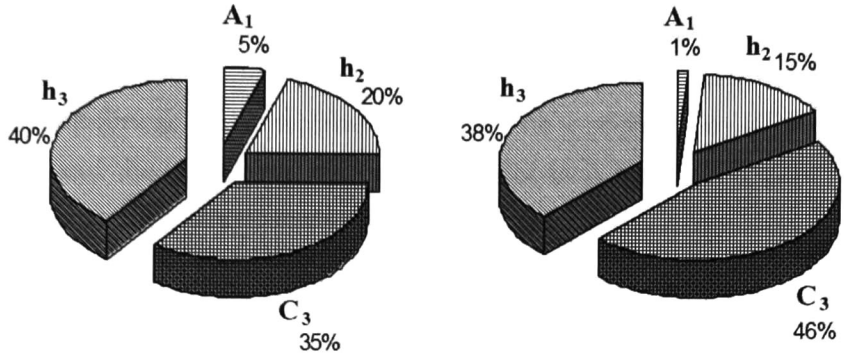
5.2.6. *Computational aspects*

Figure 5.11 compares the memory storage required for NF analysis using the conventional approach [2] and the proposed procedure as a percentage of the total memory usage. An attractive feature of the proposed procedure is that reduces the amount of computer memory required to compute the terms $\mathbf{C}_3$, since in the conventional approach, it is necessary to determine a larger number of residual terms; memory requirements increase substantially when the order of the transformation increases. This, in turn, causes an increase in CPU time thus limiting its applicability to the study of realistic systems.

A few remarks are in order:

- The number of residuals term used by the recursive approach is higher than those generated by the proposed technique; as shown in Figure 5.11; this leads to a different usage of computer memory.

- The non-recursive approach results in faster computational times and provides a means to characterize the effects of higher order terms.



a) Non-recursive b) CNF

Figure 5.11. Comparison of memory storage requirements.

5.3 Discussion and conclusions

In this chapter, the application of the higher order normal form theory to analyze nonlinear modal interaction for both, resonance and off-resonance conditions has been examined. The present results generalize to higher orders the observations of Betancourt *et al.* [4].

The results obtained indicate that the non-recursive approach presents several advantages over recursive approaches. At the same order of approximation, the accuracy of the SNF approach is higher than the accuracy of the conventional methods. The usefulness of this analysis, stems from the ability it gives us to define nonlinear measures of mode state participations and the capacity to characterize highly stressed operating conditions.

Apart from their theoretical interest, higher order normal form analysis may be used to design and place system controllers, particularly under uncertain or highly stressed operating conditions.

The following general observations can be made drawn from the analysis of two case studies.

- The method avoids the use of center manifold reduction thus enabling the study of resonant singularities.

- Apart form its simplicity, the method is though to have potentially important applications for dynamic analysis of stressed behavior in nonlinear systems. These include the computation of normal forms and the associated coefficients under near strong resonant conditions and the study of higher-order nonlinear modal interaction.

- Mode coupling involving third order effects is shown to be significant, especially under heavy stress conditions

- From a computational point of view, the proposed method reduces the amount of memory used to compute the normal form transformations.

This approach has a sound analytical basis and results in real formulations which simplify the computation of initial conditions in normal form space and provide a more accurate characterization of nonlinearity in physical coordinates. Study experience with more complex systems using the generalized Schur decomposition enables to confirm the above findings.

The generalization of this approach to account higher order resonant conditions deserves further investigation.

5.4 References

[1] E. Barocio, A.R Messina, J. Arroyo, "Analysis of factors affecting normal form results," *Electric Power Systems Research*, vol. 45, pp. 2349-1234, Dec. 2004.

[2] M. Klein, G. Rogers, P. Kundur, "A fundamental study of interarea oscillations in power systems", *IEEE Trans. on Power Systems*, vol. 6, no. 3, pp. 914-921, August 1991.

[3] E. Barocio, A. R. Messina, "Normal form analysis of stressed power systems: incorporation of SVC models," *Electrical Power and Energy Systems*, vol. 25, 2003, pp. 79-90.

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Chapter 6

Conclusions and Future Research

In this work, a novel analytical technique based on normal form theory for the study of nonlinear inter-area oscillations in large electric power systems has been presented. The current normal form analysis methods have been extended considerably to include treatment of unconventional normal forms and the study of more general vector fields. In many applications, this may offer an advantage over conventional techniques which may result in an inaccurate approximation to system's dynamic behavior.

In sharp contrast with much of the existing work in dynamical systems theory, the method employs a single, one-step normal form transformation to annihilate the desired order nonlinear terms and allows the study of high-dimensional systems in an efficient manner. Unlike these approaches, the terms of k th-order in the nonlinear transformation are used to simplify not only the k th-order terms in the system, but also used to annihilate higher-order nonlinear terms. As a result, the method leads to the computation of the simplest normal form. Furthermore, this results in an efficient non-recursive formulation that overcomes some of the limitations of existing approaches.

The proposed method represents a significant improvement over conventional methodologies since it can be extended to analyze higher order nonlinear dynamic behavior in the neighborhood of resonant power systems. This formulation is generally applicable to a wide range of nonlinear models and dynamic

processes that exhibit strong and nonlinear resonant conditions and can be used to study bifurcations and other dynamic phenomena. Also, this development provides a new class of nonlinear analysis techniques for the analysis of complex system representations with multiple, higher order nonlinearities. This aspect is entirely new. It is shown that most previous methods for normal analysis are a particular case of the technique.

Applications to complex test power systems have demonstrated the accuracy and efficiency of the developed technique.

6.1 General conclusions

In light of the results presented above, the following general conclusions can be drawn.

- Non-recursive formulations offer the potential to quantify neglected dynamics and result in formulations in which second –and higher resonance conditions can be identified. The work on normal form theory outlined here differs in several respects from existing approaches. A notable feature of this approach is that the proposed non-recursive normal form approach allows reducing the number of residual terms. As a result, the simplest normal form representation can be constructed.

- The methods developed in this work may be applied rather generally to model the dynamics of both, resonant and non-resonant vector fields. They are gaining acceptance in many areas in addition to power system dynamics, including mechanical oscillations, nonlinear optics, and chemical processes. The combined use of non-recursive formulations and the real normal form approach permits the efficient study of both, resonant and non-resonant vector fields.

- Simulation studies suggest that higher dimensional representations may be needed to fully extract system nonlinear power system behav-

ior especially under heavy stress operating conditions. More specifically, experience with the study of complex systems shows that third order nonlinear modal interaction, becomes significant when high stressed conditions are present in the nonlinear process. An interesting aspect arising from these studies is that, under some circumstances, third order nonlinear interaction can be more significant that the second order. This may have important implications for the design and location of power system controllers.

- The presence of near strong resonant conditions has a significant effect on the magnitude of the coefficients of the convention normal form transformations. By contrast, the real normal transformation coefficients present a much more smooth behavior; this facilitates the analysis and interpretation of mode interaction and feature extraction. In addition, the technique avoids the use of physical scaling to remove the effects of physical units and results in physically meaningful interaction indices

6.2 Suggestions for future work

There are several aspects of the application of our procedures that require additional study and which will be addressed in future. Among them, the following issues deserve further investigation.

- Little work has been done on the analysis of higher order resonances. The use of the developed methods for the identification of higher order nonlinear resonances in the behavior of the power system dynamics seems promising. In particular, the effect of second order resonances has not been investigated.

- Error analysis and techniques for assessing the accuracy and robustness of the method are currently being actively investigated. More specifically, the quantification of errors introduced by the several sim-

plifying assumptions in both, conventional normal form theory and non-recursive approaches are needed.

- Treatment of more general system configurations, especially in conjunction with more practical systems. The generalization of modal interaction and nonlinearity indices to take into account the presence of linear and higher-order resonances are also the subject of current investigations.

- The analysis of dynamic bifurcations arising from model-dependent formulations

- The improvement of computational efficiency and modeling capabilities of existing formulations.

- The extension of the proposed algorithms to treat bifurcations arising from model dependent parameters.

Appendix A

Data for the two-area four-machine test power system

A.1 Base operating case

The base operating case is summarized in Table A.1. Values are in p.u. on 100 MVA base.

Table A.1: Load flow solution

BUS #	BUS VOLTAGE (P.U.)	ANGLE (DEG)	GENERATION REAL (P.U.)	GENERATION REACTIVE (P.U.)	LOAD REAL (P.U.)	LOAD REACTIVE (P.U.)
1	1.03	26.849	7	1.7919	0	0
2	1.01	17.1	7	2.2026	0	0
3	1.03	0	7	1.8521	0	0
4	1.01	-10.158	7	0	0	0
5	0.9736	1.956	0	2.5	11.67	1
6	0.9763	-25.258	0	2.5	15.67	1

A.2 Network data

Network data is given in Table A.2 in p.u. on a 100 MVA base.

Table A.2: Transmission line data

BUS #	BUS #	R	X	b
1	2	0.0025	0.025	0.04375
2	5	0.0010	0.010	0.01750
5	6	0.0100	0.220	0.38500
6	4	0.0010	0.010	0.01750
4	3	0.0025	0.025	0.04375

A.3 Machine and excitation system data

Synchronous machine and excitation data are given in Table A.3 in p.u. on a 100 MVA base.

Table A.3: Synchronous machine and exciter data

BMVA	x_d	x_q	x'	T'_{d0}	T'_{q0}	H	D	K_{exc}	T_{exc}
900	1.8	1.7	0.3	8	0.4	6.5	9 2.5 11 2	110	0.02

Appendix B

B.1 Solution of homological equations using real Schur transformation

Solution to homological equations for non-diagonalizable cases, can be determined by transforming the linear equations of (3.) into an upper (lower) triangular form trough the use of the Schur transformation.

Consider to this end, the next equivalences

$$\mathbf{A}_1 = \mathbf{U}\mathbf{T}\mathbf{U}^T \quad \text{with} \quad \mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{B.1}$$

where $\mathbf{T}$ is a upper triangular matrix and $\mathbf{U}$ is a unitary matrix.[]

Use of these identities in the homological equations yields

$$\begin{aligned} &\mathbf{g}_2 + \mathbf{h}_2[\mathbf{U}\mathbf{T}\mathbf{U}^T \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{U}\mathbf{T}\mathbf{U}^T] - \mathbf{U}\mathbf{T}\mathbf{U}^T\mathbf{h}_2 = \mathbf{A}_2 \\ &\qquad\qquad \vdots \end{aligned} \tag{B.2}$$

or, equivalently,

$$\begin{aligned} &\mathbf{g}_2 + \mathbf{h}_2[\mathbf{U}\mathbf{T}\mathbf{U}^T \otimes \mathbf{U}\mathbf{U}^T + \mathbf{U}\mathbf{U}^T \otimes \mathbf{U}\mathbf{T}\mathbf{U}^T] - \mathbf{U}\mathbf{T}\mathbf{U}^T\mathbf{h}_2 = \mathbf{A}_2 \\ &\qquad\qquad \vdots \end{aligned} \tag{B.3}$$

After some algebra, (B.3) reduces further to

$$\begin{aligned} &\mathbf{g}_2 + \mathbf{h}_2\mathbf{U}^{(2)}[\mathbf{T} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{T}]\mathbf{U}^{T^{(2)}} - \mathbf{U}\mathbf{T}\mathbf{U}^T\mathbf{h}_2 = \mathbf{A}_2 \\ &\qquad\qquad \vdots \end{aligned} \tag{B.4}$$

Now post-multiplying by $\mathbf{U}^{(2)}, \mathbf{U}^{(3)} \cdots, \mathbf{U}^{(k)}$ and pre-multiplying by $\mathbf{U}^T$

$$\begin{aligned} &\mathbf{U}^T\mathbf{g}_2\mathbf{U}^{(2)} + \mathbf{U}^T\mathbf{h}_2\mathbf{U}^{(2)}[\mathbf{T} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{T}] - \mathbf{T}\mathbf{U}^T\mathbf{h}_2\mathbf{U}^{(2)} = \mathbf{U}^T\mathbf{A}_2\mathbf{U}^{(2)} \\ &\qquad\qquad \vdots \end{aligned} \tag{B.5}$$

or, in compact form

$$\begin{aligned} &\breve{\mathbf{g}}_2 + \breve{\mathbf{h}}_2[\mathbf{T} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{T}] - \mathbf{T}\breve{\mathbf{h}}_2 = \breve{\mathbf{A}}_2 \\ &\qquad\qquad \vdots \end{aligned} \tag{B.6}$$

where $\breve{\mathbf{g}}_2 = \mathbf{U}^T\mathbf{g}_2\mathbf{U}^{(2)}$, $\breve{\mathbf{h}}_2 = \mathbf{U}^T\mathbf{h}_2\mathbf{U}^{(2)}$, $\breve{\mathbf{A}}_2 = \mathbf{U}^T\mathbf{A}_2\mathbf{U}^{(2)}$

The linear system (B.6) has an upper triangular form structure, and can be solved in direct form by backward substitution.

Appendix C

C.1 Normal form analysis near singularities

In this section, we generalize our previous formulations to treat the computation of normal forms near singularities.

C.2 The case of two repeated eigenvalues

Let consider the next nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathbf{A}_k \mathbf{x}^{[k]} \tag{C.1}$$

Now assume that only second order nonlinearities appear, then the homological equation to be solved is

$$\mathbf{g}_{2S}^T + \left[\mathbf{I} \otimes \left[\mathbf{I} \otimes \mathbf{A}^T + \mathbf{A}^T \otimes \mathbf{I}\right] - \mathbf{A} \otimes \mathbf{I}^2\right] \mathbf{h}_{2S}^T = \mathbf{A}_{2S}^T \tag{C.2}$$

or, in explicit form,

$$\begin{bmatrix} g_{211} \\ g_{212} \\ g_{213} \\ g_{214} \\ g_{221} \\ g_{222} \\ g_{223} \\ g_{224} \end{bmatrix} + \begin{bmatrix} \lambda & & & & -1 & & & \\ 1 & \lambda & & & & -1 & & \\ 1 & & \lambda & & & & -1 & \\ & 1 & 1 & \lambda & & & & -1 \\ & & & & \lambda & & & \\ & & & & 1 & \lambda & & \\ & & & & 1 & & \lambda & \\ & & & & & 1 & 1 & \lambda \end{bmatrix} \begin{bmatrix} h_{211} \\ h_{212} \\ h_{213} \\ h_{214} \\ h_{221} \\ h_{222} \\ h_{223} \\ h_{224} \end{bmatrix} = \begin{bmatrix} A_{211} \\ A_{212} \\ A_{213} \\ A_{214} \\ A_{221} \\ A_{222} \\ A_{223} \\ A_{224} \end{bmatrix} \tag{C.3}$$

Solving for the second-order nonlinear transformation coefficients yields

$$\mathbf{h}_2 = \begin{bmatrix} \frac{A_{211}}{\lambda} + \frac{A_{221}}{\lambda^2} & -\frac{A_{211}}{\lambda^2} + \frac{A_{212}}{\lambda} - \frac{2A_{221}}{\lambda^3} + \frac{A_{222}}{\lambda^2} & -\frac{A_{211}}{\lambda^2} + \frac{A_{213}}{\lambda} - \frac{2A_{221}}{\lambda^3} + \frac{A_{223}}{\lambda^2} & \frac{2A_{211}}{\lambda^3} - \frac{A_{212}}{\lambda^2} - \frac{2A_{213}}{\lambda^2} + \frac{A_{214}}{\lambda} + \frac{6A_{221}}{\lambda^4} - \frac{2A_{222}}{\lambda^3} - \frac{2A_{223}}{\lambda^3} + \frac{A_{224}}{\lambda^2} \\ \frac{A_{221}}{\lambda} & -\frac{A_{221}}{\lambda^2} + \frac{A_{222}}{\lambda} & -\frac{A_{221}}{\lambda^2} + \frac{A_{223}}{\lambda} & \frac{2A_{221}}{\lambda^3} + \frac{A_{222}}{\lambda^2} - \frac{A_{223}}{\lambda^2} + \frac{A_{224}}{\lambda} \end{bmatrix}$$

C.3 The case of purely imaginary eigenvalues (Hopf bifurcation)

Consider a nonlinear system of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathbf{A}_k \mathbf{x}^{[k]} \tag{C.4}$$

This case have two purely imaginary eigenvalues $\pm j\omega$, for this case the homological equation to be solved is

$$\begin{bmatrix} g_{211} \\ g_{212} \\ g_{213} \\ g_{214} \\ g_{221} \\ g_{222} \\ g_{223} \\ g_{224} \end{bmatrix} + \begin{bmatrix} & \omega & \omega & & \omega & & & \\ -\omega & & & \omega & & \omega & & \\ -\omega & & & \omega & & & \omega & \\ & -\omega & -\omega & & & & & \omega \\ -\omega & & & & & \omega & \omega & \\ & -\omega & & & -\omega & & & \omega \\ & & -\omega & & -\omega & & & \omega \\ & & & -\omega & & -\omega & -\omega & \end{bmatrix} \begin{bmatrix} h_{211} \\ h_{212} \\ h_{213} \\ h_{214} \\ h_{221} \\ h_{222} \\ h_{223} \\ h_{224} \end{bmatrix} = \begin{bmatrix} A_{211} \\ A_{212} \\ A_{213} \\ A_{214} \\ A_{221} \\ A_{222} \\ A_{223} \\ A_{224} \end{bmatrix} \qquad \text{(C.5)}$$

Obtaining the solution for this linear system, the second order transformations is given by

$$\mathbf{h}_2 = \frac{1}{3\omega}\begin{bmatrix} -A_{2_{12}} - A_{2_{13}} - A_{2_{21}} - 2A_{2_{24}} & A_{2_{11}} - A_{2_{14}} - A_{2_{22}} + 2A_{2_{23}} & A_{2_{11}} - A_{2_{14}} + 2A_{2_{22}} - A_{2_{23}} & A_{2_{12}} + A_{2_{13}} - 2A_{2_{21}} - A_{2_{24}} \\ A_{2_{11}} + 2A_{2_{14}} - A_{2_{22}} - A_{2_{23}} & A_{2_{12}} - 2A_{2_{13}} + A_{2_{21}} - A_{2_{24}} & -2A_{2_{12}} + A_{2_{13}} + A_{2_{21}} - A_{2_{24}} & 2A_{2_{11}} + A_{2_{14}} + A_{2_{22}} + A_{2_{23}} \end{bmatrix}$$

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Análisis de formas normales de alto orden en sistemas de potencia resonantes usando enfoques no recursivos

del (la) C.

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