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Aspects of brane mechanics

Geometric intrinsic models

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Dedication and acknowledgements

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Resumen

La mecánica de branas se entiende como el estudio de la dinámica de objetos extendidos y desempeña un papel esencial en varios contextos de la física. Por ejemplo, en el marco de los escenarios de mundos de branas, donde el universo de cuatro dimensiones se considera como un objeto extendido incrustado en un fondo de mayor dimensión; en la teoría M, donde las branas se consideran objetos fundamentales. También se pueden mencionar otras aplicaciones en el campo de la astrofísica y la física de agujeros negros, donde los grados de libertad físicos están localizados en subvariedades del espacio-tiempo.

En este trabajo, definimos y discutimos diversas herramientas variacionales que nos permiten calcular eficientemente variaciones de orden superior de la acción para objetos extendidos. Implementamos varios de estos métodos variacionales en un modelo geométrico introducido por Regge y Teitelboim en la década de 1970. En este modelo, el universo se propone como un objeto extendido incrustado en un espacio plano de mayor dimensión, y sus grados de libertad son las llamadas funciones de incrustación. Estudiamos las ecuaciones de movimiento y las correspondientes ecuaciones de Jacobi, que nos permiten examinar la estabilidad de cualquier solución particular de este modelo. Además, realizamos un estudio hamiltoniano de este modelo, considerándolo como un sistema singular de orden superior.

Abstract

Brane mechanics is understood as the study of the dynamics of extended objects and plays an essential role in various contexts of physics. For example, in the framework of brane world scenarios, where the four-dimensional universe is considered as an extended object embedded in a higher-dimensional background; in M-theory, where branes are considered fundamental objects. Other applications can also be mentioned in the field of astrophysics and black hole physics, where the physical degrees of freedom are localized on submanifolds of spacetime.

In this work, we define and discuss various variational tools that allow us to efficiently calculate higher-order variations of the action for extended objects. We implement several of these variational methods in a geometric model introduced by Regge and Teitelboim in the 1970s. In this model, the universe is proposed as an extended object embedded in a higher-dimensional flat space, and its degrees of freedom are the so-called embedding functions. We study the equations of motion and the corresponding Jacobi equations, which allow us to examine the stability of any particular solution of this model. Additionally, we perform a Hamiltonian study of this model, considering it as a higher-order singular system.

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Chapter 1

Introduction

Brane mechanics is a field dedicated to studying the dynamics of extended objects known as branes. These branes generalize the concepts of particles and strings. By varying the functional action, one can determine the behavior of branes. This action is formulated using quantities that remain invariant under reparametrizations of the brane's worldvolume, which is a submanifold embedded in a higher-dimensional manifold known as ambient spacetime, background spacetime, or target spacetime [72].

Branes play a crucial role in the investigation of various physical systems, where their degrees of freedom are confined to specific submanifolds. For instance, in classical mechanics, when dealing with systems conserving energy, the configuration space describing the system's dynamics can be mapped to a geodesic curve on a new manifold, which incorporates the Jacobi metric. This metric encodes relevant physical quantities of the system. This mapping allows for the correspondence of the Euler-Lagrange equations of the physical system with the geodesic equations [92, 105]. Another significant application arises in M-theory, where branes are considered fundamental objects [102]. Moreover, branes play a crucial role in the braneworld scenario, which conceptualizes the universe as a brane embedded in a higher-dimensional ambient spacetime [63, 72, 95]. Branes are also employed in astrophysics to describe inhomogeneities in the early universe, potentially originating from quantum fluctuations [107]. They have even been utilized to explain the formation of primordial black holes [49]. The study of surfaces and boundary entropy also draws motivation from branes [2, 42, 104]. The widespread presence of branes in models aimed at elucidating diverse physical systems underscores the importance of developing comprehensive methodologies for understanding the dynamics of extended objects [72].

While the first variation of the action yields equations of motion that describe the evolution of a physical system, it does not provide information regarding solution stability. To obtain a more complete description, it becomes necessary to compute the second variation of the action, leading to the Jacobi equation. These equations have multiple applications, such as analyzing the stability of specific solutions, approximating solutions for highly nonlinear systems through Jacobi fields, and investigating chaotic behavior by establishing connections between Jacobi fields and Lyapunov coefficients [26]. In the context of a free particle in curved spacetime, the Jacobi equation coincides with the geodesic deviation equation, shedding light on the separation or convergence of infinitesimally neighboring geodesics due to the curvature background. This understanding extends to string theory as well.

In 1971, Tullio Regge and Claudio Teitelboim proposed a geometric model that considers the universe as a brane embedded in a larger spacetime [95]. This model, initially referred to as "gravity à la string" due to its inspiration from string theory, later became known as geodetic brane gravity (GBG) introduced by Davidson et al. [66] . It serves as an extension of general relativity (GR), encompassing all GR solutions. In this thesis, we will refer to this model as the RT model or simply GBG. The degrees of freedom of the brane in this model are described by background functions known as embedding functions, which specify and describe the brane's worldvolume [66].

An interesting feature of the GBG model is that its cosmology equations, when derived, can be reduced to the Friedman equations of standard cosmology, but with an additional term in the energy-momentum tensor. This additional term has been suggested to represent dark matter, in other words, the dark matter could have a geometric origin. [66]. However, the proposal initially faced criticism due to gauge dependency and the lack of an appropriate Hamiltonian formulation [36]. Nevertheless, significant progress has been made in addressing these concerns, with several authors advancing the formulation and analysis of GBG, including quantum aspects [16, 28, 31, 86, 88]. In this work, we conduct a Hamiltonian analysis of the RT model, treating it as a system of high-order derivatives, using the Hamiltonian extension introduced by Ostrogradsky.

Furthermore, we analyze the model using the variational tools discussed in the earlier sections of this thesis.

The organization of this thesis is as follows. In the first section, we introduce the notation employed throughout the work and present relevant geometric objects in the context of brane mechanics. We then discuss different variational derivatives that enable us to calculate action variations, obtaining covariant equations for both world volume reparametrizations and background spacetime diffeomorphisms. Higher-order variations of the action are also explored, along with the utility of the resulting equations. Specific examples are provided to illustrate the application of variational tools. In the second part of this thesis, we study and analyze the equations of the RT model, employing the variational derivatives discussed earlier. We linearize the equations of motion to obtain and examine the Jacobi equations. By considering the Jacobi equations in the RT model, we investigate the stability of a four-dimensional Schwarzschild black hole embedded in a six-dimensional spacetime. Our stability analysis focuses on determining the quasi-normal modes' oscillation frequencies, which can be expressed in terms of deformation fields or Jacobi fields. To achieve this, numerical analysis is employed to obtain these oscillation frequencies. Subsequently, we conduct a Hamiltonian analysis of GBG, treating it as a system with singular derivative high-order terms, utilizing the theory developed by Ostrogradsky. The constraints of the system are determined, ensuring the correct count of degrees of freedom and facilitating the construction of Dirac brackets, which are essential for canonical quantization. Finally, we provide a brief discussion, draw conclusions from this work, and outline potential future projects derived from this thesis.

CHAPTER 2

BRANE MECHANICS GEOMETRY

In order to make this work easy to understand to most people who feel attracted to study of physical systems using geometric models. I use a familiar language and notation for the majority of physicists, mathematicians, and people in similar areas. I hope that this chapter serves as an introduction to the geometry behind brane mechanics and that it will be of great use to that amateur student who has concerns about this fascinating topic. If the reader wants to delve more into some concepts exposed here, the following references could help him [17, 21–24].

In this work, we will study relativistic systems; for this reason, we are going to work with pseudo-Riemannian manifolds[82]. These are diferrentiable manifolds where the requirement of positive-defiteness is relaxed. In brane mechanics the evolution of the brane is described by a pseudo-Riemannian manifold called world volume. On the another hand, the brane is a differential Riemannian manifold because its dimensions are only spatial and the spatial metric is always positive-definite. For example, a particle is a brane with spatial dimension zero, and its world volume is the worldline of the particle[110], the following immediate example is a string that has one only spatial dimension, and the world volume is known as the string worldsheet [116]. In summary, a brane is a spacial object that generalizes the concept of particle and string.

2.1 Notation

Consider that the world volume that describes the evolution of a brane is a $p+1$ dimensional manifold, and we denote it by m. This is embedded in a bigger ambient spacetime $\mathscr{M}$ of dimension $N+1$. $\mathscr{M}$, as well as m, is a pseudo-Riemannian manifold and it has associated a metric $\eta_{\mu\nu}$ with a signature $\{-,+,..,+\}$ (all Greek index runs from zero to N), evidently $N>p$. In general, the ambient spacetime can be curved, and one can define a covariant derivative $\mathscr{D}_\mu$, such that, this is compatible with the metric of $\mathscr{M}$, i.e.

$$\mathscr{D}_\mu \eta_{\alpha\beta} = 0, \tag{2.1}$$

if this derivative applies to a vector $V^\mu \in \mathscr{M}$, one gets

$$\mathscr{D}_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\nu{}_{\mu\alpha} V^\alpha, \tag{2.2}$$

where $\Gamma^\mu{}_{\alpha\beta}$ are the Christoffel symbols of ambient spacetime. Now, if one applies the commutator,constructed by covariant derivatives, to a vector V^μ, one obtains

$$\left[\mathscr{D}_\mu, \mathscr{D}_\nu\right] V^\alpha = \mathbf{R}^\alpha{}_{\mu\nu\beta} V^\beta, \tag{2.3}$$

this last equation can be generalized by taking covariant derivative along arbitrary directions $\mathscr{D}_V = V^\mu \mathscr{D}_\mu$, then

$$\left([\mathscr{D}_U, \mathscr{D}_W] - \mathscr{D}_{[U,W]}\right) V^\alpha = \mathbf{R}^\alpha{}_{\sigma\mu\nu} U^\mu W^\nu V^\sigma, \tag{2.4}$$

here $[U,W]$ is the Lie bracket of two vectors fields, $\mathbf{R}_{\alpha\beta\mu\nu}$ are the Riemann tensor components of $\mathscr{M}$, and these can be written using Christoffel symbols as follows

$$\mathbf{R}^\rho{}_{\sigma\mu\nu} = 2\left(\partial_{[\mu} \Gamma^\rho{}_{\nu]\sigma} + \Gamma^\rho{}_{\delta[\mu} \Gamma^\delta{}_{\nu]\sigma}\right), \tag{2.5}$$

where square brackets represent anticommutation between indexes ($A_{[\mu\nu]} = 1/2\left(A_{\mu\nu} - A_{\nu\mu}\right)$).

Because m is embedded in a larger spacetime, the world volume can be described by $y^\mu = X^\mu(x^a)$, where y^μ are the coordinates of $\mathscr{M}$, X^μ are known as embedding functions, and x^a are the coordinates on world volume (the latin indexes $a,b,c,...$ run from zero to p). And one can build tangent vectors to m, only taking the derivative of the embedding function with respect to the coordinates x^a

$$X^\mu_a := \partial_a X^\mu = \frac{\partial X^\mu}{\partial x^a}. \tag{2.6}$$

Due to the fact tangent vectors are built through a variation of embedding functions along each coordinate on m. They form a vectorial basis of the world volume at each point. Now, if one takes the inner product between these vectors, which is defined by metric $\eta_{\mu\nu}$, one gets components g_{ab} of a induced metric. This metric has a signature $\{-,+,...,+\}$. Usually, it is said $\mathscr{M}$ induces a metric on m in the following way

$$g_{ab} = \eta_{\mu\nu} X^{\mu}_{a} X^{\nu}_{b} = X_a \cdot X_b, \tag{2.7}$$

where $\ll \cdot \gg$ denotes the inner product from ambient spacetime. For geometric reasons (immersion theorems), it is known to induce a general metric g_{ab}. It is necessary that the background spacetime has at most dimension $N = p(p+1)/2$. In addition, if worldvolume has isometries, i.e., the induced metric admits Killing vectors, this number can be reduced [20, 45, 64]. Note that even though metric g_{ab} is a $(0,2)$-tensor on the world volume, this transforms as a scalar under diffeomorphism of ambient spacetime. One can calculate the inverse of the induced metric that we denote by g^{ab}, and this is used to up the indexes of the tensors of m, while g_{ab} lows them. The metric determinant is g, which is less than zero since the world volume is a timelike manifold, i.e., its normal vectors are spacelike, a spacelike vector V fulfills $V \cdot V > 1$ [12] . We denote the normal vectors to m as n^i, where the index i refers to the i-th normal vector. There are $N - p$ normal vectors, and they fulfill the following properties

$$n^i \cdot X_a = 0 \quad n^i \cdot n^j = \delta^{ij}, \tag{2.8}$$

these equations define vectorial normal fields until one rotation $O(N-p)$ and a sign. The indexes i, j of normal vectors are lowered and raised by δ_{ij} and δ^{ij}, respectively. The composed set by tangent and normal vectors, $\{X_a, n^i\}$, forms a vectorial basis of ambient spacetime. Thus, metric tensor $\eta_{\mu\nu}$ can be written as

$$\eta_{\mu\nu} = h_{\mu\nu} + \Pi_{\mu\nu}, \tag{2.9}$$

where $h^{\mu\nu} = g^{ab} X^{\mu}_{a} X^{\nu}_{b}$ and $\Pi_{\mu\nu} = n_{\mu}{}^{i} n_{\nu i}$. Thus, one can build the following tangential and normal projectors, respectively

$$h^{\mu}{}_{\nu} := h^{\mu\alpha} \eta_{\alpha\nu} = h^{\mu\alpha} h_{\alpha\nu}, \quad \Pi^{\mu}{}_{\nu} := \Pi^{\mu\alpha} \eta_{\alpha\nu} = n^{\mu}{}_{i} n_{\nu}{}^{i}. \tag{2.10}$$

One defines a covariant derivative along the coordinates of the world volume, such that, this is also compatible with the background metric. One projects the covariant derivative of $\mathscr{M}$ along tangent vectors of m, $\mathscr{D}_a := X^{\mu}_{a} \mathscr{D}_{\mu}$. Now, if one takes the gradient of vectors

$\{X_a, n^i\}$ using this derivative, it is obtained

$$\mathscr{D}_a X_b = \gamma^c{}_{ab} X_c - K_{ab}{}^i n_i, \tag{2.11a}$$

$$\mathscr{D}_a n^i = K_{ab}{}^i X_c g^{bc} + \omega_a{}^{ij} n_j. \tag{2.11b}$$

These equations are commonly referred to as the Gauss-Weingarten equations, which allow for the description of the full extrinsic geometry of a world volume. Here, $\gamma^c{}_{ab}$ represents the Christoffel symbols associated with the induced metric. These symbols provide valuable information regarding the tangential variations of tangent vectors within the world volume. Their explicit expressions are given by

$$\gamma^c{}_{ab} = g^{cd} X_d \cdot \mathscr{D}_a X_b = \gamma_{ab}{}^c. \tag{2.12}$$

However, they must also satisfy the following

$$\gamma^c{}_{ab} = \frac{1}{2} g^{cd} (\partial_b g_{da} + \partial_a g_{db} - \partial_d g_{ab}). \tag{2.13}$$

On the other hand, $K_{ab}{}^i$ is the $i-th$ curvature extrinsic associated with $i-th$ normal vector. In the literature, it is also called the second fundamental form. This geometric object provides us with information about tangential changes when one makes a variation of normal vectors along tangent vectors to m

$$K_{ab}{}^i = X_a \cdot \mathscr{D}_b n^i = -\mathscr{D}_a X_b \cdot n^i = -\mathscr{D}_b X_a \cdot n^i = K^i_{ba}, \tag{2.14}$$

tt is important to observe that the extrinsic curvature is symmetric when the indices a and b are exchanged, as a consequence of the property described in equation (2.8). Additionally, the quantity $\omega_a{}^{ij}$ represents a twist potential associated with the freedom to rotate the normal vectors at each point of the world volume [73]. In the literature, this twist potential is often denoted by the symbol T [57]. Notably, under these rotations, ω_a transforms as a connection.

$$\omega_a \to \mathscr{O} \omega_a \mathscr{O}^{-1} + \mathscr{D}_a \mathscr{O} \mathscr{O}^{-1}. \tag{2.15}$$

One can define a covariant derivative torsionless and compatible with induced metric g_{ab}. We denote it by ∇_a. However, this covariant derivative does not consider the freedom that normal vectors can be rotated at every point of m. Then, one can introduce another covariant derivative $\widetilde{\nabla}_a$ that considers it. Thereby, when it applies to an object that transforms as a $(1,1)$-tensor under these rotations, we have

$$\widetilde{\nabla}_a \Psi_j{}^i = \nabla_a \Psi_j{}^i - \omega_a{}^{ik} \Psi_{kj} - \omega_{ajk} \Psi^{ik}. \tag{2.16}$$

Note that, induced metric transforms as a scalar under these rotations, then $\widetilde{\nabla}_a g_{cd} = \nabla_a g_{cd}$. On the other hand, extrinsic curvature $K_{ab}{}^i$ transforms like a vector under the same rotations due to free index i.
If now one takes the commutator of these covariant derivatives and applies it to an object that transforms like a vector under these rotations, one obtains the following

$$[\widetilde{\nabla}_a, \widetilde{\nabla}_b]\Psi^i = \Omega_{ab}{}^{ij}\Psi_j, \tag{2.17}$$

where Ω is the curvature associated with twist potential, and it is defined by

$$\Omega_{ab}{}^{ij} := \nabla_b \omega_a{}^{ij} - \nabla_a \omega_a{}^{ij} - \omega_b{}^{ik}\omega_{ak}{}^j + \omega_a{}^{ik}\omega_{bk}{}^j. \tag{2.18}$$

this curvature is the conformal invariant traceless part of the squared extrinsic curvature [21, 22].

Moreover, certain relations must be satisfied to ensure the consistency of the embedded functions. These integrability conditions establish connections between the extrinsic and intrinsic geometries of the world volume and the background geometry. They can be expressed as follows

$$\mathbf{R}_{\alpha\beta\mu\nu} X^\alpha_a X^\beta_b X^\mu_c X^\nu_d = \mathscr{R}_{abcd} - K_{ac}{}^i K_{bdi} + K_{ad}{}^i K_{bci}, \tag{2.19a}$$

$$\mathbf{R}_{\alpha\beta\mu\nu} n^{\alpha i} X^\beta_a X^\mu_b X^\nu_c = \widetilde{\nabla}_c K_{ab}{}^i - \widetilde{\nabla}_b K_{ac}{}^i, \tag{2.19b}$$

$$\mathbf{R}_{\alpha\beta\mu\nu} n^{\alpha j} n^{\beta i} X^\mu_b X^\nu_a = \Omega_{ab}{}^{ij} - K_{ac}{}^i K_b{}^{cj} + K_{bc}{}^i K_a{}^{cj}. \tag{2.19c}$$

These are the integrability conditions of Gauss-Codazzi, Codazzi-Minardi, and Ricci, respectively. Here $\mathscr{R}^a{}_{bcd}$ is the Riemann tensor of world volume, and it can be written in terms of Christoffel symbols of m, analogous to the equation (2.5).
In this brief section, we have shown the notation used throughout this work, and we counted the most relevant geometric objects in brane mechanics. Besides, we reviewed important identities that will be useful to calculate variations of important composed geometric objects.

CHAPTER 3

Variation in Brane Mechanics

This chapter is split into two principal parts. In the first, we discuss the variation of the intrinsic and extrinsic geometry of the brane, taking into account the deformation of embedding function along only normal directions to m. We calculate, for example, the variations of induced metric, extrinsic curvature, and other important geometric objects. This kind of variation is so helpful to find the covariant equation of motion under the transformation of the world volume.
In the second part, we take a totally covariant approach from the ambient spacetime, and we make variations along a completely arbitrary vector (not only along the normal vectors). Due to this fact, we obtain covariant equations of motion under diffeomorphisms of ambient spacetime.

3.1 Normal variation to m

When dealing with extended objects, it is crucial that the world volume, which characterizes the brane's dynamics, remains invariant under reparametrization. This requirement ensures that the brane's behavior does not depend on the choice of coordinates employed on the world volume. Constructing the action functional for the brane necessitates utilizing geometric quantities that preserve this symmetry. These quantities are constructed using intrinsic and extrinsic geometric objects such as g_{ab} and ${K_{ab}}^{i}$. Thus, it is essential to have a comprehensive understanding of the variations of these geometric objects to effectively construct the action functional.

The world volume of a brane is described by embedding functions that provide information about its geometric structure within the ambient spacetime. When these embedding functions undergo deformations, it leads to changes in the geometric properties. In other words, a geometric object constructed based on these functions will be altered. We can interpret the deformed embedding functions as representing the embedding functions of a neighboring region of the world volume m. Mathematically, these deformed embedding functions can be expressed as

$$X'^{\mu} = X^{\mu} + s\delta X^{\mu}, \tag{3.1}$$

where s is a small parameter and δX^{μ} is the vector along which one makes the deformation. And this can be decomposed in the basis $\{X_a, n^i\}$ in the following way

$$\delta X^{\mu} = \phi^a X^{\mu}_a + \phi^i n^{\mu}{}_i, \tag{3.2}$$

here ϕ^a is a vector under coordinates changes on the world volume while ϕ^i is a vector under rotations of normal fields. Since the tangential part of the deformations can be understood as a reparametrization of the world volume, and we will work with geometric models that are invariant under these reparametrizations, then the tangential deformations can be omitted, and we only consider that

$$\delta X \equiv \delta = n_i \phi^i, \tag{3.3}$$

$N-p$ fields ϕ^i determine the normal deformations . Thus, the variations of geometric objects of m are a combination of ϕ^i and their derivatives. In this first subsection of this chapter, we consider the deformation of the intrinsic geometry of world volume, and it is helpful to define a covariant variation along the vector δ given by

$$\mathscr{D}_{\delta} := \delta^{\mu}\mathscr{D}_{\mu}. \tag{3.4}$$

Taking the gradient of tangent vector X_a along deformation vector δ and splitting it in the background basis $\{n^i, X_a\}$, we obtain the following

$$\mathscr{D}_{\delta} X_a = \beta_{ab} g^{bc} X_c + J_{ai} n^i, \tag{3.5}$$

where β_{ab} and J_{aj} are given by

$$\beta_{ab} = \mathscr{D}_{\delta} X_a \cdot X_b = \beta_{ba}, \quad J_{ai} = \mathscr{D}_{\delta} X_a \cdot n_i = -X_a \cdot \mathscr{D}_{\delta} n_i. \tag{3.6}$$

Note that, the term β_{ab}, in the equation (3.5), plays a similar role to Christoffel symbols. While $J_a{}^i$ transforms like a vector under rotation of normal vectors.

It is possible to write β_{ab} and J_{ai} in terms of ϕ^i and its derivatives. However, we must assume that if one deforms the world volume, the tangent vectors are still tangent, i.e., the Lie derivative of tangent vectors along deformation vector δ is equal to zero, $\mathscr{L}_\delta X_a = 0$. This last implies that

$$\mathscr{D}_\delta X_a = \mathscr{D}_a \delta. \tag{3.7}$$

Using (3.7) we can write β_{ab} and J_{ai} in the following way

$$\beta_{ab} = \mathscr{D}_\delta X_a \cdot X_b = \mathscr{D}_a(n_i \phi^i) \cdot X_b = K_{abi}\phi^i \tag{3.8}$$

$$J_{ai} = \mathscr{D}_a(n^j \phi_j) \cdot n_i = (\mathscr{D}_a n^j \cdot n_i)\phi_j + \nabla_a \phi_i = \widetilde{\nabla}_a \phi_i, \tag{3.9}$$

with these terms, one can calculate the variation of induced metric as follows

$$\mathscr{D}_\delta g_{ab} = \mathscr{D}_\delta (X_a \cdot X_b) = 2X_a \cdot \mathscr{D}_\delta X_b = 2\beta_{ab} = 2K_{ab}{}^i \phi_i, \tag{3.10}$$

and taking into account that $g^{ac} g_{cb} = \delta^a{}_b$, we have the variation of the inverse of the induced metric is

$$\mathscr{D}_\delta g^{ab} = -2K^{ab}{}_i \phi^i. \tag{3.11}$$

then the variation of the root of the determinant is given by

$$\mathscr{D}_\delta \sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{ab} \mathscr{D}_\delta g^{ab} = \sqrt{-g}\, g_{ab} K^{abi} \phi_i, \tag{3.12}$$

If one works with the geometric model that only depends on the volume of m[54, 78], the previous variations are enough. However, one also can work with action more generals that depend on Ricci scalar, or combinations of it. For example, geodetic brane gravity action [66, 95] or $f(R)$ theories [103]. We can also have actions with invariant reparametrization terms that are specific combinations of contraction of Riemann tensor, such that, when these kinds of actions are varied, one obtains second-order equations of motion [71]. Therefore, it is important to calculate the variation of Riemann, Ricci tensor, and Ricci scalar. Taking into account that the Riemann tensor is built from the Christoffel symbols and their derivatives, it is convenient, in the first instance, to calculate the variation of the Christoffel symbols, which is given by

$$\begin{aligned} \mathscr{D}_\delta \gamma^c{}_{ab} &= \frac{1}{2} g^{cd} \left[\nabla_b (\mathscr{D}_\delta g_{ad}) + \nabla_a (\mathscr{D}_\delta g_{bd}) - \nabla_b (\mathscr{D}_\delta g_{ab})\right] \\ &= g^{cd} \left[\nabla_b \left(K_{ad}{}^i \phi_i\right) + \nabla_a \left(K_{bd}{}^i \phi_i\right) - \nabla_d \left(K_{ab}{}^i \phi_i\right)\right], \end{aligned} \tag{3.13}$$

where we have used the variation of induced metric in (3.12). One can take the result (3.13) and substitute it into the variation of Riemann and Ricci tensor,

$$\mathscr{D}_\delta \mathscr{R}^a{}_{bcd} = \nabla_c \left(\mathscr{D}_\delta \gamma^a{}_{bd}\right) - \nabla_d \left(\mathscr{D}_\delta \gamma^a{}_{bc}\right), \tag{3.14}$$

$$\mathscr{D}_\delta \mathscr{R}_{ab} = \nabla_c \left(\mathscr{D}_\delta \gamma^c{}_{ab}\right) - \nabla_b \left(\mathscr{D}_\delta \gamma^c{}_{ac}\right), \tag{3.15}$$

then, one can calculate the Ricci variation as follows

$$\begin{aligned}
\mathscr{D}_\delta \mathscr{R} &= \mathscr{R}_{ab} \mathscr{D}_\delta g^{ab} + g^{ab} \mathscr{D}_\delta \mathscr{R}_{ab} \\
&= -2\mathscr{R}_{ab} K^{ab}{}_i \phi^i + \nabla_c \left(g^{ab} \mathscr{D}_\delta \gamma^c{}_{ab} - g^{ac} \mathscr{D}_\delta \gamma^b{}_{ba}\right) \\
&= -2\mathscr{R}_{ab} K^{ab}{}_i \phi^i + \nabla_a \left(\left(g^{ac} g^{bd} - g^{ab} g^{cd}\right) \nabla_b \mathscr{D}_\delta g_{cd}\right) \\
&= -2\mathscr{R}_{ab} K^{ab}{}_i \phi^i + 2\nabla_a \left(\left(g^{ac} g^{bd} - g^{ab} g^{cd}\right) \nabla_b (K_{cd}{}^i \phi_i)\right)
\end{aligned} \tag{3.16}$$

modulo a divergence, the variation of the Ricci scalar is

$$\mathscr{D}_\delta \mathscr{R} = -2\mathscr{R}_{ab} K^{ab}{}_i \phi^i \tag{3.17}$$

These results encode the variation of the intrinsic geometry of the brane world volume. However, sometimes it is essential to calculate the variation of a geometric object that corresponds to the extrinsic geometry of m, such as extrinsic curvature. Since some actions can be built using this object, for example, the Lovelock type brane gravity [4, 30], where the action is built through specific contraction of this geometric object, and one obtains second-order equations of motion, similar to lovelock theory [71].

The gradient along the deformation vector of normal vectors can also be expanded on the basis $\{e_a, n^i\}$

$$\mathscr{D}_\delta n_i = -J_{ai} g^{ab} X_b + \gamma_{ij} n^j, \tag{3.18}$$

note that, J_{ai} is defined in (3.6), while γ_{ij} is given by

$$\gamma_{ij} = \mathscr{D}_\delta n_i \cdot n_j = -\gamma_{ji} \tag{3.19}$$

γ_{ij} plays a similar role as twist potential $\omega_a{}^{ij}$ actually, it also transforms as a connexion under rotation of normal fields

$$\gamma \longrightarrow \mathscr{O} \gamma \mathscr{O} + (\mathscr{D}_\delta \mathscr{O}) \mathscr{O}^{-1} \tag{3.20}$$

Considering the previous sentence, one can define a covariant variation $\widetilde{\mathscr{D}}_\delta$ that takes into account this gauge symmetry, where one can rotate normal vectors at every point of world volume. And when one applies this covariant variation to a $(1,1)$-tensor under these rotations, it is obtained

$$\widetilde{\mathscr{D}}_\delta \Psi^i{}_j = \mathscr{D}_\delta \Psi^i{}_j - \gamma^i{}_k \Psi^k{}_j - \gamma_j{}^k \Psi^i{}_k. \tag{3.21}$$

If we want to have fully covariant equations of motion under coordinates changes, we must use this covariant variation. Then, the covariant variation of the normal vector can be written as

$$\widetilde{\mathscr{D}}_\delta n_i = J_{ai} g^{ab} X_b = -\left(\tilde{\nabla}^a \phi_i\right) X_a. \tag{3.22}$$

Using the definition of extrinsic curvature tensor, one can calculate its variations. Remember that $K_{ab}{}^i$ transforms as a vector under the rotations $\mathscr{O}$, then $\widetilde{\mathscr{D}}_\delta$ should be applied

$$\widetilde{\mathscr{D}}_\delta K_{ab}{}^i = -\widetilde{\mathscr{D}}_\delta n^i \cdot \mathscr{D}_a X_b - n^i \cdot \mathscr{D}_\delta \mathscr{D}_a X_b \tag{3.23}$$

where if one uses the variation of normal vectors and Gauss-Weingarten equations (2.11a), the first term of the last equation can be written as

$$-\widetilde{\mathscr{D}}_\delta n^i \cdot \mathscr{D}_a X_b = \gamma^c{}_{ab} J_c{}^i \tag{3.24}$$

for the second term on the right-hand side of (3.23) , we utilize the Ricci identity given by

$$[\mathscr{D}_\delta, \mathscr{D}_a] X^\mu_b - \mathscr{D}_{[\delta, X_a]} X^\mu_a = \mathbf{R}^\mu{}_{\nu\alpha\beta} n^\alpha{}_j X^\beta_a X^\nu_b \phi^j, \tag{3.25}$$

using the equation (3.7) one has that $[\delta, X_a] = 0$, and hence the term proportional to $\mathscr{D}_{[\delta, X_a]} X_b$ in (3.25) vanishes, so it is obtained the following

$$\begin{aligned} -n^i \cdot \mathscr{D}_\delta \mathscr{D}_a X_b &= -\mathbf{R}^\mu{}_{\nu\alpha\beta} n_\mu{}^i n^\alpha{}_j X^\beta_a X^\nu_b \phi^j - n^i \cdot \mathscr{D}_a \mathscr{D}_\delta X_b, \\ &= -\mathbf{R}^\mu{}_{\nu\alpha\beta} n_\mu{}^i n^\alpha{}_j X^\beta_a X^\nu_b \phi^j - \mathscr{D}_a \left(n^i \cdot \mathscr{D}_\delta X_b\right) + \mathscr{D}_a n^i \cdot \mathscr{D}_\delta X_b, \\ &= -\mathbf{R}^\mu{}_{\nu\alpha\beta} n_\mu{}^i n^\alpha{}_j X^\beta_a X^\nu_b \phi^j - \tilde{\nabla}_a \tilde{\nabla}_b \phi^i + K_{bcj} K_a{}^{ci} \phi^j. \end{aligned} \tag{3.26}$$

Therefore, the variation of extrinsic curvature tensor is

$$\widetilde{\mathscr{D}}_\delta K_{ab}{}^i = -\tilde{\nabla}_a \tilde{\nabla}_b \phi^i + K_{bcj} K_a{}^{ci} \phi^j + \mathbf{R}^\mu{}_{\nu\alpha\beta} n_\mu{}^i n^\beta{}_j X^\alpha_a X^\nu_b \phi^j. \tag{3.27}$$

From this last result, it is easy to see that the variation of the mean extrinsic curvature ,$K^i = g^{ab} K_{ab}{}^i$, is given by

$$\widetilde{\mathscr{D}}_\delta K^i = -\widetilde{\Delta} \phi^i - K_{ab}{}^i K^{ab}{}_j \phi^j + \mathbf{R}^\mu{}_{\nu\alpha\beta} n_\mu{}^i n^\beta{}_j h^{\alpha\nu} \phi^j, \tag{3.28}$$

where $\widetilde{\Delta} = g^{ab}\widetilde{\nabla}_a\widetilde{\nabla}_b$.

Having under control the covariant variation of these geometric objects will allow us to quickly calculate a large number of models that are invariant under reparametrizations of the world volume.

3.1.1 Examples of geometric models in brane mechanics

As we have already commented the functional action is built with terms that are invariant under symmetries of the physical system. In the case of extended objects, one must have invariant terms under reparametrizations of the world volume of the brane. The first action with this characteristic is the Dirac-Nambu-Goto action (DNG)

$$S_{DNG}[X] = -\alpha_0 \int_m \sqrt{-g}. \tag{3.29}$$

For convenience, we have absorbed the noncovariant volume term $d^{p+1}x$ into the integral symbol, i.e., $\int_m d^{p+1}x \longrightarrow \int_m$. Note that (3.29) is an action proportional to the volume of world volume, where α_0 is a proportionality constant, which in the case of $p = 1$ relates to the tension in a relativistic string [94, 116]. In the early 1970s, Nambu and Gotto took this action to describe the strong interaction [54, 78]. However, Paul Dirac had already considered a similar action since 1962, in [39], to model the electron as an extensible object. His aim was to explain the muon as a perturbation of this electron like-bubble. Subsequently, this term has also been taken into a account to model topological defects that could be originated in the early universe and could be responsible for forming structures [6, 48, 108]. Additionally, this action was used to study the probability of black hole formation through the collapse of perfectly circular strings [49]. Several more models have been built through this action, see [5, 34, 35, 47, 69].
By performing the variation of the action (3.29) and using the results obtained in the last sections, one gets

$$\widetilde{\mathscr{D}}_\delta S_{DNG}[X] = -\alpha_0 \int_m \widetilde{\mathscr{D}}_\delta \sqrt{-g} = -\alpha_0 \int_m \sqrt{-g}\, g_{ab} K^{abi} \phi^i, \tag{3.30}$$

where in the second equality, one has considered the equation (3.12). If one assumes that the action, S_{DNG}, extremizes for any field ϕ^i, i.e. $\widetilde{\mathscr{D}}_\delta S_{DNG} = 0$, then the resulting equations are

$$g_{ab} K^{abi} = K^i = 0. \tag{3.31}$$

That is to say that the mean extrinsic curvatures of the world volume of the extended object are identically zero. In the non-relativistic case, they are the equations for minimum surfaces [41]. One can use the results of last subsection to calculate the linearization of the equations of motion. In this case, the linearized equations coincide with the result obtained in (3.28), when one equalizes them to zero

$$-\widetilde{\Delta}\phi^i - K_{ab}{}^i K^{ab}{}_j\phi^j + \mathbf{R}^\mu{}_{\nu\alpha\beta} n_\mu{}^i n^\beta{}_j h^{\alpha\nu}\phi^j = 0. \tag{3.32}$$

Notice that, the fields ϕ^i fulfill these differential equations. Later, we will delve into the geometric meaning of linearizing the equations of motion. Besides, we will use them to obtain more information about the being studied physical system.

Another action that one can take is

$$S_2 = \sigma \int_m \sqrt{-g} K^i K_i, \tag{3.33}$$

this action is the relativistic extension of Canham-Helfrich energy for an elastic membrane [13, 60]. By calculating the variation of S_2, one obtains the following

$$\begin{aligned}
\widetilde{\mathscr{D}}_\delta S_2 =& \sigma \int_m \widetilde{\mathscr{D}}_\delta \sqrt{-g} K^i K_i + 2\sqrt{-g} K_i \widetilde{\mathscr{D}}_\delta K^i, \\
=& \sigma \int_m \sqrt{-g} \left[2K_i \left(-\tilde{\Delta}\phi^i - K_{ab}{}^i K^{ab}{}_j \phi^j + \mathscr{R}^\mu{}_{\nu\alpha\beta} n_\mu{}^i n^\beta{}_j h^{\alpha\nu}\phi^j \right) + K^i K_i K^j \phi_j \right], \\
=& \sigma \int_m \sqrt{-g} \left\{ \left[K^i K_i K_j - 2K_i K^i{}_{ab} K^{ab}{}_j + 2\mathbf{R}^\mu_{\nu\alpha\beta} n_\mu{}^i n^\beta{}_j h^{\alpha\nu} \right] \phi^j - 2K_i \widetilde{\Delta}\phi^i \right\}.
\end{aligned} \tag{3.34}$$

By integrating by parts in the last term of the last equality in (3.34) and extremizing the action S_2, one gets the equations of motion

$$-\widetilde{\Delta} K^j + \frac{1}{2} K^i K_i K^j - K_i K_{ab}{}^i K^{abj} + \mathbf{R}^\mu{}_{\nu\alpha\beta} n_\mu{}^i n^{\beta j} h^{\alpha\nu} K_i = 0 \tag{3.35}$$

Unlike the DNG model, here, these last equations of motion depend on background curvature. It is owing to the fact S_2 is a higher derivative action while S_{DNG} only depends on the first derivatives of embedded functions X^μ. Afterward, we will discuss another action that satisfies the invariance under reparametrization and is proportional to the integral of Ricci scalar $\mathscr{R}$. Nonetheless, this action plays a critical role in this work, therefore it will be discussed in another chapter.

We have reviewed geometric models for branes where we have applied the variations of different objects that were obtained in the last subsection.

3.2 Variational Covariant Approach

Now a completly variational covariant approach from ambient spacetime is showed in this chapter, here one does not split the perturbation into tangential and normal parts concerning worldvolume m. This access establishes the brane mechanics as a covariant fields theory. Where the covariance is regarding both diffeomorphisms of background and reparametrization of m. The principal tool used here is a variational derivative, which is inspired by a Bażański work [7]. In calculating the standard variation, the variations of a field involve an infinitesimal parameter. We denote it by s. This parameter defines a uniparametric family of embedding functions denoted by $X^\mu(x^a, s)$. One assumes that $X(x^a, s=0) = X^\mu(x^a)$ are the embedding functions that define m. Thus if we consider an infinitesimal variation around $X^\mu(x^a)$, we have

$$X^\mu \longrightarrow X^\mu + s\delta X^\mu. \tag{3.36}$$

Note that even though embedding functions transform as scalar functions under background diffeomorphism. As we had already seen, δX^μ is a vector that can be understood as the vector along which the deformation of the embedding functions of m is made. It is given by [50, 68]

$$\delta X^\mu = \left(\frac{\partial X^\mu}{\partial s}\right)_{s=0}.$$

In the variation of fields, it is usually convenient to define a covariant variation that considers the system's underlying symmetries. For this reason, we propose a fully covariant variation, defined by

$$\mathscr{D}_X = \delta X^\mu \mathscr{D}_\mu \tag{3.37}$$

These types of covariant variation have been implemented in [8, 17]. By defining a covariant variation as in (3.37) , conveniently, $\mathscr{D}_X \eta_{\mu\nu} = 0$ is fulfilled because the background metric is compatible with $\mathscr{D}_\mu$. In addition, we assume at the beginning that the variation is conserved along world volume. Geometrically, this translates that the Lie derivative of deformation vector δX^μ vanishes along tangent vectors. This implies

$$[\mathscr{D}_X, \mathscr{D}_a] X^\mu = \mathscr{D}_X X^\mu_a - \mathscr{D}_a \delta X^\nu \tag{3.38}$$

This can be understood owing to the fact that we are considering a one-parameter family of embedding functions, labeled by parameter s, and infenetesimally partial derivatives of these functions commutt, as required to have a foliation [50]. Unlike we did in the last section, we resist the temptation to divide variation δX^μ into its normal and tangential parts. Although this is highly convenient at the first variation, because the tangential variation can be associated with a reparametrization, and in the absence of borders, these can be safely neglected as pure gauge, at higher orders, this early gauge fixing meddles the general structure of the perturbation theory. Thus, if one maintains this covariant structure under the background spacetime transformations, the higher-order variations are more accessible. Carter has emphasized the utility of this covariant treatment of spacetime in the brane mechanics, see [25].

We will consider invariant actions under diffeomorphisms of ambient spacetime and reparametrization of m. Nevertheless, we will only focus on models that solely depends on the first derivative of embedded functions, $\partial_a X^\mu$. This coincides with the components of the vectors tangent to m, X^μ_a. Abusing the language, we can write these kinds of actions as

$$S[X] = \int_m \mathscr{L}(X^\mu_a), \tag{3.39}$$

models such as $S_2[X]$ in (3.33) are out of this variational approach. However, this description still involves a large number of physics models. In (3.39), $\mathscr{L}$ is the Lagrangian density of weight one. Anew one has absorbed the noncovariant volume in the integral symbol. By varying the action (3.39), for this one uses the covariant variation that has been defined in (3.37). Then, one obtains

$$\mathscr{D}_X S[X] = \int_m \mathscr{D}_X \mathscr{L}\left(X^\mu_a\right), \tag{3.40}$$

by using the identity (3.38)

$$\mathscr{D}_X \mathscr{L} = \frac{\partial \mathscr{L}}{\partial X^\mu_a} \mathscr{D}_X X^\mu_a = \frac{\partial \mathscr{L}}{\partial X^\mu_a} \mathscr{D}_a \delta X^\mu. \tag{3.41}$$

By integrating by parts

$$\mathscr{D}_X S[X] = \int_m \frac{\partial \mathscr{L}}{\partial X^\mu_a} \mathscr{D}_a \delta X^\mu = \int_m \mathscr{E}_\mu(\mathscr{L}) \delta X^\mu + \int_m \mathscr{D}_a \mathscr{Q}^a, \tag{3.42}$$

where we identify current density of Noether as

$$\mathscr{Q}^a = \frac{\partial \mathscr{L}}{\partial X^\mu_a} \delta X^\mu = \mathscr{P}_\mu{}^a \delta X^\mu, \tag{3.43}$$

here $\mathscr{P}_\mu{}^a$ is the canonical linear momentum. It is a one-form from background and vectorial density from world volume, and this is given by

$$\mathscr{P}_\mu{}^a = \frac{\partial \mathscr{L}}{\partial X^\mu_a}. \tag{3.44}$$

The Euler-Lagrange equations are obtained when one equalizes the action variation to zero. And these read as

$$\mathscr{E}_\mu(\mathscr{L}) = -\mathscr{D}_a\left(\frac{\partial \mathscr{L}}{\partial X^\mu_a}\right) = -\mathscr{D}_a \mathscr{P}_\mu{}^a = 0, \tag{3.45}$$

notice that, these equations are a conservation law, because the action is invariant under reparametrization. Besides, theses resulting equation are second order. Theses characteristics are also shared by equations of motion of branes types Lovelock [30].

Coming up next, we will implement this method using as example the DNG action, defined in (3.29). Taking into account that

$$\frac{\partial}{\partial X^\mu_a}\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{cd}\frac{\partial}{\partial X^\mu_a} g_{cd} = \sqrt{-g}\, g^{ab}\eta_{\mu\nu}X^\nu_b, \tag{3.46}$$

then, linear momentum reads

$$\mathscr{P}_\mu{}^a = -\alpha_0\sqrt{-g}\,\eta_{\mu\nu}g^{ab}X^\nu_b. \tag{3.47}$$

Note that, linear momentum is tangential to m. So equations of motions are given by vanishing of first variation

$$-\mathscr{D}_a\mathscr{P}_\mu{}^a = \alpha_0\mathscr{D}_a\left(\sqrt{-g}\,\eta_{\mu\nu}g^{ab}X^\nu_b\right) = 0 \tag{3.48}$$

These equations are covariant under diffeomorphisms of background spacetime, and they look different from obtained equations in (3.31). However, they give us the same information. Remember that in the last section, we obtained the equations of motion splitting the deformation in the normal and tangential part to m, and we only chose a deformation offshore normal fields since the tangential part does not contribute to the dynamics. So it is possible to recover the equations obtained in (3.31) by projecting the last equations along normal fields

$$-n^{\mu i}\mathscr{D}_a\mathscr{P}_\mu{}^a = \alpha_0\sqrt{-g}\,\eta_{\mu\nu}g^{ab}n^{\mu i}\mathscr{D}_a X^\nu_b = -\alpha\sqrt{-g}K^i = 0, \tag{3.49}$$

we have used the Gauss-Weingarten equation (2.11a) in the second equality. One can show that the projection of equations of motion (3.48) along tangent vectors is identically zero

$$\begin{aligned}
-X^\mu_c\mathscr{D}_a\mathscr{P}_\mu{}^a &= X^\mu_c\left(-\alpha_0\mathscr{D}_a\left(\sqrt{-g}\,\eta_{\mu\nu}g^{ab}X^\nu_b\right)\right)\\
&= -\alpha_0\left[\partial_a\left(\sqrt{-g}\,g^{ab}\right)g_{cd} + \sqrt{-g}\,g^{ab}\gamma^d{}_{ab}g_{cd}\right]\\
&= \sqrt{-g}\,g_{bc}\nabla_a g^{ab}\\
&= 0
\end{aligned} \tag{3.50}$$

In the second equality, it has been used the fact that the induced metric transforms as a scalar under background transformations, then $\mathscr{D}_a\left(\sqrt{-g}\,g^{ab}\right) = \partial_a\left(\sqrt{-g}\,g^{ab}\right)$. In addition, the equation (2.11a) was used too. Subsequently, in the last equality, we used that covariant derivative ∇_a is compatible with the metric g_{ab}. Both the normal variations, which we discussed in the first part of this chapter, and covariant general variations, provide us with the tools to obtain the dynamics of a geometric model. However, the completely covariant variational method turns out to be more efficient for calculating high-order variations. We will see it in the next chapter, where we will fully exploit the properties of the covariant derivative that we have defined, this will allow us to obtain the respective equations for the different variations in a simple way.

CHAPTER 4

Higher-Order Variations

This chapter is based in the papers Covariant higher order perturbations of branes in curved spacetime and A covariant simultaneous action for branes. Where we discuss the linearization of equations of motion (LEM).The LEM are a first-order expansion of Euler-Lagrange's equations around one of their solutions. One can obtain them directly by varying the equations of motion. However, it is well known that they can also be obtained through the second variation of the action. Thus, we can implement the completely covariant approach reviewed in the last chapter to calculate them. Additionally, we will go further and make a third variation of the action by exploiting this variational approach. We will also review the importance of this third variation in the analysis of a physical system. Finally, we will show a variational principle that we have developed in the context of relativistic branes, which allows us to obtain the first and second variations of the action at the same time in a straightforward manner.

4.1 Linearized Equations of Motion

The primary objective of a theoretical physicist is to construct models that can explain the fundamental and essential characteristics of a physical system. However, due to the complexity of certain systems, it is sometimes necessary to develop idealized models that are only valid within a specific range and capture only a few aspects of the system. For instance, in the 1920s, Alexander Friedman, Georges Lemaitre, Howard Robertson, and Arthur Walker (FLRW) proposed a solution to the general relativity Einstein equations, which describes an isotropic homogeneous universe in expansion. However, this model, also known as the standard model of cosmology [10], cannot account for the formation of structures in our universe, such as galaxies and galaxy clusters. To improve upon this model, one may consider a slightly modified version that more accurately describes our universe. This can be achieved by linearizing the Einstein equation and deriving differential equations for the model's variations. By solving these equations, we can construct approximate solutions that provide a more precise description of the universe [77].

In a broad sense, the LEM are useful in constructing new approximate solutions that can provide a better description of a specific physical system. They are also valuable in studying the linear stability of each solution of the physical system. The LEM represent differential equations for perturbations of a particular solution of the equations of motion, and by solving them, we can determine whether these perturbations increase or decrease, resulting in an unstable or stable solution, respectively. Given the crucial role of the LEM in studying physical systems, it is essential to approach them in the context of extended objects. One can achieve this by using the covariant variation method, which involves varying the system equations of motion along normal deformations, as discussed in the first section of the previous chapter. If we denote the equations of motion by $\mathscr{E}^i = 0$, then the LEM are

$$\widetilde{\mathscr{D}}_\delta \mathscr{E}^i = 0. \tag{4.1}$$

The Linearized Einstein's Equations (LEM) for the DNG action in equation (3.32) have been computed. The calculation process for this particular action was straightforward. However, when dealing with other actions like Equation (3.35), it becomes a tedious task, leading to the derivation of significantly large equations [17]. Despite these challenges, all the methods described in Section 3.1 can be directly applied to linearize the equations of motion for the brane.

However, there are situations where utilizing the fully covariant formalism from the ambient space proves to be comparatively simpler. Specifically, when confronted with higher-order variations, the fully covariant formalism becomes particularly advantageous, offering a more efficient approach to manage the intricate equations implicated in such scenarios.

Through the investigation of this approach, there is an opportunity to enhance our comprehension of the Linear Equations of Motion (LEM) and how they can be obtained in a manner that is both efficient and systematic. This exploration holds substantial implications for the examination of physical systems and the developed of novel theoretical models. For this reason, in the next section, we will see that LEM can also be obtained using this approach.

4.2 Second Covariant Variation and Jacobi Equations

As we know, the equations of motion of a system can be deduced by varying the action. Remember that the action S is a functional, i.e., it is a mathematical device that, in this particular case, takes the embedding functions X and results in a number. When we speak of varying the action, we are essentially introducing new functions $X' = X + s\zeta$ close to the embedding functions. With s being a small parameter ($s < 1$), and ζ being an arbitrary function. The resulting change in the action is the difference between the numerical value obtained from evaluating the action for the original and the neighbor embedding functions.

$$\Delta S \equiv S[X + s\zeta] - S[X], \tag{4.2}$$

the variation above is finite, and since the parameter s is small, one can perform an expansion of ΔS around embedding functions X, resulting in the following

$$\Delta S = s\delta S + \frac{s^2}{2!}\delta^2 S + \frac{s^3}{3!}\delta^3 S + O(s^4), \tag{4.3}$$

where we have that

$$\delta^{(n)} S = \left(\frac{d^n S}{ds^n}\right)_{s=0}, \tag{4.4}$$

The first variation is the expansion to the first order of the action when a slight change in the embedding functions is performed. By using the chain rule, this first variation can be written as

$$\delta S = \left(\frac{dS}{ds}\right)_{s=0} = \left(\frac{\partial X^\mu(x^a,s)}{\partial s}\partial_\mu S\right)_{s=0} = \left(\frac{\partial X^\mu(x^a,s)}{\partial s}\mathscr{D}_\mu S\right)_{s=0} = \mathscr{D}_X S, \tag{4.5}$$

where in the second equality, we have used that the action is invariant under transformations of background spacetime. Here $\mathscr{D}_X$ is the covariant variation that was defined in the section 3.2. Surely the meaning of the second variation for the reader is now evident. It is understood as the second-order expansion of the action under small changes of its X

argument functions. It reads as

$$
\begin{aligned}
\delta^2 S &= \left(\frac{d^2 S}{ds^2}\right)_{s=0} \\
&= \left(\frac{\partial X^\mu(x^a,s)}{\partial s}\mathscr{D}_\mu\left(\frac{\partial X^\nu(x^a,s)}{\partial s}\mathscr{D}_\nu S\right)\right)_{s=0} \\
&= \delta X^\mu \mathscr{D}_\mu \delta X^\nu \mathscr{D}_\nu S \\
&= \mathscr{D}_X \mathscr{D}_X S \\
&= \mathscr{D}_X^2 S.
\end{aligned} \tag{4.6}
$$

While the practical significance of calculating the second variation of the action may not be immediately apparent, it plays a crucial role in various physical contexts. Regrettably, it tends to be overlooked in both undergraduate and postgraduate courses, where the focus is typically only on the first variation of the action to derive the dynamic equations of the system. However, the second variation of the action holds great relevance in numerous scenarios. Its importance becomes particularly pronounced when examining the formation of structures in the Universe or studying anisotropies in the cosmic microwave background. In the context of brane scenarios, the second variation is indispensable for analyzing brane stability and understanding the behavior of their intrinsic geometry under deformation. Additionally, the second variation is critical for assessing quantum corrections at the one-loop level when considering potential quantization of the system using the path integral approach.

By equating the second variation to zero while assuming the fulfillment of the equations of motion, we derive a collection of differential equations that govern the deformations δX of the system, referred to as the Jacobi equations. In order to preserve covariance under background diffeomorphisms, we can employ the variational covariant approach outlined in the preceding chapter and compute the second variation of the action utilizing the operator $\mathscr{D}_X$. This methodology allows us to obtain the Jacobi equations in their fully covariant form, which can be expressed as

$$\mathscr{J}^\mu(\delta X) = 0, \tag{4.7}$$

if one projects the Jacobi equations on the basis $\{n^i, X_a\}$, the tangential projections are identically zero, while normal projections are reduced to linearized equations that are covariant under reparametrization of world volume, which were discussed in the previous section, i.e.

$$n_\mu{}^i \mathscr{J}^\mu(\delta X) = \widetilde{\mathscr{D}}_\delta \mathscr{E}^i. \tag{4.8}$$

In the second variation of the geometric model in (3.39), We choose to utilize the expression (3.42) for the first variation. For the sake of simplicity, we assume that the world volume is no boundaries, $\partial m = 0$, or that appropriate boundary conditions have been selected, and we neglect the contribution of the Noether current for the moment. Thus, the second variation can be written as

$$\mathscr{D}_X^2 S[X] = \int_m \left[\mathscr{D}_X \mathscr{E}_\mu \delta X^\mu + \mathscr{E}_\mu \delta^2 X^\mu\right], \tag{4.9}$$

where

$$\delta^2 X = \frac{\partial^2 X(x^a, s)}{\partial s^2}. \tag{4.10}$$

Assuming that equations of motion are fulfilled, we obtain

$$\mathscr{D}_X^2 S[X]|_{[0]} = \int \left[\mathscr{D}_X \mathscr{E}_\mu|_{[0]}\right] \delta X^\mu, \tag{4.11}$$

the subscript [0] serves as a reminder that we are considering the on-shell condition. It is evident that evaluating the second variation is tantamount to linearizing the equations of motion. It is important to note that, in this context, the linearized equations of motion preserve covariance solely in relation to the ambient spacetime. Nonetheless, as observed earlier, it is possible to transform them into covariant equations of motion with respect to world volume reparametrizations through the application of a normal projection.
In contrast to (3.7), it should be emphasized that when $[\mathscr{D}_X, \mathscr{D}_a]$ is applied to a tensor quantity, the result is nonzero. This is a consequence of the variation operator being a covariant directional derivative. Here is where background spacetime curvature comes into play, as shown explicitly below. We have

$$\begin{aligned} \mathscr{D}_X \mathscr{E}_\mu|_{[0]} &= -\mathscr{D}_X \mathscr{D}_a \mathscr{P}_\mu{}^a \\ &= -[\mathscr{D}_X, \mathscr{D}_a]\mathscr{P}_\mu{}^a - \mathscr{D}_a \mathscr{D}_X \mathscr{P}_\mu{}^a \\ &= -\mathbf{R}_{\alpha\beta\mu}{}^\rho \delta X^\alpha X_a^\beta \mathscr{P}_\rho{}^a - \mathscr{D}_a \mathscr{D}_X \mathscr{P}_\mu{}^a. \end{aligned} \tag{4.12}$$

We have used Bianchi's identity (2.4) and remember that $[\delta X, X_a] = 0$. Therefore, the second variation, (4.11) , takes the form

$$\mathscr{D}_X^2 S[X]|_{[0]} = -\int_m \left[\mathscr{D}_a \mathscr{D}_X \mathscr{P}_\mu{}^a + \mathbf{R}_{\alpha\beta\mu}{}^\rho \delta X^\alpha X_a^\beta \mathscr{P}_\rho{}^a\right] \delta X^\mu. \tag{4.13}$$

If one uses the linear momentum definition that is given in (3.44), the first term on the right-hand side of the above expression can be written as its variation or linearization

$$\begin{aligned} \mathscr{D}_X \mathscr{P}_\mu{}^a &= \frac{\partial \mathscr{L}}{\partial X_b^\nu \partial X_a^\mu} \mathscr{D}_X X_b^\nu \\ &= \mathscr{H}_{\nu\mu}^{ba} \mathscr{D}_X X_b^\nu \\ &= \mathscr{H}_{\nu\mu}^{ba} \mathscr{D}_a \delta X^\nu, \end{aligned} \tag{4.14}$$

where we have defined the Hessian matrix in the following way

$$\mathscr{H}^{ba}_{\nu\mu} = \frac{\partial^2 \mathscr{L}}{\partial X^\nu_b \partial X^\mu_a}. \tag{4.15}$$

It should be noted that the Hessian exhibits degeneracy, allowing for null eigenvectors, as a result of the gauge freedom associated with reparameterization invariance. Furthermore, the Hessian possesses symmetry among pairs of its indices, which can be expressed as follows

$$\mathscr{H}^{ba}_{\nu\mu} = \mathscr{H}^{ab}_{\mu\nu}. \tag{4.16}$$

By inserting the equation (4.14) in the second variation (4.13), one obtains

$$\mathscr{D}^2_X S[X]|_{[0]} = -\int_m \{\mathscr{D}_a \left[\mathscr{H}^{ba}_{\nu\mu} \left(\mathscr{D}_b \delta X^\nu\right)\right] + \mathbf{R}_{\alpha\beta\mu}{}^{\rho} \delta X^\alpha X^\beta_a \mathscr{P}_\rho{}^a\} \delta X^\mu. \tag{4.17}$$

At this juncture, δX assumes an arbitrary nature and denotes a linear perturbation around any embedding on-shell. Nonetheless, if our interest lies in perturbing from one on-shell configuration to another, it becomes necessary to select δX^μ in a manner that renders the second variation null. Upon inspecting the aforementioned expression, the equations governing the vector δX^μ become apparent and can be expressed as

$$-\mathscr{D}_a \left[\mathscr{H}^{ba}_{\nu\mu} \mathscr{D}_b \eta^\nu\right] - \mathbf{R}_{\nu\beta\mu}{}^{\rho} X^\beta_a \mathscr{P}_\rho{}^a \eta^\nu = 0. \tag{4.18}$$

To maintain consistency with the notation used in [14, 15], we introduce $\delta X^\mu = \eta^\mu$. The vector η is interpreted as the connecting vector between neighboring branes during their evolution. This concept aligns with the geodesic deviation equation, as discussed in [7, 9, 12, 110]. The first term, (4.18), is second-order, as expected since the equations of motion are second-order. In classical mechanics terminology, the Hessian assumes the role of a mass matrix. The second term, which involves the Riemann tensor, can be understood as an external force. This interpretation aligns with our intuition based on the geodesic deviation equation in general relativity. In the case of a flat background, the Jacobi equations take the form of a conservation law, similar to a divergence-free equation for linearized canonical momentum.

By applying the results obtained for the DNG action, where we have previously computed its canonical linear momentum in the preceding chapter, we proceed to calculate its

Hessian for the purpose of evaluating the second variation.

$$
\begin{aligned}
\mathscr{H}^{ba}_{\nu\mu} &= \frac{\partial}{\partial X^\nu_b}\left(-\alpha_0\sqrt{-g}\,\eta_{\mu\lambda}\gamma^{ac}X^\lambda_c\right), \\
&= -\alpha_0\sqrt{-g}\left[X^b_\nu X^a_\mu + \eta_{\mu\nu}g^{ab} - \eta_{\mu\lambda}X^\lambda_c\left(g^{ba}X^c_\nu + g^{bc}X^a_\nu\right)\right], \\
&= -\alpha_0\sqrt{-g}\left(X^b_\nu X^a_\mu + \eta_{\mu\nu}g^{ab} - g^{ab}h_{\mu\nu} - X^b_\mu X^a_\nu\right), \\
&= -\alpha_0\sqrt{-g}\left(g^{ab}\Pi_{\mu\nu} + X^{ab}_{\mu\nu}\right),
\end{aligned}
\tag{4.19}
$$

where we have defined

$$
\begin{aligned}
X^a_\mu &\equiv g^{ab}\eta_{\mu\nu}X^\nu_a, \\
X^{ab}_{\mu\nu} &\equiv X^a_\mu X^b_\nu - X^b_\nu X^a_\mu.
\end{aligned}
\tag{4.20}
$$

For this case the Jacobi equations are

$$
\alpha_0\{\mathscr{D}_a\left[\sqrt{-g}\left(g^{ab}\Pi_{\mu\nu} + X^{ab}_{\mu\nu}\right)\mathscr{D}_b\eta^\nu\right] + \sqrt{-g}\,\mathbf{R}_{\alpha\beta\mu\rho}\eta^\alpha h^{\beta\rho}\} = 0. \tag{4.21}
$$

Note that the normal and tangential projections of the deviation vector arise naturally in the expression, showing the mixture of both contributions. These equations provide a generalization to the branes of the well-known geodesic deviation equation for particles.

4.3 Third Covariant Variation

In order to demonstrate the efficacy of a covariant variational approach in perturbation theory for brane dynamics, we expand our analysis to the third order. Additionally, one motivation for considering the third variation of the geometric model is the potential degeneracy of the second-order contribution among strain modes. As anticipated, the computational complexity noticeably escalates. Nevertheless, employing a covariant variational approach provides a deeper understanding of the inherent mathematical variational structure and facilitates its practical application when incorporating second-order perturbations. Furthermore, this covariant approach elucidates the interplay between first and second-order perturbations.

Following the discussion at the beginning of section 4.2, it is not difficult for the reader to understand that the third variation of the action is the third-order expansion of the finite variation in (4.3). Thus, considering expression (4.11) for the second variation. An additional variation is given by

$$
\mathscr{D}^3_X S[X]|_{[0]} = \int_m \{[\mathscr{D}^2_X\mathscr{E}_\mu|_{[0]}]\,\delta X^\mu + [\mathscr{D}_X\mathscr{E}_\mu|_{[0]}]\,\delta^2 X^\mu\}. \tag{4.22}
$$

Assuming that both the equations of motion (3.45) and the Jacobi equations (4.18) are satisfied, then the second term vanishes, and the third variation (4.22) reduces to

$$\mathscr{D}_X^3 S[X]|_{[0,1]} = \int_m \left[\mathscr{D}_X \mathscr{E}_\mu(\mathscr{L})|_{[0,1]}\right], \tag{4.23}$$

In this case, the subscript $[0,1]$ serves as a reminder that both the equations of motion and the Jacobi equations are presumed to be satisfied. Next, we proceed to examine the second variation of the equations of motion. To facilitate this analysis, it is advantageous to initially consider a flat background where the Riemann tensor is zero. Subsequently, we can introduce the additional complexity of a curved background.

4.3.1 Flat Background Spacetime

In the specific scenario of a Minkowski spacetime, background derivatives exhibit commutation, enabling us to utilize, similar to the second variation, the fact that the equations of motion can be expressed as a conservation law. Concerning the linear momentum, the second variation can be represented as shown in Equation (4.9). Consequently, employing Equation (3.44), the third variation can be expressed as follows

$$\mathscr{D}_X^3 S[X]|_{[0,1]} = \int_m \left[\left(\mathscr{D}_a \mathscr{D}_X^2 \mathscr{P}_\mu{}^a\right)\delta X^\mu\right]|_{[0,1]}, \tag{4.24}$$

Considering the commutation between partial derivatives and variations, we proceed to unpack the second variation of the linear momentum. Referring to Equation (4.14) for the first variation of the linear momentum, it is worth noting that when the variation acts on the first variation of the shape functions δX^μ, it yields a second variation of the shape functions $\delta^2 X^\mu$. Similarly, when the variation acts on the Hessian matrices of the energy density, it produces a source term that exhibits a quadratic dependence on the first variations δX^μ. It is important to bear in mind that the assumption of the satisfaction of the Jacobi equation is tantamount to assuming the given nature of the first-order perturbations δX^μ. In this context, we are referring to a source term. Consequently, the second variation of the linear stress tensor, as obtained from the variation of Equation (4.14), can be expressed as the sum of

$$\mathscr{D}_X^2 \mathscr{P}_\mu{}^a\left(\delta^2 X, \delta X\right) = \mathscr{D}_X \mathscr{P}_\mu{}^a\left(\delta^2 X\right) + \mathscr{S}_\mu{}^a(\delta X), \tag{4.25}$$

the first term in the expression depends on the second variation $\delta^2 X$ and exhibits the same structure as the first-order variation. It corresponds to the Jacobi operator with δX replaced by $\delta^2 X$ in Equation (4.14). This term does not necessitate any additional effort as the Jacobi operator is already known, as shown in Equation (4.18). The second term in Equation (4.25) represents a source term that is contingent upon the first variation δX due to the variation of the Hessian matrix.

$$\mathscr{S}_\mu{}^a(\delta X) = \left(\frac{\partial^2 \mathscr{D}_X \mathscr{L}}{\partial X^\mu_a \partial X^\nu_c}\right) \mathscr{D}_c \delta X^\nu, \tag{4.26}$$

this term does necessitate additional effort. By employing Equation (3.41) for $\mathscr{D}_X \mathscr{L}$, it assumes a formidable appearance.

$$\mathscr{S}_\mu{}^a(\delta X) = \mathscr{T}^{cba}_{\rho\nu\mu} \mathscr{D}_c \delta X^\nu \mathscr{D}_b \delta X^\rho, \tag{4.27}$$

where we have defined the tensor of third order derivatives analogous to the Hessian

$$\mathscr{T}^{cba}_{\rho\nu\mu} = \frac{\partial^3 \mathscr{L}}{\partial X^\rho_c \partial X^\nu_b \partial X^\mu_a}. \tag{4.28}$$

Returning to the third variation of the action (4.24), we find therefore that it can be written in the deceivingly simple form

$$\mathscr{D}^3_X S[X]|_{[0,1]} = \int_m \left\{\left[\mathscr{J}_\mu\left(\delta^2 X\right) + \mathscr{D}_a \mathscr{S}_\mu{}^a(\delta X)\right] \delta X^\mu\right\}|_{[0,1]}, \tag{4.29}$$

in this equation, the Jacobi operator,

$$\mathscr{J}_\mu\left(\delta^2 X\right) = \mathscr{D}_a \mathscr{D}_X \mathscr{P}_\mu{}^a\left(\delta^2 X\right),$$

incorporates the second-order perturbations, while the source term $\mathscr{D}_a \mathscr{S}_\mu{}^a(\delta X)$ depends on the first order perturbations. Notably, the expression for the third variation reveals that the higher-order Jacobi equation exhibits a distinct structure compared to the second-order Jacobi equation. Specifically, it is non-homogeneous due to the presence of a source term.

4.3.2 Curved Background Spacetime

Now, let's examine the modification of the third variation (4.29) when the brane evolves in a curved background, where partial derivatives and variations no longer commute. We begin with Equation (4.17), representing the second variation with a non-vanishing curvature term. Introducing an additional third variation, we obtain

$$\begin{aligned} \mathscr{D}_X S[X]|_{[0,1]} = -\int_m \Big\{ & \mathscr{D}_a \left[\mathscr{D}_X \left(\mathscr{H}^{ba}_{\nu\mu} \mathscr{D}_b \delta X^\nu\right)\right] + \mathbf{R}_{\alpha\beta\nu}{}^\rho \delta X^\alpha e^\beta_a \mathscr{D}_X \mathscr{P}_\rho{}^a \\ & + \mathscr{D}_X\left(\mathbf{R}_{\alpha\beta\mu}{}^\rho \delta X^\alpha e^\beta_a \mathscr{P}_\rho{}^a\right)\Big\} \delta X^\mu, \end{aligned} \tag{4.30}$$

where we used the Bianchi identity yet again in the first line. The first term can be written in terms of the canonical momentum using (4.14)

$$
\begin{aligned}
\mathscr{D}_X\left(\mathscr{H}^{ba}_{\nu\mu}\mathscr{D}_b\delta X^\nu\right) &= \mathscr{D}^2_X\mathscr{P}_\mu{}^a \\
&= \mathscr{T}^{cba}_{\rho\nu\mu}\left(\mathscr{D}_c\delta X^\rho\right)\left(\mathscr{D}_b\delta X^\nu\right) + \mathscr{H}^{ba}_{\nu\mu}\mathscr{D}_X\left(\mathscr{D}_b\delta X^\nu\right) \\
&= \mathscr{T}^{cba}_{\rho\nu\mu}\left(\mathscr{D}_c\delta X^\rho\right)\left(\mathscr{D}_b\delta X^\nu\right) + \mathscr{H}^{ba}_{\nu\mu}\mathbf{R}^\nu{}_{\alpha\rho\beta}\delta X^\alpha\delta X^\rho e^\beta_b \\
&\quad + \mathscr{H}^{ba}_{\nu\mu}\mathscr{D}_b\delta^2 X^\nu,
\end{aligned}
\tag{4.31}
$$

where the tensor $\mathscr{T}$ is defined in (4.28). In the last step, we commuted the derivative and the variation producing a background Riemann tensor projection. Let us now use the Leibniz rule to unpack the third term in (4.30),

$$
\begin{aligned}
\mathscr{D}_X\left(\mathbf{R}_{\alpha\beta\nu}{}^\rho\delta X^\alpha X^\beta_a\mathscr{P}_\rho{}^a\right) &= \left(\mathscr{D}_\sigma\mathbf{R}_{\alpha\beta\mu}{}^\rho\right)\delta X^\sigma\delta X^\alpha X^\beta_a\mathscr{P}_\rho{}^a + \mathbf{R}_{\alpha\beta\mu}{}^\rho\left(\delta^2 X^\alpha\right)X^\beta_a\mathscr{P}_\rho{}^a \\
&\quad + \mathbf{R}_{\alpha\beta\mu}{}^\rho\delta X^\alpha\mathscr{D}_a\left(\delta X^\beta\right)\mathscr{P}_\rho{}^a + \mathbf{R}_{\alpha\beta\mu}{}^\rho\delta X^\alpha X^\beta_a\left(\mathscr{D}_X\mathscr{P}_\rho{}^a\right),
\end{aligned}
\tag{4.32}
$$

where we applied Equation (3.7) in the second term. It is important to observe that the presence of numerous curvature terms, along with a derivative of the Riemann tensor, significantly complicates the expression compared to the scenario of a flat background spacetime. By substituting Equations (4.32) and (4.31) into Equation (4.30), we get

$$
\begin{aligned}
\mathscr{D}^3_X S[X]|_{[0,1]} = -\int_m \Big\{&\mathscr{D}_a\Big[\mathscr{T}^{cba}_{\rho\nu\mu}\left(\mathscr{D}_c\delta X^\rho\right)\left(\mathscr{D}_b\delta X^\nu\right) \\
&+\mathscr{H}^{ba}_{\nu\mu}\mathscr{D}_b\delta^2 X^\nu + \mathscr{H}^{ba}_{\nu\mu}\mathbf{R}^\nu{}_{\alpha\rho\beta}\delta X^\alpha\delta X^\rho X^\beta_b\Big] \\
&+ 2\mathbf{R}_{\alpha\beta\mu}{}^\rho X^\beta_a\delta X^\alpha\mathscr{D}_X\mathscr{P}_\rho{}^a + \left(\mathscr{D}_\sigma\mathbf{R}_{\alpha\beta\mu}{}^\rho\right)\delta X^\sigma\delta X^\alpha X^\beta_a\mathscr{P}_\rho{}^a \\
&+\mathbf{R}_{\alpha\beta\mu}{}^\rho\left(\delta^2 X^\alpha\right)X^\beta_a\mathscr{P}_\rho{}^a + \mathbf{R}_{\alpha\beta\mu}{}^\rho\delta X^\alpha\left(\mathscr{D}_a\delta X^\beta\right)\mathscr{P}_\rho{}^a\Big\},
\end{aligned}
\tag{4.33}
$$

As we can see, it is apparent that the first two lines represent a conservation law, specifically for the second variation of the canonical momentum. Furthermore, there is a term that is proportional to both the curvature and the first variation of the momentum, as well as a term that is linear in the second-order deviation vector. Notably, we now encounter quadratic terms and derivatives of the first-order deviation vector. However, it is assumed that such a vector is already known.

The vanishing of the previous integral leads to the equation

$$
\mathscr{D}_a\left[\mathscr{D}^2_X\mathscr{P}_\mu{}^a\right] + \mathbf{R}_{\alpha\beta\mu}{}^\rho\delta X^\alpha X^\beta_a\mathscr{D}_X\mathscr{P}_\rho{}^a + \mathscr{D}_X\left(\mathbf{R}_{\alpha\beta\mu}{}^\rho\delta X^\alpha X^\beta_a\mathscr{P}_\rho{}^a\right) = 0,
\tag{4.34}
$$

the above equation, we have conveniently expressed it in terms of the canonical momentum to avoid unnecessarily long expressions and to demonstrate the underlying structure of the equations, as promised. It is important to note that the embeddings are known from the equations of motion, and the variation vector δX^ν satisfies the Jacobi equation (4.18) with $\eta = \delta X^\nu$. Equation (4.34) now determines the second-order variation vector $\delta^2 X^\nu$. Similar to the previous section, the variation (4.33) can be organized as follows

$$\mathscr{D}^3_X S[X]|_{[0,1]} = -\int_m \left[\mathscr{J}_\mu(\delta^2 X) + \mathscr{F}_\mu(\delta X)\right]\delta X^\mu, \tag{4.35}$$

where $\mathscr{F}_\mu$ is the source term given by

$$\begin{aligned}\mathscr{F}_\mu =& \mathscr{D}_a\left[\mathscr{T}^{cba}_{\rho\nu\mu}\left(\mathscr{D}_c\delta X^\rho\right)\left(\mathscr{D}_b\delta X^\nu\right) + \mathscr{H}^{ba}_{\nu\mu}\mathbf{R}^\nu{}_{\alpha\rho\beta}\delta X^\alpha\delta X^\rho X^\beta_b\right]\\ &+\mathbf{R}_{\alpha\beta\mu}{}^\rho\left[2\delta X^\alpha X^\beta_a\mathscr{H}^{ba}_{\nu\rho}\mathscr{D}_b\delta X^\nu + \delta X^\alpha\mathscr{D}_a\delta X^\beta\mathscr{P}_\rho{}^a\right]\\ &+\left(\mathscr{D}_\sigma\mathbf{R}_{\alpha\beta\mu}{}^\rho\right)\delta X^\sigma\delta X^\alpha X^\beta_a\mathscr{P}_\rho{}^a.\end{aligned} \tag{4.36}$$

We can see that equation (4.34) corresponds to a Jacobi equation for the second-order variation field $\delta^2 X^\mu$ with a source term $\mathscr{F}_\mu$, akin to Equation (4.29). However, in this case, the source term is modified by the curvature of the spacetime background. It is worth noting that the expression in Equation (4.34) has previously been derived in [70], albeit with a focus on normal perturbation modes and direct perturbations in the equations of motion. Here, we present a fully covariant expression from a variational perspective. This expression provides a second-order approximation to the deviation of neighboring on-shell branes in a fixed arbitrary background. Expanding to second-order perturbations is not only a natural progression following the first order but also serves to overcome degeneracies that arise at the first order. Moreover, certain physical quantities exhibit a leading second-order contribution. For a more comprehensive list, refer to, for instance, Ref. [75].

We can apply these results to the example of DNG action. For this, we need to obtain the tensor $\mathscr{T}$ defined in (4.28). When the algebraic dust settles down, we obtain

$$\mathscr{T}^{cba}_{\tau\nu\mu} = \mu\sqrt{-g}\left[\Pi_{\nu\tau}\left(g^{bc}X^a_\mu - 2g^{ab}X^c_\mu\right) + 2\Pi_{\mu\tau}\left(g^{ac}X^b_\nu - g^{ab}X^c_\nu - g^{bc}X^a_\nu\right) + X^{abc}_{\mu\nu\tau}\right], \tag{4.37}$$

where there is an abundance of raised and lowered indices, and now we have a tangential antisymmetric trivector

$$X^{abc}_{\mu\nu\tau} := 3!X^{[a}_\mu X^b_\nu X^{c]}_\tau. \tag{4.38}$$

Putting it all together, we use the expressions (3.47), (4.19), and (4.37) to insert them into (4.35), taking into account (4.36). Then, we can write the full third variation for the

DNG action

$$
\begin{aligned}
\mathscr{D}^3_X S_{DNG}|_{[0,1]} = \int_m \Big\{ & \mathscr{J}_\mu\left(\delta^2 X\right) + \mu \mathscr{D}_a \Big[\sqrt{-g}\Big(\Pi_{\nu\tau}\left(g^{bc}X^a_\mu - 2g^{ab}X^c_\mu\right) \\
& + 2\Pi_{\nu\tau}\left(g^{ac}X^b_\nu - g^{ab}X^c_\nu - g^{bc}X^a_\nu\right) + X^{abc}_{\mu\nu\tau}\Big)\left(\mathscr{D}_c \delta X^\tau\right)\left(\mathscr{D}_b \delta X^\nu\right) \\
& + \sqrt{-g}\left(g^{ab}\Pi_{\nu\mu} + X^{ab}_{\mu\nu}\right)\mathbf{R}^\nu{}_{\alpha\rho\beta}\delta X^\alpha \delta X^\rho X^\beta_b\Big] \\
& + \mu\sqrt{-g}\left(\mathscr{D}_\omega \mathbf{R}_{\alpha\beta\mu\rho}\right)\delta X^\omega \delta X^\alpha h^{\beta\rho} + \mu\sqrt{-g}\mathbf{R}_{\alpha\beta\mu}{}^\rho \\
& \times \Big[2\delta X^\rho X^\beta_a \mathscr{D}_b\left(\delta X^\nu\right)\left(g^{ab}\Pi_{\rho\nu} + X^{ab}_{\rho\nu}\right) \\
& + \delta X^\alpha \mathscr{D}_a\left(\delta X^\beta\right)\eta_{\rho\nu}g^{ac}X^\nu_c\Big]\Big\}\delta X^\mu.
\end{aligned}
\tag{4.39}
$$

Of course, the vanishing of the last integral gives us the second order perturbation to the equations of motion

$$
\begin{aligned}
\mathscr{J}_\mu\left(\delta^2 X\right) = -\mu\Big\{ & \mathscr{D}_a\Big[\sqrt{-g}\Big(\Pi_{\nu\tau}\left(g^{bc}X^a_\mu - 2g^{ab}X^c_\mu\right) + 2\Pi_{\nu\tau}\Big(g^{ac}X^b_\nu - g^{ab}X^c_\nu \\
& - g^{bc}X^a_\nu\Big) + X^{abc}_{\mu\nu\tau}\Big)\left(\mathscr{D}_c\delta X^\tau\right)\left(\mathscr{D}_b\delta X^\nu\right) \\
& + \sqrt{-g}\left(g^{ab}\Pi_{\nu\mu} + X^{ab}_{\mu\nu}\right)\mathbf{R}^\nu{}_{\alpha\rho\beta}\delta X^\alpha \delta X^\rho X^\beta_b\Big] \\
& + \sqrt{-g}\mathbf{R}_{\alpha\beta\mu}{}^\rho\Big[2\delta X^\rho X^\beta_a \mathscr{D}_b\left(\delta X^\nu\right)\left(g^{ab}\Pi_{\rho\nu} + X^{ab}_{\rho\nu}\right) \\
& + \delta X^\alpha \mathscr{D}_a\left(\delta X^\beta\right)\eta_{\rho\nu}g^{ac}X^\nu_c\Big] \\
& + \sqrt{-g}\left(\mathscr{D}_\omega \mathbf{R}_{\alpha\beta\mu\rho}\right)\delta X^\omega \delta X^\alpha h^{\beta\rho}\Big\},
\end{aligned}
\tag{4.40}
$$

In this case, $\mathscr{J}_\mu\left(\delta^2 X\right)$ corresponds to the left-hand side of Equation (4.21), with $\delta X \to \delta^2 X$. The right-hand side of Equation (4.40) represents the source term in Jacobi's non-homogeneous equation. Notably, the source term is influenced by the Riemann tensor of the background spacetime and its covariant derivative. One might assume that the covariant derivative of the Riemann tensor should vanish since the background spacetime is fixed. However, when introducing variations, local changes in the embeddings must be taken into account, which justifies the inclusion of this term.

4.4 The Covariant Simultaneous Action

This section presents a covariant simultaneous action for branes in an arbitrary curved background spacetime. Here, we show an action that depends on a pair of independent field variables, the brane embedding functions, through the canonical momentum of a reparametrization invariant geometric model for the brane and an auxiliary vector field. The form of the action is analogous to a symplectic potential. Extremization of the simultaneous action produces at once the equations of motion and the Jacobi equations for the brane geometric model, and it also provides a convenient shortcut toward its second variation.

The conventional variational approach involves considering the first variation of the action concerning an infinitesimal variation of the fields, which in this case are the embedding functions. By setting this variation to zero under appropriate boundary conditions, we obtain the equations of motion for the brane. The second variation of the action, assuming the satisfaction of the equations of motion, leads to a quadratic form that determines the stability of the field configurations. Furthermore, the vanishing of the second variation yields the Jacobi equation for the brane, known as the geodesic deviation equation in the case of a relativistic particle. The simultaneous action encapsulates these principles concisely and elegantly.

$$S_S[X,\eta] = \int_m \mathscr{P}_\mu{}^a \mathscr{D}_a \eta^\mu. \tag{4.41}$$

It is important to mention that we denote simultaneity with the subscript S. In Equation (4.41), the two independent fields are $X^\mu(x^a)$, representing the world volume embedding functions, and an auxiliary vector field $\eta^\mu(x^a)$. The operator $\mathscr{D}_a$ corresponds to the same covariant derivative defined in subsection 2.1, and the linear momentum $\mathscr{P}_\mu{}^a$ is given by equation (3.44). It is worth noting the analogy between the term inside the integral and a symplectic potential of the form pdq. The symmetries of the simultaneous action include world volume diffeomorphisms, or reparametrization invariance, as well as a constant translation of the auxiliary field. Additionally, the action remains invariant under background diffeomorphisms, or Poincaré transformations in the case of a flat Minkowski background, given that it is a scalar.

The first variation of the simultaneous action with respect to an infinitesimal variation of the auxiliary field $\eta^\mu \to \eta^\mu + \delta\eta^\mu$, where $\delta\eta^\mu$ denotes a infinitesimal change, keeping the embedding functions fixed, is simply

$$\delta_\eta S_S\left[X,\eta\right]|_X = \int_m \mathscr{P}_\mu{}^a \mathscr{D}_a \delta\eta^\mu = -\int_m \left(\mathscr{D}_a \mathscr{P}_\mu{}^a\right)\delta\eta^\mu = \int_m \mathscr{E}_\mu \delta\eta^\mu, \tag{4.42}$$

in the given equation, we have performed an integration by parts and, for simplicity, temporarily omitted a boundary term. The vanishing of this first variation under an arbitrary variation of the auxiliary field, $\delta_\eta S\left[X,\eta\right]|_X = 0$, thus leads to the equations of motion for the brane.

$$\mathscr{E}_\mu = -\mathscr{D}_a \mathscr{P}_\mu{}^a = 0. \tag{4.43}$$

It is important to highlight that this aspect of the simultaneous variational principle is independent of the specific model and does not rely on the particular form of $\mathscr{L}\left(X^\mu_a\right)$. Instead, it capitalizes on the fact that the equations of motion can be expressed as a conservation law [3, 25]. This conservation law is essentially a dimensionally reduced version of the conservation of the stress-energy tensor in relativistic theories.
Shifting focus to the variation of the simultaneous action concerning a variation of the embedding functions, $X^\mu \to X^\mu + \delta X^\mu$, we employ the same covariant variational derivative as in Equation (3.37). The notable advantage is that $\mathscr{D}_X \eta_{\mu\nu} = 0$ since the background covariant derivative is metric compatible. With the auxiliary field η^μ held fixed, we obtain

$$\mathscr{D}_X S\left[X,\eta\right]|_\eta = \int_m \left[\left(\mathscr{D}_X \mathscr{P}_\mu{}^a\right)\mathscr{D}_a \eta^\mu + \mathscr{P}_\mu{}^a \mathscr{D}_X \mathscr{D}_a \eta^\mu\right]. \tag{4.44}$$

By utilizing the definition of the canonical momentum given in Equation (3.44), the first term can be written as

$$\begin{aligned}\left(\mathscr{D}_X \mathscr{P}_\mu{}^a\right)\mathscr{D}_a \eta^\mu &= \frac{\delta^2 \mathscr{L}}{\partial X^\nu_b \partial X^\mu_a}\left(\mathscr{D}_X X^\nu_b\right)\mathscr{D}_a \eta^\mu, \\ &= \mathscr{H}^{ba}_{\nu\mu}\left(\mathscr{D}_X X^\nu_b\right)\mathscr{D}_a \eta^\mu = \mathscr{H}^{ba}_{\nu\mu}\left(\mathscr{D}_b \delta X^\nu\right)\mathscr{D}_a \eta^\mu,\end{aligned} \tag{4.45}$$

where we have defined the Hessian in (4.15) and in the second line we have used the fact that variation and partial derivative commute, $\mathscr{D}_X X^\nu_b = \mathscr{D}_b \delta X^\nu$. Note that the Hessian is degenerate, meaning it possesses null eigenvectors. This is a result of the gauge freedom associated with reparametrization invariance.

To derive the second term in Equation (4.44), we consider the dependence of the covariant derivative on the embedding functions and make use of the Bianchi identity. Additionally, we take into account the independence of the field variables X^μ and η, specifically that $\mathscr{D}_X \eta = 0$. This allows us to obtain the following expression

$$\mathscr{P}_\mu{}^a \mathscr{D}_X \mathscr{D}_a \eta^\nu = \mathscr{P}_\mu{}^a \left[\mathscr{D}_X, \mathscr{D}_a\right]\eta^\mu = -\mathbf{R}_{\rho\sigma\nu}{}^\mu \delta X^\rho X^\sigma_a \eta^\nu \mathscr{P}_\mu{}^a. \tag{4.46}$$

Inserting (4.15) and (4.46) in the variation (4.44) results immediately in

$$\mathscr{D}_X S\left[X,\eta\right]|_\eta = \int_m \left[\mathscr{H}^{ba}_{\nu\mu}\left(\mathscr{D}_a \delta X^\nu\right)\mathscr{D}_a \eta^\mu - \mathbf{R}_{\rho\sigma\nu}{}^\mu \delta X^\rho X^\sigma_a \eta^\nu \mathscr{P}_\mu{}^a\right]. \tag{4.47}$$

At this stage, by identifying the auxiliary field as the variation of the embedding functions, $\eta^\mu = \delta X^\mu$, we have successfully obtained a concise and general expression for the second variation of the geometric model (3.39). This highlights the advantage of the simultaneous action approach, especially when compared to more laborious approaches found in the literature. Furthermore, the significance of the Hessian becomes apparent as it plays a central role in the formulation. In order to derive the Jacobi equations for the brane, one simply needs to integrate by parts the first term in (4.47), neglecting a boundary term. This procedure yields

$$\mathscr{D}_X S\left[X,\eta\right]|_\eta = -\int_m \{\left[\mathscr{D}_b\left(\mathscr{H}^{ba}_{\nu\mu}\mathscr{D}_a\eta^\mu\right)\right] + \mathbf{R}_{\nu\sigma\rho}{}^\mu X^\sigma_a \eta^\rho \mathscr{P}_\mu{}^a\}\delta X^\nu, \tag{4.48}$$

the vanishing of this variation gives the Jacobi equations for the brane

$$\left[\mathscr{D}_b\left(\mathscr{H}^{ba}_{\nu\mu}\mathscr{D}_a\eta^\mu\right)\right] + \mathbf{R}_{\nu\sigma\rho}{}^\mu X^\sigma_a \eta^\rho \mathscr{P}_\mu{}^a = 0, \tag{4.49}$$

To summarize, the simultaneous approach allows us to derive both the equations of motion for the embedding functions and the Jacobi equations for the deviation vector. It is interesting to note the interchange of tasks: the variation with respect to the deviation vector yields the equations of motion for the embedded functions, while the variation with respect to the embedding functions yields the Jacobi equations for the deviation vector. An added advantage of the simultaneous approach is that there is no need to manually impose the evaluation of the Jacobi equation on-shell; it happens automatically.

To illustrate the general formalism, let us consider the DNG model (3.29) in an arbitrary curved spacetime background. This can be compared to the more conventional treatment presented by Carter in a covariant approach [23]. Although the conclusions are the same, they are expressed in a different manner. By employing the DNG canonical linear momentum in (3.47), the simultaneous action (4.41) takes the following form

$$S_{S_{DNG}}[X,\eta] = -\alpha_0 \int_m \sqrt{-g}\,\eta_{\mu\nu} g^{ab} X^\nu_b \mathscr{D}_a \eta^\mu. \tag{4.50}$$

By setting the first variation of this simultaneous action with respect to η to zero, as depicted in Equation (4.42), we obtain the DNG equations of motion, which are analogous to Equation (4.43).

$$-\mathscr{D}_a\mathscr{P}_\mu{}^a = \alpha_0 \mathscr{D}_a\left(\sqrt{-g}\,\eta_{\mu\nu} g^{ab} X^\nu_b\right), \tag{4.51}$$

its normal projection gives

$$-n^\mu{}_i \mathscr{D}_a \mathscr{P}_\mu{}^a = -\alpha_0\sqrt{-g}K_i = 0. \tag{4.52}$$

As previously mentioned, this corresponds to the relativistic formulation of the equilibrium condition for a minimal surface with an arbitrary codimension.

Now, let's proceed with the variation of the simultaneous action concerning a variation of the embedding functions. To obtain the expression for the Hessian defined in Equation (4.15), we need to perform a short calculation. Taking into account Equation (4.19), we find

$$\mathscr{H}^{ba}_{\nu\mu} = -\alpha_0\sqrt{-g}\left(g^{ab}\Pi_{\mu\nu} + X^{ab}_{\mu\nu}\right), \tag{4.53}$$

Inserting (4.53) in the first variation (4.47) gives immediately

$$\begin{aligned}\mathscr{D}_X S_{S_{DNG}}\left[X,\eta\right]|_\eta = -\,\alpha_0\int_m \sqrt{-g}\,\Big[&\left(g^{ab}\Pi_{\mu\nu} + X^{ab}_{\mu\nu}\right)\left(\mathscr{D}_b\delta X^\nu\right)\mathscr{D}_a\eta^\mu \\ &-\mathbf{R}_{\rho\sigma\nu}{}^{\mu}\delta X^\rho h^\sigma{}_\mu\eta^\nu\Big].\end{aligned} \tag{4.54}$$

If at this point we identify the auxiliary field η^μ with the variation of the embedding functions δX^μ, the above expression represents the second variation of the DNG action considering variations of the embedding functions. By integrating the first term by parts and disregarding a boundary term, we obtain

$$\begin{aligned}\mathscr{D}_X S\left[X,\eta\right]|_\eta = \alpha_0\int_m \{&\mathscr{D}_b\sqrt{-g}\left[\left(g^{ab}\Pi_{\mu\nu} + X^{ab}_{\mu\nu}\right)\mathscr{D}_a\eta^\mu\right]\delta X^\nu \\ &+\sqrt{-g}\,\mathbf{R}_{\rho\sigma\nu}{}^{\mu}\delta X^\rho h^\sigma{}_\mu\eta^\nu\}.\end{aligned} \tag{4.55}$$

Setting this variation to vanish, $\mathscr{D}_X S_{S_{DNG}}\left[X,\eta\right]|_\eta = 0$ gives the Jacobi equations for the DNG brane

$$\alpha_0\mathscr{D}_a\sqrt{-g}\left[\left(g^{ab}\Pi_{\mu\nu} + X^{ab}_{\mu\nu}\right)\mathscr{D}_a\eta^\mu\right] + \alpha_0\sqrt{-g}\,\mathbf{R}_{\nu\sigma\mu}{}^{\rho}h^\sigma{}_\rho\eta^\mu = 0. \tag{4.56}$$

The first kinetic term in the Jacobi equations differs from the case of a particle due to the presence of a "friction" term caused by the tangential bivector. It is worth noting that these Jacobi equations have been derived before in a different, yet ultimately equivalent form in the work of Pavšič [91]. These equations provide a generalization of the well-known geodesic deviation equation for particles to branes. In the special case of a relativistic particle, which corresponds to a degenerate brane of dimension zero, the simultaneous action was introduced by Bazanski in [8, 9], building upon a general formalism developed in [7].

In conclusion, this section has introduced a covariant simultaneous action that, for any reparametrization-invariant local geometric model describing a relativistic brane, yields both the equations of motion and the Jacobi equations for the model simultaneously. Moreover, the action provides a convenient shortcut for studying the second variation of the geometric model, offering an alternative path for stability analyses. The simultaneous action can be readily extended to incorporate additional fields residing on the brane or include "pressure terms" that arise in fundamental branes. However, it is important to acknowledge that such extensions would impact the simple elegance of the covariant simultaneous principle, yet they would be valuable in scenarios where external forces are relevant.

CHAPTER 5

VARIATIONAL ANALYSIS OF GODETIC BRANE GRAVITY

In the last chapters, we have discussed the variation of the functional action of a general system by using both variations along only normal vectors and arbitrary variations. We also reviewed how to calculate the LEM of a system, and we discussed their different utilities. In this chapter, we will apply the majority of these viewed tools to meticulously analyze a geometric model known as the Regge-Teitelboim (RT) model, which we also refer to as Geodetic Brane Gravity (GBG). This chapter is primarily based on the paper Jacobi equation of geodetic brane gravity.

5.1 Regge-Teitelboim model

In the 1970s, T. Regge and C. Teitelboim introduced a geometric model for our universe, considering it as the world volume of a three-dimensional brane evolving geodesically in a fixed higher-dimensional background Minkowski spacetime. Their motivation, as indicated by the title of their proceedings contribution "Gravity 'a la string: a progress report," was closely related to string theory [95]. The action they considered in their geometric model is equivalent to the Einstein-Hilbert action of general relativity. However, the crucial difference lies in the choice of field variables. Instead of the spacetime metric as in general relativity, the Regge-Teitelboim (RT) model employs the embedding functions of the world volume as the field variables, making the world volume metric a composite field variable. The equations of motion of the RT model are second-order in derivatives and weaker than the Einstein equations. This characteristic of geometric models with second-order equations of motion is shared by a broader class of models, including Lovelock branes, to which the RT model belongs [4, 30, 53]. While all solutions of the Einstein equations are also solutions of the RT model, the solution space of the latter is more extensive [74, 90, 106]. The additional part of the RT model can be interpreted as "geometric dark matter" [32], providing an alternative perspective to the ongoing efforts to describe dark matter/energy by incorporating exotic terms in the energy-momentum tensor [11, 27], or modifying the geometric sector through theories like $f(R)$ theories [33, 81, 103]. In this context, it is noteworthy that the inclusion of a world volume cosmological constant is equivalent, from a brane viewpoint, to incorporating a DNG term in the action.

It is worth emphasizing that, from a basic geometric standpoint, the local existence of an embedding framework requires a maximum of $N+1=(p+1)(p+2)/2$ dimensions for the ambient spacetime background. For instance, for $p+1=4$, a maximum of 10 dimensions is necessary. Furthermore, if the world volume metric possesses Killing vectors, this number can be further reduced [20, 45, 64, 97]. It is important to note that not every solution of the Einstein equations can be embedded as a hypersurface. For example, the embedding of the Schwarzschild solution necessitates at least a co-dimension of two [67, 87]. This serves as a significant motivation to consider arbitrary co-dimensions, despite the increased complexity it entails. Naturally, this has implications for the stability of such geometric configurations, as it becomes necessary to analyze the conditions for their stability. In particular, higher co-dimensions require the utilization of geometric structures that account for the rotational freedom in the normal fields to the world volume. Surprisingly, this aspect has been overlooked in many discussions of the RT model and does not appear to have been addressed previously.

The RT model involves the integral over the trajectory of a p-dimensional brane Σ, which depends on the scalar curvature $\mathscr{R}$ of the world volume m obtained from the induced metric g_{ab}[95],

$$S_{RT}\left[X,\varphi_M\right]=\int_m \sqrt{-g}\left[\frac{1}{2\kappa}\mathscr{R}+L_M\left(\varphi_M,X\right)\right], \tag{5.1}$$

where κ is a constant, typically set to $\kappa=8\pi G_N$ to relate it to general relativity. The field variables are the embedding functions $X^\mu(x^a)$ and the matter fields $\varphi_M(x^a)$ living on the brane, with a matter Lagrangian $L_M(\varphi_M,X)$. We assume the world volume to be without boundary for simplicity. The symmetries of the action include world volume reparametrizations, the Poincaré symmetry of the background Minkowski spacetime, and invariance under rotations of the normal vectors adapted to the world volume, as $p+1 < N+1$.

The equations of motion for the RT model can be obtained by varying the action S_{RT} using either the variation $\widetilde{\mathscr{D}_\delta}$, which is along normal vectors only, or the arbitrary covariant variation $\mathscr{D}_X$, which yields covariant equations under transformations of the background spacetime. Another approach is to construct the simultaneous action for this model, which allows us to derive the dynamical equations by varying it with respect to the auxiliary field η. The choice of variational methods depends on the specific model and the most appropriate approach for studying it. For example, the $\mathscr{D}_X$ variation is efficient when the spacetime is curved, as it utilizes the Ricci identity and reduces the complexity of calculations. Additionally, the resulting equations are invariant under diffeomorphisms of the ambient spacetime, making them useful for studying the system from an external perspective. However, in the case of the RT model, where the ambient spacetime is flat and the equations should be invariant under Poincaré transformations, the advantages of the $\mathscr{D}_X$ variation are somewhat reduced. Nevertheless, we will use it in the RT model as an example to demonstrate its application.

The first variation of the action (5.1), using $\mathscr{D}_X$, reads

$$\begin{aligned} \mathscr{D}_X S_{RT} &= \int_m \left[\frac{\sqrt{-g}}{2\kappa} G_{ab} + \frac{\delta\left(\sqrt{-g} L_M\right)}{\delta g^{ab}} \right] \mathscr{D}_X g^{ab} \\ &= -\frac{1}{\kappa} \int_m \sqrt{-g}\, [G_{ab} - \kappa T_{ab}] X^a \cdot \mathscr{D}^b \delta X, \end{aligned} \tag{5.2}$$

in this context, G_{ab} represents the Einstein tensor of the world volume, and T_{ab} is the energy-momentum tensor defined as $T_{ab} = -2\frac{\delta(\sqrt{-g} L_M)}{\delta g^{ab}}$. In the second line of (5.2), we have used the variation of the inverse metric and the symmetry of G_{ab} and T_{ab} under index exchange. After integrating by parts in (5.2) and considering appropriate boundary conditions, we obtain

$$\mathscr{D}_X S_{RT} = -\frac{1}{\kappa} \int_m \mathscr{D}_a \left[\sqrt{-g} \left(G^{ab} - \kappa T^{ab} \right) X_a \right] \cdot \delta X, \tag{5.3}$$

by extremizing the action, $\mathscr{D}_X S = 0$, we get

$$-\frac{1}{\kappa} \partial_b \left[\sqrt{-g} \left(G^{ab} - \kappa T^{ab} \right) X^\mu_a \right] = 0, \tag{5.4}$$

note that directional covariant derivative $\mathscr{D}_a$ is reduced to ∂_a because of null curvature of background spacetime. Because the divergence of Einstein and energy momentum tensor vanishes then the last equation can be written as

$$-\frac{1}{\kappa}\sqrt{-g}\left(G^{ab}-\kappa T^{ab}\right)\nabla_b X^\mu_a = 0 \tag{5.5}$$

Using the Gauss-Weingarten equations (2.11a) and (2.11b) we have that tangential projection of the equation of motion (5.5) is identically zero. However, if one takes the normal projection of (5.5), this is reduced to

$$\frac{1}{\kappa}\sqrt{-g}\left(G^{ab}-\kappa T^{ab}\right)K_{ab}{}^i = 0 \tag{5.6}$$

Another way to arrive at this equation is through the variation of the simultaneous action, which was discussed in section 4.4. Remember to build such action it is important to calculate the momentum $\mathscr{P}_\mu{}^a$ defined in (3.44). For this case, we have

$$\mathscr{P}_\mu{}^a = \frac{1}{\kappa}\sqrt{-g}\left(G^{ab}-\kappa T^{ab}\right)X^\mu_a, \tag{5.7}$$

thus the simultaneous action reads

$$S^{(S)}_{RT} = \int_m \sqrt{-g}\left(G^{ab}-\kappa T^{ab}\right)X^\mu_a\partial_a\eta^\mu, \tag{5.8}$$

so if one varies the simultaneous action concerning auxiliary field η, it is obtained

$$\begin{aligned}\delta S^{(S)}_{RT} &= \frac{1}{\kappa}\int_m\left[\sqrt{-g}\eta_{\mu\nu}\left(G^{ab}-\kappa T^{ab}\right)X^\mu_a\right]\partial_b\delta\eta^\nu \\ &= -\frac{1}{\kappa}\int_m\left[\sqrt{-g}\eta_{\mu\nu}\left(G^{ab}-\kappa T^{ab}\right)\nabla_b X^\mu_a\right]\delta\eta^\nu\end{aligned} \tag{5.9}$$

where we have done a integration by parts in the second equality, then, taking $\delta S^{(S)}_{RT} = 0$, one obtains the equations

$$\sqrt{-g}\left(G^{ab}-\kappa T^{ab}\right)\nabla_b X^\mu_a = 0 \tag{5.10}$$

as we saw if these equations are projected along tangent fields then these vanish. However, if we project them along normal vector the resulting equations are the same to (5.6).

Lastly, one can also use the covariant variation $\widetilde{\mathscr{D}}_\delta$ to obtain the dynamics equation of extended object, by taking $\widetilde{\mathscr{D}}_\delta = 0$.

$$\begin{aligned}\widetilde{\mathscr{D}}_\delta S_{RT} &= \int_m \widetilde{\mathscr{D}}_\delta\left[\sqrt{-g}\left(\frac{1}{2\kappa}\mathscr{R}+L_M\right)\right] \\ &= -\frac{1}{2\kappa}\int_m\sqrt{-g}\,[G_{ab}-\kappa T_{ab}]\widetilde{\mathscr{D}}_\delta g^{ab} \\ &= \frac{1}{\kappa}\int_m\sqrt{-g}\,[G_{ab}-\kappa T_{ab}]K^{ab}{}_i\phi^i,\end{aligned} \tag{5.11}$$

where we have used the variation of inverse metric (3.11) in the last equality of (5.11). Thus, if one takes into account that the variation of action is zero for any field variation ϕ^i, we have

$$\mathscr{E}_i = (G_{ab} - \kappa T_{ab}) K^{ab}{}_i = 0, \tag{5.12}$$

these equations are of second-order in derivatives of the fields X^μ, with the extrinsic curvature tensor playing the role of an acceleration. This type of gravity has a built-in Einstein limit since every solution of Einstein equations, $G_{ab} - \kappa T_{ab} = 0$, is necessarily a solution of geodetic brane gravity. On the other hand, equations (5.12) are weaker, in the sense that a more general solution of the form $G_{ab} - \kappa T_{ab} = \tau_{ab}$ may exist as long as

$$\tau_{ab} K^{ab}{}_i = 0, \quad \text{and} \quad \tau_{ab} \neq 0. \tag{5.13}$$

Indeed, it has been suggested in [66] that the geometric structure τ_{ab} can be interpreted as a non-ordinary matter contribution, often referred to as "dark matter," as it is distinct from the standard matter contribution represented by T_{ab}. It is worth noting that in the absence of matter, akin to classical string theory, the equations of motion (5.12) can be seen as a generalization of the condition for extremal surfaces. This generalization manifests through the vanishing of a trace of the extrinsic curvature, where the Einstein tensor G_{ab} assumes the role of the induced metric.

5.2 Jacobi Equation of RT Model

In the previous section, we have used the three different ways, which we reviewed in the last chapters, to obtain the equations of motion of the RT model. However, as we mentioned, each of these methods may be more appropriate than another depending on the geometric model. This also happens when calculating the Jacobi equations of the system.

Let us remember that the RT model tries to describe our universe, where one assumes it is a brane embedded in a larger flat spacetime. Thus, one would like to have equations that are covariant under the reparametrizations of the brane world volume . Using, for example, the covariant variation $\mathscr{D}_X$ to calculate the Jacobi equations of this system is hugely laborious. Because firstly, one should calculate the second variation of the action, $\mathscr{D}_X^2 S_{RT} = 0$, taking into account that the equations of motion are satisfied. The Jacobi equations $\mathscr{J}^\mu(\delta X) = 0$, which are covariant under diffeomorphism, are obtained.

However, remember that we want equations that are covariant under reparametrization of m, then, additionally one has to project to $\mathscr{J}^\mu(\delta X) = 0$ along normal vector.
On the another hand, if one uses the simultaneous action to obtain the Jacobi equations, one has to vary concerning embedded functions and also project along of normal fields. Then, for this particular case, it is more convenient to use a *direct approach*, by using the covariant variation $\widetilde{\mathscr{D}}_\delta$ and taking into account the variation of fundamental forms as induced metric and curvature extrinsic tensor, see (3.10) and (3.23).
As formally discussed in the section 3.1, for co-dimension higher than one, the application of the normal deformation operator $\widetilde{\mathscr{D}}_\delta$ to the equations of motion together with the assumption that the equations of motion $\mathscr{E}^i = 0$ are fulfilled, afford the linearized equations, and we obtain the Jacobi equations that are covariant under reparametrization of world volume. Hence, we focus on the expressions

$$\widetilde{\mathscr{D}}_\delta \left(G_{ab} K^{ab}{}_i - \kappa T^{ab} K_{abi} \right) = 0. \tag{5.14}$$

Not all solutions of the equations of motion (5.12) lead to stable configurations of the extended object. In this sense, the Jacobi equations provide conditions to explore this issue. Certainly, their solutions, also named Jacobi fields, help us to understand more deeply how the geometry behaves under deformations of the embedding functions in the background spacetime. Since we are interested in obtaining a covariant expression for such equations, a convenient strategy is to directly linearize the equations of motion (5.12).

To evaluate the variation $\widetilde{\mathscr{D}}_\delta(G_{ab}K^{ab}{}_i)$, we will break it down into several steps for clarity. Starting with the explicit expression of the world volume Einstein tensor, we find that it can be written as follows

$$\begin{aligned} \widetilde{\mathscr{D}}_\delta \left(G_{ab} K^{ab}{}_i \right) = &\left(\frac{1}{2} \mathscr{R} K_{abi} - \frac{1}{2} \mathscr{R}_{ab} K_i + 2 G_a{}^c K_{cbi} \right) \mathscr{D}_\delta g^{ab} + G^{ab} \widetilde{\mathscr{D}}_\delta K_{abi} \\ &+ K^{ab}{}_i \mathscr{D}_\delta \mathscr{R}_{ab} - \frac{1}{2} K_i g^{ab} \mathscr{D}_\delta \mathscr{R}_{ab}. \end{aligned} \tag{5.15}$$

Before proceeding further, let us calculate the variations involving the Ricci tensor. We can express the Ricci tensor $\mathscr{R}_{ab}$ in terms of the extrinsic curvatures using equation (2.19a). After performing a direct calculation, we obtain the following expression

$$g^{ab} \mathscr{D}_\delta \mathscr{R}_{ab} = 2 \left(g^{ab} K_j - K^{ab}{}_j \right) \widetilde{\mathscr{D}}_\delta K_{ab}{}^j + \mathscr{R}_{ab} \mathscr{D}_\delta g^{ab}, \tag{5.16}$$

$$K^{ab}{}_i \mathscr{D}_\delta \mathscr{R}_{ab} = \left(g^{ab} K^{cd}{}_i K_{cdj} - 2 K^{ac}{}_i K^b{}_{cj} + K^{ab}{}_i K_j \right) \widetilde{\mathscr{D}}_\delta K_{ab}{}^j + \mathscr{R}_{acbd} K^{cd}{}_i \mathscr{D}_\delta g^{ab}. \tag{5.17}$$

Inserting these expressions into (5.15) yield

$$
\begin{aligned}
\widetilde{\mathscr{D}}_\delta \left(G_{ab}K^{ab}{}_i\right) =& \left(\mathscr{R}_{acbd}K^{cd}{}_i - \mathscr{R}_{ab}K_i + \frac{1}{2}\mathscr{R}K_{abi} + 2G_a{}^cK_{bci}\right)\mathscr{D}_\delta g^{ab} \\
& \left[K_iK^{ab}{}_j + K_jK^{ab}{}_i - K^a{}_{ci}K^{bc}{}_j - K^a cjK^{bc}{}_i \right.\\
& \left. -g^{ab}\left(K_iK_j - K^{cd}{}_iK_{cdj}\right) + G^{ab}\delta_{ij}\right]\widetilde{\mathscr{D}}_\delta K_{ab}{}^j,
\end{aligned} \tag{5.18}
$$

in turn, this expression suggests to introduce the geometric tensor

$$
\mathbb{G}^{ab}{}_{ij} = K_iK^{ab}{}_j + K_jK^{ab}{}_i - K^a{}_{ci}K^{bc}{}_j - K^a cjK^{bc}{}_i - g^{ab}\left(K_iK_j - K^{cd}{}_iK_{cdj}\right). \tag{5.19}
$$

Note that it is symmetric in both pairs of indices $\mathbb{G}^{ab}{}_{ij} = \mathbb{G}^{ba}{}_{ij} = \mathbb{G}^{ab}{}_{ji}$.
An important fact that will be utilized later is that this tensor, which is quadratic in the extrinsic curvature, is divergence-free with respect to the tangential indices.

$$
\widetilde{\nabla}_a \mathbb{G}^{ab}{}_{ij} = 0 \tag{5.20}
$$

This is not self-evident, and it requires the use of the contracted relations (2.19), as shown explicitly in Appendix A. Note that for a hypersurface, using the contracted Gauss-Codazzi equations (2.19), $\mathbb{G}^{abi}ij$ takes the simple form $\mathbb{G}^{ab} = 2G^{ab}$, a fact that emphasizes its geometrical nature, for codimension higher than one. By inserting (5.18) into (5.14), and taking into account the definition (5.19), we get

$$
\begin{aligned}
& \left[\mathbb{G}^{ab}{}_{ij} + \left(G^{ab} - \kappa T^{ab}\right)\delta_{ij}\right]\widetilde{\mathscr{D}}_\delta K_{ab}{}^j - \kappa K_{abi}\mathscr{D}_\delta T^{ab} \\
& + \left(\mathscr{R}_{acbd}K^{cd}{}_i - \mathscr{R}_{ab}K_i + \mathscr{R}_a{}^cK_{bci} + G_a{}^cK_{bci}\right)\mathscr{D}_\delta g^{ab} = 0
\end{aligned} \tag{5.21}
$$

The variations given in (3.11) and (3.27) are reduced due to flat background spacetime

$$
\mathscr{D}_\delta g^{ab} = -K^{ab}{}_i\phi^i, \tag{5.22}
$$

$$
\widetilde{\mathscr{D}}_\delta K_{ab}{}^i = -\widetilde{\nabla}_a\widetilde{\nabla}_b\phi^i + K_{ac}{}^iK^c{}_{bj}\phi^j. \tag{5.23}
$$

At this stage, we are ready to insert the needed normal deformations of the first and second fundamental into (5.21) to obtain

$$
\begin{aligned}
& \left[\mathbb{G}^{ab}{}_{ij} + \left(G^{ab} - \kappa T^{ab}\right)\delta_{ij}\right]\widetilde{\nabla}_a\widetilde{\nabla}_b\phi^j + \left\{2\left(\mathscr{R}_{acbd}K^{cd}{}_i - \mathscr{R}_{ab}K_i + \mathscr{R}_a{}^cK_{bci}\right.\right. \\
& \left.\left. + G_a{}^cK_{bci}\right)K^{ab}{}_l - \left[\mathbb{G}^{ab}{}_{ij} + \left(G^{ab} - \kappa T^{ab}\right)\delta_{ij}\right]K_a{}^{cj}K_{bcl}\right\}\phi^l + \kappa K_{abi}\mathscr{D}_\delta T^{ab} = 0.
\end{aligned} \tag{5.24}
$$

To provide a concise representation of the structure of these equations, which can appear daunting in both form and content, it is helpful to introduce the tensor

$$
\mathbb{M}^{ab}{}_{ij} := \mathbb{G}^{ab}{}_{ij} + \left(G^{ab} - \kappa T^{ab}\right)\delta_{ij}, \tag{5.25}
$$

by virtue of this definition, we have the useful identity

$$\begin{aligned}\mathbb{M}^{ab}{}_{ij}K_a{}^{cj}K_{cbl} = &\left(\mathscr{R}_{acbd}K^{cd}{}_i - \mathscr{R}_{ab}K_i + \mathscr{R}_a{}^cK_{bci} + G_a{}^cK_{bci}\right)K^{ab}{}_l \\ &- \kappa T_a{}^cK_{bci}K^{ab}{}_l,\end{aligned} \tag{5.26}$$

that coincides with the terms appearing on the r.h.s. of (5.24) This identity allows us to rewrite (5.24) in a compact form as

$$\mathbb{M}^{ab}{}_{ij}\widetilde{\nabla}_a\widetilde{\nabla}_b\phi^j + \left(\mathbb{M}^{ab}{}_{il}K_a{}^{cl}K_{bcj} + 2\kappa T_a{}^cK_{bci}K^{ab}{}_j\right)\phi^j + \kappa K_{abi}\mathscr{D}_\delta T^{ab} = 0. \tag{5.27}$$

These equations represent the Jacobi equations for geodetic brane gravity, describing small deformations of the worldvolume in the normal direction. When arbitrary matter fields confined to the worldvolume are included, the dynamics of the Jacobi fields are affected by the derivatives of the matter fields. It is important to note that (5.27) are second-order partial differential equations for the unknown functions ϕ^i, a characteristic feature of brane theories with second-order derivative equations of motion [4]. The solutions to the Jacobi equations provide insight into the stability of the system through the nature of the normal modes ϕ^i, and appropriate boundary conditions must be considered. However, in the case where there are no brane matter fields ($T_{ab} = 0$) and assuming the equations of motion are satisfied, we obtain a more concise and elegant expression for the Jacobi equations in a pure RT geometrical model. In this case, we define a new tensor

$$\mathscr{M}^{ab}{}_{ij} := \mathbb{G}^{ab}{}_{ij} + G^{ab}\delta_{ij}, \tag{5.28}$$

we can write (5.27) in the form

$$\mathscr{M}^{ab}{}_{ij}\widetilde{\nabla}_a\widetilde{\nabla}_b\phi^j + \mathscr{V}_{ij}\phi^j = 0, \tag{5.29}$$

where we identify a geometrical “potential”

$$\mathscr{V}_{ij} := \mathscr{M}^{ab}{}_{il}K_a{}^{cl}K_{bcj}. \tag{5.30}$$

The resemblance of (5.29) to a set of Klein-Gordon equations is remarkable. It is worth noting that the matrix structure (5.30) is symmetric in the normal indices. This arrangement opens the possibility of formulating an auxiliary variational problem. Moreover, under these conditions, we observe that the "mass matrix" $\mathbb{M}^{ab}{}_{ij}$ is divergence-free, as can be deduced from the geometric identity (5.20) and the divergence-free property of the Einstein tensor.

$$\widetilde{\nabla}_a\mathscr{M}^{ab}{}_{ij} = 0. \tag{5.31}$$

The accessory action can be written then as

$$S_{\mathrm{RT}}[\phi] = -\frac{1}{2}\int_m \sqrt{-g}\left[\mathscr{M}^{ab}{}_{ij}\tilde{\nabla}_a\phi^i\tilde{\nabla}_b\phi^j - \mathscr{V}_{ij}\phi^i\phi^j\right]. \tag{5.32}$$

Up to a boundary term, variation with respect to the normal deformations gives the Jacobi equations in the form (5.29) as its Euler-Lagrange equations. It is interesting to note that the accessory principle, up to factor of one half, gives the *index* of the RT geometric model,

$$I_{\mathrm{RT}}[\phi] = \int_m \mathscr{I}(\phi,\phi). \tag{5.33}$$

As all accessory variational principles, (5.32) is a quadratic expression in the field variables. This property renders it amenable to quantization using a path integral approach, enabling the study of the effect of quantum fluctuations [115].

Regarding the hypersurface case, by considering the reductions $\mathbb{G}^{ab} = 2G^{ab}$ and $\mathscr{M}^{ab} = 3G^{ab}$, the Jacobi equations (5.29) specialize to

$$G^{ab}\left(\nabla_a\nabla_b\phi + K_a{}^cK_{bc}\phi\right) = 0. \tag{5.34}$$

This result aligns with the findings in [4], where the RT model is regarded as a specific instance of Lovelock branes. It is evident that the Jacobi equation (5.34) can be derived from the extremization of the action functional.

$$S[\phi] = -\frac{1}{2}\int_m \sqrt{-g}\left[G^{ab}\nabla_a\phi\nabla_b\phi - G^{ab}K_a{}^cK_{bc}\,\phi^2\right], \tag{5.35}$$

When the action is varied with respect to the ϕ field, it yields the following result. If we consider a worldvolume in which $G_{ab} \sim g_{ab}$, meaning it is an Einstein manifold, the action reduces to that of a massive scalar field, where the variable mass term is proportional to $K_{ab}K^{ab}$. This outcome is equivalent to the linearization of the equation in the DNG model in a flat background, as demonstrated in [56].

Another interesting case is provided by the inclusion of the DNG action, playing the role of a cosmological constant Λ, in our development. In such a case, $L_{\mathrm{m}} = \Lambda$ so that $T_{ab} = \Lambda g_{ab}$. The form of the Jacobi equations, (5.29), remains unchanged except that the matrix $\mathbb{M}^{ab}{}_{ij}$ now becomes

$$\mathbb{M}^{ab}{}_{ij} = \mathbb{G}^{ab}{}_{ij} + \left(G^{ab} - \kappa\Lambda g^{ab}\right)\delta_{ij}. \tag{5.36}$$

Notice that we still have at hand the divergenceless property $\tilde{\nabla}_a\mathbb{M}^{ab}{}_{ij} = 0$.

5.2.1 Linear stability of Schwarzschild geometry in $\mathscr{M}^6$

To illustrate the formalism developed earlier, we consider the case of a Schwarzschild geometry for the worldvolume embedded in a 6-dimensional Minkowski spacetime, $\mathscr{M}^6$, without any brane matter fields. It is worth noting that embedding a Schwarzschild black hole in a flat background requires at least a co-dimension of two. It is important to mention that a Schwarzschild solution in general relativity automatically satisfies the equations of motion (5.12). We utilize the Jacobi equations to analyze its linear local stability. For the specific case with $G_{ab} = R_{ab} = T_{ab} = 0$, the Jacobi equations (5.29) simplify to the following form

$$\mathbb{G}^{abij}\left(\widetilde{\nabla}_a \widetilde{\nabla}_b \phi_j + K_{acj} K^c{}_{bl} \phi^l\right) = 0 \tag{5.37}$$

Among the different embeddings for a 4-dim Schwarzschild geometry, see *e.g.* [87], we choose to consider the Fronsdal embedding [46] given by

$$\begin{aligned}
X^1 &= 2R\sqrt{1-\frac{R}{r}}\sinh\left(\frac{t}{2R}\right),\\
X^2 &= 2R\sqrt{1-\frac{R}{r}}\cosh\left(\frac{t}{2R}\right),\\
X^3 &= \int \sqrt{\frac{R}{r}+\left(\frac{R}{r}\right)^2+\left(\frac{R}{r}\right)^3}\,dr,\\
X^4 &= r\sin\theta\sin\phi,\\
X^5 &= r\sin\theta\cos\phi,\\
X^6 &= r\cos\theta,
\end{aligned} \tag{5.38}$$

The coordinates $\{t, r, \theta, \phi\}$ represent the local brane coordinates, and R corresponds to the event horizon. It is important to verify that the embedding functions satisfy the correct conditions. To do so, one can calculate the components of the induced metric using (2.7) and confirm that they match the Schwarzschild metric in spherical coordinates [100]. Similarly, by employing (2.8), we can determine the two normal vectors and subsequently obtain the non-vanishing components of the extrinsic curvature tensor for this parametrization (5.38). Specifically, we find that the components of two extrinsic curvature are

$$\begin{aligned}
K_{ab}{}^1 &= \mathrm{diag}\left(0, ba' - ab', -rb, -rb\sin^2\theta\right),\\
K_{ab}{}^2 &= \mathrm{diag}\left(-\frac{\gamma^2}{2Ra}, \frac{2aR^2}{r^3 - Rr^2}, -\frac{aR^2}{r}, -\frac{aR^2}{r\sin^2\theta}\right),
\end{aligned} \tag{5.39}$$

where we have introduced

$$a = \sqrt{\frac{r^3}{r^3 + r^2R + rR^2 + R^3}},$$
$$b = \sqrt{\frac{R(r^2 + rR + R^2)}{r^3 + r^2R + rR^2 + R^3}}, \quad (5.40)$$
$$\gamma = \sqrt{1 - \frac{R}{r}}.$$

By considering the ansatz $\Phi = e^{-i\omega t} Y_{lm}(\theta, \phi)\rho(r)$, where the normal deformation field is $\Phi = (\phi^1, \phi^2)$, and ρ is also be considered as a vector field, $\rho = (\rho^1, \rho^2)$, the Jacobi equations (5.29) are separable, and can be written as a matrix arrangement of radial equations in the form

$$\mathbf{A}\rho'' + \mathbf{B}\rho' + (\mathbf{C} - \omega^2 \mathbf{D})\rho = 0, \quad (5.41)$$

where $\rho' = d\rho/dr$. Here, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are 2×2 matrices that depend on the radial coordinate r. Their explicit components are given in Appendix B. It is worth mentioning that the matrix $\mathbf{C}$ is the only one that contains the angular momentum information through l. Multiplying equation (5.41) by $\mathbf{D}^{-1}$, and introducing the tortoise-like radial coordinate $r_* = \int (dr/f_g)$ with $f_g = br^2\gamma^2/\sqrt{3}aR^2$, and considering $\rho = \mathbf{M}\chi$, where $\mathbf{M}$ is a matrix defined in such a way that the term proportional to $d\chi/dr_*$ vanishes, for more detail on calculus review Appendix A. Then, the system of equations (5.41) acquires a form familiar in black hole theory stability analysis

$$\frac{d^2\chi}{dr_*^2} + \omega^2\chi - \mathbf{V}\chi = 0, \quad (5.42)$$

where, as in (5.41), χ must be understood as a vector. The matrix potential $\mathbf{V}$ is explicitly expressed in terms of matrices $\mathbf{B}, \mathbf{C}$, and, $\mathbf{D}$ in Appendix A. The system of equations (5.42) describes a system of coupled harmonic oscillators with quasi-normal frequencies $\omega = \omega_R + i\omega_I$. By examining this frequencies ω, we can investigate the stability of this configuration.

Regarding the asymptotic behavior of the fields, at spatial infinity for non zero angular momentum, $l \neq 0$, the diagonal components of the potential matrix $\mathbf{V}$ diverge. Therefore, it can be assumed that the field χ tends to zero as r approaches infinity. On the other hand, at the event horizon R, the matrix potential $\mathbf{V}$ vanishes. As a result, the solution to the equation takes the form $\chi \sim e^{-i\omega r_*} + e^{i\omega r_*}$, where the exponential term with a minus sign represents an incoming wave while the one with a plus sign represents an outgoing wave. However, since nothing can escape from the black hole, the presence of an outgoing wave is not permissible. Thus, the field χ must have the form $\chi = e^{-i\omega r_*}\psi$. By substituting this expression into (5.42), we obtain the following set of equations

$$f_g \frac{d}{dr}\left(f_g \frac{d\psi}{dr}\right) - 2i\omega f_g \frac{d\psi}{dr} - \mathbf{V}\psi = 0. \tag{5.43}$$

We divide this equation by f_g, and then multiply it by $\psi^\dagger$ so that we get

$$\psi^\dagger \frac{d}{dr}\left(f_g \frac{d\psi}{dr}\right) - 2i\omega\psi^\dagger \frac{d\psi}{dr} - \psi^\dagger \mathbf{V}_g \psi = 0, \tag{5.44}$$

where $\mathbf{V}_g = \mathbf{V}/f_g$. Now, integrating by parts (5.44), and taking into account that $\psi(\infty) = 0$ and $f_g(R) = 0$, we obtain

$$\int_R^\infty dr \left[f_g \left|\frac{d\psi}{dr}\right|^2 + 2i\omega\psi^\dagger \frac{d\psi}{dr} + \psi^\dagger \mathbf{V}_g \psi \right] = 0. \tag{5.45}$$

The transpose and conjugate operations applied to the last equation result in

$$\int_R^\infty dr \left[f_g \left|\frac{d\psi}{dr}\right|^2 - 2i\omega^* \frac{d\psi^\dagger}{dr}\psi + \psi^\dagger \mathbf{V}_g^\dagger \psi \right] = 0. \tag{5.46}$$

We proceed to integrate by parts the second term of the previous equation, and then we take the difference of the result with (5.45). We get

$$\int_R^\infty dr \left[(\omega - \omega^*)\psi^\dagger \psi' - \frac{i}{2}\psi^\dagger \left(\mathbf{V}_g - \mathbf{V}_g^\dagger\right)\psi \right] = \omega^* \left|\psi(R)\right|^2, \tag{5.47}$$

From (5.47) we can solve for $\psi^\dagger\psi'$ and substituting this in the (5.45), we obtain

$$\int_R^\infty dr \left[f_g \left|\psi'\right|^2 + \frac{i\omega_R}{2\omega_I}\psi^\dagger \left(\mathbf{V}_g - \mathbf{V}_g^\dagger\right)\psi + \frac{1}{2}\psi^\dagger \left(\mathbf{V}_g + \mathbf{V}_g^\dagger\right)\psi \right] = -\frac{|\omega|^2 \left|\psi(R)\right|^2}{\omega_I}, \tag{5.48}$$

where we assume that $\omega_I \neq 0$. If the integral in (5.48) is positive definite, then the imaginary part of frequency must be negative, that is an indication of having stable deformations of this black hole geometry. Indeed, since the first term in (5.48) is positive, it only remains to analyse the nature of the second term

$$\begin{aligned}
v :=& \frac{i\omega_R}{2\omega_I}\psi^\dagger \left(\mathbf{V}_g - \mathbf{V}_g^\dagger\right)\psi + \frac{1}{2}\psi^\dagger \left(\mathbf{V}_g + \mathbf{V}_g^\dagger\right)\psi \\
=& V_{g11}\left|\psi_1\right|^2 + V_{g22}\left|\psi_2\right|^2 - \left|\left(V_{g12} + V_{g21}\right) + \frac{\omega_R}{\omega_I}\left(V_{g12} - V_{g21}\right)\right| |\psi_1\psi_2| \\
&+ \left(\left|\psi_1\right|^2 + \left|\frac{1}{2}\left(V_{g12} + V_{g21}\right)\psi_2\right|^2 + \left|V_{g12} + V_{g21}\right| |\psi_1\psi_2| \right. \\
&\left. - \left|\psi_1 - \frac{1}{2}\left(V_{g12} + V_{g21}\right)\psi_2\right|^2 \right) + \left(\left|\psi_1\right|^2 + \left|\frac{\omega_R}{2\omega_I}\left(V_{g12} - V_{g21}\right)\psi_2\right|^2 \right. \\
&\left. + \left|\frac{\omega_R}{\omega_I}\right| |V_{g12} - V_{g21}||\psi_1\psi_2| - \left|\psi_1 - \frac{i\omega_R}{2\omega_I}\left(V_{g12} - V_{g21}\right)\psi_2\right|^2 \right),
\end{aligned} \tag{5.49}$$

where a matrix multiplication has been performed in the second line, and we have used the triangle inequality. Note that the two terms that appear in the parentheses are positive definite. Therefore, the sign of ν depends strongly on the values of V_{g11} and V_{g22}. In this sense, ν could be negative if V_{g11} and V_{g22} are both negative enough. Indeed, to illustrate this fact, in Figure 5.1 we show the functions V_{g11} and V_{g22} for $l = 1$ and $R = 2$.

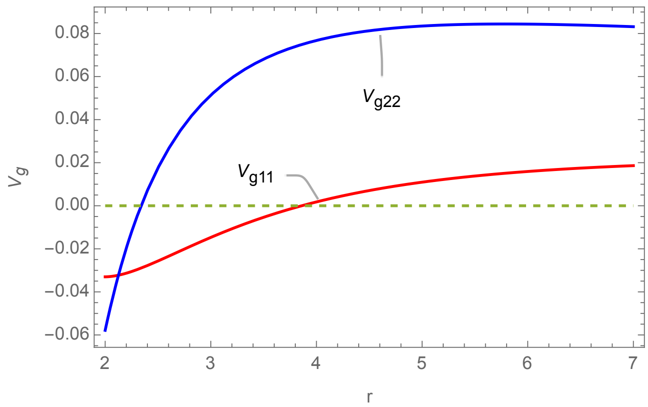

Figure 5.1: $V_{g11}(r)$ and $V_{g22}(r)$ for $l = 1$ and $R = 2$.

This observation suggests that for $r > 2$, there may exist negative values of V_{g11} and V_{g22}, indicating the potential presence of frequencies with positive imaginary parts associated with unstable oscillation modes. However, this analysis alone is not conclusive as the overall positivity of the integral in (5.48) needs to be determined. To further investigate the frequencies, numerical methods are a suitable strategy. Various numerical techniques have been employed in similar problems, including continued fraction and series methods [44, 62, 84, 85], among others. The choice of method depends on the asymptotic behavior of the potential at spatial infinity and the event horizon.

In our case, the matrix potential **V** behaves similarly to the potential of a Schwarzschild black hole in anti-de Sitter spacetime in the respective regions. Thus, we can refer to the numerical method employed in [62] for guidance, as they performed a similar analysis. In their work, they used a Taylor expansion of the components of the field χ given by

$$\chi_i = e^{-i\omega r_*} \sum_n a_n^{(i)}(\omega)(r-R)^n \,, \tag{5.50}$$

by substituting this expansion into (5.42) and performing a Taylor series expansion around the event horizon, we can obtain a set of algebraic equations. Solving these equations order by order allows us to determine the coefficients $a_n^{(i)}(\omega)$. It is important to note that these coefficients depend on the initial coefficients $a_0^{(1)}$ and $a_0^{(2)}$. The solutions can be expressed as

$$\begin{aligned} \chi_1 &= a_0^{(1)}\chi_1^{(1)} + a_0^{(2)}\chi_1^{(2)}, \\ \chi_2 &= a_0^{(1)}\chi_2^{(1)} + a_0^{(2)}\chi_2^{(2)}. \end{aligned} \tag{5.51}$$

The solutions for $\chi_i^{(1)}$ are obtained by considering $a_0^{(1)} = 1$ and $a_0^{(2)} = 0$, while the solutions for $\chi_i^{(2)}$ are obtained by setting $a_0^{(1)} = 0$ and $a_0^{(2)} = 1$. In order to satisfy the condition that these functions must be zero at spatial infinity, we require

$$\det \begin{vmatrix} \chi_1^{(1)} & \chi_1^{(2)} \\ \chi_2^{(1)} & \chi_2^{(2)} \end{vmatrix}_{\lim_{r\to\infty}} = 0. \tag{5.52}$$

Hence, the oscillation frequencies can be determined by finding the roots of equation (5.52). The procedure described in (5.52) can be implemented using software such as MATHEMATICA, as demonstrated in Appendix C, allowing us to compute the lowest eigenfrequencies for different values of l and R. Specifically, by performing the variable change $r \to Rr$ in (5.42), we obtain the values presented in Table 5.1.

l	$R\,\omega_R$	$R\,\omega_I$
1	~ 0	0.83
2	1.01	0.36
3	1.62	0.56
4	2.45	0.75

Table 5.1: Frequencies for different values of l.

The real part ω_R corresponds to the oscillation frequency of the deformation, while the imaginary part ω_I is related to its decay or growth. It is worth noting that the imaginary part of ω is always positive for any value of R. These frequencies correspond to unstable deformations, as the field Φ behaves as $\Phi \sim e^{-i\omega t}$, which diverges when $t \to \infty$. While a four-dimensional Schwarzschild black hole is stable in general relativity [96, 109], our findings indicate the presence of linear instabilities in the embedded Schwarzschild black hole that satisfies the geodetic brane gravity equation. Similar instabilities have been observed in the study of higher-dimensional black holes [55].

Chapter 6

Regge-Teitelboim Model as a Singular Second-Order System

In this chapter, we will address the study of the Regge-Teitelboim model as a second-order singular system. We will heavily rely on the theory presented in Appendix E . Additionally, several of the variation concepts introduced in the first part of this thesis will be utilized. Overall, this chapter will provide a more comprehensive of the Regge-Teitelboim model and its properties. This part of the thesis is based on the paper Ostrogradsky-Hamilton approach to Geodetic Brane Gravity.

6.1 Ostrogradsky-Hamilton approach to Geodetic Brane Gravity

In this section, we will revisit the canonical formalism of the RT model, focusing solely on the geometric aspect of the action. We will do this by utilizing the Ostrogradsky-Hamilton [83] framework, which is specifically designed for analyzing singular systems, as discussed in the Appendix E. This approach offers several advantages, including the ability to fully capture the geometric essence of the RT model in any codimension and to account for the effects of all geometric terms. To achieve this, we will employ the Hamiltonian formulation for relativistic extended objects, which was previously developed in [18, 19] and inspired by the Arnowitt-Deser-Misner (ADM) Hamiltonian formulation of General Relativity.

The canonical analysis of the RT model does not involve the reduction by eliminating a total divergence. The resulting Lagrangian remains linear in the accelerations of the extended object, allowing it to be a second-order derivative theory. However, according to the Ostrogradsky approach, the canonical approach requires doubling the number of phase space variables. Despite this, the advantage of this treatment is that it preserves the original geometric nature of the model. Moreover, it highlights the important role played by both the momenta and the Hamiltonian constraints within the canonical structure.

6.1.1 Geodetic brane gravity without matter term

Geodetic brane gravity, when $L_{\rm m}=0$, is described by RT model defined by

$$S_{\rm RT}[X^\mu] = \frac{1}{2}\int_m d^{p+1}x\,\sqrt{-g}\,\mathscr{R}. \tag{6.1}$$

It should be noted that setting $L_{\rm m}=0$ does not impact the geometric aspects discussed in this section, which specifically pertain to the curvature contribution. Since we are not considering any coupling to brane matter fields, we can set $\alpha = 1$. By performing the first variation of the action, we can derive the equations of motion. The classical trajectories of the brane are determined by the $N-p$ compact relations

$$G^{ab}K^i_{ab} = 0. \tag{6.2}$$

These equations of motion are of second order in derivatives of the field variables X^μ because of the presence of the extrinsic curvature. Additionally, there are $p+1$ tangential vanishing expressions related to the equations of motion, reflecting the reparametrization invariance of the action (6.1). Indeed, these are given by the divergence-free condition $\nabla_a G^{ab} = 0$. Another way of expressing the eom is by using the definition of the extrinsic curvature. In addition, the eom (6.2) can also be written as a set of projected conservation laws

$$-(\nabla_a \mathscr{P}^a)\cdot n^i = 0, \tag{6.3}$$

where in this case that $L_{\rm m}=0$, the conserved stress tensor ${\mathscr{P}_\mu}^a$, is given by

$${\mathscr{P}_\mu}^a := -\sqrt{-g}\,G^{ab}X_{\mu b}, \tag{6.4}$$

The stress tensor given by (6.4) is part of a class of conserved stress tensors that arise in second-order derivative geometric models, known as Lovelock branes [30]. With an eye towards the Hamiltonian framework, when Σ is viewed as a spacelike manifold immersed into m (see D), the associated timelike unit normal, ξ^a, helps to construct the linear momentum density on Σ with

$$\pi_\mu := N^{-1}\xi_a \mathscr{P}_\mu{}^a, \tag{6.5}$$

In this context, N denotes the lapse function, which appears in the ADM decomposition for geometric models of extended objects that depend on the extrinsic curvature.

6.2 The ADM Lagrangian for geodetic brane gravity

A modified version of the ADM framework for General Relativity, adapted for branes, is presented in detail in [18, 19]. When assuming that m is globally hyperbolic, it becomes possible to foliate it into a collection of spacelike hypersurfaces Σ_t. This motivates the decomposition of relevant geometric quantities on the worldvolume into spatial and temporal derivatives, similar to the ADM formulation of General Relativity.

We describe Σ_t using an embedding formulation. First, using the embedding $y^\mu = X^\mu(t = \text{const}, u^A)$, we split the $p+1$ worldvolume coordinates x^a into an arbitrary time parameter t and p coordinates u^A with $(A, B = 1, 2 \ldots, p)$., for Σ_t. In this sense, Σ_t is viewed as the spacelike extended object Σ at fixed t. Secondly, it can be described also by its embedding in m itself, $x^a = X^a(u^A)$. Both descriptions are related by composition. Indeed, in one picture, the tangent vectors to Σ_t are $\epsilon^\mu{}_A = X^\mu{}_A = \partial X^\mu / \partial u^A$, and then the induced metric on Σ_t is $h_{AB} = X_A \cdot X_B$. On the other hand, the tangent vectors to Σ_t are $\epsilon^a{}_A = X^a{}_A = \partial X^a / \partial u^A$ and the induced metric is $h_{AB} = g_{ab} X^a{}_A X^b{}_B$. Notice that $h_{AB} = X_A \cdot X_B = (X_a \cdot X_b) X^a{}_A X^b{}_B$, and we see that $\epsilon^\mu{}_A = X^\mu{}_a \epsilon^a{}_A$, from composition. Accordingly, the choice of the hypersurface vector basis depends on the particular description we are interested in. For the first description we have $\{\epsilon^\mu{}_A, n^\mu{}_i, \xi^\mu\}$, whereas for the second one we have $\{\epsilon^a{}_A, \xi^a\}$, where the appearance of the unit timelike vector accounts for the causal structure on Σ_t. Note that ξ^μ is defined implicitly by $\epsilon_A \cdot \xi = 0$, $n_i \cdot \xi = 0$ and $\xi \cdot \xi = -1$, and in the second description we have a single unit timelike normal vector, ξ^a, defined implicitly by $g_{ab}\epsilon^a{}_A \xi^b = 0$ and $g_{ab}\xi^a \xi^b = -1$, up to a sign. Furthermore, note that $g_{ab}\epsilon^a{}_A \xi^b = (X_a \cdot X_b)\epsilon^a{}_A \xi^b = \epsilon_A \cdot (\xi^b e_b) = 0$ so that $\xi^\mu = \xi^a X^\mu{}_a$. In both descriptions, h^{AB} and h denotes the inverse metric and the determinant of h_{AB}, respectively. We also define $\mathscr{D}_A$ as the torsion-less covariant derivative compatible with h_{AB}, see Appendix D.

It is advantageous to define the following projections of the extrinsic curvature of m:,

$$L^i_{AB} = \epsilon^a{}_A \epsilon^b{}_B K^i_{ab} = -n^i \cdot \mathscr{D}_A \epsilon_B, \tag{6.6}$$

$$L_A{}^i = \epsilon^a{}_A \eta^b K^i_{ab} = -n^i \cdot \mathscr{D}_A \xi, \tag{6.7}$$

in addition to

$$k_{AB} = -g_{ab}\xi^a \mathscr{D}_A \epsilon^b{}_B = k_{BA}, \tag{6.8}$$

that is the Σ_t extrinsic curvature associated with the embedding of Σ_t in m given by $x^a = \chi^a(u^A)$. In a similar manner, in this geometrical framework the velocity vector, $\dot{X}^a = \partial_t X^\mu$, is tangent to the world volume m. In terms of the basis $\{\epsilon^a{}_A, \xi^a\}$ the velocity can be written as

$$\dot{X}^a = N\xi^a + N^A \epsilon^a{}_A, \tag{6.9}$$

where, using familiar ADM terminology, N and N^A are the *lapse* and the *shift vector*, respectively. Since the lapse and the shift vector are expressed in terms of the derivatives of X^μ, *i.e.* $N = -g_{ab}\dot{X}^a \xi^b$ and $N^A = g_{ab} h^{AB} \dot{X}^a \epsilon^b{}_B$, neither N nor N^A is a canonical field variable. Indeed, contrary to what happens in the ADM treatment for the general relativity, in the treatment adopted for extended objects both the lapse function and the shift vector are functions of the phase space, and not Lagrange multipliers.

When examining the progression of Σ_t, it is advantageous to initially select the coordinate basis $\{\epsilon^a{}_A, \dot{X}^a\}$. Consequently, the projections of the worldvolume metric g_{ab} onto this basis promptly yield its ADM form. Therefore, we obtain

$$\begin{aligned} g_{00} &= g_{ab}\dot{X}^a \dot{X}^b = -N^2 + N^A N^B h_{AB}, \\ g_{0A} &= g_{ab}\dot{X}^a \epsilon^b{}_A = N^B h_{AB}, \\ g_{AB} &= g_{ab}\epsilon^a{}_A \epsilon^b{}_B = h_{AB}. \end{aligned} \tag{6.10}$$

In matrix form, the induced metric and its inverse are given by

$$(g_{ab}) = \begin{pmatrix} -N^2 + N^A N^B h_{AB} & N^A h_{AB} \\ N^B h_{AB} & h_{AB} \end{pmatrix}, \tag{6.11}$$

and

$$(g^{ab}) = \frac{1}{N^2} \begin{pmatrix} -1 & N^A \\ N^A & N^2 h_{AB} - N^A N^B \end{pmatrix}, \tag{6.12}$$

respectively. The metric determinant is given by $g = -N^2 h$. Applying a similar approach, we can decompose the extrinsic curvature as follows

$$
\begin{aligned}
K^i_{00} &= K^i_{ab}\dot{X}^a\dot{X}^b = -n^i \cdot \ddot{X}, \\
K^i_{0A} &= K^i_{ab}\dot{X}^a \epsilon^b{}_A = -n^i \cdot \mathscr{D}_A \dot{X}, \\
K^i_{AB} &= K^i_{ab}\epsilon^a{}_A \epsilon^b{}_B = -n^i \cdot \mathscr{D}_A \mathscr{D}_B X = L^i_{AB}.
\end{aligned} \tag{6.13}
$$

In matrix form we have

$$
K^i_{ab} = -\begin{pmatrix} n^i \cdot \ddot{X} & n^i \cdot \mathscr{D}_A \dot{X} \\ n^i \cdot \mathscr{D}_A \dot{X} & -L^i_{AB} \end{pmatrix}. \tag{6.14}
$$

The mean extrinsic curvature, $K^i = g^{ab}K^i_{ab}$, using (6.12) and (6.14), is given by

$$
K^i = \frac{1}{N^2}\left[(n^i \cdot \ddot{X}) - 2N^A(n^i \cdot \mathscr{D}_A \dot{X}) + (N^2 h^{AB} - N^A N^B)L^i_{AB}\right]. \tag{6.15}
$$

The first term in the expression shows the linear dependence between K^i and the accelerations of the extended object. It is worth mentioning that when considering pure normal evolution with $N^A = 0$, the previous expression simplifies to

$$
N^2 K^i = n^i \cdot \ddot{X} + N^2 L^i, \tag{6.16}
$$

where $L^i := h^{AB}L^i_{AB} = -n^i \cdot \mathscr{D}^A \mathscr{D}_A X$, that emphasizes the linear dependance on the acceleration.

By taking into account the contracted integrability conditions related to the Gauss-Weingarten equations (D.1), we can represent the worldvolume Ricci scalar as the combination of a first-order function and a divergence term.

$$
\mathscr{R} = \overline{R} + k_{AB}k^{AB} - k^2 + 2\nabla_a(k\xi^a - \xi^b \nabla_b \xi^a), \tag{6.17}
$$

In the expression above, $k = h^{AB}k_{AB}$ and $\overline{R}$ represents the Ricci scalar defined on Σ. The inclusion of the final term, which is a total divergence, should not come as a surprise. In General Relativity, this term is commonly known as the Gibbons-Hawking-York boundary term [51, 113, 114]. It can either be subtracted from the outset, or kept as in the proof of the positivity of energy theorems by Schoen and Yau [98, 99]. Alternatively, by employing the integrability conditions associated with the Gauss-Weingarten equations (2.11a,2.11b), the induced scalar curvature can be expressed as a single second-order function when the boundary term is kept.

$$
\mathscr{R} = 2L_i K^i - G^{ABCD}\,\Pi_{\mu\nu}\mathscr{D}_A \mathscr{D}_B X^\mu \mathscr{D}_C \mathscr{D}_D X^\nu - 2h^{AB}\delta_{ij}\widetilde{\mathscr{D}}_A n^i \cdot \widetilde{\mathscr{D}}_B n^j. \tag{6.18}
$$

The first term in equation (6.15) conceals the linear dependence on the acceleration through K^i. In this context, $\widetilde{\mathscr{D}}_A$ represents the covariant derivative associated with the connection $\sigma_A{}^{ij} := \epsilon^a{}_A \omega_a^{ij}$, which accounts for the rotational freedom of the normal vector fields (refer to Appendix D). The geometric object involved is

$$G^{ABCD} := h^{AB}h^{CD} - \frac{1}{2}(h^{AC}h^{BD} + h^{AD}h^{BC}), \tag{6.19}$$

is a Wheeler-DeWitt like metric associated to h_{AB}. It is important to note that the normals, $n^{\mu,i} = n^{\mu,i}(X^\alpha, \dot{X}^\alpha)$, depend on the functions X^α and their derivatives. This function dependence should be taken into account. It is evident that the canonical formulation, which is based on the expression (6.17) and typically neglects the divergence term, assumes a boundary-free brane, resulting in a different starting point compared to the expression (6.18) that includes the divergence term.

Given our focus on providing an original portrayal of the RT model while honoring its second-order nature, we proceed with the ADM decomposition of each term in the action (6.1) in the following manner

$$S_{\rm RT}[X^\mu] = \int_{\mathbb{R}} dt\, L_{\rm RT}(X^\mu{}_A, \dot{X}^\mu, \dot{X}^\mu{}_A, \ddot{X}^\mu), \tag{6.20}$$

where we recall that $\dot{X}^\mu$ belongs to the configuration space from the Ostrogradsky-Hamilton viewpoint, and

$$L_{\rm RT} = \int_\Sigma d^p u\, \mathscr{L}_{\rm RT} = \int_\Sigma \mathscr{L}_{\rm RT}. \tag{6.21}$$

For ease of notation, moving forward, the differential $d^p u$ will be incorporated whenever an integration over Σ is carried out. Thus, the Lagrangian density can be expressed as follows

$$\mathscr{L}_{\rm RT} = \frac{1}{2}N\sqrt{h}\left[2L_i K^i - G^{ABCD}\,\Pi_{\mu\nu}\mathscr{D}_A\mathscr{D}_B X^\mu \mathscr{D}_C\mathscr{D}_D X^\nu\; - 2h^{AB}\delta_{ij}\widetilde{\mathscr{D}}_A n^i \cdot \widetilde{\mathscr{D}}_B n^j\right]. \tag{6.22}$$

A few remarks are worth noting regarding the structure of this Lagrangian density. In the first term, the linear acceleration dependence is concealed by the mean extrinsic curvature, as evident in the first term of (6.15). The second term incorporates both the superspace metric resembling Wheeler-De Witt and the previously defined normal projector $\Pi^{\mu\nu} = n^{\mu i} n^\nu_i$. Lastly, the last term precisely represents a nonlinear sigma model constructed from $n^{\mu,i}$, exhibiting an $O(N-p)$ symmetry that reflects the rotational invariance of the normal vectors $n^\mu{}_i = n^\mu{}_i(X^\alpha, \dot{X}^\alpha)$, subject to the constraint $n^i \cdot n^j = \delta^{ij}$.

The Lagrangian density (6.22) serves as our initial reference point for obtaining the Hamiltonian formulation of GBG.

Regarding the ADM decomposition of the linear momentum density (6.5), using the tangential projector from m onto the hypersurface Σ defined as $\overline{h}^{ab} = h^{AB}\epsilon^a{}_A\epsilon^b{}_B = g^{ab} + \xi^a\xi^b$, we have

$$\pi_\mu = -\sqrt{h}\,\xi^a G_{ab} g^{bc} X_{\mu c} = \sqrt{h}\left[(\xi^a G_{ab}\xi^b)\xi_\mu - (\xi^a G_{ab}\epsilon^{bB})\epsilon_{\mu B}\right], \tag{6.23}$$

where we have taken into account the relationship $\sqrt{-g} = N\sqrt{h}$. Additionally, considering the integrability conditions associated with (2.19), we can express the projections of the worldvolume Einstein tensor as follows.

$$\xi^a G_{ab}\xi^b = \frac{1}{2}\left(\overline{R} - k_{AB}k^{AB} + k^2\right), \tag{6.24}$$

$$\xi^a G_{ab}\epsilon^{bB} = \mathscr{D}_A(k^{AB} - h^{AB}k) = -(L_i^{AB} - h^{AB}L_i)L_A{}^i, \tag{6.25}$$

$$\epsilon^a{}_A G_{ab}\epsilon^b{}_B = K_i L^i_{AB} - L^i_{AC}L^C{}_{Bi} + L^i_A L_{Bi} - \frac{1}{2}\mathscr{R} h_{AB}, \tag{6.26}$$

where we recall that $\overline{R}$ denotes the Ricci scalar of the hypersurface Σ_t.

6.3 Ostrogradsky-Hamilton approach

In accordance with the Ostrogradsky-Hamilton formulation, which was discussed in the initial section of this chapter, we have a phase space of dimension $4N$ that is spanned by two pairs of conjugate variables. $\{X^\mu, p_\mu; \dot{X}^\mu, P_\mu\}$ where the momenta p_μ and P_μ, conjugate to X^μ and $\dot{X}^\mu$ respectively, are defined in terms of the Σ basis as

$$P_\mu = \frac{\delta L_{\rm RT}}{\delta \ddot{X}^\mu} = \frac{\sqrt{h}}{N} L^i n_{\mu i}, \tag{6.27}$$

$$p_\mu = \frac{\delta L_{\rm RT}}{\delta \dot{X}^\mu} - \partial_t P_\mu = \pi_\mu + \partial_A\left(N^A P_\mu + \sqrt{h}\,h^{AB} L_B{}^i n_{\mu i}\right), \tag{6.28}$$

Here, we consider π_μ as defined in (6.5). It is important to note that the momenta P_μ and p_μ have a spatial weight of one due to the presence of the factor $\sqrt{h}$. Additionally, when integrating over a closed spatial geometry, the difference between the momenta p_μ and π_μ arises from a boundary term. In this regard, while the momenta P_μ are explicitly normal to the worldvolume, the momenta p_μ are tangential to the worldvolume, with the exception of a spatial divergence. To maintain a broad scope in our analysis, we will not impose restrictions on closed geometry and will allow for arbitrary boundary conditions.

In this extended phase space, the appropriate Legendre transformation is given by $\mathscr{H}_c := p \cdot \dot{X} + P \cdot \ddot{X} - \mathscr{L}_{\rm RT}$, and it provides the canonical Hamiltonian density of weight one

$$\mathscr{H}_c = p \cdot \dot{X} + 2N^A (P \cdot \mathscr{D}_A \dot{X}) + (N^2 h^{AB} - N^A N^B)(P \cdot \mathscr{D}_A \mathscr{D}_B X) + N\sqrt{h}\, h^{AB} \delta_{ij} \widetilde{\mathscr{D}}_A n^i \cdot \widetilde{\mathscr{D}}_B n^j + \frac{1}{2} N \sqrt{h}\, G^{ABCD}\, \Pi_{\mu\nu} \mathscr{D}_A \mathscr{D}_B X^\mu \mathscr{D}_C \mathscr{D}_D X^\nu, \tag{6.29}$$

so that the canonical Hamiltonian reads

$$H_c[X^\mu, p_\mu; \dot{X}^\mu, P_\mu] = \int_\Sigma \mathscr{H}_c(X^\mu, p_\mu, \dot{X}^\mu, P_\mu). \tag{6.30}$$

It is important to note that the canonical Hamiltonian exhibits a linear dependence on the momenta p_μ and P_μ. In classical dynamics, the physical momenta p_μ can take both positive and negative values in phase space, resulting in an unbounded canonical Hamiltonian from below. This implies that the well-known Ostrogradsky instabilities may be present in the dynamics of the theory, as discussed in previous works (e.g., [112]).

Furthermore, it is worth mentioning the absence of a quadratic term P^2, which would indicate a genuine second-order derivative brane model [1, 19, 80]. Additionally, the canonical Hamiltonian $\mathscr{H}_c$ exhibits a highly nonlinear dependence on the configuration variables X^μ and $\dot{X}^\mu$, including the lapse and shift functions as well as the last two terms in (6.29).

The presence of local symmetries is reflected in the existence of constraints on the phase space variables. In principle, we can determine these constraints by computing the null eigenvectors of the Hessian matrix. However, in this case, the Hessian matrix vanishes identically, indicating the presence of additional constraints.

$$\mathscr{H}_{\mu\nu} = \frac{\delta^2 L_{\rm RT}}{\delta \ddot{X}^\mu \delta \ddot{X}^\nu} = 0. \tag{6.31}$$

This characteristic is a distinguishing feature of theories that are affine in acceleration [1, 29]. The fact that the rank of the Hessian matrix is zero indicates that the phase space is fully constrained, meaning that we have N primary constraints. It is evident that we cannot express any of the accelerations $\ddot{X}^\mu$ in terms of the phase space variables. Thus, the definition of the momenta P_μ (6.27) itself gives rise to a set of N primary constraint densities.

$$\mathscr{C}_\mu := P_\mu - \frac{\sqrt{h}}{N} L^i n_{\mu i} = 0. \tag{6.32}$$

A more manageable approach to the computations with these constraints, without affecting their content, is to exploit the intrinsic geometric nature of the system. Indeed, using the tangential projector from $\mathscr{M}$ onto the hypersurface Σ, $\overline{h}^{\mu\nu} = h^{AB}\epsilon^{\mu}{}_{A}\epsilon^{\nu}{}_{B} = \eta^{\mu\nu} + \xi^{\mu}\xi^{\nu} - n^{\mu i}n^{\nu}{}_{i}$, written in terms of the hypersurface Σ basis $\{\dot{X}^{\mu}, \epsilon^{\mu}{}_{A}, n^{\mu}{}_{i}\}$ we can rewrite them as $\mathscr{C}_{\mu} = \xi_{\mu\nu}\mathscr{C}^{\nu} = \mathscr{C}_1\dot{X}_{\mu} + \mathscr{C}_A\,\epsilon_{\mu}{}^{A} + \mathscr{C}_i\,n_{\mu}{}^{i} = 0$, where we have $\xi^{\mu} = X^{\mu}{}_{a}\xi^{a} = (\dot{X}^{\mu} - N^{A}\epsilon^{\mu}{}_{A})/N$. This linear combination helps to identify a set of equivalent primary constraints densities

$$\mathscr{C}_1 := P\cdot\dot{X} = 0, \tag{6.33}$$

$$\mathscr{C}_A := P\cdot\partial_A X = 0, \tag{6.34}$$

$$\mathscr{C}_i := P\cdot n_i - \frac{\sqrt{h}}{N}L_i = 0. \tag{6.35}$$

We will see below that these constraints do generate the expected local gauge transformations.

It is convenient to turn these constraints densities into functions in the phase space Γ. To do this, we smear out the constraints (6.33), (6.34) and (6.35) by test fields λ, λ^{A} and ϕ^{i} defined on Σ_t, and then we integrate them over the entire spatial hypersurface Σ with

$$\mathscr{S}_{\lambda} := \int_{\Sigma}\lambda P\cdot\dot{X}, \tag{6.36}$$

$$\mathscr{V}_{\vec{\lambda}} := \int_{\Sigma}\lambda^{A}P\cdot\partial_A X, \tag{6.37}$$

$$\mathscr{W}_{\vec{\phi}} := \int_{\Sigma}\phi^{i}\left[P\cdot n_i + \frac{\sqrt{h}}{N}L_i\right]. \tag{6.38}$$

Following the Dirac-Bergmann procedure for constrained systems, the dynamics in the phase space Γ is governed by the total Hamiltonian, which can be expressed as

$$H[X^{\mu}, p_{\mu}; \dot{X}^{\mu}, P_{\mu}] = H_0 + \mathscr{S}_{\lambda} + \mathscr{V}_{\vec{\lambda}} + \mathscr{W}_{\vec{\phi}}. \tag{6.39}$$

The time evolution of any phase space function F is given by

$$\partial_t F = \dot{F} \approx \{F, H\}, \tag{6.40}$$

where we have used the Ostrogradsky-Poisson bracket [PB] appropriate for second-order derivative theories

$$\{F, G\} = \int_{\Sigma}\left[\frac{\delta F}{\delta X}\cdot\frac{\delta G}{\delta p} + \frac{\delta F}{\delta\dot{X}}\cdot\frac{\delta G}{\delta P} - (F \longleftrightarrow G)\right], \tag{6.41}$$

with $F,G \in \Gamma$. By utilizing (6.40), we can effectively calculate the time evolution of any constraint function. However, it is important to note that the primary constraint functions (6.36-6.38), under the Poisson bracket (PB) structure, exhibit involution with each other. Thus, we have

$$
\begin{aligned}
&\{\mathscr{S}_\lambda, \mathscr{S}_{\lambda'}\} = 0, && && \{\mathscr{V}_{\vec{\lambda}}, \mathscr{V}_{\vec{\lambda}'}\} = 0, \\
&\{\mathscr{S}_\lambda, \mathscr{V}_{\vec{\lambda}}\} = \mathscr{V}_{\vec{\lambda}'}, && \lambda'^A = \lambda\lambda^A, && \{\mathscr{V}_{\vec{\lambda}}, \mathscr{W}_{\vec{\phi}}\} = 0, \\
&\{\mathscr{S}_\lambda, \mathscr{W}_{\vec{\phi}}\} = \mathscr{W}_{\vec{\phi}'}, && \phi'^i = \lambda\phi^i, && \{\mathscr{W}_{\vec{\phi}}, \mathscr{W}_{\vec{\phi}'}\} = 0,
\end{aligned} \tag{6.42}
$$

here, we have employed the variational derivative of the primary constraints as given in F. To ensure consistency of the primary constraints according to the Dirac-Bergmann procedure, we demand that their time evolution becomes zero. Following this process, we obtain the densities of the secondary constraints as

$$C_1 := \mathscr{H}_0 = 0, \tag{6.43}$$

$$C_A := p \cdot \partial_A X + P \cdot \partial_A \dot{X} = 0, \tag{6.44}$$

$$C_i := p \cdot n_i - n_i \cdot \partial_A \left(N^A P + \sqrt{h}\, h^{AB} L_B{}^j n_j \right) = 0. \tag{6.45}$$

It is worth mentioning that these constraints can also be obtained by projecting the momenta p_μ given by (6.28), along the Σ_t basis $\{\dot{X}^\mu, \epsilon^\mu{}_A, n^\mu{}_i\}$.

As above, we turn the local secondary constraints into secondary constraint functions in the phase space Γ by smearing them by the test fields Λ, Λ^A and Φ^i, defined on Σ_t, and integrating over Σ,

$$S_\Lambda := \int_\Sigma \Lambda \mathscr{H}_0, \tag{6.46}$$

$$V_{\vec{\Lambda}} := \int_\Sigma \Lambda^A \left(p \cdot \partial_A X + P \cdot \partial_A \dot{X} \right), \tag{6.47}$$

$$W_{\vec{\Phi}} := \int_\Sigma \Phi^i \left[p \cdot n_i - n_i \cdot \partial_A \left(N^A P + \sqrt{h}\, h^{AB} L_B{}^j n_j \right) \right]. \tag{6.48}$$

We would like to make the following observations. The constraints (6.33) and (6.34) are characteristic of second-order derivative brane models and solely involve the momenta P_μ. From a geometric perspective, these constraints can be interpreted as a consequence of the orthonormality of the worldvolume basis.

In contrast, the constraints (6.43) and (6.44) involve all the phase space variables. The constraint (6.43) indicates the vanishing of the canonical Hamiltonian, which is expected due to the invariance under worldvolume reparametrization in the theory. It generates diffeomorphisms from Σ onto the worldvolume. On the other hand, the constraint (6.44) generates diffeomorphisms tangential to Σ. This can be verified by examining the PB with the phase space variables, as we will see shortly. These two constraints should be familiar to readers acquainted with the ADM formulation of General Relativity [110].

Regarding the remaining constraints (6.35) and (6.45), the first constraint represents a way to express the trace of the spatial-spatial projection of the extrinsic curvature, L^i, in terms of the phase space variables. The second constraint reflects the orthogonality between the physical momenta π_μ and the normal vectors to the worldvolume. In other words, the constraints (6.35) and (6.45) are characteristic of brane models that are linear in accelerations.

6.3.1 Hamilton's equations

Here we obtain the field equations in the Hamiltonian formulation. This computation is helpful in order to fix some Lagrange multipliers that appear as test functions in the definition of the constraints as functions in phase space in terms of the phase space variables. In addition it provides a check, as it reproduces the form of the momenta P_μ, p_μ given by (6.27) and (6.28), respectively.

By considering the functional derivatives in appendix F as well as the Hamiltonian (6.39) we have first that

$$\partial_t X^\mu = \{X^\mu, H\} = \frac{\delta H_0}{\delta p_\mu} = \dot{X}^\mu . \tag{6.49}$$

This result is obvious since the only dependence on p_μ is through the term $p \cdot \dot{X}$ appearing in H_c. Secondly, we compute

$$\begin{aligned} \partial_t \dot{X}^\mu &= \{\dot{X}^\mu, H\} = \frac{\delta H_0}{\delta P_\mu} + \frac{\delta \mathscr{S}_\lambda}{\delta P_\mu} + \frac{\delta \mathscr{V}_{\vec{\lambda}}}{\delta P_\mu} + \frac{\delta \mathscr{W}_{\vec{\phi}}}{\delta P_\mu}, \\ &= 2N^A \mathscr{D}_A \dot{X}^\mu + (N^2 h^{AB} - N^A N^B) \mathscr{D}_A \mathscr{D}_B X^\mu + \lambda \dot{X}^\mu + \lambda^A \epsilon^\mu{}_A + \phi^i n^\mu{}_i . \end{aligned} \tag{6.50}$$

Upon contracting (6.50) with the momenta P_μ and taking into account the identity (6.15), as well as the primary constraint densities (6.33), (6.34), and (6.35), we can establish the following identification

$$\phi^i = N^2 K^i . \tag{6.51}$$

To determine the remaining Lagrange multipliers, it is helpful to recall a significant identity that relates the acceleration in terms of the Σ_t basis [19],

$$\begin{aligned}\ddot{X}^\mu =&(\dot{N}_A + N\mathscr{D}_A N - N^B\mathscr{D}_A N_B)\epsilon^{\mu A} + (\dot{N} + N^A\mathscr{D}_A N + N^A N^B k_{AB})\xi^\mu \\ &+ (n^i\cdot\ddot{X})n^\mu{}_i.\end{aligned} \tag{6.52}$$

As before, by considering (6.52) and the primary constraints, when contracting (6.50) with ξ and ϵ_A yields

$$\lambda = \mathscr{D}_A N^A - \frac{N^2}{\sqrt{h}}\xi^a\nabla_a\left(\frac{\sqrt{h}}{N}\right) = \frac{1}{N}\left(\dot{N} - N^A\mathscr{D}_A N - N^2 k\right), \tag{6.53}$$

$$\lambda^A = N\mathscr{D}^A N - N^A\mathscr{D}_B N^B + \frac{N^2}{\sqrt{h}}\xi^a\nabla_a\left(\frac{\sqrt{h}N^A}{N}\right), \tag{6.54}$$

where we have used the time derivative of the spatial metric, $\dot{h}_{AB} = 2Nk_{AB} + 2\mathscr{D}_{(A}N_{B)}$ and its determinant, $\partial_t(\sqrt{h}) = \sqrt{h}(NK + \mathscr{D}_A N^A)$. It is worthwhile to mention that the Lagrange multipliers (6.53) and (6.54) are inherent to second-order derivative brane models [19, 76].

We turn now to compute the time evolution of the momenta P_μ. We obtain the lengthy expression

$$\begin{aligned}\partial_t P_\mu = \{P_\mu, H\} =& -\frac{\delta H_0}{\delta\dot{X}^\mu} - \frac{\delta\mathscr{S}_\lambda}{\delta\dot{X}^\mu} - \frac{\delta\mathscr{W}_{\vec{\phi}}}{\delta\dot{X}^\mu}, \\ =& -p_\mu - 2(P\cdot\mathscr{D}_A\dot{X})h^{AB}\epsilon_{\mu B} + \mathscr{D}_A(2N^A P_\mu) + 2Nh^{AB}(P\cdot\mathscr{D}_A\mathscr{D}_B X)\xi_\mu \\ &+2N^B(P\cdot\mathscr{D}_A\mathscr{D}_B X)h^{AC}\epsilon_{\mu C} + \frac{1}{2}\sqrt{h}G^{ABCD}\Pi_{\alpha\beta}\mathscr{D}_A\mathscr{D}_B X^\alpha\mathscr{D}_C\mathscr{D}_D X^\beta\xi_\mu \\ &-\sqrt{h}G^{ABCD}L^i_{AB}k_{CD}n_{\mu i} + \sqrt{h}h^{AB}\delta_{ij}\left(\widetilde{\mathscr{D}}_A n^i\cdot\widetilde{\mathscr{D}}_B n^j\right)\xi_\mu \\ &+\left[\widetilde{\mathscr{D}}_A\left(2N\sqrt{h}h^{AB}\delta_{ij}\widetilde{\mathscr{D}}_B n^j\right)\cdot\xi\right]\frac{1}{N}n_\mu{}^i - \lambda P_\mu + \phi^i\frac{\sqrt{h}}{N^2}L_i\xi_\mu + \phi^i\frac{\sqrt{h}}{N^2}k\,n_{\mu i}.\end{aligned} \tag{6.55}$$

By inserting (6.51) and (6.53) into the previous expression we get

$$\begin{aligned}p_\mu = \Big\{&\alpha\sqrt{h}\Big[L_i K^i - 2L_i L^i + \frac{1}{2}G^{ABCD}\Pi_{\alpha\beta}\mathscr{D}_A\mathscr{D}_B X^\alpha\mathscr{D}_C\mathscr{D}_D X^\beta \\ &+ h^{AB}\delta_{ij}\left(\widetilde{\mathscr{D}}_A n^i\cdot\widetilde{\mathscr{D}}_B n^j\right)\Big]\xi_\mu \\ &+\sqrt{h}\left[kK^i + \widetilde{\mathscr{D}}_A\left(2Nh^{AB}\delta_{ij}\widetilde{\mathscr{D}}_B n^j\right)\cdot\left(\frac{1}{N}\eta\right) - G^{ABCD}L^i_{AB}k_{CD}\right]n_{\mu i} \\ &+2\sqrt{h}h^{AB}L_i L_A{}^i\epsilon_{\mu B} - \frac{1}{N}(\dot{N} - N^A\mathscr{D}_A N - N^2 k)P_\mu + \mathscr{D}_A(2N^A P_\mu)\Big\} \\ &-\partial_t P_\mu.\end{aligned} \tag{6.56}$$

This expression coincides with the definition of p_μ (6.28) in a higher derivative theory, when we identify the term within the curly brackets on the right-hand side as $\delta L_{\rm RT}/\delta \dot{X}^\mu$. Additionally, this expression demonstrates the linear relationship between the momenta p_μ and the accelerations of the extended object [76]. Thus far, the Hamilton's equations successfully reproduce the expressions for the momenta, as well as the expressions for the velocity and accelerations of the extended object. Finally, the time evolution of the momenta p_μ is given by

$$\partial_t p_\mu = \{p_\mu, H\} = -\frac{\delta H_0}{\delta X^\mu} - \frac{\delta \mathscr{S}_\lambda}{\delta X^\mu} - \frac{\delta \mathscr{V}_{\vec{\lambda}}}{\delta X^\mu} - \frac{\delta \mathscr{W}_{\vec{\phi}}}{\delta X^\mu}, \tag{6.57}$$

the expressions provided above correspond to the field equations of the model (6.2) in its canonical form. This can be demonstrated through a lengthy yet straightforward calculation, involving the explicit introduction of the Lagrange multipliers (6.51), (6.53), and (6.54).

6.4 First- and second-class constraints

To classify the constraint surface, it is necessary to distinguish between the primary and secondary constraints, categorizing them as either first or second class constraints. To initiate this process, we reassign labels to the constraint functions in the following manner.

$$\varphi_I := \{\mathscr{W}_{\vec{\phi}}, \mathscr{S}_\lambda, \mathscr{V}_{\vec{\lambda}}, S_\Lambda, V_{\vec{\Lambda}}, W_{\vec{\Phi}}\}, \qquad I = 1,2,\ldots,6., \tag{6.58}$$

where we have chosen a convenient order for them. Then, we turn to construct the antisymmetric matrix composed of the PB of all the constraint functions, $W'_{IJ} := \{\varphi_I, \varphi_J\}$. Explicitly, the matrix W'_{IJ} reads, weakly on the constraint surface,

$$(W'_{IJ}) \approx \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \mathscr{C} \\ 0 & 0 & 0 & 0 & 0 & \mathscr{A} \\ 0 & 0 & 0 & 0 & 0 & \mathscr{B} \\ 0 & 0 & 0 & 0 & 0 & \mathscr{D} \\ 0 & 0 & 0 & 0 & 0 & \mathscr{E} \\ -\mathscr{C} & -\mathscr{A} & -\mathscr{B} & -\mathscr{D} & -\mathscr{E} & \mathscr{F} \end{pmatrix}, \tag{6.59}$$

where the nonvanishing entries $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \mathscr{E}$ and $\mathscr{F}$ are defined in appendix G. The rank of this matrix is 2, thus pointing out the existence of two second-class constraint functions. To select these it is necessary to determine first the 4 zero modes $\omega^I_{(u)}$ with $z = 1,2,3,4$, so that $W'_{IJ}\omega^J_{(u)} = 0$. These can be taken as follows

$$\omega^I_{(1)} = \begin{pmatrix} -\mathscr{A}/\mathscr{C} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^I_{(2)} = \begin{pmatrix} -\mathscr{B}/\mathscr{C} \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^I_{(3)} = \begin{pmatrix} -\mathscr{D}/\mathscr{C} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^I_{(4)} = \begin{pmatrix} -\mathscr{E}/\mathscr{C} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{6.60}$$

With these the functions $\gamma_u := \omega^I_{(u)}\varphi_I$ are first-class constraints,

$$\begin{aligned} \gamma_1 &= \mathscr{S}_\lambda - \tfrac{\mathscr{A}}{\mathscr{C}}\mathscr{W}_\phi, & \gamma_2 &= \mathscr{V}_{\vec{\lambda}} - \tfrac{\mathscr{B}}{\mathscr{C}}\mathscr{W}_\phi, \\ \gamma_3 &= S_\Lambda - \tfrac{\mathscr{D}}{\mathscr{C}}\mathscr{W}_\phi, & \gamma_4 &= V_{\vec{\Lambda}} - \tfrac{\mathscr{E}}{\mathscr{C}}\mathscr{W}_\phi. \end{aligned} \tag{6.61}$$

To identify the second-class constraints formally, we can follow these steps. By selecting a set of linearly independent vectors, denoted as $\omega^I_{(u')}$ with $u' = 5,6$, which are independent of the vectors $\omega^I_{(u)}$, and satisfying the condition $\det(\omega^I_{(I')}) \neq 0$ with $I' = (u,u')$, we can define the functions $\chi_{u'} := \omega^I_{(u')}\varphi_I$ as second-class constraints [52]. This can be achieved by choosing

$$\omega^I_{(5)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \omega^I_{(6)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{6.62}$$

we observe that the previously mentioned conditions are satisfied. Then,

$$\chi_5 = \mathscr{W}_{\vec{\phi}}, \tag{6.63}$$

$$\chi_6 = W_{\vec{\Phi}}, \tag{6.64}$$

are second-class constraints.

The constraints γ_u and $\chi_{u'}$ establish an equivalent representation of the constrained phase space. Within this new framework, we can introduce the matrix elements $C_{u'v'} := \chi_{u'}, \chi_{v'}$ with $u', v' = 5,6$, and its inverse matrix components $(C^{-1})^{u'v'}$, which are determined by

$$(C_{u'v'}) = \begin{pmatrix} 0 & \mathscr{C} \\ -\mathscr{C} & \mathscr{F} \end{pmatrix}, \quad \text{and} \quad \left((C^{-1})^{u'v'}\right) = \frac{1}{\mathscr{C}^2}\begin{pmatrix} \mathscr{F} & -\mathscr{C} \\ \mathscr{C} & 0 \end{pmatrix}, \tag{6.65}$$

respectively. In accordance with the theory of constrained systems, the matrix $(C^{-1})^{u'v'}$ provides a means to introduce the Dirac bracket using the standard approach.

$$\{F,G\}_D := \{F,G\} - \{F,\chi_{u'}\}(C^{-1})^{u'v'}\{\chi_{v'},G\}. \tag{6.66}$$

After formally identifying the second-class constraints, we can impose them strongly to zero, thereby reducing them to identities that express certain phase space variables in terms of others. Consequently, the first-class constraints (6.61) are simplified to

$$\begin{aligned} \gamma_1 &= \mathscr{S}_\lambda, & \gamma_3 &= S_\Lambda, \\ \gamma_2 &= \mathscr{V}_{\vec{\lambda}}, & \gamma_4 &= V_{\vec{\Lambda}}, \end{aligned} \tag{6.67}$$

As anticipated, the first-class constraint functions γ_2 and γ_4 each consist of p primary constraints and p secondary constraints, respectively. Likewise, the second-class constraint functions χ_5 and χ_6 incorporate $(N-p)$ primary constraints and $(N-p)$ secondary constraints, respectively. This implies that the count of physical degrees of freedom (dof) is as follows: 2 dof = (total number of canonical variables) - 2 (number of first-class constraints) - (number of second-class constraints). Hence, dof $= N - p = i$. Therefore, there exist i degrees of freedom, with one corresponding to each normal vector of the worldvolume. This count aligns with the number of physical transverse motions $\sigma^i := n^i \cdot \delta X$ that characterize first-order derivative brane models, as expected.

With support with the gauge transformations that generate the first-class constraints, it is convenient to name $\mathscr{S}_\lambda$ the *shift constraint* while $\mathscr{V}_{\vec{\lambda}}$ will be referred to as the *primary vector constraint*. In the same spirit, S_Λ and $V_{\vec{\Lambda}}$ may be thought of as being the *scalar* and *secondary vector* constraint, respectively, in comparison to the ones appearing in a canonical analysis of the Dirac-Nambu-Goto model.

6.4.1 Algebra of constraints

Under the Dirac bracket, the algebra spanned by the first-class constraints is

$$\{\mathscr{S}_\lambda, \mathscr{S}_{\lambda'}\}_D = 0, \tag{6.68a}$$

$$\{\mathscr{S}_\lambda, \mathscr{V}_{\vec{\lambda}}\}_D = \mathscr{V}_{\vec{\lambda}_1}, \tag{6.68b}$$

$$\{\mathscr{S}_\lambda, S_\Lambda\}_D = -\mathscr{S}_{\lambda_1} - S_{\Lambda_1}, \tag{6.68c}$$

$$\{\mathscr{S}_\lambda, V_{\vec{\Lambda}}\}_D = -\mathscr{S}_{\mathscr{L}_{\vec{\Lambda}}\lambda}, \tag{6.68d}$$

$$\{\mathscr{V}_{\vec{\lambda}}, \mathscr{V}_{\vec{\lambda}'}\}_D = 0, \tag{6.68e}$$

$$\{\mathscr{V}_{\vec{\lambda}}, S_\Lambda\}_D = \mathscr{S}_{\mathscr{L}_{\vec{\lambda}}\Lambda} - \mathscr{V}_{\vec{\lambda}_2} - V_{\vec{\Lambda}_1}, \tag{6.68f}$$

$$\{\mathscr{V}_{\vec{\lambda}}, V_{\vec{\Lambda}}\}_D = \mathscr{V}_{[\vec{\lambda},\vec{\Lambda}]}, \tag{6.68g}$$

$$\{S_\Lambda, S_{\Lambda'}\}_D = \mathscr{S}_{\lambda_2}, \tag{6.68h}$$

$$\{S_\Lambda, V_{\vec{\Lambda}}\}_D = -S_{\mathscr{L}_{\vec{\Lambda}}\Lambda} + \mathscr{V}_{\vec{\lambda}_3}, \tag{6.68i}$$

$$\{V_{\vec{\Lambda}}, V_{\vec{\Lambda}'}\}_D = V_{[\vec{\Lambda},\vec{\Lambda}']}. \tag{6.68j}$$

where we have introduced

$$\begin{aligned}
&\lambda_1^A = \lambda\lambda^A, && \lambda_2^A = 2\Lambda N^B \mathscr{D}_B \lambda^A,\\
&\lambda_1 = 2\Lambda \mathscr{L}_{\vec{N}}\lambda, && \lambda_2 = (N^2 h^{AB} - N^A N^B)(\Lambda \mathscr{D}_A \mathscr{D}_B \Lambda' - \Lambda' \mathscr{D}_A \mathscr{D}_B \Lambda),\\
&\Lambda_1^A = \Lambda\lambda^A, && \lambda_3^A = \Lambda(N^2 h^{AB} - N^A N^B)(\mathscr{D}_A \mathscr{D}_B \Lambda^C + R_{ADB}{}^C \Lambda^D),\\
&\Lambda_1 = \lambda\Lambda.
\end{aligned} \tag{6.69}$$

This algebra is equivalent to the algebra under under the PB, once we apply the property $\{F, \gamma_u\} \approx \{F, \gamma_u\}_D$, for any phase space function F.

The geometric interpretation of this algebra can be illustrated as follows. Starting with (6.68h), we observe that different orderings of the scalar constraints only differ by a shift transformation, indicating that the time evolution with the scalar constraint is unique up to a rescaling. From (6.68i), we note that the Poisson bracket (PB) of a vector with a scalar constraint yields a scalar constraint with a test field given by the Lie derivative of the parameter Λ along the vector field $\vec{\lambda}$, accompanied by tangential deformations provided by the primary vector constraint. Relation (6.68j) demonstrates that the secondary vector constraints form their own proper subalgebra, exhibiting invariance under reparametrizations of the theory. As for (6.68a) and (6.68e), they show that shift and primary vector transformations each form their own sub-algebra, with both sub-algebras being Abelian. Expression (6.68b) depicts how the primary vector constraint changes under the shift transformation, revealing that although the vector constraint is preserved, the test field is modified. Relationships (6.68c) and (6.68d) demonstrate how the shift transformations change under the scalar and vector constraints. At this point, it becomes evident that the scalar and vector constraints serve as generators of diffeomorphisms, both tangential and orthogonal, to Σ_t. Similarly, (6.68f) and (6.68g) determine how the primary vector constraint changes under the scalar and vector constraints. It is important to note that despite the constraint algebra being closed under the Dirac bracket, it is an open algebra since several of the test fields (6.69) depend on some of the phase space variables. Furthermore, this constraint algebra differs from those encountered in usual gauge theories. This fact poses a challenge for the standard canonical quantization of GBG within the framework considered.

6.4.2 Infinitesimal canonical transformations

In order to further illustrate the role of the constraints in the theory, in this subsection we consider infinitesimal canonical transformations.

It is worth remembering that, for any classical observable $F \in \Gamma$, the Hamiltonian vector field

$$X_F := \int_\Sigma \left(\frac{\delta F}{\delta p} \cdot \frac{\delta}{\delta X} + \frac{\delta F}{\delta P} \cdot \frac{\delta}{\delta \dot{X}} - \frac{\delta F}{\delta X} \cdot \frac{\delta}{\delta p} - \frac{\delta F}{\delta \dot{X}} \cdot \frac{\delta}{\delta P} \right), \tag{6.70}$$

generates a one-parameter family of canonical transformations $G \longrightarrow G + \delta_F G$, where $\delta_F G := \epsilon\{G, F\}$, with ϵ being an infinitesimal dimensionless quantity. The Hamiltonian vector fields associated with the first-class constraints (6.67) induce the infinitesimal canonical transformations

$$X_{\gamma_1} \longrightarrow \begin{cases} \delta_{\mathscr{S}_\lambda} X^\mu = 0, \\ \delta_{\mathscr{S}_\lambda} \dot{X}^\mu = \epsilon_1 \lambda \dot{X}^\mu, \\ \delta_{\mathscr{S}_\lambda} p_\mu = 0, \\ \delta_{\mathscr{S}_\lambda} P_\mu = -\epsilon_1 \lambda P_\mu, \end{cases} \qquad X_{\gamma_3} \longrightarrow \begin{cases} \delta_{S_\Lambda} X^\mu = \epsilon_3 \Lambda \dot{X}^\mu, \\ \delta_{S_\Lambda} \dot{X}^\mu = \epsilon_3 \frac{\delta S_\Lambda}{\delta P_\mu}, \\ \delta_{S_\Lambda} p_\mu = -\epsilon_3 \frac{\delta S_\Lambda}{\delta X^\mu}, \\ \delta_{S_\Lambda} P_\mu = -\epsilon_3 \frac{\delta S_\Lambda}{\delta \dot{X}^\mu}, \end{cases}$$
$$X_{\gamma_2} \longrightarrow \begin{cases} \delta_{\mathcal{V}_{\vec{\lambda}}} X^\mu = 0, \\ \delta_{\mathcal{V}_{\vec{\lambda}}} \dot{X}^\mu = \epsilon_2 \mathscr{L}_{\vec{\lambda}} X^\mu, \\ \delta_{\mathcal{V}_{\vec{\lambda}}} p_\mu = \epsilon_2 \mathscr{L}_{\vec{\lambda}} P_\mu, \\ \delta_{\mathcal{V}_{\vec{\lambda}}} P_\mu = 0, \end{cases} \qquad X_{\gamma_4} \longrightarrow \begin{cases} \delta_{V_{\vec{\Lambda}}} X^\mu = \epsilon_4 \mathscr{L}_{\vec{\Lambda}} X^\mu, \\ \delta_{V_{\vec{\Lambda}}} \dot{X}^\mu = \epsilon_4 \mathscr{L}_{\vec{\Lambda}} \dot{X}^\mu, \\ \delta_{V_{\vec{\Lambda}}} p_\mu = \epsilon_4 \mathscr{L}_{\vec{\Lambda}} p_\mu, \\ \delta_{V_{\vec{\Lambda}}} P_\mu = \epsilon_4 \mathscr{L}_{\vec{\Lambda}} P_\mu, \end{cases} \tag{6.71}$$

where ϵ_u, with $u = 1, \ldots, 4$, denotes arbitrary gauge parameters corresponding to each of the first-class constraints γ_u, respectively. For instance,

$$\dot{X}^\mu \mapsto \dot{X}^\mu + \epsilon_1 \lambda \dot{X}^\mu, \qquad \text{and} \qquad P_\mu \mapsto P_\mu - \epsilon_1 \lambda P_\mu,$$

are the gauge transformations induced by the gauge function λ. From (6.71) we infer that the constraint $V_{\vec{\Lambda}}$ generates diffeomorphisms tangential to Σ_t, while S_Λ is the generator of diffeomorphisms out of Σ_t onto the worldvolume m. On the other hand, $\mathscr{S}_\lambda$ is the generator of a momentum reflection in the sub-sector of Γ given by $\{\dot{X}^\mu;P_\mu\}$ that is, the sector associated to the second-order derivative dependence; from another view point, this constraint generates *shift transformations* only in the velocity sector of the phase space. Finally, the constraint $\mathscr{V}_{\vec{\lambda}}$ only acts on the sub-sector $\{\dot{X}^\mu;p_\mu\}$ by generating displacements in the orthogonal complement of this sub-sector, that is, in the sub-sector $\{X^\mu;P_\mu\}$.

CHAPTER 7

Discussion and Conclusions

In the first part of this work, our main objective was to develop covariant variational tools that are valuable for studying the dynamics of extended objects, with a focus on obtaining covariant equations from a variational principle. For this purpose, we introduced different covariant variations, each with its own advantages and disadvantages depending on the specific physical system under consideration.

If we approach a physical system from the perspective of an ambient spacetime, the covariant variation discussed in Section 3.2 is suitable. It allows us to derive covariant equations that are invariant under the background symmetries. Moreover, the calculation of the first, second, and third variations of the action is relatively straightforward. It is important to note that dealing with a curved ambient spacetime does not pose a significant challenge within this approach.

However, the resulting equations obtained using this covariant variation contain non-dynamic parts. To eliminate these non-dynamic components, it is necessary to project the equations along the normal fields. In contrast, the covariant variation presented in Section 3.1 provides equations that are covariant under full dynamical reparametrizations of the extended object's world volume. However, calculating subsequent variations of the action using this approach can be significantly more complex and time-consuming, particularly when considering a curved background spacetime.

In addition to the above variations, we introduced a simultaneous covariant action for extended objects. This action, yields both the equations of motion and the Jacobi equations. The simultaneous action offers a convenient alternative approach for stability studies, providing a direct path to the second variation of the geometric model. Furthermore, the simultaneous action can be extended to incorporate additional brane-living fields or "pressure terms" that may arise in fundamental branes. Although these additions may affect the simplicity and elegance of the covariant simultaneous principle, they prove beneficial in applications where external forces play a significant role.

In the second part of this thesis, our focus was specifically on the Regge-Teitelboim geometric model, utilizing the variational tools discussed earlier. Through these tools, we derived dynamic equations as well as Jacobi equations, with the latter serving as a crucial tool for studying the stability of specific solutions to the equations of motion. As expected, these equations are explicitly covariant under rotations of the normal fields. Within this geometric framework, conserved geometric structures play a fundamental role in expressing the Jacobi equations in the form of wave-like equations, allowing for the extraction of geometric "mass" terms.
Having established the general form of the Jacobi equations, we directed our attention to a specific solution of the equations of motion, namely a four-dimensional Schwarzschild geometry embedded in a six-dimensional Minkowski spacetime. By exploiting the symmetries of this solution, we derived a set of equations that determine the quasi-normal modes of the system. Our findings indicate the presence of instability for this configuration in the absence of matter. While an analytical study of the Jacobi equations for this case provides valuable insights, it is not conclusive. However, numerical analysis supports the existence of unstable small deformations.

Furthermore, the results presented suggest the potential extension of these findings to the entire class of Lovelock branes, representing the next step in understanding this type of geometric model for branes. It is important to note that a crucial assumption, both in physical and geometric terms, is the embedding of the world volume in a flat background spacetime. Generalizing this framework to arbitrary background spacetimes poses challenges, but for maximally symmetric ambient spacetimes such as de Sitter or anti-de Sitter backgrounds, it appears to be achievable.

Additionally, we have conducted a comprehensive Ostrogradsky-Hamilton canonical analysis of this model, wherein a crucial aspect of our investigation involves constructing an ADM Lagrangian density for the model that is linear in the acceleration of the embedding functions. Typically, this term is disregarded as a boundary contribution. However, by retaining it, we treat the RT model as a higher derivative theory, despite its second-order equations of motion. Following the Ostrogradsky-Hamilton canonical formulation, we introduce an extended phase space with positions and velocities as configuration canonical variables, along with their corresponding conjugate momenta. We have derived the canonical Hamiltonian density for the model, which includes terms linear in the conjugate momenta. This indicates the well-known Ostrogradsky instability in higher derivative theories, where the Hamiltonian is unbounded from below. Nonetheless, we remain hopeful that a suitable canonical transformation can be found to address this issue and obtain a Hamiltonian that is bounded from below. Paul [89] has suggested a potential strategy, involving the resolution of second-class constraints, but it seems to be a non-trivial task in the present scenario. Another alternative is to employ a path integral quantization program tailored to second-order singular systems, which incorporate second-class constraints in the theory [101]. However, further investigation is required in this regard. Consistent with the theory's reparametrization invariance symmetry, the Hamiltonian is a linear combination of constraints. We have identified the complete set of constraints and categorized them as first- and second-class constraints. The presence of second-class constraints is the consequence of including a linear term in the Lagrangian's acceleration, but their form is quite manageable. Moreover, we explicitly demonstrate how the constraints generate the expected gauge transformations, and we have successfully determined the correct count of physical degrees of freedom. Additionally, we have verified that Hamilton's equations reproduce the Euler-Lagrange equations of the theory. It should be noted that the codimension remains arbitrary based on the expressions obtained for the Lagrangian, Hamiltonian, and constraint densities. Many of the features observed in the RT model extend to the broader class of theories that are linear affine in accelerations [29]. In principle, starting from our classical formulation, it is possible to implement a formal canonical quantization program, which would fulfill Regge and Teitelboim's original motivation. Notably, in the context of quantum gravity, a significant technical advantage lies in the existence of a fixed background, which aids in formal quantization. The phase space variables would be promoted to operators within a suitable Hilbert space. Consistent with a theory featuring second-class constraints, the Dirac brackets would be replaced by commutators for these operators. However, certain challenges must be addressed, such as finding appropriate gauge fixing conditions to obtain a space of physical states and resolving the issue of obtaining a Hamiltonian that is bounded from below, while avoiding the presence of ghosts and the violation of unitarity. Ideally, deriving a Hamiltonian constraint quadratic in the momenta p_μ would be desirable. Furthermore, another complication arises from the fact that the constraint algebra obtained is not a genuine Lie algebra and deviates from the typical algebra encountered in gauge theories. All of these considerations would also arise in a path integral or BRST quantization of the model. Despite being aware of the forthcoming difficulties, we believe that our Ostrogradsky-Hamilton treatment of geodetic brane gravity serves as a reliable foundation for further investigation.

Several ongoing projects in which we are actively involved have emerged from this thesis. One such project focuses on the extension of the simultaneous action to high derivative systems. This extension is particularly valuable due to its elegant formulation and ease of implementation, making it conducive to studying more intricate systems. Within the context of the Regge-Teitelboim model, we are currently engaged in a project aimed at deriving its Jacobi equations, which are geodesic deviation equations applicable to arbitrary ambient spacetimes. This endeavor holds great potential in physical models involving higher dimensions. Furthermore, we have been investigating the boundary terms associated with the aforementioned model, as they play a crucial role in determining appropriate conditions. In general relativity, for instance, the inclusion of the Gibbons-Hawking-York term is necessary when considering boundaries. However, in the geodetic brane gravity framework, it is not guaranteed that the boundary term follows the same structure, given the distinct nature of the degrees of freedom involved.

On a the other hand, using the Jacobi equation derived for the Regge-Teitelboim model, we are interested in exploring the stability of other significant solutions, such as asymptotically de Sitter black holes and Kerr black holes. Our objective is to compile a catalog of classically linearly stable and unstable solutions. Lastly, we are considering the quantization of the deformation modes of the Schwarzschild black hole. Although we have shown that these modes are characteristic of unstable deformations, it remains uncertain whether we can surpass these instabilities at the quantum level, as is the case in electromagnetism.

APPENDIX

A

PROOF FOR THE CONSERVATION LAW (5.20).

The tensor $\mathbb{G}_{ab}{}^{ij}$ is divergenceless. To prove this we define the tensorial matrices

$$\mathbb{G}_{ab}{}^{ij} := 2\mathbb{R}_{ab}{}^{ij} - g_{ab}\mathbb{R}^{ij}, \tag{A.1}$$

where

$$\mathbb{R}_{ab}{}^{ij} := K^{(i}K_{ab}{}^{j)} - K_a{}^{c(i}K_{bc}{}^{j)}, \quad \text{and} \quad \mathbb{R}^{ij} := K^iK^j - K^{cd\,i}K_{cd}{}^j. \tag{A.2}$$

The divergence of (A.1) is

$$\widetilde{\nabla}^a\mathbb{G}^{ij}_{ab} = 2\widetilde{\nabla}^a\mathbb{R}^{ij}_{ab} - g_{ab}\widetilde{\nabla}^a\mathbb{R}^{ij}.$$

On the one hand, we have for the first term

$$\begin{aligned}\widetilde{\nabla}^a\mathbb{R}_{ab}{}^{ij} &= (\widetilde{\nabla}^aK^{(i})K_{ab}{}^{j)} + (\widetilde{\nabla}^aK_{ab}{}^{(i})K^{j)} - (\widetilde{\nabla}^aK_a{}^{c(i})K_{bc}{}^{j)} - K_a{}^{c(i}\widetilde{\nabla}^aK^{j)}_{bc}\\ &= (\widetilde{\nabla}^aK^{(i})K_{ab}{}^{j)} + (\widetilde{\nabla}_bK^{(i})K^{j)} - (\widetilde{\nabla}^cK^{(i})K^{j)}_{bc} - K_a{}^{c(i}\widetilde{\nabla}^aK^{j)}_{bc}\\ &= (\widetilde{\nabla}_bK^{(i})K^{j)} - K_a{}^{c(i}\widetilde{\nabla}^aK^{j)}_{bc},\end{aligned} \tag{A.3}$$

where we have used (2.19b) to obtain the second line. Now, for the second term

$$\widetilde{\nabla}_b\mathbb{R}^{ij} = 2\left[(\widetilde{\nabla}_bK^{(i})K^{j)} - K_a{}^{c(i}\widetilde{\nabla}^aK^{j)}_{bc}\right]. \tag{A.4}$$

The difference between twice (A.3) and (A.4) reads

$$2\widetilde{\nabla}^a\mathbb{R}_{ab}{}^{ij} - g_{ab}\widetilde{\nabla}^a\mathbb{R}^{ij} = \widetilde{\nabla}^a\mathbb{G}_{ab}{}^{ij} = 0. \tag{A.5}$$

Appendix B

Explicit form of transformation matrix M and effective potential V.

The matrices appearing in (5.41) are

$$\text{(B.1)}\quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & -a_{11} \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & -b_{11} \end{pmatrix},\ \mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & -c_{11} - \frac{6R^2}{r^6} \end{pmatrix},\ \mathbf{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & -d_{11} \end{pmatrix}.$$

Notice that **A**, **B** and **D** are traceless symmetric matrices while **C** is not. In fact, this is responsible for not being able to decouple the system of equations (5.41). The explicit components of these matrices are

(B.2)
$$
\begin{aligned}
a_{11} &= \frac{2R\left(r^3-R^3\right)}{r^3\left(r^3+r^2R+rR^2+R^3\right)},\\
a_{12} &= -\frac{(r-R)(r^3+r^2R+rR^2-R^3)\sqrt{r^3R(r^2+rR+R^2)}}{r^5R(r^3+r^2R+rR^2+R^3)},\\
b_{11} &= \frac{2R\left(-r^6+r^4R^2+6r^3R^3+3r^2R^4+2rR^5+R^6\right)}{r^4\left(r^3+r^2R+rR^2+R^3\right)^2},\\
b_{12} &= \left[2r^3\sqrt{r^3R\left(r^2+rR+R^2\right)}\left(r^3+r^2R+rR^2+R^3\right)^2\right]^{-1}\times\\
&\quad \left(-r^9-3r^8R-6r^7R^2-22r^6R^3-20r^5R^4-16r^4R^5+6r^3R^6+6r^2R^7+5rR^8+3R^9\right),\\
c_{11} &= l(l+1)\frac{R(r-R)\left(r^2+2rR+3R^2\right)}{r^5\left(r^3+r^2R+rR^2+R^3\right)}-\left[2r^6\left(r^3+r^2R+rR^2+R^3\right)^3\right]^{-1}\times\\
&\quad R^2\left(9r^9+27r^8R+54r^7R^2+42r^6R^3+17r^5R^4-21r^4R^5+4rR^8+12R^9\right),\\
c_{12} &= l(l+1)\left[4r^7R\left(r^3+r^2R+rR^2+R^3\right)\right]^{-1}\sqrt{\frac{r^3R}{r^2+rR+R^2}}\times\\
&\quad \left(r^6+2r^5R+3r^4R^2+16r^3R^3+15r^2R^4+14rR^5-3R^6\right)\\
&\quad -\left[4r^{11/2}\sqrt{R\left(r^2+rR+R^2\right)}\left(r^3+r^2R+rR^2+R^3\right)^3\right]^{-1}\times\\
&\quad R\left(3r^{11}+12r^{10}R+30r^9R^2+90r^8R^3+182r^7R^4+296r^6R^5\right.\\
&\quad \left.+254r^5R^6+150r^4R^7-5r^3R^8-56r^2R^9-56rR^{10}-36R^{11}\right),\\
d_{11} &= -\frac{6R^4}{r^6-r^2R^4},\\
d_{12} &= \frac{3R^2\left(r^3+r^2R+rR^2-R^3\right)\sqrt{r^3R\left(r^2+rR+R^2\right)}}{\left(r^7-r^4R^3\right)\left(r^3+r^2R+rR^2+R^3\right)}.
\end{aligned}
$$

On account of the definition

(B.3)
$$\mathbf{h} := -\left(\frac{\mathbf{D}^{-1}\mathbf{B}}{f_g}+\mathbf{I}\frac{df_g}{dr}\right),$$

where $\mathbf{I}$ denotes the 2×2 unit matrix, we find that the matrix $\mathbf{M}$ can be written as

(B.4)
$$\mathbf{M} = \exp\left(-\frac{1}{2}\int\frac{\mathbf{h}}{f_g}dr\right).$$

In the same way, the potential matrix $\mathbf{V}$ in terms of these matrices, becomes

(B.5)
$$\mathbf{V} = \frac{f_g}{2}\frac{d\mathbf{h}}{dr}+\frac{1}{4}\,\mathbf{h}^2+\mathbf{M}^{-1}\,\mathbf{D}^{-1}\mathbf{C}\mathbf{M}.$$

Appendix C

Mathematica Code

```
(******************************************************************************************************************************)
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(*We define the following geometric objects*)

ξ:={t,r,θ,ϕ}(*coordinates on worldvolume*);
g={{-γ[r]^2,0,0,0},{0,1/γ[r]^2,0,0},{0,0,r^2,0},{0,0,0,r^2 Sin[θ]^2}};(*induced metric*)
Ig=Inverse[g](*Inveverse of induced metric*);
Γ[a1_,b1_,c_]:=Γ[a1,b1,c]=Simplify[(1/2)*Sum[(Ig[[a1,d]]*(D[g[[d,b1]],ξ[[c]]]+D[g[[d,c]],ξ[[b1]]]-D[g[[b1,c]],ξ[[d]]])),{d,1,4}]](*Crhistoffel Symbols*);
Riemann[a1_,b1_,c1_,d1_]:=Riemann[a1,b1,c1,d1]=Sum[(g[[a1,e1]]×(D[Γ[e1,d1,b1],ξ[[c1]]]-D[Γ[e1,c1,b1],ξ[[d1]]]
+Sum[(Γ[e1,c1,f1]×Γ[f1,d1,b1]-Γ[e1,d1,f1]×Γ[f1,c1,b1]),{f1,1,4}])),{e1,1,4}](*Riemann Tensor*);
K1={{0,0,0,0},{0,(D[a[r],r]×b[r]-a[r]×D[b[r],r])/a[r],0,0},{0,0,-r b[r],0},{0,0,0,-r b[r]Sin[θ]^2}}(*Extrinsic Curvature 1 with low indices*);

K2={{-γ[r]^2/(2 R a[r]),0,0,0},{0,2 a[r]R^2/(r^2(r-R)),0,0},{0,0,-a[r]R^2/r,0},{0,0,0,-a[r]R^2 Sin[θ]^2/r}}(*Extrinsic Curvature 2 with low indices*);
K1U={{0,0,0,0},{0,γ[r]^4 (b[r] a'[r]/a[r]-b'[r]),0,0},{0,0,-b[r]/r^3,0},{0,0,0,-b[r] Csc[θ]^2/r^3}}(*Extrinsic Curvature 1 with up indices*);
K2U={{-1/(2 R a[r] γ[r]^2),0,0,0},{0,2 R^2 a[r] γ[r]^4/(r^2 (r-R)),0,0},{0,0,-R^2 a[r]/r^5,0},{0,0,0,-R^2 a[r] Csc[θ]^2/r^5}}(*Extrinsic Curvature 2 with up indices*);
k1=-2 b[r]/r+γ[r]^2 (b[r] a'[r]/a[r]-b'[r])(*Mean Extrinsic 1 Curvature*);
k2=1/(2 R a[r])+2 R^2 a[r] (-r+R+r γ[r]^2)/(r^3 (r-R))(*Mean Extrinsic 2 Curvature*);
Kt={K1,K2}(*Extrinsic curvatures with low indices*);
KUt={K1U,K2U}(*Extrinsic curvatures with up indices*);
kt={k1,k2}(*Mean Extrinsic Curvatures*);
ωt{{{0,0,0,0},{0,(R^2/r^2)(a[r]×D[b[r],r]-b[r]×D[a[r],r]),0,0}},{{0,-(R^2/r^2)(a[r]×D[b[r],r]-b[r]×D[a[r],r]),0,0},{0,0,0,0}}}(*Gauge Field ω_a*);
ϕt={ϕ1[t,r,θ,ϕ],ϕ2[t,r,θ,ϕ]}(*Deformation fields*);
G[a1_,b1_,i_,j_]:=G[a1,b1,i,j]=kt[[i]]×KUt[[j,a1,b1]]+kt[[j]]×KUt[[i,a1,b1]]-(Sum[Sum[(KUt[[i,b1,d1]]×KUt[[j,a1,c1]]×g[[c1,d1]]+KUt[[j,b1,d1]]×KUt[[i,a1,c1]]×g[[c1,d1]]),{d1,1,4}],{c1,1,4}])
+Ig[[a1,b1]]((Sum[Sum[(Kt[[i,c1,d1]]×KUt[[j,c1,d1]]),{d1,1,4}],{c1,1,4}])-kt[[i]]×kt[[j]])(*Tensor G^ab_ij*);
DDϕ[c1_,d1_,i_]:=D[(D[ϕt[[i]],ξ[[d1]]]-Sum[(ωt[[i,j,d1]]*ϕt[[j]]),{j,1,2}]),ξ[[c1]]]-Sum[(Γ[e1,c1,d1]×D[ϕt[[i]],ξ[[e1]]]),{e1,1,4}]+(Sum[(Γ[e1,c1,d1]×Sum[(ωt[[i,j,e1]]×ϕt[[j]]),{j,1,2}]),{e1,1,4}])
-(Sum[(ωt[[i,j,c1]]*D[ϕt[[j]],ξ[[d1]]]),{j,1,2}])+Sum[Sum[(ωt[[i,j,c1]]×ωt[[j,k,d1]]×ϕt[[k]]),{k,1,2}],{j,1,2}](*This term is the second covariant derivative of deformation fields, ∇_a∇_bϕ^i*);

(******************************************************************************************************************************)
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```

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(*The equations of motion are given by*)
```

$$\text{Equation[i1\_]:=}\sum_{a1=1}^{4}\sum_{b1=1}^{4}\sum_{j1=1}^{2}\left(G[a1,b1,i1,j1]\times\left(DD\phi[a1,b1,j1]+\sum_{c1=1}^{4}\sum_{d1=1}^{4}\sum_{l1=1}^{2}Kt[\![j1,a1,c1]\!]\times Ig[\![c1,d1]\!]\times Kt[\![l1,d1,b1]\!]\times \phi t[\![l1]\!]\right)\right)\ \text{(*Equation of Motion*)}$$

```
(*it is convenient to have each one of the components of these differential equations separately *)
α1=Simplify[Expand[Coefficient[E1,ϕ1[t,r,θ,ϕ]]],r>R>0];
α2=Simplify[Expand[Coefficient[E1,ϕ1^(1,0,0,0)]],r>R>0];
α3=Simplify[Expand[Coefficient[E1,ϕ1^(2,0,0,0)[t,r,θ,ϕ]]],r>R>0];
α4=Simplify[Expand[Coefficient[E1,ϕ1^(0,1,0,0)[t,r,θ,ϕ]]],r>R>0];
α5=Simplify[Expand[Coefficient[E1,ϕ1^(0,2,0,0)[t,r,θ,ϕ]]],r>R>0];
α6=Simplify[Expand[Coefficient[E1,ϕ1^(0,0,1,0)[t,r,θ,ϕ]]],r>R>0];
α7=Simplify[Expand[Coefficient[E1,ϕ1^(0,0,2,0)[t,r,θ,ϕ]]],r>R>0];
α8=Simplify[Expand[Coefficient[E1,ϕ1^(0,0,0,1)[t,r,θ,ϕ]]],r>R>0];
α9=Simplify[Expand[Coefficient[E1,ϕ1^(0,0,0,2)[t,r,θ,ϕ]]],r>R>0];
β1=Simplify[Expand[Coefficient[E1,ϕ2[t,r,θ,ϕ]]],r>R>0];
β2=Simplify[Expand[Coefficient[E1,ϕ2^(1,0,0,0)[t,r,θ,ϕ]]],r>R>0];
β3=Simplify[Expand[Coefficient[E1,ϕ2^(2,0,0,0)[t,r,θ,ϕ]]],r>R>0];
β4=Simplify[Expand[Coefficient[E1,ϕ2^(0,1,0,0)[t,r,θ,ϕ]]],r>R>0];
β5=Simplify[Expand[Coefficient[E1,ϕ2^(0,2,0,0)[t,r,θ,ϕ]]],r>R>0];
β6=Simplify[Expand[Coefficient[E1,ϕ2^(0,0,1,0)[t,r,θ,ϕ]]],r>R>0];
β7=Simplify[Expand[Coefficient[E1,ϕ2^(0,0,2,0)[t,r,θ,ϕ]]],r>R>0];
β8=Simplify[Expand[Coefficient[E1,ϕ2^(0,0,0,1)[t,r,θ,ϕ]]],r>R>0];
β9=Simplify[Expand[Coefficient[E1,ϕ2^(0,0,0,2)[t,r,θ,ϕ]]],r>R>0];
δ1=Simplify[Expand[Coefficient[E2,ϕ1[t,r,θ,ϕ]]],r>R>0];
δ2=Simplify[Expand[Coefficient[E2,ϕ1^(1,0,0,0)[t,r,θ,ϕ]]],r>R>0];
δ3=Simplify[Expand[Coefficient[E2,ϕ1^(2,0,0,0)[t,r,θ,ϕ]]],r>R>0];
δ4=Simplify[Expand[Coefficient[E2,ϕ1^(0,1,0,0)[t,r,θ,ϕ]]],r>R>0];
δ5=Simplify[Expand[Coefficient[E2,ϕ1^(0,2,0,0)[t,r,θ,ϕ]]],r>R>0];
δ6=Simplify[Expand[Coefficient[E2,ϕ1^(0,0,1,0)[t,r,θ,ϕ]]],r>R>0];
δ7=Simplify[Expand[Coefficient[E2,ϕ1^(0,0,2,0)[t,r,θ,ϕ]]],r>R>0];
δ8=Simplify[Expand[Coefficient[E2,ϕ1^(0,0,0,1)[t,r,θ,ϕ]]],r>R>0];
δ9=Simplify[Expand[Coefficient[E2,ϕ1^(0,0,0,2)[t,r,θ,ϕ]]],r>R>0];
ϵ1=Simplify[Expand[Coefficient[E2,ϕ2[t,r,θ,ϕ]]],r>R>0];
ϵ2=Simplify[Expand[Coefficient[E2,ϕ2^(1,0,0,0)[t,r,θ,ϕ]]],r>R>0];
ϵ3=Simplify[Expand[Coefficient[E2,ϕ2^(2,0,0,0)[t,r,θ,ϕ]]],r>R>0];
ϵ4=Simplify[Expand[Coefficient[E2,ϕ2^(0,1,0,0)[t,r,θ,ϕ]]],r>R>0];
ϵ5=Simplify[Expand[Coefficient[E2,ϕ2^(0,2,0,0)[t,r,θ,ϕ]]],r>R>0];
ϵ6=Simplify[Expand[Coefficient[E2,ϕ2^(0,0,1,0)[t,r,θ,ϕ]]],r>R>0];
ϵ7=Simplify[Expand[Coefficient[E2,ϕ2^(0,0,2,0)[t,r,θ,ϕ]]],r>R>0];
ϵ8=Simplify[Expand[Coefficient[E2,ϕ2^(0,0,0,1)[t,r,θ,ϕ]]],r>R>0];
ϵ9=Simplify[Expand[Coefficient[E2,ϕ2^(0,0,0,2)[t,r,θ,ϕ]]],r>R>0];

(*where the explicit form of funcions a, b, γ is *)

a[r]=Sqrt[r^3/(r^3+r^2 R+r R^2+R^3)];
a'[r]=D[a[r],r];
a''[r]=D[a'[r],r];
b[r]=Sqrt[R(r^2+r R+R^2)/(r^3+r^2 R+r R^2+R^3)];
b'[r]=D[b[r],r];
b''[r]=D[b'[r],r];
γ[r]=Sqrt[1-R/r];
γ'[r]=D[γ[r],r];
γ''[r]=D[γ'[r],r];

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(*Using these coefficients one can construct the following matrices*)
II:={{1,0},{0,1}};
A1:={{α5,β5},{β5,-α5}}//Simplify;
B1:={{α4,β4},{β4,-α4}}//Simplify;
C1:={{α1-λ α7,β1-λ β7},{β1-λ β7,ϵ1+λ α7}}//Simplify;
D1:={{α3,β3},{β3,-α3}}//Simplify;
C2:={{α1-λ α7,β1-λ β7},{β1-λ β7,-α1+λ α7}}//Simplify;
Z1={{0,0},{0,α1+ϵ1}}//Simplify;
C11=C2+Z1//Simplify;
A=-Inverse[D1].A1//Simplify;
B:=-Inverse[D1].B1//Simplify;
F:=-Inverse[D1].C2//Simplify;
Z=-Inverse[D1].Z1//Simplify;
A2=A;
B2=((1/r)A+B);
C3=-(A/(4 r^2))+(B/(2r))+F+Z;
fg=Simplify[Sqrt[((r-R)^2 (r^2+r R+R^2))/(3 r R^3)],r-R>0];
H1=-Simplify[Expand[1/(2 fg)(B2/fg-D[fg,r]II)],{r>R>0}];
h=B2/fg-D[fg,r]II//Simplify;
M={{Cos[h12[r]],Sin[h12[r]]},{-Sin[h12[r]],Cos[h12[r]]}};

(*Considering these matrices the potential matrix can be written as *)
V=Simplify[(h.h)/4+fg D[h,r]/2-Inverse[M].C3.M,r>R>0];

(*Here we calculate the limit of the matrix potential in the space boundaries *)
Limit[(V/.h12[r]→d),r→∞]

Limit[(V/.h12[r]→d),r→R]

(*It is convenient to make a change of variables from r to 1/z and from R to 1*)

B4=fg(h-2 ⅈ ω II+D[fg,r] II)//Simplify;
C4=C3-ⅈ ω h//Simplify;
fgz=fg/.{r→1/z,R→1};
B4z=B4/.{r→1/z,R→1};
C4z=Simplify[C4/.{r→1/z,R→1},0<z<1];
Az=Simplify[z^4 fgz^2,0<z<1];
Bz=Simplify[2 z^3 fgz^2-z^2 B4z,0<z<1];

(*If we make the previous change of variable, the equations of motion are*)

Eq1=Simplify[Az D[ϕ1[z],z,z]+Bz⟦1,1⟧×D[ϕ1[z],z]+Bz⟦1,2⟧×D[ϕ2[z],z]+C4z⟦1,1⟧×ϕ1[z]+C4z⟦1,2⟧×ϕ2[z],0<z<1];
Eq2=Simplify[Az D[ϕ2[z],z,z]+Bz⟦2,2⟧×D[ϕ2[z],z]+Bz⟦2,1⟧×D[ϕ1[z],z]+C4z⟦2,2⟧×ϕ2[z]+C4z⟦2,1⟧×ϕ1[z],0<z<1];

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(*Here we expand the equation of motion in Tylor series*)
SERIES[ω0_,λ0_,q0_,φ0_,ORDH_,z0_]:=(                              (*λ0 is l(l+1) where l is angular momentum of the deformation*)

param={};
param={ω→ω0,qh[0]→q0,φh[0]→φ0,λ→λ0};
ϕ1[z_]:=Sum[qh[i](z-1)^i,{i,0,ORDH+20}]//.param;
ϕ2[z_]:=Sum[φh[i](z-1)^i,{i,0,ORDH+20}]//.param;
Eq1;
Eq2;
ss=Series[{Eq1,Eq2}//.param,{z,1,ORDH}];
yh=Union[Table[qh[i],{i,1,ORDH}],Table[φh[i],{i,1,ORDH}]];
eqs=Union[Table[SeriesCoefficient[ss[[1]],i]==0,{i,1,ORDH}],Table[SeriesCoefficient[ss[[2]],i]==0,{i,1,ORDH}]];
systH=Solve[eqs,yh][[1]];

qn=Sum[qh[i](z-1)^i,{i,0,ORDH}]//.Union[param,systH];
φn=Sum[φh[i](z-1)^i,{i,0,ORDH}]//.Union[param,systH];
{qn,φn}/.z→z0
);


DET[ω0_,λ0_,ORDH_,z0_]:=(                     (*This is te determinat of the matrix S*)
S10=SERIES[ω0,λ0,1,0,ORDH,z0];
a11=S10[[1]];
a12=S10[[2]];
S01=SERIES[ω0,λ0,0,1,ORDH,z0];
a21=S01[[1]];
a22=S01[[2]];
Det[{{a11,a12},{a21,a22}}]
);
(*We build a loop where at each turn we solve the equation detS=0, finding a
frequency that tends to a value the larger the order of expansion *)

xi=20;
xf=101;
dim=xf-xi;
dx=(xf-xi)/dim;
TT=Table[Null,{dim}];
xx=xi;
λ00=2;
Xg=0.4+.1 i(*guess frequency*);
Monitor[
    For[ii=1,ii<dim+1,
    (****)
    ff[XX_?NumberQ]:=DET[XX,λ00,xx,10^(-10)];
    Z=FindRoot[ff[XX],{XX,Xg}];
    Xg=XX/.Z;
    check=Abs[ff[Xg]];
    TT[[ii]]={N[xx],Xg,check};
    ii++;
    xx=xx+dx],{ii,N[xx],Xg,check}];

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APPENDIX

D

INTEGRABILITY CONDITIONS

Depending on the viewpoint, we have integrability conditions to describe the geometry of an extended object at a fixed time, Σ_t, once this undergo an ADM split.

D.0.1 Σ_t embedded in m

If Σ_t is embedded into m, $x^a = X^a(u^A)$, with u^A being the local coordinates in Σ_t and $A = 1,2,\dots,p.$, the orthonormal basis is provided by $\{\epsilon^a{}_A = \partial_A X^a, \xi^a\}$. This satisfies $g_{ab}\epsilon^a{}_A\xi^b = 0$, $g_{ab}\xi^a\xi^b = -1$ and $g_{ab}\epsilon^a{}_A\epsilon^b{}_B = h_{AB}$ where h_{AB} is the spacelike metric associated to Σ. The corresponding Gauss-Weingarten (GW) equations are

$$
\begin{aligned}
\nabla_A\epsilon^a{}_B &= \Gamma^C_{AB}\epsilon^a{}_C + k_{AB}\xi^a, \\
\nabla_A\xi^a &= k_{AB}h^{BC}\epsilon^a{}_C,
\end{aligned}
\tag{D.1}
$$

where $\nabla_A = \epsilon^a{}_A\nabla_a$, $k_{AB} = k_{BA}$ is the extrinsic curvature of Σ_t associated to the normal ξ^a and Γ^C_{AB} stands for the connection compatible with h_{AB}.

The intrinsic and extrinsic geometries for the embedding under consideration must satisfy the integrability conditions

$$\mathscr{R}_{abcd}\epsilon^a{}_A\epsilon^b{}_B\epsilon^c{}_C\epsilon^d{}_D = R_{ABCD} - k_{AD}k_{BC} + k_{AC}k_{BD}, \tag{D.2a}$$

$$\mathscr{R}_{abcd}\epsilon^a{}_A\epsilon^b{}_B\epsilon^c{}_C\xi^d = \mathscr{D}_A k_{BC} - \mathscr{D}_B k_{AC}, \tag{D.2b}$$

where R_{ABCD} is the Riemann tensor associated to the spacelike manifold Σ_t and $\mathscr{D}_A$ is the covariant derivative compatible with h_{AB}.

D.0.2 Σ_t embedded in $\mathscr{M}$

If Σ_t is embedded into $\mathscr{M}$, $x^\mu = X^\mu(u^A)$, the orthonormal basis is provided by $\{\epsilon^\mu{}_A = \partial_A X^\mu, \xi^\mu, n^\mu{}_i\}$. This satisfies $\epsilon_A \cdot \xi = \epsilon_A \cdot n_i = \xi \cdot n_i = 0$, $\xi \cdot \xi = -1$, $n_i \cdot n_j = \delta_{ij}$ and $\epsilon_A \cdot \epsilon_B = h_{AB}$. The corresponding GW equations are

$$
\begin{aligned}
D_A \epsilon^\mu{}_B &= \Gamma^C_{AB} \epsilon^\mu{}_C + k_{AB} \xi^\mu - L_A{}^i n^\mu{}_i, \\
D_A \xi^\mu &= k_{AB} h^{BC} \epsilon^\mu{}_C - L_A{}^i n^\mu{}_i, \\
D_A n^{\mu i} &= L^i_{AB} h^{BC} \epsilon^\mu{}_C - L_A{}^i \xi^\mu + \sigma_A{}^{ij} n^\mu{}_j,
\end{aligned} \tag{D.3}
$$

where $D_A = \epsilon^\mu{}_A \mathscr{D}_\mu$ and $\mathscr{D}_\mu$ being the background covariant derivative, $L^i_{AB} = L^i_{BA}$ is the extrinsic curvature of Σ_t associated to the normal $n^\mu{}_i$. Additionally, we have introduced $L_A{}^i := \epsilon^a{}_A \xi^b K^i_{ab}$ and $\sigma_A{}^{ij} := \epsilon^a{}_A \omega_a{}^{ij}$. Observe that the tangent-normal projection of the worldvolume extrinsic curvature is in fact a piece of a non-trivial twist potential given by $L_A{}^i = n^i \cdot D_A \xi$.

The intrinsic and extrinsic geometries for the embedding under consideration must satisfy the integrability conditions

$$0 = -R_{ABCD} - k_{AC} k_{BD} + k_{BC} k_{AD} + L^i_{AC} L_{BD\,i} - L^i_{BC} L_{AD\,i}, \tag{D.4a}$$

$$0 = \mathscr{D}_A k_{BC} - \mathscr{D}_B k_{AC} + L_A{}^i L_{BC\,i} - L_B{}^i L_{AC\,i}, \tag{D.4b}$$

$$0 = \widetilde{\mathscr{D}}_A L^i_{BC} - \widetilde{\mathscr{D}}_B L^i_{AC} + L_A{}^i k_{BC} - L_B{}^i k_{AC}, \tag{D.4c}$$

$$0 = \widetilde{\mathscr{D}}_A L_B{}^i - \widetilde{\mathscr{D}}_B L_A{}^i + L_A{}^{Ci} k_{BC} - L_B{}^{Ci} k_{AC}, \tag{D.4d}$$

$$0 = -\Omega^{ij}_{AB} + L_A{}^{Ci} L^j_{BC} - L_B{}^{Ci} L^j_{AC} - L_A{}^i L_B{}^j + L_B{}^i L_A{}^j, \tag{D.4e}$$

where $\Omega^{ij}_{AB} := \widetilde{\mathscr{D}}_B \sigma_A{}^{ij} - \widetilde{\mathscr{D}}_A \sigma_B{}^{ij}$ is the curvature tensor associated with the gauge field $\sigma_A{}^{ij}$ and $\widetilde{\mathscr{D}}_A$ is the $O(N-p-1)$ covariant derivative acting on the normal indices associated with the connection $\sigma_A{}^{ij}$.

APPENDIX E

SINGULAR SECOND-ORDER SYSTEMS

E.1 Dirac Method

In this section we discuss brieflly the Dirac Method for Hamiltonian systems with constrains in a field theory, following the references [37, 38, 40, 58, 61] .
We consider the action

$$S\left[\Phi^A(x)\right] = \int dtL = \int dtd^p x \mathscr{L}\left(\Phi^A, \partial_a \Phi^A\right), \tag{E.1}$$

where L is the Lagrangian of the system, $\mathscr{L}$ is the density Lagrangian and, Φ^A are the field variables. Define the conjugate canonical momentum associated to fields Φ^A as

$$P_A = \frac{\delta L}{\delta \dot{\Phi}^A}, \tag{E.2}$$

where the dot over P_A represents the derivative concerning the parameter t. Performing a Legandre transformation ,one can write the action in (E.1) in term of a Hamiltonian density $\mathscr{H}_C = P_A \dot{\Phi}^A - \mathscr{L}$,

$$S = \int dtd^p x \left(P_A \dot{\Phi}^A - \mathscr{H}_C\right) = \int dtH_c, \tag{E.3}$$

where $Hc = \int d^p x \mathscr{H}$ is the canonical Hamiltonian.
By defining the Poisson brackets of two functions, A and B, of phase space as

$$\{A,B\} = \frac{\delta A}{\delta \Phi^A}\frac{\delta B}{\delta P_A} - \frac{\delta A}{\delta P_A}\frac{\delta B}{\delta \Phi^A}. \tag{E.4}$$

We see that H_c generates the temporal evolution of canonical variables, i.e.

$$\begin{aligned}\dot{\Phi}^A &= \{\Phi^A, H_c\} = \frac{\delta H_c}{\delta P_A},\\ \dot{P}_A &= \{P_A, H_c\} = -\frac{\delta H_c}{\partial \Phi^A}.\end{aligned} \tag{E.5}$$

We are interested in working with singular systems. In this kind of systems the determinant of Hessian matrix, $\mathscr{H}_{AB}$, is equal to zero

$$\det|\mathscr{H}_{AB}| = \det\left|\frac{\delta^2 L}{\partial\dot{\Phi}^A \partial\dot{\Phi}^B}\right| = 0, \tag{E.6}$$

this implies that accelerations $\dot{\Phi}^A$ cannot be only determined by fields Φ^A and its derivatives. Additional, the vanishing of this determinant leads to not all linear momenta being independent and there are some relations

$$\varphi_m\left(\Phi^A, P_A, \partial_i\Phi^A, \partial_i P_A\right) \approx 0 \quad m = 1, \cdots, M. \tag{E.7}$$

They are called primary constrictions for emphasizing that one does not use the equation of motion to arrive at them but only momenta definition (E.2). Here, ∂_i denotes the spacial derivative and the symbol $\approx$ reads as weekly zero and means that is equal to zero on the sub-manifold, Γ, of phase space that is defined by the primary constrains.
Consider there are M' independent relations of the form (E.2). So M' rows of Hessian matrix must be null, therefore

$$M' = N - \text{Rank}(\mathscr{H}_{AB}). \tag{E.8}$$

Assuming that the rank of the Hessian matrix is constant, this implies that the number of primary constraints is fixed. On the other hand, since there are no restrictions to (E.2) that demand that they are independent, in principle, $M' \leq M$. Moreover, independent constraints, $\varphi_{m'}$ do not have to come directly from (E.2), as it is for φ_m. They can be combinations of these latter. So it is essential to calculate the null vector of the Hessian matrix. By denoting, $V^m_{m'}$ as the components of null vectors and φ_m as the primary constraints, then we have the primary independent constraints are written as follows

$$\varphi_{m'} = V^m_{m'}\varphi_m. \tag{E.9}$$

The Hamiltonian H_c is not unique and it has to be replaced by a effective Hamiltonian $\tilde{H}$, called primary Hamiltonian [61],

$$\tilde{H} = H_c + u^m\varphi_m \approx H_c, \tag{E.10}$$

$\tilde{H}$ generates new equation of motion

$$\begin{aligned}\dot{\Phi}^A &= \{\Phi^A, \tilde{H}\} = \frac{\delta \tilde{H}}{\delta P_A} + u^m \frac{\delta \varphi_m}{\delta P_A},\\ \dot{P}_A &= \{P_A, \tilde{H}\} = -\frac{\delta \tilde{H}}{\partial \Phi^A} - u^m \frac{\delta \varphi_m}{\delta \Phi^A},\\ \phi_m &= 0,\end{aligned} \tag{E.11}$$

these equations of motion are consistent with variation $\delta\Phi^A$ and δP_A because they preserve the constrains $\varphi_m \approx 0$. Now consistency conditions have to be applied to avoid that when the system evolves, the sub-manifold Γ changes. Thus, one assumes that

$$\dot{\varphi}_n = \{\varphi_n, \tilde{H}\} \approx \{\varphi_n, H_c\} + u^m \{\varphi_n, \varphi_m\} \approx 0, \tag{E.12}$$

the equation in (E.12) can be represented as a linear system equation

$$h_n + W_{nm} u^m \approx 0, \tag{E.13}$$

or in a matrix form as

$$h + W u \approx 0, \tag{E.14}$$

here h and u are vectors and W is a matrix. There are the following cases for this matrix system:

- (i) If $h \neq 0$ and $\det W \neq 0$, we can determine all Lagrangian multipliers since there exists the inverse of W, denoted by W^{-1}, and then

$$u^m = -\left(W^{-1}\right)^{mn} h_n. \tag{E.15}$$

- (ii) If $h \neq 0$ and $\det W = 0$. In this case, the number of multipliers, u^m, that can be determined is equal to the rank of the matrix W, $K = \text{rank}(W)$. In contrast, N' multipliers cannot be determined, where N' is the nullity of matrix W. If one calculates the null vector, V^i $(i = 1, \cdots, N')$, of matrix W. By definition, they satisfy $WV^i \approx 0$. This implies that

$$hV^i \approx 0. \tag{E.16}$$

these are new relations that are satisfied by the canonical variables, i.e., there are new constrains. They are known as secondary constraints.

- (iii) If $h = 0$ and $\det W \neq 0$. Taking into account the equation (E.14) , we directly obtain

  (E.17) $$Wu \approx 0,$$

  the only solution to the last equation is the trivial solution, leading to $\tilde{H} = H_c$.

- (iv) If $h = 0$ and $\det W = 0$. For this case, like the previous one, we have

  (E.18) $$Wu \approx 0.$$

  However, due to the fact that $\det W = 0$, only some multipliers can be weakly determined. Noteworthy that the fact that $h = 0$ could come from null canonical Hamiltonian.

By virtue of consistency conditions, secondary constraints must also be preserved over time. This will involve iterating over the previous steps applied to the secondary constraints. This algorithm must be repeated if new restrictions appear until no new ones. This process ends after a finite number of iterations. Once the information from the consistency relations on the primary constraints is exhausted and with the addition of secondary constraints, a total Hamiltonian can be constructed, which will contain information on all the constraints found so far, both primary and secondary, this will be

(E.19) $$H_T := H_c + u^p \varphi_p,$$

where φ are all constraints, and the equations of motion take the following form

(E.20) $$\begin{aligned} \dot{\Phi}^A &\approx \{\Phi^A, H_T\}, \\ \dot{P}_A &\approx \{P_A, H_T\}. \end{aligned}$$

Taking the consistency conditions

(E.21) $$\dot{\varphi}_p \approx \{\varphi_p, H_T\} \approx \{\varphi_p, H_c\} + u^q \{\varphi_p, \varphi_q\} \approx 0,$$

where $p, q = 1, \cdots, T$ with T the number of all constraints. As we have already mentioned, the consistency relations give a linear equations system to multipliers u, whose general solution can be written as follows

(E.22) $$u^q = U^q + V^q,$$

U^q is the particular solution to the inomogeneous equations and V^q is the general solution to homogeneous system,

(E.23) $$V^q \{\varphi_p, \varphi_q\} \approx 0.$$

In addition, V^q can be written as a linear combination of independent solutions of homogeneous system, i.e., $V^q = v^i V_i^q$, where $i = 1, \cdots, k$, where k is the number of independent solutions of homogeneous system, then

$$u^q = U^q + v^i V_i^q. \tag{E.24}$$

since v^i are arbitrary, then u^q can be separated in two parts, one is fixed by consistency conditions and the another is indeterminate. This must be reflected in the total Hamiltonian and, of course, in the equations of motion. By substituting (E.24) in the total Hamiltonian, we have

$$H_T = H_c + U^q \phi_q + v^i \varphi_i, \tag{E.25}$$

where $\varphi_i = V_i^q \varphi_q$. If we write the temporal evolution of any function, F, of phase space taking into account (E.24), we get

$$\dot{F} \approx \{F, H_T\} \approx \{F, H'\} + v^i \{F, \varphi_i\}, \tag{E.26}$$

where we have defined $H' := H_c + U^q \varphi_q$. These equations have k arbitrary functions and are equivalent to the Lagrangian equations. The fact that arbitrary functions appear makes a fundamental difference since the initial conditions no longer have a unique evolution but remain indeterminate until arbitrary functions.

E.1.1 First and second class constraints

Until now, it has not been necessary to make a substantial difference between primary and secondary constraints since they are treated at the same level. Notwithstanding, it will be handy to split the constraints into first and second class.
Let be a functional of the space phase, F. It is first class if its Poisson bracket with all constraints is weakly zero, i.e.

$$\{F, \varphi_p\} \approx 0. \tag{E.27}$$

On the contrary F is a second class constraint. Remember $p = 1, \cdots, j$ and j the total number of constraints.
If F is a first class constraint, then $\{F, \varphi_p\}$ must be strongly equal to zero to any linear combination of constraints φ because these are the only independent quantities that are weakly zero, therefore

$$\{F, \varphi_p\} = f_p^q \varphi_q. \tag{E.28}$$

By using the definitions of a first or second class constrain, and taking the following Poisson bracket

$$\{\varphi_i, \varphi_p\} = \{V_i^j \varphi_j, \varphi_p\} = V_i^j \{\varphi_j, \varphi_p\} + \{V_i^j, \varphi_p\}\varphi_j \approx V_i^j \{\varphi_j, \varphi_p\}, \tag{E.29}$$

taking into account that V_i^j is a solution of homogeneous system equation for multipliers u, this implies that

$$\{\varphi_i, \varphi_p\} \approx 0, \tag{E.30}$$

thus, $\varphi_i = V_i^j \varphi_j$ is a first class constraint. One can proof that H' is also a first class constraint.
We denote the first class constraint as γ, and the second class are indicated by χ. Keep in mind that the first-class constraints do not have to be directly one of the primary or secondary constraints. In general, they can be combinations of these. To find them correctly, it will be necessary to calculate the null vectors of the matrix whose entries are the Poisson brackets between the constraints. Later, one contracts null vectors with the j constraints, and finally, one will find the correct first-class constraints. For the above, let us look at the matrix,

$$W'_{pq} = \{\varphi_p, \varphi_q\}, \tag{E.31}$$

this has dimension $j \times j$ and φ_p are all constraints both secondary and primary. If $\det W' \approx 0$, then there are $j - R$ first class constraints (R is the rank of W').

Proof:

If $\det W' = 0$ in Γ, then the nullity of W' must be $j - R$. Therefore, $j - R$ null vectors can be found, we denote them by ω_i with $i = 1, \cdots, j - R$, such that

$$\omega_i^p \{\varphi_p, \varphi_q\} \approx 0, \tag{E.32}$$

implying

$$\{\omega_i^p \varphi_p, \varphi_q\} \approx 0, \tag{E.33}$$

evidently, the constraints $\omega_i^p \varphi_p$ form a set of $j - R$ first class constraints. Thus, all first class constrains of a theory can be written as

$$\gamma_i := \omega_i^p \varphi_p. \tag{E.34}$$

It is a systematic way to find the first class constraints. On the another hand, the matrix W_{pq} can be written in the following way

$$W_{pq} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{\alpha\beta} \end{pmatrix}_{j\times j}, \tag{E.35}$$

where $C_{\alpha,\beta} = \{\chi_\alpha, \chi_\beta\}$, this is a $R \times R$ matrix, R is the number of second class constraints and it coincides with the rank of the matrix W'. In addition, one can proof that R must be even.

E.1.2 Gauge Transformations

Up to this point, ways to find the multipliers, u, associated with the theory have been addressed. However, the presence of such arbitrary multipliers, either in the equations of motion or in the solutions, implies that the phase space variables q^i and p_i can not be uniquely determined from initial conditions, so they have no physical meaning. However, these are deterministic theories at the classical level, so two states that share the same initial conditions should not be physically different. When two different functions provide the same physical state, it said they are related under a gauge transformation.
Consider a dynamical variable F, with initial value F_0, its value at time δt is

$$F(\delta t) = F_0 + \delta t \left[\{F, H'\} + v^i \{F, \gamma_i\} \right], \tag{E.36}$$

remember that v^i are arbitrary, suppose we take another value of it. That would give a different $F'(\delta t)$, the difference being

$$\Delta F(\delta t) = F(\delta t) - F'(\delta t) = \delta t \left(v^i - v'^i \right) \{F, \gamma_i\} = \delta v^i \{F, \gamma_i\}, \tag{E.37}$$

the last expression represents a transformation of dynamical variable F, where generating functions are the first class constraints γ_i. Dirac postulated that all first class constrains are generating functions of transformations, that lead to changes in the canonical variables Φ^A and P_A, that do not affect the physical state. They are known as gauge transformations.
Now if we apply two gauge transformation with generating functions γ_i and γ'_i and after we apply the same transformation in the reverse order and calculate the difference, we obtain

$$\Delta F = v^i v'^{i'} \left[\{\{F, \gamma_i\}, \gamma'_{i'}\} - \{\{F, \gamma'_{i'}\}, \gamma_i\} \right] = v^i v'^{i'} \{F, \{\gamma_i, \gamma'_{i'}\}\}, \tag{E.38}$$

taking into account a theorem which proof that Poisson bracket of two first class constraints is also a first class constraint and the Dirac postulate, we arrive that Poisson bracket of two first class constraints also generates a gauge transformation.

E.1.3 Degrees of Freedom Count

With all concepts that have been given in this section, it is possible to count of degrees of freedom (DOF) of the system. Remember that DOF of a system is the number of physical independent variables that are necessary to describe it.
Taking as a guide the count in the elemental mechanics, the number of generalized coordinates minus number of independent restrictions. Extrapolating to phase space, the count of DOF is

$$\#DOF = \frac{1}{2}\left[(\#\text{CV}) - (\#\text{SCC}) - 2(\#\text{FCC})\right], \tag{E.39}$$

where the number of canonical variables, second class constraints, first class constraints are denoted by #CV ,#SCC ,#FCC respectively. The factor 1/2 appears to compensate that the DOF only refer to the generalized coordinates, the number 2 accompanies the first-class restrictions because they have a double role, both as a constraint and generator of a gauge transformation.

E.1.4 Dirac Bracket

As we have mentioned, the total hamiltonian H_T contains all system constraints. However, there is no distinction between first and second class constraints when one writes this Hamiltonian. Writing a new function called Extended Hamiltonian ($H_E = H' + v^i \gamma_i$), which takes an explicit separation between first and second constraints, the evolution of a function of phase space is given by

$$\dot{F} \approx \{F, H_E\}. \tag{E.40}$$

Additionally, one can build called Dirac bracket. For this consider two function of phase space, F_1 and F_2, we define Dirac bracket as follow

$$\{F_1, F_2\}_D = \{F_1, F_2\} - \{F_1, \chi_\alpha\} \left(C^{-1}\right)^{\alpha\beta} \{\chi_\beta, F_2\}. \tag{E.41}$$

The equation of motion of dynamic variable can be written using Dirac bracket as follows

$$\dot{F} = \{F, H_E\}_D, \tag{E.42}$$

similarly, a gauge transformation of a function F can be rewritten as

$$\{F,\gamma_i\} \approx \{F,\gamma_i\}_D, \tag{E.43}$$

Dirac bracket plays a crucial role in the quantization of a gauge theory since it is the one that is promoted to a commutator instead of Poisson bracket.

E.2 Ostrogradsky-Hamilton Approach

The Hamiltonian formalism, used to study dynamic systems, is restricted to systems that depend solely on first derivatives. Ostrogradsky expanded this analysis to higher-order systems [83]. He discovered an instability in such systems, now known as Ostrogradsky instability, because the Hamiltonian is unbounded below. Consequently, negative energy values can arise, making it impossible to find a minimum energy state. These higher-order models are flawed at the quantum level due to the existence of ghost states with negative norms, which can potentially violate unitarity [59]. However, they possess appealing properties, such as improved convergence of Feynman diagrams. Consequently, higher-order theories such as [43, 65, 93] have gained significant interest.
In this section, we demonstrate how to utilize the Ostrogradsky method, following the references [79, 111, 112].
Consider a system with an action given by

$$S[\Phi^A] = \int dtL = \int dtd^px\mathscr{L}\left(\Phi^A,\partial_i\Phi^A,\partial_i^2\Phi^A,\dot{\Phi}^A,\ddot{\Phi}^A\right). \tag{E.44}$$

To conduct a Hamiltonian analysis, it is possible to apply a Legendre transformation. In this particular case, it is necessary to take into account $4n$ canonical coordinates due to the dependence of the Lagrangian function on second-order time derivatives. They are given in the following way

$$\begin{aligned} \Phi_1^A &= \Phi^A, \quad p_A = \frac{\delta L}{\delta\dot{\Phi}^A} - \dot{P}_A, \\ \Phi_2^A &= \dot{\Phi}^A, \quad P_A = \frac{\delta L}{\delta\ddot{\Phi}^A}, \end{aligned} \tag{E.45}$$

the Hamiltonian function can be written as follows

$$H_c = \Phi_2^A p_A + \dot{\Phi}_2^A P_A - L. \tag{E.46}$$

It is important to observe that the final Hamiltonian function exhibits linearity with respect to the momentum p_A. This characteristic indicates the instability of the system,

as p_A can assume negative values, resulting in the absence of a lower bound for the energy.
The equation of motion can be expressed using this canonical Hamiltonian function in the following manner:

$$\dot{\Phi}^A = \frac{\partial H_c}{\partial p_A}, \qquad \ddot{\Phi}^A = \frac{\partial H_c}{\partial P_A}, \\ \dot{p}_A = -\frac{\partial H_c}{\partial \Phi^A}, \quad \dot{P}_A = -\frac{\partial H_c}{\partial \dot{\Phi}^A}, \tag{E.47}$$

in a similar manner to a first-order system, we can establish a Poisson bracket in this case as well and utilize it to formulate the equations of motion. Let's consider two functions, F and G, defined on the phase space. The Poisson bracket between them is defined as follows

$$\{F,G\} = \frac{\delta F}{\delta \Phi^A}\cdot\frac{\delta G}{\delta p_A} + \frac{\delta F}{\delta \dot{\Phi}^A}\cdot\frac{\delta G}{\delta P_A} - \frac{\delta F}{\delta p_A}\cdot\frac{\delta G}{\delta \Phi^A} - \frac{\delta F}{\delta P_A}\cdot\frac{\delta G}{\delta \dot{\Phi}^A}. \tag{E.48}$$

Therefore the equation of motion can be written as

$$\dot{\Phi}^A = \{\Phi^A, H_c\}, \quad \ddot{\Phi}^A = \{\dot{\Phi}^A, H_c\}, \\ \dot{p}_A = \{p_A, H_c\}, \quad \dot{P}_A = \{P_A, H_c\}. \tag{E.49}$$

Thus far, we have made the assumption that the Lagrangian function is non-degenerate, meaning that the determinant of the Hessian matrix is nonzero. Under this condition, it is not possible to eliminate the second-time derivative of any field Φ^A through integration by parts.
If the second-order system is singular, it will involve constraints. Similar to a first-order system, it is not possible to express all accelerations $\ddot{\Phi}^A$ solely in terms of canonical variables p_A, P_A, Φ^A, and $\dot{\Phi}^A$. Consequently, some equations of the form

$$P_A = \frac{\delta L}{\delta \ddot{\Phi}^A}, \tag{E.50}$$

imply relations between canonical variables that we can write as follows

$$\varphi_m\left(P_A, p_A, \Phi^A, \dot{\Phi}^A\right) = 0, \tag{E.51}$$

Here, the variables φ_m represent the primary constraints. At this stage, the Dirac algorithm can be applied in a similar manner to the approach employed in the previous section for first-order systems. As a result, primary, secondary, first-class, and second-class constraints can be obtained. It is important to consider the Poisson bracket in equation (E.48) when constructing the Dirac bracket. Thus,

$$\#DOF = \frac{1}{2}\left[(\#\mathrm{CV}) - (\#\mathrm{SCC}) - 2(\#\mathrm{FCC})\right], \tag{E.52}$$

By employing an extension of the Dirac algorithm, it is possible to perform the corresponding Hamiltonian analysis for singular second-order systems.
Dirac's technique is particularly suitable for investigating the Hamiltonian formulation of a theory where states are defined on a general spacelike surface, even in higher-order systems. The fundamental concept involves introducing a set of curvilinear spacetime coordinates, $x^a = (x^0, \cdots, x^n)$, into the theory. The equation $x^0 = \text{cte}$ defines a generic spacelike surface, while also introducing arbitrary coordinates on that surface. This step becomes necessary when working within a general Riemannian manifold where no natural choice for spatial coordinates is available. The insights gained from this comprehensive analysis will prove invaluable in the subsequent section, as we apply them to study the Regge-Teitelboim model as a second-order system.

APPENDIX

F

DICTIONARY OF VARIATION AND FUNTIONAL DERIVATIVES OF CONSTRAINTS

In this appendix we present the variation of important geometric objects in the study of Regge-Teitelboim à la ADM approach. In this approach, we take into account X^μ and $\dot{X}^\mu$ are independent variables.

F.1 Variation different geometric objects

The variation of the different geometric objects is given in terms of the variation of the variable set $\{X^\mu, \dot{X}\}$

$$\delta\xi^\mu = -(\xi\cdot\mathscr{D}_A\delta X)h^{AB}\epsilon^\mu{}_B + \left(\frac{n^j}{N}\cdot\delta\dot{X} - \frac{N^A}{N}n^j\cdot\mathscr{D}_A\delta X\right)n_j^\mu \tag{F.1}$$

$$\tilde{\delta}n^{\mu i} = -\left(n^i\cdot\mathscr{D}_A\delta X\right)h^{AB}\epsilon^\mu{}_B + \left(\frac{n^i}{N}\cdot\delta\dot{X} - \frac{N^A}{N}n^i\cdot\mathscr{D}_A\delta X\right)\xi^\mu \tag{F.2}$$

$$\delta h_{AB} = 2\epsilon_{(A}\cdot\mathscr{D}_{B)}\delta X \tag{F.3}$$

$$\delta h^{AB} = -2h^{AC}h^{BD}\epsilon_{(B}\cdot\mathscr{D}_{B)}\delta X \tag{F.4}$$

$$\delta h = 2hh^{AB}\epsilon_A\cdot\mathscr{D}_A\delta X \tag{F.5}$$

$$\delta N = N^A\eta\cdot\mathscr{D}_A\delta X - \xi\cdot\delta\dot{X} \tag{F.6}$$

$$\delta N^A = -2h^{A(C}N^{D)}(\epsilon\cdot\mathscr{D}_D\delta X) + h^{AB}\dot{X}\cdot\mathscr{D}_B\delta X + h^{AB}\epsilon_B\cdot\delta\dot{X} \tag{F.7}$$

$$\tilde{\delta}L_{AB}{}^i = -\frac{N^C}{N}\left(n^i\cdot\mathscr{D}_C\delta X\right)k_{AB} + \frac{1}{N}\left(n^i\cdot\delta\dot{X}\right)k_{AB} - n^i\cdot\mathscr{D}_A\mathscr{D}_B\delta X \tag{F.8}$$

$$\tilde{\delta}L^i = -2(\epsilon_B\cdot\mathscr{D}_A\delta X)L^{BAi} - \frac{N^C}{N}\left(n^i\cdot\mathscr{D}_C\delta X\right)k + \frac{1}{N}\left(n^i\cdot\delta\dot{X}\right)k - h^{AB}\left(n^i\cdot\mathscr{D}_A\mathscr{D}_B\delta X\right) \tag{F.9}$$

F.2 Functional derivatives of the constraints

F.2.1 functional derivatives of primary constraints

$$\frac{\delta \mathscr{S}_\lambda}{\delta X^\mu} = 0, \tag{F.10}$$

$$\frac{\delta \mathscr{S}_\lambda}{\delta \dot{X}^\mu} = \lambda P_\mu, \tag{F.11}$$

$$\frac{\delta \mathscr{S}_\lambda}{\delta p_\mu} = 0, \tag{F.12}$$

$$\frac{\delta \mathscr{S}_\lambda}{\delta P_\mu} = \lambda \dot{X}^\mu, \tag{F.13}$$

$$\frac{\delta \mathscr{V}_{\vec{\lambda}}}{\delta X^\mu} = -\partial_A(\lambda^A P_\mu) = -\mathscr{L}_{\vec{\lambda}} P_\mu, \tag{F.14}$$

$$\frac{\delta \mathscr{V}_{\vec{\lambda}}}{\delta \dot{X}^\mu} = 0 \tag{F.15}$$

$$\frac{\delta \mathscr{V}_{\vec{\lambda}}}{\delta p_\mu} = 0 \tag{F.16}$$

$$\frac{\delta \mathscr{V}_{\vec{\lambda}}}{\delta P_\mu} = \lambda^A \partial_A X^\mu = \mathscr{L}_{\vec{\lambda}} X^\mu, \tag{F.17}$$

$$\begin{aligned} \frac{\delta \mathscr{W}_{\vec{\phi}}}{\delta X^\mu} = &-\mathscr{D}_A\left(\phi^i \frac{\sqrt{h}}{N^2} N^A L_i \xi_\mu\right) + \mathscr{D}_A\left(\phi^i \frac{\sqrt{h}}{N} h^{AB} L_i \epsilon_{\mu B}\right) \\ &-\mathscr{D}_A\left(2\phi^i \frac{\sqrt{h}}{N} L_i^{AB} \epsilon_{\mu B}\right) - \mathscr{D}_A\left(\phi^i \frac{\sqrt{h}}{N^2} N^A k\, n_{\mu i}\right) \\ &+\mathscr{D}_A \mathscr{D}_B\left(\phi^i \frac{\sqrt{h}}{N} h^{AB} n_{\mu i}\right), \end{aligned} \tag{F.18}$$

$$\frac{\delta \mathscr{W}_{\vec{\phi}}}{\delta \dot{X}^\mu} = -\phi^i \frac{\sqrt{h}}{N^2} L_i \xi_\mu - \phi^i \frac{\sqrt{h}}{N^2} k\, n_{\mu i}, \tag{F.19}$$

$$\frac{\delta \mathscr{W}_{\vec{\phi}}}{\delta p_\mu} = 0 \tag{F.20}$$

$$\frac{\delta \mathscr{W}_{\vec{\phi}}}{\delta P_\mu} = \phi^i n^\mu{}_i, \tag{F.21}$$

F.2.2 functional derivatives of secondary constraints

(F.22) $$\frac{\delta S_\Lambda}{\delta X^\mu} = \mathscr{D}_A \widetilde{T}^A,$$

(F.23) $$\frac{\delta S_\Lambda}{\delta \dot{X}^\mu} = \Lambda p_\mu + 2\Lambda(P\cdot\mathscr{D}_A\dot{X})h^{AB}\epsilon_{\mu B} - \mathscr{D}_A(2\Lambda N^A P_\mu)$$

(F.24) $$-2\Lambda N h^{AB}(P\cdot\mathscr{D}_A\mathscr{D}_B X)\xi_\mu - 2\Lambda N^B(P\cdot\mathscr{D}_A\mathscr{D}_B X)h^{AC}\epsilon_{\mu C}$$

(F.25) $$-\frac{1}{2}\sqrt{h}\,\Lambda G^{ABCD}\Pi_{\alpha\beta}\mathscr{D}_A\mathscr{D}_B X^\alpha \mathscr{D}_C\mathscr{D}_D X^\beta \xi_\mu$$

(F.26) $$+\Lambda\sqrt{h}\,G^{ABCD}L^i_{AB}k_{CD}\,n_{\mu i} - \Lambda\sqrt{h}\,h^{AB}\delta_{ij}\left(\widetilde{\mathscr{D}}_A n^i\cdot\widetilde{\mathscr{D}}_B n^j\right)\xi_\mu$$

(F.27) $$-\left[\widetilde{\mathscr{D}}_A\left(2\Lambda N\sqrt{h}\,h^{AB}\delta_{ij}\widetilde{\mathscr{D}}_B n^j\right)\cdot\xi\frac{1}{N}\right]n_\mu{}^i,$$

(F.28) $$\frac{\delta S_\Lambda}{\delta p_\mu}, = \Lambda\dot{X}^\mu,$$

(F.29) $$\frac{\delta S_\Lambda}{\delta P_\mu} = 2\Lambda N^A\mathscr{D}_A\dot{X}^\mu + \Lambda(N^2h^{AB} - N^AN^B)\mathscr{D}_A\mathscr{D}_B X^\mu,$$

(F.30) $$\frac{\delta V_{\vec\Lambda}}{\delta X^\mu} = -\partial_A(\Lambda^A p_\mu) = -\mathscr{L}_{\vec\Lambda}p_\mu,$$

(F.31) $$\frac{\delta V_{\vec\Lambda}}{\delta \dot{X}^\mu} = -\partial_A(\Lambda^A P_\mu) = -\mathscr{L}_{\vec\Lambda}P_\mu,$$

(F.32) $$\frac{\delta V_{\vec\Lambda}}{\delta p_\mu} = \Lambda^A\partial_A X^\mu = \mathscr{L}_{\vec\Lambda}X^\mu,$$

(F.33) $$\frac{\delta V_{\vec\Lambda}}{\delta P_\mu} = \Lambda^A\partial_A \dot{X}^\mu = \mathscr{L}_{\vec\Lambda}\dot{X}^\mu,$$

(F.34) $$\frac{\delta W_{\vec\Phi}}{\delta X^\mu} = \mathscr{D}_A\widetilde{W}^A,$$

(F.35) $$\frac{\delta W_{\vec\Phi}}{\delta \dot{X}^\mu} = (P\cdot n_i)\widetilde{\mathscr{D}}_A\Phi^i h^{AB}\epsilon_{\mu B} + \frac{1}{N}\widetilde{\mathscr{D}}_A\left(\widetilde{\mathscr{D}}_B\Phi^i\sqrt{h}\,h^{AB}\right)n_{\mu i}$$
$$+\frac{\sqrt{h}}{N}\Phi_i L_A{}^i L^A{}_j\, n_\mu{}^j + \frac{1}{N}(p\cdot\xi)\Phi_i\, n_\mu{}^i$$
$$-\frac{1}{N}\left[N^A L_A{}^j(P\cdot n_j) + \alpha\sqrt{h}\,L_A{}^j L^A{}_j\right]\Phi_i\, n_\mu{}^i,$$

(F.36) $$\frac{\delta W_{\vec\Phi}}{\delta p_\mu} = \Phi^i n^\mu{}_i$$

(F.37) $$\frac{\delta W_{\vec\Phi}}{\delta P_\mu} = N^A\mathscr{D}_A(\Phi^i n^\mu{}_i) = \mathscr{L}_{\vec N}(\Phi^i n^\mu{}_i),$$

where

$$\begin{aligned}\widetilde{W}^A \;=\;& h^{AB}(p\cdot\epsilon_B)\Phi_i n_\mu{}^i + \frac{N^A}{N}(p\cdot\xi)\Phi_i n_\mu{}^i + \widetilde{\mathscr{D}}_B\Phi_i(P\cdot n^i)h^{AB}\dot{X}_\mu\\ &-\left[h^{AB}N^C L^j_{BC}(P\cdot n_j) + \sqrt{h}\,h^{AB}L^C{}_j L^j_{BC}\right]\Phi_i n_\mu{}^i + 2N^{(A}h^{B)C}(P\cdot n^i)\widetilde{\mathscr{D}}_C\Phi_i\epsilon_{\mu B}\\ &-\left[\frac{N^AN^B}{N}L_B{}^j(P\cdot n_j) + \sqrt{h}\frac{N^A}{N}L^B{}_j L_B{}^j\right]\Phi_i n_\mu{}^i + 2\sqrt{h}\,h^{B(C}L^{A)}{}_i\widetilde{\mathscr{D}}_B\Phi^i\epsilon_{\mu C}\\ &-\sqrt{h}\,h^{AB}L^C{}_i\widetilde{\mathscr{D}}_C\Phi^i\epsilon_{\mu B} + \sqrt{h}\,h^{AB}h^{CD}L^i_{BC}\widetilde{\mathscr{D}}_D\Phi_i\xi_\mu - \sqrt{h}\,k^{AB}\widetilde{\mathscr{D}}_B\Phi_i n_\mu{}^i\\ &+\sqrt{h}\frac{N^A}{N}h^{BC}\widetilde{\mathscr{D}}_B\widetilde{\mathscr{D}}_C\Phi_i n_\mu{}^i + \sqrt{h}\,h^{AB}L^C{}_j L^i_{BC}\Phi_i n_\mu{}^j + \sqrt{h}\frac{N^A}{N}L^B{}_j L_B{}^i\Phi_i n_\mu{}^j.\end{aligned}\tag{F.38}$$

and

$$
\begin{aligned}
\widetilde{T}^A \quad := \quad & -2\Lambda N h^{AB}(P\cdot\mathscr{D}_B\dot{X})\xi_\mu + 2\Lambda N^A h^{BC}(P\cdot\mathscr{D}_B\dot{X})\epsilon_{\mu C} - 2\Lambda N^A h^{BC} N(P\cdot\mathscr{D}_B\mathscr{D}_C X)\xi_\mu \\
+ \quad & 2\Lambda N^C h^{AB} N(P\cdot\mathscr{D}_B\mathscr{D}_C X)\xi_\mu - 2\Lambda N^B h^{CD} N^A (P\cdot\mathscr{D}_B\mathscr{D}_C X)\epsilon_{\mu D} \\
+ \quad & 2\Lambda N^2 h^{AC} h^{BD}(P\cdot\mathscr{D}_B\mathscr{D}_C X)\epsilon_{\mu D} + \mathscr{D}_B\left[\Lambda(N^2 h^{AB} - N^A N^B)P_\mu\right] \\
- \quad & \frac{1}{2}\Lambda\sqrt{h}N^A G^{BCDE}\Pi_{\alpha\beta}\mathscr{D}_B\mathscr{D}_C X^\alpha \mathscr{D}_D\mathscr{D}_E X^\beta \xi_\mu \\
- \quad & \frac{1}{2}\Lambda N\sqrt{h}G^{CDEF}\Pi_{\alpha\beta}\mathscr{D}_C\mathscr{D}_D X^\alpha \mathscr{D}_E\mathscr{D}_F X^\beta h^{AB}\epsilon_{\mu B} + 2\Lambda N\sqrt{h}(L_i^{AB}L^i - L_i^{AC}L_C{}^{Bi})\epsilon_{\mu B} \\
+ \quad & \Lambda\sqrt{h}N^A G^{BCDE}L^i_{BC}k_{DE}\, n_{\mu i} + \mathscr{D}_B(\Lambda N\sqrt{h}G^{ABCD}\Pi_{\mu\nu}\mathscr{D}_C\mathscr{D}_D X^\nu) \\
- \quad & \Lambda\sqrt{h}h^{BC}\delta_{ij}\left(\widetilde{\mathscr{D}}_B n^i\cdot\widetilde{\mathscr{D}}_C n^j\right)N^A\xi_\mu - \Lambda N\sqrt{h}h^{AB}h^{CD}\delta_{ij}\left(\widetilde{\mathscr{D}}_C n^i\cdot\widetilde{\mathscr{D}}_D n^j\right)\epsilon_{\mu B} \\
+ \quad & 2\Lambda N\sqrt{h}h^{AB}h^{CD}\delta_{ij}\left(\widetilde{\mathscr{D}}_B n^i\cdot\widetilde{\mathscr{D}}_C n^j\right)\epsilon_{\mu D} - \left[\widetilde{\mathscr{D}}_C\left(2\Lambda N\sqrt{h}h^{CD}\delta_{ij}\widetilde{\mathscr{D}}_D n^j\right)\cdot\epsilon_B h^{AB}\right]n_\mu{}^i \\
- \quad & \left[\widetilde{\mathscr{D}}_C\left(2\Lambda N\sqrt{h}h^{BC}\delta_{ij}\widetilde{\mathscr{D}}_B n^j\right)\cdot\xi\frac{N^A}{N}\right]n_\mu{}^i. \qquad \text{(F.39)}
\end{aligned}
$$

APPENDIX G

CONSTRAINT ALGEBRA

Primary-primary constraints

$$
\begin{aligned}
&\{\mathscr{S}_\lambda, \mathscr{S}_{\lambda'}\} = 0, && && \{\mathscr{V}_{\vec{\lambda}}, \mathscr{V}_{\vec{\lambda}'}\} = 0,\\
&\{\mathscr{S}_\lambda, \mathscr{V}_{\vec{\lambda}}\} = \mathscr{V}_{\vec{\lambda}'} && \lambda'^A = \lambda\lambda^A, && \{\mathscr{V}_{\vec{\lambda}}, \mathscr{W}_{\vec{\phi}}\} = 0,\\
&\{\mathscr{S}_\lambda, \mathscr{W}_{\vec{\phi}}\} = \mathscr{W}_{\vec{\phi}'} && \phi'^i = \lambda\phi^i, && \{\mathscr{W}_{\vec{\phi}}, \mathscr{W}_{\vec{\phi}'}\} = 0,
\end{aligned}
\tag{G.1}
$$

Primary-secondary constraints

$$
\begin{aligned}
&\{\mathscr{S}_\lambda, S_\Lambda\} = -\mathscr{S}_{\lambda_1} - S_{\Lambda_1}, && \{\mathscr{W}_{\vec{\phi}}, S_\Lambda\} = \mathscr{S}_{\lambda_2} + \mathscr{V}_{\vec{\lambda}_2} - \mathscr{W}_{\vec{\phi}_1} - W_{\vec{\Phi}_1},\\
&\{\mathscr{S}_\lambda, V_{\vec{\Lambda}}\} = -\mathscr{S}_{\mathscr{L}_{\vec{\Lambda}}\lambda}, && \{\mathscr{W}_{\vec{\phi}}, V_{\vec{\Lambda}}\} = \mathscr{S}_{\lambda_3} - \mathscr{V}_{\vec{\lambda}_3} - \mathscr{W}_{\vec{\phi}_2},\\
&\{\mathscr{S}_\lambda, W_{\vec{\Phi}}\} = \mathscr{A}, && \{\mathscr{W}_{\vec{\phi}}, W_{\vec{\Phi}}\} = \mathscr{W}_{\vec{\phi}_3} + \mathscr{C},\\
&\{\mathscr{V}_{\vec{\lambda}}, S_\Lambda\} = \mathscr{S}_{\mathscr{L}_{\vec{\lambda}}\Lambda} - \mathscr{V}_{\vec{\lambda}_1} - V_{\vec{\Lambda}_1}, &&\\
&\{\mathscr{V}_{\vec{\lambda}}, V_{\vec{\Lambda}}\} = \mathscr{V}_{[\vec{\lambda},\vec{\Lambda}]}, &&\\
&\{\mathscr{V}_{\vec{\lambda}}, W_{\vec{\Phi}}\} = \mathscr{B}, &&
\end{aligned}
\tag{G.2}
$$

where

$$\lambda_1 = 2\Lambda\mathscr{L}_{\vec{N}}\lambda, \qquad \phi_1^i = \Lambda\phi^i Nk - \phi^i N^A \mathscr{D}_A\Lambda + \Lambda N^A \widetilde{\mathscr{D}}_A \phi^i, \tag{G.3}$$

$$\Lambda_1 = \lambda\Lambda, \qquad \Phi_1^i = \Lambda\phi^i, \tag{G.4}$$

$$\lambda_1^A = 2\Lambda N^B \mathscr{D}_B \lambda^A, \qquad \lambda_3 = \tfrac{\Lambda^A}{N}\phi_i L_A{}^i, \tag{G.5}$$

$$\Lambda_1^A = \Lambda\lambda^A, \qquad \lambda_3^A = \tfrac{N^A}{N}\phi_i \Lambda^B L_B{}^i + \phi_i \Lambda^B L_B{}^{A\,i}, \tag{G.6}$$

$$\lambda_2^A = \Lambda\phi_i\left(N^A L^i - N^B L_B{}^{A\,i} - \frac{N^A N^B}{N} L_B{}^i\right), \qquad \phi_2^i = \Lambda^A \widetilde{\mathscr{D}}_A \phi^i, \tag{G.7}$$

$$\lambda_2^A = \Lambda\phi_i\left(N^A L^i - N^B L_B{}^{A\,i} - \frac{N^A N^B}{N} L_B{}^i\right), \qquad \phi_3^i = \tfrac{N^A}{N}\phi^i \Phi_j L_A{}^i, \tag{G.8}$$

and

$$\text{(G.9)} \qquad \mathscr{A} = \int_\Sigma \lambda \widetilde{\mathscr{D}}_A \Phi^i \sqrt{h} h^{AB}(n_i \cdot \mathscr{D}_B \dot{X}),$$

$$\text{(G.10)} \qquad \mathscr{B} = -\int_\Sigma \sqrt{h} h^{AB} \lambda^C \widetilde{\mathscr{D}}_A \Phi_i L^i_{BC},$$

$$\text{(G.11)} \qquad \mathscr{C} = \int_\Sigma \left[\frac{\sqrt{h}}{N^2} \phi^i \Phi_i L_A{}^j (N^A L_j + N L^A{}_j) - \frac{1}{N} \phi^i \Phi_i (p \cdot \xi) + \frac{\sqrt{h}}{N} \phi^i \Phi_j (L_i^{AB} L^j_{AB} - L_i L^j) \right]$$

Secondary-secondary constraints

$$\text{(G.12)} \qquad \begin{aligned} &\{S_\Lambda, S_{\Lambda'}\} = \mathscr{S}_{\lambda_4} + \mathscr{W}_{\vec{\phi}_4}, && \{V_{\vec{\Lambda}}, V_{\vec{\Lambda}'}\} = V_{[\vec{\Lambda},\vec{\Lambda}']}, \\ &\{S_\Lambda, V_{\vec{\Lambda}}\} = \mathscr{V}_{\vec{\lambda}_4} - \mathscr{S}_{\mathscr{L}_{\vec{\Lambda}}\Lambda}, && \{V_{\vec{\Lambda}}, W_{\vec{\Phi}}\} = \mathscr{S}_{\lambda_5} + \mathscr{V}_{\vec{\lambda}_5} + \mathscr{W}_{\vec{\phi}_4} + W_{\vec{\Phi}_4} + \mathscr{E}, \\ &\{S_\Lambda, W_{\vec{\Phi}}\} = \mathscr{S}_{\lambda_6} + \mathscr{V}_{\vec{\lambda}_6} + \mathscr{D}, && \{W_{\vec{\Phi}}, W_{\vec{\Phi}'}\} = -\mathscr{W}_{\vec{\phi}_5} + \mathscr{F}, \end{aligned}$$

where

$$\text{(G.13)} \qquad \lambda_4 = (N^2 h^{AB} - N^A N^B)(\Lambda \mathscr{D}_A \mathscr{D}_B \Lambda' - \Lambda' \mathscr{D}_A \mathscr{D}_B \Lambda),$$

$$\text{(G.14)} \qquad \phi_4^i = 2N^3 h^{AB} L_A{}^i (\Lambda \mathscr{D}_B \Lambda' - \Lambda' \mathscr{D}_B \Lambda),$$

$$\text{(G.15)} \qquad \lambda_4^A = \Lambda (N^2 h^{BC} - N^B N^C)(\mathscr{D}_B \mathscr{D}_C \Lambda^A + R_{BDC}{}^A \Lambda^D),$$

$$\text{(G.16)} \qquad \lambda_5 = \frac{1}{N}\left[\Lambda^A N^B k_A{}^C L^i_{BC} \Phi_i - \mathscr{L}_{\vec{\Lambda}}(N^A L_A{}^i)\Phi_i - \Lambda^A L_A{}^i \mathscr{L}_{\vec{N}} \Phi_i\right],$$

$$\text{(G.17)} \qquad \lambda_5^C = \Lambda^A \widetilde{\mathscr{D}}_A (N^B L_B{}^{Ci}) \Phi_i + \Lambda^A \widetilde{\mathscr{D}}_A (N^B L_B{}^i) \frac{N^C}{N} \Phi_i$$

$$\text{(G.18)} \qquad + \; \Lambda^A \frac{N^B}{N} \left[N \widetilde{\mathscr{D}}_B \Phi_i L_A{}^{Ci} - k_A{}^D L^i_{BD} N^C \Phi_i + \frac{N^B}{N}(L_A{}^i \widetilde{\mathscr{D}}_B \Phi_i N^C \right.$$

$$\text{(G.19)} \qquad \left. - \; N k_A{}^C L_B{}^i \Phi_i) \right],$$

$$\text{(G.20)} \qquad \phi_4^i = 2\Lambda^{[A} N^{B]} \left(L_A{}^{Cj} L^i_{BC} - L_A{}^j L_B{}^i \right) \Phi_j,$$

$$\text{(G.21)} \qquad \Phi_4^i = \Lambda^A \widetilde{\mathscr{D}}_A \Phi^i,$$

$$\text{(G.22)} \qquad \phi_5^i = N h^{AB} K_B{}^j (\Phi_j \widetilde{\mathscr{D}}_A \Phi'^{\,i} - \Phi'_j \widetilde{\mathscr{D}}_A \Phi^i),$$

$$\text{(G.23)} \qquad \lambda_6^C = \Lambda \Phi_i \left[\left(N^2 h^{AB} + N^A N^B \right) \left(\widetilde{\mathscr{D}}_A L^{iC}{}_B - L^i{}_B k_A{}^C \right) + N N^C \widetilde{\mathscr{D}}_A L^{Ai} \right.$$

$$\text{(G.24)} \qquad \left. + \; 2N^A L_B{}^{Ci} \mathscr{D}_A N^B + 2\frac{N^A N^C}{N} L^i{}_B \mathscr{D}_A N^B + \frac{N^A N^B N^C}{N} \widetilde{\mathscr{D}}_A L^i{}_B \right]$$

$$\text{(G.25)} \qquad + \; \Lambda \widetilde{\mathscr{D}}_A \Phi_i \left[2\left(h^{AB} N^2 + N^A N^B \right) \left(L^{iC}{}_B + \frac{L^i{}_B N^C}{N} \right) \right],$$

$$\text{(G.26)} \qquad \lambda_6 = -\Lambda \Phi_i \left[\left(h^{AB} N^2 + N^A N^B \right) \frac{\widetilde{\mathscr{D}}_A L^i{}_B}{N} + 2N^A L^i{}_B \mathscr{D}_A N^B \right]$$

$$\text{(G.27)} \qquad + \; \Lambda \widetilde{\mathscr{D}}_A \Phi_i \left[-2 \left(h^{AB} N^2 + N^A N^B \right) \frac{L^i{}_B}{N} \right],$$

and

$$\mathscr{D} = \int_\Sigma \Big\{ \Lambda\Phi_i \Big[2NL_A{}^i \left(P\cdot\mathscr{D}^A\dot{X}\right) - 2N\left(N^B L^{A\,i} + NL^{AB\,i}\right)(P\cdot\mathscr{D}_A\mathscr{D}_B X) \tag{G.28}$$

$$- \Big[N^2\left(L^i{}_A L^{jA} - L^i{}_{AB}L^{jAB}\right) - NN^A\left(L^i L^j{}_A - L^i{}_B L^{jB}{}_A\right)\Big]\left(P\cdot n_j\right) \tag{G.29}$$

$$+\sqrt{h}\Big[\frac{N}{2}L^iL^jL_j + \frac{N}{2}L^iL^{jCD}L_{jCD} - NL^jL_j{}^{AB}L^i{}_{AB} - NL_j{}^AL^i{}_{AD}L^{jD} \tag{G.30}$$

$$-NL_j^{AC}L_C{}^{B\,j}L^i_{AB} + 4NL_{jA}L^{(i}{}_BL^{j)AB} + 2N^CL_j{}^AL^{[i}{}_AL^{j]}{}_C - 2NL^iL_j{}^AL^j{}_A\Big]\Big] \tag{G.31}$$

$$+\Lambda\widetilde{\mathscr{D}}_A\Phi_i\Big[N^A\left(p\cdot n^i\right) - \left(P\cdot n^i\right)\left(N^A\mathscr{D}_CN^C + N\mathscr{D}^AN\right) - \widetilde{\mathscr{D}}_B\left(P\cdot n^i\right)\left(h^{AB}N^2 + N^AN^B\right) \tag{G.32}$$

$$+\sqrt{h}N\left(\left(k^{AB} - h^{AB}k\right)L^i{}_B + \widetilde{\mathscr{D}}_BL^{iAB}\right) + 2\sqrt{h}N^{[B}\widetilde{\mathscr{D}}_BL^{|i|A]} - \sqrt{h}N^C\widetilde{\mathscr{D}}^AL^i{}_C\Big] \tag{G.33}$$

$$+\Lambda\widetilde{\mathscr{D}}_A\widetilde{\mathscr{D}}_B\Phi_i\left(2\sqrt{h}L^{i[B}N^{A]}\right)\Big\}, \tag{G.34}$$

$$\begin{aligned}
\mathscr{E} &= \int_\Sigma\sqrt{h}\Big\{\frac{2}{N}\Lambda^{[A}N^{B]}(L_A{}^{C\,i}L^j_{BC} - L_A{}^iL_B{}^j)L_j\Phi_i - 2\Lambda^CL_C{}^{[i}L_A{}^{j]}L^A{}_j\Phi_i \\
&+ 2h^{A[B}h^{D]C}\mathscr{D}_B\Lambda_CL_D{}^i\widetilde{\mathscr{D}}_A\Phi_i + \Lambda^D(L^i_{CD}k^{CA} - k_{DC}L^{CA\,i})\widetilde{\mathscr{D}}_A\Phi_i \\
&+ 2\Lambda^DL^A{}_jh^{BC}L^{[i}_{BD}L^{j]}_{AC}\Phi_i + 2h^{A[B}h^{D]C}\Lambda_CL_D{}^i\widetilde{\mathscr{D}}_A\widetilde{\mathscr{D}}_B\Phi_i\Big\},
\end{aligned} \tag{G.35}$$

$$\begin{aligned}
\mathscr{F} &= \int_\Sigma\Big\{\Big[-h^{AB}(p\cdot\partial_AX) + N^AL^l_{AC}h^{CB}(P\cdot n_l) + \sqrt{h}L_A{}^lL^{AB}_l\Big]\times \\
&\quad\delta_{ij}\left(\Phi^i\widetilde{\mathscr{D}}_B\Phi'^j - \Phi'^i\widetilde{\mathscr{D}}_B\Phi^j\right) \\
&- 2\sqrt{h}L_{A(i}\left(h^{AB}L_{j)} - L^{AB}_{j)}\right)\left(\Phi^i\widetilde{\mathscr{D}}_B\Phi'^j - \Phi'^i\widetilde{\mathscr{D}}_B\phi^j\right)\Big\}.
\end{aligned} \tag{G.36}$$

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